recall : $\operatorname{card}(P(X)) = 2^{\operatorname{card}(X)} > \operatorname{card}(X)$

ex $X = \mathbb{N}$

define $\varphi : P(\mathbb{N}) \to [0,1]$

$$A \mapsto \sum_{k=1}^{\infty} a_k 2^{k}, \quad a_k = \begin{cases} 1, & k \in A \\ 0, & k \notin A \end{cases}$$

φ surj

so $\operatorname{card}(P(\mathbb{N})) \geq \operatorname{card}([0,1]) = \operatorname{card}(\mathbb{R})$

Want : "measure" of how large a subset is

ex $X = \{1, \dots, n\}$

$\mu : P(X) \longrightarrow [0, \infty)$

$$A \mapsto \sum_{x \in A} \mu(\{x\})$$

can do the same thing for ctbl set and specify $\mu(\{x\})$ for each $x \in X$

But what if $X = \mathbb{R}$

want $\mu : \underbrace{P(\mathbb{R})}_{\text{large}} \to [0, \infty)$

We want μ to satisfy

(i) $\mu(\emptyset) = 0$

(ii) $\mu((a,b)) = b - a$ (and reasonably $\mu(A+t) = \mu(A)$ $\forall A, t$)

(iii) $\mu\left(\bigsqcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$

Thm such μ does not exist

(Pf 395 IBL covered ; Folland 1.17, by axiom of choice)

remedy define μ not on $P(\mathbb{R})$, but on a suitable subset of $P(\mathbb{R})$

σ-Algebra ("σ": means ctbl sum/union)

Def Suppose X a set, $S \subseteq P(X)$, call S a σ-algebra on X if:

(1) $\emptyset \in S$ (closed under complement)

(2) if $E \in S$ then $X \setminus E \in S$

(3) if $E_1, E_2, \dots$ in S then $\bigcup_{k=1}^{\infty} E_k \in S$ (closed under ctbl union)

(σ-algebra 即 closed under complement 和 ctbl union 的子集簇)

ex X any set

$S = \{\emptyset, X\}$ is a σ-algebra on X (trivial)

$S = P(X)$ is a σ-algebra on X (power)

Thm σ-algebra also closed under difference & symmetric diff & ctbl intersection

Let S be a σ-algebra on set X

$\Rightarrow$ (a) $X \in S$

(b) if $D, E \in S \Rightarrow D \cup E,\ D \cap E,\ D \setminus E \in S$

(c) if $D, E \in S \Rightarrow D \Delta E \in S$ $= (E \setminus F) \cup (F \setminus E)$

(d) if $A_1, A_2, \dots \in S \Rightarrow \bigcap_{i=1}^{\infty} A_i \in S$

Pf (a) $\emptyset \in S \Rightarrow X \setminus \emptyset = X \in S$

(b) $D \cup E = D \cup E \cup \emptyset \cup \dots$ (seq of $\emptyset$)

$X \setminus (D \cap E) = \underbrace{(X \setminus D)}_{\in S} \cup \underbrace{(X \setminus E)}_{\in S} \in S$

$\Rightarrow D \cap E \in S$

$D \setminus E = D \cap (X \setminus E) \in S$

(d) $X \setminus \bigcap_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} \underbrace{(X \setminus E_n)}_{\in S} \in S$

$\Rightarrow \bigcap_{n=1}^{\infty} E_n \in S$ $\square$

Rmk 因而 σ-algebra :

(a) $\emptyset$, X 都在其中

(b) up to ctblly closed under 交并补

看起来有点像 topology, 但 σ-algebra 并不保证是 topology (没有任意并)

topology 也并不保证是 σ-algebra (没有 ctbl 交)

Lemma 任意 σ-algebra 的交仍是个 σ-algebra

if $\{A_i\}_{i \in I}$ is a family of σ-algebra on X

$\Rightarrow \bigcap_{i \in I} A_i$ is a σ-algebra on X

Pf 显然, 两个 σ-algebra 的交仍是 X 上的 σ-algebra

Corollary unique smallest σ-algebra containing a collection of subsets

Given $\mathcal{E} \subseteq P(X)$,

$$\boxed{\bigcap_{\substack{S \subseteq P(X) \\ S \supseteq \mathcal{E} \\ S \text{ is } \sigma\text{-algebra on } X}} S}$$ is a σ-algebra on X

Def $\langle \mathcal{E} \rangle$, the σ-algebra generated by $\mathcal{E}$