Math 597, Midterm Exam, February 26, 2025, 6:00–8:00 pm

Instructions

1. **Do not open this exam until you are told to do so**.
2. This examination booklet contains 6 problems on 7 sheets of paper including the front cover.
3. If needed, you may continue your solution on the back side of a page. If this is not enough, you may also use scratch paper, assuming everything is clearly marked.
4. This is a closed book exam. Electronic devices, calculators and note-cards are not allowed.

Name:

UMID:

Point distribution of the problems on the exam

Question:	1	2	3	4	5	6	Total
Points:	10	10	10	10	10	10	60

This page will not be graded

1. (10 points) Let $(X, \mathcal{A})$ be a measurable space, and $f\colon X \to \mathbb{R}$ a nonnegative function. Set

$$V(f) := \{(x, y) \mid 0 < y < f(x)\} \subset X \times \mathbb{R}.$$

   (a) Prove that if f is Borel measurable, then $V(f) \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$.

   (b) Prove that if $V(f) \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$, then f is Borel measurable.
   Hint: consider the rectangle $R_c := \{(x, y) \mid f(x) > c, y > 0\} \subset X \times \mathbb{R}$ for $c > 0$.

For (a), note that the function $(x, y) \mapsto y$ is Borel measurable by construction. Similarly, $(x, y) \mapsto x$ is measurable, and hence the composition $(x, y) \mapsto x \to f(x)$ is Borel measurable. It follows that the function $F\colon X \times \mathbb{R} \to \mathbb{R}$ given by $F(x, y) = y - f(x)$ is the difference of Borel measurable functions, and hence Borel measurable. Thus $V(f) = \{y > 0\} \cap \{F(x, y) > 0\}$ is measurable.

For (b) it suffices to show that the set $f^{-1}((c, \infty))$ is measurable for all $c \in \mathbb{R}$. This is trivial when $c < 0$, and $f^{-1}((0, \infty) = \bigcup_{n \geq 1} f^{-1}((\frac{1}{n}, \infty))$, so it suffices to consider $c > 0$.

Now, the set $f^{-1}((c, \infty))$ is measurable iff the rectangle R_c is measurable. Note that $(x, y) \in R_c$ iff $y > 0$ and there exists $n \geq 1$ such that $0 < \frac{1}{n}y + c < f(x)$. In other words, $(x, y) \in R_c$ iff $y > 0$ and $(x, \frac{1}{n}y + c) \in V(f)$ for some $n \geq 1$. Now, for any $n \geq 1$, the map $T\colon X \times Y \to X \times Y$ given by $T(x, y) = (x, \frac{1}{n}y + c)$ is measurable (the preimage of a rectangle is a rectangle), so since $V(f)$ is measurable, the set $\{(x, y) \mid (x, \frac{1}{n}y + c) \in V(f)\}$ is measurable. Thus R_c is a countable union of measurable sets, and hence measurable.

An quicker solution for (b), found by some of you, goes as follows. Suppose $V(f) \in \mathcal{A} \otimes \mathcal{B}_{\mathbb{R}}$. Then $V(f)^y \in \mathcal{A}$ for all $y \in \mathbb{R}$. But if $y > 0$, then $V(f)^y = \{x \in X \mid f(x) > y\} = f^{-1}(y, \infty))$. Thus $f^{-1}((y, \infty)) \in \mathcal{A}$ for all $y > 0$, which as noted above implies that f is measurable.

Extra writing area for Question 1

2. (10 points) Let $K \subset \mathbb{R}$ be a nonempty compact subset with the property that for any $x \in K$ and any open interval $I \subset \mathbb{R}$ containing x, the set $I \setminus \{x\}$ has nonempty intersection with both K and K^c.

   Prove that there exists a function $f : \mathbb{R} \to \mathbb{R}$ with the following properties:

   (a) the restriction of f to any open interval $I \subset \mathbb{R}$ fails to be Lebesgue measurable;

   (b) f is differentiable at every point $x \in K$, and $f'(x) = 1$.

   You may use without proof the existence of a set $V \subset \mathbb{R}$ such that, for any nonempty open interval $I \subset \mathbb{R}$, $V \cap I$ is not Lebesgue measurable.

   Set $h(x) = \inf\{|x - y| \mid y \in K\}$. Then h is continuous, and $h(x) \geq 0$ with equality iff $x \in K$. Set $g = \mathbf{1}_V h^2$. If I is a nonempty open interval, then I contains a nonempty interval J disjoint from K. If $g|_I$ is measurable, then $g|_J$ is measurable, which implies that $\mathbf{1}_V|_J = (g/h^2)|_J$ is measurable, a contradiction. On the other hand, if $x \in K$, then $g(y) \leq |y - x|$ for all y, so $|(h(x + t) - h(x))/t| \leq |t|$ for $t \neq 0$, which implies $h'(x) = 0$.

   Now the function $f(x) = g(x) + x$ will have the requested properties.

Extra writing area for Question 2

3. (10 points) Let $(X, \mathcal{A}, \mu)$ be a measure space with $\mu(X) < \infty$, and $f\colon X \to [0, \infty)$ a bounded measurable function. Prove that

$$\int f\, d\mu = \inf \left\{ \int \psi\, d\mu \mid \psi \text{ simple function such that } f(x) \le \psi(x) \text{ for all } x \in X \right\}. \qquad (\star)$$

Also give examples showing that $(\star)$ may fail if $\mu(X) = \infty$ or if f is (finite-valued but) unbounded.

Let R be the right-hand side of $(\star)$. By monotonicity, $\int f \le R$. By assumption, there exists $M \in [0, \infty)$ such that $0 \le f \le M$. Pick any $\epsilon > 0$. By definition of the integral, there exists a simple function ϕ such that $0 \le \phi \le M - f$ and $\int \phi + \epsilon \ge \int (M - f) = M\mu(X) - \int f$. If we set $\psi = M - \phi$, then ψ is a simple function, $0 \le \psi \le f$, and $\int \psi = M\mu(X) - \int \phi \le \int f + \epsilon$. Thus $R \le \int \phi + \epsilon$, so letting $\epsilon \to 0$ we get $\int f = R$.

For an example with f bounded but $\mu(X) = \infty$, take $X = \mathbb{R}$ (with Lebesgue measure) and $f(x) = \frac{1}{1+x^2}$. Then $\int f < \infty$, but since $f > 0$, any simple function $\psi \ge f$ must satisfy $\psi \ge c$ for some constant $c > 0$, and hence $\int \psi = \infty$; hence $R = \infty$.

For an example when $\mu(X) < \infty$ but f is unbounded, take $X = (0, 1)$ and $f(x) = \frac{1}{\sqrt{x}}$. Then $\int f < \infty$, but there is no simple function ψ satisfying $\psi \ge f$, so $R = \infty$.

Extra writing area for Question 3

4. (10 points) Let $(x_n)_1^\infty$ be any sequence of real numbers. Set

$$f_n(x) := \cos(n\pi(x - x_n))^{n^2}.$$

Prove that there exists a compact set $K \subset [0,1]$ with $m(K^c) \le 5^{-97}$, and a sequence $1 \le n_1 < n_2 < \dots$ such that $|f_{n_j}(x)| \le j^{-1360}$ for $x \in K$ and $j \ge 1$.

We claim that $f_n \to 0$ in measure. This will imply that a subsequence f_{n_j} tends to zero a.e. on $[0,1]$. Hence $f_{n_j} \to 0$ almost uniformly by Egoroff's theorem. Thus we can find a measurable set E such that $m(E^c) \le 5^{-98}$ and such that $f_{n_j} \to 0$ uniformly on E. By inner regularity, we can pick $K \subset E$ compact such that $m(K^c) \le 5^{-97}$. We still have $f_{n_j} \to 0$ uniformly on K, which leads to $|f_{n_j}| \le j^{-1360}$ on K, after passing to a further subsequence.

It remains to prove the claim, so fix $\epsilon > 0$, and set $A_n := \{|f_n| \ge \epsilon\}$. We must show that $\lim_n m(A_n) = 0$. Now A_n is the intersection of $[0,1]$ with the disjoint union of the intervals $(x_n + \frac{j}{n} - \delta_n, x_n + \frac{j}{n} + \delta_n)$, where $j \in \mathbb{Z}$ and $\delta_n \in (0, \frac{1}{2n})$ is the solution to $\cos(n\pi\delta_n) = \epsilon^{1/n^2}$. As $\epsilon^{1/n^2} \to 1$, we have $n\pi\delta_n \to 0$. Now there are at most $n+1$ intervals $I_{n,j}$ intersecting $(0,1)$ and each of them has Lebesgue measure $2\delta_n$, so $m(A_n) \le 2(n+1)\delta_n \to 0$.

Extra writing area for Question 4

5. (10 points) Let (x_n) be an enumeration of the rational numbers in $(0,1)$, and set

$$f_n(x) = \frac{\sin nx}{\sqrt{x}} \prod_{j=1}^{n} (x - x_j).$$

Compute $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

If $x \in (0,1)$, then $|x - x_j| < 1$ for all j, and $|x - x_j| \le \frac{1}{2}$ for infinitely many j. As $|\sin nx| \le 1$, it follows that $f_n(x) \to 0$ for all $x \in (0,1)$. Now $|f_n(x)| \le g(x)$, where $g(x) = \frac{1}{\sqrt{x}}$, and $\int_0^1 g(x)\,dx < \infty$, so by dominated convergence we have $\lim_n \int_0^1 f_n(x)\,dx = 0$.

Extra writing area for Question 5

6. (10 points) Let $f\colon [0,1] \to [0,1]$ be a Lebesgue measurable function. Prove that there exists a Lebesgue measurable set $E \subset [0,1]$ such that $m(E) \geq \frac{1}{597}$ and

$$\int_{[0,1]} |f(x) - y|^{-1/2}\, dx \leq 597$$

for all $y \in E$. *Hint*: integrate with respect to $y \in [0,1]$.

The function $F\colon [0,1]^2 \to [0,+\infty]$ defined by $F(x,y) = |f(x) - y|^{-1/2}$ is measurable. By Tonelli we have

$$\int_0^1 \int_0^1 |f(x) - y|^{-1/2}\, dx dy = \int F\, dm = \int_0^1 \int_0^1 |f(x) - y|^{-1/2}\, dy dx$$

If $b \in [0,1]$ we have

$$\int_0^1 |b - y|^{-1/2}\, dy = \int_0^b (b-y)^{-1/2}\, dy + \int_b^1 (y-b)^{-1/2}\, dy \leq 2 \int_0^1 y^{-1/2}\, dy = 4.$$

If we set $g(y) := \int_0^1 |f(x) - y|^{-1/2}\, dx$, we therefore get $\int_0^1 g(y)\, dy \leq 4$. By Chebyshev/Markov, the set $E := \{g \leq 597\}$ therefore satisfies $m(E^c) \leq \frac{4}{597}$, so $m(E) \geq 1 - \frac{4}{597} \geq \frac{1}{597}$.

Extra writing area for Question 6