Stat 4202: Mathematical Statistics II

Homework 1

Spring 2025

Question 1. *Use the result of Example 8.4 on page 253 to show that for random samples of size $n = 3$, the median is a biased estimator of the parameter θ of an exponential population.*

Question 2. *Show that $\frac{X+1}{n+2}$ is a biased estimator of the binomial parameter θ. Is this estimator asymptotically unbiased?*

Question 3. *Show that the sample proportion $\frac{X}{n}$ is a minimum variance unbiased estimator of the binomial parameter θ. (Hint: Treat $\frac{X}{n}$ as the mean of a random sample of size n from a Bernoulli population with the parameter θ.)*

Question 4. *If $\hat{\theta}_1$ and $\hat{\theta}_2$ are independent unbiased estimators of a given parameter θ and $var(\hat{\theta}_1) = 3 \cdot var(\hat{\theta}_2)$, find the constants a_1 and a_2 such that $a_1\hat{\theta}_1 + a_2\hat{\theta}_2$ is an unbiased estimator with minimum variance for such a linear combination.*

Question 5. *Show that the mean of a random sample of size n from an exponential population is a minimum variance unbiased estimator of the parameter θ.*

Question 6. *The information about θ in a random sample of size n is also given by*

$$-n \cdot E\left[\frac{\partial^2 \ln f(X)}{\partial \theta^2}\right]$$

where $f(x)$ is the value of the population density at x, provided that the extremes of the region for which $f(x) \neq 0$ do not depend on θ. The derivation of this formula takes the following steps:

(a) Differentiating the expressions on both sides of

$$\int f(x)\,dx = 1$$

with respect to θ, show that

$$\int \frac{\partial \ln f(x)}{\partial \theta} \cdot f(x)\,dx = 0$$

by interchanging the order of integration and differentiation.

(b) Differentiating again with respect to θ, show that

$$E\left[\left(\frac{\partial \ln f(X)}{\partial \theta}\right)^2\right] = -E\left[\frac{\partial^2 \ln f(X)}{\partial \theta^2}\right].$$

Question 7. *If $\bar{X}_1$ is the mean of a random sample of size n from a normal population with the mean μ and the variance σ_1^2, $\bar{X}_2$ is the mean of a random sample of size n from a normal population with the mean μ and the variance σ_2^2, and the two samples are independent, show that*

(a) $\omega \cdot \bar{X}_1 + (1-\omega) \cdot \bar{X}_2$*, where* $0 \leq \omega \leq 1$*, is an unbiased estimator of* μ*;*

(b) the variance of this estimator is a minimum when $\omega = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.

Question 8. *With reference to Exercise 21, find the efficiency of the estimator of part (a) with* $\omega = \frac{1}{2}$ *relative to this estimator with* $\omega = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$.