Math 4547: Real Analysis I

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1 January 6, 2025

Professor Margolis introduced the course and discussed the syllabus. The course will cover the following topics: Here are some common number systems

1.1 What is Analysis?

Analysis is the branch of mathematics that deals with the rigorous study of limits, functions, derivatives, integrals, and infinite series. It provides the foundation for calculus and extends its concepts to more abstract settings.

1.2 Analysis Study Tips

- Attend all lectures and take good notes.
- Read the textbook and work through the examples.
- Do the homework problems.
- Study with others.
- Ask questions.
- Practice, practice, practice.

Theorem 1
Every convergent squence is bounded.

1.3 The Real Numbers

1.3.1 What are the reals?

- The **natural Numbers** $\mathbb{N} = \{1, 2, 3, \ldots\}$

- The **integers** $\mathbb{Z}$ = {0, 1, -1, 2, -2, $\cdots$ }
- The **rational Numbers** $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}$ and $q \neq 0\}$
- The **real Numbers** $\mathbb{R}$
- The **complex Numbers** $\mathbb{C}$: $= \{\ a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$

Theorem 2
There is no rational number x, such that $x^2 = 2$.

Proof. We assume for contradiction that such an x exists. Then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. We can assume that p and q have no common factors. Then, $\frac{p^2}{q^2} = 2$, which implies

$$p^2 = 2q^2$$

Thus, p^2 is even. As the square of an odd number is odd, it follows p must even. Therefore, $p = 2k$ for an integer k. We have $2q^2 = p^2 = (2k)^2 = 4k^2$, and so $q^2 = 2k^2$. Thus, q^2 is even. Since p and q are both even, this contradicts our assumption that p and q have no common factors. Therefore, there is no rational number x such that $x^2 = 2$. $\square$

This theorem implies, if we visualize $\mathbb{Q}$ as points lying on a number line, there is a 'hole' where $\sqrt{2}$ is. (There are many more 'holes' e.g. π, e, $\sqrt{3}$, $\ldots$)

The key property that $\mathbb{R}$ possesses, but $\mathbb{Q}$ doesn't is that $\mathbb{R}$ has "no holes" (formally, $\mathbb{R}$ is complete.)

In this class, we will rigorously deduce all properties of $\mathbb{R}$ from the axioms of the real numbers.

The axioms are in three groups.

1. Field Axioms (addition and multiplication)
2. Order axioms (needed to describe properties concerning inequalities)
3. Completeness Axiom

1.4 Field axioms

1.4.1 Addition axioms

1. For every pair $a, b \in \mathbb{R}$, we can associate a real number $a + b$ called their **sum**.
2. For every real number a, there is a real number $-a$ called its **negative** or **additive inverse**.
3. There is a special real number 0 called zero or the additive identity such that for all a, b, c real numbers:
   (a) $a + b = b + a$
   (b) $a + (b + c) = (a + b) + c$
   (c) $a + 0 = a$
   (d) $a + (-a) = 0$

2 January 8, 2025