Math 4547: Real Analysis I

Lecturer: **Professor Alex Margolis**
Notes by: Farhan Sadeek

Spring 2025

1 January 6, 2025

Professor Margolis introduced the course and discussed the syllabus. The course will cover the following topics: Here are some common number systems

1.1 What is Analysis?

Analysis is the branch of mathematics that deals with the rigorous study of limits, functions, derivatives, integrals, and infinite series. It provides the foundation for calculus and extends its concepts to more abstract settings.

1.2 Analysis Study Tips

- Attend all lectures and take good notes.
- Read the textbook and work through the examples.
- Do the homework problems.
- Study with others.
- Ask questions.
- Practice, practice, practice.

Theorem 1
Every convergent squence is bounded.

1.3 The Real Numbers

1.3.1 What are the reals?

- The **natural Numbers** $\mathbb{N} = \{1, 2, 3, ...\}$

- The **integers** $\mathbb{Z} = \{0, 1, -1, 2, -2, \cdots\}$
- The **rational Numbers** $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
- The **real Numbers** $\mathbb{R}$
- The **complex Numbers** $\mathbb{C}$: $= \{\, a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$

Theorem 2
There is no rational number x, such that $x^2 = 2$.

Proof. We assume for contradiction that such an x exists. Then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. We can assume that p and q have no common factors. Then, $\frac{p^2}{q^2} = 2$, which implies

$$p^2 = 2q^2$$

Thus, p^2 is even. As the square of an odd number is odd, it follows p must even. Therefore, $p = 2k$ for an integer k. We have $2q^2 = p^2 = (2k)^2 = 4k^2$, and so $q^2 = 2k^2$. Thus, q^2 is even. Since p and q are both even, this contradicts our assumption that p and q have no common factors. Therefore, there is no rational number x such that $x^2 = 2$. $\square$

This theorem implies, if we visualize $\mathbb{Q}$ as points lying on a number line, there is a 'hole' where $\sqrt{2}$ is. (There are many more 'holes' e.g. π, e, $\sqrt{3}$, ...)

The key property that $\mathbb{R}$ possesses, but $\mathbb{Q}$ doesn't is that $\mathbb{R}$ has "no holes" (formally, $\mathbb{R}$ is complete.)

In this class, we will rigorously deduce all properties of $\mathbb{R}$ from the axioms of the real numbers.

The axioms are in three groups.

1. Field Axioms (addition and multiplication)
2. Order axioms (needed to describe properties concerning inequalities)
3. Completeness Axiom

1.4 Addition axioms

1. For every pair $a, b \in \mathbb{R}$, we can associate a real number $a + b$ called their **sum**.
2. For every real number a, there is a real number $-a$ called its **negative** or **additive inverse**.
3. There is a special real number 0 called zero or the additive identity such that for all $a, b, c, x, y, z \cdots$ are real numbers unless otherwise stated:
   (a) $a + b = b + a$
   (b) $a + (b + c) = (a + b) + c$
   (c) $a + 0 = a$
   (d) $a + (-a) = 0$

2 January 8, 2025

We will use the axioms to deduce all the properties of $\mathbb{R}$. From these axioms, we can derive many more properties of the real numbers.

Proposition 3
If $x + a$ for all $a \in \mathbb{R}$, then $a = 0$.

Proof. We know that

$$\begin{aligned} x &= x + 0 && \text{(A3)} \\ &= 0 + x && \text{(by assumption on } x\text{)} \end{aligned}$$

$\square$

Proposition 4
Left cancellation of addition If $a + x = a + y$, then $x = y$.

Proof. We start with the given equation $a + x = a + y$. By the additive identity property (A3), we have:

$$\begin{aligned} y &= y + 0 && \text{(A3)} \\ &= y + (a + (-a)) && \text{(A4)} \\ &= (y + a) + (-a) && \text{(A2)} \\ &= (a + y) + (-a) && \text{(A1)} \\ &= (a + x) + (-a) && \text{(given)} \\ &= x + (a + (-a)) && \text{(A1)} \\ &= x + 0 && \text{(A4)} \\ &= x && \text{(A3)} \end{aligned}$$

Therefore, $x = y$. $\square$

Proposition 5
$-(-a) = a$

Proof. We need to show that $-(-a) = a$. Consider the following:

$$\begin{aligned} (-a) + (-(-a)) &= 0 && \text{(by definition of additive inverse)} \\ (-a) + a &= 0 && \text{(since } -(-a) = a\text{)} \\ a + (-a) &= 0 && \text{(by commutativity of addition)} \\ (-a) + (-(-a)) &= a + (-a) && \text{(by substitution)} \\ (-(-a)) &= a && \text{(by left cancellation of addition)} \end{aligned}$$

Therefore, $-(-a) = a$. $\square$

Proposition 6

-(a + b) = (-a) + (-b)

Proof. We need to show that the additive inverse of $(a + b)$ is equal to the sum of the additive inverses of a and b. Consider the following:

$$\begin{aligned}
(a+b)+(-(a+b)) &= 0 \quad \text{(by definition of additive inverse)} \\
(a+b)+((-a)+(-b)) &= a+(b+((-a)+(-b))) \quad \text{(by associativity of addition)} \\
&= a+((b+(-a))+(-b)) \quad \text{(by associativity of addition)} \\
&= a+((-a)+(b+(-b))) \quad \text{(by commutativity of addition)} \\
&= a+((-a)+0) \quad \text{(by definition of additive inverse)} \\
&= a+(-a) \quad \text{(by identity property of addition)} \\
&= 0 \quad \text{(by definition of additive inverse)}
\end{aligned}$$

Therefore, $-(a + b) = (-a) + (-b)$. $\square$

Proposition 7

-0 = 0

Proof. We need to show that the additive inverse of 0 is 0. Consider the following:

$$\begin{aligned}
0+0 &= 0 \quad \text{(by the identity property of addition, A3)} \\
0+(-0) &= 0 \quad \text{(by the definition of additive inverse, A4)}
\end{aligned}$$

Therefore, we have:

$$0 + 0 = 0 + (-0)$$

By the left cancellation property of addition, it follows that:

$$0 = -0$$

Therefore, $-0 = 0$. $\square$

2.1 Multiplication Axioms

Definition 8

For all $a, b \in \mathbb{R}$, we can associate a real number $a \times b$ called their **product**.

Definition 9

For every $a \in \mathbb{R}$, there is some $a^{-1} \in \mathbb{R}$ called its **multiplicative inverse** or **reciprocal** such that for all $a \neq 0$, $a \times a^{-1} = 1$.

Definition 10

There is a number 1 called **one** or the **multiplicative identity** such that for all $a \in \mathbb{R}$, $a \times 1 = a$.

Definition 11

For all $a, b, c \in \mathbb{R}$, we have the following properties of multiplication:

- For all $a, b \in \mathbb{R}$, $a \times b = b \times a$.
- For all $a, b, c \in \mathbb{R}$, $a \times (b \times c) = (a \times b) \times c$.
- For all $a \in \mathbb{R}$, $a \times 0 = 0$.
- For all $a, b \in \mathbb{R}$, $a \times (b + c) = a \times b + a \times c$.

Proposition 12

If $a \times a \times b = a$, and $a \in \mathbb{R}$ then $b = 1$.

Proposition 13

If $a \neq 0$ and $a \times b = a \times c$, then $b = c$.

Proposition 14

If $a \neq 0$ and $a^{-1} \neq 0$ and $(a^{-1})^{-1} = a$.

Proposition 15

If $a \neq 0$, $b \neq 0$, and $a \times b \neq 0$, then $(a \times b)^{-1} = a^{-1} \times b^{-1}$.

Proposition 16

If $a, b, c \in \mathbb{R}$, then $(a + b) \times c = (a \times b) + (a \times c)$

Proof.

$$
\begin{aligned}
(a + b) \times c &= c \times (a + b) \\
&= c \times a + c \times b \\
&= a \times c + b \times c
\end{aligned}
$$

□

Proposition 17

For all, $a \in \mathbb{R}$, $a \times 0 = 0$

Proof.

$$
\begin{aligned}
a \times 0 &= a \times 0 + 0 \\
&= a \times (0 + 0) \\
&= a \times 0 + a \times 0 \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

□

Proposition 18

If $a \times b = 0$, then either $a = 0$ or $b = 0$ or both.

Proposition 19

$a \times (-b) = (a \times b)$. In particular, $a \times (-1) = - - a$

Proof.

$$
\begin{aligned}
a \times (-b) + a \times b &= a \times (b + (-b)) \\
&= a \times 0 \\
&= 0 \\
&= a \times b + (-(a \times b))
\end{aligned}
$$

Hence, additive inverse of $a \times b = -(a \times b)$. □

Proposition 20

$(-1) = -1$ and $(-1) \times (-1) = 1$

Proof.

$$\begin{aligned}(-1) \times (-1) &= -(-1) \times 1 \\ &= -(-1) \times (1 + 0) \\ &= -(-1) \times (1 + (-1)) \\ &= -(-1) \times 0 \\ &= 0 \\ &= (-1) + (-1)\end{aligned}$$

□