

# Math 4547: Real Analysis I

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## 1 January 6, 2025

Professor Margolis introduced the course and discussed the syllabus. The course will cover the following topics:  
Here are some common number systems

### 1.1 What is Analysis?

Analysis is the branch of mathematics that deals with the rigorous study of limits, functions, derivatives, integrals, and infinite series. It provides the foundation for calculus and extends its concepts to more abstract settings.

#### Theorem 1

Every convergent sequence is bounded.

### 1.2 The Real Numbers

#### 1.2.1 What are the reals?

- The **natural Numbers**  $\mathbb{N} = \{1, 2, 3, \dots\}$
- The **integers**  $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$
- The **rational Numbers**  $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
- The **real Numbers**  $\mathbb{R}$
- The **complex Numbers**  $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ , where  $i^2 = -1$

#### Theorem 2

There is no rational number  $x$ , such that  $x^2 = 2$ .

*Proof.* We assume for contradiction that such an  $x$  exists. Then  $x = \frac{p}{q}$  for some  $p, q \in \mathbb{Z}$  and  $q \neq 0$ . We can assume that  $p$  and  $q$  have no common factors. Then,  $\frac{p^2}{q^2} = 2$ , which implies

$$p^2 = 2q^2$$

Thus,  $p^2$  is even. As the square of an odd number is odd, it follows  $p$  must be even. Therefore,  $p = 2k$  for an integer  $k$ . We have  $2q^2 = p^2 = (2k)^2 = 4k^2$ , and so  $q^2 = 2k^2$ . Thus,  $q^2$  is even. Since  $p$  and  $q$  are both even, this contradicts our assumption that  $p$  and  $q$  have no common factors. Therefore, there is no rational number  $x$  such that  $x^2 = 2$ .  $\square$

This theorem implies, if we visualize  $\mathbb{Q}$  as points lying on a number line, there is a ‘hole’ where  $\sqrt{2}$  is. (There are many more ‘holes’ e.g.  $\pi$ ,  $e$ ,  $\sqrt{3}$ , ...)

The key property that  $\mathbb{R}$  possesses, but  $\mathbb{Q}$  doesn’t is that  $\mathbb{R}$  has “no holes” (formally,  $\mathbb{R}$  is complete.)

In this class, we will rigorously deduce all properties of  $\mathbb{R}$  from the axioms of the real numbers.

The axioms are in three groups.

1. Field Axioms (addition and multiplication)
2. Order axioms (needed to describe properties concerning inequalities)
3. Completeness Axiom

## 1.3 Addition axioms

1. For every pair  $a, b \in \mathbb{R}$ , we can associate a real number  $a + b$  called their **sum**.
2. For every real number  $a$ , there is a real number  $-a$  called its **negative** or **additive inverse**.
3. There is a special real number  $0$  called zero or the additive identity such that for all  $a, b, c, x, y, z \dots$  are real numbers unless otherwise stated:

(a)  $a + b = b + a$

(b)  $a + (b + c) = (a + b) + c$

(c)  $a + 0 = a$

(d)  $a + (-a) = 0$

## 2 January 8, 2025

In this lecture, we will use the axioms to deduce various properties of the real numbers  $\mathbb{R}$ . From these axioms, we can derive many more properties of the real numbers.

### Proposition 3

If  $x + a = x$  for all  $a \in \mathbb{R}$ , then  $a = 0$ .

*Proof.* We know that

$$\begin{aligned}x &= x + 0 && \text{(A3)} \\&= x + a && \text{(by assumption on } a\text{)}\end{aligned}$$

By the left cancellation property of addition, it follows that  $a = 0$ . □

**Proposition 4** (Left cancellation of addition)

If  $a + x = a + y$ , then  $x = y$ .

*Proof.* We start with the given equation  $a + x = a + y$ . By the additive identity property (A3), we have:

$$\begin{aligned}y &= y + 0 && \text{(A3)} \\&= y + (a + (-a)) && \text{(A4)} \\&= (y + a) + (-a) && \text{(A2)} \\&= (a + y) + (-a) && \text{(A1)} \\&= (a + x) + (-a) && \text{(given)} \\&= x + (a + (-a)) && \text{(A1)} \\&= x + 0 && \text{(A4)} \\&= x && \text{(A3)}\end{aligned}$$

Therefore,  $x = y$ . □

**Proposition 5**

$$-(-a) = a$$

*Proof.* We need to show that  $-(-a) = a$ . Consider the following:

$$\begin{aligned}(-a) + (-(-a)) &= 0 && \text{(by definition of additive inverse)} \\(-a) + a &= 0 && \text{(since } -(-a) = a\text{)} \\a + (-a) &= 0 && \text{(by commutativity of addition)} \\(-a) + (-(-a)) &= a + (-a) && \text{(by substitution)} \\(-(-a)) &= a && \text{(by left cancellation of addition)}\end{aligned}$$

Therefore,  $-(-a) = a$ . □

**Proposition 6**

$$-(a + b) = (-a) + (-b)$$

*Proof.* We need to show that the additive inverse of  $(a + b)$  is equal to the sum of the additive inverses of  $a$  and  $b$ . Consider the following:

$$\begin{aligned}
 (a + b) + (-(a + b)) &= 0 \quad (\text{by definition of additive inverse}) \\
 (a + b) + ((-a) + (-b)) &= a + (b + ((-a) + (-b))) \quad (\text{by associativity of addition}) \\
 &= a + ((b + (-a)) + (-b)) \quad (\text{by associativity of addition}) \\
 &= a + ((-a) + (b + (-b))) \quad (\text{by commutativity of addition}) \\
 &= a + ((-a) + 0) \quad (\text{by definition of additive inverse}) \\
 &= a + (-a) \quad (\text{by identity property of addition}) \\
 &= 0 \quad (\text{by definition of additive inverse})
 \end{aligned}$$

Therefore,  $-(a + b) = (-a) + (-b)$ . □

### Proposition 7

$$-0 = 0$$

*Proof.* We need to show that the additive inverse of 0 is 0. Consider the following:

$$\begin{aligned}
 0 + 0 &= 0 \quad (\text{by the identity property of addition, A3}) \\
 0 + (-0) &= 0 \quad (\text{by the definition of additive inverse, A4})
 \end{aligned}$$

Therefore, we have:

$$0 + 0 = 0 + (-0)$$

By the left cancellation property of addition, it follows that:

$$0 = -0$$

Therefore,  $-0 = 0$ . □

## 2.1 Multiplication Axioms

### Definition 8

For all  $a, b \in \mathbb{R}$ , we can associate a real number  $a \times b$  called their **product**.

### Definition 9

For every  $a \in \mathbb{R}$ , there is some  $a^{-1} \in \mathbb{R}$  called its **multiplicative inverse** or **reciprocal** such that for all  $a \neq 0$ ,  $a \times a^{-1} = 1$ .

**Definition 10**

There is a number 1 called **one** or the **multiplicative identity** such that for all  $a \in \mathbb{R}$ ,  $a \times 1 = a$ .

**Definition 11**

For all  $a, b, c \in \mathbb{R}$ , we have the following properties of multiplication:

- For all  $a, b \in \mathbb{R}$ ,  $a \times b = b \times a$ .
- For all  $a, b, c \in \mathbb{R}$ ,  $a \times (b \times c) = (a \times b) \times c$ .
- For all  $a \in \mathbb{R}$ ,  $a \times 0 = 0$ .
- For all  $a, b \in \mathbb{R}$ ,  $a \times (b + c) = a \times b + a \times c$ .

**Proposition 12**

If  $a \times b = a$ , and  $a \in \mathbb{R}$  then  $b = 1$ .

*Proof.* We start with the given equation  $a \times b = a$ . By the multiplicative identity property, we have:

$$\begin{aligned} a \times b &= a \times 1 \\ b &= 1 \quad (\text{by left cancellation of multiplication}) \end{aligned}$$

Therefore,  $b = 1$ . □

**Proposition 13**

If  $a \neq 0$  and  $a \times b = a \times c$ , then  $b = c$ .

*Proof.* We start with the given equation  $a \times b = a \times c$ . By the multiplicative inverse property, we have:

$$\begin{aligned} a^{-1} \times (a \times b) &= a^{-1} \times (a \times c) \\ (a^{-1} \times a) \times b &= (a^{-1} \times a) \times c \\ 1 \times b &= 1 \times c \\ b &= c \end{aligned}$$

Therefore,  $b = c$ . □

**Proposition 14**

If  $a \neq 0$  and  $a^{-1} \neq 0$ , then  $(a^{-1})^{-1} = a$ .

*Proof.* We need to show that the multiplicative inverse of  $a^{-1}$  is  $a$ . Consider the following:

$$\begin{aligned}a^{-1} \times a &= 1 \quad (\text{by definition of multiplicative inverse}) \\(a^{-1})^{-1} \times a^{-1} &= 1 \quad (\text{by definition of multiplicative inverse}) \\(a^{-1})^{-1} &= a\end{aligned}$$

Therefore,  $(a^{-1})^{-1} = a$ . □

**Proposition 15**

If  $a \neq 0$ ,  $b \neq 0$ , and  $a \times b \neq 0$ , then  $(a \times b)^{-1} = a^{-1} \times b^{-1}$ .

*Proof.* We need to show that the multiplicative inverse of  $a \times b$  is  $a^{-1} \times b^{-1}$ . Consider the following:

$$\begin{aligned}(a \times b) \times (a^{-1} \times b^{-1}) &= a \times (b \times (a^{-1} \times b^{-1})) \\&= a \times ((b \times a^{-1}) \times b^{-1}) \\&= a \times (1 \times b^{-1}) \\&= a \times b^{-1} \\&= 1\end{aligned}$$

Therefore,  $(a \times b)^{-1} = a^{-1} \times b^{-1}$ . □

**Proposition 16**

If  $a, b, c \in \mathbb{R}$ , then  $(a + b) \times c = (a \times c) + (b \times c)$ .

*Proof.* We need to show that the product of  $(a + b)$  and  $c$  is equal to the sum of the products of  $a$  and  $c$ , and  $b$  and  $c$ . Consider the following:

$$\begin{aligned}(a + b) \times c &= c \times (a + b) \\&= c \times a + c \times b \\&= a \times c + b \times c\end{aligned}$$

Therefore,  $(a + b) \times c = (a \times c) + (b \times c)$ . □

**Proposition 17**

For all  $a \in \mathbb{R}$ ,  $a \times 0 = 0$ .

*Proof.* We need to show that the product of any real number  $a$  and 0 is 0. Consider the following:

$$\begin{aligned}a \times 0 &= a \times (0 + 0) \\&= a \times 0 + a \times 0 \\&= 0 + 0 \\&= 0\end{aligned}$$

Therefore,  $a \times 0 = 0$ . □

**Proposition 18**

If  $a \times b = 0$ , then either  $a = 0$  or  $b = 0$  or both.

*Proof.* We need to show that if the product of  $a$  and  $b$  is 0, then either  $a$  or  $b$  or both must be 0. Consider the following:

$$a \times b = 0$$

If  $a \neq 0$ , then  $b = 0$  by the multiplicative inverse property. If  $b \neq 0$ , then  $a = 0$  by the multiplicative inverse property. Therefore, if  $a \times b = 0$ , then either  $a = 0$  or  $b = 0$  or both. □

**Proposition 19**

$a \times (-b) = (-a) \times b$ . In particular,  $a \times (-1) = -a$ .

*Proof.* We need to show that the product of  $a$  and  $-b$  is equal to the product of  $-a$  and  $b$ . Consider the following:

$$\begin{aligned}a \times (-b) + a \times b &= a \times (b + (-b)) \\&= a \times 0 \\&= 0 \\&= a \times b + (-(a \times b))\end{aligned}$$

Hence, the additive inverse of  $a \times b$  is  $-(a \times b)$ . Therefore,  $a \times (-b) = (-a) \times b$ . □

**Proposition 20**

$(-1) \times (-1) = 1$

*Proof.* We need to show that the product of  $-1$  and  $-1$  is  $1$ . Consider the following:

$$\begin{aligned}
 (-1) \times (-1) &= -(-1) \times 1 \\
 &= -(-1) \times (1 + 0) \\
 &= -(-1) \times (1 + (-1)) \\
 &= -(-1) \times 0 \\
 &= 0 \\
 &= (-1) + (-1)
 \end{aligned}$$

Therefore,  $(-1) \times (-1) = 1$ . □

### 3 January 10, 2025

For all  $a, b \in \mathbb{R}$ , we write:

- $ab$  or  $a \cdot b$  for  $a \times b$ .
- $a - b$  for  $a + (-b)$ .
- $\frac{1}{a}$  for  $a^{-1}$  if  $a \neq 0$ .
- $\frac{a}{b}$  for  $ab^{-1}$  if  $b \neq 0$ .

For  $a \neq 0$ , we write:

- $a^0$  for  $1$ .
- $a^{k+1}$  for  $a^k \cdot a$  for  $k = 0, 1, 2, \dots$
- $a^{-1}$  or  $(a^l)^{-1}$  for  $l = 1, 2, 3$

#### Definition 21

Any set equipped with operations  $+$  and  $\times$  satisfying A1 - A4, M1 - M4, Z, D is a **field**.

#### Fact 22

Some facts about the fields:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$  are all fields.
- $\mathbb{Z}$  is not a field (M4 isn't satisfied).
- $\mathbb{N}$  is not a field (A4, M4) are not satisfied.
- $\frac{\mathbb{Z}}{p\mathbb{Z}}$  (integers mod  $p$  for prime  $p$ ) is a field.



### 3.1 The order axioms

The order axioms are: There is a subset of  $P \subset \mathbb{R}$  called the set of **positive numbers**.

• If  $a, b \in \mathbb{P}$ , then  $a + b \in \mathbb{P}$ . (P1)

• If  $a, b \in \mathbb{P}$ , then  $a \times b \in \mathbb{P}$ . (P2)

• For each  $a \in \mathbb{R}$ , exactly one of the following is true:  $a \in \mathbb{P}$ ,  $a = 0$ , or  $-a \in \mathbb{P}$ .  $\leftarrow$  Law of Trichotomy (P3)

**P3** is the most powerful axiom about the positive numbers.

#### Proposition 23

Prove that  $1 \in \mathbb{P}$

*Proof.* According to **P3**, either

- $1 \in \mathbb{P}$
- $1 = 0$
- $-1 \in \mathbb{P}$

We will prove (b) and (c) are false by contradiction and then show that  $1 \in \mathbb{P}$ . If **(b)** holds,  $1 = 0$ , which contradicts **Z**. Assume for contradiction **(c)** holds. We know from last lecture that  $1 = -(-1)$ . Since  $-1 \in \mathbb{P}$ , by (P2),  $(-1) \times (-1) \in \mathbb{P}$ . But  $(-1) \times (-1) = 1$ , so  $1 \in \mathbb{P}$ .  $\therefore$ ,  $1 \in \mathbb{P}$  and  $-1 \in \mathbb{P}$  contradicts **P3**. Since, **(b)** and **(c)** cannot hold, therefore, **(a)** must hold.  $\square$

#### Fact 24

For all  $a, b \in \mathbb{R}$ , we write

- $a < b$  if  $b - a \in \mathbb{P}$
- $a > b$  if  $a - b \in \mathbb{P}$
- $a \leq b$  if  $b - a \in \mathbb{P} \cup \{0\}$
- $a \geq b$  if  $a - b \in \mathbb{P} \cup \{0\}$

#### Proposition 25

$a > b$  if and only if  $-a < -b$ . In particular,  $x > 0 \iff -x < 0$

*Proof.*

$$\begin{aligned} a > b &\iff a - b \in \mathbb{P} \\ &\iff -(-a) - b \in \mathbb{P} \\ &\iff -b - (-a) \in \mathbb{P} \\ &\iff -a < -b \end{aligned}$$

□

**Proposition 26**

For all  $x, y, z \in \mathbb{R}$  the following holds:

- $x \leq x$
- If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .
- If  $x \leq y$  and  $y \leq z$ , then  $x \leq z$ .

*Proof.*

□

**Proposition 27**

If  $x, t, z \in \mathbb{R}$  and  $x < y$ , then  $x + z < y + z$ .

*Proof.* Since  $x < y$ , we have  $x - y \in \mathbb{P}$ . By the properties of addition (A1-A4), we know that:

$$(y + z) - (x + z) = y - x$$

Since  $y - x \in \mathbb{P}$ , it follows that:

$$(y + z) - (x + z) \in \mathbb{P}$$

Hence,  $x + z < y + z$ .

□

**Proposition 28**

If  $x, y, z \in \mathbb{R}$  and  $x < y$  and  $z > 0$ , then  $xz < yz$ .

*Proof.*  $zy = zx = z(y - x)$ . Now,  $z \in \mathbb{P}$  and  $y - x \in \mathbb{P}$ , therefore  $zy - zx \in \mathbb{P}$ . Therefore,  $xz < yz$ .

□

**Corollary 29**

If  $x, y, z \in \mathbb{R}$  and  $x < y$  and  $z < 0$ , then  $xz > yz$ .

*Proof.*

□

**Corollary 30**

For all,  $a \in \mathbb{R}$ ,  $a^2 \geq 0$ .

*Proof.* By P3, either  $a > 0$ ,  $a = 0$  or  $a < 0$ .

- If  $a > 0$ , then  $a^2 = a \times a > 0$ .
- If  $a = 0$ , then  $a^2 = 0 \geq 0$ .

(P2)

- If  $a < 0$ , then  $-a > 0$  and  $(-a)^2 = a^2 > 0$ .

□

### Proposition 31

If  $x \in \mathbb{P}$ , then  $x^{-1} \in \mathbb{P}$ .

*Proof.* Since,  $x \in \mathbb{P}$ ,  $x \neq 0$ . Therefore,  $x^{-1}$  exists. By P3,  $x^{-1} > 0$ ,  $x^{-1} = 0$ , or  $x^{-1} < 0$ . If  $x^{-1} = 0$ , then  $1 = x \times x^{-1} = x \times 0 = 0$  [Contradiction] Assume  $x^{-1} < 0$ . Then  $-x^{-1} \in \mathbb{P}$  by P3. Then  $x \times (-x^{-1}) \in \mathbb{P}$  by P2. But  $x \times (-x^{-1}) = -1$ , which contradicts P3 since  $-1 \notin \mathbb{P}$ . Therefore,  $x^{-1} \in \mathbb{P}$ . □

### Corollary 32

If  $x, y \in \mathbb{P}$ , and  $x < y$ , then  $\frac{1}{y} < \frac{1}{x}$ .

## 4 January 13, 2025

Homework 1 is due on January 21, 2025.

### Definition 33

We define  $\max: \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$  by

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b \geq a \end{cases}$$

### Definition 34

We define  $\max: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases}$$

### Definition 35

We define  $|x|: \mathbb{R} \rightarrow \mathbb{R}$  as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

### Proposition 36

For all  $x \in \mathbb{R}$ ,  $|-x| = |x|$ .

*Proof.* By P3,  $x > 0$ ,  $x = 0$ , or  $x < 0$ .

- Case 1: If  $x > 0$ , then  $|x| = x$  and  $|-x| = -(-x) = x$ . Thus,  $|x| = |-x|$ .
- Case 2: If  $x = 0$ , then  $|x| = 0$  and  $|-x| = -0 = 0$ . Thus,  $|x| = |-x|$ .
- Case 3: If  $x < 0$ , then  $|x| = -x$  and  $|-x| = -(-x) = x$ . Thus,  $|x| = |-x|$ .

□

### Theorem 37 (The Triangle Δ Inequality)

For all  $a, b, \in \mathbb{R}$

$$|a + b| \leq |a| + |b|$$

with equality if and only if either  $a \geq 0$  and  $b \geq 0$  or  $a \leq 0$  and  $b \leq 0$ .

*Proof.* By P3, one of the following 8 Cases must hold:

	a	b	a + b
1	$\geq 0$	$\geq 0$	Row 2, Col 4
2	$\geq 0$	$\geq 0$	Row 3, Col 4
3	$\geq 0$	$< 0$	Row 4, Col 4
4	$\geq 0$	$< 0$	Row 5, Col 4
5	$< 0$	Row 6, Col 3	Row 6, Col 4
6	$< 0$	Row 7, Col 3	Row 7, Col 4
7	$< 0$	Row 8, Col 3	Row 8, Col 4
8	$< 0$	Row 8, Col 3	Row 8, Col 4

Case 2 and 7 is not possible. But we will prove the rest of the cases:

$$(1) |a| = a, |b| = b, |a + b| = a + b, \therefore |a + b| = a + b = |a| + |b|$$

(3)

$$(4) |a| = a, |b| = -b, |a + b| = a + b$$

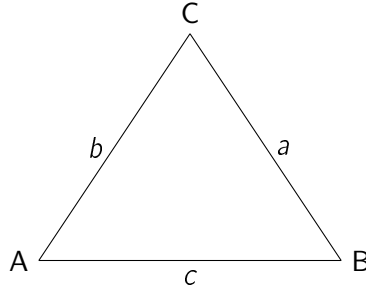
$$\begin{aligned}
 |a + b| &= -a - b = -0 - b = (-a \leq 0) \\
 &\leq a + 0 \quad (\text{since } b < 0) \\
 &= |a| + |b|
 \end{aligned}$$

(5) Follows the similarly by symmetry.

Finish this for exercise.

□

The picture of the Triangle identity



$$|a| = \|\vec{BC}\| \leq |b| + |c| = \|\vec{AC}\| + \|\vec{AB}\|$$

### Proposition 38

For all  $a, b, \in \mathbb{R}$

$$|ab| = |a| \cdot |b|$$

*Proof.* If  $a = 0$  or  $b = 0$ , then  $|ab| = |0| = 0 = |a| \cdot |b|$ . Let's assume that  $a \neq 0, b \neq 0$ .

- $a > 0, b > 0$  Then P2 implies  $ab > 0$  so,  $|ab| = ab = |a||b|$
- $a < 0, b > 0$ . Then  $ab < 0$ . Then  $|ab| = -ab = (-a)b = |a||b|$
- $a > 0, b < 0$ . This follows from case 2 by symmetry.
- $a < 0, b < 0$ . Then  $ab > 0$ . Hence,  $|ab| = ab = (-a)(-b) = |a||b|$

□

### Theorem 39 (Bernoulli's Inequality)

For all  $x \in \mathbb{R}$  with  $x > -1$  and  $n \in \mathbb{N}$ , if  $n \geq 1$ , then

$$(1 + x)^n \geq 1 + nx$$

*Proof.* We proceed by induction on  $n$ .

Base Case:  $n = 1 : (1 + x)^1 = 1 + x = 1 + 1 \cdot x$

Inductive Step: Assume

$$(1 + x)^N \geq 1 + Nx$$

We want to show  $(1 + x)^{N+1} \geq 1 + (N + 1) \cdot x$ . First, since  $x > -1$ ,  $x + 1 > 0$ .

Multiplying both sides by  $x + 1$ ,

$$\begin{aligned} (1 + x)(1 + x)^N &\geq (1 + Nx)(1 + x) \\ &= 1 + (N + 1)x + Nx^2 \quad (\text{field axioms}) \\ &\geq 1 + (N + 1)x \quad (\text{since } N > 0, x^2 > 0) \end{aligned}$$

Hence,  $(1 + x)^{N+1} \geq 1 + (N + 1)x$ .

□

## 5 January 15, 2025

### 5.1 The completeness axiom

Let  $B \subseteq \mathbb{R}$ . We say the following:

- We say  $b_1 \in B$  is a least element or minimum of  $B$  if
  - $b_1 \in B$ , and
  - $b_1 \leq b$  for all  $b \in B$ .

We write  $b_1 = \min B$ .

- We say  $b_1 \in B$  is a least element or minimum of  $B$  if
  - $b_1 \in B$ , and
  - $b_1 \leq b$  for all  $b \in B$ .

We write  $b_1 = \min B$ .

#### Example 40

Let  $B = \{1, 2, 3\}$ . Then  $\min B = 1$  and  $\max B = 3$ .

#### Proposition 41

Let  $B \subseteq \mathbb{R}$ . The maximum of  $B$  (if it exists) is unique. Similarly, the minimum of  $B$  is unique.

*Proof.* Suppose  $a, b \in \mathbb{R}$  are both maximum of  $B$ . Since  $a \in B$  and  $b$  is a max of  $B$ ,  $a \leq b$ . Similarly,  $b \leq a$ . Since  $a \leq b$  and  $b \leq a$ ,  $a = b$ .  $\square$

#### Definition 42

Let  $B \subseteq \mathbb{R}$ .

- We say  $h$  is a **lower bound** of  $B$  if  $h \leq b$  for all  $b \in B$ .
- We say  $h$  is an **upper bound** of  $B$  if  $b \leq h$  for all  $b \in B$ .

#### Example 43

Let  $B = [1, 2)$ . Then 1 is a lower bound of  $B$  and 2 is an upper bound of  $B$ . Note that 1 is the minimum of  $B$ , but 2 is not the maximum of  $B$  since  $2 \notin B$ .

#### Definition 44

Let  $B \subseteq \mathbb{R}$ . We say  $B$  is

- **bounded above** if there exists an upper bound of  $B$ .
- **bounded below** if there exists a lower bound of  $B$ .
- **bounded** if there exists an upper bound and a lower bound of  $B$ .

#### Fact 45

**Example 46** •  $\mathbb{N}$  is bounded below, but not bounded above.

- $(-\infty, 1]$  is bounded above but not bounded below
- $(1, 3)$  is bounded.

## 5.2 Completeness Axiom

#### Definition 47

A set  $B \subseteq \mathbb{R}$  is said to be **bounded above** if there exists a real number  $M$  such that  $b \leq M$  for all  $b \in B$ . The number  $M$  is called an **upper bound** of  $B$ .

#### Definition 48

A real number  $s$  is called the **supremum** or **least upper bound** of a set  $B \subseteq \mathbb{R}$  if:

1.  $s$  is an upper bound of  $B$ .
2. If  $u$  is any upper bound of  $B$ , then  $s \leq u$ .

We denote the supremum of  $B$  by  $\sup B$ .

#### Theorem 49 (Completeness Axiom)

Every non-empty set  $B \subseteq \mathbb{R}$  that is bounded above has a supremum.

#### Example 50

$$2 = \sup([1, 2])$$

*Proof.* Suppose,  $t$  is an upper bound of  $B$ . Suppose towards a contradiction that  $t < 2$ . Then

$$1 \leq t < \frac{t+2}{2} < \frac{2+2}{2} = 2$$

So,  $\frac{t+2}{2}$  is an element of the set  $B$  that is strictly bigger than  $t$  but we arrived to a contradiction. Therefore,  $2 \leq t$ . Since 2 is an upper bound of  $B = \sup(B)$ .  $\square$

### Example 51

The empty set  $\emptyset \in \mathbb{R}$  has no supremum since  $\mathbb{R}$  is the set of upper bounds of  $\emptyset$  and  $\mathbb{R}$  has no least element.

### Proposition 52

If  $B \in \mathbb{R}$  and  $\max(B)$  exists then  $\max(B) = \sup(B)$ .

*Proof.* Let  $A = \max(B)$ . Then  $B$  is non-empty and bounded above by  $a$ . If  $C = \sup(B)$ , then  $a \leq c$ . Since  $c$  is an upper bound  $a \in B$ . Also  $c \leq a$ . Since,  $c$  is the least upper bound.  $\therefore a = c$   $\square$

### Proposition 53 (The approximation property of suprema)

Let  $B \subseteq \mathbb{R}$  be non-empty and bounded above. For any  $\epsilon > 0$ , there exists an element  $b \in B$  such that  $\sup B - \epsilon < b \leq \sup B$ .

*Proof.* Suppose for contradiction there was some  $\epsilon > 0$  such that no  $b$  as above exists. Then for all  $c \in B$ ,  $c \leq \sup(B) - \epsilon$  i.e.  $\sup(B) - \epsilon$  is an upper bound of  $B$ . But  $\sup(B)$  is the least upper bound of  $B$ , so  $\sup(B) - \epsilon < \sup(B)$ . This is a contradiction. Therefore, there must exist some  $b \in B$  such that  $\sup(B) - \epsilon < b \leq \sup(B)$ .  $\square$

**Remark 54.** To prove  $a$  is a supremum of  $B$ , we need show the following:

- $a$  is an upper bound of  $B$ .
- If  $c$  is an upper bound of  $B$ , then  $a \leq c$ .

### Theorem 55

Suppose,  $F \subseteq \mathbb{R}$  is non-empty and bounded below. Then, there exist a **greatest lower bound** of  $F$ , called **infimum** of  $F$ , denoted  $\inf(F)$ .

*Proof.* Let  $B = \{x \in \mathbb{R} \mid -x \in F\}$ . We will show:

- $B$  is bounded above and non-empty, hence  $a = \sup(B)$  exists.
- $-a$  is a lower bound of  $F$ .
- If  $c$  is a lower bound of  $F$ , then  $c \leq -a$ .



- Since  $F$  is non-empty,  $B$  is non-empty. Suppose  $c$  is a lower bound of  $F$ . Let  $x \in B$ . Then  $-x \in F$ , so  $-x \geq c \rightarrow x \leq -c$ . Hence,  $c$  is an upper bound of  $B$ . Therefore,  $B$  is bounded above.
- Let  $f \in F$ . Then  $-f \in B$ , so  $-f \leq a$  (since  $a = \sup(B)$ ). Therefore,  $f \geq -a$ . Hence,  $-a$  is a lower bound of  $F$ .
- Let  $c$  be a lower bound of  $F$ . Let  $b \in B$ . Then  $-b \in F$ , so  $-b \geq c \rightarrow b \leq -c$ . Hence,  $-c$  is an upper bound of  $B$ . Therefore,  $-a \leq -c \rightarrow c \leq -a$ .

□

## 6 January 17, 2025

### Corollary 56

Let  $F$  be a non-empty and bounded below for each  $\epsilon > 0$ , there exists  $f \in F$  such that

$$\inf(F) \leq f < \inf(F) + \epsilon$$

### Theorem 57

There is a unique positive number  $\alpha$  such that  $\alpha^2 = 2$ .

*Proof.* Let  $E = \{x \in \mathbb{R} \mid x^2 < 2\}$ . Since,  $x^1 = 1 < 2$ , so  $1 \in E$ , so  $E \neq \emptyset$ .

Suppose,  $x \geq 2$ . Then  $x^2 \geq 2^2 = 4 > 2$ . Hence,  $x \notin E$ . Therefore, if  $x \in E$ , then  $x < 2$ .

Hence,  $E \neq \emptyset$ , and bounded above by 2.

Let  $\alpha = \sup(E)$ .

We know that  $1 \leq \alpha \leq 2$ .

If  $\alpha^2 \neq 2$ , then either  $\alpha^2 > 2$ , then either  $\alpha^2 > 0$  or  $\alpha^2 < 2$ . We'll show that both these cases lead to contradiction!

Case 1: When  $\alpha^2 < 2$  Let  $h = \frac{1}{2} \min(\alpha, \frac{2-\alpha^2}{3\alpha})$ . Note  $h > 0, h < \alpha$  and  $h < \frac{2-\alpha^2}{3\alpha}$ . Note that  $(\alpha + h)^2 = \alpha^2 + 2\alpha h + h^2 < \alpha^2 + 3\alpha h < \alpha^2 + 3\alpha \frac{2-\alpha^2}{3\alpha} = 2$ . So  $\alpha + h \in E$ . Since  $\alpha + h > \alpha$ , this contradicts  $\alpha = \sup(E)$ .

Case 2: When  $\alpha^2 > 2$  We set  $h = \frac{1}{2} \frac{\alpha^2 - 2}{\alpha} > 0$ . Since  $\alpha - h < \alpha$ , the approximation property says that there exists  $e \in E$  such that

$$\alpha - h < e \leq \alpha$$

. Then  $(\alpha - h)^2 < e^2$ . Then  $\alpha^2 - 2\alpha h + h^2 < 2 \Rightarrow \alpha^2 - 2\alpha h < 2$ . Therefore,  $h > \frac{\alpha^2 - 2}{2\alpha}$ . And we arrived at a contradiction. Since  $\alpha^2 < 2$  and  $\alpha^2 > 2$  cannot hold, so  $\alpha^2 = 2$ .

Suppose  $\alpha^2 = \beta^2 = 2$  and  $\alpha, \beta > 0$ . Then  $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = 0$ . Since  $\alpha + \beta > 0, \alpha - \beta = 0$ . Hence,  $\alpha = \beta$  □

**Remark 58.** The same proof shows  $\{\alpha \in \mathbb{Q} \mid \alpha^2 = 2\}$  cannot have a supremum in  $\mathbb{Q}$ , i.e. completeness doesn't hold.

**Remark 59.** We denote  $\alpha$  as above using  $\sqrt{2}$  by modifying the previous proof.

### Theorem 60

For any positive real number  $x$ , there exist unique positive real number, denoted  $\sqrt{x}$ , such that  $(\sqrt{x})^2 = x$ .

### Theorem 61 (Archimedean Property for Real Numbers)

Let  $x \in \mathbb{R}$ . Then there exists a natural number  $n \in \mathbb{N}$  such that  $n > x$ .

*Proof.* Suppose this theorem is not true for some  $x \in \mathbb{R}$ . Then  $x$  is an upper bound of  $\mathbb{N}$ . Since  $\mathbb{N} \neq \emptyset$ ,  $\alpha = \sup(\mathbb{N})$  exists. By approximation theorem, there exists some  $n \in \mathbb{N}$  such that

$$\alpha - 1 < n \leq \alpha$$

. Hence,  $\alpha < n + 1$ . Since  $n + 1$  is a natural number, this contradicts  $\alpha = \sup(\mathbb{N})$  □

### Corollary 62

Let  $\epsilon > 0$ . Then there exists some  $n \in \mathbb{N}$  such that  $\frac{1}{n} < \epsilon$ .

*Proof.* Apply the Archimedean property to  $\frac{1}{\epsilon}$  □

### Definition 63

A **sequence** is a function  $a: \mathbb{N} \rightarrow \mathbb{R}$ . We denote the  $n$ th term of the sequence as  $a_n$ . We usually write  $a(n)$  as  $a_n$

We write  $a$  as  $((a_n)_{n=1})^\infty$  or simply as  $a_n$ . Let  $a_n$  and  $b_n$  be sequences and  $c \in \mathbb{R}$  then we define  $a_n$

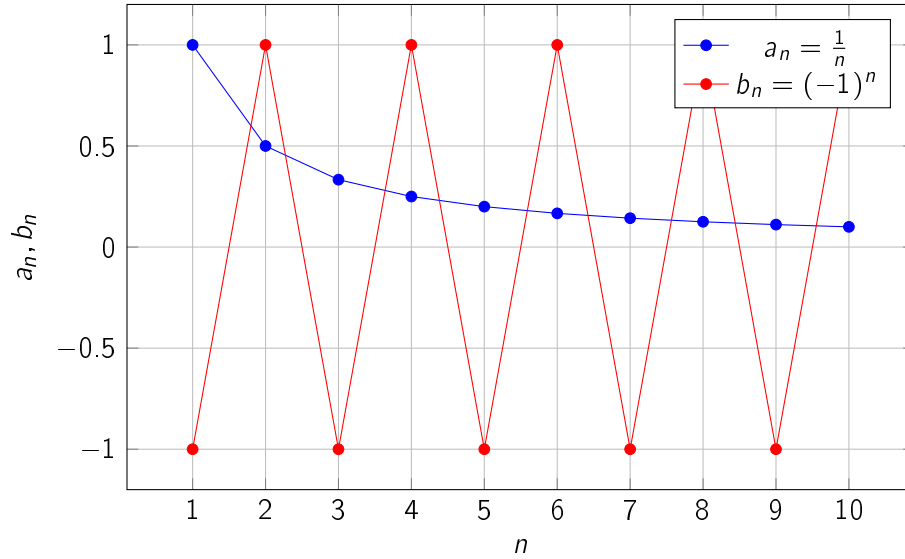
## 7 January 22, 2025

### Definition 64

A sequence  $a_n$  converges to some  $L \in \mathbb{R}$  or  $(a_n)$  tends to  $L$ ,  $a_n \rightarrow L$ ,  $\lim_{n \rightarrow \infty} a_n = L$ ,  $\lim a_n = L$  if  $\forall \epsilon > 0, \exists N \in \mathbb{N}$  such that  $\forall n \geq N, |a_n - L| < \epsilon$

### Example 65

Let  $a_n = \frac{1}{n}$ ,  $\epsilon = \frac{1}{1000}$  For all,  $n \geq 1000$ ,  $|a_n - 0| = |a_n| = \frac{1}{n} < \frac{1}{1000}$



### Definition 66

We say that a sequence  $(a_n)$  is convergent if it converges for some  $L \in \mathbb{R}$ . i.e.

$$\exists L \in \mathbb{R}, \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |a_n - L| < \epsilon$$

### Definition 67

If  $a_n \rightarrow L$ , then  $L$  is a limit of  $a_n$ .

### Definition 68

A sequence  $(a_n)$  is divergent if it is not convergent, i.e.

$$\forall L \in \mathbb{R}, \exists \epsilon > 0, \forall N \in \mathbb{N}$$

such that  $|a_n - L| \geq \epsilon$

### Example 69

Let  $a_n = \frac{2^n - 1}{2^n}$ . Then  $a_n \rightarrow 1$

*Proof.* Let  $\epsilon > 0$ . Then  $|a_n - 1| = \left| \frac{2^n - 1}{2^n} \right| = \left| \frac{-1}{2^n} \right| = \frac{1}{2^n}$ . We want to prove that  $\exists N \in \mathbb{N}$  such that  $\forall n \geq N, |a_n - 1| < \epsilon$ . By Bernoulli's Inequality,  $2^n = (1 + 1)^n$ . By Archimedean property, there exists  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ .  $\forall n \geq N, |a_n - 1| = \frac{1}{2^n} < \frac{1}{n} \leq \frac{1}{N} < \epsilon$ . Hence,  $a_n \rightarrow 1$   $\square$

Given  $a_n$  and  $L \in \mathbb{R}$ , players A and B a game.

### Example 70

Let

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4}$$

. Then  $a_n$  is convergent.

First we need to come up the limit of this sequence.

$$a_n = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{4}{n^2}}$$

So  $\frac{1}{3}$  seems like a choice of limit  $L$ .

*Proof.* Let  $\epsilon > 0$ . Then

$$\left| a_n - \frac{1}{3} \right|$$

By the Archimedean property,  $\exists N \in \mathbb{N}$  such that  $\frac{1}{N} < \epsilon$ . Hence for all,  $n \geq N$ ,  $\left| a_n - \frac{1}{3} \right| < \frac{1}{n} < \frac{1}{N} < \epsilon$   $\square$

### Example 71

Let

$$a_n = \frac{(-1)^n n^2}{n^2 + 1}$$

. Then  $a_n$  is divergent.

For large even  $n$ ,

$$a_n = \frac{n^2}{n^2 + 1} = \frac{1}{1 + \frac{1}{n^2}}$$

For large odd  $n$ ,

$$a_n = \frac{-n^2}{n^2 + 1} = -\frac{1}{1 + \frac{1}{n^2}}$$

## 8 January 27, 2025

### Theorem 72 (Limits respect Weak Inequalities)

Let  $(a_n)$  and  $(b_n)$  be sequences such that  $(a_n) \rightarrow L$  and  $(b_n) \rightarrow M$ . If  $a_n \geq b_n$  for all  $n$ , then  $L \geq M$ .

*Proof.* Suppose, for a contradiction, that  $L < M$ . Set  $\epsilon = \frac{M-L}{2} > 0$ . Since  $a_n \rightarrow L$ , there exists  $N_1$  such that  $n \geq N_1 \implies |a_n - L| < \epsilon$ . Similarly, since  $b_n \rightarrow M$ , there exists  $N_2$  such that  $n \geq N_2 \implies |b_n - M| < \epsilon$ . Therefore, for  $n \geq \max(N_1, N_2)$ , we have:

$$a_n < L + \epsilon = \frac{L + M}{2} \quad \text{and} \quad b_n > M - \epsilon = \frac{L + M}{2}$$

Hence, for  $n \geq \max(N_1, N_2)$ , we have  $a_n < \frac{L+M}{2} < b_n$ , which contradicts  $a_n \geq b_n$  for all  $n$ .  $\square$

**Remark 73.** Clearly,  $\lim$  does not respect strict inequalities: e.g.  $\frac{1}{n} > 0$  for all  $n \geq 1$  but  $0 = \lim \frac{1}{n} > \lim 0 = 0$  is false.

#### Theorem 74 (Squeeze Theorem)

Suppose that  $x_n \leq a_n \leq y_n$  for all  $n$  and that  $L = \lim x_n = \lim y_n$ . Then  $a_n \rightarrow L$  as  $n \rightarrow \infty$ .

*Proof.* Let  $\epsilon > 0$ . Then there exist  $N_1$  and  $N_2$  such that  $x_n - L > -\epsilon$  for all  $n \geq N_1$ , and  $y_n - L < \epsilon$  for all  $n \geq N_2$ . So for  $n \geq \max(N_1, N_2)$ , we have:

$$-\epsilon < x_n - L \leq a_n - L \leq y_n - L < \epsilon$$

which shows that  $a_n \rightarrow L$  also.  $\square$

## 8.1 The Algebra of Limits

Most sequences can be built up from simpler ones using addition, multiplication, etc. The algebra of limits tells us how the corresponding limits behave. Throughout the following,  $(a_n)$  and  $(b_n)$  denote sequences.

#### Proposition 75 (AOL: Constants)

If  $a_n = a$  for all  $n$ , then  $a_n \rightarrow a$ .

*Proof.* For any  $\epsilon > 0$ , take  $N = 1$ ;  $n \geq N \implies |a_n - a| = 0 < \epsilon$ .  $\square$

#### Proposition 76 (AOL: Sums)

If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a_n + b_n \rightarrow a + b$ .

*Proof.* Let  $\epsilon > 0$ . Then  $\frac{\epsilon}{2} > 0$  and so:

$$\exists N_1 : n \geq N_1 \implies |a_n - a| < \frac{\epsilon}{2}, \quad \exists N_2 : n \geq N_2 \implies |b_n - b| < \frac{\epsilon}{2}$$

Put  $N_3 = \max(N_1, N_2)$ . Then for all  $n \geq N_3$ , we have:

$$|(a_n + b_n) - (a + b)| \leq |a_n - a| + |b_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\square$

#### Proposition 77 (AOL: Scalar Products)

If  $a_n \rightarrow a$  as  $n \rightarrow \infty$  and  $\lambda \in \mathbb{R}$ , then  $\lambda a_n \rightarrow \lambda a$ .

*Proof.* Let  $\epsilon > 0$ . Then  $\frac{\epsilon}{|\lambda|+1} > 0$  and so there exists  $N$  such that  $|a_n - a| < \frac{\epsilon}{|\lambda|+1}$  for all  $n \geq N$ . Hence:

$$|\lambda a_n - \lambda a| = |\lambda| |a_n - a| \leq \frac{|\lambda| \epsilon}{|\lambda| + 1} < \epsilon$$

for all  $n \geq N$ . □

**Corollary 78 (AOL: Differences)**

If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a_n - b_n \rightarrow a - b$ .

**Corollary 79 (AOL: Translations)**

If  $a_n \rightarrow a$  and  $c \in \mathbb{R}$ , then  $a_n + c \rightarrow a + c$ .

**Lemma 80**

If  $x_n \rightarrow 0$  and  $y_n \rightarrow 0$ , then  $x_n y_n \rightarrow 0$ .

*Proof.* Given  $\epsilon > 0$ , let  $\epsilon_1 = \min(1, \epsilon) > 0$ . Then:

$$\exists N_1 : n \geq N_1 \implies |x_n| < \epsilon_1, \quad \exists N_2 : n \geq N_2 \implies |y_n| < \epsilon_1$$

So if  $n \geq \max(N_1, N_2)$ , we have:

$$|x_n y_n| \leq |x_n| |y_n| < \epsilon_1^2 \leq \epsilon$$

which completes the proof. □

**Proposition 81 (AOL: Products)**

If  $a_n \rightarrow a$  and  $b_n \rightarrow b$ , then  $a_n b_n \rightarrow ab$ .

*Proof.* Note that:

$$a_n b_n - ab = (a_n - a)(b_n - b) + b(a_n - a) + a(b_n - b)$$

By the previous lemma,  $(a_n - a)(b_n - b) \rightarrow 0$ , and by Proposition 2.2.3,  $b(a_n - a) \rightarrow 0$  and  $a(b_n - b) \rightarrow 0$ . Hence  $a_n b_n \rightarrow ab$  by Proposition 2.2.2. □

## 9 January 29, 2025

**Proposition 82 (AOL: Reciprocals)**

If  $a_n \rightarrow a$  and  $a_n \neq 0$  for all  $n$  and  $a \neq 0$ , then  $\frac{1}{a_n} \rightarrow \frac{1}{a}$ .

*Proof.* Let  $\epsilon > 0$ . As  $a \neq 0$  then  $\frac{|a|}{2} > 0$ . So there exists  $N_1$  such that for  $n \geq N_1$  we have  $|a_n - a| < \frac{|a|}{2}$ . By the Triangle Inequality

$$|a| \leq |a_n| + |a - a_n| = |a_n| + |a_n - a|$$

and so  $|a_n| > \frac{|a|}{2}$  and  $\left| \frac{1}{a_n} \right| < \frac{2}{|a|}$ . Further, as  $\frac{|a|\epsilon}{2} > 0$  then there exists  $N_2$  such that for  $n \geq N_2$

$$|a_n - a| < \frac{|a|\epsilon}{2}.$$

Set  $N_3 = \max(N_1, N_2)$  so that for  $n \geq N_3$

$$\left| \frac{1}{a_n} - \frac{1}{a} \right| = \frac{|a_n - a|}{|a_n||a|} < \frac{\frac{|a|\epsilon}{2}}{\frac{|a|}{2}|a|} = \epsilon.$$

□

### Corollary 83 (AOL: Quotients)

If  $a_n \rightarrow a$ ,  $b_n \rightarrow b$ , and  $b_n \neq 0$  for all  $n$  and  $b \neq 0$ , then  $\frac{a_n}{b_n} \rightarrow \frac{a}{b}$ .

### Proposition 84 (AOL: Modulus)

If  $a_n \rightarrow a$  then  $|a_n| \rightarrow |a|$ .

*Proof.* Exercise.

□

### Example 85

$$a_n = \frac{n^2 + n + 1}{3n^2 + 4} \rightarrow \frac{1}{3}$$

*Proof.* We write

$$\frac{n^2 + n + 1}{3n^2 + 4} = \frac{1 + \frac{1}{n} + \frac{1}{n^2}}{3 + \frac{4}{n^2}} \rightarrow \frac{1 + 0 + 0}{3 + 0} = \frac{1}{3}$$

noting

- $\frac{1}{n} \rightarrow 0$  by the Archimedean Property;
- $\frac{1}{n^2} \rightarrow 0$  by Proposition 2.2.7;
- $1 \rightarrow 1$ ;
- $1 + \frac{1}{n} + \frac{1}{n^2} \rightarrow 1$  by Proposition 2.2.2;
- $3 + \frac{4}{n^2} \rightarrow 3$  by Proposition 2.2.2;
- $\frac{1}{3 + \frac{4}{n^2}} \rightarrow \frac{1}{3}$  by Corollary 2.2.9;
- $a_n \rightarrow \frac{1}{3}$  by Proposition 2.2.7.

□

**Example 86**

Suppose  $a_1 = 1$ ,  $a_2 = 1$ , and we recursively define  $a_{n+2} = a_{n+1} + a_n$ , for  $n \geq 1$ .

**Proposition 87**

$(\frac{a_{n+1}}{a_n})$  is convergent.

*Proof.* By induction,  $a_n \geq 1$  for all  $n$ . So for  $n \geq 1$

$$\frac{a_{n+2}}{a_{n+1}} = 1 + \frac{a_{n+1}}{a_n}.$$

Write  $x_n = \frac{a_{n+1}}{a_n}$  for  $n \geq 1$ . Note that  $x_n > 0$  for all  $n$ . Then  $x_1 = 1$  and  $x_{n+1} = 1 + \frac{1}{x_n}$ . Suppose that we did have convergence and that  $x_n \rightarrow x$ . Since  $x_{n+1}$  is a tail of  $x_n$ , we see that  $(x_{n+1}) \rightarrow x$  and so  $1 + \frac{1}{x_n} \rightarrow 1 + \frac{1}{x}$  by AOL. So

$$x = 1 + \frac{1}{x}$$

by the Uniqueness of Limits. Hence  $x^2 - x - 1 = 0$  giving  $x = \frac{1 \pm \sqrt{5}}{2}$ . But  $x_n \geq 0$  for all  $n$ , and so  $x \geq 0$  by Theorem 2.1.10 giving

$$x = \frac{1 + \sqrt{5}}{2} > 1.$$

At this point we don't yet know the sequence is convergent. We only know that if the sequence converges, then it must converge to the real number  $\frac{1+\sqrt{5}}{2}$ , which we will denote  $\phi$ . Now we will show that  $(x_n)$  does indeed converge to  $\phi$ .  $\phi$  is called the Golden Ratio.

$$x_{n+1} - \phi = 1 + \frac{1}{x_n} - \phi = 1 + \frac{1}{x_n} - 1 - \frac{1}{\phi} = \frac{1}{x_n} - \frac{1}{\phi} = \frac{x_n - \phi}{x_n \phi}$$

as  $\phi^2 = \phi + 1$ . So

$$\frac{x_{n+1} - \phi}{x_n - \phi} = \frac{1}{|x_n| |\phi|} = \frac{1}{\phi x_n} \leq \frac{1}{\phi}$$

as  $x_n \geq 1$  for all  $n$ . By induction, we get

$$-\frac{1}{\phi^n} \leq x_n - \phi \leq \frac{1}{\phi^n}.$$

and are done by the Squeeze Theorem, since  $\phi > 1$  and so  $(\pm \frac{1}{\phi^n}) \rightarrow 0$ . □

**10 January 31, 2025**

**11 February 3, 2025**

**Example 88**

Let  $a_n = n$  and  $b_n = (2n + 1)^2$ . Then  $(a_n)$  and  $(b_n)$  are strictly increasing.



**Definition 89**

We say that a sequence  $(a_n)$  is:

1. bounded above if the set  $S = \{a_n \mid n \in \mathbb{N}\}$  is bounded above.
2. bounded below if the set  $S = \{a_n \mid n \in \mathbb{N}\}$  is bounded below.
3. bounded if the set  $S = \{a_n \mid n \in \mathbb{N}\}$  is bounded.

**Theorem 90 (Monotone Convergence Theorem)**

The power of this theorem is that we can show a sequence converges without first calculating its limit.

1. Let  $(a_n)$  be an increasing, bounded above sequence. Then  $(a_n)$  converges.
2. Let  $(a_n)$  be a decreasing, bounded below sequence. Then  $(a_n)$  converges.

*Proof.* If  $(a_n)$  is decreasing and bounded below, then  $(-a_n)$  is increasing and bounded above, so it suffices to prove (1). We thus assume  $(a_n)$  is increasing and bounded above. Let  $L = \sup\{a_n \mid n \in \mathbb{N}\}$ ; this exists by the completeness axiom as the set is bounded above and non-empty. Let  $\epsilon > 0$ . By the Approximation Property there exists  $N \in \mathbb{N}$  such that

$$L - \epsilon < a_N \leq L.$$

As the sequence is increasing, then for any  $n \geq N$

$$L - \epsilon < a_N \leq a_n \leq L,$$

which implies  $\forall n \geq N \mid a_n - L \mid < \epsilon$ . That is  $(a_n) \rightarrow L$ . □

**Corollary 91**

Let  $(a_n)$  be a bounded monotone sequence. Then  $(a_n)$  converges.

**Example 92**

Let  $x > 0$ . Set  $a_1 = 1$  and  $a_{n+1} = \frac{1}{2} \left( a_n + \frac{x}{a_n} \right)$  for each  $n$ . Then  $(a_n)$  converges to some positive number  $a$  such that  $a^2 = x$ .

*Proof.* By induction, it follows that  $a_n$  is defined and  $a_n > 0$  for all  $n \geq 1$ . We remark that for all  $n$ , we have  $2a_n a_{n+1} = a_n^2 + x$ . Therefore,

$$0 \leq (a_n - a_{n+1})^2 = a_n^2 - 2a_n a_{n+1} + a_{n+1}^2 = a_{n+1}^2 - x,$$

and so  $a_{n+1}^2 \geq x$  for all  $n \geq 1$ . We now show that  $(a_{n+1})$  is decreasing. Indeed, for  $n \geq 2$ ,

$$a_n - a_{n+1} = a_n - \frac{1}{2} \left( a_n + \frac{x}{a_n} \right) = \frac{a_n^2 - x}{2a_n} \geq 0.$$

Therefore,  $a_n \geq a_{n+1}$  for all  $n \geq 2$ . Therefore the sequence  $(a_{n+1})$  is decreasing and bounded below by 0, hence converges to some  $a \geq 0$ . Since  $a_{n+1}^2 \geq x$  for all  $n \geq 1$ , AOL implies  $a^2 \geq x > 0$ , and so  $a > 0$ . Moreover, since  $(a_{n+1})$  is a tail of  $(a_n)$ , it follows that  $(a_n)$  also converges to  $a$ . Since  $a_{n+1} = \frac{1}{2}(a + \frac{x}{a})$ ,  $a > 0$  and  $a > 0$ , AOL implies that

$$a = \frac{1}{2} \left( a + \frac{x}{a} \right).$$

Rearranging gives  $x = a^2$ , as required. □

## 11.1 Subsequences

### Example 93

Let  $a_n = \frac{1}{n^2}$ , i.e.

$$(a_n) = \left( 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots \right).$$

We can get new sequences by looking at:

- everything after the second place  $(\frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \dots)$
- all odd terms  $(1, \frac{1}{9}, \frac{1}{25}, \dots)$
- all prime terms  $(\frac{1}{4}, \frac{1}{9}, \frac{1}{25}, \dots)$

### Definition 94

Let  $(a_n)$  be a sequence. We say that a sequence  $(b_n)$  is a subsequence of  $(a_n)$  if there is a strictly increasing sequence of natural numbers  $(f(n))$  such that  $(b_n) = (a_{f(n)})$ . (There may be more than one such function  $f$ .) Often we write  $n_r$  for  $f(r)$  and write a subsequence as  $(a_{n_r})$  or  $(a_{n_r})_{r=1}^{\infty}$ .

### Example 95

Let

$$(a_n) = (n^2) = (1, 4, 9, 16, \dots)$$

$$(b_n) = (0) = (0, 0, 0, 0, \dots)$$

$$(f(n)) = (2n) = (2, 4, 6, 8, \dots)$$

$$(g(n)) = (2n - 1) = (1, 3, 5, 7, \dots)$$

Then

$$(a_{f(n)}) = (a_{2n}) = (4, 16, 36, 64, \dots)$$

$$(a_{g(n)}) = (a_{2n+1}) = (1, 9, 25, 49, \dots)$$

$$(b_{f(n)}) = (b_{2n}) = (0, 0, 0, 0, \dots)$$

$$(b_{g(n)}) = (b_{2n+1}) = (0, 0, 0, 0, \dots)$$

**Proposition 96**

Suppose that the sequence  $(a_n)$  converges to  $L$ . Then every subsequence  $(a_{n_r})$  also converges to  $L$ .

*Proof.* Let  $\epsilon > 0$ . Then there exists  $N$  such that

$$n \geq N \implies |a_n - L| < \epsilon.$$

As  $r \mapsto n_r$  is increasing then  $n_r \geq r$  for all  $r$  and so

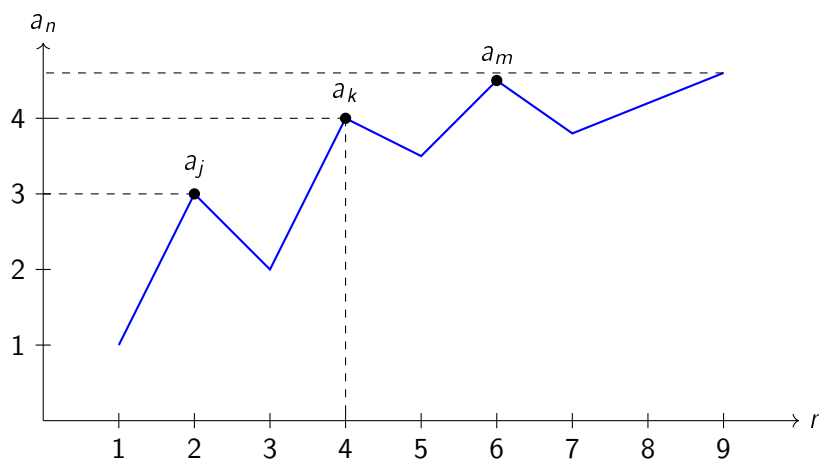
$$r \geq N \implies n_r \geq N \implies |a_{n_r} - L| < \epsilon.$$

□

## 12 February 5, 2025

**Theorem 97 (Monotone Subsequence Theorem)**

Let  $(a_n)$  be a sequence. Then  $(a_n)$  has a monotone subsequence which means the subsequence is either increasing or decreasing over that bound.

**Definition 98 (Vantage Point)**

Here,  $a_k$ ,  $a_m$ , and  $a_j$  are vantage points.

*Proof.* We say that  $k \in \mathbb{N}$  is a 'vantage point' of  $(a_n)$  if  $a_m < a_k$  for all  $m > k$ . Let  $V = \{k \in \mathbb{N} \mid k \text{ is a vantage point}\}$ .

- **Case 1:  $V$  is infinite:**

Let  $V = \{k_1, k_2, \dots\}$  where  $k_1 < k_2 < k_3 < \dots$ . Each  $k_i$  is a vantage point. If  $m > n$ , then  $a_{k_m} < a_{k_n}$ . Hence  $(a_{k_n})_{n=1}^{\infty}$  is a decreasing subsequence.

• **Case 1:  $V$  is finite:**

There exist  $k_1$  such that  $\forall n \geq k_1, a_n$  is not a vantage point,  $\exists k_2 > k_1$  such that  $k_2 > k_1$ ) Since,  $k_2 > k_1$  is also not a vantage point. hence  $\exists k_3 > k_2$  such that  $a_{k_3} \geq a_{k_2}$ .

We continue in this way, defining a strictly increasing subsequence.

$$k_1 < k_2 < k_3 < \dots$$

such that

$$a_{k_1} \leq a_{k_2} \leq a_{k_3} \leq \dots$$

. Hence  $(a_{k_n})_{n=1}^{\infty}$  is an increasing subsequence. Therefore,  $(a_{k_n})_{n=1}^{\infty}$  is monotone.

□

**Theorem 99 (Bolzano-Weierstrass Theorem)**

Let  $(a_n)$  be a bounded sequence. Then  $(a_n)$  has a convergent-subsequence.

*Proof.* By Theorem 97,  $(a_n)$  has a monotone subsequence,  $(a_{k_n})_{n=1}^{\infty}$ . Since  $(a_n)$  is bounded, so is  $(a_{k_n})$ . By the Montone Convergence Theorem,  $(a_{k_n})$  is convergent. □

*Proof.* Let  $(a_n)$  be a bounded sequence. Then there exist  $A_1$  and  $B_1$  such that  $A_1 \leq a_n \leq B_1$  for all  $n$ . We construct a nested sequence of intervals  $[A_n, B_n]$  as follows: Let  $[A_2, B_2]$  be one of the intervals  $[A_1, \frac{A_1+B_1}{2}]$  or  $[\frac{A_1+B_1}{2}, B_1]$  such that  $\{n \in \mathbb{N} \mid a_n \in [A_2, B_2]\}$  is infinite. Similarly, let  $[A_3, B_3]$  be one of the intervals  $[A_2, \frac{A_2+B_2}{2}]$  or  $[\frac{A_2+B_2}{2}, B_2]$  such that  $\{n \in \mathbb{N} \mid a_n \in [A_3, B_3]\}$  is infinite. We continue this process, ensuring that each interval  $[A_n, B_n]$  contains infinitely many terms of the sequence  $(a_n)$ . We then pick a subsequence  $k_1 < k_2 < \dots < k_n$  such that  $a_{k_n} \in [A_n, B_n]$ . Since  $B_n - A_n = \frac{B_1 - A_1}{2^{n-1}}$ , the length of the intervals  $[A_n, B_n]$  tends to 0 as  $n \rightarrow \infty$ . Since  $(A_n)$  and  $(B_n)$  are monotone and bounded, they converge to the same limit  $L$ . By the Squeeze Theorem, the subsequence  $(a_{k_n})$  converges to  $L$ . □