Math 4580 Lecture 1: Functions and Relations

January 8, 2025

Functions

Definition: Let A and B be sets. A function $f : A \to B$ assigns exactly one output $f(a) \in B$ to every input $a \in A$.

- The set A is called the **domain** of f.
- The set B is called the **codomain** of f.

Note: The domain A, codomain B, and the assignment of outputs $f(a)$ to every input $a \in A$ are all part of the data defining a function. Just writing a formula like $f(x) = e^x$ does not determine a function, as the domain and codomain are not specified. For example:

- $f : \mathbb{R} \to \mathbb{R}, f(x) = e^x$.
- $f : \mathbb{Q} \to \mathbb{Q}, f(x) = e^x$.

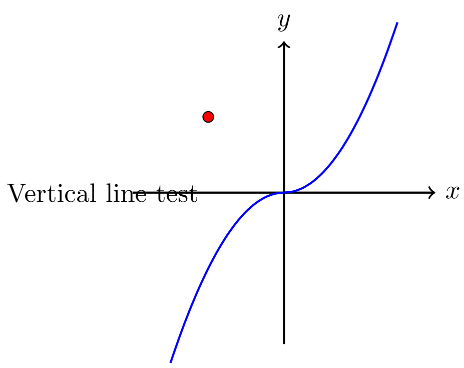
Although these functions use the same formula, their meanings are completely different because their domains and codomains differ.

Graphs

A function $f : A \to B$ is often identified with its **graph** in $A \times B$:

$$\text{graph}(f) = \{(a, b) \in A \times B : b = f(a)\}.$$

Lemma: Let $f : A \to B$ be a function. Its graph, graph(f), passes the **vertical line test**: For every $a \in A$, $V_a := \{(a, b) \in A \times B : b \in B\}$ intersects graph(f) in exactly one element.



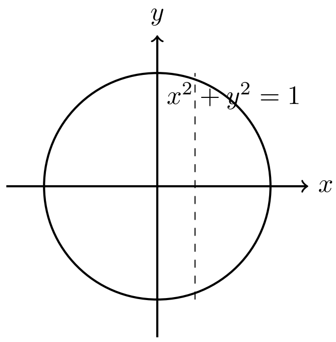
Proposition: Let $G \subseteq A \times B$ be any subset passing the vertical line test, i.e., for all $a \in A$, $V_a \cap G$ consists of exactly one element. Then $G = \text{graph}(f)$ for a unique function $f : A \to B$.

Proof: If $G = \{(a, b) \mid b \in B\}$ satisfies the vertical line test, define $f : A \to B$ by $f(a) = b$. Then $G = \text{graph}(f)$.

Definition: A subset $R \subseteq A \times B$ is called a **relation**. The vertical line test distinguishes graphs of functions from more general relations.

Examples

- Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (the unit circle). This is a relation but not the graph of a function because it fails the vertical line test: The vertical line $x = 0$ intersects the circle at two points.
- Visual depiction of a unit circle:



- Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$. The number of functions from A to B is $2^3 = 8$, corresponding to the 8 associated graphs in $A \times B$.
- The number of relations from A to B is $2^{|A| \cdot |B|} = 2^{3 \cdot 2} = 64$, containing the 8 graphs of functions from A to B.

Conclusion: The notion of relation is much more permissive than the notion of functions.

Visualizing Functions as Directed Edges

A function $f : A \to B$ can be visualized as a collection of directed edges $(a, f(a)) \in A \times B$. Each element of A has exactly one outgoing edge in the graph.