Stat 4202: Mathematical Statistics II

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STAT 4202 will rely a lot on STAT 4201. So we need to have a pretty good understanding of those concepts.

1.1 Review of Probability Theory

> **Definition 1**
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> The **Sample Space**, denoted by $\mathcal{S}$, is the set of all outcomes from an experiment.

> **Definition 2**
>
> An **Event**, usually denoted by a capital letter such as A or B, is a subset of the Sample Space.

The probability function

- $P(A) \geq 0$

- $P(\mathcal{S}) = 1$

- For disjoint sets A_1, A_2, $\cdots$, A_n:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

If an event A is a subset of another event B, then the probability of A is less than or equal to the probability of event B. That is to say, if $A \subseteq B$, then $P(A) \leq P(B)$

The complement of an event A, denoted by A^c, has a probability equal to one minus the probability of the event A. That is,

$$P(A^c) = 1 - P(A)$$

A partition of a sample space $\mathcal{S}$ is an exhaustive, non-overlapping collection of events A_1, A_2, $\cdots$, A_n that is exhaustive and mutually exclusive:

$$\bigcup_{i=1}^{n} A_i = \mathcal{S}$$

and
$$A_i \cap A_j = \emptyset \quad \forall i \neq j$$

For any partition, we have
$$\sum_{i=1}^{n} P(A_i) = 1$$

Two events A and B are **independent** if the outcome of one doesn't affect the likelihood of the occurrence of the other. For two independent events, we have
$$P(A \cap B) = P(A)P(B)$$

The **conditional probability** of A given B is given by
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Lemma 3

Note that if A and B are independent, then
$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(A)P(B)}{P(B)} \\ &= P(A) \end{aligned}$$

Corollary 4

If A and B are independent, then $P(A|B) = P(A)$ and $P(B|A) = P(B)$

1.2 Random Variables

Definition 5

A random variable is a function that takes outcomes from the sample space $\mathcal{S}$ to the real numbers $\mathbb{R}$. That is, a random variable is a function $X : \mathcal{S} \rightarrow \mathbb{R}$.

We then use a probability mass function (pmf) in the discrete case or a probability density function (pdf) in the continuous case:

pmf: $\qquad f_X(x) = P(X = x)$ $\qquad\qquad$ when X is discrete

pdf: $\qquad \int_a^b f_X(x)\mathrm{d}x = P(a \leq X \leq b)$ $\qquad\qquad$ when X is continuous

The cumulative distribution function (cdf) gives the probability of observing a value less than or equal to a given value x:

$$F_X(x) = P(X \leq x)$$

When X is a continuous random variable, the pdf is the derivative of the cdf:

$$f_X(x) = F_X'(x)$$

1.3 Expected Value and Variance

For random variable X, the **expected value** is denoted by $E(X)$ and is given by:

$$E(X) = \begin{cases} \sum_x x f_X(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} x f_X(x) \mathrm{d}x & \text{if } X \text{ is continuous} \end{cases}$$

The **variance** of a random variable X is denoted by $\mathrm{Var}(X)$ and is given by:

$$\mathrm{Var}(X) = E\left[(X - E\{X\})^2\right]$$

1.4 Covariance

The **covariance** of two random variables X and Y is denoted by:

$$\mathrm{Cov}(X, Y) = E\left[(X - E(X))(Y - E(Y))\right]$$

If two random variables X and Y are independent, then

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$$

So, we will be using these formulas to estimate the mean and the variance throughout the semester.