Math 4547: Real Analysis I

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1 The Real Numbers

1.1 The Real Numbers

Theorem 1.1.1

There is no rational number x such that $x^2 = 2$.

1.2 Field Axioms

1.2.1 Addition Axioms

Proposition 1.2.1

If $a + x = a$ for all real numbers a, then $x = 0$. This means that 0 is the only additive identity.

Proposition 1.2.2

If $a + x = a + y$ then $x = y$. In particular, this means that additive inverses are unique.

Proposition 1.2.3

$-(-a) = a$

Proposition 1.2.4

$-(a + b) = (-a) + (-b)$

Proposition 1.2.5

$-0 = 0$

1.2.2 Multiplication Axioms

Proposition 1.2.6

If $a \times b = a$ for all real numbers a, then $b = 1$.

Proposition 1.2.7

If $a \neq 0$ and $a \times b = a \times c$, then $b = c$.

Proposition 1.2.8

If $a \neq 0$, then $a^{-1} \neq 0$ and $\frac{1}{\left(\frac{1}{a}\right)} = a$.

Proposition 1.2.9

If $a \neq 0$, $b \neq 0$, and $a \times b \neq 0$, then $\frac{1}{(a \times b)} = \left(\frac{1}{a}\right) \times \left(\frac{1}{b}\right)$.

Proposition 1.2.10

$(a + b) \times c = a \times c + b \times c$

Proposition 1.2.11

$a \times 0 = 0$

Proposition 1.2.12

If $a \times b = 0$, then either $a = 0$ or $b = 0$ (or both).

Proposition 1.2.13

$a \times (-b) = -(a \times b)$. In particular, note that $a \times (-1) = -a$.

Proposition 1.2.14

$(-1) \times (-1) = 1$

1.3 The Order Axioms

Proposition 1.3.1

$1 \in P$

Proposition 1.3.2

$a > b$ if and only if $-a < -b$. In particular, $x > 0$ if and only if $-x < 0$.

Proposition 1.3.3

For all real numbers x, y, z:

1. If $x > 0$ and $y > 0$, then $x + y > 0$.
2. If $x > 0$ and $y > 0$, then $x \times y > 0$.
3. If $x > y$ and $y > z$, then $x > z$.

Proposition 1.3.4 (Inequalities Shift I)

Let x, y, z be reals such that $x < y$. Then $x + z < y + z$.

Proposition 1.3.5 (Inequalities Shift II)

Let x, y, z be reals such that $x < y$ and $0 < z$. Then $zx < zy$.

Corollary 1.3.6

Let x, y, z be reals such that $x < y$ and $z < 0$. Then $zx > zy$.

Corollary 1.3.7

$a^2 \geq 0$ for any real a.

Proposition 1.3.8

If $x \in P$, then $1/x \in P$.

Corollary 1.3.9

If $x, y \in P$ and $x < y$ then $1/y < 1/x$.

Example 1.3.10

$\max(x, y) = -\min(-x, -y)$

Theorem 1.3.11 (The Triangle Inequality)
For any real numbers a, b, we refer to this as the Δ inequality.

$$|a+b| \leq |a| + |b|$$

with equality if and only if ($a \geq 0$ and $b \geq 0$) or ($a < 0$ and $b < 0$).

Proposition 1.3.12
$|ab| = |a||b|$

Theorem 1.3.13 (Bernoulli's Inequality)
Let x be a real number with $x > -1$ and let n be a positive integer. Then

$$(1+x)^n \geq 1 + nx$$

1.4 Completeness Axiom

Proposition 1.4.1
A maximum (if it exists) of a set B is unique. Similarly, a minimum is unique.

Proposition 1.4.2
If $E \subseteq \mathbb{R}$ has a maximum, then $\max E = \sup E$.

Proposition 1.4.3 (The Approximation Property)
Let $E \subseteq \mathbb{R}$ be bounded above and non-empty and let $\epsilon > 0$. Then there exists $x \in E$ such that

$$\sup E - \epsilon < x \leq \sup E$$

Corollary 1.4.4
Let E be bounded above and non-empty. There is a function $a : \mathbb{N} \to \mathbb{R}$, such that for all n we have

$$\sup E - \frac{1}{n} < a(n) \leq \sup E.$$

Theorem 1.4.5
Let F be a non-empty set which is bounded below. Then the set of lower bounds of F has a greatest element. This element is known as the greatest lower bound or infimum of F and is written $\inf F$.