Physics 2301: Intermediate Mechanics II

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Spring 2025

1 January 7, 2025

1.1 Course Introduction

Dr. Boveia talked about the course and we will head towards relativistic mechanics. In this course, we will master the concepts from Physics 1250 but more broadly. **Tuesday** is the lecture day and **Wednesday,** Thursday, and Friday are the problem-solving days. The grading scale would be on the **Standard Ohio State** grading scale. There might be a curve but it will only curve up, not down. Our grade would rely on **Quizzes (20%), Midterm (20%), Final (20%), Homework (30%)**, and **Participation (10%)**. The exams are **open-book, open-notes, and open-internet**.

1.2 Review of Vectors and Matrices

Definition 1

A **scalar quantity** is a quantity with only magnitude. Examples include mass, temperature, and time.

Definition 2

A **vector quantity** is a quantity with both magnitude and direction. Examples include displacement, velocity, and force. Vectors follow a different set of rules compared to scalars. For example, there are two multiplication rules for vectors: **Dot Product** and **Cross Product**. The dot product results in a scalar quantity while the cross product results in a vector quantity.

$$|\vec{v}| = \text{Length and} \quad \vec{v} = \frac{|\vec{v}|}{\sqrt{3}}(V_x, V_y, V_z) \text{ if } \vec{v} = \langle V_x, V_y, V_z \rangle$$

Scalar $\times$ Vector = Vector

Vector $\cdot$ Vector = Scalar (Dot Product or Inner Product)

$$\vec{v} \cdot \vec{v} = V_x \cdot V_x + V_y \cdot V_y + V_z \cdot V_z = |\vec{v}|^2$$

$$\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z = |\vec{A}||\vec{B}| \cos\theta$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$\hat{x}$ = 'Unit Vector' $|\hat{x}| = 1$

$$\vec{A} \cdot \vec{B} = (A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \cdot (B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) = A_xB_x + A_yB_y + A_zB_z$$

$\hat{x}, \hat{y}, \hat{z}$ form an orthogonal normal basis, meaning that $\hat{x} \cdot \hat{y} = 0$

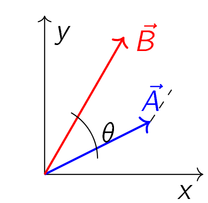
$$\hat{x} \cdot \hat{y} = 0, \quad \hat{y} \cdot \hat{z} = 0, \quad \hat{z} \cdot \hat{x} = 0$$

Here, Ortho means that $\hat{x} \cdot \hat{y} = 0$ and Normal means that $\hat{x} \cdot \hat{x} = 1$

Vector $\times$ Vector = Cross Product = Vector

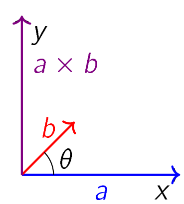
$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \hat{n}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



Dot Product

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta$



Cross Product

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\theta$

Figure 1: Geometric interpretation of dot and cross products

Dot product is just the magnitude of the projection of $\vec{A}$ onto $\vec{B}$.

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta\hat{n}$$

Dot product is the component of $\vec{B}$ along $\vec{A}$. The dot product is a scalar quantity.

The cross product is a vector that is perpendicular to both $\vec{A}$ and $\vec{B}$ ($\vec{A} \perp \vec{B}$). The magnitude of the cross product is the area of the parallelogram formed by $\vec{A}$ and $\vec{B}$. The direction of the cross product follows the right-hand rule.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

1.3 Matrices

A **matrix** is a rectangular array of numbers arranged in rows and columns. Matrices are used to represent linear transformations and solve systems of linear equations.

Definition 3

A matrix A with m rows and n columns is denoted as $A \in \mathbb{R}^{m\times n}$. Each element of the matrix is denoted as a_{ij}, where i is the row index and j is the column index.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \tag{1}$$

Matrix Addition: Two matrices A and B of the same dimension can be added by adding their corresponding elements.

$$(A + B)_{ij} = a_{ij} + b_{ij} \tag{2}$$

Matrix Multiplication: The product of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ is a matrix $C \in \mathbb{R}^{m\times p}$ where each element is given by the dot product of the corresponding row of A and column of B.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{3}$$

Identity Matrix: The identity matrix I is a square matrix with ones on the diagonal and zeros elsewhere. It acts as the multiplicative identity for matrices.

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{4}$$

Transpose of a Matrix: The transpose of a matrix A, denoted A^T, is obtained by swapping its rows and columns.

$$(A^T)_{ij} = a_{ji} \tag{5}$$

Determinant of a Matrix: The determinant is a scalar value that can be computed from the elements of a square matrix and encodes certain properties of the matrix. For a 2×2 matrix, the determinant is given by:

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{6}$$

Inverse of a Matrix: The inverse of a square matrix A, denoted A^{-1}, is the matrix such that $AA^{-1} = A^{-1}A = I$. A matrix is invertible if and only if its determinant is non-zero.

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) \tag{7}$$

where adj(A) is the adjugate of A.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Figure 2: Example of a matrix

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2.1 Angular Momentum of a Body

Theorem 4

The angular momentum (relative to the origin) of a body can be found by treating the body as a point mass located at the center of mass (CM) and finding the angular momentum of this point mass relative to the origin, and by then adding on the angular momentum of the body relative to the CM.

Proof. Let the coordinates of the CM be $R = (X, Y)$, and let the coordinates of a given point relative to the CM be $r' = (x', y')$. Then the given point has coordinates $r = R + r'$. Let the velocity of the CM be V, and let the velocity relative to the CM be v'. Then $v = V + v'$. Let the body rotate with angular speed ω' around the CM. Then $v' = \omega' r'$.

The angular momentum relative to the origin is:

$$L = \int (r \times v)\, dm = \int ((R + r') \times (V + v'))\, dm$$

Expanding the cross product:

$$L = \int (R \times V)\, dm + \int (R \times v')\, dm + \int (r' \times V)\, dm + \int (r' \times v')\, dm$$

By the definition of the CM, $\int r'\, dm = 0$, so the cross terms vanish:

$$L = MR \times V + \int (r' \times v')\, dm$$

Since $v' = \omega' r'$:

$$L = MR \times V + \left(\int r'^2\, dm \right) \omega' \hat{z} = MR \times V + I_{CM} \omega' \hat{z}$$

Thus, the angular momentum of the body is the sum of the angular momentum of the CM and the angular momentum relative to the CM. □

2.2 Kinetic Energy of a Body

Theorem 5

The kinetic energy of a body can be found by treating the body as a point mass located at the CM, and by then adding on the kinetic energy of the body due to the motion relative to the CM.

Proof. The kinetic energy is:

$$T = \int \frac{1}{2} v^2\, dm = \int \frac{1}{2} |V + v'|^2\, dm$$

Expanding the square:

$$T = \int \frac{1}{2} (V^2 + 2V \cdot v' + v'^2)\, dm$$

By the definition of the CM, $\int v'\, dm = 0$, so the cross term vanishes:

$$T = \frac{1}{2} MV^2 + \int \frac{1}{2} v'^2\, dm$$

Since $v' = \omega' r'$:

$$T = \frac{1}{2} MV^2 + \int \frac{1}{2} (\omega' r')^2\, dm = \frac{1}{2} MV^2 + \frac{1}{2} I_{CM} \omega'^2$$

Thus, the kinetic energy of the body is the sum of the kinetic energy of the CM and the kinetic energy relative to the CM. □

2.3 Parallel-Axis Theorem

Theorem 6

The moment of inertia of a body about any axis parallel to an axis through the CM is equal to the moment of inertia about the CM axis plus the mass of the body times the square of the distance between the two axes.

Proof. Consider a body with mass M and let I_{CM} be the moment of inertia about an axis through the CM. Let R be the distance between the CM and the new axis. The moment of inertia about the new axis is:

$$I = \int (r^2 + R^2)\, dm = \int r^2\, dm + \int R^2\, dm$$

Since R is constant:

$$I = I_{CM} + R^2 \int dm = I_{CM} + MR^2$$

Thus, the moment of inertia about the new axis is the sum of the moment of inertia about the CM axis and MR^2. □

2.4 Perpendicular-Axis Theorem

Theorem 7

For a planar object, the moment of inertia about an axis perpendicular to the plane is equal to the sum of the moments of inertia about two perpendicular axes in the plane.

Proof. Consider a planar object in the x-y plane. The moments of inertia about the x-axis, y-axis, and z-axis are:

$$I_x = \int (y^2 + z^2)\, dm, \quad I_y = \int (z^2 + x^2)\, dm, \quad I_z = \int (x^2 + y^2)\, dm$$

Since the object is planar, $z = 0$:

$$I_x = \int y^2\, dm, \quad I_y = \int x^2\, dm, \quad I_z = \int (x^2 + y^2)\, dm$$

Thus:

$$I_z = I_x + I_y$$

Therefore, the moment of inertia about the z-axis is the sum of the moments of inertia about the x-axis and y-axis. □

2.5 Non-planar Objects

In the previous discussion, we restricted the discussion to pancake objects in the x-y plane. However, nearly all the results we derived carry over to non-planar objects, provided that the axis of rotation is parallel to the z-axis, and provided that we are concerned only with L_z, and not L_x or L_y. So let's drop the pancake assumption and run through the results we obtained above.

First, consider an object rotating around the z-axis. Let the object have extension in the z direction. If we imagine slicing the object into pancakes parallel to the x-y plane, then the equations correctly give L_z for each pancake. And since the L_z of the whole object is simply the sum of the L_z's of all the pancakes, we see that the I_z of the whole object is simply the sum of the I_z's of all the pancakes. The difference in the z values of the pancakes is irrelevant. Therefore, for any object, we have

$$I_z = \int (x^2 + y^2)\, dm, \quad \text{and} \quad L_z = I_z \omega,$$

where the integration runs over the entire volume of the body. We will calculate I_z for many non-planar objects.

Even though the equation gives the L_z for an arbitrary object, the analysis is still not completely general because (1) we are restricting the axis of rotation to be the (fixed) z-axis, and (2) even with this restriction, an object outside the x-y plane might have nonzero x and y components of L; we found only the z-component. This second fact is strange but true. Ponder it for now; we'll deal with it later.

As far as the kinetic energy goes, the T for a non-planar object rotating around the z-axis is still given by the same equation, because we can obtain the total T by simply adding up the T's of each of the pancake slices.

Also, the equations continue to hold for a non-planar object in the case where the CM is translating while the object is spinning around it (or more precisely, spinning around an axis parallel to the z-axis and passing through the CM). The velocity V of the CM can actually point in any direction, and these two equations will still be valid. But we'll assume that all velocities are in the x-y plane.

Lastly, the parallel-axis theorem still holds for non-planar objects. But the perpendicular-axis theorem does not. This is the one instance where we need the planar assumption.

2.6 Finding the Center of Mass

The center of mass has come up repeatedly. For example, when we used the parallel-axis theorem, we needed to know where the CM was. In some cases, such as with a stick or a disk, the location is obvious. But in other cases, it isn't so clear. So let's get a little practice calculating the location of the CM. Depending on whether the mass distribution is discrete or continuous, the position of the CM is defined by

$$R_{CM} = \frac{\sum m_i r_i}{M}, \quad \text{or} \quad R_{CM} = \frac{\int r\, dm}{M},$$

where M is the total mass.

Example 8

Find the location of the CM of a hollow hemispherical shell, with uniform mass density and radius R.

Proof. By symmetry, the CM is located on the line above the center of the base. So our task reduces to finding the height, y_{CM}. Let the mass density be σ. We'll slice the hemisphere up into horizontal rings, described by the angle θ above the horizontal. If the angular thickness of a ring is $d\theta$, then its mass is

$$dm = \sigma\, dA = \sigma(\text{length})(\text{width}) = \sigma(2\pi R\cos\theta)(R\, d\theta).$$

All points on the ring have a y value of $R\sin\theta$. Therefore,

$$y_{CM} = \frac{1}{M}\int y\, dm = \frac{1}{(2\pi R^2)\sigma}\int_0^{\pi/2}(R\sin\theta)(2\pi R^2\sigma\cos\theta\, d\theta) = R\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta = \frac{R\sin^2\theta}{2}\bigg|_0^{\pi/2} = \frac{R}{2}.$$

□

2.7 Torque

We will now show that (under certain conditions, stated below) the rate of change of angular momentum is equal to a certain quantity, τ, which we call the torque. That is, $\tau = \frac{dL}{dt}$. This is the rotational analog of our old friend $F = \frac{dp}{dt}$ involving linear momentum. The basic idea here is straightforward, but there are two subtle issues. One deals with internal forces within a collection of particles. The other deals with origins (the points relative to which the angular momentum is calculated) that are not fixed. To keep things straight, we'll prove the general theorem by dealing with three increasingly complicated situations.

Our derivation of $\tau = \frac{dL}{dt}$ here holds for completely general motion; we can take the result and use it in the following chapter, too. If you wish, you can construct a more specific proof of $\tau = \frac{dL}{dt}$ for the special case where the axis of rotation is parallel to the z-axis. But since the general proof is no more difficult, we'll present it here in this chapter and get it over with.

2.7.1 Point mass, fixed origin

Consider a point mass at position $\vec{r}$ relative to a fixed origin. The time derivative of the angular momentum, $L = \vec{r} \times \vec{p}$, is

$$\frac{dL}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times (m\vec{v}) + \vec{r} \times \vec{F} = 0 + \vec{r} \times \vec{F},$$

where $\vec{F}$ is the force acting on the particle. This is the same proof as in Theorem 6.1, except that here we are considering an arbitrary force instead of a central one. If we define the torque on the particle as

$$\tau \equiv \vec{r} \times \vec{F},$$

then the equation becomes

$$\tau = \frac{dL}{dt}.$$

2.7.2 Extended mass, fixed origin

In an extended object, there are internal forces acting on the various pieces of the object, in addition to whatever external forces exist. For example, the external force on a given atom in a body might come from gravity, while the internal forces come from the adjacent atoms. How do we deal with these different types of forces?

In what follows, we will deal only with internal forces that are central forces, so that the force between two objects is directed along the line between them. This is a valid assumption for the pushing and pulling forces between molecules in a solid. (It isn't valid, for example, when dealing with magnetic forces. But we won't be interested in such things here.) We will invoke Newton's third law, which says that the force that particle 1 applies to particle 2 is equal and opposite to the force that particle 2 applies to particle 1.

For concreteness, let us assume that we have a collection of N discrete particles labeled by the index i. The total angular momentum of the system is

$$L = \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i.$$

The force acting on each particle is $\vec{F}_{\text{ext},i} + \vec{F}_{\text{int},i} = \frac{d\vec{p}_i}{dt}$. Therefore,

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_i \vec{r}_i \times \vec{p}_i\right) = \sum_i \frac{d\vec{r}_i}{dt}\times\vec{p}_i + \sum_i \vec{r}_i \times \frac{d\vec{p}_i}{dt} = \sum_i \vec{v}_i \times (m_i\vec{v}_i) + \sum_i \vec{r}_i \times (\vec{F}_{\text{ext},i} + \vec{F}_{\text{int},i}) = 0 + \sum_i \vec{r}_i \times \vec{F}_{\text{ext},i} = \sum_i \tau_{\text{ext},i}$$

The second-to-last line follows because $\vec{v}_i \times \vec{v}_i = 0$, and also because $\sum_i \vec{r}_i \times \vec{F}_{\text{int},i} = 0$, as you can show in Problem 8. In other words, the internal forces provide no net torque. This is quite reasonable. It basically says that a rigid object with no external forces won't spontaneously start rotating.

Note that the right-hand side involves the total external torque acting on the body, which may come from forces acting at many different points. Note also that nowhere did we assume that the particles were rigidly connected to each other. The equation still holds even if there is relative motion among the particles.

2.7.3 Extended mass, non-fixed origin

Let the position of the origin be $\vec{r}_0$, and let the positions of the particles be $\vec{r}_i$. The vectors $\vec{r}_0$ and $\vec{r}_i$ are measured with respect to a given fixed coordinate system. The total angular momentum of the system, relative to the (possibly moving) origin $\vec{r}_0$, is

$$L = \sum_i (\vec{r}_i - \vec{r}_0) \times m_i(\dot{\vec{r}}_i - \dot{\vec{r}}_0).$$

Therefore,

$$\frac{dL}{dt} = \frac{d}{dt}\left(\sum_i (\vec{r}_i - \vec{r}_0) \times m_i(\dot{\vec{r}}_i - \dot{\vec{r}}_0)\right) = \sum_i (\dot{\vec{r}}_i - \dot{\vec{r}}_0) \times m_i(\dot{\vec{r}}_i - \dot{\vec{r}}_0) + \sum_i (\vec{r}_i - \vec{r}_0) \times m_i(\ddot{\vec{r}}_i - \ddot{\vec{r}}_0) = 0 + \sum_i (\vec{r}_i - \vec{r}_0) \times (\vec{F}_{\text{ext},i} + \vec{F}_{\text{int},i}$$

because $m_i\ddot{\vec{r}}_i$ is the net force (namely $\vec{F}_{\text{ext},i} + \vec{F}_{\text{int},i}$) acting on the ith particle. But a quick corollary of Problem 8 is that the term involving $\vec{F}_{\text{int},i}$ vanishes (show this). And since $\sum m_i\vec{r}_i = M\vec{R}$ (where $M = \sum m_i$ is the total mass, and $\vec{R}$ is the position of the center of mass), we have

$$\frac{dL}{dt} = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_{\text{ext},i} - M(\vec{R} - \vec{r}_0) \times \ddot{\vec{r}}_0.$$

The first term is the external torque, relative to the origin $\vec{r}_0$. The second term is something we wish would go away. And indeed, it usually does. It vanishes if any of the following three conditions is satisfied:

1. $\vec{R} = \vec{r}_0$. That is, the origin is the center of mass.
2. $\ddot{\vec{r}}_0 = 0$. That is, the origin is not accelerating.
3. $(\vec{R} - \vec{r}_0)$ is parallel to $\ddot{\vec{r}}_0$. This condition is rarely invoked.

If any of these conditions is satisfied, then we are free to write

$$\frac{dL}{dt} = \sum_i (\vec{r}_i - \vec{r}_0) \times \vec{F}_{\text{ext},i} \equiv \sum_i \tau_{\text{ext},i}.$$

In other words, we can equate the total torque with the rate of change of the total angular momentum. An immediate corollary of this result is:

Corollary 9

If the total torque on a system is zero, then its angular momentum is conserved. In particular, the angular momentum of an isolated system (one that is subject to no external forces) is conserved.

Everything up to this point is valid for arbitrary motion. The particles can be moving relative to each other, and the various L_i's can point in different directions, etc. But let's now restrict the motion. In the present chapter, we are dealing only with cases where $\hat{L}$ is constant (taken to point in the z-direction). Therefore, $\frac{dL}{dt} = \frac{d(L\hat{L})}{dt} = \left(\frac{dL}{dt}\right)\hat{L}$. If in addition we consider only rigid objects (where the relative distances among the particles are fixed) that undergo pure rotation around a given point, then $L = I\omega$, which gives $\frac{dL}{dt} = I\dot{\omega} \equiv I\alpha$. Taking the magnitude of both sides of the equation then gives

$$\tau = I\alpha.$$

Invariably, we will calculate angular momentum and torque around either a fixed point or the center of mass. These are "safe" origins, in the sense that the equation holds. As long as you vow to always use one of these safe origins, you can simply apply the equation and not worry much about its derivation.

Example 10

A string wraps around a uniform cylinder of mass M, which rests on a fixed plane. The string passes up over a massless pulley and is connected to a mass m. Assume that the cylinder rolls without slipping on the plane, and that the string is parallel to the plane. What is the acceleration of the mass m? What is the minimum value of M/m for which the cylinder accelerates down the plane?

Proof. The friction, tension, and gravitational forces are shown in the figure below. Define positive a_1, a_2, and α as shown. These three accelerations, along with T and F, are five unknowns. We therefore need to produce five equations. They are:

1. $F = ma$ on $m \Rightarrow T - mg = ma_2$.
2. $F = ma$ on $M \Rightarrow Mg\sin\theta - T - F = Ma_1$.
3. $\tau = I\alpha$ on M (around the center of mass) $\Rightarrow FR - TR = \left(\frac{MR^2}{2}\right)\alpha$.
4. Non-slipping condition $\Rightarrow \alpha = \frac{a_1}{R}$.
5. Conservation of string $\Rightarrow a_2 = 2a_1$.

A few comments on these equations: The normal force and the gravitational force perpendicular to the plane cancel, so we can ignore them. We have picked positive F to point up the plane, but if it happens to point down the plane and thereby turn out to be negative, that's fine (but it won't); we don't need to worry about which way it really points. In (3), we are using the center of mass of the cylinder as our origin, but we can also use a fixed point; see the remark below. In (5), we have used the fact that the top of a rolling wheel moves twice as fast as the center. This is true because it has the same speed relative to the center as the center had relative to the ground.

We can go about solving these five equations in various ways. Three of the equations involve only two variables, so it's not so bad. (3) and (4) give $F - T = \frac{Ma_1}{2}$. Adding this to (2) gives $Mg\sin\theta - 2T = \frac{3Ma_1}{2}$. Using (1) to eliminate T, and using (5) to write a_1 in terms of a_2, then gives

$$Mg\sin\theta - 2(mg + ma_2) = \frac{3Ma_2}{4} \Rightarrow a_2 = \frac{(M\sin\theta - 2m)g}{\frac{3M}{4} + 2m}.$$

And $a_1 = \frac{a_2}{2}$. We see that a_1 is positive (that is, the cylinder rolls down the plane) if $M/m > \frac{2}{\sin\theta}$.

Remark: In using $\tau = \frac{dL}{dt}$, we can also pick a fixed point as our origin, instead of the center of mass. The most sensible point is one located somewhere along the plane. The $Mg\sin\theta$ force now provides a torque, but the friction does not. The angular momentum of the cylinder with respect to a point on the plane is $I\omega + MvR$, where the second term comes from the L due to the object being treated like a point mass at the center of mass. So $\tau = \frac{dL}{dt}$ becomes $(Mg\sin\theta)R - T(2R) = I\alpha + Ma_1R$. This is simply the sum of the third equation plus R times the second equation above. We therefore obtain the same result. □

2.8 Collisions

In Section 4.7, we looked at collisions involving point particles (or otherwise nonrotating objects). The fundamental ingredients we used to solve a collision problem were conservation of momentum and conservation of energy (if the collision was elastic). With conservation of angular momentum now at our disposal, we can extend our study of collisions to ones that contain rotating objects. The additional fact of conservation of L will be compensated for by the new degree of freedom for the rotation. Therefore, provided that the problem is set up properly, we will still have the same number of equations as unknowns.

In an isolated system, conservation of energy can be used only if the collision is elastic (by definition). But conservation of angular momentum is similar to conservation of momentum, in that it can always be used. However, conservation of L is a little different from conservation of p, because we have to pick an origin before we can proceed. In view of the three conditions that are necessary for Corollary 7.3 to hold, we must pick our origin to be either a fixed point or the CM of the system (we'll ignore the third condition, since it's rarely used). If we choose some other point, then $\tau = \frac{dL}{dt}$ does not hold, so we have no right to claim that $\frac{dL}{dt}$ equals zero just because the torque is zero (as it is for an isolated system).

There is, of course, some freedom in choosing an origin from among the legal possibilities of fixed points or the CM. And since it is generally the case that one choice is better than the others (in that it makes the calculations easier), you should take advantage of this freedom.

Let's do two examples. First, an elastic collision, and then an inelastic one.

2.8.1 Elastic Collision

Example 11

A mass m travels perpendicularly to a stick of mass m and length ℓ, which is initially at rest. At what location should the mass collide elastically with the stick, so that the mass and the center of the stick move with equal speeds after the collision?

Proof. Let the initial speed of the mass be v_0. We have three unknowns in the problem, namely the desired distance from the middle of the stick, h; the final (equal) speeds of the stick and the mass, v; and the final angular speed of the stick, ω. We can solve for these three unknowns by using our three available conservation laws:

Conservation of p:

$$mv_0 = mv + mv \implies v = \frac{v_0}{2}$$

Conservation of E:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}m\left(\frac{v_0}{2}\right)^2 + \frac{1}{2}\left(\frac{m\ell^2}{12}\right)\omega^2 \implies \omega = \frac{\sqrt{6}v_0}{\ell}$$

Conservation of L: Let's pick our origin to be the fixed point in space that coincides with the initial location of the center of the stick. Then conservation of L gives:

$$mv_0h = m\left(\frac{v_0}{2}\right)h + \left(\frac{m\ell^2}{12}\right)\omega + 0$$

The zero here comes from the fact that the CM of the stick moves directly away from the origin, so there is no contribution to L from the first of the two parts in Theorem 7.1. Plugging the ω from the previous equation into this equation gives:

$$\frac{1}{2}mv_0h = \left(\frac{m\ell^2}{12}\right)\left(\frac{\sqrt{6}v_0}{\ell}\right) \implies h = \frac{\ell}{\sqrt{6}}$$

□

2.8.2 Inelastic Collision

Example 12

A mass m travels at speed v_0 perpendicularly to a stick of mass m and length ℓ, which is initially at rest. The mass collides completely inelastically with the stick at one of its ends, and sticks to it. What is the resulting angular velocity of the system?

Proof. The first thing to note is that the CM of the system is $\frac{\ell}{4}$ from the end. The system will rotate about the CM as the CM moves in a straight line. Conservation of momentum quickly tells us that the speed of the CM is $\frac{v_0}{2}$. Also, using the parallel-axis theorem, the moment of inertia of the system about the CM is:

$$I_{CM} = I_{\text{stick, CM}} + I_{\text{mass, CM}} = \left(\frac{m\ell^2}{12} + m\left(\frac{\ell}{4}\right)^2\right) + m\left(\frac{\ell}{4}\right)^2 = \frac{5}{24}m\ell^2$$

There are now many ways to proceed, depending on what point we choose as our origin.

First method: Choose the origin to be the fixed point that coincides with the location of the CM right when the collision happens (that is, the point $\frac{\ell}{4}$ from the end of the stick). Conservation of L says that the initial L of the ball must equal the final L of the system. This gives:

$$mv_0\left(\frac{\ell}{4}\right) = \left(\frac{5}{24}m\ell^2\right)\omega + 0 \implies \omega = \frac{6v_0}{5\ell}$$

The zero here comes from the fact that the CM of the stick moves directly away from the origin, so there is no contribution to L from the first of the two parts in Theorem 7.1. Note that we didn't need to use conservation of p in this method.

Second method: Choose the origin to be the fixed point that coincides with the initial center of the stick. Then conservation of L gives:

$$mv_0\left(\frac{\ell}{2}\right) = \left(\frac{5}{24}m\ell^2\right)\omega + (2m)\left(\frac{v_0}{2}\right)\left(\frac{\ell}{4}\right) \implies \omega = \frac{6v_0}{5\ell}$$

Third method: Choose the origin to be the CM of the system. This point moves to the right with speed $\frac{v_0}{2}$, along the line a distance $\frac{\ell}{4}$ below the top of the stick. Relative to the CM, the mass m moves to the right, and the stick moves to the left, both with speed $\frac{v_0}{2}$. Conservation of L gives:

$$m\left(\frac{v_0}{2}\right)\left(\frac{\ell}{4}\right) + \left(0 + m\left(\frac{v_0}{2}\right)\left(\frac{\ell}{4}\right)\right) = \left(\frac{5}{24}m\ell^2\right)\omega \implies \omega = \frac{6v_0}{5\ell}$$

The zero here comes from the fact that the stick initially has no L around its center.

A fourth reasonable choice for the origin is the fixed point that coincides with the initial location of the top of the stick. You can work this one out for practice. □

2.9 Angular Impulse

In Section 4.5.1, we defined the impulse, I, to be the time integral of the force applied to an object, which is the net change in linear momentum. That is,

$$I \equiv \int_{t_1}^{t_2} F(t)\,dt = \Delta p$$

We now define the angular impulse, I_θ, to be the time integral of the torque applied to an object, which is the net change in angular momentum. That is,

$$I_\theta \equiv \int_{t_1}^{t_2} \tau(t)\,dt = \Delta L$$

These are just definitions, devoid of any content. The place where the physics comes in is the following. Consider a situation where $F(t)$ is always applied at the same position relative to the origin around which $\tau(t)$ is calculated. Let this position be R. Then we have $\tau(t) = R \times F(t)$. Plugging this into the previous equation, and taking the constant R outside the integral, gives $I_\theta = R \times I$. That is,

$$\Delta L = R \times (\Delta p) \quad \text{(for } F(t) \text{ applied at one position)}$$

This is a very useful result. It deals with the net changes in L and p, and not with their changes at any particular instant. Even if F is changing in some arbitrary manner as time goes by, so that we have no idea what Δp and ΔL are, we still know that they are related by the previous equation. Also, note that the derivation of the previous equation was completely general, so we can apply it in the next chapter, too.

In many cases, we don't have to worry about the cross product in the previous equation, because the lever arm R is perpendicular to the change in momentum Δp. In such cases, we have

$$|\Delta L| = R|\Delta p|$$

Also, in many cases the object starts at rest, so we don't have to bother with the Δ's. The following example is a classic application of angular impulse and the previous equation.

2.9.1 Striking a Stick

Example 13

A stick of mass m and length ℓ, initially at rest, is struck with a hammer. The blow is made perpendicular to the stick, at one end. Let the blow occur quickly, so that the stick doesn't move much while the hammer is in contact. If the CM of the stick ends up moving at speed v, what are the velocities of the ends, right after the blow?

Proof. We have no idea exactly what $F(t)$ looks like, or for how long it is applied, but we do know from the previous equation that $\Delta L = \left(\frac{\ell}{2}\right)\Delta p$, where L is calculated relative to the CM (so the lever arm is $\ell/2$). Therefore, $\left(\frac{m\ell^2}{12}\right)\omega = \left(\frac{\ell}{2}\right)mv$. Hence, the final v and ω are related by $\omega = \frac{6v}{\ell}$.

The velocities of the ends are obtained by adding (or subtracting) the rotational motion to the CM's translational motion. The rotational velocities of the ends are $\pm\omega\left(\frac{\ell}{2}\right) = \pm\left(\frac{6v}{\ell}\right)\left(\frac{\ell}{2}\right) = \pm 3v$. Therefore, the end that was hit moves with velocity $v + 3v = 4v$, and the other end moves with velocity $v - 3v = -2v$ (that is, backwards). $\square$