18.701: Algebra I

Lecturer: Professor Mike Artin

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(Professor Artin wants us to call him Mike).

First of all, it's good for us to read through the syllabus. On the back page, there is a diagnostic problem that is pretty simple; we should have it done by Monday. It will count for our grade, but it is not required. Apparently, the problem sets are hard: we should not expect to finish them quickly.

Fact 1

The two main topics of this semester will be **group theory** and **ring theory**.

When professor Artin was young, he wanted to learn the "general axioms". But it's better to use examples and use those to understand the axioms when we're learning mathematics.

Definition 2

The **general linear group**, denoted as GL_n, consists of the invertible $n \times n$ matrices with the operation of matrix multiplication.

(The definition of a **group** was given on the next day.) To make examples, easier to write down, we'll take $n = 2$. Matrix multiplication looks like the following:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \cdot 5 + 2 \cdot 7 & 1 \cdot 6 + 2 \cdot 8 \\ 3 \cdot 5 + 4 \cdot 7 & 3 \cdot 6 + 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

The definition of matrix multiplication seems kind of complicated, but it turns out we can come up with a natural explanation. One way to explain this definition is by looking at column vectors, with matrices "acting from the left and side:" if V is a space of 2-dimensional column vectors, we can treat our matrix A as a linear operator on V, where a vector $v \in V$ gets sent to Av.

Fact 3

Given two matrices A and B, it is generally not true that $AB = BA$. (For example, the matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ do not commute.) However, matrix multiplication is associative (that is $A(BC) = (AB)C$), and we know this because we're just computing three transformations in the same order: C, then B, then A.

In this class, we'll generally deal with invertible matrices (because they make our group operations nicer.) By the way, if we don't know this already, the inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Definition 4

An element A in a group has **order** n if A^n is an identity element.

Example 5

Consider a matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}$. Since $A^6 = I$, A is an element of GL_2 with order 6.

Elements can have infinite order as well, but it turns out the space of 2×2 matrix is nice:

Theorem 6

If entries of a 2×2 matrix are rational and the order is finite, it must be 1, 2, 3, 4, or 6.

(We'll prove this much later on.) Professor Artin like to use dots instead of zeros in matrices because they looke cleaner, but I will not do this in the notes.

Example 7

The following matrix just cycles the indices of a vector, so it has order n if it is n-dimensional.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{n \times n}$$

There are three kinds of **elementary matrices**, which basically change the identity matrix by a tiny bit. (This is idea of **row reducing**.) We have the matrices

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$$

which add a times the second row and vice versa, the matrices

$$\begin{bmatrix} c & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix}$$

which multiplies one of the two rows by c, and the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which swaps two rows.

Theorem 8

The elementary matrices generate GL_2. In other words, every A in GL_2 is a product of the above elementary matrices.

Let's say we start with an arbitrary matrix A say

$$M = \begin{bmatrix} 8 & 5 \\ 4 & 6 \end{bmatrix}$$

It's hard to randomly find matrices $E_1, E_2, \cdots$ that multiply M. Instead, we should work backwards, and try to write

$$E_k \cdots E_2 E_1 M = I$$

Then we know that $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$, since all elementary matrices have elementary matrix inverses. I'm not going to include how to do this here, but we basically work one column at a time and try to get the matrix to the identity.