Math 4547: Real Analysis I

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1 January 6, 2025

Professor Margolis introduced the course and discussed the syllabus. The course will cover the following topics: Here are some common number systems

1.1 What is Analysis?

Analysis is the branch of mathematics that deals with the rigorous study of limits, functions, derivatives, integrals, and infinite series. It provides the foundation for calculus and extends its concepts to more abstract settings.

Theorem 1
Every convergent squence is bounded.

1.2 The Real Numbers

1.2.1 What are the reals?

- The **natural Numbers** $\mathbb{N} = \{1, 2, 3, ...\}$
- The **integers** $\mathbb{Z} = \{0, 1, \text{-}1, 2, \text{-}2, \cdots\}$
- The **rational Numbers** $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
- The **real Numbers** $\mathbb{R}$
- The **complex Numbers** $\mathbb{C}: = \{\ a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$

Theorem 2
There is no rational number x, such that $x^2 = 2$.

Proof. We assume for contradiction that such an x exists. Then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. We can assume that p and q have no common factors. Then, $\frac{p^2}{q^2} = 2$, which implies

$$p^2 = 2q^2$$

Thus, p^2 is even. As the square of an odd number is odd, it follows p must even. Therefore, $p = 2k$ for an integer k. We have $2q^2 = p^2 = (2k)^2 = 4k^2$, and so $q^2 = 2k^2$. Thus, q^2 is even. Since p and q are both even, this contradicts our assumption that p and q have no common factors. Therefore, there is no rational number x such that $x^2 = 2$. $\square$

This theorem implies, if we visualize $\mathbb{Q}$ as points lying on a number line, there is a 'hole' where $\sqrt{2}$ is. (There are many more 'holes' e.g. π, e, $\sqrt{3}$, ...)

The key property that $\mathbb{R}$ possesses, but $\mathbb{Q}$ doesn't is that $\mathbb{R}$ has "no holes" (formally, $\mathbb{R}$ is complete.)

In this class, we will rigorously deduce all properties of $\mathbb{R}$ from the axioms of the real numbers.

The axioms are in three groups.

1. Field Axioms (addition and multiplication)
2. Order axioms (needed to describe properties concerning inequalities)
3. Completeness Axiom

1.3 Addition axioms

1. For every pair $a, b \in \mathbb{R}$, we can associate a real number $a + b$ called their **sum**.
2. For every real number a, there is a real number $-a$ called its **negative** or **additive inverse**.
3. There is a special real number 0 called zero or the additive identity such that for all $a, b, c, x, y, z \cdots$ are real numbers unless otherwise stated:
   (a) $a + b = b + a$
   (b) $a + (b + c) = (a + b) + c$
   (c) $a + 0 = a$
   (d) $a + (-a) = 0$

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In this lecture, we will use the axioms to deduce various properties of the real numbers $\mathbb{R}$. From these axioms, we can derive many more properties of the real numbers.

Proposition 3

If $x + a = x$ for all $a \in \mathbb{R}$, then $a = 0$.

Proof. We know that

$$\begin{aligned} x &= x + 0 \quad \text{(A3)} \\ &= x + a \quad \text{(by assumption on } a\text{)} \end{aligned}$$

By the left cancellation property of addition, it follows that $a = 0$. □

Proposition 4 (Left cancellation of addition)
If $a + x = a + y$, then $x = y$.

Proof. We start with the given equation $a + x = a + y$. By the additive identity property (A3), we have:

$$\begin{aligned} y &= y + 0 && \text{(A3)} \\ &= y + (a + (-a)) && \text{(A4)} \\ &= (y + a) + (-a) && \text{(A2)} \\ &= (a + y) + (-a) && \text{(A1)} \\ &= (a + x) + (-a) && \text{(given)} \\ &= x + (a + (-a)) && \text{(A1)} \\ &= x + 0 && \text{(A4)} \\ &= x && \text{(A3)} \end{aligned}$$

Therefore, $x = y$. □

Proposition 5
$-(-a) = a$

Proof. We need to show that $-(-a) = a$. Consider the following:

$$\begin{aligned} (-a) + (-(-a)) &= 0 \quad \text{(by definition of additive inverse)} \\ (-a) + a &= 0 \quad \text{(since } -(-a) = a\text{)} \\ a + (-a) &= 0 \quad \text{(by commutativity of addition)} \\ (-a) + (-(-a)) &= a + (-a) \quad \text{(by substitution)} \\ (-(-a)) &= a \quad \text{(by left cancellation of addition)} \end{aligned}$$

Therefore, $-(-a) = a$. □

Proposition 6
$-(a + b) = (-a) + (-b)$

Proof. We need to show that the additive inverse of $(a+b)$ is equal to the sum of the additive inverses of a and b. Consider the following:

$$\begin{aligned}
(a+b)+(-(a+b)) &= 0 \quad \text{(by definition of additive inverse)} \\
(a+b)+((-a)+(-b)) &= a+(b+((-a)+(-b))) \quad \text{(by associativity of addition)} \\
&= a+((b+(-a))+(-b)) \quad \text{(by associativity of addition)} \\
&= a+((-a)+(b+(-b))) \quad \text{(by commutativity of addition)} \\
&= a+((-a)+0) \quad \text{(by definition of additive inverse)} \\
&= a+(-a) \quad \text{(by identity property of addition)} \\
&= 0 \quad \text{(by definition of additive inverse)}
\end{aligned}$$

Therefore, $-(a+b)=(-a)+(-b)$. □

Proposition 7

$-0=0$

Proof. We need to show that the additive inverse of 0 is 0. Consider the following:

$$\begin{aligned}
0+0 &= 0 \quad \text{(by the identity property of addition, A3)} \\
0+(-0) &= 0 \quad \text{(by the definition of additive inverse, A4)}
\end{aligned}$$

Therefore, we have:

$$0+0=0+(-0)$$

By the left cancellation property of addition, it follows that:

$$0=-0$$

Therefore, $-0=0$. □

2.1 Multiplication Axioms

Definition 8

For all $a, b \in \mathbb{R}$, we can associate a real number $a \times b$ called their **product**.

Definition 9

For every $a \in \mathbb{R}$, there is some $a^{-1} \in \mathbb{R}$ called its **multiplicative inverse** or **reciprocal** such that for all $a \neq 0$, $a \times a^{-1} = 1$.

Definition 10

There is a number 1 called **one** or the **multiplicative identity** such that for all $a \in \mathbb{R}$, $a \times 1 = a$.

Definition 11

For all $a, b, c \in \mathbb{R}$, we have the following properties of multiplication:

- For all $a, b \in \mathbb{R}$, $a \times b = b \times a$.
- For all $a, b, c \in \mathbb{R}$, $a \times (b \times c) = (a \times b) \times c$.
- For all $a \in \mathbb{R}$, $a \times 0 = 0$.
- For all $a, b \in \mathbb{R}$, $a \times (b + c) = a \times b + a \times c$.

Proposition 12

If $a \times b = a$, and $a \in \mathbb{R}$ then $b = 1$.

Proof. We start with the given equation $a \times b = a$. By the multiplicative identity property, we have:

$$\begin{aligned} a \times b &= a \times 1 \\ b &= 1 \quad \text{(by left cancellation of multiplication)} \end{aligned}$$

Therefore, $b = 1$. □

Proposition 13

If $a \neq 0$ and $a \times b = a \times c$, then $b = c$.

Proof. We start with the given equation $a \times b = a \times c$. By the multiplicative inverse property, we have:

$$\begin{aligned} a^{-1} \times (a \times b) &= a^{-1} \times (a \times c) \\ (a^{-1} \times a) \times b &= (a^{-1} \times a) \times c \\ 1 \times b &= 1 \times c \\ b &= c \end{aligned}$$

Therefore, $b = c$. □

Proposition 14

If $a \neq 0$ and $a^{-1} \neq 0$, then $(a^{-1})^{-1} = a$.

Proof. We need to show that the multiplicative inverse of a^{-1} is a. Consider the following:

$$\begin{aligned} a^{-1} \times a &= 1 && \text{(by definition of multiplicative inverse)} \\ (a^{-1})^{-1} \times a^{-1} &= 1 && \text{(by definition of multiplicative inverse)} \\ (a^{-1})^{-1} &= a \end{aligned}$$

Therefore, $(a^{-1})^{-1} = a$. $\square$

Proposition 15

If $a \neq 0$, $b \neq 0$, and $a \times b \neq 0$, then $(a \times b)^{-1} = a^{-1} \times b^{-1}$.

Proof. We need to show that the multiplicative inverse of $a \times b$ is $a^{-1} \times b^{-1}$. Consider the following:

$$\begin{aligned} (a \times b) \times (a^{-1} \times b^{-1}) &= a \times (b \times (a^{-1} \times b^{-1})) \\ &= a \times ((b \times a^{-1}) \times b^{-1}) \\ &= a \times (1 \times b^{-1}) \\ &= a \times b^{-1} \\ &= 1 \end{aligned}$$

Therefore, $(a \times b)^{-1} = a^{-1} \times b^{-1}$. $\square$

Proposition 16

If $a, b, c \in \mathbb{R}$, then $(a + b) \times c = (a \times c) + (b \times c)$.

Proof. We need to show that the product of $(a + b)$ and c is equal to the sum of the products of a and c, and b and c. Consider the following:

$$\begin{aligned} (a + b) \times c &= c \times (a + b) \\ &= c \times a + c \times b \\ &= a \times c + b \times c \end{aligned}$$

Therefore, $(a + b) \times c = (a \times c) + (b \times c)$. $\square$

Proposition 17

For all $a \in \mathbb{R}$, $a \times 0 = 0$.

Proof. We need to show that the product of any real number a and 0 is 0. Consider the following:

$$\begin{aligned} a \times 0 &= a \times (0 + 0) \\ &= a \times 0 + a \times 0 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $a \times 0 = 0$. □

Proposition 18

If $a \times b = 0$, then either $a = 0$ or $b = 0$ or both.

Proof. We need to show that if the product of a and b is 0, then either a or b or both must be 0. Consider the following:

$$a \times b = 0$$

If $a \neq 0$, then $b = 0$ by the multiplicative inverse property. If $b \neq 0$, then $a = 0$ by the multiplicative inverse property. Therefore, if $a \times b = 0$, then either $a = 0$ or $b = 0$ or both. □

Proposition 19

$a \times (-b) = (-a) \times b$. In particular, $a \times (-1) = -a$.

Proof. We need to show that the product of a and $-b$ is equal to the product of $-a$ and b. Consider the following:

$$\begin{aligned} a \times (-b) + a \times b &= a \times (b + (-b)) \\ &= a \times 0 \\ &= 0 \\ &= a \times b + (-(a \times b)) \end{aligned}$$

Hence, the additive inverse of $a \times b$ is $-(a \times b)$. Therefore, $a \times (-b) = (-a) \times b$. □

Proposition 20

$(-1) \times (-1) = 1$

Proof. We need to show that the product of -1 and -1 is 1. Consider the following:

$$\begin{aligned}(-1) \times (-1) &= -(-1) \times 1 \\ &= -(-1) \times (1 + 0) \\ &= -(-1) \times (1 + (-1)) \\ &= -(-1) \times 0 \\ &= 0 \\ &= (-1) + (-1)\end{aligned}$$

Therefore, $(-1) \times (-1) = 1$. $\square$

3 January 10, 2025

For all $a, b \in \mathbb{R}$, we write:

- ab or $a \cdot b$ for $a \times b$.
- $a - b$ for $a + (-b)$.
- $\frac{1}{a}$ for a^{-1} if $a \neq 0$.
- $\frac{a}{b}$ for ab^{-1} if $b \neq 0$.

For $a \neq 0$, we write:

- a^0 for 1.
- a^{k+1} for $a^k \cdot a$ for $k = 0, 1, 2, \ldots$
- $a^{-1} for (a^l)^{-1}$ for $l = 1, 2, 3$

Definition 21

Any set equipped with operations $+$ and $\times$ staisfying A1 - A4, M1 - M4, Z, D is a **field**.

Fact 22

Some facts about the fields:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are all fields.
- $\mathbb{Z}$ is not a field (M4 isn't satisified).
- $\mathbb{N}$ is not a field ($A4$, $M4$) are not satsified.
- $\frac{\mathbb{Z}}{p\mathbb{Z}}$ (integers mod p for prime p) is a field.

3.1 The order axioms

The order axioms are: There is as subset of $P \subset \mathbb{R}$ called the set of **positive numbers**.

- If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$. (P1)
- If $a, b \in \mathbb{P}$, then $a \times b \in \mathbb{P}$. (P2)
- For each $a \in \mathbb{R}$, exactly one of the following is true: $a \in \mathbb{P}$, $a = 0$, or $-a \in \mathbb{P}$. $\leftarrow$ Law of Trichotomy (P3)

P3 is the most powerful aximom about the positive numbers.

Proposition 23
Prove that $1 \in \mathbb{P}$

Proof. According to **P3**, either

- $1 \in \mathbb{P}$
- $1 = 0$
- $-1 \in \mathbb{P}$

We will prove (b) and (c) are false by contradiction and then show that $1 \in \mathbb{P}$. If **(b)** holds, $1 = 0$, which contradicts **Z**. Assume for contradiction **(c)** holds. We know from last lecture that $1 = -(-1)$. Since $-1 \in \mathbb{P}$, by (P2), $(-1) \times (-1) \in \mathbb{P}$. But $(-1) \times (-1) = 1$, so $1 \in \mathbb{P}$. $\therefore$, $1 \in \mathbb{P}$ and $-1 \in \mathbb{P}$ contradicts **P3**. Since, **(b)** and **(c)** cannot hold, therefore, **(a)** must hold. $\square$

Fact 24
For all $a, b \in \mathbb{R}$, we write

- $a < b$ if $b - a \in \mathbb{P}$
- $a > b$ if $a - b \in \mathbb{P}$
- $a \leq b$ if $b - a \in \mathbb{P} \cup \{0\}$
- $a \geq b$ if $a - b \in \mathbb{P} \cup \{0\}$

Proposition 25
$a > b$ if and only if $-a < -b$. In particular, $x > 0 \iff -x < 0$

Proof.

$$\begin{aligned} a > b &\iff a - b \in \mathbb{P} \\ &\iff -(-(a)) - b \in -\mathbb{P} \\ &\iff -b - (-a) \in -\mathbb{P} \\ &\iff -a < -b \end{aligned}$$

$\square$

Proposition 26

For all $x, y, z \in \mathbb{R}$ the following holds:

- $x \leq x$
- If $x \leq y$ and $y \leq z$, then $x \leq z$.
- If $x \leq y$ and $y \leq z$, then $x \leq z$.

Proof. $\square$

Proposition 27

If $x, t, z \in \mathbb{R}$ and $x < y$, then $x + z < y + z$.

Proof. Since $x < y$, we have $x - y \in \mathbb{P}$. By the properties of addition (A1-A4), we know that:

$$(y + z) - (x + z) = y - x$$

Since $y - x \in \mathbb{P}$, it follows that:

$$(y + z) - (x + z) \in \mathbb{P}$$

Hence, $x + z < y + z$. $\square$

Proposition 28

If $x, y, z \in \mathbb{R}$ and $x < y$ and $z > 0$, then $xz < yz$.

Proof. $zy = zx = z(y - x)$. Now, $z \in \mathbb{P}$ and $y - x \in \mathbb{P}$, therefore $zy - zx \in \mathbb{P}$. Therefore, $xz < yz$. $\square$

Corollary 29

If $x, y, z \in \mathbb{R}$ and $x < y$ and $z < 0$, then $xz > yz$.

Proof. $\square$

Corollary 30

For all, $a \in \mathbb{R}$, $a^2 \geq 0$.

Proof. By P3, either $a > 0, a = 0$ or $g < 0$.

- If $a > 0$, then $a^2 = a \times a > 0$. (P2)
- If $a = 0$, then $a^2 = 0 \geq 0$.

- If $a < 0$, then $-a > 0$ and $(-a)^2 = a^2 > 0$.

$\square$

Proposition 31

If $x \in \mathbb{P}$, then $x^{-1} \in \mathbb{P}$.

Proof. Since, $x \in \mathbb{P}, x \neq 0$. Therefore, x^{-1} exists. By P3, $x^{-1} > 0, x^{-1} = 0$, or $x^{-1} < 0$. If $x^{-1} = 0$, then $1 = x \times x^{-1} = x \times 0 = 0$ [Contradiction] Assume $x^{-1} < 0$. Then $-x^{-1} \in \mathbb{P}$ by P3. Then $x \times (-x^{-1}) \in \mathbb{P}$ by P2. But $x \times (-x^{-1}) = -1$, which contradicts P3 since $-1 \notin \mathbb{P}$. Therefore, $x^{-1} \in \mathbb{P}$. $\square$

Corollary 32

If $x, y \in \mathbb{P}$, and $x < y$, then $\frac{1}{y} < \frac{1}{x}$.

4 January 13, 2025

Homework 1 is due on January 21, 2025.

Definition 33

We define $\max\colon \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{R}$ by

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b \geq a \end{cases}$$

Definition 34

We define $\max\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\max(a, b) = \begin{cases} a & \text{if } a \geq b \\ b & \text{if } b > a \end{cases}$$

Definition 35

We define $|x| \ : \mathbb{R} \to \mathbb{R}$ as follows:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Proposition 36

For all $x \in \mathbb{R}$, $|-x| = |x|$.

Proof. By P3, x > 0, x = 0, or x < 0.

- Case 1: If $x > 0$, then $|x| = x$ and $|-x| = -(-x) = x$. Thus, $|x| = |-x|$.
- Case 2: If $x = 0$, then $|x| = 0$ and $|-x| = -0 = 0$. Thus, $|x| = |-x|$.
- Case 3: If $x < 0$, then $|x| = -x$ and $|-x| = -(-x) = x$. Thus, $|x| = |-x|$.

□

Theorem 37 (The Triangle Δ Inequality)
For all $a, b, \in \mathbb{R}$

$$|a+b| =\leq |a| + |b|$$

with equality if and only if either $a \geq 0$ and $b \geq 0$ or $a \leq 0$ and $b \leq 0$.

Proof. By P3, oen of the following 8 Cases must hold:

	a	b	a + b
1	≥ 0	≥ 0	Row 2, Col 4
2	≥ 0	≥ 0	Row 3, Col 4
3	≥ 0	< 0	Row 4, Col 4
4	≥ 0	< 0	Row 5, Col 4
5	< 0	Row 6, Col 3	Row 6, Col 4
6	< 0	Row 7, Col 3	Row 7, Col 4
7	< 0	Row 8, Col 3	Row 8, Col 4
8	< 0	Row 8, Col 3	Row 8, Col 4

Case 2 and 7 is not possible. But we will prove the rest of the cases:

(1) $|a| = a$, $|b| = b$, $|a+b| = a+b$, $\therefore$ $|a+b| = a+b = |a| + b$

(3)

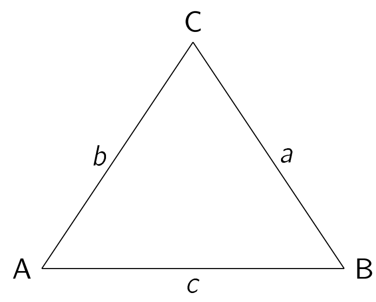
(4) $|a| = a, |b| = -b, |a+b| = a+b$

$$\begin{aligned} |a+b| = -a - b = -0 - b = (-a \leq 0) \\ &\leq a + 0 \quad (\text{since } b < 0) \\ &= |a| + |b| \end{aligned}$$

(5) Follows the similarly by symmetry.

Finish this for exercise. □

The picture of the Triangle identity



$$|a| = ||\vec{BC}|| \leq |b| + |c| = ||\vec{AC}|| + ||\vec{AB}||$$

Proposition 38

For all $a, b, \in \mathbb{R}$

$$|ab| = |a| \cdot |b|$$

Proof. If $a = 0$ or $b = 0$, then $|ab| = |0| = 0 = |a| \cdot |b|$ Let's assume that $a \neq 0, b \neq 0$.

- $a > 0, b > 0$ Then P2 implies $ab > 0$ so, $|ab| = ab = |a||b|$
- $a < 0, b > 0$. Then $ab < 0$. Then $|ab| = -ab = (-a)b = |a||b|$
- $a > 0, b < 0$. This follows from case 2 by symmetry.
- $a < 0, b < 0$.Then $ab > 0$. Hence, $|ab| = ab = (-a)(-b) = |a||b|$

□

Theorem 39 (Bernouli's Inequality)

For all $x \in \mathbb{R}$ with $x > -1$ and $n \in \mathbb{N}$, if $n \geq 1$, then

$$(1 + x)^n \geq 1 + nx$$

Proof. We proceed by induction on n.
Base Case: $n = 1 : (1 + x)^1 = 1 + x = 1 + 1 \cdot x$
Inductive Step: Assume

$$(1 + x)^N \geq 1 + Nx$$

We want to show $(1 + x)^{N+1} \geq 1 + (N + 1) \cdot x$. First, since $x > -1$, $x + 1 > 0$.
Multiplying both sides by $x + 1$,

$$\begin{aligned}(1 + x)(1 + x)^N &\geq (1 + Nx)(1 + x) \\ &= 1 + (N + 1)x + Nx^2 \quad \text{(field axioms)} \\ &\geq 1 + (N + 1)x \quad (\text{since } N > 0,\ x^2 > 0)\end{aligned}$$

Hence, $(1 + x)^{N+1} \geq 1 + (N + 1)x$. □

5 January 15, 2025

5.1 The completeness axiom

Let $B \subseteq \mathbb{R}$. We say the following:

- We say $b_1 \subseteq$ is a <u>least element</u> or <u>minimum</u> of B if
  - $b_1 \in B$, and
  - $b_1 \leq b$ for all $b \in B$.

  We write $b_1 = \min B$.
- We say $b_1 \subseteq$ is a <u>least element</u> or <u>minimum</u> of B if
  - $b_1 \in B$, and
  - $b_1 \leq b$ for all $b \in B$.

  We write $b_1 = \min B$.

Example 40

Let $B = \{1, 2, 3\}$. Then $\min B = 1$ and $\max B = 3$.

Proposition 41

Let $B \subseteq R$. The maximum of B (if it exists) is unique. Let Similarly, the minimum of B is unique.

Proof. Suppose $a, b \in \mathbb{R}$ are both maximum of B. Since $a \in B$ and b is a max of B, $a, \leqslant b$. Similarly, $b \leqslant a$.Since $a \leqslant b$ and $b \leqslant a$, $a = b$ □

Definition 42

Let $B \subseteq \mathbb{R}$.

- We say h is a **lower bound** of B if $h \leq b$ for all $b \in B$.
- We say h is an **upper bound** of B if $b \leq h$ for all $b \in B$.

Example 43

Let $B = [1, 2)$. Then 1 is a lower bound of B and 2 is an upper bound of B. Note that 1 is the minimum of B, but 2 is not the maximum of B since $2 \notin B$.

Definition 44
Let $B \in \mathbb{R}$. We say B is

- **bounded above** if there exists an upper bound of B.
- **bounded below** if there exists a lower bound of B.
- **bounded** if there exists an upper bound and a lower bound of B.

Fact 45

Example 46

- $\mathbb{N}$ is bounded below, but not bounded above.
- $(-\infty, 1]$ is bounded above but not bounded below
- $(1, 3)$ is bounded.

5.2 Completeness Axiom

Definition 47
A set $B \subseteq \mathbb{R}$ is said to be **bounded above** if there exists a real number M such that $b \leq M$ for all $b \in B$. The number M is called an **upper bound** of B.

Definition 48
A real number s is called the **supremum** or **least upper bound** of a set $B \subseteq \mathbb{R}$ if:

1. s is an upper bound of B.
2. If u is any upper bound of B, then $s \leq u$.

We denote the supremum of B by $\sup B$.

Theorem 49 (Completeness Axiom)
Every non-empty set $B \subseteq \mathbb{R}$ that is bounded above has a supremum.

Example 50
$2 = \sup([1, 2])$

Proof. Suppose, t is an upper bound of B. Suppose towards a contradiction that $t < 2$. Then

$$1 \leqslant t < \frac{t+2}{2} < \frac{2+2}{2} = 2$$

So, $\frac{t+2}{2}$ is an element of the set B that is strictly bigger than t but we arrived to a contradictiopn. Therefore, $2 \leqslant t$. Since 2 is an upper bound of $B = \sup(B)$. $\square$

Example 51

The empty set $\phi \in \mathbb{R}$ has no supremeum since $\mathbb{R}$ is the set of upper bounds of ϕ and $\mathbb{R}$ has no leas element.

Proposition 52

If $B \in \mathbb{R}$ and $\max(B)$ exists then $\max(B) = \sup(B)$.

Proof. Let $A = \max(B)$. Then B is non-empty and bounded is above by a. If $C = \sup(B)$, then $a \leq c$. Since c is an upper bound $a \in B$. Also $c \leqslant a$. Since, c is the least upper bound. $\therefore a = c$ $\square$

Proposition 53 (The approximation property of suprema)

Let $B \subseteq \mathbb{R}$ be non-empty and bounded above. For any $\epsilon > 0$, there exists an element $b \in B$ such that $\sup B - \epsilon < b \leq \sup B$.

Proof. Suppose for contradicton there was some $\epsilon > 0$ such that no b as above exists. Then for all $c \in B, c \leqslant \sup(B) - \epsilon$ i.e. $\sup(B) - \epsilon$ is an upper bound of B. But $\sup(B)$ is the least upper bound of B, so $\sup(B) - \epsilon < \sup(B)$. This is a contradiction. Therefore, there must exist some $b \in B$ such that $\sup(B) - \epsilon < b \leq \sup(B)$. $\square$

Remark 54. *To prove a is a supremum of B, we need show the following:*

- *a is an upper bound of B.*
- *If c is an upper bound of B, then $a \leqslant c$.*

Theorem 55

Suppose, $F \subseteq \mathbb{R}$ is non-empty and bounded below. Then, there exist a **greatest lower bound** of $\mathbb{F}$, called **infimum** of F, denoted $\inf(F)$.

Proof. Let $B = \{x \in \mathbb{R} \mid -x \in F\}$. We will show:

- B is bounded above and non-empty, hence $a = \sup(B)$ exists.
- $-a$ is a lower bound of F.
- If c is a lower bound of F, then $c \leqslant -a$.

- Since F is non-empty, B is non-empty. Suppose c is a lower bound of F. Let $x \in B$. Then $-x \in F$, so $-x \geqslant c \to x \leqslant -c$. Hence, c is an upper bound of B. Therefore, B is bounded above.
- Let $f \in F$. Then $-f \in B$, so $-f \leq a$ (since $a = \sup(B)$). Therefore, $f \geq -a$. Hence, $-a$ is a lower bound of F.
- Let c be a lower bound of F. Let $b \in B$. Then $-b \in F$, so $-b \geq c \to b \leq -c$. Hence, $-c$ is an upper bound of B. Therefore, $-a \leq -c \to c \leq -a$.

□

6 January 17, 2025

Corollary 56

Let F be a non-empty and bounded below for each $\epsilon > 0$, there exists $f \in F$ such that

$$\inf(F) \leqslant f < \inf(F) + \epsilon$$

Theorem 57

There is a unique positive number α such that $\alpha^2 = 2$.

Proof. Let $E = \{x \in \mathbb{R} \mid x^2 < 2\}$. Since, $x^1 = 1 < 2$, so $1 \in E$, so $E \neq 0$.
Suppose, $x \geqslant 2$. Then $x^2 \geqslant 2^2 = 4 > 2$. Hence, $x \notin E$. Therefore, if $x \in E$, then $x < 2$.
Hence, $E \neq 0$, and bounded above by 2.
Let $\alpha = \sup(E)$.
We know that $1 \leqslant \alpha \leqslant 2$.
If $\alpha^2 \neq 2$, then either $\alpha^2 = 2$, then either $\alpha^2 > 0$ or $\alpha^2 < 2$. We'll show that both these cases lead to contradiction!

Case 1: When $\alpha^2 < 2$ Let $h = \frac{1}{2}\min(\alpha, \frac{2-\alpha^2}{3\alpha})$. Note $h > 0, h < \alpha$ and $h < \frac{2-\alpha^2}{3\alpha}$. Note that $(\alpha + h)^2 = \alpha^2 + 2\alpha h + h^2 < \alpha^2 + 3\alpha h < \alpha^2 + 3\alpha\frac{2-\alpha^2}{3\alpha} = 2$. So $\alpha + h \in E$. Since $\alpha + h > \alpha$, this contradicts $\alpha = \sup(E)$.

Case 2: When $\alpha^2 > 2$ We set $h = \frac{1}{2}\frac{\alpha^2-2}{2\alpha} > 0$. Since $\alpha - h < \alpha$, the approximation property says that there exists $e \in E$ such that

$$\alpha - h < e \leqslant \alpha$$

. Then $(\alpha - h)^2 < e^2$. Then $\alpha^2 - 2\alpha h + h^2 < 2 \Rightarrow \alpha^2 - 2\alpha h < 2$. Therefore, $h > \frac{\alpha^2-2}{2\alpha}$. And we arrived at a contradiction. Since $\alpha^2 < 2$ and $\alpha^2 > 2$ cannot hold, so $\alpha^2 = 2$.

Suppose $\alpha^2 = \beta^2 = 2$ and $\alpha, \beta > 0$. Then $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = 0$. Since $\alpha + \beta > 0, \alpha - \beta = 0$. Hence, $\alpha = \beta$ □

Remark 58. *The same proof shows $\{\alpha \in \mathbb{Q} \mid \alpha^2 = 2\}$ cannot have to a supremum in $\mathbb{Q}$ i.e. completeness doesn't hold.*

Remark 59. *We denote α as above using $\sqrt{2}$ by modifying the previous proof.*

Theorem 60

For any positive real number x, there exist unique postitive real number, denoted $\sqrt{2}$, such that $(\sqrt{x})^2 = x$.

Theorem 61 (Archimedean Property for Real Numbers)

Let $x \in \mathbb{R}$. Then there exists a natural number $n \in \mathbb{N}$ such that $n > x$.

Proof. Suppose this theorem is not true for some $x \in \mathbb{R}$. Then x is an upper bound of $\mathbb{N}$. Since $\mathbb{N} \neq 0, \alpha = \sup(\mathbb{N})$ exists. By approximation theorem, there exists some $n \in \mathbb{N}$ such that

$$\alpha - 1 < n \leqslant \alpha$$

. Hence, $\alpha < n + 1$. Since $n + 1$ is a natural number, this contradicts $\alpha = \sup(\mathbb{N})$ $\square$

Corollary 62

Let $\epsilon > 0$. Then there exists some $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$.

Proof. Apply the Archimedian property to $\frac{1}{\epsilon}$ $\square$

Definition 63

A **sequence** is a function $a \colon \mathbb{N} \to \mathbb{R}$. We denote the nth term of the sequence as a_n. We usually write $\alpha(n)$ as α_n

We write α as $((\alpha_n)_{n=1})^\infty$ or simply as α_n. Let a_n and b_n be sequencces and $c \in \mathbb{R}$ then we define a_n