Math 4580: Abstract Algebra I

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1 January 6, 2025

We didn't have any, but Dr. Lipnowski did post a module on `carmen` about the syllabus and the course. This semester we will be covering the first few chapters of the book *Abstract Algebra: Theory and Applications* by Thomas Judson.

Definition 1

Set: A collection of distinct objects, considered as an object in its own right.
Axioms: A collection of objects S with assumed structural rules is defined by axioms.
Statement: In logic or mathematics, an assertion that is either true or false.
Hypothesis and Conclusion: In the statement "If P, then Q", P is the hypothesis and Q is the conclusion.
Mathematical Proof: A logical argument that verifies the truth of a statement.
Proposition: A statement that can be proven true.
Theorem: A proposition of significant importance.
Lemma: A supporting proposition used to prove a theorem or another proposition.
Corollary: A proposition that follows directly from a theorem or proposition with minimal additional proof.

2 January 8, 2025

Professor Lipnowski discussed Sam Lloyd's 15 puzzle. Each lecture will include a mystery digit, contributing up to 5% bonus to the final grade based on correct guesses.

Certain course expectations:

- All assignments (one every two weeks) and exams (one midterm and one final exam) will be take-home.
- All the problems from the course textbook.
- Collaboration is encouraged, but the work should be your own.
- For the exams, we are not supposed to talk to other friends.

2.1 Functions

Definition 2

Let A and B be sets. A function $f : A \to B$ assigns exactly one output $f(a) \in B$ to every input $a \in A$.

- The set A is called the **domain** of f.
- The set B is called the **codomain** of f.

Fact 3

The domain A, codomain B, and the assignment of outputs $f(a)$ to every input $a \in A$ are all part of the data defining a function. Just writing a formula like $f(x) = e^x$ does not determine a function, as the domain and codomain are not specified.

For example:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.
- $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = e^x$.

Although these functions use the same formula, their meanings are completely different because their domains and codomains differ.

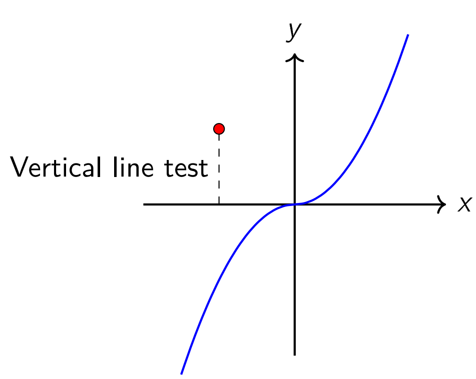
2.2 Graphs

A function $f : A \to B$ is often identified with its **graph** in $A \times B$:

$$\operatorname{graph}(f) = \{(a, b) \in A \times B : b = f(a)\}.$$

Lemma 4

Let $f : A \to B$ be a function. Its graph, $\operatorname{graph}(f)$, passes the **vertical line test**: For every $a \in A$, $V_a := \{(a, b) \in A \times B : b \in B\}$ intersects $\operatorname{graph}(f)$ in exactly one element.



Proposition 5

Let $G \subseteq A \times B$ be any subset passing the vertical line test, i.e., for all $a \in A$, $V_a \cap G$ consists of exactly one element. Then $G = \text{graph}(f)$ for a unique function $f : A \to B$.

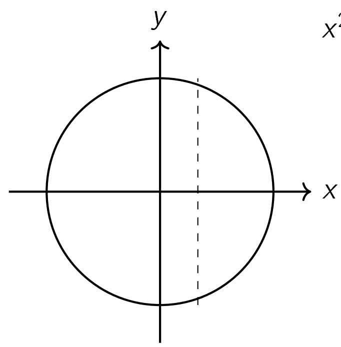
Proof. If $G = \{(a, b) \mid b \in B\}$ satisfies the vertical line test, define $f : A \to B$ by $f(a) = b$. Then $G = \text{graph}(f)$. $\square$

Definition 6

A subset $R \subseteq A \times B$ is called a **relation**. The vertical line test distinguishes graphs of functions from more general relations.

2.3 Examples

- Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (the unit circle). This is a relation but not the graph of a function because it fails the vertical line test: The vertical line $x = 0$ intersects the circle at two points.
- Visual depiction of a unit circle:



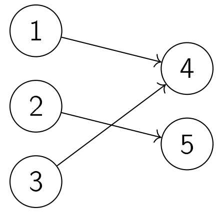
- Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$. The number of functions from A to B is $2^3 = 8$, corresponding to the 8 associated graphs in $A \times B$.
- The number of relations from A to B is $2^{|a| \cdot |b|} = 2^{3 \cdot 2} = 64$, containing the 8 graphs of functions from A to B.

Fact 7

The notion of relation is much more permissive than the notion of functions.

2.4 Visualizing Functions as Directed Edges

A function $f : A \to B$ can be visualized as a collection of directed edges $(a, f(a)) \in A \times B$. Each element of A has exactly one outgoing edge in the graph.



3 January 10, 2025

3.1 Injection and Surjection

Let $f : A \to B$ be a function.

Definition 8 (Injectivity (One-to-One))

f is injective (one-to-one) if:

$$\forall x, y \in A,\ f(x) = f(y) \implies x = y$$

Equivalently:

$$x \neq y \implies f(x) \neq f(y)$$

Fact 9

Distinct inputs have distinct outputs.

Definition 10 (Surjectivity (Onto))

f is surjective (onto) if:

$$\forall b \in B,\ \exists a \in A \text{ such that } f(a) = b.$$

Fact 11

Every $b \in B$ is an output of something through f."

Example 12

Here are a few examples of injectivity and surjectivity:

- Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ and $f : A \to B$ with $f(1), f(2), f(3)$ as elements of B. If B has only two elements, at least two of $f(1), f(2), f(3)$ must coincide (e.g., $f(1) = f(2)$). Thus, f is not injective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f : A \to B$ where:

$$f(1) = 4,\ f(2) = 7,\ f(3) = 5.$$

  Distinct inputs have distinct outputs, so f is injective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f : A \to B$ where:

$$f(1) = 4,\ f(2) = 4,\ f(3) = 6.$$

  Here, $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f(1) = f(2)$ but $1 \neq 2$, so f is not injective.
- Let $f : A \to B$ where B has size 4 and $f(1), f(2), f(3)$ are distinct elements of B. If $B \setminus \{f(1), f(2), f(3)\}$ is non-empty, then $b \neq f(a)$ for all $a \in A$, implying f is non surjective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ and $f : A \to B$ with $f(1) = 4, f(2) = 5, f(3) = 4$. f is surjective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ and $f : A \to B$ with $f(1) = 4, f(2) = 4, f(3) = 4$. f is not surjective.

3.2 Bijection and Range

Definition 13 (Bijectivity)

f is bijective if f is both injective and surjective.

Definition 14

Let $f : A \to B$ be a function. The *range* of f is the subset of B defined as:

$$\text{range}(f) := \{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$

Thus, $f : A \to B$ is surjective $\iff$ range$(f) = B$.

- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ and $f : A \to B$ where:

$$f(1) = 6,\ f(2) = 5,\ f(3) = 4.$$

  f is a bijection.

- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f : A \to B$ where:

$$f(1) = 4,\ f(2) = 4,\ f(3) = 56$$

f is neither injective nor surjective.

Question. *Let A and B be finite sets of the same size. Prove that the following are equivalent:*

1. *$f : A \to B$ is injective.*
2. *$f : A \to B$ is bijective.*
3. *$f : A \to B$ is surjective.*

Demonstrate that (1), (2), and (3) are not necessarily equivalent if $A = B = \mathbb{N}$.

Example 15

Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ -\frac{(n+1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

is a bijection from $\mathbb{N}$ to $\mathbb{Z}$.

Proof. We will prove injectivity first. Suppose $f(n_1) = f(n_2)$. Then: If $f(n_1) = f(n_2) > 0$, then n_1 and n_2 must be even, and

$$\frac{n_1}{2} = f(n_1) = f(n_2) = \frac{n_2}{2} \implies n_1 = n_2.$$

If $f(n_1) = f(n_2) < 0$, then n_1 and n_2 must be odd, and

$$-\frac{n_1 + 1}{2} = f(n_1) = f(n_2) = -\frac{n_2 + 1}{2} \implies n_1 = n_2.$$

In all cases, $n_1 = n_2$. It follows that f is injective.

Now let's prove surjectivity. Let $n \in \mathbb{Z}$. If $n > 0$, then

$$n = f(2n).$$

If $n < 0$, then

$$n = f(-2n - 1).$$

Therefore, f is surjective. □

Theorem 16 (Taylor's Theorem)

Let f be a function that is n-times differentiable at a. Then for each x in the interval containing a, there exists a ξ between a and x such that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}.$$

Proof. By the mean value theorem, for each x in the interval containing a, there exists a ξ between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x),$$

where $R_{n+1}(x)$ is the remainder term. The remainder term can be expressed as

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Therefore, we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

□

4 January 15, 2025

4.1 Equivalence Relations and Equivalence Classes

Definition 17

Let $\sim$ be an equivalence relation on a set X. Let $x \in X$. The equivalence class of x is

$$[x] := \{y \in X : y \sim x\} \subset X$$

An equivalence class in X is a subset of X of the form $[x]$ for some $x \in X$.

Fact 18

The equivalence classes of X partition X into disjoint subsets. This partition completely encapsulates the equivalence relation.

Proposition 19

Let $a, b \in X$. Either:

- $[a]$ and $[b]$ are disjoint
- $[a] = [b]$

Proof. Suppose $[a]$ and $[b]$ are not disjoint. Let $t \in [a] \cap [b]$. Then $t \sim a$ and $t \sim b$.

$$\Rightarrow a \sim t \text{ and } t \sim b \quad \text{(by symmetry)}$$

$$\Rightarrow a \sim b \quad \text{(by transitivity)}$$

This implies that $[a] = [b]$:

$$\text{If } y \sim a, \text{ by } (a \sim b) \text{ and transitivity, } y \sim b \text{ too.}$$
$$\text{If } y \sim b, \text{ by } (b \sim a) \text{ and symmetry, } y \sim a.$$

It follows that

$$[a] = \{y \in X : y \sim a\} = \{y \in X : y \sim b\} = [b]$$

The latter proposition shows that equivalence classes on X partition X:

$$X = \bigsqcup_{i \in I} A_i$$

□

Definition 20

Let $X = \bigsqcup_{i \in I} A_i$ be the partition of X into equivalence classes for $\sim$. We call any subset $S \subset X$ a complete set of equivalence class representatives if it contains exactly one element $x_i \in A_i$ for every $i \in I$, i.e., "exactly one element per equivalence class".

In practice, understanding an equivalence relation amounts to understanding its associated equivalence classes and complete sets of equivalence class representatives.

4.2 Examples of Equivalence Classes

1. Let $X = \mathbb{R}$ and define the equivalence relation $\sim$ by $x \sim y$ if and only if $x - y \in 2\pi \cdot \mathbb{Z}$.

   The equivalence class of x is:
   $$[x] = \{x + 2\pi k : k \in \mathbb{Z}\} \subset \mathbb{R}$$

   Every $z \in \mathbb{R}$ lies in an equivalence class, namely $[z]$. If $[x]$ and $[y]$ contain a common element t, then there exist $k, l \in \mathbb{Z}$ such that:

   $$x + 2\pi k = t = y + 2\pi l \implies x - y = 2\pi(l - k) \implies x \sim y$$

   This implies $[x] = [y]$. Therefore, we have:

   $$\mathbb{R} = \bigsqcup_{[z]} [z]$$

   The interval $[0, 2\pi)$ is a complete set of equivalence class representatives.

2. Let X be the set of all 2×2 matrices, and define the equivalence relation $\sim$ by $x \sim y$ if there exists a continuous path $p : [0, 1] \to X$ with $p(0) = x$ and $p(1) = y$.

The equivalence classes are the connected components of X. For example, if X consists of three disjoint disks $\mathbb{D}_1, \mathbb{D}_2, \mathbb{D}_3$, then:

$$X = \mathbb{D}_1 \sqcup \mathbb{D}_2 \sqcup \mathbb{D}_3$$

A complete set of equivalence class representatives is $\{\pi_1, \pi_2, \pi_3\}$, where $\pi_i \in \mathbb{D}_i$ for $i = 1, 2, 3$.

3. Let $X = \mathbb{R}^2$ and define the equivalence relation $\sim$ by $(a, b) \sim (c, d)$ if and only if $a^2 + b^2 = c^2 + d^2$.

   The equivalence class of (a, b) is the set of all points in $\mathbb{R}^2$ that lie on the circle centered at the origin with radius $\sqrt{a^2 + b^2}$.

Problem 21

Verify that the above defines an equivalence relation.

Equivalence classes:

$$[(a, b)] = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = a^2 + b^2\}$$

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = a^2 + b^2\}$$

is the collection of points in $\mathbb{R}^2$ having the same distance from $(0, 0)$ as (a, b), i.e., it is the circle in $\mathbb{R}^2$ centered at $(0, 0)$ passing through (a, b).

Equivalence classes for $\sim$ on $\mathbb{R}^2$: circles centered at $(0, 0)$.

$$\mathbb{R}^2 = \bigsqcup_{a \in \mathbb{R}_{\geq 0}} [(a, 0)]$$

and $\{(a, 0) : a \in \mathbb{R}_{>0}\}$ is a complete set of equivalence class representatives.

5 January 17, 2025

5.1 Mathematical Induction

Definition 22

Let $\{P(n)\}_{n \in \mathbb{N}}$ be statements indexed by $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Suppose

(a) $P(0)$ is true

(b) $P(m)$ true $\Rightarrow P(m + 1)$ true for all $m \in \mathbb{N}$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Fact 23

The following are true for a mathematical induction:

- (a) is the base case of the induction
- (b) is the inductive step
- Assuming $P(m)$ is true (in order to prove that $P(m+1)$ is true) is the inductive hypothesis.

5.1.1 Visualizing Induction

Picture the statements $P(0), P(1), P(2), \ldots$ as dominoes $0, 1, 2, \ldots$ lined up in some way. Our goal is to prove that all $P(n), n \in \mathbb{N}$ are true, amounting to toppling over every domino.

0	→ 1	→ 2	→ 3	→ 4	→ 5
0+1	1+1	2+1	3+1	4+1	5+1

Base case ⇔ we push over domino 0.

Inductive step ⇔ if domino m topples, then domino $m+1$ topples too.

Inductive hypothesis ⇔

Remark 24. *The inductive step is usually the hardest part of an inductive argument. However, as the above analogy shows, the base case is essential too: if no domino is pushed over, none will topple!*

5.2 Examples

1. Prove that

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Proof. Let $P(n) := 1 + \cdots + n = \frac{n(n+1)}{2}$.

Base case: When $n = 0$, the LHS $= 0$ (since the sum is empty) and the RHS $= 0$ too. So $P(0)$ is true.

Inductive Step: Suppose $P(m)$ is true, i.e.,

$$1 + \cdots + m = \frac{m(m+1)}{2}$$

Then

$$\begin{aligned}1+\cdots+m+(m+1) &= (1+\cdots+m)+(m+1)\\ &= \frac{m(m+1)}{2}+(m+1) \quad \text{(by our inductive hypothesis)}\\ &= (m+1)\left(\frac{m}{2}+1\right)\\ &= (m+1)\left(\frac{m+2}{2}\right)\\ &= \frac{(m+1)(m+2)}{2}\end{aligned}$$

So $P(m+1)$ is true too.

It follows, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, i.e.,

$$1+2+\cdots+n = \frac{n(n+1)}{2}$$

□

2. Let $f_n = n^{\text{th}}$ Fibonacci number, defined as the n^{th} term of the sequence defined recursively by:

$$\begin{cases} f_0 = 0 \\ f_1 = 1 \\ f_n = f_{n-1} + f_{n-2} \text{ if } n \geq 2 \end{cases}$$

n	0	1	2	3	4	5	6	7	8
f_n	0	1	1	2	3	5	8	13	21

Now that

$$f_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Note: $T_\pm := \frac{1\pm\sqrt{5}}{2}$ are the two roots of the quadratic equation $x^2 = x+1$. T_+ is known as the golden ratio.

Proof. Let $P(n)$ denote the statement

$$f_n = \frac{1}{\sqrt{5}}\left(T_+^n - T_-^n\right)$$

We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction:

Base case: $n = 0$:

$$\begin{aligned} f_0 = 0 &= \frac{1}{\sqrt{5}}\left(T_+^0 - T_-^0\right) \\ f_1 = 1 &= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right) \\ &= \frac{1}{\sqrt{5}}\left(T_+^1 - T_-^1\right) \end{aligned}$$

Inductive step: Suppose $P(k)$ is true for all $k < m$. We will prove that $P(m)$ is true too:

If $m = 0$ or $m = 1$, we verified that $P(m)$ is true in our base case. Suppose $m \geq 2$.

$$\begin{aligned} f_m &= f_{m-1} + f_{m-2} \quad \text{(defining recursion for } f_m\text{)} \\ &= \frac{1}{\sqrt{5}}\left(T_+^{m-1} - T_-^{m-1}\right) \quad \text{(since } P(m-1) \text{ is true, by hypothesis)} \\ &+ \frac{1}{\sqrt{5}}\left(T_+^{m-2} - T_-^{m-2}\right) \quad \text{(since } P(m-2) \text{ is true, by hypothesis)} \\ &= \frac{1}{\sqrt{5}}\left(T_+^{m-1} + T_+^{m-2}\right) - \frac{1}{\sqrt{5}}\left(T_-^{m-1} + T_-^{m-2}\right) \\ &= \frac{1}{\sqrt{5}}\left(T_+^{m-2}(T_+ + 1)\right) - \frac{1}{\sqrt{5}}\left(T_-^{m-2}(T_- + 1)\right) \\ &= \frac{1}{\sqrt{5}}\left(T_+^{m-2} \cdot T_+^2\right) - \frac{1}{\sqrt{5}}\left(T_-^{m-2} \cdot T_-^2\right) \\ &= \frac{1}{\sqrt{5}}\left(T_+^m - T_-^m\right) \end{aligned}$$

Thus, $P(m)$ is true too. It follows that $P(n)$ is true for all $n \in \mathbb{N}$, i.e.,

$$f_n = \frac{1}{\sqrt{5}}\left(T_+^n - T_-^n\right) \text{ for all } n \in \mathbb{N}$$

□

The above proof uses the strong form of mathematical induction.

Theorem 25 (Principle of Mathematical Induction (strong form))
Let $\{P(n)\}_{n \in \mathbb{N}}$ be statements indexed by $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Suppose

- (a) $P(0)$ is true
- (b) $P(0), P(1), \ldots, P(m) \Rightarrow P(m+1)$ true for all $m \in \mathbb{N}$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $Q(n)$ be the statement that

$$P(0), P(1), \ldots, P(n) \text{ are all true.}$$

$Q(0)$ is true $P(0)$ is true. Suppose $Q(m)$ is true, i.e.,

$$P(0), \ldots, P(m) \text{ are all true.}$$

By (b) (the strong inductive step), $P(m+1)$ is true.

Thus, $P(0), \ldots, P(m), P(m+1)$ are all true by (b). It follows that $Q(m+1)$ is true too. By induction, $Q(n)$ is true for all $n \in \mathbb{N}$, implying that $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

6 January 22, 2025

6.1 Well-Ordering Principle

Theorem 26 (Well-ordering principle)

Let $S \subset \mathbb{N}$ be non-empty. Then S contains a least element t, i.e.,

- $t \in S$
- $t \leq s$ for all $s \in S$

Proof. Let $t \in S$. Consider the subset $S' = \{s \in S : s \leq t\} = S \cap \{0, \ldots, t\}$. Since S' is a non-empty subset of $\{0, \ldots, t\}$, it is finite. Therefore, S' has a least element, say t'. By construction, $t' \in S'$ and $t' \leq s$ for all $s \in S'$. Since $S' \subset S$, it follows that $t' \in S$ and $t' \leq s$ for all $s \in S$. Thus, t' is the least element of S. $\square$

Corollary 27

$t' \in S$ is a minimal element of S.

Proof. By construction, $t' \in S$ and $t' \leq t$. For any $s \in S$, if $s \leq t$, then $s \in S'$. By the definition of t', we have $t' \leq s$. If $s \notin S'$, then $s > t$, and since $t \geq t'$, it follows that $s > t'$. Therefore, $t' \leq s$ for all $s \in S$.

This shows that t' is the least element of S.

To prove that every finite subset of $\mathbb{N}$ contains a least element, we use mathematical induction. We will show that the well-ordering principle implies the strong form of induction. $\square$

6.2 Connection between the Well-Ordering Principle and Induction

Theorem 28

Assume the well-ordering principle holds. Then the strong form of induction holds too: Suppose $\{P(n)\}_{n\in\mathbb{N}}$ are statements for which:

(a) $P(0)$ is true

(b) $P(0), \ldots, P(m-1)$ true $\Rightarrow P(m)$ true for all $m \in \mathbb{N}_{>0}$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$. We want to prove that S is empty.

Suppose S is non-empty. Let $t \in S$ be a least element. Since $P(0)$ is true, $0 \notin S$. Therefore, $t \neq 0$, i.e., $t \geq 1$. Since $0, 1, \ldots, t-1 < t$, it follows that $0 \notin S, 1 \notin S, \ldots, t-1 \notin S$, i.e., $P(0), P(1), \ldots, P(t-1)$ are all true. By assumption (b), it follows that $P(t)$ is true, i.e., $t \notin S$. This contradicts $t \in S$.

It follows that S is empty, i.e., $P(n)$ is true for all $n \in \mathbb{N}$. □

The well-ordering principle perspective often reveals what you should take as the base case for an inductive argument.

6.3 Examples

1.
$$\begin{cases} F_0 = 0 \\ F_1 = 1 & \text{for } n \geq 2. \\ F_n = F_{n-1} + F_{n-2} \end{cases}$$

Prove that

$$F_n = \frac{1}{\sqrt{5}}\left(T_+^n - T_-^n\right) \text{ for all } n \in \mathbb{N}.$$

$$T_\pm = \frac{1 \pm \sqrt{5}}{2}, \text{ the roots of } x^2 = x + 1$$

Proof. Let $S = \{n \in \mathbb{N} : F_n \neq \frac{1}{\sqrt{5}}\left(T_+^n - T_-^n\right)\}$. We want to prove that S is empty.

Suppose S is non-empty. Let t be the least element of S.

- Suppose $t \geq 2$. Then
  - (a) $F_{t-1} = \frac{1}{\sqrt{5}}\left(T_+^{t-1} - T_-^{t-1}\right)$ since $t-1 \in \mathbb{N} \setminus S$
  - (b) $F_{t-2} = \frac{1}{\sqrt{5}}\left(T_+^{t-2} - T_-^{t-2}\right)$ since $t-2 \in \mathbb{N} \setminus S$
- Note: We assume $t \geq 2$ here. Otherwise, $t-1$ and $t-2$ are not both natural numbers.

$$\begin{aligned} F_t &= F_{t-1} + F_{t-2} \quad \text{(by the recursive definition of Fibonacci numbers)} \\ &= \frac{1}{\sqrt{5}}\left(T_+^{t-1} + T_+^{t-2}\right) - \frac{1}{\sqrt{5}}\left(T_-^{t-1} + T_-^{t-2}\right) \\ &= \frac{1}{\sqrt{5}}\left(T_+^{t-2}(T_+ + 1)\right) - \frac{1}{\sqrt{5}}\left(T_-^{t-2}(T_- + 1)\right) \\ &= \frac{1}{\sqrt{5}}\left(T_+^{t-2} \cdot T_+^2\right) - \frac{1}{\sqrt{5}}\left(T_-^{t-2} \cdot T_-^2\right) \\ &= \frac{1}{\sqrt{5}}\left(T_+^t - T_-^t\right) \end{aligned}$$

Thus, $F_t = \frac{1}{\sqrt{5}}\left(T_+^t - T_-^t\right)$, implying $t \notin S$. This contradicts $t \in S$. It follows that $t = 0$ or $t = 1$.

□

Remark 29. *Three "leftover cases" form our base case, since our main argument above did not address either of these edge cases.*

- If $t = 0$,
$$F_0 = 0 = \frac{1}{\sqrt{5}}\left(T_+^0 - T_-^0\right), \text{ so } 0 \notin S$$
- If $t = 1$,
$$F_1 = 1 = \frac{1}{\sqrt{5}}\left(T_+^1 - T_-^1\right), \text{ so } 1 \notin S$$

We've shown:

- If $t \geq 2$, then t cannot be a least element of S.
- If $t = 0$ or $t = 1$, then $t \notin S$.

Thus, S contains no least element. This contradicts S being non-empty (by the well-ordering principle). It follows that S is empty, i.e.,

$$F_n = \frac{1}{\sqrt{5}}\left(T_+^n - T_-^n\right) \text{ for all } n \in \mathbb{N}$$

This perspective is also helpful for rooting out false statements you might try to prove by induction.

2. Let $P(n)$ be the statement:

$$P(n) : \text{All collections of } n \text{ boxes are the same color.}$$

We know, from life experience, this statement is false.

Let's see why:

Let $S = \{n \in \mathbb{N} : P(n) \text{ is false}\}$.

Suppose S is non-empty. Let t be the least element of S. Suppose $t \geq 3$. Then $P(1)$ and $P(2)$ are true (since $1, 2 \notin S$ by minimality of t). Let $\{1, \ldots, t\}$ be any collection of t boxes. Divide them into two sets

$$A = \{1, \ldots, t-1\} \text{ and } B = \{2, \ldots, t\}$$

Since t is minimal, $P(t-1)$ is true. So all boxes in A are some common color, call it a. Likewise, all boxes in B are some common color, call it b. Since $t \geq 3$, the sets A and B overlap. Thus $a = b$. It follows that $\{1, 2, \ldots, t\}$ are all the same color, i.e., $P(t)$ is true. Thus $t \notin S$, contradicting $t \in S$. Thus, if $t \geq 3$, t cannot be a minimal element of S.

For $t = 1$, $P(1)$ is clearly true. So $1 \notin S$. For $t = 2$, $P(2)$ is not necessarily true. So at this very last step, our argument breaks down!

7 January 24, 2025

7.1 Arithmetic of $\mathbb{Z}$

We turn from counting properties of $\mathbb{Z}$ and $\mathbb{N}$—these feature prominently in induction:

$$0 \underset{\text{next}}{\rightarrow} 1 \underset{\text{next}}{\rightarrow} 2 \underset{\text{next}}{\rightarrow} 3$$

to the basic arithmetic operations in $\mathbb{Z} : +, -, \times$. What about division??

Definition 30

Let $a, b \in \mathbb{Z}$. We say that b divides a / a is a multiple of b / a is divisible by b if $a = bk$ for some $k \in \mathbb{Z}$. We write that as following

$$b \mid a$$

Example 31

The following could be an example:

- Every integer b divides 0.
- Every integer is divisible by 1.

Fact 32

If $b \neq 0$, then b divides a iff the rational number $\frac{a}{b}$ is actually an integer.

Example 33

$$\frac{50}{7} = 7.14 \quad \text{(not an integer. So 7 does not divide 50.)}$$

7.2 The Division Algorithm

Theorem 34 (Division Algorithm)

Let $a, b \in \mathbb{Z}$, $b \neq 0$. Then there exist

- $k \in \mathbb{Z}$
- $r \in \mathbb{Z}$ with $|r| < |b|$

satisfying:

$$a = bk + r$$

Proof. Let $\frac{a}{b} = k + \alpha$ for some $k \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$ where $0 \le \alpha < 1$. Multiplying both sides by b, we get:

$$a = kb + \alpha b$$

Define $r = \alpha b$. Then:

$$a = kb + r$$

Since $0 \le \alpha < 1$, it follows that $0 \le r < |b|$. Therefore, r is an integer satisfying $0 \le r < |b|$. Thus, we have:

$$a = kb + r$$

where $k \in \mathbb{Z}$ and $r \in \mathbb{Z}$ with $0 \le r < |b|$.

$\square$

The result follows.

Remark 35. *In the above proof, we could take* $-\frac{1}{2} \le \alpha \le \frac{1}{2}$ *(as opposed to* $0 \le \alpha < 1$*). For* $r = a - kb = b\alpha$*,*

$$\begin{aligned} |r| &= |\alpha b| \\ &\le \frac{|b|}{2} \end{aligned}$$

7.3 Common Divisors

Definition 36

Let $a, b \in \mathbb{Z}$. A common divisor d of a and b is an integer $d \in \mathbb{Z}$ for which:

- $d \mid a$
- $d \mid b$

Example 37

Let's consider the following examples:

- $a =$ anything, $b = 0$
$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of } a \text{ and } b = 0 \end{array} \right\} = \{\text{divisors of } a\}$$
- $a = 26 = 2 \cdot 13$
$$b = 65 = 5 \cdot 13$$
$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of 26 and 65} \end{array} \right\} = \{\pm 1, \pm 13\}$$
- $a = 91, b = 15$
$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of 91 and 15} \end{array} \right\} = \{\pm 1\}$$
- $a = 32 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
$$b = 16 = 2 \cdot 2 \cdot 2 \cdot 2$$
$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of 32 and 16} \end{array} \right\} = \{\pm 1, \pm 2, \pm 4, \pm 8, \pm 16\}$$

In all of these examples, observe that there is a common divisor d of a and b divisible by all other common divisors.

Definition 38

$d \in \mathbb{Z}$ is a greatest common divisor of $a, b \in \mathbb{Z}$ if:

1. d is a common divisor of a and b
2. if $e \in \mathbb{Z}$ is a common divisor of a and b, then $e \mid d$.

Lemma 39

Let $a, b \in \mathbb{Z}$. Let e, d be greatest common divisors of a and b. Then $d = \pm e$.

Proof. If a and b both equal 0, then 0 is a greatest common divisor of a and b and is the only one. If not both a and b equal 0, then e and d are necessarily non-zero (since 0 does not divide any non-zero integer).

Since d is a greatest common divisor of a and b, it follows that $d \mid e$. Therefore, there exists some integer $k \in \mathbb{Z}$ such that:
$$e = kd$$

Similarly, since e is also a greatest common divisor of a and b, it follows that $e \mid d$. Therefore, there exists some integer $j \in \mathbb{Z}$ such that:
$$d = je$$

Combining these two equations, we get:

$$d = je = j(kd) = d \cdot jk$$

This implies:

$$d(1 - jk) = 0$$

Since $d \neq 0$, it follows that:

$$1 - jk = 0$$

Hence:

$$jk = 1$$

This means that j and k must be ± 1. Therefore:

$$d = je = \pm e$$

Thus, d and e are equal up to a sign. $\square$

7.4 Euclidean Algorithm

Fact 40

Let $a, b \in \mathbb{Z}$. Then

$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of } a \text{ and } b \end{array} \right\} = \left\{ \begin{array}{l} \text{common divisors} \\ \text{of } a - b \text{ and } b \end{array} \right\}$$

Proof.

- Suppose d is a common divisor of a and b. Then $a = jd$ and $b = kd$ for some $j, k \in \mathbb{Z}$.

$$\begin{aligned} a - b &= jd - kd \\ &= (j - k)d \\ &\Rightarrow d \text{ divides } a - b \end{aligned}$$

and

$$b = kd \Rightarrow d \text{ divides } b.$$

Thus, d is a common divisor of $a - b$ and b. It follows that

$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of } a \text{ and } b \end{array} \right\} \subset \left\{ \begin{array}{l} \text{common divisors} \\ \text{of } a - b \text{ and } b \end{array} \right\}$$

Suppose d divides $a - b$ and b. Then $a - b = jd$ and $b = kd$ for some $j, k \in \mathbb{Z}$.

$$\begin{aligned} a &= (a - b) + b \\ &= jd + kd \\ &\Rightarrow d \text{ divides } a \end{aligned}$$

and

$$b = kd \Rightarrow d \text{ divides } b.$$

- Thus, d is a common divisor of a and b.

$\square$

It follows that

$$\left\{\begin{array}{l}\text{common divisors}\\ \text{of } a-b \text{ and } b\end{array}\right\} \subset \left\{\begin{array}{l}\text{common divisors}\\ \text{of } a \text{ and } b\end{array}\right\}$$

Combining the latter two containments:

$$\left\{\begin{array}{l}\text{common divisors}\\ \text{of } a \text{ and } b\end{array}\right\} = \left\{\begin{array}{l}\text{common divisors}\\ \text{of } a-b \text{ and } b\end{array}\right\}$$

More generally, the exact same proof technique may be used to prove:

$$\left\{\begin{array}{l}\text{common divisors}\\ \text{of } a \text{ and } b\end{array}\right\} = \left\{\begin{array}{l}\text{common divisors}\\ \text{of } a-kb \text{ and } b\end{array}\right\}$$

for every integer k.

7.4.1 Euclidean Algorithm:

Let $CD(a, b)$ denote the set of common divisors of $a, b \in \mathbb{Z}$.

Input: (a, b), $a, b \in \mathbb{Z}$ with $b \neq 0$ and $|b| \leq |a|$.

Output: A pair $(d, 0)$ with

$$CD(a, b) = CD(d, 0)$$

Note:

- Since $d \in CD(d, 0) = CD(a, b)$, d is a common divisor of a and b.
- If $e \in CD(a, b) = CD(d, 0)$, then e divides d and e divides 0.
- Thus, d is a greatest common divisor of a and b.

The Algorithm:

1. If $b = 0$, return $(a, 0)$.
2. Otherwise, find $A \in \mathbb{Z}$ for which

$$r = a - Ab \text{ satisfies } |r| < |b|.$$

   (By the division algorithm, this is always possible)

3. Replace (a, b) by $(a^*, b^*) := (b, r)$.

   - Go to (1) if $b^* = 0$

- Go to the start of step (2) if $b^* \neq 0$

Proposition 41

The Euclidean algorithm terminates.

Proof. Let (a_n, b_n) be the n^{th} pair calculated in the process of running the Euclidean algorithm. The pair

$$(a_0, b_0), (a_1, b_1), (a_2, b_2), \ldots (a, b)$$

satisfy:

- $|a_m| \geq |b_m|$
- $(a_{m+1}, b_{m+1}) = (a_m^*, b_m^*)$

By construction,

$$|b_m^*| < |b_m|.$$

So $|b_0| > |b_1| > \ldots$ is a strictly decreasing sequence of natural numbers. Therefore, the sequence must terminate at by going to step (1) and outputting $(a_n, b_n) = (a_n, 0)$ for some (finite) $n \in \mathbb{N}$. This proves the algorithm terminates. □

Remark 42. *Given $x, y \in \mathbb{Z}$, we've seen that we can find $A \in \mathbb{Z}$ for which $r = x - Ay$ satisfies $|r| \leq |y|/2$. Applying this choice of r consistently throughout the running of the Euclidean algorithm, Euclidean_Algorithm(a, b) runs in time $O(\log_2 |b|)$.*

7.5 Examples

1. Let's find the gcd of 576 and 243.

$$\begin{aligned}
(576, 243) &= (243, 576 - 2 \cdot 243) \\
&= (243, 90) \\
&= (90, 243 - 2 \cdot 90) \\
&= (90, 63) \\
&= (63, 90 - 1 \cdot 63) \\
&= (63, 27) \\
&= (27, 63 - 2 \cdot 27) \\
&= (27, 9) \\
&= (9, 27 - 3 \cdot 9) \\
&= (9, 0)
\end{aligned}$$

Thereofore,

$$\gcd(576, 243) = 9$$

2. Let's find the gcd of 101 and 66.

$$\begin{aligned}(101, 66) &= (66, 101 - 1 \cdot 66)\\ &= (66, 35)\\ &= (35, 66 - 1 \cdot 35)\\ &= (35, 31)\\ &= (31, 35 - 1 \cdot 31)\\ &= (31, 4)\\ &= (4, 31 - 7 \cdot 4)\\ &= (4, 3)\\ &= (3, 4 - 1 \cdot 3)\\ &= (3, 1)\\ &= (1, 3 - 3 \cdot 1)\\ &= (1, 0)\end{aligned}$$

Thereofore,

$$\gcd(101, 66) = 1$$

3. Let's find the gcd of 104 and 80.

$$\begin{aligned}(104, 80) &= (80, 104 - 1 \cdot 80)\\ &= (80, 24)\\ &= (24, 80 - 3 \cdot 24)\\ &= (24, 8)\\ &= (8, 24 - 3 \cdot 8)\\ &= (8, 0)\end{aligned}$$

Thereofore,

$$\gcd(104, 80) = 8$$

We describe an enhanced version of the Euclidean algorithm that allows us to solve the equation

$$xa + yb = d \quad \text{for} \quad x, y \in \mathbb{Z}, \quad d = \gcd(a, b)$$

Proposition 43

Let $a, b \in \mathbb{Z}$. Suppose there are integers $x, y \in \mathbb{Z}$ for which

$$x \cdot a + y \cdot b = d$$

for some common divisor d of a and b. Then d is a greatest common divisor of a and b.

Proof. By assumption, d is a common divisor of a and b.

- Suppose $e \mid a$ and $e \mid b$. Then

$$e \mid xa \quad \text{and} \quad e \mid yb \implies e \mid (xa + yb) = d.$$

It follows that d is a greatest common divisor of a and b.

$\square$

7.6 The Algorithm

Let $a, b \in \mathbb{Z}$ with $|a| \geq |b|$.

1. Form a 3-column table:

$$\begin{array}{c|c|c} d & x & y \\ \hline & & \\ & & \\ & & \end{array}$$

2. Initialize the first two rows as:

e	x	y
a	1	0
b	0	1

3. Note: $xa + yb = e$ where (e, x, y) forms a row in this table.

4. Run the Euclidean algorithm in the left column of the table:

e	x	y
e'	x'	y'
e''	x''	y''

In particular,

$$e' = x'a + y'b$$
$$e'' = x''a + y''b$$

By the division algorithm, we can find $k \in \mathbb{Z}$ for which $e''' := e' - ke''$ satisfies $|e'''| \leq |e''|$.

Add the new bottom row

$$R''' := R' - kR''$$

to our table:

e	x	y
e'	x'	y'
e''	x''	y''
e'''	x'''	y'''

Note that the relation $x'''a + y'''b = e'''$ holds for the new bottom row of our table too, since it holds for the second-to-bottom and third-to-bottom rows too:

$$\begin{aligned} x'''a + y'''b &= (x' - kx'')a + (y' - ky'')b \\ &= (x'a + y'b) - k(x''a + y''b) \quad \text{(regrouping terms)} \\ &= e' - k \cdot e'' \\ &= e''' \end{aligned}$$

5. Stop adding new rows once the bottom two rows become.

The output, i.e., the last two rows $\Rightarrow$ By the theory of the Euclidean algorithm (which we've just run in the left (e) - column of our table),

$$d = \gcd(a, b)$$

Furthermore, since $xa + yb = e$ for every row (e, x, y) from our table, it follows that

$$x_0 \cdot a + y_0 \cdot b = d$$

Problem 44

Consider the following problems:

- Prove that $\gcd(x_1, y_1) = 1$.
- (HARD) Prove that $a = \pm d \cdot y_1$ and $b = \mp d \cdot x_1$.

7.7 Examples

1. Extended Euclidean algorithm for $(596, 243)$:

e	x	y
596	1	0
243	0	1
90	1	-2
63	-2	5

2. Extended Euclidean algorithm for $(3587, 1819)$:

e	x	y
3587	1	0
1819	0	1
-51	1	-2
34	35	-69
-17	36	-71
0	107	-211

We read off:

$$\begin{cases} -17 = 36 \times 3587 + (-71) \times 1819 & \text{(from the next to last row)} \\ 3587 = 17 \times 211 \\ 1819 = 17 \times 107 \end{cases}$$

8 January 31, 2025

We proved:

Proposition 45

Let $a, b \in \mathbb{Z}$. Let $d = \gcd(a, b)$. There exist integers $x, y \in \mathbb{Z}$ such that

$$xa + yb = d.$$

Not only did we prove this abstract existence statement, but we saw how to extract x, y from the output of the Extended Euclidean Algorithm.

8.1 Ideals in $\mathbb{Z}$

$I = \{xayb : x, y \in \mathbb{Z}\} \subset \mathbb{Z}$ is an ideal in the ring $\mathbb{Z}$ if and only if:

- I is closed under $+$, $-$, and $0 \in I$.
- $r \cdot i \in I$ for all $i \in I$ and $r \in \mathbb{Z}$.

The above proposition showed that every ideal in $\mathbb{Z}$ consists of multiples of a single element. Thus, $\mathbb{Z}$ is a so-called principal ideal domain. More on this later.

8.2 An important application of the above proposition:

Lemma 46

Let $a, b \in \mathbb{Z}$, $n \in \mathbb{Z}$ with $n \neq 0$. Suppose

- $n \mid ab$
- $\gcd(a, n) = 1$.

Then $n \mid b$.

Proof. Since $\gcd(a, n) = 1$, we can find integers x, y such that

$$1 = x \cdot a + y \cdot n$$

Multiply both sides of (f) by b:

$$\begin{aligned} b &= (x \cdot a + y \cdot n) \cdot b \\ &= x \cdot (ab) + (yb) \cdot n \Rightarrow b \text{ is a multiple of } n \text{ by (i)}. \end{aligned}$$

□

8.3 Application to primes and prime factorization

Definition 47

Let $p \in \mathbb{Z}$, $p \leq -1$. p is prime if

$$\{\text{divisors of } p\} = \{\pm 1, \pm p\}.$$

Example 48

- Prime: $2, 3, 5, 7, 11, 13, 17, 19, \ldots$
- Not prime: $4 = 2 \times 2, 6 = 2 \times 3, 9 = 3 \times 3, 91 = 13 \times 7$

Fact 49

Non-prime integers are otherwise known as composite.

8.4 Sieve of Eratosthenes

(An algorithm to list all primes in $\{2, 3, \ldots, N\}$)

1. Begin with $L = \{2, 3, \ldots, N\}, P = \phi$.
2. Add the smallest element s of L to P and then remove s and all of its multiples from L.

3. Continue doing this until all elements are removed from L.

Problem 50

The final P consists of all prime numbers in $\{2, \ldots, N\}$.

8.5 Factorization into primes

Proposition 51

Let $n \in \mathbb{N}$ with $n \neq 0$. Then n factors as a product of primes.

Proof. We prove this by induction on n.

Base case: $n = 1$. Then $n = 1$ is the empty product of primes.

Inductive step: Let $m \geq 2$. Suppose that for $1 \leq k < m$, k can be expressed as a product of primes.

- If m is prime, $m = m$ expresses m as a product of 1 prime.
- If m is not prime, $m = ab$ for some $1 < a, b < m$.

Since $1 \leq a = m/b < m$ and $1 \leq b = m/a < m$, we can express a and b as products of primes:

$$a = p_1 \ldots p_j \quad p_1, \ldots, p_j \text{ prime}$$
$$b = q_1 \ldots q_t \quad q_1, \ldots, q_t \text{ prime}$$

Then $m = ab = (p_1 \ldots p_j)(q_1 \ldots q_t)$ expresses m as a product of primes, thus completing the inductive step.

It follows, by induction, that every integer $n \geq 1$ can be expressed as a product of primes. $\square$

As an application, we can prove the infinitude of primes:

Theorem 52

There are infinitely many primes $p \in \mathbb{Z}$.

Proof. Let $n \in \mathbb{Z}_{>1}$.

Consider $n! + 1$, where $n! = n \times (n-1) \times \cdots \times 2 \times 1$.

Since $n!$ is a product of integers from 1 to n, any prime factor p of $n! + 1$ must satisfy $p \mid n! + 1$.

Claim: $p > n$.

Suppose for contradiction that $p \leq n$.

Since $p \leq n$, p must divide $n!$. Therefore, $p \mid n!$.

But $p \mid n! + 1$ and $p \mid n!$ imply $p \mid (n! + 1) - n! = 1$, which is a contradiction since no prime number divides 1.

Hence, $p > n$ as claimed.

Therefore, for every $n \in \mathbb{Z}_{>1}$, there exists a prime number $p > n$. This implies that there are infinitely many primes. $\square$

8.6 An important characterization of primes

Theorem 53

$p \in \mathbb{Z}$ is prime $\Leftrightarrow$ for all $a, b \in \mathbb{Z}$, $p \mid ab$ implies $p \mid a$ or $p \mid b$.

Proof. ($\Leftarrow$) Suppose p is not prime. Then $p = ab$ for some $a, b \in \mathbb{Z}$ with $a, b \neq \pm 1$. Then $p \mid p = ab$ but $p \nmid a$ and $p \nmid b$.

($\Rightarrow$) Suppose p is prime. Suppose $p \mid ab$. Note that

$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of } a \text{ and } p \end{array} \right\} \subset \left\{ \begin{array}{l} \text{divisors of} \\ p \end{array} \right\} = \{\pm 1, \pm p\}$$

Since $\pm p$ are not divisors of a,

$$\left\{ \begin{array}{l} \text{common divisors} \\ \text{of } a \text{ and } p \end{array} \right\} = \{\pm 1\}, \text{ i.e., } \gcd(a, p) = \pm 1$$

By our earlier key lemma, since $p \mid ab$ and $\gcd(a, p) = \pm 1$, it follows that $p \mid b$. $\square$

Theorem 54

Let $p \in \mathbb{Z}$ be prime. Let $a_1, \ldots, a_n \in \mathbb{Z}$ be integers for which $p \mid a_1 \ldots a_n$. Then $p \mid a_1$ or $p \mid a_2 \ldots a_n$.

Proof. We prove this by induction on n.

Base case: $n = 2$. This is the previous case, which states that if $p \mid a_1 a_2$, then $p \mid a_1$ or $p \mid a_2$.

Inductive step: Suppose the statement is true for some $n \geq 2$. That is, if $p \mid a_1 \ldots a_n$, then $p \mid a_1$ or $p \mid a_2 \ldots a_n$.

We need to show that the statement is true for $n + 1$. Suppose $p \mid a_1 a_2 \ldots a_n a_{n+1}$. By the inductive hypothesis, applied to the product $a_1 a_2 \ldots a_n$, we have $p \mid a_1$ or $p \mid a_2 \ldots a_n$.

- If $p \mid a_1$, we are done.
- If $p \mid a_2 \ldots a_n$, then by the base case applied to the product $(a_2 \ldots a_n) a_{n+1}$, we have $p \mid a_2 \ldots a_n$ implies $p \mid a_2$ or $p \mid a_3 \ldots a_n$.

Continuing this process, we eventually conclude that $p \mid a_1$ or $p \mid a_2$ or $\ldots$ or $p \mid a_{n+1}$.

Therefore, by induction, the statement is true for all $n \geq 2$. $\square$

We use the latter characterization of primes to prove uniqueness of prime factorization.

Theorem 55

Every integer $n \neq 0$ can be written in a unique way as a product of primes.

More formally, if

$$n = p_1^{e_1} \cdots p_k^{e_k} \quad p_1, \ldots, p_k \text{ distinct primes } e_1, \ldots, e_k \in \mathbb{Z}_{\geq 1}$$
$$n = q_1^{f_1} \cdots q_l^{f_l} \quad q_1, \ldots, q_l \text{ distinct primes } f_1, \ldots, f_l \in \mathbb{Z}_{\geq 1}$$

Then $k = l$ and $(q_1, \ldots, q_l)$ is a rearrangement of $(p_1, \ldots, p_k)$, i.e., $q_i = p_{\sigma(i)}$ for some bijection $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$ and $f_j = e_{\sigma(j)}$.

Proof. We prove this by induction on n.

Base case: $n = 1$. $n = 1$ can only be factored as the empty product over primes. Thus, its factorization into primes is unique.

Inductive step: Let $m \geq 2$. Suppose every $1 \leq k < m$ can be factored uniquely as a product of primes.

Suppose

$$m = p_1^{e_1} \cdots p_k^{e_k} \quad p_1, \ldots, p_k \text{ distinct primes } e_1, \ldots, e_k \in \mathbb{Z}_{\geq 1}$$
$$m = q_1^{f_1} \cdots q_l^{f_l} \quad q_1, \ldots, q_l \text{ distinct primes } f_1, \ldots, f_l \in \mathbb{Z}_{\geq 1}$$

are two factorizations of m. Let $p = p_1$.

By (i), $p \mid m$. By (ii), $p \mid m = q_1^{f_1} \cdots q_l^{f_l}$. By our product characterization of primes, (i) implies $p \mid q_1$ or $\ldots$ or $p \mid q_l$.

Since the q's are prime, $p \mid q_i$ is equivalent to $p = q_i$.

Thus, $p = q_1$ or $\ldots$ or $p = q_l$.

Suppose WLOG that $p_1 = p = q_1$.

Then

$$m/p = p_1^{e_1 - 1} p_2^{e_2} \cdots p_k^{e_k} = q_1^{f_1 - 1} q_2^{f_2} \cdots q_l^{f_l}$$

Continuing by the same argument (and letting q_1 play the role of p_1 too), we can prove that

$$p_1 = p = q_1$$
$$e_1 = f_1$$

Consider

$$m/p^{e_1} = p_2^{e_2} \cdots p_k^{e_k}$$
$$m/q_1^{f_1} = q_2^{f_2} \cdots q_l^{f_l}$$

By inductive hypothesis (since $1 \leq m/p^{e_1} < m$),

$k-1 = l-1$

$= (q_2, \ldots, q_l)$ is a rearrangement of $(p_2, \ldots, p_k)$ via a bijection $\sigma : \{2, \ldots, k\} \to \{2, \ldots, k\}$

$q_j = p_{\sigma(j)}$ for $j = 2, \ldots, l$

$f_j = e_{\sigma(j)}$ for $j = 2, \ldots, k$

The inductive step follows from this:

$$k-1 = l-1 \Rightarrow k = l$$

$$= (q_2, \ldots, q_l) \text{ a rearrangement of } (p_2, \ldots, p_k) \text{ via } \sigma : \{2, \ldots, k\} \to \{2, \ldots, k\}$$

$$\Rightarrow (q_1, \ldots, q_l) \text{ is a rearrangement of } (p_1, \ldots, p_k) \text{ via } \tilde{\sigma} : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, k\}$$

$$\tilde{\sigma}(x) = \begin{cases} \sigma(x) \text{ if } x \neq 1 \\ 1 \quad \text{if } x = 1 \end{cases}$$

$$f_j = e_{\sigma(j)} \text{ for } j = 2, \ldots, k$$

$$\Rightarrow f_j = e_{\sigma(j)} \text{ for } j = 1, \ldots, k \quad (\text{since } \sigma(1) = 1).$$

By induction, unique factorization in $\mathbb{Z}$ follows. $\square$