Physics 2301, Spring 2025
Intermediate Mechanics II

Homework 4
Due Monday, Feb 3 before 11:59 pm in Gradescope. No late work will be accepted.

Problem 1 (10 points) Identical masses *m* are located at the eight corners of a square cube, $x = \pm 1$, $y = \pm 1$, $z = \pm 1$. Find the moment of inertia matrix and show that the three eigenvalues are identical.

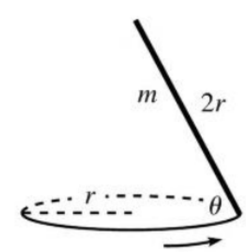
Problem 2 (10 points) Calculate any of the eigenvalues of the inertia matrix for a spherical shell of inner radius *a*, outer radius *b*, and mass *M*, assuming the mass density is constant.

Problem 3 (10 points) Consider a rigid triangular frame with three point masses. The first, with mass *m, is* located at $\vec{r}_A = a\hat{x} + 2a\hat{y} - a\hat{z}$. The second, a mass *2m,* is located at $\vec{r}_B = a\hat{x} - a\hat{y} - a\hat{z}$. The third, a mass *3m,* is located at $\vec{r}_C = -a\hat{x} + a\hat{z}$. The three displacement vectors indicate position relative to the center of mass.

Express the inertia matrix for each of the three masses, and add them to get the matrix for the whole system.

Someone claims that the principal directions will be $\hat{y}$, $\hat{x} + \hat{z}$, and $\hat{x} - \hat{z}$. Check if this is true, and if so, give the associated moments of inertia.

Problem 4 (15 points) ***Morin 9.45*** A stick of mass *m* and length *2r* is arranged to make a constant angle θ with the horizontal, with its bottom end sliding in a circle on a frictionless ring of radius *r*. What is the frequency of this motion?



It turns out that there is a minimum θ for which this motion is possible; what is it?

If the radius of the ring is now *R*, what is the largest value of *r/R* for which this motion is possible for $\theta \to 0$?

Problem 5 (15 points) A uniform square cube, each edge of length l, balances on one edge. That is, one edge is in contact with a horizontal plane, but this is an unstable equilbrium—an impulse will knock the cube over. Suppose the cube is given a tiny displacement and allowed to fall. Show that the angular velocity of the cube when one face strikes the plane is given by

$$\omega^2 = A\frac{g}{l}(\sqrt{2} - 1)$$

where $A = \frac{3}{2}$ if the edge cannot slide on the plane and $A = \frac{12}{5}$ if sliding can occur without friction.

(continued on the next page...)

Problem 6 (10 points) To rotate a vector $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ by an angle θ around the $\hat{z}$ axis, we apply the matrix

$$R_z(\theta) \equiv \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To rotate a matrix M, the rotated matrix M_R is given by the matrix multiplication $M_R = R \cdot M \cdot R^T$ where R^T is the transposition of R where its rows turn into columns (and vice versa): $(R^T)_{ij} = R_{ji}$.

Write out each of these 3x3 matrices and use the above to rotate the following:

(a) The moment of inertia matrix M_0 for a mass m at $\vec{r} = a\hat{x}$. Note that $\hat{x}$ is a principal direction of M_0 with eigenvalue 0. Verify that the corresponding $R_z \cdot \hat{x}$ is a principal direction of the rotated matrix $M_R \equiv R_z \cdot M_0 \cdot R^T$.

(b) Let an antisymmetric matrix Ω, corresponding to the rotation vector $\vec{\omega}$, be defined by the rule that for any input $\vec{r}$, $\Omega \cdot \vec{r} = \vec{\omega} \times \vec{r}$. Take $\vec{\omega} = \omega\hat{x}$, and verify that the rotated version of Ω is the same matrix that you would get from the rotated vector $R_z \cdot \omega\hat{x}$.