a messy manner. Since there are no torques acting on the system (after the initial blow), we know that **L** forever remains constant. It turns out that $\boldsymbol{\omega}$ moves around **L** while the masses rotate around this changing $\boldsymbol{\omega}$. These matters are the subject of Section 9.6, although in that discussion we restrict ourselves to symmetric tops, that is, ones with two equal moments. But these issues aside, it's good to know that we can, without too much difficulty, determine what's going on immediately after the blow.

3. The object in this problem was assumed to be floating freely in space. If we instead have an object that is pivoted at a given fixed point, then we should use this pivot as our origin. There is then no need to perform the last step of adding on the velocity of the origin (which was the CM, above), because this velocity is now zero. Equivalently, just consider the pivot to be an infinite mass, which is therefore the location of the (motionless) CM. ♣

9.4.2 Frequency of motion due to a torque

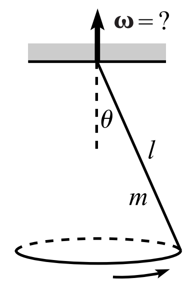

Fig. 9.20

Problem: Consider a stick of length ℓ, mass m, and uniform mass density. The stick is pivoted at its top end and swings around the vertical axis. Assume that conditions have been set up so that the stick always makes an angle θ with the vertical, as shown in Fig. 9.20. What is the frequency, ω, of this motion?

Solution: Our strategy will be to find the principal moments and then the angular momentum of the system (in terms of ω), and then find the rate of change of **L**, and then calculate the torque and equate it with $d\mathbf{L}/dt$. We will choose the pivot to be the origin.[8] Again, there are five standard steps that we must perform.

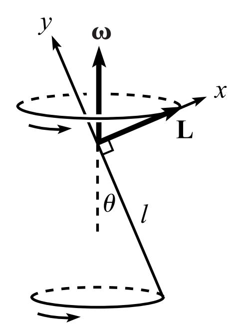

Fig. 9.21

- **Calculate the principal moments:** The principal axes are the axis along the stick, along with any two orthogonal axes perpendicular to the stick. So let the x and y axes be as shown in Fig. 9.21. The positive z axis then points out of the page. The moments (relative to the pivot) are $I_x = m\ell^2/3, I_y = 0$, and $I_z = m\ell^2/3$ (which won't be needed).
- **Find L:** The angular velocity vector points vertically (however, see the third remark following this solution), so in the basis of the principal axes, the angular velocity vector is $\boldsymbol{\omega} = (\omega \sin\theta, \omega\cos\theta, 0)$, where ω is yet to be determined. The angular momentum of the system (relative to the pivot) is therefore

$$\mathbf{L} = (I_x\omega_x,\ I_y\omega_y,\ I_z\omega_z) = \big((1/3)m\ell^2\omega\sin\theta, 0, 0\big). \tag{9.34}$$

- **Find $d\mathbf{L}/dt$:** The vector **L** in Eq. (9.34) points up to the right, along the x axis (at the instant shown in Fig. 9.21), with magnitude $L = (1/3)m\ell^2\omega\sin\theta$. As the stick rotates around the vertical axis, **L** traces out the surface of a cone. That is, the tip of **L** traces out a horizontal circle. The radius of this circle is the horizontal component of **L**, which is $L\cos\theta$. The speed of the tip (which is the magnitude of $d\mathbf{L}/dt$) is therefore

[8] This is a better choice than the CM because this way we won't have to worry about any messy forces acting at the pivot when computing the torque. The task of Exercise 9.41 is to work through the more complicated solution which has the CM as the origin.

$(L\cos\theta)\omega$, because $\mathbf{L}$ rotates around the vertical axis with the same frequency as the stick. So $d\mathbf{L}/dt$ has magnitude

$$\left|\frac{d\mathbf{L}}{dt}\right| = (L\cos\theta)\omega = \frac{1}{3}m\ell^2\omega^2\sin\theta\cos\theta, \tag{9.35}$$

and it points into the page.

Remark: With more complicated objects where $I_y \neq 0$, $\mathbf{L}$ won't point nicely along a principal axis, so the length of its horizontal component (the radius of the circle that $\mathbf{L}$ traces out) won't immediately be obvious. In this case, you can either explicitly calculate the horizontal component (see the spinning-top example in Section 9.7.5), or you can just do things the formal way by finding the rate of change of $\mathbf{L}$ via the expression $d\mathbf{L}/dt = \boldsymbol{\omega}\times\mathbf{L}$, which holds for all the same reasons that $\mathbf{v}\equiv d\mathbf{r}/dt = \boldsymbol{\omega}\times\mathbf{r}$ holds. In the present problem, we obtain

$$\begin{aligned} d\mathbf{L}/dt &= (\omega\sin\theta, \omega\cos\theta, 0)\times\big((1/3)m\ell^2\omega\sin\theta, 0, 0\big) \\ &= \big(0, 0, -(1/3)m\ell^2\omega^2\sin\theta\cos\theta\big), \end{aligned} \tag{9.36}$$

in agreement with Eq. (9.35). And the direction is correct, because the negative z axis points into the page. Note that we calculated this cross product in the principal-axis basis. Although these axes are changing in time, they present a perfectly good set of basis vectors at any instant. ♣

- **Calculate the torque:** The torque (relative to the pivot) is due to gravity, which effectively acts on the CM of the stick. So $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$ has magnitude

$$\tau = rF\sin\theta = (\ell/2)(mg)\sin\theta, \tag{9.37}$$

and it points into the page.

- **Equate $\boldsymbol{\tau}$ with $d\mathbf{L}/dt$:** The vectors $d\mathbf{L}/dt$ and $\boldsymbol{\tau}$ both point into the page, which is good, because they had better point in the same direction. Equating their magnitudes gives

$$\frac{m\ell^2\omega^2\sin\theta\cos\theta}{3} = \frac{mg\ell\sin\theta}{2} \quad\Longrightarrow\quad \omega = \sqrt{\frac{3g}{2\ell\cos\theta}}. \tag{9.38}$$

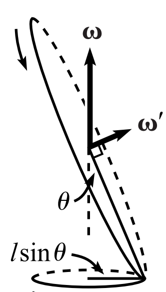

Fig. 9.22

Remarks:

1. This frequency is slightly larger than the frequency that would arise if we instead had a mass on the end of a massless stick of length ℓ. From Problem 9.12, the frequency in that case is $\sqrt{g/\ell\cos\theta}$. So, in some sense, a uniform stick of length ℓ behaves like a mass on the end of a massless stick of length $2\ell/3$, as far as these rotations are concerned.
2. As $\theta\to\pi/2$, the frequency goes to ∞, which makes sense. And as $\theta\to 0$, it approaches $\sqrt{3g/2\ell}$, which isn't so obvious.
3. As explained in Problem 9.1, the instantaneous $\boldsymbol{\omega}$ is not uniquely defined in some situations. At the instant shown in Fig. 9.20, the stick is moving directly into the page. What if someone else wants to think of the stick as (instantaneously) rotating around the $\boldsymbol{\omega}'$ axis perpendicular to the stick (the x axis, in the above notation), instead of the vertical axis, as shown in Fig. 9.22. What is the angular speed ω'?

   Well, if ω is the angular speed of the stick around the vertical axis, then we may view the tip of the stick as instantaneously moving in a circle of radius $\ell\sin\theta$ around the vertical axis $\boldsymbol{\omega}$. So $\omega(\ell \sin\theta)$ is the speed of the tip of the stick. But we may also view the tip of the stick as instantaneously moving in a circle of radius ℓ around $\boldsymbol{\omega}'$, as shown. The speed of the tip is still $\omega(\ell \sin\theta)$, so the angular speed around this axis is given by $\omega'\ell = \omega(\ell \sin\theta)$. Hence $\omega' = \omega \sin\theta$, which is simply the x component of $\boldsymbol{\omega}$ that we found above, right before Eq. (9.34). The moment of inertia around $\boldsymbol{\omega}'$ is $m\ell^2/3$, so the angular momentum has magnitude $(m\ell^2/3)(\omega \sin\theta)$, in agreement with Eq. (9.34). And the direction is along the x axis, as it should be.

Note that although $\boldsymbol{\omega}$ is not uniquely defined at any instant, $\mathbf{L} \equiv \int (\mathbf{r} \times \mathbf{p})\, dm$ certainly is.[9] Choosing $\boldsymbol{\omega}$ to point vertically, as we did in the above solution, is in some sense the natural choice, because this $\boldsymbol{\omega}$ doesn't change with time. ♣

9.5 Euler's equations

Consider a rigid body instantaneously rotating around an axis $\boldsymbol{\omega}$. This $\boldsymbol{\omega}$ may change as time goes on, but all we care about for now is what it is at a given instant. The angular momentum is given by Eq. (9.8) as $\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$, where $\mathbf{I}$ is the inertia tensor, calculated with respect to a given origin and a given set of axes (and $\boldsymbol{\omega}$ is written in the same basis, of course).

As usual, things are much nicer if we use the principal axes (relative to the chosen origin) as the basis vectors of our coordinate system. Since these axes are fixed with respect to the rotating object, they will rotate with respect to the fixed reference frame. In this basis, $\mathbf{L}$ takes the nice form,

$$\mathbf{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3), \tag{9.39}$$

where ω_1, ω_2, and ω_3 are the components of $\boldsymbol{\omega}$ along the principal axes. In other words, if you take the vector $\mathbf{L}$ in space and project it onto the instantaneous principal axes, then you get the components in Eq. (9.39).

On one hand, writing $\mathbf{L}$ in terms of the rotating principal axes allows us to write it in the nice form of Eq. (9.39). But on the other hand, writing $\mathbf{L}$ in this way makes it nontrivial to determine how it changes in time, because the principal axes themselves are changing. However, it turns out that the benefits outweigh the detriments, so we will invariably use the principal axes as our basis vectors.

The goal of this section is to find an expression for $d\mathbf{L}/dt$, and to then equate this with the torque. The result will be Euler's equations in Eq. (9.45).

Derivation of Euler's equations

If we write $\mathbf{L}$ in terms of the body frame, which we'll choose to be described by the principal axes painted on the body, then $\mathbf{L}$ can change (relative to the lab frame) due to two effects. It can change because its coordinates in the body frame change, and it can also change because of the rotation of the body frame. To be precise, let $\mathbf{L}_0$ be the vector $\mathbf{L}$ at a given instant. At this instant, imagine painting the vector $\mathbf{L}_0$ onto the body frame, so that $\mathbf{L}_0$ then rotates with the body. The rate

[9] The nonuniqueness of $\boldsymbol{\omega}$ arises from the fact that $I_y = 0$ here. If all the moments are nonzero, then $(L_x, L_y, L_z) = (I_x\omega_x, I_y\omega_y, I_z\omega_z)$ uniquely determines $\boldsymbol{\omega}$, given $\mathbf{L}$.