Math 4580: Abstract Algebra I

Lecturer: **Professor Michael Lipnowski**
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1 January 6, 2025

We didn't have any, but Dr. Lipnowski did post a module on `carmen` about the syllabus and the course. This semester we will be covering the first few chapters of the book *Abstract Algebra: Theory and Applications* by Thomas Judson.

Definition 1

Set: A collection of distinct objects, considered as an object in its own right.
Axioms: A collection of objects S with assumed structural rules is defined by axioms.
Statement: In logic or mathematics, an assertion that is either true or false.
Hypothesis and Conclusion: In the statement "If P, then Q", P is the hypothesis and Q is the conclusion.
Mathematical Proof: A logical argument that verifies the truth of a statement.
Proposition: A statement that can be proven true.
Theorem: A proposition of significant importance.
Lemma: A supporting proposition used to prove a theorem or another proposition.
Corollary: A proposition that follows directly from a theorem or proposition with minimal additional proof.

2 January 8, 2025

Professor Lipnowski discussed Sam Lloyd's 15 puzzle. Each lecture will include a mystery digit, contributing up to 5% bonus to the final grade based on correct guesses.

Certain course expectations:

- All assignments (one every two weeks) and exams (one midterm and one final exam) will be take-home.
- All the problems from the course textbook.
- Collaboration is encouraged, but the work should be your own.
- For the exams, we are not supposed to talk to other friends.

2.1 Functions

Definition 2
Let A and B be sets. A function $f : A \to B$ assigns exactly one output $f(a) \in B$ to every input $a \in A$.

- The set A is called the **domain** of f.
- The set B is called the **codomain** of f.

Fact 3
The domain A, codomain B, and the assignment of outputs $f(a)$ to every input $a \in A$ are all part of the data defining a function. Just writing a formula like $f(x) = e^x$ does not determine a function, as the domain and codomain are not specified.

For example:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.
- $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = e^x$.

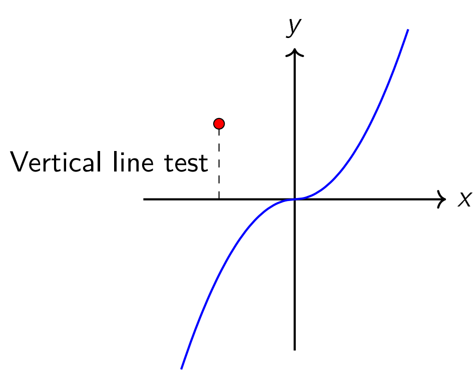
Although these functions use the same formula, their meanings are completely different because their domains and codomains differ.

2.2 Graphs

A function $f : A \to B$ is often identified with its **graph** in $A \times B$:

$$\mathrm{graph}(f) = \{(a, b) \in A \times B : b = f(a)\}.$$

Lemma 4
Let $f : A \to B$ be a function. Its graph, $\mathrm{graph}(f)$, passes the **vertical line test**: For every $a \in A$, $V_a := \{(a, b) \in A \times B : b \in B\}$ intersects $\mathrm{graph}(f)$ in exactly one element.



Proposition 5

Let $G \subseteq A \times B$ be any subset passing the vertical line test, i.e., for all $a \in A$, $V_a \cap G$ consists of exactly one element. Then $G = \text{graph}(f)$ for a unique function $f : A \to B$.

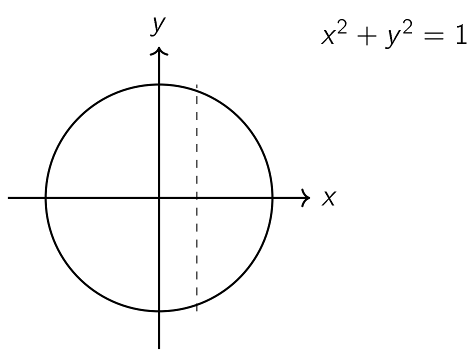
Proof. If $G = \{(a, b) \mid b \in B\}$ satisfies the vertical line test, define $f : A \to B$ by $f(a) = b$. Then $G = \text{graph}(f)$. $\square$

Definition 6

A subset $R \subseteq A \times B$ is called a **relation**. The vertical line test distinguishes graphs of functions from more general relations.

Examples

- Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (the unit circle). This is a relation but not the graph of a function because it fails the vertical line test: The vertical line $x = 0$ intersects the circle at two points.
- Visual depiction of a unit circle:



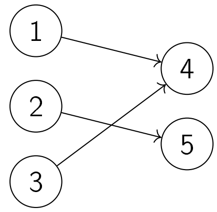
- Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$. The number of functions from A to B is $2^3 = 8$, corresponding to the 8 associated graphs in $A \times B$.
- The number of relations from A to B is $2^{|A| \cdot |B|} = 2^{3 \cdot 2} = 64$, containing the 8 graphs of functions from A to B.

Fact 7

The notion of relation is much more permissive than the notion of functions.

Visualizing Functions as Directed Edges

A function $f : A \to B$ can be visualized as a collection of directed edges $(a, f(a)) \in A \times B$. Each element of A has exactly one outgoing edge in the graph.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