Math 4580: Abstract Algebra I

Lecturer: **Professor Michael Lipnowski**
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Spring 2025

1 January 6, 2025

We didn't have any, but Dr. Lipnowski did post a module on `carmen` about the syllabus and the course. This semester we will be covering the first few chapters of the book *Abstract Algebra: Theory and Applications* by Thomas Judson.

Definition 1

Set: A collection of distinct objects, considered as an object in its own right.
Axioms: A collection of objects S with assumed structural rules is defined by axioms.
Statement: In logic or mathematics, an assertion that is either true or false.
Hypothesis and Conclusion: In the statement "If P, then Q", P is the hypothesis and Q is the conclusion.
Mathematical Proof: A logical argument that verifies the truth of a statement.
Proposition: A statement that can be proven true.
Theorem: A proposition of significant importance.
Lemma: A supporting proposition used to prove a theorem or another proposition.
Corollary: A proposition that follows directly from a theorem or proposition with minimal additional proof.

2 January 8, 2025

Professor Lipnowski discussed Sam Lloyd's 15 puzzle. Each lecture will include a mystery digit, contributing up to 5% bonus to the final grade based on correct guesses.

Certain course expectations:

- All assignments (one every two weeks) and exams (one midterm and one final exam) will be take-home.
- All the problems from the course textbook.
- Collaboration is encouraged, but the work should be your own.
- For the exams, we are not supposed to talk to other friends.

2.1 Review of Set Theory

In this course, we will study **functions** and **equivalence relations**.

Definition 2

A **function** from a set A to a set B is a relation that assigns to each element x in A exactly one element y in B. We write $f : A \to B$ to denote a function f from A to B. Here, A is the **domain** and B is the **codomain**.

Domain: The set of all possible input values for the function.

Codomain: The set of all possible output values for the function.

Example 3

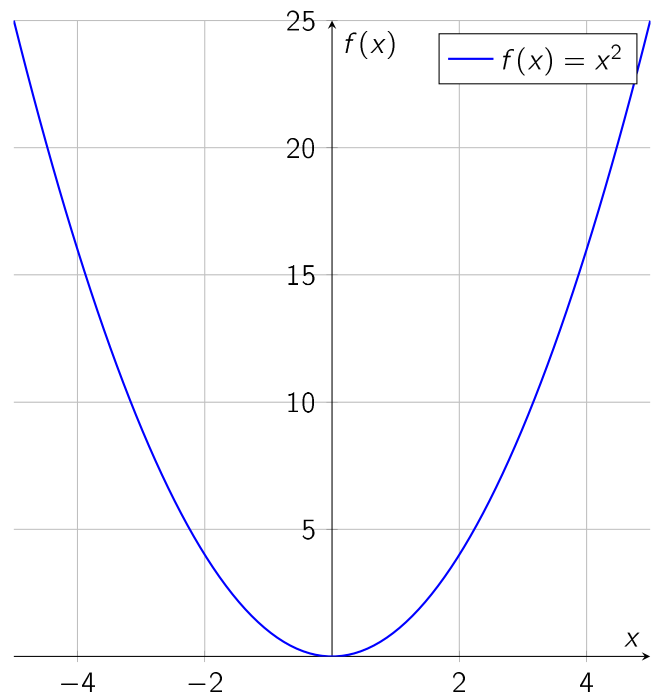
Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^2$. Here, the domain is the set of all integers $\mathbb{Z}$, and the codomain is also the set of all integers $\mathbb{Z}$. For example, $f(2) = 4$ and $f(-3) = 9$.

Example 4

Another example is the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$. Here, the domain is the set of all non-negative real numbers $\mathbb{R}_{\geq 0}$, and the codomain is the set of all real numbers $\mathbb{R}$. For example, $g(4) = 2$ and $g(9) = 3$.

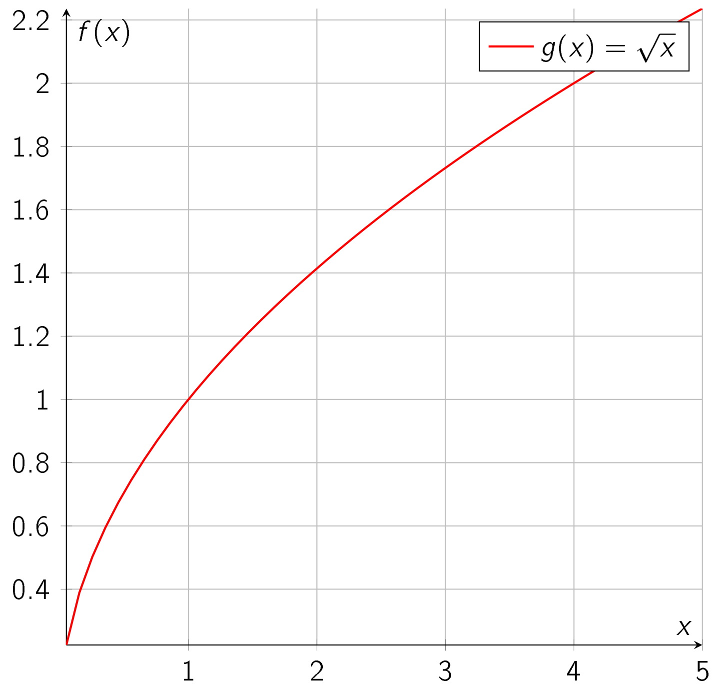
Fact 5

A function must have a unique output for each input in its domain. This means that for every $x \in A$, there is exactly one $y \in B$ such that $f(x) = y$.



Fact 6

A function f can be represented as a special kind of subset of the Cartesian product $A \times B$. Specifically, the **graph** of f is the set of ordered pairs $\{(a, b) \mid a \in A, b = f(a)\}$.

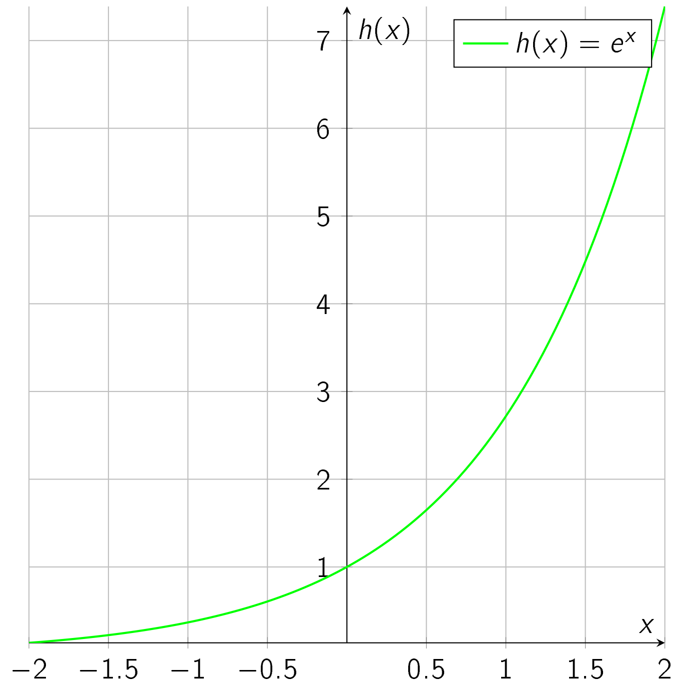


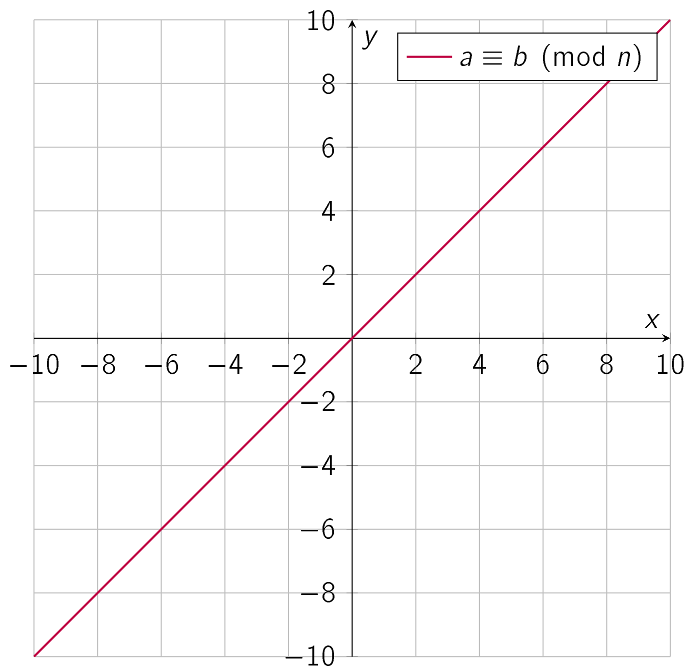
Proposition 7

A set $G \subseteq A \times B$ is the graph of a function $f : A \to B$ if and only if G intersects every vertical line $x = a$ at exactly one point.

Example 8

Consider the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = e^x$. The graph of this function is the set of ordered pairs $\{(x, e^x) \mid x \in \mathbb{R}\}$.



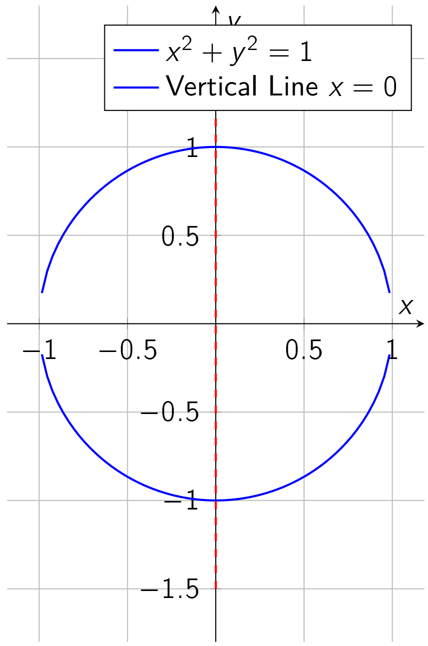


Definition 9

The **vertical line test** is a visual way to determine if a curve in the coordinate plane represents a function. A curve represents a function if and only if no vertical line intersects the curve more than once. This is because a function can only have one output for each input.

Example 10

Consider the relation defined by the equation $x^2 + y^2 = 1$, which represents a circle with radius 1 centered at the origin. This relation does not pass the vertical line test because there are vertical lines that intersect the circle at two points. For example, the vertical line $x = 0$ intersects the circle at the points $(0, 1)$ and $(0, -1)$. Therefore, the circle is not the graph of a function.



In this case the circle is not the graph of a function because it fails the vertical line test.