I encourage collaboration when solving these problems, however you should write up solutions independently and in your own words. Solutions should be presented clearly and methodically. Even if you get the correct answer, you may lose points if you fail to show your working and explain your solutions. Solutions should be justified by Definitions and Theorems in lectures.

You should submit your solutions via Gradescope by **Monday 3rd February at 11:59pm**. You are responsible for ensuring submitted work is readable. If you have technical problems with Gradescope, please submit work to me at `margolis.93@osu.edu`. Please email if you need any clarification.

In all the following questions, you should make use of the definition of $\lim_{n\to\infty} a_n = L$ as defined in class. For this homework, you should *not* make use of any algebra of limits results/limit theorems, i.e. you should not use any results in or after Section 2.2 of the lectures notes.

1. (7 points) In Lucy's Analysis exam, she is asked to define what it means for a sequence to be convergent. She writes (incorrectly) that:

   A sequence (a_n) is convergent if there exists some $L \in \mathbb{R}$ and some $N \in \mathbb{N}$ such that for all $\epsilon > 0$ and all $n \geq N$, we have $|a_n - L| < \epsilon$.[1]

   (a) Explain how Lucy's definition is different from the definition given in class.

   We call a sequence (a_n) *Lucy-convergent* if it satisfies the above "definition" in Lucy's answer, and *convergent* if it satisfies the definition of convergence defined in class.

   (b) Is every Lucy-convergent sequence a convergent sequence? Either prove your answer or provide a counterexample.

   (c) Is every convergent sequence a Lucy-convergent sequence? Either prove your answer or provide a counterexample.

2. (4 points) Let $a_n = \frac{1}{n^2+3n+4}$. Show that (a_n) converges to zero.

3. (4 points) Let $a_n = \frac{4n-1}{n+8}$. Show that (a_n) is convergent.

4. (6 points) Carefully prove that neither of the following sequences are convergent:

   (a) (a_n), where $a_n = \frac{(-1)^n 2n}{n+1}$.

   (b) (b_n), where $b_n = 2n + 7$.

5. (9 points) Let (a_n) be sequence and let $L \in \mathbb{R}$. For each of the following statements, either give a proof or a counterexample.

   (a) If $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$.

   (b) If $\lim_{n\to\infty} |a_n| = |L|$, then $\lim_{n\to\infty} a_n = L$.

[1] This definition is incorrect. Do not use it instead of the correct definition.

(c) If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

The following bonus question is optional. Answering the bonus question gives you an opportunity to make up for lost points in the remaining questions, but you cannot receive more than 100%.

6. (4 points (bonus)) Let $B \subseteq \mathbb{R}$ and suppose β **is an upper bound of** B. Prove the following are equivalent:

   (I) $\beta = \sup(B)$

   (II) There exists a sequence (b_n) such that:

   - $b_n \in B$ for all $n \in \mathbb{N}$.
   - (b_n) converges to β.