Math 4580: Abstract Algebra I

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1 January 6, 2025

We didn't have any, but Dr. Lipnowski did post a module on carmen about the syllabus and the course. This semester we will be covering the first few chapters of the book *Abstract Algebra: Theory and Applications* by Thomas Judson.

Definition 1

Set: A collection of distinct objects, considered as an object in its own right.
Axioms: A collection of objects S with assumed structural rules is defined by axioms.
Statement: In logic or mathematics, an assertion that is either true or false.
Hypothesis and Conclusion: In the statement "If P, then Q", P is the hypothesis and Q is the conclusion.
Mathematical Proof: A logical argument that verifies the truth of a statement.
Proposition: A statement that can be proven true.
Theorem: A proposition of significant importance.
Lemma: A supporting proposition used to prove a theorem or another proposition.
Corollary: A proposition that follows directly from a theorem or proposition with minimal additional proof.

2 January 8, 2025

Professor Lipnowski discussed Sam Lloyd's 15 puzzle. Each lecture will include a mystery digit, contributing up to 5% bonus to the final grade based on correct guesses.

Certain course expectations:

- All assignments (one every two weeks) and exams (one midterm and one final exam) will be take-home.
- All the problems from the course textbook.
- Collaboration is encouraged, but the work should be your own.
- For the exams, we are not supposed to talk to other friends.

2.1 Functions

Definition 2
Let A and B be sets. A function $f : A \to B$ assigns exactly one output $f(a) \in B$ to every input $a \in A$.

- The set A is called the **domain** of f.
- The set B is called the **codomain** of f.

Fact 3
The domain A, codomain B, and the assignment of outputs $f(a)$ to every input $a \in A$ are all part of the data defining a function. Just writing a formula like $f(x) = e^x$ does not determine a function, as the domain and codomain are not specified.

For example:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.
- $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = e^x$.

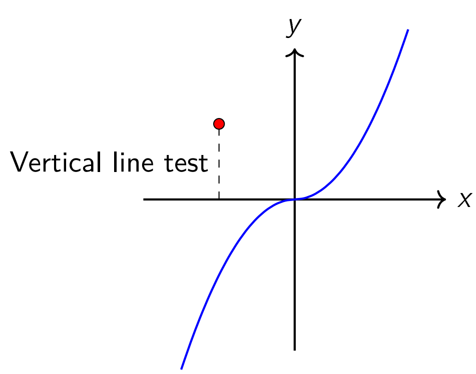
Although these functions use the same formula, their meanings are completely different because their domains and codomains differ.

2.2 Graphs

A function $f : A \to B$ is often identified with its **graph** in $A \times B$:

$$\mathrm{graph}(f) = \{(a, b) \in A \times B : b = f(a)\}.$$

Lemma 4
Let $f : A \to B$ be a function. Its graph, $\mathrm{graph}(f)$, passes the **vertical line test**: For every $a \in A$, $V_a := \{(a, b) \in A \times B : b \in B\}$ intersects $\mathrm{graph}(f)$ in exactly one element.



Proposition 5

Let $G \subseteq A \times B$ be any subset passing the vertical line test, i.e., for all $a \in A$, $V_a \cap G$ consists of exactly one element. Then $G = \text{graph}(f)$ for a unique function $f : A \to B$.

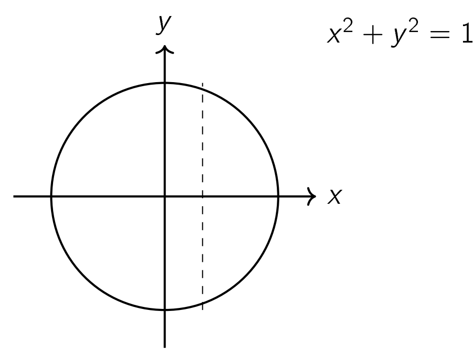
Proof. If $G = \{(a, b) \mid b \in B\}$ satisfies the vertical line test, define $f : A \to B$ by $f(a) = b$. Then $G = \text{graph}(f)$. $\square$

Definition 6

A subset $R \subseteq A \times B$ is called a **relation**. The vertical line test distinguishes graphs of functions from more general relations.

2.3 Examples

- Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (the unit circle). This is a relation but not the graph of a function because it fails the vertical line test: The vertical line $x = 0$ intersects the circle at two points.
- Visual depiction of a unit circle:



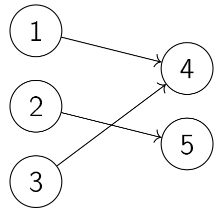
- Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$. The number of functions from A to B is $2^3 = 8$, corresponding to the 8 associated graphs in $A \times B$.
- The number of relations from A to B is $2^{|a| \cdot |b|} = 2^{3 \cdot 2} = 64$, containing the 8 graphs of functions from A to B.

Fact 7

The notion of relation is much more permissive than the notion of functions.

2.4 Visualizing Functions as Directed Edges

A function $f : A \to B$ can be visualized as a collection of directed edges $(a, f(a)) \in A \times B$. Each element of A has exactly one outgoing edge in the graph.



3 January 10, 2025

3.1 More about function

$f : A \to B$ is a function.

- injective (one-to-one)
- surjective (onto)
- bijective (one-to-one and onto)

Definition 8

Let $f : A \to B$ be a function. f is injective if for all $x, y \in A$, if $f(x) = f(y)$ then $x = y$.

Example 9

Consider the function $f : \{1, 2, 3\} \to \{a, b, c\}$ defined by the following assignments:

$$f(1) = a, \quad f(2) = b, \quad f(3) = c$$

This function is bijective because it is both injective and surjective.

Example 10

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$. We claim that f is injective.

To prove this, suppose $f(a) = f(b)$ for some $a, b \in \mathbb{R}$. Then:

$$2a + 3 = 2b + 3$$

Subtracting 3 from both sides, we get:

$$2a = 2b$$

Dividing both sides by 2, we obtain:

$$a = b$$

Therefore, f is injective.

Example 11

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. We claim that f is not injective.

To see this, observe that $f(2) = 4$ and $f(-2) = 4$. Since $f(2) = f(-2)$ but $2 \neq -2$, the function f is not injective.

Example 12

Consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Define the function $f : A \to B$ by the following assignments:

$$f(1) = 4, \quad f(2) = 5, \quad f(3) = 4$$

This function is not injective because $f(1) = f(3) = 4$ but $1 \neq 3$.

Here we have two elements in set B but there are three elements in input in set A. That's why there has to be a collision. Since $|a| > |b|$, there must be a collision. If $f(1), f(2), f(3) \in B$, and $|b| = 2$ Tow of those must be the same. That means that f is not one-to-one.

Definition 13

A function $f : A \to B$ is called **surjective** (or **onto**) if for every element $b \in B$, there exists at least one element $a \in A$ such that $f(a) = b$.

Example 14

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$. This function is surjective because for every $y \in \mathbb{R}$, we can find an $x \in \mathbb{R}$ such that $f(x) = y$ (specifically, $x = y$).

Example 15

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. This function is not surjective because there is no $x \in \mathbb{R}$ such that $f(x) = -1$.

Example 16

Consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Define the function $f : A \to B$ by the following assignments:

$$f(1) = 4, \quad f(2) = 4, \quad f(3) = 4$$

This function is surjective because every element in B is mapped to by at least one element in A. However, it is not injective because $f(1) = f(2) = f(3) = 4$ but $1 \neq 2 \neq 3$.

Example 17

Consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Define the function $f : A \to B$ by the following assignments:

$$f(1) = 4, \quad f(2) = 4, \quad f(3) = 5$$

This function is surjective because every element in B is mapped to by at least one element in A. However, it is not injective because $f(1) = f(2) = 4$ but $1 \neq 2$.

Example 18

Consider the sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$. Define the function $f : A \to B$ by the following assignments:

$$f(1) = 4, \quad f(2) = 5, \quad f(3) = 6$$

This function is neither injective nor surjective. It is not injective because $|a| < |b|$, and it is not surjective because the element $7 \in B$ is not mapped to by any element in A.

Definition 19

A function $f : A \to B$ is called **range** if it is the set of all possible outputs of f. Formally, the range of f is defined as:

$$\text{range}(f) = \{f(a) \mid a \in A\}$$

Fact 20

The range of a function $f : A \to B$ is a subset of the codomain B. A function is surjective if and only if its range is equal to its codomain.

Example 21

Consider the function $f : \{1, 2, 3\} \to \{4, 5\}$ defined by the following assignments:

$$f(1) = 4, \quad f(2) = 5, \quad f(3) = 4$$

The range of f is $\{4, 5\}$, which is equal to the codomain $\{4, 5\}$. Therefore, f is surjective.

Example 22

Consider the function $f : \{1, 2, 3\} \to \{4, 5, 6\}$ defined by the following assignments:

$$f(1) = 4, \quad f(2) = 5, \quad f(3) = 4$$

The range of f is $\{4, 5\}$, which is a subset of the codomain $\{4, 5, 6\}$ but not equal to it. Therefore, f is not surjective.

Fact 23

f is **surjective** means that the **range** of f is equal to the **codomain** of f.

Definition 24

A function $f : A \to B$ is called **bijective** (or a **bijection**) if it is both injective and surjective or (one-to-one correspondence). This means that every element in B is mapped to by exactly one element in A.

Example 25

Consider the function $f : \{1, 2, 3\} \to \{4, 5, 6\}$ defined by the following assignments:

$$f(1) = 5, \quad f(2) = 4, \quad f(3) = 6$$

This function is bijective because it is both injective (no two elements in A map to the same element in B) and surjective (every element in B is mapped to by some element in A).

Here we can notice that we have a matching between the elements of A and B, every element of set A has one and only mapping into the set B.

Example 26

Consider the function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(x) = x$, where $\mathbb{N}$ is the set of natural numbers and $\mathbb{Z}$ is the set of integers. We claim that f is injective but not surjective.

To prove that f is injective, suppose $f(a) = f(b)$ for some $a, b \in \mathbb{N}$. Then:

$$a = b$$

Therefore, f is injective because no two different elements in $\mathbb{N}$ map to the same element in $\mathbb{Z}$.

However, f is not surjective because there are elements in $\mathbb{Z}$ that are not in the range of f. For example, there is no $x \in \mathbb{N}$ such that $f(x) = -1$. Therefore, f is not surjective.

Example 27

Consider the function $f : \mathbb{Z} \to \mathbb{N}$ defined by:

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2|n| - 1 & \text{if } n < 0 \end{cases}$$

We claim that f is bijective.

To prove that f is injective, suppose $f(a) = f(b)$ for some $a, b \in \mathbb{Z}$. We need to show that $a = b$.

- If $a \geq 0$ and $b \geq 0$, then $f(a) = 2a$ and $f(b) = 2b$. Since $f(a) = f(b)$, we have $2a = 2b$, which implies $a = b$.
- If $a < 0$ and $b < 0$, then $f(a) = 2|a| - 1$ and $f(b) = 2|b| - 1$. Since $f(a) = f(b)$, we have $2|a| - 1 = 2|b| - 1$, which implies $|a| = |b|$ and hence $a = b$.
- If $a \geq 0$ and $b < 0$, then $f(a) = 2a$ and $f(b) = 2|b| - 1$. Since $f(a) = f(b)$, we have $2a = 2|b| - 1$, which is a contradiction because $2a$ is even and $2|b| - 1$ is odd. Therefore, this case cannot occur.
- If $a < 0$ and $b \geq 0$, then $f(a) = 2|a| - 1$ and $f(b) = 2b$. Since $f(a) = f(b)$, we have $2|a| - 1 = 2b$, which is a contradiction because $2|a| - 1$ is odd and $2b$ is even. Therefore, this case cannot occur.

Therefore, f is injective.

To prove that f is surjective, let $m \in \mathbb{N}$. We need to find $n \in \mathbb{Z}$ such that $f(n) = m$.

- If m is even, say $m = 2k$ for some $k \in \mathbb{N}$, then $f(k) = 2k = m$.
- If m is odd, say $m = 2k+1$ for some $k \in \mathbb{N}$, then $f(-(k+1)) = 2|-(k+1)| - 1 = 2(k+1) - 1 = m$.

Therefore, f is surjective.

Since f is both injective and surjective, it is bijective.

Example 28

Consider the function $f : \mathbb{Z} \to \mathbb{N}$ defined by:

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2(-n - 1) + 1 & \text{if } n < 0 \end{cases}$$

We claim that f is bijective.

Example 29

Consider the function $f : \mathbb{Z} \to \mathbb{N}$ defined by:

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ 2(-n - 1) + 1 & \text{if } n < 0 \end{cases}$$

We claim that f is bijective.