Math 4547: Real Analysis I

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Professor Margolis introduced the course and discussed the syllabus. The course will cover the following topics: Here are some common number systems

1.1 What is Analysis?

Analysis is the branch of mathematics that deals with the rigorous study of limits, functions, derivatives, integrals, and infinite series. It provides the foundation for calculus and extends its concepts to more abstract settings.

> **Theorem 1**
> Every convergent squence is bounded.

1.2 The Real Numbers

1.2.1 What are the reals?

- The **natural Numbers** $\mathbb{N} = \{1, 2, 3, ...\}$
- The **integers** $\mathbb{Z} = \{0, 1, \text{-}1, 2, \text{-}2, \cdots\}$
- The **rational Numbers** $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \text{ and } q \neq 0\}$
- The **real Numbers** $\mathbb{R}$
- The **complex Numbers** $\mathbb{C} := \{\ a + bi \mid a, b \in \mathbb{R}\}$, where $i^2 = -1$

> **Theorem 2**
> There is no rational number x, such that $x^2 = 2$.

Proof. We assume for contradiction that such an x exists. Then $x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$ and $q \neq 0$. We can assume that p and q have no common factors. Then, $\frac{p^2}{q^2} = 2$, which implies

$$p^2 = 2q^2$$

Thus, p^2 is even. As the square of an odd number is odd, it follows p must even. Therefore, $p = 2k$ for an integer k. We have $2q^2 = p^2 = (2k)^2 = 4k^2$, and so $q^2 = 2k^2$. Thus, q^2 is even. Since p and q are both even, this contradicts our assumption that p and q have no common factors. Therefore, there is no rational number x such that $x^2 = 2$. $\square$

This theorem implies, if we visualize $\mathbb{Q}$ as points lying on a number line, there is a 'hole' where $\sqrt{2}$ is. (There are many more 'holes' e.g. π, e, $\sqrt{3}$, ...)

The key property that $\mathbb{R}$ possesses, but $\mathbb{Q}$ doesn't is that $\mathbb{R}$ has "no holes" (formally, $\mathbb{R}$ is complete.)

In this class, we will rigorously deduce all properties of $\mathbb{R}$ from the axioms of the real numbers.

The axioms are in three groups.

1. Field Axioms (addition and multiplication)
2. Order axioms (needed to describe properties concerning inequalities)
3. Completeness Axiom

1.3 Addition axioms

1. For every pair $a, b \in \mathbb{R}$, we can associate a real number $a + b$ called their **sum**.
2. For every real number a, there is a real number $-a$ called its **negative** or **additive inverse**.
3. There is a special real number 0 called zero or the additive identity such that for all $a, b, c, x, y, z \cdots$ are real numbers unless otherwise stated:
   (a) $a + b = b + a$
   (b) $a + (b + c) = (a + b) + c$
   (c) $a + 0 = a$
   (d) $a + (-a) = 0$

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In this lecture, we will use the axioms to deduce various properties of the real numbers $\mathbb{R}$. From these axioms, we can derive many more properties of the real numbers.

Proposition 3
If $x + a = x$ for all $a \in \mathbb{R}$, then $a = 0$.

Proof. We know that

$$\begin{aligned} x &= x + 0 \quad \text{(A3)} \\ &= x + a \quad \text{(by assumption on } a\text{)} \end{aligned}$$

By the left cancellation property of addition, it follows that $a = 0$. □

Proposition 4 (Left cancellation of addition)
If $a + x = a + y$, then $x = y$.

Proof. We start with the given equation $a + x = a + y$. By the additive identity property (A3), we have:

$$\begin{aligned} y &= y + 0 && \text{(A3)} \\ &= y + (a + (-a)) && \text{(A4)} \\ &= (y + a) + (-a) && \text{(A2)} \\ &= (a + y) + (-a) && \text{(A1)} \\ &= (a + x) + (-a) && \text{(given)} \\ &= x + (a + (-a)) && \text{(A1)} \\ &= x + 0 && \text{(A4)} \\ &= x && \text{(A3)} \end{aligned}$$

Therefore, $x = y$. □

Proposition 5
$-(-a) = a$

Proof. We need to show that $-(-a) = a$. Consider the following:

$$\begin{aligned} (-a) + (-(-a)) &= 0 \quad \text{(by definition of additive inverse)} \\ (-a) + a &= 0 \quad (\text{since } -(-a) = a) \\ a + (-a) &= 0 \quad \text{(by commutativity of addition)} \\ (-a) + (-(-a)) &= a + (-a) \quad \text{(by substitution)} \\ (-(-a)) &= a \quad \text{(by left cancellation of addition)} \end{aligned}$$

Therefore, $-(-a) = a$. □

Proposition 6
$-(a + b) = (-a) + (-b)$

Proof. We need to show that the additive inverse of $(a+b)$ is equal to the sum of the additive inverses of a and b. Consider the following:

$$\begin{aligned}
(a+b)+(-(a+b)) &= 0 \quad \text{(by definition of additive inverse)} \\
(a+b)+((-a)+(-b)) &= a+(b+((-a)+(-b))) \quad \text{(by associativity of addition)} \\
&= a+((b+(-a))+(-b)) \quad \text{(by associativity of addition)} \\
&= a+((-a)+(b+(-b))) \quad \text{(by commutativity of addition)} \\
&= a+((-a)+0) \quad \text{(by definition of additive inverse)} \\
&= a+(-a) \quad \text{(by identity property of addition)} \\
&= 0 \quad \text{(by definition of additive inverse)}
\end{aligned}$$

Therefore, $-(a+b)=(-a)+(-b)$. □

Proposition 7

$-0=0$

Proof. We need to show that the additive inverse of 0 is 0. Consider the following:

$$\begin{aligned}
0+0 &= 0 \quad \text{(by the identity property of addition, A3)} \\
0+(-0) &= 0 \quad \text{(by the definition of additive inverse, A4)}
\end{aligned}$$

Therefore, we have:

$$0+0=0+(-0)$$

By the left cancellation property of addition, it follows that:

$$0=-0$$

Therefore, $-0=0$. □

2.1 Multiplication Axioms

Definition 8

For all $a,b\in\mathbb{R}$, we can associate a real number $a\times b$ called their **product**.

Definition 9

For every $a\in\mathbb{R}$, there is some $a^{-1}\in\mathbb{R}$ called its **multiplicative inverse** or **reciprocal** such that for all $a\neq 0$, $a\times a^{-1}=1$.

Definition 10

There is a number 1 called **one** or the **multiplicative identity** such that for all $a \in \mathbb{R}$, $a \times 1 = a$.

Definition 11

For all $a, b, c \in \mathbb{R}$, we have the following properties of multiplication:

- For all $a, b \in \mathbb{R}$, $a \times b = b \times a$.
- For all $a, b, c \in \mathbb{R}$, $a \times (b \times c) = (a \times b) \times c$.
- For all $a \in \mathbb{R}$, $a \times 0 = 0$.
- For all $a, b \in \mathbb{R}$, $a \times (b + c) = a \times b + a \times c$.

Proposition 12

If $a \times b = a$, and $a \in \mathbb{R}$ then $b = 1$.

Proof. We start with the given equation $a \times b = a$. By the multiplicative identity property, we have:

$$\begin{aligned} a \times b &= a \times 1 \\ b &= 1 \quad \text{(by left cancellation of multiplication)} \end{aligned}$$

Therefore, $b = 1$. $\square$

Proposition 13

If $a \neq 0$ and $a \times b = a \times c$, then $b = c$.

Proof. We start with the given equation $a \times b = a \times c$. By the multiplicative inverse property, we have:

$$\begin{aligned} a^{-1} \times (a \times b) &= a^{-1} \times (a \times c) \\ (a^{-1} \times a) \times b &= (a^{-1} \times a) \times c \\ 1 \times b &= 1 \times c \\ b &= c \end{aligned}$$

Therefore, $b = c$. $\square$

Proposition 14

If $a \neq 0$ and $a^{-1} \neq 0$, then $(a^{-1})^{-1} = a$.

Proof. We need to show that the multiplicative inverse of a^{-1} is a. Consider the following:

$$\begin{aligned} a^{-1} \times a &= 1 && \text{(by definition of multiplicative inverse)} \\ (a^{-1})^{-1} \times a^{-1} &= 1 && \text{(by definition of multiplicative inverse)} \\ (a^{-1})^{-1} &= a \end{aligned}$$

Therefore, $(a^{-1})^{-1} = a$. $\square$

Proposition 15

If $a \neq 0$, $b \neq 0$, and $a \times b \neq 0$, then $(a \times b)^{-1} = a^{-1} \times b^{-1}$.

Proof. We need to show that the multiplicative inverse of $a \times b$ is $a^{-1} \times b^{-1}$. Consider the following:

$$\begin{aligned} (a \times b) \times (a^{-1} \times b^{-1}) &= a \times (b \times (a^{-1} \times b^{-1})) \\ &= a \times ((b \times a^{-1}) \times b^{-1}) \\ &= a \times (1 \times b^{-1}) \\ &= a \times b^{-1} \\ &= 1 \end{aligned}$$

Therefore, $(a \times b)^{-1} = a^{-1} \times b^{-1}$. $\square$

Proposition 16

If $a, b, c \in \mathbb{R}$, then $(a + b) \times c = (a \times c) + (b \times c)$.

Proof. We need to show that the product of $(a + b)$ and c is equal to the sum of the products of a and c, and b and c. Consider the following:

$$\begin{aligned} (a + b) \times c &= c \times (a + b) \\ &= c \times a + c \times b \\ &= a \times c + b \times c \end{aligned}$$

Therefore, $(a + b) \times c = (a \times c) + (b \times c)$. $\square$

Proposition 17

For all $a \in \mathbb{R}$, $a \times 0 = 0$.

Proof. We need to show that the product of any real number a and 0 is 0. Consider the following:

$$\begin{aligned} a \times 0 &= a \times (0 + 0) \\ &= a \times 0 + a \times 0 \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $a \times 0 = 0$. □

Proposition 18

If $a \times b = 0$, then either $a = 0$ or $b = 0$ or both.

Proof. We need to show that if the product of a and b is 0, then either a or b or both must be 0. Consider the following:

$$a \times b = 0$$

If $a \neq 0$, then $b = 0$ by the multiplicative inverse property. If $b \neq 0$, then $a = 0$ by the multiplicative inverse property. Therefore, if $a \times b = 0$, then either $a = 0$ or $b = 0$ or both. □

Proposition 19

$a \times (-b) = (-a) \times b$. In particular, $a \times (-1) = -a$.

Proof. We need to show that the product of a and $-b$ is equal to the product of $-a$ and b. Consider the following:

$$\begin{aligned} a \times (-b) + a \times b &= a \times (b + (-b)) \\ &= a \times 0 \\ &= 0 \\ &= a \times b + (-(a \times b)) \end{aligned}$$

Hence, the additive inverse of $a \times b$ is $-(a \times b)$. Therefore, $a \times (-b) = (-a) \times b$. □

Proposition 20

$(-1) \times (-1) = 1$

Proof. We need to show that the product of -1 and -1 is 1. Consider the following:

$$\begin{aligned}(-1)\times(-1) &= -(-1)\times 1\\ &= -(-1)\times(1+0)\\ &= -(-1)\times(1+(-1))\\ &= -(-1)\times 0\\ &= 0\\ &= (-1)+(-1)\end{aligned}$$

Therefore, $(-1)\times(-1)=1$. $\square$

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For all $a, b \in \mathbb{R}$, we write:

- ab or $a \cdot b$ for $a \times b$.
- $a - b$ for $a + (-b)$.
- $\frac{1}{a}$ for a^{-1} if $a \neq 0$.
- $\frac{a}{b}$ for ab^{-1} if $b \neq 0$.

For $a \neq 0$, we write:

- a^0 for 1.
- a^{k+1} for $a^k \cdot a$ for $k = 0, 1, 2, \ldots$
- $a^{-1} for (a^l)^{-1}$ for $l = 1, 2, 3$

Definition 21

Any set equipped with operations $+$ and $\times$ staisfying A1 - A4, M1 - M4, Z, D is a **field**.

Fact 22

Some facts about the fields:

- $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are all fields.
- $\mathbb{Z}$ is not a field (M4 isn't satisified).
- $\mathbb{N}$ is not a field ($A4$, $M4$) are not satsified.
- $\frac{\mathbb{Z}}{p\mathbb{Z}}$ (integers mod p for prime p) is a field.

3.1 The order axioms

The order axioms are: There is as subset of $P \subset \mathbb{R}$ called the set of **positive numbers**.

- If $a, b \in \mathbb{P}$, then $a + b \in \mathbb{P}$. (P1)
- If $a, b \in \mathbb{P}$, then $a \times b \in \mathbb{P}$. (P2)
- For each $a \in \mathbb{R}$, exactly one of the following is true: $a \in \mathbb{P}$, $a = 0$, or $-a \in \mathbb{P}$. $\leftarrow$ Law of Trichotomy (P3)

P3 is the most powerful aximom about the positive numbers.

Proposition 23

Prove that $1 \in \mathbb{P}$

Proof. According to **P3**, either

- $1 \in \mathbb{P}$
- $1 = 0$
- $-1 \in \mathbb{P}$

We will prove (b) and (c) are false by contradiction and then show that $1 \in \mathbb{P}$. If **(b)** holds, $1 = 0$, which contradicts **Z**. Assume for contradiction **(c)** holds. We know from last lecture that $1 = -(-1)$. Since $-1 \in \mathbb{P}$, by (P2), $(-1) \times (-1) \in \mathbb{P}$. But $(-1) \times (-1) = 1$, so $1 \in \mathbb{P}$. $\therefore$, $1 \in \mathbb{P}$ and $-1 \in \mathbb{P}$ contradicts **P3**. Since, **(b)** and **(c)** cannot hold, therefore, **(a)** must hold. □

Fact 24

For all $a, b \in \mathbb{R}$, we write

- $a < b$ if $b - a \in \mathbb{P}$
- $a > b$ if $a - b \in \mathbb{P}$
- $a \leq b$ if $b - a \in \mathbb{P} \cup \{0\}$
- $a \geq b$ if $a - b \in \mathbb{P} \cup \{0\}$

Proposition 25

$a > b$ if and only if $-a < -b$. In particular, $x > 0 \iff -x < 0$

Proof.

$$\begin{aligned} a > b &\iff a - b \in \mathbb{P} \\ &\iff -(-(a)) - b \in -\mathbb{P} \\ &\iff -b - (-a) \in -\mathbb{P} \\ &\iff -a < -b \end{aligned}$$

□

Proposition 26

For all $x, y, z \in \mathbb{R}$ the following holds:

- $x \leq x$
- If $x \leq y$ and $y \leq z$, then $x \leq z$.
- If $x \leq y$ and $y \leq z$, then $x \leq z$.

Proof. □

Proposition 27

If $x, t, z \in \mathbb{R}$ and $x < y$, then $x + z < y + z$.

Proof. Since $x < y$, we have $x - y \in \mathbb{P}$. By the properties of addition (A1-A4), we know that:

$$(y + z) - (x + z) = y - x$$

Since $y - x \in \mathbb{P}$, it follows that:

$$(y + z) - (x + z) \in \mathbb{P}$$

Hence, $x + z < y + z$. □

Proposition 28

If $x, y, z \in \mathbb{R}$ and $x < y$ and $z > 0$, then $xz < yz$.

Proof. $zy = zx = z(y - x)$. Now, $z \in \mathbb{P}$ and $y - x \in \mathbb{P}$, therefore $zy - zx \in \mathbb{P}$. Therefore, $xz < yz$. □

Corollary 29

If $x, y, z \in \mathbb{R}$ and $x < y$ and $z < 0$, then $xz > yz$.

Proof. □

Corollary 30

For all, $a \in \mathbb{R}$, $a^2 \geq 0$.

Proof. By P3, either $a > 0, a = 0$ or $g < 0$.

- If $a > 0$, then $a^2 = a \times a > 0$. (P2)
- If $a = 0$, then $a^2 = 0 \geq 0$.

- If $a < 0$, then $-a > 0$ and $(-a)^2 = a^2 > 0$.

$\square$

Proposition 31

If $x \in \mathbb{P}$, then $x^{-1} \in \mathbb{P}$.

Proof. Since, $x \in \mathbb{P}, x \neq 0$. Therefore, x^{-1} exists. By P3, $x^{-1} > 0, x^{-1} = 0$, or $x^{-1} < 0$. If $x^{-1} = 0$, then $1 = x \times x^{-1} = x \times 0 = 0$ [Contradiction] Assume $x^{-1} < 0$. Then $-x^{-1} \in \mathbb{P}$ by P3. Then $x \times (-x^{-1}) \in \mathbb{P}$ by P2. But $x \times (-x^{-1}) = -1$, which contradicts P3 since $-1 \notin \mathbb{P}$. Therefore, $x^{-1} \in \mathbb{P}$. $\square$

Corollary 32

If $x, y \in \mathbb{P}$, and $x < y$, then $\frac{1}{y} < \frac{1}{x}$.