Math 4580: Abstract Algebra I

Lecturer: **Professor Michael Lipnowski**

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1 January 6, 2025

We didn't have any, but Dr. Lipnowski did post a module on carmen about the syllabus and the course. This semester we will be covering the first few chapters of the book *Abstract Algebra: Theory and Applications* by Thomas Judson.

Definition 1

Set: A collection of distinct objects, considered as an object in its own right.

Axioms: A collection of objects S with assumed structural rules is defined by axioms.

Statement: In logic or mathematics, an assertion that is either true or false.

Hypothesis and Conclusion: In the statement "If P, then Q", P is the hypothesis and Q is the conclusion.

Mathematical Proof: A logical argument that verifies the truth of a statement.

Proposition: A statement that can be proven true.

Theorem: A proposition of significant importance.

Lemma: A supporting proposition used to prove a theorem or another proposition.

Corollary: A proposition that follows directly from a theorem or proposition with minimal additional proof.

2 January 8, 2025

Professor Lipnowski discussed Sam Lloyd's 15 puzzle. Each lecture will include a mystery digit, contributing up to 5% bonus to the final grade based on correct guesses.

Certain course expectations:

- All assignments (one every two weeks) and exams (one midterm and one final exam) will be take-home.
- All the problems from the course textbook.
- Collaboration is encouraged, but the work should be your own.
- For the exams, we are not supposed to talk to other friends.

2.1 Functions

Definition 2

Let A and B be sets. A function $f : A \to B$ assigns exactly one output $f(a) \in B$ to every input $a \in A$.

- The set A is called the **domain** of f.
- The set B is called the **codomain** of f.

Fact 3

The domain A, codomain B, and the assignment of outputs $f(a)$ to every input $a \in A$ are all part of the data defining a function. Just writing a formula like $f(x) = e^x$ does not determine a function, as the domain and codomain are not specified.

For example:

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$.
- $f : \mathbb{Q} \to \mathbb{Q}$, $f(x) = e^x$.

Although these functions use the same formula, their meanings are completely different because their domains and codomains differ.

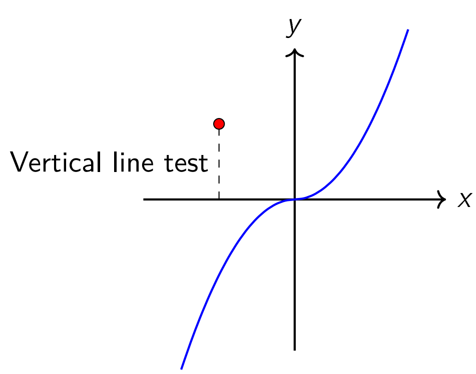
2.2 Graphs

A function $f : A \to B$ is often identified with its **graph** in $A \times B$:

$$\mathrm{graph}(f) = \{(a, b) \in A \times B : b = f(a)\}.$$

Lemma 4

Let $f : A \to B$ be a function. Its graph, $\mathrm{graph}(f)$, passes the **vertical line test**: For every $a \in A$, $V_a := \{(a, b) \in A \times B : b \in B\}$ intersects $\mathrm{graph}(f)$ in exactly one element.



Proposition 5

Let $G \subseteq A \times B$ be any subset passing the vertical line test, i.e., for all $a \in A$, $V_a \cap G$ consists of exactly one element. Then $G = \text{graph}(f)$ for a unique function $f : A \to B$.

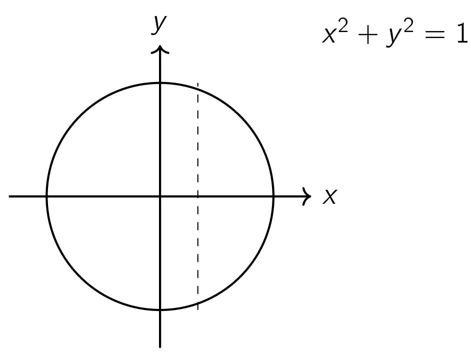
Proof. If $G = \{(a, b) \mid b \in B\}$ satisfies the vertical line test, define $f : A \to B$ by $f(a) = b$. Then $G = \text{graph}(f)$. □

Definition 6

A subset $R \subseteq A \times B$ is called a **relation**. The vertical line test distinguishes graphs of functions from more general relations.

2.3 Examples

- Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (the unit circle). This is a relation but not the graph of a function because it fails the vertical line test: The vertical line $x = 0$ intersects the circle at two points.
- Visual depiction of a unit circle:



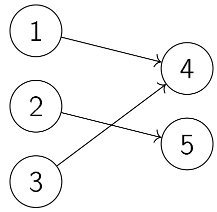
- Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$. The number of functions from A to B is $2^3 = 8$, corresponding to the 8 associated graphs in $A \times B$.
- The number of relations from A to B is $2^{|a| \cdot |b|} = 2^{3 \cdot 2} = 64$, containing the 8 graphs of functions from A to B.

Fact 7

The notion of relation is much more permissive than the notion of functions.

2.4 Visualizing Functions as Directed Edges

A function $f : A \to B$ can be visualized as a collection of directed edges $(a, f(a)) \in A \times B$. Each element of A has exactly one outgoing edge in the graph.



3 January 10, 2025

3.1 Injection and Surjection

Let $f : A \to B$ be a function.

Definition 8 (Injectivity (One-to-One))

f is injective (one-to-one) if:

$$\forall x, y \in A,\ f(x) = f(y) \implies x = y$$

Equivalently:

$$x \neq y \implies f(x) \neq f(y)$$

Fact 9

Distinct inputs have distinct outputs.

Definition 10 (Surjectivity (Onto))

f is surjective (onto) if:

$$\forall b \in B,\ \exists a \in A \text{ such that } f(a) = b.$$

Fact 11

Every $b \in B$ is an output of something through f."

Example 12

Here are a few examples of injectivity and surjectivity:

- Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ and $f : A \to B$ with $f(1), f(2), f(3)$ as elements of B. If B has only two elements, at least two of $f(1), f(2), f(3)$ must coincide (e.g., $f(1) = f(2)$). Thus, f is not injective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f : A \to B$ where:
$$f(1) = 4,\ f(2) = 7,\ f(3) = 5.$$
Distinct inputs have distinct outputs, so f is injective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f : A \to B$ where:
$$f(1) = 4,\ f(2) = 4,\ f(3) = 6.$$
Here, $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f(1) = f(2)$ but $1 \neq 2$, so f is not injective.
- Let $f : A \to B$ where B has size 4 and $f(1), f(2), f(3)$ are distinct elements of B. If $B \setminus \{f(1), f(2), f(3)\}$ is non-empty, then $b \neq f(a)$ for all $a \in A$, implying f is non surjective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ and $f : A \to B$ with $f(1) = 4, f(2) = 5, f(3) = 4$. f is surjective.
- Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$ and $f : A \to B$ with $f(1) = 4, f(2) = 4, f(3) = 4$. f is not surjective.

3.2 Bijection and Range

Definition 13 (Bijectivity)

f is bijective if f is both injective and surjective.

Definition 14

Let $f : A \to B$ be a function. The *range* of f is the subset of B defined as:
$$\text{range}(f) := \{b \in B \mid b = f(a) \text{ for some } a \in A\}.$$
Thus, $f : A \to B$ is surjective $\iff \text{range}(f) = B$.

- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$ and $f : A \to B$ where:
$$f(1) = 6,\ f(2) = 5,\ f(3) = 4.$$
f is a bijection.

- Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$ and $f : A \to B$ where:

  $$f(1) = 4,\ f(2) = 4,\ f(3) = 56$$

  f is neither injective nor surjective.

Question. *Let A and B be finite sets of the same size. Prove that the following are equivalent:*

1. *$f : A \to B$ is injective.*
2. *$f : A \to B$ is bijective.*
3. *$f : A \to B$ is surjective.*

Demonstrate that (1), (2), and (3) are not necessarily equivalent if $A = B = \mathbb{N}$.

Example 15

Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even}, \\ -\frac{(n+1)}{2} & \text{if } n \text{ is odd}. \end{cases}$$

is a bijection from $\mathbb{N}$ to $\mathbb{Z}$.

Proof. **Injectivity:** Suppose $f(n_1) = f(n_2)$. Then:

- If $f(n_1) = f(n_2) > 0$, then n_1, n_2 must be even and

  $$\frac{n_1}{2} = f(n_1) = f(n_2) = \frac{n_2}{2} \implies n_1 = n_2.$$

- If $f(n_1) = f(n_2) < 0$, then n_1, n_2 must both be odd and

  $$-\frac{n_1+1}{2} = f(n_1) = f(n_2) = -\frac{n_2+1}{2} \implies n_1 = n_2.$$

In all cases, $n_1 = n_2$.

It follows that f is injective.

Surjectivity: Let $n \in \mathbb{Z}$.

- If $n > 0$, then

  $$n = f(2n).$$

- If $n < 0$, then

  $$n = f(-2n - 1).$$

$$\therefore\ f \text{ is surjective.}$$

$\square$

Theorem 16 (Taylor's Theorem)

Let f be a function that is n-times differentiable at a. Then for each x in the interval containing a, there exists a ξ between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Proof. By the mean value theorem, for each x in the interval containing a, there exists a ξ between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{n+1}(x),$$

where $R_{n+1}(x)$ is the remainder term. The remainder term can be expressed as

$$R_{n+1}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

Therefore, we have

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

$\square$