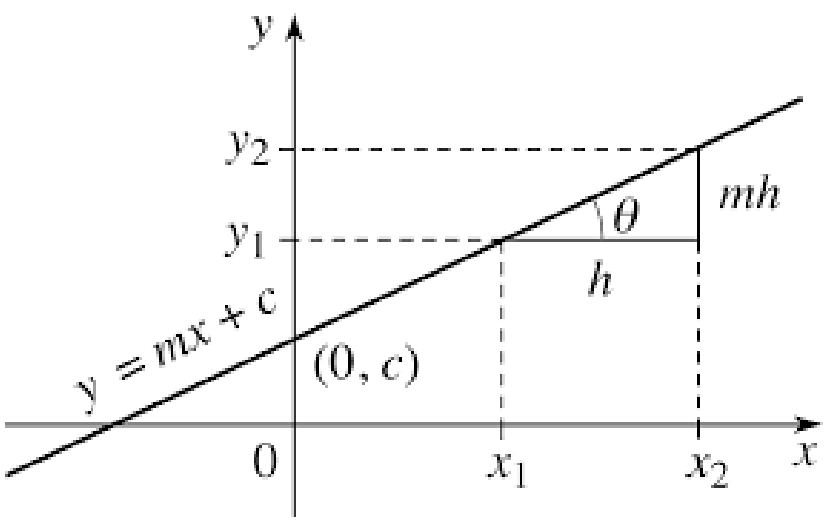
Coordinate Geometry

Outline

- Equation of a straight line
- Parallel lines
- Perpendicular lines
- Distance between two points
- Mid point between two points
- Perpendicular Distance from a point to a line
- Equation of a circle
- Polar Coordinates
- Tangent to a circle

Linear Functions

- A **Linear Function** is a function of the form: $f(x) = mx + c$ where ***m*** and ***c*** are **real numbers** and ***m*** is the **slope** and ***c*** is the **intercept**.



- The **domain** and **range** of a **linear function** are **all real numbers**.

Question

Q) Identify the slope and the intercept of the following:

1. $5x - 6y = -12$

2. $x - 6y = -11$

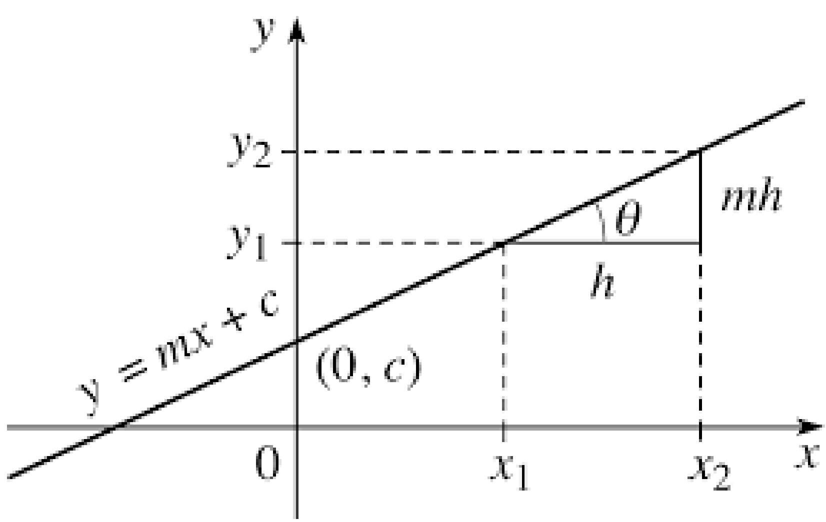
3. $y - 5x = 20$

4. $-3x = -0.5y + 7$

5. $\frac{x+3}{2y-5} = -4$

- The slope **m** is also called as the **gradient / Average rate of change** of the line.
- The **average rate of change** of a linear function is **defined by** $\dfrac{\Delta y}{\Delta x}$

$$m = \tan(\theta) = \frac{y_2 - y_1}{x_2 - x_1}$$



- The angle theta is measured counter clockwise from the positive x-axis.

When **$f(x) = -3x+4$**

Slope: $m = -3$

Intercept: $c = 4$

The **average rate of change** is the **constant** $m = -3$

Since $m =-3$ is negative, the graph is slanted downwards. Thus the function is decreasing

When **$f(x) = 3$**

$f(x)=0x + 3$

Slope: $m = 0$

Intercept: $c = 3$

- The average rate of change is 0
- The function is constant neither increasing or decreasing

Plotting a graph from the line equation

Methods

1. Plot the line by identifying x and y intercepts
2. Plot the line by identifying the slope and the intercept from the given equation

Note: If the slope is not given, at least two points are required to plot a line.

Plotting a line by identifying x and y intercepts

Example: Plot the graph of $y = 2x - 4$

- In order to find the y intercept, set x=0 in the equation
- In order to find the x intercept, set y=0 in the equation

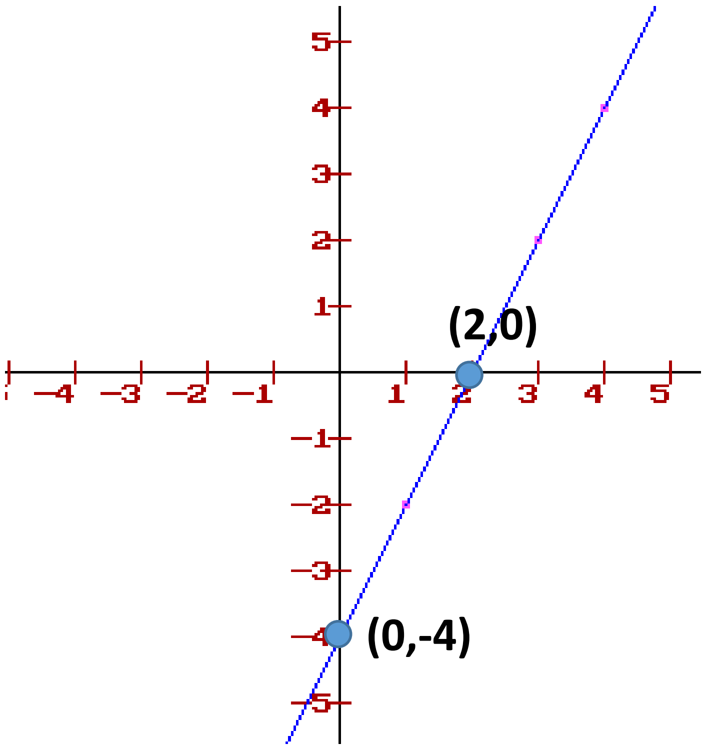
when $x = 0, \quad y = -4, \qquad (0, -4)$

when $y = 0, \quad x = 2, \qquad (2,\ \ 0)$

- Now use the two points (0,-4) and (2, 0) to plot the line

Plotting a line by identifying x and y intercepts

- Plot the graph of $y = 2x - 4$



Plotting a line by identifying the slope and the intercept

- Plot the graph of $y = x - 4$

- Slope: $m = 1,$ $\tan(\theta) = 1$

$$\theta = \tan^{-}(1)$$
$$\theta = 45^{o}$$

- Intercept:

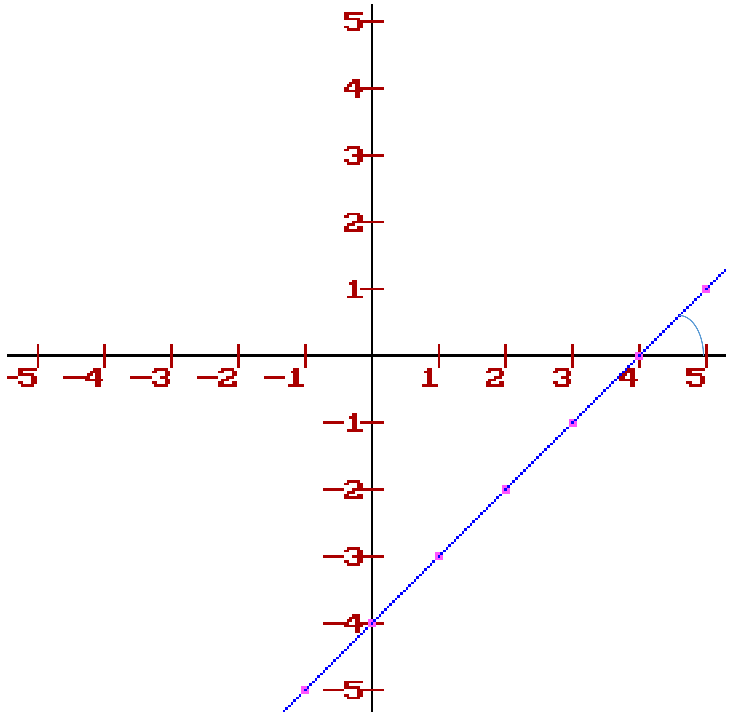
$$c = -4,$$

- Now, plot the graph using m and c.

Plotting a line by identifying the slope and the intercept

- Plot the graph of $y = x - 4$

$$m = 1$$
$$c = -4$$



(45°)

- Draw rough sketches of the following graphs.

i) $y = mx$

ii) $y = mx + c$

iii) $y = mx - c$

iv) $y = -mx$

v) $y = -mx + c$

vi) $y = -mx - c$

Qs-

Exercise:

Sketch $y = -2x + 7$ using x and y intercepts and plot the same line identifying the slope and the intercept.

Identify whether the following data fits a linear function or not

- Q1.

x	y=f(x)
-1	-9
0	-7
1	-5

Q3.

X	y=f(x)
-1	5
0	5
1	5

- Q2.

x	y=f(x)
-1	-6
0	-7
1	-6

Q4.

X	y=f(x)
-1	-4
0	-1
1	6

- NOTE: Slope of a linear function is a constant regardless of what points are used to calculate it.

Equation of a line

a) Finding the line equation when the **slope** and **a point** on the line **is given**:

Lets consider the slope to be 'm' and the point A (x_1, y_1) to be on the line. Then, the line equation is given by:

$$y - y_1 = m(x - x_1)$$

Example:

Q) Find the equation of a line that goes through the point (3, 4) with a slope =-2.

Equation of a line

b) Finding the line equation **when two points** are **known**.

Suppose A (x_1, y_1) and B (x_2, y_2) are on the line. Then, the line equation is given by:

$$y - y_1 = \frac{(y_2 - y_1)}{(x_2 - x_1)}(x - x_1)$$

Example:

Q) Find the equation of a line through points (3, 4) and (-1, 6).

Parallel lines

Parallel lines have the **same slope**.

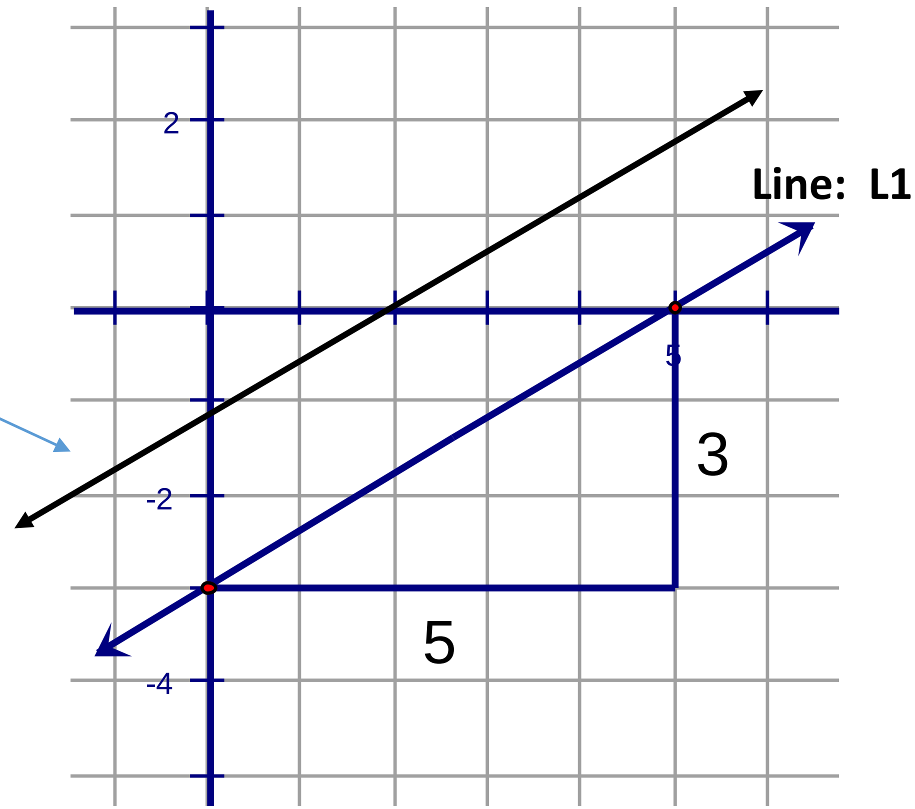
Q) Graph a line parallel to the given line L1 and through point (0, -1):

Slope of the parallel line: 3/5

Line equation:

$$(y-(-1)) = \frac{3}{5}(x-0)$$

$$y = \frac{3}{5}x - 1$$



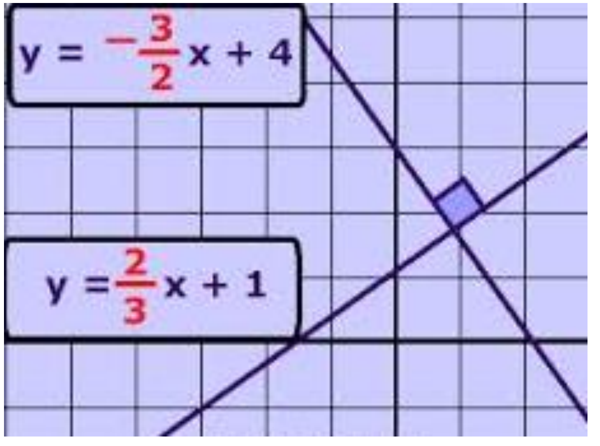
Perpendicular lines

Perpendicular lines have the opposite reciprocal slopes

Suppose the slope of a line L_1 is m_1 and the line drawn perpendicular to L_1 has a slope of m_2. Then, the following relationship holds:

$$m_1 \times m_2 = -1$$

For E.g.



Example:

Find the equation of a line(L_1) through points (3, 4) and (-4, -6). Now write the equation of the line perpendicular to L_1 containing point (2,3)

Summary: Parallel and Perpendicular lines

Consider two lines L_1 and L_2 with slopes m_1 and m_2 .

a) If L_1 is parallel to L_2

$$m_1 = m_2$$

b) If L_1 is perpendicular to L_2

$$m_1 \times m_2 = -1$$

Identify the following pairs of lines are parallel, perpendicular or not

1. $x - y + 1 = 0$

   $x + y - 6 = 0$

2. $-51x + 23y + 40 = 0$

   $-51\, x + 23y - 19 = 0$

3. $\mathrm{a}x + by + c = 0$

   $\mathrm{a}x + by - c = 0$

4. $-3x + 4y + 1 = 0$

   $4x + 3y - 6 = 0$

5. $-3x + 4y + 1 = 0$

   $4x - 3y - 6 = 0$

6. $\mathrm{a}x + by + c = 0$

   $bx - ay - c = 0$

Distance between two points

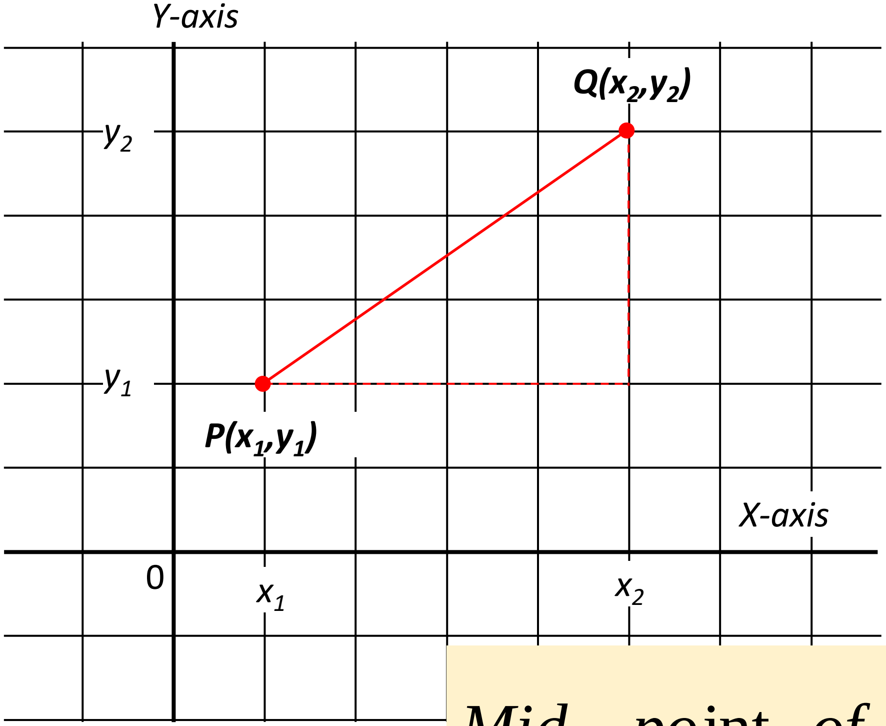
The distance between point A (x_1,y_1) and point B(x_2,y_2) is given by the following expression.

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Proof: Use Pythagoras theorem to prove.

Q) Find the distance between points (-1,-2) and (3,4).

Mid Point between two points P & Q



$$Mid-point \ \ of \ \ PQ: \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

Q) Find the mid point between (-1,-2) and (3,4)

Identifying the intersection point of two intersecting lines

Consider L_1 to be a line with the line equation $y=m_1x+c_1$ and L_2 to be another line with the line equation $y=m_2x+c_2$.

If the lines L_1 and L_2 intersect, the intersecting point could be found by solving the line equations simultaneously.

Solve:

$$L_1 \rightarrow y=m_1x+c_1$$

$$L_2 \rightarrow y=m_2x+c_2$$

Find the intersection point of the following lines:

1. $x + y = 5$

   $x - y = 2$

2. $x - y = -1$

   $3x + 5y = -1$

3. $4x + 7y = 20$

   $21x - 13y = 21$

Identifying whether a given point is on a defined line

Consider L_1 to be a line with the line equation y=mx+c.

If a point (x_1,y_1) is on the line L_1:

$$\boxed{y_1 = mx_1 + c}$$

The point (x_1,y_1) satisfies the line equation.

Q: Check whether the following points are on the given lines

1. $x + y = 5$ A(2,3) B(-2,-3) C(1,4) D(3,2)

2. $3x - 2y = 1$ A(1,1) B(-1,-2) C(2/3,0) D(0,0.5)

3. $3x + 5y = -1$ A(1,-1) B(-2,1) C(2,-1) D(5,-3)

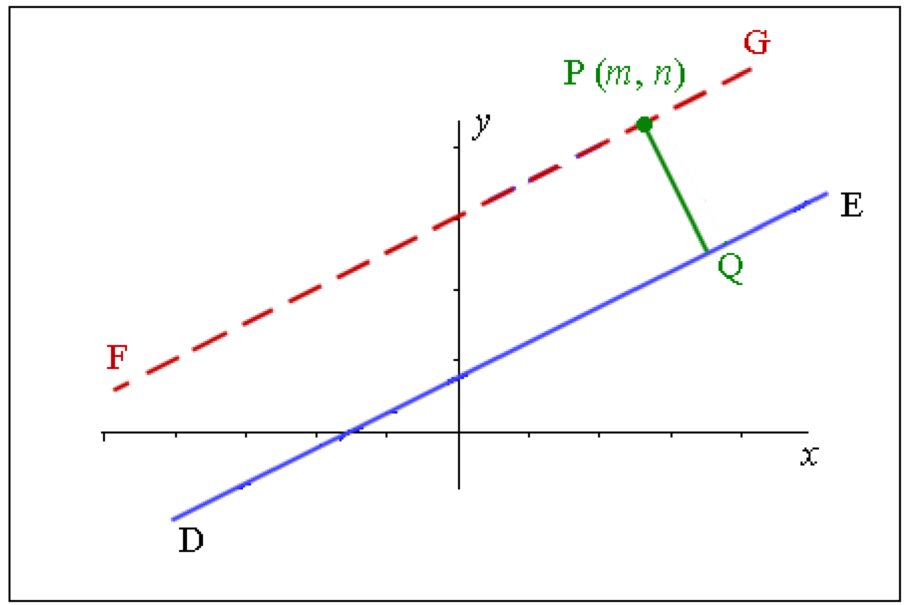
4. $4x + 7y = 20$ A(1.5,2) B(-2,4) C(2,2) D(7,-1)

5. $2x + 7y = 11$ A(2,1) B(-2,2) C(-1.5,2) D(-5,3)

Perpendicular distance from a point to a line

Perpendicular distance from a point P(m,n) to the line DE (ax+by+c=0) is given by:

$$PQ = \frac{|am + bn + c|}{\sqrt{a^2 + b^2}}$$

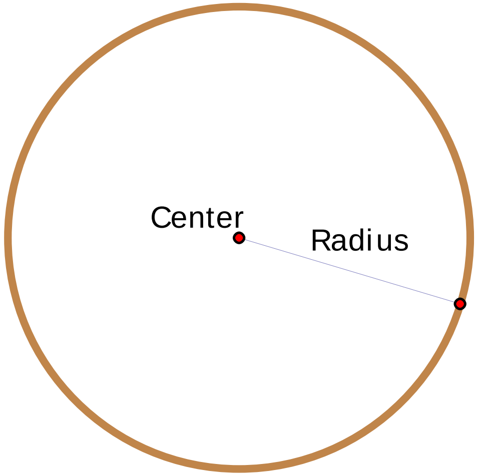


Example: What is the distance *d* of the point *P*(-6, -7) from the line *L* with equation 3*x* +4*y* = 11?

Circles

Definitions

- Circle: The set of all points that are equidistant from a fixed point.
- Center: the fixed point
- Radius: a segment whose endpoints are the center and a point on the circle



Standard equation of a circle

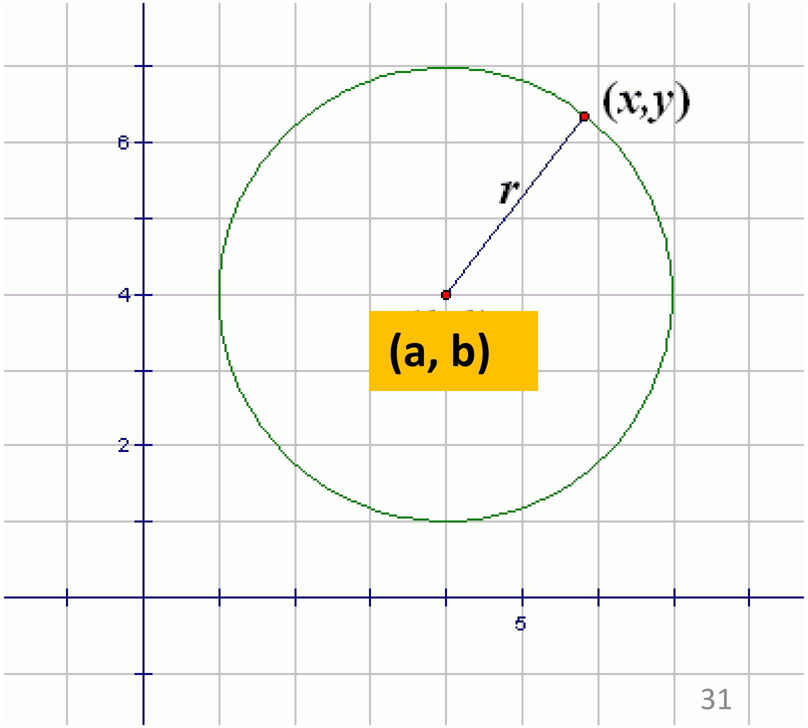
If the circle is at the origin

$x^2 + y^2 = r^2$ r is the radius

If the circle is not at the origin

$$(x-a)^2 + (y-b)^2 = r^2$$

The center is at (a, b)



Writing the equation of a circle given the center and the radius

1. Write the equation of a circle given the center C(-5,0) and radius 5
2. Write the equation of a circle given the center C(2,3) and radius 5

$$(x-a)^2+(y-b)^2=r^2$$

Writing the equation of a circle given the center and a point on the circle

1. Write the equation of a circle given the center C(-3,0) and the point A(0,-4)
2. Write the equation of a circle given the center C(2,-3) and the point A(8,5)

$$(x-a)^2+(y-b)^2=r^2$$

Checking whether a given point is out, in or on the circle

$(x - 13)^2 + (y - 6)^2 = 5^2$

Tell if the point (5, 6) is inside or outside the circle.

Check if $(5 - 13)^2 + (6 - 6)^2$ is <, >, or = to 5^2

$(- 8)^2 + 0^2 = 64 > 25$

Greater than 5^2 indicates the point is **outside** the circle.

Less than 5^2 indicates the point is inside the circle.

Equal to 5^2 indicates the point is on the circle.

Checking whether a given point is out, in or on the circle

$(x - 13)^2 + (y - 6)^2 = 5^2$

Tell if the following points are inside, outside or on the circle.

A (5, 6)

B (14,8)

C (20, 9)

D (16, 2)

General equation of a circle

When g, f, c are constants, the general equation of a circle is given by:

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Center: $(-g, -f)$

Radius: $\sqrt{g^2 + f^2 - c}$

Q: Find the center and the radius of the following

1. $x^2 + y^2 - 2x - 4y + 1 = 0$

2. $x^2 + y^2 + 2x - 10y - 10 = 0$

3. $x^2 + y^2 - x - 8y + 16 = 0$

Finding the equation of a circle given 3 points on the circle

1. Find the equation of the circle that goes through points (0,0), (1,0) and (0,1)

2. Find the equation of the circle that goes through points (4,2), (2,0) and (0,2)

$x^2 + y^2 + ax + by + c = 0$

$16 + 4 + 4a + 2b + c = 0$

$4 + 2a + c = 0$

$0 + 4 + 2b + c = 0$

$$4a + 2b + c = -20$$

$$2a + c = -4$$

$$2b + c = -4$$

$$a = \frac{-4 - c}{2} \qquad \frac{-(4+c)}{2}$$

$$b = \frac{-4 - c}{2} \qquad \frac{-(4+c)}{2}$$

$$4 \times \frac{-(4+c)}{\cancel{2}} + \frac{\cancel{2} \times -(4+c)}{\cancel{2}} + c = -20$$

$$-8 - 2c - 4 - \cancel{c} + \cancel{c} = -20$$

$$-12 - 2c = -20$$

$$-2c = -8$$

$$c = 4$$

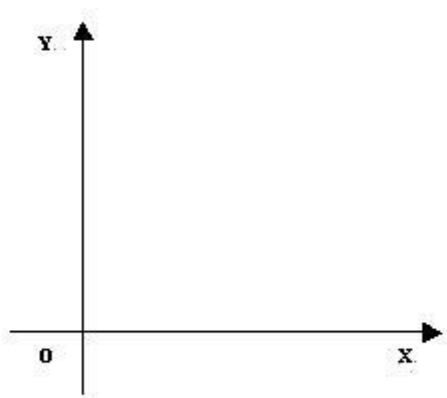
$$a = -4$$

$$b = -4$$

$$x^2 + y^2 - 4x - 4y + 4 = 0$$

Cartesian Coordinates

- A coordinate system represents a point in the plane by an ordered pair of numbers called coordinates.
- Usually, we use Cartesian coordinates, which are directed distances from two perpendicular axes.

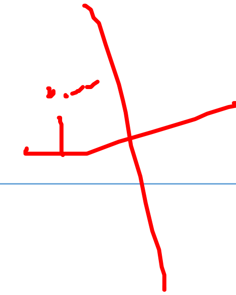


Polar Coordinates

- We choose a point in the plane that is called the pole (or origin) and is labeled *O.*
- Then, we draw a ray (half-line) starting at *O* called the polar axis. This axis is usually drawn horizontally to the right corresponding to the positive *x*-axis in Cartesian coordinates.

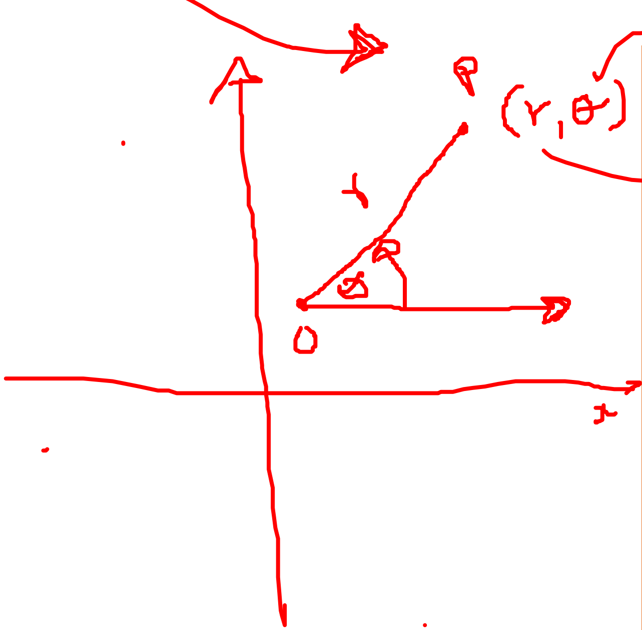
(x, y)

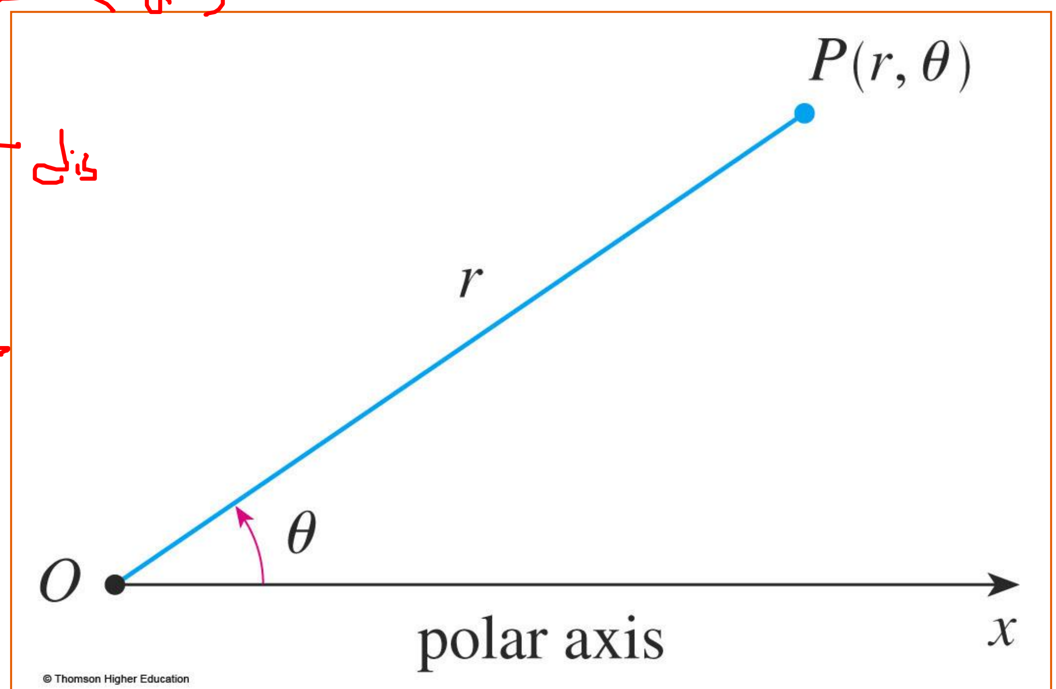
(x, y)



Polar Coordinates

- If P is any other point in the plane, let:
  - r be the distance from O to P.
  - ϑ be the angle (usually measured in radians) between the polar axis and the line OP.



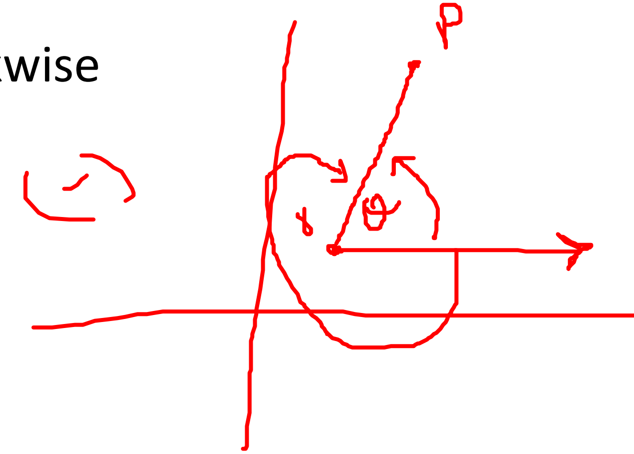


Polar Coordinates

- *P* is represented by the ordered pair (r, ϑ).
r, ϑ are called **polar coordinates** of *P*.

- We use the convention that an angle is:
    - Positive—if measured in the counterclockwise / anti direction from the polar axis.
    - Negative—if measured in the clockwise direction from the polar axis.

(+)

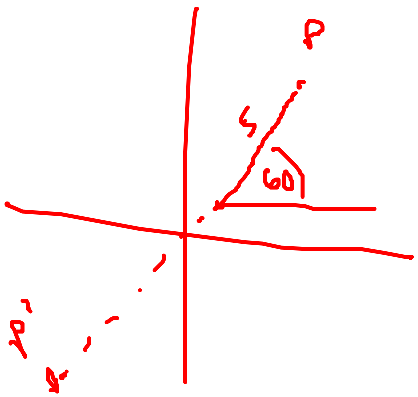


Polar Coordinates

$\pi + \theta$

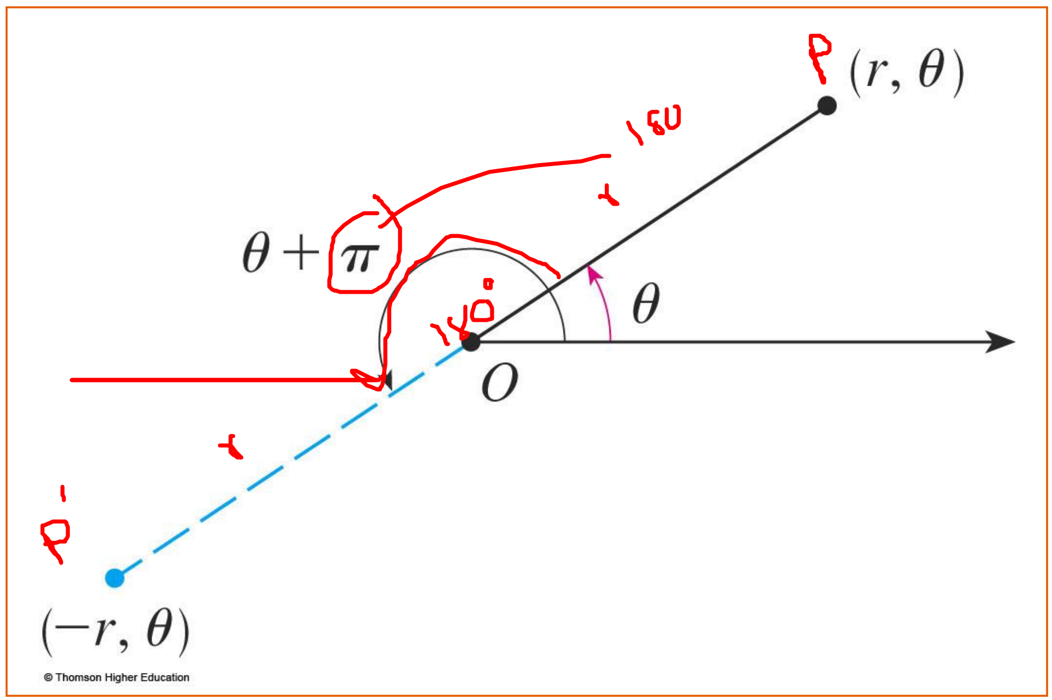
θ

- As shown below, the points $(–r, \vartheta)$ and (r, ϑ) lie on the same line through *O* and at the same distance $| r |$ from *O*, but on opposite sides of *O*.



$(-r, \theta)$

$(-5, 60^\circ)$



hypotense

adj

op

polar ⟷ Cartesian → Trigonometry

(r, θ) (x, y)

h

op

adj

Polar and Cartesian coordinates

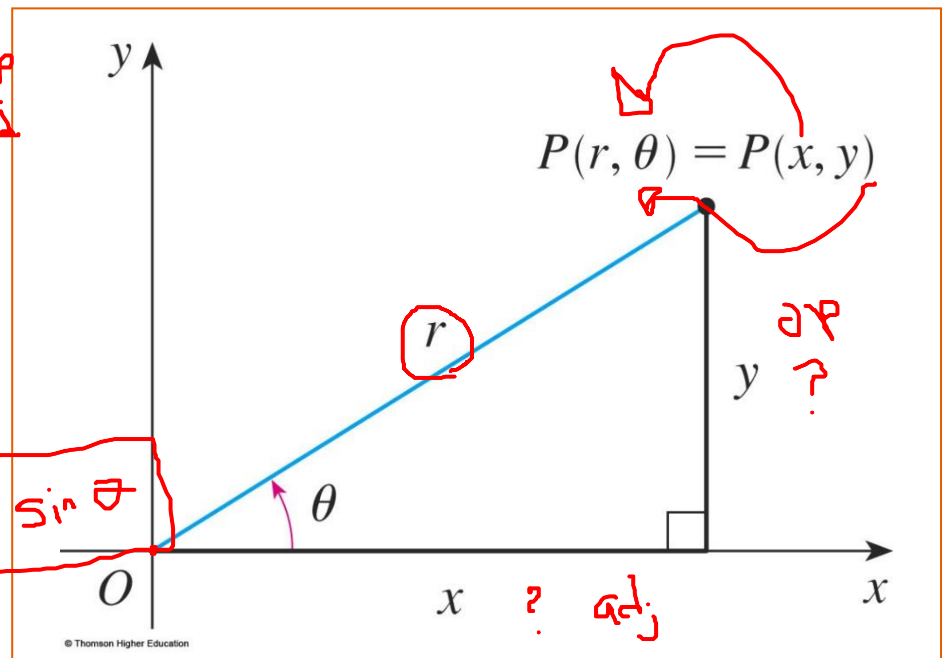
- The connection between polar and Cartesian coordinates can be seen here.
  - The pole corresponds to the origin.
  - The polar axis coincides with the positive *x*-axis.

$\sin() = \frac{op}{hyp}$

$\tan() = \frac{op}{adj}$

$\cos() = \frac{ad}{hp}$

$$x = r\cos\theta$$
$$y = r\sin\theta$$



$r \times \cos\theta = \frac{x}{r} \times r$

$\boxed{x = r\cos\theta}$

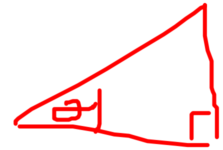
$\boxed{y = r\sin\theta}$

$r \times \sin\theta = \frac{y}{r} \times r$

$x = r\cos\theta$

$y = r\sin\theta$

SHORT BREAK
10.55 – 11.05



Find the Cartesian coordinates of following polar coordinates

1. (10,30) $= (8.65, 5)$
2. (5,60) $= (2.5, 4.33)$
3. (14.90) $= (0, 14)$

$\boxed{1}$ $(10, 30^\circ)$ with $r = 10$, $\theta = 30^\circ$ =

$x = 10 \times \cos 30^\circ$

$= 10 \times 0.865$

$\cancel{= 8.65}$

$y = 10 \times \sin 30^\circ$

$y = 10 \times 0.5$

$y = 5$

$x = 14 \times \cancel{\cos 90}$

$\cancel{= 0}$

$y = \cancel{14 \times \sin 90^\circ}$

$= 14$

Equation of a Circle in Polar Coordinates

$$\boxed{(x-a)^2+(y-b)^2=r^2}$$

Circle Equations

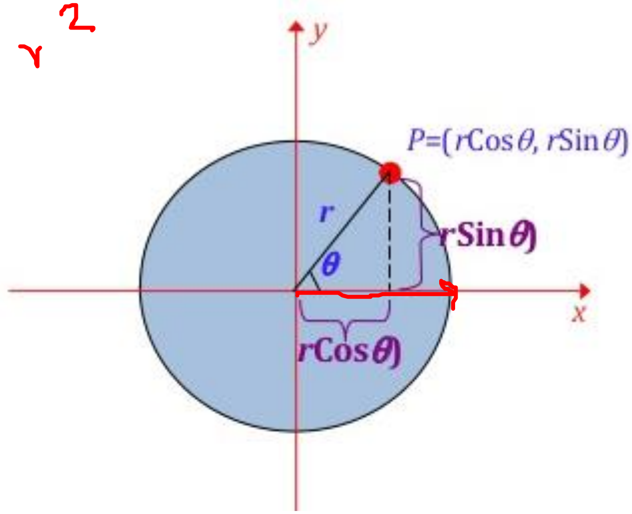
- Polar form

$$\boxed{\begin{aligned} x &= r\text{Cos}\,\theta \\ y &= r\text{Sin}\,\theta \end{aligned}}$$

(r = radius of circle)

$$(x-0)^2+(y-0)^2=r^2$$

$$x^2+y^2=r^2$$



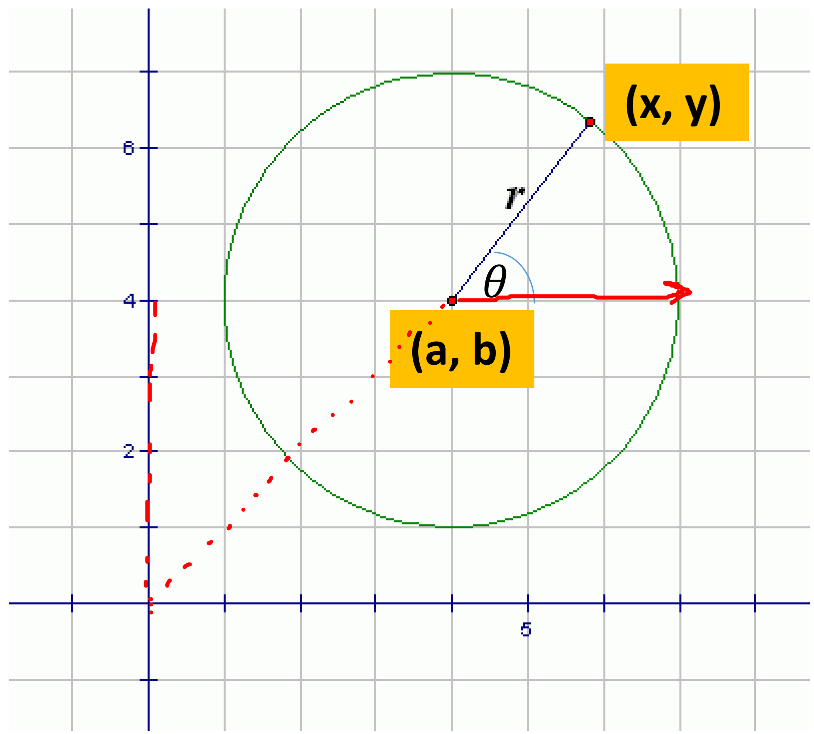
$$x = r\cos\theta$$

$$y = r\sin\theta$$

General Equation of a Circle in Polar Coordinates

- General Equation is given by:

$$x = a + r\cos\theta$$
$$y = b + r\sin\theta$$



$x = 0 + r\cos\theta$

$y = 0 + r\sin\theta$

$x = r\cos\theta$

$y = r\sin\theta$

Find the Cartesian points of following polar coordinates with center coordinates

1. (6,35) -- center(2,3) $= (6.91, 6.44)$
2. (10,45) -- center(-4,5) $= (3.07, 12.07)$

① $x = 2 + 6 \times \cos 35 = 6.91$

$y = 3 + 6 \times \sin 35 = 6.44$

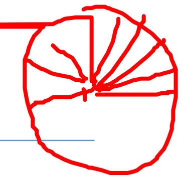
② $x = -4 \times 10 \times \cos 45 = 3.07$

$y = 5 + 10 \times \sin 45 = 12.07$

$(5.13, 4)$ //

$(-g, -f)$

$r = \sqrt{g^2 + f^2 - c}$



Q) Find 3 points on the circle defined by:

$$x^2 + y^2 - 8x - 6y + 21 = 0$$

$+2x \; -4x \qquad +2x \; -3$

$$\boldsymbol{x = a + r\cos\theta}$$
$$\boldsymbol{y = b + r\sin\theta}$$

Center ?

radius ?

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$g = -4$

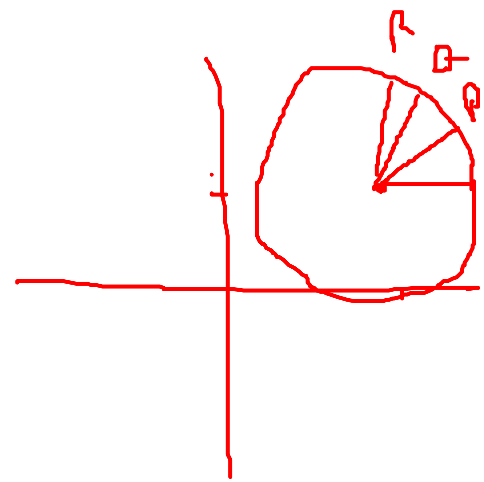
$f = -3$

$\boxed{(4, 3)}$

$r = \sqrt{16 + 9 - 21}$

$= \sqrt{4}$

$\boxed{r = 2}$



$\theta = 30^\circ$

$x = a + r\cos\theta$

$= 4 + 2 \times$

$= 5.13$

$y = b + r\sin\theta$

$y = 3 + 2 \times 0.5$

$y = 4$

Thank you