Contest Math Selection Test

October Math Circle

November 30, 2022

§1 Problems

Problem 1.1. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$

Problem 1.2. Determine all triples of real numbers (x, y, z) such that

$$\begin{aligned} xyz &= 8 \\ x^2y + y^2z + z^2x &= 73 \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 8 \end{aligned}$$

Problem 1.3. Compute

$$\sum_{k=0}^{n} \frac{(4k+1)k!}{(2k+1)!}$$

Problem 1.4. Let x and y be positive real numbers and θ be an angle which is not an integer multiple of $\pi/2$. Suppose

$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y} \quad \text{and} \quad \frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{7\sin(2\theta)}{x^3y + xy^3}$$

Compute

$$\frac{x}{y} + \frac{y}{x}$$

Problem 1.5. Let P be a point inside circle Γ. Consider the set of chords of Γ that contain P. Prove that their midpoints all lie on a circle.

Problem 1.6. Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote its circumcenter and incenter. Then $OI^2 = R(R - 2r)$. In particular, $R \geq 2r$.

Problem 1.7. Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.

Problem 1.8. Point P is located inside triangle ABC so that angles PAB, PBC, and $PCA$are all congruent. The sides of the triangle have lengths $AB = 13$, $BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Problem 1.9. Find the ordered pairs (p, q) for prime numbers p and q such that

$$pq \mid (5^p - 2^p)(5^q - 2^q)$$

Problem 1.10. Prove that if $p \equiv 3 \mod 4$ is a prime number, such that p divides $a^2 + b^2$ for positive integers a and b, then $p \mid a$ and $p \mid b$.

Problem 1.11. Find the last two digits of the number

$$\left\lfloor \frac{10^{93}}{10^{31} + 3} \right\rfloor$$

Problem 1.12. Find all integral solutions of the equation

$$a^2 + b^2 + c^2 = a^2 b^2$$

Problem 1.13. Let A be a set formed by choosing 20 numbers arbitrarily from the arithmetic sequence 1, 4, 7, . . . , 100. Prove that there must be two numbers in A such that their sum is 104.

Problem 1.14. For any non-empty finite set A of real numbers, let $s(A)$ denote the sum of the elements in A. If there are exactly 61 subsets A of the set $\{1, \ldots, 23\}$ that satisfy
(1) They contain exactly three elements.
(2) $s(A) = 36$
Find the number of subsets A that satisfy (1) and $s(A) < 36$

Problem 1.15. Prove the following formulas:
(a)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

(b)

$$\sum_{k=0}^{n} \binom{n}{2k} = \sum_{k=0}^{n} \binom{n}{2k+1} = 2^{n-1}$$

Problem 1.16. We draw n straight lines in the plane, no two of which are parallel and no three of which pass through the same point. These lines divide the plane into a number of regions. Determine the number of these regions.