Calculus Selection Test

October Math Circle

November 30, 2022

§1 Problems

Problem 1.1. Compute the integral

$$\int_2^{\infty} \frac{1}{(x-1)(x-2)^{\frac{1}{2}}(3x-2)^{\frac{1}{2}}} dx$$

Problem 1.2. Define the functions f and g such that $f(x) = e^x \sin x$ and $g(x) = e^x \cos x$. Compute

$$\left(f^{(n)}(x)\right)^2 + \left(g^{(n)}(x)\right)^2$$

where $f^{(n)}$ denotes the n^{th} derivative of the function f.

Problem 1.3. Let the surface area of the revolution of the curve e^{-x} around the x-axis from a to b equal $A(a, b)$. Compute
(a) $A(0, b)$
(b) $A(0, \infty)$

Problem 1.4. Compute the integrals
(a)

$$\int x \log(\sqrt{1 + x^2}) dx$$

(b)

$$\int \frac{dx}{2 + \tan(x)}$$

(c)

$$\int \frac{(x^2 - 1)dx}{(x^4 + 3x^2 + 1) \arctan\left(\frac{x^2+1}{1}\right)}$$

(d)

$$\int \sqrt{\frac{x-1}{x+1}} \cdot \frac{1}{x^2} dx$$

(e)

$$\int \frac{e^{\sin x}\left(x \cos^3 x - \sin x\right) dx}{\cos^2 x}$$

Problem 1.5. A right triangle with hypotenuse of length a is rotated about one of its legs to generate a right circular cone. Find the greatest possible volume of such a cone.

Problem 1.6. Find a reduction formula for

$$I_n = \int \sec^n dx$$

Problem 1.7. Let f be a continuous function on $[a, b]$, such that f is twice differentible on (a, b), and $f'(x) > 0$ and $f''(x) > 0$.
(a) Find the second derivative of the inverse function g.
(b) Prove that g bends in the opposite direction to f.

Problem 1.8. Compute the limits
(a)

$$\lim_{x\to\infty} \frac{\left(\int_0^x e^{x^2} dx\right)^2}{\int_0^x e^{2x^2} dx}$$

(b)

$$\lim_{n\to\infty} \frac{\pi}{n}\left(\sin\frac{\pi}{n} + \sin\frac{2\pi}{n} + \cdots + \sin\frac{(n-1)\pi}{n} + \sin\frac{n\pi}{n}\right)$$

(c)

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

(d)

$$\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Problem 1.9. Calculate the definite integrals
(a)

$$\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$$

(b)

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x} dx$$

(c)

$$\int_{-4}^{-5} e^{(x+5)^5} dx + 3\int_{1/3}^{2/3} e^{9(x-\frac{2}{3})^2} dx$$

(d)

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$$

Problem 1.10. Find the areas enclosed between the curves
(a)

$$x^2 + y^2 - 2x + 4y - 11 = 0 \quad \text{and the curve} \quad y = -x^2 + 2x + 1 - 2\sqrt{3}$$

(b)

$$x = a\sin t \quad \text{and} \quad y = b\sin 2t$$

Problem 1.11. Consider the polynomial

$$P(x) = (x-a_1)^2 + \ldots (x-a_n)^2$$

for constants $a_i s$. Find the number of values of x that minimizes the polynomial and compute them.

Problem 1.12. Consider the functions

$$C(r) = 2\pi r \quad \text{and} \quad A(r) = \pi r^2$$

(a) These are the circumference and area functions for a circle of radius r, respectively. Give a reasoning for why is $A'(r) = C(r)$.
(b) Generalize this to a sphere of a radius r.
(c) Is this true for all solids and planar figures? Give examples or counterexamples.

Problem 1.13. Consider the infinite series given by

$$S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$$

Prove that $x^2 S''(x) + xS'(x) + x^2 S(x) = 0$

Problem 1.14. Consider the infinite series

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin(n^2 x)}{n^2}$$

(a) Test the series for **absolute convergence**.
(b) Determine if the function S is continuous.

Problem 1.15. Prove the inequality

$$(n-1)! \leq n^n e^{-n} e \leq n!$$

Using the inequality or otherwise, compute

$$\lim_{n\to\infty} \left(\frac{(3n)!}{n^{3n}}\right)^{1/n}$$

Problem 1.16. Prove that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

You must prove convergence whenever you use it.