Calculus Camp Admission Test

October Math Circle

June 16, 2022

§1 Problems

Problem 1.1. Prove that for angles of a triangle α, β, and γ. The relation

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)\cos\left(\frac{\gamma}{2}\right)$$

holds.

Problem 1.2. Compute

$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right)$$

Problem 1.3. Determine all values of x for which the equation

$$\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} - \frac{1}{\tan^2 x} - \frac{1}{\cot^2 x} - \frac{1}{\sec^2 x} - \frac{1}{\csc^2 x} = -3$$

holds.

Problem 1.4. (A) Transform the equation

$$r = \frac{2}{2 - \cos\theta}$$

from polar coordinates to rectangular coordinates.
(B) Write the point $(4, -3)$ in rectangular coordinates.

Problem 1.5. Prove that for $0 < x < 1$, and

$$\alpha = 2\arctan\left(\frac{1+x}{1-x}\right), \quad \beta = \arcsin\left(\frac{1-x^2}{1+x^2}\right)$$

the relation $\alpha + \beta = \pi$ holds true.

Problem 1.6. Solve the inequality

$$\frac{\sin^2 x - \frac{1}{4}}{\sqrt{3} - (\sin x + \cos x)} > 0$$

Problem 1.7. Determine all values of a such that the inequality

$$-3 < \frac{x^2 + ax - 2}{x^2 - x + 1} < 2$$

holds true for all values of real x.

Problem 1.8. Prove that

$$\frac{\log_a(x)}{\log_{ab}(x)} = 1 + \log_a(b)$$

Problem 1.9. Two cars simultaneously start out from a point and proceed in the same direction, one of them going at a speed of 50 km/hr and the other at 40 km/hr. In half an hour a third car starts out from the same point and overtakes the first car 1.5 hours after catching up with the second car. Determine the speed of the third car.

Problem 1.10. Prove that all functions may be written as a sum of odd and even functions.

Problem 1.11. Prove that

$$|x+y| \geq |x| - |y|$$

Hence express

$$||a+b| - |a| - |b||$$

with one less pair of absolute value signs.

Problem 1.12. Use mathematical induction to prove that

$$1 + 3 + 6 + 10 + \cdots + \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

Problem 1.13. Consider the equation

$$f(x + 1/x) = x^3 + 1/x^3$$

Find $f(x)$.

Problem 1.14. Let $P(x) = x^2$, $Q(x) = 2^x$, and $R(x) = \sin(x)$. Find the functions
(a) $(P \circ Q)(y)$
(b) $(P \circ Q \circ R)(y) + (R \circ Q)(y)$

Problem 1.15. In a square $ABCD$ with side $AB = 2$, a stright line MN is drawn perpendicularly to AC. If the distance from the vertex A to the line MN is x. Find the portion of the area of triangle AMN cut off from the square by the line MN as a function of x.

Problem 1.16. Find the inverse function f^{-1} for the function f defined by the rule

$$f(x) = \log_a\left(x + \sqrt{x^2+1}\right)$$