Contest Math Selection Test

October Math Circle

June 12, 2022

§1 Problems

Problem 1.1. Define the sequence $\{a_n\}$ such that

$$a_1 \quad \text{and} \quad a_n = \frac{\sqrt{3}a_{n-1}+1}{\sqrt{3}-a_n} \quad \text{for} \quad n \geq 1$$

Compute $\sum_{m=1}^{2022} a_n$

Problem 1.2. Consider five real numbers k_1, k_2, k_3, k_4, k_5, and define a function f such that

$$f(n) = \frac{k_1}{n^2+1} + \frac{k_2}{n^2+2} + \frac{k_3}{n^2+3} + \frac{k_4}{n^2+4} + \frac{k_5}{n^2+5}$$

If $f(n) = \frac{1}{n}$ for $n = 1, 2, 3, 4, 5$. Compute

$$\frac{k_1}{37} + \frac{k_2}{38} + \frac{k_3}{39} + \frac{k_4}{40} + \frac{k_5}{41}$$

Problem 1.3. Find triplets of real numbers (a, b, c) satisfying the system of equations

$$\begin{cases} ab &= \frac{c^2}{1+c^2} \\ bc &= \frac{a^2}{1+a^2} \\ ca &= \frac{b^2}{1+b^2} \end{cases}$$

Problem 1.4. Take three arbitrary real numbers a, b, c such that a triangle exists with these side-lengths. Prove that

$$\frac{1}{\sqrt{a+b-c}} + \frac{1}{\sqrt{b+c-a}} + \frac{1}{\sqrt{c+b-a}} \geq \frac{9}{ab+bc+ca}$$

Problem 1.5. Let $ABCD$ be a convex quadrilateral. Prove that AC is perpendicular on BD if and only if $AB^2 + CD^2 = AD^2 + BC^2$

Problem 1.6. A circle has center on the side $AB$of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

Problem 1.7. Let $ABCD$ be a quadrilateral. If a circle can be inscribed† in it, prove that $AB + CD = BC + DA$.

Problem 1.8. Let $ABCDE$ be a convex pentagon such that $\angle BAC = \angle CAD = \angle DAE$ and $\angle ABC = \angle ACD = \angle ADE$. Diagonals BD and CE meet at P. Prove that ray AP bisects CD

Problem 1.9. Prove that for distinct a, b, c, the relation

$$\gcd(ab+1, bc+1, ca+1) \leq \frac{a+b+c}{3}$$

holds true.

Problem 1.10. Choose natural numbers a, b such that $ab(a+b)$ is divisible by a^2+ab+b^2. Prove that $|a-b| > \sqrt[3]{ab}$

Problem 1.11. Determine all triplets of positive numbers (a, b, c) that satisfy the equation

$$\operatorname{lcm}(a, b, c) = \frac{ab+bc+ca}{4}$$

Problem 1.12. Let a and b be natural numbers satisfying

$$\frac{a}{b} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}$$

Problem 1.13. Define a good ordered 6-tuples of positive integers to be a 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$ such that

$$a_1 + a_2 + a_3 + 3a_4 + 3a_5 + 5a_6 = 21$$

Find the number of such good 6-tuples.

Problem 1.14. Define a set S made up of n elements. Suppose that S has k disjoint subsets A_i. What's the maximum value of k?

Problem 1.15. 2022 circles divide the plane into N regions such that any pair of circles intersects at two points and no point lies on three circles. Compute N.

Problem 1.16. Given a 10×10 board, we want to remove n of the 100 squares so that no 4 of the remaining squares form the corners of a rectangle with sides parallel to the sides of the board. Determine the minimum value of n.