Contest Math Selection Test

October Math Circle

June 12, 2022

§1 Problems

Problem 1.1. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a+b)^2$

Solution. Let the roots of $P(x)$ be m and n, then we can write

$$P(x) = x^2 - (m+n)x + mn$$

The fact that $P(P(x)) = 0$ has solutions $x = 3, 4, a, b$ implies that some combination of 2 of these are the solution to $P(x) = m$, and the other 2 are the solution to $P(x) = n$. It's fairly easy to see there are only 2 possible such groupings: $P(3) = P(4) = m$ and $P(a) = P(b) = n$, or $P(3) = P(a) = m$ and $P(4) = P(b) = n$ (Note that a, b are interchangeable, and so are m and n). We now casework: If $P(3) = P(4) = m$, then

$$9 - 3(m+n) + mn = 16 - 4(m+n) + mn = m \implies m + n = 7$$

$$a^2 - a(m+n) + mn = b^2 - b(m+n) + mn = n \implies a + b = m + n = 7$$

so this gives $(a+b)^2 = 7^2 = 49$. Next, if $P(3) = P(a) = m$, then

$$9 - 3(m+n) + mn = a^2 - a(m+n) + mn = m \implies a + 3 = m + n$$

$$16 - 4(m+n) + mn = b^2 - b(m+n) + mn = n \implies b + 4 = m + n$$

Subtracting the first part of the first equation from the first part of the second equation gives

$$7 - (m+n) = n - m \implies 2n = 7 \implies n = \frac{7}{2} \implies m = -3$$

Hence, $a + b = 2(m+n) - 7 = 2 \cdot \frac{1}{2} - 7 = -6$, and so $(a+b)^2 = (-6)^2 = 36$. Therefore, the solution is 85 □

Problem 1.2. Determine all triples of real numbers (x, y, z) such that

$$\begin{aligned} xyz &= 8 \\ x^2y + y^2z + z^2x &= 73 \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 8 \end{aligned}$$

Solution. The ugliest expression in this system is the last one, so let's expand it to see if it is simplifiable. Expanding, we get

$$x^2y + y^2z + z^2zx + x^2z + y^2x + z^2y - 6xyz = 98$$

Using the other two equations, we get $x^2z + y^2x + z^2y = 98 + 6 \cdot 8 - 73 = 73$. So, we have

$$x^2y + y^2z + z^2x = xy^2 + yz^2 + zx^2 \Rightarrow (x-y)(x-z)(y-z) = 0$$

As the equations are cyclic, we can assume that $x = y$. Now, the first two equations become

$$x^2z = 8, x^3 + x^2z + z^2x = 73$$

The second equation rearranges as

$$z^2 = \frac{65}{x} - x^2$$

and substituting into the first equation yields $65x^3 - x^6 = 64$. and so $x^3 = 1, 64$. Hence $x = 1, 4$, and we get the ordered pairs $(1, 1, 8), (4, 4, \frac{1}{8})$ and other cyclic permutations as solutions. □

Problem 1.3. Compute

$$\sum_{k=0}^{n} \frac{(4k+1)k!}{(2k+1)!}$$

Solution. A good way to start telescoping is to try to note the relationship between consecutive terms. Note that

$$\frac{k!}{(2k+1)!} = \frac{(k+1)!}{(2k+3)!} \cdot \frac{(2k+3)(2k+2)}{(2k+1)} = (4k+6) \cdot \frac{(k+1)!}{(2k+3)!}$$

This means that we can write $\frac{(4k+1)k!}{(2k+1)!}$ as

$$\frac{(4k+2)k!}{(2k+1)!} - \frac{k!}{(2k+1)!} = \frac{(4k+2)k!}{(2k+1)!} - \frac{(4k+6)(k+1)!}{(2k+3)!}$$

Now, notice that is is a telescoping series. Hence the answer

$$2 - \frac{n!}{(2n+1)!}$$

□

Problem 1.4. Let x and y be positive real numbers and θ be an angle which is not an integer multiple of $\pi/2$. Suppose

$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y} \quad \text{and} \quad \frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{7\sin(2\theta)}{x^3y + xy^3}$$

Compute

$$\frac{x}{y} + \frac{y}{x}$$

Solution. From the first equation, we have $x = k\sin\theta, y = k\cos\theta$. Then substitution into the second equation yields:

$$\frac{\cos^4\theta}{\sin^4\theta} + \frac{\sin^4\theta}{\cos^4\theta} = \frac{97 \cdot 2\sin\theta\cos\theta}{\sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)} = 194$$

We see that if $t = \frac{x}{y} + \frac{y}{x}$, then $\frac{\cos^4\theta}{\sin^4\theta} + \frac{\sin^4\theta}{\cos^4\theta} = (t^2-2)^2 - 2 = 194$ so solving, we have $t = 4$ □

Problem 1.5. Let P be a point inside circle Γ. Consider the set of chords of Γ that contain P. Prove that their midpoints all lie on a circle.

Solution. Let $\overline{AB}$ be the diameter that passes through P. Let O be the center. Draw any other chord that passes through P. Let M be its midpoint. It is easy to see that $\angle PMO = 90°$. It follows that all midpoints lie on the circle with diameter $\overline{PO}$.

□

Problem 1.6. Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote its circumcenter and incenter. Then $OI^2 = R(R-2r)$. In particular, $R \geq 2r$.

Solution. Refer to Chen, E. (2021). Euclidean Geometry in Mathematical Olympiads. MAA Press, an imprint of the American Mathematical Society. Pages 36-37. □

Problem 1.7. Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.

Solution. Notice that $[ACDF] = [ABDE]$ tells us that $[ABD] = [ACD]$, so B and C are the same distance from AD. Therefore, $AD||BC$. Suppose we let $x = [CGB]$ and $x + 10 = [AGB]$. Then,

$$[DGA] = \frac{DG}{GB}[AGB] = AGGC(x+10) = \frac{(x+10)^2}{x}$$

Thus,

$$[BAD] = [BGA] + [DGA] = (x+10) + \frac{(x+10)^2}{x} = 84$$

Thus, we have $x^2 - 27x + 50 = 0$ Thus, $x = 2, 25$. Now, note that $[ABCD] = [ACD] + [ABC] = 84 + x + (x+10)$, so by letting $x = 2$, this is at least $84 + 2 + 12 = 98$. Now, let O be the midpoint of AD. We note that, because $ACDF$ and $ABDE$ are both parallelograms, O is also the midpoint of BE and CF. This tells us that $ABCD$ and $DEFA$ are symmetric about O, so in particular, they have the same area. Thus, we have $[ABCDEF] = 2[ABCD] = 2(98) = 196$ □

Problem 1.8. Point P is located inside triangle ABC so that angles PAB, PBC, and $PCA$are all congruent. The sides of the triangle have lengths $AB = 13$, $BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Let $\angle PAB = \angle PBC = \angle PCA = x$. Then, using Law of Cosines on the three triangles containing vertex P, we have

$$\begin{aligned} b^2 &= a^2 + 169 - 26a\cos x \\ c^2 &= b^2 + 196 - 28b\cos x \\ a^2 &= c^2 + 225 - 30c\cos x. \end{aligned}$$

Add the three equations up and rearrange to obtain

$$(13a + 14b + 15c)\cos x = 295.$$

Also, using $[ABC] = \frac{1}{2}ab\sin\angle C$ we have

$$[ABC] = [APB]+[BPC]+[CPA] = \frac{\sin x}{2}(13a+14b+15c) = 84 \iff (13a+14b+15c)\sin x = 168.$$

Divide the two equations to obtain $\tan x = \frac{168}{295} \iff \boxed{463}$. □ □

Problem 1.9. Find the ordered pairs (p, q) for prime numbers p and q such that

$$pq \mid (5^p - 2^p)(5^q - 2^q)$$

Solution. let's assume $p \geq q$. By Fermat's Little Theorem,

$$\begin{aligned} 5^p - 2^p &\equiv 3 \mod p \\ 5^q - 2^q &\equiv 3 \mod q \end{aligned}$$

If $q = 3$, a possible value for p is 4. If $p > 4$, then $p \mid 5^3 - 2^3 = 117$ or $p = 13$. Thus we get the pairs $(3, 3), (3, 13), (13, 3)$. Now assume that $p \geq q > 3$. Then $\gcd(q, 5^q - 2^q) = 1$, Hence

$$5^p \equiv 2^p \mod q$$

From Fermat's Little Theorem we also know that $5^{q-1} \equiv 2^{q-1} \equiv 1 \mod q$. Because $\gcd(p, q-1) = 1$ we get that $ap + b(q-1) = 1$. Now notice that

$$5^{ap+b(q-1)} = (5^p)^a \cdot (5^{q-1})^b = (2^p)^a \cdot (2^{q-1})^b = 2^{ap+b(q-1)} \mod q$$

Which implies that $5 \equiv 2 \mod q$. Hence $q = 3$, a contradiction. □

Problem 1.10. Prove that if $p \equiv 3 \mod 4$ is a prime number, such that p divides $a^2 + b^2$ for positive integers a and b, then $p \mid a$ and $p \mid b$.

Solution. Suppose that one of the numbers a, b is not divisible by p. Then the other one is not divisible by p, either. Thus from Fermat's little theorem we get $a^{p-1} \equiv 1 \mod p$ and $b^{p-1} \equiv 1 \mod p$. This implies $a^{p-1} + b^{p-1} \equiv 2 \mod p$.

On the other hand, the number $a^{p-1} + b^{p-1} \equiv a^{4k+2} + b^{4k+2} = (a^2)^{2k+1} + (b^2)^{2k+1}$ is divisible by $a^2 + b^2$, and thus it is divisible by p. □

Problem 1.11. Find the last two digits of the number

$$\left\lfloor \frac{10^{93}}{10^{31} + 3} \right\rfloor$$

Solution. Let $t = 10^{31}$. Then the number is

$$\left\lfloor \frac{t^3}{t+3} \right\rfloor$$

We use a simple trick, $t^3 = (t^3 + 3^3) - 3^3$. Thus

$$\left\lfloor \frac{t^3+3^3}{t+3} - \frac{3^3}{t+3} \right\rfloor = t^2 - 3t + 9 - \left\lfloor \frac{3^3}{t+3} \right\rfloor$$

Clearly

$$-1 < \frac{3^3}{t+3} < 0 \Rightarrow \left\lfloor \frac{-3^3}{t+3} \right\rfloor = -1$$

Then the number is $10^{31}(10^{31} - 3) + 8$. Hence the last digit is 8. □

Problem 1.12. Find all integral solutions of the equation

$$a^2 + b^2 + c^2 = a^2b^2$$

Solution. There's a trivial solution, $(a, b, c) = (0, 0, 0)$, which is in fact the only one. We assume that there's another solution for the sake of contradiction. Then, $a \neq 0$ and $b \neq 0$, whence $a^2 - 1 \geq 0$ and $b^2 - 1 \geq 0$. If a or b is even, then $a^2 - 1 \equiv 3 \mod 4$ or $b^2 - 1 \equiv 3 \mod 4$. Hence, $(a^2 - 1)(b^2 - 1) = c^2 + 1$ is divisible by a prime natural number $p \equiv 3 \mod 4$. Thus, $c^2 \equiv -1 \mod p$, which is a contradiction as $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = -1$. Therefore, a and b must be odd. Thus, $c^2 = a^2b^2 - a^2 - b^2 \equiv 1 - 1 - 1 = -1 \mod 8$, which is a contradiction. □

Problem 1.13. Let A be a set formed by choosing 20 numbers arbitrarily from the arithmetic sequence 1, 4, 7, . . . , 100. Prove that there must be two numbers in A such that their sum is 104.

Solution. Note that the numbers have the form $3n + 1$ for $n = 0, 1, \ldots, 33$. We seek $3n+1$, $3m+1$ so that $n+m = 34$. Evidently $n = 0$ and $n = 17$ do not help. The other 32 numbers form 16 pairs with the required sum. Now we use Pigeon hole principle for the 16 pairs. So if we take 19 numbers then we are sure to get two from the same pair. □

Problem 1.14. For any non-empty finite set A of real numbers, let $s(A)$ denote the sum of the elements in A. If there are exactly 61 subsets A of the set $\{1, \ldots, 23\}$ that satisfy
(1) They contain exactly three elements.
(2) $s(A) = 36$
Find the number of subsets A that satisfy (1) and $s(A) < 36$

Solution. For every 3-subset (a, b, c) link this 3-subset to the 3-subset $(24-a, 24-b, 24-c)$. This will make a bijection between the 3-subsets with sum less than 36 to the 3-subsets with sum greater than 36, hence the quantity of them are equal. So we just have to decrease the number of 3-subsets with sum exactly equal to 36, from the number of all 3-subsets and divide it by 2. The total number of 3-element sets is $\binom{23}{3} = 1771$ and so

$$\frac{1771 - 61}{2} = 855$$

is the total number of sets A. □

Problem 1.15. Prove the following formulas:
(a)
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
(b)
$$\sum_{k=0}^{n} \binom{n}{2k} = \sum_{k=0}^{n} \binom{n}{2k+1} = 2^{n-1}$$

Solution. (a) Think about $(1+1)^n$.
(B) Add (a) to $(1-1)^n$ □

Problem 1.16. We draw n straight lines in the plane, no two of which are parallel and no three of which pass through the same point. These lines divide the plane into a number of regions. Determine the number of these regions.

Solution. Every new n-th line intesects the other $n-1$ lines, producing $n-1$ points of intersection that divide each region into two new ones. Hence the number of regions increases by n. This basically transforms the problem into finding the closed-form formula for the sequence $a_0 = 1$ and $a_n = a_{n-1} + n$. By recursion, this is leads to $a_n = n + \sum\limits_{k=1}^{n} k = \frac{n^2+n+2}{2}$ □