$\mathcal{OMCC}$'s Grade 10 Math Tournament

October Math Circle ($\mathcal{OMCC}$)

May 3, 2022

§1 Problems

Problem 1.1. Compute

$$\lfloor 1 \rfloor + \lfloor 1.7 \rfloor + \lfloor 2.4 \rfloor + \cdots + \lfloor 99 \rfloor$$

Problem 1.2. Suppose $\triangle ABC$ is a right triangle at A and has side lengths a, b, c, and $c \leq b$. If the maximum value of

$$\frac{a-b}{c}$$

is $\sqrt{m} - n$. Find m.

Problem 1.3. Two squares $ABCD$ and $EFGC$ are constructed such that point D is the origin and that the parabola defined by $y = k\sqrt{x}$ passes through points B and F. If $\frac{FG}{BC} = \frac{a+\sqrt{b}}{c}$, compute $\frac{a+b}{c}$.

Problem 1.4. Let $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. If $C = ABA^T$. Compute the sum of elements of $A^T C^{2022} A$.

Problem 1.5. Let the sum of the solutions to the equation

$$x^{x\sqrt{x}} = (x\sqrt{x})^x$$

be S. If S has the form $\frac{m}{n}$. Find $m+n$.

Problem 1.6. Define a polynomial P such that

$$P(x) = (x^2 - 5x + 1)(x^2 - 5x + 2) \ldots (x^2 - 5x + 101)$$

Find the sum of the roots of the polynomial.

Problem 1.7. Let x and y be real numbers satisfying $(x-3)^2 + 4(y-1)^2 = 4$. Find the maximum of

$$z = \frac{x+y-3}{x-y+1}$$

Problem 1.8. Suppose that P is a polynomial of the second degree, and let P^n (for $n \geq 2$ denote the composition of P with itself n times, so that $P^2 = P(P(x))$. If the sum of the roots of P^{2022} is 2^{17}. If the sum of the roots of P is S. Compute $|\log_2(S)|$

Problem 1.9. Define a complex number z, and let $\Im(z)$ denote the imaginary part of z. If the minimum of

$$\frac{\Im(z^5)}{(\Im(z))^5}$$

is M, Compute $|M|$.

Problem 1.10. Let $f(x) = x^2$ and $g(x) = \sin(x)$ for all real x. Find the solution set for the equation

$$(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$$

(A) $\pm\sqrt{n\pi}, \quad n \in \{0, 1, 2, \dots\}$
(B) $\pm\sqrt{n\pi}, n \in \{1, 2, 3, \dots\}$
(C) $\frac{\pi}{2} + 2n\pi, \quad n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$
(D) $2n\pi, \quad n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$

Problem 1.11. Define the matrix $A = \begin{pmatrix} x & y & -z \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}$ for natural numbers x, y, and z. If $|\mathbf{adj}(\mathbf{adj}(\mathbf{adj}(\mathbf{adj}\ A)))| = 2^{32} \cdot 6^{16}$, Find $x + y + z$.

Problem 1.12. Let α and β be the roots of $x^2 - 6x - 2$ such that $\alpha > \beta$. Suppose $a_n = \alpha^n - \beta^n$ for $n \geq 1$. Compute $\frac{a_{10} - 2a_8}{2a_9}$

Problem 1.13. Define the polynomials P, Q, R, and S such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$$

Find $P(1)$.

Problem 1.14. Define x, y, and z to be real numbers such that

$$\begin{cases} \tan x + \tan y + \tan z & = 6 - (\cot x + \cot y + \cot z) \\ \tan^2 x + \tan^2 y + \tan^2 z & = 6 - (\cot^2 x + \cot^2 y + \cot^2 z) \\ \tan^3 x + \tan^3 y + \tan^3 z & = 6 - (\cot^3 x + \cot^3 y + \cot^3 z) \end{cases}$$

Evaluate $\frac{\tan x}{\tan y} + \frac{\tan y}{\tan z} + \frac{\tan z}{\tan x} + 3 \tan x \tan y \tan z$

Problem 1.15. From the vertex of a parabola draw two mutually perpendicular chords of lengths 8 and 27. If the length of the latus rectum of the parabola is $\frac{a\sqrt{b}}{c}$. Compute $a + b + c$.

Problem 1.16. If the solution set of the equation

$$x^2 - 40\lfloor x \rfloor + 39 = 0$$

is in the form $\{a, \sqrt{b}, \sqrt{c}, d\}$, compute $a + b + c + d$.

Problem 1.17. Consider the two sets

$$A = \{\sin \alpha, \sin 2\alpha, \sin 3\alpha\}$$

and

$$B = \{\cos \alpha, \cos 2\alpha, \cos 3\alpha\}$$

If the family of solutions for α making the two sets equal is $\frac{\pi}{a} + \frac{k\pi}{b}$. Compute $a - b$.

Problem 1.18. If

$$\cot\left(\operatorname{arccot}(3) + \operatorname{arccot}(7) + \cdots + \operatorname{arccot}(23 \cdot (24) + 1)\right) = \frac{m}{n}$$

Find $m + n$.

Problem 1.19. Consider the equation

$$\log_b \log_b \log_b(x) = \log_{b^2} \log_{b^2} \log_{b^2}(x)$$

If $b = 128$, then the equation has a unique solution $> b^2$ in the form a^n. Find $a + n$.

Problem 1.20. The system of equations

$$\begin{cases} 11x^4 + 4x^3y + 4xy^3 + x - 11y^4 & = 0 \\ 2x^4 - 22x^3y - 22xy^3 + y - 2y^4 & = 0 \end{cases}$$

defines a non-zero complex number $z = a + bi$, such that $\Re(z)$ and $\Im(z)$ are both rational. Find $\frac{\Im(z)}{\Re(z)}$.