Calculus Admission Test

October Math Circle

November 30, 2022

§1 Calculus Admission Test Problems

Problem 1.1. The value of the integral

$$I = \int_{\pi/4}^{\pi/3} \frac{1}{\tan\theta + \cot\theta} d\theta$$

can be written as $\frac{a}{b}$ where a and b are relatively prime. Find $a + b$

Problem 1.2. Evaluate

$$\lim_{n\to\infty} \left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{n}{2n^2} \right)$$

Problem 1.3. Consider two twice-differentiable f and g, such that, $f(x)g(x) = 1$ for all x, and $f'(x)$ and $g'(x)$ are never zero. Find $\frac{f''(x)}{f'(x)} - \frac{g''(x)}{g'(x)}$. Choose all that's correct.

$(a)\frac{-f'(x)}{f(x)}$

$(b)\frac{-2f'(x)}{f(x)}$

$(c)\frac{-2g'(x)}{g(x)}$

$(d)\frac{2f'(x)}{f(x)}$

Problem 1.4. Find

$$\lim_{n\to\infty} \frac{1^p + 2^p + 3^p + \cdots + n^p}{n^{p+1}}$$

Problem 1.5. Compute the integrals

(a)

$$I = \int \frac{\cos(2x)}{\cos(x)} dx$$

(b)

$$I = \int \ln(1+x) dx$$

(c)

$$I = \int \sqrt{\tan x} \quad dx$$

Problem 1.6. If $a_1 = 1$ and $a_n = n(a_{n-1} + 1)$ for $n \geq 2$. Compute

$$\lim_{n\to\infty} \left(\left(1 + \frac{1}{a_1}\right)\left(1 + \frac{1}{a_2}\right)\left(1 + \frac{1}{a_3}\right) \cdots \left(1 + \frac{1}{a_n}\right) \right)$$

Problem 1.7. (a) Prove that the tangent line to the graph of f defined as $f(x) = \sqrt{1-x^2}$ at $(a, \sqrt{1-a^2})$ intersects the graph only at that point.
(b) Prove similarly that the tangent lines to an ellipse or hyperbola intersect these sets only once.

Problem 1.8. Find the length of the curve defined by $x = \cos^3\theta$ and $y = \sin^3\theta$ for $0 \leq \theta \leq \pi/2$

Problem 1.9. Find the surface and the volume of revolution done by the curve defined as $y = \frac{1}{x}$ from 1 to ∞. Explain.

Problem 1.10. Let $f(x)$ be a function which has $n+1$ continuous derivatives in an open interval containing the origin, and assume that the $(n+1)$-th derivative is bounded by a constant M on this interval. Let $P_n(x)$ be the Taylor polynomial of degree n for $f(x)$. What are the following limits?
(a) $\lim\limits_{x\to 0} \frac{f(x)-P_n(x)}{x^2}$ for $n \geq 3$.
(b) $\lim\limits_{x\to 0} \frac{f(x)-P_n(x)}{x^{n-1}}$
(c) $\lim\limits_{x\to 0} \frac{f(x)-P_n(x)}{x^n}$

Problem 1.11. Test the following series for convergence.
(a)

$$\sum_{n=1}^{\infty} \frac{\log(n)}{n}$$

(b)

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{1+\alpha}}$$

for a positive real number α.
(c)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/n}}$$

(d) Find the radius of convergence for

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$$

Problem 1.12. Let $J_k(x)$ denote the power series

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+k)!} \left(\frac{x}{2}\right)^{2n+k}$$

for any positive k. Compute

$$x^2 J_k''(x) + x J_k'(x) + (x^2 - k^2) J_k$$

Problem 1.13. On the curve defined as $y = ax^2 + bx + c$, with all the coefficients being natural numbers, four points are marked such that $A(-2,3)$, $B(-1,1)$, $C(\alpha, \beta)$, and $D(2,7)$. These points make a convex quadrilateral of maximum area. Find the minimum of $a + b + c + 2\alpha + 4\beta$

Problem 1.14. Find the following improper integrals if they exist.
(a)

$$\int_1^\infty x^n e^{-x} dx$$

for positive integers n
(b)

$$\int_3^\infty \frac{1}{x(\log(x))^4} dx$$

Problem 1.15. Prove that
(a)

$$\int \sin^m x \cos^n x dx = \frac{-1}{m+n} \sin^{m-1} x \cos^{n+1} x + \frac{m-1}{m+n} \int \sin^{m-2} x \cos^n x dx$$

(b)

$$\int x^m (\ln x)^n dx = \frac{1}{m+1} x^{n+1} \ln(x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$$

Problem 1.16. Define two differentiable functions f and g, such that,

$$\frac{f'(x)}{g'(x)} = e^{f(x)-g(x)}$$

for all x, and $f(0) = g(2003) = 1$. The maximum value of c that satisfies $f(2003) > c$ for all functions f and g has the form $a - \ln(b)$, compute $a + b$.

§2 Contest Math Problems

Problem 2.1. Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x)) = 0$ has four distinct solutions, $x = 3, 4, a, b$. Find the sum of all possible values of $(a + b)^2$

Problem 2.2. Determine all triples of real numbers (x, y, z) such that

$$\begin{aligned} xyz &= 8 \\ x^2y + y^2z + z^2x &= 73 \\ x(y-z)^2 + y(z-x)^2 + z(x-y)^2 &= 8 \end{aligned}$$

Problem 2.3. Compute

$$\sum_{k=0}^{n} \frac{(4k+1)k!}{(2k+1)!}$$

Problem 2.4. Let x and y be positive real numbers and θ be an angle which is not an integer multiple of $\pi/2$. Suppose

$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y} \quad \text{and} \quad \frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{7\sin(2\theta)}{x^3y + xy^3}$$

Compute

$$\frac{x}{y} + \frac{y}{x}$$

Problem 2.5. Let P be a point inside circle Γ. Consider the set of chords of Γ that contain P. Prove that their midpoints all lie on a circle.

Problem 2.6. Let ABC be a triangle. Let R and r denote its circumradius and inradius, respectively. Let O and I denote its circumcenter and incenter. Then $OI^2 = R(R-2r)$. In particular, $R \geq 2r$.

Problem 2.7. Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.

Problem 2.8. Point P is located inside triangle ABC so that angles PAB, PBC, and $PCA$are all congruent. The sides of the triangle have lengths $AB = 13$, $BC = 14$, and $CA = 15$, and the tangent of angle PAB is m/n, where m and n are relatively prime positive integers. Find $m+n$.

Problem 2.9. Find the ordered pairs (p, q) for prime numbers p and q such that

$$pq \mid (5^p - 2^p)(5^q - 2^q)$$

Problem 2.10. Prove that if $p \equiv 3 \mod 4$ is a prime number, such that p divides $a^2 + b^2$ for positive integers a and b, then $p \mid a$ and $p \mid b$.

Problem 2.11. Find the last two digits of the number

$$\left\lfloor \frac{10^{93}}{10^{31}+3} \right\rfloor$$

Problem 2.12. Find all integral solutions of the equation

$$a^2 + b^2 + c^2 = a^2 b^2$$

Problem 2.13. Let A be a set formed by choosing 20 numbers arbitrarily from the arithmetic sequence 1, 4, 7, . . . , 100. Prove that there must be two numbers in A such that their sum is 104.

Problem 2.14. For any non-empty finite set A of real numbers, let $s(A)$ denote the sum of the elements in A. If there are exactly 61 subsets A of the set $\{1, \ldots, 23\}$ that satisfy
(1) They contain exactly three elements.
(2) $s(A) = 36$
Find the number of subsets A that satisfy (1) and $s(A) < 36$

Problem 2.15. Prove the following formulas:
(a)

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

(b)

$$\sum_{k=0}^{n} \binom{n}{2k} = \sum_{k=0}^{n} \binom{n}{2k+1} = 2^{n-1}$$

Problem 2.16. Suppose that 10 boys and 15 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. Find the average value of S (if all possible orders of these 25 people are considered).

Problem 2.17. Suppose that n people each know exactly one piece of information, and all n pieces are different. Every time person A phones person B, A tells B everything that A knows, while B tells A nothing. What is the minimum number of phone calls between pairs of people needed for everyone to know everything? Prove your answer is a minimum