# MARKOV TRACE ON A TOWER OF AFFINE TEMPERLEY-LIEB ALGEBRAS OF TYPE $\tilde{A}_n$

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ABSTRACT. We define a tower of affine Temperley-Lieb algebras of type $\tilde{A}_n$. We prove that there exists a unique Markov trace on this tower, this trace comes from the Markov-Ocneanu-Jones trace on the tower of Temperley-Lieb algebras of type $A_n$. We define an invariant of special kind of links as an application of this trace.

## 1. INTRODUCTION

About 30 years ago, V. Jones discovered one of the most famous invariants of oriented knots and links [10]. The remarkable feature of the construction of this invariant is that it works in a purely algebraic setting: it arises from certain trace functions (Markov traces) on Temperley-Lieb algebras of type $A$. Later V. Jones, himself, redefined his traces on Iwahori-Hecke algebras of type $A$, with a parameter "z" in the ground ring. Temperley-Lieb algebras of type $A$ are quotients of Iwahori-Hecke algebras of type $A$, which themselves are finite-dimensional quotients of the group algebras of Artin's braid groups (braid groups of type $A$). The other remarkable feature of the above-mentioned invariant is that: it is easy to compute, yet it does not "distinguish" totally between oriented links, in other words there exist at least two nonequivalent links having the same Jones' polynomial.

Since then many generalizations of the construction of Markov traces beyond the type $A$ have been achieved, yet have been restricted to the "spherical cases", that is, braid groups, Iwahori-Hecke algebras and Temperley-Lieb algebras associated with finite Weyl groups. A classification of Markov traces on Iwahori-Hecke algebras of type $B$ and $D$ was given by Geck and Lambropoulou in [2] from which we can determine the Markov traces factoring by the corresponding Temperley-Lieb algebras by noticing that the space of all trace functions on a finite-dimensional Iwahori-Hecke algebra is spanned by the characters of the irreducible representations of this algebra, these characters are essentially known [3].

The $\tilde{A}$-type affine braid group is the braid group under question in this work (we call its elements affine braids). Geometrically, one can see several presentations in the literature, among which we choose the one corresponding to the $B$-type braid group (we call its elements $B$-braids) [9]. In [5] we see that $B$-type braid group is a semi-direct product of $\tilde{A}$-type affine braid group with a normal subgroup generated by one element (acting on the affine braids as the "Dynkin automorphism" which is to be defined in the third section). In particular every affine braid is a $B$-braid.

In [9] we give a definition of an affine oriented link to be the closure of an affine braid seen as a $B$-braid.

Or equivalently to be: an oriented link in a solid torus which makes the same number of positive and negative rounds around the "middle hole".

Or equivalently to be: an oriented braid in $S^3$ in which the number of positive rounds equals the number of negative rounds, around a fixed circle (links which do not make rounds are counted here, in other terms every oriented link -in the usual sense- is an affine oriented link). Hence, an invariant of oriented affine links is an invariant of oriented links.

In this paper, we will essentially work on the images of the affine braids in the affine Temperley-Lieb algebra of type $\tilde{A}$ (in literature sometimes it is called non-extended affine Temperley-Lieb algebra in order to distinguish it from the extended affine Temperley-Lieb algebra which is a slightly larger structure, see [5] and [4]) and use purely algebraic tools.

We build a tower of affine Temperley-Lieb algebra of type $\tilde{A}$ and define what should be the Markov conditions in the affine case. We then use the main result of [6], which is a classification of fully commutative elements in the affine Weyl group of type $\tilde{A}$, to classify a large class of traces on the affine Temperley-Lieb algebra of type $\tilde{A}$ – recall that a basis of this algebra is indexed by those fully commutative elements. Finally we prove the existence and uniqueness of the Markov trace on the tower of affine Temperley-Lieb algebras of type $\tilde{A}$, that is Theorem 5.5.

This affine Markov trace thus defines the unique Jones-like invariant of oriented affine links, when composed with the following path:

$$\text{Affine oriented links} \longrightarrow \text{Affine braid groups} \longrightarrow \text{Affine T-L algebras}$$

## 2. NOTATIONS

Let $K$ be an integral domain of characteristic 0. Suppose that $q$ is a square invertible element in $K$ of which we fix a root $\sqrt{q}$. For $x, y$ in a given ring we define $V(x, y) := xyx + xy + yx + x + y + 1$. We mean by algebra in what follows $K$-algebra.

We denote by $B(\tilde{A}_n)$ (resp. $W(\tilde{A}_n)$) the affine braid (resp. affine Coxeter) group with $n+1$ generators of type $\tilde{A}$, while we denote by $B(A_n)$ (resp. $W(A_n)$) the braid (resp. Coxeter) group with $n$ generators of type $A$, where $n \geq 0$. By definition $B(\tilde{A}_n)$ has $\{\sigma_1, \sigma_2, \ldots, \sigma_n, a_{n+1}\}$ as a set of generators together with the following defining relations:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ where $1 \leq i, j \leq n$ when $|i-j| \geq 2$,

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ when $1 \leq i \leq n-1$,

(3) $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ when $2 \leq i \leq n-1$,

(4) $\sigma_1 a_{n+1}\sigma_1 = a_{n+1}\sigma_1 a_{n+1}$,

(5) $\sigma_n a_{n+1}\sigma_n = a_{n+1}\sigma_n a_{n+1}$ for $n \geq 2$,

while $B(A_n)$ is generated by $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$. We let $W^c(\tilde{A}_n)$ (resp. $W^c(A_n)$) be the set of fully commutative elements in $W(\tilde{A}_n)$ (resp. $W(A_n)$).

Let $n \geq 2$. We define $\widehat{TL}_{n+1}(q)$ to be the algebra with unit given by a set of generators $\left\{g_{\sigma_1}, ..., \ g_{\sigma_n}, g_{a_{n+1}}\right\}$, with the following relations [4]:

- $g_{\sigma_i} g_{\sigma_j} = g_{\sigma_j} g_{\sigma_i}$, for $1 \leq i, j \leq n$ and $|i - j| \geq 2$.
- $g_{\sigma_i} g_{a_{n+1}} = g_{a_{n+1}} g_{\sigma_i}$, for $2 \leq i \leq n-1$.
- $g_{\sigma_i} g_{\sigma_{i+1}} g_{\sigma_i} = g_{\sigma_{i+1}} g_{\sigma_i} g_{\sigma_{i+1}}$, for $1 \leq i \leq n-1$.
- $g_{\sigma_i} g_{a_{n+1}} g_{\sigma_i} = g_{a_{n+1}} g_{\sigma_i} g_{a_{n+1}}$, for $i = 1, n$.
- $g_{\sigma_i}^2 = (q-1) g_{\sigma_i} + q$, for $1 \leq i \leq n$.
- $g_{a_{n+1}}^2 = (q-1) g_{a_{n+1}} + q$,
- $V(g_{\sigma_i}, g_{\sigma_{i+1}}) = V(g_{\sigma_1}, g_{a_{n+1}}) = V(g_{\sigma_n}, g_{a_{n+1}}) = 0$, for $1 \leq i \leq n-1$.

The set $\left\{g_w : w \in W^c(\tilde{A}_n)\right\}$ is well defined in the usual sense of the theory of Hecke algebra and it is a $K$-basis [1, §2]. We set $T_{a_{n+1}}$ (resp. $T_{\sigma_i}$ for $1 \leq i \leq n$) to be $\sqrt{q} g_{a_{n+1}}$ (resp. $\sqrt{q} g_{\sigma_i}$ for $1 \leq i \leq n$). Hence, $T_w$ is well defined for $w \in W^c(\tilde{A}_n)$, it equals $q^{\frac{l(w)}{2}} g_w$. The multiplication associated to the basis $\left\{T_w : w \in W^c(\tilde{A}_n)\right\}$, is given as follows:

$$
\begin{aligned}
&T_w T_v = T_{wv} && \text{whenever } l(wv) = l(w) + l(v). \\
&T_s T_w = \sqrt{q}(q-1) T_w + q^2 T_{sw} && \text{whenever } l(sw) = l(w) - 1,
\end{aligned}
$$

for $w, v$ in $W^c(\tilde{A}_n)$ and $s$ in $\{\sigma_1, ..., \ \sigma_n, a_{n+1}\}$.

In what follows we suppose that $q+1$ is invertible in $K$, we set $\delta = \frac{1}{2+q+q^{-1}} = \frac{q}{(1+q)^2}$ in $K$. In view of [5] , for $1 \leq i \leq n$ we set $f_{\sigma_i} := \frac{g_{\sigma_i}+1}{q+1}$ and $f_{a_{n+1}} := \frac{g_{a_{n+1}}+1}{q+1}$. In other terms $g_{\sigma_i} = (q+1) f_{\sigma_i} - 1$, and $g_{a_{n+1}} = (q+1) f_{a_{n+1}} - 1$. The set $\left\{f_w : w \in W^c(\tilde{A}_n)\right\}$ is well defined and it is a $K$-basis for $\widehat{TL}_{n+1}(q)$.

We define the Temperley-Lieb algebra of type $A$ with $n$ generators $TL_n(q)$, as the subalgebra of $\widehat{TL}_{n+1}(q)$ generated by $\{g_{\sigma_1}, ..., \ g_{\sigma_n}\}$, with $\{g_w : w \in W^c(A_n)\}$ as $K$-basis.

Now for $TL_0(q) = K$, we consider the following tower:

$$
TL_0(q) \ \subset TL_1(q) \ \ ... \subset TL_{n-1}(q) \quad \subset TL_n(q) \ \ ...
$$

**Theorem 2.1.** *[10] There is a unique collection of traces $(\tau_{n+1})_{0\leq n}$ on $(TL_n)_{0\leq n}$, such that:*

(1) $\tau_1(1) = 1$.

(2) *For $1 \leq n$, we have $\tau_{n+1}(hT_{\sigma_n}^{\pm 1}) = \tau_n(h)$, for any $h$ in $TL_{n-1}(q)$.*

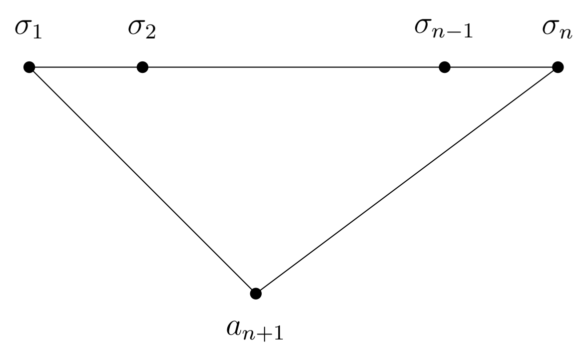
The collection $(\tau_{n+1})_{0\leq n}$ is called a Markov trace. Moreover, for any $a, b$ and $c$ in $TL_n(q)$ and for $n \geq 1$, every $\tau_{n+1} : TL_n(q) \longrightarrow K$ verifies:

$$\tau_{n+1}(bT_{\sigma_n}c) = \tau_n(bc) \text{ and } \tau_{n+1}(a) = -\frac{1+q}{\sqrt{q}}\tau_n(a).$$

## 3. The tower of affine Temperley-Lieb algebras and affine Markov trace

In this section we define a tower of affine Temperley-Lieb algebras, we show that this tower "surjects" onto the tower of Temperley-Lieb algebras mentioned in the introduction, and we define the affine Markov trace.

We consider the Dynkin diagram of the group $B(\tilde{A}_n)$.




We denote the Dynkin automorphism $(\sigma_1 \mapsto \sigma_2 \mapsto \cdots \mapsto \sigma_n \mapsto a_{n+1} \mapsto \sigma_1)$ by $\psi_{n+1}$. We have the following injection:

$$\begin{aligned} G_n : K[B(\tilde{A_{n-1}})] &\longrightarrow K[B(\tilde{A}_n)] \\ \sigma_i &\longmapsto \sigma_i \qquad \text{for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1} \end{aligned}$$

We prove in [7], to which we refer for details, that $G_n$ induces an injective homomorphism of the corresponding Hecke algebras [7, Proposition 4.3.3]. This homomorphism induces in turn a map at the Temperley-Lieb level that is an algebra homomorphism [7, §5.2.3], but, in the affine case, the possible lack of injectivity forces us to use careful notations in the following Proposition: indeed we use the generic letter $t$ instead of our previous $g$ for the generators of $\widehat{TL}_n(q)$.

**Proposition 3.1.** *The injection $G_n$ induces the following morphism of algebras:*

$$\begin{aligned} F_n : \widehat{TL}_n(q) &\longrightarrow \widehat{TL}_{n+1}(q) \\ t_{\sigma_i} &\longmapsto g_{\sigma_i} \textit{ for } 1 \leq i \leq n-1 \\ t_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}. \end{aligned}$$

Notice that the conjugation by $\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}$ in $B(\tilde{A}_n)$ acts on the image of $B(\tilde{A_{n-1}})$ as $\psi_n^{-1}$: $(\sigma_1 \mapsto a_n \mapsto \sigma_{n-1} \mapsto \sigma_{n-2} \mapsto ..\sigma_2 \mapsto \sigma_1)$. We denote by $\Psi$ the automorphism of $G_n(B(\tilde{A_{n-1}}))$ image of $\psi_n$. We thus write $(\sigma_n..\sigma_1 a_{n+1})^d h = \Psi^{-d}(h)(\sigma_n..\sigma_1 a_{n+1})^d$, for any $h$ in $G_n(B(\tilde{A_{n-1}}))$. This convention is specially important when used for the affine Temperley-Lieb algebra. Indeed $\psi_{n+1}$ also acts as an automorphism of the algebra $\widehat{TL}_{n+1}(q)$ and the conjugation by $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ in $\widehat{TL}_{n+1}(q)$ stabilizes the image of $\widehat{TL}_n(q)$, on which it acts like the image $\Psi$ of $\psi_n$ under $F_n$. We then write likewise:

$$g^d_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\, h = \Psi^{-d}(h)\, g^d_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}} \qquad \text{for all } h \in F_n(\widehat{TL}_n(q)) \text{ and } d \in \mathbb{Z}. \tag{1}$$

We also prove in [7, §5.2.2]:

**Proposition 3.2.** *The following map is a surjection of algebras*

$$\begin{aligned} E_n : \widehat{TL}_{n+1}(q) &\longrightarrow TL_n(q) \\ g_{\sigma_i} &\longmapsto g_{\sigma_i} \textit{ for } 1 \leq i \leq n \\ g_{a_{n+1}} &\longmapsto g_{\sigma_1}\cdots g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-1}}^{-1}\cdots g_{\sigma_1}^{-1}. \end{aligned}$$

*Moreover, the following diagram commutes:*

$$\begin{array}{ccc} \widehat{TL}_n(q) & \xrightarrow{\quad F_n \quad} & \widehat{TL}_{n+1}(q) \\ {\scriptstyle E_{n-1}}\Big\downarrow & & \Big\downarrow{\scriptstyle E_n} \\ TL_{n-1}(q) & \hookrightarrow & TL_n(q) \end{array}$$

Note that $E_n$ composed with the natural inclusion of $TL_n(q)$ into $\widehat{TL}_{n+1}(q)$, gives $Id_{TL_n(q)}$.

In view of Proposition 3.1 we can consider the tower of affine T-L algebras (it is not known whether it is a tower of faithful arrows or not):

$$\widehat{TL}_1(q) \xrightarrow{F_1} \widehat{TL}_2(q) \xrightarrow{F_2} \widehat{TL}_3(q) \longrightarrow \quad ... \quad \widehat{TL}_n(q) \xrightarrow{F_n} \widehat{TL}_{n+1}(q) \longrightarrow ...$$

**Definition 3.3.** *We call* $(\hat{\tau}_n)_{1\leq n}$ *an affine Markov trace, if every* $\hat{\tau}_n$ *is a trace function on* $\widehat{TL}_n(q)$ *with the following conditions:*

- $\hat{\tau}_1(1) = 1$, *(here* $\widehat{TL}_1(q) = K$*).*
- $\hat{\tau}_{n+1}(F_n(h)T_{\sigma_n}^{\pm 1}) = \hat{\tau}_n(h)$, *for all* $h \in \widehat{TL}_n(q)$ *and for* $n \geq 1$.
- $\hat{\tau}_n$ *is invariant under the Dynkin automorphism* $\psi_n$ *for all* $n$.

**Remark 3.4.** *We notice that the second condition gives us that* $\hat{\tau}_{n+1}\Big(F_n(h)T_{\sigma_n}^{-1}\Big) = \hat{\tau}_n\Big(h\Big)$, *which means that:*

$$\hat{\tau}_{n+1}\Big(F_n(h)[\frac{1}{q^2}T_{\sigma_n} - \frac{q-1}{q\sqrt{q}}]\Big) = \hat{\tau}_n\Big(h\Big). \text{ Thus } \hat{\tau}_{n+1}\Big(F_n(h)\Big) = -\frac{q+1}{\sqrt{q}}\hat{\tau}_n\Big(h\Big).$$

**Remark 3.5.** *The third condition of Definition 3.3 is, in fact, not independent, i.e., it results from the first and second conditions. We just have to see that if we have two elements in* $\widehat{TL}_n(q)$, *say* $x$ *and* $y$, *such that* $\psi_n(x) = y$, *then* $F_n(x)$ *and* $F_n(y)$ *are conjugate in* $\widehat{TL}_{n+1}(q)$, *by some power of the element* $g_{\sigma_n \dots \sigma_1 a_{n+1}}$, *which results from (1) above. Nevertheless, we will keep viewing it as a condition and study traces invariant under the Dynkin automorphism.*

## 4. On the space of traces on $\widehat{TL}_{n+1}(q)$

4.1. **Traces on $\widehat{TL}_2(q)$.** In this subsection we parametrize all the traces on the algebra $\widehat{TL}_2(q)$, which have the same value over the two generators of $\widehat{TL}_2(q)$, i.e., traces which are fixed under the action of the diagram automorphism.

The algebra $\widehat{TL}_2(q)$ is generated by two elements: $g_{\sigma_1}, g_{a_2}$, with only Hecke quadratic relations. That is:

$$g_{\sigma_1}^2 = (q-1)g_{\sigma_1} + q, \text{ and } g_{a_2}^2 = (q-1)g_{a_2} + q.$$

Making the same changes as in the introduction, we set $\mathbf{T}_{\sigma_1} := \sqrt{q}g_{\sigma_1}$, and $\mathbf{T}_{a_2} := \sqrt{q}g_{a_2}$. Hence $\mathbf{T}_w = \Big(\sqrt{q}\Big)^{l(w)} g_w$ for any $w \in W(\tilde{A}_1)$. The set $\Big\{\mathbf{T}_w; w \in W(\tilde{A}_1)\Big\}$ is another $K$-basis of $\widehat{TL}_2(q)$. The multiplication law of the new basis takes the form:

$$\mathbf{T}_{\sigma_1}^2 = \sqrt{q}(q-1)\mathbf{T}_{\sigma_1} + q^2 \text{ thus: } \mathbf{T}_{\sigma_1}^{-1} = \frac{1}{q^2}(\mathbf{T}_{\sigma_1} - \sqrt{q}(q-1)).$$

$$\mathbf{T}_{a_2}^2 = \sqrt{q}(q-1)\mathbf{T}_{a_2} + q^2 \text{ thus: } \mathbf{T}_{a_2}^{-1} = \frac{1}{q^2}(\mathbf{T}_{a_2} - \sqrt{q}(q-1))$$

We consider the changed basis mentioned in the introduction. $\widehat{TL}_2(q)$ is generated by $f_{\sigma_1}$ and $f_{a_2}$ with relations $f_{\sigma_1}^2 = f_{\sigma_1}$ and $f_{a_2}^2 = f_{a_2}$. Moreover, $\widehat{TL}_2(q)$ has $\Big\{f_w; w \in W(\tilde{A}_1)\Big\}$

as a $K$-basis. The aim is to parametrize all traces over this algebra, which are invariant under the action of the Dynkin automorphism $\psi_2$, which exchanges $\mathbf{T}_{\sigma_1}$ and $\mathbf{T}_{a_2}$, (that is exchanging $f_{\sigma_1}$ and $f_{a_2}$). Clearly, any trace that has the same value on $f_{\sigma_1}$ and $f_{a_2}$ is invariant under the Dynkin automorphism $\psi_2$.

**Proposition 4.1.** *Let $A_0, A_1$ and $(\alpha_i)_{i\geq 1}$ be arbitrary elements in the ground field. Then, there exists a unique trace $t$ on $\widehat{TL}_2(q)$, invariant by the action of $\psi_2$, in such a way that: $A_0 = t(1)$, $A_1 = t(f_{\sigma_1})$ and $\alpha_i = t\Big((f_{\sigma_1 a_2})^i\Big)$.*

*Proof.* We start by the existence. Let $t$ be the linear function given by:

$$\begin{aligned} t : \widehat{TL}_2(q) &\longrightarrow K \\ t(1) &= A_0 \\ t(f_{\sigma_1}) &= t(f_{a_2}) = A_1 \\ t\Big((f_{\sigma_1 a_2})^s\Big) &= t\Big((f_{a_2\sigma_1})^s\Big) = t\Big((f_{\sigma_1 a_2})^s f_{\sigma_1}\Big) = t\Big((f_{a_2\sigma_1})^s f_{a_2}\Big) = \alpha_s. \end{aligned}$$

Where $A_0, A_1$ and $\alpha_i$ are arbitrary elements in the ground field for $i \geq 1$.

We show that this linear function is a trace. First we see that $t$ is, by definition, invariant under the Dynkin automorphism $\psi_2$. In order to show that $t$ is a trace, we show that $t(xy) = t(yx)$ for any $x$ and $y$ in $\widehat{TL}_2(q)$. The way to do so, is to show that it is true when $x$ is any element of the left column, and $y$ is any element of the right column, in the following table:

| | |
|---|---|
| $[1](f_{\sigma_1 a_2})^k$ | $[1'](f_{\sigma_1 a_2})^h$ |
| $[2](f_{a_2\sigma_1})^k$ | $[2'](f_{a_2\sigma_1})^h$ |
| $[3](f_{\sigma_1 a_2})^k f_{\sigma_1}$ | $[3'](f_{\sigma_1 a_2})^h f_{\sigma_1}$ |
| $[4](f_{a_2\sigma_1})^k f_{a_2}$ | $[4'](f_{a_2\sigma_1})^h f_{a_2}$ |

The only cases to consider are **[1-2']**, **[1-3']**, **[1-4']** **and** **[3-4']**, up to applying $\psi_2$.

**[1-2']**:

Here, $t\Big(xy\Big) = t\Big((f_{\sigma_1 a_2})^k (f_{a_2\sigma_1})^h\Big) = t\Big((f_{\sigma_1 a_2})^k f_{a_2\sigma_1} (f_{a_2\sigma_1})^{h-1}\Big) = t\Big((f_{\sigma_1 a_2})^k (f_{\sigma_1 a_2})^{h-1} f_{\sigma_1}\Big)$

$$= t\Big((f_{\sigma_1 a_2})^{k+h-1} f_{\sigma_1}\Big) = \alpha_{k+h-1},$$

$$\text{while, } t\Big(yx\Big) = t\Big((f_{a_2\sigma_1})^h (f_{\sigma_1 a_2})^k\Big) = t\Big((f_{a_2\sigma_1})^h f_{\sigma_1 a_2} (f_{\sigma_1 a_2})^{k-1}\Big) = t\Big((f_{a_2\sigma_1})^h (f_{a_2\sigma_1})^{k-1} f_{a_2}\Big)$$

$$= \alpha_{k+h-1}.$$

**[1-3']**:

$$\text{Here, } t(xy) = t\Big((f_{\sigma_1 a_2})^k (f_{\sigma_1 a_2})^h f_{\sigma_1}\Big) = t\Big((f_{\sigma_1 a_2})^{k+h} f_{\sigma_1}\Big), \text{ which is equal to } \alpha_{k+h},$$

$$\text{while, } t(yx) = t\Big((f_{\sigma_1 a_2})^h f_{\sigma_1} (f_{\sigma_1 a_2})^k\Big) = t\Big((f_{\sigma_1 a_2})^{h+k}\Big) = \alpha_{k+h}.$$

**[1-4']**:

$$\text{Here, } t\Big(xy\Big) = t\Big((f_{\sigma_1 a_2})^k (f_{a_2\sigma_1})^h f_{a_2}\Big) = t\Big((f_{\sigma_1 a_2})^k f_{a_2} (f_{\sigma_1 a_2})^h\Big) = t\Big((f_{\sigma_1 a_2})^{k+h}\Big) = \alpha_{k+h},$$

$$\text{while, } t\Big(yx\Big) = t\Big((f_{a_2\sigma_1})^h f_{a_2} (f_{\sigma_1 a_2})^k\Big) = t\Big(f_{a_2} (f_{\sigma_1 a_2})^h (f_{\sigma_1 a_2})^k\Big) = t\Big(f_{a_2} (f_{\sigma_1 a_2})^{h+k}\Big) = \alpha_{k+h}.$$

**[3-4']**:

$$\text{We see that: } t\Big(xy\Big) = t\Big((f_{\sigma_1 a_2})^k f_{\sigma_1} (f_{a_2\sigma_1})^h f_{a_2}\Big) = t\Big((f_{\sigma_1 a_2})^{k+h+1}\Big) = \alpha_{k+h+1},$$

$$\text{while, } t\Big(yx\Big) = t\Big((f_{a_2\sigma_1})^h f_{a_2} (f_{\sigma_1 a_2})^k f_{\sigma_1}\Big) = t\Big((f_{a_2\sigma_1})^{h+k+1}\Big) = \alpha_{k+h+1}.$$

Now, we end the proof by showing the uniqueness. Let $t$ be a $\psi_2$-invariant trace on $\widehat{TL}_2(q)$. We have necessarily $t(f_{\sigma_1}) = t(f_{a_2})$, since $t$ is a $\psi_2$-invariant, call this value $A_1$. For every $s \geq 1$ we have $t\Big((f_{\sigma_1 a_2})^s\Big) = t\Big((f_{a_2\sigma_1})^s\Big)$, since $t$ is a trace, call this value $\alpha_s$. Finally, we have $\alpha_s = t\Big((f_{\sigma_1 a_2})^s f_{\sigma_1}\Big) = t\Big((f_{a_2\sigma_1})^s f_{a_2}\Big)$, since $t$ is a trace, and $f_{a_2},\ f_{\sigma_1}$ are idempotent. Call $t(1) = A_0$, thus, $t$ is uniquely determined by $A_0,\ A_1$ and $\alpha_s$, for $s \geq 1$.

$\square$

4.2. **Traces on $\widehat{TL}_3(q)$.** In this subsection, we parametrize all the traces over $\widehat{TL}_3(q)$, which are invariant under the action of the Dynkin automorphism $\psi_3$.

The affine Temperley-Lieb algebra in three generators $g_{\sigma_1}, g_{\sigma_2}$ and $g_{a_3}$ can be presented by those generators with the relations of Hecke algebra, together with:

$$V(g_{\sigma_1}, g_{\sigma_2}) = V(g_{\sigma_1}, g_{a_3}) = V(g_{\sigma_2}, g_{a_3}) = 0.$$

We consider the same change of generators as in the case of $\widehat{TL}_2(q)$. Hence, $f_{\sigma_i} = \frac{g_{\sigma_i}+1}{q+1}$ with $g_{\sigma_i} = (q+1)f_{\sigma_i} - 1$ for $i = 1, 2$ the same for $f_{a_3}$. $\widehat{TL}_3(q)$ is presented by these three generators and the following relations:

$$f_{\sigma_i}^2 = f_{\sigma_i} \quad \text{for } i = 1, 2 \text{ and } f_{a_3}^2 = f_{a_3},$$

$$f_{\sigma_i} f_{a_3} f_{\sigma_i} = \delta f_{\sigma_i} \text{ and } f_{a_3} f_{\sigma_i} f_{a_3} = \delta f_{a_3}, \quad \text{for } i = 1, 2$$

$$f_{\sigma_1} f_{\sigma_2} f_{\sigma_1} = \delta f_{\sigma_1} \text{ and } f_{\sigma_2} f_{\sigma_1} f_{\sigma_2} = \delta f_{\sigma_2},$$

here we will use the $K$-basis $\Big\{ f_w; w \in W^c(\tilde{A}_2) \Big\}$.

**Lemma 4.2.** *Let $h$ and $k$ be two positive integers. Then:*

$$\Big(f_{\sigma_2\sigma_1 a_3}\Big)^k \Big(f_{\sigma_1\sigma_2 a_3}\Big)^h = \begin{cases} \delta^{3h} \Big(f_{\sigma_2\sigma_1 a_3}\Big)^{k-h} & \text{for } h < k. \\ \delta^{3k-1} f_{\sigma_2 a_3} \Big(f_{\sigma_1\sigma_2 a_3}\Big)^{h-k} & \text{for } h \geq k. \end{cases}$$

$$\Big(f_{\sigma_1\sigma_2 a_3}\Big)^h \Big(f_{\sigma_2\sigma_1 a_3}\Big)^k = \begin{cases} \delta^{3k} \Big(f_{\sigma_1\sigma_2 a_3}\Big)^{h-k} & \text{for } h > k. \\ \delta^{3h-1} f_{\sigma_1 a_3} \Big(f_{\sigma_2\sigma_1 a_3}\Big)^{k-h} & \text{for } h \leq k. \end{cases}$$

*Proof.* By induction, with a direct computation the Lemma follows. □

Now we parametrize all the traces on $\widehat{TL}_3(q)$, which are invariant by the Dynkin automorphism $\psi_3$. We know that any element of the $K$-basis $\Big\{ f_w; w \in W^c(\tilde{A}_2) \Big\}$ can be written as follows see [6]:

$$\begin{array}{c} 1 \\ f_{a_3} \\ f_{\sigma_1 a_3} \end{array} \Bigg\rangle (f_{\sigma_2\sigma_1 a_3})^k \Bigg\langle \begin{array}{c} 1 \\ f_{\sigma_2} \\ f_{\sigma_2\sigma_1} \end{array} \qquad \text{or} \qquad \begin{array}{c} 1 \\ f_{a_3} \\ f_{\sigma_2 a_3} \end{array} \Bigg\rangle (f_{\sigma_1\sigma_2 a_3})^k \Bigg\langle \begin{array}{c} 1 \\ f_{\sigma_1} \\ f_{\sigma_1\sigma_2} \end{array}$$

**Lemma 4.3.** *Let $k$ be a positive integer, then for any $w$, such that $l(w) = 3k$, the element $f_w$ is the image, under some power of the Dynkin automorphism $\psi_3$, of one of the following* *elements* $(f_{\sigma_2\sigma_1 a_3})^k$ *or* $(f_{\sigma_1\sigma_2 a_3})^k$. *Similarly for any* $u$ *of length* $3k+1$ *(resp.* $3k+2$*), the element* $f_u$ *is the image under a power of* $\psi_3$ *of one of the following elements* $(f_{\sigma_2\sigma_1 a_3})^k f_{\sigma_2}$ *or* $(f_{\sigma_1\sigma_2 a_3})^k f_{\sigma_1}\Big($*resp.* $(f_{\sigma_2\sigma_1 a_3})^k f_{\sigma_2\sigma_1}$ *or* $(f_{\sigma_1\sigma_2 a_3})^k f_{\sigma_1\sigma_2}\Big)$.

$$(f_{\sigma_2\sigma_1 a_3})^k \left\langle \begin{array}{l} 1 \\ f_{\sigma_2} \\ f_{\sigma_2\sigma_1} \end{array} \right. \qquad \text{and} \qquad (f_{\sigma_1\sigma_2 a_3})^k \left\langle \begin{array}{l} 1 \\ f_{\sigma_1} \\ f_{\sigma_1\sigma_2} \end{array} \right.$$

*Proof.* The proof is direct, by induction over $k$. $\square$

**Proposition 4.4.** *For* $i \geq 1$*, let* $B_0, B_1, B_2$ *and* $\beta_i$ *be in* $K$*. Then, there exists a unique,* $\psi_3$*-invariant, trace over* $\widehat{TL}_3(q)$*, say* $s$*, such that:* $B_0 = s(1)$, $B_1 = s(f_{\sigma_1})$, $B_2 = s(f_{\sigma_1\sigma_2})$, $\beta_1 = s(f_{\sigma_1\sigma_2 a_3})$, $\beta_k = s((f_{\sigma_1\sigma_2 a_3})^k f_{\sigma_1})$ *and* $\beta_k = \frac{1}{\delta} s((f_{\sigma_1\sigma_2 a_3})^k f_{\sigma_1\sigma_2})$*, for* $k \geq 1$.

*Proof.* For the existence, we consider the following linear map, we can show, using Lemma 4.3, that it is indeed a $\psi_3$-invariant trace.

$$\begin{aligned} s \text{ is given as follows, } s : \widehat{TL}_3(q) &\longrightarrow K \\ s(1) &= B_0, \\ s(f_{\sigma_1}) &= s(f_{\sigma_2}) = s(f_{a_3}) = B_1, \\ s(f_u) &= B_2 \text{ for any } u \text{ in } W^c(\tilde{A}_2) \text{ with } l(u) = 2, \end{aligned}$$

$$\text{and } s(f_v) = \begin{cases} \beta_k & \text{when } \ l(v) = 3k, \ \text{ or } \ l(v) = 3k+1. \\ \delta\beta_k & \text{when } \ l(v) = 3k+2, \quad \text{ for } \ k \geq 1. \end{cases}$$

Where $\beta_k$ (for $1 \leq k$), $B_0, B_1$ and $B_2$ are arbitrary in the field $K$. While for the uniqueness, we follow the steps of the proof of Proposition 4.1

$\square$

4.3. **Markov elements.** We consider $F_n : \widehat{TL}_n(q) \longrightarrow \widehat{TL}_{n+1}(q)$ of Proposition 3.1. In this subsection we set $F := F_n$. We give a definition of Markov elements in $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then we show that any trace over $\widehat{TL}_{n+1}(q)$ is uniquely determined by its values on those elements.

**Definition 4.5.** *For* $F$ *as above, and* $n \geq 2$*, a Markov element in* $\widehat{TL}_{n+1}(q)$ *is any element of the form* $A g_{\sigma_n}^{\epsilon} B$*, where* $A$ *and* $B$ *are in* $F(\widehat{TL}_n(q))$ *and* $\epsilon \in \{0, 1\}$.

The aim of this subsection is to prove the following Theorem.

**Theorem 4.6.** *Let $\tau_{n+1}$ be any trace over $\widehat{TL}_{n+1}(q)$ for $2 \leq n$. Then, $\tau_{n+1}$ is uniquely defined by its values on the Markov elements in $\widehat{TL}_{n+1}(q)$.*

The proof of Theorem 4.6 is divided into two parts. In the first we show some general facts, in the second we prove the above Theorem for $3 \leq n$, while for $n = 2$ we will not give the proof, as it is pretty long and available in [8].

**Part 1**

In this part, we suppose that $\tau_{n+1}$ is any trace on $\widehat{TL}_{n+1}(q)$. We will apply $\tau_{n+1}$ to $\widehat{TL}_{n+1}(q)$ assuming that $2 \leq n$, and show that $\tau_{n+1}$ is uniquely determined on $\widehat{TL}_{n+1}(q)$ by its values on the positive powers of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, in addition to its values on Markov elements. From now on we denote by $w$: an arbitrary element in $W^c(\tilde{A}_n)$.

**Lemma 4.7.** *In $\widehat{TL}_{n+1}(q)$ we have:*

$$(1)\ g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k + \sum_{i=1}^{i=k-1} f_i (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i + A\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k g_{\sigma_n} \prod_{j=0}^{j=k-1} \Psi^j\Big(F((t_{a_n})^{-1})\Big),$$

$$(2)\ (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n} = (q-1)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k + \sum_{i=1}^{i=k-1} h_i (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i + A \prod_{j=0}^{j=k-1} \Psi^{-j}\Big((g_{\sigma_{n-1}})^{-1}\Big) g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F(t_{a_n})\Big)^k,$$

*with $A$ in the ground field and $f_i, h_i$ in $F(\widehat{TL}_n(q))$.*

*Proof.*

$$\begin{aligned} g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k &= \Big(q-1\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k \\ &\quad + q g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F\Big(t_{a_n}\Big) g_{\sigma_n} F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1} \\ &= \Big(q-1\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k + \\ &\quad q g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F\Big(t_{a_n}\Big) g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1} \Psi^{k-1}\Big(F((t_{a_n})^{-1})\Big). \end{aligned}$$

So, by induction on $k$, (1) follows. In the very same way we deal with (2), by noticing that: $g_{a_{n+1}} g_{\sigma_n} = g_{\sigma_n}^{-1} F(t_{a_n}) g_{\sigma_n}^2 = (q-1) g_{a_{n+1}} + q g_{\sigma_n}^{-1} F(t_{a_n})$.

□

A main result in [6] is to give a general form for "fully commutative braids", from which we deduce that any element of the basis of $\widehat{TL}_{n+1}(q)$ (where we have the convention $\sigma_{n+1} = 1$ in $W(\tilde{A_n})$ thus $g_{\sigma_n\sigma_{n-1}..\sigma_i} = 1$ when $i = n+1$), is either of the form

$$c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}$$

or of the form

$$g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k d g_{\sigma_n\sigma_{n-1}..\sigma_i}$$

where $c$ and $d$ are in $F(\widehat{TL}_n(q))$, $1 \leq i \leq n+1$ and $0 \leq i_0 \leq n-1$ .

By Lemma 4.7 $c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}$ is of the form:

$$\sum_{j=1}^{j=h} c_j(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^j + M.$$

Where $h \leq k$, $c_j$ is in $F(\widehat{TL}_n(q))$ for any $j$ and $M$ is a Markov element.

Now we deal with the second form:

$$\tau_{n+1}\Big(g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_i}\Big) = \tau_{n+1}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_i} g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big).$$

For any possible value for $i_0$ or $i$, we see that:

$$g_{\sigma_n\sigma_{n-1}..\sigma_i} g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}} c(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = c' g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^s c'',$$

where $c', c''$ are in $F(\widehat{TL}_n(q))$ and $s \leq k+1$. By Lemma 4.7 we see that this element is of the form:

$$\sum_{j=1}^{j=h} f_j(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^j + M,$$

where $h \leq k+1$, $f_j$ is in $F(\widehat{TL}_n(q))$ for any $j$ and $M$ is a Markov element .

Hence, we see that in order to define $\tau_{n+1}$ uniquely it is enough to have its values on Markov elements and its values on $\Omega(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, where $1 \leq k$ (since if $k$ is equal to 0 then we are again in the case of a Markov element) and $\Omega$ is in $F\Big(\widehat{TL}_n(q)\Big)$.

**Lemma 4.8.** *Let $2 \leq n$ then $\tau_{n+1}$ is uniquely defined by its values on Markov elements, in addition to its values on $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, with $0 \leq k$ .*

*Proof.* In order to determine $\tau_{n+1}\Big(h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$, with a positive $k$ and an arbitrary $h$ in $F\Big(\widehat{TL}_n(q)\Big)$, it is enough to treat $\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$, with $x$ in $W^c(\tilde{A_{n-1}})$, but the fact that $\tau_{n+1}$ is a trace, in addition to the fact that $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ acts as a Dynkin automorphism on $F\Big(\widehat{TL}_n(q)\Big)$, authorizes us to suppose that $x$ has a reduced expression which ends with $\sigma_{n-1}$.

Now we show by induction on $l(x)$, that $\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)$ is a sum of values of $\tau_{n+1}$ over $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$, elements of the form $h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^i$ with $i<k$ and Markov elements, (of course with coefficients in the ground ring which might be zeros).

For $l(x)=0$ the property is true. Take $l(x)>0$, and let $x=z\sigma_{n-1}$ be a reduced expression, hence:

$$\begin{aligned}\tau_{n+1}\Big(F(t_x)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big) &= \tau_{n+1}\Big(F(t_z)F(t_{\sigma_{n-1}})g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\ &= \tau_{n+1}\Big(F(t_z)\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}_{=-V(g_{\sigma_{n-1}},g_{\sigma_n})}g_{\sigma_{n-2}..\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big).\end{aligned}$$

Recalling that $V(g_{\sigma_{n-1}},g_{\sigma_n})=0$, this is equal to the following sum:

$$\begin{aligned}&-\tau_{n+1}\Big(F(t_z)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}g_{a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-2}..\sigma_1 a_{n+1}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}\underbrace{g_{\sigma_n}g_{a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\\ &-\tau_{n+1}\Big(F(t_z)g_{\sigma_{n-2}..\sigma_1}\underbrace{g_{\sigma_n}g_{a_{n+1}}}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big).\end{aligned}$$

Now we apply the induction hypothesis to the first term. The second and the third terms are equal to:

$$\begin{aligned}&\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1}\right)\\ &\quad+\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^{k-1}\right),\end{aligned}$$

which is equal to:

$$\begin{aligned}&\tau_{n+1}\left(\Psi^{1-k}\Big(F\Big((t_{a_n})^{-1}\Big)\Big)F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right)\\ &\quad+\tau_{n+1}\left(\Psi^{1-k}\Big(F\Big((t_{a_n})^{-1}\Big)\Big)F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right).\end{aligned}$$

The fourth and the fifth terms are equal to:

$$\tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-1}}g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right)$$

$$+\ \tau_{n+1}\left(F\Big(t_z\Big)g_{\sigma_{n-2}..\sigma_1}F\Big(t_{a_n}\Big)\Big(g_{\sigma_n}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}\Big)\right).$$

Thus, Lemma 4.7 tells us that the property is true for those four terms. This step is to be applied repeatedly, to the powers of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ down to an element of the form $\tau_{n+1}\Big(h(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^1\Big)$, arriving to the sum of:

$$\tau_{n+1}(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})$$

and

$$\tau_{n+1}(h' g_{\sigma_{n-1}..\sigma_1 a_{n+1}}),$$

which is the sum of values of $\tau_{n+1}$ on Markov elements, since $h, h' \in F\Big(\widehat{TL}_n(q)\Big)$. □

We end this part by the following Lemma:

**Lemma 4.9.** *Let $1 \leq k$. Then $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k$ is a sum of two kinds of elements:*

(1) $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^j g_{\sigma_n}h$, *with $j \leq k$.*
(2) $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^i g_{\sigma_n}f$, *with $i < k$,*

*with $h, f$ in $F\Big(\widehat{TL}_n(q)\Big)$ and $2 \leq n$.*

*Moreover, in the first type we have one, and only one element, with $j = k$, in which we have:*

$$h = \prod_{i=0}^{i=k-1}\Psi^{-i}\Big(F(t_{a_n}^{-1})\Big).$$

*Proof.* Suppose that $k = 1$. Then,

$$g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}} = g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)g_{\sigma_n}F\Big(t_{a_n}\Big)^{-1},$$

the property is true.

Suppose the property is true for $k-1$, then, with $2 \leq k$, we have:

$$(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k = (g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}.$$

We apply the property to $(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^{k-1}$, which gives two cases:

(1) $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}g_{\sigma_n}hg_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, with $j'\leq k-1$ which is: $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Psi^{-1}\Big(h\Big)$, which is equal to:

$$qg_{\sigma_n}\ \Big(\ g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'+1}g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Psi^{-1}\Big(h\Big)$$

$$+\,(q-1)g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Psi^{-1}\left(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\right)\Psi^{-1}\left(h\right).$$

Since, $j'+1\leq k$, the first term is clear to be of the first type, while the second term is equal to:

$$\Big(q-1\Big)qg_{\sigma_{n-1}..\sigma_1}F\Big(t_{a_n}\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Psi^{-1}\Big(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\Big)\Psi^{-1}\Big(h\Big)+$$

$$\Big(q-1\Big)^2g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Psi^{-1}\Big(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{j'}\Big)\Psi^{-1}\Big(h\Big).$$

Here, the first term is of the second type (with $i=1<k$), and the second term is of the first type (with $j=1$).

(2) $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'}g_{\sigma_n}fg_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$, with $i'<k-1$, which is:

$$\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'}g_{\sigma_n}g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Psi^{-1}\Big(f\Big)=$$

$$q\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'+1}g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Psi^{-1}\Big(f\Big)+$$

$$\Big(q-1\Big)g_{\sigma_n}\Big(g_{\sigma_{n-1}..\sigma_1}F(t_{a_n})\Big)g_{\sigma_n}F\Big((t_{a_n})^{-1}\Big)\Psi^{-1}\Big(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^{i'}\Big)\Psi^{-1}\Big(f\Big).$$

Since $i'+1<k$, the first term is of the second type, while the second term is of the first type with $j=1$. The Lemma is proven.

(By induction over $k$ again, the last formula is easy).

□

**Part 2**

In this part we treat Theorem4.6 when $n\geq 3$. As said at the beginning of Part 1, for $n\geq 3$, and by sending the "fully commutative braids" onto $\widehat{TL}_n(q)$, we get that any element of the basis of $\widehat{TL}_{n+1}(q)$ is a linear combination of two kinds of elements, namely:

$$I=F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

$$II=g_{\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}}F_n(t_u)(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

here, $u$ is in $W^c(\tilde{A}_{n-1})$, where $1 \leq s \leq n+1$ with $0 \leq i_0 \leq n-1$ and $0 \leq k$.

Using Lemma 4.7 we see that:

$$\begin{aligned}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k g_{\sigma_n} = (q-1)&\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k \\ &+ \sum_{i=1}^{i=k-1} h_i\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i \\ &+ A\prod_{j=0}^{j=k-1}\Psi^{-j}\Big(\Big(g_{\sigma_{n-1}}\Big)^{-1}\Big)g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k,\end{aligned}$$

but, $I = F_n(t_u)\underbrace{(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}})^k g_{\sigma_n}}\, g_{\sigma_{n-1}..\sigma_s}$, that is:

$$\begin{aligned}I = (q-1)&F_n\Big(t_u\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k g_{\sigma_{n-1}..\sigma_s} \\ &+ \sum_{i=1}^{i=k-1} h_i F_n\Big(t_u\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i g_{\sigma_{n-1}..\sigma_s} \\ &+ A\prod_{j=0}^{j=k-1} F_n\Big(t_u\Big)\Psi^{-j}\Big(\Big(g_{\sigma_{n-1}}\Big)^{-1}\Big)g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_{n-1}..\sigma_s}.\end{aligned}$$

Using the action of $g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}$ on $F_n(\widehat{TL}_n(q))$, we see that:

$$I = \sum_{i=1}^{i=k} F_n\Big(t_{b_i}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i + \sum_j F_n\Big(t_{b_j}\Big)g_{\sigma_n}F_n\Big(t_{d_j}\Big),$$

where $b_j$, $c_j$ and $d_i$ are in $W^c(\tilde{A}_{n-1})$, for every $i$ and $j$.

Now, we see, as well, that:

$$\begin{aligned}II &= g_{\sigma_{i_0}..\sigma_2\sigma_1}g_{a_{n+1}}F_n\Big(t_u\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k g_{\sigma_n\sigma_{n-1}..\sigma_s} \\ &= g_{\sigma_{i_0}..\sigma_2\sigma_1}F_n\Big(t_{a_n}\Big)\underbrace{g_{\sigma_n}\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k}\Psi^k\Big(F_n\Big(t_{a_n}^{-1}\Big)F_n\Big(t_u\Big)\Big)g_{\sigma_n\sigma_{n-1}..\sigma_s},\end{aligned}$$

since $g_{a_{n+1}} = F_n(t_{a_n})g_{\sigma_n}F_n(t_{a_n}^{-1})$.

By Lemma 4.7, we see that $II$ is equal to:

$$(q-1)g_{\sigma_{i_0}..\sigma_2\sigma_1}F_n\Big(t_{a_n}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^k\Psi^k\Big(F_n\Big(t_{a_n}^{-1}\Big)F_n(t_u)\Big)g_{\sigma_n\sigma_{n-1}..\sigma_s}$$

$$+\sum_{i=1}^{i=k-1} g_{\sigma_{i_0}..\sigma_2\sigma_1}F_n\Big(t_{a_n}\Big)f_i\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i\Psi^k\Big(F_n(t_{a_n}^{-1})F_n(t_u)\Big)g_{\sigma_n\sigma_{n-1}..\sigma_s}$$

$$+Ag_{\sigma_{i_0}..\sigma_2\sigma_1}F_n\Big(t_{a_n}\Big)\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}\prod_{j=0}^{j=k-1}\Psi^j\Big(F\Big((t_{a_n})^{-1}\Big)\Big)\Psi^k\Big(F_n(t_{a_n}^{-1})F_n(t_u)\Big)g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

which is equal to:

$$\sum_{i=1}^{i=k}F_n\Big(t_{x'_i}\Big)\underbrace{\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i g_{\sigma_n}}\,g_{\sigma_{n-1}..\sigma_s}$$

$$+Ag_{\sigma_{i_0}..\sigma_2\sigma_1}F_n\Big(t_{a_n}\Big)\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}\prod_{j=0}^{j=k-1}\Psi^j\Big(F\Big((t_{a_n})^{-1}\Big)\Big)\Psi^k\Big(F_n(t_{a_n}^{-1})F_n(t_u)\Big)g_{\sigma_n\sigma_{n-1}..\sigma_s},$$

where $x'_i$ is in $W^c(\tilde{A_{n-1}})$ for all $i$.

Now we repeat the same step as for $I$, to get the next corollary.

**Corollary 4.10.** *Let $3 \leq n$. Let $w$ be in $W^c(\tilde{A}_n)$ .*

*Then there exist $0 \leq k$ and $1 \leq s \leq n+1$, there exist $x_i$ , $y_j$ and $z_j$ in $W^c(\tilde{A_{n-1}})$ $\Big[$With the convention $W^c(\tilde{A}_2) = W(\tilde{A}_2)\Big]$ such that:*

$$g_w = \sum_{i=1}^{i=k}F_n\Big(t_{x_i}\Big)\Big(g_{\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}}\Big)^i + \sum_j F_n\Big(t_{y_j}\Big)g_{\sigma_n}F_n\Big(t_{z_j}\Big)g_{\sigma_n\sigma_{n-1}..\sigma_s}.$$

Now we suppose that $3 \leq n$. Consider the following sequence:

$$\widehat{TL}_{n-1}(q) \longrightarrow \widehat{TL}_n(q) \longrightarrow \widehat{TL}_{n+1}(q).$$

We keep using $t_{\sigma_i}$ (resp. $g_{\sigma_i}$) as generators of $\widehat{TL}_n(q)$ (resp. $\widehat{TL}_{n+1}(q)$). We use $e_{\sigma_i}$ for $\widehat{TL}_{n-1}(q)$. With a simple computation, we see that $g_{\sigma_n}$ commutes with $F_nF_{n-1}(e_{\sigma_i})$, for $1 \leq i \leq n-2$, and with $F_nF_{n-1}(e_{a_{n-1}})$, hence it commutes with every element in $F_nF_{n-1}(\widehat{TL}_{n-1}(q))$.

Lemma 4.8 and Lemma 4.9 confirm that $\tau_{n+1}$ is uniquely defined over $\widehat{TL}_{n+1}(q)$ by its values on $g_{\sigma_n}(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n}))^k g_{\sigma_n}h$, for a positive $k$ and an arbitrary $h$ in $F(\widehat{TL}_n(q))$ beside its values on Markov elements. In other terms: $\tau_{n+1}$ is uniquely defined over $\widehat{TL}_{n+1}(q)$

by its values over $g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(t_v\Big)$, with a positive $k$ and an arbitrary $v$ in $W^c(\tilde{A_{n-1}})$, besides the values on Markov elements.

$$\text{Set } I := \tau_{n+1}\Bigg(g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(t_v\Big)\Bigg),$$

by Corollary 4.10 we see that:

$$\begin{aligned} t_v = & \sum_{i=1}^{i=h} \underbrace{F_{n-1}\Big(e_{x_i}\Big)\Big(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n}\Big)^i}_{C} \\ & + \sum_j \underbrace{F_{n-1}\Big(e_{y_j}\Big)t_{\sigma_{n-1}}F_{n-1}\Big(e_{z_j}\Big)t_{\sigma_{n-1}\sigma_{n-2}..\sigma_s}}_{B} \\ & + \sum_j \underbrace{F_{n-1}\Big(e_{y'_j}\Big)t_{\sigma_{n-1}}F_{n-1}\Big(e_{z'_j}\Big)}_{A}, \end{aligned}$$

where $0 \le h$ and $1 \le s \le n-1$. With $x_i$ , $y_j$, $z_j$, $y'_j$ and $z'_j$ are in $W^c(\tilde{A_{n-2}})$.

We have added the third term $C$ to the two terms of Corollary 4.10, because we had to take into account here, the case of $s = n+1$, i.e., $g_{\sigma_{n+1}} = 1$ for $W^c(\tilde{A_{n-1}})$.

For terms of **Type (A)**, we see that:

$$\begin{aligned} I_1 := & \tau_{n+1}\Bigg(g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F_n(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y'_j})t_{\sigma_{n-1}}F_{n-1}(e_{z'_j})\Big)\Bigg) \\ = \ & \tau_{n+1}\Bigg(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F_n(t_{a_n})\Big)^k F_n\Big(F_{n-1}(e_{y'_j})\Big)g_{\sigma_n}F_n\Big(t_{\sigma_{n-1}}\Big)g_{\sigma_n}F_n\Big(F_{n-1}(e_{z'_j})\Big)\Bigg) \\ = & \tau_{n+1}\Bigg(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k F_n\Big(F_{n-1}(e_{y'_j})\Big)\underbrace{g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n}}_{r}F_n\Big(F_{n-1}(e_{z'_j})\Big)\Bigg) \\ = & \tau_{n+1}\Bigg(\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k F_n\Big(F_{n-1}(e_{y'_j})\Big)\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}_{r}F_n\Big(F_{n-1}(e_{z'_j})\Big)\Bigg), \end{aligned}$$

which is clearly, the sum of values of $\tau_{n+1}$ on Markov elements.

For terms of **Type (B)**, we see that:

$$I_2 := \tau_{n+1}\Big[g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y_j})t_{\sigma_{n-1}}F_{n-1}(e_{z_j})t_{\sigma_{n-1}\sigma_{n-2}..\sigma_s}\Big)\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y_j})t_{\sigma_{n-1}}F_{n-1}(e_{z_j})t_{\sigma_{n-1}}t_{\sigma_{n-2}..\sigma_s}\Big)\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}F_nF_{n-1}\Big(e_{\sigma_{n-2}..\sigma_s}\Big)\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(F_{n-1}(e_{y_j})t_{\sigma_{n-1}}F_{n-1}(e_{z_j})t_{\sigma_{n-1}}\Big)\Big].$$

Now, we set ${}_r^mF := F_mF_{m-1}..F_r$.

We call $\delta$ the image of $F_{n-1}(e_{\sigma_{n-2}..\sigma_s})$ under the action of $\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k$, thus:

$$I_2 = \tau_{n+1}\Big[g_{\sigma_n}\Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1}F(t_{a_n})\Big)^k g_{\sigma_n}F_n\Big(\delta\Big)\Big({}_{n-1}^{n}F(e_{y_j})\Big)g_{\sigma_{n-1}}\Big({}_{n-1}^{n}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}\Big(F_n(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})\Big)^k g_{\sigma_n}F_n\Big(\delta\Big)\Big({}_{n-1}^{n}F(e_{y_j})\Big)g_{\sigma_{n-1}}\Big({}_{n-1}^{n}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big].$$

Now consider $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k$. We apply Lemma 4.9 to this element in $\widehat{TL}_n(q)$, hence, it is the sum of two kind of elements: (1) Markov elements (2) elements of the form $t_{\sigma_{n-1}}(e_{\sigma_{n-2}\sigma_{n-3}..\sigma_1 a_{n-1}})^j t_{\sigma_{n-1}}\delta$, where $j \le k$, and $\delta$ in $F_{n-1}\Big(\widehat{TL}_{n-1}(q)\Big)$. In the case (1) we are done. If we are in case (2), then we apply the Lemma 4.9 on $(e_{\sigma_{n-2}\sigma_{n-3}..\sigma_1 a_{n-1}})^j$. We keep going in the same manner, by applying Lemma 4.9 repeatedly (in fact $n-2$ times), we arrive to:

$$t_{\sigma_{n-1}}t_{\sigma_{n-2}}..t_{\sigma_2}\ \Big(\ F_{n-1}F_{n-2}..F_2({}^2g_{\sigma_1 a_2})^j\Big)t_{\sigma_2}..t_{\sigma_{n-2}}t_{\sigma_{n-1}}\lambda$$

$$= t_{\sigma_{n-1}}t_{\sigma_{n-2}}..t_{\sigma_2}\Big({}_2^{n-1}F({}^2g_{\sigma_1 a_2})^j\Big)t_{\sigma_2}..t_{\sigma_{n-2}}t_{\sigma_{n-1}}\lambda,$$

where $\lambda$ is in ${}_{n-1}^{n}F(\widehat{TL}_{n-1}(q))$. Leaving aside Markov elements, we get:

$$I_2 = \tau_{n+1}\Big[g_{\sigma_n}F_n\Big(t_{\sigma_{n-1}}t_{\sigma_{n-2}}..t_{\sigma_2}\Big({}_2^{n-1}F({}^2g_{\sigma_1 a_2})^j\Big)t_{\sigma_2}..t_{\sigma_{n-2}}t_{\sigma_{n-1}}\lambda\Big)g_{\sigma_n}F_n\Big(\delta\Big)$$
$$\Big({}_{n-1}^{n}F(e_{y_j})\Big)g_{\sigma_{n-1}}\Big({}_{n-1}^{n}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big]$$

$$= \tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F({}^2g_{\sigma_1 a_2})^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}F_n(\lambda\delta)$$
$$\Big({}_{n-1}^{n}F(e_{y_j})\Big)g_{\sigma_{n-1}}\Big({}_{n-1}^{n}F(e_{z_j})\Big)g_{\sigma_{n-1}}\Big].$$

We set $M' := F_n\Big(\lambda\delta\Big)\Big(\prod_{n-1}^{n} F(e_{y_j})\Big) g_{\sigma_{n-1}} \Big(\prod_{n-1}^{n} F(e_{z_j})\Big)$, which is a Markov element in $\widehat{TL}_{n-1}(q)$. Hence, we have:

$$\tau_{n+1}\left[ g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{n-2}} .. g_{\sigma_2} \Big(\prod_{2}^{n-1} F({}^2 g_{\sigma_1 a_2})^j\Big) g_{\sigma_2} .. g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' g_{\sigma_{n-1}} \right]$$

$$= \tau_{n+1}\left[ \underbrace{g_{\sigma_{n-1}} g_{\sigma_n} g_{\sigma_{n-1}}} g_{\sigma_{n-2}} .. g_{\sigma_2} \Big(\prod_{2}^{n-1} F({}^2 g_{\sigma_1 a_2})^j\Big) g_{\sigma_2} .. g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' \right].$$

We apply the TL relations. The cases corresponding to 1 and $g_{\sigma_{n-1}}$ are obvious.

For the terms corresponding to $g_{\sigma_{n-1}} g_{\sigma_n}$, we have:

$$\tau_{n+1}\left[ g_{\sigma_{n-1}} g_{\sigma_n} \underbrace{g_{\sigma_{n-2}} .. g_{\sigma_2} \Big(\prod_{2}^{n-1} F({}^2 g_{\sigma_1 a_2})^j\Big) g_{\sigma_2} .. g_{\sigma_{n-2}}} g_{\sigma_{n-1}} g_{\sigma_n} M' \right]$$

$$= \tau_{n+1}\left[ g_{\sigma_{n-1}} g_{\sigma_{n-2}} .. g_{\sigma_2} \Big(\prod_{2}^{n-1} F({}^2 g_{\sigma_1 a_2})^j\Big) g_{\sigma_2} .. g_{\sigma_{n-2}} \underbrace{g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_n}} M' \right].$$

We are done, since it is a sum of values of $\tau_{n+1}$ on Markov elements, (the same for the term corresponding to $g_{\sigma_n}$) .

For the terms corresponding to $g_{\sigma_n} g_{\sigma_{n-1}}$, we have:

$$\tau_{n+1}\left[ g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{n-2}} .. g_{\sigma_2} \Big(\prod_{2}^{n-1} F({}^2 g_{\sigma_1 a_2})^j\Big) g_{\sigma_2} .. g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_n} M' \right],$$

which is the case of term (A), since $M'$ is a Markov element in $\widehat{TL}_{n-1}(q)$.

For terms of **Type (C)**, we see that:

$$I_3 := \tau_{n+1}\left[ g_{\sigma_n} \Big(g_{\sigma_{n-1}\sigma_{n-2}..\sigma_1} F_n(t_{a_n})\Big)^k g_{\sigma_n} F_n\Big(F_{n-1}(e_{x_i})(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i\Big) \right]$$

$$= \tau_{n+1}\left[ g_{\sigma_n} F_n\Big((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k\Big) g_{\sigma_n} F_n\Big(F_{n-1}(e_{x_i})(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i\Big) \right].$$

Call $\gamma$ the image of $F_{n-1}(e_{x_i})$ under the action of $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i$. Thus:

$$I_3 = \tau_{n+1}\left[ g_{\sigma_n} F_n\Big((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k\Big) g_{\sigma_n} F_n\Big((t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i \gamma\Big) \right].$$

As we have seen in the case (B), $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^k$ can be written as a sum of elements of the form (up to Markov elements in $\widehat{TL}_n(q)$ ):

$$t_{\sigma_{n-1}} t_{\sigma_{n-2}} .. t_{\sigma_2} \Big(\prod_{2}^{n-1} F({}^2 g_{\sigma_1 a_2})^j\Big) t_{\sigma_2} .. t_{\sigma_{n-2}} t_{\sigma_{n-1}} \lambda,$$

where $j \le k$, and $\lambda$ is in ${}_{n-1}^{\;n}F\Big(\widehat{TL}_{n-1}(q)\Big)$.

Call $\eta$ the image of $\lambda$ under the action of $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i$.

The determination of $I_3$ can be reduced to computing the following value:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}F_n\Big(\Big(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n}\Big)^i\eta\gamma\Big)\Big].$$

We repeat the same algorithm to $(t_{\sigma_{n-1}\sigma_{n-2}..\sigma_1 a_n})^i$. Hence, we get some $l \le i$, and some $\Delta$ in ${}_{n-1}^{\;n}F\Big(\widehat{TL}_{n-1}(q)\Big)$, such that we are reduced to compute:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}\underbrace{g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-1}}}g_{\sigma_{n-2}}..g_{\sigma_2}$$
$$\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}\Delta\Big].$$

We see, after using the T-L relations, that the terms corresponding to 1 and $g_{\sigma_{n-1}}$ are values of $\tau_{n+1}$ on Markov elements.

The term corresponding to $g_{\sigma_{n-1}}g_{\sigma_n}$ is:

$$\tau_{n+1}\Big[g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_n}g_{\sigma_{n-2}}..g_{\sigma_2}$$
$$\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}\Delta\Big]$$
$$= \tau_{n+1}\Big[g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^j\Big)g_{\sigma_2}..g_{\sigma_{n-2}}g_{\sigma_{n-1}}g_{\sigma_{n-2}}..g_{\sigma_2}$$
$$\Big({}_2^{n-1}F\big({}^2g_{\sigma_1 a_2}\big)^l\Big)g_{\sigma_2}..g_{\sigma_{n-2}}\underbrace{g_{\sigma_n}g_{\sigma_{n-1}}g_{\sigma_n}}\Delta\Big].$$

The term in square brackets is clearly a Markov element (the same thing with the term corresponding to $g_{\sigma_n}$) .

The term corresponding to $g_{\sigma_n} g_{\sigma_{n-1}}$ is:

$$\tau_{n+1}\Bigg[g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_{2}^{n-1}F({}^{2}g_{\sigma_1 a_2})^{j}\Big) g_{\sigma_2}..g_{\sigma_{n-2}} g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_{n-2}}..g_{\sigma_2}$$

$$\Big({}_{2}^{n-1}F({}^{2}g_{\sigma_1 a_2})^{l}\Big) g_{\sigma_2}..g_{\sigma_{n-2}} g_{\sigma_{n-1}}\Delta\Bigg]$$

$$= \tau_{n+1}\Bigg[\underbrace{g_{\sigma_n} g_{\sigma_{n-1}} g_{\sigma_n}} g_{\sigma_{n-2}}..g_{\sigma_2}\Big({}_{2}^{n-1}F({}^{2}g_{\sigma_1 a_2})^{j}\Big) g_{\sigma_2}..g_{\sigma_{n-2}} g_{\sigma_{n-1}} g_{\sigma_{n-2}}..g_{\sigma_2}$$

$$\Big({}_{2}^{n-1}F({}^{2}g_{\sigma_1 a_2})^{l}\Big) g_{\sigma_2}..g_{\sigma_{n-2}} g_{\sigma_{n-1}}\Delta\Bigg].$$

It is a Markov element, Theorem4.6 follows.

## 5. Affine Markov trace: existence and uniqueness

**5.1. Existence.** Now, consider the following commutative diagram:

$$\begin{tikzcd}
\widehat{TL}_1(q) \arrow[r] \arrow[d, twoheadrightarrow] & \widehat{TL}_2(q) \arrow[r] \arrow[d, twoheadrightarrow] & \dots \arrow[r] & \widehat{TL}_n(q) \arrow[r] \arrow[d, twoheadrightarrow] & \widehat{TL}_{n+1}(q) \arrow[d, twoheadrightarrow] \\
TL_0(q) \arrow[r, hookrightarrow] \arrow[drr, "\tau_1"'] & TL_1(q) \arrow[r, hookrightarrow] \arrow[dr, "\tau_2"] & \dots \arrow[r, hookrightarrow] & TL_{n-1}(q) \arrow[r, hookrightarrow] \arrow[dl, "\tau_n"'] & TL_n(q) \arrow[dll, "\tau_{n+1}"] \\
 & & K & &
\end{tikzcd}$$

Set $\rho_{n+1}$ to be the trace over $\widehat{TL}_{n+1}(q)$ induced by $\tau_{n+1}$ over $TL_n(q)$ for $0 \le n$.

**Proposition 5.1.** *Under the above notations, with the notations of Theorem2.1 we have:*

- $\rho_{n+1}(F_n(h)T_{\sigma_n}^{\pm 1}) = \rho_n(h)$, *for all* $h \in \widehat{TL}_n(q)$, *where* $1 \le n$.
- $\rho_i$ *is invariant under the action of* $\phi_i$ *for all* $i$.

*Proof.* We have: $\rho_{n+1}\Big(F_n(h)T_{\sigma_n}^{\pm 1}\Big)$ equals $\tau_{n+1}\Bigg(E_n\Big(F_n(h)\Big)E_n\Big(T_{\sigma_n}^{\pm 1}\Big)\Bigg)$.

$$\text{Hence, } \rho_{n+1}\Big(F_n(h)T_{\sigma_n}^{\pm 1}\Big) = \tau_{n+1}\Bigg(x_n\Big(E_{n-1}(h)\Big)T_{\sigma_n}^{\pm 1}\Bigg) = \tau_n\Big(E_{n-1}(h)\Big) = \rho_n\Big(h\Big).$$

We made use of the fact that the following diagram commutes, together with the fact that $(\tau_n)_{1\leq n}$ is a Markov trace:

$$\begin{tikzcd}
\widehat{TL_n(q)} \arrow[rrrr, "F_n"] \arrow[dr, two heads, "E_{n-1}"] \arrow[ddrr, two heads, "\rho_n"'] & & & & \widehat{TL_{n+1}(q)} \arrow[dl, two heads, "E_n"'] \arrow[ddll, two heads, "\rho_{n+1}"] \\
& TL_{n-1}(q) \arrow[rr, hook, "x_n"] \arrow[dr, two heads, "\tau_n"] & & TL_n(q) \arrow[dl, two heads, "\tau_{n+1}"'] & \\
& & K & &
\end{tikzcd}$$

For the second statement, we show that $\rho_n\Big(h\Big) = \rho_n\Big([h]\Big)$, where $[h]$ is the image of $h$ under $\phi_n^{-1}$. So we start from $\rho_n\Big(h\Big) = \tau_n\Big(E_{n-1}(h)\Big)$. But since $\tau_n$ is the $n$-th Markov trace, we have $\tau_n\Big(E_{n-1}(h)\Big) = -\frac{\sqrt{q}}{1+q}\tau_{n+1}\Big(x_n(E_{n-1}(h))\Big)$, which is equal to $-\frac{\sqrt{q}}{1+q}\tau_{n+1}\Bigg(E_n\Big(F_n(h)\Big)\Bigg)$, since the diagram $T$ commutes, this term is equal to $-\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(F_n(h)\Big)$, hence to:

$$-\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(g_{\sigma_n..\sigma_1 a_{n+1}}F_n(h)g^{-1}_{\sigma_n..\sigma_1 a_{n+1}}\Big) = -\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(F_n([h])\Big).$$

Now we consider the same steps in the opposite direction, that is:

$$-\frac{\sqrt{q}}{1+q}\rho_{n+1}\Big(F_n([h])\Big) = -\frac{\sqrt{q}}{1+q}\tau_{n+1}\Bigg(E_n\Big(F_n([h])\Big)\Bigg) = \rho_n\Big([h]\Big).$$

$\square$

**Corollary 5.2.** *With the above notations, in the sense of Definition 3.3: $(\rho_i)_{1\leq i}$ is an affine Markov trace over $\Big(\widehat{TL}_i(q)\Big)_{1\leq i}$.*

5.2. **Uniqueness.** Consider the following homomorphism:

$$\begin{aligned}
F_2 : \widehat{TL}_2(q) &\longrightarrow \widehat{TL}_3(q)\\
g_{\sigma_1} &\longmapsto g_{\sigma_1}\\
g_{a_2} &\longmapsto g_{\sigma_2}g_{a_3}g^{-1}_{\sigma_2}
\end{aligned}$$

where, for possible lack of injectivity (see the comments preceding Proposition 3.1), we use slanted letters $g$, $f$ in $\widehat{TL}_2(q)$ while we use the usual style $g$, $f$ in $\widehat{TL}_3(q)$.

We set $F := F_2$ in order to simplify in what follows. $F$ can be expressed by the following form considering the "f" generators, we see that $F\Big(f_{a_2}\Big) = F\Big(\frac{g_{a_2}+1}{q+1}\Big)$, which is equal to $\frac{1}{q+1}g_{\sigma_2}g_{a_3}g_{\sigma_2}^{-1} + \frac{1}{q+1}$, hence to:

$$\frac{1}{q+1}\Bigg[\Big((q+1)f_{\sigma_2}-1\Big)\Big((q+1)f_{a_3}-1\Big)\Big(\frac{1}{q}\Big((q+1)f_{\sigma_2}-1\Big)+\frac{1-q}{q}\Big)\Bigg]+\frac{1}{q+1}.$$

Thus, we see that:

$$\begin{aligned} F : \widehat{TL}_2(q) &\longrightarrow \widehat{TL}_3(q) \\ f_1 &\longmapsto f_{\sigma_1} \\ f_{a_2} &\longmapsto -\frac{q+1}{q}f_{a_3\sigma_2} - (q+1)f_{\sigma_2 a_3} + f_{\sigma_2} + f_{a_3}. \end{aligned}$$

Notice that $F(f_{a_2})f_{\sigma_2}F(f_{a_2}) = \delta F(f_{a_2})$, and $f_{\sigma_2}F(f_{a_2})f_{\sigma_2} = \delta f_{\sigma_2}$. Since we are interested with viewing $F\Big(\widehat{TL}_2(q)\Big)$ in $\widehat{TL}_3(q)$, we will investigate in what follows, the elements $\Big(F(f_{\sigma_1}f_{a_2})\Big)^k$ and $\Big(F(f_{a_2}f_{\sigma_1})\Big)^k$, for $k$ a positive integer.

$$\text{Set } x_1 := F(f_{\sigma_1}f_{a_2}) = f_{\sigma_1}F(f_{a_2}) = -\frac{q+1}{q}f_{\sigma_1 a_3\sigma_2} - (q+1)f_{\sigma_1\sigma_2 a_3} + f_{\sigma_1\sigma_2} + f_{\sigma_1 a_3}.$$

And for $1 \le i$, we set:

$$\begin{aligned} x_i := \Big(-1\Big)^i \Big(\frac{q+1}{q}\Big)^i f^i_{\sigma_1 a_3\sigma_2} &+ \Big(-1\Big)^i\Big(q+1\Big)^i f^i_{\sigma_1\sigma_2 a_3} \\ + \Big(-1\Big)^{i-1}\Big(\frac{q+1}{q}\Big)^{i-1} f^{i-1}_{\sigma_1 a_3\sigma_2}f_{\sigma_1 a_3} &+ \Big(-1\Big)^{i-1}\Big(q+1\Big)^{i-1} f^{i-1}_{\sigma_1\sigma_2 a_3}f_{\sigma_1\sigma_2}. \end{aligned}$$

Notice that $x_1^2 = 3\delta x_1 + x_2$. It is easy to show that:

$$x_1x_i = \delta^2 x_{i-1} + 2\delta x_i + x_{i+1}, \text{ for } 2 \le i,$$

thus, for $1 \le k$, we have $x_1^k = \sum_{i=1}^{i=k-1}\gamma_i x_i + x_k$, here $\gamma_i$ is a polynomial in $\delta$, for all $i$.

Notice that $x_1x_j = x_jx_1$ for $j = 1, 2$. For $j = 1$ it is clear, while for $j = 2$ we have $x_2 = x_1^2 - 3\delta x_1$. Now suppose that $3 \le j$. We have $x_j = x_1x_{j-1} - \delta^2 x_{j-2} - 2\delta x_{j-1}$, hence we see by induction on $j$, that $x_1x_j = x_jx_1$, for all $j$ .

We define the $\mathbb{Q}$-linear map $\chi : \widehat{TL}_3(q) \longrightarrow \widehat{TL}_3(q)$ which sends 1 to 1, and for any $u = s_1s_2..s_r$ reduced expression of any element $u$ in $W^c(\tilde{A}_2)$, it sends $f_u$ to $f_{s_rs_{r-1}..s_1}$, with $q$ sent to $\frac{1}{q}$.

Set $z_1 := F(f_{a_2} f_{\sigma_1})$. Then

$$z_1 = F(f_{a_2}) f_{\sigma_1} = -\frac{q+1}{q} f_{a_3\sigma_2\sigma_1} - (q+1) f_{\sigma_2 a_3 \sigma_1} + f_{\sigma_2\sigma_1} + f_{a_3\sigma_1}.$$

And for $1 \leq i$, we set

$$\begin{aligned} z_i: &= \Big(-1\Big)^i \quad \Big(\frac{q+1}{q}\Big)^i \quad f^i_{a_3\sigma_2\sigma_1} \qquad + \Big(-1\Big)^i \Big(q+1\Big)^i f^i_{\sigma_2 a_3\sigma_1} \\ &+ \Big(-1\Big)^{i-1} \Big(\frac{q+1}{q}\Big)^{i-1} f_{\sigma_2\sigma_1} f^{i-1}_{a_3\sigma_2\sigma_1} \quad + \Big(-1\Big)^{i-1}\Big(q+1\Big)^{i-1} f_{a_3\sigma_1} f^{i-1}_{\sigma_2 a_3\sigma_1}. \end{aligned}$$

Notice that $\chi(x_i) = z_i$ for all $i$. Now $\chi(x_1 x_i) = \chi(x_i x_1) = z_1 z_i$. We see that $\chi(\delta) = \delta$. Moreover, $z_1 z_j = \chi(x_1 x_j) = \chi(\delta^2 x_{i-1} + 2\delta x_i + x_{i+1}) = \delta^2 z_{i-1} + 2\delta z_i + z_{i+1}$. And in the same way, by acting by $\chi$, we find that $z_1^k = \sum_{i=1}^{i=k-1} \gamma_i z_i + z_k$, where $\gamma_i$ is as above.

Consider $x_i f_{\sigma_2}$ for $1 \leq i$, we see that it is equal to:

$$\begin{aligned} &\Big(-1\Big)^i \quad \Big(\frac{q+1}{q}\Big)^i \quad f^i_{\sigma_1 a_3\sigma_2} \qquad + \Big(-1\Big)^i \Big(q+1\Big)^i f^i_{\sigma_1\sigma_2 a_3} f_{\sigma_2} \\ &+ \Big(-1\Big)^{i-1} \Big(\frac{q+1}{q}\Big)^{i-1} f^{i-1}_{\sigma_1 a_3\sigma_2} f_{\sigma_1 a_3} f_{\sigma_2} \quad + \Big(-1\Big)^{i-1}\Big(q+1\Big)^{i-1} f^{i-1}_{\sigma_1\sigma_2 a_3} f_{\sigma_1\sigma_2}, \end{aligned}$$

which is:
$$\begin{aligned} &\Big(-1\Big)^i \ \Big(\frac{q+1}{q}\Big)^i \quad f^i_{\sigma_1 a_3\sigma_2} \qquad + \delta\Big(-1\Big)^i \Big(q+1\Big)^i f^{i-1}_{\sigma_1\sigma_2 a_3} f_{\sigma_1\sigma_2} \\ &+ \Big(-1\Big)^i \Big(\frac{q+1}{q}\Big)^{i-1} f^i_{\sigma_1 a_3\sigma_2} \qquad + \Big(-1\Big)^{i-1}\Big(q+1\Big)^{i-1} f^{i-1}_{\sigma_1\sigma_2 a_3} f_{\sigma_1\sigma_2}. \end{aligned}$$

Hence, $x_i f_{\sigma_2} = \Big[(-1)^{i-1} \dfrac{(q+1)^{i-1}}{q^i}\Big] f^i_{\sigma_1 a_3\sigma_2} + \Big[(-1)^{i-1}(q+1)^{i-2}\Big] f^{i-1}_{\sigma_1\sigma_2 a_3} f_{\sigma_1\sigma_2}$, for $1 \leq i$.

In particular $x_1 f_{\sigma_2} = -\dfrac{q+1}{q} f_{\sigma_1 a_3\sigma_2} - (q+1) f_{\sigma_1\sigma_2 a_3} f_{\sigma_2} + f_{\sigma_1\sigma_2} + f_{\sigma_1 a_3} f_{\sigma_2}$,

thus,

$$x_1 f_{\sigma_2} = \frac{-1}{q} f_{\sigma_1 a_3\sigma_2} + \frac{1}{q+1} f_{\sigma_1\sigma_2}.$$

Now we apply $\chi$ to $x_i f_{\sigma_2}$. Hence

$$f_{\sigma_2} z_i = \Big[(-1)^i q (q+1)^{i-1}\Big] f^i_{\sigma_2 a_3\sigma_1} + \Big[(-1)^{i-1} (\frac{q+1}{q})^{i-2}\Big] f_{\sigma_2\sigma_1} f^{i-1}_{a_3\sigma_2\sigma_1}, \text{ for } 1 \leq i.$$

In particular $f_{\sigma_2} z_1 = -q f_{\sigma_2 a_3\sigma_1} + \frac{q}{q+1} f_{\sigma_2\sigma_1}$.

We consider Propositions 4.1 and 4.4, take $t$ to be any $\psi_2$-invariant trace over $\widehat{TL}_2(q)$, determined by $A_0, A_1$ and $(\alpha_i)_{1\leq i}$. Let $s$ be any $\psi_3$-invariant trace over $\widehat{TL}_3(q)$, determined by $B_0, B_1, B_2$ and $(\beta_i)_{1\leq i}$. We show in what follows that:

*there are a unique $t$ and a unique $s$, such that $t$ is the second component and $s$ is the third component of a Markov trace.*

In order to simplify, we set $\hat{\tau}_2 := t$ and $\hat{\tau}_3 := s$.

At first, being a first component of a Markov trace, forces $\hat{\tau}_2$ to have the value 1 on $\mathbf{T}_{\sigma_1}$ and $\mathbf{T}_{a_2}$, but $f_{\sigma_1} = \frac{1+g_{\sigma_1}}{1+q} = \frac{1}{1+q} + \frac{\mathbf{T}_{\sigma_1}}{\sqrt{q}(1+q)}$. Hence, $A_1 = -\frac{\sqrt{q}}{1+q}$. Moreover, $\hat{\tau}_2(1) = -\frac{1+q}{\sqrt{q}}\hat{\tau}_1(1)$. Thus, $A_0 = -\frac{1+q}{\sqrt{q}}$.

Now, we have:

$$B_0 = \hat{\tau}_3\Big(1\Big) = -\frac{1+q}{\sqrt{q}}\hat{\tau}_2\Big(1\Big) = \Big(-\frac{1+q}{\sqrt{q}}\Big)^2,$$

$$\text{and } B_1 = \hat{\tau}_3\Big(f_{\sigma_1}\Big) = -\frac{1+q}{\sqrt{q}}\hat{\tau}_2\Big(f_{\sigma_1}\Big) = \frac{1+q}{\sqrt{q}}\frac{\sqrt{q}}{1+q} = 1.$$

**Remark 5.3.** *$\hat{\tau}_3$ must verify $\hat{\tau}_3\Big(F(h)T_{\sigma_2}\Big) = \hat{\tau}_2\Big(h\Big)$, for every $h$ in $\widehat{TL}_2(q)$.*

$$\textit{But, } \hat{\tau}_3\Big(F(h)T_{\sigma_2}\Big) = \sqrt{q}\hat{\tau}_3\Big(F(h)g_{\sigma_2}\Big) = \sqrt{q}\hat{\tau}_3\Bigg(F(h)\Big[(q+1)f_{\sigma_2} - 1\Big]\Bigg).$$

$$\textit{So, } \sqrt{q}\hat{\tau}_3\Bigg(F(h)\Big[(q+1)f_{\sigma_2} - 1\Big]\Bigg) = \sqrt{q}\Big(q+1\Big)\hat{\tau}_3\Big(F(h)f_{\sigma_2}\Big) - \sqrt{q}\hat{\tau}_3\Big(F(h)\Big)$$

$$\sqrt{q}\Big(q+1\Big)\hat{\tau}_3\Big(F(h)f_{\sigma_2}\Big) + \sqrt{q}\frac{1+q}{\sqrt{q}}\hat{\tau}_2\Big(h\Big).$$

*Hence, our condition becomes*

$$\sqrt{q}\Big(q+1\Big)\hat{\tau}_3\Big(F(h)f_{\sigma_2}\Big) = -\sqrt{q}\frac{1+q}{\sqrt{q}}\hat{\tau}_2\Big(h\Big) + \hat{\tau}_2\Big(h\Big) = -q\hat{\tau}_2\Big(h\Big).$$

*Thus, we must have*

$$\hat{\tau}_3\Big(F(h)f_{\sigma_2}\Big) = -\frac{\sqrt{q}}{(q+1)}\hat{\tau}_2\Big(h\Big), \ \textit{as an "f" equivalent to } \hat{\tau}_3\Big(F(h)T_{\sigma_2}\Big) = \hat{\tau}_2\Big(h\Big).$$

Now, we have:

$$B_2 = \hat{\tau}_3(f_{\sigma_1\sigma_2}) = -\frac{\sqrt{q}}{1+q}\hat{\tau}_2(f_{\sigma_1}) = (\frac{\sqrt{q}}{1+q})^2.$$

So, under the assumption that our two traces are the second and the third components of a given Markov trace, we get the following:

$$A_1 = -\frac{\sqrt{q}}{1+q},\ A_0 = -\tfrac{1+q}{\sqrt{q}}.$$

$$B_2 = \Big(\frac{\sqrt{q}}{1+q}\Big)^2,\ B_1 = 1 \text{ and } B_0 = \Big(\tfrac{1+q}{\sqrt{q}}\Big)^2.$$

In particular, we have for all $1 \leq i$:

$$\hat{\tau}_3\Big(x_1^i f_{\sigma_2}\Big) = -\frac{\sqrt{q}}{(q+1)}\hat{\tau}_2\Big((f_{\sigma_1 a_2})^i\Big), \quad \text{and} \quad \hat{\tau}_3\Big(z_1^i f_{\sigma_2}\Big) = -\frac{\sqrt{q}}{(q+1)}\hat{\tau}_2\Big((f_{a_2\sigma_1})^i\Big).$$

In other terms, for all $i$ we have:

$$\hat{\tau}_3(x_1^i f_{\sigma_2}) = -\frac{\sqrt{q}}{(q+1)}\alpha_i, \quad \text{and} \quad \hat{\tau}_3(f_{\sigma_2} z_1^i) = -\frac{\sqrt{q}}{(q+1)}\alpha_i.$$

Since $\hat{\tau}_3$ is determined by $\beta_i$, we can view these equalities as system of equations in $\beta_i$ and $\alpha_i$. In what follows, we show that this system has at most one solution: $(\alpha_i, \beta_i)_{1\leq i}$.

For $i = 1$, we see that we have two equations:

$$\hat{\tau}_3(\frac{-1}{q} f_{\sigma_1 a_3 \sigma_2} + \frac{1}{q+1} f_{\sigma_1\sigma_2}) = -\frac{\sqrt{q}}{(q+1)}\alpha_1, \quad \text{and} \quad \hat{\tau}_3(-q f_{\sigma_2 a_3 \sigma_1} + \frac{q}{q+1} f_{\sigma_2\sigma_1}) = -\frac{\sqrt{q}}{(q+1)}\alpha_1,$$

$$\text{that is} \quad \frac{-1}{q}\beta_1 + \frac{1}{q+1}B_2 = -\frac{\sqrt{q}}{(q+1)}\alpha_1, \quad \text{and} \quad -q\beta_1 + \frac{q}{q+1}B_2 = -\frac{\sqrt{q}}{(q+1)}\alpha_1,$$

$$\text{that is} \quad \frac{-1}{q}\beta_1 + \frac{q}{(q+1)^3} = -\frac{\sqrt{q}}{(q+1)}\alpha_1, \quad \text{and} \quad -q\beta_1 + \frac{q^2}{(q+1)^3} = -\frac{\sqrt{q}}{(q+1)}\alpha_1.$$

Clearly, those two linear equations are independent, hence, they determine a unique solution $(\alpha_1, \beta_1)$. Let us see the equations when $i = 2$, we have:

$$\hat{\tau}_3(x_1^2 f_{\sigma_2}) = -\frac{\sqrt{q}}{(q+1)}\alpha_2, \quad \text{and} \quad \hat{\tau}_3(f_{\sigma_2} z_1^2) = -\frac{\sqrt{q}}{(q+1)}\alpha_2.$$

We see that:

$$\begin{aligned} x_1^2 f_{\sigma_2} &= 3\delta x_1 f_{\sigma_2} + x_2 f_{\sigma_2} = 3\frac{-1}{q}\delta f_{\sigma_1 a_3 \sigma_2} + 3\frac{1}{q+1}\delta f_{\sigma_1\sigma_2} - \frac{(q+1)}{q^2} f^2_{\sigma_1 a_3 \sigma_2} - f_{\sigma_1\sigma_2 a_3} f_{\sigma_1\sigma_2} \\ &= \frac{-3}{(1+q)^2} f_{\sigma_1 a_3 \sigma_2} + \frac{3}{(1+q)^3} f_{\sigma_1\sigma_2} - \frac{(q+1)}{q^2} f^2_{\sigma_1 a_3\sigma_2} - f_{\sigma_1\sigma_2 a_3 \sigma_1\sigma_2}, \end{aligned}$$

$$\begin{aligned} \text{hence, } \hat{\tau}_3(x_1^2 f_{\sigma_2}) &= \frac{-3}{(1+q)^2}\beta_1 + \frac{3}{(1+q)^3}B_2 - \frac{(q+1)}{q^2}\beta_2 - \delta\beta_1 \\ &= \frac{3}{(1+q)^3}B_2 - \frac{3+q}{(1+q)^2}\beta_1 - \frac{(q+1)}{q^2}\beta_2. \end{aligned}$$

Now, $f_{\sigma_2} z_1^2 = \chi(x_1^2 f_{\sigma_2})$

$$= \chi\left(\frac{-3}{(1+q)^2}\right) f_{\sigma_2 a_3 \sigma_1} + \chi\left(\frac{3}{(1+q)^3}\right) f_{\sigma_2 \sigma_1} - \chi\left(\frac{(q+1)}{q^2}\right) f^2_{\sigma_2 a_3 \sigma_1} - f_{\sigma_2 \sigma_1 a_3 \sigma_2 \sigma_1},$$

$$\text{so } f_{\sigma_2} z_1^2 = \frac{-3q^2}{(1+q)^2} f_{\sigma_2 a_3 \sigma_1} + \frac{3q^3}{(1+q)^3} f_{\sigma_2 \sigma_1} - q(q+1) f^2_{\sigma_2 a_3 \sigma_1} - f_{\sigma_2 \sigma_1 a_3 \sigma_2 \sigma_1}.$$

$$\begin{aligned} \text{Now, we apply the trace } \hat{\tau}_3\Big(f_{\sigma_2} z_1^2\Big) &= \frac{-3q^2}{(1+q)^2}\beta_1 + \frac{3q^3}{(1+q)^3} B_2 - q(q+1)\beta_2 - \delta\beta_1 \\ &= \frac{3q^3}{(1+q)^3} B_2 - \frac{3q^2+q}{(1+q)^2}\beta_1 - q(q+1)\beta_2. \end{aligned}$$

In other terms, we have the two equations:

$$\begin{aligned} -\frac{(q+1)}{q^2}\beta_2 - \frac{3+q}{(1+q)^2}\beta_1 + \frac{3q}{(1+q)^5} &= -\frac{\sqrt{q}}{(q+1)}\alpha_2, \\ -q(q+1)\beta_2 - \frac{3q^2+q}{(1+q)^2}\beta_1 + \frac{3q^4}{(1+q)^5} &= -\frac{\sqrt{q}}{(q+1)}\alpha_2. \end{aligned}$$

Which indeed determine a unique $(\alpha_2, \beta_2)$ as a solution.

Now, have:

$$x_1^k = \sum_{i=1}^{i=k-1} \gamma_i x_i + x_k,$$

$$\text{hence, } x_1^k f_{\sigma_2} = \sum_{i=1}^{i=k-1} \gamma_i x_i f_{\sigma_2} + x_k f_{\sigma_2},$$

thus

$$\begin{aligned} x_1^k f_{\sigma_2} = \sum_{i=1}^{i=k-1} \gamma_i \left[\Big(-1\Big)^{i-1} \frac{(q+1)^{i-1}}{q^i}\right] f^i_{\sigma_1 a_3 \sigma_2} + \gamma_i \left[\Big(-1\Big)^{i-1} (q+1)^{i-2}\right] f^{i-1}_{\sigma_1 \sigma_2 a_3} f_{\sigma_1 \sigma_2} \\ + \left[\Big(-1\Big)^{k-1} \frac{(q+1)^{k-1}}{q^k}\right] f^k_{\sigma_1 a_3 \sigma_2} + \left[\Big(-1\Big)^{k-1} (q+1)^{k-2}\right] f^{k-1}_{\sigma_1 \sigma_2 a_3} f_{\sigma_1 \sigma_2}. \end{aligned}$$

Now we apply $\hat{\tau}_3$, we get:

$$-\frac{\sqrt{q}}{(q+1)}\alpha_k = \sum_{i=1}^{i=k-1}\gamma_i\Big[\Big(-1\Big)^{i-1}\frac{(q+1)^{i-1}}{q^i}\Big]\beta_i + \delta\gamma_i\Big[\Big(-1\Big)^{i-1}(q+1)^{i-2}\Big]\beta_{i-1}$$

$$+\delta\Big[\Big(-1\Big)^{k-1}(q+1)^{k-2}\Big]\beta_{k-1} + \Big[\Big(-1\Big)^{k-1}\frac{(q+1)^{k-1}}{q^k}\Big]\beta_k.$$

It is clear that the coefficient of $\beta_k$ is not zero, since $\beta_k$ does not appear in:

$$A := \sum_{i=1}^{i=k-1}\gamma_i\Big[\Big(-1\Big)^{i-1}\frac{(q+1)^{i-1}}{q^i}\Big]\beta_i + \delta\gamma_i\Big[\Big(-1\Big)^{i-1}(q+1)^{i-2}\Big]\beta_{i-1}$$

$$+\delta\Big[\Big(-1\Big)^{k-1}(q+1)^{k-2}\Big]\beta_{k-1}.$$

Now, we repeat the same steps with $z_i$, namely:

$$z_1^k = \sum_{i=1}^{i=k-1}\gamma_i d_i + d_k,$$

$$\text{hence, } f_{\sigma 2}z_1^k = \sum_{i=1}^{i=k-1}\gamma_i f_{\sigma_2}d_i + f_{\sigma_2}d_k.$$

Thus,

$$f_{\sigma_2}z_1^k = \sum_{i=1}^{i=k-1}\gamma_i\Big[\Big(-1\Big)^{i}q(q+1)^{i-1}\Big]f^i_{\sigma_2 a_3\sigma_1} + \gamma_i\Big[\Big(-1\Big)^{i-1}(\frac{q+1}{q})^{i-2}\Big]f_{\sigma_2\sigma_1}f^{i-1}_{a_3\sigma_2\sigma_1}$$

$$+\Big[\Big(-1\Big)^{k}q(q+1)^{k-1}\Big]f^k_{\sigma_2 a_3\sigma_1} + \Big[\Big(-1\Big)^{k-1}(\frac{q+1}{q})^{k-2}\Big]f_{\sigma_2\sigma_1}f^{k-1}_{a_3\sigma_2\sigma_1}.$$

Now we apply $\hat{\tau}_3$, we get:

$$-\frac{\sqrt{q}}{(q+1)}\alpha_k = \sum_{i=1}^{i=k-1}\gamma_i\Big[\Big(-1\Big)^{i}q(q+1)^{i-1}\Big]\beta_i + \gamma_i\delta\Big[\Big(-1\Big)^{i-1}(\frac{q+1}{q})^{i-2}\Big]\beta_{i-1}$$

$$+\delta\Big[\Big(-1)^{k-1}(\frac{q+1}{q})^{k-2}\Big]\beta_{k-1} + \Big[\Big(-1\Big)^{k}q(q+1)^{k-1}\Big]\beta_k.$$

The coefficient of $\beta_k$ is not zero, since $\beta_k$ does not appear in

$$B := \sum_{i=1}^{i=k-1} \gamma_i \Big[ \big( -1\big)^i q(q+1)^{i-1} \Big] \beta_i + \gamma_i \delta \Big[ \big( -1\big)^{i-1} (\frac{q+1}{q})^{i-2} \Big] \beta_{i-1}$$

$$+\delta \Big[ \big( -1)^{k-1} (\frac{q+1}{q})^{k-2} \Big] \beta_{k-1}.$$

In other terms, we have the two following equations, in $\beta_k$ and $\alpha_k$:

$$-\frac{\sqrt{q}}{(q+1)} \alpha_k = A + \Big[ \big( -1\big)^{k-1} \frac{(q+1)^{k-1}}{q^k} \Big] \beta_k,$$

$$-\frac{\sqrt{q}}{(q+1)} \alpha_k = B + \Big[ \big( -1\big)^k q(q+1)^{k-1} \Big] \beta_k.$$

Those are two independent linear equations in $\beta_k$ and $\alpha_k$, with non-zero coefficients, by induction over $k$ (that is: assuming that $(\alpha_i, \beta_i)$ is unique for $i < k$ then $(\alpha_k, \beta_k)$ is unique) we get the following corollary.

**Corollary 5.4.** *Suppose that $(\hat{\tau}_i)_{1 \leq i}$ is a Markov trace over the tower of $\tilde{A}$-type T-L algebras, then $\hat{\tau}_i = \rho_i$ for $i = 1, 2, 3$.*

Finally, we sum up the proof of the main theorem: we know, by Corollary 5.2 that there exists, at least, one affine Markov trace. Now, Corollary 5.4 says that in any given affine Markov trace, the three first components are $\rho_1, \rho_2$ and $\rho_3$ (of Corollary 5.2), while Theorem4.6 affirms that a third component in a given Markov trace determines a unique fourth component, and so on for any $\hat{\tau}_i$ with $i \geq 3$. Hence, we get our main theorem:

**Theorem 5.5.** *There exists a unique affine Markov trace over the tower of $\tilde{A}$-type Temperley-Lieb algebras, namely $(\rho_i)_{1 \leq i}$.*

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