TOWER OF FULLY COMMUTATIVE ELEMENTS OF TYPE $\tilde{A}$ AND APPLICATIONS

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Abstract. Let $W^c(\tilde{A}_n)$ be the set of fully commutative elements in the affine Coxeter group $W(\tilde{A}_n)$ of type $\tilde{A}$. We classify the elements of $W^c(\tilde{A}_n)$ and give a normal form for them. We give a first application of this normal form to fully commutative affine braids. We then use this normal form to define two injections from $W^c(\tilde{A}_{n-1})$ into $W^c(\tilde{A}_n)$ and examine their properties. We finally consider the tower of affine Temperley-Lieb algebras of type $\tilde{A}$ and use the injections above to prove the injectivity of this tower.

Braid groups; affine Coxeter groups; affine Temperley-Lieb algebra; fully commutative elements.

1. Introduction

Let (W, S) be a Coxeter system. We say that w in W is *fully commutative* if any reduced expression for w can be obtained from any other using only commutation relations among the members of the set S. If W is simply laced then the fully commutative elements of W are those with no sts factor in any reduced expression, where t and s are any non-commuting generators.

In this paper we are interested with the affine Coxeter group of type $\tilde{A}$ which has an infinite set of fully commutative elements, as proved in [19] where Stembridge assigns to each fully commutative element w a unique labeled partial order, called the heap of w, whose linear extensions encode the reduced expressions for w. The notion of heap was used frequently as a way to approach affine fully commutative elements, while other notions, for example abacus diagrams, were used in [16]. In this work we only use algebraic methods to deal with them, such as the affine length (see Definition 2.6).

In a given Coxeter group the subset of fully commutative elements is indeed an interesting set with many remarkable properties, in particular relating to Kazhdan-Lusztig polynomials (see for example [3]) hence relating to μ-coefficients. Moreover, they play the most important role in the M-coefficients notion, see [14]. Under certain conditions they are compatible with the classical Kazhdan-Lusztig cells, in the sense that the set of fully commutative elements is a union of cells [13, 18]. There is also an intrinsic notion of cell coming from the structure of Temperley-Lieb algebra, those cells are classified in [8].

This paper is divided into two parts. The first part establishes a classification of affine fully commutative elements in type $\tilde{A}$: they are depicted by a normal form given in Theorem 2.11. This form is similar to Stembridge's description of fully commutative elements in the Coxeter groups of finite type A, B, D [20], although a classification of fully commutative elements of type A was given by Jones in [17] before even the official definition of fully commutative elements in the 90's, see for example [6, 10].

Classification is interesting in itself, nevertheless, since affine fully commutative elements in type $\tilde{A}$ index a basis of the affine Temperley-Lieb algebra [7], it is to have consequences on the structure of the affine Temperley-Lieb algebra, on the tower of affine Temperley-Lieb algebras defined in [1] and on the traces on this algebra. This is precisely the point of the second part, which is divided into two applications.

The first application is to give a general form for "fully commutative braids" as follows: we lift the fully commutative elements to elements having the same expression in the $\tilde{A}$-type braid group $B(\tilde{A}_n)$, or: *fully commutative braids* (in this work we use the same symbols for the generators of the braid group and their images in the corresponding Coxeter group). Regarding $B(\tilde{A}_{n-1})$ as a subgroup of $B(\tilde{A}_n)$ by means of an injective homomorphism R_n, we give in Theorem 3.5 a general form for these fully commutative braids in terms of elements of $B(\tilde{A}_{n-1})$ and the lift c_n of a certain Coxeter element to $B(\tilde{A}_n)$. The tower of affine braid groups:

$$\{1\}\longrightarrow B(\tilde{A}_1) \xrightarrow{R_1} \cdots \xrightarrow{R_{n-1}} B(\tilde{A}_{n-1}) \xrightarrow{R_n} B(\tilde{A}_n) \xrightarrow{R_{n+1}} \cdots$$

gives rise to an analogous injective tower of the group algebras $K[B(\tilde{A}_n)]$ over an integral domain K of characteristic 0. Let q be an invertible element in K. The affine Temperley-Lieb algebra $\widehat{TL}_{n+1}(q)$ is a quotient of the braid group algebra $K[B(\tilde{A}_n)]$ and we get (see Section 5) a tower of affine Temperley-Lieb algebras:

$$\widehat{TL}_1(q) \xrightarrow{R_1} \widehat{TL}_2(q) \longrightarrow \cdots \xrightarrow{R_{n-1}} \widehat{TL}_n(q) \xrightarrow{R_n} \widehat{TL}_{n+1}(q) \xrightarrow{R_{n+1}} \cdots$$

The images by the quotient map of the fully commutative braids in $K[B(\tilde{A}_n)]$ make up a basis of $\widehat{TL}_{n+1}(q)$ and the form for fully commutative braids given in Theorem 3.5 is the key to the definition of *Markov elements* in $\widehat{TL}_{n+1}(q)$, and the key to proving that any trace on the affine Temperley-Lieb algebra is uniquely defined by its values on Markov elements [2, Theorem 4.6]. This in turn leads to the existence and uniqueness of the affine Markov trace [2] and on the other hand is a step towards Green's conjectures (Property B) [14].

The second application is to prove the faithfulness of the arrows of the tower of affine Temperley-Lieb algebras (Theorem 5.4). This was one of the most interesting questions since defining this tower in [1]. The faithfulness has consequences on the affine knot invariant defined in [1], and on the parabolic-like presentation defined in [1] on the level of affine Hecke algebra and recently for the affine Temperley-Lieb algebra.

The paper is organized as follows:

In Section 2, we give some general definitions, then we state and prove our main result, Theorem 2.11: a normal form for affine fully commutative elements in type $\tilde{A}$. This is the affine version of Theorem 2.3.

In Section 3, we define the tower of affine braid groups and establish its faithfulness. We then define fully commutative braids and, using our normal form and the fact that the lift c_n of a certain Coxeter element to $B(\tilde{A}_n)$ acts as a Dynkin automorphism on $B(\tilde{A}_{n-1})$ (Lemma 3.2), we find a general form for fully commutative braids in Theorem 3.5.

In Section 4, we show that the set $W^c(\tilde{A}_{n-1})$ of fully commutative elements in the Coxeter group with n generators of type $\tilde{A}$ injects into $W^c(\tilde{A}_n)$ in two different ways (Theorem 4.2). The existence of these two injections I and J depends totally on the normal form of Theorem 2.11. The intersection of their images is the image of $W^c(A_{n-1})$ on which they coincide.

In Section 5, we define the tower of affine Temperley-Lieb algebras coming from the tower of affine braid groups, then we prove in Theorem 5.4 the faithfulness of the arrows of this tower, using in a crucial way the injections I and J of the previous section.

2. A normal form for affine fully commutative elements

Let (W, S) be a Coxeter system with associated Dynkin Diagram Γ. For s, t in S we let m_{st} be the order of st in W. Let $w \in W$. We denote by $l(w)$ the length of a (any) reduced expression of w. We call *support of* w and denote by $Supp(w)$ the subset of S consisting of all generators appearing in a (any) reduced expression of w. We define $\mathscr{L}(w)$ to be the set of $s \in S$ such that $l(sw) < l(w)$, in other words s appears at the left edge of some reduced expression of w. We define $\mathscr{R}(w)$ similarly.

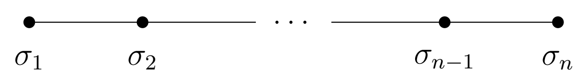
We know that from a given reduced expression of w we can arrive to any other reduced expression only by applying braid relations [4, §1.5 Proposition 5]. Among these relations there are commutation relations: those that correspond to generators t and s with $m_{st} = 2$.

Definition 2.1. *Elements for which one can pass from any reduced expression to any other one only by applying commutation relations are called* fully commutative elements. *We denote by W^c the set of fully commutative elements in W.*

The center of our interest in this work is fully commutative elements in $\tilde{A}$-type Coxeter groups. In this case fully commutative elements have some additional elegant properties, in particular:

Proposition 2.2. [15, Lemma 3.1] *Let (W, S) be a Coxeter system such that m_{st} is odd or 2 for any s, t in S. Let $w \in W$. Then w is fully commutative if and only if every s in Supp(w) occurs the same number of times in any reduced expression of w.*

Consider the A-type Coxeter group with n generators $W(A_n)$, with the following Dynkin diagram:



We let:

$$\begin{aligned} \lfloor i, j \rfloor &= \sigma_i \sigma_{i-1} \dots \sigma_j \ \text{ for } n \geq i \geq j \geq 1 \ \text{ and } \ \lfloor 0, 1 \rfloor = 1, \\ \lceil i, j \rceil &= \sigma_i \sigma_{i+1} \dots \sigma_j \ \text{ for } 1 \leq i \leq j \leq n \ \text{ and } \ \lceil n+1, n \rceil = 1, \\ h(i, r) &= \lfloor i, 1 \rfloor \lceil r, n \rceil \quad \text{ for } 0 \leq i < r \leq n+1 \text{ and } (i, r) \neq (0, 1), \end{aligned}$$

hence

$$\begin{aligned} h(i, r) &= \sigma_i \sigma_{i-1} \dots \sigma_1 \sigma_r \sigma_{r+1} \dots \sigma_n \quad \text{ for } 1 \leq i < r \leq n, \\ h(0, r) &= \lceil r, n \rceil \text{ for } 2 \leq r \leq n, \\ h(i, n+1) &= \lfloor i, 1 \rfloor \text{ for } 1 \leq i \leq n, \\ h(0, n+1) &= 1. \end{aligned}$$

Considering right classes of $W(A_{n-1})$ in $W(A_n)$, Stembridge has described canonical reduced words for elements of $W(A_n)$, namely:

$$\lceil m_1, n_1 \rceil \lceil m_2, n_2 \rceil \dots \lceil m_r, n_r \rceil$$

where $n \geq n_1 > \cdots > n_r \geq 1$ and $n_i \geq m_i \geq 1$ [20, p.1288]. He also proved [20, Corollary 5.8] that fully commutative elements are those for which the canonical reduced word satisfies $m_1 > \cdots > m_r$. The set of fully commutative elements is stable under the inverse map; taking inverses we get:

Theorem 2.3. [20, Corollary 5.8] *$W^c(A_n)$ is the set of elements of the form:*

$$\lfloor l_1, g_1 \rfloor \lfloor l_2, g_2 \rfloor \dots \lfloor l_s, g_s \rfloor, \;\; \textit{with} \begin{cases} 1 \le l_1 < \dots < l_s \le n, \\ 1 \le g_1 < \dots < g_s \le n, \\ l_t \ge g_t \textit{ for } 1 \le t \le s. \end{cases} \tag{1}$$

Inspecting the inequalities above, we see that the only term in expression (1) in which σ_n can occur is the s-th term. If σ_n does occur, then l_s must be equal to n and, whether or not g_s is equal to n, σ_n occurs only once. Similarly, if σ_1 does occur in expression (1), then $g_1 = 1$ and $\sigma_1 = \sigma_{g_1}$ will appear only once.

Definition 2.4. *An element u in $W^c(A_n)$ is called* extremal *if both σ_n and σ_1 belong to $Supp(u)$.*

Lemma 2.5. *An extremal element different from $\lfloor n, 1 \rfloor$ can be written as*

$$h(i, r)\, x \quad \textit{with} \;\; 1 \le i < r \le n \quad \textit{and} \;\; Supp(x) \subseteq \{\sigma_2, \dots, \sigma_{n-1}\}.$$

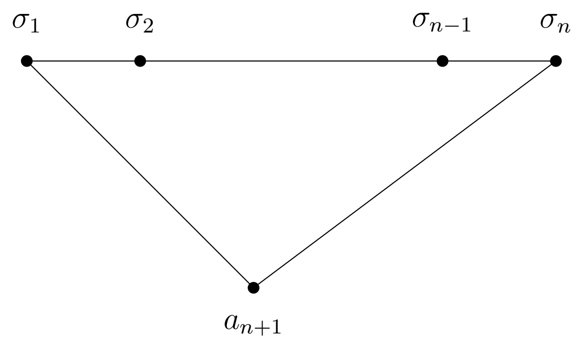
Proof. An extremal element u has a reduced expression of the form (1) above with $g_1 = 1$ and $l_s = n$. If $s = 1$ we have $u = \lfloor n, 1 \rfloor = h(n, n+1)$, the only extremal element for which the leftmost term in the reduced expression (1) is σ_n.

Assume $u \neq \lfloor n, 1 \rfloor$. For $n = 1$ we have $\sigma_1 = \lfloor 1, 1 \rfloor$. For $n = 2$ the element $\sigma_1 \sigma_2$ is the only extremal element different from $\lfloor 2, 1 \rfloor$ and it is equal to $h(1, 2)$. Assume now $n \ge 3$.

The rightmost term in (1) is $\lfloor n, g_s \rfloor$ with $g_s > 1$, so the generators on the right of σ_n, if any, belong to $\{\sigma_2, \dots, \sigma_{n-1}\}$. The generator σ_n commutes with any element in $\{\sigma_2, \dots, \sigma_{n-2}\}$, so, using the commutation relation $\sigma_i \sigma_n = \sigma_n \sigma_i$ for $2 \le i \le n-2$, we can repeatedly push σ_n to the left in expression (1) above until the element on the left of σ_n is either σ_{n-1} or σ_1. In this process all generators σ_i that are pushed to the right of σ_n belong to $\{\sigma_2, \dots, \sigma_{n-1}\}$.

If we arrive at a subexpression $a = \sigma_{n-1} \sigma_n$, which happens if and only if $l_{s-1} = n - 1$, then again it commutes with any element in $\{\sigma_2, \dots, \sigma_{n-3}\}$ so we can push a to the left until the element on the left of a is either σ_{n-2} or σ_1. We continue in this way as long as $l_{s-t} = n - t$ until we reach σ_1, and obtain the final expression $\lfloor l_1, 1 \rfloor \lceil n-k, n \rceil x$, with $k = \max\{t \mid 0 \le t < n-1 \text{ and } l_{s-t} = n-t\}$, as announced. □

Now let $W(\tilde{A}_n)$ be the affine Coxeter group of $\tilde{A}$-type with $n+1$ generators, with the following Dynkin diagram:



Our notation encapsulates the fact that we view $W(A_n)$ as the parabolic subgroup of $W(\tilde{A}_n)$ generated by $\sigma_1, \ldots, \sigma_n$. Recalling Proposition 2.2 we make the following definition.

Definition 2.6. *We define the* affine length *of u in $W^c(\tilde{A}_n)$ to be the number of times a_{n+1} occurs in a (any) reduced expression of u. We denote it by $L(u)$.*

Lemma 2.7. *Let w be a fully commutative element in $W(\tilde{A}_n)$ with $L(w) = m \geq 2$. Fix a reduced expression of w as follows:*

$$w = u_1 a_{n+1} u_2 a_{n+1} \ldots u_m a_{n+1} u_{m+1}$$

with u_i, for $1 \leq i \leq m+1$, a reduced expression of a fully commutative element in $W^c(A_n)$. Then $u_2, \ldots, u_m$ are extremal elements and there is a reduced expression of w of the form:

$$w = h(i_1, r_1) a_{n+1} h(i_2, r_2) a_{n+1} \ldots h(i_m, r_m) a_{n+1} \, v_{m+1} \tag{2}$$

where $v_{m+1} \in W^c(A_n)$, $0 \leq i_1 < r_1 \leq n+1$, $(i_1, r_1) \neq (0, 1)$ and, for $2 \leq t \leq m$, we have either $1 \leq i_t < r_t \leq n$ or $(i_t, r_t) = (n, n+1)$.

Proof. We first remark that in the same manner as in the previous lemma, we can write any fully commutative element in $W(A_n)$ as $h(i, r)\, x$ with $0 \leq i < r \leq n+1$, $(i, r) \neq (0, 1)$ and $\mathrm{Supp}(x) \subseteq \{\sigma_2, \ldots, \sigma_{n-1}\}$, in particular x commutes with a_{n+1}. Writing in this way $u_1 = h(i_1, r_1)x_1$, we can push x_1 to the right of a_{n+1}, obtaining a new term u_2 that we in turn write $h(i_2, r_2)x_2$ with x_2 commuting to a_{n+1}. Proceeding from left to right, we obtain formally form (2).

It remains to show that the elements u_i, $2 \leq i \leq m$, are extremal. Indeed, if the support of some such u_i was contained in $\{\sigma_2, \ldots, \sigma_{n-1}\}$, this u_i would commute with a_{n+1} and we would get a reduced expression containing $a_{n+1}a_{n+1}$, a contradiction. Now if some such u_i contained only one of σ_1, σ_n, then, using the commutation relations, $a_{n+1} u_i a_{n+1}$ could be written $\ldots a_{n+1}\sigma_1 a_{n+1} \ldots$ or $\ldots a_{n+1}\sigma_n a_{n+1} \ldots$, hence would contain a braid, which is impossible in a reduced expression for a fully commutative element. $\square$

Lemma 2.8. *Let $1 \leq l \leq n$ and $0 \leq i < r \leq n+1$, $(i,r) \neq (0,1)$. Then $w = h(i,r)\ a_{n+1}\ \sigma_l$ is a reduced fully commutative element if and only if one of the following holds:*

(1) $l = r-1 = i$;
(2) $i < l < r$.

Proof. Assume first that $r \leq n$ and write $w = \lfloor i,1 \rfloor \lceil r,n \rceil a_{n+1}\ \sigma_l$. Using commutation relations we push σ_l to the left as long as it commutes with its left neighbour.

- If $l \geq r$ we will arrive at the braid $\sigma_l \sigma_{l+1} \sigma_l$ if $l < n$, $\sigma_n a_{n+1} \sigma_n$ if $l = n$: w is not fully commutative.
- Assume $l < r$. If $i = 0$, indeed w is reduced fully commutative. We proceed with $i \geq 1$.
  - If $l < r-1$ and $l \leq i$, again we get a braid $\sigma_l \sigma_{l-1} \sigma_l$ by pushing σ_l leftmost hence w is not fully commutative.
  - If $i < l$ indeed w is reduced fully commutative.
  - If $r-1 > 1$, then σ_{r-1} commutes with a_{n+1} so for $l = r-1$ we get $\sigma_i \ldots \sigma_1 \sigma_r \sigma_{r-1} \sigma_{r+1} \cdots \sigma_n a_{n+1}$ which is reduced fully commutative.
  - Finally if $r-1 = 1 = l$, then σ_1 cannot get past a_{n+1} on the left and again we have a reduced fully commutative element.

If $r = n+1$ and either $l < i$ or $l = i < n$, the same process produces a braid $\sigma_l \sigma_{l-1} \sigma_l$ or (if $l = 1$) $\sigma_1 a_{n+1} \sigma_1$, hence w is not fully commutative, while, for $i < l$ or $l = i = n$, w is reduced fully commutative. □

Lemma 2.9. *Let $h(i,r)$ and $h(i',r')$ be extremal elements different from $\lfloor n,1 \rfloor$. Then*

$$w = h(i,r)\ a_{n+1}\ h(i',r')$$

is a reduced fully commutative element if and only if one of the following holds:

(1) $i < i' < r' < r$;
(2) $i \leq i'$ and $r' = r = i'+1$.

Proof. We have by assumption $1 \leq i < r \leq n$ and $1 \leq i' < r' \leq n$. We write

$$w = \lfloor i,1 \rfloor \lceil r,n \rceil\ a_{n+1}\ \lfloor i',1 \rfloor \lceil r',n \rceil.$$

Assume w is reduced fully commutative. From the previous lemma we must have $i' = r-1 = i$ or $i < i' < r$. We know examine r', after noticing that if $r' > i'+1$, then $\lfloor i',1 \rfloor \sigma_{r'} = \sigma_{r'} \lfloor i',1 \rfloor$ so Lemma 2.8 imposes $r' = r-1 = i$ or $i < r' < r$.

If $i' = r-1 = i$, then $r' > i'+1 = r$ is impossible by the previous remark, while $r' = i'+1 = r$ produces a reduced fully commutative w.

If $i < i' < r$ then

- if $r' > i'+1$, the previous remark gives $r' = r-1$ or $i < r' < r$, whence $i < i' < r' < r$, and under this condition w is reduced fully commutative;
- if $r' = i'+1 \leq r$, we can write
$$w = \lfloor i,1 \rfloor \lceil r,n \rceil \; a_{n+1} \; \sigma_{i'}\sigma_{i'+1} \lfloor i'-1,1 \rfloor \lceil i'+2,n \rceil.$$
We claim that no braid relation involving $\sigma_{i'}$ or $\sigma_{i'+1}$ can occur. On the right of the product $\sigma_{i'}\sigma_{i'+1}$ in the expression just above, this is clear. This same product $\sigma_{i'}\sigma_{i'+1}$ can be pushed to its left as long as it commutes with its left neighbour. If $r' = i'+1 < r-1$ we arrive at
$$w = \lfloor i,1 \rfloor \sigma_{i'}\sigma_{i'+1} \lceil r,n \rceil \; a_{n+1} \; \lfloor i'-1,1 \rfloor \lceil i'+2,n \rceil$$
where we see that our claim holds. If $r' = i'+1 = r-1$ we arrive at
$$w = \lfloor i,1 \rfloor \sigma_{r-2}\sigma_r\sigma_{r-1} \lceil r+1,n \rceil \; a_{n+1} \; \lfloor i'-1,1 \rfloor \lceil i'+2,n \rceil$$
and our claim holds again. Finally if $r' = i'+1 = r$ we arrive at
$$w = \lfloor i,1 \rfloor \sigma_r\sigma_{r-1}\sigma_{r+1}\sigma_r \lceil r+2,n \rceil \; a_{n+1} \; \lfloor i'-1,1 \rfloor \lceil i'+2,n \rceil \quad \text{if } r < n,$$
$$w = \lfloor i,1 \rfloor \sigma_n \; a_{n+1} \; \sigma_{n-1}\sigma_n \lfloor n-2,1 \rfloor \quad \text{if } r = n,$$
and our claim is proved, hence w is reduced fully commutative.

□

Lemma 2.10. *Let $w \in W^c(\tilde{A}_n)$ with $L(w) = m \geq 2$. Write w as in (2) and assume that $h(i_t, r_t) \neq \lfloor n,1 \rfloor$ for $2 \leq t \leq m$. There exist nonnegative integers p and k satisfying $p+k = m$ and an integer $j \in \{1,\dots,n-1\}$ such that w has the following form:*

$$\begin{aligned} &\text{if } p = 0 : w = (h(j,j+1)\; a_{n+1})^k \; w_r, \\ &\text{if } p > 0 : w = h(i_1,r_1)\; a_{n+1} \dots h(i_p,r_p)\; a_{n+1}(h(j,j+1)\; a_{n+1})^k \; w_r, \end{aligned} \tag{3}$$

with $w_r \in W^c(A_n)$ and, if $p > 0$:
- $1 \leq i_2 < \dots < i_p < r_p < \dots < r_2 \leq n$ *and* $r_p - i_p \geq 2$,
- *if $k > 0$: either $i_p < j < j+1 < r_p$ or $j+1 = r_p$.*

The element w_r can be described as follows:
- *if $k > 0$: for some $z \in \{0,\dots,n\}$ such that $j+z-1 \leq n$, we have*
  - *if $z = 0$: $w_r = 1$;*
  - *if $z \geq 1$: $w_r = \lfloor j,d_1 \rfloor \lfloor j+1,d_2 \rfloor \dots \lfloor j+z-1,d_z \rfloor$ with $1 \leq d_1 < \dots < d_z \leq n$ and $j+c \geq d_{c+1}$ for $0 \leq c \leq z-1$;*
- *if $k = 0$ (hence $p > 0$): for some $t \in \{0,\dots,n\}$ we have*
  - *if $t = 0$: $w_r = 1$;*

– *if* $t \geq 1$*:* $w_r = \lfloor l_1, g_1 \rfloor \lfloor l_2, g_2 \rfloor \dots \lfloor l_t, g_t \rfloor$ *with:*
  * $1 \leq l_1 < \cdots < l_t \leq n$,
  * $1 \leq g_1 < \cdots < g_t \leq n$,
  * $l_i \geq g_i$ *for* $1 \leq i \leq t$,
  * $i_p < l_1 < r_p$,
  * *for any* i*,* $2 \leq i \leq t$*, such that* $l_i > l_{i-1} + 1$ *we have* $l_i < r_p$.

Proof. We apply Lemma 2.9 repeatedly from left to right, i.e. letting t increase in form (2) (Lemma 2.7). Case (1) forces the inequality $i < i' < r' < r$, it cannot happen more than $[n/2]$ times. Case (2) can only be followed by $h(i', i'+1)$ again. We thus get (3).

To determine w_r we use form (1) (Theorem 2.3) and we repeatedly apply Lemma 2.8, remembering that if $l_i > l_{i-1} + 1$, there is a reduced expression for w_r that begins with σ_{l_i}. If $k > 0$ any reduced expression for w_r has to begin with σ_j on the left, we thus obtain a simple condition. □

We can now start the classification. Let $w \in W(\tilde{A}_n)$ with $L(w) = m \geq 1$, written as in (2) in Lemma 2.7. We discuss according to the first term $h(i_1, r_1)$ which can have one, and only one, of the following forms:

(1) $i_1 = n, r_1 = n + 1$ (extremal, equal to $\lfloor n, 1 \rfloor$);
(2) $1 \leq i_1 < r_1 \leq n$ (extremal, different from $\lfloor n, 1 \rfloor$);
(3) $i_1 = 0$ and $2 \leq r_1 \leq n$ (σ_n appears but not σ_1);
(4) $1 \leq i_1 \leq n - 1$ and $r_1 = n + 1$ (σ_1 appears but not σ_n);
(5) $i_1 = 0$ and $r_1 = n + 1$.

We now examine each case.

(1) By Lemma 2.8, after $h(n, n+1) = \sigma_n \sigma_{n-1} \dots \sigma_1$ there is only one choice for $h(i_2, r_2)$, namely $h(n, n+1)$ itself, and this repeats until we reach the rightmost term. This term must be 1 or some $\lfloor n, i \rfloor$. So w has the form:
$$(h(n, n+1)\ a_{n+1})^k\ w_r \text{ with } w_r = 1 \text{ or } w_r = \lfloor n, i \rfloor.$$
(2) By Lemma 2.8 all elements $h(i_t, r_t)$ must also be extremal and different from $\lfloor n, 1 \rfloor$, hence the previous discussion applies and we get the forms in Lemma 2.10, either with $p = 0$ (if $r_1 = i_1 + 1$), or with $p > 0$. We only have to extend the condition of the lemma to (i_1, r_1), that is:
$1 \leq i_1 < \cdots < i_p < r_p < \cdots < r_1 \leq n$, $r_p - i_p \geq 2$ and, if $k > 0$, either $i_p < j < j + 1 < r_p$ or $j + 1 = r_p$.
(3) Assume first $m \geq 2$. By Lemma 2.8 again we must have $i_2 < r_1$ hence, as before, all elements $h(i_t, r_t)$ in form (2) must be extremal and different from $\lfloor n, 1 \rfloor$. Furthermore, if $r_2 - i_2 > 1$, then σ_{r_2} can be pushed to the left of

$h(i_2, r_2)$ so Lemma 2.8 implies $r_2 < r_1$. We obtain the possible forms from Lemma 2.10 with $p > 0$, with the condition:

$0 = i_1 < i_2 < \cdots < i_p < r_p < \cdots < r_2 < r_1 \leq n$, $r_p - i_p \geq 2$ and, if $k > 0$, either $i_p < j < j+1 < r_p$ or $j+1 = r_p$.

If $m = 1$, we have to describe the rightmost term v_2 in expression (2). Writing v_2 in form (1) as in the proof of Lemma 2.10, we find exactly the same expression as in the case $k = 0$ for w_r in this lemma, with $p = 1$.

(4) If $m \geq 2$, Lemma 2.8 gives $i_2 > i_1$. If $i_2 = n$, from the same lemma we obtain

$$w = h(i_1, n+1)a_{n+1}(h(n, n+1)a_{n+1})^k w_r$$

with $k > 0$ and $w_r = 1$ or $w_r = \lfloor n, i \rfloor$ for some i, $1 \leq i \leq n$.

If $i_2 < n$ we are back to the case of extremal elements different from $\lfloor n, 1 \rfloor$ and we obtain the possible forms from Lemma 2.10 with $p > 0$, with the condition:

$1 \leq i_1 < i_2 < \cdots < i_p < r_p < \cdots < r_2 < r_1 = n+1$, $r_p - i_p \geq 2$ and, if $k > 0$, either $i_p < j < j+1 < r_p$ or $j+1 = r_p$.

If $m = 1$, Lemma 2.10 provides, for the rightmost term, the same expression as in the case $k = 0$ for w_r in this lemma, with $p = 1$.

(5) Here if $m \geq 2$, in case $i_2 < n$, then in the notation of Lemma 2.10 we have $h(i_1, r_1) = 1$, and in case $i_2 = n$, we have a form similar to case (1), namely

$$a_{n+1}(h(n, n+1)\ a_{n+1})^k\ w_r \text{ with } w_r = 1 \text{ or } w_r = \lfloor n, i \rfloor.$$

If $m = 1$ the rightmost term can be any fully commutative element w_r in $W(A_n)$. Actually the description of w_r given in Lemma 2.10, with $k = 0$, $p = 1$, $i_1 = 0$ and $r_1 = n+1$, applies here as well.

We subsume this discussion in the following theorem:

Theorem 2.11. *Let $w \in W^c(\tilde{A}_n)$ with $L(w) \geq 1$. Then w can be written in a unique way as a reduced word of the following form, for nonnegative integers p and k and for $j \in \{1, \ldots, n\}$:*

$$\begin{aligned} &\text{if } p = 0 : w = (h(j, j+1)\ a_{n+1})^k\ w_r\ (\text{with } k > 0), \\ &\text{if } p > 0 : w = h(i_1, r_1)\ a_{n+1} \ldots h(i_p, r_p)\ a_{n+1}(h(j, j+1)\ a_{n+1})^k\ w_r, \end{aligned} \tag{4}$$

with $w_r \in W^c(A_n)$ and, if $p > 0$:

- $0 \leq i_1 < \cdots < i_p < r_p < \cdots < r_1 \leq n+1$ *and* $r_p - i_p \geq 2$,
- *if $k > 0$: either $i_p < j < j+1 < r_p$ or $j+1 = r_p$.*

In particular we have $p \leq [\frac{n+1}{2}]$.

The element w_r can be described as follows:

- *If $k > 0$: for some $z \in \{0, \dots, n\}$ such that $j + z - 1 \leq n$, we have*
  - *if $z = 0$: $w_r = 1$;*
  - *if $z \geq 1$: $w_r = \lfloor j, d_1 \rfloor \lfloor j+1, d_2 \rfloor \dots \lfloor j+z-1, d_z \rfloor$ with $1 \leq d_1 < \cdots < d_z \leq n$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z-1$.*
- *If $k = 0$ (hence $p > 0$): for some $t \in \{0, \dots, n\}$ we have*
  - *if $t = 0$: $w_r = 1$;*
  - *if $t \geq 1$: $w_r = \lfloor l_1, g_1 \rfloor \lfloor l_2, g_2 \rfloor \dots \lfloor l_t, g_t \rfloor$ with:*
    - *$1 \leq l_1 < \cdots < l_t \leq n$,*
    - *$1 \leq g_1 < \cdots < g_t \leq n$,*
    - *$l_i \geq g_i$ for $1 \leq i \leq t$,*
    - *$i_p < l_1 < r_p$;*
    - *for any i, $2 \leq i \leq t$, such that $l_i > l_{i-1} + 1$ we have $l_i < r_p$.*

Conversely, any word written as above is reduced and fully commutative.

Example 2.12. *Let w be in $W^c(\tilde{A}_2)$. Then there exists $k \geq 0$ such that w has one and only one of the following forms:*

$$\begin{array}{c}1\\ a_3\\ \sigma_1 a_3\end{array} \Big\rangle\, (\sigma_2\sigma_1 a_3)^k \,\Big\langle \begin{array}{c}1\\ \sigma_2\\ \sigma_2\sigma_1\end{array} \qquad\qquad \begin{array}{c}1\\ a_3\\ \sigma_2 a_3\end{array} \Big\rangle\, (\sigma_1\sigma_2 a_3)^k \,\Big\langle \begin{array}{c}1\\ \sigma_1\\ \sigma_1\sigma_2\end{array}$$

We finish this section with simple remarks. Firstly, the affine length of a fully commutative element w written in form (4) is $p+k$. Secondly, the family of elements corresponding to the case $k > 0$ and $j = n$ is particularly simple: their form is

$$h(i, n+1)\ a_{n+1}(h(n, n+1)\ a_{n+1})^t\ w_r$$

for some i such that $0 \leq i \leq n$ and with $w_r = 1$ or $\lfloor n, l \rfloor$ with $n \geq l \geq 1$.

3. Fully commutative affine braids

We denote by $B(\tilde{A}_n)$ the affine braid group with $n+1$ generators of type $\tilde{A}$, while we denote by $B(A_n)$ the braid group with n generators of type A, where $n \geq 1$. By definition $B(\tilde{A}_n)$ has $\{\sigma_1, \sigma_2, \ldots, \sigma_n, a_{n+1}\}$ as a set of generators together with the following defining relations:

(1) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $1 \leq i, j \leq n$ and $|i - j| \geq 2$,

(2) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-1$,

(3) $\sigma_i a_{n+1} = a_{n+1}\sigma_i$ for $2 \leq i \leq n-1$,

(4) $\sigma_1 a_{n+1}\sigma_1 = a_{n+1}\sigma_1 a_{n+1}$ for $n \geq 2$,

(5) $\sigma_n a_{n+1}\sigma_n = a_{n+1}\sigma_n a_{n+1}$ for $n \geq 2$,

while $B(A_n)$ is generated by $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ with relations (1) and (2).

Lemma 3.1. *The following map:*

$$\begin{aligned} R_n : B(\tilde{A}_{n-1}) &\longrightarrow B(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \qquad \textit{for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1}\sigma_n^{-1} \end{aligned}$$

is a group monomorphism.

Proof. Graham and Lehrer give in [12, §2] a presentation of the B-type braid group $B(B_{n+1})$ with generators $\{\tau_{n+1}, \sigma_1, \ldots, \sigma_n, a_{n+1}\}$ and relations (1) to (5) above, plus

$$\tau_{n+1}\sigma_i\tau_{n+1}^{-1} = \sigma_{i+1} \text{ for } 1 \leq i \leq n-1, \quad \tau_{n+1}\sigma_n\tau_{n+1}^{-1} = a_{n+1}, \quad \tau_{n+1}a_{n+1}\tau_{n+1}^{-1} = \sigma_1.$$

This presentation is related to the usual presentation of the B-type braid group, with generators $\{t, \sigma_1, \sigma_2, \ldots, \sigma_n\}$ and braid relations, by

$$\tau_{n+1} = t\sigma_1\sigma_2\ldots\sigma_n, \quad a_{n+1} = \tau_{n+1}\sigma_n\tau_{n+1}^{-1} = t\sigma_1\sigma_2\ldots\sigma_n\sigma_{n-1}^{-1}\ldots\sigma_1^{-1}t^{-1}.$$

They show that the subgroup of $B(B_{n+1})$ generated by $\{\sigma_1, \ldots, \sigma_n, a_{n+1}\}$ is isomorphic to $B(\tilde{A}_n)$ and fits into the exact sequence [12, Corollary 2.7]

$$1 \longrightarrow B(\tilde{A}_n) \longrightarrow B(B_{n+1}) \longrightarrow \mathbb{Z} \longrightarrow 1.$$

We thus view $B(\tilde{A}_n)$ as a subgroup of the braid group $B(B_{n+1})$ for $n \geq 1$.

There is a natural injection from $B(B_n)$ generated by $\{t, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}\}$ into $B(B_{n+1})$ generated by $\{t, \sigma_1, \sigma_2, \ldots, \sigma_n\}$ (see e.g. [9, §2.2]). We claim that this injection restricts to R_n on $B(\tilde{A}_{n-1})$. To prove this claim we only need to check that the image of a_n is $\sigma_n a_{n+1} \sigma_n^{-1}$. Since σ_n commutes with t, σ_1, …, σ_{n-2}, we have:

$$\begin{aligned}\sigma_n a_{n+1} \sigma_n^{-1} &= \sigma_n t \sigma_1 \sigma_2 \dots \sigma_n \sigma_{n-1}^{-1} \dots \sigma_1^{-1} t^{-1} \sigma_n^{-1} \\ &= t\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_n\sigma_{n-1}\sigma_n\sigma_{n-1}^{-1}\sigma_n^{-1}\sigma_{n-2}^{-1} \dots \sigma_1^{-1} t^{-1} \\ &= t\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}\sigma_n\sigma_{n-1}\sigma_{n-1}^{-1}\sigma_n^{-1}\sigma_{n-2}^{-1} \dots \sigma_1^{-1} t^{-1} \\ &= t\sigma_1\sigma_2 \dots \sigma_{n-2}\sigma_{n-1}\sigma_{n-2}^{-1} \dots \sigma_1^{-1} t^{-1} \\ &= a_n\end{aligned}$$

as announced. □

Using the injective morphism R_n we now view $B(\tilde{A}_{n-1})$ as a subgroup of $B(\tilde{A}_n)$. Let ψ_n be the Dynkin automorphism of $B(\tilde{A}_{n-1})$ that shifts the generators of the Dynkin diagram one step counterclockwise ($\sigma_1 \mapsto a_n \mapsto \sigma_{n-1} \mapsto \cdots \mapsto \sigma_2 \mapsto \sigma_1$). It generates a subgroup of $\mathrm{Aut}(B(\tilde{A}_{n-1}))$ of order n. We simply refer to it by ψ in what follows.

Lemma 3.2. *The element $c_n = \sigma_n\sigma_{n-1} \dots \sigma_1 a_{n+1}$ of $B(\tilde{A}_n)$ normalizes $B(\tilde{A}_{n-1})$. It acts by conjugacy on $B(\tilde{A}_{n-1})$ in the same way as ψ. We will write:*

$$c_n^t h = \psi^t\,[h]\, c_n^t \text{ for any } h \in B(\tilde{A}_{n-1}).$$

Proof. Applying braid relations in $B(\tilde{A}_n)$ we see that, for $2 \leq i \leq n-1$, we have:

$$\begin{aligned}\sigma_n\sigma_{n-1} \dots \sigma_1 a_{n+1}\sigma_i &= \sigma_n \dots \sigma_{i+1}\ \sigma_i\sigma_{i-1}\sigma_i \dots \sigma_1 a_{n+1} \\ &= \sigma_n \dots \sigma_{i+1}\sigma_{i-1}\sigma_i\sigma_{i-1} \dots \sigma_1 a_{n+1} \\ &= \sigma_{i-1}\sigma_n\sigma_{n-1} \dots \sigma_1 a_{n+1}.\end{aligned}$$

In a similar way, we use the braid relations involving a_{n+1} and the following relations worth keeping in mind:

$$a_{n+1}a_n = \sigma_n^{-1}a_n\sigma_n a_n = \sigma_n^{-1}\sigma_n a_n \sigma_n = a_n\sigma_n = \sigma_n a_{n+1} \tag{5}$$

to obtain:

$$\begin{aligned}\sigma_n\sigma_{n-1} \dots \sigma_1 a_{n+1}\sigma_1 &= a_n\sigma_n\sigma_{n-1} \dots \sigma_1 a_{n+1}, \\ \sigma_n\sigma_{n-1} \dots \sigma_1 a_{n+1} a_n &= \sigma_{n-1}\sigma_n\sigma_{n-1} \dots \sigma_1 a_{n+1}\end{aligned}$$

hence our result. □

Let $n \geq 1$. Recall from [4, IV.1, Proposition 5] that there exists a map

$$g : W(\tilde{A}_n) \longrightarrow B(\tilde{A}_n)$$

such that, given any reduced expression of $w \in W(\tilde{A}_n)$, the image $g(w)$ of w is defined by the same expression in the braid group (keeping the same symbols for the generators of the affine braid group and their images via the natural surjection onto the affine Coxeter group).

Definition 3.3. *We call* fully commutative braids *the images under g of the fully commutative elements in $W(\tilde{A}_n)$, i.e. the elements of $\{g(w) \mid w \in W^c(\tilde{A}_n)\}$.*

We now proceed to give a general form for fully commutative braids in which the element c_n is singled out, as a consequence of the normal form for fully commutative elements in Theorem 2.11. We use the notation in Section 2, equally valid for the braid groups, with an additional index n in $B(\tilde{A}_n)$ and $n-1$ in $B(\tilde{A}_{n-1})$.

Lemma 3.4. *Consider the element $y = h_n(j, j+1)\, a_{n+1}$ of $B(\tilde{A}_n)$ where $1 \leq j \leq n$. Let $k \geq 1$ be an integer and write $k = m(n-j+1)+r$ with $0 \leq r < n-j+1$. Then if $j = n$ we have $y^k = c_n^k$ while for $j < n$ we have (with the convention $\lfloor n, n+1-r \rfloor = 1$ if $r = 0$):*

(1) *if $m = 0$:* $y^k = (h_{n-1}(j, j+1)a_n)^r \lfloor n, n+1-r \rfloor$;

(2) *if $m > 0$:*

$$y^k = \left(\prod_{i=0}^{i=m-1} \psi^i \left[(h_{n-1}(j, j+1)a_n)^{n-j} \lceil j, n-1 \rceil \right] \right) \psi^m \left[(h_{n-1}(j, j+1)a_n)^r \right] c_n^m \lfloor n, n+1-r \rfloor.$$

Proof. We have for $j < n$, using relations (5):

$$y = \sigma_j \ldots \sigma_1 \sigma_{j+1} \ldots \sigma_{n-1} \sigma_n a_{n+1} = \sigma_j \ldots \sigma_1 \sigma_{j+1} \ldots \sigma_{n-1} a_n \sigma_n = h_{n-1}(j, j+1) a_n \sigma_n.$$

Observe that for $j+1 < s \leq n$ we have $\sigma_s y = y \sigma_{s-1}$. Hence if $j+1 < n$:

$$y^2 = h_{n-1}(j, j+1) a_n \sigma_n y = h_{n-1}(j, j+1) a_n y \sigma_{n-1} = (h_{n-1}(j, j+1) a_n)^2 \sigma_n \sigma_{n-1}.$$

Continuing this way, we can see that whenever $0 \leq t \leq n-j$ we have

$$y^t = (h_{n-1}(j, j+1) a_n)^t \lfloor n, n+1-t \rfloor.$$

This holds in particular for $t = n-j$ thus:

$$y^{n-j+1} = (h_{n-1}(j, j+1) a_n)^{n-j} \sigma_n \sigma_{n-1} \ldots \sigma_{j+1} \sigma_j \ldots \sigma_2 \sigma_1 \sigma_{j+1} \ldots \sigma_{n-1} \sigma_n a_{n+1}.$$

We now use: $\sigma_n \sigma_{n-1} \ldots \sigma_1 \sigma_u = \sigma_{u-1} \sigma_n \sigma_{n-1} \ldots \sigma_1$ for $2 \leq u \leq n$, and get:

$$y^{n-j+1} = (h_{n-1}(j, j+1) a_n)^{n-j} \lceil j, n-1 \rceil c_n.$$

An easy induction using Lemma 3.2 leads to:

$$y^{m(n-j+1)} = \left(\prod_{i=0}^{i=m-1} \psi^i \left[(h_{n-1}(j, j+1) a_n)^{n-j} \lceil j, n-1 \rceil \right] \right) c_n^m.$$

Finally, let $k = m(n-j+1)+r$, where $0 \leq r < n-j+1$. We have:

$$y^k = \left(\prod_{i=0}^{i=m-1} \psi^i \left[(h_{n-1}(j,j+1)a_n)^{n-j}\lceil j, n-1\rceil\right]\right) \; c_n^m (h_{n-1}(j,j+1)a_n)^r \lfloor n, n+1-r\rfloor$$

$$= \left(\prod_{i=0}^{i=m-1} \psi^i \left[(h_{n-1}(j,j+1)a_n)^{n-j}\lceil j, n-1\rceil\right]\right) \; \psi^m \left[(h_{n-1}(j,j+1)a_n)^r\right] c_n^m \lfloor n, n+1-r\rfloor$$

as claimed. ☐

Theorem 3.5. *Let $n \geq 2$. Let w be a fully commutative braid in $B(\tilde{A}_n)$. Then w can be written in one and only one of the following two forms:*

$$\begin{aligned} & u\; c_n^t\; v \\ \textit{or} \quad \lfloor i, 1\rfloor a_{n+1}\; & u\; c_n^t\; v, \end{aligned}$$

with $u \in B(\tilde{A}_{n-1})$, $v \in B(A_n)$, $t \geq 0$ and $0 \leq i \leq n-1$.

Proof. We have $w = g(\bar{w})$ with $\bar{w}$ in $W^c(\tilde{A}_n)$, and we use the reduced expression of w corresponding to the normal form of $\bar{w}$ given by Theorem 2.11, of which we use the notations. We may and do ignore the rightmost term w_r that belongs to $B(A_n)$ and we remark that if $p = 0$ the element has indeed the first claimed form, by Lemma 3.4. We proceed with $p > 0$ and set

$$x = h_n(i_1, r_1)\; a_{n+1} \dots h_n(i_p, r_p)\; a_{n+1}$$

so that $w = xy^k$ where $y = h_n(j, j+1)\, a_{n+1}$ as in the previous lemma.

We remark that for $1 \leq i \leq p$ we have $r_i \leq r_1 - i + 1$; since $r_1 \leq n+1$ this gives: $r_i \leq n - i + 2$. We have two main cases to consider:

- $r_1 \leq n$, that is σ_n belongs to the support of $\sigma_{r_1} \dots \sigma_{n-1}\sigma_n$;
- $r_1 = n+1$, that is σ_n does not belong to the support of $\sigma_{r_1} \dots \sigma_{n-1}\sigma_n$.

We start with the first case $r_1 \leq n$, so that $r_i \leq n - i + 1$. Using first the relation $\sigma_n a_{n+1} = a_n \sigma_n$, then the braid relations $\sigma_u\sigma_{u-1}\sigma_u = \sigma_{u-1}\sigma_u\sigma_{u-1}$ for $u = n$, $n-1$, $\dots$, $n-p+2$, we push to the right the first σ_n from the left and get:

$$x = \lfloor i_1, 1\rfloor \lceil r_1, n-1\rceil a_n h(i_2, r_2)\; a_{n+1} \dots h(i_p, r_p)\; a_{n+1}\sigma_{n-(p-1)}.$$

We repeat this operation with the first σ_n from the left in this new expression, and so on. Proceeding from $i = 1$ to $i = p$, we obtain:

$$x = h_{n-1}(i_1, r_1)\; a_n \dots h_{n-1}(i_p, r_p)\; a_n\sigma_n\sigma_{n-1} \dots \sigma_{n-(p-1)}.$$

We set $\rho = h_{n-1}(i_1, r_1)\ a_n \dots h_{n-1}(i_p, r_p)\ a_n$, an element of $B(\tilde{A}_{n-1})$. If $k = 0$ we have $w = x = \rho\lfloor n, n-(p-1)\rfloor$ and our claim holds. Let $k \geq 1$ and set $\epsilon = n-(p-1)$. We have $\epsilon \geq j+1$ since $j+1 \leq r_p \leq n-(p-1)$ and we can write:

$$w = \rho\ \sigma_n\sigma_{n-1}\dots\sigma_\epsilon\ y^k,\ \text{ with } \rho \in B(\tilde{A}_{n-1}) \text{ and } \epsilon \geq j+1.$$

We recall that $y = h_n(j, j+1)\, a_{n+1}$ acts on σ_i in the following way: $\sigma_i y = y\sigma_{i-1}$ for $j+1 < i \leq n$. We distinguish two cases:

(1) $1 \leq k \leq \epsilon - (j+1)$,

(2) $\epsilon - (j+1) < k$.

We start with (1). We have: $\sigma_n\sigma_{n-1}\dots\sigma_\epsilon y^k = y^k\sigma_{n-k}\sigma_{n-1-k}\dots\sigma_{\epsilon-k}$, with

$$k \leq \epsilon - (j+1) = n-j-p < n-j.$$

Thus we are in case (1) of Lemma 3.4, that is:

$$w = \rho(h_{n-1}(j, j+1)a_n)^k\lfloor n, n+1-k\rfloor\lfloor n-k, \epsilon-k\rfloor\ \in\ B(\tilde{A}_{n-1})\ B(A_n)$$

as required (this is the first form in the theorem with $t = 0$).

Now we deal with case (2) and set $h = k-(\epsilon-(j+1)) \geq 1$. Using the computation from case (1) for $\epsilon-(j+1)$ we get, using again Lemma 3.4 (1):

$$\begin{aligned}
\sigma_n\sigma_{n-1}\dots\sigma_\epsilon\ y^k &= \sigma_n\sigma_{n-1}\dots\sigma_\epsilon\ y^{\epsilon-(j+1)}y^h = y^{\epsilon-(j+1)}\sigma_{j+p}\sigma_{j+p-1}\dots\sigma_{j+1}\ y^h\\
&= (h_{n-1}(j, j+1)a_n)^{\epsilon-(j+1)}\lfloor n, j+p+1\rfloor\lfloor j+p, j+1\rfloor\ y^h\\
&= (h_{n-1}(j, j+1)a_n)^{\epsilon-(j+1)}\lfloor n, j+1\rfloor\ y\ y^{h-1}.
\end{aligned}$$

We now compute:

$$\begin{aligned}
\lfloor n, j+1\rfloor\ y\ y^{h-1} &= (\sigma_n\dots\sigma_{j+1})(\sigma_j\dots\sigma_1)(\sigma_{j+1}\dots\sigma_n)a_{n+1}y^{h-1}\\
&= (\sigma_n\dots\sigma_1)(\sigma_{j+1}\dots\sigma_n)a_{n+1}y^{h-1}\\
&= (\sigma_j\dots\sigma_{n-1})(\sigma_n\dots\sigma_1)a_{n+1}y^{h-1}
\end{aligned}$$

since $(\sigma_n\dots\sigma_1)\sigma_k = \sigma_{k-1}(\sigma_n\dots\sigma_1)$ for $2 \leq k \leq n$. Setting

$$\eta = \rho(h_{n-1}(j, j+1)a_n)^{\epsilon-(j+1)}\sigma_j\dots\sigma_{n-1} \in B(\tilde{A}_{n-1}),$$

we get $w = \eta c_n y^{h-1}$.

(a) If $h-1 \leq n-j$, we see, using Lemma 3.4, that:

$$\begin{aligned}
w &= \eta c_n(h_{n-1}(j, j+1)a_n)^{h-1}\lfloor n, n+1-(h-1)\rfloor\\
&= \eta\ \psi\left[(h_{n-1}(j, j+1)a_n)^{h-1}\right]c_n\lfloor n, n+1-(h-1)\rfloor,
\end{aligned}$$

which is of the first claimed form.

(*b*) If $n-j<h-1$, we write $h-1=m(n-j+1)+r$ with $0\leq r<n-j+1$ as in Lemma 3.4 and get:

$$\begin{aligned} w &= \eta c_n \left(\prod_{i=0}^{i=m-1} \psi^i \left[(h_{n-1}(j,j+1)a_n)^{n-j}\lceil j,n-1\rceil\right]\right)\psi^m\left[(h_{n-1}(j,j+1)a_n)^r\right] \\ &\qquad\qquad c_n^m \lfloor n,n+1-r\rfloor \\ &= \eta\left(\prod_{i=1}^{i=m} \psi^i \left[(h_{n-1}(j,j+1)a_n)^{n-j}\lceil j,n-1\rceil\right]\right)\psi^{m+1}\left[(h_{n-1}(j,j+1)a_n)^r\right] \\ &\qquad\qquad c_n^{m+1} \lfloor n,n+1-r\rfloor \end{aligned}$$

which has indeed the required form (first form in the theorem).

We are left with with the second main case $r_1=n+1$: the element under study has a normal form $w=\lfloor i_1,1\rfloor a_{n+1}w'$ with $0\leq i_1\leq n$. In fact $i_1=n$ is the case of positive powers of $c_n=\sigma_n\ldots\sigma_1 a_{n+1}$ that have the first form in the theorem. For $0\leq i_1<n$ the element w' actually belongs to one of the previous cases thus we get the second form in the theorem. □

Using the natural surjection of $B(\tilde{A}_n)$ onto $W(\tilde{A}_n)$, we deduce from Theorem 3.5 two possible forms for fully commutative elements in $W(\tilde{A}_n)$. In the above notation, the image of $v\in B(A_n)$ belongs to $W(A_n)$ and, considering the left classes of $W(A_{n-1})$ in $W(A_n)$ as in [20, p.1288], this image can be written uniquely as a product $v'\sigma_n\sigma_{n-1}\ldots\sigma_s$ with $v'\in W(A_{n-1})$ and $1\leq s\leq n+1$ (with the convention that for $s=n+1$, the product $\sigma_n\sigma_{n-1}\ldots\sigma_s$ is 1). Since c_n normalizes $W(\tilde{A}_{n-1})$ the element v' can be moved to the left of c_n^t into an element that incorporates with the image of u. We obtain:

Corollary 3.6. *Let w be a fully commutative element in $W(\tilde{A}_n)$ for $n\geq 2$. Then w can be written in one and only one of the following two forms:*

$$\begin{aligned} & d\, c_n^t\sigma_n\sigma_{n-1}\ldots\sigma_s \\ \textit{or}\quad & \lfloor i,1\rfloor a_{n+1}\, d\, c_n^t\sigma_n\sigma_{n-1}\ldots\sigma_s, \end{aligned}$$

where d is in $W(\tilde{A}_{n-1})$ and t, s and i are integers with $t\geq 0$, $1\leq s\leq n+1$ and $0\leq i\leq n-1$.

4. The tower of fully commutative elements

The injection of braid groups $R_n : B(\tilde{A}_{n-1}) \longrightarrow B(\tilde{A}_n)$ of Lemma 3.1 induces the group homomorphism:

$$\begin{aligned} R_n : W(\tilde{A}_{n-1}) &\longrightarrow W(\tilde{A}_n) \\ \sigma_i &\longmapsto \sigma_i \text{ for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n. \end{aligned}$$

Lemma 4.1. *The morphism $R_n : W(\tilde{A}_{n-1}) \longrightarrow W(\tilde{A}_n)$ is an injection.*

Proof. This fact is most easily seen using the description of $W(\tilde{A}_n)$ as the group of $(n+1)$-periodic permutations of $\mathbb{Z}$ with total shift equal to 0, which we briefly recall. A permutation u of $\mathbb{Z}$ is m-periodic if

$$u(i+m) = u(i) + m \text{ for any } i \in \mathbb{Z}.$$

We define the total shift of an m-periodic permutation u to be:

$$\frac{1}{m} \sum_{i=1}^{i=m} \Big(u(i) - i \Big).$$

We set ${}^m_0\mathbb{Z}$ to be the set of m-periodic permutations with total shift equal to 0. It forms a subgroup of the group of permutations of $\mathbb{Z}$. Let i be an integer such that $1 \leq i \leq m$ and let ${}^m s_i$ be the m-periodic permutation defined by: ${}^m s_i(k) = k+1$ for $k \equiv i \pmod{m}$, ${}^m s_i(k) = k-1$ for $k \equiv i+1 \pmod{m}$, ${}^m s_i(k) = k$ for $k \not\equiv i, i+1 \pmod{m}$. Then

$$\begin{aligned} \sigma_i &\longmapsto {}^{n+1}s_i \quad \text{for } 1 \leq i \leq n \\ a_{n+1} &\longmapsto {}^{n+1}s_{n+1} \end{aligned} \tag{6}$$

is an isomorphism from $W(\tilde{A}_n)$ onto ${}^{n+1}_0\mathbb{Z}$ (see e.g. [5, Proposition 3.2]).

It is easy to check that ${}^n_0\mathbb{Z}$ injects into ${}^{n+1}_0\mathbb{Z}$ as follows. We define an injection $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}$ by letting $\phi(i + kn) = i + k(n+1)$ for $1 \leq i \leq n$ and $k \in \mathbb{Z}$. We then map an n-periodic permutation v to the $(n+1)$-periodic permutation v' defined by $v'(\phi(x)) = \phi(v(x))$ $(x \in \mathbb{Z})$ and $v'(k(n+1)) = k(n+1)$ $(k \in \mathbb{Z})$. For $v \in^n_0 \mathbb{Z}$ and $1 \leq i \leq n$, write $v(i) = j_i + k_i n$ with $1 \leq j_i \leq n$; the total shift of v is $(\sum_{i=1}^{i=n} k_i) + \frac{1}{n} \sum_{i=1}^{i=n} (j_i - i)$. Since v is an n-periodic permutation it induces a permutation mod n, hence $\sum_{i=1}^{i=n} j_i = \sum_{i=1}^{i=n} i$ and consequently $\sum_{i=1}^{i=n} k_i = 0$. Now the total shift of v' is

$$\frac{1}{n+1}\sum_{i=1}^{i=n+1}(v'(i)-i) = \frac{1}{n+1}\left(\sum_{i=1}^{i=n}(\phi(v(i))-i)\right) + \frac{1}{n+1}(n+1-(n+1))$$
$$= \frac{1}{n+1}\sum_{i=1}^{i=n}(j_i + k_i(n+1) - i) = 0$$

This injection maps ${}^{n}s_i$ to ${}^{n+1}s_i$ for $1 \leq i \leq n-1$, and ${}^{n}s_n$ to ${}^{n+1}s_n \, {}^{n+1}s_{n+1} \, {}^{n+1}s_n$. Going back to $W(\tilde{A}_{n-1})$ and $W(\tilde{A}_n)$ through the isomorphisms (6) above, we find the morphism R_n, hence itself an injection. $\square$

The Coxeter group $W(A_{n-1})$ with Coxeter generators $(\sigma_1, \cdots, \sigma_{n-1})$ is a parabolic subgroup of $W(A_n)$. This is no longer the case for $W(\tilde{A}_{n-1})$ and $W(\tilde{A}_n)$, indeed proper parabolic subgroups of $W(\tilde{A}_n)$ are finite. This is an important difficulty when dealing with the affine case. $W(A_n)$ though, with Coxeter generators $(\sigma_1, \cdots, \sigma_n)$, is a parabolic subgroup of $W(\tilde{A}_n)$ and fully commutative elements of $W(A_n)$ are fully commutative in $W(\tilde{A}_n)$ as well. As for $W(\tilde{A}_{n-1})$, the injection $R_n : W(\tilde{A}_{n-1}) \longrightarrow W(\tilde{A}_n)$ of Lemma 4.1 is a group homomorphism, but it does not preserve full commutativity, as can be seen directly on the image of a_n, namely $\sigma_n a_{n+1} \sigma_n$. When dealing with fully commutative elements, the notion of homomorphism of groups may thus become irrelevant. We introduce the following maps:

Theorem 4.2. *For $w \in W^c(\tilde{A}_{n-1})$, let $I(w)$ (resp. $J(w)$) be the element of $W(\tilde{A}_n)$ obtained by substituting $\sigma_n a_{n+1}$ (resp. $a_{n+1}\sigma_n$) to a_n in the normal form for w. Both expressions are reduced and both $I(w)$ and $J(w)$ are fully commutative, with the same affine length as w. The maps:*

$$I, J : W^c(\tilde{A}_{n-1}) \longrightarrow W^c(\tilde{A}_n)$$

are injective and satisfy $l(I(w)) = l(J(w)) = l(w) + L(w)$. Furthermore the images of I and J intersect exactly on $W^c(A_{n-1})$.

Proof. We use the notation in Section 2 with an additional index n in $W(\tilde{A}_n)$ and $n-1$ in $W(\tilde{A}_{n-1})$. We then see that $I(h_{n-1}(i,j)\, a_n) = h_n(i,j)\, a_{n+1}$.

Let $w \in W^c(\tilde{A}_{n-1})$. By inspection of the normal forms in $W^c(\tilde{A}_{n-1})$ and $W^c(\tilde{A}_n)$ given in Theorem 2.11 one sees directly that $I(w)$ is the normal form of a fully commutative element in $W(\tilde{A}_n)$ and that this process is injective. Indeed we have:

$$\begin{aligned}&I(h_{n-1}(i_1, r_1)\ a_n \ldots h_{n-1}(i_p, r_p)\ a_n(h_{n-1}(j, j+1)\ a_n)^k\ w_r)\\ &\qquad\qquad = h_n(i_1, r_1)\ a_{n+1} \ldots h_n(i_p, r_p)\ a_{n+1}(h_n(j, j+1)\ a_{n+1})^k\ w_r.\end{aligned}$$

As for $J(w)$, the corresponding normal form is obtained as follows. We first observe that

$$J(h_{n-1}(j, j+1)\ a_n) = \lfloor j, 1\rfloor \lceil j+1, n-1\rceil a_{n+1}\sigma_n = \lfloor j, 1\rfloor a_{n+1}\lceil j+1, n\rceil,$$

which implies, since $\sigma_j, \ldots, \sigma_1$ commute with $\sigma_{j+2}, \ldots, \sigma_n$:

$$\begin{aligned}J((h_{n-1}(j, j+1)\ a_n)^2) &= \lfloor j, 1\rfloor a_{n+1}\lceil j+1, n\rceil \lfloor j, 1\rfloor a_{n+1}\lceil j+1, n\rceil\\ &= \lfloor j, 1\rfloor a_{n+1}\lfloor j+1, 1\rfloor \lceil j+2, n\rceil a_{n+1}\lceil j+1, n\rceil,\end{aligned}$$

and inductively:

$$J((h_{n-1}(j, j+1)\ a_n)^k) = \lfloor j, 1\rfloor a_{n+1}\,(h_n(j+1, j+2)a_{n+1})^{k-1}\ \lceil j+1, n\rceil.$$

For $k > 0$ we thus get

$$J((h_{n-1}(j, j+1)a_n)^k w_r) = h_n(j, n+1)a_{n+1}\,(h_n(j+1, j+2)a_{n+1})^{k-1}\ \lceil j+1, n\rceil w_r \tag{7}$$

and we observe that either $w_r = 1$, in which case $\lceil j+1, n\rceil$ has the shape required by Theorem 2.11, or $w_r = \lfloor j, d_1\rfloor \lfloor j+1, d_2\rfloor \ldots \lfloor j+z-1, d_z\rfloor$, in which case

$$\lceil j+1, n\rceil w_r = \lfloor j+1, d_1\rfloor \lfloor j+2, d_2\rfloor \ldots \lfloor j+z, d_z\rfloor \lceil j+z+1, n\rceil$$

which is again the shape of the rightmost element required in Theorem 2.11. The expression (7) is thus the normal form of $J(w)$ when $p = 0$ in Theorem 2.11 (4).

If p is positive in Theorem 2.11 (4), we have:

$$\begin{aligned}J(h_{n-1}(i_p, r_p)\ a_n)\lfloor j, 1\rfloor a_{n+1} &= \lfloor i_p, 1\rfloor \lceil r_p, n-1\rceil a_{n+1}\sigma_n \lfloor j, 1\rfloor a_{n+1}\\ &= \lfloor i_p, 1\rfloor a_{n+1}\lceil r_p, n\rceil \lfloor j, 1\rfloor a_{n+1}.\end{aligned}$$

Thus if $j < r_p - 1$:

$$\begin{aligned}J(h_{n-1}(i_p, r_p)\ a_n)\lfloor j, 1\rfloor a_{n+1} &= \lfloor i_p, 1\rfloor a_{n+1}\lfloor j, 1\rfloor \lceil r_p, n\rceil a_{n+1}\\ &= \lfloor i_p, 1\rfloor a_{n+1}h_n(j, r_p)a_{n+1}\end{aligned}$$

while if $j = r_p - 1$:

$$\begin{aligned}J(h_{n-1}(i_p, r_p)\ a_n)\lfloor j, 1\rfloor a_{n+1} &= \lfloor i_p, 1\rfloor a_{n+1}\lfloor r_p, 1\rfloor \lceil r_p+1, n\rceil a_{n+1}\\ &= \lfloor i_p, 1\rfloor a_{n+1}h_n(j+1, j+2)a_{n+1}.\end{aligned}$$

We proceed in the same way from right to left, noticing that when going from (i_t, r_t) to (i_{t-1}, r_{t-1}) the case $i_t = r_{t-1} - 1$ cannot occur, so we get:

$$J(h_{n-1}(i_{t-1}, r_{t-1})\ a_n)\lfloor i_t, 1\rfloor a_{n+1} = \lfloor i_{t-1}, 1\rfloor a_{n+1} h_n(i_t, r_{t-1}) a_{n+1}.$$

We can now write the normal form of $J(w)$ when $p > 0$ in Theorem 2.11 (4):

$J(h_{n-1}(i_1, r_1)\ a_n \dots h_{n-1}(i_p, r_p)\ a_n(h_{n-1}(j, j+1)\ a_n)^k\ w_r)$ is equal to:

$$\begin{aligned}
&h_n(i_1, n{+}1)a_{n+1} \dots h_n(i_p, r_{p-1})a_{n+1}h_n(j, r_p)a_{n+1}(h_n(j{+}1, j{+}2)a_{n+1})^{k-1}\lceil j{+}1, n\rceil w_r \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } k > 0 \text{ and } j < r_p - 1, \\
&h_n(i_1, n{+}1)a_{n+1} \dots h_n(i_p, r_{p-1})a_{n+1}(h_n(j{+}1, j{+}2)a_{n+1})^{k}\lceil j{+}1, n\rceil w_r \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } k > 0 \text{ and } j = r_p - 1, \\
&h_n(i_1, n{+}1)a_{n+1} \dots h_n(i_p, r_{p-1})a_{n+1}\lceil r_p, n\rceil w_r \qquad\qquad \text{if } k = 0.
\end{aligned}$$

We see as before, by a suitable right shift of $\sigma_{r_p}, \dots, \sigma_n$, that for $k = 0$ the rightmost term $\lceil r_p, n\rceil w_r$ has again the shape required by Theorem 2.11.

The fact that the substitution process adds to the original length the number of occurrences of a_n, i.e. the affine length, is clear. As for the intersection of the images, one only needs to notice that if a reduced expression of a fully commutative element contains an element σ_n to the left of the first a_{n+1} from left to right, then all reduced expressions for this element have the same property. $\square$

We remark that the injection I is well defined only on the set of fully commutative elements. Indeed, substituting $\sigma_n a_{n+1}$ to a_n in the two reduced expressions $\sigma_{n-1}a_n\sigma_{n-1}$ and $a_n\sigma_{n-1}a_n$ gives rise to different elements of $W(\tilde{A}_n)$. It might be the case that I is well defined on the set of elements for which the number of occurrences of a_n in any reduced expression is the same, but as we do not need this, we will not examine it further.

5. The tower of Temperley-Lieb algebras

Let K be an integral domain of characteristic 0 and q be an invertible element in K. We mean by algebra in what follows K-algebra. For x, y in a given ring with identity we define

$$V(x, y) = xyx + xy + yx + x + y + 1.$$

For $n \geq 2$, we define $\widehat{TL}_{n+1}(q)$ to be the algebra with unit given by the set of generators $\left\{g_{\sigma_1}, \dots, g_{\sigma_n}, g_{a_{n+1}}\right\}$, with the following relations (see [11, 0.1, 0.5]):

$$(8)\qquad \begin{cases} g_{\sigma_i}g_{\sigma_j} = g_{\sigma_j}g_{\sigma_i} \text{ for } 1 \leq i,j \leq n \text{ and } |i-j| \geq 2, \\ g_{\sigma_i}g_{a_{n+1}} = g_{a_{n+1}}g_{\sigma_i} \text{ for } 2 \leq i \leq n-1, \\ g_{\sigma_i}g_{\sigma_{i+1}}g_{\sigma_i} = g_{\sigma_{i+1}}g_{\sigma_i}g_{\sigma_{i+1}} \text{ for } 1 \leq i \leq n-1, \\ g_{\sigma_i}g_{a_{n+1}}g_{\sigma_i} = g_{a_{n+1}}g_{\sigma_i}g_{a_{n+1}} \text{ for } i = 1, n, \\ g_{\sigma_i}^2 = (q-1)g_{\sigma_i} + q \text{ for } 1 \leq i \leq n, \\ g_{a_{n+1}}^2 = (q-1)g_{a_{n+1}} + q, \\ V(g_{\sigma_i}, g_{\sigma_{i+1}}) = V(g_{\sigma_1}, g_{a_{n+1}}) = V(g_{\sigma_n}, g_{a_{n+1}}) = 0 \text{ for } 1 \leq i \leq n-1. \end{cases}$$

We set $\widehat{TL}_1(q) = K$. For $n = 1$, the algebra $\widehat{TL}_2(q)$ is generated by two elements: g_{σ_1}, g_{a_2}, with only Hecke quadratic relations. That is:

$$g_{\sigma_1}^2 = (q-1)g_{\sigma_1} + q \quad \text{and} \quad g_{a_2}^2 = (q-1)g_{a_2} + q.$$

Let w be a fully commutative element in $W(\tilde{A}_n)$. Pick a reduced expression of w, say $w = s_1 \cdots s_k$ with $s_i \in \{\sigma_1, ..., \sigma_n, a_{n+1}\}$ for $1 \leq i \leq k$. Then $g_w := g_{s_1} \cdots g_{s_k}$ is a well defined element in $\widehat{TL}_{n+1}(q)$ that does not depend on the choice of a reduced expression of w, and the set $\left\{g_w \mid w \in W^c(\tilde{A}_n)\right\}$ is a K-basis of $\widehat{TL}_{n+1}(q)$ [7, Proposition 1]. The multiplication associated to this basis satisfies, for w, v in $W^c(\tilde{A}_n)$ and s in $\{\sigma_1, ..., \sigma_n, a_{n+1}\}$:

$$\begin{aligned} &g_w g_v = g_{wv} && \text{whenever } l(wv) = l(w) + l(v) \text{ and } wv \in W^c(\tilde{A}_n), \\ &g_s g_w = (q-1)g_w + q g_{sw} && \text{whenever } l(sw) = l(w) - 1. \end{aligned}$$

The classical Temperley-Lieb algebra of type A with n generators, $TL_n(q)$, can be regarded as the subalgebra of $\widehat{TL}_{n+1}(q)$ generated by $\{g_{\sigma_1}, ..., \, g_{\sigma_n}\}$. A K-basis of $TL_n(q)$ is given by $\{g_w \mid w \in W^c(A_n)\}$. We set $TL_0(q) = K$.

The affine Temperley-Lieb algebra $\widehat{TL}_{n+1}(q)$ is the quotient of the affine braid group algebra $K[B(\tilde{A}_n)]$ by the ideal generated by Relations (8). The injection of braid groups $R_n : B(\tilde{A}_{n-1}) \longrightarrow B(\tilde{A}_n)$ of Lemma 3.1 induces the injection of group algebras:

$$\begin{aligned} G_n : K[B(\tilde{A}_{n-1})] &\longrightarrow K[B(\tilde{A}_n)] \\ \sigma_i &\longmapsto \sigma_i \qquad \text{for } 1 \leq i \leq n-1 \\ a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1} \end{aligned}$$

which, as we will see shortly, induces an algebra homomorphism at the Temperley-Lieb level. Yet the possible lack of injectivity of this homomorphism forces us to use different notations for the generators of $\widehat{TL}_n(q)$ and $\widehat{TL}_{n+1}(q)$. In the following proposition we use $\left\{t_{\sigma_1}, ..., t_{\sigma_{n-1}}, t_{a_n}\right\}$ as the set of generators for $\widehat{TL}_n(q)$ satisfying relations (8) (with t replacing g and n replacing $n+1$).

Proposition 5.1. *The injection G_n induces the following morphism of algebras:*

$$\begin{aligned} R_n : \widehat{TL}_n(q) &\longrightarrow \widehat{TL}_{n+1}(q) \\ t_{\sigma_i} &\longmapsto g_{\sigma_i} \text{ for } 1 \le i \le n-1 \\ t_{a_n} &\longmapsto g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1}. \end{aligned}$$

The restriction of R_n to $TL_{n-1}(q)$ is an injective morphism into $TL_n(q)$ and satisfies $R_n(t_w) = g_{I(w)} = g_{J(w)}$ for $w \in W^c(A_{n-1})$.

Proof. We first have to show that the defining relations (8) for $\widehat{TL}_n(q)$, with t replacing g and n replacing $n+1$, are preserved for the images in $TL_n(q)$. This is immediate for those relations that do not involve a_n, using directly relations (8) for $\widehat{TL}_{n+1}(q)$. For the others, this is easily checked. For instance we have:

$$\begin{aligned} R_n(V(t_{\sigma_1}, t_{a_n})) &= g_{\sigma_n} V(g_{\sigma_1}, g_{a_{n+1}}) g_{\sigma_n}^{-1} = 0, \\ R_n(V(t_{\sigma_{n-1}}, t_{a_n})) &= g_{\sigma_n} g_{\sigma_{n-1}} V(g_{\sigma_n}, g_{a_{n+1}}) g_{\sigma_{n-1}}^{-1} g_{\sigma_n}^{-1} = 0. \end{aligned}$$

The last assertion comes from the fact that $W(A_{n-1})$ is parabolic in $W(A_n)$: fully commutative elements of type A_{n-1} embed in those of type A_n and this embedding preserves the normal form. $\square$

Remark 5.2. *The injection $G_n : K[B(\tilde{A}_{n-1})] \longrightarrow K[B(\tilde{A}_n)]$ also induces a homomorphism of the corresponding Hecke algebras. We have shown in [1, Proposition 4.3.3] that this homomorphism is injective for $K = \mathbb{Z}[q, q^{-1}]$ where q is an indeterminate. We will not need this fact in what follows.*

Proposition 5.3. *For any $w \in W^c(\tilde{A}_{n-1})$ of positive affine length, the element $R_n(t_w)$ has the following form:*

$$R_n(t_w) = (-1)^{L(w)} g_{I(w)} + (-\frac{1}{q})^{L(w)} g_{J(w)} + \sum_{\substack{l(x) < l(I(w)) \\ L(x) \le L(w)}} \alpha_x g_x \qquad (\alpha_x \in K).$$

Proof. We prove the statement by induction on the affine length $L(w)$ of an element $w \in W^c(\tilde{A}_{n-1})$. We first assume that $L(w) = 1$. We can write $w = u a_n v$ where u

and v belong to $W^c(A_{n-1})$. We compute:

$$
\begin{aligned}
R_n(t_{a_n}) &= g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n}^{-1} = g_{\sigma_n} g_{a_{n+1}} \left(\frac{1}{q} g_{\sigma_n} + (\frac{1}{q}-1) g_1 \right) \\
&= \frac{1}{q} g_{\sigma_n} g_{a_{n+1}} g_{\sigma_n} + (\frac{1}{q}-1) g_{\sigma_n} g_{a_{n+1}} \\
&= \ -\frac{1}{q} g_{\sigma_n} g_{a_{n+1}} - \frac{1}{q} g_{a_{n+1}} g_{\sigma_n} - \frac{1}{q} g_{\sigma_n} - \frac{1}{q} g_{a_{n+1}} - \frac{1}{q} g_1 + (\frac{1}{q}-1) g_{\sigma_n} g_{a_{n+1}} \\
&= \ - g_{\sigma_n} g_{a_{n+1}} - \frac{1}{q} g_{a_{n+1}} g_{\sigma_n} - \frac{1}{q} g_{\sigma_n} - \frac{1}{q} g_{a_{n+1}} - \frac{1}{q} g_1 .
\end{aligned}
$$

Since R_n is a homomorphism of algebras we have $R_n(t_w) = R_n(t_u) R_n(t_{a_n}) R_n(t_v)$ and, by Proposition 5.1, we have $R_n(t_v) = g_{I(v)} = g_{J(v)}$. Furthermore, multiplying on the left the element $v \in W^c(A_{n-1})$ by any element of $\{\sigma_n a_{n+1}, a_{n+1}\sigma_n, \sigma_n, a_{n+1}\}$ produces a reduced fully commutative word, hence:

$$
R_n(t_{a_n}) R_n(t_v) = \ - g_{\sigma_n a_{n+1} I(v)} - \frac{1}{q} g_{a_{n+1}\sigma_n I(v)} - \frac{1}{q} g_{\sigma_n I(v)} - \frac{1}{q} g_{a_{n+1} I(v)} - \frac{1}{q} g_{I(v)} .
$$

Finally, since u also belongs to $W^c(A_{n-1})$ we get, as claimed:

$$
\begin{aligned}
R_n(t_w) &= g_{I(u)} \ \left(-g_{\sigma_n a_{n+1} I(v)} - \frac{1}{q} g_{a_{n+1}\sigma_n I(v)} - \frac{1}{q} g_{\sigma_n I(v)} - \frac{1}{q} g_{a_{n+1} I(v)} - \frac{1}{q} g_{I(v)} \right) \\
&= \ - g_{I(w)} - \frac{1}{q} g_{J(w)} - \frac{1}{q} \sum_{\substack{l(x)<l(I(w)) \\ L(x)\leq 1}} \alpha_x g_x \qquad (\alpha_x \in K).
\end{aligned}
$$

We now assume that the property holds for any u of positive affine length at most k. Any $w \in W^c(\tilde{A}_{n-1})$ with $L(w) = k+1$ can be written as $w = u a_n v$ where $u \in W^c(\tilde{A}_{n-1})$, $L(u) = k$, $v \in W^c(A_{n-1})$, and $l(w) = l(u) + l(v) + 1$. We have $R_n(t_w) = R_n(t_u) R_n(t_{a_n}) R_n(t_v)$. Using our previous computation of $R_n(t_{a_n}) R_n(t_v)$ and the induction hypothesis we write:

$$
\begin{aligned}
R_n(t_w) = \Big((-1)^{L(u)} g_{I(u)} + (-\frac{1}{q})^{L(u)} g_{J(u)} + \sum_{\substack{l(x)<l(I(u)) \\ L(x)\leq L(u)}} \alpha_x g_x \Big) \\
\Big(- g_{\sigma_n a_{n+1} I(v)} - \frac{1}{q} g_{a_{n+1}\sigma_n I(v)} - \frac{1}{q} g_{\sigma_n I(v)} - \frac{1}{q} g_{a_{n+1} I(v)} - \frac{1}{q} g_{I(v)} \Big).
\end{aligned}
$$

We know from Theorem 4.2 that $I(w) = I(u)\sigma_n a_{n+1} I(v)$ and $J(w) = J(u) a_{n+1}\sigma_n I(v)$ and that both are reduced fully commutative words, hence $g_{I(u)} g_{\sigma_n a_{n+1} I(v)} = g_{I(w)}$ and

$g_{J(u)}g_{a_{n+1}\sigma_n I(v)} = g_{J(w)}$. We obtain the two leading terms

$$(-1)^{L(w)}g_{I(w)} + (-\frac{1}{q})^{L(w)}g_{J(w)}$$

in the formula that we are looking for.

We now observe that the other terms in the development of the product above have affine length at most $L(w)$ and Coxeter length at most $l(I(w))$. The only terms that might have length $l(I(w))$ come from $I(u)$ and $J(u)$ in the first parenthesis, together with $\sigma_n a_{n+1} I(v)$ and $a_{n+1}\sigma_n I(v)$ in the second. The cases of $I(w)$ and $J(w)$ being settled, it remains to prove that $g_{I(u)}g_{a_{n+1}\sigma_n I(v)}$ and $g_{J(u)}g_{\sigma_n a_{n+1} I(v)}$ are linear combinations of basis elements g_x where the length of $x \in W^c(\tilde{A}_n)$ is strictly less than $l(I(w))$.

Remember from Lemma 2.7 that between two consecutive appearances of a_{n+1} we must see one and only one occurrence of σ_n. So the word $I(u)a_{n+1}\sigma_n I(v)$ either is not reduced, hence of length strictly less than $l(I(w))$, or is not fully commutative. In the latter case it contains a braid so the corresponding product $g_{I(u)}g_{a_{n+1}\sigma_n I(v)}$ decomposes, in the Temperley Lieb algebra, into a linear combination of elements g_z with $l(z) < l(I(w))$. Similarly $J(u)\sigma_n a_{n+1} I(v)$ has two occurrences of σ_n between the rightmost occurrence of a_{n+1} and the previous one on the left: it cannot be fully commutative reduced hence the product $g_{J(u)}g_{\sigma_n a_{n+1} I(v)}$ decomposes as before into terms of strictly smaller length. The result follows. □

Theorem 5.4. *The tower of affine Temperley-Lieb algebras*

$$\widehat{TL}_1(q) \xrightarrow{R_1} \widehat{TL}_2(q) \xrightarrow{R_2} \widehat{TL}_3(q) \longrightarrow \cdots \longrightarrow \widehat{TL}_n(q) \xrightarrow{R_n} \widehat{TL}_{n+1}(q) \longrightarrow \cdots$$

is a tower of faithful arrows.

Proof. We need to show that R_n is an injective homomorphism of algebras. A basis for $\widehat{TL}_n(q)$ is given by the elements t_w where w runs over $W^c(\tilde{A}_{n-1})$. Assume that there are non trivial dependence relations between the images of these basis elements. Pick one such relation, say $\sum_w \lambda_w R_n(t_w) = 0$, and let $k = \max\{l(w)+L(w) \mid \lambda_w \neq 0\}$. Using Proposition 5.3 we can write this relation as follows:

$$\sum_{\substack{l(w)+L(w)=k \\ L(w)>0}} \lambda_w((-1)^{L(w)}g_{I(w)} + (-\frac{1}{q})^{L(w)}g_{J(w)}) + \sum_{\substack{l(w)+L(w)=k \\ L(w)=0}} \lambda_w g_{I(w)} + \sum_{l(x)<k} \lambda'_x g_x = 0$$

for suitable coefficients λ'_x (where the x's are elements of $W^c(\tilde{A}_n)$ and $l(x)$ is the length in $W(\tilde{A}_n)$). Since the elements g_y for $y \in W^c(\tilde{A}_n)$ form a basis of $\widehat{TL}_{n+1}(q)$, and since I and J are injective and the intersection of their images is $W^c(A_{n-1})$, we see that all the coefficients λ_w for $l(w) + L(w) = k$ must be 0, a contradiction. $\square$

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