A CLASSIFICATION OF AFFINE FULLY COMMUTATIVE ELEMENTS

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Abstract. We classify fully commutative elements in the affine Coxeter group of type $\tilde{A_n}$. We give a normal form for such elements, then we propose an application of this normal form: we lift these fully commutative elements to the affine braid group of type $\tilde{A_n}$ and we get a form for "fully commutative braids".

1. Introduction and notation

Let (W, S) be a Coxeter system with associated Dynkin Diagram Γ. Let $w \in W$. We know that from a given reduced expression of w we can arrive to any other reduced expression only by applying braid relations [2]. Among these relations there are commutation relations corresponding to the non-neighbours (precisely, t and s with $m_{st} = 2$).

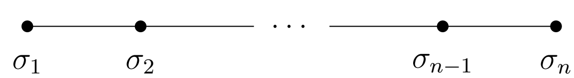
Definition 1.1. *Let w be in W. We call support of w the subset of S consisting of all generators appearing in a (any) reduced expression of w . It is to be denoted by $Supp(w)$.*

We define $\mathscr{L}(w)$ to be the set of $s \in S$ such that $l(sw) < l(w)$, in other terms s appears at the left edge of some reduced expression of w. Similarly we define $\mathscr{R}(w)$.

Definition 1.2. *Elements for which one can pass from any reduced expression to any other one only by applying commutation relations are called fully commutative elements. Usually we denote the set of fully commutative elements by W^c.*

Other than the classification itself, a normal form for fully commutative elements is interesting in its own right since they index a canonical basis of the Temperley-Lieb algebra.

Consider the A-type Coxeter group with n generators $W(A_n)$, with the following Dynkin diagram:



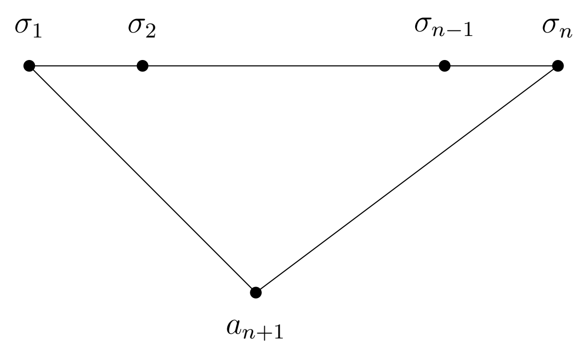
We set $W^c(A_n)$ to be the set of its fully commutative elements, its cardinality is the Catalan number $\frac{1}{n+2}\binom{2(n+1)}{n+1}$. One can prove easily by induction on n $\Big($considering right classes of $W(A_{n-1})$ in $W(A_n)\Big)$ the following well known theorem.

Theorem 1.3. *Let u be any fully commutative element in $W(A_n)$. Then there is a unique reduced expression of u of the form:*

$$u = \sigma_{i_1}\sigma_{i_1-1}...\sigma_{j_1} \quad \sigma_{i_2}\sigma_{i_2-1}...\sigma_{j_2} \quad ... \quad \sigma_{i_p}\sigma_{i_p-1}...\sigma_{j_p},$$

where $1 \leq i_1 < i_2.. < i_p \leq n$, $1 \leq j_1 < j_2.. < j_p \leq n$ and $j_k \leq i_k$ for every $1 \leq k \leq p$.

Now let $W(\tilde{A_n})$ be the affine Coxeter group of $\tilde{A}$-type with $n+1$ generators, with the following Dynkin diagram:



The main result of this paper is presented in theorem 2.4 which is the affine version of theorem 1.3.

In the second section, we give some general definitions. Then we state and prove our general result about the affine fully commutative elements. In the third section we give a consequence of our classification. We lift the fully commutative elements to elements having the same expression in the $\tilde{A}$-type braid group $B(\tilde{A_n})$, or : "fully commutative braids" which will be the key, in a forthcoming paper, to classify traces on the associated affine Temperley-Lieb algebra [3].

2. Fully commutative elements

Let (W, S) be a Coxeter system such that any two elements in S are conjugate in W, in this case fully commutative elements have some additional elegant properties, for example we can reformulate the definition as follows.

Proposition 2.1. *Let (W, S) be such that any two elements in S are conjugate in W. Let $w \in W$. Then w is fully commutative if and only if every s in $Supp(w)$ occurs the same number of times in any reduced expression of w.*

Proof. We omit the proof. $\square$

Hence, in this case, for a fully commutative element w, we can talk of the multiplicity of a simple reflexion in $Supp(w)$. That is if s is in $Supp(w)$, we call the multiplicity of s in w

the number of times s appears in a (hence every) reduced expression of w. The center of our interest in this work is fully commutative elements in $\tilde{A}$-type Coxeter groups, which is an example of Coxeter groups in which any two elements in S are conjugate.

Notice that, in theorem 1.3, if σ_n belongs to $supp(u)$, then σ_n will certainly appear only once, and it is to be equal to σ_{i_p}. Similarly for σ_1: if it belongs to $supp(u)$, then σ_1 will certainly appear only once, and it is equal to σ_{j_1}.

Definition 2.2. *An element u in $W^c(A_n)$ is called full if and only if both σ_n and σ_1 belong to $Supp(w)$. In this case u has a reduced expression of the form:*

$$u = \sigma_{i_1}..\sigma_1\sigma_{i_2}..\sigma_{j_2}\ ...\ \sigma_n..\sigma_{j_p},$$

where $1 \leq i_1 < i_2.. < i_{p-1} < n$, $1 < j_2.. < j_p \leq n$ and $j_k \leq i_k$ for every $1 \leq k \leq n$.

Definition 2.3. *Suppose that u is full, i.e., $u = \sigma_{i_1}..\sigma_1\sigma_{i_2}..\sigma_{j_2}\ ...\ \sigma_n..\sigma_{j_p}$. We say that σ_n is on the left (in u), if and only if $u = \sigma_n\ ...\ \sigma_2\sigma_1$. In all other cases we say that σ_n is on the right.*

2.1. **Classification of $W^c(\tilde{A}_n)$: a normal form.** In this subsection we prove the following theorem, considering remark 2.13.

Theorem 2.4. *Let $2 \leq n$. Let $w \in W(\tilde{A}_n)$ be a fully commutative element such that $a_{n+1} \in supp(w)$. Then, there exists a unique reduced expression of w of the following form:*

$$w = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1}\ ...\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^k u,$$

where: $0 \leq i_1 < i_2\ ...\ < i_p < r_p < r_2\ ...\ < r_1 \leq n+1$, $r_p - i_p \geq 2$, $i_p < j \leq r_p - 1$, $i_1 \leq n$ and $0 \leq k$,

and where u has one of the following forms

- *If $k = 0$, then:*
  $$u = a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2}\ ...\ \sigma_{l_t}..\sigma_{g_t},$$
  where $1 \leq l_1 < l_2.. < l_t \leq n$, $1 \leq g_1 < g_2.. < g_t \leq n$ and $g_f \leq l_f$, for any $1 \leq f \leq t$. With $i_p < l_1$ and $l_1 < r_p$.

- *If $k \geq 1$, then:*
  $$u = a_{n+1}\sigma_j\ ...\ \sigma_{d_1}\sigma_{j+1}\ ...\ \sigma_{d_2}\sigma_{j+2}\ ...\ \sigma_{d_3}\ ...\ \sigma_{j+z}\ ...\ \sigma_{d_{z+1}},$$
  where $d_1 < d_2\ ...\ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

Definition 2.5. *We define the affine length of u in $W^c(A_n)$ to be the multiplicity of a_{n+1} in Supp(u). We denote it by $L(u)$.*

Suppose that w is a fully commutative element in $W(\tilde{A}_n)$. Clearly $L(w) = 0$ expresses the case where a_{n+1} is not in *supp*(w), in other terms w is a fully commutative element in $W(A_n)$. Suppose that $L(w) = m$ where m is positive. Any reduced expression of w is of the form:

$$w = u_1 a_{n+1} u_2 a_{n+1} \ \dots \ u_m a_{n+1} u_{m+1},$$

where u_i is in $W^c(A_n)$, for $1 \leq i \leq m+1$. Moreover, suppose that $L(w) \geq 2$. Then u_i must be full for $2 \leq i \leq m$, otherwise w is not fully commutative.

Before treating the general case, we classify fully commutative elements of $W(\tilde{A}_2)$. This gives an idea about the general proof, in its simplest form.

Theorem 2.6. *Let w be in $W^c(\tilde{A}_2)$. Then there exists $0 \leq k$, such that w has one and only one of the following forms:*

$$\begin{tikzcd} 1 \arrow[dr, dash] & & 1 \arrow[dl, dash] \\ a_3 \arrow[r, dash] & (\sigma_2\sigma_1a_3)^k & \sigma_2 \arrow[l, dash] \\ \sigma_1a_3 \arrow[ur, dash] & & \sigma_2\sigma_1 \arrow[ul, dash] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[dr, dash] & & 1 \arrow[dl, dash] \\ a_3 \arrow[r, dash] & (\sigma_1\sigma_2a_3)^k & \sigma_1 \arrow[l, dash] \\ \sigma_2a_3 \arrow[ur, dash] & & \sigma_1\sigma_2 \arrow[ul, dash] \end{tikzcd}$$

Proof. As we saw above $w = u_1 a_3 u_2 a_3 \ \dots \ u_m a_3 u_{m+1}$, where u_i is in $W^c(A_2)$. If $L(w)$ is 0 or 1 it is clear that we can get it from the tree formulas above. Suppose that $2 \leq L(w)$. Hence u_i is full for $2 \leq i \leq m$. In particular u_2 is full. Actually there are not many choices for u_2, since the only full elements in $W^c(A_2)$ are $\sigma_1\sigma_2$ and $\sigma_2\sigma_1$. The first possibility is that $u_2 = \sigma_1\sigma_2$. Now being a full element, u_3 is definitely equal to $\sigma_1\sigma_2$, otherwise we would have, in w, the following subword $u_1a_3\sigma_1\underbrace{\sigma_2a_3\sigma_2}\sigma_1$. This is not possible since w is fully commutative, thus $u_3 = u_2 = \sigma_1\sigma_2$. The same holds for every u_i for $i \leq m$, i.e., if u_2 is equal to $\sigma_1\sigma_2$ then $w = u_1a_3(\sigma_1\sigma_2a_3)^{m-1}u_{m+1}$.

It is clear that u_1 is in $W^c(A_2)$, and does not end with σ_1, hence u_1 is equal to σ_2 or 1. In the same way, we see that u_{m+1} is in $W^c(A_2)$, it cannot begin with σ_2, so u_{m+1} is equal to σ_1,$\sigma_1\sigma_2$ or 1. In other terms, if u_2 is equal to $\sigma_1\sigma_2$ we get the second tree.

Now suppose that $u_2 = \sigma_2\sigma_1$, then $w = u_1a_3(\sigma_2\sigma_1a_3)^{m-1}u_{m+1}$. With a similar discussion about the first choice of u_2, we see that when $u_2 = \sigma_2\sigma_1$ we get the first tree.

□

In order to simplify, we suppose now that $n \geq 3$ (although many propositions in what follows are valid in $W(\tilde{A}_2)$).

Remark 2.7. *Let u be a full element : $u = \sigma_{i_1}..\sigma_1\sigma_{i_2}..\sigma_{j_2} \ ... \ \sigma_n..\sigma_{j_p}$. Assume that σ_n is on the right in u, hence, by pushing σ_n to the left we see easily that*

$$u = \sigma_i..\sigma_2\sigma_1\sigma_r \ ... \ \sigma_{n-1}\sigma_n x,$$

where $1 \leq i \leq n-1$, $1 \leq r \leq n$ and $i < r$, while $supp(x) \subseteq \{\sigma_2, \sigma_3..\sigma_{n-1}\}$ if x is not 1.

Lemma 2.8. *Let w be in $W^c(\tilde{A}_n)$ such that $L(w) = m \geq 2$. Say:*

$$w = u_1 a_{n+1} u_2 a_{n+1} \ ... \ u_m a_{n+1} u_{m+1}.$$

Assume that σ_n is on the right in u_h, for $2 \leq h \leq m$. Then w has one of the three following forms:

$$w_1 = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)}$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_{z+1}},$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$,

with $i_p < j$ and $j+1 < r_p$,

while $d_1 < d_2 \ ... \ < d_{z+1}$ and $j + c \geq d_{c+1}$, for $0 \leq c \leq z$.

$$w_2 = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)}$$
$$a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_z}$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ r_1 \leq n$, $r_p - i_p \geq 4$, and $p < n/2$,

with $i_p < j$ and $j+2 < r_p$,

while $d_1 < d_2 \ ... \ < d_z$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

$$w_3 = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}$$
$$a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ ... \ \sigma_{l_t}..\sigma_{g_t}.$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$,

with $1 \leq l_1 < l_2.. < l_t \leq n$ and $1 \leq g_1 < g_2.. < g_t \leq n$,

while $i_p < l_1$, $l_1 < r_p$ and $g_k \leq l_k$ for any $1 \leq k \leq n$.

Proof. Before starting with the details of the proof, we call the reader's attention to the fact that our assumption that σ_n is on the right in u_h for $2 \leq h \leq m$ is legitimate, since we know that these u_h are full by the discussion above. Using the discussion above we can write:

$$u_{h-1} = \sigma_{i_h}..\sigma_2\sigma_1\sigma_{r_h} \ ... \ \sigma_{n-1}\sigma_n x_h, \text{ for } 3 \leq h \leq m+1.$$

As above $1 \leq i_h \leq n-1$, $1 \leq r_h \leq n$ and $i_h < r_h$, with $Supp(x_h) \subseteq \{\sigma_2, \sigma_3, ..\sigma_{n-1}\}$. Since a_{n+1} commutes with x_h for all h, we can write $x_i a_{n+1} u_{i+1}$ as $a_{n+1}u'_{i+1}$ with u'_{i+1} full, in which σ_n is on the right. Applying this inductively, we can write w as follows:

$$w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_{m-1}}..\sigma_2\sigma_1\sigma_{r_{m-1}} \ ... \\ ... \ \sigma_{n-1}\sigma_n a_{n+1} u_{m+1},$$

with u_1, u_{m+1}, i_h and r_h as above. Now we have 3 main cases to consider:

(1) $r_1 - i_1 = 1$, i.e., $r_1 = i_1 + 1$.

In this case we do not have many choices for the full elements on the right of u_2: we have one and only one choice, $i_h = i_1$ for all $h \leq m-1$. Thus $j = i_1$. We have:

$$w = u_1(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-1}a_{n+1}u_{m+1}.$$

Here we see that u_{m+1} is a fully commutative element, which need not to be full, yet this element cannot have a reduced expression starting by any simple reflection in $W(A_n)$ but σ_{i_1}. If $u_{m+1} \neq 1$, we can thus, express it as follows:

$$u_{m+1} = \sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_{z+1}},$$

where $d_1 < d_2 \ ... \ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

(2) $r_1 - i_1 = 2$, i.e., $r_1 = i_1 + 2$.

In this case we have, as well, only one choice for the full element on the right of u_2, namely (we set $i_1 = j$):

$$w = u_1 a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-2}a_{n+1}u_{m+1},$$

with conditions on u_{m+1} analogous to those of case (1), that is:

$$u_{m+1} = \sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{i_1+3}..\sigma_{d_3}..\sigma_{j+z}..\sigma_{d_z},$$

where $d_1 < d_2 \ ... \ < d_z$ and $j + c \geq d_c$ for $1 \leq c \leq z$.

(3) $r_1 - i_1 > 2$.

Say $w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n$
$a_{n+1} \ ... \ \sigma_{i_{m-1}}..\sigma_2\sigma_1\sigma_{r_{m-1}} \ ... \ \sigma_{n-1}\sigma_n a_{n+1}u_{m+1}$,

we see that we have to choose r_2 and i_2 such that $i_1 < i_2 < r_2 \leq r_1$, with either $i_2+1 < r_2 < r_1$ or $i_2+1 = r_2 \leq r_1$.Hence, after a finite number of steps, we will face one of the cases (1) or (2). Thus we have one of the next forms:

(1') This is the case related to (1), i.e., we have:

$$w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)} a_{n+1}u_{m+1},$$

here $i_1 < i_2 \ ... \ < i_p < r_p < \ ... \ r_2 < r_1 \leq n$ and $r_p - i_p \geq 3$. We have necessarily $p < n/2$, while u_{m+1} is as in case (1).

(2') This case is related to (2), i.e., we have:

$$w = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)} a_{n+1}u_{m+1},$$

here $i_1 < i_2 \ ... \ < i_p < r_p < \ ... \ r_2 < r_1 \leq n$ and $r_p - i_p \geq 4$. We have necessarily $p < n/2$, while u_{m+1} is as in case (2).

(3') This case is related to some "short" elements (with respect to L):

suppose that we stopped picking pairs (i, r) before having a difference of 1 or 2 between them, hence w is equal to:

$$u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n a_{n+1}u_{m+1},$$

with $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$ and $r_p - i_p \geq 3$. We have necessarily $p < n/2$.

In this case, the choice of u_{m+1} is much more complicated than in the other two cases. It has the form:

$$\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ ... \ \sigma_{l_t}..\sigma_{g_t},$$

where $1 \leq l_1 < l_2.. < l_t \leq n$, $1 \leq g_1 < g_2.. < g_t \leq n$ and $g_k \leq l_k$ for any $1 \leq k \leq n$. And in addition we have $i_p < l_1$ and $l_1 < r_p$.

Actually we do not need a full characterization of the element u_{m+1} in the case of $k = 0$, yet, the conditions given are necessary.

□

Definition 2.9. *In elements of type w_1, the following element is called the short block:*

$$a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n.$$

We call $(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)}$ the convergent block of w_1.

We call $a_{n+1}u_{m+1}$ the residue block of w_1.

Hence we can write $w_1 = u_1$. short block. convergent block. residue block. (We do the same thing for elements of type w_2, in which for example, the convergent block is $(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)}$).

Definition 2.10. *An element of the last two types is called short, if and only if its convergent block is equal to 1.*

Remark 2.11. *It is easy to see that w_1 and w_2 could be unified in the following form:*

$$w_1 = u_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n$$
$$\Big(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n\Big)^{m-\Big(1+L(\textit{the short block})\Big)}$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{i_1+2}..\sigma_{d_3} \ \dots \ \sigma_{i_1+z}..\sigma_{d_{z+1}},$$

where: $i_1 < i_2 \ \dots \ < i_p < r_p < r_2 \ \dots \ < r_1 \leq n$, $r_p - i_p \geq 2$, and p necessarily lesser than $n/2$.

while $i_p < j$, $j+1 < r_p$, $d_1 < d_2 \ \dots \ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

Nevertheless, for the moment, we will go on keeping looking at them as two different forms .

We see that the set of short elements is of finite cardinal, because of the fact that the affine length L of such elements is bounded. Special cases of the last lemma, which come from the 3 types above when $u_{m+1} = 1$, are included in the general formula.

Now we classify the elements of $W^c(\tilde{A}_n)$ with $n \geq 3$.

Consider an arbitrary w in $W^c(\tilde{A}_n)$ with $L(w) \geq 2$, written as:

$$w = u_1 a_{n+1} u_2 a_{n+1} \ \dots \ u_m a_{n+1} u_{m+1}.$$

We start the classification, depending on the choice of u_1 which can be assumed to have one, and only one of the following forms:

(a) u_1 is full, with σ_n on the left.

(b) u_1 is full, with σ_n on the right.

(c) σ_n belongs to $supp(u_1)$ and σ_1 does not.

(d) σ_1 belongs to $supp(u_1)$ and σ_n does not.

(e) $u_1 = 1$.

Suppose that we are in case (a).

We have $u_1 = \sigma_n\sigma_{n-1} \ldots \sigma_2\sigma_1$. In this case there is only one choice for the full elements u_i with $2 \leq i \leq m$, which is to be equal to u_1, hence $w = (\sigma_n\sigma_{n-1} \ldots \sigma_2\sigma_1 a_{n+1})^m u_{m+1}$. Here u_{m+1} is either 1 or $\sigma_n\sigma_{n-1} \ldots \sigma_j$, thus we have two possible types:

$$x_1 = (\sigma_n\sigma_{n-1} \ldots \sigma_2\sigma_1 a_{n+1})^m \sigma_n\sigma_{n-1} \ldots \sigma_i, \text{ for } 1 \leq i \leq n.$$
$$x_2 = (\sigma_n\sigma_{n-1} \ldots \sigma_2\sigma_1 a_{n+1})^m.$$

Suppose that we are in case (b).

Set $u_1 := \sigma_{i_0}..\sigma_2\sigma_1\sigma_{r_0} \ldots \sigma_{n-1}\sigma_n x_0$. It is clear that u_i, for $2 \leq i$, cannot be equal to $\sigma_n \ldots \sigma_1$, hence all the full elements u_i, for $2 \leq i \leq m$, have σ_n on the right. Here we can use the same discussion as in lemma 2.8. We arrive to the possible types, by replacing $u_1 a_{n+1}$ (in which w starts) by 1. Thus we have three possible types (modulo maybe a shift of index to the left):

$$\begin{aligned} x_3 = {} & \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ldots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n \\ & (a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-p} \\ & a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ldots \sigma_{j+z}..\sigma_{d_{z+1}}, \end{aligned}$$

where $i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$,

with $i_p < j$ and $j + 1 < r_p$,

while $d_1 < d_2 \ldots < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

$$\begin{aligned} x_4 = {} & \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ldots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}..\sigma_{n-1}\sigma_n \\ & a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+1)} \\ & a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ldots \sigma_{j+z}..\sigma_{d_z}, \end{aligned}$$

where $i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n$, $r_p - i_p \geq 4$ and $p < n/2$,

with $i_p < j$ and $j + 2 < r_p$,

while $d_1 < d_2 \ldots < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$.

$$\begin{aligned} x_5 = {} & \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ldots \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \\ & a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ldots \sigma_{l_t}..\sigma_{g_t}, \end{aligned}$$

where $i_1 < i_2 \ldots < i_p < r_p < r_2 \ldots < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$,

with $1 \leq l_1 < l_2.. < l_t \leq n$ and $1 \leq g_1 < g_2.. < g_t \leq n$,

while $i_p < l_1$, $l_1 < r_p$ and $g_k \leq l_k$ for any $1 \leq k \leq n$.

Of course, we keep in mind the three special cases x'_3, (resp. x'_4 and x'_5), which are obtained from x_3, (resp. x_4 and x_5) by replacing u_{m+1} by 1.

Suppose that we are in case (c).

Here, u_1 can be written as $u_1 = \sigma_h\sigma_{h+1}..\sigma_{n-1}\sigma_n y$, where $2 \leq h \leq n$, y is in $W(A_{n-1})$ and $\sigma_1 \notin supp(y)$. Hence, we can write w as follows:

$$w = \sigma_h..\sigma_n a_{n+1} u_2 a_{n+1} \ ... \ u_m a_{n+1} u_{m+1}.$$

We see that u_2 (thus every u_i with $2 \leq i \leq m+1$) cannot start with σ_n. That means each u_i, with $2 \leq i \leq m$, is a full element in which σ_n is on the right. By using the lemma 2.8 w is one of the three following elements:

$$\begin{aligned} x_6 = {} & \sigma_h..\sigma_n a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n \\ & \left(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n\right)^{m-(1+p)} \\ & a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_{z+1}}, \end{aligned}$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$,

with $i_p < j$, $j+1 < r_p$, $d_1 < d_2 \ ... \ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$,

while $i_1 < h$, and if $r_1 - i_1 > 1$, then $r_1 < h$.

$$\begin{aligned} x_7 = {} & \sigma_h..\sigma_n a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n \\ & a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{i_1+2}..\sigma_{n-1}\sigma_n \\ & \left(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n\right)^{m-(p+2)} \\ & a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_z}, \end{aligned}$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 4$ and $p < n/2$,

with $i_p < i_1$, $i_1 + 2 < r_p$, $d_1 < d_2 \ ... \ < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$,

while $i_1 < h$, and if $r_1 - i_1 > 1$ then $r_1 < h$.

$$\begin{aligned} x_8 = {} & \sigma_h..\sigma_n a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \\ & a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ ... \ \sigma_{l_t}..\sigma_{g_t}, \end{aligned}$$

here $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$,

where $1 \leq l_1 < l_2.. < l_t \leq n$ and $1 \leq g_1 < g_2.. < g_t \leq n$,

with $g_k \leq l_k$ for any $1 \leq k \leq n$, $i_p < l_1$ and $l_1 < r_p$,

while $i_1 < h$, and if $r_1 - i_1 > 1$ then $r_1 < h$.

As before we keep in mind the three special cases x'_6, (resp. x'_7 and x'_8), which are obtained from x_6, (resp. x_7 and x_8) by replacing u_{m+1} by 1.

Suppose that we are in case (d).

Here, u_1 can be written $\sigma_h\sigma_{h-1}..\sigma_1 y$, where y is in $W(A_{n-1})$, with $\sigma_1 \notin supp(y)$ and $1 \leq h \leq n-1$. Hence we can suppose that

$$w = \sigma_h..\sigma_1 a_{n+1} u_2 a_{n+1} \ ... \ u_m a_{n+1} u_{m+1}.$$

Here, we have two main choices for u_2. The first one is that σ_n is on the left, then w has the following form:

$$x_9 = \sigma_h..\sigma_1 a_{n+1}(\sigma_n\sigma_{n-1} \ ... \ \sigma_2\sigma_1 a_{n+1})^m \sigma_n \ ... \ \sigma_i, \text{ where } 1 \leq h \leq n-1, \text{ and } 1 \leq i \leq n.$$

The second choice is that u_i, for $2 \leq i \leq n$, has σ_n on the right. As above we have three forms, namely:

$$x_{10} = \sigma_h..\sigma_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(1+p)}$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_{z+1}},$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$ and $r_p - i_p \geq 3$ and $p < n/2$,

with $i_p < i_1$, $i_1 + 1 < r_p$, $d_1 < d_2 \ ... \ < d_{z+1}$ and $i_1 + c \geq d_{c+1}$ for $0 \leq c \leq z$,

while $h < i_1$.

$$x_{11} = \sigma_h..\sigma_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{i_1+2}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+2)}$$
$$a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_z},$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 4$ and $p < n/2$,

with $i_p < i_1$, $i_1 + 2 < r_p$, $d_1 < d_2 \ ... \ < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$,

while $h < i_1$.

$$x_{12} = \sigma_h..\sigma_1 a_{n+1}\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}$$
$$a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ ... \ \sigma_{l_t}..\sigma_{g_t},$$

where $i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$, $r_p - i_p \geq 3$ and $p < n/2$,

with $1 \leq l_1 < l_2.. < l_t \leq n$ and $1 \leq g_1 < g_2.. < g_t \leq n$,

while $g_k \leq l_k$, for any $1 \leq k \leq n$, with $i_p < l_1$, $l_1 < r_p$ and $h < i_1$.

Still, we keep in mind the three special cases x'_9, (resp. x'_{10} and x'_{11}), which are obtained from x_9, (resp. x_{10} and x_{11}) by replacing u_{m+1} by 1.

Suppose that we are in case (e).

This case will be a particular case of the above cases. We use the following notation in $W(\tilde{A}_n)$: $\sigma_0 = \sigma_{n+1} = 1$. With this notation we see that types x_1, x_2 and x_9 could be unified in one form, say c_1.

Moreover, x_3 (resp. x_4 and x_5) can be unified in one form with x_6 (resp. x_7 and x_8), when $i_1 = 0$.

Similarly, x_3 (resp. x_4 and x_5) can be unified in one form with x_{10} (resp. x_{11} and x_{12}), when $r_1 = n + 1$.

From what precedes, we formulate our classification by the following corollary.

Corollary 2.12. *Let $n \geq 3$. Let w be in $W^c(\tilde{A}_n)$, such that $L(w) = m \geq 2$. Then w has one of the following forms:*

$$c_1 = \sigma_j \ ... \ \sigma_2\sigma_1(\sigma_n\sigma_{n-1} \ ... \ \sigma_2\sigma_1 a_{n+1})^m \sigma_n\sigma_{n-1} \ ... \ \sigma_i,$$

where $1 \leq i \leq n+1$ and $0 \leq j \leq n$.

$$c_2 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^{m-(p)}$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_{z+1}},$$

where $0 \leq i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n+1$ and $r_p - i_p \geq 3$,

with $i_p < j$ and $j + 1 < r_p$,

while $d_1 < d_2 \ ... \ < d_{z+1}$ and $j + c \geq d_{c+1}$ for $0 \leq c \leq z$.

$$c_3 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n$$
$$a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_{j+1}..\sigma_2\sigma_1\sigma_{j+2}..\sigma_{n-1}\sigma_n)^{m-(p+1)}$$
$$a_{n+1}\sigma_{j+1}..\sigma_{d_1}\sigma_{j+2}..\sigma_{d_2}\sigma_{j+3}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_z},$$

where $0 \leq i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n+1$ and $r_p - i_p \geq 4$,

with $i_p < j$ and $j + 2 < r_p$,

while $d_1 < d_2 \ ... \ < d_z$ and $i_1 + c \geq d_c$ for $1 \leq c \leq z$.

$$c_4 = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p}$$
$$a_{n+1}\sigma_{l_1}..\sigma_{g_1}\sigma_{l_2}..\sigma_{g_2} \ ... \ \sigma_{l_t}..\sigma_{g_t},$$

where: $0 \leq i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n$ *and* $r_p - i_p \geq 3$,

with $1 \leq l_1 < l_2.. < l_t \leq n$ *and* $1 \leq g_1 < g_2.. < g_t \leq n$,

while $i_p < l_1$, $l_1 < r_p$ *and* $g_k \leq l_k$ *for any* $1 \leq k \leq n$.

In all cases p is necessarily bounded by $n/2$ *.*

In order to get to the final form of our classification we shall do one more step, explained in the following remark:

Remarks 2.13. *We set* $\sigma_t = 1$ *when* $0 \geq t$ *or* $t \geq n+1$. *We can actually unify cases* c_1, c_2 *and* c_3 *with the following formula:*

$$\sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ ... \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ ... \ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^K$$
$$a_{n+1}\sigma_j..\sigma_{d_1}\sigma_{j+1}..\sigma_{d_2}\sigma_{j+2}..\sigma_{d_3} \ ... \ \sigma_{j+z}..\sigma_{d_{z+1}},$$

where $0 \leq i_1 < i_2 \ ... \ < i_p < r_p < r_2 \ ... \ < r_1 \leq n+1$ *and* $r_p - i_p \geq 2$,

with $i_p < j$, $j \leq r_p - 1$, $i_1 \leq n$ *and* $1 \leq K$,

while $d_1 < d_2 \ ... \ < d_{z+1}$ *and* $j + c \geq d_{c+1}$ *for* $0 \leq c \leq z$.

Moreover, we see that our formula expresses the elements of $W^c(\tilde{A}_2)$. With this last remark, the proof of theorem 2.4 is completed, after noticing that the way in which we get the general form, ensures the unicity of this form.

3. Fully commutative affine braids

Now we consider the tower of affine braid groups:

$$B(\tilde{A}_0) \xrightarrow{F_0} B(\tilde{A}_1) \xrightarrow{F_1} \ ... \ B(\tilde{A_{n-1}}) \xrightarrow{F_n} B(\tilde{A}_n) \xrightarrow{F_{n+1}} \ ...$$

where $B(\tilde{A}_0)$ is the trivial group. Via F_n, every $B(\tilde{A_{n-1}})$ injects into $B(\tilde{A}_n)$, where F_n is induced by the injection of the B-type braid groups $B(B_n) \hookrightarrow B(B_{n+1})$, noticing that $B(\tilde{A}_n)$ is a subgroup of $B(B_{n+1})$ for $n \geq 0$ (see [1]). The injection F_n is given as follows:

$$
\begin{aligned}
F_n : B(\tilde{A_{n-1}}) &\longrightarrow B(\tilde{A}_n)\\
\sigma_i &\longmapsto \sigma_i \qquad \text{for } 1 \leq i \leq n-1\\
a_n &\longmapsto \sigma_n a_{n+1} \sigma_n^{-1}.
\end{aligned}
$$

We are interested with viewing $B(\tilde{A}_n)$ containing $B(\tilde{A_{n-1}})$. The following computations are done in view of understanding the tower of affine Temperley-Lieb algebras defined in [3].

In what follows we give a kind of normal form for the lift, in $B(\tilde{A}_n)$, of fully commutative elements in $W(\tilde{A}_n)$ see corollary 3.2. We keep the same symbols for the generators of the affine braid group and their images via the natural surjection onto the corresponding affine Coxeter group.

Let $1 \leq n$. Let $\bar{w}$ be in $W^c(\tilde{A}_n)$. The general form of $\bar{w}$ is

$$
\begin{aligned}
\bar{w} = {} & \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots \ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots \ \sigma_{n-1}\sigma_n\\
& (a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^k \bar{u},\\
& \text{where } 0 \leq i_1 < i_2 \ \dots \ < i_p < r_p < \ \dots \ r_2 < r_1 \leq n+1 \text{ and } r_p - i_p \geq 2,\\
& \text{with } i_p < j,\ j \leq r_p - 1,\ i_1 \leq n \text{ and } 0 \leq k,\\
& \text{while } \bar{u} = a_{n+1}\bar{v}, \text{ with } \bar{v} \text{ fully commutative in } W(A_n).
\end{aligned}
$$

We lift $\bar{w}$ (resp. $\bar{u}$ and $\bar{v}$) to w (resp. u and v) in $B(\tilde{A}_n)$, recalling that this lift is legitimate (see [2]). Assume that $j < n$, i.e., w is not of the form $v(a_{n+1}\sigma_n..\sigma_2\sigma_1)^k u$. We show that w has the form:

$h(\sigma_n..\sigma_1 a_{n+1})^m x$, where x is in $B(A_n)$ and h is in $B(\tilde{A_{n-1}})$.

We show as well that $\bar{w}$ has the form:

$\bar{f}(\sigma_n..\sigma_1 a_{n+1})^m \sigma_n..\sigma_i$, where $1 \leq i \leq n+1$, m is a positive integer and $\bar{f}$ is in $W(\tilde{A_{n-1}})$.

Notice that $\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}$ acts on $B(\tilde{A_{n-1}})$ by conjugation, as an automorphism, say ψ, that shifts generators as follows ($\sigma_1 \mapsto a_n \mapsto \sigma_{n-1} \mapsto \sigma_{n-2} \mapsto ..\sigma_2 \mapsto \sigma_1$). We write $(\sigma_n..\sigma_1 a_{n+1})^d h = \psi^d[h](\sigma_n..\sigma_1 a_{n+1})^d$, for any h in $B(\tilde{A_{n-1}})$. The automorphism ψ is of order n. We keep in mind that $a_{n+1}a_n = a_n\sigma_n = \sigma_n a_{n+1}$.

Lemma 3.1. *Let y be $\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1}$ in $B(\tilde{A}_n)$. Let $w = y^k$ with $1 \leq j \leq n$ and $k \geq 1$. Suppose that $k = m(n-j+1)+r$ where $0 \leq r < n-j+1$. Then:*

(1) *If* $m = 0$ *we have:*

$$w = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$

(2) *if* $0 < m$ *we have:*

$$w = \prod_{i=0}^{i=m-1}\psi^i\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right]\psi^m\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\right]$$
$$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1})^m\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$

Proof.

$$\begin{aligned}
\text{We have } y &= \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1} = \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n\sigma_n.\\
\text{Observe that for } j+1 < s \leq n &\text{ we have } \sigma_sy = y\sigma_{s-1}.\\
\text{Hence, } y^2 &= \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n\sigma_n\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1}\\
&= \sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1}\sigma_{n-1}\\
&= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^2\sigma_n\sigma_{n-1} \text{ if } 2 \leq n-j.
\end{aligned}$$

Continuing this way, we can see that whenever $0 \leq r \leq n-j$:

$$y^r = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\sigma_n\sigma_{n-1}..\sigma_s, \text{ with } s+r = n+1. \text{ in particular:}$$

$$y^{n-j} = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_n\sigma_{n-1}..\sigma_{j+1}. \text{ Thus:}$$

$$y^{n-j+1} = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_n\sigma_{n-1}..\sigma_{j+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_na_{n+1}$$

$$= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1}. \text{ We see that:}$$

$$y^{2(n-j+1)} = \psi^0\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right]$$
$$.\psi^1\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right](\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1})^2.$$

In the same way, for $m \geq 0$, considering the action of $\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1}$ on $B(\tilde{A}_{n-1})$, we see that:

$$y^{m(n-j+1)} = \prod_{i=0}^{i=m-1}\psi^i\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right](\sigma_n\sigma_{n-1}..\sigma_2\sigma_1a_{n+1})^m.$$

Finally, let $k = m(n-j+1)+r$, where $0 \leq r < n-j+1$. We have:

$$y^k = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right]$$

$$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^m(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$ Thus,

$$y^k = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right] \psi^m\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\right]$$

$$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^m\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$

□

In particular, for $j=1$, i.e., $w = (\sigma_1\sigma_2..\sigma_n a_{n+1})^k$, we have:

$$y^k = \prod_{i=0}^{i=m-1} \psi^i \left[(\sigma_1..\sigma_{n-1}a_n)^{n-1}\sigma_1..\sigma_{n-1}\right] \psi^m\left[(\sigma_1..\sigma_{n-1}a_n)^r\right](\sigma_n..\sigma_1 a_{n+1})^m\sigma_n\sigma_{n-1}..\sigma_{n+1-r}.$$

Now we go back to the general form of $\bar{w}$, that is:

$$\bar{w} = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots\ \sigma_{n-1}\sigma_n$$
$$(a_{n+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n)^k\bar{u},$$

where $0 \leq i_1 < i_2 \ \dots\ < i_p < r_p < r_2 \ \dots\ < r_1 \leq n+1$ and $r_p - i_p \geq 2$,

with $i_p < j$, $j \leq r_p - 1$, $i_1 \leq n$ and $0 \leq k$,

while $\bar{u} = a_{n+1}\bar{v}$, with $\bar{v}$ fully commutative in $W(A_n)$.

Then, $w = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots\ \sigma_{n-1}\sigma_n a_{n+1}$
$(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1})^k v$,

with conditions similar to those above.

We see that $r_i \leq r_1 - i + 1$, but $r_1 \leq n+1$. which gives: $r_i \leq n - i + 2$ for all i.

Here we treat two main cases:

- $r_1 \leq n$, that is σ_n belongs to the support of $\sigma_{r_1}..\sigma_{n-1}\sigma_n$.
- $r_1 = n+1$, that is σ_n does not belong to the support of $\sigma_{r_1}..\sigma_{n-1}\sigma_n$, (this case covers the case where $p = 0$).

We start by the first case $r_1 \leq n$ (we suppose that $p > 0$).

Set $x := \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}\sigma_n a_{n+1}\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \ \dots\ \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \ \dots\ \sigma_{n-1}\sigma_n a_{n+1}$.

Thus, $w = x(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1})^k v$. Repeating the first step of lemma 3.1 (keeping in mind that $r_i \leq n-i+1$ if $r_1 \leq n$), we see that:

$$\begin{aligned} x &= \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}\sigma_n a_{n+1} \; ... \; \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \; ... \; \sigma_{n-1}\sigma_n a_{n+1}\sigma_{n-(p-1)} \\ &= \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}a_n \; ... \; \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \; ... \; \sigma_{n-1}\sigma_n a_{n+1}\sigma_{n-p+2}\sigma_{n-(p-1)}. \end{aligned}$$

After p steps we see that:

$$x = \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}a_n..\sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \; ... \; \sigma_{n-1}a_n\sigma_n\sigma_{n-1}..\sigma_{n-(p-1)}.$$

Now, we set $\epsilon := n-(p-1)$. We show that $\epsilon \geq j+1$ as follows.

$$j+1 \leq r_p \leq n-(p-1), \text{ that is } j+1 \leq \epsilon.$$

Set $\rho := \sigma_{i_1}..\sigma_2\sigma_1\sigma_{r_1}..\sigma_{n-1}a_n\sigma_{i_2}..\sigma_2\sigma_1\sigma_{r_2}..\sigma_{n-1}a_n \; ... \; \sigma_{i_p}..\sigma_2\sigma_1\sigma_{r_p} \; ... \; \sigma_{n-1}a_n$. Thus,

$$w = \rho\sigma_n\sigma_{n-1}..\sigma_\epsilon(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}\sigma_n a_{n+1})^k v, \text{ with } \rho \in B(\tilde{A_{n-1}}) \text{ and } \epsilon > j+1.$$

Now we have $w = \rho\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k v$, where y is as in lemma 3.1. Every y acts on σ_i in the following way: $\sigma_i y = y\sigma_{i-1}$, for $j+1 < i \leq n$.

If $k=0$, the job is done (this case is included in the general form).

Let $1 \leq k$. We have two main cases:

(1) $1 \leq k < \epsilon - (j+1)$.
(2) $\epsilon - (j+1) \leq k$.

We start by (1). Set $e := \epsilon - k$. We have:

$$\begin{aligned} &\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = y^k\sigma_{e+(n-\epsilon)}\sigma_{e+(n-\epsilon)-1}..\sigma_e. \text{ That is,} \\ &\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = y^k\sigma_{n-k}\sigma_{n-1-k}..\sigma_{\epsilon-k}. \end{aligned}$$

We have $k < \epsilon-(j+1) = n-(p-1)-(j+1) = n-j-p < n-j$. Thus, in the terms of lemma **??**, we are in case (1), (even with the same y). That is:

$$y^k = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^k\sigma_n\sigma_{n-1}..\sigma_{n+1-k}.$$

$$\text{Thus } w = \rho(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^k\sigma_n\sigma_{n-1}..\sigma_{n-k}\sigma_{n-1-k}..\sigma_{\epsilon-k}v$$

which is basically the case $m=0$.

Now we deal with case (2), where $\epsilon-(j+1) \leq k$.

We see that for $k = \epsilon - (j+1)$, we have:

$\sigma_n \sigma_{n-1}..\sigma_\epsilon y^k = y^k \sigma_{n-k}\sigma_{(n-k)-1}..\sigma_{j+1}$. In other terms:

$$\sigma_n \sigma_{n-1}..\sigma_\epsilon (y)^{\epsilon-(j+1)} = (y)^{\epsilon-(j+1)} \sigma_{j+p}\sigma_{j+p-1}..\sigma_{j+1}.$$

for $k \geq \epsilon - (j+1)$ we have $\sigma_n \sigma_{n-1}..\sigma_\epsilon y^k = \sigma_n \sigma_{n-1}..\sigma_\epsilon y^{\epsilon-(j+1)} y^{k-(\epsilon-(j+1))}$.

Now, $\epsilon - (j+1) = n - (p-1) - (j+1) = n - j - p < n - j$. Here, we can apply lemma 3.1 (again we are in the first case). Precisely :

$$\begin{aligned} (y)^{\epsilon-(j+1)} &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{\epsilon-(j+1)} \sigma_n\sigma_{n-1}..\sigma_{n+1-(\epsilon-(j+1))} \\ &= (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{\epsilon-(j+1)} \sigma_n\sigma_{n-1}..\sigma_{j+p+1}. \end{aligned}$$

Set $h := k - (\epsilon - (j+1))$. The case $h = 0$ is easy, if $h \geq 1$ we get:

$$\sigma_n\sigma_{n-1}..\sigma_\epsilon y^k = (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{\epsilon-(j+1)} \sigma_n\sigma_{n-1}..\sigma_{j+p+1}\sigma_{j+p}..\sigma_{j+1} y y^{h-1}.$$

But, $\sigma_n\sigma_{n-1}..\sigma_{j+p+1}\sigma_{j+p}..\sigma_{j+1} y y^{h-1} = \sigma_n..\sigma_{j+1}\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_n a_{n+1} y^{h-1}$,

which is equal to $\sigma_n\sigma_{n-1}..\sigma_1 \underbrace{\sigma_{j+1}..\sigma_{n-1}\sigma_n} a_{n+1} y^{h-1}$,

Which is $\underbrace{\sigma_j..\sigma_{n-2}\sigma_{n-1}} \sigma_n\sigma_{n-1}..\sigma_1 a_{n+1} y^{h-1}$.

Now set $\eta := \rho(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{\epsilon-(j+1)} \sigma_j..\sigma_{n-2}\sigma_{n-1} \in B(\tilde{A_{n-1}})$.
We get $w = \eta\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1} y^{h-1} v$.

(a) If $h - 1 \leq n - j$, we see that:

$$\begin{aligned} w &= \eta\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1} (\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{h-1} \sigma_n\sigma_{n-1}..\sigma_{n+1-(h-1)} v \\ &= \eta\psi\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{h-1}\right] \sigma_n\sigma_{n-1}..\sigma_1 a_{n+1}\sigma_n\sigma_{n-1}..\sigma_{n+1-(h-1)} v. \end{aligned}$$

(b) If $n - j < h - 1$, we see that:

$$w = \eta\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1} \prod_{i=0}^{i=m-1} \psi^i\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right]$$
$$\psi^m\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\right](\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^m \sigma_n\sigma_{n-1}..\sigma_{n+1-r} v.$$

Thus, $w = \eta \prod_{i=1}^{i=m} \psi^i\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^{n-j}\sigma_j..\sigma_{n-1}\right] \psi^{m+1}\left[(\sigma_j..\sigma_2\sigma_1\sigma_{j+1}..\sigma_{n-1}a_n)^r\right]$
$(\sigma_n\sigma_{n-1}..\sigma_2\sigma_1 a_{n+1})^{m+1} \sigma_n\sigma_{n-1}..\sigma_{n+1-r} v$,

where $h-1=m(n-j+1)+r$, with $0 \leq r < n-j+1$.

So in this case (namely, $r_1 \leq n$), we see that w is written as

$$c(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k \sigma_n\sigma_{n-1}..\sigma_i v,$$

where c is in $B(\tilde{A_{n-1}})$, v is in $B(A_n)$, $1 \leq i \leq n+1$ and $0 \leq k$.

Now, we deal with the second main case: $r_1 = n+1$. Here σ_n is not in the support of $\sigma_{r_1}..\sigma_{n-1}\sigma_n$ (which is equal in this case to 1). Hence, we can suppose that the element in question is of the form $\sigma_{i_0}..\sigma_2\sigma_1 a_{n+1} w$, where $0 \leq i_0 < r_1$. Here we have $i_0 < n$, since $i_0 = n$ is the case of positive powers of $\sigma_n..\sigma_1\sigma_1 a_{n+1}$. Moreover, when $i_0 = 0$, then the element in question is of the form $a_{n+1}w$, which is the case $p = 0$. As a consequence of this discussion we get the following corollary.

Corollary 3.2. *Let $\bar{w}$ be fully commutative in $W(\tilde{A}_n)$, where $2 \leq n$. Let w be the corresponding element in $B(\tilde{A}_n)$, as above. Then w can be written in one and only one of the following two forms:*

$$c(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k v,$$
$$\text{or } \sigma_{i_0}..\sigma_2\sigma_1 a_{n+1} c(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k v.$$

Here, c is in $B(\tilde{A_{n-1}})$, while v is in $B(A_n)$.

Moreover, $\bar{w}$ can be written in one, and only one of the following two forms (deduced from the above two forms, considering the left classes of $W(A_{n-1})$ in $W(A_n)$):

$$\bar{d}(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k \sigma_n\sigma_{n-1}..\sigma_i,$$
$$\text{or } \sigma_{i_0}..\sigma_2\sigma_1 a_{n+1}\bar{d}(\sigma_n\sigma_{n-1}..\sigma_1 a_{n+1})^k \sigma_n\sigma_{n-1}..\sigma_i.$$

Here $\bar{d}$ is in $W(\tilde{A_{n-1}})$, where $1 \leq i \leq n$, with $0 \leq i_0 \leq n-1$ and $0 \leq k$.

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