Assignment 3

a) The assignments have to be done individually.

b) The assignments have to be answered in ENGLISH.

c) The answers have to be uploaded to Toledo trough ”assignments”, not using any other Toledo tool and not by email.

d) Clearly indicate your name and student number in the uploaded answer.

e) Indicate the time that you have (approximately) spend on the assignment. This will not be taken into account in any way for the quotation but will give us an idea of the load of the assignments. We estimate the load of maximally two hours and half per assignment.

f) You will be rewarded for correct answers, not for the format of your answer. Scanned handwritten answers are therefore preferred. Do not spend time in typesetting the document. Just make sure that everything is readable and clear! As the final exam is handwritten, this is a useful exercise.

g) The name of the uploaded document should be **rXXXXXXX.pdf** where you replace XXXXXXX with your student number.

h) You have to upload your solution through the assignment, **not using the file exchange**.

i) Clearly indicate the **final answer** of the question by placing it in a $\boxed{\text{box}}$.

j) **Explicitly write down all reasoning used to get to an answer.**

Sum-product algorithm

Consider a probabilistic model of students' performance during multiple weeks $i \in \{1, 2, 3\}$:

- the variable C_i indicates whether a student attended to the lesson or not (respectively $C_i = 1$ and $C_i = 0$);
- the variable K_i indicates whether a student is knowledgeable on the week's topic or not (respectively $K_i = 1$ and $K_i = 0$);
- the variable A_i indicates whether a student passes the weekly assignment or not (respectively $A_i = 1$ and $A_i = 0$).

Without additional evidence, we assume a prior $P(C_i = 1) = 0.7$.

If the student is knowledgeable about the previous week's topic, or during the first week (we assume that the student meets the prerequisites of the course), the probability of being knowledgeable is 0.8 if he/she attended the lesson and 0.4 if he/she didn't attend the lesson (some knowledge about the general topic of the course carry over). If instead the student is not knowledgeable about the previous week's topic, then the probability of being knowledgeable drops to 0.5 if he/she attended the lesson and to 0.1 if he/she didn't attend the lesson.

Finally, the probability of passing an assignment is 0.9 if the student is knowledgeable on the respective topic and 0.01 otherwise.

Note: *In this exercise we assume that the model is fully unrolled, that is, the parameters are considered different even if they share the same value according to the description above. For instance,* $\theta_{K_2=1|K_1=1,C_2=1} = P(K_2 = 1|K_1 = 1, C_2 = 1)$ *and* $\theta_{K_3=1|K_2=1,C_3=1} = P(K_3 = 1|K_2 = 1, C_3 = 1)$ *are different, although they both have value* 0.8.

1. Draw
   - a the belief network of the model above
   - b the corresponding factor graph (give a name to the factors too!)
2. Write
   - a the factorization of the joint probability $P(C_1, C_2, C_3, K_1, K_2, K_3, A_1, A_2, A_3)$
   - b how many parameters are needed for this model.
3. We are interested in computing the probability that a student is knowledgeable about the third week's topic ($P(K_3 = 1)$) using the sum-product algorithm **on the factor graph**.
   - a Which node should be chosen as the root in order to minimise the number of messages?
   - b How many messages would then be sent?
   - c How many messages would be needed to compute every marginal instead?
   - d Compute $P(K_3 = 1)$
   - e Compute the probability of the same event when observing that the student attended to the first lesson only, i.e. $P(K_3 = 1|C_1 = 1, C_2 = C_3 = 0)$

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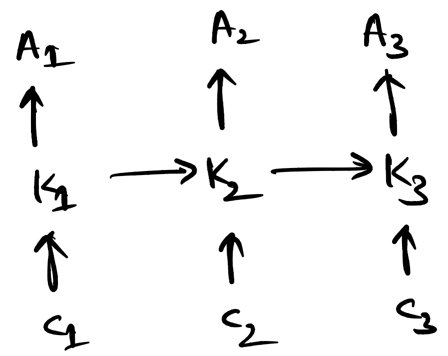
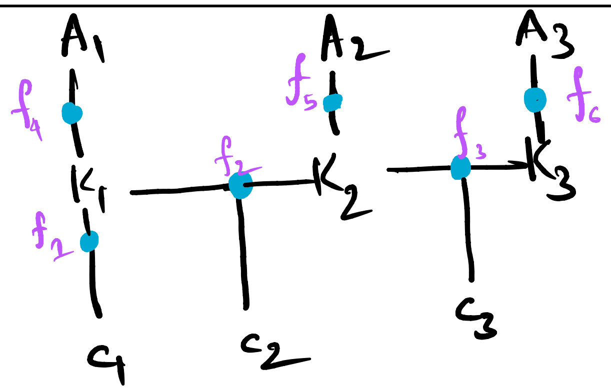
Solution

Fill in each answer in the respective box. Intermediate computations can be done on a separate page.

1a



1b



2a $P(\{C_i\}_{i=1}^3, \{A_i\}_{i=1}^3, \{K_i\}_{i=1}^3) = P(C_1)P(C_2)P(C_3)\,P(A_1|K_1)P(A_2|K_2)P(A_3|K_3)P(K_1|C_1)\,P(K_2|K_1,C_2)\,P(K_3|K_2,C_3)$

2b **15**

3a K_3

3b 14

3c 28

3d $P(K_3=1) = 0.5552$

3e $P(K_3=1 | C_1=1, C_2=0, C_3=0) = 0.181$

1 .a)

K_{i-1}	K_i	C_i	$P(K_i \mid K_{i-1}, C_i)$
0	1	0	0.1
1	1	0	0.4
0	1	1	0.5
1	1	1	0.8
0	0	0	0.9
1	0	0	0.6
0	0	1	0.5
1	0	1	0.2

$\forall i = 2, 3$

K_1, K_2, K_3

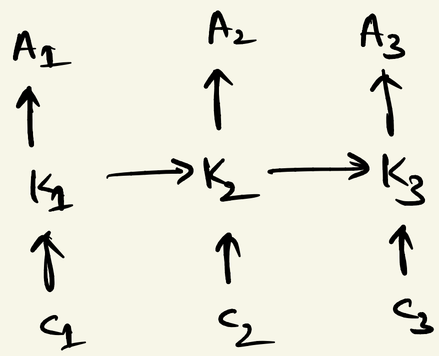
$K_2 = 1$

A_i \ K_i	0	1
0	0.99	0.1
1	0.01	0.9

$\forall\ i = 1, 2, 3$

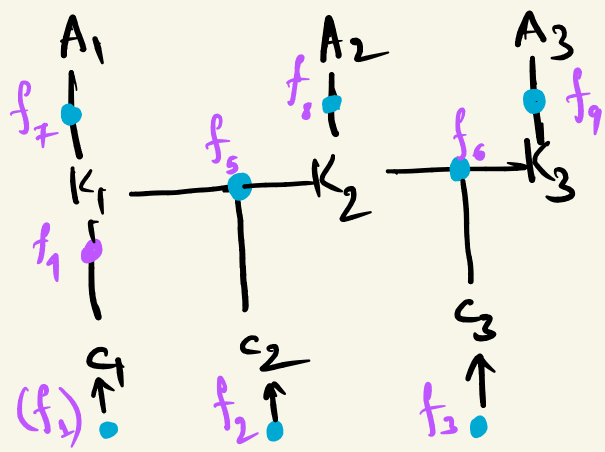
2) a)

$$P(C_1, C_2, C_3, K_1, K_2, K_3, A_1, A_2, A_3)$$



$$= P(C_1)P(C_2)P(C_3)P(A_1|K_1)P(A_2|K_2)$$
$$P(A_3|K_3)P(K_1|C_1)\ P(K_2|K_1, C_2)$$
$$P(K_3|K_2, C_3)$$

$$2\cdot 4 + 2^2\cdot 2 + 3$$
$$= 8 + 8 + 3 = 19$$



$$= \frac{1}{Z}\Big(f_1(K_1, C_1)\, f_2(K_1, K_2, C_2)\, f_3(K_2, C_3, K_3)$$
$$f_4(A_1, K_1)\, f_5(A_2, K_2)\, f_6(A_3, K_3)\Big)$$

where

$f_4(A_1, K_1) = P(A_1\|K_1)$	$f_1(K_1, C_1) = P(K_1\|C_1)$
$f_5(A_2, K_2) = P(A_2\|K_2)$	$f_2(K_1, K_2, C_2) = P(K_2\|K_1, C_2)$
$f_6(A_3, K_3) = P(A_3\|K_3)$	$f_3(K_2, K_3, C_3) = P(K_3\|K_2, C_2)$

where Z is a normalizing constant, $(Z = 1)$

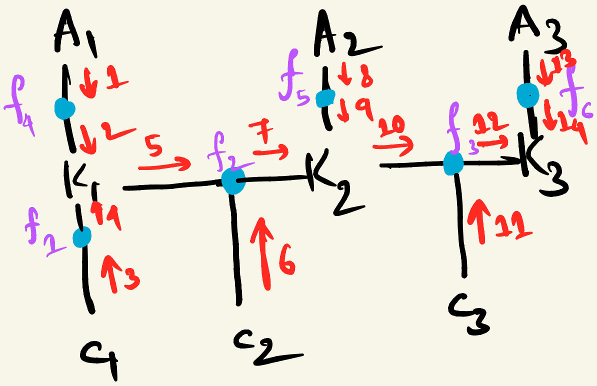
b) Parameters needed

$$1+1+1+1+1+1+1+2^2+2^2$$

$$= 15$$

3) Let $X = \{C_1, C_2, C_3, K_1, K_2, K_3, A_1, A_2, A_3\}$

a>



① $$\mu_{A_1\to f_4}(A_1=1) = \mu_{A_1\to f_4}(A_1=0) = 1$$

⑧ $$\mu_{A_2\to f_5}(A_2=1) = \mu_{A_2\to f_5}(A_2=0) = 1$$

⑬ $$\mu_{A_3\to f_6}(A_3=1) = \mu_{A_3\to f_6}(A_3=0) = 1$$

② $$\mu_{f_4\to K_1}(K_1=1) = \sum_{A_1} f_4(1,A_1)\,\mu_{A_1\to f_4}(A_1)$$
$$= f_4(1,0) + f_4(1,1)$$
$$= 0.9 + 0.1 = 1$$

③, ⑥, ⑪

$$\mu_{C_i\to f_{3+i}}(C_i=0) = \mu_{f_i\to C_i}(C_i=0) = 0.3 \quad \forall i=1,2,3$$

$$\mu_{C_i\to f_{3+i}}(C_i=1) = \mu_{f_i\to C_i}(C_i=1) = 0.7$$

④

$$\mu_{f_1\to K_1}(K_1) = \sum_{C_1} f_1(K_1,C_1)\,\mu_{C_1\to f_1}(C_1)$$

$$\mu_{f_1\to K_1}(1) = f_1(1,0)\,\mu_{C_1\to f_1}(0) + f_1(1,1)\,\mu_{C_1\to f_1}(1)$$
$$= 0.3\times 0.4 + 0.8\times 0.7 = 0.12 + 0.56 = 0.68$$

$$\mu_{f_1 \to K_1}(0) = f_1(0,0)\,\mu_{C_1 \to f_1}(0) + f_1(0,1)\,\mu_{C_1 \to f_1}(1)$$
$$= 0.6 \times 0.3 + 0.2 \times 0.7$$
$$= 0.18 + 0.14$$
$$= 0.32$$

5

$$\mu_{K_1 \to f_2}(K_1 = 1) = \mu_{f_4 \to K_1}(K_1 = 1)\,\mu_{f_1 \to K_1}(K_1 = 1)$$
$$= 1 \times 0.68 = 0.68$$
$$\mu_{K_1 \to f_2}(K_1 = 0) = \mu_{f_4 \to K_1}(K_1 = 0)\,\mu_{f_1 \to K_1}(K = 0)$$
$$= 0.32$$

7

$$\mu_{f_2 \to K_2}(K_2) = \sum_{K_1, C_2} f_2(K_1, K_2, C_2)\,\mu_{K_1 \to f_2}(K_1)\,\mu_{C_2 \to f_2}(C_2)$$

$$\mu_{f_2 \to K_2}(0) = f_2(0,0,0) \times 0.32 \times 0.3 +$$
$$f_2(0,0,1) \times 0.32 \times 0.7 +$$
$$f_2(1,0,0) \times 0.68 \times 0.3 +$$
$$f_2(1,0,1) \times 0.68 \times 0.7$$
$$= 0.9 \times 0.32 \times 0.3 + 0.5 \times 0.32 \times 0.7 +$$
$$0.6 \times 0.68 \times 0.3 + 0.2 \times 0.68 \times 0.7$$
$$= 0.416$$

$$\mu_{f_2 \to K_2}(1) = 0.32 \times 0.3 \times 0.1 + 0.21 \times 0.5 + 0.68 \times 0.3 \times 0.4$$
$$+ 0.68 \times 0.7 \times 0.8$$
$$= 0.584$$

9, 14

$$\mu_{f_5 \to K_2}(K_2) = \sum_{A_2} f_5(A_2, K_2) \cdot \mu_{A_2 \to f_5}(A_2)$$
$$= \sum_{A_2} f_5(A_2, K_2)$$

$$\mu_{f_5 \to K_2}(1) = f_5(0,1) + f_5(1,1) = 1$$
$$\mu_{f_5 \to K_2}(0) = f_5(1,0) + f_5(0,0) = 1$$
$$\mu_{f_6 \to K_3}(1) = \mu_{f_6 \to K_3}(0) = 1$$

10

$$\mu_{K_2 \to f_3}(K_2) = \mu_{f_5 \to K_2}(K_2) \; \mu_{f_2 \to K_2}(K_2)$$

$$\mu_{K_2 \to f_3}(0) = \mu_{f_5 \to K_2}(0) \; \mu_{f_2 \to K_2}(0)$$
$$= 0.416$$

$$\mu_{K_2 \to f_3}(1) = 0.584$$

12

$$\mu_{f_3 \to K_3}(K_3) =$$

$$\sum_{K_2, C_3} f_3(K_2, K_3, C_3)\, \mu_{K_2 \to f_3}(K_2)\, \mu_{C_3 \to f_3}(C_3)$$

$$\mu_{f_3 \to K_3}(1) = f_3(0,1,0) \times 0.416 \times 0.3 + f_3(0,1,1) \times 0.416 \times 0.7 + f_3(1,1,0) \times 0.584 \times 0.3 + f_3(1,1,1) \times 0.584 \times 0.7$$

K_{i-1}	K_i	C_i	$P(K_i \mid K_{i-1}, C_i)$
0	1	0	0.1
1	1	0	0.4
0	1	1	0.5
1	1	1	0.8
0	0	0	0.9
1	0	0	0.6
0	0	1	0.5
1	0	1	0.2

$$= 0.1 \times 0.416 \times 0.3 + 0.5 \times 0.416 \times 0.7 + 0.4 \times 0.584 \times 0.3 + 0.8 \times 0.584 \times 0.7$$

$$= 0.416 \times (0.03 + 0.35) + 0.584 \times (0.12 + 0.56)$$

$$= 0.416 \times 0.38 + 0.584 \times 0.68$$

$$= 0.5552$$

$$\mu_{f_3 \to K_3}(0) = 0.416 \times (f_6(0,0,0) \times 0.3 + f_6(0,0,1) \times 0.7) + 0.584 \times (f_6(1,0,0) \times 0.3 + f_6(1,0,1) \times 0.7)$$

$$= 0.416 \times (0.27 + 0.35) + 0.584 \times (0.18 + 0.14)$$

$$= 0.4448$$

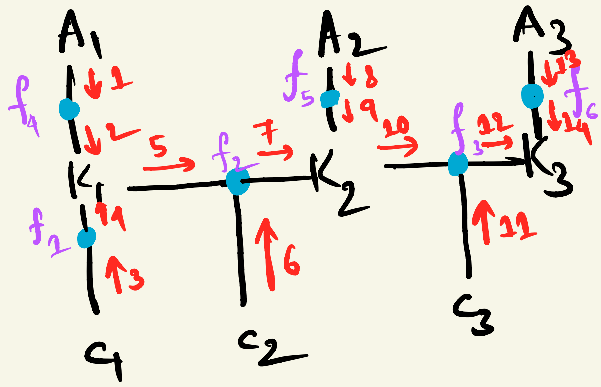
$$P(K=1) = \mu_{f_3 \to K_3}(1) = 0.5552$$

3) ey

$$P(K_3=1 \mid c_1=1, c_2=0, c_3=0)$$

4

$$\mu_{f_1 \to K_1}(K_1) = f_1(K_1, 1)\, \mu_{c_1 \to f_1}(1)$$



$$\mu_{f_1 \to K_1}(1) = f_1(1,1)\, \mu_{c_1 \to f_1}(1)$$
$$= 0.8 \times 0.7 = 0.56$$

$$\mu_{f_1 \to K_1}(0) = f_1(0,1)\, \mu_{c_1 \to f_1}(1)$$
$$= 0.2 \times 0.7$$
$$= 0.14$$

5

$$\mu_{K_1 \to f_2}(K_1) = \mu_{f_4 \to K_1}(K_1) \cdot \mu_{f_1 \to K_1}(K_1)$$

$$\mu_{K_1 \to f_2}(1) = 0.56$$
$$\mu_{K_1 \to f_2}(0) = 0.14$$

9

$$\mu_{f_2 \to k_2}(k_2) = \sum_{k_1, c_2} f_2(k_1, k_2, 0)\, \mu_{k_1 \to f_2}(k_1)\, \mu_{c_2 \to f_2}(0)$$

$$\mu_{f_2 \to k_2}(0) = f_2(0,0,0) \times 0.14 \times 0.3 +$$

$$f_2(1,0,0) \times 0.56 \times 0.3$$

$$= 0.9 \times 0.32 \times 0.3 +$$

$$0.6 \times 0.68 \times 0.3 = 0.1944$$

$$\mu_{f_2 \to k_2}(1) = f_2(0,1,0) \times 0.14 \times 0.3 +$$

$$f_2(1,1,0) \times 0.56 \times 0.3$$

$$= 0.03 \times 0.14 + 0.12 \times 0.56$$

$$= 0.0714$$

10

$$\mu_{k_2 \to f_3}(k_2) = \mu_{f_5 \to k_2}(k_2)\, \mu_{f_2 \to k_2}(k_2)$$

$$\mu_{k_2 \to f_3}(0) = \mu_{f_5 \to k_2}(0)\, \mu_{f_5 \to k_2}(0)$$

$$= 0.1944$$

$$\mu_{k_2 \to f_3}(1) = 0.0714$$

12

$$\mu_{f_3 \to k_3}(k_3) =$$

$$\sum_{k_2} f_3(k_2, k_3, 0)\, \mu_{k_2 \to f_3}(k_2)\, \mu_{c_3 \to f_3}(0)$$

$$\mu_{f_3 \to k_3}(1) = f_3(0,1,0)\, \mu_{k_2 \to f_3}(0)\, \mu_{c_3 \to f_3}(0)$$

$$+ f_3(1,1,0)\, \mu_{k_2 \to f_3}(1)\, \mu_{c_3 \to f_3}(0)$$

$$= 0.1 \times 0.1944 \times 0.3 + 0.4 \times 0.0714 \times 0.3$$

$$= 0.0144$$

$$\mu_{f_6 \to k_3}(0) = f_3(0,0,0) \times 0.1944 \times 0.3 + f_3(1,0,0) \times 0.0714 \times 0.3$$

$$= 0.27 \times 0.1944 + 0.18 \times 0.0714$$

$$= 0.06534$$

$$P(k_3 = 1 \mid c_1 = 1, c_2 = 0, c_3 = 0) = \frac{0.0144}{0.0144 + 0.06534}$$

$$= 0.181$$