Assignment 5

a) The assignments have to be done individually.

b) The assignments have to be answered in ENGLISH.

c) The answers have to be uploaded to Toledo trough "assignments", not using any other Toledo tool and not by email.

d) Clearly indicate your name and student number in the uploaded answer.

e) Indicate the time that you have (approximately) spend on the assignment. This will not be taken into account in any way for the quotation but will give us an idea of the load of the assignments. We estimate the load of maximally two hours and half per assignment.

f) You will be rewarded for correct answers, not for the format of your answer. Scanned handwritten answers are therefore preferred. Do not spend time in typesetting the document. Just make sure that everything is readable and clear! As the final exam is handwritten, this is a useful exercise.

g) The name of the uploaded document should be **rXXXXXXX.pdf** where you replace XXXXXXX with your student number.

h) You have to upload your solution through the assignment, **not using the file exchange**.

i) Clearly indicate the **final answer** of the question by placing it in a box.

1 Direct Sampling

Use direct sampling on $p(X, Y, Z)$.

$p(X, Y, Z = 0)$	$Y = 0$	$Y = 1$
$X = 0$	0.1	0.05
$X = 1$	0.2	0.15

$p(X, Y, Z = 1)$	$Y = 0$	$Y = 1$
$X = 0$	0.1	0.3
$X = 1$	0.05	0.05

Compute 5 samples from $p(X, Y, Z)$. Use the following sequence as uniform random number generator:

```
[0.90, 0.61 , 0.42, 0.23, 0.33]
```

Furthermore, use the following ordering:

$$
\begin{aligned}
&(X = 0, Y = 0, Z = 0)\\
&(X = 1, Y = 0, Z = 0)\\
&(X = 0, Y = 1, Z = 0)\\
&(X = 1, Y = 1, Z = 0)\\
&(X = 0, Y = 0, Z = 1)\\
&(X = 1, Y = 0, Z = 1)\\
&(X = 0, Y = 1, Z = 1)\\
&(X = 1, Y = 1, Z = 1)
\end{aligned}
$$

2 Rejection Sampling

Given the trapezoidal distribution $p(x)$ in Figure 1, perform rejection sampling. Use the uniform distribution $\mathcal{U}(0, b)$ as proposal distribution $q(x)$. In other words we have: $p^*(x) \leq M \times \mathcal{U}(0, b)$

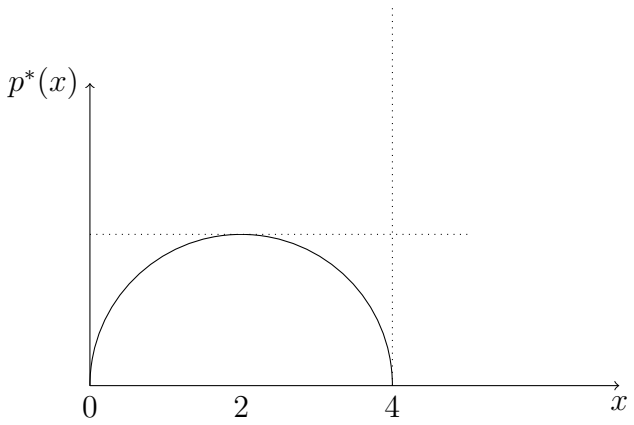

Figure 1: Semi-Circle distribution

$$
p^*(x) = \begin{cases} \sqrt{4 - (x-2)^2} & \text{if } 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases} \tag{1}
$$

- Find a suitable value for b and for the coefficient M that is needed to rescale the proposal. In particular, make the choice that allows to waste as few samples as possible.
- Use the following sequence as uniform random number generator.

  `[0.08, 0.09, 0.71, 0.94, 0.7 , 0.53, 0.56, 0.32, 0.38, 0.16]`

- Graphically illustrate the rejection sampling process trying to get a sample 5 times. Shade the acceptance and rejection regions and plot the samples obtained from the given random numbers.

- Clearly indicate which samples were accepted and which samples were rejected.

3 Ancestral Sampling

Consider the bayesian network in Figure 2. Let a, b, c, d be 4 Bernoulli random

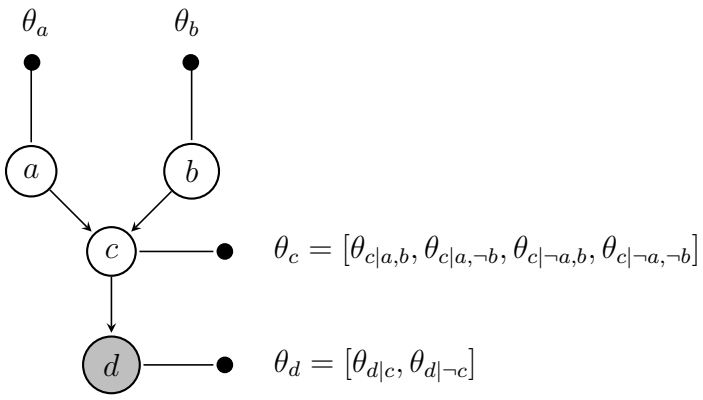

Figure 2

variables and their probability distributions can be expressed according to the following Conditional Probability Tables (CPTs):

$$p(a=1) = 0.7 \quad p(d=1|c=0) = 0.3$$
$$p(b=1) = 0.4 \quad p(d=1|c=1) = 0.8$$

$$p(c=1|a=0,b=0) = 0.1 \quad p(c=1|a=1,b=0) = 0.2$$
$$p(c=1|a=0,b=1) = 0.6 \quad p(c=1|a=1,b=1) = 0.9$$

Perform ancestral sampling in order to sample from $p(a, b, c|d = 1)$. Attempt to sample 5 times and show which samples are rejected and which samples are accepted. Use the following sequence as uniform random number generator:

```
[0.91, 0.25, 0.10, 0.01,
0.51, 0.34, 0.52, 0.77,
0.35, 0.20, 0.16, 0.28,
0.17, 0.24, 0.32, 0.71,
0.44, 0.01, 0.59, 0.15]
```

Assume also that the variables are sampled in lexicographic order. More concretely: the first sample 0.91 is for a and the second 0.25 is for b.

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Solution

Fill in each answer in the respective box. Intermediate computations can be done on a separate page.

1. Sample from $p(X, Y, Z)$:

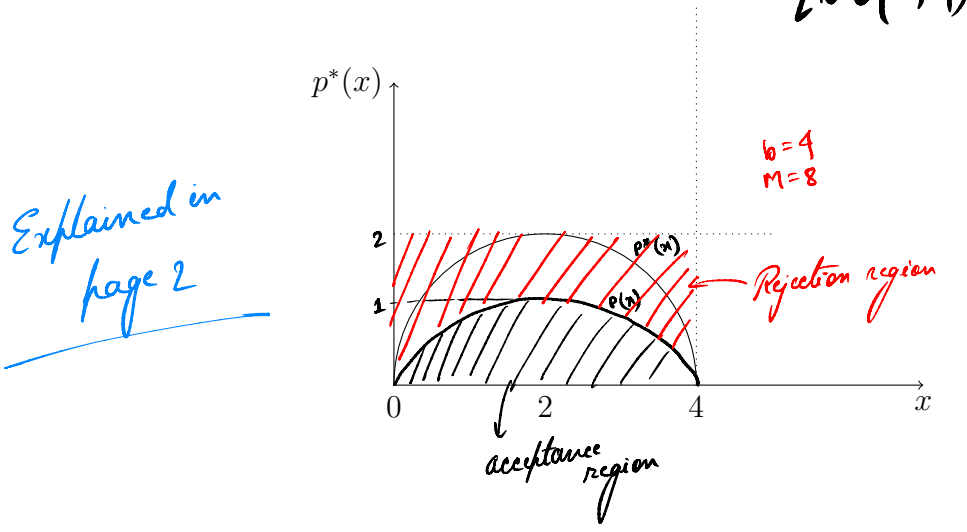
$u \in \texttt{seq}$	(X, YZ)
0.90	(0, 1, 1)
0.61	(1, 0, 1)
0.42	(1, 1, 0)
0.23	(1, 0, 0)
0.33	(0, 1, 0)

2.1 $b = 4$

2.2 $M = 8$

2.3 Graphically illustrate the rejection sampling process trying to get a sample 5 times. Shade the acceptance and rejection regions and plot the samples obtained from the given random numbers.

$q \sim U(0, 4)$



Explained in page 2

2.4. Clearly indicate which samples where accepted and which samples were rejected:

$u \in \texttt{seq}$	Sample Accepted (Yes/No)
[0.08, 0.09]	yes
[0.71, 0.94]	NO
[0.7, 0.53]	yes
[0.56, 0.32]	yes
[0.38, 0.16]	yes

3. Sample from $p(a, b, c|d = 1)$:

$u \in$ `seq`	a	b	$c\|a, b$	$d\|c$	Accepted (Yes/No)
$[0.91, 0.25, 0.10, 0.01]$	0	1	1	1	yes
$[0.51, 0.34, 0.52, 0.77]$	1	1	1	1	yes
$[0.35, 0.20, 0.16, 0.28]$	1	1	1	1	yes
$[0.17, 0.24, 0.32, 0.71]$	1	1	1	1	yes
$[0.44, 0.01, 0.59, 0.15]$	1	1	1	1	yes

1.

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$c_0 = 0$

$c_{000} = P(X=0, Y=0, Z=0) = 0.1$

$c_{100} = P(X=1, Y=0, Z=0) = 0.1 + 0.2 = 0.3$

$c_{010} = P(X=0, Y=1, Z=0) = 0.35$

$c_{110} = P(X=1, Y=1, Z=0) = 0.5$

$c_{001} = P(X=0, Y=0, Z=1) = 0.6$

$c_{101} = P(X=1, Y=0, Z=1) = 0.65$

$c_{011} = P(X=0, Y=1, Y=1) = 0.95$

$c_{111} = P(X=1, Y=1, Z=1) = 1$

The five samples corresponding to the sequence of random numbers are

$(0,1,1), (1,0,1), (1,1,0), (1,0,0), (0,1,0)$

2) $p^*(x) = \begin{cases} \sqrt{4-(x-2)^2} & \text{if } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$

$\therefore \boxed{b = 4}$

$x_{max} = \underset{x}{\text{argmax}}\, p^*(x)$

$= 2$

$\boxed{M = b \times x_{max} = 8}$ (Since $M \times q(x) = 2$)

Sampling Process

(i) $x^1 = 0.08$

$u = 0.09$

$a = \dfrac{p^*(x^1)}{M \cdot q(x^1)} = 0.28 > 2$

accepted

(ii) $x^2 = 0.71$

$u = 0.94$

$a = 0.76 < u$

rejected

(iii) $x^3 = 0.7$

$u = 0.53$

$a = 0.76 > 0.53$

accepted

iii) $x^4 = 0.56$

$u = 0.32$

$a = 0.69 > 0.32$

accepted

iv) $x^5 = 0.38$

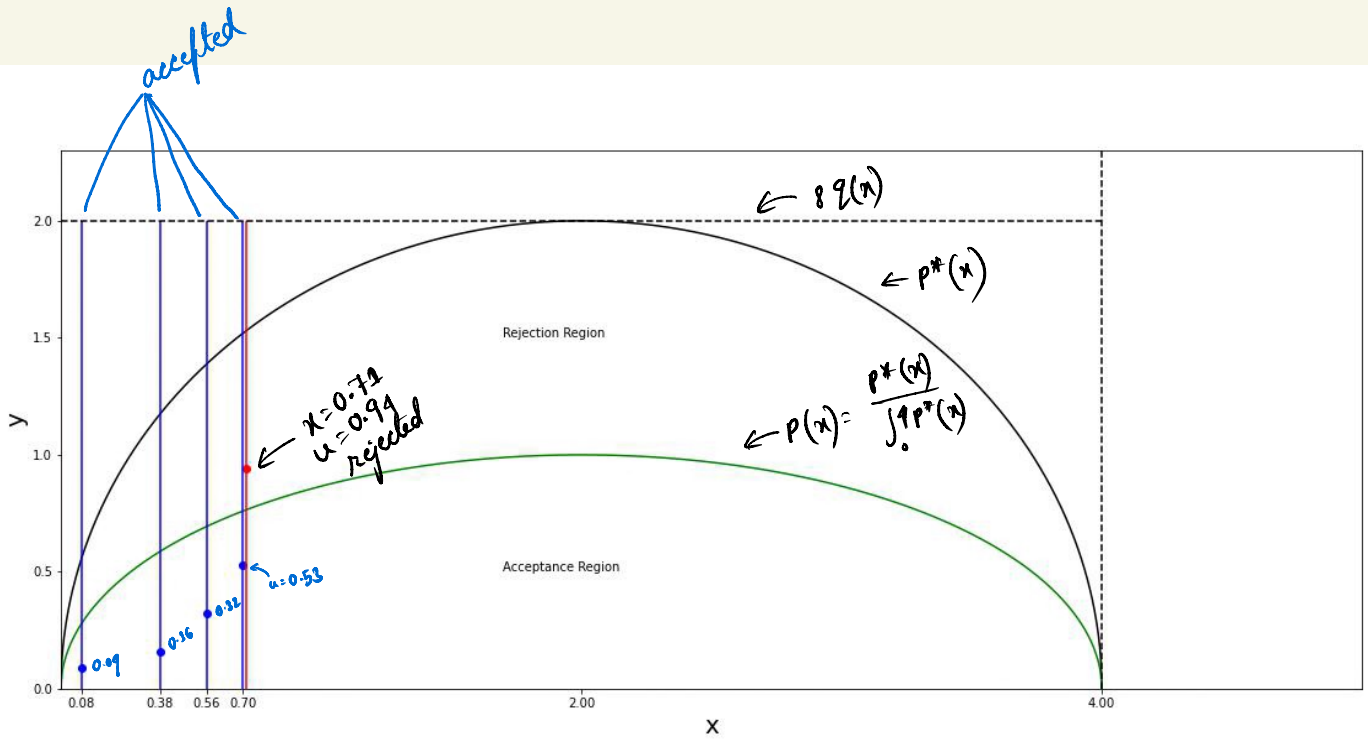
$u = 0.16$

$a = 0.59 > 0.16$

accepted.

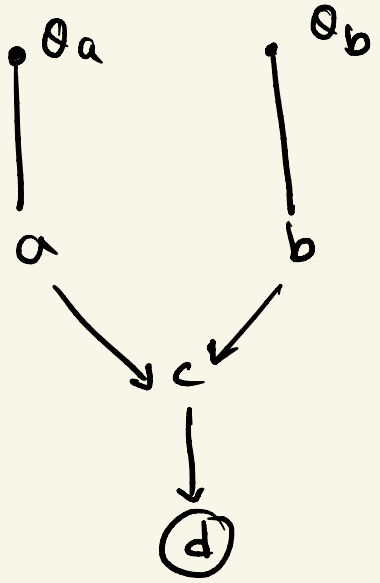
So the sample set is

$$\{0.08, 0.7, 0.56, 0.38\}$$



The region bounded by green curve is acceptance region given we choose $q(x)$ as our proposal function.

3.



(i) $u_a^1 = 0.91$

$a = 0$

$u_b^1 = 0.25$

$b = 1$

$u_c^1 = 0.10$

$P(c=1 \mid a=0, b=1) = 0.6$

$c = 1$

$u_d^1 = 0.01$

$P(d=1 \mid c=1) = 0.8$

$d = 1$

Sample $(0, 1, 1, 1)$ accepted.

ii) $u_a^2 = 0.51$

$a = 1$

$u_b^2 = 0.34$

$b = 1$

$u_c^2 = 0.52$

$P(c=1 \mid a=1, b=1) = 0.9$

$c = 1$

$u_d^2 = 0.77$

$P(d=1 \mid c=1) = 0.8$

$d = 1$

Sample (1, 1, 1, 1) accepted

iii) $u_a^3 = 0.35$

$a = 1$

$u_b^3 = 0.20$

$b = 1$

$u_c^3 = 0.16$

$c = 1$

$u_d^3 = 0.28, \quad d = 1$

∴ Sample (1, 1, 1, 1) accepted.

iv) $u_a^4 = 0.17$

$a = 1$

$u_b^4 = 0.29$

$b = 1$

$u_c^4 = 0.32$

$c = 1$

$u_d^4 = 0.71$

$d = 1$

Sample (1,1,1,1) accepted.

v) $u_a^5 = 0.44$

$a = 1$

$u_b^5 = 0.01$

$b = 1$

$u_c^5 = 0.59$

$c = 1$

$u_d^5 = 0.15$

$d = 1$

Sample (1,1,1,1) accepted.