Masters DSC/MLDM/CPS2

Optimization & Operational Research - Exam

(26/03/2019) 2h00 : personal documents allowed

Exercise 1 : A sum of two functions (8 points)

We consider two parameters $\gamma, \delta \in \mathbb{R}$. The aim of this exercise is to study function $h_{\gamma,\delta} : \mathbb{R}^4 \to \mathbb{R}$ defined by :

$$h_{\gamma,\delta}(x_1, x_2, x_3, x_4) = f_\gamma(x_1, x_2) + g_\delta(x_3, x_4),$$

where $f_\gamma, h_\delta : \mathbb{R}^2 \to \mathbb{R}$ are defined by :

$$f_\gamma(x_1, x_2) = \frac{1}{2}\left(2x_1^2 + 4x_2^2 - 2\gamma x_1 x_2\right) + x_1 - x_2 + 3,$$
$$h_\delta(x_3, x_4) = \delta x_3^3 + \frac{1}{2}\left(x_3^2 + x_4^2 + 0.5x_3x_4\right) - 6x_3 + 2x_4.$$

Part A : Study of f_γ

This part focuses on the function f_γ.

1. Study the convexity of the function f_γ.
2. Give the solution of *Euler's Equation*, i.e. the solution of the linear system $\nabla f_\gamma(x, y) = (0, 0)$, for all values of γ.
3. Give the nature of the previous extrema (i.e. say if it is local/global minimum/maximum of the function). The nature of the extremum depends on γ.

Part B : Study of g_δ

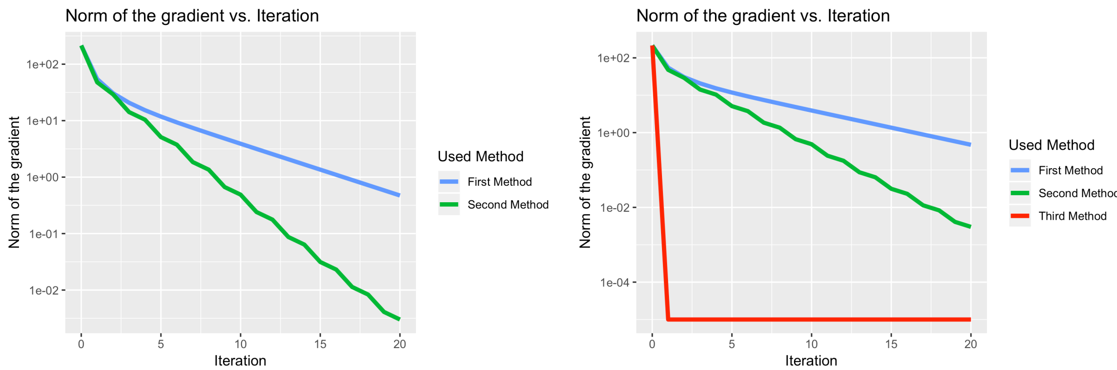
This pat focuses on the function g_δ.

1. Study the convexity of the function g_δ.
2. Give the solution of *Euler's Equation*, i.e. the solution of the linear system $\nabla g_\delta(x, y) = (0, 0)$, for all values of δ.
3. Give the nature of the previous extrama (i.e. say if it is local/global minimum/maximum of the function). The nature of the extremum depends on δ.

Part C : Minimization of $h_{\gamma,\delta}$

1. Show that the sum of two convex functions is convex.
2. Using the previous parts, show that the function $h_{\gamma,\delta}$ is convex if and only if $\gamma \in [-2\sqrt{2}, 2\sqrt{2}]$ and $\delta = 0$.

3. In the following, we consider that both γ and δ are equal to 0. Thus, the function $h_{0,0}$ is convex. The figure below on the left illustrates the convergence of two variations of gradient descent algorithm : (i) *gradient descent with a fixed step* and (ii) *gradient descent with with an optimal step*



   (a) Recall what the gradient descent algorithm consists of in general.
   (b) Explain which curve corresponds to which method. Justify your answer.
   (c) The figure on the right illustrates another variation of the gradient descent algorithm represented by the red curve (mentioned in class). In your opinion, which algorithm is that ? (Bonus)
   (d) If you have found the right algorithm, try to explain why it converges in only one iteration in this case. (Bonus+)

Exercise 2 : Linear programming (6 points)

Part A : Formulate and solve an optimization problem

A chemical firm makes two types of industrial solvents, S_1 and S_2. Each solvent is a mixture of three chemicals. Each kL of S_1 requires 12L of chemical **A**, 9L of chemical **B**, and 30L of chemical **C**. Each kL of S_2 requires 24L of chemical **A**, 5L of chemical **B**, and 30L of chemical **C**. The profit per kL of S1 is \$100, and the profit per kL of S2 is \$85. The inventory of the company shows 480 L of chemical **A**, 180 L of chemical **B**, and 720 L of chemical **C**. Assuming the company can sell all the solvent it makes, find the number of kL of each solvent that the company should make to maximize profit.

1. Formulate the corresponding optimization problem and explain the meaning behind the introduced variables ;
2. Do one step of the simplex algorithm. Explain which is the entering / leaving variable and why. Give the tableau before and after this step ;
3. Finally, give the new feasible solution and explain what will happen next (e.g., this is the optimal solution, there is no optimal solution, ...).

Part B : Understanding simplex method

1. What are the basic variables of the problem given below ? What is the value of nonbasic variables ?

B			X1	X2	X3	X4	X5
X ?	4	=	0	1/3	-1	1	0
X ?	20	=	1	2/3	-1	0	0
X ?	80	=	0	-1	3	0	1
	Z-3	=	0	-25/3	0	40/3	0

2. What can you say about the solution of the following minimization problem ?

B			X1	X2	X3	X4
X3	24	=	0	1	1	0
X4	20	=	-2	2	0	1
	Z-0	=	-40	-35	0	0

Exercise 3 : Constrained optimization (6 points)

Consider the following constrained optimization problem with an arbitrary constant c_1 :

$$\begin{array}{ll} \min\limits_{x_1,x_2,x_3} & c_1x_1 - 4x_2 - 2x_3 \\ \text{subject to} & x_1^2 + x_2^2 \leq 2 \\ & x_2^2 + x_3^2 \leq 2 \\ & x_1^2 + x_3^2 \leq 2 \end{array}$$

1. Write down the KKT-conditions for the problem.
2. Are there any values for the constant c_1 which make the point $\mathbf{x} = (1.4, 0.2, 0.2)^T$ an optimal solution to the problem ? If there are any such values, determine all of these values for c_1.
3. Reply to the previous question with $\mathbf{x} = (1, 1, 1)^T$.
4. Assume that $c_1 = -6$. Compute the value of the dual objective function $g(\lambda)$ in the point $\lambda^* = (1, 1, 1)^T$, and determine whether it is an optimal solution to the dual problem.

① $h_{r,\delta}(x_1,x_2,x_3,x_4) = f_r(x_1,x_2) + g_\delta(x_1,x_2)$

$$f_r(x_1,x_2) = \frac{1}{2}\left(2x_1^2+4x_2^2-2rx_1x_2\right)+x_1-x_2+3$$

$$g_\delta(x_1,x_2) = \delta x_3^3+\frac{1}{2}\left(x_3^2+x_4^2+0.5x_3x_4\right)-6x_3+2x_4$$

Part A

$$\partial f_{x_1} = \frac{1}{2}\left(4x_1-2rx_2\right)+1$$

$$\partial f_{x_2} = \frac{1}{2}\left(8x_2-2rx_1\right)-1$$

$$\frac{1}{2}\left(2x_1^2-2\cdot2\sqrt{2}x_1x_2+4x_2^2\right)+x_1-x_2+3$$

$$\frac{1}{2}\left(\sqrt{2}x_1-2x_2\right)^2+(x_1-x_2)+3$$

$\therefore \partial^2 f_{x_1} = 2$ $\quad \therefore \frac{\partial^2 f}{\partial x_1 \partial x_2} = -r$

$\therefore \partial^2 f_{x_2} = 4$

$2x_1-2\sqrt{2}x_2+1=0$

$4x_2-2\sqrt{2}x_1-1=0$

$\Rightarrow 2x_1-2\sqrt{2}x_2=-1$

$2\sqrt{2}x_1-4x_2=-1$

$\Rightarrow 2\sqrt{2}x_1-4x_2=-\sqrt{2}$

$$\therefore H = \begin{vmatrix} 2 & -r \\ -r & 4 \end{vmatrix}$$

$$\det H = 8-r^2 \geq 0$$

$r^2 \leq 8 \quad -2\sqrt{2} \leq r \leq 2\sqrt{2}$

f_r is convex for $-2\sqrt{2} \leq r \leq 2\sqrt{2}$

② $\boxed{\begin{matrix} 2x_1-rx_2+1=0 \\ 4x_2-rx_1-1=0 \end{matrix}}$ $\Rightarrow 2x_1-rx_2=-1$, $\Rightarrow rx_1-4x_2=-1$

$\therefore 2rx_1-r^2x_2=-r$

$\therefore 2rx_1-8x_2=-2$

$(8-r^2)x_2=(2-r)$

$\Rightarrow x_2 = \frac{2-r}{8-r^2}$, $r^2 \neq 8$

$2x_1-\frac{2r-r^2}{8-r^2}=-1$

$\Rightarrow 2x_1=\frac{2r-r^2}{8-r^2}-1$

$=\frac{2r-\cancel{r^2}-8+\cancel{r^2}}{8-r^2}$

$=$

$\Rightarrow x_1=\frac{r-4}{8-r^2}$

$\left(\frac{r-4}{8-r^2}, \frac{2-r}{8-r^2}\right)$ is a global minimum for $r^2<8$

When $r^2=8$. $r=\pm2\sqrt{2}$

$$f_r(x_1,x_2) = \frac{1}{2}\left(\sqrt{2}x_1-2x_2\right)^2+x_1-x_2+3$$

$x_1 \leq x_2$

$x_1=\sqrt{2}x_2$ $\quad \therefore x_2 \leq \sqrt{2}x_2$

$\therefore x_1-x_2 \geq x_1-\sqrt{2}x_2$

Part of g_δ

① $h_\delta(x_3,x_4) = \delta x_3^3+\frac{1}{2}\left(x_3^2+x_4^2+0.5x_3x_4\right)-6x_3+2x_4$

$$\frac{\partial h}{\partial x_3} = 3\delta x_3^2+x_3-6+0.5x_4$$

$$\frac{\partial^2 h}{\partial x_3^2} = 6x_3\delta+1$$

$$\frac{\partial h}{\partial x_4} = x_4+0.5x_3+2$$

$$\frac{\partial^2 h}{\partial x_4^2} = 1$$

$$\frac{\partial^2 h}{\partial x_3 \partial x_4} = 0.5$$

$$\begin{vmatrix} 6x_3\delta+1 & 0.5 \\ 0.5 & 1 \end{vmatrix}$$

$\therefore 6x_3\delta+2 \geq 0$

$\Rightarrow x_3\delta \geq -\frac{1}{3}$

$6x_3\delta+1 \geq 0$

$\Rightarrow x_3\delta \geq -\frac{1}{6}$

g_δ is convex when $x_3\delta \geq -\frac{1}{6} > -\frac{1}{3}$

$3\delta x_3^2+x_3+0.5x_4-6=0$

$x_4+0.5x_3+2=0 \Rightarrow x_4=-2-0.5x_3$

$3\delta x_3^2+x_3+0.5(-2-0.5x_3)-6=0$

$\Rightarrow 3\delta x_3^2+x_3+(1-0.25x_3)-6=0$

$\Rightarrow 3\delta x_3^2+0.75x_3-7=0$

$$x_3 = \frac{-0.75 \pm \sqrt{(0.75)^2+4\cdot3\delta\cdot7}}{6\delta}$$

$$= \frac{-0.75 \pm \sqrt{(0.75)^2+4\cdot3\delta\cdot7}}{6\delta}$$

$84\delta+(0.75)^2 \geq 0$

$\Rightarrow \delta \geq -\frac{(0.75)^2}{84} \Rightarrow \delta \geq -\frac{(0.25)(0.75)}{28}$

For $\delta \geq -\frac{(0.25)(0.75)}{28}$

$$x_4 = -2-0.5\left(\frac{-0.75 \pm \sqrt{(0.75)^2+84\delta}}{6\delta}\right)$$

$$= -2-\left(\frac{-0.75 \pm \sqrt{0.75^2+84\delta}}{12\delta}\right)$$

Part C

① $f_1 \in C$

$f_2 \in C$

$g = f_1 + f_2$

$$g(\lambda x_1 + (1-\lambda)x_2) = (f_1 + f_2)(\lambda x_1 + (1-\lambda)x_2)$$

$$= f_1(\lambda x_1 + (1-\lambda)x_2) + f_2(\lambda x_1 + (1-\lambda)x_2)$$

$$\leq \lambda f_1(x_1) + (1-\lambda) f_1(x_2) + \lambda f_2(x_1) + (1-\lambda) f_2(x_2)$$

$$\leq \lambda (f_1 + f_2)(x_1) + (1-\lambda)(f_1 + f_2)(x_2)$$

$$\leq \lambda g(x_1) + (1-\lambda) g(x_2)$$

2.

$$H_g = \begin{vmatrix} 1 & 0.5 \\ 0.5 & 1 \end{vmatrix}$$

g_8 is convex as H_g is PSD.

③ a) Gradient Descent algorithm.

Initial step u_0

While $|d_K| \leq \epsilon$

$d_K = \nabla f(u_K)$

$$\rho_{K+1} = \arg\min_{\rho} f(u_K - \rho\, d_K)$$

$$u_{K+1} = u_K - \rho_{K+1} d_K$$

ⓑ The first method is the a)

The second method is b)

The green curve is more efficient since we are optimizing with optimal step so that the f decreases more at each level which means, $\nabla f \to 0$ as we iterate.

ⓒ Conjugate gradient Descent

ⓓ
$$A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \\ -6 \\ 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0.5 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 4-\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0.5 \\ 0 & 0 & 0.5 & 1-\lambda \end{pmatrix}$$

$$\therefore \{(2-\lambda)(4-\lambda)\}\{(1-\lambda)^2 - (0.5)^2\} = 0$$

Part 2

A $\max z = 100 S_1 + 85 S_2$

s.t

$12 S_1 + 24 S_2 \le 480$

$9 S_1 + 5 S_2 \le 180$

$30 S_1 + 30 S_2 \le 720$

(S_1) S_1 $(12L)$ (S_2) S_2 (242)

A

$-Z$	S_1	S_2	x_1	x_2	x_3	B	
1	100	85	0	0	0	0	
0	12	24	1	0	0	480	40
0	9	5	0	1	0	180	20
0	30	30	0	0	1	720	24

$x_1 = x_2 = x_3 = 0$

Basic variables $-Z, x_1, x_2, x_3$

Leaving variable x_2

$\frac{765-500}{9} = \frac{265}{9}$

$-Z$	S_1	S_2	x_1	x_2	x_3	B
1	0	$\frac{265}{9}$	0	$-\frac{100}{9}$	0	-2000
0	0	$\frac{52}{3}$	1	$-\frac{4}{3}$	0	240
0	1	$\frac{5}{9}$	0	$\frac{1}{9}$	0	20
0	0	$30 - \frac{150}{9}$ ($\frac{50}{3}$)	0	$-\frac{30}{9}$	1	120

$Z = 2000$

$S_1 = 20$

Next step S_2 will enter, x_3 will leave.

(1) $x_1, x_4, x_5, -Z$

x_2, x_3 are non basic.

$x_2 = x_3 = 0$

$-z' + 40x_1 + 35x_2$

$z' - 40x_1 - 35x_2 = 0$

(2) $\min Z = -40x_1 - 35x_2$

$\max z' = 40x_1 + 35x_2$ $\quad x_2 \le 2$

$-z'$	x_1	x_2	x_3	x_4	B
1	40	35	0	0	0
0	0	1	1	0	24 ← 2
0	-2	2	0	1	20 ← 10
1	75	0	0	$-\frac{35}{2}$	-350
0	1	0	1	-1	14
0	-1	1	0	$\frac{1}{2}$	10

$x_2 = 10$, $x_1 = 0$

$x_1 = 14$, $x_2 = 24$

$$\min_{x_1,x_2,x_3} c x_1 - 4x_2 - 2x_3$$

s.t

$x_1^2 + x_2^2 \le 2$

$x_2^2 + x_3^2 \le 2$

$x_1^2 + x_3^2 \le 2$

$$\therefore L = c x_1 - 4x_2 - 2x_3 + \lambda_1(x_1^2+x_2^2-2) + \lambda_2(x_2^2+x_3^2-2) + \lambda_3(x_1^2+x_3^2-2)$$

$\lambda_1, \lambda_2, \lambda_3 \ge 0$

$\lambda_1(x_1^2+x_2^2-2) = 0$

$\lambda_2(x_2^2+x_3^2-2) = 0$

$\lambda_3(x_1^2+x_3^2-2) = 0$

$\nabla L(x_1), \nabla L(x_2)$, $\nabla L(x_3) = 0$

$c + 2\lambda_1 x_1 + 2\lambda_3 x_1 = 0$

$-4 + 2\lambda_1 x_2 + 2\lambda_2 x_2 = 0$

$-2 + 2\lambda_2 x_3 + 2\lambda_3 x_3 = 0$

$$\begin{pmatrix} x_1 & 0 & x_1 \\ x_2 & x_2 & 0 \\ 0 & x_3 & x_3 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -c/2 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1.4 & 0 & 1.4 \\ 0.2 & 0.2 & 0 \\ 0 & 0.2 & 0.2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} -c/2 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0.2 & 0.2 & 0 \\ 0 & 0.2 & 0.2 \end{pmatrix} = \begin{pmatrix} -\frac{c}{2.8} \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0.2 & -0.2 \\ 0 & 0.2 & 0.2 \end{pmatrix} = \begin{pmatrix} -\frac{c}{2.8} \\ 2+\frac{c}{14} \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0.2 & -0.2 \\ 0 & 0 & 0.4 \end{pmatrix} = \begin{pmatrix} -\frac{c}{2.8} \\ 2+\frac{c}{14} \\ 1-\frac{c}{14} \end{pmatrix}$$

$$\lambda_3 = \frac{1}{0.4}\left(1 - \frac{c}{14}\right) \ge 0$$

$$\lambda_2 = \frac{1}{0.2}\left(\frac{5}{2} + \frac{c}{28}\right)$$

$\lambda_1 = \lambda_3 - \frac{c}{28}$

$= -\frac{1}{0.4}\left(1 - \frac{c}{28}\right)$

$\frac{c}{14} - 1 \le 0$

$\Rightarrow c \le 14$

$\therefore c \ge 0$

$c = 0$

$$0.2\lambda_2 - \frac{1}{2}\left(1 - \frac{c}{14}\right) = 2 + \frac{c}{14}$$

$$\Rightarrow 0.2\lambda_2 - \frac{1}{2} + \frac{c}{28} = 2 + \frac{c}{14}$$

$$\Rightarrow 0.2\lambda_2 = \frac{5}{2} + \frac{c}{14} - \frac{c}{28} \quad \Rightarrow 0.2\lambda_2 = \left(\frac{5}{2} + \frac{c}{28}\right)$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -c \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -c \\ 2+c \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -c \\ 2+c \\ 1-c \end{pmatrix}$$

$$\Rightarrow \lambda_3 = \left(\frac{1-c}{2}\right) \geq 0$$

$\therefore c \leq 1$

$\therefore c \geq 0$

$\therefore \lambda_2 - \lambda_3 = 2+c$

$\Rightarrow \lambda_2 = \frac{1-c}{2} + (2+c)$

$= \frac{1-c+4+2c}{2} = \frac{c+5}{2}$

$\lambda_1 = -c - \frac{1-c}{2}$

$= \frac{-2c-1+c}{2} = \frac{c-1}{2}$

$c = 1$

(4) $g(\lambda) = \min\limits_{x_1, x_2, x_3} L(x_1, x_2, x_3)$

$\lambda_1 = 1$

$\lambda_2 = 1$

$\lambda_3 = 1$

$9 + 2\lambda_1 x_1 + 2\lambda_3 x_1 = 0 \Rightarrow x_1 = \frac{6}{2(\lambda_1+\lambda_3)} = \frac{3}{\lambda_1+\lambda_3}$

$-4 + 2\lambda_1 x_2 + 2\lambda_2 x_2 = 0 \Rightarrow x_2 = \frac{4}{2(\lambda_2+\lambda_1)} = \frac{2}{\lambda_1+\lambda_2}$

$-2 + 2\lambda_2 x_3 + 2\lambda_3 x_3 = 0$

$\Rightarrow x_3 = \frac{2}{2(\lambda_2+\lambda_3)} = \frac{1}{\lambda_2+\lambda_3}$

$x_1 = \frac{3}{2}, \; x_2 = 1, \; x_3 = \frac{1}{2}$

For $(\lambda_1, \lambda_2, \lambda_3)$ be optimal, it has to satisfy KKT.

$\lambda_1(x_1^2 + x_2^2 - 2) = 0 \longleftarrow$ Not satisfied

$\lambda_2(x_2^2 + x_3^2 - 2) = 0$

So, $(1,1,1)$ isn't optimal.

$\lambda_3(x_1^2 + x_3^2 - 2) = 0$

$9 + 2\lambda_1 x_1 + 2\lambda_3 x_1 = 0$

$-4 + 2\lambda_1 x_2 + 2\lambda_2 x_2 = 0$

$-2 + 2\lambda_2 x_3 + 2\lambda_3 x_3 = 0$

$= (-9-4-1) + \left(\frac{5}{4}\right)$

$+ \left(\frac{-3}{4}\right) + \left(\frac{10}{4} - 2\right)$

$= -14 + \frac{2}{4} + \frac{2}{4} = -13$

$f\left(\frac{3}{2}, 1, \frac{1}{2}\right) = -\cancel{6}^{3} \cdot \frac{3}{\cancel{2}} - 4 - \cancel{2} \cdot \frac{1}{\cancel{2}}$

$= -9 - 4 - 1$

$= -14$