Sheet 1: Introduction - Exercices

Advanced Algorithms - Master DSC/MLDM/CPS2

Recap

Classic series:

- $\sum_{i=1}^{n} i = 1 + \cdots + n = \frac{n(n+1)}{2}$
- $a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{j=0}^{n-1} ar^j = a\frac{1-r^n}{1-r}$.

1 Exercise

Let f and g be two functions that take non negative values and suppose that f is $O(g)$, show that g is $\Omega(f)$.
Repeat the same process for proving (easy?):

- If $f \in O(g)$ and $g \in O(h)$ then $f \in O(h)$
- If $f \in \Omega(g)$ and $g \in \Omega(h)$ then $f \in \Omega(h)$
- If $f \in \theta(g)$ and $g \in \theta(h)$, then $f \in \theta(h)$
- If $f \in O(h)$ and $g \in O(h)$ then $f + g \in O(h)$
- If $g \in O(f)$ then $f + g \in \theta(f)$

2 Exercise

Assume you have functions f and g such that $f(n)$ is $O(g(n))$. For each of the following statements, decide whether you think is true or false and give a proof or a counter example

1. $\log_2 f(n)$ is $O(\log_2 g(n))$
2. $2^{f(n)}$ is $O(2^{g(n)})$
3. $f(n)^2$ is $O(g(n)^2)$

3 Exercise

Arrange the following list of functions in ascending order of growth rate. That is, if function $g(n)$ immediately follows function $f(n)$ in your list than it should be the case that $f(n)$ is $O(g(n))$.

1. $f_1(n) = 10^n$
2. $f_2(n) = n^{1/3}$
3. $f_3(n) = n^n$
4. $f_4(n) = \log_2 n$
5. $f_5(n) = 2^{\sqrt{\log_2 n}}$

4 Recurrences

Can you find the solution for each recurrence?

1. $T(n) = T(n-1) + n$, $n \geq 2$ and $T(1) = 1$.
2. $T(n) = T(n/2) + n$, $n \geq 2$, $T(1) = 0$ and n a power of 2.
3. $T(n) = 2T(n/2) + n^2$, $n \geq 2$, $T(1) = 0$ and n a power of 2.
4. $T(n) = 2T(\sqrt{n}) + \log_2 n$, with $n \geq 4$ and $T(2) = 1$.
5. $T(n) = T(\sqrt{n}) + \log_2 \log_2 n$, with $n \geq 4$ and $T(2) = 1$.

Bonus recurrence: $T(n) = 3T(n/2) - 2T(n/4) + \log n$, $T(2) = 3$, $T(1) = 3$. This problem is hard and out of the scope of the class: this is a non homogeneous linear relation, for your culture the solution will be given in the solution sheet. You must solve the equation of the form: $F(k) = \alpha \times l_1^k + \beta \times l_2^k + \gamma \times k^2 + \delta \times k$, with l_1 and l_2 solutions of the characteritic equation: $x^2 - 3x + 2 = 0$. Try to see how to come to the formulation $F(k)$, and then to get the parameters of $\alpha, \beta, \gamma, \delta$ and then solve the equation.

5 Exercise

We have a set of electronic chips, and we have a tool for testing if two chips are equivalent in $O(1)$ (*e.g.* they have the same function). We would like to design an algorithm that is able to answer *yes* if in a set of n chips, there are **strictly more** than $n/2$ chips that are equivalent to one another, in other words we are looking for a chip such that it is similar to at least $(n/2) + 1$ other chips. The only possible operation is to pick two chips and use the testing tool.

Propose an algorithm able to answer this question in $O(n \log n)$. Can you propose a better algorithm?

Note on Master theorems

*Simple formulation 1

Theorem 1 (Master Theorem 1)**.** *If $T(n) \leq aT(n/b) + O(n^d)$ for some positive constants a, b, d then*

1. *$T(n) \in O(n^d)$ if $a < b^d$*
2. *$T(n) \in O(n^d \log n)$ if $a = b^d$*
3. *$T(n) \in O(n^{\log_b a})$ if $a > b^d$.*

*More complex formulation 2

Theorem 2 (Master theorem 1)**.** *Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function and let $T(n)$ be defined on the non negative integers by the recurrence*

$$T(n) = aT(n/b) + f(n)$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$, then $T(n)$ can be bounded asymptotically as follows:

1. *if $f(n) \in O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) \in \theta(n^{\log_b a})$*
2. *if $f(n) \in \theta(n^{\log_b a} \log^k n)$ with with $k \geq 0$ a constant, then $T(n) \in \theta(n^{\log_b a} \log^{k+1} n)$*
3. *if $f(n) \in \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) \in \theta(f(n))$.*

① $f = O(g)$ then $\forall c_1, n_0 \in \mathbb{R}$ s.t

$$f(n) \le c_1 g(n) \quad \forall n \ge n_0$$

$$\Rightarrow g(n) \ge \frac{1}{c_1} f(n) \quad \forall n \ge n_0$$

$$\ge c_2 f(n) \quad \forall n \ge n_0$$

② $f \in O(g) \Rightarrow f(n) \le c_1 g(n) \quad \forall n > n_0$

$g \in O(h) \Rightarrow g(n) \le c_2 h(n) \quad \forall n > n_1$

$$f(n) \le c_1 c_2 h(n) \quad \forall n \ge \max(n_1, n_2)$$

$$\therefore f(n) \le c\, h(n)$$

$\therefore$ Then $f \in O(h)$

② ① Let $f = 2$
$g = 1$

Then $f \in O(g)$

But $\log f = \log 2$
$\log g = 0$

$\therefore$ So, $\log f \notin \log g$

② $f(n) = 2n$
$g(n) = n$

$\therefore$ So, $2^{f(n)} = 2^{2n} = 4^n$

$= 2^g = 2^n$

③ $f^2(n) \le c^2 g^2(n) \le c^2 g^2(n) + g^2(n)$

$$\le (c^2+1)\, g^2(n)$$

$\therefore$ So, $f^2(n) \in O(g^2(n))$

③ $f_1(n) = 10^n$

$n^n > 10^n \quad \forall n > 10$

$\therefore f_1 < f_3$

$(\log 10) \cdot 10^n - \frac{1}{n} < 0$

$(\log 10) \cdot 10^n - \frac{1}{3} \cdot x^{-2/3} < 0$

$x^{1/3} - 10^x$

$x^{2/3} < x$

$f_2 < f_3$

$f_3 > f_1 > f_4$

$f_1 > f_4$

$\frac{1}{3}\left(\frac{3}{x} - \frac{1}{x^{2/3}}\right) > 0$

$\frac{3x^{2/3} - x}{x^{5/3}}$

$\therefore A = 10^n$

$\therefore \log A = n \log 10$

$\therefore \frac{1}{A} \cdot \frac{dA}{dn} = \log 10 \Rightarrow \frac{dA}{dn} = (\log 10) \cdot 10^n$

$3x^{2/3} > x$

$\therefore \frac{3}{x} > \frac{1}{x^{2/3}} \Rightarrow \frac{3}{x^{1/3}} > 1$

$\Rightarrow$

$\Rightarrow x^{1/3} < 3$

$\Rightarrow x < 27$

$\therefore f_2 - f_4$

$f(n) = n^{\frac{1}{3}} - \log_2 n$

$\Rightarrow f'(n) = \frac{1}{3} n^{-2/3} - \frac{1}{n}$

$\Rightarrow \frac{1}{3} \cdot \frac{1}{n^{\frac{2}{3}}} - \frac{1}{n} > 0$

$\Rightarrow \frac{1}{3} > \frac{1}{n} \Rightarrow n > 27$

$f_2 > f_4$

$\therefore f_3 > f_1 > f_2 > f_4$

$$\underline{f_5 - f_4}$$

$\therefore f_4 = \ln n$

$\Rightarrow \dfrac{df}{dn} = \dfrac{1}{n}$

$A = 2^{\sqrt{\log_2 n}}$

$\Rightarrow \ln A = \sqrt{\log n}\, \ln 2$

$\Rightarrow \dfrac{1}{A} \cdot \dfrac{dA}{dn} = \dfrac{1}{2} (\log n)^{-1/2} \cdot \dfrac{1}{n} \ln 2$

$\Rightarrow \dfrac{dA}{dn} = \dfrac{2^{\sqrt{\log n}}}{2n} \cdot \dfrac{\ln 2}{\sqrt{\log n}}$

$\dfrac{2^{\sqrt{\ln n}}}{2n} \dfrac{\ln 2}{\sqrt{\ln n}} - \dfrac{1}{n}$

$\Rightarrow c_1 \dfrac{2^{\sqrt{\ln n}}}{\sqrt{\ln n}} - \dfrac{1}{n}$

$\dfrac{c_1 n 2^{\sqrt{\ln n}} - \sqrt{\ln n}}{n\sqrt{\ln n}} > 0$

4 (i) $T(n) = T(n-1) + n$, $n \geq 2$, $T(1) = 1$

$\Rightarrow T(n) = T(n-2) + (n-1) + n$

$= n + (n-1) + (n-2) + \text{———} + 1$

$= \dfrac{n(n+1)}{2} = \dfrac{n^2+n}{2} = O(n^2)$

② $T(n) = T\left(\frac{n}{2}\right) + n$

Let $n = 2^K$

$T(2^K) = T(2^{K-1}) + 2^K$

$= T(2^{K-2}) + 2^{K-1} + 2^K$

$= 2^K + 2^{K-1} + T(2^{K-2}) = 2^K + 2^{K-1} + 2^{K-2} + \cdots 1$

$= \frac{1(2^{K+1} - 1)}{1} = (2n - 1) = O(n)$

③ $T(n) = 2T\left(\frac{n}{2}\right) + n^2$

Let $n = 2^K$

$T(2^K) = 2T(2^{K-1}) + 2^{2K}$

$= 2^2 T(2^{K-1}) + 2 \cdot 2^{2(K-1)} + 2^{2K}$

$= 2^2 T(2^{K-1}) + 2^{2K-1} + 2^{2K}$

$= 2^{2K} + 2^{2K-1} + \cdots 2^K$

$= \frac{2^K \cdot (2^{K+1} - 1)}{2-1} = 2 \cdot 2^{2K} - 2^K = 2n^2 - n = O(n^2)$

④ $T(n) = 2T(\sqrt{n}) + \log_2 n$, $n \geq 4$, $T(2) = 1$

Let $n = 2^K \Rightarrow \lg n = K$

$\therefore T(2^K) = 2T(2^{\frac{K}{2}}) + \ln 2^K = 2T(2^{K/2}) + K \log_2 2$

$\therefore F(K) = T(2^K)$

$\therefore F(K) = 2F\left(\frac{K}{2}\right) + K \log_2 2$

$= 2F\left(\frac{K}{2}\right) + K$

$= 2^2 F\left(\frac{K}{2^2}\right) + K + K$

$= K \log K$

$F(K) = O(K \log K) = O(\lg n \lg \lg n)$

⑤ $T(n) = T(\sqrt{n}) + \lg \lg n$

$\because n = 2^{2^K} \Rightarrow \lg n = 2^K \Rightarrow \lg \lg n = K$

$T(2^{2^K}) = T(2^{2^{K-1}}) + K$

$\therefore F(K) = F(K-1) + K = K + (K-1) + \cdots + 1$

$= \frac{K(K+1)}{2} = O(K^2) = O((\lg \lg n)^2)$

5 $T(n) = 3T\left(\frac{n}{2}\right) - 2T\left(\frac{n}{4}\right) + \lg n$

$\therefore n = 2^k$

$\therefore T(2^k) = 3T(2^{k-1}) - 2T(2^{k-2}) + k$

$\therefore F(k) = 3F(k-1) - 2F(k-2) + k$

$\therefore f_n = Af_{k-1} + Bf_{n-2} + g(k)$

$\therefore a_n = Aa_{n-1} + Ba_{n-2} + g(n) \rightarrow$

So, $a_n = a_n^{(h)} + a_n^{(P)}$

$g(n) = t^n(\mathrm{Pol}(\deg N))$

$\therefore$ If t isn't a root of $s^2 - As - B = 0$

$\therefore$ If t is a root of $s^2 - As - B = 0$

$\therefore a_n^{(P)} = t^n(P_0 + P_1 n + P_2 n^2 + \dots + P_N n^N)$

$\therefore a_n^{(P)} = t^n n^m(P_0 + P_1 n + P_2 n^2 + \dots + P_N n^N)$

So,

$g(k) = 1^k(k)$

$a_n^{(P)} = k(\gamma k + \delta)$

$\therefore \alpha 1^k + \beta 2^k$