

Lecture 5: Model-Free Control

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Outline

- 1 Introduction
- 2 On-Policy Monte-Carlo Control
- 3 On-Policy Temporal-Difference Learning
- 4 Off-Policy Learning
- 5 Summary

Model-Free Reinforcement Learning

- Last lecture:
 - Model-free prediction
 - Estimate the value function of an *unknown* MDP
- This lecture:
 - Model-free control
 - Optimise the value function of an *unknown* MDP

Uses of Model-Free Control

Some example problems that can be modelled as MDPs

- Elevator
- Parallel Parking
- Ship Steering
- Bioreactor
- Helicopter
- Aeroplane Logistics
- Robocup Soccer
- Quake
- Portfolio management
- Protein Folding
- Robot walking
- Game of Go

For most of these problems, either:

- MDP model is unknown, but experience can be sampled
- MDP model is known, but is too big to use, except by samples

Model-free control can solve these problems

On and Off-Policy Learning

- **On-policy** learning

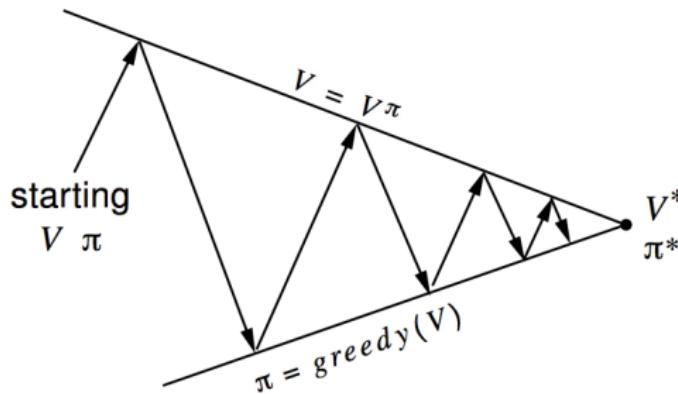
- “Learn on the job”
- Learn about policy π from experience sampled from π

- **Off-policy** learning

- “Look over someone’s shoulder”
- Learn about policy π from experience sampled from μ

Robot learning from human observation

Generalised Policy Iteration (Refresher)

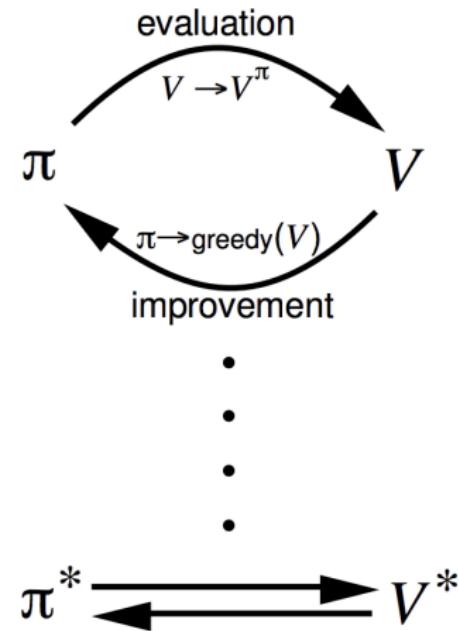


Policy evaluation Estimate v_π

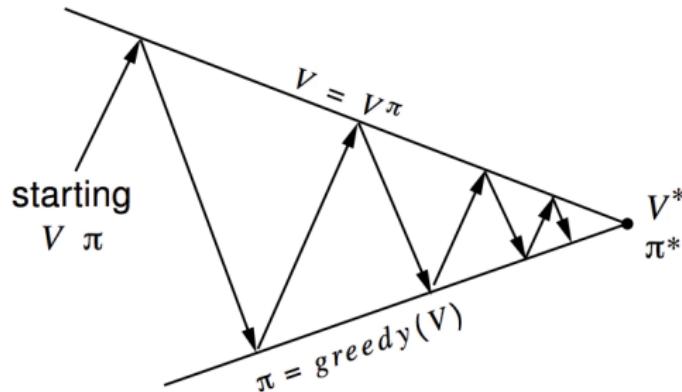
e.g. Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$

e.g. Greedy policy improvement



Generalised Policy Iteration With Monte-Carlo Evaluation



Policy evaluation Monte-Carlo policy evaluation, $V = v_\pi$?

Policy improvement Greedy policy improvement?

→ exploration issue.

Model-Free Policy Iteration Using Action-Value Function

- Greedy policy improvement over $V(s)$ requires model of MDP

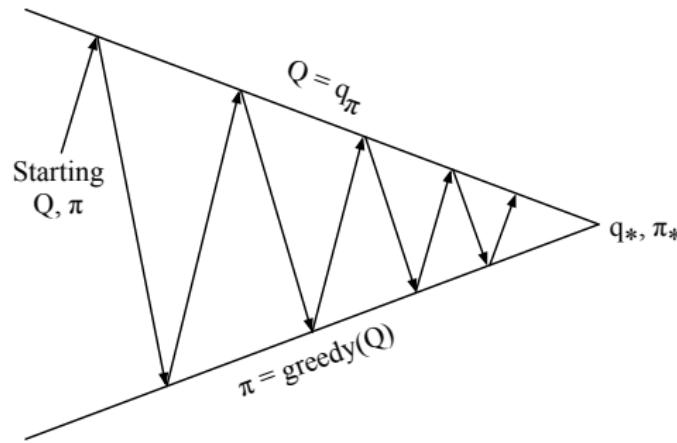
$$\pi'(s) = \operatorname{argmax}_{a \in \mathcal{A}} \mathcal{R}_s^a + \mathcal{P}_{ss'}^a V(s')$$

*not available
in model-free*

- Greedy policy improvement over $Q(s, a)$ is model-free

$$\pi'(s) = \operatorname{argmax}_{a \in \mathcal{A}} Q(s, a)$$

Generalised Policy Iteration with Action-Value Function



Policy evaluation Monte-Carlo policy evaluation, $Q = q_\pi$

Policy improvement Greedy policy improvement?

Example of Greedy Action Selection



"Behind one door is tenure - behind the other is flipping burgers at McDonald's."

- There are two doors in front of you.
- You open the left door and get reward 0
 $V(\text{left}) = 0$
- You open the right door and get reward +1
 $V(\text{right}) = +1$
- You open the right door and get reward +3
 $V(\text{right}) = +2$
- You open the right door and get reward +2
 $V(\text{right}) = +2$
- ⋮
- Are you sure you've chosen the best door?

ϵ -Greedy Exploration

- Simplest idea for ensuring continual exploration
- All m actions are tried with non-zero probability
- With probability $1 - \epsilon$ choose the greedy action
- With probability ϵ choose an action at random

$$\pi(a|s) = \begin{cases} \epsilon/m + 1 - \epsilon & \text{if } a^* = \underset{a \in \mathcal{A}}{\operatorname{argmax}} Q(s, a) \\ \epsilon/m & \text{otherwise} \end{cases}$$

ϵ -Greedy Policy Improvement

Theorem

For any ϵ -greedy policy π , the ϵ -greedy policy π' with respect to q_π is an improvement, $v_{\pi'}(s) \geq v_\pi(s)$

$$\begin{aligned}
 q_\pi(s, \pi'(s)) &= \sum_{a \in \mathcal{A}} \pi'(a|s) q_\pi(s, a) \\
 v_{\pi'}(s) &= \epsilon/m \sum_{a \in \mathcal{A}} q_\pi(s, a) + (1 - \epsilon) \max_{a \in \mathcal{A}} q_\pi(s, a) \\
 &\geq \epsilon/m \sum_{a \in \mathcal{A}} q_\pi(s, a) + (1 - \epsilon) \sum_{a \in \mathcal{A}} \frac{\pi(a|s) - \epsilon/m}{1 - \epsilon} q_\pi(s, a) \\
 &= \sum_{a \in \mathcal{A}} \pi(a|s) q_\pi(s, a) = v_\pi(s)
 \end{aligned}$$

Therefore from policy improvement theorem, $v_{\pi'}(s) \geq v_\pi(s)$

$$V_{\pi}(s) = \frac{S_{\pi}(n)}{N(s)}$$

$$V_{\pi}(s) = q_{\pi}(n, a) \quad V_{\pi}(s^+) = V_{\pi}(s^+) + \alpha(a^+ - V_{\pi}(s^+))$$

$$\pi'(s) = \underset{a}{\operatorname{argmax}} \ g_{\pi}(s, a)$$

$$q_{\pi}(s, \pi'(s)) = \sum_a \pi'(a|s) q_{\pi}(s, a)$$

$$= \sum_m \sum_a q_{\pi}(s/a) + (1-\epsilon) \max_a q_{\pi}(s/a)$$

$$\geq \sum_m \sum_a q_{\pi}(n/a) + (1-\epsilon) \sum_a \frac{\pi(a/s) - \epsilon/m}{1-\epsilon} q_{\pi}(n/a)$$

$$\geq \sum_a \pi(a/s) q_{\pi}(n/a)$$

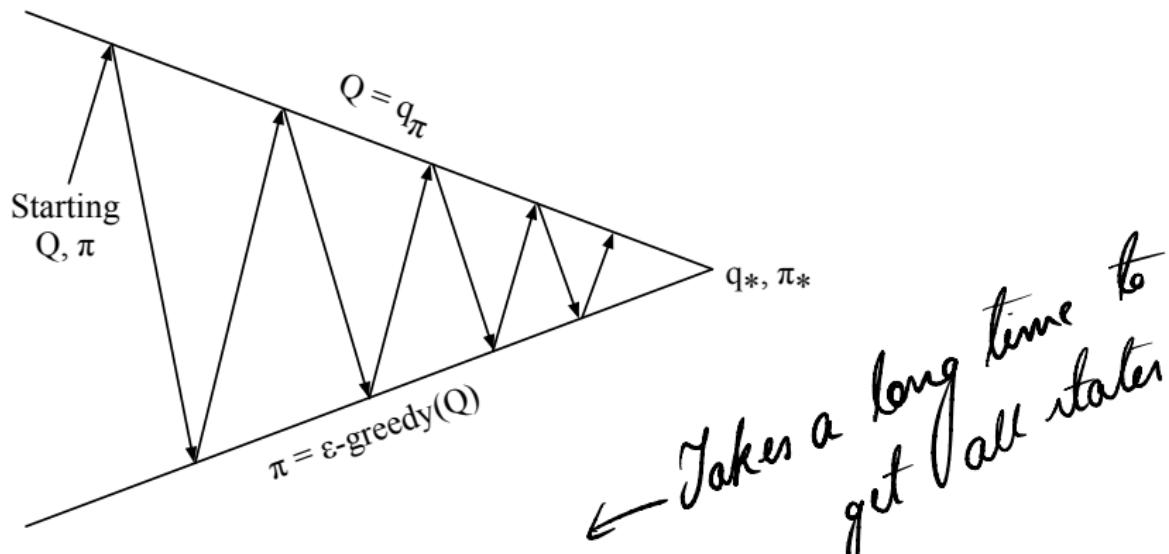
$$\geq V_{\pi}(n)$$

Monte - carlo Prediction π

$$E^k = (s^1, a^1, r^1, \dots, s^T)$$

$$\frac{N(s^k) + 1}{N(s^k, a^t) + 1}$$

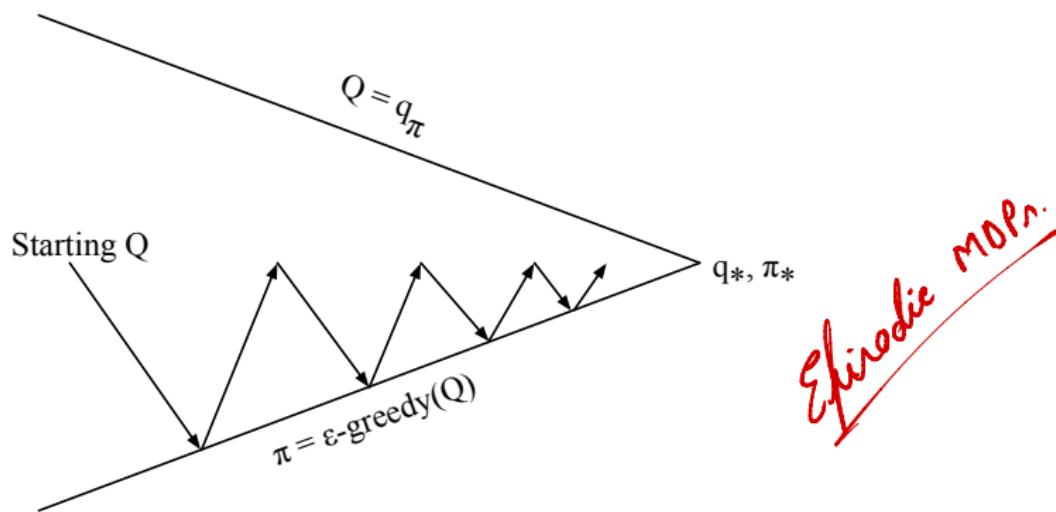
Monte-Carlo Policy Iteration



Policy evaluation Monte-Carlo policy evaluation, $Q = q_\pi$

Policy improvement ϵ -greedy policy improvement

Monte-Carlo Control



Every episode:

Policy evaluation Monte-Carlo policy evaluation, $Q \approx q_\pi$

Policy improvement ϵ -greedy policy improvement

GLIE

Definition

Greedy in the Limit with Infinite Exploration (GLIE)

- All state-action pairs are explored infinitely many times,

$$\lim_{k \rightarrow \infty} N_k(s, a) = \infty$$

- The policy converges on a greedy policy,

$$\lim_{k \rightarrow \infty} \pi_k(a|s) = \mathbf{1}(a = \operatorname{argmax}_{a' \in \mathcal{A}} Q_k(s, a'))$$

- For example, ϵ -greedy is GLIE if ϵ reduces to zero at $\epsilon_k = \frac{1}{k}$

GLIE Monte-Carlo Control

- Sample k th episode using π : $\{S_1, A_1, R_2, \dots, S_T\} \sim \pi$
- For each state S_t and action A_t in the episode,

$$N(S_t, A_t) \leftarrow N(S_t, A_t) + 1$$

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{1}{N(S_t, A_t)} (G_t - Q(S_t, A_t))$$

- Improve policy based on new action-value function

$$\epsilon \leftarrow 1/k$$

$$\pi \leftarrow \epsilon\text{-greedy}(Q)$$

Theorem

GLIE Monte-Carlo control converges to the optimal action-value function, $Q(s, a) \rightarrow q_(s, a)$*

Lecture 5: Model-Free Control

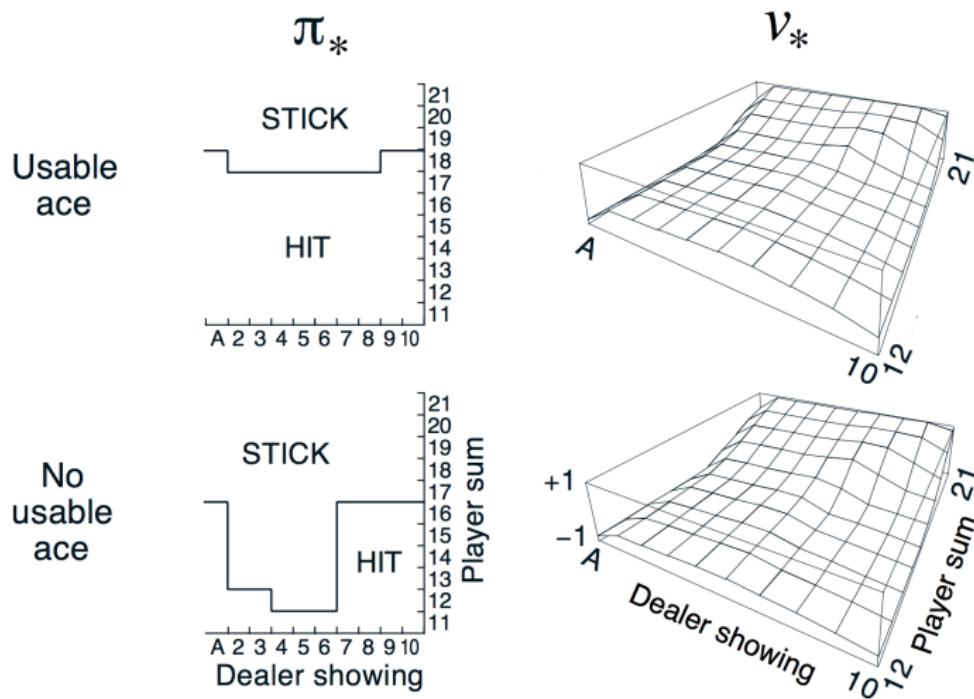
└ On-Policy Monte-Carlo Control

└ Blackjack Example

Back to the Blackjack Example



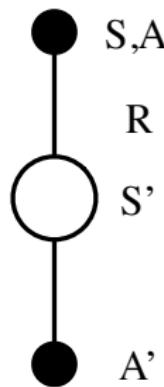
Monte-Carlo Control in Blackjack



MC vs. TD Control

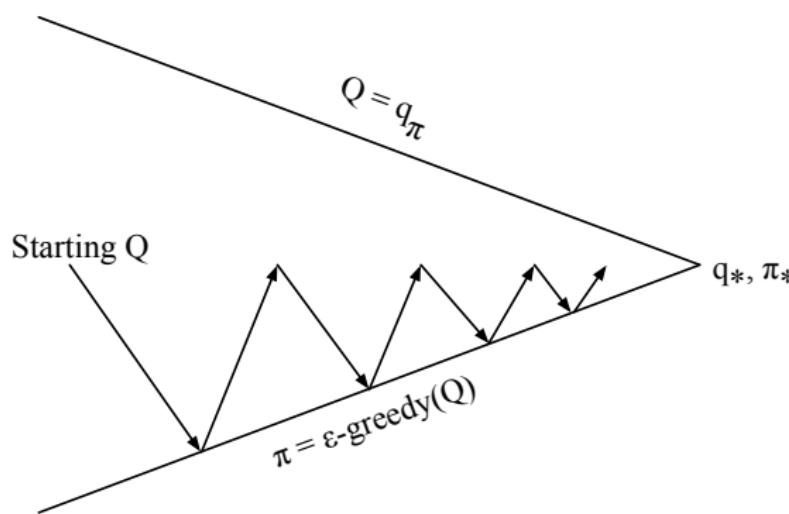
- Temporal-difference (TD) learning has several advantages over Monte-Carlo (MC)
 - Lower variance
 - Online
 - Incomplete sequences
- Natural idea: use TD instead of MC in our control loop
 - Apply TD to $Q(S, A)$
 - Use ϵ -greedy policy improvement
 - Update every time-step

Updating Action-Value Functions with Sarsa



$$Q(S, A) \leftarrow Q(S, A) + \alpha (R + \gamma Q(S', A') - Q(S, A))$$

On-Policy Control With Sarsa



Every time-step:

Policy evaluation **Sarsa**, $Q \approx q_\pi$

Policy improvement ϵ -greedy policy improvement

Sarsa Algorithm for On-Policy Control

Initialize $Q(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\text{terminal-state}, \cdot) = 0$
Repeat (for each episode):

 Initialize S

 Choose A from S using policy derived from Q (e.g., ε -greedy)

 Repeat (for each step of episode):

 Take action A , observe R, S'

 Choose A' from S' using policy derived from Q (e.g., ε -greedy)

$$Q(S, A) \leftarrow Q(S, A) + \alpha [R + \gamma Q(S', A') - Q(S, A)]$$

$S \leftarrow S'; A \leftarrow A'$;

 until S is terminal

Convergence of Sarsa

Theorem

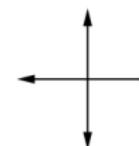
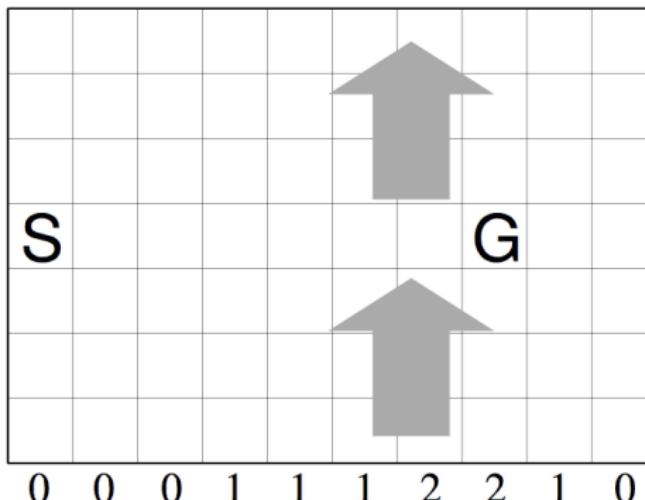
*Sarsa converges to the optimal action-value function,
 $Q(s, a) \rightarrow q_*(s, a)$, under the following conditions:*

- GLIE sequence of policies $\pi_t(a|s)$
- Robbins-Monro sequence of step-sizes α_t

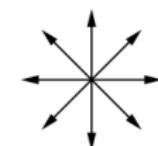
$$\sum_{t=1}^{\infty} \alpha_t = \infty$$

$$\sum_{t=1}^{\infty} \alpha_t^2 < \infty$$

Windy Gridworld Example



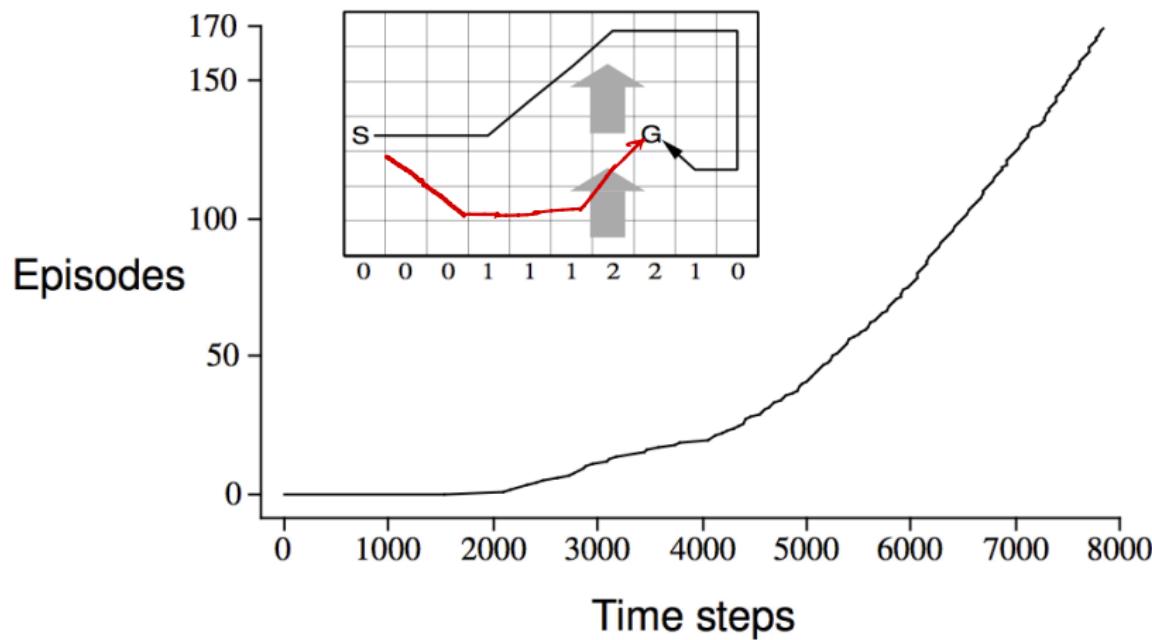
standard moves



king's moves

- Reward = -1 per time-step until reaching goal
- Undiscounted

Sarsa on the Windy Gridworld



n -Step Sarsa

- Consider the following n -step returns for $n = 1, 2, \infty$:

$$n = 1 \quad (\text{Sarsa}) \quad q_t^{(1)} = R_{t+1} + \gamma Q(S_{t+1})$$

$$n = 2 \quad q_t^{(2)} = R_{t+1} + \gamma R_{t+2} + \gamma^2 Q(S_{t+2})$$

$$\vdots$$

$$\vdots$$

$$n = \infty \quad (\text{MC}) \quad q_t^{(\infty)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{T-1} R_T$$

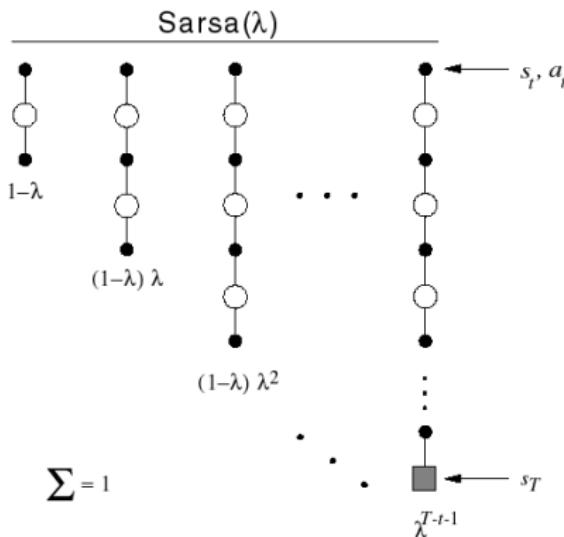
- Define the n -step Q-return

$$q_t^{(n)} = R_{t+1} + \gamma R_{t+2} + \dots + \gamma^{n-1} R_{t+n} + \gamma^n Q(S_{t+n})$$

- n -step Sarsa updates $Q(s, a)$ towards the n -step Q-return

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left(q_t^{(n)} - Q(S_t, A_t) \right)$$

Forward View Sarsa(λ)



- The q^λ *return* combines all n -step Q>Returns $q_t^{(n)}$
- Using weight $(1 - \lambda)\lambda^{n-1}$

$$q_t^\lambda = (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} q_t^{(n)}$$

- Forward-view Sarsa(λ)

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha \left(q_t^\lambda - Q(S_t, A_t) \right)$$

Backward View Sarsa(λ)

- Just like TD(λ), we use **eligibility traces** in an online algorithm
- But Sarsa(λ) has one eligibility trace for each state-action pair

$$E_0(s, a) = 0$$

$$E_t(s, a) = \gamma \lambda E_{t-1}(s, a) + \mathbf{1}(S_t = s, A_t = a)$$

- $Q(s, a)$ is updated for every state s and action a
- In proportion to TD-error δ_t and eligibility trace $E_t(s, a)$

$$\delta_t = R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)$$

$$Q(s, a) \leftarrow Q(s, a) + \alpha \delta_t E_t(s, a)$$

Sarsa(λ) Algorithm

Initialize $Q(s, a)$ arbitrarily, for all $s \in \mathcal{S}, a \in \mathcal{A}(s)$

Repeat (for each episode):

$$E(s, a) = 0, \text{ for all } s \in \mathcal{S}, a \in \mathcal{A}(s)$$

Initialize S, A

Repeat (for each step of episode):

Take action A , observe R, S'

Choose A' from S' using policy derived from Q (e.g., ε -greedy)

$$\delta \leftarrow R + \gamma Q(S', A') - Q(S, A)$$

$$E(S, A) \leftarrow E(S, A) + 1$$

For all $s \in \mathcal{S}, a \in \mathcal{A}(s)$:

$$Q(s, a) \leftarrow Q(s, a) + \alpha \delta E(s, a)$$

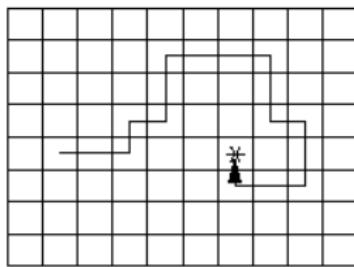
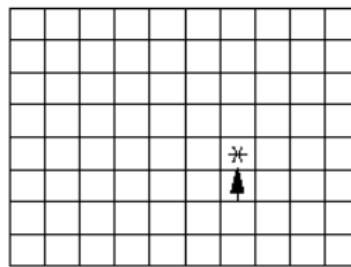
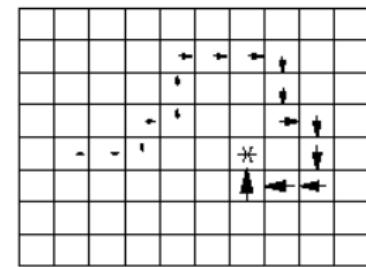
$$E(s, a) \leftarrow \gamma \lambda E(s, a)$$

$$S \leftarrow S'; A \leftarrow A'$$

until S is terminal

Sarsa(λ) Gridworld Example

Path taken

Action values increased
by one-step SarsaAction values increased
by Sarsa(λ) with $\lambda=0.9$ 

Off-Policy Learning

- Evaluate target policy $\pi(a|s)$ to compute $v_\pi(s)$ or $q_\pi(s, a)$
- While following behaviour policy $\mu(a|s)$

$$\{S_1, A_1, R_2, \dots, S_T\} \sim \mu$$

- Why is this important?
 - Learn from observing humans or other agents
 - Re-use experience generated from old policies $\pi_1, \pi_2, \dots, \pi_{t-1}$
 - Learn about *optimal* policy while following *exploratory* policy
 - Learn about *multiple* policies while following *one* policy

Importance Sampling

- Estimate the expectation of a different distribution

$$\begin{aligned}\mathbb{E}_{X \sim P}[f(X)] &= \sum P(X)f(X) \\ &= \sum Q(X)\frac{P(X)}{Q(X)}f(X) \\ &= \mathbb{E}_{X \sim Q}\left[\frac{P(X)}{Q(X)}f(X)\right]\end{aligned}$$

Importance Sampling for Off-Policy Monte-Carlo

- Use returns generated from μ to evaluate π
- Weight return G_t according to similarity between policies
- Multiply importance sampling corrections along whole episode

Not good

$$G_t^{\pi/\mu} = \frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} \frac{\pi(A_{t+1}|S_{t+1})}{\mu(A_{t+1}|S_{t+1})} \cdots \frac{\pi(A_T|S_T)}{\mu(A_T|S_T)} G_t$$

- Update value towards *corrected* return

$$V(S_t) \leftarrow V(S_t) + \alpha \left(G_t^{\pi/\mu} - V(S_t) \right)$$

- Cannot use if μ is zero when π is non-zero
- Importance sampling can dramatically increase variance

Importance Sampling for Off-Policy TD

- Use TD targets generated from μ to evaluate π
- Weight TD target $R + \gamma V(S')$ by importance sampling
- Only need a single importance sampling correction

$$V(S_t) \leftarrow V(S_t) + \alpha \left(\frac{\pi(A_t|S_t)}{\mu(A_t|S_t)} (R_{t+1} + \gamma V(S_{t+1})) - V(S_t) \right)$$

- Much lower variance than Monte-Carlo importance sampling
- Policies only need to be similar over a single step

Q-Learning

- We now consider off-policy learning of action-values $Q(s, a)$
- **No** importance sampling is required
- Next action is chosen using behaviour policy $A_{t+1} \sim \mu(\cdot | S_t)$
- But we consider alternative successor action $A' \sim \pi(\cdot | S_t)$
- And update $Q(S_t, A_t)$ towards value of alternative action

$$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \alpha (R_{t+1} + \gamma Q(S_{t+1}, A') - Q(S_t, A_t))$$

<https://towardsdatascience.com/simple-reinforcement-learning-q-learning-fcddc4b6fe56>

Off-Policy Control with Q-Learning

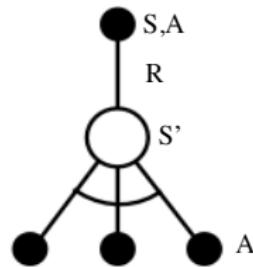
- We now allow both behaviour and target policies to **improve**
- The target policy π is **greedy** w.r.t. $Q(s, a)$

$$\pi(S_{t+1}) = \operatorname{argmax}_{a'} Q(S_{t+1}, a')$$

- The behaviour policy μ is e.g. **ϵ -greedy** w.r.t. $Q(s, a)$
- The Q-learning target then simplifies:

$$\begin{aligned} & R_{t+1} + \gamma Q(S_{t+1}, A') \\ &= R_{t+1} + \gamma Q(S_{t+1}, \operatorname{argmax}_{a'} Q(S_{t+1}, a')) \\ &= R_{t+1} + \max_{a'} \gamma Q(S_{t+1}, a') \end{aligned}$$

Q-Learning Control Algorithm



$$Q(S, A) \leftarrow Q(S, A) + \alpha \left(R + \gamma \max_{a'} Q(S', a') - Q(S, A) \right)$$

Theorem

Q-learning control converges to the optimal action-value function,
 $Q(s, a) \rightarrow q_*(s, a)$

Q-Learning Algorithm for Off-Policy Control

Initialize $Q(s, a), \forall s \in \mathcal{S}, a \in \mathcal{A}(s)$, arbitrarily, and $Q(\text{terminal-state}, \cdot) = 0$

Repeat (for each episode):

 Initialize S

 Repeat (for each step of episode):

 Choose A from S using policy derived from Q (e.g., ε -greedy)

 Take action A , observe R, S'

$$Q(S, A) \leftarrow Q(S, A) + \alpha [R + \gamma \max_a Q(S', a) - Q(S, A)]$$

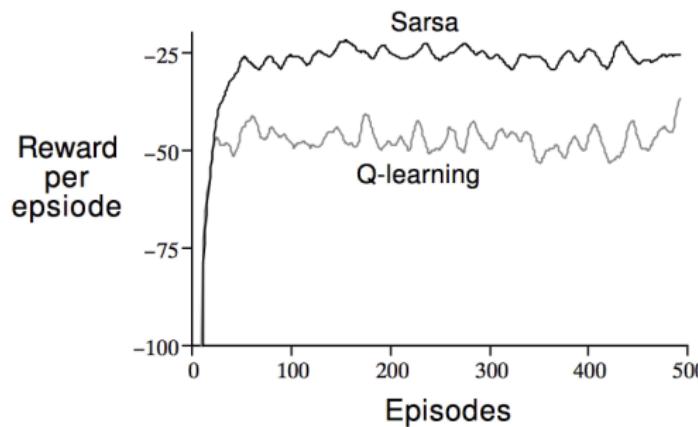
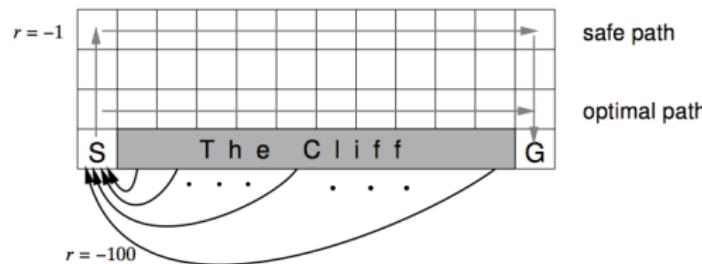
$S \leftarrow S'$;

 until S is terminal

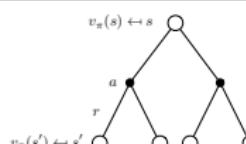
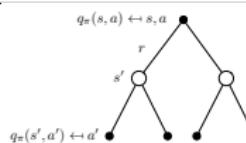
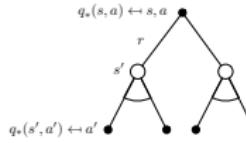
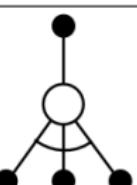
Q-Learning Demo

Q-Learning Demo

Cliff Walking Example



Relationship Between DP and TD

	<i>Full Backup (DP)</i>	<i>Sample Backup (TD)</i>
Bellman Expectation Equation for $v_\pi(s)$	$v_\pi(s) \leftarrow s$  <p>Iterative Policy Evaluation</p>	 <p>TD Learning</p>
Bellman Expectation Equation for $q_\pi(s, a)$	$q_\pi(s, a) \leftarrow s, a$  <p>Q-Policy Iteration</p>	 <p>Sarsa</p>
Bellman Optimality Equation for $q_*(s, a)$	$q_*(s, a) \leftarrow s, a$  <p>Q-Value Iteration</p>	 <p>Q-Learning</p>

Relationship Between DP and TD (2)

<i>Full Backup (DP)</i>	<i>Sample Backup (TD)</i>
Iterative Policy Evaluation $V(s) \leftarrow \mathbb{E}[R + \gamma V(S') \mid s]$	TD Learning $V(S) \xleftarrow{\alpha} R + \gamma V(S')$
Q-Policy Iteration $Q(s, a) \leftarrow \mathbb{E}[R + \gamma Q(S', A') \mid s, a]$	Sarsa $Q(S, A) \xleftarrow{\alpha} R + \gamma Q(S', A')$
Q-Value Iteration $Q(s, a) \leftarrow \mathbb{E} \left[R + \gamma \max_{a' \in \mathcal{A}} Q(S', a') \mid s, a \right]$	Q-Learning $Q(S, A) \xleftarrow{\alpha} R + \gamma \max_{a' \in \mathcal{A}} Q(S', a')$

where $x \xleftarrow{\alpha} y \equiv x \leftarrow x + \alpha(y - x)$

Questions?