LIMIT & CONTINUITY

Definition of Limit: If the values of **f (x)** become arbitrarily close to a single number **l** as the values of a variable **x** approaches to **a** from both sides of **a** (right and left), then **l** is called the limit of the function $\mathbf{f(x)}$. It is denoted by $\lim\limits_{x\to a} f(x) = l$.

Ex. 1) $\lim\limits_{x\to 1}(x^2 - 2x + 1)$

$= \lim\limits_{x\to 1}(x^2 - 2x + 1)$

$= 1 - 2 + 1 = 0$

Ex. 2) $\lim\limits_{x\to 3} \frac{x^2 - 2x}{x + 1}$

$= \lim\limits_{x\to 3} \frac{x^2 - 2x}{x + 1}$

$= \frac{9 - 6}{3 + 1} = \frac{3}{4}$

Ex. 3) $\lim\limits_{x\to -1} \frac{x + 1}{x^2 - 1}$

$= \lim\limits_{x\to -1} \frac{x + 1}{x^2 - 1}$

$= \lim\limits_{x\to -1} \frac{(x + 1)}{(x + 1)(x - 1)}$

$= \lim\limits_{x\to -1} \frac{1}{x - 1}$

$= -\frac{1}{2}$

Ex. 4) $\lim\limits_{x\to \frac{\pi}{2}} \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}}$

$= \lim\limits_{x\to \frac{\pi}{2}} \frac{x^3 - \frac{\pi^3}{8}}{x - \frac{\pi}{2}}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\left(x - \frac{\pi}{2}\right)\left(x^2 + x.\frac{\pi}{2} + \frac{\pi^2}{4}\right)}{\left(x - \frac{\pi}{2}\right)}$$

$$= \lim_{x \to \frac{\pi}{2}} \left(x^2 + x.\frac{\pi}{2} + \frac{\pi^2}{4}\right)$$

$$= \frac{\pi^2}{4} + \frac{\pi^2}{4} + \frac{\pi^2}{4} = \frac{3\pi^2}{4}$$

Ex. 5) $\lim_{x \to -\infty} \frac{\sqrt{3x^4 + x}}{x^2 - 8}$

$$= \lim_{x \to -\infty} \frac{\sqrt{x^4 \left(3 + \frac{x}{x^4}\right)}}{x^2 \left(1 - \frac{8}{x^2}\right)}$$

$$= \lim_{x \to -\infty} \frac{x^2 \sqrt{\left(3 + \frac{1}{x^3}\right)}}{x^2 \left(1 - \frac{8}{x^2}\right)}$$

$$= \lim_{x \to -\infty} \frac{\sqrt{3 + \frac{1}{x^3}}}{1 - \frac{8}{x^2}}$$

$$= \frac{\sqrt{3 + 0}}{1 - 0}$$

$$= \sqrt{3}$$

H.W

1) $\lim_{x \to 1} \frac{x^4 - 1}{x - 1}$

2) $\lim_{x \to \infty} \frac{5x^2 + 7}{3x^2 - x}$

3) $\lim_{x \to \infty} \frac{2^x - 2^{-x}}{2^x + 2^{-x}}$

4) $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$

Continuity: If $\mathbf{f(a)}$ is defined, $\lim\limits_{\mathbf{x\to a}} \mathbf{f(x)}$ exists and $\lim\limits_{\mathbf{x\to a}} \mathbf{f(x)} = \mathbf{f(a)}$ then $\mathbf{f(x)}$ is called continuous at $\mathbf{x = a}$. Here, $\lim\limits_{\mathbf{x\to a}} \mathbf{f(x)}$ means the value of limit of $\mathbf{f\ (x)}$ at $\mathbf{x = a}$ and $\mathbf{f(a)}$ means the value of $\mathbf{f(x)}$ at $\mathbf{x = a}$.

If $\lim\limits_{\mathbf{x\to a_-}} \mathbf{f(x)} = \mathbf{f\ (a)}$, then $\mathbf{f(x)}$ is called left continuous at $\mathbf{x = a}$.

If $\lim\limits_{\mathbf{x\to a_+}} \mathbf{f(x)} = \mathbf{f\ (a)}$, then $\mathbf{f(x)}$ is called right continuous at $\mathbf{x = a}$.

If $\mathbf{f(x)}$ is not continuous at $\mathbf{x = a}$ then it is said to be discontinuous.

Ex. 1) At $x = 1$ and $x = 0$ discuss the continuity of the function $f: R \to R$,

$$\text{where, } f(x) = \begin{cases} x^2 + 1 \text{ when } x < 0 \\ x \quad \text{when } 0 \le x \le 1 \\ \dfrac{1}{x} \text{ when } x > 1 \end{cases}$$

Solution:

At $x = 1$,

$$\lim_{x\to 1_+} f(x) = \lim_{x\to 1_+} \frac{1}{x} = \frac{1}{1} = 1$$

$$\lim_{x\to 1_-} f(x) = \lim_{x\to 1_-} (x) = 1$$

and,

when, $x = 1$ then, $f(x) = f(1) = 1$

$$\therefore \lim_{x\to 1_+} f(x) = \lim_{x\to 1_-} f(x) = f(1)$$

So, the function $f(x)$ is continuous at $x = 1$.

At $x = 0$,

$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+} (x) = 0$$

$$\lim_{x\to 0_-} f(x) = \lim_{x\to 0_-} (x^2 + 1) = 1$$

Since, $\lim\limits_{x\to 0_+} f(x) \neq \lim\limits_{x\to 0_-} f(x)$

So, the function $f(x)$ is not continuous at $x = 0$.

Ex. 2) Investigate the continuity of the function at $x = 0$ and $x = \frac{3}{2}$,

$$\text{where, } f(x) = \begin{cases} 3 + 2x \text{ when } -\frac{3}{2} \leq x < 0 \\ 3 - 2x \text{ when } 0 \leq x \leq \frac{3}{2} \\ -3 - 2x \quad \text{when } x \geq \frac{3}{2} \end{cases}$$

Solution:

At $x = 0$,

$$\lim_{x \to 0_+} f(x) = \lim_{x \to 0_+} (3 - 2x) = 3$$

$$\lim_{x \to 0_-} f(x) = \lim_{x \to 0_-} (3 + 2x) = 3$$

and,

when, $x = 0$ then, $f(x) = f(0) = 3 - 0 = 3$

$$\therefore \lim_{x \to 0_+} f(x) = \lim_{x \to 0_-} f(x) = f(0)$$

So, the function $f(x)$ is not continuous at $x = 0$.

At $x = \frac{3}{2}$,

$$\lim_{x \to \frac{3}{2}_+} f(x) = \lim_{x \to \frac{3}{2}_+} (-3 - 2x) = -3 - 2.\frac{3}{2} = -3 - 3 = -6$$

$$\lim_{x \to \frac{3}{2}_-} f(x) = \lim_{x \to \frac{3}{2}_-} (3 - 2x) = 3 - 2.\frac{3}{2} = 3 - 3 = 0$$

Since, $\lim_{x \to \frac{3}{2}_+} f(x) \neq \lim_{x \to \frac{3}{2}_-} f(x)$

So, the function $f(x)$ is not continuous at $x = \frac{3}{2}$.

Ex. 3) Show that, the function f (x) is discontinuous at x = 2.

Also Define the function **f** in such a way so that it is continuous at x = 2.

$$\text{where, } f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{when } x \neq 2 \\ 3 & \text{when } x = 2 \end{cases}$$

Solution:

At x = 2,

$$\lim_{x \to 2_+} f(x) = \lim_{x \to 2_+} \left(\frac{x^2 - 4}{x - 2} \right) = \lim_{x \to 2_+} (x + 2) = 4$$

$$\lim_{x \to 2_-} f(x) = \lim_{x \to 2_-} \left(\frac{x^2 - 4}{x - 2} \right) = \lim_{x \to 2_-} (x + 2) = 4$$

when, $x = 2$ then, $f(x) = f(2) = 3$

Since, $\lim_{x \to 2_+} f(x) \neq \lim_{x \to 2_-} f(x) \neq f(2)$. So, the function f(x) is not continuous at x = 2.

For continuity f(x)is defined at x = 2 in the following way,

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$$

<u>H.W:</u>

1) Show that, the function f (x) is discontinuous at x = 2.

$$\text{where, } f(x) = \begin{cases} x + 2 & \text{when } x < 2 \\ x^2 - 1 & \text{when } x \geq 2 \end{cases}$$

2) Show that, the function f (x) is continuous at x = 0 and discontinuous at x = 1,

$$\text{where, } f(x) = \begin{cases} -x & \text{when } x \leq 0 \\ x & \text{when } 0 < x < 1 \\ 1 - x & \text{when } x \geq 1 \end{cases}$$

3) Investigate the continuity of the function at x = 1,

$$\text{where, } f(x) = \begin{cases} x & \text{when } x \leq 1 \\ 2x - 1 & \text{when } x > 1 \end{cases}$$

4) Show that, the function f (x) is continuous at $x = 0$ and discontinuous at $x = -1$.

$$\text{where, } f(x) = \begin{cases} 1 + x & \text{when } -4 \leq x < -1 \\ 4 & \text{when } -1 \leq x \leq 0 \\ 4 + x^2 & \text{when } 0 < x \leq 4 \end{cases}$$

5) Investigate the continuity of the function at $x = 0,1,2$

$$\text{where, } f(x) = \begin{cases} -x^2 & \text{when } x \leq 0 \\ 5x - 4 & \text{when } 0 < x \leq 1 \\ 4x^2 - 3x & \text{when } 1 < x < 2 \\ 3x + 4 & \text{when } x \geq 2 \end{cases}$$

6) Investigate the continuity of the function at $x = 1$ and $x = 2$,

$$\text{where, } f(x) = \begin{cases} \ln x & \text{when } 0 < x \leq 1 \\ 0 & \text{when } 1 < x \leq 2 \\ 1 + x^2 & \text{when } x > 2 \end{cases}$$