Maclaurin's and Taylor's Theorem

Maclaurin's Theorem: If at $x = 0$, $f(x), f'(x). f''(x), \ldots \ldots \ldots \ldots, f^n(x)$ exists then, maclaurin's polynomial of n-th order,

$$P_n(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + \frac{x^n}{n!}f^n(0)$$

Taylor's Theorem: If at $x = x_0$, $f(x), f'(x), f''(x), \ldots \ldots \ldots \ldots, f^n(x)$ exists then, taylor's polynomial of n-th order,

$$P_n(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) + \cdots \ldots \ldots \ldots \ldots + \frac{(x - x_0)^n}{n!}f^n(x_0)$$

Ex. 1) Find the Maclaurin's polynomial $P_n(x)$ for $\frac{1}{1-x}$

Solution: Let,

$$f(x) = \frac{1}{1-x} \qquad \Rightarrow f(0) = 1$$

$$f'(x) = \frac{1}{(1-x)^2} \qquad \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{2}{(1-x)^3} \qquad \Rightarrow f''(0) = 2 = 2!$$

$$f'''(x) = \frac{2.3}{(1-x)^4} \qquad \Rightarrow f'''(0) = 6 = 3!$$

$\ldots \ldots \ldots \ldots \ldots \ldots$

$\ldots \ldots \ldots \ldots \ldots \ldots$

$$f^n(x) = \frac{n!}{(1-x)^{n+1}} \qquad \Rightarrow f^n(0) = n!$$

We know, from the Maclaurin's series,

$$P_n(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + \frac{x^n}{n!}f^n(0)$$

$$\therefore P_n(x) = 1 + x + x^2 + x^3 + \cdots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots + x^n$$

Ex.2) Proof by Maclaurin's Theorem,

$$\mathbf{\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5(1+c)^5}}$$

Solution: Let,

$f(x) = \ln(1+x) \qquad \Rightarrow f(0) = 0$

$f'(x) = \dfrac{1}{1+x} \qquad \Rightarrow f'(0) = 1$

$f''(x) = \dfrac{-1}{(1+x)^2} \qquad \Rightarrow f''(0) = -1$

$f'''(x) = \dfrac{2}{(1+x)^3} \qquad \Rightarrow f'''(0) = 2$

$f^{iv}(x) = \dfrac{-6}{(1+x)^4} \qquad \Rightarrow f^{iv}(0) = -6$

$f^{v}(x) = \dfrac{24}{(1+x)^5} \qquad \Rightarrow f^{v}(c) = \dfrac{24}{(1+c)^5}$

We know, from the Maclaurin's series,

$$P_n(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(x) + \frac{x^5}{5!}f^{v}(x)$$

$$\therefore P_n(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5(1+c)^5}$$

Ex.3) Find the first six Taylor polynomial of $\ln x$ at $x = 2$

Solution: Let,

$f(x) = \ln x \qquad \Rightarrow f(2) = \ln 2$

$f'(x) = \dfrac{1}{x} \qquad \Rightarrow f'(2) = \dfrac{1}{2}$

$f''(x) = \dfrac{-1}{x^2} \qquad \Rightarrow f''(2) = -\dfrac{1}{4}$

$f'''(x) = \dfrac{2}{x^3} \qquad \Rightarrow f'''(2) = \dfrac{1}{4}$

$$f^{iv}(x) = \frac{-6}{x^4} \qquad \Rightarrow f^{iv}(2) = -\frac{3}{8}$$

$$f^{v}(x) = \frac{24}{x^5} \qquad \Rightarrow f^{v}(2) = \frac{3}{4}$$

We know by the Taylor's theorem,

$$P_n(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) + \frac{(x - x_0)^4}{4!}f^{iv}(x_0) + \frac{(x - x_0)^5}{5!}f^{v}(x_0)$$

$$\Rightarrow P_n(x) = f(2) + (x - 2)f'(2) + \frac{(x - 2)^2}{2!}f''(2) + \frac{(x - 2)^3}{3!}f'''(2) + \frac{(x - 2)^4}{4!}f^{iv}(2) + \frac{(x - 2)^5}{5!}f^{v}(2)$$

$$\Rightarrow P_n(x) = \ln 2 + \frac{x - 2}{2} + \frac{(x - 2)^2}{2!}\left(-\frac{1}{4}\right) + \frac{(x - 2)^3}{3!}\left(\frac{1}{4}\right) + \frac{(x - 2)^4}{4!}\left(-\frac{3}{8}\right) + \frac{(x - 2)^5}{5!}\left(\frac{3}{4}\right)$$

$$\therefore P_n(x) = \ln 2 + \frac{x - 2}{2} - \frac{(x - 2)^2}{8} + \frac{(x - 2)^3}{24} - \frac{(x - 2)^4}{64} + \frac{(x - 2)^5}{160}$$

Ex. 4) Find the first eight Taylor polynomial of $\cos x$ at $x = 0$

Solution: Let,

$$f(x) = \cos x \qquad \Rightarrow f(0) = 1$$

$$f'(x) = -\sin x \qquad \Rightarrow f'(0) = 0$$

$$f''(x) = -\cos x \qquad \Rightarrow f''(0) = -1$$

$$f'''(x) = \sin x \qquad \Rightarrow f'''(0) = 0$$

$$f^{iv}(x) = \cos x \qquad \Rightarrow f^{iv}(0) = 1$$

$$f^{v}(x) = -\sin x \qquad \Rightarrow f^{v}(0) = 0$$

$$f^{vi}(x) = -\cos x \qquad \Rightarrow f^{vi}(0) = -1$$

$$f^{vii}(x) = \sin x \qquad \Rightarrow f^{vii}(0) = 0$$

We know by the Taylor's theorem,

$$P_n(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \frac{(x - x_0)^3}{3!}f'''(x_0) + \frac{(x - x_0)^4}{4!}f^{iv}(x_0) + \frac{(x - x_0)^5}{5!}f^{v}(x_0) + \frac{(x - x_0)^6}{6!}f^{vi}(x_0) + \frac{(x - x_0)^7}{7!}f^{vii}(x_0)$$

$$\Rightarrow P_n(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \frac{x^5}{5!}f^{v}(0) + \frac{x^6}{6!}f^{vi}(0) + \frac{x^7}{7!}f^{vii}(0)$$

$$\therefore P_n(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

H.W:

1) Find the first eight Taylor polynomial of $\sin x$ at $x = 0$

2) Find the first eight Taylor polynomial of e^x at $x = 0$

3) Find the first six Taylor polynomial of $\ln(x+1)$ at $x = 0$