Applications of Linear Algebra

1. Area of any triangle ABC through three given points in a plane.

For three vertices, $A\ (x_1, y_1), B\ (x_2, y_2)$ and $C\ (x_3, y_3)$, Area $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

Ex. 1) Find the area of the triangle whose vertices are $A(2, 1), B(-2, 4)$ and $C(0, -2)$.

Solution: According to theorem, the area of the triangle ABC whose vertices are $A\ (x_1, y_1), B\ (x_2, y_2)$ and $C\ (x_3, y_3)$ can be written in determinant form as,

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Therefore, the area of the triangle ABC whose vertices are $A(2,1), B(-2,4)$ and $C(0, -2)$ is given by,

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ -2 & 4 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 9 \text{ square units}$$

2. The equation of a straight line through two given points in a plane.

The equation of a straight line passing through two given points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ is given by the determinant equation $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Ex. 2) Find the equation of the straight line passing through the points $(-1, -1)$ and $(5, 7)$.

Solution: According to theorem, the equation of a straight line passing through two given points $A\ (x_1, y_1)$ and $B\ (x_2, y_2)$ is given by the determinant equation $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

Therefore, the equation of a straight line passing through the points $(-1, -1)$ and $(5,7)$ is given by,

$$\begin{vmatrix} x & y & 1 \\ -1 & -1 & 1 \\ 5 & 7 & 1 \end{vmatrix} = 0$$

or, $4x - 3y + 1 = 0$; which is the required equation of the line.

3. Condition for three straight lines to meet at a point.

The condition that the three straight lines $a_1x + b_1y + c_1 = 0, a_2x + b_2y + c_2 = 0$ and

$a_3x + b_3y + c_3 = 0$ meet at a point can be written in determinant form as,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Ex. 3) Proof that the straight lines $\mathbf{3x + 5y - 14 = 0, 3x - y - 2 = 0}$ and $\mathbf{3x - 7y + 10 = 0}$ meet in a point.

Solution: According to theorem, the condition that the three straight lines $a_1x + b_1y + c_1 = 0$,

$a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ meet at a point can be written in determinant form as,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Now, for the given three straight lines we have,

$$\begin{vmatrix} 3 & 5 & -14 \\ 3 & -1 & -2 \\ 3 & -7 & 10 \end{vmatrix} = -252 + 252 = 0;$$ so, the straight lines meet in a point.

4. Condition for a pair of straight lines.

The condition for the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

to represent a pair of straight lines can be expressed in determinant form as,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Ex. 4) Proof that the second degree equation $\mathbf{3x^2 - 14xy - 5y^2 - 54x - 2y + 51 = 0}$ represents a pair straight lines.

Solution: According to theorem, the condition for the general equation of second degree

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines can be expressed

in determinant form as, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

For the given second degree equation, $a = 3, b = -5, h = -7, g = -27, f = -1, c = 51$

$$\Delta=\begin{vmatrix} 3 & -7 & -27 \\ -7 & -5 & -1 \\ -27 & -1 & 51 \end{vmatrix} = -3456 + 3456 = 0$$

So, the given second degree equation represent a pair of straight lines.

5. The equation of a circle through three given points not all lying on a straight line in a plane.

Let, A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) be the three points not all lying in a straight line.

Then the equation of a circle through these three given points can be expressed in determinant

equation as, $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ {x_1}^2+{y_1}^2 & x_1 & y_1 & 1 \\ {x_2}^2+{y_2}^2 & x_2 & y_2 & 1 \\ {x_3}^2+{y_3}^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$

Ex. 5) Find the equation of the circle passing through three points $(1, 3)$, $(-1, 1)$and $(2, -1)$.

Solution: According to theorem, the equation of a circle passing through these three given points

can be expressed in determinant equation as, $\begin{vmatrix} x^2+y^2 & x & y & 1 \\ {x_1}^2+{y_1}^2 & x_1 & y_1 & 1 \\ {x_2}^2+{y_2}^2 & x_2 & y_2 & 1 \\ {x_3}^2+{y_3}^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$

Therefore, the equation of the circle passing through three points $(1,3)$, $(-1,1)$and $(2, -1)$

is given by the determinant equation,

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 10 & 1 & 3 & 1 \\ 2 & -1 & 1 & 1 \\ 5 & 2 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x^2+y^2-10 & x-1 & y-3 & 0 \\ 8 & 2 & 2 & 0 \\ -3 & -3 & 2 & 0 \\ 5 & 2 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x^2+y^2-10 & x-1 & y-3 \\ 8 & 2 & 2 \\ -3 & -3 & 2 \end{vmatrix} = 0$$

$\Rightarrow 5x^2 + 5y^2 - 11x - 9y - 12 = 0$; which is the required equation.

6. The equation of a plane through three non-collinear points in space.

The plane in three dimensional space with equation $ax + by + cz + d = 0$ that passes through three non $-$ collinear points $A\ (x_1, y_1, z_1), B\ (x_2, y_2, z_2)$ and $C\ (x_3, y_3, z_3)$ is given by the determinant equation as, $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

Ex. 6) Find the equation of the plane which passes through the three non $-$ collinear points $(1, 1, 0), (2, 0, -1)$ and $(2, 9, 2)$.

Solution: According to theorem, the equation the plane which passes through the three non $-$ collinear points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is given by the determinant equation as, $\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$

Therefore, the equation of the plane passing through three non $-$ collinear points $(1,1,0)$, $(2,0,-1)$ and $(2,9,2)$ is given by the determinant equation,

$$\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & -1 & 1 \\ 2 & 9 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -9 & -3 & 0 \\ 2 & 9 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-1 & y-1 & z \\ -1 & 1 & 1 \\ 0 & -9 & -3 \end{vmatrix} = 0$$

$\Rightarrow 2x - y + 3z - 1 = 0$; which is the required equation of the plane.

Applications of Determinant (Cramer's Rule) in Business and Economics

Ex. 1) A company produces three products everyday. Their total production on a certain day is 45 tons. It is found that the production of the third product exceeds the production of the first product by 8 tons while the total production of first and third products is twice the production of the second product. Determine the production level of each product using Cramer's Rule.

Solution: Let, the production level of the three products be x, y and z respectively. Then we have the following system of linear equations:

$$\begin{cases} x+y+z=45 \\ z=x+8 \\ x+z=2y \end{cases} \Rightarrow \begin{cases} x+y+z=45 \\ -x+z=8 \\ x-2y+z=0 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

Let, $\Delta =$ Determinant of co – efficients of $x, y, z = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 6 \neq 0$

Since, $\Delta \neq 0$, the unique solution of the linear system (i) is given by Cramer's Rule,

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

Where, $\Delta_1 = \begin{vmatrix} 45 & 1 & 1 \\ 8 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = 66, \Delta_2 = \begin{vmatrix} 1 & 45 & 1 \\ -1 & 8 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 90, \Delta_3 = \begin{vmatrix} 1 & 1 & 45 \\ -1 & 0 & 8 \\ 1 & -2 & 0 \end{vmatrix} = 114$

$$\therefore x = \frac{\Delta_1}{\Delta} = 1, y = \frac{\Delta_2}{\Delta} = 15, z = \frac{\Delta_3}{\Delta} = 19$$

Ex. 2) An automibile company uses three types of steel S_1, S_2, S_3 for producing three types of cars C_1, C_2, C_3. Steel requirement for each type of car is given below:

		CARS		
STEEL		C_1	C_2	C_3
	S_1	2	3	4
	S_2	1	1	2
	S_3	3	2	1

Determine the number of cars of each type which can be produced using 29, 13 and 16 tons of steel of three types respectively.

Solution: Let, the production of C_1, C_2 and C_3 types of cars be x, y and z respectively. Then we have the following system of linear equations:

$$\begin{cases} 2x + 3y + 4z = 29 \\ x + y + 2z = 13 \\ 3x + 2y + z = 16 \end{cases} \quad \dots\dots\dots\dots\dots\dots\dots (i)$$

Let, $\Delta =$ Determinant of co − efficients of x, y, z $= \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 5 \neq 0$

Since, $\Delta \neq 0$, the unique solution of the linear system (i)is given by Cramer′s Rule,

$$x = \frac{\Delta_1}{\Delta}, y = \frac{\Delta_2}{\Delta}, z = \frac{\Delta_3}{\Delta}$$

Where, $\Delta_1 = \begin{vmatrix} 29 & 3 & 4 \\ 13 & 1 & 2 \\ 16 & 2 & 1 \end{vmatrix} = 10, \Delta_2 = \begin{vmatrix} 2 & 29 & 4 \\ 1 & 13 & 2 \\ 3 & 16 & 1 \end{vmatrix} = 15, \Delta_3 = \begin{vmatrix} 2 & 3 & 29 \\ 1 & 1 & 13 \\ 3 & 2 & 16 \end{vmatrix} = 20$

$$\therefore x = \frac{\Delta_1}{\Delta} = 2, y = \frac{\Delta_2}{\Delta} = 3, z = \frac{\Delta_3}{\Delta} = 4$$

Ex. 3) A farm produces three products A, B and C which it sells in two markets.

Annual sales in units are given below:

Markets	**Units Sold**		
	A	**B**	**C**
I	600	300	1200
II	500	1400	700

If the prices per unit of A, B, C are Rs. 4, 2 and 1.50 respectively and the costs per unit are Rs. 2,1,0.50 respectively, Find the total profit in each market by using matrix Algebra.

Solution: We consider the matrices, $P = \begin{bmatrix} 4 \\ 2 \\ 1.50 \end{bmatrix}$ and $Q = \begin{bmatrix} 2 \\ 1 \\ 0.50 \end{bmatrix}$

where the matrix P represents the selling price per unit and Q represents the cost price per unit of A, B and C. Therefore, the profit per unit is,

$$P - Q = \begin{bmatrix} 4 \\ 2 \\ 1.50 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 0.50 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Thus, the total profit is given by the product of the matrices,

$\begin{bmatrix} 600 & 300 & 1200 \\ 500 & 1400 & 700 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2700 \\ 3100 \end{bmatrix}$; Profit of 1st market = 2700 and 2nd market = 3100.

Ex. 4) The equilibrium conditions for three realted markets are given by,

$$\begin{cases} 3x + 2y - z = 20 \\ 2x + 3y - 3z = 7 \\ x - y + 6z = 41 \end{cases}$$; using matrix algebra find the equilibrium price for each market.

Solution: The given system can be written in matrix form as,

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & 3 & -3 \\ 1 & -1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 7 \\ 41 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B$$

Now, $|A| = 20 \neq 0$; So, A^{-1} exists, $A^{-1} = \frac{1}{|A|}\text{ Adj } A = \frac{1}{20}\begin{bmatrix} 15 & -11 & -3 \\ -15 & 19 & 7 \\ -5 & 5 & 5 \end{bmatrix}$

Hence, the solution is given by,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \frac{1}{20}\begin{bmatrix} 15 & -11 & -3 \\ -15 & 19 & 7 \\ -5 & 5 & 5 \end{bmatrix}\begin{bmatrix} 20 \\ 7 \\ 41 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$$

Thus, $x = 5, y = 6, z = 7$.

Ex. 4) An investor deposited Tk. 40000 in a bank, some at a half yearly rate of 5% and the rest at a annual rate of 12%. At the end of the year, he recieved annual interest of Tk. 4500. Using matrix method, find out how much he deposited at 5%.

Solution: Let, x and y represent the investments at the rates of interest of 10% and 12% per annual respectively. Then according to the given data we have,

$$\begin{cases} x + y = 40000 \\ \frac{10}{100}x + \frac{12}{100}y = 4500 \end{cases} \Rightarrow \begin{cases} x + y = 40000 \\ 5x + 6y = 225000 \end{cases}$$

The above system can be written in matrix form as,

$$\begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 40000 \\ 225000 \end{bmatrix} \Rightarrow AX = B \Rightarrow X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B$$

Now, $|A| = 1 \neq 0$; So, A^{-1} exists, $A^{-1} = \frac{1}{|A|}\text{ Adj } A = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}$

Hence, the solution is given by,

$$X = \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}B = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 40000 \\ 225000 \end{bmatrix} = \begin{bmatrix} 15000 \\ 25000 \end{bmatrix}$$

Thus, $x = 15000, y = 25000$; He deposited Tk. 15000 at a half yearly rate of 5%.

H.W

1) A transport company uses 3 types of trucks T_1, T_2, T_3 to transport 3 types of vehicles V_1, V_2, V_3.

The capacity of each truck in terms of 3 types of vehicles is given below:

	V_1	V_2	V_3
T_1	1	3	2
T_2	2	2	3
T_3	3	2	2

Using the matrix method find,

(i) The number of trucks of each type required to transport 85, 105 and 110 vehicles of V_1, V_2, V_3.

(ii) The number of vehicles of each type which can be transported if the company has 10, 20 and 30 trucks of each type respectively.

ANS: (i) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 10 \end{bmatrix}$; (ii) $\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 140 \\ 130 \\ 140 \end{bmatrix}$

2) There are two families A and B. There are 4 men, 3 women and 2 children in family A and 3 men, 2 women and 2 children in family B. The recommended daily allowance

For calories is, Men: 2000 gm; Women: 1500 gm; Children: 12000 gm

For proteins is, Men: 50 gm; Women: 40 gm; Children: 30 gm

Represent the above information by matrices. Using matrix multiplication, calculate the total requirement of calories and proteins for each of the two families.

ANS: $\begin{array}{cc} \text{Calories} & \text{Proteins} \\ \end{array}$ $\begin{bmatrix} 14900 & 380 \\ 11400 & 290 \end{bmatrix} \begin{array}{c} A \\ B \end{array}$