<u>Eigen Values and Eigen Vectors</u>

Ex. 1) Find the Eigen values and corresponding Eigen Vectors of the matrix, $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$

Solution: The characteristic matrix of A is,

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{bmatrix}$$

Now, the determinant,

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{vmatrix} = (\lambda - 2)(\lambda - 4) - 3$$

Therefore, the characteristic equation of A is,

$$(\lambda - 2)(\lambda - 4) - 3 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 8 - 3 = 0$$

$$\Rightarrow \lambda^2 - 6\lambda + 5 = 0$$

$$\Rightarrow (\lambda - 5)(\lambda - 1) = 0$$

$\Rightarrow \lambda = 1$ and $\lambda = 5$; which are the Eigen values of A.

Now, let $X = \begin{bmatrix} x \\ y \end{bmatrix}$ is an Eigen vector of A.

Then,

$$(\lambda I - A)X = 0$$

$$\Rightarrow \begin{bmatrix} \lambda - 2 & -3 \\ -1 & \lambda - 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \text{(i)}$$

If $\lambda = 1$ then equation (i) becomes,

$$\begin{bmatrix} -1 & -3 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -x - 3y = 0 \\ -x - 3y = 0 \end{cases}$$

$$\Rightarrow x + 3y = 0$$

We have one equation with two unknowns.

Let, the free variable, $y = a$ and $x = -3a$.

Therefore, the eigenvector corresponding to $\lambda = 1$ is, $X = \begin{bmatrix} -3a \\ a \end{bmatrix}$

If $\lambda = 5$ then equation (i) becomes,

$$\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} 3x - 3y = 0 \\ -x + y = 0 \end{cases}$$

$$\Rightarrow x - y = 0$$

We have one equation with two unknowns.

Let, $y = b$ and $x = b$.

Therefore, the eigenvector corresponding to $\lambda = 5$ is, $X = \begin{bmatrix} b \\ b \end{bmatrix}$

Ex.2) Find the Eigen values and corresponding Eigen Vectors of the matrix, $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$

Solution: The characteristic matrix of A is,

$$\lambda I - A = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda + 2 & 0 \\ 0 & 5 & \lambda - 2 \end{bmatrix}$$

Now, the determinant,

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda + 2 & 0 \\ 0 & 5 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda + 2)(\lambda - 2)$$

Therefore, the characteristic equation of A is,

$$(\lambda - 1)(\lambda + 2)(\lambda - 2) = 0$$

$\Rightarrow \lambda = 1$, $\lambda = -2, \lambda = 2$; which are the Eigen values of A.

Now, let $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is an Eigen vector of A.

Then,

$(\lambda I - A)X = 0$

$$\Rightarrow \begin{bmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda + 2 & 0 \\ 0 & 5 & \lambda - 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (i)$$

If $\lambda = 1$ then equation (i) becomes,

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -2y + z = 0 \\ 3y = 0 \\ 5y - z = 0 \end{cases}$$

Solving we get, $y = z = 0$.

Let, the free variable $x = a$.

Therefore, the eigenvector corresponding to $\lambda = 1$ is, $X = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

If $\lambda = -2$ then equation (i) becomes,

$$\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} -3x - 2y + z = 0 \\ 5y - 4z = 0 \end{cases}$$

We have two equations with three unknowns.

Let, the free variable, $z = b$.

Then, $y = \frac{4b}{5}, x = \frac{-b}{5}$

Therefore, the eigenvector corresponding to $\lambda = -2$ is, $X = \begin{bmatrix} \frac{-b}{5} \\ \frac{4b}{5} \\ b \end{bmatrix}$

If $\lambda = 2$ then equation (i) becomes,

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 4 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x - 2y + z = 0 \\ 4y = 0 \\ 5y = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x - 2y + z = 0 \\ y = 0 \end{cases}$$

We have two equations with three unknowns.

Let, the free variable, $z = c$.

Then, $y = 0, x = -c$

Therefore, the eigenvector corresponding to $\lambda = -2$ is, $X = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$