LU DECOMPOSITION METHOD

Every square matrix can be expressed as the product of a Lower Triangular Matrix and an Upper Triangular Matrix.

Working Rule: Consider the system of equations,

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{32}z = b_3 \end{cases};$$

Then, $AX = B \ldots\ldots\ldots\ldots$ (i) where, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Let, $A = L\,.\,U$; where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Then, from (i), $LUX = B \ldots\ldots\ldots\ldots\ldots$ (ii)

Let, $UX = Y \ldots\ldots\ldots\ldots\ldots\ldots$ (iii)

Therefore from (ii), $LY = B \ldots\ldots\ldots\ldots$ (iv) where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ (Solve to find y_1, y_2, y_3)

Then, from (iii), $UX = Y$, we can find $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (required solution)

Ex. 1) Solve the system of linear equations by LU Decomposition Method:

$$\begin{cases} \mathbf{x + y + z = 1} \\ \mathbf{4x + 3y - z = 6} \\ \mathbf{3x + 5y + 3z = 4} \end{cases}$$

Solution: Let us write the given system of linear equations in matrix form,

$A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix};$

$AX = B \ldots\ldots\ldots\ldots\ldots$ (i) Let, $A = LU$

Now, $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}.u_{11} & l_{21}.u_{12} + u_{22} & l_{21}.u_{13} + u_{23} \\ l_{31}.u_{11} & l_{31}.u_{12} + l_{32}.u_{22} & l_{31}.u_{13} + l_{32}.u_{23} + u_{33} \end{bmatrix}$$

Now,

$u_{11} = 1; \qquad u_{12} = 1; \qquad u_{13} = 1$

$l_{21}.u_{11} = 4 \Rightarrow l_{21} = 4$

$l_{31}.u_{11} = 3 \Rightarrow l_{31} = 3$

$l_{21}.u_{12} + u_{22} = 3 \Rightarrow 4.1 + u_{22} = 3 \Rightarrow u_{22} = 3 - 4 = -1$

$l_{21}.u_{13} + u_{23} = -1 \Rightarrow 4.1 + u_{23} = -1 \Rightarrow u_{23} = -1 - 4 = -5$

$l_{31}.u_{12} + l_{32}.u_{22} = 5 \Rightarrow 3.1 + l_{32}.(-1) = 5 \Rightarrow -l_{32} = 5 - 3 = 2 \Rightarrow l_{32} = -2$

$l_{31}.u_{13} + l_{32}.u_{23} + u_{33} = 3 \Rightarrow 3.1 + (-2).(-5) + u_{33} = 3 \Rightarrow u_{33} = 3 - 3 - 10 \Rightarrow u_{33} = -10$

Now,

$A = LU$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix}$$

From (i), $LUX = B$

Let, $UX = Y$; where, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Therefore, $LY = B$; where, $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

Now, $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

$$\Rightarrow \begin{cases} y_1 = 1 \\ 4y_1 + y_2 = 6 \\ 3y_1 - 2y_2 + y_3 = 4 \end{cases}$$

$\therefore y_1 = 1, y_2 = 2, y_3 = 5$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\text{Now}, UX = Y$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x + y + z = 1 \\ -y - 5z = 2 \\ \quad -10z = 5 \end{cases}$$

$$\therefore z = \frac{5}{-10} = -\frac{1}{2}$$

$$-y - 5\left(-\frac{1}{2}\right) = 2 \Rightarrow -y = 2 - \frac{5}{2} \Rightarrow y = \frac{1}{2}$$

$$x + y + z = 1 \Rightarrow x = 1 - \frac{1}{2} + \frac{1}{2} = 1$$

Therefore, the required solution,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix}$$