Ex. 1) Evaluate the Detereminant: $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} & \mathbf{2} \\ \mathbf{3} & \mathbf{0} & \mathbf{1} & \mathbf{5} \\ \mathbf{1} & \mathbf{-2} & \mathbf{0} & \mathbf{3} \\ \mathbf{-2} & \mathbf{-4} & \mathbf{1} & \mathbf{6} \end{vmatrix}$

Solution: Let, D be the determinant. Then,

$$D = \begin{vmatrix} 1 & 2 & -1 & 2 \\ 3 & 0 & 1 & 5 \\ 1 & -2 & 0 & 3 \\ -2 & -4 & 1 & 6 \end{vmatrix}$$

$$= 1\begin{vmatrix} 0 & 1 & 5 \\ -2 & 0 & 3 \\ -4 & 1 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 & 5 \\ 1 & 0 & 3 \\ -2 & 1 & 6 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 0 & 5 \\ 1 & -2 & 3 \\ -2 & -4 & 6 \end{vmatrix} - 2\begin{vmatrix} 3 & 0 & 1 \\ 1 & -2 & 0 \\ -2 & -4 & 1 \end{vmatrix}$$

$$= (0 - 0 - 10) - 2(-9 - 12 + 5) - (0 - 0 - 40) - 2(-6 - 0 - 8)$$

$$= -10 + 32 + 40 + 28 = 90$$

Ex. 2) Solve the following linear equations with the help of matrices: $\begin{cases} \mathbf{2x + y = 1} \\ \mathbf{x - 2y = 3} \end{cases}$

Solution: The system of linear equations can be written in matrix form as,

$$\begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Let, $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, L = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, then from (i)we get,

$$AX = L$$

$$\Rightarrow X = A^{-1}L \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (ii)$$

Let, D be the determinant of the matrix A. Then,

$D = \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = -4 - 1 = -5 \neq 0; A^{-1}$ exists.

Co-factors are,

$$A_{11} = -2, A_{12} = -1, A_{21} = -1, A_{22} = 2$$

$\therefore \text{Adj } A = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{D} \text{ Adj } A = \frac{1}{-5} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$

From (ii),

$X = A^{-1}L$

$\Rightarrow X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore x = 1, y = -1$

Ex. 3) Solve the following linear equations with the help of matrices: $\begin{cases} \mathbf{3x + 5y - 7z = 13} \\ \mathbf{4x + y - 12z = 6} \\ \mathbf{2x + 9y - 3z = 20} \end{cases}$

Solution: The system of linear equations can be written in matrix form as,

$\begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$

Let, $A = \begin{bmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, L = \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix}$, then from (i) we get,

$AX = L$

$\Rightarrow X = A^{-1}L \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}$

Let, D be the determinant of the matrix A. Then,

$D = \begin{vmatrix} 3 & 5 & -7 \\ 4 & 1 & -12 \\ 2 & 9 & -3 \end{vmatrix} = 17 \neq 0; A^{-1}$ exists.

Co-factors are,

$A_{11} = 105$

$A_{12} = -12$

$A_{13} = 34$

$A_{21} = -48$

$A_{22} = 5$

$A_{23} = -17$

$A_{31} = -53$

$A_{32} = 8$

$A_{33} = -17$

$$\therefore \text{Adj A} = \begin{bmatrix} 105 & -12 & 34 \\ -48 & 5 & -17 \\ -53 & 8 & -17 \end{bmatrix}^T = \begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{Adj A} = \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix}$$

From (ii),

$X = A^{-1}L$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 105 & -48 & -53 \\ -12 & 5 & 8 \\ 34 & -17 & -17 \end{bmatrix} \begin{bmatrix} 13 \\ 6 \\ 20 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 1365 - 288 - 1060 \\ -156 + 30 + 160 \\ 442 - 102 - 340 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} 17 \\ 34 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore x = 1, y = 2, z = 0$

$\mathbf{Ex. 4)}$ **Solve the following linear equations by row canonical form**: $\begin{cases} \mathbf{3x + 5y - 7z = 13} \\ \mathbf{4x + y - 12z = 6} \\ \mathbf{2x + 9y - 3z = 20} \end{cases}$

Solution: The system of linear equations can be written as,

$$[AL] = \left[\begin{array}{ccc|c} 3 & 5 & -7 & 13 \\ 4 & 1 & -12 & 6 \\ 2 & 9 & -3 & 20 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 3 & 5 & -7 & 13 \\ 0 & -17 & -6 & -34 \\ 2 & 9 & -3 & 20 \end{array}\right] \qquad R_2' \rightarrow R_2 - 2R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 2 & 9 & -3 & 20 \end{array}\right] \qquad R_1' \rightarrow R_1 - R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 17 & 5 & 34 \end{array}\right] \qquad R_3' \rightarrow R_3 - 2R_1$$

$$= \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 0 & -1 & 0 \end{array}\right] \qquad R_3' \rightarrow R_3 + R_2$$

$$= \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & -17 & -6 & -34 \\ 0 & 0 & -1 & 0 \end{array}\right] \qquad R_2' \rightarrow \frac{R_2}{-17}, R_3' \rightarrow -R_3$$

$$= \left[\begin{array}{ccc|c} 1 & -4 & -4 & -7 \\ 0 & 1 & \frac{6}{17} & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore,

$$\begin{cases} x - 4y - 4z = -7 \\ \qquad y + \dfrac{6}{17}z = 2 \\ \qquad\qquad z = 0 \end{cases}$$

Hence,

$z = 0$

$y + 0 = 2 \Rightarrow y = 2$

$x - 8 - 0 = -7 \Rightarrow x = 1$

$\therefore x = 1, y = 2, z = 0$

H.W:

1) Solve the following linear equations with the help of matrices

(i) $\begin{cases} 5x - 6y + 4z = 15 \\ 7x + 4y - 3z = 19 \\ 2x + y + 6z = 46 \end{cases}$ $(x = 3, y = 4, z = 6)$

(ii) $\begin{cases} 2x - 3y + 4z = 1 \\ 3x + 4y - 5z = 10 \\ 5x - 7y + 2z = 3 \end{cases}$ $(x = 2, y = 1, z = 0)$

2) Solve the following linear equations by row canonical form:

(i) $\begin{cases} 2x - y + z = 1 \\ x + 4y - 3z = -2 \\ 3x + 2y - z = 0 \end{cases}$ $(x = 0, y = 1, z = 2)$