Vector Spaces

Basis of a vector space: Let, V be a vector space over the field F and

$v_1, v_2, \dots \dots \dots, v_n \in V$. Then, $\{v_1, v_2, \dots \dots \dots, v_n\}$ is a basis for V if and only if,

(i) $\{v_1, v_2, \dots \dots \dots, v_n\}$ is linearly independent.

(ii) $\{v_1, v_2, \dots \dots \dots, v_n\}$ spans V.

Dimension of a vector space: The dimension of a vector space is equal to the maximum number of linearly independent vectors contained in it.

Ex. 1) Prove that the vectors, $(1, 2, 0)$, $(0, 5, 7)$ and $(-1, 1, 3)$ form a basis of $\mathbb{R}^3$.

Solution: The given vectors will be a basis of $\mathbb{R}^3$ if and only if they are linearly independent and every vector in $\mathbb{R}^3$ can be written as a linear combination of $(1,2,0)$, $(0,5,7)$ and $(-1,1,3)$.

First we shall prove that the vectors are linearly independent.

Form the matrix whose rows are the given vectors we get,

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 5 & 7 \\ -1 & 1 & 3 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 5 & 3 \\ 0 & 2 & 0 \end{bmatrix}$$

This system is in echelon form and has exactly three equations in three unknowns.

So, it has only the zero solution i.e. $x = 0, y = 0, z = 0$.

Hence, the given vectors are linearly independent.

To show that the vectors spans $\mathbb{R}^3$

We must show that an arbitrary vector $v = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as a linear combination $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ of the vectors v_1, v_2 and v_3.

Expressing this equation in terms of components gives,

$$(\mathrm{a}, \mathrm{b}, \mathrm{c}) = \alpha_1(1,2,0) + \alpha_2(0,5,7) + \alpha_3(-1,1,3)$$

$$\Rightarrow (\mathrm{a}, \mathrm{b}, \mathrm{c}) = (\alpha_1, 2\alpha_1, 0) + (0, 5\alpha_2, 7\alpha_2) + (-\alpha_3, \alpha_3, 3\alpha_3)$$

$$\Rightarrow (\mathrm{a}, \mathrm{b}, \mathrm{c}) = (\alpha_1 - \alpha_3, 2\alpha_1 + 5\alpha_2 + \alpha_3, 7\alpha_2 + 3\alpha_3)$$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 - \alpha_3 = \mathrm{a} \\ 2\alpha_1 + 5\alpha_2 + \alpha_3 = \mathrm{b} \\ 7\alpha_2 + 3\alpha_3 = \mathrm{c} \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{cases} \alpha_1 - \alpha_3 = \mathrm{a} \\ 5\alpha_2 + 3\alpha_3 = \mathrm{b} - 2\mathrm{a} \\ 2\alpha_2 = \mathrm{c} - \mathrm{b} + 2\mathrm{a} \end{cases}$$

$$\therefore \alpha_2 = \frac{1}{2}(\mathrm{c} - \mathrm{b} + 2\mathrm{a}), \alpha_3 = \frac{1}{6}(-14\mathrm{a} + 7\mathrm{b} - 5\mathrm{c}), \alpha_1 = \frac{1}{6}(-8\mathrm{a} + 7\mathrm{b} - 5\mathrm{c}).$$

Therefore, $(\mathrm{a}, \mathrm{b}, \mathrm{c}) = \frac{1}{6}(-8\mathrm{a} + 7\mathrm{b} - 5\mathrm{c})\ \mathrm{v}_1 + \frac{1}{2}(\mathrm{c} - \mathrm{b} + 2\mathrm{a})\ \mathrm{v}_2 + \frac{1}{6}(-14\mathrm{a} + 7\mathrm{b} - 5\mathrm{c})\ \mathrm{v}_3.$

Therefore, $\mathrm{v}_1, \mathrm{v}_2$ and v_3 generate $\mathbb{R}^{\mathbf{3}}$.

Hence, the vectors (1,2,0), (0,5,7) and (– 1,1,3) form a basis of $\mathbb{R}^{\mathbf{3}}$.

Ex. 2) Find the solution space W of the following homogeneous system of linear equations:

$$\begin{cases} \mathbf{x + 2y - z + 4t = 0} \\ \mathbf{2x - y + 3z + 3t = 0} \\ \mathbf{4x + y + 3z + 9t = 0} \\ \mathbf{y - z + t = 0} \\ \mathbf{2x + 3y - z + 7t = 0} \end{cases}$$

Solution: Forming a co − efficient matrix we get,

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & -1 & 3 & 3 \\ 4 & 1 & 3 & 9 \\ 0 & 1 & -1 & 1 \\ 2 & 3 & -1 & 7 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This system is in echelon form having two equations in 4 unknowns.

So, the system has $4 - 2 = 2$ free variables which are z and t.

Let, $z = a$ and $t = b$. Then we have, $y = a - b$ and $x = -a - 2b$.

Hence, the required solution space is, $W = \{a, b \in \mathbb{R} : (-a - 2b, a - b, a, b)\}$

Ex. 3) Let, U be the subspace of $\mathbb{R}^3$ spanned by the vectors $(1, 2, 1), (0, -1, 0)$ and $(2, 0, 2)$.

Find a basis and dimension of U.

Solution: Form the matrix whose rows are the given vectors we get,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, the matrix is in row echelon form and the non – zero rows in the matrix are

$(1,2,1)$ and $(0,-1,0)$. These non – zero rows form a basis and Basis of $U = \{(1,2,1), (0,-1,0)\}$

and dimension of U, $\dim U = 2$.

Ex. 4) Let, W be the subspace of $\mathbb{R}^3$ spanned by the polynomials

$\mathbf{p_1(t) = t^3 + 2t^2 - 2t + 1, p_2(t) = t^3 + 3t^2 - t + 4, p_3(t) = 2t^3 + t^2 - 7t - 7}$

Find a basis and dimension of W.

Solution: Clearly, W is a subspace of the vector space of the polynomials in t of degree ≤ 3.

Now forming a matrix with the co – efficients of the given vectors $p_1(t), p_2(t)$ and $p_3(t)$ we get,

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & 3 & -1 & 4 \\ 2 & 1 & -7 & -7 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 2 & -2 & 1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the matrix is in row echelon form and the non $-$ zero rows in the matrix are

$(1,2,-2,1)$ and $(0,1,1,3)$.

These non $-$ zero rows form a basis of the vector space generated by the co $-$ ordinate vectors

and so the set of corresponding polynomials is $\{(t^3 + 2t^2 - 2t + 1), (t^2 + t + 3)\}$.

and dimension of W, $\dim W = 2$.