Vector Spaces

Definition of Vector Space: A vector space over an arbitrary field F is a non − empty set V, whose elements are called vectors for which two operations are prescribed

The first operation called vector addition, assigns to each pair of vectors u and v in V, denoted by $u + v$ called their sum.

The second operation, called scalar multiplication assigns to each vector in V and each scalar α in F, a vector denoted by αv.

Linear Combination of vectors: Let V be a vector space over the field F and let $v_1, v_2, \dots\dots\dots, v_n \in V$. Then any vector $v \in V$ is called a linear combination of $v_1, v_2, \dots\dots\dots, v_n$ if and only if there exists scalar $\alpha_1, \alpha_2, \dots\dots\dots, \alpha_n$ in F such that,

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \dots \dots + \alpha_n v_n$

Ex. 1) Consider the vectors $\mathbf{v_1 = (2, 1, 4), v_2 = (1, -1, 3)}$ and $\mathbf{v_3 = (3, 2, 5)}$in $\mathbb{R}^3$

Show that, $\mathbf{v = (5, 9, 5)}$is a linear combination of $\mathbf{v_1, v_2}$ and $\mathbf{v_3}$.

Solution: To show that, v is a linear combination of v_1, v_2 and v_3, there must be scalars $\alpha_1, \alpha_2, \alpha_3$ in F such that, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 \dots\dots\dots\dots\dots\dots$ (i)

Now,

$(5,9,5) = \alpha_1(2,1,4) + \alpha_2(1,-1,3) + \alpha_3(3,2,5)$

$\Rightarrow (5,9,5) = (2\alpha_1, \alpha_1, 4\alpha_1) + (\alpha_2, -\alpha_2, 3\alpha_2) + (3\alpha_3, 2\alpha_3, 5\alpha_3)$

$\Rightarrow (5,9,5) = (2\alpha_1 + \alpha_2 + 3\alpha_3, \alpha_1 - \alpha_2 + 2\alpha_3, 4\alpha_1 + 3\alpha_2 + 5\alpha_3)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} 2\alpha_1 + \alpha_2 + 3\alpha_3 = 5 \\ \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 4\alpha_1 + 3\alpha_2 + 5\alpha_3 = 5 \end{cases} \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(ii)}$$

Reducing the system to echelon form we get,

$$\begin{cases} \alpha_1 - \alpha_2 + 2\alpha_3 = 9 \\ 3\alpha_2 - \alpha_3 = -13 \\ -\dfrac{2}{3}\alpha_3 = -\dfrac{2}{3} \end{cases}$$

$\therefore \alpha_3 = 1, \alpha_2 = -4, \alpha_1 = 3.$

Hence, from (i), $v = 3v_1 - 4v_2 + v_3$.

Therefore, v is a linear combination of v_1, v_2 and v_3.

Ex. 2) Is the vector $\mathbf{v} = (2, -5, 3)$ in $\mathbb{R}^3$ is a linear combination of the vectors $\mathbf{v_1} = (1, -3, 2), \mathbf{v_2} = (2, -4, -1)$ and $\mathbf{v_3} = (1, -5, 7)$

Solution: Let, $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ where $\alpha_1, \alpha_2, \alpha_3$ are unknown scalars.

Now,

$(2, -5,3) = \alpha_1(1, -3,2) + \alpha_2(2, -4, -1) + \alpha_3(1, -5,7)$

$\Rightarrow (2, -5,3) = (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -4\alpha_2, -\alpha_2) + (\alpha_3, -5\alpha_3, 7\alpha_3)$

$\Rightarrow (2, -5,3) = (\alpha_1 + 2\alpha_2 + \alpha_3, -3\alpha_1 - 4\alpha_2 - 5\alpha_3, 2\alpha_1 - \alpha_2 + 7\alpha_3)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 2 \\ -3\alpha_1 - 4\alpha_2 - 5\alpha_3 = -5 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)} \\ 2\alpha_1 - \alpha_2 + 7\alpha_3 = 3 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{cases} \alpha_1 + 2\alpha_2 + \alpha_3 = 2 \\ 2\alpha_2 - 2\alpha_3 = 1 \\ 0 = \dfrac{3}{2} \end{cases}$$

The system has an equation of the form $0 = \dfrac{3}{2}$, which is not true.

Hence the above system is inconsistent and it has no solution.

Therefore, the vector v is not a linear combination of v_1, v_2 and v_3.

Ex. 3) For which value of $\boldsymbol{\lambda}$ will be the vector $\mathbf{v} = (\mathbf{1}, \boldsymbol{\lambda}, \mathbf{5})$ is a linear combination of the vectors $\mathbf{v_1} = (\mathbf{1}, -\mathbf{3}, \mathbf{2})$ and $\mathbf{v_2} = (\mathbf{2}, -\mathbf{1}, \mathbf{1})$.

Solution: Let, $v = \alpha_1 v_1 + \alpha_2 v_2$ where α_1 and α_2 are unknown scalars.

Now,

$(1, \lambda, 5) = \alpha_1(1, -3, 2) + \alpha_2(2, -1, 1)$

$\Rightarrow (1, \lambda, 5) = (\alpha_1, -3\alpha_1, 2\alpha_1) + (2\alpha_2, -\alpha_2, \alpha_2)$

$\Rightarrow (1, \lambda, 5) = (\alpha_1 + 2\alpha_2, -3\alpha_1 - \alpha_2, 2\alpha_1 + \alpha_2)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 + 2\alpha_2 = 1 \\ -3\alpha_1 - \alpha_2 = \lambda \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)} \\ 2\alpha_1 + \alpha_2 = 5 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{cases} \alpha_1 + 2\alpha_2 = 1 \\ 5\alpha_2 = \lambda + 3 \\ \alpha_2 = -1 \end{cases}$$

The system has solution for $\lambda = -8$ and the solution is $\alpha_1 = 3$ and $\alpha_2 = -1$.

Therefore, the vector v is a linear combination of v_1 and v_2 if $\lambda = -8$.

Ex. 4) Write the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{3} & -\mathbf{1} \\ \mathbf{1} & -\mathbf{2} \end{bmatrix}$ as a linear combination of the matrices $\mathbf{A_1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & -\mathbf{1} \end{bmatrix}, \mathbf{A_2} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}$ and $\mathbf{A_3} = \begin{bmatrix} \mathbf{1} & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

Solution: Let, $A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$ is a linear combination of A_1, A_2 and A_3 where $\alpha_1, \alpha_2, \alpha_3$ are unknown scalars.

Now,

$$\begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ 0 & -\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & \alpha_2 \\ -\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} \alpha_3 & -\alpha_3 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & -1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 - \alpha_3 \\ 0 - \alpha_2 + 0 & -\alpha_1 + 0 + 0 \end{bmatrix}$$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 3 \\ \alpha_1 + \alpha_2 - \alpha_3 = -1 \\ -\alpha_2 = 1 \\ -\alpha_1 = -2 \end{cases} \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}$$

The solution of the system is $\alpha_1 = 2, \alpha_2 = -1, \alpha_3 = 2$

Therefore, $A = 2A_1 - A_2 + 2A_3$. Thus, A is a linear combination of A_1, A_2 and A_3.

Ex. 5) Write the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{1} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}$ as a linear combination of the matrices

$\mathbf{A_1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}, \mathbf{A_2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$ **and** $\mathbf{A_3} = \begin{bmatrix} \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{-1} \end{bmatrix}$

Solution: Let, $A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$ is a linear combination of A_1, A_2 and A_3

where $\alpha_1, \alpha_2, \alpha_3$ are unknown scalars.

Now,

$$\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \alpha_2 & \alpha_2 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_3 \\ 0 & -\alpha_3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 + 2\alpha_3 \\ \alpha_1 + \alpha_2 & \alpha_2 - \alpha_3 \end{bmatrix}$$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 = 3 \\ \alpha_1 + 2\alpha_3 = 1 \\ \alpha_1 + \alpha_2 = 1 \\ \alpha_2 - \alpha_3 = -1 \end{cases} \quad \dots\dots\dots\dots\dots\dots\dots\dots \text{(i)}$$

The solution of the system is $\alpha_1 = 3, \alpha_2 = -2, \alpha_3 = -1$

Therefore, $A = 3A_1 - 2A_2 - A_3$. Thus, A is a linear combination of A_1, A_2 and A_3.