Vector Spaces

Theorem: $\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$

Ex. 1) Let, U and W be the subspaces of $\mathbb{R}^4$ generated by the set of vectors:

$\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$ and $\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$ respectively.

Find (i) $\dim(U+W)$ (ii) $\dim(U \cap W)$

Solution: **(i)** $U + W$ is a subspace spanned by all given six vectors

Hence, forming the matrix whose rows are the given six vectors, we get,

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, the matrix is in row echelon form having three non $-$ zero rows $(1,1,0,-1)$, $(0,1,3,1)$

and $(0,0,-1,-2)$ which will form a basis of $U+W$.

Thus, $\dim(U+W) = 3$

(ii) Let us first find the $\dim U$ and $\dim W$. Forming the matrix whose rows are the generators of U,

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form having two $non-zero$ rows $(1,1,0,-1)$ and $(0,1,3,1)$ which will form a basis of U.

Thus, $\dim U = 2$

Again forming the matrix whose rows are the generators of W,

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 2 & 2 & -2 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form having two $non-zero$ rows $(1,2,2,-2)$ and $(0,-1,-2,1)$ which will form a basis of W.

Thus, $\dim W = 2$

Now, by theorem we have,

$\dim (U + W) = \dim U + \dim W - \dim (U \cap W)$

$\Rightarrow \dim(U \cap W) = \dim U + \dim W - \dim(U + W) = 2 + 2 - 3 = 1$

Ex. 2) Let, V be the vector space of 2×2 matrices over the real field $\mathbb{R}$. Find a Basis and Dimension of the subspace W of V spanned by,

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{-1} & \mathbf{3} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{1} & \mathbf{-1} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{5} & \mathbf{12} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{-2} & \mathbf{5} \end{bmatrix}$$

Solution: The $co-ordinate$ vectors of the given matrices relative to the usual basis of V are as follows:

$[A] = (1,2,-1,3), [B] = (2,5,1,-1), [C] = (5,12,1,1), [D] = (3,4,-2,5)$

Forming a matrix whose rows are the $co-ordinate$ vectors, we get

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & -1 \\ 5 & 12 & 1 & 1 \\ 3 & 4 & -2 & 5 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & -7 \\ 0 & 0 & 7 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in row echelon form having three non $-$ zero rows $(1,2,-1,3), (0,1,3,-7)$ and $(0,0,7,-18)$ which are linearly independent.

Hence, the corresponding matrices $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 3 & -7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 7 & -18 \end{bmatrix}$ form a basis of W.

Therefore, $\dim \mathrm{W} = 3$

Ex. 3) Prove that the vectors, $\mathbf{u_1} = (\mathbf{1}, \mathbf{0}, \mathbf{2}), \mathbf{u_2} = (-\mathbf{1}, \mathbf{1}, \mathbf{0})$ and $\mathbf{u_3} = (\mathbf{0}, \mathbf{2}, \mathbf{3})$ form a basis of $\mathbb{R}^3$.

Find the co $-$ ordinates of the vectors $\mathbf{v} = (\mathbf{1}, -\mathbf{1}, \mathbf{1})$ and $\mathbf{w} = (-\mathbf{1}, \mathbf{8}, \mathbf{11})$ relative to the basis.

Solution:

First Portion: The given vectors will be a basis of $\mathbb{R}^3$ if and only if they are linearly independent and every vector in $\mathbb{R}^3$ can be written as a linear combination of $\mathrm{u}_1 = (1,0,2), \mathrm{u}_2 = (-1,1,0)$ and $\mathrm{u}_3 = (0,2,3)$.

First we shall prove that the vectors are linearly independent.

Form the matrix whose rows are the given vectors we get,

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 2 & 3 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

This system is in echelon form and has exactly three equations in three unknowns.

So, it has only the zero solution i. e. $\mathrm{x} = 0, \mathrm{y} = 0, \mathrm{z} = 0$.

Hence, the given vectors are linearly independent.

To show that the vectors spans $\mathbb{R}^3$

We must show that an arbitrary vector $v = (a, b, c)$ in $\mathbb{R}^3$ can be expressed

as a linear combination $v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$ of the vectors u_1, u_2 and u_3.

Expressing this equation in terms of components gives,

$(a, b, c) = \alpha_1(1,0,2) + \alpha_2(-1,1,0) + \alpha_3(0,2,3)$

$\Rightarrow (a, b, c) = (\alpha_1, 0, 2\alpha_1) + (-\alpha_2, \alpha_2, 0) + (0, 2\alpha_3, 3\alpha_3)$

$\Rightarrow (a, b, c) = (\alpha_1 - \alpha_2, \alpha_2 + 2\alpha_3, 2\alpha_1 + 3\alpha_3)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 - \alpha_2 = a \\ \alpha_2 + 2\alpha_3 = b \\ 2\alpha_1 + 3\alpha_3 = c \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{cases} \alpha_1 - \alpha_2 = a \\ \alpha_2 + 2\alpha_3 = b \\ -\alpha_3 = c - 2b - 2a \end{cases}$$

$\therefore \alpha_3 = 2a + 2b - c, \alpha_2 = -4a - 3b + 2c, \alpha_1 = -3a - 3b + 2c.$

Therefore, $(a, b, c) = (-3a - 3b + 2c)\ u_1 + (-4a - 3b + 2c)\ u_2 + (2a + 2b - c)\ u_3$.

Therefore, u_1, u_2 and u_3 generate $\mathbb{R}^3$. Hence, the vectors u_1, u_2 and u_3 form a basis of $\mathbb{R}^3$.

Second Portion: Let,

$v = (1, -1, 1) = x_1 u_1 + x_2 u_2 + x_3 u_3$

$\Rightarrow (1, -1, 1) = x_1(1,0,2) + x_2(-1,1,0) + x_3(0,2,3)$

$\Rightarrow (1, -1, 1) = (x_1, 0, 2x_1) + (-x_2, x_2, 0) + (0, 2x_3, 3x_3)$

$\Rightarrow (1, -1, 1) = (x_1 - x_2, x_2 + 2x_3, 2x_1 + 3x_3)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} x_1 - x_2 = 1 \\ x_2 + 2x_3 = -1 \\ 2x_1 + 3x_3 = 1 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{cases} x_1 - x_2 = 1 \\ x_2 + 2x_3 = -1 \\ -x_3 = 1 \end{cases}$$

$\therefore x_3 = -1, x_2 = 1, x_1 = 2$

Therefore, $v = (1,-1,1) = 2u_1 + u_2 - u_3$. So, the vector v has co – ordinates $(2,1,-1)$.

Similarly Let,

$w = (-1,8,11) = y_1u_1 + y_2u_2 + y_3u_3$

$\Rightarrow (-1,8,11) = y_1(1,0,2) + y_2(-1,1,0) + y_3(0,2,3)$

$\Rightarrow (-1,8,11) = (y_1, 0, 2y_1) + (-y_2, y_2, 0) + (0, 2y_3, 3y_3)$

$\Rightarrow (-1,8,11) = (y_1 - y_2, y_2 + 2y_3, 2y_1 + 3y_3)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} y_1 - y_2 = -1 \\ y_2 + 2y_3 = 8 \\ 2y_1 + 3y_3 = 11 \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{cases} y_1 - y_2 = -1 \\ y_2 + 2y_3 = 8 \\ -y_3 = -3 \end{cases}$$

$\therefore y_3 = 3, y_2 = 2, y_1 = 1$

Therefore, $w = (-1,8,1) = u_1 + 2u_2 + 3u_3$. So, the vector w has co – ordinates $(1,2,3)$.

H.W:

1) Let, $V(R)$ be the vector space of all polynomials of degree ≤ 3. Determine whether the following polynomials are linearly dependent or independent.

(i) $p_1(t) = t^3 + 2t^2 + 4t - 1, p_2(t) = 2t^3 - t^2 - 3t + 5, p_3(t) = t^3 - 4t^2 + 2t + 3$

(ii) $p_1(t) = t^3 - 3t^2 - 2t + 3, p_2(t) = 2t^3 - 5t^2 - 5t + 7, p_3(t) = t^3 - 2t^2 - 3t + 3$

2) Prove that the vectors, $u_1 = \left(2, -\frac{1}{2}, 1\right), u_2 = (3,2,1)$ and $u_3 = (0,1,1)$ form a basis of $\mathbb{R}^3$.

3) Extend $\{(2,0,0,-1), (1,3,-1,0)\}$ to a basis of $\mathbb{R}^3$.

4) Let, U and V be the subspaces of $\mathbb{R}^4$ generated by the set of vectors:

$\{(1,0,2,3)$ and $(0,1,-1,2)\}$ and $\{(1,2,3,4),(-1,-1,5,0),(0,0,0,1)\}$ respectively.

Find (i) $\dim(\mathrm{U}+\mathrm{V})$ (ii) $\dim(\mathrm{U}\cap\mathrm{V})$

5) Let, W be the subspace of $\mathbb{R}^3$ generayed by the polynomials $v_1(t)=t^3-2t^2+4t+1$,

$v_2(t)=2t^3-3t^2+9t-1, v_3(t)=t^3+6t-5, v_4(t)=2t^3-5t^2++t+5$

Find a basis and dimension of W.

6) Let, S be the following basis of the vector space W of matrices: $\left\{\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix},\begin{bmatrix}4 & 1\\ 1 & 0\end{bmatrix},\begin{bmatrix}3 & -2\\ -2 & 1\end{bmatrix}\right\}$

Find the co $-$ ordinate vector of the matrix $A \in W$ relative to the above basis where

(a) $A=\begin{bmatrix}1 & -5\\ -5 & 5\end{bmatrix}$ and (b) $A=\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$