Vector Spaces

Linear span of a subset of a vector space: If S a non empty subset of a vector space V, then the linear span of S, is the set of all linear combinations of elements of S.

Ex. 1) Show that, the vectors $\mathbf{u} = (\mathbf{1}, \mathbf{2}, \mathbf{3}), \mathbf{v} = (\mathbf{0}, \mathbf{1}, \mathbf{2})$ and $\mathbf{w} = (\mathbf{0}, \mathbf{0}, \mathbf{1})$ span or generate $\mathbb{R}^3$

Solution: We must determine whether an arbitrary vector $v = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as a linear combination $v = \alpha_1 u + \alpha_2 v + \alpha_3 w$ of the vectors u, v and w.

Expressing this equation in terms of components gives,

$(a, b, c) = \alpha_1(1,2,3) + \alpha_2(0,1,2) + \alpha_3(0,0,1)$

$\Rightarrow (a, b, c) = (\alpha_1, 2\alpha_1, 3\alpha_1) + (0, \alpha_2, 2\alpha_2) + (0,0, \alpha_3)$

$\Rightarrow (a, b, c) = (\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 + \alpha_3)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 = a \\ 2\alpha_1 + \alpha_2 = b \\ 3\alpha_1 + 2\alpha_2 + \alpha_3 = c \end{cases}$$

$$\Rightarrow \begin{cases} 3\alpha_1 + 2\alpha_2 + \alpha_3 = c \\ 2\alpha_1 + \alpha_2 = b \\ \alpha_1 = a \end{cases}$$

The above system is in echelon form and is consistent. In fact, the system has the solution,

$\alpha_1 = a, \alpha_2 = b - 2a, \alpha_3 = c - 2b + a$

Therefore, u, v and w generate $\mathbb{R}^3$.

Ex. 2) Determine whether the vectors $\mathbf{v_1} = (\mathbf{2}, -\mathbf{1}, \mathbf{3}), \mathbf{v_2} = (\mathbf{4}, \mathbf{1}, \mathbf{2})$ and $\mathbf{v_3} = (\mathbf{8}, -\mathbf{1}, \mathbf{8})$ span or generate $\mathbb{R}^3$.

Solution: We must determine whether an arbitrary vector $v = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as a linear combination $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ of the vectors v_1, v_2 and v_3.

Expressing this equation in terms of components gives,

$(a, b, c) = \alpha_1(2, -1,3) + \alpha_2(4,1,2) + \alpha_3(8, -1,8)$

$\Rightarrow (a, b, c) = (2\alpha_1, -\alpha_1, 3\alpha_1) + (4\alpha_2, \alpha_2, 2\alpha_2) + (8\alpha_3, -\alpha_3, 8\alpha_3)$

$\Rightarrow (a, b, c) = (2\alpha_1 + 4\alpha_2 + 8\alpha_3, -\alpha_1 + \alpha_2 - \alpha_3, 3\alpha_1 + 2\alpha_2 + 8\alpha_3)$

Equating corresponding components and forming linear system we get,

$$\begin{cases} 2\alpha_1 + 4\alpha_2 + 8\alpha_3 = a \\ -\alpha_1 + \alpha_2 - \alpha_3 = b \\ 3\alpha_1 + 2\alpha_2 + 8\alpha_3 = c \end{cases}$$

Now, this system will be consistent for all a, b and c, if and only if the matrix of co − efficients,

$A = \begin{bmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{bmatrix}$ is invertible. [Invertible: $AA^{-1} = I$]

Now, the determinant of A is, $|A| = \begin{vmatrix} 2 & 4 & 8 \\ -1 & 1 & -1 \\ 3 & 2 & 8 \end{vmatrix} = 20 + 20 - 40 = 0$

Hence, the co − efficient matrix A is not invertible and thus v_1, v_2 and v_3 do not span $\mathbb{R}^3$.

Ex. 3) Determine whether the vectors $\mathbf{v_1} = \left(\mathbf{1}, \frac{\mathbf{1}}{\mathbf{2}}, \frac{\mathbf{1}}{\mathbf{4}}\right), \mathbf{v_2} = (-\mathbf{2}, -\mathbf{4}, -\mathbf{8})$ and $\mathbf{v_3} = (\mathbf{3}, \mathbf{9}, \mathbf{27})$ span or generate $\mathbb{R}^3$.

Solution: We must determine whether an arbitrary vector $v = (a, b, c)$ in $\mathbb{R}^3$ can be expressed as a linear combination $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ of the vectors v_1, v_2 and v_3.

Expressing this equation in terms of components gives,

$$(a, b, c) = \alpha_1 \left(1, \frac{1}{2}, \frac{1}{4}\right) + \alpha_2(-2, -4, -8) + \alpha_3(3, 9, 27)$$

$$\Rightarrow (a, b, c) = \left(\alpha_1, \frac{1}{2}\alpha_1, \frac{1}{4}\alpha_1\right) + (-2\alpha_2, -4\alpha_2, -8\alpha_2) + (3\alpha_3, 9\alpha_3, 27\alpha_3)$$

$$\Rightarrow (a, b, c) = \left(\alpha_1 - 2\alpha_2 + 3\alpha_3, \frac{1}{2}\alpha_1 - 4\alpha_2 + 9\alpha_3, \frac{1}{4}\alpha_1 - 8\alpha_2 + 27\alpha_3\right)$$

Equating corresponding components and forming linear system we get,

$$\begin{cases} \alpha_1 - 2\alpha_2 + 3\alpha_3 = a \\ \frac{1}{2}\alpha_1 - 4\alpha_2 + 9\alpha_3 = b \\ \frac{1}{4}\alpha_1 - 8\alpha_2 + 27\alpha_3 = c \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1 - 2\alpha_2 + 3\alpha_3 = a \\ \alpha_1 - 8\alpha_2 + 18\alpha_3 = 2b \\ \alpha_1 - 32\alpha_2 + 108\alpha_3 = 4c \end{cases}$$

Reducing the system to echelon form we get,

$$\begin{cases} \alpha_1 - 2\alpha_2 + 3\alpha_3 = a \\ -6\alpha_2 + 15\alpha_3 = 2b - a \\ 30\alpha_3 = 4a - 10b + 4c \end{cases}$$

$$\therefore \alpha_3 = \frac{1}{15}(2a - 5b + 2c), \alpha_2 = \frac{1}{6}(3a - 7b + 2c), \alpha_1 = \frac{1}{15}(2a - 5b + 2c).$$

Therefore, $(a, b, c) = \frac{1}{15}(2a - 5b + 2c)\, v_1 + \frac{1}{6}(3a - 7b + 2c)\, v_2 + \frac{1}{15}(2a - 5b + 2c)\, v_3$.

Therefore, v_1, v_2 and v_3 generate $\mathbb{R}^{\mathbf{3}}$.

Linear Dependence and Linear Independence: Let, V be a vector space over the field F.

The vectors $v_1, v_2, \dots\dots\dots, v_n \in V$ are said to be Linearly Dependent over F if there exists a

non − trivial linear combination of them equal to the zero vector 0.

For Dependence, there should be at least one zero row in the echelon form. Otherwise, if we find consistent system it will be linearly independent.

Ex. 1) Prove that the set of vectors, $\{(2, 1, 2), (0, 1, -1), (4, 3, 3)\}$ is linearly dependent.

Solution: Form the matrix whose rows are the given vectors we get,

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 4 & 3 & 3 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

This system is in echelon form and has a zero row. Hence, the given vectors are linearly dependent.

Ex. 2) Prove that the set of vectors, $\{(2, -1, 4), (3, 6, 2), (2, 10, -4)\}$ is linearly independent.

Solution: Form the matrix whose rows are the given vectors we get,

$$\begin{bmatrix} 2 & -1 & 4 \\ 3 & 6 & 2 \\ 2 & 10 & -4 \end{bmatrix}$$

Reducing the system to echelon form we get,

$$\begin{bmatrix} 1 & 5 & -2 \\ 0 & -9 & 8 \\ 0 & 0 & -\frac{16}{9} \end{bmatrix}$$

Since, the echelon matrix has no zero − row. Hence, the given vectors are linearly independent.

H.W:

1) Write $(5,6,0)$ as a linear combination of $(-1,2,0)$, $(3,1,2)$, $(4,-1,0)$ and $(0,1,-1)$.

2) Determine whether or not the vector $(3,9,-4,-2)$ is a linear combination of the vector set $\{(1,-2,0,3), (2,3,0,-1), (2,-1,2,1)\}$.

3) Determine whether or not the following vectors span $\mathbb{R}^3$

(i) $u_1 = (1,1,2), u_2 = (1,-1,2)$ and $u_3 = (1,0,1)$

(ii) $u_1 = (-1,1,0), u_2 = (-1,0,1)$ and $u_3 = (1,1,1)$

4) Show that the space generated by the vectors $u_1 = (1,2,-1,3), u_2 = (2,,4,1,-2)$, $u_3 = (3,6,3,-7)$ and the space generated by the vectors $v_1 = (1,2,-4,11)$ and $v_2 = (2,4,-5,14)$ is a linear combination of the vector set

5) Show that, $\mathbb{R}^3 = \{(1,2,1), (2,1,0), (1,-1,2)\}$.

6) Find a condition on a, b, c so that, $v = (a, b, c)$ is a linear combination of $v_1 = (1,-3,2)$ and $v_2 = (2,-1,-1)$, so that v belongs to span (v_1, v_2)

7) Test the dependency of the following sets:

(i) $\{(1,2,-3), (2,0,-1), (7,6,-11)\}$.

(ii) $\{(2,0,-1), (1,1,0), (0,-1,1)\}$

8) Let, V be the vector space of all 2×3 matrices oer the real field $\mathbb{R}$. Show that,

$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 & 4 \\ 4 & 5 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -8 & 7 \\ 2 & 10 & -1 \end{bmatrix}$ are linearly dependent.