Symmetric Matrix: A matrix which is equal to its transpose $(A = A^T)$ is called symmetric matrix.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 5 \\ 3 & 5 & 7 \end{bmatrix}$

Skew- symmetric Matrix: A matrix which is equal to the negative of its transpose $(A = -A^T)$ is called skew- symmetric matrix.

Ex: $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

Ex. 1) Find the symmetric and skew symmetric parts of the matrix, $\mathbf{A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}}$

Solution: Given,

$A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ $\therefore A^T = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$

The symmetric part of A,

$$A = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right) = \begin{bmatrix} 1 & 4 & \frac{7}{2} \\ 4 & 8 & 3 \\ \frac{7}{2} & 3 & 7 \end{bmatrix}$$

The skew symmetric part of A,

$$A = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}\right) = \begin{bmatrix} 0 & -2 & \frac{1}{2} \\ 2 & 0 & -2 \\ -\frac{1}{2} & 2 & 0 \end{bmatrix}$$

Ex. 2) If $\mathbf{A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix}}$ **and** $\mathbf{B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix}}$**, then prove that,** $\mathbf{\overline{A+B} = \overline{A} + \overline{B}}$

Proof: Given,

$$A = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix} \qquad \therefore \overline{A} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix}$$

$$B = \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix} \qquad \therefore \overline{B} = \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix}$$

$$\overline{A} + \overline{B} = \begin{bmatrix} 1 & 1-i \\ 2+3i & -i \end{bmatrix} + \begin{bmatrix} 2+i & -i \\ 1-5i & 3 \end{bmatrix} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix}$$

$$\text{Now, } A + B = \begin{bmatrix} 1 & 1+i \\ 2-3i & i \end{bmatrix} + \begin{bmatrix} 2-i & i \\ 1+5i & 3 \end{bmatrix} = \begin{bmatrix} 3-i & 1+2i \\ 3+2i & 3+i \end{bmatrix}$$

$$\therefore \overline{A+B} = \begin{bmatrix} 3+i & 1-2i \\ 3-2i & 3-i \end{bmatrix}$$

Involutory Matrix: A matrix is called involutory matrix if $A^2 = I$

Ex. 3) Show that the matrix, $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involutory.

Proof: Given,

$$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} . \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ (proved)}$$

Ex. 4) If $A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$, Find the adjoint matrix of A (Adj A)

Solution: Given,

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$$

Co-factors of A are,

$$A_{11} = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5$$

$$A_{12} = -\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10$$

$$A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8$$

$$A_{21} = -\begin{vmatrix} 2 & -3 \\ -2 & 5 \end{vmatrix} = -4$$

$$A_{22} = \begin{vmatrix} -1 & -3 \\ 4 & 5 \end{vmatrix} = 7$$

$$A_{23} = -\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix} = 6$$

$$A_{31} = \begin{vmatrix} 2 & -3 \\ 1 & 0 \end{vmatrix} = 3$$

$$A_{32} = -\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6$$

$$A_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$$

$$\therefore \text{Adj } A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

Ex. 5) If $\mathbf{A} = \begin{bmatrix} \mathbf{-1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} & \mathbf{0} \\ \mathbf{4} & \mathbf{-2} & \mathbf{5} \end{bmatrix}$ **, Find the inverse matrix of A.**

Solution: Given,

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{bmatrix}$$

Let, $D = |A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 1 & 0 \\ 4 & -2 & 5 \end{vmatrix} = -1 \neq 0.$ So, A^{-1} exists.

Co-factors of A are,

$$A_{11} = \begin{vmatrix} 1 & 0 \\ -2 & 5 \end{vmatrix} = 5$$

$$A_{12} = -\begin{vmatrix} 2 & 0 \\ 4 & 5 \end{vmatrix} = -10$$

$$A_{13} = \begin{vmatrix} 2 & 1 \\ 4 & -2 \end{vmatrix} = -8$$

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$$A_{32} = -\begin{vmatrix} -1 & -3 \\ 2 & 0 \end{vmatrix} = -6$$

$$A_{33} = \begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} = -5$$

$$\therefore \text{Adj } A = \begin{bmatrix} 5 & -10 & -8 \\ -4 & 7 & 6 \\ 3 & -6 & -5 \end{bmatrix}^T = \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{D} \text{ Adj } A = \frac{1}{-1} \begin{bmatrix} 5 & -4 & 3 \\ -10 & 7 & -6 \\ -8 & 6 & -5 \end{bmatrix} = \begin{bmatrix} -5 & 4 & -3 \\ 10 & -7 & 6 \\ 8 & -6 & 5 \end{bmatrix}$$