Vector Calculus

1) What is nabla?

Ans. Del or **nabla**, is an operator used in mathematics, in particular in vector calculus, as a vector differential operator, usually represented by the **nabla** symbol ∇.

$$\left[\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right]$$

2) Gradient: $\text{grad}\,\emptyset = \nabla\emptyset = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\emptyset = \left(\frac{\partial\emptyset}{\partial x}\mathbf{i} + \frac{\partial\emptyset}{\partial y}\mathbf{j} + \frac{\partial\emptyset}{\partial z}\mathbf{k}\right)$

3) Divergence: Suppose $V(x, y, z) = V_1(x, y, z)\mathbf{i} + V_2(x, y, z)\mathbf{j} + V_3(x, y, z)\mathbf{k}$

$$\mathbf{div\,V} = \nabla.\mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right).(V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

4) Curl: Suppose, $V(x, y, z) = V_1(x, y, z)\mathbf{i} + V_2(x, y, z)\mathbf{j} + V_3(x, y, z)\mathbf{k}$

$$\text{curl}\,\mathbf{V} = \nabla \times \mathbf{V} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times \left((V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k})\right)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \mathbf{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \mathbf{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \mathbf{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

Ex: 1) If $\emptyset = 3xyz - x^2y^2z^2$ then find $\nabla\emptyset$ at the point $(1, 1, 1)$.

Solution: Given, $\emptyset = 3xyz - x^2y^2z^2$

$$\therefore \nabla\emptyset = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(3xyz - x^2y^2z^2)$$

$$= \mathbf{i}\,\frac{\partial}{\partial x}(3xyz - x^2y^2z^2) + \mathbf{j}\frac{\partial}{\partial y}(3xyz - x^2y^2z^2) + \mathbf{k}\,\frac{\partial}{\partial z}(3xyz - x^2y^2z^2)$$

$$= \mathbf{i}(3yz - 2xy^2z^2) + \mathbf{j}(3xz - 2y\,x^2z^2) + \mathbf{k}(3xy - 2z\,x^2y^2)$$

At the point, $(1,1,1)$, $\nabla\emptyset = \mathbf{i} + \mathbf{j} + \mathbf{k}$

Ex. 2) If $\mathbf{A} = [2x^2, -3yz, xz^2]$ and $\emptyset = 2z - x^3y$ then prove at $(1, -1, 1)$

(i) $\nabla . \mathbf{A} = 3$ **(ii) $\nabla \times \mathbf{A} = [-3, -1, 0]$**

(iii) $\mathbf{A} . \nabla\emptyset = 5$ **(iv) $\mathbf{A} \times \nabla\emptyset = [7, -1, -11]$**

Solution: (**i**) Given, $A = [2x^2, -3yz, xz^2]$

We know, $\nabla . A = \dfrac{\partial A_1}{\partial x} + \dfrac{\partial A_2}{\partial y} + \dfrac{\partial A_3}{\partial z}$

$= \dfrac{\partial}{\partial x}(2x^2) + \dfrac{\partial}{\partial y}(-3yz) + \dfrac{\partial}{\partial z}(xz^2) = 4x - 3z + 2xz$

At the point $(1, -1, 1)$, $\nabla . A = 4 - 3 + 2 = 6 - 3 = 3$ (proved)

(**ii**) Given, $A = [2x^2, -3yz, xz^2]$

We know,

$$\nabla \times A = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 & -3yz & xz^2 \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial}{\partial y}(xz^2) - \frac{\partial}{\partial z}(-3yz)\right\} - \mathbf{j}\left\{\frac{\partial}{\partial x}(xz^2) - \frac{\partial}{\partial z}(2x^2)\right\} + \mathbf{k}\left\{\frac{\partial}{\partial x}(-3yz) - \frac{\partial}{\partial y}(2x^2)\right\}$$

$= \mathbf{i}(0 + 3y) - \mathbf{j}(z^2 - 0) + \mathbf{k}(0 - 0) = 3y\mathbf{i} - z^2\mathbf{j}$

At $(1, -1, 1)$, $\nabla \times A = [-3, -1, 0]$ (proved)

(**iii**)Given, $A = [2x^2, -3yz, xz^2]$ and $\emptyset = 2z - x^3y$

We know, $\nabla\emptyset = \mathbf{i}\dfrac{\partial}{\partial x}(2z - x^3y) + \mathbf{j}\dfrac{\partial}{\partial y}(2z - x^3y) + \mathbf{k}\dfrac{\partial}{\partial z}(2z - x^3y) = -3x^2y\mathbf{i} - x^3\mathbf{j} + 2\mathbf{k}$

$A . \nabla\emptyset = [2x^2, -3yz, xz^2] . (-3x^2y\mathbf{i} - x^3\mathbf{j} + 2\mathbf{k}) = -6x^4y + 3x^3yz + 2xz^2 = 6 - 3 + 2 = 5$

(**iv**) $\nabla\emptyset = \mathbf{i}\dfrac{\partial}{\partial x}(2z - x^3y) + \mathbf{j}\dfrac{\partial}{\partial y}(2z - x^3y) + \mathbf{k}\dfrac{\partial}{\partial z}(2z - x^3y) = -3x^2y\mathbf{i} - x^3\mathbf{j} + 2\mathbf{k}$

$A \times \nabla\emptyset = [2x^2, -3yz, xz^2] \times [-3x^2y, -x^3, 2]$

$$\Rightarrow \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & 2 \end{vmatrix} = \mathbf{i}(-6yz + x^4z^2) - \mathbf{j}(4x^2 + 3x^3yz^2) + \mathbf{k}(-2x^5 - 9x^2y^2z^2)$$

$$= 7i - j - 11k = [7, -1, - - 1]$$

Ex. 3) If $\mathbf{V = [x + 3y, y - 2z, x + az]}$ is solenoidal, then find the value of a.

Solution: If V is solenoidal then, $\nabla . V = 0$

$$\Rightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + az) = 0 \Rightarrow 1 + 1 + a = 0 \Rightarrow a = -2$$

Ex. 4) Find the unit tangent vector to any point on the curve $\mathbf{x = t^2 + 1}$,

$\mathbf{y = 4t - 3, z = 2t^2 - 6t}$ and determine the unit tangent at the point $\mathbf{t = 2}$.

Solution: Let, $r = [x, y, z] = [t^2 + 1, 4t - 3, 2t^2 - 6t]$

$$\therefore \frac{dr}{dt} = [2t, 4, 4t - 6]$$

$$\left|\frac{dr}{dt}\right| = \sqrt{4t^2 + 16 + (4t - 6)^2}$$

$$\text{Then, the unit tangent vector} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \frac{[2t, 4, 4t - 6]}{\sqrt{4t^2 + 16 + (4t - 6)^2}}$$

At $t = 2$,

$$\text{The unit tangent vector} = \frac{\frac{dr}{dt}}{\left|\frac{dr}{dt}\right|} = \frac{[4,4,2]}{\sqrt{16 + 16 + 4}} = \frac{[4,4,2]}{6} = \frac{4}{6}i + \frac{4}{6}j + \frac{2}{6}k$$

Ex. 5) Find the directional derivative of $\mathbf{f(x,y) = \tan^{-1}\frac{y}{x}}$

at the point $(-2,2)$ in the direction of $\mathbf{a = -i - j}$.

Solution: Given,

$$f(x,y) = \tan^{-1}\frac{y}{x}$$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right)\left(\tan^{-1}\frac{y}{x}\right) = \mathbf{i}\,\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) + \mathbf{j}\,\frac{1}{1+\frac{y^2}{x^2}}\,\frac{1}{x}$$

$$= -\frac{\mathbf{i}\,y}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x^2}\right) + \frac{\mathbf{j}}{\frac{x^2+y^2}{x^2}}\,\frac{1}{x} = -\frac{\mathbf{i}\,y}{x^2+y^2} + \frac{\mathbf{j}\,x}{x^2+y^2}$$

At the point $(-2,2)$

$$\nabla f = -\frac{\mathbf{i}\,y}{x^2+y^2} + \frac{\mathbf{j}\,x}{x^2+y^2} = -\frac{\mathbf{i}\,2}{8} + \frac{\mathbf{j}\,(-2)}{8} = -\frac{\mathbf{i}}{4} - \frac{\mathbf{j}}{4}$$

Hence, the directional derivative of $f = \nabla f\,\frac{a}{|a|} = \left(-\frac{\mathbf{i}}{4} - \frac{\mathbf{j}}{4}\right)\frac{-i-j}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{2\sqrt{2}}$

Ex. 6) Find the directional derivative of $\mathbf{f(x,y) = x^2 + xy}$ at the point $(1,2)$

in the direction of the unit vector $\hat{\mathbf{a}} = \frac{\mathbf{i}}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}}$.

Solution: Given,

$$f(x,y) = x^2 + xy$$

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right)(x^2 + xy) = \mathbf{i}\,(2x + y) + \mathbf{j}x$$

At the point $(1,2)$

$$\nabla f = 4\mathbf{i} + \mathbf{j}$$

Hence, the directional derivative of $f = \nabla f\,\hat{a} = (4\mathbf{i} + \mathbf{j})\left(\frac{\mathbf{i}}{\sqrt{2}} + \frac{\mathbf{j}}{\sqrt{2}}\right) = \frac{4}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{5}{\sqrt{2}}$

Ex. 7) Find the angle between the surfaces $\mathbf{xy^2z = 3x + z^2}$ and $\mathbf{3x^2 - y^2 + 2z = 1}$ at the point $(\mathbf{1}, -\mathbf{2}, \mathbf{1})$.

Solution: Let,

$\emptyset = xy^2z - 3x - z^2$ and $\psi = 3x^2 - y^2 + 2z - 1$

Angle between the surfaces, $\cos\theta = \frac{\nabla\emptyset . \nabla\psi}{|\nabla\emptyset| . |\nabla\psi|}$

$\nabla\emptyset = \mathbf{i}(y^2z - 3) + \mathbf{j}(2xyz) + \mathbf{k}(xy^2 - 2z)$

$\nabla\psi = \mathbf{i}(6x) + \mathbf{j}(-2y) + \mathbf{k}(2)$

At $(1, -2, 1)$, $\nabla\emptyset = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\nabla\psi = 6\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$

$\nabla\emptyset . \nabla\psi = 6 - 16 + 4 = -6$

$|\nabla\emptyset| = \sqrt{21}$ and $|\nabla\psi| = 2\sqrt{14}$

Hence, the angle between the surfaces, $\cos\theta = \frac{\nabla\emptyset . \nabla\psi}{|\nabla\emptyset| . |\nabla\psi|} = \frac{-6}{\sqrt{21} . 2\sqrt{14}} = \frac{-3}{\sqrt{21} . \sqrt{14}}$

$\Rightarrow \cos\theta = \frac{-3}{\sqrt{294}} \Rightarrow \theta = \cos^{-1}\left(\frac{-3}{\sqrt{294}}\right)$

H.W:

1) If $\emptyset = 3x^2 - 4y^3 + z^2$ then find $\nabla\emptyset$ at the point $(1,2,3)$ and $|\nabla\emptyset|$ Ans. $6i - 48j + 6k$

2) If $A = 2xz^2 i - yzj + 3xz^3 k$, Then find the value of $\nabla \times (\nabla \times A)$ at $(1,1,1)$ Ans. $5i + 3k$

3) If $A = 3xz^2 i - yzj + (x + 2z)k$, then find curl (curl A) Ans. $-6xi + (6z - 1)k$

4) Find the directional derivative of $f(x, y) = xe^y - ye^x$ at the point $(0,0)$ in the direction of $a = 5i - 2j$. Ans. $\frac{7}{\sqrt{29}}$

5) Find the directional derivative of $f(x, y) = x^3z - yz^2 + z^2$ at the point $(2, -1, 1)$ Ans. $5\sqrt{14}$

in the direction of $a = 3i - j + 2k$.

6) Find the directional derivative of $f(x, y) = 4xz^3 - 3x^2y^2z$ at the point $(2, -1, 2)$ Ans. $\frac{376}{7}$

in the direction of $a = 2i - 3j + 6k$.

7) Find the angle between the surfaces $x^2y + z = 3$ and $x \ln z + y^2 = 4$

at the point $(-1, 2, 1)$ Ans. $\theta = \cos^{-1}\left(\frac{1}{\sqrt{34}}\right)$