Beta-Gamma Function

Beta-Function:

The Beta function is denoted by $\beta\ (m, n)$ is defined as

$\beta\ (m, n) = \int_0^1 x^{m-1}\,(1-x)^{n-1} dx, \quad m > 0, n > 0$.

Gamma-Function:

The Gamma function is denoted by $\Gamma(n)$ is defined as:

$\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1} dx\,, \quad n > 0$

Important Formula
1. $\Gamma 1 = 1$ 2. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ 3. $\Gamma(n+1) \ = n\,\Gamma n$ 4. $\beta\ (m, n) = \frac{\Gamma m\ \Gamma n}{\Gamma m+n}$ 5. $\int_0^{\frac{\pi}{2}} \sin^p\theta\ \cos^q\theta\ d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\ \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$

Question 1. : Prove that $\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$

Solution: Let $I = \int_0^\infty e^{-t^2}\, dt$ … … … … (1)

Putting $\quad t^2 = x \quad \dots\dots\dots \quad \dots \quad \dots$ (2)

$$t = \sqrt{x}$$

Differentiating with respect to x,

$$\frac{dt}{dx} = \frac{1}{2\sqrt{x}} \qquad \left\{ since\ \frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}\, x^{\left(\frac{1}{2}-1\right)} = \frac{1}{2}\, x^{-\frac{1}{2}} \right.$$

$$=> \ dt = \frac{1}{2} x^{-\frac{1}{2}}\, dx$$

⇨ Limit: when $t = o, \quad x = 0 \quad \{\ putting\ t = 0\ at\ equation(2)$

$t = \infty, \quad x = \infty \qquad \{\ putting\ t = \infty\ at\ equation(2)$

Thus from equation (1) we have,

$$I = \int_0^\infty e^{-x}\, . \frac{1}{2} x^{-\frac{1}{2}}\, dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x}\, . x^{\frac{1}{2}-1}\, dx$$

$$= \frac{1}{2} . \Gamma\left(\frac{1}{2}\right) \qquad [Since\ \ \Gamma(n) = \ \int_0^\infty e^{-x}\, x^{n-1} dx,]$$

$$= \frac{\sqrt{\pi}}{2} \qquad \text{(Proved).}$$

Question 2. Evaluate $\int_0^\infty e^{-y^2} y^5\, dy$ … … … . . (1)

Solution: Given,

$\int_0^\infty e^{-y^2} y^5\, dy$ … … … . . (1)

Putting $\ y^2 = z \quad \dots\dots\dots$ (2)

$so, \quad y = \sqrt{z} \quad \dots\dots\dots$ (3)

Differentiating with respect to y,

$$2y = \frac{dz}{dy}$$

$$=> dy = \frac{dz}{2y}$$

$$=> dy = \frac{dz}{2\sqrt{z}}$$ [putting , $y = \sqrt{z}$ from equation (3)]

Limit:

$when\ y = 0,\ z = 0$ $\quad [\,putting\ y = 0\ in\ eqn\ (2)]$

$when\ y = \infty,\ z = \infty\ [\,putting\ y = \infty\ \ in\ eqn\ (2)]$

Now from equation 1,

$$\int_0^\infty e^{-y^2} y^5\, dy \ = \ \int_0^\infty e^{-y^2} y^4\, y\, dy$$

$$= \ \int_0^\infty e^{-z} z^2 \sqrt{z}\, \frac{dz}{2\sqrt{z}} \quad \{\, since, y^2 = z\,, so,\ y^4 = \ (y^2)^2 = z^2$$

$$= \frac{1}{2} \int_0^\infty e^{-z} z^{3-1}\ dz$$

$$= \frac{1}{2}.\Gamma(3) \quad \{ Since\ \ \Gamma(n) = \ \int_0^\infty e^{-x}\, x^{n-1} dx,$$

$$= \frac{1}{2}.\Gamma(2+1)$$

$$= \frac{1}{2}.2\Gamma(2) \quad \{ Since\,, \Gamma(n+1) = n\Gamma n\,,\ \ so\ \Gamma(2+1) = 2\Gamma(2)$$

$$= \ \frac{1}{2}\ 2\Gamma(1+1)$$

$$= \frac{1}{2}\ 2\ \times 1 \times \Gamma(1)$$

$$= \frac{1}{2}\ 2\ \times 1 \times 1$$

$$= 1$$

$(Ans.)$

Question 3. Express the integral $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ in terms of Beta function

Solution: Let $I = \int_0^1 \frac{dx}{\sqrt{1-x^3}} \dots \dots \dots (1)$

By the definition of beta function we have

$$\beta\,(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx, \qquad m > 0, n > 0$$

Putting $x^3 = t$ -----------(2) [So, $x = t^{\frac{1}{3}}$ and $x^2 = t^{\frac{2}{3}}$]

Differentiating with respect to x,

$$3x^2 = \frac{dt}{dx}$$

$$=> dx = \frac{dt}{3x^2}$$

$$=> dx = \frac{dt}{3t^{\frac{2}{3}}} \quad \left\{ since\ we\ find\ before\ x^2 = t^{\frac{2}{3}} \right.$$

Limit:

When x=0, t=0; [putting x=0 in eqn(2)]

when x=1, t=1 [putting x=1 in eqn(2)]

Now from (1) we get,

$$I = \int_0^1 \frac{dt}{3t^{\frac{2}{3}}}\ \frac{1}{\sqrt{1-t}}$$

$$= \frac{1}{3}\int_0^1 t^{-\frac{2}{3}}.\ (1-t)^{-\frac{1}{2}}\ dt$$

$$= \frac{1}{3}\int_0^1 t^{\frac{1}{3}-1}.\ (1-t)^{\frac{1}{2}-1}\ dt$$

$$= \frac{1}{3}\beta\left(\frac{1}{3},\frac{1}{2}\right) \qquad (ans.)$$

Question 4. Evaluate $\int_0^1 x^5(1-x)^{\frac{5}{2}}\,dx$

Take $\mathbf{k\text{-}1} = \frac{5}{2}$,

$\Rightarrow \mathbf{k} = \frac{5}{2} + 1 = \frac{7}{2}$

So we are writing, $\frac{5}{2} = \frac{7}{2} - 1$

Solution: Given,

$$\int_0^1 x^5(1-x)^{\frac{5}{2}}\,dx$$

$$=\int_0^1 x^{6-1}(1-x)^{\frac{7}{2}-1}\,dx$$

$$=\beta\left(6,\ \frac{7}{2}\right) \qquad \left[as\ \beta\ (m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx\right]$$

$$=\frac{\Gamma(6)\ \Gamma\left(\frac{7}{2}\right)}{\Gamma\left(6+\frac{7}{2}\right)} \qquad \left[as\ \beta\ (m,n) = \frac{\Gamma(m)\,\Gamma\,(n)}{\Gamma(m+n)}\right]$$

$$=\frac{5.4.3.2.1.\ \Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{19}{2}\right)}$$

$$=\frac{5.4.3.2.1.\Gamma\left(\frac{7}{2}\right)}{\frac{17}{2}.\frac{15}{2}.\frac{13}{2}.\frac{11}{2}.\frac{9}{2}.\frac{7}{2}.\Gamma\left(\frac{7}{2}\right)} \qquad \left\{since,\ \Gamma\left(\frac{19}{2}\right) = \Gamma\left(\frac{17}{2}+1\right) = \frac{17}{2}\Gamma\left(\frac{17}{2}\right)\ and\ so\ on\right.$$

$$=\frac{4.2.2.2.2.2.2.2}{17.13.11.9.7}$$

$$=\frac{2^9}{153153} \qquad (ans.)$$

Question 5. Show that $\int_0^{\frac{\pi}{2}} \sin^4\theta\ \cos^6\theta\ d\theta = \frac{3\pi}{512}$ ... ... (1)

Solution: We know, $\int_0^{\frac{\pi}{2}} \sin^p\theta\ \cos^q\theta\ d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$ … … …(2)

Comparing with eqn(2), p=4, q=6 in eqn (1) we get,

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\ \cos^6\theta\ d\theta = \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{4+6+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma\,(6)}$$

$$= \frac{\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{2.5.4.3.2.1} \quad [Applying\,, \Gamma n+1 = n\Gamma \text{n}]$$

$$= \frac{3.\sqrt{\pi}.\sqrt{\pi}}{2.2.2.2.2.2.4.2}$$

$$= \frac{3\pi}{512} \qquad \text{(ans.)}$$

Exercise:

1. Evaluate $\int_0^1 x^7(1-x)^{\frac{3}{2}}\,dx$

   Hints: Write, $x^7 = x^{8-1}$ and $(1-x)^{\frac{3}{2}} = (1-x)^{\frac{5}{2}-1}$ then similar as Question 4

2. Evaluate $\int_0^1 x^6(1-x)^4\,dx$

   Hints: Write x^6=x^{7-1} and $(1-x)^4 = (1-x)^{5-1}$ then similar as Ques. 4

3. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5\theta\ \cos^7\theta\ d\theta$

   Hints: Almost similar to Question 5. Just follow that.