Gamma function: Improper integral $\int_0^{\infty} e^{-x} x^{n-1} dx$ is called gamma function or second eularian. It is denoted by $\Gamma n$

So, $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$

# Prove that $\Gamma n+1 = n\Gamma n$

Proof: we know $\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$ ……………….(1)

Replacing n by n+1 we get,

$$\Gamma n+1 = \int_0^{\infty} e^{-x} x^{n} dx = \left[x^n e^{-x}\right]_0^{\infty} + n\int_0^{\infty} e^{-x} x^{n-1} dx = 0 + n\Gamma n = n\Gamma n$$

So, $\Gamma n+1 = n\Gamma n$ (proved)

# Prove that $\Gamma n+1 = n!$

Proof: we know $\Gamma n+1 = n\Gamma n$

Replacing n by (n-1),(n-2),(n-3),……3,2,1 respectively in (1) we get

$\Gamma n = (n-1)\Gamma(n-1)$

$\Gamma(n-1) = (n-2)\Gamma(n-2)$

$\Gamma(n-2) = (n-3)\Gamma(n-3)$

…………………….

…………………….

………………………..

$\Gamma 3 = 2\Gamma(2$

$\overline{)2} = 1\overline{)1}$

Using these values in (1) we get

$\overline{)n+1} = n\overline{)n}$

$= n(n-1)\overline{)(n-1)}$

$= n(n-1)(n-2)\overline{)(n-2)}$

..........................

$= n(n-1)(n-2)............3.2.\overline{)1}$

$= n(n-1)(n-2)............3.2.1$

$= n!$

$\therefore \overline{)n+1} = n!$ (proved)

Beta Function: : Integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called betaa function or first eularian. It is denoted by $\beta(m,n)$

So, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

# Prove that $\beta(m,n) = \beta(n,m)$

Proof: we know $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$ ..........(1)

Let,

$y = 1-x$

$\Rightarrow dy = -dx$

$\therefore dx = -dy$

If $x \to 0$ then $y \to 1$

And If $x \to 1$ then $y \to 0$

So, from (1)

$$\beta(m,n)=\int_1^0 (1-y)^{m-1}y^{n-1}(-dy)$$

$$=\int_0^1 y^{n-1}(1-y)^{m-1}dy$$

$$=\beta(n,m)$$

So, $\beta(m,n)=\beta(n,m)$ (proved)

# Prove the relation between Betta and Gamma function, i.e, $\beta(m,n)=\dfrac{\overline{)m}\,\overline{)n}}{\overline{)m+n}}$

Proof: We know $\overline{)m}=\int_0^\infty e^{-x}x^{m-1}dx$ …………(1)

Let $x=\lambda y \quad \therefore dx=\lambda dy$

If $x\to 0$ then $y\to 0$

And If $x\to\infty$ then $y\to\infty$

Now, from (1)

$$\overline{)m}=\int_0^\infty e^{-\lambda y}(\lambda y)^{m-1}\lambda dy$$

$$=\int_0^\infty e^{-\lambda y}\lambda^m y^{m-1}\lambda dy$$

$$\overline{)m}=\int_0^\infty e^{-\lambda y}\lambda^m y^{m-1}\lambda dy \text{ ………………….(2)}$$

Again we know, $\overline{)n}=\int_0^\infty e^{-\lambda}\lambda^{n-1}d\lambda$ ………..(3)

Multiplying )2) and (3) we get,

$$\overline{)m}\,\overline{)n}=\int_0^\infty\int_0^\infty e^{-\lambda(1+y)}\lambda^{m+n-1}y^{m-1}d\lambda dy$$

$$=\int_0^\infty\left[\int_0^\infty e^{-\lambda(1+y)}\lambda^{m+n-1}d\lambda\right]y^{m-1}dy$$

$$=\int_0^\infty\frac{\overline{)m+n}}{(1+y)^{m+n}}y^{m-1}dy$$

$$= \int_0^\infty \frac{\Gamma(m+n)}{(1+y)^{m+n}} y^{m-1} dy$$

$$= \Gamma(m+n) \int_0^\infty \frac{1}{(1+y)^{m+n}} y^{m-1} dy$$

$$= \Gamma(m+n) \;\; \beta(m,n)$$

So, $\Gamma(m)\Gamma(n) = \Gamma(m+n) \; \beta(m,n)$

$\therefore \beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ (proved)

# prove that $\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

Proof: We know $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$

Let $x = \sin^2\theta$ so, $dx = 2\sin\theta\cos\theta d\theta$

If $x \to 0$ then $\theta \to 0$

And If $x \to 1$ then $\theta \to \dfrac{\pi}{2}$

$\therefore \beta(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta . 2\sin\theta\cos\theta d\theta$

$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta . d\theta$.................(1)

Let 2m-1=p and 2n-1=q

So, $m = \dfrac{p+1}{2}$ *and* $n = \dfrac{q+1}{2}$

From (1)

$$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta . d\theta$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta . d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

We also know that $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

So, $\displaystyle\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$ (proved)