Multiple Integrals and Chain Rule

$\mathbf{Formula}$: If $u = f(x, y, z)$ then, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

Ex. 1) If $\mathbf{f(x, y) = c}$, then show that, $\mathbf{\frac{dy}{dx} = -\frac{f_x}{f_y}}$

Proof: Here,

$f(x, y) = c$

$\Rightarrow d[f(x, y)] = d(c)$

$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

$\Rightarrow f_x dx + f_y dy = 0$

$\Rightarrow f_y dy = -f_x dx$

$\Rightarrow \frac{dy}{dx} = -\frac{f_x}{f_y}$

Ex. 2) If $\mathbf{u = x^2 - 5xy + 6y^2 + 7x + 8y + 5}$, then find du.

Solution: Given,

$u = x^2 - 5xy + 6y^2 + 7x + 8y + 5$

$\frac{\partial u}{\partial x} = 2x - 5y + 7$ and $\frac{\partial u}{\partial y} = -5x + 12y + 8$

Therefore, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = (2x - 5y + 7)dx + (-5x + 12y + 8)dy$

Ex. 3) If $\mathbf{x^2 + y^2 = 4}$, then find $\mathbf{\frac{dy}{dx}}$.

Solution: Let,

$f(x, y) = x^2 + y^2 = 4$

Now, $f_x = 2x$ and $f_y = 2y$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x}{2y} = -\frac{x}{y}$$

Ex. 4) If $\mathbf{ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0}$, then find $\mathbf{\frac{dy}{dx}}$.

Solution: Let,

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Now, $f_x = 2ax + 2hy + 2g$

and $f_y = 2hx + 2by + 2f$

$$\therefore \frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2ax + 2hy + 2g}{2hx + 2by + 2f} = -\frac{ax + hy + g}{hx + by + f}$$

Ex. 5) If $\mathbf{u = x^2 + y^2 + 4xy}$ and $\mathbf{x^2 - y^2 + 7x + 8y - 5 = 0}$, then find $\mathbf{\frac{du}{dx}}$ and $\mathbf{\frac{du}{dy}}$.

Solution: Given,

$u = x^2 + y^2 + 4xy$

$$\frac{\partial u}{\partial x} = 2x + 4y \text{ and } \frac{\partial u}{\partial y} = 2y + 4x$$

$$\text{Therefore, } du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\Rightarrow \frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = (2x + 4y) + (2y + 4x)\frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Again,

$$x^2 - y^2 + 7x + 8y - 5 = 0$$

$$\Rightarrow 2x - 2y\frac{dy}{dx} + 7 + 8\frac{dy}{dx} = 0$$

$$\Rightarrow 2x + 7 - (2y - 8)\frac{dy}{dx} = 0$$

$$\Rightarrow (2y - 8)\frac{dy}{dx} = 2x + 7$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x + 7}{2y - 8}$$

From (i),

$$\frac{du}{dx} = (2x + 4y) + (2y + 4x)\frac{2x + 7}{2y - 8}$$

Ex. 6) If $\mathbf{u = \ln(x^2 + y^2)}$ and $\mathbf{x = e^{-t}}$ and $\mathbf{y = e^{t}}$, then show that, $\mathbf{\frac{du}{dt} = 2\frac{e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}}}$

Proof: Given,

$$u = \ln(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

We know,

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$\Rightarrow \frac{du}{dt} = \frac{2x}{x^2 + y^2} . \frac{d}{dt}(e^{-t}) + \frac{2y}{x^2 + y^2} \frac{d}{dt}(e^t)$$

$$\Rightarrow \frac{du}{dt} = \frac{2x}{x^2 + y^2}(-e^{-t}) + \frac{2y}{x^2 + y^2}(e^t)$$

$$\Rightarrow \frac{du}{dt} = \frac{-2xe^{-t} + 2ye^t}{x^2 + y^2}$$

$$\Rightarrow \frac{du}{dt} = \frac{-2xe^{-t} + 2ye^t}{e^{2t} + e^{-2t}} = 2\frac{e^{2t} - e^{-2t}}{e^{2t} + e^{-2t}} \qquad \text{(proved)}$$