Area in polar form

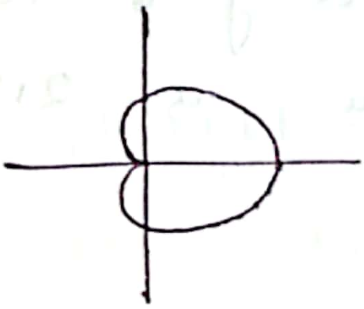
In polar form $x = r\cos\theta$ | $\therefore r = \sqrt{x^2+y^2}$
$y = r\sin\theta$ | $\theta = \tan^{-1} y/x$

$$\text{Area} = \frac{1}{2}\int r^2 \, d\theta.$$

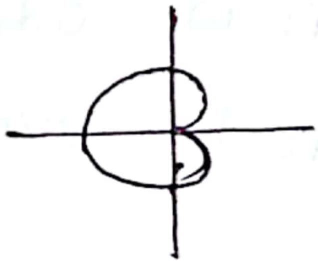
Some special forms of curves

1. Cardiode:-

(a) $r = a(1+\cos\theta)$



(b) $r = a(1-\cos\theta)$

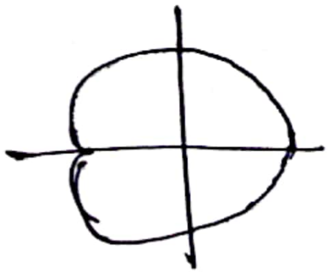


$\theta = 0$ to π

2. Limacon:-

$r = a + b\cos\theta$

if $a=b$ then it will be cardiode



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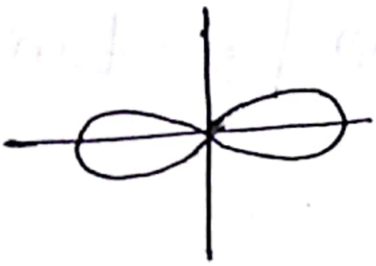
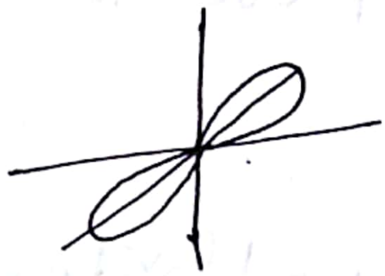
3. Lemniscate:-

(i) $r^2 = a^2\cos 2\theta$



(ii) $r^2 = a^2\sin 2\theta$



(4) Loop:- $r = a\sin n\theta$ or $r = a\cos n\theta$

if n is odd we will get n loops

if n is eveen " " " $2n$ "

1. Find the area of one loop of the curve $r = a\cos 3\theta$. Also find total area

Sol:- Here, $r = a\cos 3\theta$ — ①

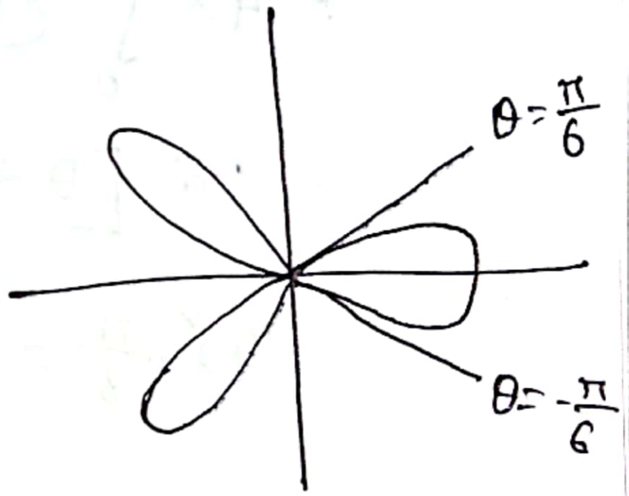
we get three loops for equation ①

if $r=0$ then $a\cos 3\theta = 0$

$$\Rightarrow \cos 3\theta = \cos\left(\pm\frac{\pi}{2}\right)$$

$$\therefore \theta = \pm\frac{\pi}{6}$$

So, one loop will remain through $-\frac{\pi}{6}$ to $\frac{\pi}{6}$.



$$\text{Area} = \frac{1}{2}\int r^2 \, d\theta$$

$$= \frac{1}{2}\int_{-\frac{\pi}{6}}^{\pi/6} a^2\cos^2 3\theta \, d\theta$$

$$= \frac{a^2}{2}\cdot 2\int_0^{\pi/6} \frac{1}{2}\cdot 2\cos^2 3\theta \, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/6} (1+\cos 6\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 6\theta}{6}\right]_0^{\pi/6}$$

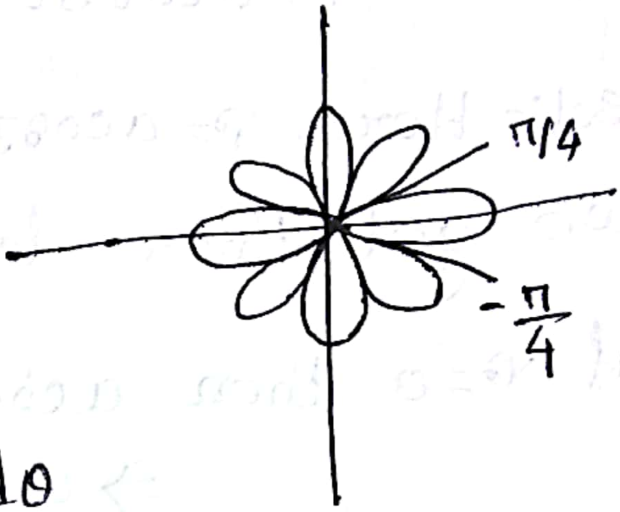
$$= \frac{a^2}{2}\cdot\frac{\pi}{6} = \frac{\pi a^2}{12}$$

Area of one loop $= \frac{\pi a^2}{12}$. Total area $= 3\cdot\frac{\pi a^2}{12} = \frac{\pi a^2}{4}$

2. Find the area of loops of equation $r = a\cos 4\theta$

Solution:-

$r = a\cos 4\theta$ — (1)



if $r = 0$ then $\theta = \pm\frac{\pi}{8}$

$$\text{Area} = 8\cdot\frac{1}{2}\int_{-\pi/8}^{\pi/8} a^2\cos^2 4\theta \, d\theta$$

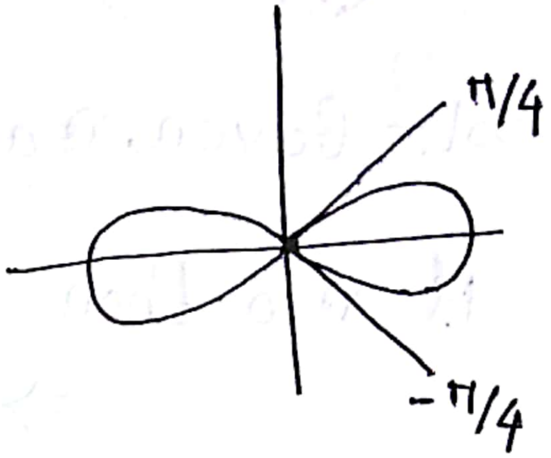
$$= 4\cdot 2a^2\int_0^{\pi/8} \frac{1}{2}\cdot 2\cos^2 4\theta \, d\theta$$

$$= 4a^2\int_0^{\pi/8} (1+\cos 8\theta)\, d\theta$$

$$= 4a^2\left[\theta + \frac{\sin 8\theta}{8}\right]_0^{\pi/8}$$

$$= 4a^2\cdot\frac{\pi}{8} = \frac{\pi a^2}{2}$$

3. Find the area of one loop and all loops of $r^2 = a^2 \cos 2\theta$



Solution:-

Here, $r^2 = a^2 \cos 2\theta$

so we get two loops

if $r = 0$ then $\theta = \pm \frac{\pi}{4}$

Area of one loop $= \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 \, d\theta$

$= \frac{1}{2} \cdot 2 \int_{0}^{\pi/4} a^2 \cos 2\theta \, d\theta$

$= a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4}$

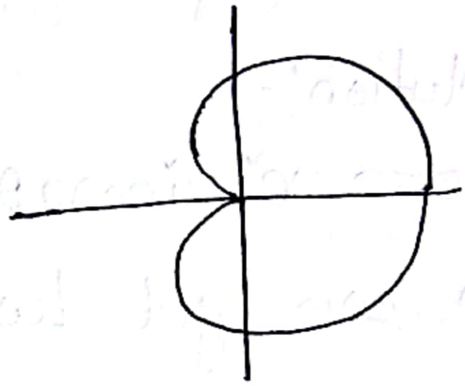
$= \frac{a^2}{2}$

$\therefore$ Total area $= 2 \cdot \frac{a^2}{2}$

$= a^2$

4. Find the area of the cardiode $r = a(1+\cos\theta)$

Sol:- Given, $r = a(1+\cos\theta)$



If $r=0$ then $1+\cos\theta=0$

$$\Rightarrow \cos\theta = -1$$

$$\therefore \theta = \pm\pi$$

$$\therefore \text{Area} = \frac{1}{2}\int_{-\pi}^{\pi} r^2\, d\theta$$

$$= 2\cdot\frac{1}{2}\int_0^{\pi} a^2(1+\cos\theta)^2\, d\theta$$

$$= a^2\int_0^{\pi}(2\cos^2\theta/2)^2\, d\theta$$

$$= 4a^2\int_0^{\pi}\cos^4\theta/2\, d\theta$$

Let $\frac{\theta}{2} = t$ $\therefore d\theta = 2dt$

if $\theta = 0$ then $t=0$

if $\theta = \pi$ then $t = \frac{\pi}{2}$

$$= 8a^2\int_0^{\pi/2}\cos^4 t\, dt$$

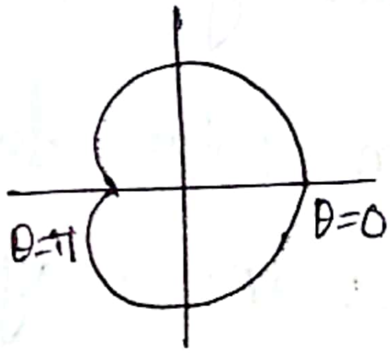
$$= 8a^2 \frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{2\,\Gamma 3}$$

$$= \frac{8a^2\,\sqrt{\pi}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{2\cdot 2\cdot 1}$$

$$= \frac{3\pi a^2}{2}$$

5. Find the area of the limacone $r = a + b\cos\theta$; $a > b$

Solution:- Given that, $r = a + b\cos\theta$



$$\therefore \text{Area} = 2\int_0^{\pi} \frac{1}{2} r^2\, d\theta$$

$$= \int_0^{\pi} (a+b\cos\theta)^2\, d\theta$$

$$= \int_0^{\pi} (a^2 + 2ab\cos\theta + b^2\cos^2\theta)\, d\theta$$

$$= \int_0^{\pi} \left[a^2 + 2ab\cos\theta + \frac{b^2}{2}(1+\cos 2\theta)\right] d\theta$$

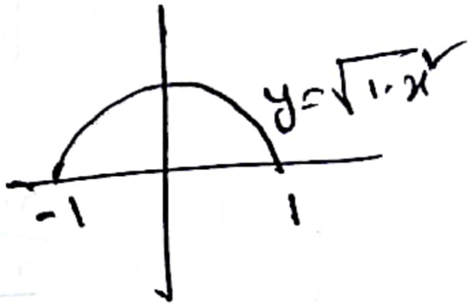
$$= \left[a^2\theta + 2ab\sin\theta + \frac{b^2\theta}{2} + \frac{b^2\sin 2\theta}{4}\right]_0^{\pi}$$

$$= a^2\pi + 0 + \frac{b^2\pi}{2} + 0 = \pi\left(a^2 + \frac{b^2}{2}\right)$$ Ans

Double integral in polar

6. Use polar co-ordinate evaluate

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx$$



Sol:- ~~HB~~ Let, $I = \iint_R (x^2+y^2)^{3/2}\, dA$

Here R is enclosed by $x=-1$ to $x=1$

and $y=0$ to $y=\sqrt{1-x^2}$

In polar form $x = r\cos\theta$, $y = r\sin\theta$

$\therefore dA = dx\,dy = r\,dr\,d\theta$

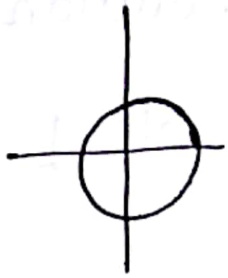
and $\theta = 0$ to $\theta = \pi$ and $r = 0$ to 1

$$\therefore I = \int_0^1\int_0^{\pi} (r^2)^{3/2} \cdot r\, dr\, d\theta = \int_0^1\int_0^{\pi} r^4\, dr\, d\theta$$

$$= \pi \cdot \left[\frac{r^5}{5}\right]_0^1 = \frac{\pi}{5}.$$

7. Use polar co-ordinates to evaluate $\iint_R e^{-(x^2+y^2)}\,dA$, Where the region is enclo-sed by the circle $x^2+y^2=1$

Sol:- Let, $I = \iint_R e^{-(x^2+y^2)}\,dA$ — (1)



Here R is enclosed by $x^2+y^2=1$

For polar co-ordinate system

$x = r\cos\theta$, $y = r\sin\theta$

$dA = dx\,dy$

$= r\,dr\,d\theta$

~~If $\theta=0$~~ The limit will be $r=0$ and $r=1$

and $\theta=0$ to $\theta=2\pi$

$$\therefore I = \int_0^1\int_0^{2\pi} e^{-r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^1 re^{-r^2}[\theta]_0^{2\pi}\,dr = 2\pi\int_0^1 re^{-r^2}\,dr$$

$r^2 = t$

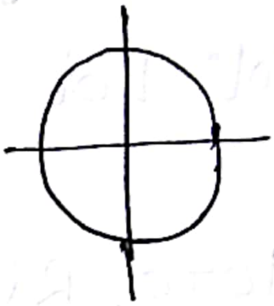
$\therefore r\,dr = \frac{dt}{2}$

limit 0 to 1

$$= 2\pi\int_0^1 \frac{e^{-t}\,dt}{2} = \pi\left[-e^{-t}\right]_0^1 = \pi\left(1-\frac{1}{e}\right)$$ Ans

8. Use polar co-ordinates to evaluate $\iint_R \frac{dA}{(1+x^2+y^2)^{3/2}}$, where the region R is enclosed by the circle $x^2+y^2=a^2$

Solution:-



Let, $I = \iint_R \frac{dA}{(1+x^2+y^2)^{3/2}}$

R is enclosed by $x^2+y^2=a^2$

$x = r\cos\theta$, $y = r\sin\theta$

$dA = dx\,dy = r\,dr\,d\theta$

Limit will be $r=0$ to a and $\theta = 0$ to 2π

$$\therefore I = \int_0^a \int_0^{2\pi} \frac{r}{(1+r^2)^{3/2}}\,dr\,d\theta$$

$$= \int_0^a \frac{r}{(1+r^2)^{3/2}}\,dr\,[\theta]_0^{2\pi}$$

$$= 2\pi \int_0^a \frac{r}{(1+r^2)^{3/2}}\,dr$$

Let $1+r^2=t$

$\therefore\ r\,dr = \dfrac{dt}{2}$

if $r=0$ then $t=1$

If $r=a$ " $t=1+a^2$

$$\therefore I = 2\pi \cdot \frac{1}{2}\int_{1}^{1+a^2} \frac{dt}{t^{3/2}}$$

$$= \pi \left[-\frac{2}{\sqrt{t}}\right]_{1}^{1+a^2}$$

$$= 2\pi\left[1-\frac{1}{\sqrt{1+a^2}}\right]$$