<u>Volume</u>

1. For Cartesian: $V = \int_a^b \pi y^2 dx \;\; or, V = \int_a^b \pi x^2 dx$
2. For Polar: $V = \int \frac{2\pi}{3} r^3 Sin\theta d\theta.$

<u>Example-01:</u> For the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ show that the volume of the solid formed by the revolution about x-axis is $\frac{32a^3}{105}$

<u>Solution:</u>

Given that,

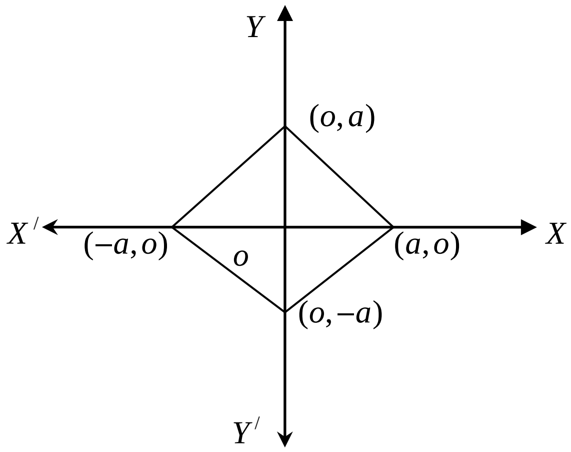
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\Rightarrow y^{\frac{2}{3}} = a^{\frac{2}{3}} - x^{\frac{2}{3}}$$

$$\Rightarrow y^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 \ldots\ldots\ldots\ldots\ldots(2)$$

$$when, x = 0 \; then, y = \pm a$$

$$when, \; y = 0, then, x = \pm a$$



Therefore, the required volume is, $V = \int_a^b \pi y^2 dx = 2\pi \int_a^b \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 dx$

Put $x = aSin^3\theta$; $dx = 3aSin^2\theta.Cosd\theta$

$when, \; x = 0, \; then, \theta = 0$

$when, \; x = 0, \; then, \; \theta = \frac{\pi}{2}$

$$\therefore V = 2\pi \int_0^{\frac{\pi}{2}} \left(a^{\frac{2}{3}} - a^{\frac{2}{3}} Sin^2\theta\right)^3 .3aSin^2\theta.Cosd\theta$$

$$= 2\pi\int_0^{\frac{\pi}{2}} a^{2.}3aSin^2\theta.Cos^7 d\theta$$

$$= 6\pi a^3\int_0^{\frac{\pi}{2}} Sin^2\theta.Cos^7 d\theta$$

By using Gamma-Beta function,

$$= 6\pi a^3 \frac{\left\lceil\frac{2+1}{2}.\left\lceil\frac{7+1}{2}\right.\right.}{2\left\lceil\frac{2+7+2}{2}\right.} = 6\pi a^3 \frac{\left\lceil\frac{3}{2}.\right\lceil 4}{2\left\lceil\frac{11}{2}\right.} = 6\pi a^3 \frac{\left\lceil\frac{3}{2}\right..3.2.1}{2.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\left\lceil\frac{3}{2}\right.}$$

$$= \frac{32\pi a^3}{105}\ (Showed.)$$

<u>Example-02:</u> Find the volume of the region obtained by revolving the curve $r = a(1+Cos\theta)$ about the initial line.

<u>Solution:</u>

Given that,

$$r = a(1+Cos\theta)...............(1)$$

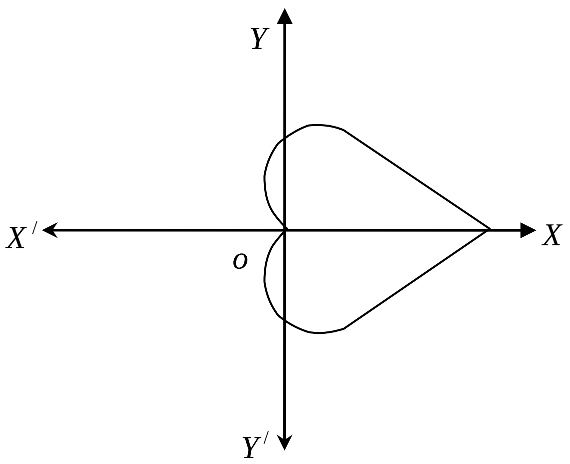
Equation (1) is symmetrical with the initial line,

$when, r = 0,\ \ then\ a(1+Cos\theta) = 0$

$$\Rightarrow Cos\theta = -1$$

$$\therefore \theta = \pm\pi$$

So the equation (1) lies between $\theta = -\pi\, to\, +\pi$



Therefore, the required volume is,

$$V = \int_0^{\pi}\frac{2}{3}\pi r^3 Sin\theta d\theta \quad = \frac{2}{3}\pi\int_0^{\pi} a^3(1+Cos\theta)^3 Sin\theta d\theta$$

$$= \frac{2\pi a^3}{3}.2\int_0^{\frac{\pi}{2}}(2Cos^2\frac{\theta}{2})^3 2Sin\frac{\theta}{2}Cos\frac{\theta}{2}d\theta \quad = \frac{2\pi a^3}{3}.32\int_0^{\frac{\pi}{2}} Sin\frac{\theta}{2}Cos^7\frac{\theta}{2}d\theta$$

$$= \frac{64\pi a^3}{3}\int_0^{\frac{\pi}{2}} Sin\frac{\theta}{2}Cos^7\frac{\theta}{2}d\theta$$

By using Gamma-beta function,

$$= \frac{6\pi a^3}{3} \frac{\Gamma\left(\frac{1+1}{2}\right) . \Gamma\left(\frac{7+1}{2}\right)}{2\Gamma\left(\frac{1+7+2}{2}\right)}$$

$$= \frac{64\pi a^3}{3} \frac{\Gamma(1) . \Gamma(4)}{2\Gamma(5)}$$

$$= 6\pi a^3 \frac{1.3.2.1}{2.4.3.2.1}$$

$$= \frac{8\pi a^3}{3} \; (Ans.)$$

<u>Example-03:</u> Find the volume of the solid generated by the revolved of an ellipse round its minor axis is $\frac{4}{3}\pi a^3 b$.

<u>Solution:</u>

The equation of ellipse is,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 .... (1)$$

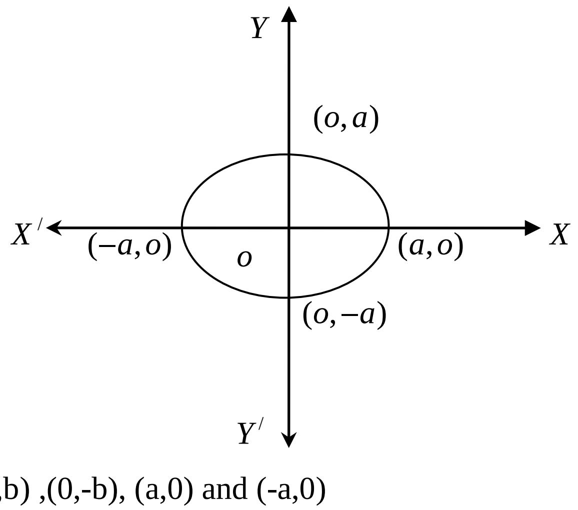
From (1),

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\Rightarrow x^2 = \frac{a^2}{b^2}(b^2 - y^2) .... (2)$$

$$when, x = 0 \; then, y = \pm b$$

$$when, \; y = 0, then, x = \pm a$$



The curve meets at (0,b) ,(0,-b), (a,0) and (-a,0)
Now, draw a graph roughly,
Therefore, the required volume is,

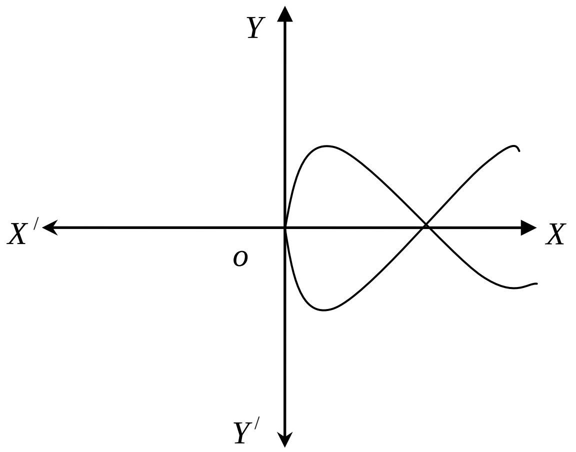
$$V = \int_a^b \pi x^2 dy$$
$$= 2\pi \int_0^b \frac{a^2}{b^2}.\left(b^2 - y^2\right)dy$$
$$= 2\pi\left[a^2\int_0^b dy - \frac{a^2}{b^2}\int_0^b y^2.dy\right] = 2\pi\left[a^2\left[y\right]_0^b - \frac{a^2}{b^2}\left[\frac{y^3}{3}\right]_a^b\right]$$
$$= 2\pi\left[a^2 b - \frac{a^2}{b^2}.\frac{b^3}{3}\right] = 2\pi\left[a^2 b - \frac{a^2 b}{3}\right]$$
$$= \frac{4\pi a^2 b}{3} \quad (Ans.)$$

<u>Example-04:</u> The curve $y^2 = x^2\left(\frac{3a - x}{a + x}\right)$ revolves about the axis of X. Find the volume generated by the loop.

<u>Solution:</u>

Given that,

$$y^2 = x^2\left(\frac{3a - x}{a + x}\right) \text{............}(1)$$
$$when, x = 0 \; then, y = 0$$
$$when, \; y = 0, then, x = 0, 3a$$



Therefore, the required volume is,

$$V = \int_0^{3a} \pi y^2 dx$$
$$= \pi\int_0^{3a} x^2\left(\frac{3a - x}{a + x}\right)dx \quad = \pi\int_0^{3a} x^2\left(\frac{a + x + 2a - 2x}{a + x}\right)dx$$

$$= \pi\int_0^{3a}\left[x^2 + \frac{2x^2(a+x-2x)}{a+x}\right]dx \;\; = \pi\int_0^{3a}\left[x^2 + 2x^2 - \frac{4x^3}{a+x}\right]dx$$

$$= \pi\int_0^{3a}\left[3x^2 - \frac{4x^3}{a+x}\right]dx \;\; = \pi\int_0^{3a}3x^2dx + \pi\int_0^{3a} - \frac{4x^3}{a+x}dx$$

$$= \pi\int_0^{3a}3x^2dx + \pi\int_0^{3a} - \frac{4x^2(a+x) + 4ax(a+x) - 4a^2(a+x) + 4a^3}{a+x}dx$$

$$= \pi\int_0^{3a}3x^2dx + \pi\int_0^{3a}\left[-4x^2 + 4ax - 4a^2 + \frac{4a^3}{a+x}\right]dx$$

$$= \pi\left[x^3\right]_0^{3a} + \pi\left[\frac{-4x^3}{3} + 2ax^2 - 4a^2x + 4a^3\ln(a+x)\right]_0^{3a}$$

$$= 27\pi a^3 - 36\pi a^3 + 16\pi a^3 - 12\pi a^3 + 4\pi a^3\ln(4a) - -4\pi a^3\ln(a)$$

$$= -3\pi a^3 + 4\pi a^3\ln(4) = -3\pi a^3 + 8\pi a^3\ln(2)$$

$$= \pi a^3(8lm2 - 3)\ (Ans.)$$

<u>Example-05:</u> The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the axis of X. Find the volume generated by the loop of the solid.

<u>Solution:</u>

Given that,

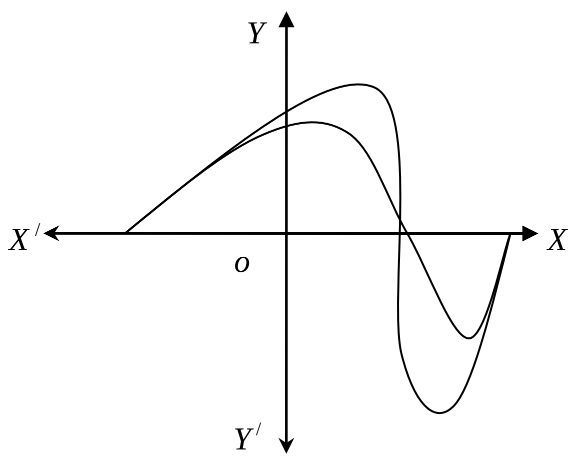
$$2ay^2 = x(x-a)^2 \ldots\ldots\ldots\ldots\ldots(1)$$

$$when, x = 0\ then, y = 0$$

$$when,\ y = 0, then, x = 0, a$$

Now,

Therefore, the required volume is,



$$V = \int_0^a \pi y^2 dx$$

$$= \pi\int_0^3 \frac{x(x-a)^2}{2a}dx \quad = \frac{\pi}{2a}\left[\int_0^a x^3dx - 2a\int_0^a x^2dx + a^2\int_0^a xdx\right]$$

$$=\frac{\pi}{2a}\left[\left[\frac{x^4}{4}\right]_0^a-2a\left[\frac{x^3}{3}\right]_0^a+a^2\left[\frac{x^2}{2}\right]_0^a\right]=\frac{\pi}{2a}\left[\frac{a^4}{4}-\frac{2a^4}{3}+\frac{a^4}{2}-0+0-0\right]$$

$$=\frac{\pi}{2a}\left[\frac{3a^4-8a^4+6a^2}{12}\right]=\frac{\pi}{2a}.\frac{a^4}{12}$$

$$=\frac{\pi a^4}{24}\quad (Ans.)$$

<u>Example-06:</u> Find the volume of the solid generated by the revolution of an ellipse round its major axis.

<u>Solution:</u>

The equation of ellipse is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1..................(1)$$

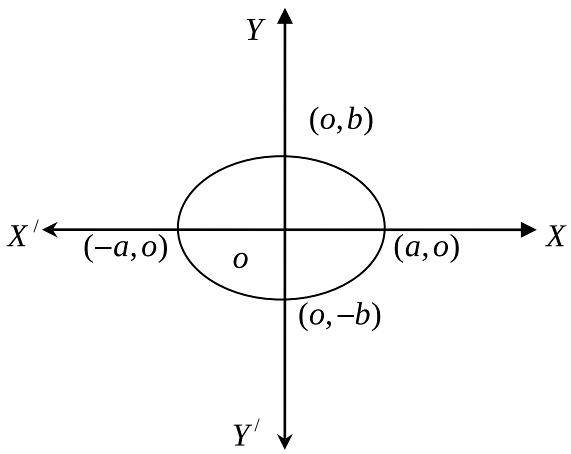
From (1),

$$\frac{y^2}{b^2}=1-\frac{x^2}{b^2}$$

$$\Rightarrow y^2=\frac{b^2}{a^2}(a^2-x^2)..........(2)$$

$$when, x=0\ then, y=\pm b$$

$$when,\ y=0, then, x=\pm a$$



The curve meets at (0,b) ,(0,-b), (a,0) and (-a,0)
Now, draw a graph roughly,
Therefore, the required volume is,

$$V=\int_0^a \pi y^2 dx$$

$$= 2\pi\int_b^a \frac{b^2}{a^2}.\left(a^2 - x^2\right)dx$$

$$= 2\pi\left[b^2\int_0^a dx - \frac{b^2}{a^2}\int_0^a x^2.dx\right] = 2\pi\left[a^2[x]_0^a - \frac{b^2}{a^2}\left[\frac{x^3}{3}\right]_0^a\right]$$

$$= 2\pi\left[b^2a - \frac{b^2}{a^2}.\frac{a^3}{3}\right] = 2\pi\left[ab^2 - \frac{ab^2}{3}\right]$$

$$= \frac{4\pi ab^2}{3} \qquad (Ans.)$$

<u>Example-07:</u> Find the volume of the region obtained by revolving the curve $r = 2a(1 + Cos\theta)$ **about the initial line.**

<u>Solution:</u>

Given that,

$$r = 2a(1 + Cos\theta)..............(1)$$

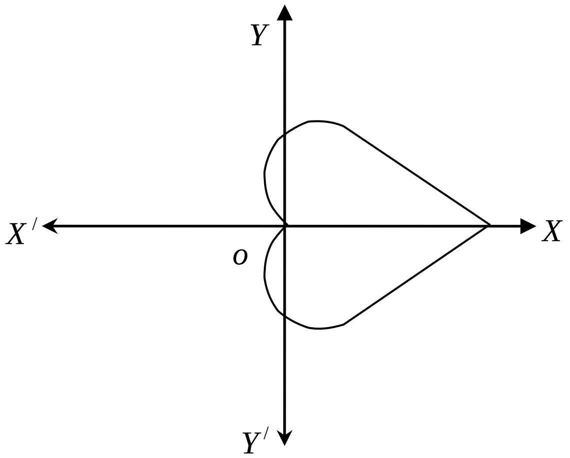
Equation (1) is symmetrical with the initial line,

$when, r = 0, \;\; then \; 2a(1 + Cos\theta) = 0$

$$\Rightarrow Cos\theta = -1$$

$$\therefore \theta = \pm\pi$$

So the equation (1) lies between $\theta = -\pi \; to \; +\pi$



Therefore, the required volume is,

$$V = \int_0^\pi \frac{2}{3}\pi r^3 Sin\theta d\theta$$

$$= \frac{2}{3}\pi\int_0^\pi 2^3.a^3(1 + Cos\theta)^3 Sin\theta d\theta \quad = \frac{16\pi a^3}{3}\int_0^{\frac{\pi}{2}}(2Cos^2\frac{\theta}{2})^3 2Sin\frac{\theta}{2}Cos\frac{\theta}{2}d\theta$$

$$= \frac{16\pi a^3}{3}.32\int_0^{\frac{\pi}{2}} Sin\frac{\theta}{2}Cos^7\frac{\theta}{2}d\theta \quad = \frac{512\pi a^3}{3}\int_0^{\frac{\pi}{2}} Sin\frac{\theta}{2}Cos^7\frac{\theta}{2}d\theta$$

By using Gamma-beta function,

$$= \frac{512\pi a^3}{3} \frac{\overline{\left| \frac{1+1}{2} \right.} \cdot \overline{\left| \frac{7+1}{2} \right.}}{2\overline{\left| \frac{1+7+2}{2} \right.}} = \frac{512\pi a^3}{3} \frac{\overline{|1} \cdot \overline{|4}}{2\overline{|5}} = \frac{512\pi a^3}{3} \frac{1.3.2.1}{2.4.3.2.1}$$

$$= \frac{64\pi a^3}{3} \ (Ans.)$$

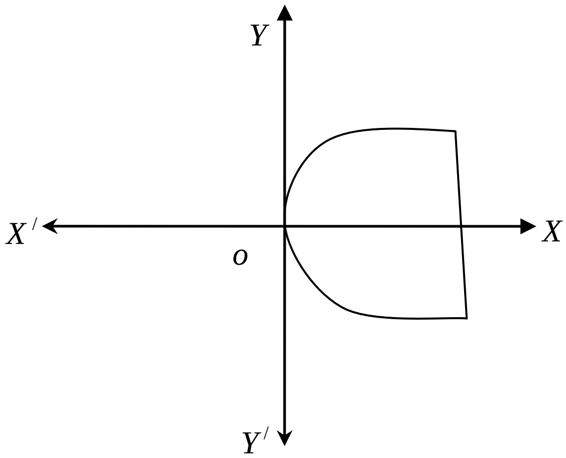
<u>Example-08</u>: Find the volume of solid obtained by rotating about x-axis. The area of the parabola $y^2 = 4ax$ cut off by its lotus rectum.

<u>Solution:</u>

Given that,

$$y^2 = 4ax ....(1)$$

Since, the distance from vertex to lotus rectum is 'a'



Therefore, the required volume is,

$$V = \int_0^a \pi y^2 dx$$

$$= \pi \int_0^a 4ax dx \ = 4a\pi \left[\frac{x^2}{2} \right]_0^a \ = 4a\pi \frac{a^2}{2}$$

$$= 2\pi a^3 \qquad (Ans.)$$

<u>Example-09:</u> Find the volume of the solid generated by revolving the area enclosed by the curve $a^2 y^2 = x^2 (x-a)(2a-x)$

<u>Solution:</u>

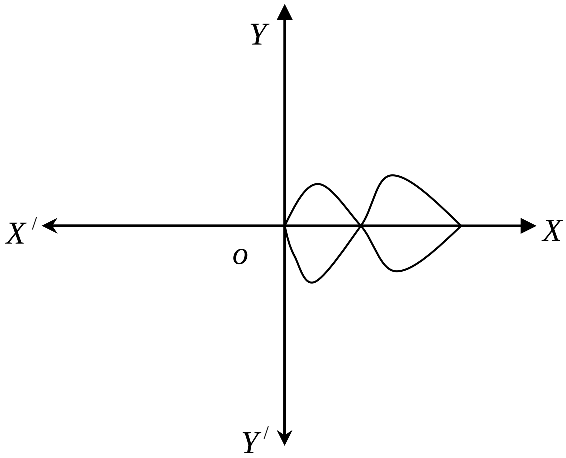
Given that,

$$a^2 y^2 = x^2 (x-a)(2a-x) ....(1)$$

$$y^2 = \frac{x^2 (x-a)(2a-x)}{a^2}$$

$$when, x = 0 \ then, y = 0$$

$$when, \ y = 0, then, x = 0, a, 2a$$



Now,

Therefore, the required volume is,

$$V=\int_0^{2a}\pi y^2dx \quad =\pi\int_0^{2\pi}\frac{x^2(x-a)(2a-x)}{a^2}dx$$

$$=\frac{\pi}{a^2}\left[\int_0^{2a}3ax^3dx-2a^2\int_0^{2a}x^2dx+a\int_0^{2a}x^3dx\right] \quad =\frac{\pi}{2a}\left[3a\left[\frac{x^4}{4}\right]_0^{2a}-\left[\frac{x^5}{5}\right]_0^{2a}2a^2\left[\frac{x^3}{3}\right]_0^{2a}+a\left[\frac{x^4}{4}\right]_0^{2a}\right]$$

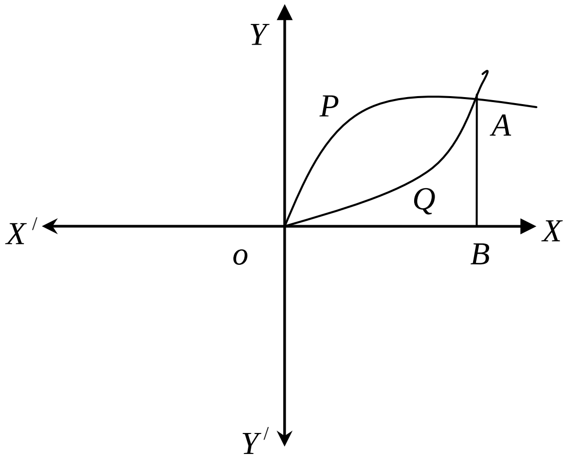
$$=\frac{\pi}{a^2}\left[12a^5-\frac{32a^5}{5}+\frac{16a^5}{3}4a^5\right]$$

$$=\frac{64\pi a^3}{15} \quad (Ans.)$$

<u>Example-10:</u> Find the volume of solid generated by revolving the area included between the curve The curve $y^2=x^3$ and $x^2=y^3$ about X-axis.
<u>Solution:</u>

Given that,

$$y^2=x^3.............(1)$$
$$and,\ x^2=y^3$$
$$\Rightarrow y^2=(x^{\frac{2}{3}})^2............(2)$$
$$when, x=0\ then, y=0$$
$$when,\ y=0, then, x=0$$



Again, from (1) we have,

$$y^2 = x^3$$
$$\Rightarrow y^2 = x^2.x \;\; \Rightarrow y^2 = y^3.x$$
$$\Rightarrow xy = 1$$
$$\therefore x = \frac{1}{y} \text{..........(3)}$$

Putting the value of 'x' in (2) we get,

$$(\frac{1}{y})^2 = y^3$$
$$\Rightarrow y = 1$$

From (3) we get, $x = 1$

So, the line intersects at (0, 0) and (1, 1)

If the required volume is V then,

$$V = rotated\ volume\ of\ OPABO - Rotated\ volume\ of\ OQABO$$
$$\Rightarrow V = V_1 - V_2 \text{...................(4) } [Let]$$

Now,

$$V_1 = \int_0^1 \pi y^2 dx = \pi \int_0^1 x^{\frac{4}{3}} dx$$
$$= \pi \left[\frac{x^{\frac{4}{3}+1}}{\frac{4}{3}+1} \right] = \frac{3\pi}{7} \left[x^{\frac{7}{3}} \right]_0^1 = \frac{3\pi}{7}[1-0] = \frac{3\pi}{7}$$

Again,

$$V_2 = \int_0^1 \pi y^2 dx = \pi \int_0^1 x^3 dx \quad = \pi \left[\frac{x^4}{4} \right]_0^1 = \pi \left[x^4 \right]_0^1 = \frac{\pi}{4}[1-0] = \frac{\pi}{4}$$

Therefore, Total Volume $V = V_1 - V_2 = \frac{3\pi}{7} - \frac{\pi}{4} = \frac{12\pi - 7\pi}{28} = \frac{5\pi}{28}$ $(Ans.)$

<u>Example-11:</u> Find the Volume of the solid generated revolving the curve $y^2(a+x) = x^2(3a-x)$.

<u>Solution:</u>

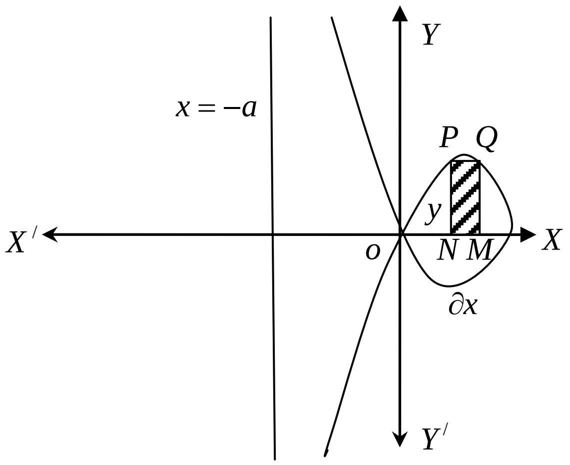
Given that,

$$y^2(a+x)=x^2(3a-x)$$

$$\therefore y^2=\frac{x^2(3a-x)}{(a+x)}\text{............}(1)$$

Equation (1) is symmetrical about X-axis

$$when, x=0\ then, y=0$$
$$when,\ y=0, then, x=3a$$



Equation (1) passing through O(0,0) and A(3a, 0)

Therefore, the required volume is,

$$V=\int_0^{3a}\pi y^2dx\quad=\pi\int_0^{3a}\frac{x^2(3a-x)}{a+x}dx=\pi\int_0^{3a}\frac{3ax^2-x^3}{a+x}dx=$$

$$=\pi\int_0^{3a}\frac{-4x^2(a+x)+4ax(a+x)-4a^2(a+x)+4a^3}{a+x}$$

$$=\pi\int_0^{3a}-4x^2+4ax-4a^2+\frac{4a^3}{a+x}\quad=\pi\left[-4\frac{x^3}{3}+2ax^2-4a^2x+4a^3\log(a+x)\right]_0^{3a}$$

$$=\pi\left[-9a^3+4a^3\log\frac{4a}{a}\right]\quad=\pi a^3\left[-3+4\log 2^2\right]$$

$$=\pi a^3\left[8\log 2-3\right]\quad(Ans.)$$

AREA

- The area bounded by a curve $y = f(x)$ the axis of x and two ordinates x=a, and y=b is given by the definite initial $\int_a^b f(x)dx$
- The area bounded by a curve x=f(y) the axis of Y and the two axis y=c and y=d is given by the definite integral.
- Cartesian Equations:
    - Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
    - Parabola: $y^2 = 4ax$
    - Cissoids: $y^2(a-x) = x^3$
    - Folium: $x^3 + y^3 = 3axy$
    - Asteroid: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^3$
    - Cycloid: $x = a(\theta \mp Sin\theta)$, $y = a(1 \mp Cin\theta)$
    - Cardioids: $r = a(1 \pm Cos\theta)$
    - Conic: $\frac{l}{r} = 1 + eCos\theta$
    - Three leaved raze: $r = aSin3\theta$
    - Four leaved raze: $r = aSin2\theta$
    - Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
    - Catenary's: $y = e\cosh(\frac{x}{c})$
    - Strophoid: $(x^2 + y^2)x = a^2(x^2 - y^2)$
    - Lemniscates : $(x^2 + y^2) = a^2(x^2 - y^2)$

Some properties:

- **For Cartesian:**
    - If y = f(x) then area $A = \int_a^b y.dx = \int_a^b f(x)dx$.
    - If x =f(y) then area $A = \int_a^b x.dy = \int_a^b f(y)dy$
    - Area bounded between two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ is $A = \int_a^b (y_2 - y_1)dx = \int_a^b \{f_2(x) - f_1(x)\}dx$
- **For Polar Form:**
    - If $r = f(\theta)$ then $A = \int_\alpha^\beta \frac{1}{2} r^2 d\theta = \frac{1}{2}\int_\alpha^\beta \{f(\theta)\}^2 d\theta$
    - If $\theta = f(r)$ then $A = \int_\alpha^\beta \frac{1}{2} \theta^2 dr = \frac{1}{2}\int_\alpha^\beta \{f(r)\}^2 dr$

- Area between bounded $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$ is

$$A = \frac{1}{2}\int_{\alpha}^{\beta}(r_2{}^2 - r_1{}^2)d\theta$$

<u>Example-01:</u> Find the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

<u>Solution:</u>

Here Given that,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1..................(1)$$

The ellipse is symmetric about both axes,

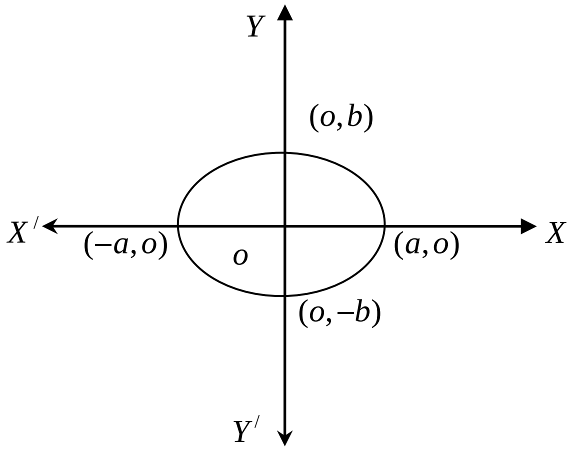
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow y^2 = b^2\left(1-\frac{x^2}{a^2}\right)$$

$$\therefore y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$when, x=0\ then, y=\pm b$$

$$when,\ y=0, then, x=\pm a$$



The curve meets at (0,b) ,(0,-b), (a,0) and (-a,0)
Now, draw a graph roughly,
Hence, the area of the given ellipse,

$$A = 4\int_0^a y dx$$

$$= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} Sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2} Sin^{-1}1 - 0 - 0\right]$$

$$= \frac{4b}{a}\left[\frac{a^2}{2}.\frac{\pi}{2}\right]$$

$$= \pi\, ab \;\; (Ans.)$$

<u>Example-02:</u> Find the area of the parabola $y^2 = 4ax$ cut off by its lotus rectum.

<u>Solution:</u>

Given that,

$$y^2 = 4ax ....(1)$$

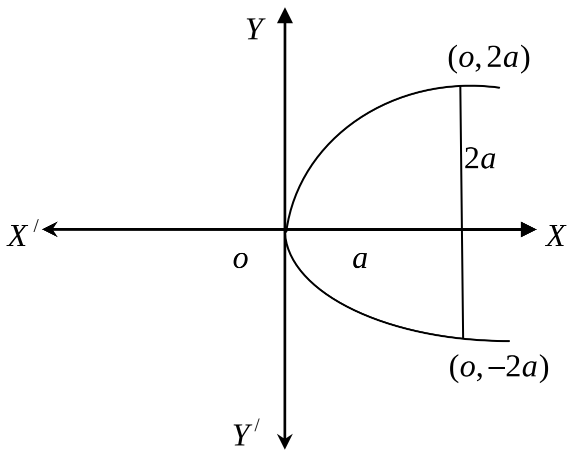
Since, the distance from vertex to lotus rectum is x = a

$$\therefore y^2 = 4a^2$$

$$or,\; y = \pm 2a$$

$$when, y = 0\; then, x = 0$$

$$when,\; x = 0, then, y = \pm 2a$$



Now, we draw the graph roughly,

Therefore, the required area of the parabola is,

$$A = 2\int_0^a y.dx$$
$$= 2\int_0^a \sqrt{4ax}dx$$
$$= 4\sqrt{a}\int_0^a x^{\frac{1}{2}}.dx$$
$$= 4\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a$$
$$= \frac{8a^2}{3} \quad (Ans.)$$

<u>Example-03:</u> Show that the area enclosed between the parabola $y^2 = 4a(x+a),\ y^2 = -4a(x-a),$ **is** $\frac{16a^2}{3}$

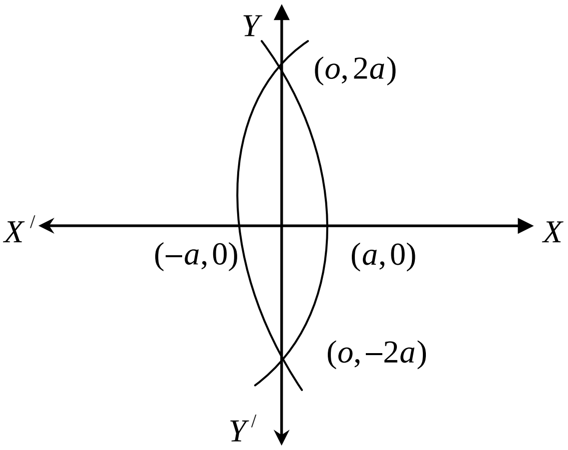
<u>Solution:</u>

Given that,

$$y^2 = 4a(x+a)\text{...........}(1)$$
$$y^2 = -4a(x-a)\text{..........}(2)$$

Now, the vertex of (1) is (-a,0) and (2) is (a,0)
From (1) and (2) we have,

$$4a(x+a) = -4a(x-a)$$
$$\therefore x = 0$$
$$when, x = 0\ then, y = \pm 2a$$
$$when,\ y = 0, then, x = \pm a$$



Therefore, the intersecting point (0,2a) and (0,-2a)
Now, we draw the curve roughly.
From (1) we get,

$$x = \frac{y^2}{4a} - a = x_1$$

From (2) we get,

$$x = a - \frac{y^2}{4a} = x_2$$

Therefore, the area is,

$$A = \int_0^{2a} (x_2 - x_1)dy$$

$$= \int_0^{2a} (a - \frac{y^2}{4a} - \frac{y^2}{4a} + a)dy$$

$$= \int_0^{2a} (2a - \frac{2y^2}{4a})dy$$

$$= \int_0^{2a} (2a - \frac{y^2}{2a})dy$$

$$= \frac{1}{2a}\int_0^{2a} (4a^2 - y^2)dy$$

$$= \frac{1}{2a}\left[4a^2 y - \frac{y^3}{3}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[4a^2 2a - \frac{8a^3}{3}\right]$$

$$= \frac{16a^3}{6a}$$

i.e, the total area,= $2.\frac{16a^3}{6a} = \frac{16a^2}{3}$ (Showed.)

<u>Example-04:</u> Find the whole area of the curve $a^2 y^2 = x^3(2a - x)$**.**

<u>Solution:</u>

Given that,

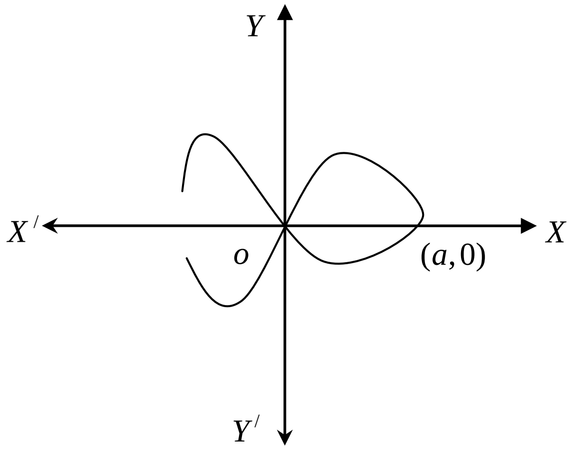
$$a^2 y^2 = x^3(2a - x)\text{...............(1)}$$

Equation (1) is symmetrical about x-axis

We have, $y = \frac{x^{\frac{3}{2}}\sqrt{2a - x}}{a}$

$when, x = 0 \; then, y = 0$

$when, \; y = 0, then, x = 0, 2a$



The curves meets at (0,0) and (2a,0)

Therefore, the area is,

$$A = \int_0^{2a} y dx$$

$$= 2\int_0^{2a} \frac{x^{\frac{3}{2}}\sqrt{2a-x}}{a} dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(2aSin^2\theta)^{\frac{3}{2}}\sqrt{2a-2a\sin^2\theta}}{a} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(2aSin^2\theta)^{\frac{3}{2}}\sqrt{2a}.Cos\theta.4aSin\theta.Cos\theta}{a} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{3}{2}} 4aSin^3\theta\sqrt{2a}.Cos\theta.4aSin\theta.Cos\theta}{a} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{(2a)^{\frac{3}{2}+\frac{1}{2}} 4aSin^4\theta.4aSin^4\theta.Cos^2\theta}{a} d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{4a^{3.}4}{a} Sin^4\theta.Cos^2\theta.d\theta$$

$$= 32a^2\int_0^{\frac{\pi}{2}} Sin^4\theta.Cos^2\theta.d\theta$$

By using Gamma-beta function,

$$= 32a^2 \frac{\Gamma\left(\frac{4+1}{2}\right).\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)}$$

$$= 32a^2 \frac{\Gamma\left(\frac{5}{2}\right).\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$$

$$= 16a^2 \frac{\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{\Gamma(3+1)}$$

$$= 16a^2 . \frac{3\pi a}{8 \times 3!}$$

$$= \pi a^2 \ \ (Ans.)$$

<u>Example-05:</u> Find the area of the loop of the curve $y^2 = x^2(x+a)$.

<u>Solution:</u>

Given that,

$$y^2 = x^2(x+a) \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

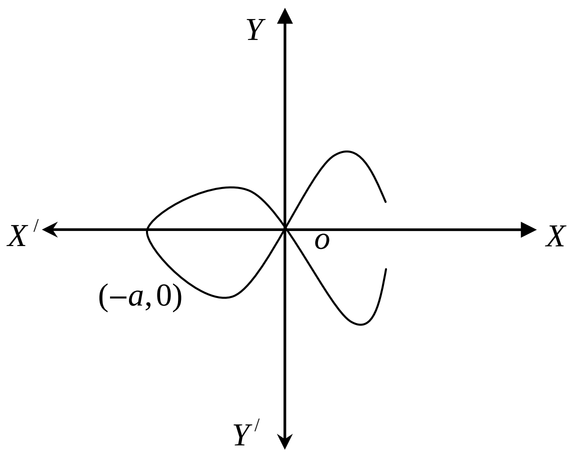
The curve (1) is symmetric about x-axis,

$$when, x = 0 \ then, y = 0$$

$$when, \ y = 0, then, x = 0, -a$$

Now, draw the graph roughly,

Therefore, the required area is,



$$A=\int_{-a}^{0} y.dx$$
$$=2\int_{-a}^{0} x.\sqrt{x+a}\,.dx$$
$$=2\int_{-a}^{0} (a+x-a).\sqrt{x+a}\,.dx$$
$$=2\int_{-a}^{0}\left\{(a+x)^{\frac{3}{2}}-a.\sqrt{x+a}\right\}dx$$
$$=2\left[\frac{2}{5}(a+x)^{\frac{5}{2}}-\frac{2}{3}a.(a+x)^{\frac{3}{2}}\right]$$
$$=2\int_{-a}^{0}\left\{(a+x)^{\frac{3}{2}}-a.\sqrt{x+a}\right\}dx$$
$$=2\left[\frac{2}{5}a^{\frac{5}{2}}-\frac{2}{3}a.a^{\frac{3}{2}}\right]$$
$$=2\left[\frac{2}{5}a^{\frac{5}{2}}-\frac{2}{3}a^{\frac{5}{2}}\right]$$
$$=2a^{\frac{5}{2}}\left(\frac{2}{5}-\frac{2}{3}\right)$$
$$=2a^{\frac{5}{2}}\left(\frac{6-10}{15}\right)$$
$$=-\frac{8a^{\frac{5}{2}}}{15}$$
$$=\frac{8a^{\frac{5}{2}}}{15}\quad (Ans.)\ [Area\ is\ always\ (+)ve]$$

Example-06: **show that the area between the parabola $y^2=4x$ and the straight line $y=2x-4$ is 9.**

Solution:

Given that,

$y^2=4x$................(1)

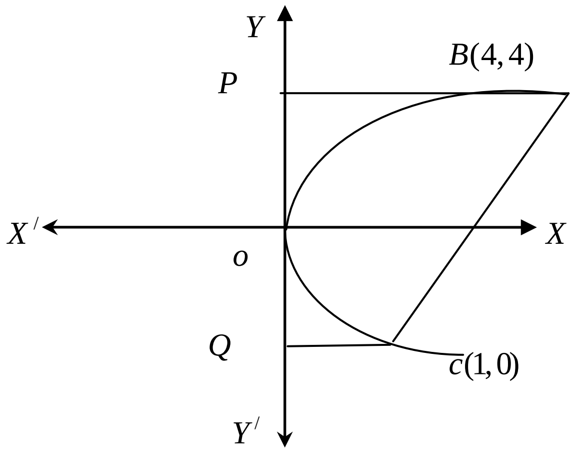
And, $y=2x-4$.............(2)

From (1) and (2) we have,

$$(2x-4)^2=4x$$
$$\Rightarrow 4x^2+16-16x-4x=0$$
$$\Rightarrow x^2-5x+4=0$$
$$\Rightarrow x^2-4x-x+4=0$$
$$\Rightarrow x(x-4)-(x-4)=0$$
$$\Rightarrow (x-4)(x-1)=0$$
$$\therefore x=4,1$$

$when, x = 4\ then, y = 4$

$when,\ x = 1, then, y = -2$



The perpendicular BP and CQ are drawn on the directory.

Trapezium BPQC,

$$= \frac{1}{2}(BP + CQ).PQ$$

$$= (4+1)(PO + OQ)$$

$$= \frac{1}{2}.5.(4+2) = 15$$

'o' is the vertex of the parabola,

Area between the parabola and the straight line,

= Trapezium-(OPB+OQC)

$$= 15 - \left[\int_{-2}^{0} xdy + \int_{0}^{4} xdy\right]$$

$$= 15 - \int_{-2}^{4} xdy = 15 - \int_{-2}^{4} \frac{y^2}{4} dy$$

$$= 15 - \frac{1}{4}.\frac{1}{3}.\left[y^3\right]_{-2}^{4}$$

$$= 15 - \frac{1}{12}\left[4^3 + 2^3\right]$$

$$= 15 - \frac{72}{12}$$

$$= 9 \quad (Showed.)$$

<u>Example-07:</u> Find the area bounded by the cardioids $r = a(1 - Cos\theta)$ **.**

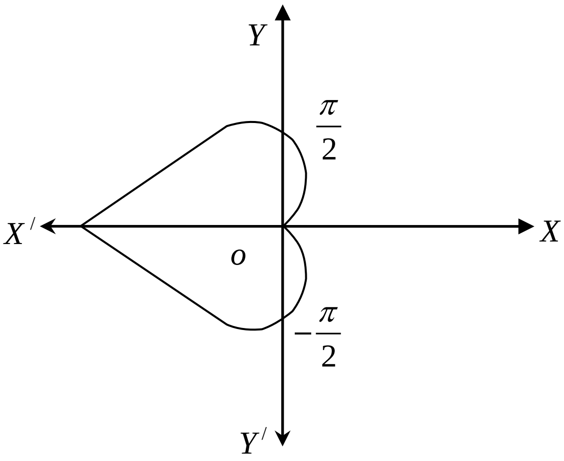
<u>Solution:</u>

Given that,

$$r = a(1 - Cos\theta)...............(1)$$

$$= a(1 - \cos 2.\frac{\theta}{2}) = 2aSin^2 \frac{\theta}{2}$$

θ	0	$\pi/2$	θ	$-\pi/2$
r	0	2a	0	-2a



Therefore, the required area is,

$$A = 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\pi r^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{2} 4a^2 Sin^4 \frac{\theta}{2}.d\theta$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} Sin^4 \frac{\theta}{2}.Cos^0 \frac{\theta}{2} d\theta$$

By using Gamma-beta function,

$$= 4a^2 \frac{\Gamma\left(\frac{4+1}{2}\right).\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{4+0+2}{2}\right)}$$

$$= 4a^2 \frac{\Gamma\left(\frac{5}{2}\right).\Gamma\left(\frac{1}{2}\right)}{2\Gamma 3}$$

$$= 4a^2 \frac{\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right).}{2.2!}$$

$$= \frac{4a^2 3\pi}{16}\ (Ans.)$$

$$= \frac{3a^2\pi}{4}\ (Ans.)$$

<u>Example-08:</u> Find the whole area of the loops of the curve (i). $a^2y^2 = x^2\left(a^2 - x^2\right)$**. (ii).** $y^2 = x^2(4-x)$ **(iii).** $r = a\cos 2\theta$ **(iv).** $r^2 = a^2\cos 2\theta$

<u>(i) Solution:</u>

Given that,

$$a^2y^2 = x^2\left(a^2 - x^2\right) \ldots\ldots\ldots\ldots\ldots(1)$$

Equation (1) is symmetrical about x-axis

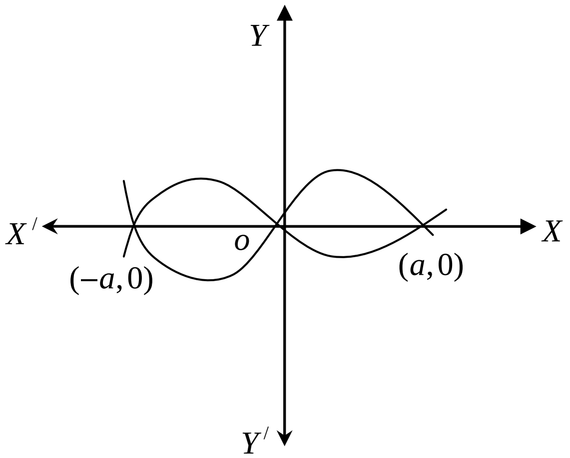
We have, $y = \pm \dfrac{x\sqrt{a^2 - x^2}}{a}$

$when, x = 0\ then, y = 0$

$when,\ y = 0, then, x = 0, \pm a$

The curves meets at (0,0), (a,0) and (-a,0)

Therefore, the area is,



$$A = \int_0^a ydx$$

$$= 4\int_0^a \frac{x\sqrt{a^2 - x^2}}{a}dx$$

$Let, x = a\sin\theta$

$dx = aCos.d\theta$

$when, x = 0,\ \theta = 0$

$x = a,\ \theta = \dfrac{\pi}{2}$

$$= \frac{4}{a}\int_0^{\frac{\pi}{2}} aSin\theta\sqrt{a^2(1 - Sin^2\theta)}.aCos\theta d\theta$$

$$= \frac{4}{a}\int_0^{\frac{\pi}{2}} aSin\theta.aCos\theta.aCos\theta\, d\theta$$

$$= \frac{4}{a}\int_0^{\frac{\pi}{2}} a^3 Sin\theta.Cos^2\theta.d\theta$$

$$= \frac{4a^3}{a}\int_0^{\frac{\pi}{2}} Sin\theta.Cos^2\theta.d\theta$$

By using Gamma-beta function,

$$= 4a^2 \frac{\Gamma\left(\frac{1+1}{2}\right).\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{1+2+2}{2}\right)}$$

$$= 4a^2 \frac{\Gamma(1)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(\frac{5}{2}\right)}$$

$$= 4a^2 \frac{.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{.2.\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{4a^2}{3}\ (Ans.)$$

(ii) Solution:

Given that,

$$y^2 = x^2(4-x)................(i)$$

The curve (1) is symmetric about x-axis,

$$when, x = 0\ then, y = 0$$

$$when,\ y = 0, then, x = 0, 4$$

Now, draw the graph roughly,

Therefore, the required area is,

$$A = \int_0^a y.dx$$

$$= 2\int_0^4 x.\sqrt{4-x}\,.dx$$

$$= 2\int_0^{\frac{\pi}{2}} 4.Sin^2\theta.2Cos.8Cos\theta.Sin\theta\, d\theta$$

$$= 64\times 2\int_0^{\frac{\pi}{2}} Sin^3\theta.Cos^2\theta.d\theta$$

$$= 128\int_0^{\frac{\pi}{2}} Sin^3\theta.Cos^2\theta.d\theta$$

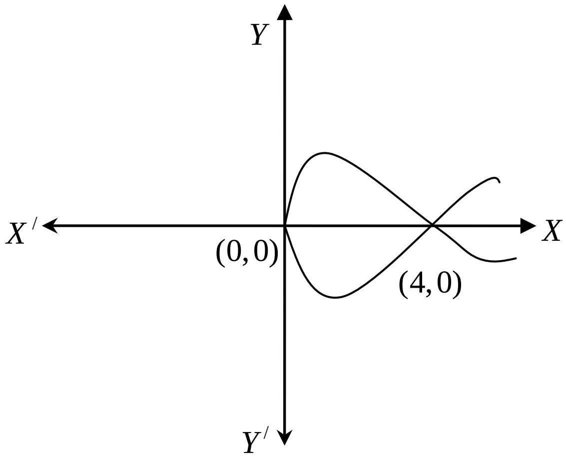
$$Let, x = 4\sin^2\theta$$

$$dx = 42Sin\vartheta.Cos\theta.d\theta$$

$$when, x = 0,\ \theta = 0$$

$$x = 4,\ \theta = \frac{\pi}{2}$$

By using Gamma-beta function,



$$= 128 \frac{\Gamma\left(\frac{3+1}{2}\right) . \Gamma\left(\frac{2+1}{2}\right)}{2\,\Gamma\left(\frac{3+2+2}{2}\right)}$$

$$= 128 \frac{\Gamma 2\, \Gamma\left(\frac{3}{2}\right)}{2\,\Gamma\left(\frac{7}{2}\right)}$$

$$= 64 \frac{.1!\, \Gamma\left(\frac{3}{2}\right)}{\frac{7}{2} . \Gamma\left(\frac{5}{2}\right)}$$

$$= 64 \frac{\Gamma\left(\frac{3}{2}\right)}{\frac{7}{2} . \frac{5}{2}\, \Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{128}{35} \ (Ans.)$$

<u>(iii). Solution:</u>

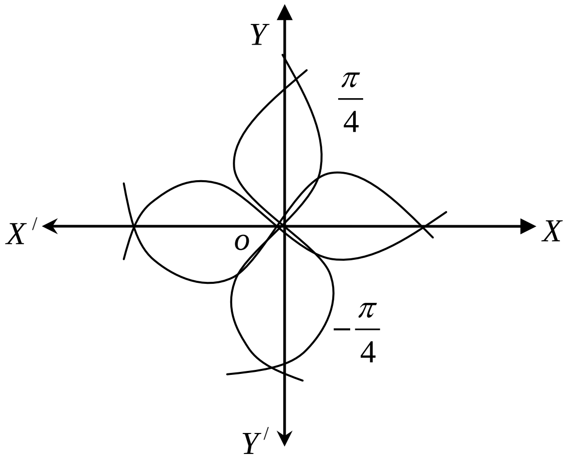
Given that,

$$r = aCos2\theta ....(1)$$

i.e, 2 is even so 2.2 = 4 loop of the curve (1)

$when, r = 0, \ \ then \ aCos2\theta = 0$

$$\therefore \theta = \pm \frac{\pi}{4}$$



Therefore, the required area is,

$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta$$

$$= 2\int_0^{\frac{\pi}{4}} a^2 Cos^2 2\theta \, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} Cos^2 2\theta \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{4}} 2Cos^2 2\theta \, d\theta$$

$$= \frac{a^2}{2} \int_0^{\frac{\pi}{4}} (1 + Cos4\theta) \, d\theta$$

$$= \frac{a^2}{2} \left[\theta + \frac{Sin4\theta}{4} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{a^2}{2} \left[\frac{\pi}{4} + \frac{Sin\pi}{4} - 0 \right]$$

$$= \frac{a^2}{2} . \frac{\pi}{4}$$

$$= \frac{\pi a^2}{8}$$

Total length $= 4a$

$$= 4.\frac{\pi a^2}{8} = \frac{\pi a^2}{2} \quad (Ans.)$$

(iv). Solution:

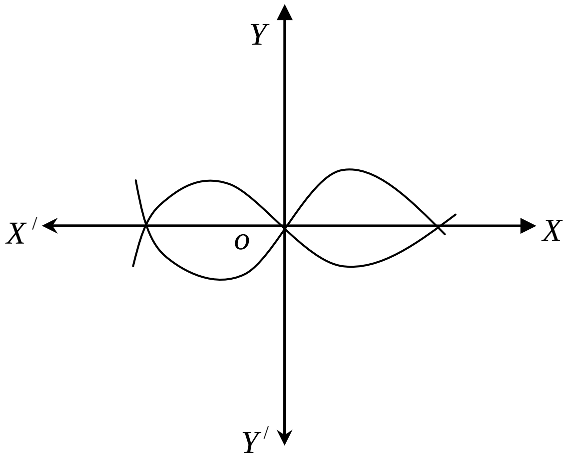
Given that,

$$r^2 = a^2 Cos2\theta ....(1)$$

The curve (1) is symmetrical about both axis,

$when, r = 0, \ then, \ aCos2\theta = 0$

$$\therefore \theta = \pm \frac{\pi}{4}$$



Therefore, the required area is,

$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} a^2 Cos^2 2\theta \, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} Cos^2 2\theta \, d\theta$$

$$= a^2 \left[\frac{Sin\, 2\theta}{2} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2} \left[Sin 2.\frac{\pi}{4} - 0 \right]$$

$$= \frac{a^2}{2}$$

Total area, $= 2.\frac{a^2}{2} = a^2\ Sq.Unit\ (Ans.)$

<u>Example-09:</u> Find the area of the region bounded by $y^2 = x(2a - x)$ **and** $y^2 = ax$**.**

<u>Solution:</u>

Given that,

$$y^2 = x(2a - x) \text{..............(1)}$$

And, $$y^2 = ax \text{................(2)}$$

From (1) we have,

$$y^2 = x(2a - x)$$

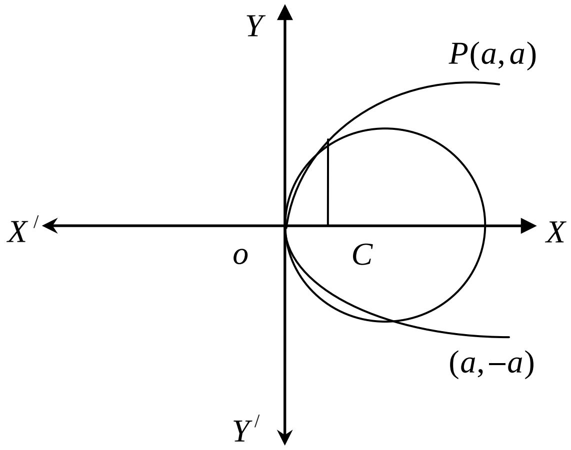
$$\Rightarrow y^2 - 2ax + x^2 = 0$$

$$\Rightarrow (x - a)^2 + (y - 0)^2 = a^2 \text{.............(3)}$$

Equation (2) represent a circle whose centre is (a,0) and radius 'a'

$$when, x = 0\, then, y = 0$$

$$when,\ x = a, then, y = \pm a$$



Hence, they intersect at p(a, a) and Q(a, -a)
Let us consider the area in the first quadrant

Area OPC = $\frac{1}{4}(area\, of\ the\, circle) = \frac{1}{4}\pi a^2$

Area bounded y the quadrant $y^2 = ax$ and the X-axis is OPC

$= \int_0^a ydx = \int_0^a \sqrt{ax}dx = \sqrt{a}\int_0^a x^{\frac{1}{2}}dx = \sqrt{a}.\frac{2}{3}.a^{\frac{3}{2}} = \frac{2}{3}.a^2$

Area between the circle and the parabola,
Since, two circles intersect at $Q(a, -a)$

$\therefore$ Area $= 2\left[\frac{1}{4}\pi a^2 - \frac{2}{2}a^2\right] = a^2\left[\frac{1}{2}\pi - \frac{4}{3}a^2\right]$ $(Ans.)$

<u>Example-10:</u> Find the whole area of the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

<u>Solution</u>:

Given that,

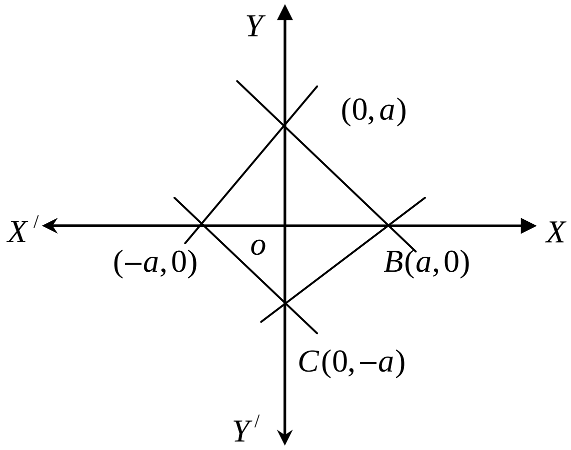
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \text{...................}(1)$$

$$\Rightarrow y^{\frac{2}{3}} = a^{\frac{2}{3}} - x^{\frac{2}{3}}$$

$$\Rightarrow y^2 = \left(a^{\frac{2}{3}} - x^{\frac{2}{3}}\right)^3 \text{................}(2)$$

$when, x = 0\ then, y = \pm a$

$when,\ y = 0, then, x = \pm a$



Points are (a,0), (-a,0) (0,a) and (0,-a)
Now, we draw curve roughly,

Therefore, the required volume is, $A=\int_0^a ydx$..........(2)

Put x= $aCos^3\theta;\ y=aSin^3\theta d\theta,\ and\ \ dx=-3aCos^2\theta.Sin\theta.d\theta$

$when,\ \ x=0,\ then, \theta=0$

$when,\ x=0,\ then,\ \theta=\frac{\pi}{2}$

$$\therefore\ A=4\int_{\frac{\pi}{2}}^{0} aSin^3\theta.(-3aCos^2\theta.Sin\theta)d\theta$$

$$=-4.3\int_{\frac{\pi}{2}}^{0} a^{2\cdot}Cos^2\theta.Sin^4\theta d\theta$$

$$=12a^2\int_{\frac{\pi}{2}}^{0} Cos^2\theta.Sin^4\theta d\theta$$

By using Gamma-Beta function,

$$=12a^2\frac{\Gamma\left(\frac{4+1}{2}\right).\Gamma\left(\frac{2+1}{2}\right)}{2\Gamma\left(\frac{4+2+2}{2}\right)}$$

$$=12a^2\frac{\Gamma\left(\frac{5}{2}\right).\Gamma\left(\frac{3}{2}\right)}{2\Gamma 4}$$

$$=12a^2\frac{\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{2.6}$$

$$=\frac{3\pi a^2}{8}\ (Ans.)$$

<u>Example-11:</u> Find the area of the region lying above x-axis and included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

<u>Solution:</u>

Given that,

$$x^2 + y^2 = 2ax............(1)$$

$$\therefore y = \sqrt{2ax - x^2}$$

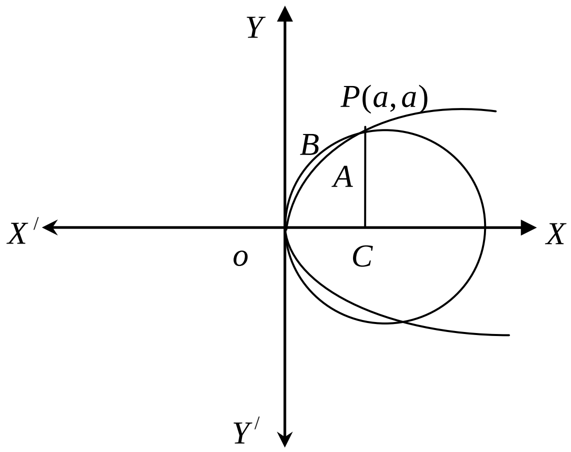
And, $$y^2 = ax................(2)$$

From (1) and (2) we have,

$$2ax - x^2 = ax$$

$$\Rightarrow x(a - x) = 0$$

$$\therefore x = 0, a \ \ and \ \ y = 0, \pm a$$



Hence, two curves intersect at (0, 0), (a, a)and (a, -a)

Area OBPCO = $\int_0^a \sqrt{ax}dx$

Therefore, the required area, $= \int_0^a \sqrt{2ax - x^2}dx - \int_0^a \sqrt{ax}dx$

Now, $\int_0^a \sqrt{2ax - x^2}dx = \int_0^a \sqrt{a^2 - (a-x)^2}dx$

Put, $a - x = aSin\theta$ then, $\int_0^a \sqrt{2ax - x^2}dx = \int_{\frac{\pi}{2}}^0 (a\cos\theta).(-aCos\theta)\,d\theta$

$$= \int_{\frac{\pi}{2}}^0 a^2 \cos^2\theta\, d\theta = a^2 . \frac{1}{2} . \frac{\pi}{2} = \frac{\pi a^2}{4}$$

Also, $\int_0^a \sqrt{ax}dx = \left|\sqrt{a}\,\frac{2}{3}.x^{\frac{3}{2}}\right|_0^a = \frac{2a^2}{3}$

Therefore, the required area = $a^2\left(\frac{\pi}{4} - \frac{2}{3}\right)$ $(Ans.)$

<u>Example-12:</u> Find the area of the surface revolution formed by revolving the curve $r = 2aCos\theta$ about the initial axis.

<u>Solution:</u>

Given that,

$$r = 2aCos\theta...............(1)$$

$$\frac{dr}{d\theta} = -2aSin\theta$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = 4a^2Sin^2\theta$$

$$\therefore r^2 + \left(\frac{dr}{d\theta}\right)^2 = 4a^2Cos^2\theta + 4a^2Sin^2\theta = 4a^2$$

θ	0	$\pi/2$	π	$-\pi/2$
r	2a	0	-2a	0

Therefore, the required area is,

$$A = \int_0^{\frac{\pi}{2}} 2\pi r\, Sin\theta \sqrt{r^2 + (\frac{dr}{d\theta})^2}\, d\theta$$

$$= 2\pi.2a\int_0^{\frac{\pi}{2}} 2aCos\theta.Sin\theta\, d\theta$$

$$= 4a\pi.2a\int_0^{\frac{\pi}{2}} Cos\theta.Sin\theta\, d\theta$$

By using Gamma-Beta function,

$$= 8a^2\pi \frac{\Gamma\frac{1+1}{2}.\Gamma\frac{1+1}{2}}{2\Gamma\frac{1+1+2}{2}}$$

$$= 8a^2\pi \frac{1.1}{2\Gamma 2}$$

$$= 4\pi a^2\ (Ans.)$$

<u>Example-14:</u> Find the whole area of the loops of the curve $r^2 = a^2\cos 2\theta$

<u>Solution:</u>

Given that,

$$r^2 = a^2Cos2\theta...............(1)$$

The curve (1) is symmetrical about both axis,

$when, r = 0,\ \ then,\ aCos2\theta = 0$

$$\therefore \theta = \pm\frac{\pi}{4}$$

Therefore, the required area is,

$$A = 2\int_0^{\frac{\pi}{4}} \frac{1}{2} r^2 d\theta$$

$$= \int_0^{\frac{\pi}{4}} a^2 Cos^2 2\theta \, d\theta$$

$$= a^2 \int_0^{\frac{\pi}{4}} Cos^2 2\theta \, d\theta$$

$$= a^2 \left[\frac{Sin 2\theta}{2} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{a^2}{2} \left[Sin 2.\frac{\pi}{4} - 0 \right]$$

$$= \frac{a^2}{2} \ (Ans.)$$

LENGTH

Rectification Length of plane curve:

- **Cartesian equation:**

$y = f(x)$ the length of the arc of the curve $y = f(x)$ included between two points whose abscissa a and b is $\frac{ds}{dx} = \int_a^b \sqrt{1+(\frac{dy}{dx})^2}.dx$

- **Expression for length of arc:**

$$\frac{ds}{dy} = \sqrt{1+(\frac{dy}{dx})^2} dx$$

$$\frac{ds}{dt} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}.d\theta$$

$$\frac{ds}{dt} = \sqrt{r^2 + (\frac{dr}{d\theta})^2}.d\theta$$

- **Cartesian equation:**

$$x = f(y)$$

$$\frac{ds}{dy} = \int_a^b \sqrt{1+(\frac{dx}{dy})^2}.dy$$

- **Parametric equation:**

$x = f(t)$, $y = \varphi(t)$. The length of the arc of the curves $x = f(t)$, $y = \varphi(t)$ included between two points whose parametric values are α, β is

$$S = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} d\theta$$

- **Polar equation:**

$$S = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

<u>Example-01:</u> For the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ show that the centre length of the curve is 6a.

<u>Solution:</u>

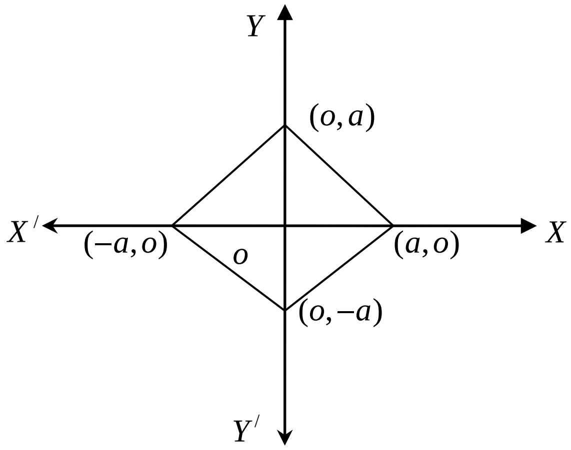
Given that,

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \text{..................}(1)$$

The curve (1) is symmetrical about both axis and curve (1) is symmetrical also x = y line.

$$when, x = 0\ then, y = \pm a$$

$$when,\ y = 0, then, x = \pm a$$



The curves meets the axis at (a,0), (-a, 0), (0,a) and (0,-a)
Now, we draw the curve roughly:
Now, differentiating (1) w. r. to ‘x’ we have,

$$\frac{2}{3}x^{\frac{2}{3}-1}+\frac{2}{3}y^{\frac{2}{3}-1}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx}=(-\frac{x}{y})^{-\frac{1}{3}}$$

Now, the required length is,

$$S=4\int_0^a\sqrt{1+(\frac{dy}{dx})^2}.dx$$

$$=4\int_0^a\sqrt{1+\left\{(-\frac{x}{y})^{-\frac{1}{3}}\right\}^2}.dx$$

$$=4\int_0^a\sqrt{1+(\frac{x}{y})^{-\frac{2}{3}}}.dx$$

$$=4\int_0^a\sqrt{1+(\frac{y}{x})^{\frac{2}{3}}}.dx$$

$$=4\int_0^a\sqrt{\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}}.dx$$

$$=4\int_0^a\sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}}.dx$$

$$=4\int_0^a(a^{\frac{2}{3}})^{\frac{1}{2}}.(x^{-\frac{2}{3}})^{\frac{1}{2}}.dx$$

$$= 4a^{\frac{1}{3}}\int_0^a x^{-\frac{1}{3}}.dx$$

$$= 4a^{\frac{1}{3}}\left[\frac{x^{-\frac{1}{3}+1}.}{-\frac{1}{3}+1}\right]_0^a$$

$$= 4a^{\frac{1}{3}}\left[\frac{x^{\frac{2}{3}}.}{\frac{2}{3}}\right]_0^a = \frac{4a^{\frac{1}{3}}.a^{\frac{2}{3}}}{\frac{2}{3}}$$

$$= 6a^{\frac{1+2}{3}} = 6a \quad (Ans.)$$

<u>Example-02:</u> Find the length of the loop of the curve $3ay^2 = x(x-a)^2$ **.**

<u>Solution:</u>

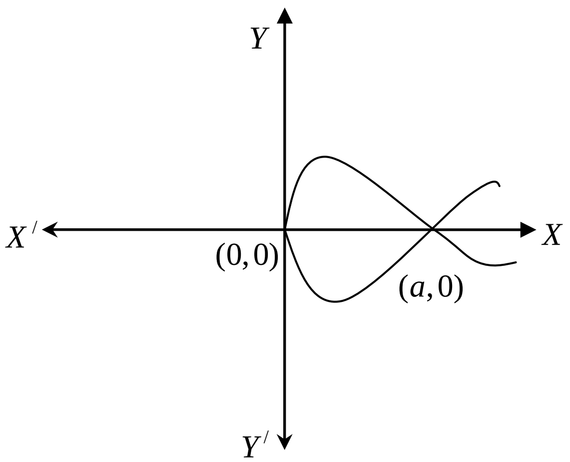
Given that,

$$3ay^2 = x(x-a)^2 ....(1)$$

Here, even power of 'y' is present. So (1) is symmetrical about x-axis.

$$when, x = 0 \ then, y = 0$$

$$when, \ y = 0, then, x = 0, a$$



Again, when $x < 0$ then y is imaginary. No part of the curve for $x < 0$.

Now, we draw the curve roughly.

Differencing (1) w. r. to 'x' we have,

$$6ay.\frac{dy}{dx} = 2x(x-a)+(x-a)^2 = (x-a)(2x+x-a)$$

$$\therefore \frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}$$

$$\therefore \left(\frac{dy}{dx}\right)^2 = \frac{(x-a)^2(3x-a)^2}{36a^2y^2}$$

$$= \frac{(3x-a)^2}{12ax}$$

$$= \frac{9x^2-6ax+a^2}{12ax}$$

Hence, the required length is,

$$S = 2\int_0^a \sqrt{1+(\frac{dy}{dx})^2}.dx$$

$$= 2\int_0^a \sqrt{1+\frac{9x^2-6ax+a^2}{12ax}}.dx$$

$$= 2\int_0^a \sqrt{\frac{12ax+9x^2+a^2-6ax}{12ax}}.dx$$

$$= 2\int_0^a \frac{\sqrt{(3x+a)^2}}{\sqrt{12a}.\sqrt{x}}dx$$

$$= \frac{2}{\sqrt{12a}}\int_0^a \frac{3x+a}{\sqrt{x}}dx$$

$$= \frac{1}{\sqrt{3a}}\int_0^a \left(3x.x^{\frac{-1}{2}} + ax^{-\frac{1}{2}}\right)$$

$$= \frac{1}{\sqrt{3a}} \int_0^a \left(3x^{\frac{1}{2}} + ax^{-\frac{1}{2}} \right) dx$$

$$= \frac{1}{\sqrt{3a}} \left[3 \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^a + a \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^a \right]$$

$$= \frac{1}{\sqrt{3a}} \left[\frac{3a^{\frac{3}{2}}}{\frac{3}{2}} + \frac{a.a^{\frac{3}{2}}}{\frac{1}{2}} \right]$$

$$= \frac{1}{\sqrt{3a}} \left(2a^{\frac{3}{2}} + 2a^{\frac{3}{2}} \right)$$

$$= \frac{4a^{\frac{3}{2}}}{\sqrt{3}a^{\frac{1}{2}}}$$

$$= \frac{4a}{\sqrt{3}} \ (Ans.)$$

<u>Example-03:</u> Find the length of the curve $8a^2y^2 = x^2(a^2 - x^2)$**.**

<u>Solution:</u>

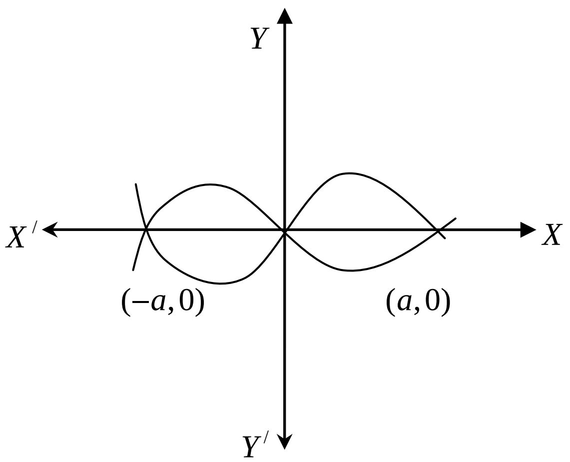
Given that,

$$8a^2y^2 = x^2(a^2 - x^2) \ldots\ldots\ldots\ldots(1)$$

Equation (1) is symmetrical about x-axis.

$$when, x = 0 \ then, y = 0$$

$$when, \ y = 0, then, x = 0, \pm a$$



Again, when $x < 0$ then y is imaginary. No part of the curve for $x < 0$.
Now, we draw the curve roughly.
Differencing (1) w. r. to ‘x’ we have,

$$2x^2(a^2-x^2)=8a^2y^2$$
$$\Rightarrow a^2x^2-x^4=8a^2y^2$$
$$\therefore a^2 2x-4x^3=8a^2.2y.\frac{dy}{dx}$$
$$\therefore \frac{dy}{dx}=\frac{a^2x-2x^3}{8a^2y}$$
$$\Rightarrow \left(\frac{dy}{dx}\right)^2=\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}$$
$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\frac{(a^2-2x^2)^2}{8a^2(a^2-x^2)}=\frac{8a^4-8a^2x^2+a^4 4x^4-4a^2x^2}{8a^2(a^2-x^2)}$$
$$=\frac{9a^4-12a^2x^2+4x^4}{8a^2(a^2-x^2)}$$
$$=\frac{(3a^2-2x^2)^2}{8a^2(a^2-x^2)}$$
$$\therefore \sqrt{1+\left(\frac{dy}{dx}\right)^2}=\frac{(3a^2-2x^2)}{2\sqrt{2}a(a^2-x^2)^{\frac{1}{2}}}$$

Hence, the required length is,

$$S=4\int_0^a\sqrt{1+(\frac{dy}{dx})^2}.dx$$
$$=4\int_0^a\frac{(3a^2-2x^2)}{2\sqrt{2}a(a^2-x^2)^{\frac{1}{2}}}.dx$$
$$=\frac{4}{2\sqrt{2}a}\int_0^a\frac{2(a^2-x^2)+x^2}{\sqrt{(a^2-x^2)}}.dx$$
$$=\frac{2\sqrt{2}}{\sqrt{2}a}\int_0^a\left[2(a^2-x^2)+\frac{x^2}{\sqrt{(a^2-x^2)}}\right].dx$$
$$=\frac{2\sqrt{2}}{a}\int_0^a\left[\frac{x\sqrt{(a^2-x^2)}}{2}+\frac{a^2}{2}Sin^{-1}\frac{x}{a}\right]_0^a+\sqrt{2}a\left[Sin^{-1}\frac{x}{a}\right]_0^a$$
$$=\frac{2\sqrt{2}}{a}\int_0^a\left[\frac{a^2}{2}Sin^{-1}1\right]+\sqrt{2}a\left[Sin^{-1}1\right]$$
$$=\frac{2\sqrt{2}}{a}\int_0^a\left[\frac{a^2}{2}.\frac{\pi}{2}\right]+\sqrt{2}a.\frac{\pi}{2}$$
$$=\frac{a\pi}{\sqrt{2}}+\frac{a\pi}{\sqrt{2}}$$
$$=\sqrt{2}a\pi$$
$$=a\pi\sqrt{2}\quad (Ans.)$$

<u>Example-04:</u> Find the length of the arc $y^2 = 4ax$. Extended from the vertex to one extremity of lotus rectum.

<u>Solution:</u>

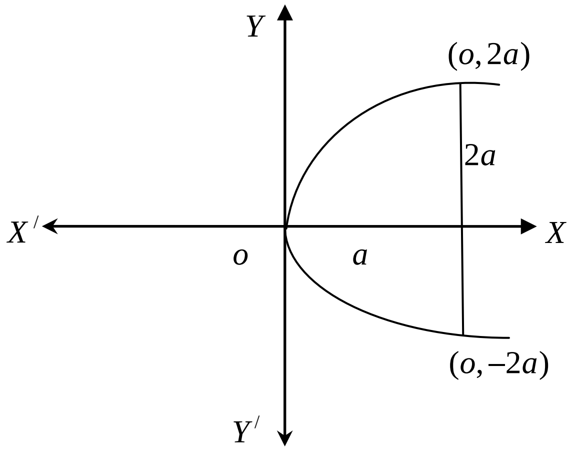
Given that,

$$y^2 = 4ax ....(1)$$

The curve (1) is present even power of y. So (1) is symmetrical about x-axis.
The vertex of the parabola is (0, 0)
The lotus rectum x = a

$$\therefore y^2 = 4a^2$$

$$\therefore y = \pm 2a$$



Therefore, the curves meets on x = a at (a. 2a) and (a, -2a)
Now, draw the graph roughly:
Differentiating (1) w. r. to 'x' we get,

$$2y = 4a\frac{dx}{dy}$$

$$\Rightarrow \left(\frac{dx}{dy}\right) = \frac{y}{2a}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{4a^2}$$

$$\therefore \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + \frac{y^2}{4a^2}} = \frac{\sqrt{4a^2 + y^2}}{2a}$$

Therefore, the required length is,

$$S = \int_0^{2a} \sqrt{1 + (\frac{dx}{dy})^2}\,.dy$$

$$= \int_0^{2a} \frac{\sqrt{4a^2 + y^2}}{2a} dy$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{4a^2 + y^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{\sqrt{4a^2 + y^2}}{2} + \frac{4a^2}{2}\log(y + \sqrt{4a^2 + y^2})\right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a\sqrt{4a^2 + 4a^2}}{2} + \frac{4a^2}{2}\log(2a + \sqrt{4a^2 + 4a^2})\right]$$

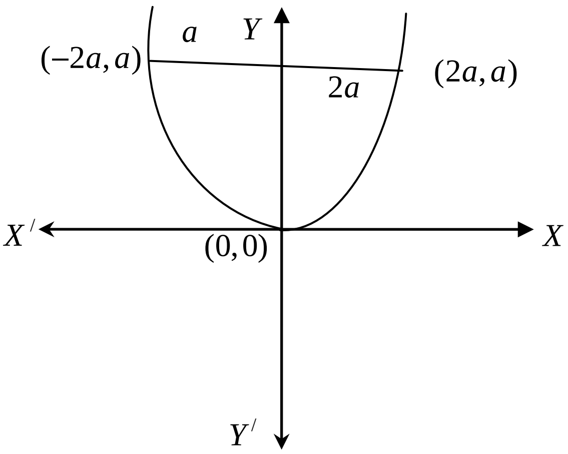
$$= \frac{1}{2a}\left[\frac{2a2\sqrt{2}a}{2} + \frac{4a^2}{2}\log(2a + 2a\sqrt{2}) - 2a^2\log 2a\right]$$

$$= \sqrt{2}a + \frac{1}{2a}.2a^2\log(2a + 2a\sqrt{2}) - a\log 2a$$

$$= a\left[\sqrt{2} + .\log\frac{2a(1+\sqrt{2})}{2a}\right]$$

$$= a\left[\sqrt{2} + .\log(1+\sqrt{2})\right] \qquad (Ans.)$$

Example-05: **Find the length of the arc** $x^2 = 4ay$ **. Extended from the vertex to one extremity of lotus rectum. (H.W.)**



Example-06: **Find the perimeter of the cardioids** $r = a(1 - Cos\theta)$ **.**

Solution:

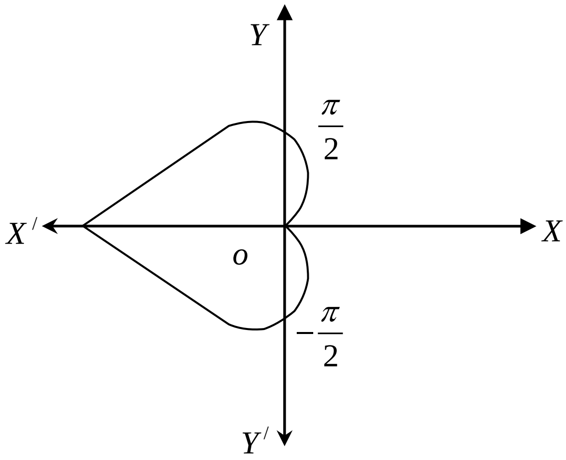
Given that,

$$r = a(1 - Cos\theta)...............(1)$$

$$= a(1 - \cos 2.\frac{\theta}{2}) = 2aSin^2\frac{\theta}{2}$$

θ	0	$\pi/2$	π	$-\pi/2$
r	0	a	2a	-a

Now, draw the graph roughly,



Therefore, the required perimeter is,

$$S = 2\int_0^{\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2}.d\theta$$

$$= 2\int_0^{\pi} \sqrt{4a^2Sin^4\frac{\theta}{2} + 4a^2Sin^2\frac{\theta}{2}Cos^2\frac{\theta}{2}}.d\theta$$

$$= 2\int_0^{\pi} \sqrt{4a^2Sin^2\frac{\theta}{2}(Sin^2\frac{\theta}{2} + Cos^2\frac{\theta}{2})}.d\theta$$

$$= 2\int_0^{\pi} \sqrt{4a^2Sin^2\frac{\theta}{2}}.d\theta$$

$$= 2\int_0^{\pi} 2aSin\frac{\theta}{2}d\theta$$

$$= 4a\int_0^{\pi} Sin\frac{\theta}{2}d\theta$$

$$= 4a\left[-Cos\frac{\theta}{2}\right]_0^{\pi}$$

$$= 8a[-0+1]$$

$$= 8a \quad (Ans.)$$

<u>Example-07:</u> Find the perimeter of the cardioids $r = a(1+Cos\theta)$ **. (H.W.)**

<u>Example-08:</u> Show that the length of the arc of the parabola $y^2 = 4ax$ **. Cut off** $y = 2x$ **. is** $a\left[\sqrt{2} + .\log(1+\sqrt{2})\right]$

<u>Solution:</u>

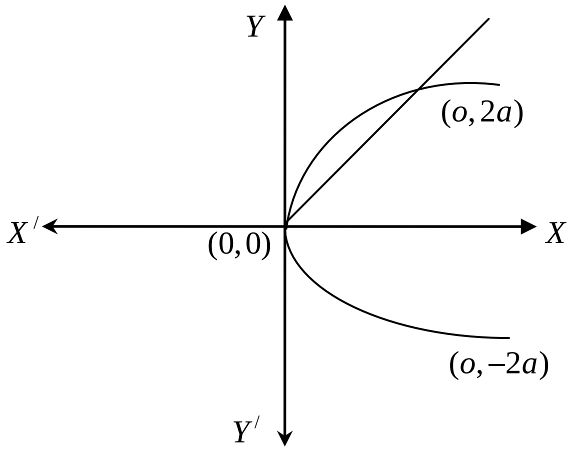
Given that,

$$y^2 = 4ax..............(1)$$

and, $y = 2x..............(2)$

From (1) and (2) we have,

$$4x^2 - 4ax = 0$$
$$\Rightarrow (x-a)4x = 0$$
$$\therefore x = 0, a$$
$$when, x = 0 \ then, y = 0$$
$$when, \ x = a, then, y = 2a$$



The curve (2) cut the parabola at (0, 0) and (a, 2a)
Now, draw the graph roughly:
Differentiating (1) w. r. to 'x' we get,

$$2y = 4a\frac{dx}{dy}$$
$$\Rightarrow \left(\frac{dx}{dy}\right) = \frac{y}{2a}$$
$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{4a^2}$$
$$\therefore \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{y^2}{4a^2}} = \frac{\sqrt{4a^2+y^2}}{2a}$$

Therefore, the required length is,

$$S = \int_0^{2a} \sqrt{1+(\frac{dx}{dy})^2}.dy$$
$$= \int_0^{2a} \frac{\sqrt{4a^2+y^2}}{2a}dy$$

$$= \frac{1}{2a}\int_0^{2a} \sqrt{4a^2 + y^2}\, dy$$

$$= \frac{1}{2a}\left[\frac{\sqrt{4a^2 + y^2}}{2} + \frac{4a^2}{2}\log(y + \sqrt{4a^2 + y^2}) \right]_0^{2a}$$

$$= \frac{1}{2a}\left[\frac{2a\sqrt{4a^2 + 4a^2}}{2} + \frac{4a^2}{2}\log(2a + \sqrt{4a^2 + 4a^2}) \right]$$

$$= \frac{1}{2a}\left[\frac{2a2\sqrt{2}a}{2} + \frac{4a^2}{2}\log(2a + 2a\sqrt{2}) - 2a^2\log 2a \right]$$

$$= \sqrt{2}a + \frac{1}{2a}.2a^2\log(2a + 2a\sqrt{2}) - a\log 2a$$

$$= a\left[\sqrt{2} + .\log\frac{2a(1+\sqrt{2})}{2a} \right]$$

$$= a\left[\sqrt{2} + .\log(1+\sqrt{2}) \right] \qquad (Ans.)$$

<u>Example-09:</u> Find the perimeter of the cardioids $x = a(\theta - Sin\theta)$ **.and** $y = a(1 - Cos\theta)$

<u>Solution:</u>

Given that,

$$x = a(\theta - Sin\theta)...............(1)$$

$$\frac{dx}{d\theta} = a(1 - \cos 2.\frac{\theta}{2}) = 2aSin^2\frac{\theta}{2}$$

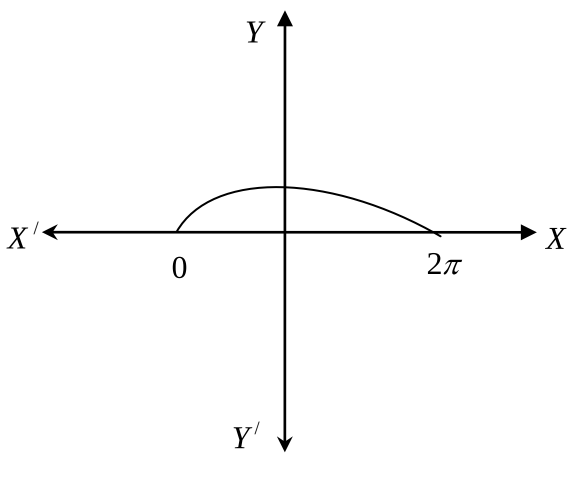
and,

$$y = a(1 - Cos\theta)$$

$$\therefore \frac{dy}{d\theta} = aSin\theta = 2aSin\frac{\theta}{2}.Cos\frac{\theta}{2}$$

$when,\, x = 0\ then, \sin\theta = 0, \theta = 0$

$when,\ y = 0, then,\ Cos\theta = 1, \theta = 2\pi$



Now, draw the graph roughly,

Therefore, the required perimeter is,

$$S = \int_0^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}.d\theta$$

$$= \int_0^{\pi} \sqrt{4a^2 Sin^4 \frac{\theta}{2} + 4a^2 Sin^2 \frac{\theta}{2} Cos^2 \frac{\theta}{2}}.d\theta$$

$$= \int_0^{\pi} \sqrt{4a^2 Sin^2 \frac{\theta}{2} (Sin^2 \frac{\theta}{2} + Cos^2 \frac{\theta}{2})}.d\theta$$

$$= \int_0^{\pi} \sqrt{4a^2 Sin^2 \frac{\theta}{2}}.d\theta$$

$$= 2a \int_0^{\pi} Sin \frac{\theta}{2} d\theta$$

$$= 4a \int_0^{\pi} Sin \frac{\theta}{2} d\theta$$

$$= 2a \left[- Cos \frac{\theta}{2} \right]_0^{\pi}$$

$$= 4a[1+1]$$

$$= 8a \quad (Ans.)$$

<u>**Example-10:**</u> **Show that the length of the arc of the parabola** $y^2 = 4ax$ **. Cut off** $3y = 8x$ **. is** $\left[\log^2 \sqrt{2} + \frac{15}{16} \right]$

<u>**Solution**</u>:

Given that,

$$y^2 = 4ax...............(1)$$

$$\text{and, } 3y = 8x...............(2)$$

From (1) and (2) we have,

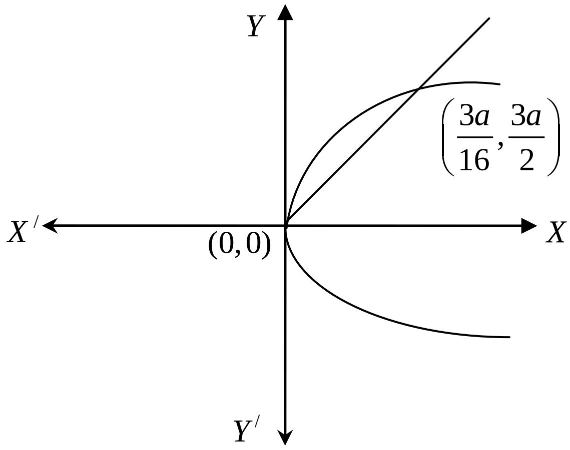
$$64x^2 - 36ax = 0$$

$$\Rightarrow (16x - 9a)4x = 0$$

$$\therefore x = 0, \frac{9a}{16}$$

$$when, x = 0 \; then, y = 0$$

$$when, \; x = \frac{9a}{16}, then, y = \frac{3a}{2}$$



The curve (2) cut the parabola at (0, 0) and (a, 2a)
Now, draw the graph roughly:
Differentiating (1) w. r. to 'x' we get,

$$2y = 4a\frac{dx}{dy}$$

$$\Rightarrow \left(\frac{dx}{dy}\right) = \frac{y}{2a}$$

$$\Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{y^2}{4a^2}$$

$$\therefore \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+\frac{y^2}{4a^2}} = \frac{\sqrt{4a^2+y^2}}{2a}$$

Therefore, the required length is,

$$S = \int_0^{\frac{3a}{2}} \sqrt{1+(\frac{dx}{dy})^2}\,.dy$$

$$= \int_0^{\frac{3a}{2}} \frac{\sqrt{4a^2+y^2}}{2a}dy$$

$$= \frac{1}{2a}\int_0^{\frac{3a}{2}} \sqrt{4a^2+y^2}\,dy$$

$$= \frac{1}{2a}\left[\frac{y\sqrt{4a^2+y^2}}{2} + \frac{4a^2}{2}\log(y+\sqrt{4a^2+y^2})\right]_0^{\frac{3a}{2}}$$

$$= \frac{1}{2a}\left[\frac{\frac{3a}{2}\sqrt{4a^2+\frac{9a^2}{2}}}{2} + \frac{4a^2}{2}\log(\frac{3a}{2}+\sqrt{4a^2+\frac{9a^2}{2}})\right]$$

$$= \frac{1}{2a}\left[\frac{3a\sqrt{25a^2}}{8} + 2a^2 \log\left(\frac{\frac{3a}{2}+\frac{5a}{2}}{2}\right)\right]$$

$$= \frac{1}{2a}\left[\frac{3a}{4}.\frac{5a}{2} + 2a^2 \log\left(\frac{8a^2}{4a}\right)\right]$$

$$= \frac{3a}{2a}.\frac{5a}{8} + \frac{2a^2}{2a}\log 2$$

$$= \frac{15a}{16} + a\log 2$$

$$= a\left[\log 2 + \frac{15}{16}\right] \qquad (Showed.)$$

<u>Example-11:</u> Find the length of the loop of the curve $9ay^2 = (x-2a)(x-5a)^2$**.**

<u>Solution:</u>

Given that,

$$9ay^2 = (x-2a)(x-5a)^2 ....(1)$$

Here, even power of 'y' is present. So (1) is symmetrical about x-axis.

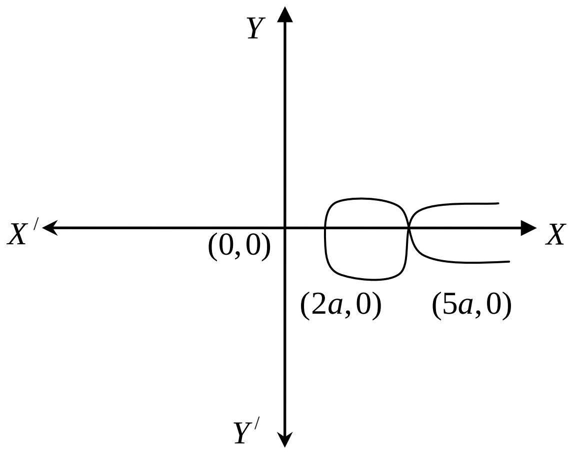
$$when,\ y = 0, then, x = 2a,\ 5a$$

Again, when $x < 0$ then y is imaginary. No part of the curve for $x < 0$.

Now, we draw the curve roughly.

Differencing (1) w. r. to 'x' we have,

$$y = (x-2a)^{\frac{1}{2}} + \frac{(x-5a)}{2\sqrt{x-2a}}$$



$$\therefore \frac{dy}{dx} = \left[(x-2a)^{\frac{1}{2}} + \frac{1}{2}(x-2a)^{-\frac{1}{2}}(x-5a) \right] . \frac{1}{3\sqrt{a}}$$

$$= \left[(x-2a)^{\frac{1}{2}} + \frac{(x-5a)}{2\sqrt{x-2a}} \right] . \frac{1}{3\sqrt{a}}$$

$$= \left[\frac{2(x-2a)+(x+5a)}{2\sqrt{x-2a}} \right] . \frac{1}{3\sqrt{a}}$$

$$= \frac{1}{3\sqrt{a}\,2\sqrt{x-2a}} \left[2x-4a+x-5a \right].$$

$$\therefore \frac{dy}{dx} = \frac{x-3a}{2\sqrt{a}\sqrt{x-2a}}$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = \frac{(x-3a)^2}{4a(x-2a)}$$

$$\Rightarrow 1 + \left(\frac{dy}{dx} \right)^2 = 1 + \frac{(x-3a)^2}{4a(x-2a)} = \frac{4a(x-2a)(x-3a)^2}{4a(x-2a)} = \frac{a^2+x^2-2ax}{4a(x-2a)}$$

$$= \frac{(x-a)^2}{4a(x-2a)}$$

$$\therefore \sqrt{1+(\frac{dy}{dx})^2} = \frac{x-a}{2\sqrt{a}\sqrt{x-2a}}$$

Hence, the required length is,

$$S = 2\int_{2a}^{5a} \sqrt{1+(\frac{dy}{dx})^2}\,.dx$$

$$= 2\int_{2a}^{5a} . \frac{x-a}{2\sqrt{a}\sqrt{x-2a}}\, dx$$

$$= 2.\frac{1}{2\sqrt{a}} \int_{2a}^{5a} . \frac{(x-2a)+a}{\sqrt{x-2a}}\, dx$$

$$= \frac{1}{\sqrt{a}} \left[\int_{2a}^{5a} . \sqrt{x-2a}\,dx + \frac{a}{\sqrt{a}} \int_{2a}^{5a} . \frac{dx}{\sqrt{x-2a}} \right]$$

$$= \frac{1}{\sqrt{a}} \left[\frac{(x-2a)^{\frac{3}{2}}}{\frac{3}{2}} \right]_{2a}^{5a} + \sqrt{a}.2\left[\sqrt{x-2a} \right]_{2a}^{5a}$$

$$= \frac{2}{3\sqrt{a}} 3a.\sqrt{3}.\sqrt{a} + 2\sqrt{a}.3.\sqrt{a}$$

$$= 2a.\sqrt{3} + 2a.\sqrt{3}$$

$$= 4a\sqrt{3} \quad (Ans.)$$

$$=\frac{1}{\sqrt{3a}}\int_0^a\left(3x^{\frac{1}{2}}+ax^{-\frac{1}{2}}\right)dx$$

$$=\frac{1}{\sqrt{3a}}\left[3\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a+a\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^a\right]$$

$$=\frac{1}{\sqrt{3a}}\left[\frac{3a^{\frac{3}{2}}}{\frac{3}{2}}+\frac{a.a^{\frac{3}{2}}}{\frac{1}{2}}\right]$$

$$=\frac{1}{\sqrt{3a}}\left(2a^{\frac{3}{2}}+2a^{\frac{3}{2}}\right)$$

$$=\frac{4a^{\frac{3}{2}}}{\sqrt{3}a^{\frac{1}{2}}}$$

$$=\frac{4a}{\sqrt{3}}\ (Ans.)$$

<u>Example-11:</u> If $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ **then show that the length of the curve is** $S=\frac{3}{2}\sqrt[3]{ax^2}$ **from (0, 0) to (x, y).**

<u>Solution:</u>

Given that,

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Differentiating (1) w. r. to ‘x’ we obtain,

$$\frac{2}{3}.x^{-\frac{1}{3}}+\frac{2}{3}.y^{-\frac{1}{3}}.\frac{dy}{dx}=0$$

$$\Rightarrow\frac{dy}{dx}=-\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}}=\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2=1+\left(\frac{y}{x}\right)^{\frac{2}{3}}=\frac{x^{\frac{2}{3}}+y^{\frac{2}{3}}}{x^{\frac{2}{3}}}=\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

Hence, the required length is,

$$S=\int_0^x\sqrt{1+\left(\frac{dy}{dx}\right)^2}.dx$$

$$= \int_0^x \sqrt{\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}}} dx$$

$$= a^{\frac{1}{3}} \int_0^x \sqrt{x^{-\frac{2}{3}}}\, dx$$

$$= a^{\frac{1}{3}} \int_0^x x^{-\frac{2}{3}\cdot\frac{1}{2}} ..dx = a^{\frac{1}{3}} \int_0^x x^{-\frac{1}{3}} ..dx$$

$$= a^{\frac{1}{3}} \left[\frac{x^{-\frac{1}{3}+1} ..}{\frac{-1}{3}+1} \right]_0^x$$

$$= \frac{a^{\frac{1}{3}} . x^{\frac{2}{3}}}{\frac{2}{3}}$$

$$= \frac{3}{2} \sqrt[3]{ax^2} \quad (Showed.)$$

<u>Example-12:</u> Find the length of the perimeter of the hypocycloid $(\frac{x}{a})^{\frac{2}{3}} + (\frac{y}{b})^{\frac{2}{3}} = 1$ **(H.W.)**

<u>Example-13:</u> Find the length of the curves $x = aCos^3 t$ **.and** $y = bSin^3 t$

<u>Solution:</u>

Given that,

$$x = a\cos^3 t \text{...............(1)}$$

$$\frac{dx}{dt} = a.3Cos^2 t - S\text{int} = -3aCos^2 tS\text{int}$$

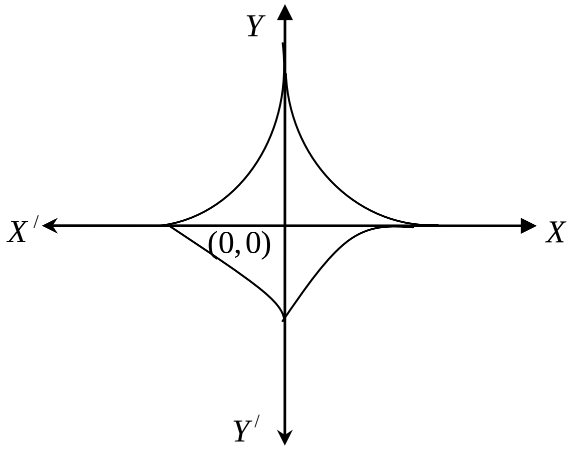
and,

$$y = bSin^3 t$$

$$\therefore \frac{dy}{dt} = 3bSin^2 t.Cost$$

$$when, x = 0 \ then, \cos t = 0 \,, t = \frac{\pi}{2}$$

$$when, \ y = 0, then, \ S\text{int} = 1, t = 0$$



Now, draw the graph roughly,

$$\therefore (\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2 = 9a^2 \cos^4 t \sin^2 t + 9b^2 Sin^4 t \cos^2 t$$
$$= 9\cos^2 t \sin^2 t\,(a^2 \cos^2 t + b^2 \sin^2 t)$$
$$= 9\cos^2 t \sin^2 t\,\{a^2(1-\sin^2 t) + b^2 \sin^2 t\}$$
$$= 9\cos^2 t \sin^2 t\,\{a^2 + (b^2 - a^2)\sin^2 t\}$$
$$\therefore \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} = 3\cos t \sin t \sqrt{a^2 + (b^2 - a^2)\sin^2 t}$$

Therefore, the required perimeter is,

$$S = 4\int_0^{2\pi} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}\,.d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} 3\cos t \sin t \sqrt{a^2 + (b^2 - a^2)\sin^2 t}$$

$Let,$

$$a^2 + (a^2 + b^2)Sin^2 t = z$$
$$2S\,\text{int}.Cost(b^2 - a^2)dt = dz$$
$$or \sin t.\cos t.dt = \frac{dz}{2(b^2 - a^2)}$$

Limit:

$$when, t = 0\ then, a^2 + (b^2 - a^2).0 = z, \therefore z = a^2$$
$$when,\ t = \frac{\pi}{2} then, z = b^2$$

$$\therefore S = 4\int_{a^2}^{b^2} 3\sqrt{z}.\frac{dz}{2(b^2-a^2)} = \frac{12}{2(b^2-a^2)}\int_{a^2}^{b^2} z^{\frac{1}{2}} dz$$

$$= \frac{12}{2(b^2-a^2)}\left[\frac{z^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_{a^2}^{b^2}$$

$$= \frac{12}{2(b^2-a^2)}.\frac{2}{3}\left(b^3-a^3\right)$$

$$= \frac{4\left(b^3-a^3\right)}{b^2-a^2}$$

$$= \frac{4(b-a)(b^2+ab+a^2)}{(b-a)(b+a)}$$

$$= \frac{4(b^2+ab+a^2)}{(b+a)} \qquad (Ans.)$$

<u>Example-14:</u> Find the perimeter of the cardioids $x = aSin2\theta(1+Cos2\theta)$ **.and** $y = aCos2\theta(1-Cos2\theta)$

<u>Solution:</u>

Given that,

$$x = aSin2\theta(1+Cos2\theta)\text{.............}(1)$$

$$\frac{dx}{d\theta} = a\left[Sin2\theta(-Sin2\theta) + (1+Cos2\theta)Cos2\theta.2\right]$$

$$= 2a\left[-Sin^2 2\theta + Cos2\theta + Cos^2 2\theta\right]$$

$$= 2a\left[Cos2\theta + (Cos^2 2\theta - Sin^2 2\theta)\right]$$

$$= 2a\left[Cos2\theta + Cos4\theta\right]$$

and,

$$y = aCos2\theta(1-Cos2\theta)$$

$$\frac{dy}{d\theta} = a\left[Cos2\theta(0+Sin2\theta.2) + (1-Cos2\theta).(Sin2\theta)2\right]$$

$$= 2a\left[Cos2\theta.Sin2\theta - Sin2\theta + Sin2\theta.Cos2\theta.\right]$$

$$= 2a\left[2Cos2\theta.\sin 2\theta - Sin2\theta\right]$$

$$= 2a\left[Sin4\theta - Sin2\theta\right]$$

Limit:

$$when, x = 0 \;\; then, \sin\theta = 0, \theta = 0$$

$$when, \;\; y = 0, then, \;\; Cos\theta = 1, \theta = \frac{\pi}{2}$$

$$\therefore \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} = \sqrt{4a^2\left[Cos^2 2\theta + Cos^2 4\theta + 2Cos2\theta.Cos4\theta + Sin^2 4\theta + Sin^2 2\theta - 2Sin4\theta.Sin2\theta\right]}$$
$$= \sqrt{8a^2\left[Cos2\theta Cos4\theta - Sin4\theta.Sin2\theta\right]}$$
$$= \sqrt{8a^2 Cos(2\theta + 4\theta)} = \sqrt{8a^2 Cos6\theta}$$

Now, draw the graph roughly,

Therefore, the required length is,

$$S = 4\int_0^{\frac{\pi}{2}} \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2}.d\theta$$
$$= 4\int_0^{\frac{\pi}{2}} 2\sqrt{2}a.Cos^{\frac{1}{2}} 6\theta.Sin^{\frac{1}{2}} 6\theta$$

By using Gamma-beta function,

$$= 8\sqrt{2}a \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right).\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma\left(\frac{\frac{1}{2}+\frac{1}{2}+2}{2}\right)}$$

$$= 8\sqrt{2}a \frac{2.\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}$$

$$= 16\sqrt{2}a \quad (Ans.)$$