Functions of several variables

Ex. 1) Find the Domain and Range of the function $\mathbf{f(x, y) = \sqrt{4 - x^2 - y^2}}$

Solution: For Domain,

F is defined for,

$4 - x^2 - y^2 \geq 0$

$\Rightarrow x^2 + y^2 \leq 4$

$\therefore$ Domain $= \{(x, y): x^2 + y^2 \leq 4\}$

Which indicates a circle whose centre is at (0,0) and radius is 2.

Again, for all the points on the circle $x^2 + y^2 = 4, f(x, y) = 0$ and the maximum value at the centre

$\sqrt{4} = 2.$ Therefore, for every value of the domain the functional value will stay within [0,2].

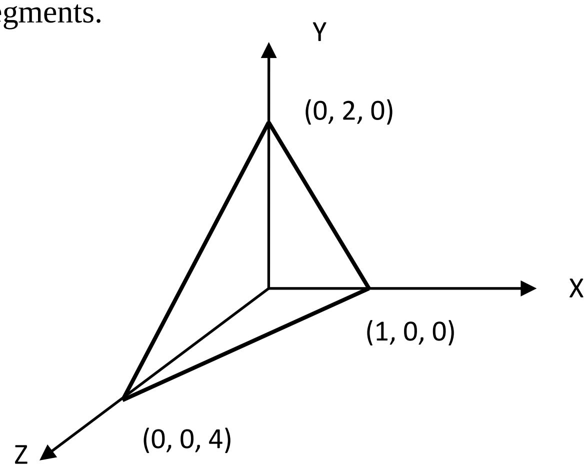
$\therefore$ Range $= [0,2]$

Ex. 2) Sketch the graph of the function $\mathbf{f(x, y) = 4 - 4x - 2y}$.

Solution: Let, $z = 4 - 4x - 2y \Rightarrow 4x + 2y + z = 4$

This is an equation of a plane, which coincides the axes at $x = 1, y = 2$ and $z = 4$.

A triangular portion of the plane can be sketched by plotting the intersections with the co-ordinate axes and joining them with the line segments.



Ex. 3) Sketch the graph of the function $\mathbf{f(x, y) = 4 - x^2 - y^2}$.

Solution: Let, $z = 4 - x^2 - y^2$

This is an equation of a paraboloid, which coincides the axes at $x = 2, y = 2$ and $z = 4$.

Limit and Continuity:

Ex. 4) If $\mathbf{f(x,y) = \dfrac{x+y-1}{x+y+2}}$, then show that, $\mathbf{\lim\limits_{x\to 0}\lim\limits_{y\to 1} f(x,y) = \lim\limits_{y\to 1}\lim\limits_{x\to 0} f(x,y)}$

Solution:

$$\text{L.H.S: } \lim_{x\to 0}\lim_{y\to 1} f(x,y) = \lim_{x\to 0}\lim_{y\to 1}\frac{x+y-1}{x+y+2} = \lim_{x\to 0}\frac{x+1-1}{x+1+2} = \lim_{x\to 0}\frac{x}{x+3} = \frac{0}{0+3} = 0$$

$$\text{R.H.S: } \lim_{y\to 1}\lim_{x\to 0} f(x,y) = \lim_{y\to 1}\lim_{x\to 0}\frac{x+y-1}{x+y+2} = \lim_{y\to 1}\frac{0+y-1}{0+y+2} = \lim_{y\to 1}\frac{y-1}{y+2} = \frac{1-1}{1+2} = 0$$

$$\therefore \lim_{x\to 0}\lim_{y\to 1} f(x,y) = \lim_{y\to 1}\lim_{x\to 0} f(x,y) \quad \text{(showed)}$$

Ex. 5) Discuss the iterated limits of the following functions at the point $\mathbf{(0,0)}$.

(i) $\mathbf{f(x,y) = \dfrac{x^3+y^3}{x^3-y^3}}$ (ii) $\mathbf{f(x,y) = \dfrac{x^2+y^2}{(x+y)^2+2x^2y^2}}$

Solution: (i) Given, $f(x,y) = \dfrac{x^3+y^3}{x^3-y^3}$

Now,

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0}\frac{x^3+y^3}{x^3-y^3} = \lim_{x\to 0}\frac{x^3}{x^3} = \lim_{x\to 0} 1 = 1$$

And,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\lim_{x\to 0}\frac{x^3+y^3}{x^3-y^3} = \lim_{y\to 0}\frac{y^3}{-y^3} = \lim_{x\to 0} -1 = -1$$

$$\therefore \lim_{x\to 0}\lim_{y\to 1} f(x,y) \neq \lim_{y\to 1}\lim_{x\to 0} f(x,y)$$

Therefore, iterated limits of the function does not exist at the point (0,0).

Ex. 6) Prove that, the iterated limits of the function $\mathbf{f(x,y) = \dfrac{xy}{x^2+y^2}}$ at the point $\mathbf{(0,0)}$ exists but simultaneous limit does not exist.

Solution:

(i) Given, $f(x,y) = \dfrac{xy}{x^2+y^2}$

Now,

$$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0}\frac{xy}{x^2+y^2} = \lim_{x\to 0} 0 = 0$$

And,

$$\lim_{y\to 0}\lim_{x\to 0} f(x,y) = \lim_{y\to 0}\frac{xy}{x^2+y^2} = \lim_{x\to 0} 0 = 0$$

$$\therefore \lim_{x\to 0}\lim_{y\to 1} f(x,y) = \lim_{y\to 1}\lim_{x\to 0} f(x,y)$$

Therefore, iterated limits of the function does not exist at the point (0,0).

Now, the simultaneous limit of the function along $y = mx$ at (0,0)

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)}\frac{x.\,mx}{x^2+m^2x^2} = \frac{m}{1+m^2}$$; it will have different values for different values of m.

Therefore, simultaneous limit of the function does not exist at the point (0,0).