Some curves and its properties for identifications:

a. Cycloid

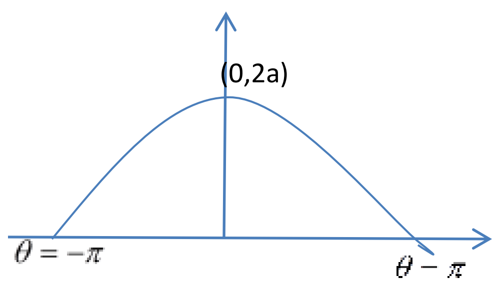
i. $x = a(\theta + \sin\theta) \quad y = a(1 + \cos\theta)$

ii. $x = a(\theta + \sin\theta) \quad y = a(1 - \cos\theta)$

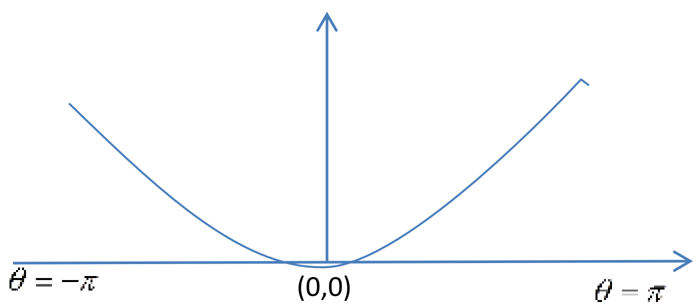
iii. $x = a(\theta - \sin\theta) \quad y = a(1 + \cos\theta)$

iv. $x = a(\theta - \sin\theta) \quad y = a(1 - \cos\theta)$

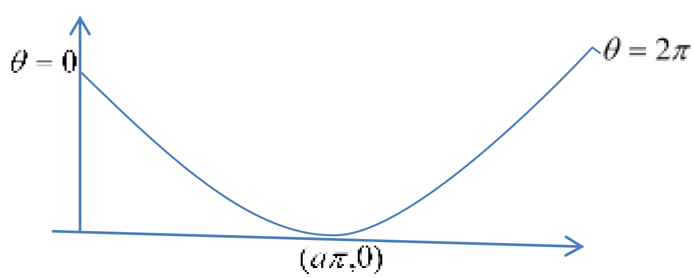
I. $x = a(\theta + \sin\theta) \quad y = a(1 + \cos\theta)$



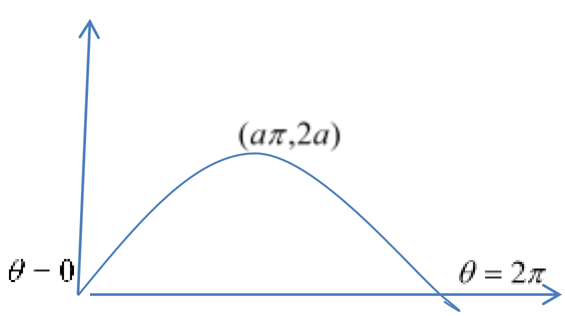
II. $x = a(\theta + \sin\theta) \quad y = a(1 - \cos\theta)$



III. $x = a(\theta - \sin\theta) \quad y = a(1 + \cos\theta)$

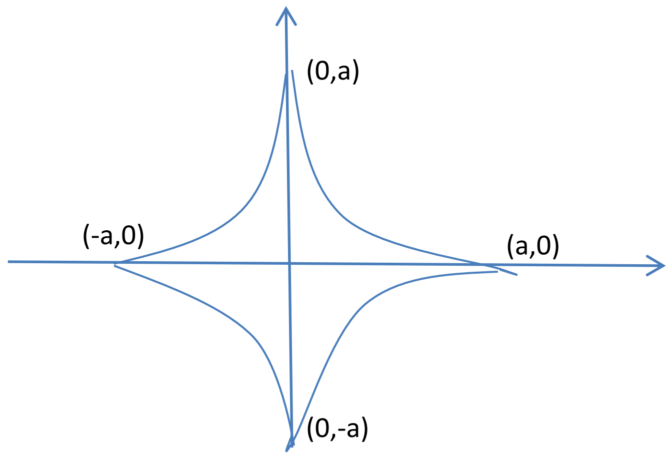


IV. $x = a(\theta - \sin\theta) \quad y = a(1 - \cos\theta)$



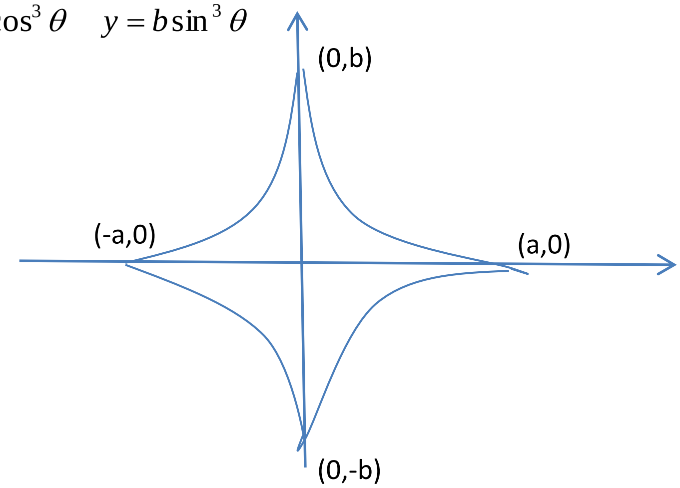
b. Astroid: $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

$x = a\cos^3\theta \quad y = a\sin^3\theta$

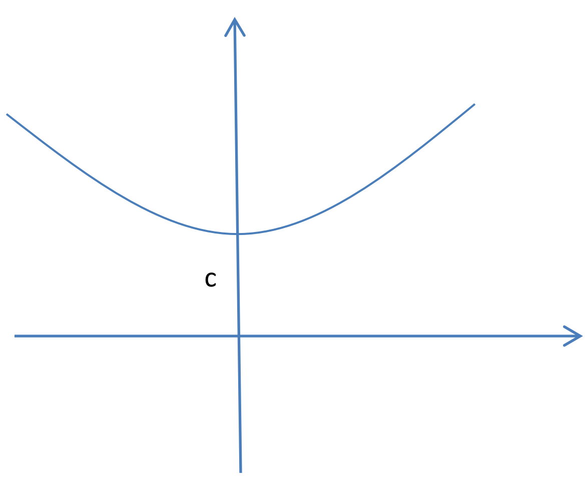


c. Four cusped hypo-cycloid: $\frac{x}{a}^{\frac{2}{3}} + \frac{y}{b}^{\frac{2}{3}} = 1$

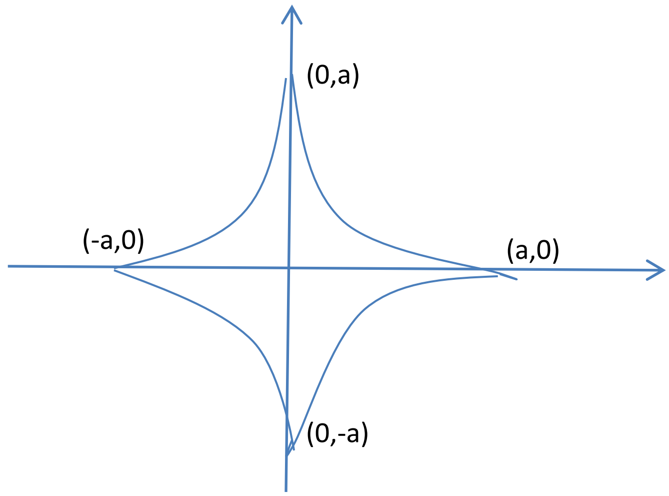
$x = a\cos^3\theta \quad y = b\sin^3\theta$



d. Catenary: $y = c\cosh\frac{x}{c}$



Exercise-1: Find the area of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$



Area $= 4\int ydx$

Here $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is a asteroid and $x = a\cos^3\theta \quad y = a\sin^3\theta$

$x = a\cos^3\theta$

$dx = -3a\cos^2\theta\sin\theta d\theta$

If x=0 then $\theta = \frac{\pi}{2}$ and if x=a then $\theta = 0$

Now, area

$4\int ydx$

$= 4\int_{\frac{\pi}{2}}^{0} -a\sin^3\theta 3a\cos^2\theta\sin\theta d\theta$

$= 12a^2\int_{0}^{\frac{\pi}{2}} \sin^4\theta\cos^3\theta d\theta$

$= 12a^2 \frac{\Gamma\frac{5}{2}\Gamma\frac{3}{2}}{2\Gamma 4} = 12a^2 \frac{\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2}.\frac{1}{2}.\Gamma\frac{1}{2}}{2.3.2.1} = 12a^2 \frac{3\sqrt{\pi}\sqrt{\pi}}{96} = \frac{3\pi a^2}{8}$

Exercise-2: find the area of the cycloid $x = a(\theta - \sin\theta) \quad y = a(1-\cos\theta)$ bounded with its base.

Solution:

Given that $x = a(\theta - \sin\theta) \quad y = a(1-\cos\theta)$

$x = a(\theta - \sin\theta) \qquad\qquad y = a(1-\cos\theta)$

$\therefore \frac{dx}{d\theta} = a(1-\cos\theta) \qquad\qquad \therefore \frac{dy}{d\theta} = a\sin\theta$

Area

$\int y dx$

$= \int_0^{2\pi} a(1-\cos\theta)a(1-\cos\theta)d\theta$

$= a^2 \int_0^{2\pi} (1-\cos\theta)^2 d\theta$

$= a^2 \int_0^{2\pi} (2\sin^2 \frac{\theta}{2})d\theta$

$= 4a^2 \int_0^{2\pi} \sin^4 \frac{\theta}{2} d\theta$

Let $\frac{\theta}{2} = x \quad \therefore d\theta = 2dx$

If $\theta = 0$ then x=0

If $\theta = 2\pi$ then $x = \pi$

So, area

$= 4a^2 \int_0^{\pi} \sin^4 x 2dx$

$= 8a^2 \int_0^{\pi} \sin^4 xdx$

$= 8a^2 .2\int_0^{\frac{\pi}{2}} \sin^4 xdx$

$= 16a^2 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} = 16a^2 \frac{\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{2.2.1} = \frac{48a^2\sqrt{\pi}\sqrt{\pi}}{16} = 3\pi a^2$

Exercise-3: find the area of the cycloid $x = a(\theta + \sin\theta) \quad y = a(1+\cos\theta)$ bounded with its base.

Solution:

Given that $x = a(\theta + \sin\theta) \quad y = a(1+\cos\theta)$

$x = a(\theta + \sin\theta) \qquad\qquad y = a(1+\cos\theta)$

$\therefore \dfrac{dx}{d\theta} = a(1+\cos\theta) \qquad\qquad \therefore \dfrac{dy}{d\theta} = -a\sin\theta$

Area

$\int y\,dx$

$$= \int_0^{2\pi} a(1+\cos\theta)a(1+\cos\theta)d\theta$$

$$= a^2\int_0^{2\pi}(1+\cos\theta)^2\,d\theta$$

$$= a^2\int_0^{2\pi}(2\cos^2\frac{\theta}{2})d\theta$$

$$= 4a^2\int_0^{2\pi}\cos^4\frac{\theta}{2}\,d\theta$$

Let $\dfrac{\theta}{2} = x \;\therefore d\theta = 2dx$

If $\theta = 0$ then x=0

If $\theta = 2\pi$ then $x = \pi$

So, area

$$= 4a^2\int_0^{\pi}\cos^4 x2dx$$

$$= 8a^2\int_0^{\pi}\cos^4 xdx$$

$$= 8a^2.2\int_0^{\frac{\pi}{2}}\cos^4 xdx$$

$$= 16a^2\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(3)} = 16a^2\frac{\frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right).\Gamma\left(\frac{1}{2}\right)}{2.2.1} = \frac{48a^2\sqrt{\pi}\sqrt{\pi}}{16} = 3\pi a^2$$