Mathematical Expectation and Generating Functions

Mathematical Expectation: Let, $x_1, x_2, x_3, \ldots \ldots \ldots \ldots, x_n$ be n observations of a discrete random variable x with probabilities $P(x_1), P(x_2), P(x_3), \ldots \ldots \ldots \ldots, P(x_n)$ respectively, then the mathematical expectation of variable x, denoted by E(x) is given by,

$$E(x) = \sum_{i=1}^{n} x_i\, P(x_i) \ldots \ldots \ldots \ldots \ldots (i)$$

For continuous random variable x having probability density function f(x) the expectation of x is defined as,

$$E(x) = \int x\, f(x) dx \ldots \ldots \ldots \ldots \ldots (ii)$$

The expectation of x is very often called the mean of x.

Ex. 1) Find the expected value of a discrete random variable x whose probability is given by $P(x) = \left({}^{1}/_{2}\right)^{x}$; where $x = 1, 2, 3$.

Solution: We have,

$$E(x) = \sum_{i=1}^{3} x_i\, P(x_i)$$

$$= \sum_{i=1}^{3} x_i \left({}^{1}/_{2}\right)^{x_i} = 1\left({}^{1}/_{2}\right) + 2\left({}^{1}/_{2}\right)^{2} + 3\left({}^{1}/_{2}\right)^{3} = \frac{11}{8} = 1.38$$

Ex. 2) Find the expected number of points in a single throw of fair dice.

Solution: The possible values of the variable are, $x = 1,2,3,4,5,6$

Each having probability, $P(x) = \frac{1}{6}$

$$E(x) = \sum_{i=1}^{6} x_i\, P(x_i) == \sum_{i=1}^{6} x_i\ {}^{1}/_{6}$$

$$= 1\left({}^{1}/_{6}\right) + 2.{}^{1}/_{6} + 3.{}^{1}/_{6} + 4.{}^{1}/_{6} + 5.{}^{1}/_{6} + 6.{}^{1}/_{6} = 3.5$$

Expectation of sum of random variables:

Theorem-1: The expectation of the sum of two random variables is equal to the sum of their expectations.

$$\text{Symbolically}, E(x + y) = E(x) + E(y)$$

Theorem-2: The expectation of the product of two random variables is equal to the product of their expectations.

$$\text{Symbolically}, E(xy) = E(x). E(y)$$

Ex. 3) The joint density function of two random variables x and y is given by,

$$f(x) = \begin{cases} \frac{xy}{96} & ; \quad 0 < x < 4, \quad 1 < y < 5 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Find (i) E(x) (ii) E(y) (iii) E(xy)

Solution:

$$\text{(i) } E(x) = \int_x \int_y x\, f(x, y)\, dx\, dy$$

$$= \int_0^4 \int_1^5 x\, \frac{xy}{96}\, dx\, dy$$

$$= \frac{1}{96} \int_0^4 x^2 dx \int_1^5 y\, dy = \frac{1}{96} \left[\frac{x^3}{3}\right]_0^4 \left[\frac{y^2}{2}\right]_1^5 = \frac{1}{96} \times \frac{64}{3} \times \left(\frac{25}{2} - \frac{1}{2}\right) = \frac{8}{3}$$

$$\text{(ii) } E(y) = \int_x \int_y y\, f(x, y)\, dx\, dy$$

$$= \int_0^4 \int_1^5 y\, \frac{xy}{96}\, dx\, dy$$

$$= \frac{1}{96} \int_0^4 x dx \int_1^5 y^2\, dy = \frac{1}{96} \left[\frac{x^2}{2}\right]_0^4 \left[\frac{y^3}{3}\right]_1^5 = \frac{1}{96} \times \frac{16}{2} \times \left(\frac{125}{3} - \frac{1}{3}\right) = \frac{31}{9}$$

$$\text{(ii) } E(xy) = E(x)E(y) = \frac{8}{3} \cdot \frac{31}{9} = \frac{248}{27}$$

Ex.4) An insurance company will sell a \$ 10,000 one-year term life insurance policy to an individual in a particular risk group for a premium of \$ 368. Find the expected value to the company of a single policy if a person in this risk group has a 97.25% chance of surviving on year.

Solution: We have:

- Policy cover =\$10,000
- Premium of the policy =\$368
- Probability that person survives $= 97.25\% = 0.9725$
- Probability that person will not survive $= 1 - 0.9725 = 0.0275$

If a person dies, the company will pay \$(10,000−368) = \$9,632\$ to the insured's nominee.
If a person survives, the company will pay \$368 as a premium.

We're calculating the expected value for the company, so let's consider the negative amount as the amount the company will give and the positive amount as the amount the company will receive:
The probability distribution is as follows:

x	P(x)	x. P(x)
\$368	0.9725	\$357.88
-\$9,632	0.0275	-\$264.88

The expected value to the company can be calculated as follows:

$$E(x) = \sum x\,P(x) = \$368 \times 0.9725 - \$9{,}632 \times 0.0275 = \$\,93$$

Therefore, the expected value to the company of a single policy if a person in this risk group has a 97.25% chance of surviving one year is $93.

H.W:

1) Given the probability model in the table below, what is the expected value of the random variable?

x	50	20	5	10	25	15
P(x)	0.1	0.3	0.3	0.7	0.4	0.7

2) The probability of a random variable x is given by,

$$f(x) = \begin{cases} \dfrac{x}{2} & ; \quad 0 < x < 2, \\ 0 & ; \quad \text{otherwise} \end{cases}$$

3) A pair of dice is thrown. The random variable, X, is defined as the sum of the obtained scores. Determine the expected value.

4) A local club plans to invest 10000 to host a baseball game. They expect to sell tickets worth $15000. But if it rains on the day of game, they won't sell any tickets and the club will lose all the money invested. If the weather forecast for the day of game is 20% possibility of rain, is this a good investment?

5) A company makes electronic gadgets. One out of every 50 gadgets is faulty, but the company doesn't know which ones are faulty until a buyer complains. Suppose the company makes a $3 profit on the sale of any working gadget, but suffers a loss of $80 for every faulty gadget because they have to repair the unit. Check whether the company can expect a profit in the long term.