Continuity of a Function of Several Variables

If limiting value and functional value be identical of a function $f(x, y)$ at any point, then the function be continuous at that point.

If f is a function of x and y, then $f(x, y)$ will be continuous at (a, b) if $\lim_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b)$

Ex. 1) Show that, $f(x, y) = x^2 + y^2$ is continuous at $(x, y) = (3, 4)$.

Solution: Here,

$$\lim_{\substack{x \to 3 \\ y \to 4}} f(x, y) = \lim_{\substack{x \to 3 \\ y \to 4}} (x^2 + y^2) = 3^2 + 4^2 = 9 + 16 = 25$$

and, $f(3,4) = 3^2 + 4^2 = 9 + 16 = 25$

So, $f(x, y)$ is continuous at $(3,4)$.

Ex. 2) Show that, $f(x, y)$ is discontinuous at $(x, y) = (0, 0)$ if f is defined by

$$\mathbf{f(x, y)} = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \textbf{when } (x, y) \neq (0, 0) \\ 0 & \textbf{when } (x, y) = (0, 0) \end{cases}$$

Solution: Here,

$$\lim_{x \to 0} \lim_{y \to 0} f(x, y) = \lim_{x \to 0} \lim_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{x^2}{x^2} = \lim_{x \to 0} 1 = 1$$

Again,

$$\lim_{y \to 0} \lim_{x \to 0} f(x, y) = \lim_{y \to 0} \lim_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} = \lim_{x \to 0} \frac{-y^2}{y^2} = \lim_{x \to 0} -1 = -1$$

Since, $\lim_{x \to 0} \lim_{y \to 0} f(x, y) \neq \lim_{y \to 0} \lim_{x \to 0} f(x, y)$, therefore $\lim_{\substack{x \to 0 \\ y \to 0}} f(x, y)$ does not exist.

So, $f(x, y)$ is discontinuous at $(0,0)$.

Ex. 3) Show that, $f(x, y)$ is continuous at $(x, y) = (0, 0)$ if f is defined by

$$\mathbf{f(x, y)} = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \textbf{when } (x, y) \neq (0, 0) \\ 0 & \textbf{when } (x, y) = (0, 0) \end{cases}$$

Solution: Here,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$$

Now, along $y = mx$

$$\lim_{x\to 0} \frac{mx^2}{\sqrt{x^2+m^2x^2}} = \lim_{x\to 0} \frac{mx}{\sqrt{1+m^2}} = 0$$

Now, along $y = x$

$$\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+x^2}} = \lim_{x\to 0} \frac{x^2}{\sqrt{2}x} = \lim_{x\to 0} \frac{x}{\sqrt{2}} = 0$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. Also, $f(0,0) = 0$

So, $f(x,y)$ is continuous at $(0,0)$.

Ex. 4) Show that, $\mathbf{f(x,y)}$ is discontinuous at $\mathbf{(x,y) = (0,0)}$ if f is defined by

$$\mathbf{f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & \text{when } (x,y) \neq (0,0) \\ \quad 0 & \text{when } (x,y) = (0,0) \end{cases}}$$

Solution: Here,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$$

Now, along $y = mx$

$$\lim_{x\to 0} \frac{x.m^3x^3}{x^2+m^6x^6} = \lim_{x\to 0} \frac{m^3x^4}{x^2+m^6x^6} = \lim_{x\to 0} \frac{m^3x^2}{1+m^6x^4} = 0$$

Now, along $y^3 = x$

$$\lim_{(x,y)\to(0,0)} \frac{x.x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ is not unique. Therefore, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

So, $f(x,y)$ is not continuous at $(0,0)$.

H.W:

1) Show that, $f(x,y)$ is continuous at $(x,y) = (0,0)$ if f is defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 \ \text{ when } (x,y) = (0,0) \end{cases}$$

2) Show that, $f(x,y)$ is discontinuous at $(x,y) = (0,0)$ if f is defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 \ \text{ when } (x,y) = (0,0) \end{cases}$$

3) Show that, $f(x,y)$ is discontinuous at $(x,y) = (0,0)$ if f is defined by

$$f(x,y) = \begin{cases} \dfrac{x^3y}{2x^6+y^2} & \text{when } (x,y) \neq (0,0) \\ 0 \ \text{ when } (x,y) = (0,0) \end{cases}$$