Cylinderical Co-ordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dz\,dy\,dx = r\,dz\,dr\,d\theta$

1. Use cylinderical co-ordinates to find the volume of the Solid bounded by the hemisphere $z = \sqrt{25 - x^2 - y^2}$, xy plane and the cylinder $x^2 + y^2 = 9$.

Solution:-

Volume $V = \iiint_G dz\,dy\,dx$ — (1)

Here Solid G is bounded by $z = \sqrt{25 - x^2 - y^2}$ plane xy and cylinder $x^2 + y^2 = 9$

For cylinderical co-ordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dz\,dy\,dx = r\,dz\,dr\,d\theta$.

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If $z=0$, $z=\sqrt{25-r^2}$, $r=0$, $r=3$, $\theta=0$ and $\theta=2\pi$

From (1)

$$V=\int_0^{2\pi}\int_0^{3}\int_0^{\sqrt{25-r^2}} r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{3} r\,[z]_0^{\sqrt{25-r^2}}\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^{3} r\sqrt{25-r^2}\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{2\pi}\int_0^{3} -2r\sqrt{25-r^2}\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{2\pi}\left[\frac{2}{3}(25-r^2)^{3/2}\right]_0^3 d\theta$$

$$=-\frac{1}{2}\cdot\frac{2}{3}\int_0^{2\pi}(4^3-5^3)\,d\theta$$

$$=\frac{1}{3}(125-64)\,[\theta]_0^{2\pi}$$

$$=\frac{61}{3}\cdot 2\pi$$

$$=\frac{122\pi}{3}$$

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2. Use cylinderical co-ordinates to find the volume of ice-cream cone bounded by the cone $z=3\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+(z-9)^2=9$.

Solution:-

Volume $V=\iiint_G dz\,dy\,dx$ —①

Here G is bounded by $z=3\sqrt{x^2+y^2}$ and $x^2+y^2+(z-9)^2=9$.

In cylinderical co-ordinate

$x=r\cos\theta$

$y=r\sin\theta$

$z=z$

$dx\,dy\,dz=r\,dz\,dr\,d\theta$

$z=3\sqrt{x^2+y^2}$

$=3r$

and $x^2+y^2+(z-9)^2=9$

$\Rightarrow r^2+(z-9)^2=9$

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$\Rightarrow (z-9)^2 = 9 - r^2$

$\therefore z = 9 + \sqrt{9-r^2}$

$\Rightarrow 3r = 9 + \sqrt{9-r^2}$

$\Rightarrow (3r-9)^2 = 9 - r^2$

$\Rightarrow 10r^2 - 54r + 72 = 0$

$\Rightarrow 5r^2 - 27r + 36 = 0$

$\Rightarrow (r-3)(5r-12) = 0$

$\therefore r = 3, \quad r \neq \frac{12}{5}$

Here, $z = 3r$, $z = 9 + \sqrt{9-r^2}$, $r=0$, $r=3$, $\theta=0$ and $\theta = 2\pi$

From ①

$$V = \int_0^{2\pi}\int_0^3\int_{3r}^{9+\sqrt{9-r^2}} r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 r\,[z]_{3r}^{9+\sqrt{9-r^2}} \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 r\left[9+\sqrt{9-r^2}-3r\right] dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left(9r + r\sqrt{9-r^2} - 3r^2\right) dr\, d\theta$$

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$$= \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{1}{2} \cdot \frac{2}{3} (9-r^2)^{3/2} - r^3 \right]_0^3 d\theta$$

$$= \int_0^{2\pi} \left[\left(\frac{81}{2} - 0 - 27 \right) - (0 - 9 - 0) \right] d\theta$$

$$= \frac{45}{2} \int_0^{2\pi} d\theta$$

$$= 2\pi \cdot \frac{45}{2}$$

$$= 45\pi \text{ (Ans)}$$

HW-

① Use cylinderical co-ordinates to find the volume of the solid bounded by $z = x^2 + y^2$, $x^2 + y^2 = 4$ and $z = 0$.

Spherical co-ordinates

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$dx\,dy\,dz = r^2 \sin\theta \, dr \, d\theta \, d\phi$

1. Use spherical co-ordinates to find the volume of the solid bounded by $z = \sqrt{x^2+y^2}$, $x^2+y^2=4$ and $z=0$.

Solution:- Volume, $V = \iiint_G dz\,dy\,dx$ —— (i)

G is bounded by. $z=0$, $z=\sqrt{x^2+y^2}$ and $x^2+y^2=4$

For spherical co-ordinates

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

$dx\,dy\,dz = r^2 \sin\theta \, dr \, d\theta \, d\phi$

Now $z=\sqrt{x^2+y^2}$

$\Rightarrow r\cos\theta=\sqrt{r^2\sin^2\theta}$

$\Rightarrow \tan\theta = 1$

$\therefore \theta=\frac{\pi}{4}$

$z=0$

$\Rightarrow r\cos\theta=0$

$\Rightarrow \theta=\frac{\pi}{2}$

$x^2+y^2=4$

$\Rightarrow r^2\sin^2\theta=4$

$\Rightarrow r^2\sin^2\frac{\pi}{4}=4$

$\Rightarrow r^2 \cdot \frac{1}{2}=4$

$\therefore r=2\sqrt{2}$

So, the solid bounded $\phi=0, \phi=2\pi, \theta=\frac{\pi}{4}, \theta=\frac{\pi}{2}$

$r=0$ and $r=2\sqrt{2}$.

From ①

$$V=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\sqrt{2}} r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$=\frac{(2\sqrt{2})^3}{3}\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\sin\theta\, d\theta\, d\phi = \frac{16\sqrt{2}}{2}\int_0^{2\pi}\left[-\cos\theta\right]_{\pi/4}^{\pi/2} d\phi$$

$$=\frac{16\sqrt{2}}{3}\cdot\frac{1}{\sqrt{2}}\int_0^{2\pi} d\phi = 2\pi\cdot\frac{16}{3}$$

$$=32\pi/3$$

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2. Use spherical co-ordinates to find the volume of solid bounded by the sphere $x^2+y^2+z^2=16$ and the cone $z=\sqrt{x^2+y^2}$

Solution:- Volume, $V = \iiint\limits_{G} dz\,dy\,dx$ — ①

Here G is bounded by $x^2+y^2+z^2=16$ and $z=\sqrt{x^2+y^2}$

For spherical $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dz\,dy\,dx = r^2\sin\theta\,dr\,d\theta\,d\phi$

Now, $x^2+y^2+z^2=16$

$\Rightarrow r^2=16$

$\therefore r=4$

$z=\sqrt{x^2+y^2}$

$\Rightarrow r\cos\theta = r\sin\theta$

$\therefore \theta = \frac{\pi}{4}$

Here G is bounded by, $\phi=0$, $\phi=2\pi$, $\theta=0$, $\theta=\frac{\pi}{4}$, $r=0$ and $r=4$

From ①

$$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{4} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/4} \left[\frac{r^3}{3}\right]_0^4 \sin\theta \, d\theta \, d\phi$$

$$= \frac{64}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\theta \, d\theta \, d\phi$$

$$= \frac{64}{3}\int_0^{2\pi} \left[-\cos\theta\right]_0^{\pi/4} d\phi$$

$$= 2\pi \cdot \frac{64}{3}\left(-\frac{1}{\sqrt{2}} + 1\right)$$

$$= \frac{64}{3}\left(2-\sqrt{2}\right) \quad Ans$$

H.W-① Use spherical co-ordinates to find the volume of the solid bounded by $z = \sqrt{x^2+y^2}$, $x^2+y^2=4$ and $z=0$.