Partial Derivatives of a Function of Two Variables

If $z = f(x, y)$ is a function of two variables, then the partial derivative of f with respect to x is the derivative that results when y is held fixed and x is allowed to vary.

It can be expressed using limit as follows,

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

Similarly, the partial derivative of f with respect to y can be expressed using limit as follows,

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y} = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k}$$

Ex. 1) If $f(x, y) = \sin(ax + by)$ then find $f_x(x, y)$ and $f_y(x, y)$ by definition.

Solution: By definition,

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} = \lim_{h \to 0} \frac{\sin(ax + ah + by) - \sin(ax + by)}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(ax + by + \frac{ah}{2}\right)\sin\frac{ah}{2}}{h} = \lim_{h \to 0} 2\cos\left(ax + by + \frac{ah}{2}\right)\frac{\sin\frac{ah}{2}}{\frac{ah}{2}\cdot\frac{2}{a}}$$

$$= a\lim_{h \to 0} \cos\left(ax + by + \frac{ah}{2}\right)\lim_{h \to 0}\left(\frac{\sin\frac{ah}{2}}{\frac{ah}{2}}\right) = a.\cos(ax + by).1 = a\cos(ax + by)$$

By definition,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(x, y+k) - f(x, y)}{k} = \lim_{k \to 0} \frac{\sin(ax + by + bk) - \sin(ax + by)}{k}$$

$$= \lim_{h \to 0} \frac{2\cos\left(ax + by + \frac{bk}{2}\right)\sin\frac{bk}{2}}{k} = \lim_{k \to 0} 2\cos\left(ax + by + \frac{bk}{2}\right)\frac{\sin\frac{bk}{2}}{\frac{bk}{2}\cdot\frac{2}{b}}$$

$$= b\lim_{k \to 0} \cos\left(ax + by + \frac{bk}{2}\right)\lim_{k \to 0}\left(\frac{\sin\frac{bk}{2}}{\frac{bk}{2}}\right) = b.\cos(ax + by).1 = b\cos(ax + by)$$

$\mathbf{Ex.\ 2)\ If\ f(x, y) = x^2 + xy\ then\ find\ f_x(1, 2)\ and\ f_y(2, 3)\ by\ definition}$.

Solution: By definition,

$$f_x(1,2) = \lim_{h \to 0} \frac{f(1 + h, 2) - f(1,2)}{h} = \lim_{h \to 0} \frac{(1 + h)^2 + (1 + h). 2 - (1^2 + 1.2)}{h}$$

$$= \lim_{h \to 0} \frac{1 + 2h + h^2 + 2 + 2h - 3}{h} = \lim_{h \to 0} \frac{h^2 + 4h}{h}$$

$$= \lim_{h \to 0} (h + 4) = 4$$

By definition,

$$f_y(x, y) = \lim_{k \to 0} \frac{f(2,3 + k) - f(2,3)}{k} = \lim_{k \to 0} \frac{2^2 + 2(3 + k) - (2^2 + 2.3)}{k}$$

$$= \lim_{h \to 0} \frac{4 + 6 + 2k - 4 - 6}{k} = \lim_{k \to 0} \frac{2k}{k} = \lim_{k \to 0} 2 = 2$$

$\mathbf{Ex.\ 3)\ If\ f(x, y) = x^2y + 5y^3, then}$

$\mathbf{(a)\ Find\ the\ slope\ of\ the\ surface\ z = f(x, y)\ in\ the\ x - direction\ at\ the\ point\ (1, -2)}$.

$\mathbf{(b)\ Find\ the\ slope\ of\ the\ surface\ z = f(x, y)\ in\ the\ y - direction\ at\ the\ point\ (1, -2)}$.

Solution: (a) Given, $f(x, y) = x^2y + 5y^3$

Differentiating f with respect to x with y held fixed yields,

$f_x(x, y) = 2xy$

Thus, the slope in the $x -$ direction at the point $(1, -2) = -4$

that is, z is decreasing at the rate of 4 units per unit increase in x.

$\mathbf{Ex.\ 4)\ Find\ the\ slope\ of\ the\ sphere\ x^2 + y^2 + z^2 = 1\ in\ the\ y - direction}$

at the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ **and** $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$.

Solution: The point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ lies in the upper hemisphere $z = \sqrt{1 - x^2 - y^2}$ and the point $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ lies in the lower hemisphere $z = \sqrt{1 - x^2 - y^2}$

To perform the implicit differentiation, we view z as a function of x and y and differentiate both sides with respect to y, taking x to be fixed.

Therefore,

$$\frac{\partial}{\partial y}[x^2 + y^2 + z^2] = \frac{\partial}{\partial y}[1]$$

$$\Rightarrow 2y + 2z\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{y}{z}$$

Substituting the y and z $co - ordinates$ of the points $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ we find that the slope at the point $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ is $-\frac{1}{2}$ and slope at $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ is $\frac{1}{2}$.