Maxima & Minima

Step-1 :- Find $\frac{\partial f}{\partial x} / f_x$ and $\frac{\partial f}{\partial y}$ or f_y

Step-2: Find critical point using conditions

$$\frac{\partial f}{\partial x} = 0, \ \frac{\partial f}{\partial y} = 0 \quad \text{or} \quad \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y}$$

Step-3: Find. $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

Step-4 :-

* If $r > 0$, $t > 0$ & $rt - s^2 > 0$ at critical point then we get minimum value at critical point.

* If $r < 0$, $t < 0$ and $rt - s^2 > 0$ at critical point then we get maximum value at critical point.

happened we get the critical point as a saddle point.

Example:- $u = x^2 - xy + y^2 + 3x - 2y + 1$, Find the maximum and minimum of the function.

Sol:- Given that, $u = x^2 - xy + y^2 + 3x - 2y + 1$

$\therefore \dfrac{\partial u}{\partial x} = 2x - y + 3$ and $\dfrac{\partial u}{\partial y} = -x + 2y - 2$

For critical point

$\dfrac{\partial u}{\partial x} = 0$ and $\dfrac{\partial u}{\partial y} = 0$

$\therefore 2x - y + 3 = 0$ — (i) $\quad -x + 2y - 2 = 0$ — (ii)

Solving (i) & (ii) we get the critical point $\left(-\frac{4}{3}, \frac{1}{3}\right)$

$\therefore r = \dfrac{\partial^2 u}{\partial x^2} = 2$, $\quad t = \dfrac{\partial^2 u}{\partial y^2} = 2$, $\quad s = \dfrac{\partial^2 u}{\partial x \partial y} = -1$

at the critical point $\left(-\frac{4}{3}, \frac{1}{3}\right)$

$r>0$, $t>0$ and $rt - s^2 = 4 - (-1)^2$

$= 4 - 1$

$= 3 > 0$

so, we get minimum value at $\left(-\frac{4}{3}, \frac{1}{3}\right)$

$\therefore u = \left(-\frac{4}{3}\right)^2 - \left(-\frac{4}{3}\right)\cdot\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)^2 + 3\left(-\frac{4}{3}\right) - 2\left(\frac{1}{3}\right) +$

$= \frac{16}{9} + \frac{4}{9} + \frac{1}{9} - 4 - \frac{2}{3} + 1$

$= \frac{7}{3} - 3 - \frac{2}{3}$

$= \frac{7 - 9 - 2}{3}$

$= \frac{-4}{3}$ (Answer)

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Example- Find the maximum and minimum value of $f(x,y) = x^3+y^3-3x-12y+20$.

Solution:- $f(x,y) = x^3+y^3-3x-12y+20$

$f_x = 3x^2-3$ and $f_y = 3y^2-12$

For critical point,

$f_x = 0$ and $f_y = 0$

$\Rightarrow 3x^2-3=0$

$\Rightarrow 3x^2=3$

$\Rightarrow x^2=1$

$\therefore x = \pm 1$

$\Rightarrow 3y^2-12=0$

$\Rightarrow 3y^2=12$

$\Rightarrow y^2=4$

$\therefore y = \pm 2$

So, critical points are $(1,2)$ $(1,-2)$ $(-1,2)$ $(-1,-2)$

Now, $r = f_{xx} = 6x$

$s = f_{xy} = 0$

$t = f_{yy} = 6y$

$\therefore rt - s^2 = 36xy$

at $(1,2)$ $\quad r = 6x = 6.1 = 6 > 0$, $\quad t = 6y = 6.2 = 12 > 0$, $\quad rt - s^2 = 36xy = 36.1.2 = 72 > 0$

Since $r > 0$, $t > 0$ and $rt - s^2 > 0$ at $(1,2)$

so, we get minimum value at $(1,2)$

$\therefore$ Minimum value. $f(1,2) = 1^3 + 2^3 - 3.1 - 12.2 + 20$
$= 1 + 8 - 3 - 24 + 20$
$= 29 - 27$
$= 2$

at $(1,-2)$ $\quad r = 6x = 6.1 = 6 > 0$ $\quad t = 6y = 6.(-2) = -12 < 0$ $\quad rt - s^2 = 36xy = -72 < 0$

Since $r > 0$, $t < 0$ and $rt - s^2 < 0$ at $(1,-2)$

so there is no maximum or minimum value.

at $(-1,2)$ $\quad r = 6x = -6 < 0$, $\quad t = 6y = 12 > 0$ $\quad rt - s^2 = 36xy = -72 < 0$

since $r < 0$, $t > 0$ and $rt - s^2 < 0$.

$\therefore$ There is no maximum value or minimum - value.

at $(-1,-2)$ $\quad r = 6x = -6 < 0 \quad t = 6y = -12 < 0 \quad rt - s^2 = 36xy = 72 > 0$

Since $r<0$, $t<0$ and $rt-s^2>0$

so we get maximum value at $(-1,-2)$

$\therefore$ Maximum value

$$f(-1,-2) = (-1)^3 + (-2)^3 - 3(-1) - 12(-2) + 20$$
$$= -1 - 8 + 3 + 24 + 20$$
$$= 38$$

$\therefore$ Maxima $= 38$, Minima $= 2$

H.W →

1. $u = x^2 + 2y^2 - 4x + 4y - 3$
2. $u = x^2 + y^2 + \frac{x}{2} + \frac{2}{y}$
3. $u = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$