Jacobian: If u=u(x,y,z), v= v(x,y,z) and w=w(x,y,z) are partial differentiable then the following determinant is called jacobian of u, v and w with respect to x,y,z in the domain function which is denoted by $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

We actually use jacobian in integration for changing the variable. Because $dudvdw = |j(x,y,z)|dxdydz$

Example 1: Evaluate $\iint(x+y)dydx$ by making the change of variables, where R is the region enclosed by the lines x=o, x+y=2,y=0 and x+y=3

Solution: let x+y=u and x=v

So, y=u-v

Now, $j(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$

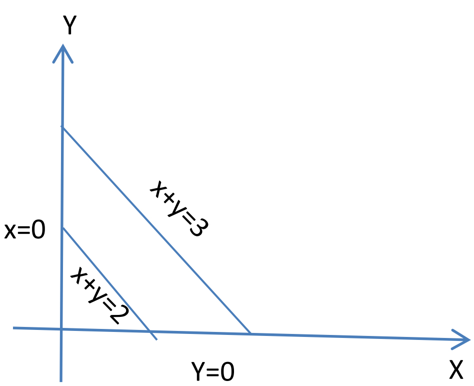
When x=0 then v=0

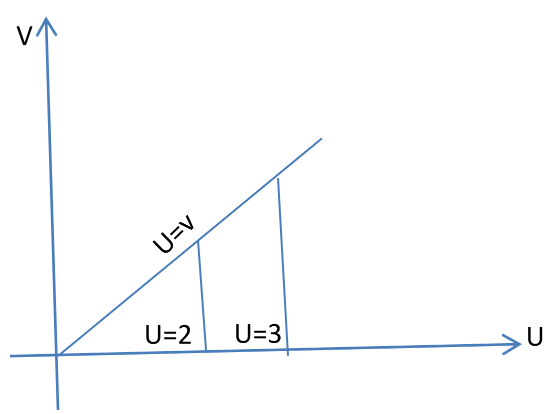
And x+y=2 then u=2

y=0 then u-v=0 so, u=v

and x+y=3 then u=3

Now the region enclosed by the lines x=o, x+y=2,y=0 and x+y=3 is replaced by the area enclosed by u=2, u=3, v=0 and v=u



$$\iint (x+y)dydx = \int_2^3 \int_0^u ududv = \int_2^3 [v]_0^u udu = \int_2^3 u^2 du = \left[\frac{u^3}{3}\right]_2^3 = \frac{19}{3}$$

Example 2: Evaluate $\iint \sin(\frac{x-y}{x+y})dydx$ by making the change of variables, where R is the region enclosed by two axis and x+y=1.

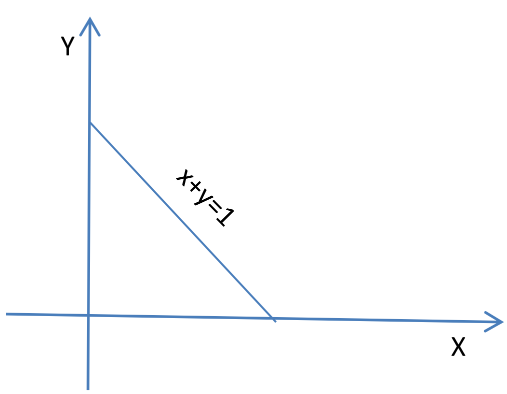
Solution: Let x-y=u andx+y=v

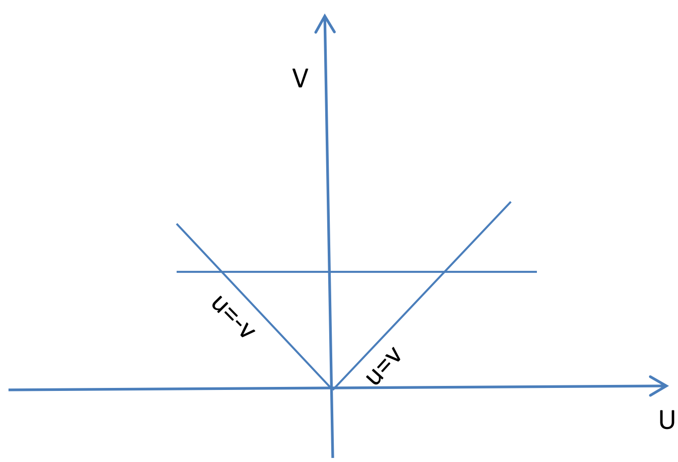
$$j(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} == \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

$$dxdy = \frac{1}{2} dudv$$

| | |
|---|---|
| When x=0 | Then u=-v |
| When y=0 | Then u=v |
| When x+y=1 | Then v=1 |

Now the region enclosed by two axis and x+y= is changed by the equations u=-v, u=v, v=0 and v=1.




$$\iint \sin(\frac{x-y}{x+y})dydx = \int_0^1 \int_{-v}^{v} \sin\left(\frac{u}{v}\right)\frac{1}{2} dudv = \frac{1}{2}\int_0^1 \left[-v\cos\frac{u}{v}\right]_{-v}^{v} dv = 0$$

Example 3: Evaluate $\iint e^{\frac{y-x}{y+x}} dydx$, where is the trapezoid with vertices (0,1), (0,2),(2,0) and (1,0)

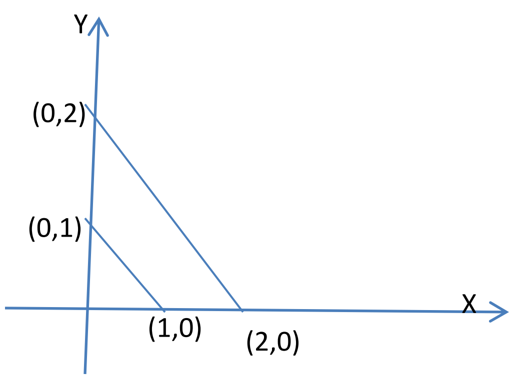
Solution: let u=y-x and v=y+x so $x = \frac{1}{2}(v-u), y = \frac{1}{2}(v+u)$

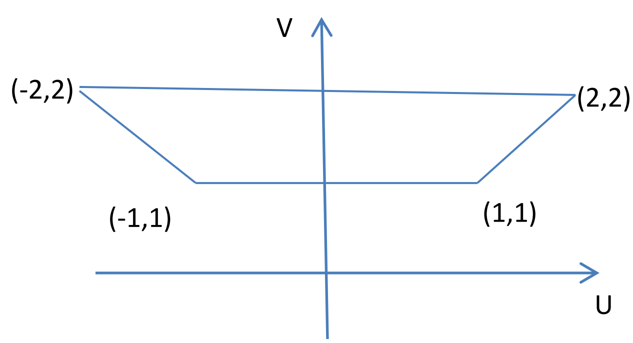
$$j(x,y) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$$

$$dxdy = \frac{1}{2} dudv$$

| (x,y) | (u,v) |
|---|---|
| (0,1) | (1,1) |
| (0,2) | (2,2) |
| (2,0) | (-2,2) |
| (1,0) | (-1,1) |

So, the new region is enclosed by the lines v=1, v=2, u=v and u=-v.




$$\iint e^{\frac{y-x}{y+x}} dydx = \frac{1}{2}\int_1^2 \int_{-v}^{v} e^{\frac{u}{v}} dudv = \frac{1}{2}\int_1^2 \left[ ve^{\frac{vu}{}} \right]_{-v}^{v} = \frac{1}{2}\int_1^2 (ev - \frac{v}{e})dv = \frac{1}{2}(e - \frac{1}{e})\int_1^2 vdv$$

$$= \frac{1}{2}(e - \frac{1}{e})\left[\frac{v^2}{2}\right]_1^2 = \frac{3}{4}(e - 1/e)$$

Example 4: use the change of variables u= x-2y , v=2x+y to evaluate the integral $\iint \frac{x-2y}{2x+y}dydx$, where is the rejoin enclosed by the lines x-2y=1, x-2y=4, 2x+y=1 and 2x+y=3.

Solution: let u=x-2y, and v=2x+y

So, $x=\frac{1}{5}(u+2v)$, $y=\frac{1}{5}(v-2u)$

$$j(x,y)=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & -2\\ 2 & 1\end{vmatrix}=5$$

$$dxdy=\frac{1}{5}dudv$$

When x-2y=1 then u=1

When x-2y=4 then u=4

When 2x+y=1 then v=1

When 2x+y=3 then v=3

Now the area is enclosed by v=1,v=3, u=1 and u=4.

$$\iint \frac{x-2y}{2x+y}dydx=\frac{1}{5}\int_1^4\int_1^3 \frac{u}{v}dudv=\frac{1}{5}\int_1^4 udu\int_1^3 \frac{1}{v}dv=\frac{1}{5}\left[\frac{u^2}{2}\right]_1^4\left[\ln v\right]_1^3=\frac{3}{2}\ln 3$$