13.2.1 DEFINITION Let f be a function of two variables, and assume that f is defined at all points of some open disk centered at (x_0, y_0), except possibly at (x_0, y_0). We will write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \tag{3}$$

if given any number $\epsilon > 0$, we can find a number $\delta > 0$ such that $f(x, y)$ satisfies

$$|f(x, y) - L| < \epsilon$$

whenever the distance between (x, y) and (x_0, y_0) satisfies

$$0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$$

Example 1 Figure 13.2.3a shows a computer-generated graph of the function

$$f(x, y) = -\frac{xy}{x^2 + y^2}$$

The graph reveals that the surface has a ridge above the line $y = -x$, which is to be expected since $f(x, y)$ has a constant value of $\frac{1}{2}$ for $y = -x$, except at $(0, 0)$ where f is undefined (verify). Moreover, the graph suggests that the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along a line through the origin varies with the direction of the line. Find this limit along

(a) the x-axis (b) the y-axis (c) the line $y = x$
(d) the line $y = -x$ (e) the parabola $y = x^2$

Solution (a). The x-axis has parametric equations $x = t$, $y = 0$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,\,y)\to(0,\,0)\\ (\text{along } y=0)}} f(x, y) = \lim_{t\to 0} f(t, 0) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 13.2.3b.

Solution (b). The y-axis has parametric equations $x = 0$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,\,y)\to(0,\,0)\\ (\text{along } x=0)}} f(x, y) = \lim_{t\to 0} f(0, t) = \lim_{t\to 0}\left(-\frac{0}{t^2}\right) = \lim_{t\to 0} 0 = 0$$

which is consistent with Figure 13.2.3b.

Solution (c). The line $y = x$ has parametric equations $x = t$, $y = t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} f(x, y) = \lim_{t\to 0} f(t, t) = \lim_{t\to 0}\left(-\frac{t^2}{2t^2}\right) = \lim_{t\to 0}\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

Solution (d). The line $y = -x$ has parametric equations $x = t$, $y = -t$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=-x)}} f(x, y) = \lim_{t\to 0} f(t, -t) = \lim_{t\to 0}\frac{t^2}{2t^2} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2}$$

which is consistent with Figure 13.2.3*b*.

Solution (e). The parabola $y = x^2$ has parametric equations $x = t$, $y = t^2$, with $(0, 0)$ corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x^2)}} f(x, y) = \lim_{t\to 0} f(t, t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

This is consistent with Figure 13.2.3*c*, which shows the parametric curve

$$x = t, \quad y = t^2, \quad z = -\frac{t}{1+t^2}$$

superimposed on the surface. ◄

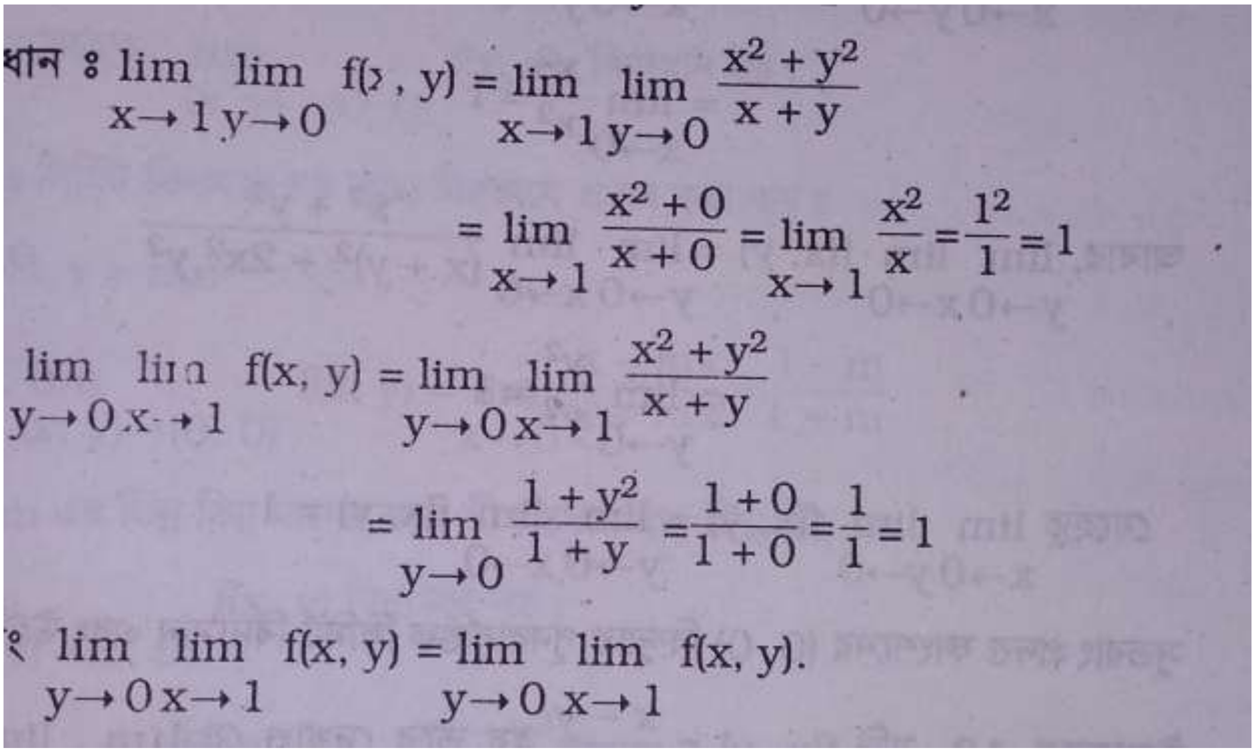
► Example 2

$$\begin{aligned}
\lim_{(x,y)\to(1,4)} [5x^3y^2 - 9] &= \lim_{(x,y)\to(1,4)} [5x^3y^2] - \lim_{(x,y)\to(1,4)} 9 \\
&= 5\left[\lim_{(x,y)\to(1,4)} x\right]^3\left[\lim_{(x,y)\to(1,4)} y\right]^2 - 9 \\
&= 5(1)^3(4)^2 - 9 = 71
\end{aligned}$$ ◄

Some more examples

1. If $f(x,y)=\dfrac{x^2+y^2}{x+y}$ then show that $\lim\limits_{x\to 1}\lim\limits_{y\to 0} f(x,y)=\lim\limits_{y\to 0}\lim\limits_{x\to 1} f(x,y)$

Solution:



ধান ঃ $\lim\limits_{x\to 1}\lim\limits_{y\to 0} f(x,y)=\lim\limits_{x\to 1}\lim\limits_{y\to 0}\dfrac{x^2+y^2}{x+y}$

$=\lim\limits_{x\to 1}\dfrac{x^2+0}{x+0}=\lim\limits_{x\to 1}\dfrac{x^2}{x}=\dfrac{1^2}{1}=1$

$\lim\limits_{y\to 0}\lim\limits_{x\to 1} f(x,y)=\lim\limits_{y\to 0}\lim\limits_{x\to 1}\dfrac{x^2+y^2}{x+y}$

$=\lim\limits_{y\to 0}\dfrac{1+y^2}{1+y}=\dfrac{1+0}{1+0}=\dfrac{1}{1}=1$

ঃ $\lim\limits_{y\to 0}\lim\limits_{x\to 1} f(x,y)=\lim\limits_{y\to 0}\lim\limits_{x\to 1} f(x,y)$.

2. If $f(x,y)=\dfrac{x-y}{x+y}$ then show that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)\neq\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ also discuss that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ exist or not.

Solution:

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)=\lim\limits_{x\to 0}\lim\limits_{y\to 0}\dfrac{x-y}{x+y}$

$=\lim\limits_{x\to 0}\dfrac{x-0}{x+0}$

$=\lim\limits_{x\to 0}\dfrac{x}{x}$

=1

And

$\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=\lim\limits_{y\to 0}\lim\limits_{x\to 0}\dfrac{x-y}{x+y}$

$= \lim_{y\to 0} \frac{0-y}{0+y}$

$= \lim_{y\to 0} \frac{-y}{y}$

= -1

So, $\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$

As, $\lim_{x\to 0}\lim_{y\to 0} f(x,y) \neq \lim_{y\to 0}\lim_{x\to 0} f(x,y)$ we can say $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist.

Again,

Let $y = mx$

$\lim_{x\to 0}\lim_{y\to 0} f(x,y) = \lim_{x\to 0}\lim_{y\to 0} \frac{x-y}{x+y}$

$= \lim_{x\to 0} \frac{x-mx}{x+mx}$

$= \frac{1-m}{1+m}$

We get different values of the limit for different values of m.

So, $\lim_{x\to 0}\lim_{y\to 0} f(x,y)$ does not exist.

Exercise

1–6 Use limit laws and continuity properties to evaluate the limit.

1. $\lim_{(x,y)\to(1,3)} (4xy^2 - x)$

2. $\lim_{(x,y)\to(0,0)} \frac{4x-y}{\sin y - 1}$

3. $\lim_{(x,y)\to(-1,2)} \frac{xy^3}{x+y}$

4. $\lim_{(x,y)\to(1,-3)} e^{2x-y^2}$

5. $\lim_{(x,y)\to(0,0)} \ln(1+x^2y^3)$

6. $\lim_{(x,y)\to(4,-2)} x\sqrt[3]{y^3+2x}$

7–8 Show that the limit does not exist by considering the limits as $(x, y) \to (0, 0)$ along the coordinate axes.

7. (a) $\lim_{(x,y)\to(0,0)} \frac{3}{x^2+2y^2}$ (b) $\lim_{(x,y)\to(0,0)} \frac{x+y}{2x^2+y^2}$

8. (a) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$ (b) $\lim_{(x,y)\to(0,0)} \frac{\cos xy}{x^2+y^2}$