BERNOULLI'S EQUATION

General Form :

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \; ; \; n \neq 0, 1.$$

01. $x^2 \frac{dy}{dx} - 2xy = 3y^4$

⇒ Sol^n :

Given,

$$x^2 \frac{dy}{dx} - 2xy = 3y^4$$

$$\Rightarrow \frac{x^2}{y^4} \cdot \frac{dy}{dx} - \frac{2xy}{y^4} = \frac{3y^4}{y^4}$$

$$\Rightarrow x^2 y^{-4} \frac{dy}{dx} - 2xy^{-3} = 3$$

$$\Rightarrow y^{-4} \cdot \frac{dy}{dx} - \frac{2y^{-3}}{x} = \frac{3}{x^2} \quad \text{——— (i)}$$

Let,

$$y^{-3} = z$$

$$\Rightarrow -3y^{-4} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^{-4} \cdot \frac{dy}{dx} = -\frac{1}{3} \cdot \frac{dz}{dx}$$

(i) becomes ⇒ $-\frac{1}{3} \cdot \frac{dz}{dx} - \frac{2z}{x} = \frac{3}{x^2}$

$$\Rightarrow \frac{dz}{dx} + \frac{6z}{x} = -\frac{9}{x^2} \quad \text{——— (ii)}$$

$$\text{I.F.} = e^{\int \frac{6}{x} dx} = e^{6 \ln x} = e^{\ln x^6} = x^6$$

Multiplying (ii) with I.F. ⇒

$$x^6 \cdot \frac{dz}{dx} + x^6 \cdot \frac{6z}{x} = -\frac{9}{x^2} \cdot x^6$$

$$\Rightarrow x^6 \cdot \frac{dz}{dx} + 6zx^5 = -9x^4$$

$$\Rightarrow \frac{d}{dx}(x^6 \cdot z) = -9x^4$$

$$\Rightarrow \int d(x^6 \cdot z) = \int (-9x^4)\, dx$$

$$\Rightarrow x^6 z = -9 \cdot \frac{x^5}{5} + C$$

$$\Rightarrow x^6 y^{-3} = -\frac{9}{5} x^5 + C \quad \text{(Ans:)}$$

BERNOULLI'S EQUATION

*** 01. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

=> Sol^n :

Given,

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \frac{1}{\cos^2 y} \cdot \frac{dy}{dx} + x \frac{\sin 2y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} + x \cdot \frac{2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} + 2x \tan y = x^3 \text{ ——— (i)}$$

Let,

$$2\tan y = z$$

$$\therefore 2\sec^2 y \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \sec^2 y \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \frac{dz}{dx}$$

$\therefore$ (i) becomes $\Rightarrow$

$$\frac{1}{2} \cdot \frac{dz}{dx} + xz = x^3$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2x^3 \quad \text{—— (ii)}$$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying (ii) with I.F. $\Rightarrow$

$$e^{x^2} \cdot \frac{dz}{dx} + e^{x^2} \cdot x z = e^{x^2} \cdot 2x^3$$

$$\Rightarrow \frac{d}{dx}\left(e^{x^2} \cdot z\right) = 2x^3 \cdot e^{x^2}$$

$$\Rightarrow \int d\left(e^{x^2} \cdot z\right) = \int \left(2x^3 \cdot e^{x^2}\right) dx$$

$$\Rightarrow e^{x^2} \cdot z = \int \left(2x^3 \cdot e^{x^2}\right) dx \quad \text{—— (iii)}$$

Let,

$$x^2 = p$$

$$\therefore 2x = \frac{dp}{dx}$$

$\therefore$ (iii) becomes $\Rightarrow$ $e^p \cdot z = \int p \cdot \frac{dp}{\cancel{dx}} \cdot e^p \cdot \cancel{dx}$

$$\Rightarrow e^p \cdot z = \int p \cdot e^p$$

$$z \cdot e^p = p \int e^p\,dp - \int \left(\frac{d}{dp}(p) \int e^p\,dp \right) dp$$

$$\Rightarrow z e^p = p e^p - \int e^p\,dp$$

$$\Rightarrow z e^p = p e^p - e^p + c$$

$$\Rightarrow z e^{x^2} = x^2 e^{x^2} - e^{x^2} + c$$

$$\Rightarrow 2\tan y \cdot e^{x^2} = x^2 e^{x^2} - e^{x^2} + c \quad \text{(Ans:)}$$

APPLICATION OF 1ST ORDER ODE

=> **Application in real life :** Population dynamics, Circuit analysis, Newton's law of cooling.

Q1. A culture initially has P_0 number of bacteria. At $t = 1h$, the number of bacteria y measured to be $3/2\,P_0$. If the rate of growth y is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

=> **Sol^n:**

Initially $t = 0$.

Accorting to the question, $\frac{dp}{dt} \propto P(t)$

$$\Rightarrow \frac{dp}{dt} = k\,P(t) \quad \dots (i)$$

With condition, $P(0) = P_0 \quad \dots$ (ii)

$$P(1) = \frac{3}{2} P_0 \quad \dots (iii)$$

$$\frac{dp}{dt} = KP$$

$$\Rightarrow \frac{dp}{p} = K\,dt$$

$$\Rightarrow \int \frac{dp}{p} = \int k\,dt$$

$$\Rightarrow \ln p = kt + \ln c$$

$$\Rightarrow \ln p - \ln c = kt$$

$$\Rightarrow \ln\left(\frac{p}{c}\right) = kt$$

$$\Rightarrow \frac{p}{c} = e^{kt}$$

$$\Rightarrow P(t) = ce^{kt} \quad \text{....(iv)}$$

Putting $t = 0$ in (iv) $\Rightarrow P(0) = C$

$$\therefore C = P_0$$

$\therefore$ (iv) becomes, $P(t) = P_0 e^{kt}$ ....(v)

Putting $t = 1$ in (v) $\Rightarrow P(1) = P_0 e^{k}$

$$\Rightarrow \frac{3}{2} P_0 = P_0 e^{k}$$

$$\Rightarrow e^{k} = \frac{3}{2}$$

$$\therefore k = \ln \frac{3}{2} = 0.406$$

The time for bacteria to triple,

$$P(t) = 3P_0$$

$$\Rightarrow P_0 e^{0.406t} = 3P_0$$

$$\therefore t = \frac{\ln 3}{0.406} = 2.71 \text{ h.} \quad (\underline{\text{Ans:}})$$

Q1. An RL circuit has an e.m.f. 5V, Resistance of $50\,\Omega$, Inductance of 1 Henry and no initial current. Find current at any time t.

⇒ Solⁿ :

According to Kirchoff's Law, $L\frac{dI}{dt} + IR = V$

$$\Rightarrow \frac{dI}{dt} + 50\,I = 5 \quad \ldots (i)$$

With the condition, $I(0) = 0 \quad \ldots$ (ii)

I.F. of (i) ⇒ $e^{\int 50\,dt} = e^{50t}$

Multiplying (i) by I.F. ⇒ $e^{50t}\frac{dI}{dt} + e^{50t}\cdot 50\,I = 5e^{50t}$

$$\Rightarrow \frac{d}{dt}(e^{50t}\cdot I) = 5e^{50t}$$

$$\Rightarrow \int d(e^{50t}\cdot I) = \int 5e^{50t}\,dt$$

$$\Rightarrow e^{50t}\cdot I = 5\cdot e^{50t}\cdot\frac{1}{50} + C$$

$$\Rightarrow e^{50t}\cdot I = \frac{1}{10}e^{50t} + C$$

$$\Rightarrow I(t) = \frac{1}{10} + Ce^{-50t} \quad \ldots (iii)$$

When, $t=0$ (iii) becomes,

$$I(0) = \frac{1}{10} + C$$

$$\therefore C = -\frac{1}{10}$$

$\therefore$ (iii) becomes $\Rightarrow$ $I(t) = \frac{1}{10} - \frac{1}{10} e^{-50t}$

$$\therefore I(t) = \frac{1}{10}\left(1 - e^{-50t}\right) \text{ (Ans:)}$$

* Newton's law of cooling $\rightarrow$ $\frac{dT}{dt} \propto (T - T_m)$

$T(t)$ = Temperature of object at time t

T_m = Ambient temperature

$\frac{dT}{dt}$ = Rate of cooling

02. When a cake is removed from oven, its temperature is measured 300°F. 3 minutes later it's temperature is 200°F. How long it will take for the cake to cool off to a temperature of 80°F. Here, room temperature is 70°F.

⇒ Solⁿ:

According to Newton's Law of Cooling, $\frac{dT}{dt} \propto (T - T_m)$

$$\Rightarrow \frac{dT}{dt} = k\,(T - \underset{70}{\underset{\downarrow}{T_m}}) \quad \ldots (i)$$

With conditions, $\left.\begin{array}{l} T(0) = 300°F \\ T(3) = 200°F \end{array}\right\} \ldots (ii)$

(i) $\Rightarrow \int \frac{dT}{T-70} = \int k\,dt$

$\Rightarrow \ln(T-70) = kt + \ln C$

$\Rightarrow \ln\left(\frac{T-70}{C}\right) = kt$

$\Rightarrow T - 70 = e^{kt} \cdot C$

$\Rightarrow T(t) = 70 + Ce^{kt} \quad \ldots (iii)$

$T(0) = 300 = 70 + C \cdot 1$

$\therefore C = 230$

(iii) $\Rightarrow T(t) = 70 + 230\,e^{kt} \quad \ldots (iv)$

when, $t = 3$, (iv) $\Rightarrow 200 = 70 + 230\,e^{3k}$

$\Rightarrow 230\,e^{3k} = 130$

$\Rightarrow 3k = \ln\left(\frac{130}{230}\right)$

$\therefore k = \frac{\ln\left(\frac{130}{230}\right)}{3} = -0.1902$

(iv) $\Rightarrow T(t) = 70 + 230\,e^{-0.1902t} \quad \ldots (v)$

$\therefore 80 = 70 + 230\,e^{-0.1902t}$

$\Rightarrow e^{-0.1902t} = \frac{10}{230}$

$\therefore t = 16.49$ min $= 16$ min 29.4 sec. (Ans.)

PRACTICE PROBLEMS

01. $(D^2 - 3D + 2)y = e^x$

02. $(D^2 - 2D + 1)y = \cos 3x$

03. $(D^2 - 3D + 2)y = e^{5x}$

HOD: Cauchy Euler Equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = R(x)$$

01. $x^2 \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = x^3$ .... (i)

$\Rightarrow$ Sol^n:

Let, $x = e^t$

$\Rightarrow t = \ln x$.

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \quad \text{[Chain Rule]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}$$

$$\Rightarrow x \frac{dy}{dx} = \frac{dy}{dt} \quad \dots \text{(ii)}$$

Diff. (ii) w.r.t. x =>

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dx\,dt}$$

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dt^2} \cdot \frac{dt}{dx}$$

$$\Rightarrow x \frac{d^2y}{dx^2} + \frac{dy}{dx} = \frac{d^2y}{dt^2} \cdot \frac{1}{x}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} + \frac{dy}{dt} = \frac{d^2y}{dt^2}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} \quad \ldots\ (iii)$$

$$(i) \Rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} - 2\frac{dy}{dt} + 2y = (e^t)^3$$

$$\Rightarrow \frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$$

Let, $y = e^{mt} \Rightarrow \frac{dy}{dt} = me^{mt} \Rightarrow \frac{d^2y}{dt^2} = m^2e^{mt}$

$\therefore$ Aux. $Sol^n \Rightarrow e^{mt}(m^2 - 3m + 2) = 0$

$\therefore m_1 = 2,\ m_2 = 1$

$\therefore y_c = C_1 e^{2t} + C_2 e^t$

$$y_p = \frac{1}{3^2 - 9 + 2} e^{3t} = \frac{1}{2} e^{3t}$$

$\therefore y = y_c + y_p$

$= C_1 e^{2t} + C_2 e^t + \frac{1}{2} e^{3t}$

$= C_1 x^2 + C_2 x + \frac{1}{2} x^3$ (Ans:)

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AFTER MID TERM EXAM

HIGHER ORDER D.E.

01. $(D^2+D-2)y = 2(1+x-x^2)$

=> Sol^n:

Aux. Sol^n => $e^{mx}(m^2+m-2)=0$

$\therefore m_1 = 1, \; m_2 = -2$

$\therefore y_c = c_1 e^x + c_2 e^{-2x}$

$$y_p = \frac{1}{\boxed{D^2+D-2}} 2(1+x-x^2)$$

→ দ্বীপদীতে নিতে হবে

$$= \frac{1}{-2\left[1-\left(\frac{D^2+D}{2}\right)\right]} 2(1+x-x^2)$$

→ Constant part common নিতে হবে

$$= -\left[1-\left(\frac{D^2+D}{2}\right)\right]^{-1}(1+x-x^2)$$

$$= -1\left[1+\frac{D+D^2}{2}+\left(\frac{D+D^2}{2}\right)^2+\dots\right](1+x-x^2)$$

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$$= -\left[1 + \frac{D}{2} + \frac{D^2}{2} + \frac{D^2 + 2D^3 + D^4}{4} + \ldots\right](1 + x - x^2)$$

$$= -\left[1 + \frac{D}{2} + \frac{3}{4}D^2 + \ldots\right](1 + x - x^2)$$

$$= -\left[(1 + x - x^2) + \frac{1}{2}(1 - 2x) + \frac{3}{4}(-2)\right]$$

$$= -\left[1 + x - x^2 + \frac{1}{2} - x - \frac{3}{2}\right]$$

$$= -\left[-x^2\right]$$

$$= x^2$$

$$\therefore y = C_1 e^x + C_2 e^{-2x} + x^2 \quad (\text{Ans:})$$

*** 02. $(D^3 - D^2 - 6D)y = x^2 + 1$

=> Sol^n :

Aux. Sol^n => $e^{mx}(m^3 - m^2 - 6m) = 0$

$$\therefore m_1 = 0,\ m_2 = -2,\ m_3 = 3$$

$$\therefore y_c = C_1 e^0 + C_2 e^{-2x} + C_3 e^{3x}$$

$$= C_1 + C_2 e^{-2x} + C_3 e^{3x}$$

$$\therefore y_p = \frac{1}{D^3 - D^2 - 6D}(x^2 + 1)$$

$$= \frac{1}{-6D\left[1 + \left(\frac{D - D^2}{6}\right)\right]}(x^2 + 1)$$

$$= -\frac{1}{6D}\left(1+\frac{D-D^2}{6}\right)^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left(1-\frac{D-D^2}{6}+\left(\frac{D-D^2}{6}\right)^2-\dots\right)(x^2+1)$$

$$= -\frac{1}{6D}\left(1-\frac{D}{6}-\frac{D^2}{6}+\frac{D^2}{36}-\frac{D^3}{18}+\frac{D^4}{36}-\dots\right)(x^2+1)$$

$$= -\frac{1}{6D}\left(1-\frac{D}{6}+\frac{7D^2}{36}\right)(x^2+1)$$

$$= -\frac{1}{6D}\left((x^2+1)-\frac{1}{6}(2x)+\frac{7}{36}(2)\right)$$

$$= -\frac{1}{6D}\left(x^2+1-\frac{1}{3}x+\frac{7}{18}\right)$$

$D \rightarrow$ Differentiation

$\frac{1}{D} \rightarrow$ Integration

$$= -\frac{1}{6D}\left(x^2-\frac{1}{3}x+\frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3}-\frac{1}{3}\cdot\frac{x^2}{2}+\frac{25}{18}\cdot x\right)$$

$$= -\frac{x^3}{18}+\frac{x^2}{36}-\frac{25x}{108}$$

$$\therefore\ y = c_1+c_2e^{-2x}+c_3e^{3x}-\frac{x^3}{18}+\frac{x^2}{36}-\frac{25x}{108}$$ (Ans:)

AFTER WINTER VACATION

LAPLACE TRANSFORMATION

*Dfⁿ for xam :

Let, F(t) be function of t specified for $t>0$. Then the Laplace transformation of F(t) denoted by $\mathcal{L}\{F(t)\}$ is defined by,

$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$, where we assume the parameter s is real.

	$F(t)$		$\mathcal{L}\{F(t)\} = f(s)$
01.	c	$\longrightarrow$	$\frac{c}{s}$
02.	t^n	$\longrightarrow$	$\frac{n!}{s^{n+1}}$
03.	e^{at}	$\longrightarrow$	$\frac{1}{s-a}$
04.	$\sin(at)$	$\longrightarrow$	$\frac{a}{s^2+a^2}$
05.	$\cos(at)$	$\longrightarrow$	$\frac{s}{s^2+a^2}$
06.	$\sin h(at)$	$\longrightarrow$	$\frac{a}{s^2-a^2}$
07.	$\cos h(at)$	$\longrightarrow$	$\frac{s}{s^2-a^2}$

→ For hyperbolic function

$$\boxed{\begin{aligned} e^{-\infty} &= 0 \\ e^{0} &= 1 \\ e^{\infty} &= \infty \end{aligned}}$$

01. Prove that, $\mathcal{L}\{c\} = \frac{c}{s}$

⇒ Sol^n :

By defitition of Laplace Transformation,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

Here,

$F(t) = c$

$$\therefore \mathcal{L}\{c\} = \int_0^\infty e^{-st} c\, dt$$

$$= c \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= c/s \left[-e^{-s \cdot \infty} + e^0\right]$$

$$= \frac{c}{s} [0+1]$$

$$= \frac{c}{s}$$ [Proved].

02. Prove that, $\mathcal{L}\{t\} = \frac{1}{s^2}$

⇒ Sol^n:

By definition of Laplace Transformation,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

Here, $F(t) = t$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

LIATE

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx$$

sign	D	I
		e^{-st}
+	t	$\frac{e^{-st}}{-s}$
−	1	$\frac{e^{-st}}{s^2}$
+	0	

* যে u যাবে D (diff.) -এ
যে v যাবে I (int.) -এ
যতক্ষণ D = 0 না হবে

$$\therefore \mathcal{L}\{t\} = \left[-\frac{t\,e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^\infty$$

$$= \frac{-t\,e^{-\infty}}{s} - \frac{e^{-\infty}}{s^2} + \frac{0 \cdot e^0}{s} + \frac{e^0}{s^2}$$

$$= \frac{1}{s^2}$$ [Proved]

* Prove that, $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$.

=> Sol^n :

By definition of Laplace Transform,

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt$$

Here, $F(t) = e^{at}$

$$\therefore \mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{-st} . e^{at}\, dt$$

$$= \int_0^{\infty} e^{-t(s-a)}\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)} \right]_0^{\infty}$$

$$= \frac{e^{-\infty}}{-(s-a)} + \frac{e^{-0}}{s-a}$$

$$= \frac{1}{s-a}$$ [Proved]

* Prove that, $\mathcal{L}\{\sin at\} = \dfrac{a}{s^2+a^2}$.

⇒ Solⁿ:

By definition of Laplace Transformation,

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st}\, F(t)\, dt$$

Here,
$F(t) = \sin(at)$

$$\therefore \mathcal{L}\{\sin(at)\} = \int_0^{\infty} e^{-st} \cdot \sin(at)\, dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2} (-s \sin at - a \cos at) \right]_0^{\infty}$$

$$= \frac{1}{s^2+a^2} (0 - 0 + 0 + a)$$

$$= \frac{a}{s^2+a^2} \quad \text{[Proved]}.$$

* $\int e^{\alpha t} \cdot \sin(\beta t)\, dt$

$$= \frac{e^{\alpha t}(\alpha \sin \beta t - \beta \cos \beta t)}{\alpha^2 + \beta^2}$$

* $\int e^{\alpha t} \cdot \cos(\beta t)\, dt$

$$= \frac{e^{\alpha t}(\alpha \cos \beta t + \beta \sin \beta t)}{\alpha^2 + \beta^2}$$

H.W. Prove that, $\mathcal{L}\{\cos(at)\} = \dfrac{s}{s^2+a^2}$.

⇒ Sol^n:

By the definition of Laplace Transformation,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

Here, $F(t) = \cos(at)$

$$\therefore \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st}.\cos(at)\,dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}\left(-s\cos(at) + a\sin(at)\right)\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left(\cancel{0} + \cancel{0} + s + 0\right)$$

$$= \frac{s}{s^2+a^2} \quad \text{[Proved]}.$$

* Prove that, $\mathcal{L}\{\cosh(at)\} = \dfrac{s}{s^2-a^2}$

⇒ Sol^n :

By defination of Laplace Transformation,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} \cdot F(t) \cdot dt$$

Here, $F(t) = \cosh(at)$

$$\therefore \mathcal{L}\{\cosh(at)\} = \int_0^\infty e^{-st} \cdot \cosh(at) \cdot dt$$

$$= \int_0^\infty \frac{e^{at}+e^{-at}}{2} \cdot e^{-st} \cdot dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-t(s-a)} + e^{-t(s+a)}\right) dt$$

$$= \frac{1}{2}\left(\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty + \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty\right)$$

$$= \frac{1}{2}\left(0 + \frac{1}{s-a} + 0 + \frac{1}{s+a}\right)$$

$$= \frac{1}{2} \cdot \frac{s-a+s+a}{(s-a)(s+a)}$$

$$= \frac{1}{2} \cdot \frac{2s}{s^2-a^2}$$

$$= \frac{s}{s^2-a^2}$$ [Proved].

* $\cosh(at) = \dfrac{e^{at}+e^{-at}}{2}$

* $\sinh(at) = \dfrac{e^{at}-e^{-at}}{2}$

H.W. Prove that, $\mathcal{L}\{\sinh(at)\} = \dfrac{a}{s^2 - a^2}$.

$\Rightarrow$ Sol^n :

By definition of Laplace Transformation,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

Here, $F(t) = \sinh(at)$

$$\therefore \mathcal{L}\{\sinh(at)\} = \int_0^\infty e^{-st} \cdot \sinh(at)\, dt$$

$$= \int_0^\infty \frac{e^{at} - e^{-at}}{2} \cdot e^{-st}\, dt$$

$$= \frac{1}{2} \int_0^\infty \left(e^{-t(s-a)} - e^{-t(s+a)} \right) dt$$

$$= \frac{1}{2} \left(\left[\frac{e^{-t(s-a)}}{-(s-a)} \right]_0^\infty - \left[\frac{e^{-t(s+a)}}{-(s+a)} \right]_0^\infty \right)$$

$$= \frac{1}{2} \left(0 + \frac{1}{s-a} - 0 - \frac{1}{s+a} \right)$$

$$= \frac{1}{2} \cdot \frac{s+a-s+a}{s^2 - a^2} = \frac{1}{2} \cdot \frac{2a}{s^2 - a^2}$$

$$= \frac{a}{s^2 - a^2}$$

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PROPERTIES :

01. LINEARITY PROPERTY :

If C_1, C_2 are any constants, while $F_1(t)$, $F_2(t)$ are functions with Laplace Transforms $f_1(s)$, $f_2(s)$ respectively, then,

$$\mathcal{L}\{C_1F_1(t) + C_2F_2(t)\} = C_1\mathcal{L}\{F_1(t)\} + C_2\mathcal{L}\{F_2(t)\} = C_1f_1(s) + C_2f_2(s)$$

(a) $\mathcal{L}\{5e^{-t} + 2\cos 5t - 3\}$

$= 5\mathcal{L}(e^{-t}) + 2\mathcal{L}(\cos 5t) - \mathcal{L}(3)$

$= 5 \cdot \frac{1}{s+1} + 2 \cdot \frac{s}{s^2+25} - \frac{3}{s}$ (Ans:)

(b) $\mathcal{L}\{(t^2+1)^2\} = \mathcal{L}\{t^4 + 2t^2 + 1\}$

$= \frac{4!}{s^5} + 2 \cdot \frac{2!}{s^3} + \frac{1}{s}$ (Ans:)

PROPERTIES OF LAPLACE TRANSFORM

02. First Translation or Shifting Property:

If $\mathcal{L}\{F(t)\} = f(s)$, then, $\mathcal{L}\{e^{at} F(t)\} = f(s-a)$

(a) $\mathcal{L}\{e^{-2t} \cos 5t\}$

$\Rightarrow$ Solⁿ: We know, $\mathcal{L}\{\cos 5t\} = \dfrac{s}{s^2+25}$

$$\therefore \mathcal{L}\{e^{-2t} \cos 5t\} = \frac{s-(-2)}{\{s-(-2)\}^2+25} = \frac{s+2}{(s+2)^2+25}$$

(b) $\mathcal{L}\{e^{5t} \sin t\}$

$\Rightarrow$ Solⁿ: We know, $\mathcal{L}\{\sin t\} = \dfrac{1}{s^2+1}$

$$\therefore \mathcal{L}\{e^{5t} \sin t\} = \frac{1}{(s-5)^2+1}$$

03. Second Translation or Shifting property:

If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$ then,

$$\mathcal{L}\{G(t)\} = e^{-as} f(s)$$

01. If $G(t) = \begin{cases} (t-2)^3, & t > 2 \\ 0, & t < 2 \end{cases}$, Find $\mathcal{L}\{G(t)\}$

$\Rightarrow$ Sol^n:

$$\mathcal{L}\{G(t)\} = e^{-2s} \cdot \frac{3!}{s^4}$$

02. If $G(t) = \begin{cases} \sin\left(t - \frac{\pi}{2}\right), & t > \frac{\pi}{2} \\ 0, & t < \frac{\pi}{2} \end{cases}$, Find $\mathcal{L}\{G(t)\}$.

$\Rightarrow$ Sol^n:

$$\mathcal{L}\{G(t)\} = e^{-\pi/2 \cdot s} \cdot \frac{1}{s^2+1}$$ (Ans:)

04. Multiplication by t^n:

If $\mathcal{L}\{F(t)\} = f(s)$, then

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \cdot \frac{d^n}{ds^n} f(s)$$

01. $\mathcal{L}\{t^2 \cos at\}$

$\Rightarrow$ Sol^n: $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

$$\therefore \mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left\{\frac{(s^2+a^2)\frac{d}{ds}(s) - s\,\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\left\{\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right\}$$

$$= \frac{d}{ds}\,\frac{a^2-s^2}{(s^2+a^2)^2}$$

$$= \frac{(s^2+a^2)^2\cdot\frac{d}{ds}(a^2-s^2) - (a^2-s^2)\frac{d}{ds}(s^2+a^2)^2}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2)^2 - (a^2-s^2)\,2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2) - 4s(a^2-s^2)}{(s^2+a^2)^3}$$

$$= \frac{-2s^3-2a^2s-4a^2s+4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3-6a^2s}{(s^2+a^2)^3}$$ (Ans:)

02. $\mathcal{L}\{t^{50} e^{2t}\}$

$\Rightarrow$ Sol^n:

we know, $\mathcal{L}\{t^{50}\} = \dfrac{50!}{s^{51}}$

$\therefore \mathcal{L}\{t^{50} e^{2t}\} = \dfrac{50!}{(s-2)^{51}}$ (Ans:)

LAPLACE TRANSFORM OF PERIODIC FUNCTION

Let, $F(t)$ is Periodic function having period T. So,

$$\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt$$

Q1. $F(t) = \begin{cases} 2t^2, & 0<t<1 \\ t, & 1<t<2 \end{cases}$ → Piecewise function

is periodic function of period 2.

Find (i) $\mathcal{L}\{F(t)\}$

(ii) Sketch $F(t)$

⇒ Solⁿ: (i) $\mathcal{L}\{F(t)\} = \dfrac{1}{1-e^{-2s}} \displaystyle\int_0^2 e^{-st} F(t)\,dt$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} 2t^2\,dt + \int_1^2 e^{-st}\, t\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[2\int_0^1 e^{-st}\, t^2\,dt + \int_1^2 e^{-st}\, t\,dt \right]$$

sign	D	I
+	t^2	e^{-st}
−	$2t$	$-\frac{e^{-st}}{s}$
+	2	$+\frac{e^{-st}}{s^2}$
−	0	$-\frac{e^{-st}}{s^3}$

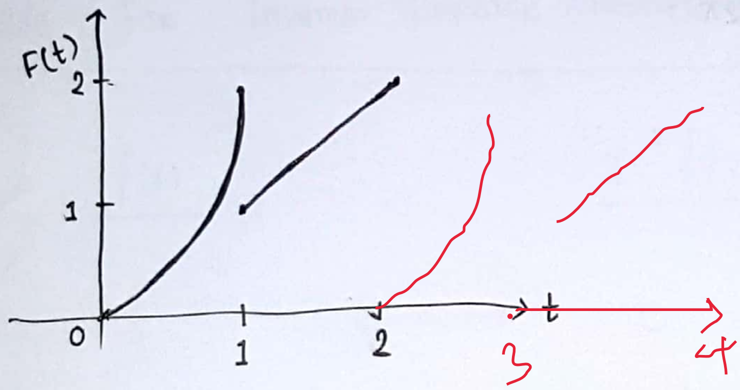
sign	D	I
+	t	e^{-st}
−	1	$-\frac{e^{-st}}{s}$
+	0	$\frac{e^{-st}}{s^2}$

$$= \frac{1}{1-e^{-2s}}\left[2\left[-\frac{t^2e^{-st}}{s}-\frac{2te^{-st}}{s^2}-\frac{2e^{-st}}{s^3}\right]_0^1+\left[-\frac{te^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_1^2\right.$$

$$= \frac{1}{1-e^{-2s}}\left[2\left[\cancel{\frac{2}{s^3}}+\left(-\frac{e^{-s}}{s}\right)-\frac{2e^{-s}}{s^2}-\frac{2e^{-s}}{s^3}+\frac{2}{s^3}\right]+\right.$$

$$\left.\left[-\frac{2e^{-2s}}{s}+\frac{e^{-s}}{s}-\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s^2}\right]\right] \text{(Ans:)}$$

• (ii)



Periodic function-এ (-)ve part দেখা লাগাবে না since 0 থেকে start হয়।

* $F(t) = \begin{cases} \sin t & , 0<t<\pi \\ 0 & , \pi<t<2\pi \end{cases}$, is periodic function of period 2π.

Find (i) $\mathcal{L}\{F(t)\}$ (ii) sketch $F(t)$

(i) $\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}F(t)dt$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\, dt + \int_{\pi}^{2\pi} e^{-st}\, 0\, dt$$

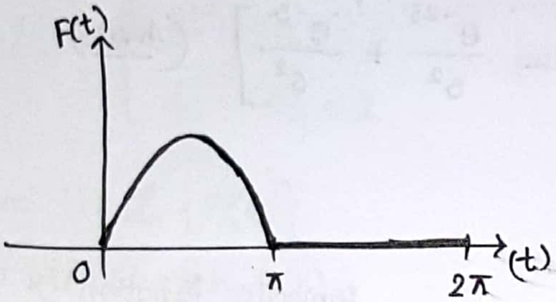
$$\int e^{\alpha t}\sin\beta t\, dt = \frac{e^{\alpha t}}{\alpha^2+\beta^2}(\alpha\sin\beta t-\beta\cos\beta t)$$

$$= \frac{1}{1-e^{-2\pi s}\,(s^2+1)}\left[-s\,e^{-st}\sin t - e^{-st}\cos t\right]_0^{\pi}$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)} \left[-e^{-\pi s} \cos \pi + e^0 \cos 0 \right]$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)} \left(e^{-\pi s} + 1 \right) \text{ (Ans:)}$$

(ii)



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INVERSE LAPLACE TRANSFORM

* Dfn for xam

If the Laplace Transform of $F(t)$ is $f(s)$, i.e. $\mathcal{L}\{F(t)\} = f(s)$, then $F(t)$ is called an inverse Laplace Transform of $f(s)$ and we write symbolically $F(t) = \mathcal{L}^{-1}\{f(s)\}$, where $\mathcal{L}^{-1}$ is called inverse Laplace Transform operator.

Table for Inverse Laplace Transformation

	$f(s)$		$\mathcal{L}^{-1}\{f(s)\} = F(t)$
01.	$\frac{c}{s}$	$\longrightarrow$	c
02.	$\frac{1}{s-a}$	$\longrightarrow$	e^{at}
03.	$\frac{1}{s^{n+1}}$	$\longrightarrow$	$\frac{t^n}{n!}$
04.	$\frac{1}{s^2+a^2}$	$\longrightarrow$	$\frac{\sin(at)}{a}$
05.	$\frac{s}{s^2+a^2}$	$\longrightarrow$	$\cos(at)$
06.	$\frac{1}{s^2-a^2}$	$\longrightarrow$	$\frac{\sinh(at)}{a}$
07.	$\frac{s}{s^2-a^2}$	$\longrightarrow$	$\cosh(at)$

* $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

* $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

* $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$

* $\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin(5t)}{5}$

* $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos(4t)$

* $\mathcal{L}^{-1}\left\{\frac{1}{s^2-25}\right\} = \frac{\sin h(5t)}{5}$

* $\mathcal{L}^{-1}\left\{\frac{s}{s^2-16}\right\} = \cos h(4t)$

PROPERTIES

01. Linearity Property:

* $\mathcal{L}^{-1}\left\{2\,\frac{1}{s^5} - \frac{s}{s^2+25} + \frac{1}{s-2} + \frac{5}{s}\right\}$

$$= 2\cdot\frac{t^4}{4!} - \cos(5t) + e^{2t} + 5 \quad \text{(Ans:)}$$

INVERSE LAPLACE TRANSFORM

CT 03-A20 20

First Translation or Shifting Property :

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at}\,F(t)$.

01. $\mathcal{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$$= 6\,\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 8\,\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6\cos 4t\, e^{2t} + 8\,\frac{\sin 4t}{4}\, e^{2t}$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t \quad \text{(Ans:)}$$

02. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-2s+5}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$= \frac{\sin 2t}{2}\, e^{t} \quad \text{(Ans:)}$$

* HEAVISIDE Expansion Formula

*Xam

Let, $P(s), Q(s)$ be polynomials, where $P(s)$ has degree less that that of $Q(s)$. suppose that $Q(s)$ has 'n' distinct zeros, a_k, $K=1,2,3,\dots n$.

↳ যাদের value এর জন্য function undefined হবে।

Then,

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{K=1}^{n} \frac{P(a_K)}{Q'(a_K)} e^{a_K t}$$

$$= \frac{P(a_1)}{Q'(a_1)} e^{a_1 t} + \frac{P(a_2)}{Q'(a_2)} e^{a_2 t} + \dots + \frac{P(a_n)}{Q'(a_n)} e^{a_n t}$$

01. $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

⇒ Sol^n:

$$P(s) = 2s^2 - 4$$

$$Q(s) = (s+1)(s-2)(s-3)$$

$$= s^3 - 4s^2 + s + 6$$

$$Q'(s) = 3s^2 - 8s + 1$$

Here, $Q(s)$ has three distinct zeros,

~~zeros~~ say, $a_1 = -1$, $a_2 = 2$, $a_3 = 3$

By HEAVISIDE Expansion formula,

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \frac{P(-1)}{Q'(-1)} e^{-t} + \frac{P(2)}{Q'(2)} e^{2t} + \frac{P(3)}{Q'(3)} e^{3t}$$

$$= \left(\frac{-2}{12}\right)e^{-t} + \left(\frac{-4}{3}\right)e^{2t} + \left(\frac{14}{4}\right)e^{3t} = -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t} \quad \text{(Ans:)}$$

PARTIAL FRACTION

01. $\mathcal{L}^{-1}\left\{\dfrac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

⇒ Solⁿ:

Let, $$\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$$

$$\Rightarrow 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$$

At $s=2$, (i) ⇒

$$2\cdot 2^2 - 4 = \cancel{A} + B(2+1)(2-3) + C(2+1)(2-2) \Rightarrow B = -\frac{4}{3}$$

At $s=3$, (i) ⇒ $2\cdot 3^2 - 4 = C(3+1)(3-2) \Rightarrow C = \dfrac{14}{4}$

At $s=-1$, (i) ⇒ $2(-1)^2-4 = A(-1-2)(-1-3) \Rightarrow A = -\dfrac{2}{12}$

$$\therefore \frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{-2}{12(s+1)} - \frac{4}{3(s-2)} + \frac{14}{4(s-3)}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{-2}{12(s+1)} - \frac{4}{3(s-2)} + \frac{14}{4(s-3)}\right\}$$

$$= -\frac{2}{12}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{14}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$ (Ans.)

INVERSE LAPLACE TRANSFORM

Partial Fraction

01. $\mathcal{L}^{-1}\left\{\dfrac{3s+1}{(s-1)(s^2+1)}\right\}$

⇒ Solⁿ:

Let, $\dfrac{3s+1}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$

$\Rightarrow 3s+1 = A(s^2+1) + (Bs+C)(s-1) \quad \ldots (i)$

At $s=1$, (i) $\Rightarrow 3 \cdot 1 + 1 = A(s^2+1) + (Bs+C)(s-1) \xrightarrow{(i)} \Rightarrow 2A = 4 \Rightarrow A = 2$

Again from (i),

$3s+1 = A(s^2+1) + Bs^2 + Cs - Bs - C$

$\Rightarrow 3s+1 = s^2(A+B) + s(C-B) + (A-C) \quad \ldots (ii)$

Equating co-efficients of s^2, s from (ii),

$A+B = 0 \Rightarrow B = -A = -2$

$C-B = 3 \Rightarrow C = 3+B = 3-2 = 1$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$= 2e^t - 2\cos t + \sin t \quad \text{(Ans:)}$$

02. $\mathcal{L}^{-1}\left\{\dfrac{s^2-2s+3}{(s-1)^2(s+1)}\right\}$

$\Rightarrow$ Sol^n:

Let, $$\frac{s^2-2s+3}{(s-1)^2(s+1)} = \frac{A}{(s+1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)}$$

$$\Rightarrow s^2-2s+3 = A(s-1)^2 + B(s+1) + C(s-1)(s+1) \quad \ldots\ldots (i)$$

At $s=1$, (i) $\Rightarrow 1-2+3 = B(1+1) \Rightarrow 2B = 2 \Rightarrow B = 1$

At $s=-1$, (i) $\Rightarrow (-1)^2 - 2(-1) + 3 = A(-1-1)^2 \Rightarrow 4A = 6 \Rightarrow A = \frac{3}{2}$

Also from (i),

$$s^2-2s+3 = C(s^2-1) + B(s+1) + A(s^2-2s+1)$$

$$\Rightarrow s^2-2s+3 = s^2(A+C) + s(B-2C) + (B-A+C)$$

Equating co-efficients of s^2 from (ii),

$A+C = 1 \Rightarrow C = 1-A = 1-\frac{3}{2} = -\frac{1}{2}$

$$\frac{s^2-2s+3}{(s-1)^2(s+1)} = \frac{-1}{2(s-1)} + \frac{1}{(s-1)^2} + \frac{3}{2(s+1)}$$

$$\mathcal{L}^{-1}\left\{\frac{s^2-2s+3}{(s-1)^2(s+1)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{2(s-1)} + \frac{1}{(s-1)^2} + \frac{3}{2(s+1)}\right\}$$

$$= -\frac{1}{2}e^t + te^t + \frac{3}{2}e^{-t}$$ (Ans.)

Laplace Transform of Derivatives:

(i) Take Laplace Transform

(ii) Apply Formula

(iii) Apply Conditions

(iv) Solve for $f(s)$

(v) Take Inverse Laplace Transform

(vi) Find $F(t)$

If $\mathcal{L}\{F(t)\} = f(s)$ then,

(i) $\mathcal{L}\{F'(t)\} = s f(s) - F(0)$

(ii) $\mathcal{L}\{F''(t)\} = s^2 f(s) - s F(0) - F'(0)$

01. $F''(t) + 4F(t) = 0$, $F(0) = 10$, $F'(0) = 0$

$\Rightarrow$ Soln:

$$F''(t) + 4F(t) = 0$$

$$\Rightarrow \mathcal{L}\{F''(t)\} + 4\mathcal{L}\{F(t)\} = \mathcal{L}(0)$$

$$\Rightarrow \left(s^2 f(s) - s F(0) - F'(0)\right) + 4 f(s) = 0$$

$$\Rightarrow s^2 f(s) - 10s + 4 f(s) = 0$$

$$\Rightarrow (s^2+4) f(s) = 10s$$

$$\Rightarrow f(s) = \frac{10s}{s^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}\{f(s)\} = 10\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$\therefore F(t) = 10 \cos 2t \text{ (Ans:)}$$

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APPLICATION OF LAPLACE TRANSFORM

01. $F'' + F = t,\ F(0) = 1,\ F'(0) = -2$

⇒ Solⁿ: Here, $F'' + F = t$

$$\Rightarrow \mathcal{L}\{F''\} + \mathcal{L}\{F\} = \mathcal{L}\{t\}$$

$$\Rightarrow \left(s^2 f(s) - sF(0) - F'(0)\right) + f(s) = \frac{1}{s^2}$$

$$\Rightarrow s^2 f(s) - s + 2 + f(s) = \frac{1}{s^2}$$

$$\Rightarrow (s^2+1) f(s) = \frac{1}{s^2} + s - 2$$

$$\Rightarrow f(s) = \frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}$$

$$\Rightarrow \mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}\right\}$$

$$\Rightarrow F(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \quad \ldots (i)$$

Let

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$\Rightarrow 1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$$

$$\Rightarrow 1 = A(s^3+s) + B(s^2+1) + (Cs^3 + Ds^2)$$

$$\Rightarrow 1 = s^3(A+C) + s^2(B+D) + As + B \quad \ldots (ii)$$

Equatting co-efficients of from both sides of (ii)

$A+C = 0 \Rightarrow C = 0$

$B+D = 0 \Rightarrow D = -B = -1$

$A = 0\ ,\ B = 1$

$$\therefore (i) \Rightarrow F\{t\} = \mathcal{L}^{-1}\left\{\frac{0}{s} + \frac{1}{s^2} + \frac{0.s-1}{s^2+1}\right\}$$

$$+\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$\therefore F\{t\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} + \cos t - 2\sin t$$

$$= t - \sin t + \cos t - 2\sin t$$

$$= t - 3\sin t + \cos t$$ (Ans:)

H.W.

01. $F'' - 3F' + 2F = 4e^{2t}$, ~~F(0)~~, $F(0) = -3$, $F'(0) = 5$

Ans: $F(t) = -7e^{t} + 4e^{2t} + 4te^{2t}$.

02. $F'' + 9F = \cos 2t$, $F(0) = 1$, $F(0) = C$

Ans: $F(t) = \frac{4}{5}\cos 3t + \frac{C}{3}\sin 3t + \frac{1}{5}\cos 2t$.

02. A Particle p of mass 2 gram moves on X-axis and is attracted toward origin o with a force numarically equal to 8 . If it is initially at rest at $X = 10$, find its position at any subsequent time assuming,

(a) no other force act,

(b) a damping force equal to 8 times the instantaneous velocity acts.

=> Soln :

(a) When no other force acts,

$$m \cdot \frac{d^2 x}{dt^2} \quad -k x$$

$$\Rightarrow 2 \frac{d^2 x}{dt^2} = -8X, \text{ with initial conditions } X(0) = 10,\ X'(0) = 0$$

$$\Rightarrow \frac{d^2 X}{dt^2} + 4X = 0$$

$$\Rightarrow \mathcal{L}\left\{\frac{d^2 X}{dt^2}\right\} + 4\mathcal{L}\{X\} = \mathcal{L}\{0\}$$

$$\Rightarrow \left(s^2 x(s) - sX(0) - X'(0)\right) + 4x(s) = 0$$

$$\Rightarrow s^2 x(s) - 10s + 4x(s) = 0$$

$$\Rightarrow (s^2+4)x(s) = 10s$$

$$\Rightarrow x(s) = \frac{10s}{s^2+4}$$

$$\Rightarrow \mathcal{L}^{-1}\{x(s)\} = \mathcal{L}^{-1}\left\{\frac{10s}{s^2+4}\right\}$$

$$\therefore X(t) = 10\cos 2t \quad \text{(Ans:)}$$

(b) When damping force acts,

$$m\frac{d^2x}{dt^2} = -Kx - a\frac{dx}{dt}$$

$$\Rightarrow 2\frac{d^2x}{dt^2} = -8x - 8\frac{dx}{dt}$$, with initial conditions $x(0)=10,\ x'(0)=0$

$$\Rightarrow \frac{d^2x}{dt^2} + 4x + 4\frac{dx}{dt} = 0$$

$$\Rightarrow \mathcal{L}\left\{\frac{d^2x}{dt^2}\right\} + 4\mathcal{L}\{x\} + 4\mathcal{L}\left\{\frac{dx}{dt}\right\} = \mathcal{L}\{0\}$$

$$\Rightarrow \left(s^2x(s) - s\,x(0) - \cancel{x'(0)}\right) + 4x(s) + 4\left(s x(s) - x(0)\right) = 0$$

$$\Rightarrow s^2x(s) - 10s + 4x(s) + 4(sx(s) - 10) = 0$$

$$\Rightarrow s^2x(s) - 10s + 4x(s) + 4sx(s) - 40 = 0$$

$$\Rightarrow (s^2+4s+4)x(s) = 10s+40$$

$$\Rightarrow x(s) = \frac{10s+40}{s^2+4s+4}$$

$$\Rightarrow \mathcal{L}^{-1}\{x(s)\} = \mathcal{L}^{-1}\left\{\frac{10\cdot s+40}{s^2+4s+4}\right\}$$

$$\Rightarrow x(t) = \mathcal{L}^{-1}\left\{\frac{10(s+2)+20}{(s+2)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{10(s+2)}{(s+2)^2}\right\} + 20\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= 10\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 20te^{-2t}$$

$$= 10e^{-2t} + 20te^{-2t}$$ (Ans:)

FOURIER SERIES

Full Range Fourier Series:

*** Xam

Let, $f(x)$ is periodic function with period, $T = 2L$ or $f(x)$ is defined on the interval $-L < x < L$. Then the Fourier series representation is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

; where a_0, a_n, b_n are Fourier coefficients.

Here,
$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Q1. Find Full Range Fourier Series for $f(x) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$; having period 2.

=> Soln:

We know, $T = 2L \Rightarrow 2L = 2 \Rightarrow L = 1$

$\therefore$ $f(x)$ is defined on $-1 < x < 1$.

* $\sin(n\pi) = 0$
* $\cos(n\pi) = (-1)^n$
* $\cos(-\theta) = \cos\theta$
* $\sin(-\theta) = -\sin\theta$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\,dx$$

$$= \int_{-1}^{0}(-1)\,dx + \int_{0}^{1}(1)\,dx$$

$$= \left[-x\right]_{-1}^{0} + \left[x\right]_{0}^{1}$$

$$= -\left[0+1\right] + \left[1-0\right]$$

$$= -1+1 = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= 1\int_{-1}^{1} f(x)\cos(n\pi x)\,dx$$

$$= \int_{-1}^{0} -\cos(n\pi x)\,dx + \int_{0}^{1}\cos(n\pi x)\,dx$$

$$= \left[\frac{-\sin(n\pi x)}{n\pi}\right]_{-1}^{0} + \left[\frac{\sin(n\pi x)}{n\pi}\right]_{0}^{1}$$

$$= -\frac{1}{n\pi}\left[\sin(n\pi x)\right]_{-1}^{0} + \frac{1}{n\pi}\left[\sin(n\pi x)\right]_{0}^{1}$$

$$= -\frac{1}{n\pi}\left[\sin 0 - \sin(-n\pi)\right] + \frac{1}{n\pi}\left[\sin(n\pi) - \sin 0\right]$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= 1\int_{-1}^{1} f(x)\,\sin(n\pi x)\,dx$$

$$= \int_{-1}^{0} -\sin(n\pi x)\,dx + \int_{0}^{1} \sin(n\pi x)\,dx$$

$$= \left[\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[\frac{-\cos n\pi x}{n\pi}\right]_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\cos(n\pi x)\right]_{-1}^{0} - \frac{1}{n\pi}\left[\cos(n\pi x)\right]_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\cos 0 - \cos(-n\pi)\right] - \frac{1}{n\pi}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{1}{n\pi}\left[\cos 0 - \cos(n\pi)\right] - \frac{1}{n\pi}\left[\cos(n\pi) - \cos 0\right]$$

$$= \frac{1}{n\pi}\left[1 - (-1)^n - (-1)^n + 1\right]$$

$$= \frac{1}{n\pi}\left[2 - 2(-1)^n\right]$$

$\therefore$ Full Range Fourier Series :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

$$= \sum_{n=1}^{\infty} \frac{1}{n\pi}\left(2-2(-1)^n\right) \sin(n\pi x) \quad \text{(Ans:)}$$

FOURIER SERIES

02. Find Full Range Fourier Series for $f(x) = e^x$, $-\pi < x < \pi$.

$\Rightarrow$ Sol^n :

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$$

$$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos(nx)\,dx$$

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$$= \frac{1}{\pi}\left[\frac{e^x(\cos(nx) + n\sin(nx))}{1+n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^x\cos(nx) + n e^x \sin(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^\pi\cos(n\pi) - e^{-\pi}\cos(-n\pi) + n e^\pi \sin(n\pi) - n e^{-\pi}\sin(-n\pi)\right]$$

$$= \frac{1}{\pi(n^2+1)}\left[e^\pi(-1)^n - e^{-\pi}(-1)^n\right]$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x(\sin(nx) - n\cos(nx))}{1+n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^x\sin(nx) - n e^x\cos(nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi(n^2+1)}\left[e^\pi\sin(n\pi) - e^{-\pi}\sin(-n\pi) - n e^\pi\cos(n\pi) + n e^{-\pi}\cos(-n\pi)\right]$$

$$= \frac{1}{\pi(n^2+1)} \left[n e^{-\pi}(-1)^n - n e^{\pi}(-1)^n \right]$$

$\therefore$ Full Range Fourier Series is :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{e^{\pi}(-1)^n - e^{-\pi}(-1)^n}{\pi(n^2+1)} \cos(nx) + \frac{n e^{-\pi}(-1)^n - n e^{\pi}(-1)^n}{\pi(n^2+1)} \sin(nx) \right]$$ (Ans:)

H.W.

Find Full Range Fourier Series for $f(x) = e^{-x}$, $-\pi < x < \pi$.

HALF RANGE FOURIER SERIES

HALF RANGE FOURIER COSINE SERIES

* Df^n A half range cosine series is a series in which cosine term is present and function, $f(x)$ is defined on the interval $0<x<L$.

The Half Range Fourier Cosine Series representation is :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$-L \quad 0 \quad L$

Here,

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$$

HALF RANGE FOURIER SINE SERIES

* Df^n A Half Range Fourier Sine Series is a series in which only sine term is present and $f(x)$ is defined on $0<x<L$. The Half Range Fourier Sine Series representation is :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) dx \; ; \text{ where, } b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

01. Find Half Range Fourier Cosine and Sine Series for $f(x) = x^2$, $0<x<3$.

⇒ Sol^n :

Half Range Cosine Series:

$$a_0 = \frac{2}{3}\int_0^3 x^2\,dx$$

$$= \frac{2}{3}\left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{9}(27-0) = 6$$

$$a_n = \frac{2}{3}\int_0^3 x^2 \cos\left(\frac{n\pi x}{3}\right) dx$$

Sign	D	I
		$\cos\frac{n\pi x}{3}$
+	x^2	$\frac{3\sin\frac{n\pi x}{3}}{n\pi}$
−	$2x$	$\frac{9\left(-\cos\frac{n\pi x}{3}\right)}{(n\pi)^2}$
+	2	$\frac{27\left(-\sin\frac{n\pi x}{3}\right)}{(n\pi)^3}$
−	0	

$$= \frac{2}{3}\left[x^2 \cdot \frac{3\sin\left(\frac{n\pi x}{3}\right)}{n\pi} - 2x\cdot\frac{9\left(-\cos\frac{n\pi x}{3}\right)}{(n\pi)^2} + \frac{27\left(-\sin\frac{n\pi x}{3}\right)}{(n\pi)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{x^3}{n\pi}\cdot 3\sin\left(\frac{n\pi x}{3}\right) + \frac{2x}{n^2\pi^2}\cdot 9\cos\left(\frac{n\pi x}{3}\right) - \frac{27\cdot 2}{n^3\pi^3}\sin\left(\frac{n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{3x^3}{n\pi}\sin\left(\frac{n\pi x}{3}\right) + \frac{18x}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right) - \frac{54}{n^3\pi^3}\sin\left(\frac{n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{54}{n^2\pi^2}\cos(n\pi) - 0\right]$$

$$= \frac{36}{n^2\pi^2}(-1)^n$$

$\therefore$ Half Range Cosine Series is :

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= 3 + \sum_{n=1}^{\infty} \frac{36}{n^2\pi^2}(-1)^n \cos\left(\frac{n\pi x}{3}\right) \quad \text{(Ans:)}$$

$$b_n = \frac{2}{3}\int_0^3 x^2 \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{2}{3}\left[-\frac{3x^2}{n\pi}\cos\left(\frac{n\pi x}{3}\right) + \frac{18x}{n^2\pi^2}\sin\left(\frac{n\pi x}{3}\right) + \frac{54}{n^3\pi^3}\cos\left(\frac{n\pi x}{3}\right)\right]_0^3$$

Sign	D	I
		$\sin\frac{n\pi x}{3}$
+	x^2	$-\frac{3}{n\pi}\cos\left(\frac{n\pi x}{3}\right)$
−	$2x$	$-\frac{9}{n^2\pi^2}\sin\left(\frac{n\pi x}{3}\right)$
+	2	$\frac{27}{n^3\pi^3}\cos\left(\frac{n\pi x}{3}\right)$
−	0	

$$= \frac{2}{3}\left[-\frac{27}{n\pi}\cos(n\pi) + \frac{54}{n^3\pi^3}\cos(n\pi) - \frac{54}{n^3\pi^3}\cos(0)\right]$$

$$= \frac{2}{3}\left[-\frac{27}{n\pi}(-1)^n + \frac{54}{n^3\pi^3}(-1)^n - \frac{54}{n^3\pi^3}\right]$$

$\therefore$ Half Range Sine Series is :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2}{3}\left[-\frac{27}{n\pi}(-1)^n + \frac{54}{n^3\pi^3}(-1)^n - \frac{54}{n^3\pi^3}\right]\sin\left(\frac{n\pi x}{3}\right) \quad \text{(Ans:)}$$

FINITE FOURIER SINE TRANSFORM

$$f_s = \int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right) dx$$

FINITE FOURIER COSINE TRANSFORM

$$f_0 = \int_0^L f(x)\,dx, \quad f_c = \int_0^L f(x)\,\cos\left(\frac{n\pi x}{L}\right) dx$$

Q1. Find Finite Fourier Cosine and Sine Transform for $f(x) = x$, $0 < x < 4$.

⇒ **Sol^n:**

Finite Cosine Transform:

$$f_0 = \int_0^4 x\,dx = \left[\frac{x^2}{2}\right]_0^4 = \frac{1}{2}\left[16 - 0\right] = 8$$

$$f_c = \int_0^4 x\cos\left(\frac{n\pi x}{4}\right) dx$$

sign	D	I
+	x	$\cos\frac{n\pi x}{4}$
−	1	$\frac{\sin\frac{n\pi x}{4}}{\frac{n\pi}{4}}$
+	0	$-\frac{\cos\frac{n\pi x}{4}}{\left(\frac{n\pi}{4}\right)^2}$

$$= \left[\frac{4x}{n\pi}\sin\left(\frac{n\pi x}{4}\right) + \cancel{\frac{n\pi}{4}}\,\frac{16}{n^2\pi^2}\cos\left(\frac{n\pi x}{4}\right)\right]_0^4$$

$$= \frac{16}{n^2\pi^2}\cos(n\pi) - \frac{16}{n^2\pi^2}\cos(0)$$

$$= \frac{16}{n^2\pi^2}(-1)^n - \frac{16}{n^2\pi^2}$$

Finite Fourier Sine ~~Si~~ Transform :

$$f_s = \int_0^4 x \sin\left(\frac{n\pi x}{4}\right) dx$$

sign	D	I
+	x	$\sin\left(\frac{n\pi x}{4}\right)$
−	1	$-\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)$
+	0	$-\frac{16}{n^2\pi^2}\sin\left(\frac{n\pi x}{4}\right)$

$$= \left[-\frac{4x}{n\pi}\cos\left(\frac{n\pi x}{4}\right) + \frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}\right]_0^4$$

$$= -\frac{16}{n\pi}\cos(n\pi) + 0$$

$$= -\frac{16}{n\pi}(-1)^n$$ (Ans:)

FOURIER TRANSFORM

Infinite Fourier ~~Fourier~~ Sine Transform

$$f_s = \int_0^\infty f(x) \sin(nx) dx$$

Infinite Fourier cosine Transform

$$f_c = \int_0^\infty f(x) \cos(nx) dx$$

01. Find Infinite Fourier Sine and Cosine Transform for $f(x) = e^{-x}, x \geq 0$.

⇒ Sol^n:

Infinite Fourier Sine Transform :

$$f_s = \int_0^{\infty} f(x) \sin(nx)\, dx$$

$$= \int_0^{\infty} e^{-x} \sin(nx)\, dx$$

$$= \left[\frac{e^{-x}\{(-\sin(nx)) - (n \cos(nx))\}}{1+n^2} \right]_0^{\infty}$$

$$= \frac{1}{1+n^2} \left[-e^{-x} \sin(nx) - n e^{-x} \cos(nx) \right]_0^{\infty}$$

$$= \frac{1}{1+n^2} \left[-e^{-\infty} \sin(nx) + e^{-0} \sin 0 - n e^{-x} \cos(nx) + n e^{-0} \cos 0 \right]$$

$$= \frac{1}{1+n^2} \quad \text{(Ans:)}$$

Infinite Fourier Cosine Transform :

$$f_c = \int_0^{\infty} e^{-x} \cos(nx)\, dx$$

$$= \left[\frac{e^{-x}\{(-\cos(nx)) + (n \sin(nx))\}}{1+n^2} \right]_0^{\infty}$$

$$= \frac{1}{1+n^2} \left[-e^{-x} \cos(nx) + n e^{-x} \sin(nx) \right]_0^{\infty}$$

$$= \frac{1}{1+n^2} \left[-e^{-\infty} \cos(nx) + e^{-0} \cos 0 + n e^{-\infty} \sin(nx) - n e^{-0} \sin 0 \right]$$

$$= \frac{1}{1+n^2} \quad \text{(Ans:)}$$

=> 1-4 compulsory

[Application of 1st order & Bernoulli] - 25 (1 set)

[Cachy - Euler

Higher Order [After Mid - Lecture of 18th Dec]

General Form

[Laplace - Inverse Laplace Df^n, Formula, Properties

[Application of Laplace

=> 5 or 6

[Laplace Periodic & Inverse Laplace (statement of Heaviside)

=> 7 or 8

[Fourier Full or Half Range, Df^n