Laplace Transformation:

Let $F(t)$ be a function of t specified for $t>0$. Then the laplace Transformation of $F(t)$ denoted by $\mathcal{L}\{F(t)\}$ is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} e^{-st} F(t)\, dt,$$

where we assume the parameter s is real

$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1. c	$\frac{c}{s}$
2. t^n	$\frac{n!}{s^{n+1}}$
3. e^{at}	$\frac{1}{s-a}$
4. $\sin at$	$\frac{a}{s^2+a^2}$
5. $\cos(at)$	$\frac{s}{s^2+a^2}$
6. $\sinh(at)$	$\frac{a}{s^2-a^2}$
7. $\cosh(at)$	$\frac{s}{s^2-a^2}$

Prove that, $\mathcal{L}\{c\} = \frac{c}{s}$

Soln: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\, dt$$

Here $F(t) = c$

$$\therefore \mathcal{L}\{c\} = \int_0^{\infty} e^{-st} . c\, dt$$

$$= c\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{c}{s}\left[-e^{-s\infty} + e^0\right]$$

$$= \frac{c}{s}(0+1) = \frac{c}{s} \quad \underline{\text{Ans:}}$$

Prove that, $\mathcal{L}\{t\} = \frac{1}{s^2}$

soln: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\,dt$$

Here, $F(t) = t$

$$\left[\begin{array}{l} \text{LIATE} \rightarrow v ? \\ \int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx \end{array}\right.$$

$$\therefore \mathcal{L}\{t\} = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$= \frac{-te^{-\infty}}{s} - \frac{e^{-\infty}}{s^2} + \frac{0.e^0}{s} + \frac{e^0}{s^2}$$

$$= \frac{1}{s^2}$$

<u>Ans:</u>

$e^0 = 1$, $e^\infty = \infty$, $e^{-\infty} = 0$.

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* Prove that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

soln: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

Here, $F(t) = e^{at}$

$$\therefore \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} . e^{at} dt$$

$$= \int_0^\infty e^{-t(s-a)} dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= \frac{e^{-\infty}}{-(s-a)} + \frac{e^0}{s-a}$$

$$= \frac{1}{s-a}$$

Ans:

* Prove that, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Solution: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

Here, $F(t) = \sin at$

$$\therefore \mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st} . \sin(at)dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$= \frac{1}{s^2+a^2}(0-0+0+a)$

$= \frac{a}{s^2+a^2}$ Ans:

Prove that $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$

soln: By defination of LT,

$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$

Here, $F(t) = \cos(at)$

$\therefore \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st} . \cos(at)\, dt$

$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$

$= \frac{1}{s^2+a^2}(s+0)$

$= \frac{s}{s^2+a^2}$ Ans!

Prove that, $\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}$

Solution: By defination of LT,

$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$

Here, $F(t) = \cosh(at)$

$\cosh(at) = \frac{e^{at}+e^{-at}}{2}$, $\sinh(at) = \frac{e^{at}-e^{-at}}{2}$

$\therefore \mathcal{L}\{\cosh(at)\} = \int_0^\infty e^{-st} \cosh(at) dt$

$$= \int_0^\infty \frac{e^{at} + e^{-at}}{2} \cdot e^{-st}\, dt$$

$$= \frac{1}{2} \int_0^\infty \left(e^{-t(s-a)} + e^{-t(s+a)}\right) dt$$

$$= \frac{1}{2} \left(\left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty + \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty \right)$$

$$= \frac{1}{2} \left(0 + \frac{1}{s-a} + 0 + \frac{1}{s+a} \right)$$

$$= \frac{1}{2} \frac{s-a+s+a}{(s-a)(s+a)}$$

$$= \frac{1}{2} \cdot \frac{2s}{s^2-a^2}$$

$$= \frac{s}{s^2-a^2}$$ Ans:

H/T Prove that sinh (at) Homework)

Properties:

Linearity property:

If c_1, c_2 are any constants, while $F_1(t), F_2(t)$ are functions with LTs $f_1(s), f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t)+c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\}+c_2\mathcal{L}\{F_2(t)\}$$
$$= c_1 f_1(s) + c_2 f_2(s)$$

① $\mathcal{L}(5e^{-t}+2\cos 5t-3\}$

$$= 5\mathcal{L}(e^{-t})+2\mathcal{L}(\cos 5t)-\mathcal{L}(3)$$
$$= 5\cdot\frac{1}{s+1}+2\cdot\frac{s}{s^2+25}-\frac{3}{s}$$
$$= \frac{5}{s+1}+\frac{2s}{s^2+25}-\frac{3}{s}$$

② $\mathcal{L}\{(t^2+1)^2\} = \mathcal{L}\{t^4+2t^2+1\}$

$$= \mathcal{L}\{t^4\}+2\mathcal{L}\{t^2\}+\mathcal{L}\{1\}$$
$$= \frac{4!}{s^5}+2\frac{2!}{s^3}+\frac{1}{s}$$

Ans:

② First Translation or shifting property:-

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}(e^{at}F(t)\} = f(s-a)$

① $\mathcal{L}\{e^{-2t}\cos 5t\}$

soln: $\mathcal{L}\{\cos 5t\} = \frac{s}{s^2+25}$

$$\mathcal{L}\{e^{-2t}\cdot\cos 5t\} = \frac{s-(-2)}{\{s-(-2)\}^2+25}$$ [replace s by s-2]

$$= \frac{s+2}{(s+2)^2+25}$$

(2) $\mathcal{L}\{e^{5t} \cdot \sin t\}$

$\therefore \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$\mathcal{L}\{e^{5t} \sin t\} = \frac{1}{(s-5)^2+1}$ **Ans:**

(1) $\mathcal{L}\{t^3 + \cos 2t + \sinh 3t\}$

(2) $\{e^{-2t} \cdot t^4\}$ (3) $\{e^{2t} \cdot 3 \cdot \sin t\}$ (4) $\{\cos t \cdot e^{2t}\}$

(4) $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

$\mathcal{L}\{e^{2t} \cos t\} = \frac{s-(-2)}{}$

Second Translation on shifting property :

If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t > a \\ 0, & t < a \end{cases}$

then $\mathcal{L}\{G(t)\} = e^{-as} \cdot f(s)$

① If $G(t) = \begin{cases} (t-2)^3, & t > 2, \\ 0, & t < 2. \end{cases}$ find $\mathcal{L}\{G(t)\}$

soln : $\mathcal{L}\{G(t)\} = e^{-2s} \frac{3!}{s^4}$

② If $G(t) = \begin{cases} \sin(t - \frac{\pi}{2}), & t > \frac{\pi}{2} \\ 0, & t < \frac{\pi}{2}, \end{cases}$ find $\mathcal{L}\{G(t)\}$

soln: $\mathcal{L}\{G(t)\} = e^{-\frac{\pi}{2}s} \frac{1}{s^2+1}$

Multiplication by t^n :

If $\mathcal{L}\{F(t)\} = f(s)$; then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

① $\mathcal{L}\{t^2 \cos at\}$

soln: $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

$\therefore \mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2} \left(\frac{s}{s^2+a^2} \right)$

$= \frac{d}{ds} \left\{ \frac{(s^2+a^2)\frac{d}{ds}(s) - s\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2} \right\}$

$= \frac{d}{ds} \left\{ \frac{s^2+a^2-2s^2}{(s^2+a^2)^2} \right\}$

$= \frac{d}{ds} \frac{a^2-s^2}{(s^2+a^2)^2}$

$$= \frac{(s^2+a^2)^2 \frac{d}{ds}(a^2-s^2) - (a^2-s^2)\frac{d}{ds}(s^2+a^2)^2}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2)^2 - 2(s^2+a^2)\,2s(a^2-s^2)}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2)^2 - 4s(a^2-s^2)}{(s^2+a^2)^3}$$

$$= \frac{-2s^3 - 2a^2s - 4a^2s + 4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6a^2s}{(s^2+a^2)^3}$$

(2) $\mathcal{L}\{t^{50} e^{2t}\}$

soln: $\mathcal{L}\{t^{50}\} = \frac{50!}{s^{51}}$

$$\mathcal{L}\{t^{50} \cdot e^{2t}\} = \frac{50!}{(s-2)^{51}}$$

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(1) $\mathcal{L}\{\cos^2 t\}$

soln: We know that,

$$\cos 2t = 2\cos^2 t - 1$$

$$\Rightarrow \cos^2 t = \frac{1}{2}(\cos 2t + 1)$$

$$\mathcal{L}\{\cos^2 t\} = \frac{1}{2}\left[\mathcal{L}(\cos 2t) + \mathcal{L}(1)\right]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right]$$

② $\mathcal{L}\{t^{-1/2}\}$

We know that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

But n is a fractional number,

$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$n = -\frac{1}{2}$, $\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{(-\frac{1}{2}+1)}}$

$= \frac{\Gamma(1/2)}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

③ $\mathcal{L}\{\sin t \cos t\}$

soln: $\frac{1}{2} \cdot 2\sin t \cos t$

$= \frac{1}{2}\sin 2t$

$\mathcal{L}(\frac{1}{2}\sin 2t)$

$= \frac{1}{2}\mathcal{L}\{\sin 2t\}$

$= \frac{1}{2}\frac{2}{s^2+4}$

$= \frac{1}{s^2+4}$ Ans:

⑤ $\mathcal{L}\{\sin 2t \cos 3t\}$

we know that,

$2\sin x \cos y = \sin(x+y) + \sin(x-y)$

$\sin 2t \cos 3t = \frac{1}{2}[2\sin 2t . \cos 3t]$

$= \frac{1}{2}[\sin(2t+3t) + \sin(2t-3t)]$

$= \frac{1}{2}[\sin 5t - \sin t]$

④ $\mathcal{L}\{\sin^3 2t\}$

we know that,

$\sin 3A = 3\sin 3A - 4\sin^3 A$

$\sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)$

$\Rightarrow \sin^3 2t = \frac{1}{4}(3\sin 2t - \sin 6t)$

$\mathcal{L}\{\sin^3 2t\} = \mathcal{L}\{\frac{1}{4}(3\sin 2t - \sin 6t)\}$

$= \frac{1}{4}\left\{3\frac{2}{s^2+4} - \frac{6}{s^2+36}\right]$

$= \frac{4}{2}\frac{3}{2(s^2+4)} - \frac{3}{2(s^2+36)}$

$= \frac{3}{2}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$

$= \frac{3}{2}\frac{32}{(s^2+4)(s^2+36)}$

$= \frac{48}{(s^2+4)(s^2+36)}$

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$$\mathcal{L}\{\quad\} = \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2-20}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{2(s^2-5)}{(s^2+25)(s^2+1)}$$

Ans.

LT of Integral

$$\mathcal{L}\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} f(s)$$

where $\mathcal{L}\{F(t)\} = f(s)$.

LT of division by t / $\left(\frac{1}{t} f(t)\right)$

If $\mathcal{L}\{f(t)\} = f(s)$, then

$$\mathcal{L}\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty f(s)\,ds$$

Ex: $\mathcal{L}\left\{\frac{\sin 2t}{t}\right\}$

soln: $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

$$\mathcal{L}\left\{\frac{\sin 2t}{t}\right\} = \int_s^\infty \frac{2}{s^2+4}\,ds$$

$$= 2 \cdot \frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1}\frac{s}{2}$$

$$= \tan^{-1}\left(\tan\frac{\pi}{2}\right) - \tan^{-1}\frac{s}{2}$$

$$= \cot^{-1}\frac{s}{2} \quad \text{Ans:}$$

Ex) $\mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\,dt\right\}$

$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1}\frac{s}{1}\right]_s^\infty$$

$= \tan^{-1}\infty - \tan^{-1}s$

$= \tan^{-1}\left(\tan\frac{\pi}{2}\right) - \tan^{-1}s$

$= \frac{\pi}{2} - \tan^{-1}s$

$= \cot^{-1}s.$

$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin t}{t}\,dt\right\} = \frac{1}{s}\cot^{-1}s$

[H.W] $\mathcal{L}\left\{e^{-4t}\frac{\sin 3t}{t}\right\}$

sol^n: $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

$\mathcal{L}\left\{\frac{\sin 3t}{t}\right\} = \int_s^\infty \frac{3}{s^2+9}\,ds$

$= 3\cdot\frac{1}{3}\left[\tan^{-1}\frac{s}{3}\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\frac{s}{3}$

$= \tan^{-1}\left(\tan\frac{\pi}{2}\right) - \tan^{-1}\frac{s}{3}$

$= \frac{\pi}{2} - \tan^{-1}\frac{s}{3}$

$= \cot^{-1}\frac{s}{3}$

$\mathcal{L}\left\{e^{-4t}\frac{\sin 3t}{t}\right\}$

$= \cot^{-1}\frac{s-(-4)}{3}$

$= \cot^{-1}\frac{s+4}{3}$

Ex $\mathcal{L}\left\{\frac{1}{t}(1-e^{t})\right\}$

$\mathcal{L}\{1-e^{t}\}$

$=\frac{1}{s}-\frac{1}{s-1}$

$\mathcal{L}\left\{\frac{1}{t}(1-e^{t})\right\}$

$=\int_s^\alpha\left(\frac{1}{s}-\frac{1}{s-1}\right)ds$

$=\left[\log s-\log(s-1)\right]_s^\alpha$

$=-\log s+\log(s-1)$

$=\log\frac{s-1}{s}$. Ans.

$\mathcal{L}\left\{\frac{1}{t}(e^{-at}-e^{-bt})\right\}$

soln: $\mathcal{L}\{e^{-at}-e^{-bt}\}=\frac{1}{s+a}-\frac{1}{s+b}$

$\mathcal{L}\left\{\frac{1}{t}(e^{-at}-e^{-bt})\right\}$

$=\int_s^\alpha\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$

$=\left[\log(s+a)-\log(s+b)\right]_s^\alpha$

$=-\log(s+a)+\log(s+b)$

$=\log\frac{s+b}{s+a}$ Ans.

LT of periodic function:

Let $F(t)$ be a periodic function of period T

so, $\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt$.

(1) $F(t) = \begin{cases} 2t^2, & 0<t<1 \\ t, & 1<t<2 \end{cases}$ is a periodic function of period 2. Find $\mathcal{L}\{F(t)\}$

Soln: $\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 e^{-st} F(t)\,dt$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot 2t^2 dt + \int_1^2 e^{-st} \cdot t\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[2 \int_0^1 e^{-st} t^2 dt + \int_1^2 e^{-st} \cdot t\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[2 \left[\frac{-t^2 e^{-st}}{s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right]_0^1 + \left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_1^2 \right.$$

$$= \frac{1}{1-e^{-2s}} \left[2 \left[\frac{e^{-s}}{s} - \frac{2e^{-s}}{s^2} - \frac{2e^{-s}}{s^3} + \frac{2}{s^3} \right] + \left[\frac{-2e^{-2s}}{s} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} \right.\right.$$

$$= \frac{1}{1-e^{-2\pi s}(s^2+1)} \left[-se^{-st} \sin t - e^{-st} \cos t \right]_0^\pi$$

$$= \frac{1}{1-e^{-2\pi s}(s^2+1)} \left[-e^{-\pi s} \cos \pi + e^0 \cos 0 \right]$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)} (e^{-\pi s} + 1)$$

Ans:

Q: $F(t) = \begin{cases} \sin t, & 0<t<\pi \\ 0, & \pi<t<2\pi \end{cases}$ is a periodic function of period 2π.

Find $\mathcal{L}\{F(t)\}$

Soln: $\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st} F(t)dt$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt + \int_{\pi}^{2\pi} e^{-st}\cdot 0\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{\pi} e^{-st}\sin t\,dt$$

$$= \frac{1}{1-e^{-2\pi s}(s^2+1)}\left[-se^{-st}\sin t - e^{-st}\cos t\right]_0^{\pi}$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)}\left[-e^{-\pi s}\cos\pi + e^{0}\cos 0\right]$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)}(e^{-\pi s}+1)$$ Ans.

Table for Inverse Laplace Transformation

$f(s)$ $\qquad\qquad\qquad\qquad$ $\mathcal{L}^{-1}\{f(s)\} = F(t)$

01. $\frac{c}{s} \longrightarrow c$

02. $\frac{1}{s-a} \longrightarrow e^{at}$

03. $\frac{1}{s^{n+1}} \longrightarrow \frac{t^n}{n!}$

04. $\frac{1}{s^2+a^2} \longrightarrow \frac{\sin(at)}{a}$

05. $\frac{s}{s^2+a^2} \longrightarrow \cos(at)$

06. $\frac{1}{s^2-a^2} \longrightarrow \frac{\sinh(at)}{a}$

07. $\frac{s}{s^2-a^2} \longrightarrow \cosh(at)$

Ex $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

Ex $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

Ex $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$

* $\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin(2t)}{5}$

* $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos(4t)$

* $\mathcal{L}^{-1}\left\{\frac{1}{s^2-25}\right\} = \frac{\sinh(5t)}{5}$

* $\mathcal{L}^{-1}\left\{s^2-16\right\} = \cosh(4t)$

PROPERTIES

01. Linearity Property:

$$\mathcal{L}^{-1}\left\{2\frac{1}{s^5} - \frac{s}{s^2+25} + \frac{1}{s-2} + \frac{5}{s}\right\}$$

$$= 2\frac{t^4}{4!} - \cos(5t) + e^{2t} + 5 \quad \text{Ans:}$$

First Translation of shifting Property:

If $\mathcal{L}^{-1}\{f(s)\} = F(t)$, then $\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$

01. $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$$= 6\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 8\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$= 6\cos 4t\, e^{2t} + 8\frac{\sin 4t}{4}e^{2t}$$

$$\cancel{=6\cos} = 6e^{2t}\cos 4t + 2e^{2t}\sin 4t \quad \text{Ans:}$$

02. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$= \frac{\sin 2t}{2}e^{t} \quad \text{Ans:}$$

HEAVISIDE Expansion Formula:

let, $P(s)$, $Q(s)$ be polynomials, where $P(s)$ has degree less that of $Q(s)$. suppose that $Q(s)$ has 'n' distinct zeros, a_K, $K = 1, 2, 3, \dots n$

যাদের value এর জন্য function undefined হবে।

Then,

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{K=1}^{n} \frac{P(a_K)}{Q(a_K)} e^{a_K t}$$

$$= \frac{P(a_1)}{Q'(a_1)} e^{a_1 t} + \frac{P(a_2)}{Q'(a_2)} e^{a_2 t} + \dots + \frac{P(a_n)}{Q'(a_n)} e^{a_n t} \dots$$

1. $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

soln: $P(s) = 2s^2 - 4$

$$\begin{aligned}
Q(s) &= (s+1)(s-2)(s-3) \\
&= (s^2-2s+s-2)(s-3) \\
&= (s^2-s-2)(s-3) \\
&= s^3-3s^2-s^2+3s-2s+6 \\
&= s^3-4s^2+s+6
\end{aligned}$$

$$Q'(s) = 3s^2 - 8s + 1$$

Here, $Q(s)$ has three distinct zeros.

say, $a_1 = -1$, $a_2 = 2$, $a_3 = 3$

By HEAVISIDE Expansion formula,

$$\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t}$$

$$= \left(\frac{-2}{12}\right)e^{-t} + \left(\frac{-4}{3}\right)e^{2t} + \left(\frac{14}{4}\right)e^{3t}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

<u>Ans:</u>

Partial Function

Q1. $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Soln: Let, $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$

$\Rightarrow 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$ — (1)

At, $s=2$, (1) $\Rightarrow$

$2.2^2-4 = A(2-2)(2-3) + B(2+1)(2-3) + C(2+1)(2-2)$

$\Rightarrow 4 = -3B$

$\Rightarrow B = -4/3$

At, $s=3$, (1) $\Rightarrow$

$2.3^2-4 = A(3-2)(3-3) + B(3+1)(3-3) + C(3+1)(3-2)$

$\Rightarrow 14 = 4C$

$\Rightarrow C = 14/4$

At, $s=-1$, (1) $\Rightarrow$

$2(-1)^2-4 = A(-1-2)(-1-3) + B(-1+1)(-1-3) + C(-1+1)(-1-2)$

$\Rightarrow -2 = 12A + 0 + 0$

$\Rightarrow A = -1/6$

$\therefore \frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{-2}{12(s+1)} - \frac{4}{3(s-2)} + \frac{14}{4(s-3)}$

$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-2}{12(s+1)} - \frac{4}{3(s-2)} + \frac{14}{4(s-3)}\right\}$

$$= -\frac{2}{12}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{14}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$ Ans:

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$$

Soln:

Let, $\frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$

$\Rightarrow 3s+1 = A(s^2+1) + (Bs+C)(s-1)$ —— (1)

At, $s=1$, $3+1 = 2A + (B+C)\times 0$

$\Rightarrow A = 4/2$

$\Rightarrow A = 2$

Again,

$3s+1 = As^2 + A + Bs^2 - Bs + Cs - C$

$3s+1 = (A+B)s^2 + (C-B)s + (A-C)$

Equating co-efficient of s^2, s from (1),

$A+B = 0$, $B = -A = -2$

$C - B = 3$

$\Rightarrow C = 3 + B$

$= 1$

$$\frac{3s+1}{(s-1)(s^2+1)} = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s-1} - \frac{2s}{s^2+1} + \frac{1}{s^2+1}\right\}$$

$$= 2e^{t} - 2\cos t + \sin t$$ Ans: