# Course Code: MTH (EEE) 203

# Course Title: Transformations & Partial Differential Equation

# Lecture-14

## Application Of Partial Differential Equation & Fourier Series

**Principal of Superposition:** If $u_1, u_2, \dots, u_n$ are solutions of linear homogeneous partial differential equation, then $c_1u_1 + c_2u_2 + \dots + c_nu_n$, where $c_1, c_2, \dots, c_n$ are constants, is also a solution.

**Example:** Solve the following boundary value problem by the method of Separation of variables

$$\frac{\partial U}{\partial t} = 4\frac{\partial^2 U}{\partial x^2}, U(0,t) = 0, U(\pi,t) = 0, U(x,0) = 2\sin 3x - 4\sin 5x$$

**Solution:** Given, $\frac{\partial U}{\partial t} = 4\frac{\partial^2 U}{\partial x^2} \dots\dots (i)$

Let, $U(x,t) = XT \dots (ii)$ is a solution of $(i)$ where $X$ is function of $x$ alone and $T$ is function of $t$ alone.

Then, $\frac{\partial U}{\partial t} = X\frac{dT}{dt}$, $\frac{\partial U}{\partial x} = T\frac{dX}{dx}$ and $\frac{\partial^2 U}{\partial x^2} = T\frac{d^2X}{dx^2}$

Putting these value in $(i)$ we get,

$$X\frac{dT}{dt} = 4T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{4T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \dots\dots (iii)$$

Since, $X$ is function of $x$ alone and $T$ is function of $t$ alone and $x$ and $t$ are independent variables, so each side of $(iii)$ must be a constant, say $-\lambda^2$.

$$\therefore \frac{1}{4T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \dots . (iv)$$

So from $(iv)$,

$$\frac{1}{4T}\frac{dT}{dt} = -\lambda^2$$

$$\Rightarrow \frac{dT}{T} = -4\lambda^2 dt$$

$$\Rightarrow \int \frac{dT}{T} = -4\lambda^2 \int dt$$

$$\Rightarrow \ln T = -4\lambda^2 t + \ln A; \text{ where } A \text{ is constant}$$

$$\Rightarrow \ln T - \ln A = -4\lambda^2 t$$

$$\Rightarrow \ln\left(\frac{T}{A}\right) = -4\lambda^2 t$$

$$\Rightarrow \frac{T}{A} = e^{-4\lambda^2 t}$$

$$\Rightarrow T = Ae^{-4\lambda^2 t}$$

Again, from $(iv)$,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} = -\lambda^2 X$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0 \,\ldots\ldots (v)$$

Let, $X = e^{mx}$ be a trail solution of $(v)$.

$$m^2 + \lambda^2 = 0 \Rightarrow m^2 = -\lambda^2 \Rightarrow m = \pm\lambda i$$

So from $(v)$,

$$X = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$$

$$= c_1(\cos\lambda x + i\sin\lambda x) + c_2(\cos\lambda x - i\sin\lambda x)$$

$$= \cos \lambda x \,(c_1 + c_2) + \sin \lambda x \, i(c_1 - c_2)$$

$$\therefore X = B \cos \lambda x + C \sin \lambda x$$

Let, $c_1 + c_2 = B$ and $i(c_1 - c_2) = C$

Thus the solution of $(i)$, from $(ii)$, is

$$U(x,t) = XT = (B \cos \lambda x + C \sin \lambda x)Ae^{-4\lambda^2 t}$$

$$= ABe^{-4\lambda^2 t} \cos \lambda x + ACe^{-4\lambda^2 t} \sin \lambda x$$

$$\therefore U(x,t) = De^{-4\lambda^2 t} \cos \lambda x + Ee^{-4\lambda^2 t} \sin \lambda x \; \ldots . (vi)$$

Where, $AB = D$ and $AC = E$

Since, $U(0,t) = 0$, So from $(vi)$,

$$U(0,t) = De^{-4\lambda^2 t} \cos 0 + Ee^{-4\lambda^2 t} \sin 0$$

$$\Rightarrow 0 = De^{-4\lambda^2 t} \Rightarrow D = 0 \left[e^{-4\lambda^2 t} \neq 0\right]$$

Thus from $(vi)$, we have, $U(x,t) = Ee^{-4\lambda^2 t} \sin \lambda x \; \ldots . (vii)$

Again, $U(\pi,t) = 0$, So from $(vii)$,

$$U(\pi,t) = Ee^{-4\lambda^2 t} \sin \lambda\pi \Rightarrow Ee^{-4\lambda^2 t} \sin \lambda\pi = 0 \; \ldots . (viii)$$

If $E = 0$, the solution is identically zero. So we must have $\sin \lambda\pi = 0$; since $e^{-4\lambda^2 t} \neq 0$ and $E \neq 0$.

So from $(viii)$, $\sin \lambda\pi = 0 \Rightarrow \sin \lambda\pi = \sin(n\pi) \Rightarrow \lambda\pi = n\pi \Rightarrow \lambda = n$

Putting the value of $\lambda = n$ in $(vii)$,

$$U(x,t) = Ee^{-4n^2 t} \sin(nx) \; \ldots . (ix)$$

In order to satisfy the last condition $U(x,0) = 2\sin 3x - 4\sin 5x$, we first use the principal of superposition to obtain the solution.

$$U(x,t) = E_1 e^{-4n_1^2 t}\sin(n_1 x) + E_2 e^{-4n_2^2 t}\sin(n_2 x) \ldots . (x)$$

Putting $t = 0$ in $(x)$, we get,

$$U(x,0) = E_1 e^{-0}\sin(n_1 x) + E_2 e^{-0}\sin(n_2 x)$$

$$\Rightarrow E_1\sin(n_1 x) + E_2\sin(n_2 x) = 2\sin 3x - 4\sin 5x$$

Which is possible if and only if

$$E_1 = 2, n_1 = 3, E_2 = -4, n_2 = 5$$

Putting these values in $(x)$, we get,

$$U(x,t) = 2e^{-4.3^2 t}\sin(3x) - 4e^{-4.5^2 t}\sin(5x)$$

$$= 2e^{-36t}\sin(3x) - 4e^{-100t}\sin(5x);$$

Which is the required solution.

**Practice:** Solve the following boundary value problem by the method of Separation of variables

$4\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, U(0,t) = 0, U(2,t) = 0, 0 < x < 2,$

$U(x,0) = 2\sin\frac{\pi x}{2} - \sin\pi x + 4\sin 2\pi x$

**Example:** Solve the following boundary value problem by the method of Separation of variables

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2}, U(0,t) = 0, U(3,t) = 0, U(x,0) = f(x)$$

**Solution:** Given, $\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2} \ldots \ldots . (i)$

Let, $U(x,t) = XT$ ... $(ii)$ is a solution of $(i)$ where $X$ is function of $x$ alone and $T$ is function of $t$ alone.

Then, $\frac{\partial U}{\partial t} = X\frac{dT}{dt}$, $\frac{\partial U}{\partial x} = T\frac{dX}{dx}$ and $\frac{\partial^2 U}{\partial x^2} = T\frac{d^2 X}{dx^2}$

Putting these value in $(i)$ we get,

$$X\frac{dT}{dt} = 2T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \dots\dots (iii)$$

Since, $X$ is function of $x$ alone and $T$ is function of $t$ alone and $x$ and $t$ are independent variables, so each side of $(iii)$ must be a constant, say $-\lambda^2$.

$$\therefore \frac{1}{2T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \dots. (iv)$$

So from $(iv)$,

$$\frac{1}{2T}\frac{dT}{dt} = -\lambda^2$$

$$\Rightarrow \frac{dT}{T} = -2\lambda^2 dt$$

$$\Rightarrow \int \frac{dT}{T} = -2\lambda^2 \int dt$$

$$\Rightarrow \ln T = -2\lambda^2 t + \ln A; \text{ where } A \text{ is constant}$$

$$\Rightarrow \ln T - \ln A = -2\lambda^2 t$$

$$\Rightarrow \ln\left(\frac{T}{A}\right) = -2\lambda^2 t$$

$$\Rightarrow \frac{T}{A} = e^{-2\lambda^2 t}$$

$$\Rightarrow T = Ae^{-2\lambda^2 t}$$

Again, from $(iv)$,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} = -\lambda^2 X$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0 \ldots\ldots (v)$$

Let, $X = e^{mx}$ be a trail solution of $(v)$.

$$m^2 + \lambda^2 = 0 \Rightarrow m^2 = -\lambda^2 \Rightarrow m = \pm\lambda i$$

So from $(v)$,

$$X = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$$

$$= c_1(\cos\lambda x + i\sin\lambda x) + c_2(\cos\lambda x - i\sin\lambda x)$$

$$= \cos\lambda x\,(c_1 + c_2) + \sin\lambda x\, i(c_1 - c_2)$$

$$\therefore X = B\cos\lambda x + C\sin\lambda x$$

Let, $c_1 + c_2 = B$ and $i(c_1 - c_2) = C$

Thus the solution of $(i)$, from $(ii)$, is

$$U(x,t) = XT = (B\cos\lambda x + C\sin\lambda x)Ae^{-2\lambda^2 t}$$

$$= ABe^{-2\lambda^2 t}\cos\lambda x + ACe^{-2\lambda^2 t}\sin\lambda x$$

$$\therefore U(x,t) = De^{-2\lambda^2 t}\cos\lambda x + Ee^{-2\lambda^2 t}\sin\lambda x \ldots.(vi)$$

Where, $AB = D$ and $AC = E$

Since, $U(0,t) = 0$, So from $(vi)$,

$$U(0,t) = De^{-2\lambda^2 t}\cos 0 + Ee^{-2\lambda^2 t}\sin 0$$

$$\Rightarrow 0 = De^{-2\lambda^2 t} \Rightarrow D = 0\left[e^{-2\lambda^2 t} \neq 0\right]$$

Thus from $(vi)$, we have, $U(x,t) = Ee^{-2\lambda^2 t}\sin\lambda x \ldots.(vii)$

Again, $U(3,t) = 0$, So from $(vii)$,

$$U(3,t) = Ee^{-2\lambda^2 t}\sin 3\lambda \Rightarrow Ee^{-2\lambda^2 t}\sin 3\lambda = 0 \ldots.(viii)$$

If $E = 0$, the solution is identically zero. So we must have $\sin 3\lambda = 0$; since $e^{-2\lambda^2 t} \neq 0$ and $E \neq 0$.

So from $(viii)$, $\sin 3\lambda = 0 \Rightarrow \sin 3\lambda = \sin(n\pi) \Rightarrow 3\lambda = n\pi \Rightarrow \lambda = \frac{n\pi}{3}$

Putting the value of $\lambda = \frac{n\pi}{3}$ in $(vii)$,

$$U(x,t) = Ee^{-2\left(\frac{n\pi}{3}\right)^2 t}\sin\left(\frac{n\pi x}{3}\right) = Ee^{\frac{-2n^2\pi^2}{9}t}\sin\left(\frac{n\pi x}{3}\right) \ldots.(ix)$$

In order to satisfy the last condition $U(x,0) = f(x)$, we are led to assume that infinitely many terms are taken. So using the principal of superposition to obtain the solution.

$$U(x,t) = \sum_{n=1}^{\infty} E_n e^{\frac{-2n^2\pi^2}{9}t}\sin\left(\frac{n\pi x}{3}\right) \ldots.(x)$$

Putting $t = 0$ in $(x)$, we get,

$$U(x,0) = \sum_{n=1}^{\infty} E_n e^{-0}\sin\left(\frac{n\pi x}{3}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{3}\right) = f(x)$$

Apply Half Range Fourier Sine Series we have,

$$E_n = \frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx$$

Putting the value of $E_n$ in $(x)$, we get,

$$U(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{3}\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx\right]e^{\frac{-2n^2\pi^2}{9}t}\sin\left(\frac{n\pi x}{3}\right)$$

$$= \frac{2}{3}\sum_{n=1}^{\infty} e^{\frac{-2n^2\pi^2}{9}t}\sin\left(\frac{n\pi x}{3}\right)\int_0^3 f(x)\sin\left(\frac{n\pi x}{3}\right)dx;$$

Which is the required solution.

**Example:** Solve the following boundary value problem by the method of Separation of variables

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, U(0,t) = 1, U(\pi,t) = 3, U(x,0) = 1$$

**Solution:** Given, $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \dots \dots (i)$

Let, $U(x,t) = V(x,t) + \varphi(x) \dots (ii)$ is a solution of $(i)$

Then, $\frac{\partial U}{\partial t} = \frac{\partial V}{\partial t}$, $\frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} + \varphi'(x)$ and $\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 V}{\partial x^2} + \varphi''(x)$

Putting these value in $(i)$ we get,

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} + \varphi''(x) \dots\dots (iii)$$

Now, $U(0,t) = 1 \Rightarrow V(0,t) + \varphi(0) = 1$

$U(\pi,t) = 3 \Rightarrow V(\pi,t) + \varphi(\pi) = 3$

$U(x,0) = 1 \Rightarrow V(x,0) + \varphi(x) = 1$

Again we suppose that, $\varphi''(x) = 0, \varphi(0) = 1, \varphi(\pi) = 3$

Since, $\varphi''(x) = 0$

$$\Rightarrow \int \varphi''(x)\, dx = \int 0\, dx \Rightarrow \varphi'(x) = c_1$$

Again, $\int \varphi'(x)\, dx = \int c_1\, dx \Rightarrow \varphi(x) = c_1 x + c_2$

Now, $\varphi(0) = c_1 \times 0 + c_2 \Rightarrow c_2 = 1\ [\varphi(0) = 1]$

Also, $\varphi(\pi) = c_1\pi + c_2 \Rightarrow 3 = c_1\pi + 1 \Rightarrow c_1\pi = 2 \Rightarrow c_1 = \frac{2}{\pi}$

Therefore, $\varphi(x) = \frac{2x}{\pi} + 1$

Now, $V(0,t) + 1 = 1 \Rightarrow V(0,t) = 0$

$V(\pi,t) + 3 = 3 \Rightarrow V(\pi,t) = 0$

$$V(x,0) + \frac{2x}{\pi} + 1 = 1 \Rightarrow V(x,0) = \frac{-2x}{\pi}$$

Since, $\varphi''(x) = 0$ then $(iii)$ becomes,

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} \ldots.(iv)$$

Let, $V(x,t) = XT \ldots (v)$ is a solution of $(iv)$ where $X$ is function of $x$ alone and $T$ is function of $t$ alone.

Then, $\frac{\partial V}{\partial t} = X\frac{dT}{dt}$, $\frac{\partial V}{\partial x} = T\frac{dX}{dx}$ and $\frac{\partial^2 V}{\partial x^2} = T\frac{d^2X}{dx^2}$

Putting these value in $(iv)$ we get,

$$X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \ldots\ldots (vi)$$

Since, $X$ is function of $x$ alone and $T$ is function of $t$ alone and $x$ and $t$ are independent variables, so each side of $(vi)$ must be a constant, say $-\lambda^2$.

$$\therefore \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \ldots\ldots (vii)$$

So from $(vii)$,

$$\frac{1}{T}\frac{dT}{dt} = -\lambda^2$$

$$\Rightarrow \frac{dT}{T} = -\lambda^2 dt$$

$$\Rightarrow \int \frac{dT}{T} = -\lambda^2 \int dt$$

$$\Rightarrow \ln T = -\lambda^2 t + \ln A; \text{ where } A \text{ is constant}$$

$$\Rightarrow \ln T - \ln A = -\lambda^2 t$$

$$\Rightarrow \ln\left(\frac{T}{A}\right) = -\lambda^2 t$$

$$\Rightarrow \frac{T}{A} = e^{-\lambda^2 t}$$

$$\Rightarrow T = Ae^{-\lambda^2 t}$$

Again, from $(vii)$,

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} = -\lambda^2 X$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0 \ldots\ldots (viii)$$

Let, $X = e^{mx}$ be a trail solution of $(viii)$.

$$m^2 + \lambda^2 = 0 \Rightarrow m^2 = -\lambda^2 \Rightarrow m = \pm\lambda i$$

So from $(viii)$,

$$X = c_3 e^{i\lambda x} + c_4 e^{-i\lambda x}$$

$$= c_3(\cos\lambda x + i\sin\lambda x) + c_4(\cos\lambda x - i\sin\lambda x)$$

$$= \cos\lambda x\,(c_3 + c_4) + \sin\lambda x\, i(c_3 - c_4)$$

$$\therefore X = B\cos\lambda x + C\sin\lambda x$$

Let, $c_3 + c_4 = B$ and $i(c_3 - c_4) = C$

Thus the solution of $(iv)$, from $(v)$, is

$$V(x,t) = XT = (B\cos\lambda x + C\sin\lambda x)Ae^{-\lambda^2 t}$$

$$= ABe^{-\lambda^2 t}\cos\lambda x + ACe^{-\lambda^2 t}\sin\lambda x$$

$$\therefore V(x,t) = De^{-\lambda^2 t}\cos\lambda x + Ee^{-\lambda^2 t}\sin\lambda x \ldots.(ix)$$

Where, $AB = D$ and $AC = E$

Since, $V(0,t) = 0$, So from $(ix)$,

$$V(0,t) = De^{-\lambda^2 t}\cos 0 + Ee^{-\lambda^2 t}\sin 0$$

$$\Rightarrow 0 = De^{-\lambda^2 t} \Rightarrow D = 0\left[e^{-\lambda^2 t} \neq 0\right]$$

Thus from $(ix)$, we have, $V(x,t) = Ee^{-\lambda^2 t}\sin\lambda x \ldots.(x)$

Again, $V(\pi,t) = 0$, So from $(x)$,

$$V(\pi,t) = Ee^{-\lambda^2 t}\sin\pi\lambda \Rightarrow Ee^{-\lambda^2 t}\sin\pi\lambda = 0 \ldots.(xi)$$

If $E = 0$, the solution is identically zero. So we must have $\sin\pi\lambda = 0$; since $e^{-\lambda^2 t} \neq 0$ and $E \neq 0$.

So from $(xi)$, $\sin\pi\lambda = 0 \Rightarrow \sin\pi\lambda = \sin(n\pi) \Rightarrow \pi\lambda = n\pi \Rightarrow \lambda = n$

Putting the value of $\lambda = n$ in $(x)$,

$$V(x,t) = Ee^{-n^2 t}\sin(nx) = Ee^{-n^2 t}\sin(nx) \ldots.(xii)$$

In order to satisfy the last condition $V(x,0) = \frac{-2x}{\pi}$, we are led to assume that infinitely many terms are taken. So using the principal of superposition to obtain the solution.

$$V(x,t) = \sum_{n=1}^{\infty} E_n e^{-n^2 t}\sin(nx) \ldots.(xiii)$$

Putting $t = 0$ in $(xiii)$, we get,

$$V(x,0) = \sum_{n=1}^{\infty} E_n e^{-0}\sin(nx)$$

$$\Rightarrow \sum_{n=1}^{\infty} E_n \sin(nx) = \frac{-2x}{\pi}$$

Apply Half Range Fourier Sine Series we have,

$$E_n = \frac{2}{\pi}\int_0^{\pi} \frac{-2x}{\pi}\sin(nx)\,dx$$

$$= \frac{-4}{\pi^2}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_0^{\pi}$$

$$= \frac{-4}{\pi^2}\left[\frac{-\pi}{n}\cos(n\pi) + 0 + \frac{1}{n^2}\sin(n\pi) - \frac{1}{n^2}\sin(0)\right] = \frac{4}{n\pi}(-1)^n$$

Putting the value of $E_n$ in $(xiii)$, we get,

$$V(x,t) = \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^n e^{-n^2 t}\sin(nx)$$

Now putting the values of $V(x,t)$ and $\varphi(x)$ in $(ii)$,

$$U(x,t) = V(x,t) + \varphi(x)$$

$$= \sum_{n=1}^{\infty} \frac{4}{n\pi}(-1)^n e^{-n^2 t}\sin(nx) + \frac{2x}{\pi} + 1;$$

Which is the required solution.

**Practice:** Solve the following boundary value problem by the method of Separation of variables

$$\frac{\partial U}{\partial t} = 2\frac{\partial^2 U}{\partial x^2}, U(0,t) = 10, U(3,t) = 40, U(x,0) = 25$$

*THANK YOU*

*STAY SAFE*