Differential Equation

An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called differential equation (DE).

⇒ Classification

01. Ordinary Differential Equation
02. Partial Differential Equation

* Classification is done based on the number of the independent variables.

Ordinary Differential Equation (ODE) :

A DE involving ordinary derivatives of one or more dependent variables with respect to single independent variables is called ODE.

xmpl - $\frac{dy}{dx} = x + \sin x$; $\frac{d^4y}{dt^4} + \left(\frac{dy}{dt}\right)^2 = 0$

Partial Differential Equation (PDE) :

A DE involving partial derivatives of one of more dependent variables with respect to more than one independent variables is called PDE.

xmpl - $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Order :

The order of DE is the highest order derivative involved in DE

Degree :

The degree of the DE is the power of the highest order derivative involved in DE.

Xmpl -

$$\frac{d^4x}{dt^4} + \left(\frac{dx}{dt}\right)^2 = 0 \rightarrow \text{4th order 1st degree.}$$

$$\left(\frac{d^2y}{dx^2}\right)^2 + 1 + \left(\frac{dy}{dx}\right)^3 = 0 \rightarrow \text{2nd order 2nd degree.}$$

$$\frac{dy}{dx} + x = 0 \rightarrow \text{1st order 1st degree.}$$

01. Find ODE by eliminating "a" from $y^2 = 4ax$.

$\Rightarrow$ Sol^n :

Given, $y^2 = 4ax$ ——(i)

$\Rightarrow 2y \cdot \frac{dy}{dx} = 4a$ [Diff. w.r.t. x]

$\Rightarrow y \cdot \frac{dy}{dx} = 2a$ ——(ii)

$\therefore a = \frac{1}{2} \cdot y \cdot \frac{dy}{dx}$ ——(iii)

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \cdot \frac{dy}{dx}$$

1st method

Putting (iii) in (i) $\rightarrow$

$$y^2 = 4 \cdot \frac{1}{2} \cdot y \cdot \frac{dy}{dx} \cdot x$$

$$\Rightarrow y = 2x \cdot \frac{dy}{dx} \quad \text{(Ans:)}$$

2nd method

Diff (ii) w.r.t. x $\rightarrow$

$$\frac{dy}{dx} \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 + y \cdot \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow y \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad \text{(Ans:)}$$

02. Find ODE by eliminating "a" from $y^2 = 4ax + 4a^2$.

⇒ Solⁿ :

Given,

$$y^2 = 4ax + 4a^2 \quad \text{—— (i)}$$

$$\Rightarrow 2y \cdot \frac{dy}{dx} = 4a + 0 \quad [\text{Diff. w.r.t. } x]$$

$$\therefore a = \frac{y}{2} \cdot \frac{dy}{dx} \quad \text{—— (ii)}$$

Putting (ii) in (i) ⇒

$$y^2 = 4 \cdot \frac{y}{2} \cdot \frac{dy}{dx} \cdot x + 4 \cdot \frac{y^2}{4} \cdot \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow y = 2x \frac{dy}{dx} + y\left(\frac{dy}{dx}\right)^2 \quad \text{(Ans:)}$$

01. Find ODE by eliminating A, B from $y = e^x(A\cos x + B\sin x)$.

$\Rightarrow$ Sol^n :

Here,

$$y = e^x(A\cos x + B\sin x) \quad \text{—(i)}$$

$\Rightarrow y' = e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)$ [Diff. w.r.t. x]

$\Rightarrow y' = y + e^x(-A\sin x + B\cos x)$ —(ii)

$\Rightarrow y'' = y' + e^x(-A\cos x - B\sin x) + e^x(-A\sin x + B\cos x)$

$\Rightarrow y'' = y' - e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) + y - y$

$\Rightarrow y'' = y' - y + y' - y$

$\Rightarrow y'' = 2y' - 2y$ (Ans:)

H.W. 01.

Find ODE by eliminating A, B from $y = Ae^{2x} + Be^{-2x}$.

=> Sol^n :

Here,

$$y = Ae^{2x} + Be^{-2x} \quad \text{—— (i)}$$

$\Rightarrow y' = 2 \cdot e^{2x} \cdot A - 2 \cdot e^{-2x} \cdot B$ [diff. w.r.t. x]

$\Rightarrow y' = 2(Ae^{2x} - Be^{-2x})$

$\Rightarrow y'' = 2(2 \cdot A \cdot e^{2x} + 2 \cdot B \cdot e^{-2x})$

$\Rightarrow y'' = 4(Ae^{2x} + Be^{-2x})$

$\Rightarrow y'' = 4y$ (Ans:)

Solution of first order ODE

=> Seperation of Variable Method :

A first order ODE of the form

$$f(x)\,dx + g(y)\,dy = 0$$

01. Solve : $\frac{dy}{dx} = \frac{x^3}{y^2}$

⇒ Sol^n :

Given,

$$\frac{dy}{dx} = \frac{x^3}{y^2}$$

$$\Rightarrow y^2 dy = x^3 dx$$

$$\Rightarrow \int y^2 dy = \int x^3 dx$$

$$\Rightarrow \frac{y^3}{3} = \frac{x^4}{4} + c$$ (Ans:)

02. $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

⇒ Sol^n :

Given,

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

$$\Rightarrow dy/dx = e^x . e^{-y} + x^2 . e^{-y}$$

$$\Rightarrow dy/dx = e^{-y}(e^x + x^2)$$

$$\Rightarrow \frac{dy}{e^{-y}} = (e^x + x^2) dx$$

$$\Rightarrow \int e^y dy = \int (e^x + x^2) dx$$

$$\Rightarrow e^y = e^x + \frac{x^3}{3} + c$$ (Ans:)

Seperation of variable by the method of substitution

01. $(x+y)^2 \frac{dy}{dx} = a^2$

⇒ **Sol^n:**

Given, $(x+y)^2 \frac{dy}{dx} = a^2$ —— (i)

Let, $(x+y) = z$ —— (ii)

$\Rightarrow y = z - x$

$\therefore \frac{dy}{dx} = \frac{dz}{dx} - 1$ —— (iii) [Diff. w.r.t. x]

Putting (ii) & (iii) in (i) ⇒

$z^2\left(\frac{dz}{dx} - 1\right) = a^2$

$\Rightarrow z^2 \frac{dz}{dx} - z^2 = a^2$

$\Rightarrow z^2 . \frac{dz}{dx} = a^2 + z^2$

$\Rightarrow \frac{z^2}{a^2+z^2} . dz = dx$

$\Rightarrow \frac{z^2+a^2-a^2}{a^2+z^2} . dz = dx$

$\Rightarrow \left(1 - \frac{a^2}{a^2+z^2}\right) dz = dx$

$\Rightarrow \int\left(1 - \frac{a^2}{a^2+z^2}\right) dz = \int dx$

$\Rightarrow z - a^2 . \frac{1}{a} \tan^{-1}\frac{z}{a} = x + c$

$\Rightarrow (x+y) - a \tan^{-1}\frac{(x+y)}{a} = x + c$ (Ans:)

02. $\frac{dy}{dx} = (4x + y + 1)^2$

$\Rightarrow$ Sol^n :

Given,

$$\frac{dy}{dx} = (4x+y+1)^2 \quad \text{——— (1)}$$

Let,

$4x+y+1 = z$ ——— (ii)

$\therefore y = z - 4x - 1$

$\therefore \frac{dy}{dx} = \frac{dz}{dx} - 4$ ——— (iii) [Diff. w.r.t. x]

Putting (ii) & (iii) in (i) $\Rightarrow$

$\frac{dz}{dx} - 4 = z^2$

$\Rightarrow \frac{dz}{dx} = z^2 + 4$

$\Rightarrow \int \frac{dz}{z^2 + 2^2} = \int dx$

$\Rightarrow \frac{1}{2} \tan^{-1} \frac{z}{2} = x + C$

$\Rightarrow \frac{1}{2} \tan^{-1} \frac{4x+y+1}{2} = x + C$ (Ans:)

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Homogeneous Function :

$y = vx \rightarrow v$ is a function of x.

01. $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow$ Sol^n :

Given, $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow 2xy\,dy = -(x^2 - 3y^2)\,dx$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{3y^2}{2xy} - \dfrac{x^2}{2xy}$

$\therefore \dfrac{dy}{dx} = \dfrac{3y}{2x} - \dfrac{x}{2y}$ ——(i)

Let, $y = vx$ ——(ii)

$\therefore \dfrac{dy}{dx} = v + \dfrac{dv}{dx}\cdot x$ ——(iii)

From (i), (ii) & (iii) $\Rightarrow$

$v + x\cdot\dfrac{dv}{dx} = \dfrac{3vx}{2x} - \dfrac{x}{2vx}$

$\Rightarrow v + x\dfrac{dv}{dx} = \dfrac{3v}{2} - \dfrac{1}{2v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{3v}{2} - \dfrac{1}{2v} - v$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{3v^2 - 1 - 2v^2}{2v}$

$\Rightarrow x\dfrac{dv}{dx} = \dfrac{v^2 - 1}{2v}$

$\Rightarrow \displaystyle\int \frac{2v}{v^2 - 1}\,dv = \int \frac{dx}{x}$

$\Rightarrow \ln(v^2 - 1) = \ln x + C$

$\Rightarrow \ln\left(\dfrac{y^2}{x^2} - 1\right) = \ln x + C$ (Ans:)

[from (ii)]

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* $xy^2 \frac{dy}{dx} = y^3 - x^3$

=> Sol^n :

Given,

$$xy^2 \frac{dy}{dx} = y^3 - x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^3 - x^3}{xy^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^3}{xy^2} - \frac{x^3}{xy^2}$$

$$\therefore \frac{dy}{dx} = \frac{y}{x} - \frac{x^2}{y^2} \quad \text{——(i)}$$

Let,

$$y = vx \quad \text{——(ii)}$$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx} \quad \text{——(iii)}$$

from (i), (ii) & (iii) =>

$$\frac{vx}{x} - \frac{x^2}{v^2x^2} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = v - \frac{1}{v^2}$$

$$\Rightarrow x\frac{dv}{dx} = -\frac{1}{v^2}$$

$$\Rightarrow \int -v^2\, dv = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{v^3}{3} = \ln x + C$$

$$\Rightarrow -\frac{1}{3} \cdot \frac{y^3}{x^3} = \ln x + C \quad \text{(Ans:)}$$

Solution of 1st order ODE

Exact Differential Method:

The first order ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact if,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

* Working Rule:

The solution of the given ODE is

$$\int M\,dx + \int (N \text{ with } x \text{ free terms})\,dy = C$$

01. $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

⇒ Sol^n :

Given, $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$

Comparing the given eq^n with $M(x,y)\,dx + N(x,y)\,dy = 0$

$M(x,y) = x^2 - 4xy - 2y^2$ & $N(x,y) = y^2 - 4xy - 2x^2$

$\therefore \frac{\partial M}{\partial y} = 0 - 4x - 4y$ & $\frac{\partial N}{\partial x} = 0 - 4y - 4x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. So, the eq^n is exact.

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$\therefore$ The sol^n is $\int M dx + \int (N \text{ with } x \text{ free term}) dy = \text{constant}$.

$$\Rightarrow \int (x^2 - 4xy - 2y^2)\, dx + \int y^2 dy = c$$

$$\Rightarrow \frac{x^3}{3} - 4y \cdot \frac{x^2}{2} - 2xy^2 + \frac{y^3}{3} = c$$

$$\Rightarrow \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c \quad \text{(Ans:)}$$

02. $(1 + e^{x/y})\, dx + e^{x/y} (1 - x/y)\, dy = 0$

$M = 1 + e^{x/y}$

$N = e^{x/y}(1 - x/y)$

$= e^{x/y} - e^{x/y} \cdot x/y$

$\therefore \frac{\partial M}{\partial y} = e^{x/y} \cdot x \cdot \left(-\frac{1}{y^2}\right)$

$= -\frac{x}{y^2} \cdot e^{x/y}$

$\therefore \frac{\partial N}{\partial x} = e^{x/y} \cdot \frac{1}{y} \cdot (1) - \frac{x}{y} \cdot e^{x/y} \cdot \frac{1}{y}(1) - e^{x/y} \cdot \frac{1}{y} (1)$

$= -\frac{x}{y^2} e^{x/y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\Rightarrow \int M\, dx + \int (N \text{ with } x \text{ free variables})\, dy = c$$

$$\Rightarrow \int (1 + e^{x/y})\, dx + 0 = c$$

$$\Rightarrow x + \left(e^{x/y} / \tfrac{1}{y}\right) = c$$

$$\Rightarrow x + y \cdot e^{x/y} = c \quad \text{(Ans:)}$$

03. $(y^2 - 2xy + 6x)\,dx - (x^2 - 2xy + 2)\,dy = 0$

$\Rightarrow$ Sol^n:

$$M = y^2 - 2xy + 6x \quad ; \quad N = -(x^2 - 2xy + 2)$$

$$\frac{\partial M}{\partial y} = 2y - 2x \quad ; \quad \frac{\partial N}{\partial x} = -2x + 2y$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \int M\,dx + \int (N \text{ with } x \text{ free variables})\,dy = C$$

$$\Rightarrow \int (y^2 - 2xy + 6x)\,dx + \int -2\,dy = C$$

$$\Rightarrow xy^2 - 2\cdot\frac{x^2}{2}\cdot y + 6\frac{x^2}{2} + (-2y) = C$$

$$\Rightarrow xy^2 - x^2y + 3x^2 - 2y = C \quad \text{(Ans:)}$$

CT - 01

Linear & Non-Linear Equation

A differential eq^n is called linear if –

01. Every dependent variable & every derivative involved occurs in the 1st degree only.

02. No products of dependent variables and/or derivatives occur.

03. In absence of trancendental functions of dependent variables (trigonometrics, exponential, logarithm).

General Form of Linear Eq^n:

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = 0$$

01. $y + \frac{dy}{dx} = x^2 \longrightarrow$ linear

02. $y^2 \frac{dy}{dx} + e^x = 0 \longrightarrow$ Non-linear

03. $x \cdot \frac{d^2y}{dx^2} + y = 0 \longrightarrow$ linear

04. $e^y \cdot \left(\frac{dy}{dx}\right)^2 + y = 0 \longrightarrow$ non-linear

First Order Linear Equation

General Form :

$$\frac{dy}{dx} + P(x,y) = Q(x) \quad \text{——(i)}$$

* Working Rule:

$$\text{Integrating Factor} = e^{\int P(x)dx}$$

Multiplying (i) by I.F. = $e^{\int P(x)dx} \cdot \frac{dy}{dx} + e^{\int P(x)dx} \cdot P(x,y) = e^{\int P(x)dx} Q(x)$

01. $(x^2+1)\frac{dy}{dx} + 4xy = x$.

=> Sol^n :

Given, $(x^2+1)\frac{dy}{dx} + 4xy = x$

$$\Rightarrow \frac{dy}{dx} + \frac{4xy}{(x^2+1)} = \frac{x}{(x^2+1)} \quad \text{——(i)}$$

$$\text{I.f.} = e^{\int \frac{4x}{x^2+1} dx}$$

$$= e^{2\ln(x^2+1)}$$

$$= e^{\ln(x^2+1)^2}$$

$$= (x^2+1)^2$$

Multiplying (i) by I.f. =>

$$(x^2+1)^2 \frac{dy}{dx} + (x^2+1)^2 \cdot \frac{4xy}{x^2+1} = (x^2+1) \cdot \frac{x}{x^2+1}$$

$$\Rightarrow (x^2+1)^2 \frac{dy}{dx} + 4xy(x^2+1) = x(x^2+1)$$

$$\Rightarrow \frac{d}{dx}\left[(x^2+1)^2 \cdot y\right] = x^3 + x$$

I.f. * Dependent Variable

$$\Rightarrow \int d\left[(x^2+1)^2 y\right] = \int (x^3+x)\,dx$$

Acts like z

Acts like dz

$$\Rightarrow (x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + C$$ (Ans:)

02. $x^4 \frac{dy}{dx} + 2x^3 y = 1.$

=> Sol^n :

Given,

$$x^4 \frac{dy}{dx} + 2x^3 y = 1$$

$$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^4} \quad \text{—— (i)}$$

$$\text{I.F.} = e^{\int \frac{2}{x} dx}$$

$$= e^{2 \ln x}$$

$$= e^{\ln x^2}$$

$$= x^2$$

(i)x I.F. => $x^2 \frac{dy}{dx} + 2xy = \frac{1}{x^2}$

$$\Rightarrow \frac{d}{dx}(x^2 \cdot y) = \frac{1}{x^2}$$

$$\Rightarrow \int d(x^2 y) = \int \frac{1}{x^2}\,dx$$

$$\Rightarrow x^2 y = -\frac{1}{x} + C$$

$$\Rightarrow x^2 y + \frac{1}{x} = C$$ (Ans:)

BERNOULLI'S EQUATION

General Form :

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \; ; \; n \neq 0, 1.$$

01. $x^2 \frac{dy}{dx} - 2xy = 3y^4$

$\Rightarrow$ **Sol^n :**

Given,

$$x^2 \frac{dy}{dx} - 2xy = 3y^4$$

$$\Rightarrow \frac{x^2}{y^4} \cdot \frac{dy}{dx} - \frac{2xy}{y^4} = \frac{3y^4}{y^4}$$

$$\Rightarrow x^2 y^{-4} \frac{dy}{dx} - 2xy^{-3} = 3$$

$$\Rightarrow y^{-4} \cdot \frac{dy}{dx} - \frac{2y^{-3}}{x} = \frac{3}{x^2} \quad \text{——— (i)}$$

Let,

$$y^{-3} = z$$

$$\Rightarrow -3y^{-4} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow y^{-4} \cdot \frac{dy}{dx} = -\frac{1}{3} \cdot \frac{dz}{dx}$$

(i) becomes $\Rightarrow -\frac{1}{3} \cdot \frac{dz}{dx} - \frac{2z}{x} = \frac{3}{x^2}$

$$\Rightarrow \frac{dz}{dx} + \frac{6z}{x} = -\frac{9}{x^2} \quad \text{——— (ii)}$$

$$\text{I.F.} = e^{\int \frac{6}{x} dx} = e^{6 \ln x} = e^{\ln x^6} = x^6$$

Multiplying (ii) with I.F. $\Rightarrow$

$$x^6 \cdot \frac{dz}{dx} + x^6 \cdot \frac{6z}{x} = -\frac{9}{x^2} \cdot x^6$$

$$\Rightarrow x^6 \cdot \frac{dz}{dx} + 6z x^5 = -9x^4$$

$$\Rightarrow \frac{d}{dx}(x^6 \cdot z) = -9x^4$$

$$\Rightarrow \int d(x^6 \cdot z) = \int (-9x^4)\, dx$$

$$\Rightarrow x^6 z = -9 \cdot \frac{x^5}{5} + C$$

$$\Rightarrow x^6 y^{-3} = -\frac{9}{5} x^5 + C \quad \text{(Ans:)}$$

01. Find ODE by eliminating "a" from $y^2 = 4ax$ & find order & degree of that ODE.

02. Solve : $(x+y)^2 \cdot \frac{dy}{dx} = a^2$

BERNOULLI'S EQUATION

*** 01. $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

=> Sol^n :

Given,

$$\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$$

$$\Rightarrow \frac{1}{\cos^2 y} \cdot \frac{dy}{dx} + x \frac{\sin 2y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} + x \cdot \frac{2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\Rightarrow \sec^2 y \cdot \frac{dy}{dx} + 2x \tan y = x^3 \text{ ——— (i)}$$

Let,

$$2\tan y = z$$

$$\therefore 2\sec^2 y \cdot \frac{dy}{dx} = \frac{dz}{dx}$$

$$\therefore \sec^2 y \cdot \frac{dy}{dx} = \frac{1}{2} \cdot \frac{dz}{dx}$$

$\therefore$ (i) becomes $\Rightarrow$

$$\frac{1}{2} \cdot \frac{dz}{dx} + xz = x^3$$

$$\Rightarrow \frac{dz}{dx} + 2xz = 2x^3 \quad \text{—— (ii)}$$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

Multiplying (ii) with I.F. $\Rightarrow$

$$e^{x^2} \cdot \frac{dz}{dx} + e^{x^2} \cdot 2xz = e^{x^2} \cdot 2x^3$$

$$\Rightarrow \frac{d}{dx}(e^{x^2} \cdot z) = 2x^3 \cdot e^{x^2}$$

$$\Rightarrow \int d(e^{x^2} \cdot z) = \int (2x^3 \cdot e^{x^2})\,dx$$

$$\Rightarrow e^{x^2} \cdot z = \int (2x^3 \cdot e^{x^2})\,dx \quad \text{—— (iii)}$$

Let,

$$x^2 = p$$

$$\therefore 2x = \frac{dp}{dx}$$

$\therefore$ (iii) becomes $\Rightarrow$ $e^p \cdot z = \int p \cdot \cancel{\frac{dp}{dx}} \cdot e^p \cdot \cancel{dx}$

$$\Rightarrow e^p \cdot z = \int p \cdot e^p \, dp$$

$$z \cdot e^p = p \int e^p\,dp - \int \left(\frac{d}{dp}(p) \int e^p\,dp \right) dp$$

$$\Rightarrow ze^p = pe^p - \int e^p\,dp$$

$$\Rightarrow ze^p = pe^p - e^p + c$$

$$\Rightarrow ze^{x^2} = x^2 e^{x^2} - e^{x^2} + c$$

$$\Rightarrow 2\tan y \cdot e^{x^2} = x^2 e^{x^2} - e^{x^2} + c \quad \text{(Ans:)}$$

APPLICATION OF 1ST ORDER ODE

=> **Application in real life:** Population dynamics, Circuit analysis, Newton's law of cooling.

Q1. A culture initially has P_0 number of bacteria. At $t = 1h$, the number of bacteria is measured to be $\frac{3}{2}P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the number of bacteria to triple.

=> **Sol^n:**

Initially $t = 0$.

According to the question, $\frac{dp}{dt} \propto P(t)$

$$\Rightarrow \frac{dp}{dt} = k\,P(t) \quad \ldots\ldots (i)$$

With condition, $P(0) = P_0 \quad \ldots\ldots (ii)$

$P(1) = \frac{3}{2}P_0 \quad \ldots\ldots (iii)$

$$\frac{dp}{dt} = kP$$

$$\Rightarrow \frac{dp}{p} = k\,dt$$

$$\Rightarrow \int \frac{dp}{p} = \int K\,dt$$

$$\Rightarrow \ln p = Kt + \ln c$$

$$\Rightarrow \ln p - \ln c = Kt$$

$$\Rightarrow \ln\left(\frac{p}{c}\right) = Kt$$

$$\Rightarrow \frac{p}{c} = e^{Kt}$$

$$\Rightarrow P(t) = ce^{Kt} \quad \ldots (iv)$$

Putting $t = 0$ in (iv) $\Rightarrow P(0) = C$

$\therefore C = P_0$

$\therefore$ (iv) becomes, $P(t) = P_0 e^{Kt} \quad \ldots (v)$

Putting $t = 1$ in (v) $\Rightarrow P(1) = P_0 e^K$

$$\Rightarrow \frac{3}{2} P_0 = P_0 e^K$$

$$\Rightarrow e^K = \frac{3}{2}$$

$$\therefore K = \ln \frac{3}{2} = 0.406$$

The time for bacteria to triple,

$$P(t) = 3P_0$$

$$\Rightarrow P_0 e^{0.406t} = 3P_0$$

$$\therefore t = \frac{\ln 3}{0.406} = 2.71 \text{ h.}$$ (Ans:)

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Q1. An RL circuit has an e.m.f. 5V, Resistance of $50\,\Omega$, Inductance of 1 Henry and no initial current. Find current at any time t.

⇒ Solⁿ :

According to Kirchoff's Law, $L\frac{dI}{dt} + IR = V$

$$\Rightarrow \frac{dI}{dt} + 50\,I = 5 \quad \ldots.(i)$$

With the condition, $I(0) = 0 \quad \ldots.(ii)$

I.F. of (i) $\Rightarrow e^{\int 50\,dt} = e^{50t}$

Multiplying (i) by I.F. $\Rightarrow e^{50t}\frac{dI}{dt} + e^{50t}\,50\,I = 5e^{50t}$

$$\Rightarrow \frac{d}{dt}(e^{50t}.\,I) = 5\,e^{50t}$$

$$\Rightarrow \int d\,(e^{50t}.\,I) = \int 5\,e^{50t}\,dt$$

$$\Rightarrow e^{50t}.\,I = 5.\,e^{50t}.\,\frac{1}{50} + C$$

$$\Rightarrow e^{50t}.\,I = \frac{1}{10}\,e^{50t} + C$$

$$\Rightarrow I(t) = \frac{1}{10} + ce^{-50t} \quad \ldots.(iii)$$

When, $t=0$ (iii) becomes,

$$I(0) = \frac{1}{10} + C$$

$$\therefore C = -\frac{1}{10}$$

$\therefore$ (iii) becomes $\Rightarrow$ $I(t) = \frac{1}{10} - \frac{1}{10} e^{-50t}$

$$\therefore I(t) = \frac{1}{10} \left(1 - e^{-50t}\right) \quad \text{(Ans:)}$$

* Newton's law of cooling $\rightarrow$ $\frac{dT}{dt} \propto (T - T_m)$

$T(t)$ = Temperature of object at time t

T_m = Ambient temperature

$\frac{dT}{dt}$ = Rate of cooling

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02. When a cake is removed from oven, its temperature is measured 300°F. 3 minutes later it's temperature is 200°F. How long it will take for the cake to cool off to a temperature of 80°F. Here, room temperature is 70°F.

⇒ Sol^n:

According to Newton's Law of Cooling, $\frac{dT}{dt} \propto (T - T_m)$

$$\Rightarrow \frac{dT}{dt} = k(T - \underset{70}{\underset{\downarrow}{T_m}}) \quad \text{....(i)}$$

With conditions,
$$\left.\begin{aligned} T(0) &= 300°F \\ T(3) &= 200°F \end{aligned}\right\} \text{....(ii)}$$

(i) $\Rightarrow \int \frac{dT}{T-70} = \int k\,dt$

$\Rightarrow \ln(T-70) = kt + \ln C$

$\Rightarrow \ln\left(\frac{T-70}{C}\right) = kt$

$\Rightarrow T - 70 = e^{kt} \cdot C$

$\Rightarrow T(t) = 70 + Ce^{kt}$ ....(iii)

$T(0) = 300 = 70 + C \cdot 1$

$\therefore C = 230$

(iii) $\Rightarrow T(t) = 70 + 230\,e^{kt}$ ....(iv)

when, $t = 3$, (iv) $\Rightarrow 200 = 70 + 230\,e^{3k}$

$\Rightarrow 230\,e^{3k} = 130$

$\Rightarrow 3k = \ln\left(\frac{130}{230}\right)$

$\therefore k = \frac{\ln\left(\frac{130}{230}\right)}{3} = -0.1902$

(iv) $\Rightarrow T(t) = 70 + 230\,e^{-0.1902t}$ ....(v)

$\therefore 80 = 70 + 230\,e^{-0.1902t}$

$\Rightarrow e^{-0.1902t} = \frac{10}{230}$

$\therefore t = 16.49$ min $= 16$ min 29.4 sec. (Ans:)

Higher Order Differential Equation

* General Form :

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{(n-1)} y}{dx^{(n-1)}} + \dots + a_{(n-1)}(x)\frac{dy}{dx} + a_n(x)\,y = \boxed{R(x) \text{ or, } 0}$$

Solution, $y = y_c + y_p$

* If R.H.S. $= 0$ then $y = y_c$
* If R.H.S. $= R(x)$ then $y = y_c + y_p$

Here,

y_c = Complementary function which contains arbitrary constant.

y_p = Particular integral which has no constant.

⇒ Distinct root → Roots are different.

⇒ Repeated root → Roots are same, repeated.

⇒ Complex root → Imaginary number as root.

01. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

$\Rightarrow$ Sol^n:

Given,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \quad \ldots\ldots (i)$$

For y_c

trial sol^n is always $y = e^{mx}$

Let, $y = e^{mx}$ be the trial sol^n of (i)

$$\Rightarrow \frac{dy}{dx} = me^{mx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = m^2e^{mx}$$

$\therefore$ Auxiliary sol^n of (i) $\Rightarrow m^2e^{mx} - 3me^{mx} + 2e^{mx} = 0$

$$\Rightarrow e^{mx}(m^2 - 3m + 2) = 0$$

e^{mx} cannot be 0

$$\therefore m^2 - 3m + 2 = 0$$

$$\therefore m_1 = 1 \; ; \; m_2 = 2.$$

$$\therefore y = y_c$$

$$= C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

$$= C_1 e^{x} + C_2 e^{2x}$$ (Ans:)

02. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

$\Rightarrow$ Sol^n:

Given,

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0 \quad \ldots\ldots (i)$$

Let,

$y = e^{mx}$

$\Rightarrow \frac{dy}{dx} = me^{mx}$

$\Rightarrow \frac{d^2y}{dx^2} = m^2 e^{mx}$

$\Rightarrow \frac{d^3y}{dx^3} = m^3 e^{mx}$

$\therefore$ Auxiliary Sol^n of (i) $\Rightarrow m^3 e^{mx} - 4m^2 e^{mx} + me^{mx} + 6e^{mx} = 0$

$\Rightarrow e^{mx}(m^3 - 4m^2 + m + 6) = 0$

$\because e^{mx} \neq 0 \quad \therefore m^3 - 4m^2 + m + 6 = 0$

$\therefore m_1 = -1 \;;\; m_2 = 2 \;;\; m_3 = 3$

$\therefore y = y_c$

$= C_1 e^{-x} + C_2 e^{2x} + C_3 e^{3x}$ (Ans:)

$m = -1$

$(m+1) = 0$

$m^3 - 4m^2 + m + 6 = 0$

$= m^3 + m^2 - 5m^2 - 5m + 6m + 6 = 0$

$= m^2(m+1) - 5m(m+1) + 6(m+1) = 0$

$(m+1)(m^2 - 5m + 6) = 0$

03. $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

$\Rightarrow$ Sol^n:

Auxiliary sol^n $\Rightarrow e^{mx}(m^3 - 4m^2 - 3m + 18) = 0$

$\because e^{mx} \neq 0 \therefore m^3 - 4m^2 - 3m + 18 = 0$

$\Rightarrow m^2(m+2) - 6m(m+2) + 9(m+2) = 0$

$\Rightarrow (m+2)(m^2 - 6m + 9) = 0$

$\Rightarrow (m+2)(m-3)^2 = 0$

$\therefore m_1 = -2 \; ; \; m_2 = 3 \; ; \; m_3 = 3$

$\therefore y_c = c_1 e^{-2x} + c_2 e^{3x} + c_3 x e^{3x}$ (Ans:)

For Repeated Root

$y_c = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \dots + c_n x^{n-1} e^{mx}$

04. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

$\Rightarrow$ Sol^n:

Auxiliary sol^n $\Rightarrow e^{mx}(m^2 - 6m + 9) = 0$

$\because e^{mx} \neq 0 \quad \therefore m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\therefore m_1 = 3 \; ; \; m_2 = 3$

$\therefore y_c = c_1 e^{3x} + c_2 x e^{3x}$ (Ans:)

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Higher Order Differential Equation

01. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

=> Soln :

Given,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0 \quad \ldots\ldots (i)$$

Auxiliary Soln => $e^{mx}(m^2 - 6m + 25) = 0$

$\because e^{mx} \neq 0 \; \therefore \; m^2 - 6m + 25 = 0$

$\therefore \; m_1 = 3 + 4i \; ; \; m_2 = 3 - 4i$

$(3 \pm 4i)$

$\alpha \pm i\beta$

Euler's Formula

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$\therefore y = y_c$

$$= c_1 e^{(3+4i)x} + c_2 e^{(3-4i)x}$$

$$= c_1 e^{3x+4ix} + c_2 e^{3x-4ix}$$

$$= e^{3x}\left(c_1 e^{4ix} + c_2 e^{-4ix}\right)$$

$$= e^{3x}\left\{c_1(\cos 4x + i\sin 4x) + c_2(\cos 4x - i\sin 4x)\right\}$$

$$= e^{3x}\left\{\cos 4x\,(c_1 + c_2) + i\sin 4x\,(c_1 - c_2)\right\}$$

$$= e^{3x}\left(c_3\cos 4x + c_4\sin 4x\right)$$

[Let, $(c_1 + c_2) = c_3$ & $i(c_1 - c_2) = c_4$]

(Ans:)

02. $\frac{d^2y}{dx^2} + y = 0$

$\Rightarrow$ Solⁿ:

Given,

$$\frac{d^2y}{dx^2} + y = 0 \;....\;(i)$$

Auxiliary Solⁿ $\Rightarrow e^{mx}(m^2+1) = 0$

$\because e^{mx} \neq 0 \;\therefore\; m^2+1 = 0$ $\quad m = \pm i \quad [\alpha \pm i\beta] \quad \alpha = 0, \; \beta = 1$

$\therefore m_1 = i \;;\; m_2 = -i$

$\therefore y_c = C_1 e^{ix} + C_2 e^{-ix}$

$= C_1(\cos x + i\sin x) + C_2(\cos x - i\sin x)$

$= \cos x\,(C_1 + C_2) + i\sin x\,(C_1 - C_2)$

$= C_3 \cos x + C_4 \sin x \quad$ [Let, $(C_1+C_2) = C_3$ & $\{i(C_1-C_2)\} = C_4$]

$m^2 + 4m + 5 = 0 \qquad a = 1, \; b = 4, \; c = 5$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm \sqrt{4i^2}}{2} = \frac{-4 \pm 2i}{2}$$

$= -2 \pm i$

$= \alpha \pm i\beta$

$\boxed{\alpha = -2 \quad \beta = 1}$

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PARTICULAR Integral (P.I.)

If $R(x) = e^{ax}$ then, P.I. $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$; $f(a) \neq 0$

01. $(D^2 + D + 5) y = e^{-2x}$ $\boxed{D = \frac{d}{dx}}$

⇒ Sol^n :

Given,

$$(D^2 + D + 5) y = e^{-2x} \quad \ldots (i)$$

Let,

$$y = e^{mx} \Rightarrow \frac{dy}{dx} = me^{mx} \Rightarrow \frac{d^2y}{dx^2} = m^2 e^{mx}$$

$\therefore$ Auxiliary Sol^n ⇒ $e^{mx}(m^2 + m + 5) = 0$

$\because e^{mx} \neq 0 \quad \therefore m^2 + m + 5 = 0$

$$\therefore m_1 = -\frac{1}{2} + \frac{\sqrt{19}}{2} i \; ; \; m_2 = -\frac{1}{2} - \frac{\sqrt{19}}{2} i$$

$$\therefore y_c = c_1 e^{\left(-\frac{1}{2} + \frac{\sqrt{19}}{2} i\right)x} + c_2 e^{\left(-\frac{1}{2} - \frac{\sqrt{19}}{2} i\right)x}$$

$$= e^{-\frac{x}{2}} \left(c_1 e^{\frac{\sqrt{19}}{2} ix} + c_2 e^{-\frac{\sqrt{19}}{2} ix} \right)$$

$$= e^{-\frac{x}{2}} \left[c_1 \left\{ \cos\left(\frac{\sqrt{19}}{2} x\right) + i \sin\left(\frac{\sqrt{19}}{2} x\right) \right\} + c_2 \left\{ \cos\left(\frac{\sqrt{19}}{2} x\right) - i \sin\left(\frac{\sqrt{19}}{2} x\right) \right\} \right]$$

$$= e^{-\frac{x}{2}} \left[\left\{ \cos\left(\frac{\sqrt{19}}{2} x\right) \right\} (c_1 + c_2) + \left\{ i \sin\left(\frac{\sqrt{19}}{2} x\right) \right\} (c_1 - c_2) \right]$$

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$$= e^{-\frac{x}{2}} \left\{ c_3 \cos\left(\frac{\sqrt{19}}{2}x\right) + c_4 \sin\left(\frac{\sqrt{19}}{2}x\right) \right\}$$ [Let, $(c_1 + c_2) = c_3$ & $i(c_1 - c_2) = c_4$]

$$y_p = \frac{1}{D^2 + D + 5} e^{-2x}$$

$$= \frac{1}{(-2)^2 + (-2) + 5} e^{-2x}$$

$$= \frac{1}{7} e^{-2x}$$

$$\therefore y = y_c + y_p$$

$$= e^{-\frac{x}{2}} \left\{ c_3 \cos\left(\frac{\sqrt{19}}{2}x\right) + c_4 \sin\left(\frac{\sqrt{19}}{2}x\right) \right\} + \frac{1}{7} e^{-2x}$$ (Ans:)

CT-02

HIGHER ORDER D.E. (P.I.)

If $R(x) = e^{ax}$ & $f(a) = 0$ then,

$$P.I. = \frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$$

n = factorize করার পরের Power.
↓
যার জন্য $f(a) = 0$

01. $(D^3 - 5D^2 + 7D - 3)\, y = \frac{1}{2}(e^{3x} + e^{x})$

=> Sol^n:

Auxiliary Sol^n => $e^{mx}(m^3 - 5m^2 + 7m - 3) = 0$

$\because e^{mx} \neq 0 \quad \therefore m^3 - 5m^2 + 7m - 3 = 0$

$\Rightarrow m^2(m-3) - 2m(m-3) + 1(m-3) = 0$

$\Rightarrow (m-3)(m^2 - 2m + 1) = 0$

$\Rightarrow (m-3)(m-1)^2 = 0$

$\therefore m_1 = 3 \; ; \; m_2 = 1 \; ; \; m_3 = 1$

$\therefore y_c = C_1 e^{3x} + C_2 e^{x} + C_3 x e^{x}$.

P.I. for $\frac{1}{2}e^{3x}$

$$y_p = \frac{1}{D^3-5D^2+7D-3}\cdot\frac{1}{2}e^{3x}$$

$$= \frac{1}{(D-3)(D-1)^2}\cdot\frac{1}{2}e^{3x}$$

$$= \frac{1}{(D-3)(3-1)^2}\cdot\frac{1}{2}e^{3x}$$

$$= \frac{1}{8}\cdot\frac{1}{D-3}e^{3x}$$

$$= \frac{1}{8}\cdot\frac{x^1}{1!}e^{3x}$$

$$= \frac{1}{8}xe^{3x}$$

P.I. for $\frac{1}{2}e^{x}$

$$y_p = \frac{1}{D^3-5D^2+7D-3}\cdot\frac{1}{2}e^{x}$$

$$= \frac{1}{(D-3)(D-1)^2}\cdot\frac{1}{2}e^{x}$$

$$= \frac{1}{(1-3)(D-1)^2}\cdot\frac{1}{2}e^{x}$$

$$= -\frac{1}{4}\cdot\frac{x^2}{2!}e^{x}$$

$$= -\frac{1}{8}x^2e^{x}$$

$\therefore\ y = y_c + y_p$

$$= C_1e^{3x} + C_2e^{x} + C_3xe^{x} + \frac{1}{8}xe^{3x} - \frac{1}{8}x^2e^{x}$$ (Ans:)

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If $R(x) = \sin(ax)$ or, $\cos(ax)$, then replace D^2 by $\{(a^2)\}$.

$$\text{P.I.} = \frac{1}{f(D)} \sin(ax) \text{ or, } \cos(ax).$$

01. $(D^2 - 3D + 2)y = \sin(3x)$

=> Sol^n:

Auxiliary sol^n => $e^{mx}(m^2 - 3m + 2) = 0$

$\because e^{mx} \neq 0 \quad \therefore m^2 - 3m + 2 = 0$

$\therefore m_1 = 1 \quad m_2 = 2$

$\therefore y_c = c_1 e^{x} + c_2 e^{2x}$

$$\therefore y_p = \frac{1}{D^2 - 3D + 2} \sin(3x)$$

$$= \frac{1}{-3^2 - 3D + 2} \sin(3x)$$

$$= \frac{1}{-3D - 7} \sin(3x)$$

$$= \frac{1\,(-3D+7)}{(-3D-7)(-3D+7)} \sin(3x)$$

$$= \frac{-3D+7}{9D^2 - 49} \sin x$$

$$= \frac{-3D+7}{9(-9) - 49} \sin x$$

$$= -\frac{1}{130}\{-3D \sin(3x) + 7 \sin(3x)\}.$$

$$= -\frac{1}{130}\left\{-3\frac{d}{dx}\sin(3x) + 7\sin(3x)\right\}$$

$$= -\frac{1}{130}(-9\cos 3x + 7 \sin 3x)$$

$\therefore y = y_c + y_p$

$= c_1 e^{x} + c_2 e^{2x} - \frac{1}{130}(-9\cos 3x + 7\sin 3x)$ (Ans:)

PRACTICE PROBLEMS

01. $(D^2 - 3D + 2)y = e^x$

02. $(D^2 - 2D + 1)y = \cos 3x$

03. $(D^2 - 3D + 2)y = e^{5x}$