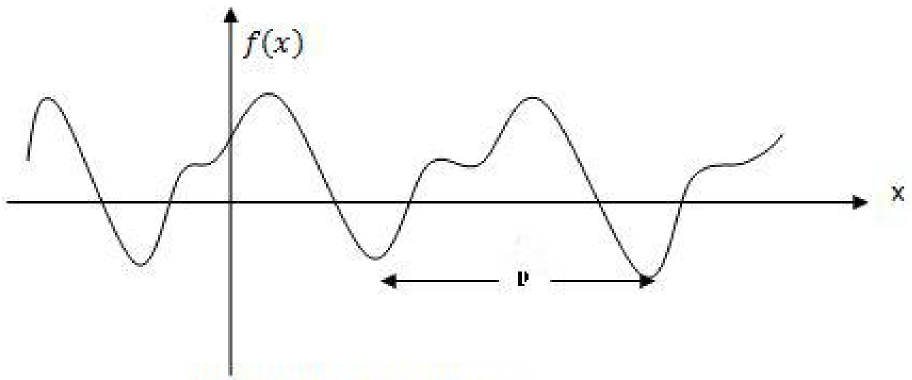
FOURIER SERIES AND FOURIER INTEGRALS

1 Introduction

Periodic functions occur frequently in engineering problems. The representation of these engineering problems in term of simple Periodic functions, such as sine and cosine is matter of great practical importance, which leads to **Fourier series**. These series, named after the physicist **JOSEPH FURIER** (1768-1830), are a very powerful tool in connection with various problems involving ordinary and partial differential equations. Here we shall discuss basic concepts, facts and techniques in connection with Fourier series. Some illustrative examples and also some important engineering applications of these series will be included.

2 Periodic functions and trigonometric series

Definition: The function $f(x)$ of a real variable x is said to be periodic if there exists a non-zero number p, independent of x, such that the equation $f(x+p) = f(x)$ holds for all values of x. The least value of $p > 0$ is called the **least period** or simply the **period** of $f(x)$.



Graph of periodic function.

Examples of periodic functions

Example 1. $f(x) = \sin x$, is a periodic function having period 2π.

Proof: $f(x) = \sin x$, $f(x+2\pi) = \sin(x+2\pi) = \sin x = f(x)$

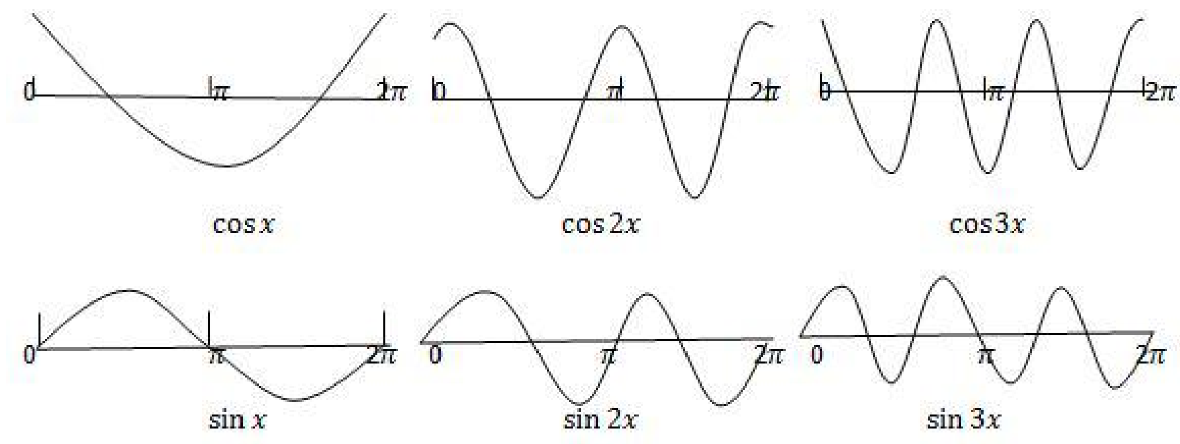
$f(x+4\pi) = \sin(x+4\pi) = \sin x = f(x)$

… … … … … … … … … … … … … … … … … … … …

$\therefore f(x) = f(x+2\pi) = f(x+4\pi) = \dots\dots\dots\dots\dots\dots = f(x+2n\pi)$

Thus $f(x) = \sin x$ is a periodic function having period 2π.

Similarly, we can show that $f(x) = \cos x$ is also periodic function having period 2π.



Graphs of cosine and sine functions having period 2π.

Example 2. $f(x) = \tan x$ is a periodic function having period π.

Proof: $f(x) = \tan x$, $f(x+\pi) = \tan(x+\pi) = \tan x$,

$f(x+2\pi) = \tan(x+2\pi) = \tan x$

… … … … … … … … … … … … … … … …

$f(x+n\pi) = \tan(x+n\pi) = \tan x$.

$\therefore f(x) = f(x+\pi) = f(x+2\pi) = \dots\dots\dots = f(x+n\pi)$

Thus $f(x) = \tan x$ is a periodic function having period π.

Definition By **Trigonometric series** we shall mean any series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \mathrm{nx} + b_n \sin \mathrm{nx})$$

Where the coefficients $a_n and\ b_n$ are constants.

3 Fourier series

Definition: The trigonometric series

$$f(x) = a_0 + a_1 \cos \mathrm{x} + a_2 \cos 2\mathrm{x} + \dots\dots\dots + a_n \cos \mathrm{nx} + \dots\dots + b_1 \sin \mathrm{x} + b_2 \sin 2\mathrm{x} + \dots\dots\dots + b_n \sin \mathrm{nx} + \dots\dots\dots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos \mathrm{nx} + b_n \sin \mathrm{nx}) \dots\dots\dots (1)$$

Is a **Fourier series** if its coefficients $a_0, a_n and\ b_n$ are given by the following formulas:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(v)dv$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\cos nv\, dv$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(v)\sin nv\, dv$$

$$\forall\, n = 1,2,3,4,5, \dots\dots\dots\dots\dots..$$

Where $f(x)$ is any singled valued function defined on the interval $(-\pi, \pi)$.

The Fourier series can also be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos \text{nx} + b_n \sin \text{nx})$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx\,, \qquad (n = 0,1,2,3 \dots\dots\dots\dots..)$$

And $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$, $(n = 1,2,3, \dots\dots\dots\dots.)$

4 The Fourier cosine sine series:

Definition Even Function:

A function $f(x)$ is called even if $f(-x) = f(x)$.

Graphically, an even function is symmetrical about the y-axis.

If $f(x)$ is an even function, then

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx$$

$$= \int_{\pi}^{0} f(-x)d(-x) + \int_{0}^{\pi} f(x)dx$$

$$= -\int_{\pi}^{0} f(x)d(x) + \int_{0}^{\pi} f(x)dx$$

$$= \int_{0}^{\pi} f(x)d(x) + \int_{0}^{\pi} f(x)dx$$

$$= 2\int_0^\pi f(x)dx$$

Thus if $f(x)$ is even, we have

$$a_0 = \frac{1}{2\pi}.2\int_0^\pi f(x)dx$$

$$= \frac{1}{\pi}\int_0^\pi f(x)dx$$

$$= \frac{1}{\pi}\int_0^\pi f(v)dv$$

Also if $f(x)$ is an even i.e. $f(-x) = f(x)$, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\, dx \quad (n = 1,2,3,\cdots\cdots)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)cosn(-x)d(-x)$$

$$= -\frac{1}{\pi}\int_{\pi}^{-\pi} f(x)cosnx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\, dx$$

$$= \frac{2}{\pi}\int_0^\pi f(x)cosnx\, dx$$

$$= \frac{2}{\pi}\int_0^\pi f(v)cosnv\, dv$$

But,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnx\, dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)sinn(-x)d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x).(-sinnx)(-\,dx)$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, sinnx\, dx$$

$= -b_n$

$\therefore \; 2b_n = 0$ or, $b_n = 0$

Therefore, if $f(x)$ is an even, then we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \, cosnx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(v)dv + \frac{2}{\pi}\sum_{n=1}^{\infty}\{\int_0^{\pi} f(v)cosnv \, dv\} \, cosnx$$

Which represents the function $f(x)$ in a series of $cosines$ and therefore it is known as $cosine$ series in the interval$(0, \pi)$.

5. Definition Odd Function:

A function $f(x)$ is called even if $f(-x) = -f(x)$.

Graphically, an odd function is symmetrical about the origin.

When $f(x)$ is odd, we have

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(-x)d(-x)$$

$$= \frac{1}{2\pi}\int_{\pi}^{-\pi} f(x)dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$= -a_0$

$\therefore \; 2a_0 = 0$ or, $a_0 = 0$

Also, if $f(x)$ is odd i.e. $f(-x) = -f(x)$, then

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx \, dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)cosn(-x)d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} -f(x)cosnx(-\,dx)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(x)cosnx\;dx$$

$$= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\;dx$$

$$= -a_n$$

$\therefore\; 2a_n = 0$ Or, $a_n = 0$

$$Finally, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnx\;dx$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} f(-x)sinn(-x)d(-x)$$

$$= \frac{1}{\pi}\int_{\pi}^{-\pi} -f(x).-sinnx.-\,dx$$

$$= -\frac{1}{\pi}\int_{\pi}^{-\pi} f(x)\;sinnx\;dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnx\;dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\;sinnx\;dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(v)\;sinnv\;dv$$

Therefore, if $f(x)$ is odd, then we have

$$f(x) = \sum_{n=1}^{\infty} b_n sinnx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\{\int_{0}^{\pi} f(v)\;sinnv\;dv\}sinnx$$

Which represents the function $f(x)$ in a series of $sines$ in the interval $(0,\pi)$ and therefore it is known as $sines$ series in the interval$(0,\pi)$.

6. Half Range Fourier Cosine and Sine Series:

When the Fourier series has only the cosine terms or only the sine terms in the expression we call the series **Half Range Fourier Cosine** or **Half Range Fourier Sine Series** respectively. When we are interested to find out a half range series corresponding to given a function, the function must be defined in the interval $(0,\pi)$, which is the half of the interval $(-\pi,\pi)$ and then the function is specified as odd or even, so that it is clearly defined in the other half of the interval, namely, $(0,\pi)$ in such a case we have

$b_n = 0, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x) cosnx\, dx$ for half range cosine series and

$a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ sinnx\, dx$ for half range sine series

7. Change of intervals:

(A) If $f(x)$ is defined in the interval $(-c,c)$ having period $2c$, the Fourier series of $f(x)$ in the interval $(-c,c)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{c} + b_n sin\frac{n\pi x}{c}\right)$$

$$Where,\ a_0 = \frac{1}{2c}\int_{-c}^{c} f(x)dx \quad (n = 0)$$

$$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\, cos\frac{n\pi x}{c} dx\ (n = 1,2,3 \cdots\cdots)$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(x)\, sin\frac{n\pi x}{c} dx\ (n = 1,2,3 \cdots\cdots)$$

(B) If $f(x)$ is defined in the interval $(\propto, \propto + 2c)$ having period $2c$, the Fourier series of $f(x)$ in the interval $(\propto, \propto + 2c)$ is given by

$$f(x) = a_0/2 + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{c} + b_n sin\frac{n\pi x}{c}\right)$$

$$Where,\ a_0 = \frac{1}{c}\int_{\propto}^{\propto+2c} f(x)dx \quad (n = 0)$$

$$a_n = \frac{1}{c}\int_{\propto}^{\propto+2c} f(x)\, cos\frac{n\pi x}{c} dx\ (n = 1,2,3\ \cdots\cdots)$$

$$b_n = \frac{1}{c}\int_{\propto}^{\propto+2c} f(x)\, sin\frac{n\pi x}{c} dx\ (n = 1,2,3\ \cdots\cdots)$$

WORKEDOUT EXAMPLES

Example 1. Obtain the Fourier series for $f(x) = e^{-x}$ in the interval $0 < x < 2\pi$.

Solution: Let

$$f(x) = e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)(1)$$

Then

$$a_{0=} \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{0}^{2\pi} e^{-x}dx$$

$$= \frac{1}{\pi}[-e^{-x}]_0^{2\pi=} \frac{1}{\pi}(1 - e^{-2\pi})$$

$$and, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx = \frac{1}{\pi}\int_{0}^{2\pi} e^{-x}cosnxdx$$

$$= \frac{1}{\pi}[-cosnx.e^{-x}]_0^{2\pi} - \frac{n}{\pi}\int_0^{2\pi} e^{-x}sinnxdx$$

$$= (1 - e^{-2\pi}) + \frac{n}{\pi}[sinnx.e^{-x}]_0^{2\pi} - \frac{n^2}{\pi}\int_0^{2\pi} e^{-x}cosnxdx$$

$$= \frac{1}{\pi}(1 - e^{-2\pi}) + 0 - n^2 a_n$$

$$\therefore (n^2 + 1)a_n = \frac{1}{\pi}(1 - e^{-2\pi})$$

$$or, \qquad a_n = \frac{1}{\pi}(1 - e^{-2\pi}).\frac{1}{n^2 + 1}$$

Finally,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}sinnxdx$$

$$= \frac{1}{\pi}[-e^{-x}sinnx]_0^{2\pi} + \frac{n}{\pi}\int_0^{2\pi} e^{-x}cosnxdx$$

$$= 0 + \frac{n}{\pi}[-e^{-x}cosnxdx]_0^{2\pi} - \frac{n^2}{\pi}\int_0^{2\pi} e^{-x}sinnxdx$$

$$or, (n^2 + 1)b_n = \frac{n}{\pi}(1 - e^{-2\pi})$$

$$\therefore b_n = \frac{1}{\pi}(1 - e^{-2\pi}).\frac{n}{n^2 + 1}$$

Now substituting the values of $a_{0,}a_n and\ b_n$ in (1), we get

$$f(x) = \frac{1}{2\pi}(1-e^{-2\pi}) + \sum_{n=1}^{\infty}\left[\frac{1}{\pi}(1-e^{-2\pi})\frac{1}{n^2+1}cosnx + \frac{1}{\pi}(1-e^{-2\pi}).\frac{n}{n^2+1}.sinnx\right]$$

$$= \frac{1}{\pi}(1-e^{-2\pi})\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}.cosnx + \sum_{n=1}^{\infty}\frac{n}{n^2+1}.sinnx\right]$$

$$= \frac{1}{\pi}(1-e^{-2\pi})$$

$$[\frac{1}{2} + \left(\frac{1}{2}cosx + \frac{1}{5}cos2x + \frac{1}{10}cos3x + \cdots\right) + \left(\frac{1}{2}sinx + \frac{2}{5}sin2x + \frac{3}{10}sin3x + \cdots\right)]$$

Example 2. Obtain the Fourier series for $f(x) = e^x$ in the interval $-\pi < x < \pi$.

Solution: Let

$$f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)(1)$$

Then

$$a_{0=}\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx$$

$$= \frac{1}{\pi}(e^{\pi} - e^{\pi})$$

$$a_o = \frac{2}{\pi}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right) = \frac{2Sinh\pi}{\pi}$$

$$and,\ \ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cosnxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x Cosnxdx$$

$$= \frac{1}{\pi}[e^{\pi} - e^{-\pi}]Cosnx + \frac{n}{\pi}[e^x Sinnx]_{-\pi}^{\pi} - \frac{n^2}{\pi}\int_{-\pi}^{\pi} e^x Cosnxdx$$

$$= \frac{2}{\pi}\left(\frac{e^{\pi}-e^{-\pi}}{2}\right)(-1)^n + 0 - n^2 a_n$$

$$or, (n^2+1)a_n = (-1)^n\frac{2Sinh\pi}{\pi}$$

$$a_n = \frac{2Sinh\pi}{\pi}\frac{(-1)^n}{(n^2+1)}$$

Finally,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinnxdx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x Sinnxdx$$

$$= \frac{1}{\pi}[e^x sinnx]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} e^x Cosnxdx$$

$$= 0 - \frac{n}{\pi}[e^x cosnx]_{-\pi}^{\pi} - \frac{n^2}{\pi}\int_{-\pi}^{\pi} e^x sinnxdx$$

$$= -n(-1)^n \frac{2sinh\pi}{\pi} - n^2 b_n$$

$$or, (n^2+1)b_n = -n.(-1)^n \frac{2Sinh\pi}{\pi}$$

$$b_n = (-1)^n \frac{2Sinh\pi}{\pi}\frac{-n}{(n^2+1)}$$

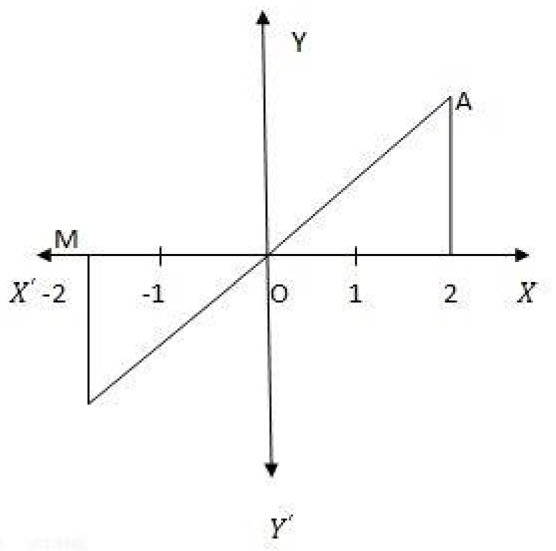
Now substituting the values of a_o, a_n *and* b_n in (1) we get

$$f(x) = \frac{1}{2\pi}.2sinh\pi + \sum_{n=1}^{\infty}\left[\frac{2Sinh\pi}{\pi}\frac{(-1)^n}{(n^2+1)}Cosnx - (-1)^n\frac{2Sinh\pi}{\pi}\frac{n}{(n^2+1)}Sinnx\right]$$

$$= \frac{2sinh\pi}{\pi}\left[\frac{1}{2} + \left(-\frac{1}{2}cosx + \frac{1}{5}cos2x - \frac{1}{10}cos3x + \cdots\right) + (\frac{1}{2}sinx - \frac{2}{5}sin2x + \frac{3}{20}sin3x - \cdots)\right]$$

Example 3: Express $f(x) = x$ as a half range sine series in the interval $0 < x < 2$.

Solution: The graph of $f(x) = x$ in $0 < x < 2$ is the line OA. Let us extend the function $f(x)$in the interval $-2 < x < 0$. So that the new function is symmetrical about the origin and therefore represents an odd function in the interval $(-2, 2)$.



Hence the Fourier series for $f(x)$ will have only the sine terms given by

$$f(x)=\sum_{n=1}^{\infty} b_n sin\frac{n\pi x}{2} \qquad (a_0=0, a_n=0)$$

Where

$b_n=\frac{2}{c}\int_0^c f(x)sin\frac{n\pi x}{2}dx$

$$=\frac{2}{2}\int_0^2 xsin\frac{n\pi x}{2}dx$$

$$=\left[-x\frac{2}{n\pi}cos\frac{n\pi x}{2}\right]_0^2+\frac{2}{n\pi}\int_0^2 cos\frac{n\pi x}{2}$$

$$=-\frac{4}{nx}cosn\pi+0+\frac{4}{n^2\pi^2}[sin\frac{n\pi x}{2}]_0^2$$

$$=-\frac{4}{n\pi}(-1)^n+0=\frac{-4(-1)^n}{n\pi}$$

Hence the Fourier sine series for $f(x)=x$ over the half

Range (0, 2) is

$$f(x)=\sum_{n=1}^{\infty}\frac{-4(-1)^n}{n\pi}sin\frac{n\pi x}{2}$$

$$=\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}sin\frac{n\pi x}{2}$$

$$=\frac{4}{\pi}[sin\frac{\pi x}{2}-\frac{1}{2}sin\frac{2\pi x}{2}+\frac{1}{3}sin\frac{3\pi x}{2}-\frac{1}{4}sin\frac{4\pi x}{2}+\cdots]$$

Example 04: Express $f(x)=x$ as a half range cosine series in the interval$0<x<2$.
Solution:

The graph of the function $f(x)=x$ in the interval $0<x<2$ is the line OA. Let us extend the function f(x) in the interval $0<x<2$.
(Shown by the line OB') so that the new function is symmetrical about the y-axis and therefore, represent an even function in the interval $0<x<2$. Hence the fourier series for f(x) over the full period (-2,2) will have only cosine terms given by,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n cos\frac{n\pi x}{2} \ (b_n=0)$$

$$Where \ a_0=\frac{2}{c}\int_0^c f(x)dx=\frac{2}{2}\int_0^2 xdx=2$$

$$and, a_n=\frac{2}{c}\int_0^c f(x)cos\frac{n\pi x}{2}dx=\frac{2}{2}\int_0^2 xcos\frac{n\pi x}{2}dx=[\frac{2x}{n\pi}sin\frac{n\pi x}{2}]_0^2-\frac{2}{n\pi}\int_0^2 cos\frac{n\pi x}{2}dx$$

$$= 0 + \frac{4}{n^2\pi^2}[sin\frac{n\pi x}{2}]_0^2 = \frac{4}{n^2\pi^2}(\cos n\pi - 1) = \frac{4}{n^2\pi^2}\{(-1)^n - 1\}$$

$$Thus\ f(x) = \frac{2}{2}\sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\{(-1)^n - 1\}\cos\frac{n\pi x}{2}\right]$$

$$= 1 + \left[-\frac{8}{1^2\pi^2}cos\frac{\pi x}{2} + 0 - \frac{8}{3^2\pi^2}cos\frac{3\pi x}{2} + 0 - \frac{8}{5^2\pi^2}cos\frac{5\pi x}{2} + \cdots\right]$$

$$= 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}cos\frac{\pi x}{2} + \frac{1}{3^2}cos\frac{3\pi x}{2} + \frac{1}{5^2}cos\frac{5\pi x}{2} + \cdots\right]$$

Example 05: Find the Fourier series expansion of the function $f(x) = x^2$ in the interval $-\pi \le x \le \pi$ and hence evaluate

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

Solution:

By definition we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$where\ \ a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx\,, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\ \ and\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

$$Now\ a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx = \frac{1}{6\pi}\{\pi^3 - (-\pi^3)\} = \frac{\pi^2}{3}$$

Again since $f(x) = x^2$ and $f(-x) = (-x^2) = x^2 = f(x)$

$\therefore f(-x) = f(x)$. Thus $f(x) = x^2$ is an even function and so sine terms will vanish i.e. $b_n = 0$

$$Finally, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}x^2\cos nx\,dx = 0 - \frac{2}{n\pi}\int_{-\pi}^{\pi}x\sin nx\,dx$$

$$= \frac{2}{n^2\pi}[\pi\cos n\pi - (-\pi)\cos n(-\pi)] - \frac{2}{n^3\pi}[\sin nx]_{-\pi}^{\pi} = \frac{2}{n^2\pi}[\pi\cos n\pi - \pi\cos n\pi] - 0$$

$= \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$. Since $\cos n\pi = (-1)^n$

Now putting the value of $a_0, a_n\ and\ b_n$ in (1) we get

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left\{\frac{4(-1)^n}{n^2}\cos nx + 0\right\}$$

$$or, f(x) = \frac{\pi^2}{3} + 4\{\frac{-\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} - \cdots\}$$

$$= \frac{\pi^2}{3} - 4\{\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \cdots\}$$

When $= 0$, we have

$$0=\frac{\pi^2}{3}\sum_{n=1}^{\infty}\left\{\frac{4(-1)^n}{n^2}\right\}$$
$$=\frac{\pi^2}{3}-4(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots)$$
$$=\frac{\pi^2}{3}-4\left\{\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right)-2\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)\right\}$$
$$=\frac{\pi^2}{3}-4\left\{\sum_{n=1}^{\infty}\frac{1}{n^2}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}\right\}$$
$$=\frac{\pi^2}{3}-2\sum_{n=1}^{\infty}\frac{1}{n^2}\quad or,2\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{3}$$

$$\therefore\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$$

Example 06:

(a) Obtain the Fourier series of the function $f(x)\begin{cases}0,-\pi\le x\le 0\\1,0\le x\le\pi\end{cases}$

Solution:

By definition we have

$$f(x)=a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx\ )$$
$$where\ \ a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx\,,a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\ \ and\ b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$
$$Now\ a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}f(x)dx+\int_{0}^{\pi}f(x)dx\right]$$
$$=\frac{1}{2\pi}\left[\int_{-\pi}^{0}0dx+\int_{0}^{\pi}1dx\right]=\frac{1}{2\pi}[0+\pi]=\frac{1}{2}$$

Again, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx\ \ (n\ne 0)$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}f(x)\cos nx\,dx+\int_{0}^{\pi}f(x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cos nx\,dx+\int_{0}^{\pi}1\cos nx\,dx\right]$$
$$=0+\frac{1}{\pi n}[\sin nx]_0^{\pi}=0+0=0$$

$$Finally, b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\sin nx\,dx + \int_{0}^{\pi} 1\sin nx\,dx\right]$$

$$= \frac{1}{\pi}[0 - \frac{1}{\text{n}}\cos nx]_0^{\pi}$$

$$= -\frac{1}{n\pi}[\cos n\pi - \cos 0]$$

$$= -\frac{1}{n\pi}[(-1)^n - 1]$$

$$= \frac{1}{n\pi}[1-(-1)^n] = \begin{cases} o\ when\ n\ is\ even \\ \dfrac{2}{n\pi}\ \ when\ n\ is\ odd \end{cases}$$

Now putting the value of $a_0, a_n\ and\ b_n$ in (1) we get

$$f(x) = \frac{1}{2} + 0 + \sum_{n=1}^{\infty} b_n \sin nx$$

$$or, f(x) = \frac{1}{2} + b_1\sin x + b_2\sin 2x + b_3\sin 3x + b_4\sin 4x + b_5\sin 5x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\sin x + 0 + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}(\sin x]] + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots)$$

Example: 7. Find the Fourier series expansion of the function,

$$f(x) = \begin{cases} 0, -\pi < x \le 0 \\ x, 0 < x \le \pi \end{cases}$$

Hence evaluate

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Solution: By the definition of fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\cos nx) \ldots (1)$$

Where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx\,, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

And

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

Now

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi} x\, dx\right]$$

$$= 0 + \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{4\pi}(\pi^2 - 0) = \frac{\pi}{4}$$

Again

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ \ dx\ (n \neq 0)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\cos nx\ \ dx + \int_{0}^{\pi} f(x)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\, dx + \int_{0}^{\pi} x\cos nx\ \ dx\right]$$

$$= 0 + \frac{1}{\pi n}\ [x\sin nx]_{\pi}^{0} - \frac{1}{\pi n}\int_{0}^{\pi} \sin nx\, dx$$

$$= 0 + \frac{1}{\pi n^2}[\cos nx]_0^{\pi}$$

$$= \frac{1}{\pi n^2}[\cos n\pi -\ \ cos 0] = \frac{1}{\pi n^2}[(-1)^n - 1]$$

Since $[\cos n\pi = (-1)^n]$

Finally,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ \ dx.$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_{0}^{\pi} f(x)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\sin nx\ \ dx + \int_{0}^{\pi} x\sin nx\, dx\right]$$

$$= 0 - \frac{1}{\pi n}[x cos\ nx]_0^{\pi} + \frac{1}{\pi n}\int_{0}^{\pi} \cos nx\ \ dx$$

$$= -\frac{1}{\pi n}\ (\pi\cos n\pi - 0) + \frac{1}{\pi n^2}\ [\sin nx]_0^{\pi}$$

$= -\frac{1}{n}\cos n\pi + 0$ since $\sin n\pi = 0, \sin 0 = 0$

$$= -\frac{1}{n}.(-1)^n = \frac{(-1)^{n+1}}{n}.$$

Now putting the values of $a_0, a_n, and\ b_n in$ (1), we get

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{\pi n^2}\{(-1)^n - 1\}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$= \frac{\pi}{4} + \left[\left(-\frac{2}{\pi 1^2}\cos x + 0 - \frac{2}{\pi 3^2}\cos 3x + 0 - \frac{2}{\pi 5^2}.\cos 5x + 0 - \dots\dots\dots\right) + (\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\dots\dots)\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\right) + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \dots\dots\right) \dots\dots..(2)$$

Putting, $x = 0$ in the above equation (2), we get

$$0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos 0}{1^2} + \frac{\cos 0}{3^2} + \frac{\cos 0}{5^2} + \dots\dots\right) + \left(\frac{\sin 0}{1} - \frac{\sin 0}{2} + \frac{\sin 0}{3} - \frac{\sin 0}{5} + \dots\dots\right)$$

$$\Rightarrow 0 = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots\right) + 0$$

$$\Rightarrow \frac{2}{\pi}\left\{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n-1)^2} + \dots\right\} = \frac{\pi}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(2n-1)^2} + \dots = \frac{\pi^2}{8}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

.

Example 8. Find the Fourier series of the function of period $-\pi < x < \pi$ as follows: $f(x) = \begin{cases} o\ when - \pi < x \le 0 \\ sinx\ \ when\ 0 < x \le \pi \end{cases}$

Solution:

By definition of the Fourier series

We have

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cos\ nx + b_n sin\ nx) \dots\dots\dots\dots\dots\dots(1)$$

Where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos\ nxdx$$

And

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin\ nxdx$$

Now

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 0.dx + \int_{0}^{\pi} sinxdx\right]$$

$$= 0 + \frac{1}{2\pi}[-cosx]_0^\pi$$

$$= -\frac{1}{2}(cos\pi - 1)$$

$$= -\frac{1}{2\pi}(-1 - 1)$$

$$= \frac{1}{\pi}$$

$$Again \;\; a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos\, nxdx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)cosndx + \int_{0}^{\pi} f(x)cosnxdx\right]$$

$$\frac{1}{\pi}\left[\int_{-\pi}^{0} 0.ndx + \int_{0}^{\pi} sinxcosnxdx\right]$$

$$= 0 + \frac{1}{2\pi}\int_{0}^{\pi} 2cosnxsinxdx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)\,x - \sin(n-1)\,x]dx$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}[\cos(n+1)\,x]_0^\pi + \frac{1}{n-1}[\cos(n-1)\,x]_0^\pi\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n+1}\{\cos(n+1)\,\pi - cos0\} + \frac{1}{n-1}\{\cos(n-1)\,\pi - cos0\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-\cos(n\pi+\pi)}{n+1} + \frac{\cos(n\pi-\pi)-1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1+cosn\pi}{n+1} - \frac{1+\mathrm{cosn}\pi}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{(n-1-n-1)(1+cosn\pi)}{n^2-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2(1+cosn\pi)}{n^2-1}\right]$$

$$= -\frac{(1+cosn\pi)}{\pi(n^2-1)}$$

$$= \frac{\{1+(-1)^n\}}{\pi(n^2-1)}\ (n \neq 1)$$

When $n = 2, a_2 = -\frac{(1+1)}{\pi(2^2-1)} = -\frac{2}{3\pi}$

When $n = 3, a_3 = 0$

When $n = 4, a_4 = -\frac{(1+1)}{\pi(4^2-1)} = -\frac{2}{15\pi}$

when $n = 5, a_5 = 0$

when $n = 6, a_4 = -\frac{(1+1)}{\pi(6^2-1)} = -\frac{2}{35\pi}$ and so on

but when

$$n = 1, a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos\ xdx$$

$$= \frac{1}{\pi}\int_0^{\pi} sincosxdx\,\frac{1}{\pi}\left[\frac{(sinx)^2}{2}\right]_0^{\pi} = 0$$

$Finally,\ \ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin\ nxdx.$

When

$$n = 1, b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin\ xdx$$

$$Or, b_1 = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)sin\ xdx + \int_0^{\pi} f(x)sin\ \ xdx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\,sin\,xdx + \int_0^{\pi} sin\,xdx\right]$$

$$= 0 + \frac{1}{2\pi}\int_0^{\pi} 2sin^2xdx.$$

$$= \frac{1}{2\pi}\int_0^{\pi}(1 - cos2x)dx$$

$$= \frac{1}{2\pi}\left[x - \frac{1}{2}sin2x\right]_0^{\pi}$$

$$= \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}$$

$$Also, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin\, nxdx \;\; (n \neq 1, n = 2,3,4 \dots \dots .)$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)sin\, nxdx + \int_0^{\pi} f(x)\, sinnxdx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\, sin\, nxdx + \int_0^{\pi} sinx\, sinnxdx\right]$$

$$= 0 + \frac{1}{2\pi}\int_0^{\pi} 2\, sinnxdx$$

$$= \frac{1}{2\pi}\int_0^{\pi}[\cos(n-1)\,x - \cos(n+1)\,x]dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{n-1}\sin(n-1)\,x - \frac{1}{n+1}\sin(n+1)\,x\right]_0^{\pi}$$

$$= \frac{1}{2\pi}.\,0 = 0\; for\; (n \neq 1, n = 2,3,4).$$

Thus $f(x) = \frac{1}{\pi} + \left(-\frac{2}{3\pi}cos2x - \frac{2}{15\pi}cos4x - \frac{2}{35\pi}cos6x - \cdots \dots \dots \dots\right) + \frac{1}{2}sinx + 0$

$$= \frac{1}{\pi} + \frac{1}{2}sinx - \frac{2}{\pi}\left(\frac{cos2x}{3} + \frac{cos4x}{15} + \frac{cos6x}{35} + \cdots\right)$$

Example 9. Find the Fourier series Expansion of the function f(x)=$|x|$ in the interval[-π,π] .Hence evaluate the sum

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$$

.

Solution:

By definition of the Fourier series, we have

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(1)$$

$$Where, a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnx\, dx$$

Now, by definition $|x| = \begin{Bmatrix} x, x > 0 \\ -x, x < 0 \end{Bmatrix}$

Hence the given function f(x)= $|x|$ is an even function and for the even function $b_n = 0, (n = 1,2,3, \ldots\ldots)$ in the fouerier series expansion (1) of f(x) and

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|dx = \frac{2}{\pi}\int_{0}^{\pi} (x)dx = \frac{2}{\pi}[\frac{x^2}{2}]_0^{\pi} = \pi$$

$$Again, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\, dx\ (n \neq 0)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|cosnx\, dx$$

$$=\frac{2}{\pi}\int_0^{\pi}|x|cosnx\, dx$$

$$=\frac{2}{\pi}\int_0^{\pi} xcosnx\, dx$$

$$=\frac{2}{\pi n}[xsinnx]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi} sinnx\, dx$$

$$=0+\frac{2}{\pi n^2}[cosnx]_0^{\pi}$$

$$=\frac{2}{\pi n^2}[cosn\pi - cos0]$$

$$=\frac{2}{\pi n^2}\{(-1)^n - 1\}$$

$$=\frac{-4}{\pi n^2}\ when\ n = 1,3,5 \ldots$$

=0, when n=2, 4, 6….

Now, substituting the values ofa_0, a_n,$and\ b_n$ in(1),we get

f(x)=$\frac{\pi}{2}$ $[\left(\frac{-4}{\pi 1^2}+0-\frac{4}{\pi 3^2}cos3x+0-\frac{4}{\pi 5^2}cos5x+\cdots\right)+0]$

=$\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{cosx}{1^2}+\frac{cos3x}{3^2}+\frac{cos5x}{5^2}+\cdots.\right]\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$

Putting x=0 in(2),we have

0=$\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{cos0}{1^2}+\frac{cos0}{3^2}+\frac{cos0}{5^2}+\cdots.\right]$

=$\frac{\pi}{2}-\frac{4}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots.\right]$

Or, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots.=\frac{\pi}{2}*\frac{\pi}{4}=\frac{\pi^2}{8}$

Or,$\sum_{n=0}^{\infty}\frac{1}{(1+2n)^2}$=$\frac{\pi^2}{8}$.

Note; the values ofa_0, a_n can also be found in the following way:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx$$

$$=\frac{1}{\pi}[\int_{-\pi}^{0}f(x)dx+\int_{0}^{\pi}f(x)dx]$$

$$=\frac{1}{\pi}[\int_{-\pi}^{0}(-x)dx+\int_{0}^{\pi}(x)dx]$$

$$=\frac{1}{\pi}[\int_{0}^{\pi}xdx+\int_{0}^{\pi}xdx]$$

$$=\frac{2}{\pi}[\int_{0}^{\pi}x\,dx]$$

$$=\frac{2}{\pi}[\frac{x^2}{2}]_0^{\pi}$$

=π

$$a_n=\int_{-\pi}^{\pi}f(x)cosnx\,dx$$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} f(x)cosnx\, dx + \int_{0}^{\pi} f(x)cosnx\, dx]$$

$$= \frac{1}{\pi}[\int_{-\pi}^{0} (-x)\, cosnx\, dx + \int_{0}^{\pi} x\, cosnx\, dx]$$

$$= \frac{1}{\pi}[\int_{0}^{\pi} x\, cosnx\, dx + \int_{0}^{\pi} x\, cosnx\, dx]$$

$$= \frac{2}{\pi}[\int_{0}^{\pi} x\, cosnx\, dx]$$

$$\frac{-4}{\pi n^2}\ when\, n = 1,3,5 \ldots$$

=0, when n=2, 4, 6….

Example 10: if $f(x) = \begin{cases} 1 \quad when\ 0 \le x \le \pi \\ -1 when - \pi \le x < 0 \end{cases}$

expand $f(x)$ in the Fourier series. Hence deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

Solution: By definition of the Fourier series we have

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n Cosnx + b_n Sinnx) \quad --- \quad --- \quad --- \quad --- \quad --- \quad (1)$$

Where $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cosnxdx \ (n \neq 0)$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinnxdx$

Now $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\, dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\, dx$$

$$= -\frac{1}{\pi}[x]_{-\pi}^{0} + \frac{1}{\pi}[x]_{0}^{\pi}$$

$$= -\frac{1}{\pi}[0 - \pi] + \frac{1}{\pi}[\pi - 0]$$

$$= 0$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\cos nx\, dx$$

$$= -\frac{1}{n\pi}[\sin nx]_{-\pi}^{0} + \frac{1}{n\pi}[\sin nx]_{0}^{\pi}$$

$$= -\frac{1}{n\pi}[\sin 0 - \sin n\pi] + \frac{1}{n\pi}[\sin n\pi - \sin 0]$$

$$= 0 \qquad Since, \sin 0 = 0$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin nx\,dx$$

$$= -\frac{1}{n\pi}[-\cos nx]_{-\pi}^{0} - \frac{1}{n\pi}[\cos nx]_{0}^{\pi}$$

$$= \frac{1}{n\pi}[\cos 0 - \cos n\pi] - \frac{1}{n\pi}[\cos n\pi - \cos 0]$$

$$= \frac{2}{n\pi}[\cos 0 - \cos n\pi]$$

$$= \frac{2}{n\pi}[1 - (-1)^n]$$

Therefore, putting the values of $a_o, a_n and\ b_n$ in (1), we get

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}[1 - (-1)^n]\right)\sin nx$$

$$= \frac{2}{\pi}\left[\frac{2\sin x}{1} + 0 + \frac{2\sin 3x}{3} + 0 + \frac{2\sin 5x}{5} + \cdots\right]$$

$$= \frac{2}{\pi}\left[\frac{2\sin x}{1} + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5} + \cdots\right] \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad (2)$$

Now since $0 < x < \pi$

If $x = \frac{\pi}{2}, f(x) = 1$, Hence putting $x = \frac{\pi}{2}$, we get

$$1 = \frac{4}{\pi}\left[\frac{2\sin\frac{\pi}{2}}{1} + \frac{2\sin\frac{3\pi}{2}}{3} + \frac{2\sin\frac{5\pi}{2}}{5} + \cdots\right]$$

or. $\frac{\pi}{4} = \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$

Example 11: if $f(x) = \begin{cases} 1 \quad when\ 0 \le x \le \pi \\ -1 when - \pi \le x < 0 \end{cases}$

expand $f(x)$ in the Fourier series. Hence deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

Solution: By definition of the Fourier series we have

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty}(a_n Cosnx + b_n Sinnx) \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad \text{---} \quad (1)$$

Where $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Cosnxdx\ (n \ne 0)$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)Sinnxdx$

Now $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\,dx$$

$$= -\frac{1}{\pi}[x]_{-\pi}^{0} + \frac{1}{\pi}[x]_{0}^{\pi}$$

$$= -\frac{1}{\pi}[0-\pi] + \frac{1}{\pi}[\pi - 0]$$

$$= 0$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\cos nx\,dx$$

$$= -\frac{1}{n\pi}[\sin nx]_{-\pi}^{0} + \frac{1}{n\pi}[\sin nx]_{0}^{\pi}$$

$$= -\frac{1}{n\pi}[\sin 0 - \sin n\pi] + \frac{1}{n\pi}[\sin n\pi - \sin 0]$$

$$= 0 \quad Since, \sin 0 = 0$$

Again, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-1)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}(1)\sin nx\,dx$$

$$= -\frac{1}{n\pi}[-\cos nx]_{-\pi}^{0} - \frac{1}{n\pi}[\cos nx]_{0}^{\pi}$$

$$= \frac{1}{n\pi}[\cos 0 - \cos n\pi] - \frac{1}{n\pi}[\cos n\pi - \cos 0]$$

$$= \frac{2}{n\pi}[\cos 0 - \cos n\pi]$$

$$= \frac{2}{n\pi}[1 - (-1)^n]$$

Therefore, putting the values of $a_o, a_n and\ b_n$ in (1), we get

$$f(x) = 0 + 0 + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}[1-(-1)^n]\right)\sin nx$$

$$= \frac{2}{\pi}\left[\frac{2\sin x}{1} + 0 + \frac{2\sin 3x}{3} + 0 + \frac{2\sin 5x}{5} + \cdots\right]$$

$$= \frac{2}{\pi}\left[\frac{2\sin x}{1} + \frac{2\sin 3x}{3} + \frac{2\sin 5x}{5} + \cdots\right] \quad --- \quad --- \quad --- \quad --- \quad --- \quad (2)$$

Now since $0 < x < \pi$

If $x = \frac{\pi}{2}, f(x) = 1$, Hence putting $x = \frac{\pi}{2}$, we get

$$1 = \frac{4}{\pi}\left[\frac{2\sin\frac{\pi}{2}}{1} + \frac{2\sin\frac{3\pi}{2}}{3} + \frac{2\sin\frac{5\pi}{2}}{5} + \cdots\right]$$

or. $\frac{\pi}{4} = \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$

Example 12: Find the Fourier series expansion of the function below, having period 2π:

$$f(x) = \begin{cases} -x \; if - \pi < x < 0 \\ \;\; x \; if \;\; 0 < x < \pi \end{cases}$$

Solution: By the definition of the Fourier series we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad (1)$$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx,$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \;\, dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \;\, dx$

Now,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x) dx + \frac{1}{\pi}\int_{0}^{\pi} x\, dx$$

$$= -\frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{0}^{\pi}$$

$$= -\frac{1}{\pi}(0 - \frac{\pi^2}{2}) + \frac{1}{\pi}(\frac{\pi^2}{2} - 0)$$

$$= \frac{\pi}{2} + \frac{\pi}{2}$$

$$= \pi$$

And $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \;\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx \;\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx \;\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-x) \cos nx \;\, dx + \frac{1}{\pi}\int_{0}^{\pi} x \cos nx \;\, dx$$

$$= -\frac{1}{\pi n}[x \sin nx]_{-\pi}^{0} + \frac{1}{\pi n}\int_{-\pi}^{0} \sin nx \;\, dx + \frac{1}{\pi n}[x \sin nx]_{0}^{\pi} - \frac{1}{\pi n}\int_{0}^{\pi} \sin nx \;\, dx$$

$$= 0 - \frac{1}{\pi n^2}[\cos nx]_{-\pi}^{0} + 0 + \frac{1}{\pi n^2}[\cos nx]_{0}^{\pi}$$

$$= -\frac{1}{\pi n^2}[\cos 0 - \cos n\pi] + \frac{1}{\pi n^2}[\cos n\pi - \cos 0]$$

$= \frac{2}{\pi n^2}[\cos n\pi - \cos 0]$

$= \frac{2}{\pi n^2}[(-1)^n - 1]$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \; dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx \; dx$

$= \frac{1}{\pi}\int_{-\pi}^{0} (-x) \sin nx \; dx + \frac{1}{\pi}\int_{0}^{\pi} x \sin nx \; dx$

$= \frac{1}{\pi n}[x \cos nx]_{-\pi}^{0} - \frac{1}{\pi n}\int_{-\pi}^{0} \cos nx \; dx - \frac{1}{\pi n}[x \cos nx]_{0}^{\pi} + \frac{1}{\pi n}\int_{0}^{\pi} \cos nx \; dx$

$= \frac{1}{\pi n}[0 + \pi \cos n\pi] - \frac{1}{\pi n^2}[\sin nx]_{-\pi}^{0} - \frac{1}{\pi n}[0 + \pi \cos n\pi] + \frac{1}{\pi n^2}[\sin nx]_{0}^{\pi}$

$= \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + 0$

$= 0$

Now putting the values of a_0 ,a_n *and* b_n in (1) we get

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n}\{(-1)^n - 1\} \cos nx + 0$$

$= \frac{\pi}{2} + \left[-\frac{2.2}{\pi 1^2} \cos x + 0 - \frac{2.2}{\pi 3^2} \cos 3x + 0 - \frac{2.2}{\pi 5^2} \cos 5x + \cdots\right]$

$= \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

<u>Example 13:</u> Find the Fourier series expansion of the function f(x) in the interval $(-\pi, \pi)$ where

$$f(x) = \begin{cases} \pi + x \; when \; -\pi < x < 0 \\ \pi - x \; when \; 0 < x < \pi \end{cases}$$

Solution: By the definition of the Fourier series we have

$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ (1)

Where $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \, dx$,

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \; dx$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \; dx$

Now,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)\, dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)\, dx$$
$$= \frac{1}{2\pi}\int_{-\pi}^{0} (\pi + x) dx + \frac{1}{2\pi}\int_{0}^{\pi} (\pi - x)\, dx$$
$$= \frac{1}{2\pi}\left[\pi x + \frac{x^2}{2}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_{0}^{\pi}$$
$$= \frac{1}{2\pi}[0 + \pi^2 - \frac{\pi^2}{2} + 0 + \pi^2 - \frac{\pi^2}{2}$$
$$= \frac{1}{2\pi} * \pi^2$$
$$= \frac{\pi}{2}$$

And $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx\ dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x) \cos nx\ dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x) \cos nx\ dx$$
$$= \frac{1}{\pi}[\pi \int_{-\pi}^{0} \cos nx\, dx + \int_{-\pi}^{0} x \cos nx\, dx + \pi \int_{0}^{\pi} \cos nx\, dx - \int_{0}^{\pi} x \cos nx\, dx]$$
$$= \frac{1}{\pi}[\frac{\pi}{n}\ [\sin nx]_{-\pi}^{0} + \frac{1}{n}[x \sin nx]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0} \sin nx\ dx + \frac{\pi}{n}[\sin nx]_{0}^{\pi} - \frac{1}{n}[x \sin nx]_{0}^{\pi} + \frac{1}{n}\int_{0}^{\pi} \sin nx\ dx]$$
$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}[\cos nx]_{-\pi}^{0} + 0 - \frac{1}{n^2}[\cos nx]_{0}^{\pi}\right]$$
$$= \frac{1}{\pi n^2}[\cos 0 - \cos n\pi - \cos n\pi + \cos 0]$$
$$= \frac{2}{\pi n^2}[1 - \cos n\pi]$$
$$= \frac{2}{\pi n^2}[1 - (-1)^n]$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx\, dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{0} (\pi + x) \sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} (\pi - x) \sin nx\, dx$$
$$= \frac{1}{\pi}\left[\pi \int_{-\pi}^{0} \sin nx\, dx + \int_{-\pi}^{0} \mathrm{x} \sin nx\, dx + \pi \int_{0}^{\pi} \sin nx\, dx - \int_{0}^{\pi} \mathrm{x} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\Big[-\frac{\pi}{n}[\cos nx\,]_{-\pi}^{0} + \frac{1}{n}[x\cos nx\,]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0}\cos nx\,dx \;-\frac{\pi}{n}[\cos nx\,]_{0}^{\pi} + \frac{1}{n}[x\cos nx\,]_{0}^{\pi} - \frac{1}{n}\int_{0}^{\pi}\cos nx\,dx\Big]$$

$$= \frac{1}{\pi}\Big[-\frac{\pi}{n} + \frac{\pi}{n}\cos n\pi + 0 - \frac{\pi}{n}\cos n\pi + \frac{1}{n^2}[\sin nx\,]_{-\pi}^{0} - \frac{\pi}{n}\cos n\pi + \frac{\pi}{n} + \frac{\pi}{n}\cos n\pi - \frac{1}{n^2}[\sin nx\,]_{0}^{\pi}\Big]$$

$$= \frac{1}{\pi}\text{ x } 0$$

$$= 0$$

$\therefore\; b_n = \;0$

Hence putting the values of a_0, a_n, b_n in equation (1), we get ,

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\{1-(-1)^n\}\cos nx + 0$$

$$= \frac{\pi}{2} + \frac{2}{\pi\,.1^2}\,.2\cos x + 0 + \frac{2}{\pi\,.3^2}\,.2\cos 3x + 0 + \frac{2}{\pi\,.5^2}\,.2\cos 5x + \cdots$$

$$= \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

Example-14: If $0 \le x \le \pi$. then show that

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{\cos 2nx}{n^2}.$$

Solution: Let, $f(x) = x(\pi - x) = \pi x - x^2$

Since the given interval is $0 \le x \le \pi$. (Half Range), the Fourier series and in the case

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \; and \; b_n = 0$

Now $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)dx$$

$$= \frac{2}{\pi}\Big[\pi.\frac{x^2}{2} - \frac{x^3}{3}\Big]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{2} - \frac{\pi^3}{3}\right]$$

$$= \frac{2}{\pi}\,.\frac{\pi^3}{6}$$

$$= \frac{\pi^3}{3}$$

And $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \quad [n \neq 0]$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\cos nx\, dx$$

$$= \frac{2}{\pi n}\left[(\pi x - x^2)\sin nx\right]_0^{\pi} - \frac{2}{\pi n}\int_0^{\pi}(\pi - 2x)\sin nx\, dx$$

$$= 0 - \frac{2}{n}\int_0^{\pi}\sin nx\, dx + \frac{4}{\pi n}\int_0^{\pi} x\sin nx\, dx$$

$$= \frac{2}{n^2}\left[cox\ nx\right]_0^{\pi} - \frac{4}{\pi n^2}[x\cos nx] + \frac{4}{\pi n^2}\int_0^{\pi}\cos nx\, dx$$

$$= \frac{2}{n^2}\left[cox\ n\pi - \cos 0\right] - \frac{4}{\pi n^2}.\pi\cos n\pi + \frac{4}{\pi n^3}\left[sin\ nx\right]_0^{\pi}$$

$$= \frac{2}{n^2}cox\ n\pi - \frac{2}{n^2} - \frac{4}{n^2}\cos n\pi + 0$$

$$= \frac{2}{n^2}[1 + (-1)^n]$$

Thus $f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[-\frac{2}{n^2}\{1 + (-1)^n\}\right]\cos nx$

$$= \frac{\pi^2}{6} + 0 - \frac{4}{2^2}\cos 2x + 0 - \frac{4}{4^2}\cos 4x + 0 - \frac{4}{6^2}\cos 6x + 0 - \frac{4}{8^2}\cos 8x + \cdots \ ....$$

$$= \frac{\pi^2}{6} - \frac{4}{2^2}\left[\cos 2x + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \frac{\cos 8x}{4^2} + \cdots\right]$$

$$= \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 2.2x}{2^2} + \frac{\cos 2.3x}{3^2} + \frac{\cos 2.4x}{4^2} + \cdots\right]$$

$$= \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{\cos 2nx}{n^2}$$

Therefore, $x(\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{\cos 2nx}{n^2}$

Example-15: Obtain the Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}\ for\ -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}\ for\quad 0 \le x \le \pi \end{cases}$$

Hence deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$.

Solution: The given function is

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}\ for\ -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}\ for\quad 0 \le x \le \pi \end{cases}$$

Since $f(-x) = 1 - \frac{2x}{\pi}\ in\ (-\pi, 0) = f(x)\ in\ (0, \pi)$

$and\ f(-x) = 1 + \frac{2x}{\pi}\ \ in\ (0, \pi) = f(x)\ in\ (-\pi, 0)$

Therefore, $f(x)$ is an even function in $(-\pi, \pi)$ and every even function is symmetrical about the $y - axis$. Thus we have,

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\ \ (1)(b_n = 0)$

Where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi} [x - \frac{2}{\pi} . \frac{x^2}{2}]_0^{\pi}$$

$$= \frac{2}{\pi}\ [\pi - \pi]$$

$$= 0$$

$and\ a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx\, dx$$

$$= \frac{2}{\pi n}\ [\ sin\ nx]_0^{\pi} + \frac{4}{\pi^2} \int_0^{\pi} x \cos nx\, dx$$

$$= 0 - \frac{4}{\pi^2 n}\ [\ x\ sin\ nx]_0^{\pi} + \frac{4}{\pi^2 n} \int_0^{\pi} \sin nx\, dx$$

$$= 0 - \frac{4}{\pi^2 n}\ [\ cos\ nx]_0^{\pi}$$

$$= -\frac{4}{\pi^2 n^2}(cosn\pi - cos0)$$

$$= -\frac{4}{\pi^2 n^2}\{(-1)^n - 1\} = \frac{4}{\pi^2 n^2}\{1 - (-1)^n\}$$

Now putting the values of a_0 and a_n in equation (1), We get

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\{1 - (-1)^n\}cosnx$$

$$= \frac{4}{\pi^2}[\frac{2cosx}{1^2} + 0 + \frac{2cos3x}{3^2} + 0 + \frac{2cos5x}{5^2} + \cdots\ \cdots\cdots\ \cdots]$$

$$= \frac{8}{\pi^2}[\frac{cosx}{1^2} + \frac{cos3x}{3^2} + \frac{cos5x}{5^2} + \cdots\ \cdots\cdots\ \cdots]\ldots\ldots\ldots\ldots\ldots\ (2)$$

Which is required Fourier series for f(x)

Now putting x=0 in (2), we have

$$f(0) = 1 = \frac{8}{\pi^2}[\frac{cos0}{1^2} + \frac{cos0}{3^2} + \frac{cos0}{5^2} + \cdots \cdots\cdots \cdots]$$

$$or \qquad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \cdots\cdots \cdots$$

Example-16: Show that in Fourier series

$$f(x) = \begin{cases} cosx \; for \; 0 \le x \le \pi \\ -cosx \; for - \pi \le x \le 0 \end{cases}$$

Is $f(x) = \frac{8}{\pi}[\frac{sin2x}{1.3} + \frac{2sin4x}{3.5} + \frac{3sin6x}{5.7} + \cdots \cdots\cdots \cdots]$

Solution: By definition of the Fourier series we have

$$f(x) = a_0 + \sum_{n=1}^{\pi} (a_n cosnx + b_n sinnx) \cdots \cdots\cdots \cdots (1)$$

Where,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$and \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$$

Now

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (-cosx)dx + \frac{1}{2\pi}\int_{0}^{\pi} cosxdx$$

$$= -\frac{1}{2\pi}[sinx]\,_{-\pi}^{0} + \frac{1}{2\pi}[sinx]\,_{0}^{\pi} = \frac{1}{2\pi} \times 0 = 0$$

and

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)cosnxdx + \frac{1}{\pi}\int_{0}^{\pi} f(x)cosnxdx$$

$= \frac{1}{\pi}\int_{-\pi}^{0}(-cosx)cosnxdx + \frac{1}{\pi}\int_{0}^{\pi} cosxcosnxdx$

$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2cosnxcosxdx + \frac{1}{2\pi}\int_{0}^{\pi} 2cosnxcosxdx$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}[\cos(n+1)\,x + \cos(n-1)\,x]dx + \frac{1}{2\pi}\int_{0}^{\pi}[\cos(n+1)\,x + \cos(n-1)\,x]dx$$

$$= -\frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_{0}^{\pi}$$

$= -\frac{1}{2\pi}.0 + \frac{1}{2\pi}.0$

$= 0 + 0$

$= 0$

Finally,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)sinnxdx + \frac{1}{\pi}\int_{0}^{\pi} f(x)sinnxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-cosx)\ sinnxdx + \frac{1}{\pi}\int_{0}^{\pi} cosx\ sinnxdx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0} 2\ sinnx\ cosx\ dx + \frac{1}{2\pi}\int_{0}^{\pi} 2\ sinnx\ cosx\ dx$$

$$= -\frac{1}{2\pi}\int_{-\pi}^{0}[\sin(n+1)\,x + \sin(n-1)\,x]dx + \frac{1}{2\pi}\int_{0}^{\pi}[\sin(n+1)\,x + \sin(n-1)\,x]dx$$

$$= \frac{1}{2\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_{-\pi}^{0} - \frac{1}{2\pi}\left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_{0}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\cos 0}{n+1} + \frac{cos 0}{n-1} - \frac{\cos(\mathrm{n}+1)\,\pi}{n+1} - \frac{\cos(\mathrm{n}-1)\,\pi}{n-1}\right] - \frac{1}{2\pi}\left[\frac{\cos(\mathrm{n}+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} - \frac{\cos 0}{n+1} - \frac{cos 0}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} - \frac{\cos(\mathrm{n}+1)\,\pi}{n+1} - \frac{\cos(\mathrm{n}-1)\,\pi}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\mathrm{n}}{n^2-1} + \frac{\cos\mathrm{n}\,\pi}{n+1} + \frac{\cos\mathrm{n}\,\pi}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2n}{n^2-1} + \frac{2n}{n^2-1} cosn\pi\right]$$

$$= \frac{1}{\pi} . \frac{2n}{n^2-1} [1 + cosn\pi]$$

$$= \frac{2n}{\pi(n^2-1)} \{1 + (-1)^n\}$$

Now putting the values of a_0, a_n and b_n in equation (1), we get

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{2n}{\pi(n^2-1)} \{1 + (-1)^n\} sinnx$$

$$= \frac{2}{\pi} [0 + \frac{2}{2^2-1} . 2sin2x + 0 + \frac{4}{4^2-1} . 2sin4x + 0 + \frac{6}{6^2-1} . 2sin6x + \cdots \cdots]$$

$$= \frac{8}{\pi}\left[\frac{sin2x}{3} + \frac{2sin4x}{15} + \frac{3sin6x}{35} + \cdots \cdots \cdots \cdots\right]$$

$$= \frac{8}{\pi}\left[\frac{sin2x}{1.3} + \frac{2sin4x}{3.5} + \frac{3sin6x}{5.7} + \cdots \cdots \cdots \cdots\right]$$

<u>Example-17:</u> Find the Fourier series for

$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \cdots \cdots \cdots$

Solution: By definition of the Fourier series

we have ,

$$f(x) = a_0 + \sum_{n=1}^{\pi} (a_n cosnx + b_n sinnx) \cdots \cdots \cdots \cdots (1)$$

Where,

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cosnxdx$$

$$and \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sinnxdx$$

Now

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} f(x)dx + \frac{1}{2\pi}\int_{0}^{\pi} f(x)dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{0} (-\pi)dx + \frac{1}{2\pi}\int_{0}^{\pi} xdx$$

$$= -\frac{\pi}{2\pi}[x]_{-\pi}^{0} + \frac{1}{2\pi}[\frac{x^2}{2}]_0^{\pi}$$

$$= -\frac{\pi}{2} + \frac{\pi}{4}$$

$$= -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-\pi)\cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\, dx$$

$$= -\frac{\pi}{n}[\sin nx]_{-\pi}^{0} + \frac{1}{n\pi}[x\sin nx]_0^{\pi} - \frac{1}{n\pi}\int_{0}^{\pi} \sin nx\, dx$$

$$= 0 + 0 + \frac{1}{\pi n^2}[\cos nx]_0^{\pi}$$

$$= \frac{1}{\pi n^2}[\cos n\pi - \cos 0]$$

$$= \frac{1}{\pi n^2}[(-1)^n - 1]\ .$$

Finally,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} (-\pi)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{n}[\cos nx]_{-\pi}^{0} - \frac{1}{\pi n}[x\cos nx]_{0}^{\pi} + \frac{1}{n\pi}\int_{0}^{\pi}\cos nx\,dx$$

$$= \frac{1}{n}[\cos 0 - \cos n\pi] - \frac{1}{n\pi}[\pi\cos nx] + \frac{1}{\pi n^2}[\sin nx]_{0}^{\pi}$$

$$= \frac{1}{n}[1 - \cos n\pi] - \frac{1}{n}\cos n\pi + 0$$

$$= \frac{1}{n}[1 - 2\cos n\pi]$$

$$= \tfrac{1}{n}[1 - 2(-1)^n].$$

Now putting the values of a_0, a_n and b_n in(1),we get

$$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{\pi n^2}\{(-1)^n - 1\}\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\{1 - 2(-1)^n\}\sin nx$$

$$= -\frac{\pi}{4} + \left[-\frac{2}{\pi.1^2}\cos x + 0 - \frac{2}{\pi.3^2}\cos 3x + 0 - \frac{2}{\pi.5^2}\cos 5x + \cdots\cdots\right] + \left[\frac{3}{1}\sin x - \frac{\sin 2x}{2} + \frac{3}{3}\sin 3x - \frac{\sin 4x}{4} + \cdots\cdots\right]$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\cdots\right] + \left[3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\cdots\right]$$

One discontinuity occurs at x=0

$$f(0) = \frac{f(0^+) + f(0^-)}{2} = -\frac{\pi}{2}\begin{cases} since\ f(0^+) = 0 \\ and\ f(0^-) = \pi \end{cases}$$

Putting $x = 0$ in the above series

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots] = -\frac{\pi}{2}$$

$$Or, -\frac{2}{\pi}[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots] = -\frac{\pi}{4}$$

$$Or, \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots] = -\frac{\pi^2}{8}$$

Example 18: Expand

$$f(x) = \begin{cases} \frac{1}{4} - x \text{ if } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} \text{ if } \frac{1}{2} < x < 1 \end{cases}$$

in a Fourier series of sine terms only.

Solution:

Since we are interested to expand $f(x)$ in a Fourier series of sine terms only.Let $f(x)$

Represent an odd function in $(1, -1)$ so that

$$f(x) = \sum_{n=1}^{\infty} b_n \ \sin n\pi x$$

Where

$$b_n = \frac{2}{c}\int_0^c f(x) sin\frac{n\pi x}{c}$$

$$= 2\int_0^1 f(x) sinn\pi x dx$$

$$= 2\int_0^{1/2} f(x) sinn\pi x dx + 2\int_{1/2}^{1} f(x) sinn\pi x dx$$

$$= 2\left[\left(\frac{1}{4} - x\right). -\frac{1}{n\pi} cosn\pi x\right]_0^{1/2} - \frac{2}{n\pi}\int_0^{1/2} cosn\pi x + 2\left[\left(x - \frac{3}{4}\right)\right]_{1/2}^{1} - \frac{1}{n\pi} cos\frac{n\pi}{2} + \frac{1}{4n\pi}$$

$$= \frac{1}{2n\pi} cos\frac{n\pi}{2} + \frac{1}{2n\pi} - \frac{2}{n^2\pi^2} sin\frac{n\pi}{2} - \frac{1}{2n\pi} cosn\pi - \frac{1}{2n\pi} cos\frac{n\pi}{2} - \frac{2}{n^2\pi^2} sin\frac{n\pi}{2}$$

$$= \frac{1}{2n\pi}(1 - cosn\pi) - \frac{4}{n^2\pi^2} sin\frac{n\pi}{2}$$

$$= \frac{1}{2n\pi}(1 - (-1)^n) - \frac{4}{n^2\pi^2} sin\frac{n\pi}{2}$$

$$\therefore b_1 = \frac{2}{2\pi} - \frac{4}{1^2\pi^2} = \frac{1}{\pi} - \frac{4}{1^2\pi^2}$$

$$b_2 = 0, b_3 = \frac{2}{2.3\pi} - \frac{4}{3^2\pi^2} sin\frac{3\pi}{2}$$

$$= \frac{1}{3\pi} + \frac{4}{3^2\pi^2}\begin{cases} sin\frac{3\pi}{2} = \sin(\pi + {}^{\pi}/_{2}) \\ = -sin\,\pi/2 = -1 \end{cases}$$

$$b_4 = 0, b_3 = \frac{2}{2.5\pi} - \frac{4}{5^2\pi^2} sin\,5\pi/2$$

$$= \frac{1}{5\pi} - \frac{4}{5^2\pi^2} \quad and\ so\ on \begin{cases} sin\frac{5\pi}{2} = \sin\left(2\pi + \frac{\pi}{2}\right) = 1 \end{cases}$$

Thus

$$f(x) = \left(\frac{1}{\pi} - \frac{4}{1^2\pi^2}\right) sin\pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right) + \cdots$$

Example 19: If $f(x) = \begin{cases} 1 \ \ for\ 0 \le x \le \frac{\pi}{2} \\ 0 \quad for\ x = \frac{\pi}{2} \\ -1\ for\ \frac{\pi}{2} \le x \le \pi \end{cases}$

Show that $f(x) = \frac{4}{\pi}\ [\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots]$

Proof:

Since the given interval is $[0, \pi]$

Which is the half of the interval $[-\pi, \pi]$, the fourier series will be a half of Range fourier cosine series and in this case

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx\ (b_n = 0)$ …… (1)

Where $a_0 = \int_0^{\pi} f(x)dx$ and $a_n = \int_0^{\pi} f(x)\cos nx\, dx$

Now $a_0 = \int_0^{\pi} f(x)dx$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} f(x)dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1dx + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} (-1)dx$$

$$= \frac{1}{\pi}[x]_0^{\frac{\pi}{2}} - \frac{1}{\pi}[x]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{2} - 0 - \pi + \frac{\pi}{2}\right]$$

$$= \frac{1}{\pi}.0$$

$= 0$

And $a_n = \int_0^{\pi} f(x) \cos nx \, dx$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x) \cos nx \, dx \; + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1 \cos nx \, dx \; + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (-1) \cos nx \, dx$$

$$= \frac{2}{\pi} . \frac{1}{n} [\sin nx]_0^{\frac{\pi}{2}} - \frac{2}{\pi} . \frac{1}{n} [\sin nx]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{2}{\pi n} [\sin \frac{n\pi}{2} - 0 - \sin n\pi + \sin \frac{n\pi}{2}]$$

$$= \frac{4}{\pi n} \sin \frac{n\pi}{2} + 0$$

$$= \frac{4}{\pi n} \sin \frac{n\pi}{2}$$

Now putting the values of a_0 and a_n in (1) we get

$$f(x) = 0 + \sum_{n=1}^{\infty} \frac{4}{\pi n} \sin \frac{n\pi}{2} \cos nx$$

$$= \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} sin \frac{n\pi}{2} \cos nx$$

$$= \frac{4}{\pi} [\frac{\sin \frac{\pi}{2}}{1} \cos x + \frac{\sin \frac{2\pi}{2}}{2} \cos 2x + \frac{\sin \frac{3\pi}{2}}{3} \cos 3x + \frac{\sin \frac{4\pi}{2}}{4} \cos 4x + \frac{\sin \frac{5\pi}{2}}{5} \cos 5x + \cdots]$$

$$= \frac{4}{\pi} [\frac{1}{1} \cos x + 0 + \frac{(-1)}{3} \cos 3x + 0 + \frac{1}{5} \cos 5x + 0 + \cdots]$$

$$= \frac{4}{\pi} [\cos x - \frac{1}{3} \cos 3x + \frac{1}{5} \cos 5x - \cdots]$$

8. Finite Fourier sine and cosine transforms

The finite Fourier sine transforms of $F(x), (0 < x < l)$ is defined as

$$f_s(n) = \int_0^{l} F(x) \, sin \frac{n\pi x}{l} \, dx \qquad (i)$$

Where n is an integer.

The function $F(x)$ is then called the inverse finite Fourier sine transforms of $f_s(n)$ and is given by

$$F(x) = \frac{2}{l}\sum_{n=1}^{\infty} f_s(n)\, sin\frac{n\pi x}{l} \qquad (ii)$$

The finite Fourier cosine transforms of $F(x), (0 < x < l)$ is defined as

$$f_c(n) = \int_0^l F(x)\, cos\frac{n\pi x}{l} dx \qquad (iii)$$

Where n is an integer.

The function $F(x)$ is then called the inverse finite Fourier cosine transforms of $f_c(n)$ and is given by

$$F(x) = \frac{1}{l} f_c(0) + \frac{2}{l}\sum_{n=1}^{\infty} f_c(n)\, cos\frac{n\pi x}{l} \qquad (iv)$$

9. Infinite Fourier sine and cosine transforms

The Infinite Fourier sine transform of a function $F(x)$ of x such that $o < x < \infty$ is denoted by $f_s(n)$ and is defined as

$$f_s(n) = \int_0^{\infty} F(x)\, sin\, nx\ dx \qquad (i)$$

The function $F(x)$ is then called the inverse finite Fourier sine transforms of $f_s(n)$ and is given by

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_s(n)\, sin\, nx\ dx \qquad (ii)$$

The Infinite Fourier cosine transform of a function $F(x)$ of x such that $o < x < \infty$ is denoted by $f_c(n)$ and is defined as

$$f_c(n) = \int_0^{\infty} F(x)\, cos\, nx\ dx \qquad (iii)$$

The function $F(x)$ is then called the inverse finite Fourier cosine transforms of $f_c(n)$ and is given by

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_c(n)\, cos\, nx\ dx \qquad (iv)$$

Note 1: The Infinite Fourier sine transform and the Infinite Fourier cosine transform are generally known as Fourier sine transform and Fourier cosine transform respectively.

Note 2: some authors also define Fourier sine transform and Fourier cosine transform in the following ways respectively:

$$(i) F_s\{f(x)\} = f_s(n) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin nx \; dx$$

$$(ii) F_c\{f(x)\} = f_c(n) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos nx \; dx$$

Note 3: some authors also define inverse Fourier sine transform and Fourier cosine transform in the following ways respectively:

$$(i) f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_s(n) \sin nx \; dn$$

$$(ii) f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_c(n) \cos nx \; dn$$

10. Fourier Integral

The Fourier Integral is very useful in the field of electrical communication and forms the basis of Cauchy's method for the solution of the partial differential equation.

General Fourier series of a periodic function $f(x)$ in the interval $(-c, c)$ is given by

$$f(x) = \frac{1}{2c} \int_{-c}^{c} f(t) dt + \sum_{n=1}^{\infty} \frac{1}{c} \int_{-c}^{c} f(t) \cos \frac{n\pi t}{c} dt . \cos \frac{n\pi x}{c}$$
$$+ \sum_{n=1}^{\infty} \frac{1}{c} \int_{-c}^{c} f(t) \sin \frac{n\pi t}{c} dt . \sin \frac{n\pi x}{c} \dots\dots (1)$$

$$\Rightarrow f(x) = \frac{1}{2c} \int_{-c}^{c} f(t) dt + \frac{1}{c} \int_{-c}^{c} f(t) \left[\sum_{n=1}^{\infty} \left\{ \cos \frac{n\pi x}{c} \cos \frac{n\pi t}{c} + \sin \frac{n\pi x}{c} \sin \frac{n\pi t}{c} \right\} \right] dt$$

$$= \frac{1}{2c} \int_{-c}^{c} f(t) dt + \frac{1}{2c} \int_{-c}^{c} f(t) \left[2 \sum_{n=1}^{\infty} \cos \left\{ \frac{n\pi(x-t)}{c} \right\} dt \right]$$

$$\Rightarrow f(x) = \frac{1}{2c} \int_{-c}^{c} f(t) \left[1 + 2 \sum_{n=1}^{\infty} \cos \left\{ \frac{n\pi(x-t)}{c} \right\} \right] dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\frac{\pi}{c} + \sum_{n=1}^{\infty} 2.\frac{\pi}{c}.\cos\left\{\frac{n\pi(x-t)}{c}\right\}\right] dt$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\frac{\pi}{c}\cos\left\{0.\frac{\pi}{c}(x-t)\right\} + \sum_{n=1}^{\infty}\frac{\pi}{c}\cos\left\{\frac{n\pi(x-t)}{c}\right\} + \sum_{n=1}^{\infty}\frac{\pi}{c}\cos\left\{\frac{-n\pi(x-t)}{c}\right\}\right] dt$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-c}^{c} f(t).\frac{\pi}{c}\left[\sum_{n=0}^{\infty}\cos\left\{\frac{n\pi(x-t)}{c}\right\} + \sum_{n=1}^{\infty}\cos\left\{\frac{-n\pi(x-t)}{c}\right\}\right] dt.$$

$$= \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{n}\frac{\pi}{c}\cos\left\{\frac{r\pi}{c}(x-t)\right\}\right] dt$$

$$\Rightarrow f(x) = \frac{1}{2\pi}\int_{-c}^{c} f(t)\left[\lim_{n\to\infty}\sum_{r=-n}^{n}\frac{1}{\frac{c}{\pi}}\cos\left\{\frac{r}{\frac{c}{\pi}}(x-t)\right\}\right] dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (2)$$

Now when, $c \to \infty, \frac{c}{\pi} \to \infty$ and we have

$$\lim_{c\to\infty}\sum_{r=-\infty}^{\infty}\frac{1}{\frac{c}{\pi}}\cos\left\{\frac{r}{\frac{c}{\pi}}(x-t)\right\}$$

$$= \lim_{\Delta u\to 0}\sum_{r=-\infty}^{\infty}\cos\{r\Delta u(x-t)\}\Delta u \quad \text{where } \Delta u = \frac{1}{\frac{c}{\pi}}$$

$$= \int_{-\infty}^{\infty}\cos\{u(x-t)\}du, \qquad \{\text{writting } r\Delta u = u \text{ and } \Delta u = du\}$$

By the definition of integral as the limit of a sum.

Substituting this value of the sum in the equation (2) we get,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)dt.\int_{-\infty}^{\infty}\cos\{u(x-t)\}du \ldots\ldots\ldots\ldots\ldots. (3)$$

This double integral is known as Fourier Integral and holds if x is a point of continuity of (x) .

The second integral in the equation (3) can be written as,

$$\int_{-\infty}^{\infty}\cos\{u(x-t)\}\, du = \int_{-\infty}^{0}\cos\{u(x-t)\}\, du + \int_{0}^{\infty}\cos\{u(x-t)\}\, du$$

$$= 2\int_0^{\infty} \cos\{u(x-t)\}du.$$

Thus equation (3) can also be written as,

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)dt\,.\int_0^{\infty} cos\{u(x-t)\}du$$

$$= \frac{1}{\pi}\int_0^{\infty} du\int_{-\infty}^{\infty} f(t)cos\{u(x-t)\}dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Which gives another form of the **Fourier integral**.

When $f(x)$an even function of x that is is, $f(-x) = f(x)$, then

$$\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\,dt = \int_{-\infty}^{0} f(t)\cos\{u(x-t)\}dt + \int_0^{\infty} f(t)\cos\{u(x-t)\}dt$$

[replacing t by $(-t)$ in the first integral]

$$= -\int_{\infty}^{0} f(-t)\cos\{u(x+t)\}dt + \int_0^{\infty} f(t)\cos\{u(x-t)\}dt$$

$$= \int_0^{\infty} f(t)\cos\{u(x+t)\}dt + \int_0^{\infty} f(t)\cos\{u(x-t)\}dt$$

$$\Rightarrow \int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\,dt = 2\int_0^{\infty} f(t)\cos ut\cos ux\,dt.$$

Substituting this result in the equation (4), then

$$f(x) = \frac{2}{\pi}\int_0^{\infty} du\int_0^{\infty} f(t)\cos ut\cos ux\,dt.$$

$$= \frac{2}{\pi}\int_0^{\infty} f(t)dt\int_0^{\infty} \cos ut\cos ux\,du \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. (5)$$

Which gives **Fourier integral, of an even function**.

Similarly when $f(x)$ is an odd function of x, that is, $f(-x) = -f(x)$ we have,

$$\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}\,dt = \int_{-\infty}^{0} f(t)\cos\{u(x-t)\}dt + \int_0^{\infty} f(t)\cos\{u(x-t)\}dt$$

On replacing t by (-t) in the first integral on the right hand side we get,

$$\int_{-\infty}^{0} f(t)\cos\{u(x-t)\}\,dt = \int_{\infty}^{0} f(-t)\cos\{u(x+t)\}\,.(-dt)$$

$$= -\int_0^{\infty} f(t)\cos\{u(x+t)\}dt$$

Thus $\int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}dt = \int_0^{\infty} f(t)[\cos\{u(x-t)\} - \cos\{u(x+t)\}]dt$

$$\Rightarrow \int_{-\infty}^{\infty} f(t)\cos\{u(x-t)\}dt = 2\int_0^{\infty} f(t)\sin ux \sin ut\, dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Substituting this relation in the equation (4) we get,

$$f(x) = \frac{2}{\pi}\int_0^{\infty} du \int_0^{\infty} f(t)\sin ut \sin ux\, dt \ = \frac{2}{\pi}\int_0^{\infty} f(t)dt \int_0^{\infty} \sin ut \sin ux\, du.$$

which is the **Fourier integral of an odd function**.

WORKED OUT EXAMPLES

Example 1: The function x^2 is periodic with period $2l$ on the interval $[-l, l]$. Find its Fourier series.

Solution: $f(x) = x^2 . f(-x) = (-x)^2 = x^2 = f(x)$ so $f(x)$ is an even function and hence sine terms will vanish. i.e. $b_n = 0$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Where $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\, dx$ $\qquad\qquad (n = 0)$

$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\frac{n\pi x}{l} dx$ $\qquad\qquad (n = 1,2,3 \ldots\ldots\ldots..)$

Since $f(x)$ is even, $a_0 = \frac{1}{l}\int_0^{l} f(v)\, dv.$ $\qquad a_n = \frac{2}{l}\int_0^{l} f(v) \cos\frac{n\pi v}{l}\, dv$

$a_0 = \frac{1}{l}\int_0^{l} x^2\, dx = \frac{1}{l}[\frac{x^3}{3}]_0^l = \frac{1}{3}l^2$

$a_n = \frac{2}{l}\int_0^{l} x^2 \cos\frac{n\pi x}{l} dx$ $\qquad$ (Integrating by part)

$= \frac{2}{l}[x^2 \sin\ \frac{n\pi x}{l}]_0^l - \frac{2}{l}.2\int_0^{l} x.\frac{l}{n\pi} \sin\frac{n\pi x}{l} dx$

$= 0 - \frac{4}{n\pi}\int_0^{l} x \sin\frac{n\pi x}{l} dx$

$= \frac{4}{n\pi}.\frac{l}{n\pi}[x\cos\frac{n\pi x}{l}]_0^l - \frac{4}{n\pi}.\frac{l}{n\pi}\int_0^{l} \cos\frac{n\pi x}{l} dx$

$= \frac{4l}{n^2\pi^2}[l\cos n\pi - 0] - \frac{4l^2}{n^3\pi^3}[\sin\frac{n\pi x}{l}]_0^l$

$= \frac{4l^2}{n^2\pi^2}(-1)^n - 0$

$= \frac{4l^2}{n^2\pi^2}(-1)^n . \therefore a_n = \frac{4l^2}{n^2\pi^2}(-1)^n$

Therefore, $f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}(-1)^n \; .\cos\frac{n\pi x}{l}$

$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \; (-1)^n \; .\cos\frac{n\pi x}{l}$

$= \frac{l^2}{3} + \frac{4l^2}{\pi^2}[-\frac{1}{1^2}\cos\frac{n\pi}{l} + \frac{1}{2^2}\cos\frac{2n\pi}{l} - \frac{1}{3^2}\cos\frac{3n\pi}{l} + \frac{1}{4^2}\cos\frac{4n\pi}{l} \dots\dots\dots\dots\dots]$

$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}[\frac{1}{1^2}\cos\frac{n\pi}{l} - \frac{1}{2^2}\cos\frac{2n\pi}{l} + \frac{1}{3^2}\cos\frac{3n\pi}{l} - \frac{1}{4^2}\cos\frac{4n\pi}{l} \dots\dots\dots\dots\dots]$

Example 8. Find the Fourier sine transform of $e^{-x}, x \geq 0$.

Solution: By the definition of Fourier sine transform of $f(x)$ for $o < x < \infty$, we have

$$f_s(n) = \int_0^{\infty} F(x) \sin nx \, dx$$

$$\mathrm{f_s(n)} = \int_0^{\infty} \mathrm{e^{-x} \sin nx \, dx} \qquad \textit{Since } F(x) = e^{-x}$$

$$= [-\frac{e^{-x}}{n}\cos nx]_0^{\infty} - \int_0^{\infty} \frac{e^{-x}\cos nx}{n} dx$$

$$= 0 + \frac{1}{n} - [\frac{e^{-x}}{n^2}\sin nx]_0^{\infty} - \int_0^{\infty} \frac{e^{-x}\sin nx}{n^2} dx$$

$$= \frac{1}{n} - 0 - \frac{1}{n^2} f_s(n)$$

$$\Rightarrow f_s(n) + \frac{1}{n^2} f_s(n) = \frac{1}{n}$$

$$\Rightarrow \left(1 + \frac{1}{n^2}\right) f_s(n) = \frac{1}{n}$$

$$\Rightarrow \left(\frac{n^2+1}{n^2}\right) f_s(n) = \frac{1}{n}$$

$$\Rightarrow f_s(n) = \frac{1}{n} . \frac{n^2}{n^2+1} = \frac{n}{n^2+1}$$

$$\therefore \; f_s(n) = \frac{n}{n^2+1}$$

.Hence the Fourier sin transform of e^{-x} is $\frac{n}{n^2+1}$.

Example 9. Find the inverse Fourier sine transform of $f_s(n) = \frac{n}{1+n^2}$

Solution: By definition of the inverse Fourier sin transform

we have

$$F(x) = \frac{2}{\pi}\int_0^\infty f_s(n)\, sin\, nxdn$$

$$= \frac{2}{\pi}\int_0^\infty \frac{n}{1+n^2} sin\, nxdn \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

From the Fourier integral formula of an odd function, we

have

$$\mathrm{f(x)} = \frac{2}{\pi}\int_0^\infty \mathrm{du}\int_0^\infty \mathrm{f(t)\, sin\, ut\, sinux\, dt}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty dn\int_0^\infty f(t)\sin nt \sin nx\, dt \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Taking

$f(x) = e^{-x}$

in (2), we have

$$e^{-x} = \frac{2}{\pi}\int_0^\infty dn\int_0^\infty e^{-t}\sin nt \sin nxdt$$

$$= \frac{2}{\pi}\int_0^\infty \sin nx\{\int_0^\infty e^{-t}\sin nt\, dt\}\ dn$$

$$= \frac{2}{\pi}\int_0^\infty \sin nx\left(\frac{n}{1+n^2}\right) dn$$

$$= \frac{2}{\pi}\int_0^\infty \frac{nsin\, nx}{1+n^2}\ dn$$

$$\Rightarrow \int_0^\infty \frac{n}{1+n^2}\sin nx\ dn = \frac{\pi}{2}e^{-x} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$$

Combining (1) and (3), we get

$$F(x) = \frac{2}{\pi}.\frac{\pi}{2}e^{-x}$$

Hence $F(x) = e^{-x}$ which is the required inverse Fourier sin transform of

$$f_s(n) = \frac{n}{1+n^2}$$

Example 10. Find the Fourier cosine transform of$e^{-x}, x \geq 0$.

Solution: By definition of Fourier cosine transform of f(x) for$0 < x < \infty$,

we have

$$f_c(n) = \int_0^{\infty} F(x) \cos nx \, dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore f_c(n) = \int_0^{\infty} e^{-x} \cos nx \, dx$$

$Since\ F(x) = e^{-x}$

$$= \left[\frac{e^{-x} \sin nx}{n}\right] + \int_0^{\infty} \frac{e^{-x} \sin nx}{n} dx$$

$$= 0 - \frac{1}{n^2}[e^{-x} Cosnx]_0^{\infty} - \frac{1}{n^2}\int_0^{\infty} e^{-x} \cos nx \, dx$$

$$= 0 + \frac{1}{n^2} - \frac{1}{n^2} f_c(n)$$

$$\Rightarrow f_c(n) + \frac{1}{n^2} f_c(n) = \frac{1}{n^2}$$

$$\Rightarrow \frac{n^2 + 1}{n^2} f_c(n) = \frac{1}{n^2}$$

$$\Rightarrow f_c(n) = \frac{1}{n^2 + 1}$$

Hence the Fourier cosine transform of e^{-x} is $\frac{1}{n^2+1}$.

Example11. Find the inverse Fourier cosine transform of $f_c(n) = \frac{1}{1+n^2}$

Solution: By definition of the inverse Fourier cosine transform

we have,

$$F(x) = \frac{2}{\pi}\int_0^{\infty} f_c(n) \cos nxdn$$

$$= \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+n^2} \cos nxdn \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$$

From the Fourier integral formula of an even function, we have

$$f(x) = \frac{2}{\pi}\int_0^{\infty} du \int_0^{\infty} f(t) \cos ut \ cosux \ dt$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} dn \int_0^{\infty} f(t) \cos nt \cos nx \ dt \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Taking

$f(x) = e^{-x}$

in (2),we have

$$e^{-x} = \frac{2}{\pi}\int_0^\infty dn \int_0^\infty e^{-t}\cos nt \cos nx dt$$

$$= \frac{2}{\pi}\int_0^\infty \cos nx\{\int_0^\infty e^{-t}\cos nt\ dt\}\ dn$$

$$= \frac{2}{\pi}\int_0^\infty \cos nx\left(\frac{1}{1+n^2}\right) dn$$

$$\therefore \int_0^\infty (\frac{1}{1+n^2}) \cos nx\ dn = \frac{\pi}{2}e^{-x} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (3)$$

Combining (1) and (3) ,

we get

$$F[x] = \frac{2}{\pi}.\frac{\pi}{2}e^{-x} = e^{-x}.$$

Hence $F[x] = e^{-x}$ which is the required inverse Fourier cosine transform of

$$F_c(n) = \frac{1}{1+n^2}$$

Example 12. Find the (a) finite Fourier sine transform (b) finite Fourier cosine transform of the function$F(x) = 2x, 0 < x < 4$.

Solution: (a) Since $l = 4$,we have

$$f_s(n) = \int_0^l F(x) sin\frac{n\pi x}{l}dx = \int_0^4 F(x) sin\frac{n\pi x}{4}dx$$

$$= \int_0^4 2x\, sin\frac{n\pi x}{4}$$

$$= [-2x.\frac{4}{n\pi}cos\frac{n\pi x}{4}]_0^4 + \frac{8}{n\pi}\int_0^4 cos\frac{n\pi x}{4}dx$$

$$= -\frac{32}{n\pi}cosn\pi + \frac{32}{n^2\pi^2}\ [sin\frac{n\pi x}{4}]_0^4$$

$$= -\frac{32}{n\pi}cosn\pi + \frac{32}{n^2\pi^2}(0-0)$$

$$= -\frac{32}{n\pi} cosn\pi$$

Which is the finite Fourier sine transform of $F(x) = 2x$

(b) If

$n > 0,$ $$f_c(n) = \int_0^l F(x)cos\frac{n\pi x}{l}dx$$

$$= \int_0^4 2x\,cos\frac{n\pi x}{4}$$

$$= [2x.\tfrac{4}{n\pi}sin\tfrac{n\pi x}{4}]_0^4 - \tfrac{8}{n\pi}\int_0^4 sin\tfrac{n\pi x}{4}dx$$

$$= 0 - \frac{8}{n\pi}(-\frac{4}{n\pi})\left[Cos\frac{nx\pi}{4}\right]_0^4$$

$= \frac{32}{n^2\pi^2}(Cosnn\pi - 1)$ which is the finite Fourier Cosine transform of $F(x) = 2x$

If $n = 0, f_c(n) = f_c(0) = \int_0^4 2xdx$

$$= 2\left[\frac{x^2}{2}\right]_0^4 = (4^2) - 0 = 16$$

$$\therefore f_c(n) = f_c(0) = 16.$$