Application OF 1st ORDER ODE

01. A culture initially has P_0 numbers of bacteria. At $t=1h$, the number of bacteria P measured to be $3/2 P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the numbers of bacteria to triple.

→ Soln:

Initially $t=0$

According to the question, $\frac{dp}{dt} \propto P(t)$

$$\Rightarrow \frac{dP}{dt} = KP(t) \text{ --- (I)}$$

with condition, $P(0) = P_0$ --- (II)

$P(1) = \frac{3}{2}P_0$ --- (III)

$$\frac{dP}{dt} = KP$$

$$\Rightarrow \frac{dP}{P} = Kdt$$

$$\Rightarrow \int \frac{dP}{P} = \int Kdt$$

$$\Rightarrow \ln p = Kt + \ln c$$

$$\Rightarrow \ln p - \ln c = Kt$$

$$\Rightarrow \ln\left(\frac{P}{c}\right) = Kt$$

$$\Rightarrow \frac{P}{c} = e^{Kt}$$

$$\Rightarrow P(t) = ce^{Kt} \text{ --- (IV)}$$

Putting $t=0$ in (IV) $\Rightarrow P(0) = c$

$\Rightarrow c = P_0$

(IV) becomes, $P(t) = P_0 e^{Kt}$ --- (V)

Putting $t=1$ in (V) $\Rightarrow P(1) = P_0 e^{K}$

$$\Rightarrow \frac{3}{2}P_0 = P_0 e^{K}$$

$$\Rightarrow e^{K} = \frac{3}{2}$$

$$\Rightarrow K = \ln\frac{3}{2}$$

$$= 0.406$$

The time for bacteria to triple,

$$P(t) = 3P_0$$

$$\Rightarrow P_0 e^{0.406t} = 3P_0$$

$$\Rightarrow t = \frac{\ln 3}{0.406} = 2.71\ h.$$

Ans:

Newton's law of cooling $\rightarrow \frac{dT}{dt} \propto (T - T_m)$

$T(t)$ = Temperature of object at time t

T_m = Ambient temperature

$\frac{dT}{dt}$ = Rate of cooling

02. When a cake is removed from oven, its temperature is measured 300°F. 3 minutes later it's temperature is 200°F. How long it will take for the cake to cool off to a temperature of 80°F. Here, room temperature is 70° F.

$\Rightarrow$ Soln:
According to Newton's Law of cooling, $\frac{dT}{dt} \propto (T - T_m)$

$$\Rightarrow \frac{dT}{dt} = K(T - \underset{70}{T_m}) \text{ --- (i)}$$

with conditions,

$$\left.\begin{array}{l} T(0) = 300°F \\ T(3) = 200°F \end{array}\right\} \text{ --- (ii)}$$

$$(i) \Rightarrow \int \frac{dT}{T-70} = \int K\,dt$$

$$\Rightarrow \ln(T-70) = Kt + \ln C$$

$$\Rightarrow \ln(T-70) - \ln C = Kt$$

$$\Rightarrow \ln\left(\frac{T-70}{C}\right) = Kt$$

$$\Rightarrow T - 70 = e^{Kt} \cdot C$$

$$\Rightarrow T(t) = 70 + Ce^{KT} \text{ --- (iii)}$$

$T(0) = 300$
$= 70 + C \cdot 1$

$C = 230$

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(iii) $\Rightarrow T(t) = 70 + 230\,e^{KT}$ —— (IV)

when, $t = 3$, (IV) $\Rightarrow 200 = 70 + 230\,e^{3K}$

$\Rightarrow 230\,e^{3K} = 130$

$\Rightarrow 3K = \ln\left(\frac{130}{230}\right)$

$\Rightarrow K = \frac{\ln(130/230)}{3}$

$\Rightarrow K = -0.1902$

(IV) $\Rightarrow T(t) = 70 + 230\,e^{-0.1902t}$ —— (V)

$\therefore 80 = 70 + 230\,e^{-0.1902t}$

$\Rightarrow e^{-0.1902t} = \frac{10}{230}$

$t = 16.49$ min $= 16$ min 29.4 sec Ans://

01. An RL circuit has an e.m.f. 5V, Resistance of 50Ω, Inductance of 1 Henry and no initial current. Find current at any time t.

⇒ Solⁿ: According to Kirchoff's Law, $L\frac{dI}{dt}+IR=V$

$$\Rightarrow \frac{dI}{dt}+50I=5 \quad \text{—— (i)}$$

with the condition,

$$I(0)=0 \quad \text{—— (ii)}$$

I.F. of (i) $\Rightarrow e^{\int 50\,dt} = e^{50t}$

Multiplying (i) by I.F. $\Rightarrow e^{50t}\frac{dI}{dt}+e^{50t}.50I=5e^{50t}$

$$\Rightarrow \frac{d}{dt}(e^{50t}.I)=5e^{50t}$$

$$\Rightarrow \int d(e^{50t}.I)=\int 5e^{50t}\,dt$$

$$\Rightarrow e^{50t}.I=5.e^{50t}.\frac{1}{50}+c$$

$$\Rightarrow e^{50t}.I=\frac{1}{10}e^{50t}+c$$

$$\Rightarrow I(t)=\frac{1}{10}+ce^{-50t} \quad \text{—— (iii)}$$

When, t=0 (iii) becomes,

$I(0)=\frac{1}{10}+c$

$\therefore c=-\frac{1}{10}$

(iii) becomes $\Rightarrow I(t)=\frac{1}{10}-\frac{1}{10}e^{-50t}$

$$I(t)=\frac{1}{10}\left(1-e^{-50t}\right)$$

Ans:

(27-02-22)

3.18 Linear Differential Equations of Second Order with constant coefficients.

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

Where P and Q are constants and R is a function of x or constant.

Differential operator. symbol D stands for the operation of differential i.e, $Dy = \frac{dy}{dx}$; $D^2y = \frac{d^2y}{dx^2}$

$\frac{1}{D}$ stands for operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ can be written in the operator form

$D^2y + \cancel{D}PDy + QY = R \Rightarrow (D^2 + PD + Q)Y = R$

3.19. Complete solution = Complementary function + Particular Integral

Case-1: Roots, Real and Different: If m_1 and m_2 are the roots then the C.F is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

Case-2: Roots, Real and Equal. If both the roots are m_1, m_1 then the C.F. is $y = (C_1 + C_2 x) e^{m_1 x}$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Soln: Given,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \quad \text{--- (i)}$$

For, y_c trial solution is always $y = e^{mx}$

Let,

$y = e^{mx}$ be the trail soln of (i)

$$\frac{dy}{dx} = me^{mx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = m^2 e^{mx}$$

Auxiliary soln of (i) $\Rightarrow m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$

$$\Rightarrow e^{mx}(m^2 - 3m + 2) = 0$$

e^{mx} cannot be 0

$\therefore m^2 - 3m + 2 = 0$

$\Rightarrow m_1 = 1 ; m_2 = 2$

$y = y_c$

$= C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$= C_1 e^{x} + C_2 e^{2x}$

Ans:

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(02) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

Soln:
$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0 \text{ ——— (i)}$$

Let, $y = e^{mx}$

$\Rightarrow \frac{dy}{dx} = me^{mx}$

$\Rightarrow \frac{d^2y}{dx^2} = m^2e^{mx}$

$\Rightarrow \frac{d^3y}{dx^3} = m^3e^{mx}$

Auxiliary soln of (i) $\Rightarrow m^3e^{mx} - 4m^2e^{mx} + me^{mx} + 6e^{mx} = 0$

$m^3e^{mx} - 4m^2e^{mx} + me^{mx} + 6e^{mx} = 0$

$\Rightarrow e^{mx}(m^3 - 4m^2 + m + 6) = 0$

$e^{mx} \neq 0 \qquad m^3 - 4m^2 + m + 6 = 0$

$\therefore m_1 = -1,\ m_2 = 2,\ m_3 = 3$

$y = y_c$

$= c_1e^{-x} + c_2e^{2x} + c_3e^{3x}$ Ans:

$m = -1$

$(m+1) = 0$

$m^3 - 4m^2 + m + 6 = 0$

$\Rightarrow m^3 + m + 5m^2 - 5m + 6m + 6$

$\Rightarrow m^2(m+1) - 5m(m+1) + 6(m+1) = 0$

$\Rightarrow (m+1)(m^2 - 5m + 6) = 0$

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④ $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Soln: Auxiliary soln ⇒

$e^{mx}(m^2 - 6m + 9) = 0$

$e^{mx} \neq 0 \quad \therefore m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\therefore m_1 = 3 \; ; m_2 = 3$

$y_c = c_1 e^{3x} + c_2 x e^{3x}$ Ans:

Q3 $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

Soln: Auxiliary soln $\Rightarrow e^{mx}(m^3 - 4m^2 - 3m + 18) = 0$

$e^{mx} \neq 0 \therefore m^3 - 4m^2 - 3m + 18 = 0$

$m^2(m+2) - 6m(m+2) + 9(m+2) = 0$

$\Rightarrow (m+2)(m^2 - 6m + 9) = 0$

$\therefore m = -2, m_2 = 3 \; ; m_3 = 3$

$y_c = c_1 e^{-2x} + c_2 e^{3x} + c_3 x e^{3x}$ Ans:

For Repeated root

$y_c = c_1 e^{mx} + c_2 x e^{mx} + c_3 x^2 e^{mx} + \dots c_n x^{n-1} e^{mx}$

Example 47 $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Here auxiliary equation is $m^2 - 8m + 15 = 0$

$(m-3)(m-5) = 0$

$\therefore m = 3, 5$

Hence, the required solution is

$y = c_1 e^{3x} + c_2 e^{5x}$

$\boxed{9}$ $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Here auxiliary equation is, $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3, 3$

Hence, the required solution is,

$y = (c_1 + c_2 x)e^{3x}$

Ans.

Exercise-3.19

1. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Here auxiliary equation is, $e^{mx}(m^2 - 3m + 2) = 0$

$e^{mx} \neq 0$; $\quad m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m_1 = 2, m_2 = 1$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{2x} + c_2 e^{x}$

Ans.

2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 30y = 0$

Here auxiliary equation is, $e^{mx}(m^2+m-30) = 0$

$e^{mx} \neq 0$; $m^2+m-30 = 0$

$\Rightarrow m^2+6m-5m-30 = 0$

$\Rightarrow m(m+6)-5(m+6) = 0$

$\Rightarrow (m+6)(m-5) = 0$

$\Rightarrow m_1 = 5$, $m_2 = -6$

$y_c = c_1 e^{5x} + c_2 e^{-6x}$ Ans:

(3) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

Here auxiliary equation is, $e^{mx}(m^2-8m+16) = 0$

$\therefore e^{mx} \neq 0$; $m^2-8m+16 = 0$

$\Rightarrow m^2-4m-4m+16 = 0$

$\Rightarrow m(m-4)-4(m-4) = 0$

$\Rightarrow (m-4)(m-4) = 0$

$\Rightarrow m = 4$, $m_2 = 4$

$y_c = c_1 e^{4x} + c_2 x e^{4x}$

$= (c_1 + c_2 x) e^{4x}$ Ans:

Case III:

For complex root, $m = \alpha \pm i\beta$

$$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$$

Solⁿ:

Given that, $y'' - 6y' + 25y = 0$ — (i)

Let the trail solution be $y = e^{mx}$

$$\therefore y' = me^{mx}$$

$$y'' = m^2 e^{mx}$$

Auxilary solution is,

$$e^{mx}(m^2 - 6m + 25) = 0$$

As $e^{mx} \neq 0$, $m^2 - 6m + 25 = 0$

$ax^2 + bx + c = 0$,

$$x = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$m = 3 \pm 4i$$

Here $\alpha = 3$, $\beta = 4$

Complementary function is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$= e^{3x}(c_1 \cos 4x + c_2 \sin 4x)$$ Ans.

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Ex $\frac{d^2y}{dx^2}+y=0$ —— (1)

Soln: $y''+y=0$ —— (1)

let the auxilary solution is

$e^{mx}(m^2+1)=0$

$e^{mx}\neq 0$; $m^2+1=0$

$\Rightarrow m^2=-1=i^2$

$\Rightarrow m=\pm i$

$=0\pm i.1$

Here, $\alpha=0, \beta=1$

Thus the complementary solution is

$y=e^{mx}(c_1\cos\beta x+c_2\sin\beta x)$

$=e^{0x}(c_1\cos x+c_2\sin x)$

$=c_1\cos x+c_2\sin x$ Ans:

Particular Integral (PI)

$\boxed{\text{If } R(x)=e^{ax}, \text{ then PI} = \frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}, f(a)\neq 0}$

Ex $(D^2+D+5)y=e^{-2x}$, $D=\frac{d}{dx}$

Soln: $(D^2+D+5)y=e^{-2x}$

For complementary function,

$D^2y+Dy+5y=0$ —— (1)

Trail sol^n is $y = e^{mx}$

$$Dy = me^{mx}$$
$$D^2y = m^2e^{mx}$$

(i) becomes

$$e^{mx}(m^2+m+5)=0$$

$e^{mx} \neq 0$; $m^2+m+5=0$

$$m = -\frac{1}{2} \pm \frac{\sqrt{19}}{2}i$$

Here $\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{19}}{2}$

$$y_c = e^{-\frac{x}{2}}\left(c_1\cos\left(\frac{\sqrt{19}}{2}x\right) + c_2\sin\left(\frac{\sqrt{19}}{2}x\right)\right)$$

$$y_p = \frac{1}{D^2+D+5}e^{-2x}$$
$$= \frac{1}{(-2)^2+(-2)+5}e^{-2x}$$
$$= \frac{1}{7}e^{-2x}$$

Complete solution,

$$y = y_c + y_p$$
$$= e^{-\frac{x}{2}}\left[c_1\cos\left(\frac{\sqrt{19}}{2}x\right) + c_2\sin\left(\frac{\sqrt{19}}{2}x\right)\right] + \frac{1}{7}e^{-2x}$$

Ans

$$\boxed{\begin{array}{l} \text{If } f(a)=0, \text{ then } \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax} \\ \text{If } f(a)=0, \text{ then } \frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax} \end{array}}$$

(*) $y'' + 6y' + 9y = 5e^{3x}$

Soln: For complementary function,

$$y'' + 6y' + 9y = 0$$

Auxilary equation is $m^2 + 6m + 9 = 0$

$$\Rightarrow (m+3)^2 = 0$$

$$\Rightarrow m = -3, -3$$

$$y_c = (c_1 + c_2 x) e^{-3x}$$

$$= c_1 e^{-3x} + c_2 e^{-3x} \cdot x$$

$$PI = \frac{1}{D^2 + 6D + 9} 5e^{3x}$$

$$y_p = 5 \frac{1}{D^2+6D+9} e^{3x}$$

$$= 5 \frac{1}{3^2+6.3+9} \cdot e^{3x}$$

$$= 5 \frac{1}{36} e^{3x}$$

Complete solution, $y = y_c + y_p$

$$= c_1 e^{-3x} + c_2 e^{-3x} \cdot x + \frac{5}{36} e^{3x}$$

Ans:

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3.20 → 1, 2, 3, 6, 7, 8, 9

$y'' - 3y' + 2y = e^{3x}$.

$y'' - 6y' + 9y = 6e^{3x} + 7e^{-2x} - \log 2$

For complementary ~~solution~~ function

$y'' - 6y' + 9y = 0$

Let the trail solution be $y = e^{mx}$

$\Rightarrow y' = me^{mx}$

$y'' = m^2 e^{mx}$

Auxiliary eq^n is $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3, 3$

$Y_c = (c_1 + c_2 x) e^{3x}$

$$P.I = \frac{1}{D^2 - 6D + 9} 6e^{3x} + \frac{1}{D^2 - 6D + 9} 7e^{-2x} + \frac{1}{D^2 - 6D + 9}(-\log 2)$$

$$= x \frac{1}{2D - 6} 6e^{3x} + \frac{1}{4 + 12 + 9} 7e^{-2x} - \log 2 \frac{1}{D^2 - 6D + 9} e^{0x}$$

$$= x^2 \cdot \frac{1}{2} \cdot 6e^{3x} + \frac{7}{25} e^{-2x} - \log 2 \left(\frac{1}{9}\right)$$

$$= 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9} \log 2$$

Complete solution is $Y = Y_c + Y_p$

$$= (c_1 + c_2 x) e^{3x} + 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9} \log 2$$

Ans.

$y'' - 3y' + 2y = e^{3x}$

For complementary function,

$$y'' - 3y' + 2y = 0$$

Let trail solution be $y = e^{mx}$

$$\Rightarrow y' = me^{mx}$$

$$\Rightarrow y'' = m^2 e^{mx}$$

Auxilary eq^n is $m^2 - 3m + 2 = 0$

$$\Rightarrow m^2 - 2m - m + 2 = 0$$

$$\Rightarrow (m-2)(m-1) = 0$$

$$\therefore m = 2, 1$$

$$Y_c = C_1 e^{x} + C_2 e^{2x}$$

$$PI = \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{3^2 - 9 + 2} e^{3x}$$

$$= \frac{1}{2} e^{3x}$$

The complete solution is $Y = Y_c + Y_p$

$$= C_1 e^{x} + C_2 e^{2x} + \frac{e^{3x}}{2}$$ Ans.

$\boxed{3.24}$ 4, 5, 6, 7, 8, 9, $\boxed{10}$

If $f(-a^2)=0$ then above rule fails.

$$\frac{1}{f(D^2)}\sin ax = x\frac{\sin ax}{f'(-a^2)}$$

If $f'(-a^2)=0$ then, $\frac{1}{f(D^2)}\sin ax = x^2\frac{\sin ax}{f''(-a^2)}$

$\boxed{54}$ $(D^2+4)y = \cos 2x$

Auxiliary equation is $m^2+4=0$

$$\Rightarrow m = \pm 2i$$

$C.F = A\cos 2x + B\sin 2x$

$$P.I = \frac{1}{D^2+4}\cos 2x$$

$$= x\cdot\frac{1}{2D}\cos 2x$$

$$= x\cdot\frac{1}{2}\frac{\sin 2x}{2}$$

$$= \frac{x}{4}\sin 2x$$

Complete solution is $y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$

Ans.

55. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$

Given, A.E is $m^3 - 3m^2 + 4m - 2 = 0$

$$\Rightarrow (m-1)(m^2-2m+2)=0$$

$$m = 1,\ 1\pm i$$

$C.F = c_1e^x + e^x(c_2\cos x + c_3\sin x)$

$$P.I. = \frac{1}{(D-1)(D^2-2D+2)}e^x + \frac{1}{D^3 - 3D^2 + 4D - 2}\cos x$$

$$= \frac{1}{(D-1)(1-2+2)}e^x + \frac{1}{(-1)D - 3(-1) + 4D - 2}\cos x$$

$$= \frac{1}{(D-1)}e^x + \frac{1}{3D+1}\cos x$$

$$= x\frac{1}{1}e^x + \frac{3D-1}{9D^2-1}\cos x$$

$$= xe^x + \frac{(3D-1)}{9(-1^2)-1}\cos x$$

$$= xe^x + \frac{3D-1}{-10}\cos x$$

$$= xe^x + \frac{-3\sin x - \cos x}{-10}$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x)$$

Here complete solution is $= c_1e^x + (c_2\cos x + c_3\sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x)$ Ans:

$\boxed{3.22}$ Homework: 1, 2, 3, 5

Particular Integral for the form,

① $y\,f(D) = x^n$

$\Rightarrow y = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$

② $y\,f(D) = e^{ax}.\phi(x)$

$\therefore y = e^{ax} \frac{1}{f(D+a)} \phi(x)$

Ex: $(D^2 - 4D + 4)y = x^3 e^{2x}$

sol^n: For the complementary function,

$(D^2 - 4D + 4)y = 0$

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2, 2$

$y_c = c_1 e^{2x} + c_2 e^{2x}.x$

$$P.I. = \frac{1}{D^2 - 4D + 4} x^3 e^{2x}$$

$$= e^{2x}. \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x}. \frac{1}{D^2} x^3$$

$$= e^{2x}. \frac{1}{D}\left[x^4/4\right]$$

$$= e^{2x}. \frac{x^5}{20}$$

Complete solution is,

$y = y_c + y_p$

$= (c_1 + c_2 x)e^{2x} + e^{2x} \cdot \frac{x^5}{20}$ Ans:

Ex-2: $y''' - 7y'' + 10y' = e^{2x} \sin x$

Soln: Given that,

$D^3y - 7D^2y + 10Dy = e^{2x} \sin x$

For complementary solution,

$D^3y - 7D^2y + 10Dy = 0$

Auxiliary equation is,

$m^3 - 7m^2 + 10m = 0$

$\Rightarrow m(m-2)(m-5) = 0$

$\therefore m = 0, 2, 5$

$y_c = c_1 + c_2 e^{2x} + c_3 e^{5x}$

$D^2 = -a^2$

$P.I. = \frac{1}{D^3 - 7D^2 + 10D} e^{2x} \sin x$

$= e^{2x} \frac{1}{(D+2)^3 - 7(D+2)^2 + 10(D+2)} \sin x$

$= e^{2x} \frac{1}{D^3 - D^2 - 6D} \sin x$ $D^2 \cdot D - D^2 - 6D$

$= e^{2x} \frac{1}{(-1)D - (-1) - 6D} \sin x$ $(-1)D - (-1) - 6D$

$= e^{2x} \frac{1}{-D+1-6D} \sin x$

$= e^{2x} \frac{1}{1-7D} \sin x$

$= e^{2x} \frac{1+7D}{(1-7D)(1+7D)} \sin x$

$= e^{2x} \frac{1+7D}{1-49D^2} \sin x$

$= e^{2x} \frac{1+7D}{50} \sin x$

$= e^{2x}/50 (\sin x + 7\cos x)$

Complete solution is $y = y_c + y_p$

$= c_1 + c_2 e^{2x} + c_3 e^{3x} + \frac{e^{2x}}{50}(\sin x + 7\cos x)$ Ans

$(D^2+6D+9)y = \frac{e^{-3x}}{x^3}$

Auxiliary equation is $m^2+6m+9=0$

$\Rightarrow (m+3)^2=0$

$\Rightarrow m = -3, -3$

$y_c = (c_1 + c_2 x)e^{-3x}$

$P.I = \frac{1}{D^2+6D+9} \frac{e^{-3x}}{x^3}$

$= e^{-3x} \frac{1}{(D-3)^2+6(D-3)+9} \frac{1}{x^3}$

$= e^{-3x} \frac{1}{D^2-6D+9+6D-18+9} \frac{1}{x^3}$

$$= e^{-3x} \frac{1}{D^2} \frac{1}{x^3}$$

$$= e^{-3x} \frac{1}{D}\left(\frac{x^{-2}}{-2}\right)$$

$$= e^{-3x} \frac{x^{-1}}{(-2)(-1)}$$

$$= e^{-3x} \frac{x^{-1}}{2} = \frac{e^{-3x}}{2x}$$

Complete solution is, $y = y_c + y_p$

$$= (c_1 + c_2 x)e^{-3x} + \frac{e^{-3x}}{2x}.$$

Ex) $(D^2 - 4)y = x^2 e^{3x}$

Ex) $(D^2 - 3D + 2)y = 2x^2 e^{4x} + 5e^{3x}$

$\frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$

① $(D^2+D-2)y = 2(1+x-x^2)$

$\Rightarrow$ For complementary function,

$$(D^2+D-2)y = 0$$

Auxiliary equation is

$$e^{mx}(m^2+m-2) = 0$$

$e^{mx} \neq 0$ $\quad m^2+m-2=0$

$\Rightarrow m = 1, -2$

$$y_c = c_1e^{x} + c_2e^{-2x}$$

Particular Integral,

$$y_p = \frac{1}{D^2+D-2} 2(1+x-x^2)$$

$$= \frac{1}{-2\left[1-\left(\frac{D^2+D}{2}\right)\right]} 2(1+x-x^2)$$

$$= -\left[1-\left(\frac{D^2+D}{2}\right)\right]^{-1}(1+x-x^2)$$

$$= -1\left[1+\frac{D+D^2}{2}+\left(\frac{D+D^2}{2}\right)^2+\cdots\right](1+x-x^2)$$

$$= -1\left[1+\frac{D}{2}+\frac{D^2}{2}+\frac{D^2+2D^3+D^4}{4}+\cdots\right](1+x-x^2)$$

$$= -1\left[1+\frac{D}{2}+\frac{3}{4}D^2+\cdots\right](1+x-x^2)$$

$$= -1\left[(1+x-x^2)+\frac{1}{2}(1-2x)+\frac{3}{4}(-2)\right]$$

Scanned with CamScanner

$= -\left[1+x-x^2+\frac{1}{2}-x-\frac{3}{2}\right]$

$= -(-x^2)$

$= x^2$

Complete solution:

$$Y = Y_c + Y_p$$

$$= c_1 e^{x} + c_2 e^{-2x} + x^2 \quad \underline{\text{Ans:}}$$

$(D^3 - D^2 - 6D)y = x^2+1$

For complementary function,

$$(D^3 - D^2 - 6D)y = 0$$

Auxiliary solution is,

$$e^{mx}(m^3 - m^2 - 6m) = 0$$

$e^{mx} \neq 0$; $\quad m^3 - m^2 - 6m = 0$

$$\Rightarrow m(m^2 - m - 6) = 0$$

$$\therefore m = 0, -2, 3.$$

$Y_c = c_1 e^0 + c_2 e^{-2x} + c_3 e^{3x}$

Particular Integral,

$$Y_p = \frac{1}{D^3 - D^2 - 6D}(x^2+1)$$

$$= \frac{1}{-6D\left[1+\left(\frac{D-D^2}{6}\right)\right]}(x^2+1)$$

$$= \frac{1}{-6D}\left[1+\frac{D-D^2}{6}\right]^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D-D^2}{6}+\left(\frac{D-D^2}{6}\right)^2 \cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left(1-\frac{D}{6}-\frac{D^2}{6}-\frac{D^3}{18}+\frac{D^4}{18}-\cdots\right)(x^2+1)$$

$$= -\frac{1}{6D}\left(1-\frac{D}{6}+\frac{7D^2}{36}\right)(x^2+1)$$

$$= -\frac{1}{6D}\left[(x^2+1)-\frac{1}{6}(2x)+\frac{7}{36}\cdot 2\right]$$

$$= -\frac{1}{6D}\left(x^2+1-\frac{1}{3}x+\frac{7}{18}\right)$$

$$= -\frac{1}{6D}\left(x^2-\frac{1}{3}x+\frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3}-\frac{1}{3}\cdot\frac{x^2}{2}+\frac{25}{18}x\right)$$

$$= -\frac{x^3}{18}+\frac{x^2}{36}-\frac{25}{108}$$ ~~Ans:~~

The complete solution is

$$Y = Y_c + Y_p$$

$$= c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{x^3}{18} + \frac{x^2}{36} - \frac{25x}{108}$$ Ans: