Differentiation

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(x) = 1$$

$$\frac{d}{dx}(c) = 1$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(c \cdot f(x)) = c \frac{d}{dx} f(x)$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(e^{mx}) = me^{mx}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\log_a e$$

$$\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$$

$$\frac{d}{dx}(u-v) = \frac{d}{dx}u - \frac{d}{dx}v$$

$$\frac{d}{dx}(uv) = u\frac{d}{dx}v + v\frac{d}{dx}u$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

Integration

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$

$\int \frac{1}{x} dx = \ln|x| + c$

$\int \frac{1}{2\sqrt{x}} dx = \sqrt{x} + c$

$\int e^x dx = e^x + c$

$\int e^{mx} dx = \frac{e^{mx}}{m} + c$

$\int a^x dx = \frac{a^x}{\ln a} + c$

$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$

$\int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c$

$\int \sin x \, dx = -\cos x + c$

$\int \cos x \, dx = \sin x + c$

$\int \tan x \, dx = \ln|\sec x| + c$

$\int \cot x \, dx = \ln|\sin x| + c$

$\int \sec x \, dx = \ln|\sec x + \tan x| + c$

$\int \operatorname{cosec} x \, dx = \ln\left|\tan\frac{x}{2}\right| + c$

$\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1} \frac{x}{a} + c$

$\int uv \, dx = u\int v \, dx - \int\left(\frac{d}{dx}(u)\int v \, dx\right) dx$

LIATE

- L — Logarithmic
- I — Inverse trigonometric
- A — Algebric
- T — Trigonometric
- E — Exponential

LIATE শব্দটিতে বর্ণের ক্রমানুসারে বামপাশের ফাংশনটিকে u ধরতে হবে।

Trigonometric Formula

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$2\cos^2 A = 1 + \cos 2A$

$2\sin^2 A = 1 - \cos 2A$

$\sin 3A = 3\sin A - 4\sin^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A$

Separation of variable by the method of substitution :

1. $(x+y)^2 \frac{dy}{dx} = a^2$

Given, $(x+y)^2 \frac{dy}{dx} = a^2$ ——— (i)

let, $x+y = z$ ——— (ii)

$\Rightarrow y = z - x$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$ ——— (iii) [Diff with r.t.x]

Putting (ii) and (iii) in (i),

$\cancel{z^2 \frac{dy}{dx}}$ $z^2\left(\frac{dz}{dx} - 1\right) = a^2$

$\Rightarrow z^2 \frac{dz}{dx} - z^2 = a^2$

$\cancel{\Rightarrow \frac{z^2}{a^2+z^2}}$ $\Rightarrow z^2 \frac{dz}{dx} = a^2 + z^2$

$\Rightarrow \frac{z^2}{a^2+z^2} dz = dx$

$\Rightarrow \frac{z^2+a^2-a^2}{a^2+z^2} dz = dx$

$\Rightarrow \left(1 - \frac{a^2}{a^2+z^2}\right) dz = dx$

$\Rightarrow \int 1\, dz - \int \frac{a^2}{a^2+z^2} dz = \int dx$

$\Rightarrow z - a^2 \cdot \frac{1}{a} \tan^{-1}\frac{z}{a} = x + c$

$\Rightarrow z - a\tan^{-1}\frac{z}{a} = x + c$

$$\Rightarrow (x+y) - a\tan^{-1}\frac{x+y}{a} = x + C$$

Ans:

2. $\frac{dy}{dx} = \left(\frac{2y+3}{4x+5}\right)^2$

$$\Rightarrow \frac{dy}{(2y+3)^2} = \frac{dx}{(4x+5)^2}$$

let, $2y+3 = u$

$\Rightarrow 2 = \frac{du}{dy}$

$\Rightarrow dy = \frac{1}{2}du$

again,

$4x+5 = v$

$\Rightarrow 4 = \frac{dv}{dx}$

$\Rightarrow dx = \frac{1}{4}dv$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u^2}du = \frac{1}{4}\int \frac{1}{v^2}dv$$

$$\Rightarrow \frac{1}{2}\left(-\frac{1}{u}\right) = \frac{1}{4}\left(-\frac{1}{v}\right) + C$$

$$\Rightarrow \frac{-1}{2u} = \frac{-1}{4v} + C$$

$$\Rightarrow \frac{1}{4v} - \frac{1}{2u} = C$$

$$\Rightarrow \frac{1}{4(4x+5)} - \frac{1}{2(2y+3)} = C$$

Ans:

3. $(1+x)dy - ~~ydx=0~~\, y\,dx = 0$

$$\Rightarrow (1+x)dy = y\,dx$$

$$\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{(1+x)}dx$$

let,

$1+x = z$

$\Rightarrow 1 = \frac{dz}{dx}$

$\Rightarrow dx = dz$

$$\Rightarrow \int \frac{1}{y}dy = \int \frac{1}{z}\,dz$$

$$\Rightarrow \ln y = \ln z + C$$

$$\Rightarrow \ln y - \ln(1+x) = C$$

$$\Rightarrow \ln\frac{y}{1+x} = C$$

Ans:

04. $\frac{dy}{dx} = (4x+y+1)^2$ ——— (i)

~~Given~~ let,

$$4x+y+1 = z \quad \text{——— (ii)}$$

$$\Rightarrow y = z - 4x - 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 4 \quad \text{——— (iii)}$$

Putting (ii) and (iii) in (i),

$$\frac{dz}{dx} - 4 = z^2$$

$$\Rightarrow \frac{dz}{dx} = z^2 + 4$$

$$\Rightarrow \frac{1}{z^2+4} dz = dx$$

$$\Rightarrow \int \frac{1}{z^2+2^2} dz = \int dx$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{z}{2} = x + c$$

$$\Rightarrow \frac{1}{2} \tan^{-1} \frac{4x+y+1}{2} = x + c$$

Ans:

05. $(4x+y)^2 \frac{dx}{dy} = 1$ ——— (i)

let. $4x+y = z$ ——— (ii)

$\cancel{\Rightarrow y = z - 4x}$ $\Rightarrow 4x = z - y$

$\cancel{\Rightarrow \frac{dy}{d}}$ $\Rightarrow 4\frac{dx}{dy} = \frac{dz}{dy} - 1$

$$\Rightarrow \frac{dx}{dy} = \frac{1}{4}\left(\frac{dz}{dy} - 1\right) \quad \text{——— (iii)}$$

Putting (ii) and (iii) in (i),

$$z^2 \cdot \frac{1}{4}\left(\frac{dz}{dy} - 1\right) = 1$$

$$\Rightarrow z^2 \frac{dz}{dy} - z^2 = 4$$

$$\Rightarrow z^2 \frac{dz}{dy} = z^2 + 4$$

$$\Rightarrow \frac{z^2}{z^2+4}\, dz = dy$$

$$\Rightarrow \frac{z^2+4-4}{z^2+4}\, dz = dy$$

$$\Rightarrow \left(1 - \frac{2^2}{z^2+2^2}\right) dz = dy$$

$$\Rightarrow \int 1\, dz - \int \frac{2^2}{z^2+2^2}\, dz = \int dy$$

$$\Rightarrow z - 2^2 \cdot \frac{1}{2} \tan^{-1} \frac{z}{2} = y$$

$$\Rightarrow z - 2\tan^{-1} \frac{z}{2} = y$$

$$\Rightarrow (4x+y) - 2\tan^{-1} \frac{4x+y}{2} = y$$

$$\Rightarrow 4x + y - y = 2\tan^{-1} \frac{4x+y}{2}$$

$$\Rightarrow \tan^{-1} \frac{4x+y}{2} = 2x + c$$

Ans:

06. $\frac{dy}{dx} = 1 + \tan(y-x)$ [Put $y-x=z$]

Given, $\frac{dy}{dx} = 1+\tan(y-x)$ —— (i)

let, $y-x = z$ —— (ii)

$\Rightarrow y = z+x$

$\Rightarrow \frac{dy}{dx} = \frac{dz}{dx} + 1$ —— (iii)

Putting (ii) and (iii) in (i),

$\frac{dz}{dx} + 1 = 1 + \tan z$

$\Rightarrow \frac{dz}{dx} = \tan z$

$\Rightarrow \frac{dz}{\tan z} = dx$

$\Rightarrow \frac{\cos z}{\sin z} dz = dx$

let, $\sin z = t$

$\Rightarrow \cos z = \frac{dt}{dz}$

$\Rightarrow \cos z \, dz = dt$

$\Rightarrow \int \frac{1}{t} dt = \int dx$

$\Rightarrow \ln t = x + c$

$\Rightarrow e^{\ln t} = e^{x+c}$

$\Rightarrow \sin z = e^{x+c}$

$\Rightarrow \sin(y-x) = e^{x+c}$

Ans:

07. Exact Differential Method :

The first order ODE $M(x,y)dx + N(x,y)dy = 0$ is exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The solution of the given ODE is

$$\int M dx + \int (N \text{ with } x \text{ free terms}) dy = c$$

01. $(x^2 - 4xy - 2y^2)dx + (y^2 - 4xy - 2x^2)dy = 0$

$M = x^2 - 4xy - 2y^2$ $\qquad$ $N = y^2 - 4xy - 2x^2$

$\frac{\partial M}{\partial y} = -4y - 4x$ $\qquad$ $\frac{\partial N}{\partial x} = -4x - 4y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

The solution of the exact equation is,

$$\int M dx + \int (\text{terms of } N \text{ not containing } x) dy = c$$

$$\Rightarrow \int (x^2 - 4xy - 2y^2) dx + \int (y^2) dy = \cancel{0}\, c$$

$$\Rightarrow \frac{x^3}{3} - 4y\frac{x^2}{2} - 2xy^2 + \frac{y^3}{3} = c$$

$$\Rightarrow \frac{x^3}{3} - 2x^2y - 2xy^2 + \frac{y^3}{3} = c$$

Ans.

02. $(1+e^{x/y})dx + e^{x/y}(1-x/y)dy = 0$

$M = (1+e^{x/y})$ $\quad N = e^{x/y}(1-x/y)$

$\frac{\partial M}{\partial y} = 0 + e^{x/y}.x.(-\frac{1}{y^2})$ $\quad = e^{x/y} - \frac{x}{y}e^{x/y}$

$= -\frac{x}{y^2}e^{x/y}$ $\quad \frac{\partial N}{\partial x} = e^{x/y}.\frac{1}{y} - e^{x/y}.\frac{1}{y}.\frac{x}{y}$

$-\frac{1}{y}e^{x/y}$

$= -\frac{x}{y^2}e^{x/y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so, the equation is exact.

So, the solution will be,

$\int M dx + \int (N \text{ with } x \text{ free terms}) dy = c$

$\Rightarrow \int (1+e^{x/y})dx + \int 0 \, dy = c$

$\Rightarrow x + e^{x/y}/\frac{1}{y} = c$

$\Rightarrow x + ye^{x/y} = c$ Ans.

03. $(5x^4 + 3x^2y^2 - 2xy^3)dx + (2x^3y - 3x^2y^2 - 5y^4)dy = 0$

$M = 5x^4 + 3x^2y^2 - 2xy^3$ $\quad N = 2x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2$ $\quad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so, the equation is exact.

So, the solution will be

$\int M\,dx + \int (N \text{ with } x \text{ free terms})\,dy = c$

$\Rightarrow \int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int (-5y^4)\,dy = c$

$\Rightarrow 5\frac{x^5}{5} + 3y^2\frac{x^3}{3} + 2y^3\frac{x^2}{2} + (-5\frac{y^5}{5}) = c$

$\Rightarrow x^5 + y^2x^3 + x^2y^3 - y^5 = c$ Ans:

04. $[1 + \log(xy)]\,dx + [1 + \frac{x}{y}]\,dy = 0$

$\Rightarrow (1 + \log x + \log y)\,dx + (1 + \frac{x}{y})\,dy = 0$

$M = 1 + \log x + \log y$ $\qquad N = 1 + \frac{x}{y}$

$\frac{\partial M}{\partial y} = \frac{1}{y}$ $\qquad \frac{\partial N}{\partial x} = \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so, the equation is exact.

So, the solution will be

$\int M\,dx + \int (N \text{ with } x \text{ free term})\,dy = c$

$\Rightarrow \int (1 + \log x + \log y)\,dx + \int 1\,dy = c$

$\Rightarrow \int 1\,dx + \int \log x\,dx + \int \log y\,dx + \int dy = c$ ——— (1)

$\int \log x\,dx = \int \log x \cdot 1\,dx$

$= \log x \int 1\,dx - \int \left(\frac{d}{dx}\log x \int 1\,dx\right)dx$

$= \log x \cdot x - \int \frac{1}{x} \cdot x \, dx$

$= x\log x - \cancel{x}\int 1 \, dx$

$= x\log x - x$

$= x(\log x - 1)$

Equation (i) becomes,

~~$x + x(\log x$~~

$x + x(\log x - 1) + x\log y + y = c$

$\Rightarrow x(1 + \log x - 1 + \log y) + y = c$

$\Rightarrow x(\log x + \log y) + y = c$ Ans:

5) $(y^2 - 2xy + 6x)dx - (x^2 - 2xy + 2)dy = 0$

$M = y^2 - 2xy + 6x$

$N = -(x^2 - 2xy + 2)$

$= 2xy - x^2 - 2$

$\frac{\partial M}{\partial y} = 2y - 2x$

$\frac{\partial N}{\partial x} = 2y - 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so the equation is exact.

$\int M \, dx + \int (N \text{ with } x \text{ free terms}) dy = c$

$\Rightarrow \int (y^2 - 2xy + 6x) dx + \int (-2) dy = c$

$\Rightarrow xy^2 - 2\frac{x^2}{2}y + 6\frac{x^2}{2} - 2y = c$

$\Rightarrow xy^2 - x^2y + 3x^2 - 2y = c$ Ans:

Exercise (3.7) (1)

$$(x+y-10)\,dx+(x-y-2)\,dy=0$$

Sol^n:

$M = x+y-10$; $N = x-y-10$

$\dfrac{\partial M}{\partial y} = 1$; $\dfrac{\partial N}{\partial x} = 1$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$; so, the eq^n is exact.

$\Rightarrow \int M\,dx + \int (N$ with x free variables$)\,dy = C$

$\Rightarrow \int (x+y-10)\,dx + \int (-y-2)\,dy = C$

$\Rightarrow \dfrac{x^2}{2} + yx - 10x - \dfrac{y^2}{2} - 2y = C$

Ans:

2. $(y^2-x^2)dx + 2xy\,dy = 0$

$M = y^2 - x^2$ $\qquad\qquad$ $N = 2xy$

$\frac{\partial M}{\partial y} = 2y$ $\qquad\qquad$ $\frac{\partial N}{\partial x} = 2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so, the eq^n is exact.

$\int M\,dx + \int (N$ with x free variables$)\,dy = c$

$\Rightarrow \int (y^2 - x^2)\,dx + \int 0\,dy = c$

$\Rightarrow \cancel{y^2x - 2x + y = c} \Rightarrow xy^2 - \frac{x^3}{3} + 0 = c \Rightarrow \frac{x^3}{3} = xy^2 + c$ Ans:

3. $(1+3e^{x/y})dx + 3e^{x/y}(1-\frac{x}{y})dy = 0$

$M = 1 + 3e^{x/y}$ $\qquad\qquad$ $N = 3e^{x/y}(1-\frac{x}{y})$

$\frac{\partial M}{\partial y} = 3e^{x/y} \cdot x \cdot (-\frac{1}{y^2})$ $\qquad$ $N = 3e^{x/y} - 3\frac{x}{y}e^{x/y}$

$= -\frac{x}{y^2} \cdot 3e^{x/y}$ $\qquad$ $\frac{\partial N}{\partial x} = 3e^{x/y} \cdot \frac{1}{y} - 3\frac{x}{y}e^{x/y} \cdot \frac{1}{y} - 3\frac{1}{y}e^{x/y}$

$= -3e^{x/y}\frac{x}{y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so, the eq^n is exact

$\int M\,dx + \int (N$ with x free variables$)\,dy = c$

$\Rightarrow \int (1+3e^{x/y})\,dx + \int 0\,dy = c$

$\Rightarrow x + 3e^{x/y}/\frac{1}{y} + 0 = c$ ~~Ans:~~ $\Rightarrow x + 3ye^{x/y} = c$

Ans:

④ $(2x-y)\,dx = (x-y)\,dy$

$\Rightarrow (2x-y)\,dx - (x-y)\,dy = 0$

$M = 2x-y$ $\qquad$ $N = -x+y$

$\dfrac{\partial M}{\partial y} = -1$ $\qquad$ $\dfrac{\partial N}{\partial x} = -1$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$; so, the eqn is exact.

$\int M\,dx + \int (N \text{ with } x \text{ free variables}) = c$

$\Rightarrow \int (2x-y)\,dx + \int y\,dy = c$

$\Rightarrow 2\cdot\dfrac{x^2}{2} - yx + \dfrac{y^2}{2} = c$

$\Rightarrow x^2 + \dfrac{y^2}{2} + c = xy$ <u>Ans:</u>

⑤ $(y\sec^2 x + \sec x\tan x)\,dx + (\tan x + 2y)\,dy = 0$

$M = y\sec^2 x + \sec x\tan x$ $\qquad$ $N = \tan x + 2y$

$\Rightarrow \dfrac{\partial M}{\partial y} = 0\cdot y + 1\cdot\sec^2 x + 0$ $\qquad$ $\dfrac{\partial N}{\partial x} = \sec^2 x + 0$

$= \sec^2 x$ $\qquad$ $= \sec^2 x$

$\therefore \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$; so, the eqn is exact.

$\int M\,dx + \int (N \text{ with } x \text{ free term})\,dy = c$

$\Rightarrow \int (y\sec^2 x + \sec x\tan x)\,dx + \int 2y\,dy = c$

$\Rightarrow y\tan x + \sec x + 2\cdot\dfrac{y^2}{2} = c$

$\Rightarrow y\tan x + \sec x + y^2 = c$ <u>Ans:</u>

(7) $(x^4 - 2xy^2 + y^4)dx - (2x^2y - 4xy^3 + \sin y)dy = 0$

$M = x^4 - 2xy^2 + y^4$ $\qquad$ $N = -2x^2y + 4xy^3 - \sin y$

$\frac{\partial M}{\partial y} = -2x \cdot 2y + 4y^3$ $\qquad$ $\frac{\partial N}{\partial x} = -2y \cdot 2x + 4y^3$

$\Rightarrow \frac{\partial M}{\partial y} = -4xy + 4y^3$ $\qquad$ $= -4xy + 4y^3$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so, the eq^n is exact.

$\int M\,dx + \int (N \text{ with } x \text{ free variables})\,dy = C$

$\Rightarrow \int (x^4 - 2xy^2 + y^4)\,dx + \int -\sin y\,dy = C$

$\Rightarrow \frac{x^5}{5} - 2y^2\frac{x^2}{2} + y^4 x + \cos y = C$

$\Rightarrow \frac{x^5}{5} - x^2y^2 + xy^4 + \cos y = C$

Ans:

(8) $(2xy + e^y)dx + (x^2 + xe^y)dy = 0$

$M = 2xy + e^y$ $\qquad$ $N = x^2 + xe^y$

$\frac{\partial M}{\partial y} = 2x + e^y$ $\qquad$ $\frac{\partial N}{\partial x} = 2x + 1.e^y + x.0$

$\frac{\partial N}{\partial x} = 2x + e^y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so, the eq^n is exact

$\int M\,dx + \int (N \text{ with } x \text{ free variables})\,dy = C$

$\Rightarrow \int (2xy + e^y)\,dx + \int 0\,dy = C$

$\Rightarrow 2y\frac{x^2}{2} + e^y . x = C \quad \Rightarrow x^2y + xe^y = C$

Ans:

(6) $(ax+hy+g)dx+(hx+by+f)dy=0$

$M=ax+hy+g$ $\qquad$ $N=hx+by+f$

$\frac{\partial M}{\partial y}=h$ $\qquad$ $\frac{\partial N}{\partial x}=h+0$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$; So,

$\int M\,dx+\int(N$ with x free variables$)dy=C$

$\Rightarrow \int(ax+hy+g)dx+\int(by+f)\,dy=C$

$\Rightarrow a\frac{x^2}{2}+hyx+gx+b\frac{y^2}{2}+fy=C$

$\Rightarrow \cancel{ax^2+hy}\ \frac{ax^2+2hyx+2gx+by^2+2fy}{2}=C$

$\Rightarrow ax^2+2hyx+2gx+by^2+2fy=C$ Ans:

correction

11. $(x^3-3xy^2)dx+(y^3-3x^2y)dy=0$; $y(0)=1$

$M=x^3-3xy^2$ $\qquad$ $N=y^3-3x^2y$

$\frac{\partial M}{\partial y}=-6xy$ $\qquad$ $\cancel{\frac{\partial N}{\partial y}=3y^2}$ $\frac{\partial N}{\partial x}=-6xy$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$; so, the eq^n is exact.

$\int M\,dx+\int(N \text{ with } x \text{ free variables})\,dy=c$

$\Rightarrow \int(x^3-3xy^2)\,dx+\int(y^3)\,dy=c$

$\Rightarrow \frac{x^4}{4}-3y^2\frac{x^2}{2}+\frac{y^4}{4}=c$

$\Rightarrow \frac{x^4}{4}-\frac{3}{2}x^2y^2+\frac{y^4}{4}=c$

$\Rightarrow \frac{x^4-6x^2y^2+y^4}{4}=c$

$\Rightarrow x^4-6x^2y^2+y^4=4c$

Putting $(x,y)=(0,1)$

$0^4-6(0)^2(1)^2+(1)^4=4c$

$\Rightarrow 0-0+1=4c$

$\Rightarrow c=1$

$\Rightarrow x^4-6x^2y^2+y^4=1$

(10) $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\sin y)\,dy=0$

$=\left[y+\frac{y}{x}+\cos y\right]dx+(x+\log x-x\sin y)\,dy=0$

$M=y+\frac{y}{x}+\cos y$

$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y$

$N=x+\log x-x\sin y$

$\frac{\partial N}{\partial x}=1+\frac{1}{x}-1.\sin y+0$

$=1+\frac{1}{x}-\sin y$

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$; so, the eq^n is exact.

$\int M\,dx+\int(N$ with x free variables$)\,dy=C$

$\Rightarrow\int\left(y+\frac{y}{x}+\cos y\right)dx+\int 0\,dy=C$

~~$\Rightarrow yx+\ldots\cos y$~~

$\Rightarrow yx+y\cdot\log x+\cos y\cdot x=C$

$\Rightarrow y(x+\log x)+x\cos y=C$ Ans:

3. $(x^2 + 2ye^{2x})dy + (2xy + 2y^2 e^{2x})dx = 0$

$M = x^2 + \cancel{2ye^{2x}}$ $\quad$ $N = \cancel{2xy + 2y^2 e^{2x}}$

$\frac{\partial M}{\partial y} = 0 + \cancel{2(1 \cdot e^{2x} + 0)}$ $\quad$ $\frac{\partial N}{\partial x} = \cancel{2y + 2}$

$= \cancel{2e^{2x}}$

$\Rightarrow (2xy + 2y^2 e^{2x})dx + (x^2 + 2ye^{2x})dy = 0$

$M = 2xy + 2y^2 e^{2x}$ $\quad$ $N = x^2 + 2ye^{2x}$

$\frac{\partial M}{\partial y} = 2x + 4ye^{2x} + 2 \cdot y^2 \cdot 0$ $\quad$ $\frac{\partial N}{\partial x} = 2x + 2(y \cdot e^{2x} \cdot 2 + 0 \cdot e^{2x})$

$= 2x + 4ye^{2x}$ $\quad$ $= 2x + 4ye^{2x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; so, the eqn is exact.

$\int M\,dx + \int (N \text{ with } x \text{ free } \cancel{\text{term}} \text{ variables})\,dy = C$

$\Rightarrow \int (2xy + 2y^2 e^{2x})dx + \int 0\,dy = C$

$\Rightarrow 2y\frac{x^2}{2} + 2y^2 \cdot e^{2x} \cdot \frac{1}{2} = C$

$\Rightarrow x^2 y + y^2 e^{2x} = C$ Ans:

Example-25: solve $(2x\log x - xy)dy + 2y\,dx = 0$

$M = 2y$ $\qquad$ $N = 2x\log x - xy$

$\frac{\partial M}{\partial y} = 2$ $\qquad$ $\cancel{\frac{\partial N}{\partial x} = 2\{\log x \frac{d}{dx}(x) \int \frac{d}{dx}(\log x)\}}$

$= 2(1+\log x) - y$

Here, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \dfrac{2-2-2\log x+y}{2x\log x - xy} = \dfrac{-(2\log x - y)}{x(2\log x - y)} = -\frac{1}{x} = f(x)$

$I.F = e^{\int f(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x}$

On multiplying the given differential equation (1) by $\frac{1}{x}$, we get,

$\frac{2y}{x}dx + (2\log x - y)dy = 0$

$\Rightarrow \int \frac{2y}{x}dx + \int -y\,dy = 0$

$\Rightarrow \cancel{\frac{2y}{x^2} - \ln y = C}$ $\quad$ ~~Ans:~~

$\Rightarrow 2y\ln x - \frac{y^2}{2} = C$

Ans:

$\textcircled{2}$ $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}(1+y^2)x\,dy=0$

$M=y+\frac{1}{3}y^3+\frac{1}{2}x^2$ $\quad N=\frac{x}{4}+\frac{xy^2}{4}$

$\frac{\partial M}{\partial y}=1+\frac{1}{3}3y^2+0$ $\quad \Rightarrow \frac{\partial N}{\partial x}=\frac{1}{4}+\frac{y^2}{4}$

Here, $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{1+y^2-\frac{1}{4}-\frac{y^2}{4}}{\frac{x}{4}+\frac{xy^2}{4}}=\frac{\frac{4-1}{4}+\left(\frac{4y^2-y^2}{4}\right)}{\frac{x}{4}(1+y^2)}$

$=\frac{\frac{3}{4}+\frac{3y^2}{4}}{\frac{x}{4}(1+y^2)}=\frac{\frac{3(1+y^2)}{4}}{\frac{x}{4}(1+y^2)}=\frac{3(1+y^2)}{4}\times\frac{4}{x(1+y^2)}$

$=\frac{3}{x}=f(x)$

$I.F=e^{\int f(x)dx}=e^{\int\frac{3}{x}dx}=e^{3\ln x}=e^{\ln x^3}=x^3$

On multiplying the given differential equation $\textcircled{1}$ by x^3, we

$\left(\frac{y}{x^3}+\frac{y^3}{3x^3}+\frac{x^2}{2x^3}\right)dx+\left(\frac{1}{4x^3}+\frac{xy^2}{4x^3}\right)dy=0$

$\Rightarrow\int\left(\frac{y}{x^3}+\frac{y^3}{3x^3}+\frac{1}{2x}\right)dx=0$ $\quad \Rightarrow\int\left(yx^3+\frac{y^3x^3}{3}+\frac{x^2x^3}{2}\right)dx=0$

x^{-3}
$=\frac{x^{-3+1}}{-3+1}$

$\Rightarrow \frac{yx^4}{4}+\frac{y^3x^4}{12}+\frac{x^6}{12}=c$

$\Rightarrow \frac{yx^4}{4}+\frac{y^3x^4}{12}+\frac{x^2}{}$ $\quad \Rightarrow \frac{yx^4}{4}+\frac{yx^4}{12}+\frac{x^6}{12}=c$

Ans:

Rule II: Solve $(y^4+2y)\,dx + (xy^3+2y^4-4x)\,dy = 0$

$M = y^4+2y \qquad\qquad N = xy^3+2y^4-4x$

$\Rightarrow \dfrac{\partial M}{\partial y} = 4y^3+2 \qquad\qquad \dfrac{\partial N}{\partial x} = y^3+0-4$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y} = \frac{-3y^3-6}{y(y^3+2)} = \frac{-3(y^3+2)}{y(y^3+2)} = \frac{-3}{y} = f(y)$$

$$I.F = e^{\int f(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3} = \frac{1}{y^3}$$

On multiplying the given equation ① by $\frac{1}{y^3}$ we get the exact,

$$\left(y^4\times\frac{1}{y^3} + 2y\times\frac{1}{y^3}\right)dx + \left(xy^3\times\frac{1}{y^3} + 2y^4\times\frac{1}{y^3} - 4x\times\frac{1}{y^3}\right)dy = 0$$

$$\Rightarrow \left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

$$\Rightarrow \int\left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = 0$$

$$\Rightarrow xy + \frac{2x}{y^2} + y^2 = c$$

$$\Rightarrow x\left(y + \frac{2}{y^2}\right) + y^2 = c$$

Ans.

3.8

(1) $(y\log y)\,dx + (x-\log y)\,dy = 0$

$M = y\log y \qquad N = x - \log y$

$\cancel{\log y \int y\,dx - \int\left(\frac{d}{dx}(\log y)\int y\,dy\right)dy}$

$\frac{\partial N}{\partial x} = 1 - 0$

$\frac{\partial M}{\partial y} = \log y + y\cdot\frac{1}{y}\log_a e \qquad \frac{\partial N}{\partial x} = 1$

$= \log y + \cancel{\log_a e}\ 1$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{1 - \log y - \cancel{\log_a e}\ 1}{y\log y} = -\frac{1}{y} = f(y)$$

$$I.F = e^{\int f(y)\,dy} = e^{\int -\frac{1}{y}dy} = e^{-\ln y} = e^{\ln y^{-1}} = \frac{1}{y}$$

On multiplying the given equation (1) by $\frac{1}{y}$ we get,

$$\left(y\log y \times \frac{1}{y}\right)dx + \left(x\times\frac{1}{y} - \log y \times \frac{1}{y}\right)dy = 0$$

$$\Rightarrow \int \log y\,dx + \int -\left(\frac{1}{y}\log y\right)dy = 0$$

$\cancel{\Rightarrow \log y \int 1\,dx - \int\left\{\frac{d}{dx}(\log y)\int 1\,dx\right\}dx + \left[\log y \frac{d}{dx}\left(-\frac{1}{y}\right)\right]}$

$\cancel{\Rightarrow \log y\, x - \int\left(\frac{1}{y}\cdot x\right)dx}$

$$\Rightarrow x\log y - \left[\log y \int \frac{1}{y}dy - \int\left(\frac{d}{dy}(\log y)\int\frac{1}{y}dy\right)dy\right] = 0$$

$$\Rightarrow x\log y - \log y \ln y - \int\left(\frac{1}{y}\log_a e\cdot \ln y\right)$$

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3.10

Rule-3: If M is of the form $M = y f_1(xy)$ and N is of the form $N = x f_2(xy)$

Then $\boxed{I.F. = \dfrac{1}{M.x - N.y}}$

Example:

Solve $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

Solution: $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$ ——— (1)

Dividing (1) by xy, we get

$$y(1+2xy)dx + x(1-xy)dy = 0 \text{ ——— (II)}$$

$$M = y f_1(xy), \quad N = x f_2(xy)$$

$$I.F. = \frac{1}{Mx - Ny}$$

$$= \frac{1}{xy(1+2xy) - xy(1-xy)}$$

$$= \frac{1}{xy + 2x^2y^2 - xy + x^2y^2} = \frac{1}{3x^2y^2}$$

On multiplying (2) by $\frac{1}{3x^2y^2}$, we have an exact differential equation

$$\frac{1}{3x^2y^2}(y + 2xy^2)dx + \frac{1}{3x^2y^2}(x - x^2y)dy = 0$$

$$\Rightarrow \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

$$\Rightarrow \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int \frac{-1}{3y}\,dy = 0$$

$\frac{x^{-2+1}}{-2+1}$

$$\Rightarrow \frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c$$

$$\Rightarrow -\frac{1}{xy} + 2\log x - \log y = b \quad \underline{\text{Ans:}}$$

Exercise – 3.10

$$1.\ (y - xy^2)dx - (x + x^2y)dy = 0 \quad \text{——— (i)}$$

$$\Rightarrow y(1 - xy)dx + \left[\{-x(1 + xy)\}dy\right] = 0 \quad \text{——— (ii)}$$

~~Dividing (ii) by xy, we get~~,

$M = yf_1(xy)$ $\qquad N = -xf_2(xy)$

$$\text{I.F.} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{x\{y(1-xy)\} - y\{-x(1+xy)\}}$$

$$= \frac{1}{xy - x^2y^2 + xy + x^2y^2}$$

$$= \frac{1}{2xy}$$

On multiplying by $\frac{1}{2xy}$

$$\frac{1}{2xy}(y - xy^2)dx + \frac{1}{2xy}(-x - x^2y)dy = 0$$

$$\Rightarrow \int\left(\frac{1}{2x} - \frac{y}{2}\right)dx + \int -\frac{1}{2y}dy = c$$

$$\Rightarrow \frac{1}{2}\ln x - \frac{xy}{2} - \frac{1}{2}\ln y = c$$

$\Rightarrow \frac{1}{2}(\ln x - \ln y) - xy = c \quad \Rightarrow (\ln x - \ln y) - xy = 2c$

$\Rightarrow \ln\left(\frac{x}{y}\right) - xy = A$

Ans:

3.9 Linear differential equations:

A differential equation of the form

$$\frac{dy}{dx} + Py = Q \quad \text{——— (P)}$$

is called a linear differential equation, where P and Q are functions of x (but not of y) or constants.

~~Working Rule:~~

Note:

$e^{\int P dx}$ is called the integrating factor.

Solution is $y \times [I.F.] = \int Q [I.F.] dx + c$

Working Rule:

Step-1: Convert the given equation to the standard form of linear differential equation i.e $\frac{dy}{dx} + Py = Q$

Step-2: Find the integrating factor i.e. $I.F. = e^{\int P dx}$

Step-3: Then the solution is $y(I.F.) = \int Q(I.F.) dx + c$

Example:

Solve: $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x+1} = e^x \frac{(x+1)^2}{(x+1)}$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$$

Integrating factor $= e^{\int\left(-\frac{1}{x+1}\right)dx}$

$$= e^{-\int \frac{dx}{x+1}}$$

$$= e^{-\ln(x+1)}$$

$$= (x+1)^{-1} = 1/(x+1)$$

The solution is:

$$y\frac{1}{x+1} = \int e^x(x+1)\cdot\frac{1}{(x+1)}\,dx$$

$$\Rightarrow y\frac{1}{x+1} = \int e^x\,dx$$

$$\Rightarrow \frac{y}{x+1} = e^x + c$$

Ans.

Example: $(x^3-x)\frac{dy}{dx} - (3x^2-1)y = x^5 - 2x^3 + x$

Given, $(x^3-x)\frac{dy}{dx} - (3x^2-1)y = x^5 - 2x^3 + x$

$$\Rightarrow \frac{dy}{dx} - \frac{(3x^2-1)}{(x^3-x)}y = \frac{x^5 - x^3 - x^3 + x}{x^3 - x}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{3x^2-1}{x^3-x}\right)y = \frac{x^3(x^2-1) - x(x^2-1)}{x^3-x}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{3x^2-1}{x^3-x}\right)y = (x^2-1)$$

$$I.F. = e^{\int -\frac{3x^2-1}{x^3-x}dx}$$

$$= e^{-\ln(x^3-x)}$$

$$= e^{\ln(x^3-x)^{-1}}$$

$$= \frac{1}{x^3-x} \cancel{= \frac{1}{x(x^2-1)}}$$

The solution is: $y(I.F.) = \int Q(I.F.)dx + C$

$$\Rightarrow y\left(\frac{1}{x^3-x}\right) = \int (x^2-1)\frac{1}{x^3-x}dx + C$$

$$\Rightarrow \frac{y}{x^3-x} = \int \frac{(x^2-1)}{x(x^2-1)}dx + C$$

$$\Rightarrow y/(x^3-x) = \int \frac{1}{x}dx + C$$

$$\Rightarrow \frac{y}{x^3-x} = \ln x + C$$

$$\Rightarrow y = (x^3-x)\ln x + (x^3-x)C$$

Ans:

Exercise - 3.5 (1)

$$\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3$$

$$I.F = e^{\int \frac{1}{x}dx}$$

$$= e^{\ln x}$$

$$= x$$

The solution is

$$y(I.F.) = \int Q(I.F.)dx + C$$

$$\Rightarrow yx = \int (x^3-3)x\,dx + C$$

$$\Rightarrow yx = \int (x^4 - 3x)dx + C$$

$$\Rightarrow yx = \frac{x^5}{5} - \frac{3x^2}{2} + C$$

Ans:

(2) $(2y-3x)\,dx + x\,dy = 0$

$\cancel{\Rightarrow dx + \frac{x}{(2y-3x)}}$

$\cancel{\Rightarrow (2y-3x)dx = -x\,dy}$

$\cancel{\Rightarrow (2y-3x)dx}$

$\Rightarrow x\,dy = -(2y-3x)\,dx$

$\Rightarrow x\frac{dy}{dx} + 2y = 3x$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = 3$

$$I.F = e^{\int \frac{2}{x}dx}$$

$$= e^{2\ln x}$$

$$= e^{\ln x^2}$$

$$= x^2$$

The solution is:

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\Rightarrow yx^2 = \int 3x^2 dx + c$$

$$\Rightarrow yx^2 = x^3 + c$$

Ans:

$\boxed{\text{Exercise} - 3.5}$

(1) $\frac{dy}{dx} + \frac{1}{x}y = x^3 - 3$

$\cancel{I.F. = e^{\int \frac{1}{x}d}}$ $I.F. = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

The sol^n is:

$$y(I.F.) = \int Q(I.F.)dx + c$$

$$\Rightarrow y.x = \int (x^3-3)x\,dx + c$$

$$\Rightarrow xy = \int (x^4-3x)\,dx + c$$

$$\Rightarrow xy = \frac{x^5}{5} - \frac{3x^2}{2} + c$$

ans:

(2) $(2y-3x)dx + xdy = 0$

$xdy = (3x-2y)dx$

$\Rightarrow x\frac{dy}{dx} = 3x - 2y$

$\Rightarrow x\frac{dy}{dx} + 2y = 3x$

$\Rightarrow \frac{dy}{dx} + \frac{2y}{x} = \frac{3x}{x}$

$\Rightarrow \frac{dy}{dx} + \frac{2}{x}y = 3$

$I.F. = e^{\int \frac{2}{x}dx} = e^{2\ln x}$

$= e^{\ln x^2} = x^2$

The sol^n is :

$y(I.F) = \int Q(I.F)dx + c$

$\Rightarrow yx^2 = \int 3x^2 dx + c$

$\Rightarrow x^2 y = x^3 + c$

Ans:

(3) $\frac{dy}{dx} + y\cot x = \cos x$

$I.F. = e^{\int \cot x dx} = e^{\ln \sin x} = \sin x$

The sol^n is : $y(I.F.) = \int Q(I.F.)dx + c$

$\Rightarrow y \sin x = \int \cos x \sin x \, dx + c$

$\Rightarrow y\sin x = \frac{1}{2}\int 2\sin x\cos x \, dx + c$

$\Rightarrow y\sin x = \frac{1}{2}\int \sin 2x \, dx + c$

$\Rightarrow y \sin x = \frac{1}{2} \times \frac{-\cos x}{2} + c$

$\Rightarrow y \sin x$

(4) $\frac{dy}{dx} + y\sec x = \tan x$

$I.F. = e^{\int \sec x dx}$

$= e^{\ln(\sec x + \tan x)}$

$= \sec x + \tan x$

The sol^n is :

$y(I.F) = \int Q(I.F)dx + c$

$\Rightarrow y(\sec x + \tan x) = \int \tan x(\sec x + \tan x)dx + c$

$\Rightarrow y(\sec x + \tan x) = \int (\tan x \sec x + \tan^2 x)dx + c$

$= \int (\tan x \sec x + \sec^2 x - 1)dx + c$

$= \sec x + \tan x - x + c$

$\Rightarrow \frac{y(\tan x + \sec x)}{(\tan x + \sec x)} = \frac{(\sec x + \tan x)}{(\sec x + \tan x)} + \frac{(c-x)}{}$

$\Rightarrow y = 1 + \frac{c - x}{(\sec x + \tan x)}$

Ans.

Application OF 1st ORDER ODE

01. A culture initially has P_0 numbers of bacteria. At $t=1h$, the number of bacteria P measured to be $3/2 P_0$. If the rate of growth is proportional to the number of bacteria $P(t)$ present at time t, determine the time necessary for the numbers of bacteria to triple.

→ Soln:

Initially $t=0$

According to the question, $\frac{dp}{dt} \propto P(t)$

$$\Rightarrow \frac{dp}{dt} = KP(t) \text{ ——— (I)}$$

with condition, $P(0) = P_0$ ——— (II)

$P(1) = \frac{3}{2} P_0$ ——— (III)

$$\frac{dp}{dt} = KP$$

$$\Rightarrow \frac{dp}{p} = Kdt$$

$$\Rightarrow \int \frac{dp}{p} = \int Kdt$$

$$\Rightarrow \ln p = Kt + \ln c$$

$$\Rightarrow \ln p - \ln c = Kt$$

$$\Rightarrow \ln\left(\frac{P}{c}\right) = Kt$$

$$\Rightarrow \frac{P}{c} = e^{Kt}$$

$$\Rightarrow P(t) = ce^{Kt} \text{ ——— (IV)}$$

Putting $t=0$ in (IV) $\Rightarrow P(0) = c$

$\Rightarrow c = P_0$

(IV) becomes, $P(t) = P_0 e^{Kt}$ ——— (V)

Putting $t=1$ in (V) $\Rightarrow P(1) = P_0 e^K$

$$\Rightarrow \frac{3}{2} P_0 = P_0 e^K$$

$$\Rightarrow e^K = \frac{3}{2}$$

$$\Rightarrow K = \ln \frac{3}{2}$$

$$= 0.406$$

The time for bacteria to triple,

$$P(t) = 3P_0$$

$$\Rightarrow P_0 e^{0.406t} = 3P_0$$

$$\Rightarrow t = \frac{\ln 3}{0.406} = 2.71 \text{ h}.$$

Ans:

Newton's law of cooling $\rightarrow \frac{dT}{dt} \propto (T - T_m)$

$T(t)$ = Temperature of object at time t.

T_m = Ambient temperature

$\frac{dT}{dt}$ = Rate of cooling

02. When a cake is removed from oven, its temperature is measured 300°F. 3 minutes later it's temperature is 200°F. How long it will take for the cake to cool off to a temperature of 80°F. Here, room temperature is 70° F.

$\Rightarrow$ Soln: According to Newton's Law of cooling, $\frac{dT}{dt} \propto (T - T_m)$

$$\Rightarrow \frac{dT}{dt} = K(T - \underbrace{T_m}_{70}) \text{ --- (i)}$$

with conditions,

$$\left.\begin{aligned} T(0) &= 300°F \\ T(3) &= 200°F \end{aligned}\right\} \text{ --- (ii)}$$

$$(i) \Rightarrow \int \frac{dT}{T-70} = \int K\,dt$$

$$\Rightarrow \ln(T-70) = Kt + \ln c$$

$$\Rightarrow \ln(T-70) - \ln c = Kt$$

$$\Rightarrow \ln\left(\frac{T-70}{c}\right) = Kt$$

$$\Rightarrow T - 70 = e^{Kt} \cdot c$$

$$\Rightarrow T(t) = 70 + ce^{KT} \text{ --- (iii)}$$

$T(0) = 300$

$= 70 + c \cdot 1$

$c = 230$

(III) $\Rightarrow T(t) = 70 + 230\, e^{KT}$ —— (IV)

When, $t = 3$, (IV) $\Rightarrow 200 = 70 + 230\, e^{3K}$

$$\Rightarrow 230\, e^{3K} = 130$$

$$\Rightarrow 3K = \ln\left(\frac{130}{230}\right)$$

$$\Rightarrow K = \frac{\ln(130/230)}{3}$$

$$\Rightarrow K = -0.1902$$

(IV) $\Rightarrow T(t) = 70 + 230\, e^{-0.1902t}$ —— (V)

$\therefore 80 = 70 + 230\, e^{-0.1902t}$

$$\Rightarrow e^{-0.1902t} = \frac{10}{230}$$

$t = 16.49$ min $= 16$ min 29.4 sec Ans://

01. An RL circuit has an e.m.f. 5V, Resistance of 50Ω, Inductance of 1 Henry and no initial current. Find current at any time t.

$\Rightarrow$ Soln: According to Kirchoff's Law, $L\frac{dI}{dt}+IR=V$

$$\Rightarrow \frac{dI}{dt}+50I=5 \quad \text{——— (i)}$$

with the condition,

$$I(0)=0 \quad \text{——— (ii)}$$

I.F. of (i) $\Rightarrow e^{\int 50dt}=e^{50t}$

Multiplying (i) by I.F. $\Rightarrow e^{50t}\frac{dI}{dt}+e^{50t}.50I=5e^{50t}$

$$\Rightarrow \frac{d}{dt}(e^{50t}.I)=5e^{50t}$$

$$\Rightarrow \int d(e^{50t}.I)=\int 5e^{50t}\,dt$$

$$\Rightarrow e^{50t}.I=5.e^{50t}.\frac{1}{50}+c$$

$$\Rightarrow e^{50t}.I=\frac{1}{10}e^{50t}+c$$

$$\Rightarrow I(t)=\frac{1}{10}+ce^{-50t} \quad \text{——— (iii)}$$

When, t=0 (iii) becomes,

$I(0)=\frac{1}{10}+c$

$\therefore c=-\frac{1}{10}$

(iii) becomes $\Rightarrow I(t)=\frac{1}{10}-\frac{1}{10}e^{-50t}$

$$I(t)=\frac{1}{10}(1-e^{-50t})$$

Ans:

(27-02-22)

3.18 Linear Differential Equations of Second Order with constant coefficients.

The general form of the linear differential equation of second order is

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$$

where P and Q are constants and R is a function of x or constant.

Differential operator. Symbol D stands for the operation of differential i.e, $Dy=\frac{dy}{dx}$; $D^2y=\frac{d^2y}{dx^2}$

$\frac{1}{D}$ stands for operation of integration.

$\frac{1}{D^2}$ stands for the operation of integration twice.

$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ can be written in the operator form

$D^2y+PDy+QY=R \Rightarrow (D^2+PD+Q)Y=R$

3.19. Complete solution = Complementary function + Particular Integral

Case-1: Roots, Real and Different: If m_1 and m_2 are the roots then the C.F is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$

Case-2: Roots, Real and Equal. If both the roots are m_1, m_1 then the C.F. is $y = (C_1 + C_2 x) e^{m_1 x}$

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

Solⁿ: Given,

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0 \text{ ——— (i)}$$

Let,

$y = e^{mx}$ be the trail solⁿ of (i)

For, y_c trial solution is always $y = e^{mx}$

$$\frac{dy}{dx} = me^{mx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = m^2 e^{mx}$$

Auxiliary solⁿ of (i) $\Rightarrow m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$

$$\Rightarrow e^{mx}(m^2 - 3m + 2) = 0$$

e^{mx} cannot be 0

$\therefore m^2 - 3m + 2 = 0$

$\Rightarrow m_1 = 1;\ m_2 = 2$

$y = y_c$

$= C_1 e^{m_1 x} + C_2 e^{m_2 x}$

$= C_1 e^{x} + C_2 e^{2x}$

Ans.

(02) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$

Soln:

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0 \quad \text{---} \quad (i)$$

Let, $y = e^{mx}$

$\Rightarrow \frac{dy}{dx} = me^{mx}$

$\Rightarrow \frac{d^2y}{dx^2} = m^2e^{mx}$

$\Rightarrow \frac{d^3y}{dx^3} = m^3e^{mx}$

Auxiliary soln of (i) $\Rightarrow m^3e^{mx} - 4m^2e^{mx} + me^{mx} + 6e^{mx} = 0$

$m^3e^{mx} - 4m^2e^{mx} + me^{mx} + 6e^{mx} = 0$

$\Rightarrow e^{mx}(m^3 - 4m^2 + m + 6) = 0$

$e^{mx} \neq 0 \qquad m^3 - 4m^2 + m + 6 = 0$

$\therefore m_1 = -1, \; m_2 = 2, \; m_3 = 3$

$y = y_c$

$= c_1e^{-x} + c_2e^{2x} + c_3e^{3x}$ Ans:

$m = -1$

$(m+1) = 0$

$m^3 - 4m^2 + m + 6 = 0$

$\Rightarrow m^3 + m^2 - 5m^2 - 5m + 6m + 6$

$\Rightarrow m^2(m+1) - 5m(m+1) + 6(m+1)$

$\Rightarrow (m+1)(m^2 - 5m + 6) = 0$

(4) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Soln: Auxiliary soln $\Rightarrow$

$e^{mx}(m^2 - 6m + 9) = 0$

$e^{mx} \neq 0 \quad \therefore m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\therefore m_1 = 3 \; ; m_2 = 3$

$y_c = C_1 e^{3x} + C_2 x e^{3x}$ Ans:

(5) $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

Soln: Auxiliary soln $\Rightarrow e^{mx}(m^3 - 4m^2 - 3m + 18) = 0$

$e^{mx} \neq 0 \; \therefore m^3 - 4m^2 - 3m + 18 = 0$

$m^2(m+2) - 6m(m+2) + 9(m+2) = 0$

$\Rightarrow (m+2)(m^2 - 6m + 9) = 0$

$\therefore m = -2, \; m_2 = 3 \; ; m_3 = 3$

$y_c = C_1 e^{-2x} + C_2 e^{3x} + C_3 x e^{3x}$ Ans:

For Repeated root

$y_c = C_1 e^{mx} + C_2 x e^{mx} + C_3 x^2 e^{mx} + \cdots C_n x^{n-1} e^{mx}$

$\boxed{\text{Example 7}}$

$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Here auxiliary equation is $m^2 - 8m + 15 = 0$

$(m-3)(m-5) = 0$

$\therefore m = 3, 5$

Hence the required solution is

$y = \cancel{C_1 e^{mx}} \; C_1 e^{3x} + C_2 e^{5x}$

[Q] $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$

Here auxiliary equation is, $m^2 - 6m + 9 = 0$

$\Rightarrow (m-3)^2 = 0$

$\Rightarrow m = 3, 3$

Hence, the required solution is,

$y = (c_1 + c_2 x)e^{3x}$

Ans:

$\boxed{\text{Exercise-3.1B}}$

1. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Here auxiliary equation is, $e^{mx}(m^2 - 3m + 2) = 0$

$e^{mx} \neq 0$;

$m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\Rightarrow m_1 = 2, m_2 = 1$

$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{2x} + c_2 e^{x}$

Ans:

2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 30y = 0$

Here auxiliary equation is, $e^{mx}(m^2+m-30) = 0$

$e^{mx} \neq 0$; $m^2+m-30=0$

$\Rightarrow m^2+6m-5m-30=0$

$\Rightarrow m(m+6)-5(m+6)=0$

$\Rightarrow (m+6)(m-5)=0$

$\Rightarrow m_1 = 5$, $m_2 = -6$

$y_c = c_1 e^{5x} + c_2 e^{-6x}$ Ans:

(3) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 16y = 0$

Here auxiliary equation is, $e^{mx}(m^2-8m+16) = 0$

$\therefore e^{mx} \neq 0$; $m^2-8m+16=0$

$\Rightarrow m^2-4m-4m+16=0$

$\Rightarrow m(m-4)-4(m-4)=0$

$\Rightarrow (m-4)(m-4)=0$

$\Rightarrow m = 4$, $m_2 = 4$

$y_c = c_1 e^{4x} + c_2 x e^{4x}$

$= (c_1 + c_2 x) e^{4x}$ Ans:

Case III :

For complex root, $m = \alpha \pm i\beta$

$$y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

* $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 25y = 0$

Soln :

Given that,

$$y'' - 6y' + 25y = 0 \quad \text{--- (i)}$$

Let the trail solution be $y = e^{mx}$

$$\therefore y' = me^{mx}$$

$$y'' = m^2 e^{mx}$$

Auxilary solution is,

$$e^{mx}(m^2 - 6m + 25) = 0$$

As $e^{mx} \neq 0$, $m^2 - 6m + 25 = 0$

$ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = 3 \pm 4i$$

Here $\alpha = 3, \beta = 4$

Complementary function is

$$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$= e^{3x}(c_1 \cos 4x + c_2 \sin 4x) \quad \text{Ans.}$$

Q) $\frac{d^2y}{dx^2} + y = 0$ —— (1)

Soln: $y'' + y = 0$ —— (1)

Let the auxilary solution is

$e^{mx}(m^2+1) = 0$

$e^{mx} \neq 0$; $m^2+1 = 0$

$\Rightarrow m^2 = -1 = i^2$

$\Rightarrow m = \pm i$

$= 0 \pm i.1$

Here, $\alpha = 0, \beta = 1$

Thus the complementary solution is

$y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$= e^{0x}(c_1 \cos x + c_2 \sin x)$

$= c_1 \cos x + c_2 \sin x$ Ans:

Particular Integral (PI)

* If $R(x) = e^{ax}$, then $PI = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, f(a) \neq 0$

Q) $(D^2+D+5)y = e^{-2x}$, $D = \frac{d}{dx}$

Soln: $(D^2+D+5)y = e^{-2x}$

For complementary function,

$D^2y + Dy + 5y = 0$ —— (1)

Trail sol^n is $y = e^{mx}$

$Dy = me^{mx}$

$D^2y = m^2 e^{mx}$

(i) becomes

$$e^{mx}(m^2+m+5)=0$$

$e^{mx} \neq 0$; $m^2+m+5=0$

$$m = -\frac{1}{2} \pm \frac{\sqrt{19}}{2} i$$

Here $\alpha = -\frac{1}{2}$, $\beta = \frac{\sqrt{19}}{2}$

$$y_c = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{19}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{19}}{2}x\right)\right)$$

$$y_p = \frac{1}{D^2+D+5} e^{-2x}$$

$$= \frac{1}{(-2)^2+(-2)+5} e^{-2x}$$

$$= \frac{1}{7} e^{-2x}$$

Complete solution,

$$y = y_c + y_p$$

$$= e^{-\frac{x}{2}}\left[c_1 \cos\left(\frac{\sqrt{19}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{19}}{2}x\right)\right] + \frac{1}{7}e^{-2x}$$

Ans.

(ii) If $f(a) = 0$, then $\frac{1}{f(D)} e^{ax} = x \frac{1}{f'(a)} e^{ax}$

If $f'(a) = 0$, then $\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax}$

(*) $y'' + 6y' + 9y = 5e^{3x}$

Soln: For complementary function,

$$y'' + 6y' + 9y = 0$$

Auxilary equation is $m^2 + 6m + 9 = 0$

$$\Rightarrow (m+3)^2 = 0$$

$$\Rightarrow m = -3, -3$$

$$y_c = (c_1 + c_2 x) e^{-3x}$$

$$= c_1 e^{-3x} + c_2 e^{-3x} . x$$

$$PI = \frac{1}{D^2 + 6D + 9} 5e^{3x}$$

$$y_p = 5 \frac{1}{D^2 + 6D + 9} e^{3x}$$

$$= 5 \frac{1}{3^2 + 6.3 + 9} . e^{3x}$$

$$= 5 \frac{1}{36} e^{3x}$$

Complete solution, $Y = Y_c + Y_p$

$$= c_1 e^{-3x} + c_2 e^{-3x} . x + \frac{5}{36} e^{3x}$$

Ans.

H.K Das

3.20 → 1, 2, 3, 6, 7, 8, 9

$y'' - 3y' + 2y = e^{3x}$.

$y'' - 6y' + 9y = 6e^{3x} + 7e^{-2x} - \log 2$

For complementary ~~solution~~ function

$$y'' - 6y' + 9y = 0$$

Let the trail solution be $y = e^{mx}$

$$\Rightarrow y' = m e^{mx}$$

$$y'' = m^2 e^{mx}$$

Auxiliary eq^n is $m^2 - 6m + 9 = 0$

$$\Rightarrow (m-3)^2 = 0$$

$$\Rightarrow m = 3, 3$$

$$Y_c = (c_1 + c_2 x) e^{3x}$$

$$P.I = \frac{1}{D^2 - 6D + 9} 6e^{3x} + \frac{1}{D^2 - 6D + 9} 7e^{-2x} + \frac{1}{D^2 - 6D + 9}(-\log 2)$$

$$= x \frac{1}{2D - 6} 6e^{3x} + \frac{1}{4 + 12 + 9} 7e^{-2x} - \log 2 \frac{1}{D^2 - 6D + 9} e^{0x}$$

$$= x^2 \cdot \frac{1}{2} \cdot 6e^{3x} + \frac{7}{25} e^{-2x} - \log 2\left(\frac{1}{9}\right)$$

$$= 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9} \log 2$$

Complete solution is $Y = Y_c + Y_p$

$$= (c_1 + c_2 x) e^{3x} + 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9} \log 2$$

Ans.

$y'' - 3y' + 2y = e^{3x}$

For complementary function,

$y'' - 3y' + 2y = 0$

Let trail solution be $y = e^{mx}$

$\Rightarrow y' = me^{mx}$

$\Rightarrow y'' = m^2 e^{mx}$

Auxilary eq^n is ~~$m^2 - 3m + 2$~~ $m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\therefore m = 2, 1$

$Y_c = C_1 e^{x} + C_2 e^{2x}$

$PI = \dfrac{1}{D^2 - 3D + 2} e^{3x}$

$= \dfrac{1}{3^2 - 3 + 2} e^{3x}$

$= \dfrac{1}{2} e^{3x}$

The complete solution is $Y = Y_c + Y_P$

$= C_1 e^{x} + C_2 e^{2x} + \dfrac{e^{3x}}{2}$ Ans.

$\boxed{3.24}$ 4, 5, 6, 7, 8, 9, $\boxed{10}$

If $f(-a^2)=0$ then above rule fails.

$$\frac{1}{f(D^2)}\sin ax = x\frac{\sin ax}{f'(-a^2)}$$

If $f'(-a^2)=0$ then, $\frac{1}{f(D^2)}\sin ax = x^2\frac{\sin ax}{f''(-a^2)}$

$\boxed{54}$ $(D^2+4)y = \cos 2x$

Auxiliary equation is $m^2+4=0$

$$\Rightarrow m = \pm 2i$$

$C.F = A\cos 2x + B\sin 2x$

$$P.I = \frac{1}{D^2+4}\cos 2x$$

$$= x\cdot\frac{1}{2D}\cos 2x$$

$$= x\cdot\frac{1}{2}\frac{\sin 2x}{2}$$

$$= \frac{x}{4}\sin 2x$$

Complete solution is $y = A\cos 2x + B\sin 2x + \frac{x}{4}\sin 2x$

Ans.

55. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$

Given, A.E is $m^3 - 3m^2 + 4m - 2 = 0$

$$\Rightarrow (m-1)(m^2-2m+2) = 0$$

$$m = 1,\ 1\pm i$$

$C.F = c_1e^x + e^x(c_2\cos x + c_3\sin x)$

$$(D-1)(D^2-2D+2)$$
$$= D^3-2D^2+2D-D^2+2D-2$$
$$= D^3-3D^2+4D-2$$

$$P.I. = \frac{1}{(D-1)(D^2-2D+2)}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x$$

$$= \frac{1}{(D-1)(1-2+2)}e^x + \frac{1}{(-1)D-3(-1)+4D-2}\cos x$$

$$= \frac{1}{(D-1)}e^x + \frac{1}{3D+1}\cos x$$

$$= x\frac{1}{1}e^x + \frac{3D-1}{9D^2-1}\cos x$$

$$= xe^x + \frac{(3D-1)}{9(-1^2)-1}\cos x$$

$$= xe^x + \frac{3D-1}{-10}\cos x$$

$$= xe^x + \frac{-3\sin x - \cos x}{-10}$$

$$= xe^x + \frac{1}{10}(3\sin x + \cos x)$$

Here complete solution is $= c_1e^x + (c_2\cos x + c_3\sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x)$ Ans:

$\boxed{3.22}$ Homework: 1, 2, 3, 5

Particular Integral for the form,

(1) $Y f(D) = x^n$

$\Rightarrow Y = \frac{1}{f(D)} x^n = [f(D)]^{-1} x^n$

(2) $Y f(D) = e^{ax} \cdot \phi(x)$

$\therefore Y = e^{ax} \frac{1}{f(D+a)} \phi(x)$

Ex: $(D^2 - 4D + 4)Y = x^3 e^{2x}$

Soln: For the complementary function,

$(D^2 - 4D + 4)Y = 0$

The auxiliary equation is

$m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = 2, 2$

$y_c = c_1 e^{2x} + c_2 e^{2x} \cdot x$

$$P.I. = \frac{1}{D^2 - 4D + 4} x^3 e^{2x}$$

$$= e^{2x} \cdot \frac{1}{(D+2)^2 - 4(D+2) + 4} x^3$$

$$= e^{2x} \cdot \frac{1}{D^2} x^3$$

$$= e^{2x} \cdot \frac{1}{D} \left[x^4/4 \right]$$

$$= e^{2x} \cdot \frac{x^5}{20}$$

Complete solution is,

$y = y_c + y_p$

$= (c_1 + c_2 x)e^{2x} + e^{2x} \cdot \frac{x^5}{20}$ Ans:

Ex-2: $y''' - 7y'' + 10y' = e^{2x} \sin x$

Soln: Given that,

$D^3 y - 7D^2 y + 10Dy = e^{2x} \sin x$

For complementary solution,

$D^3 y - 7D^2 y + 10Dy = 0$

Auxiliary equation is,

$m^3 - 7m^2 + 10m = 0$

$\Rightarrow m(m-2)(m-5) = 0$

$\therefore m = 0, 2, 5$

$y_c = c_1 + c_2 e^{2x} + c_3 e^{5x}$

$D^2 = -a^2$

$P.I. = \frac{1}{D^3 - 7D^2 + 10D} e^{2x} \sin x$

$= e^{2x} \frac{1}{(D+2)^3 - 7(D+2)^2 + 10(D+2)} \sin x$

$= e^{2x} \frac{1}{D^3 - D^2 - 6D} \sin x$ $D^2 \cdot D - D^2 - 6D$

$= e^{2x} \frac{1}{(-1)D - (-1) - 6D} \sin x$ $(-1)D - (-1) - 6D$

$$= e^{2x} \frac{1}{-D+1-6D} \sin x$$

$$= e^{2x} \frac{1}{1-7D} \sin x$$

$$= e^{2x} \frac{1+7D}{(1-7D)(1+7D)} \sin x$$

$$= e^{2x} \frac{1+7D}{1-49D^2} \sin x$$

$$= e^{2x} \frac{1+7D}{50} \sin x$$

$$= e^{2x}/50 (\sin x + 7\cos x)$$

Complete solution is $y = y_c + y_p$

$$= c_1 + c_2 e^{2x} + c_3 e^{5x} + \frac{e^{2x}}{50}(\sin x + 7\cos x)$$

Ans

$$(D^2+6D+9)y = \frac{e^{-3x}}{x^3}$$

Auxiliary equation is $m^2+6m+9=0$

$$\Rightarrow (m+3)^2 = 0$$

$$\Rightarrow m = -3, -3$$

$$y_c = (c_1 + c_2 x)e^{-3x}$$

$$P.I = \frac{1}{D^2+6D+9} \frac{e^{-3x}}{x^3}$$

$$= e^{-3x} \frac{1}{(D-3)^2+6(D-3)+9} \frac{1}{x^3}$$

$$= e^{-3x} \frac{1}{D^2-6D+9+6D-18+9} \frac{1}{x^3}$$

$$= e^{-3x} \frac{1}{D^2} \frac{1}{x^3}$$

$$= e^{-3x} \frac{1}{D} \left(\frac{x^{-2}}{-2}\right)$$

$$= e^{-3x} \frac{x^{-1}}{(-2)(-1)}$$

$$= e^{-3x} \frac{x^{-1}}{2} = \frac{e^{-3x}}{2x}$$

Complete solution is, $Y = Y_c + Y_P$

$$= (C_1 + C_2 x)e^{-3x} + \frac{e^{-3x}}{2x}.$$

H.W. $(D^2 - 4)Y = x^2 e^{3x}$

H.W. $(D^2 - 3D + 2)Y = 2x^2 e^{4x} + 5e^{3x}$

$$\frac{1}{f(D)}x^n = [f(D)]^{-1}x^n$$

① $(D^2+D-2)y = 2(1+x-x^2)$

$\Rightarrow$ For complementary function,

$$(D^2+D-2)y = 0$$

Auxiliary equation is

$$e^{mx}(m^2+m-2) = 0$$

$e^{mx} \neq 0$ $\quad\quad$ $m^2+m-2 = 0$

$$\Rightarrow m = 1, -2$$

$$y_c = c_1e^{x} + c_2e^{-2x}$$

Particular Integral,

$$y_p = \frac{1}{D^2+D-2}\,2(1+x-x^2)$$

$$= \frac{1}{-2\left[1-\left(\frac{D^2+D}{2}\right)\right]}\,2(1+x-x^2)$$

$$= -\left[1-\left(\frac{D^2+D}{2}\right)\right]^{-1}(1+x-x^2)$$

$$= -1\left[1+\frac{D+D^2}{2}+\frac{(D+D^2)^2}{2}+\cdots\right](1+x-x^2)$$

$$= -1\left[1+\frac{D}{2}+\frac{D^2}{2}+\frac{D^2+2D^3+D^4}{4}+\cdots\right](1+x-x^2)$$

$$= -1\left[1+\frac{D}{2}+\frac{3}{4}D^2+\cdots\right](1+x-x^2)$$

$$= -1\left[(1+x-x^2)+\frac{1}{2}(1-2x)+\frac{3}{4}(-2)\right]$$

$$= -\left[1 + x - x^2 + \frac{1}{2} - x - \frac{3}{2}\right]$$

$$= -(-x^2)$$

$$= x^2$$

Complete solution:

$$Y = Y_c + Y_p$$

$$= c_1 e^{x} + c_2 e^{-2x} + x^2$$ Ans:

(4) $(D^3 - D^2 - 6D)y = x^2 + 1$

For complementary function,

$$(D^3 - D^2 - 6D)y = 0$$

Auxiliary solution is,

$$e^{mx}(m^3 - m^2 - 6m) = 0$$

$e^{mx} \neq 0$; $\quad m^3 - m^2 - 6m = 0$

$$\Rightarrow m(m^2 - m - 6) = 0$$

$$\therefore m = 0, -2, 3$$

$$Y_c = c_1 e^{0} + c_2 e^{-2x} + c_3 e^{3x}$$

Particular Integral,

$$Y_p = \frac{1}{D^3 - D^2 - 6D}(x^2 + 1)$$

$$= \frac{1}{-6D\left[1 + \left(\frac{D - D^2}{6}\right)\right]}(x^2 + 1)$$

$$= \frac{1}{-6D}\left[1+\frac{D-D^2}{6}\right]^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1-\frac{D-D^2}{6}+\left(\frac{D-D^2}{6}\right)^2 \cdots\right](x^2+1)$$

$$= -\frac{1}{6D}\left(1-\frac{D}{6}-\frac{D^2}{36}-\frac{D^3}{18}+\frac{D^4}{36}\cdots\right)(x^2+1)$$

$$= -\frac{1}{6D}\left(1-\frac{D}{6}+\frac{7D^2}{36}\right)(x^2+1)$$

$$= -\frac{1}{6D}\left[(x^2+1)-\frac{1}{6}(2x)+\frac{7}{36}\cdot 2\right]$$

$$= -\frac{1}{6D}\left(x^2+1-\frac{1}{3}x+\frac{7}{18}\right)$$

$$= -\frac{1}{6D}\left(x^2-\frac{1}{3}x+\frac{25}{18}\right)$$

$$= -\frac{1}{6}\left(\frac{x^3}{3}-\frac{1}{3}\cdot\frac{x^2}{2}+\frac{25}{18}x\right)$$

$$= -\frac{x^3}{18}+\frac{x^2}{36}-\frac{25}{108} \quad \text{Ans:}$$

The complete solution is

$$y = y_c + y_p$$

$$= c_1 + c_2 e^{-2x} + c_3 e^{3x} - \frac{x^3}{18} + \frac{x^2}{36} - \frac{25x}{108} \quad \text{Ans:}$$

Laplace Transformation:

Let $F(t)$ be a function of t specified for $t>0$. Then the laplace Transformation of $F(t)$ denoted by $\mathcal{L}\{F(t)\}$ is defined by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt,$$

where we assume the parameter s is real

$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1. c	$\frac{c}{s}$
2. t^n	$\frac{n!}{s^{n+1}}$
3. e^{at}	$\frac{1}{s-a}$
4. $\sin at$	$\frac{a}{s^2+a^2}$
5. $\cos(at)$	$\frac{s}{s^2+a^2}$
6. $\sinh(at)$	$\frac{a}{s^2-a^2}$
7. $\cosh(at)$	$\frac{s}{s^2-a^2}$

Prove that, $\mathcal{L}\{c\} = \frac{c}{s}$

Sol^n: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

Here $F(t) = c$

$$\therefore \mathcal{L}\{c\} = \int_0^\infty e^{-st}.c\,dt$$

$$= c\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{c}{s}\left[-e^{-s\infty} + e^{0}\right]$$

$$= \frac{c}{s}(0+1) = \frac{c}{s}$$ Ans.

Prove that, $\mathcal{L}\{t\} = \frac{1}{s^2}$

soln: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t)\,dt$$

Here, $F(t) = t$

$$\left[\begin{array}{l} \text{LIATE} \rightarrow v? \\ \int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx \end{array}\right.$$

$$\therefore \mathcal{L}\{t\} = \left[-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$= \frac{-te^{-\infty}}{s} - \frac{e^{-\infty}}{s^2} + \frac{0.e^{0}}{s} + \frac{e^{0}}{s^2}$$

$$= \frac{1}{s^2}$$

Ans.

$e^0 = 1$, $e^\infty = \infty$, $e^{-\infty} = 0$.

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* Prove that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

sol^n: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

Here, $F(t) = e^{at}$

$$\therefore \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} . e^{at} dt$$

$$= \int_0^\infty e^{-t(s-a)} dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= \frac{e^{-\infty}}{-(s-a)} + \frac{e^0}{s-a}$$

$$= \frac{1}{s-a}$$

Ans:

$\int e^{ax}\sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$

$\int e^{ax}\cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$

$L\{u(t-a)\} = \frac{e^{-as}}{s}$

* Prove that, $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$

Solution: By defination of LT,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$$

Here, $F(t) = \sin at$

$$\therefore \mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st} . \sin(at)dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}(-s\sin at - a\cos at)\right]_0^\infty$$

$= \frac{1}{s^2+a^2}(0-0+0+a)$

$= \frac{a}{s^2+a^2}$ Ans:

Prove that $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$

soln: By defination of LT,

$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$

Here $F(t) = \cos(at)$

$\therefore \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st} \cdot \cos(at)\, dt$

$= \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$

$= \frac{1}{s^2+a^2}(s+0)$

$= \frac{s}{s^2+a^2}$ Ans:

Prove that, $\mathcal{L}\{\cosh(at)\} = \frac{s}{s^2-a^2}$

Solution: By defination of LT,

$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)dt$

Here, $F(t) = \cosh(at)$

$\cosh(at) = \frac{e^{at}+e^{-at}}{2}$, $\sinh(at) = \frac{e^{at}-e^{-at}}{2}$

$\therefore \mathcal{L}\{\cosh(at)\} = \int_0^\infty e^{-st} \cosh(at)dt$

$$= \int_0^{\infty} \frac{e^{at}+e^{-at}}{2} \cdot e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} \left(e^{-t(s-a)} + e^{-t(s+a)}\right) dt$$

$$= \frac{1}{2}\left(\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} + \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}\right)$$

$$= \frac{1}{2}\left(0 + \frac{1}{s-a} + 0 + \frac{1}{s+a}\right)$$

$$= \frac{1}{2} \frac{s-a+s+a}{(s-a)(s+a)}$$

$$= \frac{1}{2} \cdot \frac{2s}{s^2-a^2}$$

$$= \frac{s}{s^2-a^2}$$ Ans:

H.W Prove that $\sinh(at)$ Homework)

Prove that,

$$\mathcal{L}\,\sinh(at) = \frac{1}{s^2-a^2}$$

Solution: By definition of LT,

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}\, F(t)\,dt$$

Here, $F(t) = \cosh(at)$

$$\cancel{\cosh(a}\ \cosh(at) = \frac{e^{at}+e^{-at}}{2}, \quad \sinh(at) = \frac{e^{at}-e^{-at}}{2}$$

$$\mathcal{L}\{\sinh(at)\} = \int_0^\infty e^{-st}\,\sinh(at)\,dt$$

$$= \int_0^\infty e^{-st}\,\frac{e^{at}-e^{-at}}{2}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st}.e^{at} - e^{-st}e^{-at}\,dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t(s-a)} - e^{-t(s+a)}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty - \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left(-0 + \frac{1}{s-a} + 0 - \frac{1}{s+a}\right)$$

$$= \frac{1}{2}\,\frac{s+a-s+a}{(s-a)(s+a)}$$

$$= \frac{1}{2}\,\frac{2a}{(s-a)(s+a)}$$

$$= \frac{a}{s^2-a^2}$$

Ans:

Properties:

Linearity property:

If c_1, c_2 are any constants, while $F_1(t), F_2(t)$ are functions with LTs $f_1(s), f_2(s)$ respectively, then

$$\mathcal{L}\{c_1F_1(t)+c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\}+c_2\mathcal{L}\{F_2(t)\}$$
$$= c_1 f_1(s) + c_2 f_2(s)$$

① $\mathcal{L}(5e^{-t}+2\cos 5t-3\}$

$$= 5\mathcal{L}(e^{-t}) + 2\mathcal{L}(\cos 5t) - \mathcal{L}(3)$$
$$= 5\cdot\frac{1}{s+1} + 2\cdot\frac{s}{s^2+25} - \frac{3}{s}$$
$$= \frac{5}{s+1} + \frac{2s}{s^2+25} - \frac{3}{s}$$

② $\mathcal{L}\{(t^2+1)^2\} = \mathcal{L}\{t^4+2t^2+1\}$

$$= \mathcal{L}\{t^4\} + 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\}$$
$$= \frac{4!}{s^5} + 2\frac{2!}{s^3} + \frac{1}{s}$$

Ans:

③ First Translation or shifting property:-

If $\mathcal{L}\{F(t)\} = f(s)$, then $\mathcal{L}(e^{at}F(t)\} = f(s-a)$

① $\mathcal{L}\{e^{-2t}\cos 5t\}$

soln: $\mathcal{L}\{\cos 5t\} = \frac{s}{s^2+25}$

$$\mathcal{L}\{e^{-2t}\cdot\cos 5t\} = \frac{s-(-2)}{\{s-(-2)\}^2+25}$$ [replace s by s-2]

$$= \frac{s+2}{(s+2)^2+25}$$

② $\mathcal{L}\{e^{5t} \cdot \sin t\}$

$\therefore \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$\mathcal{L}\{e^{5t} \sin t\} = \frac{1}{(s-5)^2+1}$ Ans:

① $\mathcal{L}\{t^3 + \cos 2t + \sinh 3t\}$

② $\{e^{-2t} \cdot t^4\}$ ③ $\{e^{3t} \cdot 3 \cdot \sin t\}$ ④ $\{\cos t \cdot e^{2t}\}$

④ $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$

$\mathcal{L}\{e^{2t} \cos t\} = \frac{s-(-2)}{\quad}$

① $\mathcal{L}\{t^3+\cos 2t+\sinh 3t\}$

$= \mathcal{L}(t^3)+\mathcal{L}(\cos 2t)+\mathcal{L}(\sinh 3t)$

$= \frac{3!}{s^4}+\frac{s}{s^2+4^2}+\frac{a}{s^2-9}$ Ans:

② $\{e^{-2t}\cdot t^4\}$

$\mathcal{L}(t^4)=\frac{4!}{s^5}$ $\mathcal{L}(e^{-2t})=\frac{1}{s+2}$

$\mathcal{L}(e^{-2t}\cdot t^4)=\frac{4!}{(s+2)^5}$

Ans:

③ $\mathcal{L}\{e^{3t}\cdot 3\sin t\}$

$\mathcal{L}(3\sin t)=3\frac{1}{s^2+1}$

$\mathcal{L}\{e^{3t}3\sin t\}=3\frac{1}{(s-3)^2+1}$ Ans:

④ $\{\cos t\cdot e^{2t}\}$

$\mathcal{L}\{\cos t\}=\frac{s}{s^2+1}$

$\mathcal{L}\{e^{2t}\cos t\}=\frac{s}{(s-2)^2+1}$ Ans:

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Second Translation or shifting property:

If $\mathcal{L}\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$

then $\mathcal{L}\{G(t)\} = e^{-as} \cdot f(s)$

① If $G(t) = \begin{cases} (t-2)^3, & t>2, \\ 0, & t<2. \end{cases}$ find $\mathcal{L}\{G(t)\}$

Soln: $\mathcal{L}\{G(t)\} = e^{-2s} \frac{3!}{s^4}$

② If $G(t) = \begin{cases} \sin(t-\frac{\pi}{2}), & t>\frac{\pi}{2} \\ 0, & t<\frac{\pi}{2} \end{cases}$, find $\mathcal{L}\{G(t)\}$

Soln: $\mathcal{L}\{G(t)\} = e^{-\frac{\pi}{2}s} \frac{1}{s^2+1}$

Multiplication by t^n:

If $\mathcal{L}\{F(t)\} = f(s)$; then $\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

① $\mathcal{L}\{t^2 \cos at\}$

Soln: $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$

$\therefore \mathcal{L}\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2} \left(\frac{s}{s^2+a^2} \right)$

$$= \frac{d}{ds} \left\{ \frac{(s^2+a^2)\frac{d}{ds}(s) - s\frac{d}{ds}(s^2+a^2)}{(s^2+a^2)^2} \right\}$$

$$= \frac{d}{ds} \left\{ \frac{s^2+a^2-2s^2}{(s^2+a^2)^2} \right\}$$

$$= \frac{d}{ds} \frac{a^2-s^2}{(s^2+a^2)^2}$$

$$= \frac{(s^2+a^2)^2 \frac{d}{ds}(a^2-s^2) - (a^2-s^2)\frac{d}{ds}(s^2+a^2)^2}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2)^2 - 2(s^2+a^2)\,2s(a^2-s^2)}{(s^2+a^2)^4}$$

$$= \frac{-2s(s^2+a^2) - 4s(a^2-s^2)}{(s^2+a^2)^3}$$

$$= \frac{-2s^3 - 2a^2s - 4a^2s + 4s^3}{(s^2+a^2)^3}$$

$$= \frac{2s^3 - 6a^2s}{(s^2+a^2)^3}$$

(3) $\mathcal{L}\{t^{50} e^{2t}\}$

sol^n: $\mathcal{L}\{t^{50}\} = \frac{50!}{s^{51}}$

$$\mathcal{L}\{t^{50} e^{2t}\} = \frac{50!}{(s-2)^{51}}$$

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(1) $\mathcal{L}\{\cos^2 t\}$

sol^n: we know that,

$$\cos 2t = 2\cos^2 t - 1$$

$$\Rightarrow \cos^2 t = \frac{1}{2}(\cos 2t + 1)$$

$$\mathcal{L}\{\cos^2 t\} = \frac{1}{2}[\mathcal{L}(\cos 2t) + \mathcal{L}(1)]$$

$$= \frac{1}{2}\left[\frac{s}{s^2+4} + \frac{1}{s}\right]$$

② $\mathcal{L}\{t^{-1/2}\}$

we know that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

But n is a fractional number,

$\mathcal{L}\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$n = -\frac{1}{2}$, $\mathcal{L}\{t^{-1/2}\} = \frac{\Gamma(-\frac{1}{2}+1)}{s^{(-\frac{1}{2}+1)}}$

$= \frac{\Gamma(1/2)}{\sqrt{s}} = \frac{\sqrt{\pi}}{\sqrt{s}}$

③ $\mathcal{L}\{\sin t \cos t\}$

soln: $\frac{1}{2} \cdot 2 \sin t \cos t$

$= \frac{1}{2} \sin 2t$

$\mathcal{L}(\frac{1}{2} \sin 2t)$

$= \frac{1}{2} \mathcal{L}\{\sin 2t\}$

$= \frac{1}{2} \frac{2}{s^2+4}$

$= \frac{1}{s^2+4}$ Ans:

⑤ $\mathcal{L}\{\sin 2t \cos 3t\}$

we know that,

$2 \sin x \cos y = \sin(x+y) + \sin(x-y)$

$\sin 2t \cos 3t = \frac{1}{2}[2 \sin 2t . \cos 3t]$

$= \frac{1}{2}[\sin(2t+3t) + \sin(2t-3t)]$

$= \frac{1}{2}[\sin 5t - \sin t]$

④ $\mathcal{L}\{\sin^3 2t\}$

we know that,

$\sin 3A = 3 \sin 3A - 4 \sin^3 A$

$\sin^3 A = \frac{1}{4}(3 \sin A - \sin 3A)$

$\Rightarrow \sin^3 2t = \frac{1}{4}(3 \sin 2t - \sin 6t)$

$\mathcal{L}\{\sin^3 2t\} = \mathcal{L}\{\frac{1}{4}(3 \sin 2t - \sin 6t)\}$

$= \frac{1}{4}\{3 \frac{2}{s^2+4} - \frac{6}{s^2+36}\}$

$= \frac{1}{2}\left[\frac{3}{2(s^2+4)} - \frac{3}{2(s^2+36)}\right]$

$= \frac{3}{2}\left[\frac{s^2+36-s^2-4}{(s^2+4)(s^2+36)}\right]$

$= \frac{3}{2} \frac{32}{(s^2+4)(s^2+36)}$

$= \frac{48}{(s^2+4)(s^2+36)}$

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$$\mathcal{L}\{\ \} = \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{4s^2-20}{(s^2+25)(s^2+1)}\right]$$

$$= \frac{2(s^2-5)}{(s^2+25)(s^2+1)}$$

Ans:

LT of Integral

$\mathcal{L}\{\int_0^t f(t)\,dt\} = \frac{1}{s} f(s)$

where $\mathcal{L}\{F(t)\} = f(s)$.

LT of division by t / $(\frac{1}{t} f(t))$

If $\mathcal{L}\{f(t)\} = f(s)$, then

$\mathcal{L}\{\frac{1}{t} f(t)\} = \int_s^\infty f(s)\,ds$

$\mathcal{L}\{\frac{\sin 2t}{t}\}$

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

soln: $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$

$\mathcal{L}\{\frac{\sin 2t}{t}\} = \int_s^\infty \frac{2}{s^2+4}\,ds$

$= 2 \cdot \frac{1}{2}\left[\tan^{-1}\frac{s}{2}\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\frac{s}{2}$

$= \tan^{-1}(\tan\frac{\pi}{2}) - \tan^{-1}\frac{s}{2} = \frac{\pi}{2} - \tan^{-1}\frac{s}{2}$

$= \cot^{-1}\frac{s}{2}$ Ans:

(ii) $\mathcal{L}\{\int_0^t \frac{\sin t}{t}\,dt\}$

$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$

$\mathcal{L}\{\frac{\sin t}{t}\} = \int_s^\infty \frac{1}{s^2+1}\,ds$

$= \left[\tan^{-1}\frac{s}{1}\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}s$

$= \tan^{-1}(\tan\frac{\pi}{2}) - \tan^{-1}s$

$= \frac{\pi}{2} - \tan^{-1}s$

$= \cot^{-1}s.$

$\therefore \mathcal{L}\{\int_0^t \frac{\sin t}{t} dt\} = \frac{1}{s}\cot^{-1}s$

4) $\mathcal{L}\{e^{-4t}\frac{\sin 3t}{t}\}$

Soln: $\mathcal{L}\{\sin 3t\} = \frac{3}{s^2+9}$

$\mathcal{L}\{\frac{\sin 3t}{t}\} = \int_s^\infty \frac{3}{s^2+9} ds$

$= 3 \cdot \frac{1}{3}\left[\tan^{-1}\frac{s}{3}\right]_s^\infty$

$= \tan^{-1}\infty - \tan^{-1}\frac{s}{3}$

$= \tan^{-1}(\tan\frac{\pi}{2}) - \tan^{-1}\frac{s}{3}$

$= \frac{\pi}{2} - \tan^{-1}\frac{s}{3}$

$= \cot^{-1}\frac{s}{3}$

$\mathcal{L}\{e^{-4t}\frac{\sin 3t}{t}\}$

$= \cot^{-1}\frac{s-(-4)}{3}$

$= \cot^{-1}\frac{s+4}{3}$

(1) $\mathcal{L}\left\{\frac{1}{t}(1-e^{t})\right\}$

$\mathcal{L}.\{1-e^{t}\}$

$= \frac{1}{s} - \frac{1}{s-1}$

$\mathcal{L}.\left\{\frac{1}{t}(1-e^{t})\right\}$

$= \int_s^\alpha \left(\frac{1}{s} - \frac{1}{s-1}\right) ds$

$= \left[\log s - \log(s-1)\right]_s^\alpha$

$= -\log s + \log(s-1)$

$= \log \frac{s-1}{s}$. Ans.

(*) $\mathcal{L}\left\{\frac{1}{t}(e^{-at} - e^{-bt})\right\}$

soln: $\mathcal{L}\{e^{-at} - e^{-bt}\} = \frac{1}{s+a} - \frac{1}{s+b}$

$\mathcal{L}\left\{\frac{1}{t}(e^{-at} - e^{-bt})\right\}$

$= \int_s^\alpha \left(\frac{1}{s+a} - \frac{1}{s+b}\right) ds$

$= \left[\log(s+a) - \log(s+b)\right]_s^\alpha$

$= -\log(s+a) + \log(s+b)$

$= \log \frac{s+b}{s+a}$ Ans.

LT of periodic function:

Let $F(t)$ be a periodic function of period T

so, $\mathcal{L}\{F(t)\} = \dfrac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt$.

① $F(t) = \begin{cases} 2t^2, & 0<t<1 \\ t, & 1<t<2 \end{cases}$ is a periodic function of period 2. Find $\mathcal{L}\{F(t)\}$

Sol^n: $\mathcal{L}\{F(t)\} = \dfrac{1}{1-e^{-2s}} \int_0^2 e^{-st} F(t)\,dt$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot 2t^2\,dt + \int_1^2 e^{-st} \cdot t\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[2\int_0^1 e^{-st} t^2\,dt + \int_1^2 e^{-st} \cdot t\,dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[2\left[\frac{t^2 e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3} \right]_0^1 + \left[\frac{-te^{-st}}{s} - \frac{e^{-st}}{s^2} \right]_1^2 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[2\left[\frac{e^{-s}}{s} - \frac{2e^{-s}}{s^2} - \frac{2e^{-s}}{s^3} + \frac{2}{s^3} \right] + \left[\frac{-2e^{-2s}}{s} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2} \right] \right]$$

$$= \frac{1}{1-e^{-2\pi s}(s^2+1)} \left[-se^{-st}\sin t - e^{-st}\cos t \right]_0^\pi$$

Ans:

$$= \frac{1}{1-e^{-2\pi s}(s^2+1)} \left[-e^{-\pi s}\cos\pi + e^0 \cos 0 \right]$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)} \left(e^{-\pi s} + 1 \right)$$

Ans:

LT of Periodic Function:

let $F(t)$ be a periodic function of period T

so, $\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} F(t)\,dt$

① $F(t) = \begin{cases} 2t^2, & 0<t<1 \\ t, & 1<t<2 \end{cases}$ is a periodic function of period 2. Find $\mathcal{L}\{F(t)\}$

Sol^n: $\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-2s}}\int_0^2 e^{-st} F(t)\,dt$

$$= \frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}\cdot 2t^2\,dt + \int_1^2 e^{-st}\, t\, dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[2\int_0^1 e^{-st}\cdot t^2\,dt + \int_1^2 e^{-st}\, t\, dt\right]$$

$$= \frac{1}{1-e^{-2s}}\left[2\left[t^2\int e^{-st}dt - \int\left(\frac{d}{dt}t^2\int e^{-st}dt\right)dt\right]_0^1 + \left[t\int e^{-st}dt - \int\left(\frac{d}{dt}t\int e^{-st}dt\right)dt\right]_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[2\left[t^2\frac{e^{-st}}{-s} - \left(\int 2t\,\frac{e^{-st}}{-s}\,dt\right)\right]_0^1 + \left[t\frac{e^{-st}}{-s} - \int\left(1\cdot\frac{e^{-st}}{-s}\right)dt\right]_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[2\left[\frac{t^2e^{-st}}{-s} + \frac{2}{s}\left(t\int e^{-st}dt - \int\left(\frac{d}{dt}t\int e^{-st}dt\right)dt\right)\right]_0^1 + \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[2\left[\frac{t^2e^{-st}}{-s} + \frac{2}{s}\left(t\,\frac{e^{-st}}{-s} - \int 1\,\frac{e^{-st}}{-s}\,dt\right)\right]_0^1 + \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[2\left[\frac{t^2e^{-st}}{-s} - \frac{2te^{-st}}{s^2} - \frac{2e^{-st}}{s^3}\right]_0^1 + \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_1^2\right]$$

$$= \frac{1}{1-e^{-2s}}\left[2\left[\frac{-e^{-s}}{s} - \frac{2e^{-s}}{s^2} - \frac{2e^{-s}}{s^3} + \frac{2}{s^3}\right] + \left[\frac{-2e^{-2s}}{s} + \frac{e^{-s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-s}}{s^2}\right]\right]$$

Ans.

Ex. $F(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$ is a periodic function of period 2π.

Find $\mathcal{L}\{F(t)\}$

Soln: $\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\,dt$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} \cdot 0\,dt$$

$$= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\,dt$$

$$= \frac{1}{1-e^{-2\pi s}(s^2+1)} \Big[-se^{-st}\sin t - e^{-st}\cos t\Big]_0^{\pi}$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)} \Big[-e^{-\pi s}\cos\pi + e^{0}\cos 0\Big]$$

$$= \frac{1}{(1-e^{-2\pi s})(s^2+1)} (e^{-\pi s}+1)$$ Ans.

Table for Inverse Laplace Transformation

$f(s)$		$\mathcal{L}^{-1}\{f(s)\} = F(t)$
01. $\frac{c}{s}$	$\longrightarrow$	c
02. $\frac{1}{s-a}$	$\longrightarrow$	e^{at}
03. $\frac{1}{s^{n+1}}$	$\longrightarrow$	$\frac{t^n}{n!}$
04. $\frac{1}{s^2+a^2}$	$\longrightarrow$	$\frac{\sin(at)}{a}$
05. $\frac{s}{s^2+a^2}$	$\longrightarrow$	$\cos(at)$
06. $\frac{1}{s^2-a^2}$	$\longrightarrow$	$\frac{\sinh(at)}{a}$
07. $\frac{s}{s^2-a^2}$	$\longrightarrow$	$\cosh(at)$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t}$

Ex: $\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}$

* $\mathcal{L}^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin(2t)}{5}$

* $\mathcal{L}^{-1}\left\{\frac{s}{s^2+16}\right\} = \cos(4t)$

* $\mathcal{L}^{-1}\left\{\frac{1}{s^2-25}\right\} = \frac{\sinh(5t)}{5}$

* $\mathcal{L}^{-1}\left\{s^2-16\right\} = \cosh(4t)$

PROPERTIES

01. Linearity Property:

$$\mathcal{L}^{-1}\left\{2\frac{1}{s^5}-\frac{s}{s^2+25}+\frac{1}{s-2}+\frac{5}{s}\right\}$$

$$=2\frac{t^4}{4!}-\cos(5t)+e^{2t}+5$$ Ans:

First Translation of shifting Property:

If $\mathcal{L}^{-1}\{f(s)\}=F(t)$, then $\mathcal{L}^{-1}\{f(s-a)\}=e^{at}F(t)$

01. $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$$=\mathcal{L}^{-1}\left\{\frac{6s-4}{(s-2)^2+16}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$$=6\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\}+8\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2+16}\right\}$$

$$=6\cos 4t\,e^{2t}+8\frac{\sin 4t}{4}e^{2t}$$

$$\cancel{=6\cos}=6e^{2t}\cos 4t+2e^{2t}\sin 4t$$ Ans:

02. $\mathcal{L}^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$

$$=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2+4}\right\}$$

$$=\frac{\sin 2t}{2}e^{t}$$ Ans:

HEAVISIDE Expansion Formula:

let, $P(s), Q(s)$ be polynomials, where $P(s)$ has degree less that of $Q(s)$. suppose that $Q(s)$ has 'n' distinct zeros, a_K, $K = 1, 2, 3, \ldots n$

যাদের value এর জন্য function undefined হবে।

Then, $\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} = \sum_{K=1}^{n} \frac{P(a_K)}{Q'(a_K)} e^{a_K t}$

$$= \frac{P(a_1)}{Q'(a_1)} e^{a_1 t} + \frac{P(a_2)}{Q'(a_2)} e^{a_2 t} + \ldots + \frac{P(a_n)}{Q'(a_n)} e^{a_n t} \ldots$$

1. $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

soln: $P(s) = 2s^2 - 4$

$$Q(s) \cancel{P(s)} = (s+1)(s-2)(s-3)$$
$$= (s^2 - 2s + s - 2)(s-3)$$
$$= \cancel{(s)}(s^2 - s - 2)(s-3)$$
$$= s^3 - 3s^2 - s^2 + 3s - 2s + 6$$
$$= s^3 - 4s^2 + s + 6$$
$$Q'(s) = 3s^2 - 8s + 1$$

Here, $Q(s)$ has three distinct zeros.

say, $a_1 = -1$, $a_2 = 2$, $a_3 = 3$

By HEAVISIDE Expansion formula,

$$\mathcal{L}^{-1}\left\{\frac{2s^2-1}{(s+1)(s-2)(s-3)}\right\} = \frac{P(-1)}{Q'(-1)}e^{-t} + \frac{P(2)}{Q'(2)}e^{2t} + \frac{P(3)}{Q'(3)}e^{3t}$$

$$= \left(\frac{-2}{12}\right)e^{-t} + \left(\frac{-4}{3}\right)e^{2t} + \left(\frac{14}{4}\right)e^{3t}$$

$$= -\frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}$$

Ans:

Partial Function

Q1. $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\}$

Soln: Let, $\frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{A}{(s+1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$

$\Rightarrow 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2)$ ... (1)

At, $s=2$, (1) $\Rightarrow$

$2\cdot 2^2-4 = A(2-2)(2-3) + B(2+1)(2-3) + C(2+1)(2-2)$

$\Rightarrow 4 = -3B$

$\Rightarrow B = -4/3$

At, $s=3$, (1) $\Rightarrow$

$2\cdot 3^2-4 = A(3-2)(3-3) + B(3+1)(3-3) + C(3+1)(3-2)$

$\Rightarrow 14 = 4C$

$\Rightarrow C = 14/4$

At, $s=-1$, (1) $\Rightarrow$

$2(-1)^2-4 = A(-1-2)(-1-3) + B(-1+1)(-1-3) + C(-1+1)(-1-2)$

$\Rightarrow -2 = 12A + 0 + 0$

$\Rightarrow A = -1/6$

$\therefore \frac{2s^2-4}{(s+1)(s-2)(s-3)} = \frac{-2}{12(s+1)} - \frac{4}{3(s-2)} + \frac{14}{4(s-3)}$

$\therefore \mathcal{L}^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{-2}{12(s+1)} - \frac{4}{3(s-2)} + \frac{14}{4(s-3)}\right\}$

$$Bs^3+As^2+Ds^3+Bs^2+Cs+E \quad \frac{1}{(s+1)(s+2)}=\frac{A}{(s+1)}+\frac{Bs+C}{(s+2)}$$

$$= -\frac{2}{12}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}-\frac{4}{3}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}+\frac{14}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= -\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$$

Ans:

④ $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}$

soln:

Let, $\frac{3s+1}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}$

$\Rightarrow 3s+1 = A(s^2+1)+(Bs+C)(s-1)$ —— (i)

At, $s=1$, $3+1 = 2A+(B+C)\times 0$

$\Rightarrow A = 4/2$

$\Rightarrow A = 2$

Again,

$3s+1 = As^2+A+Bs^2-Bs+Cs-C$

$3s+1 = (A+B)s^2+(C-B)s+(A-C)$

Equating co-efficient of s^2, s from (ii),

$A+B=0$, $B=-A=-2$

$C-B=3$

$\Rightarrow C=3+B$

$=1$

$$\frac{3s+1}{(s-1)(s^2+1)}=\frac{2}{s-1}+\frac{-2s+1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}=\mathcal{L}^{-1}\left\{\frac{2}{s-1}-\frac{2s}{s^2+1}+\frac{1}{s^2+1}\right\}$$

$= 2e^t - 2\cos t + \sin t$ Ans:

Laplace Transformation of Derivatives:

If $\mathcal{L}\{F(t)\} = f(s)$ then,

(i) $\mathcal{L}\{F'(t)\} = sf(s) - F(0)$

(ii) $\mathcal{L}\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

Working Rule:

(i) Take Laplace Transform

(ii) Apply Formula

(iii) Apply conditions

(iv) solve for $f(s)$

(v) Take Inverse Laplace Transformation

(vi) Find $F(t)$

01. $F''(t) + 4F(t) = 0$, $F(0) = 10$, $F'(0) = 0$

Soln: $F''(t) + 4F(t) = 0 \Rightarrow \mathcal{L}\{F''(t)\} + 4\mathcal{L}\{F(t)\} = \mathcal{L}(0)$

$\Rightarrow \{s^2 f(s) - sF(0) - F'(0)\} + 4f(s) = 0$

$\Rightarrow s^2 f(s) - 10s - 0 + 4f(s) = 0$

$\Rightarrow f(s)(s^2+4) = 10s$

$\Rightarrow f(s) = \frac{10s}{s^2+4}$

$\Rightarrow \mathcal{L}^{-1}\{f(s)\} = 10\,\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\}$

$\Rightarrow F(t) = 10\cos 2t$ Ans:

Application of LAPLACE Transformation:

1. $F'' + F = t$, $F(0) = 1$, $F'(0) = -2$

Here, $F'' + F = t$

$\Rightarrow \mathcal{L}\{F''\} + \mathcal{L}\{F\} = \mathcal{L}\{t\}$

$\Rightarrow \left(s^2 f(s) - sF(0) - F'(0)\right) + f(s) = \frac{1}{s^2}$

$\Rightarrow s^2 f(s) - s + 2 + f(s) = \frac{1}{s^2}$

$\Rightarrow (s^2+1) f(s) = \frac{1}{s^2} + s - 2$

$\Rightarrow f(s) = \frac{1}{(s^2+1)}\left(\frac{1}{s^2} + s - 2\right)$

$\Rightarrow f(s) = \frac{1}{s^2(s^2+1)} + \frac{s-2}{(s^2+1)}$

$\Rightarrow \mathcal{L}^{-1} f(s) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)} + \frac{s-2}{s^2+1}\right\}$

$\Rightarrow F(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} + \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} - 2\mathcal{L}^{-1}\left(\frac{1}{(s^2+1)}\right)$ — (I)

Let, $\frac{1}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$

$\Rightarrow 1 = As(s^2+1) + B(s^2+1) + (Cs+D)s^2$

$\Rightarrow 1 = As^3 + As + Bs^2 + B + Cs^3 + Ds^2$ —— (II)

Calculating co-efficient from both side,

$A+C=0$; $B+D=0$; $A=0$; $B=1$

$\Rightarrow C=0$ $\Rightarrow D=-1$

$$(1) \Rightarrow F(t) = \mathcal{L}^{-1}\left\{\frac{0}{s}+\frac{1}{s^2}+\frac{0.s-1}{s^2+1}\right\}+\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\}-2\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}-\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}+\cos t-2\sin t$$

$$=t-\sin t+\cos t-2\sin t$$

$$=t-3\sin t+\cos t$$ Ans:

Application of Partial Differential Equation

$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$, $u(0,t)=0$, $u(\pi,t)=0$, $u(x,0)=2\sin 3x - 4\sin 5x$

Solution: Given, $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ —— (i)

Let, $u(x,t) = XT$ —— (ii) is a solution of (i) when X is function of x alone and T is function of t alone.

then, $\frac{\partial u}{\partial t} = X\frac{dT}{dt}$

$\frac{\partial u}{\partial x} = T\frac{dX}{dx}$ and $\frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$

Putting these value in (i) we get,

$$X\frac{dT}{dt} = 4T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{1}{4T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \quad \text{—— (iii)}$$

Since, X is function of x alone and T is function of t alone and x and t are independent variables, so each side of (iii) must be constant, say $-\lambda^2$

$$\therefore \frac{1}{4T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \longrightarrow (\text{iv})$$

So from (iv),

$$\frac{1}{4T}\frac{dT}{dt} = -\lambda^2$$

$$\Rightarrow \int \frac{1}{T}\,dT = \int -4\lambda^2\,dt$$

$$\Rightarrow \ln T = -4\lambda^2 t + \ln A \text{ ; where } A \text{ is a constant}$$

$$\Rightarrow \ln T - \ln A = -4\lambda^2 t$$

$$\Rightarrow \ln\left(\frac{T}{A}\right) = -4\lambda^2 t$$

$$\Rightarrow \frac{T}{A} = e^{-4\lambda^2 t}$$

$$\Rightarrow T = Ae^{-4\lambda^2 t}$$

Again from (iv),

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} = -\lambda^2 X$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0 \longrightarrow (\text{v})$$

Let, $X = e^{mx}$ be a trial solution of (v)

$$m^2e^{mx} + \lambda^2 e^{mx} = 0$$

$$\Rightarrow e^{mx}(m^2 + \lambda^2) = 0$$

$$\Rightarrow m^2 + \lambda^2 = 0$$

$\Rightarrow m^2 = -\lambda^2$

$\Rightarrow m = \pm \lambda i$

$X = e^{0x}(B\cos\lambda x + C\sin\lambda x)$

$= B\cos\lambda x + C\sin\lambda x$

~~Thus the solution of (i) from (ii) is~~

$U(x,t) = XT$

$= (B\cos\lambda x + C\sin\lambda x) Ae^{-4\lambda^2 t}$

$= ABe^{-4\lambda^2 t}\cos\lambda x + ACe^{-4\lambda^2 t}\sin\lambda x$

$U(x,t) = De^{-4\lambda^2 t}\cos\lambda x + Ee^{-4\lambda^2 t}\sin\lambda x$ ——— (vi)

where, $AB = D$ and $AC = E$

Since, $U(0,t) = 0$, so from (vi),

$U(0,t) = De^{-4\lambda^2 t}\cos 0 + Ee^{-4\lambda^2 t}\sin 0$

$\Rightarrow 0 = De^{-4\lambda^2 t}$

$\Rightarrow D = 0$

So, (vi) $\Rightarrow$

$U(x,t) = Ee^{-4\lambda^2 t}\sin\lambda x$ ——— (vii)

Again, $U(\pi, t) = 0$, so from (VII)

$$U(\pi, t) = Ee^{-4\lambda^2 t}\sin\lambda\pi$$

$$\Rightarrow 0 = Ee^{-4\lambda^2 t}\sin\lambda\pi \longrightarrow \text{(VIII)}$$

If $E = 0$, the solution is identically zero. So we must have $\sin\lambda\pi = 0$; since $e^{-4\lambda^2 t} \neq 0$ and $E \neq 0$

so from, $\sin\lambda\pi = 0$

$$\Rightarrow \sin\lambda\pi = \sin(n\pi)$$

$$\Rightarrow \lambda\pi = n\pi$$

$$\Rightarrow n = \lambda$$

Putting the value of $\lambda = n$ in (VII)

$$U(x,t) = Ee^{-4n^2 t}\sin(nx) \longrightarrow \text{(IX)}$$

$$U(x,t) = E_1 e^{-4n_1^2 t}\sin(n_1 x) + E_2 e^{-4n_2^2 t}\sin(n_2 x) \longrightarrow \text{(X)}$$

Putting $t = 0$ in (X) we get,

$$U(x,0) = E_1 e^{-0}\sin(n_1 x) + E_2 e^{-0}\sin(n_2 x)$$

$$\Rightarrow E_1\sin(n_1 x) + E_2\sin(n_2 x) = 2\sin 3x - 4\sin 5x$$

which is possible if and only if
$E_1=2,\ n_1=3,\ E_2=-4,\ n_2=5$

Putting these values in (x) we get,

$$v(x,t) = 2e^{-4.3^2t}\sin 3x - 4e^{-4.5^2.t}\sin(5x)$$
$$= 2e^{-36t}\sin 3x - 4e^{-100t}\sin 5x$$

which is the solution.

Practice:

Solve the following boundary value problem by the method of separation of variables.

$$4\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ u(0,t)=0,\ u(2,t)=0,\ 0<x<2$$

$$u(x,0) = 2\sin\frac{\pi x}{2} - \sin\pi x + 4\sin 2\pi x$$

Solution:

Given,

$$4\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \text{———} (i)$$

Let, $u(x,t) = XT$ ——— (ii) is a solution of (i) where X is function of x alone and T is function of t alone.

Then, $\frac{\partial u}{\partial t} = X\frac{dT}{dt}$,

$$\frac{\partial u}{\partial x} = T\frac{dX}{dx};\quad \frac{\partial^2 u}{\partial x^2} = T\frac{d^2X}{dx^2}$$

Putting these values in (i),

$$4X\frac{dT}{dt} = T\frac{d^2X}{dx^2}$$

$$\Rightarrow \frac{4}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} \quad \text{———} (iii)$$

Since, X is a function of x alone and T is a function of t alone and x and t are independent variables, so each side of (iii) must be ~~equal~~ a constant, say $-\lambda^2$

$$\therefore \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \text{ ——— (iv)}$$

so from (iv)

$$\frac{1}{T}\frac{dT}{dt} = -\lambda^2$$

$$\Rightarrow \int \frac{1}{T}\, dT = \int -\lambda^2/4 \, dt$$

$$\Rightarrow \ln T = -\frac{\lambda^2 t}{4} + \ln A$$

$$\Rightarrow \ln\left(\frac{T}{A}\right) = -\frac{\lambda^2}{4}t$$

$$\Rightarrow \frac{T}{A} = e^{-\frac{\lambda^2}{4}t}$$

$$\Rightarrow T = Ae^{-\frac{\lambda^2}{4}t}$$

Again from (iv)

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\Rightarrow \frac{d^2X}{dx^2} = -\lambda^2 X$$

$$\Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$$

Let, $X = e^{mx}$ be a trail solution of (v).

$$\Rightarrow m^2e^{mx} + \lambda^2 e^{mx} = 0$$

$$\Rightarrow e^{mx}(m^2+\lambda^2) = 0$$

$$m^2 = -\lambda^2$$

$$\Rightarrow m = \pm \lambda i$$

$$X = e^{0x}(B\cos\lambda x + C\sin\lambda x)$$

$$= B\cos\lambda x + C\sin\lambda x$$

$$U(x,t) = XT$$

$$= Ae^{-\frac{\lambda^2}{4}t}(B\cos\lambda x + C\sin\lambda x)$$

$$= ABe^{-\frac{\lambda^2}{4}t}\cos\lambda x + ACe^{-\frac{\lambda^2}{4}t}\sin\lambda x.$$

$$U(x,t) = De^{-\frac{\lambda^2}{4}t}\cos\lambda x + Ee^{-\frac{\lambda^2}{4}t}\sin\lambda x \quad \text{--- (VI)}$$

where, $AB = D$ and $AC = E$

since, $U(0,t) = 0$ so from (VI)

$$U(0,t) = De^{-\frac{\lambda^2}{4}t}\cos\lambda 0 + Ee^{-\frac{\lambda^2}{4}t}\sin\lambda 0$$

$$\Rightarrow 0 = De^{-\frac{\lambda^2}{4}t}$$

$$\Rightarrow D = 0$$

Thus from (vi), we have, $U(x,t) = Ee^{-\frac{\lambda^2}{4}t} \sin \lambda x$ —— (vii)

Again, $U(2,t) = Ee^{-\frac{\lambda^2}{4}t} \sin 2\lambda$

$\Rightarrow 0 = Ee^{-\frac{\lambda^2}{4}t} \sin 2\lambda$ —— (viii)

If $E=0$, the solution is identically zero. so we must have $\sin 2\lambda = 0$; since $e^{-\frac{\lambda^2}{4}t} \neq 0$ and $E \neq 0$

$\sin 2\lambda = 0$

$\Rightarrow \sin 2\lambda = \sin(n\pi)$

$\Rightarrow n\pi = 2\lambda$

~~$\Rightarrow n = \frac{2\lambda}{\pi}$~~ $\Rightarrow \lambda = \frac{n\pi}{2}$

$U(x,t) = Ee^{-\frac{n^2\pi^2}{16}t} \sin \frac{n\pi}{2} x$

$U(x,0) = 2\sin\frac{\pi x}{2} - \sin \pi x + 4 \sin 2\pi x$

$U(x,t) = E_1 e^{-\frac{n_1^2\pi^2}{16}t} \sin \frac{n_1\pi}{2} x + E_2 e^{-\frac{n_2^2\pi^2}{16}t} \sin \frac{n_2\pi}{2} x + E_3 e^{-\frac{n_3^2\pi^2}{16}t} \sin \frac{n_3\pi}{2} x$

Putting, $t=0$,

$U(x,0) = E_1 e^{-0} \sin \frac{n_1\pi}{2} x + E_2 e^{-0} \sin \frac{n_2\pi}{2} x + E_3 e^{-0} \sin \frac{n_3\pi}{2} x$

which is possible if and only if,

$E_1 = 2$, $E_2 = -1$, $E_3 = 4$

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