Bisection Method

by

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Basis of Bisection Method/ binary search method

Theorem An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_l and x_u if $f(x_l)\, f(x_u) < 0$.

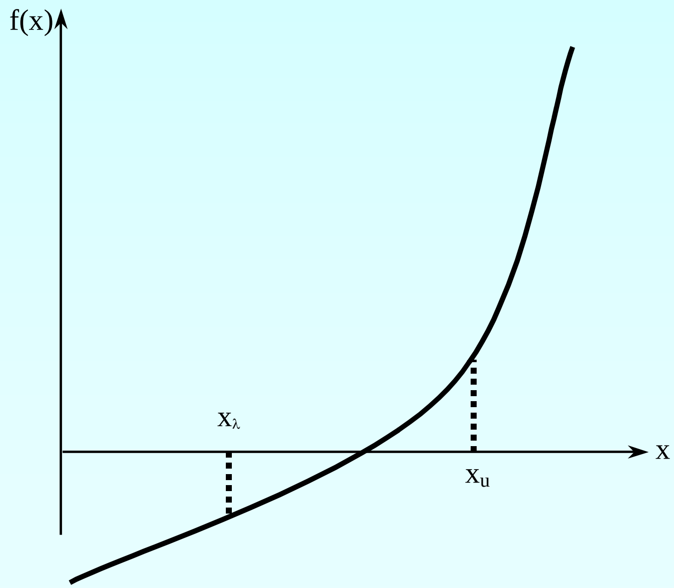


Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

Basis of Bisection Method

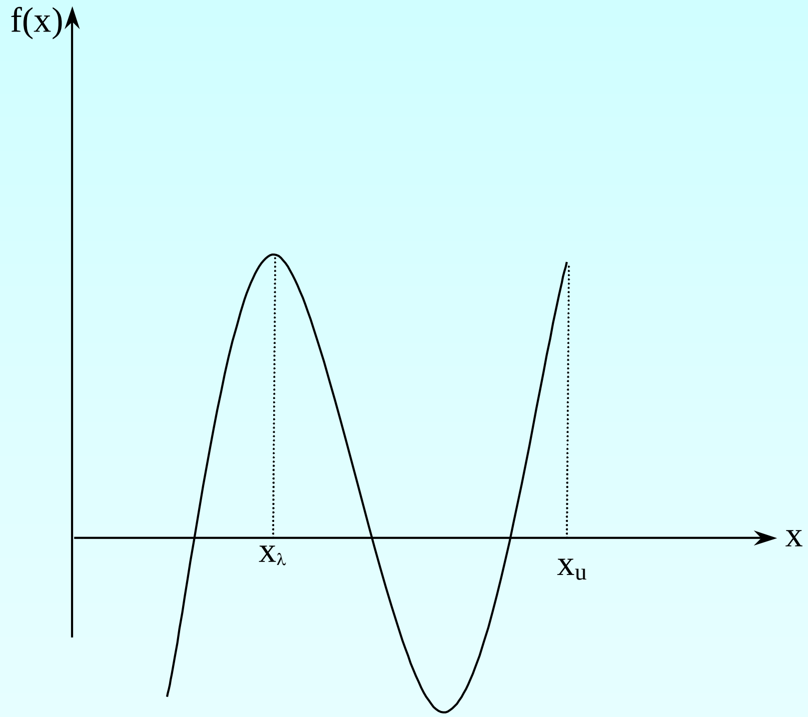


Figure 2 If function $f(x)$ does not change sign between two points, roots of the equation $f(x)=0$ may still exist between the two points.

Basis of Bisection Method

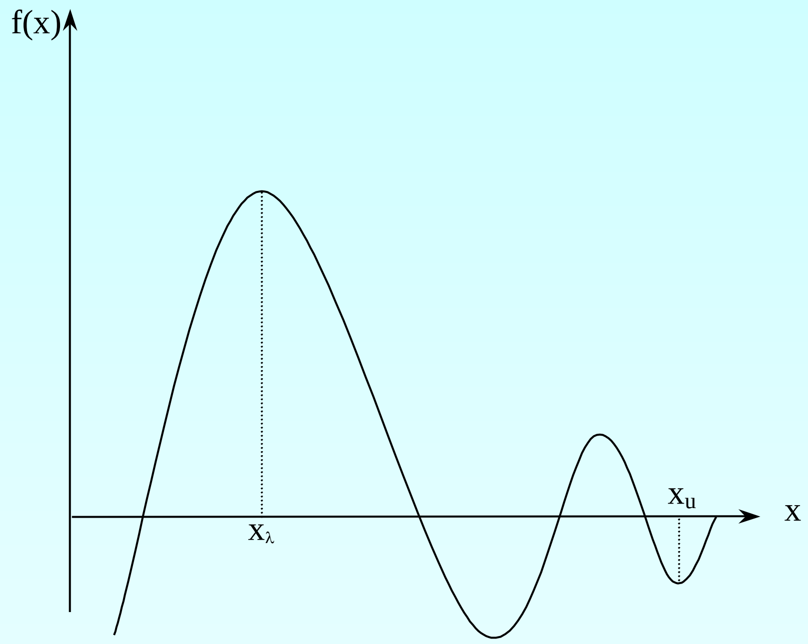


Figure 4 If the function $f(x)$ changes sign between two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Algorithm for Bisection Method

Step 1

Choose x_λ and x_u as two guesses for the root such that $f(x_\lambda)\, f(x_u) < 0$, or in other words, $f(x)$ changes sign between x_λ and x_u. This was demonstrated in Figure 1.

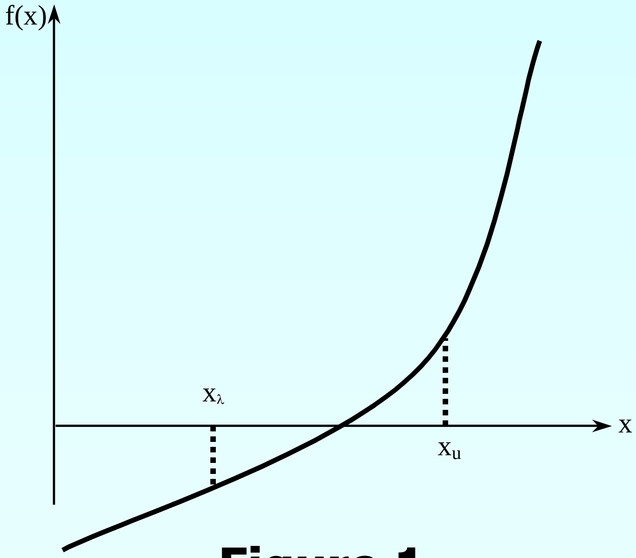


Figure 1

Step 2

Estimate the root, x_m of the equation f (x) = 0 as the mid point between x_λ and x_u as

$$x_m = \frac{x_\lambda + x_u}{2}$$

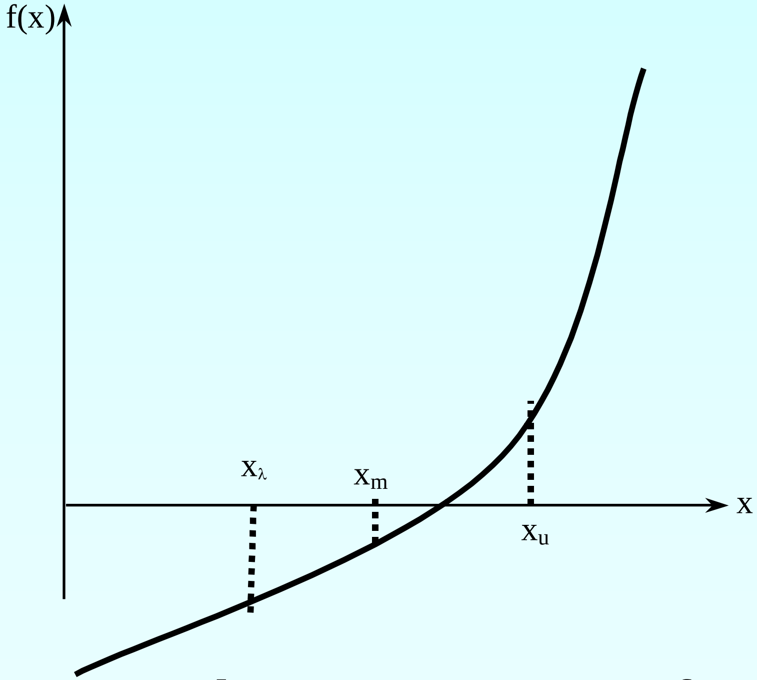


Figure 5 Estimate of x_m

Step 3

Now check the following

a) If $f(x_l)f(x_m) < 0$, then the root lies between x_λ and x_m; then $x_\lambda = x_\lambda$; $x_u = x_m$.

b) If $f(x_l)f(x_m) > 0$, then the root lies between x_m and x_u; then $x_\lambda = x_m$; $x_u = x_u$.

c) If $f(x_l)f(x_m) = 0$; then the root is $x_{m.}$ Stop the algorithm if this is true.

Step 4

Find the new estimate of the root

$$x_m = \frac{x_\lambda + x_u}{2}$$

Find the absolute relative approximate error

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

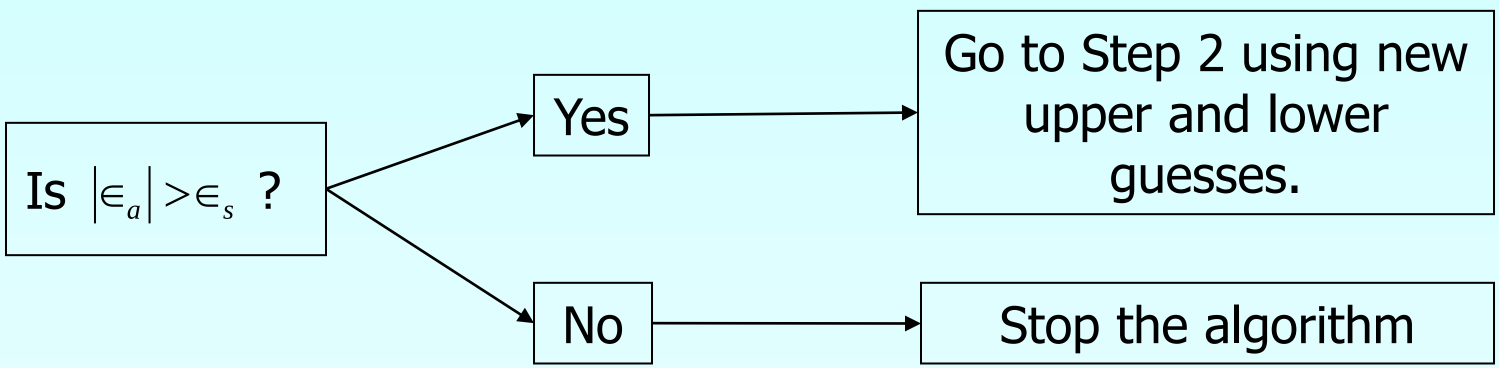
where

$$x_m^{old} = \text{previous estimate of root}$$

$$x_m^{new} = \text{current estimate of root}$$

Step 5

Compare the absolute relative approximate error $\left|\in_a\right|$ with the pre-specified error tolerance $\in_s$.



Note: one should also check whether the number of iterations is more than the maximum number of iterations allowed. If so, one needs to terminate the algorithm and notify the user about it.

Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

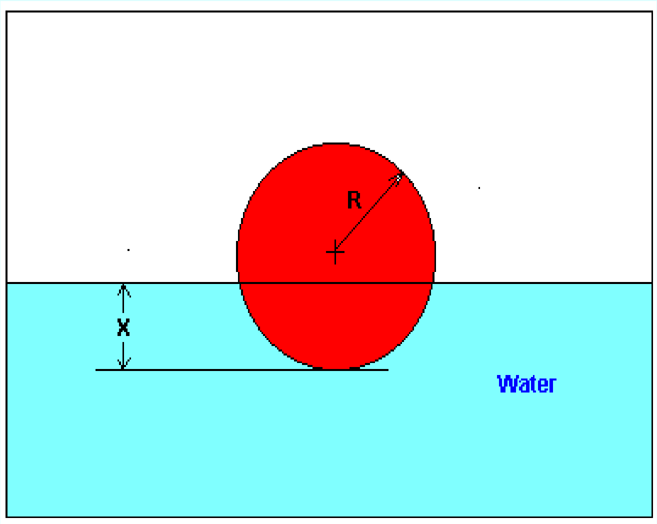


Figure 6 Diagram of the floating ball

Example 1 Cont.

The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

a) Use the bisection method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.

b) Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the end of each iteration.

Example 1 Cont.

From the physics of the problem, the ball would be submerged between $x = 0$ and $x = 2R$,

where R = radius of the ball,

that is

$$0 \le x \le 2R$$

$$0 \le x \le 2(0.055)$$

$$0 \le x \le 0.11$$

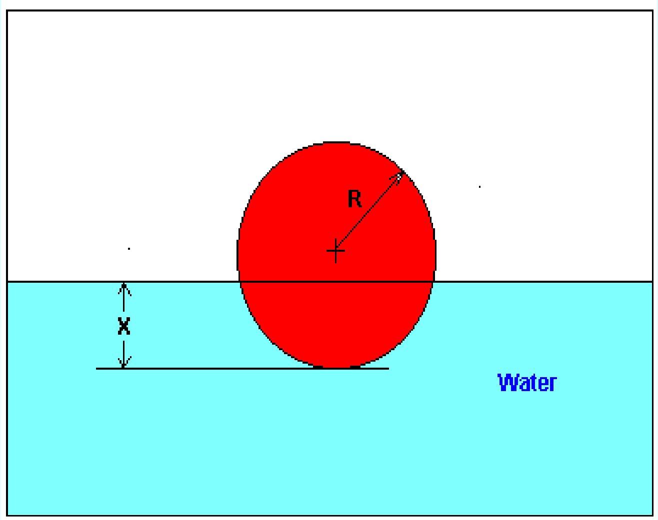


Figure 6 Diagram of the floating ball

Example 1 Cont.

Solution

To aid in the understanding of how this method works to find the root of an equation, the graph of f(x) is shown to the right,

where

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$

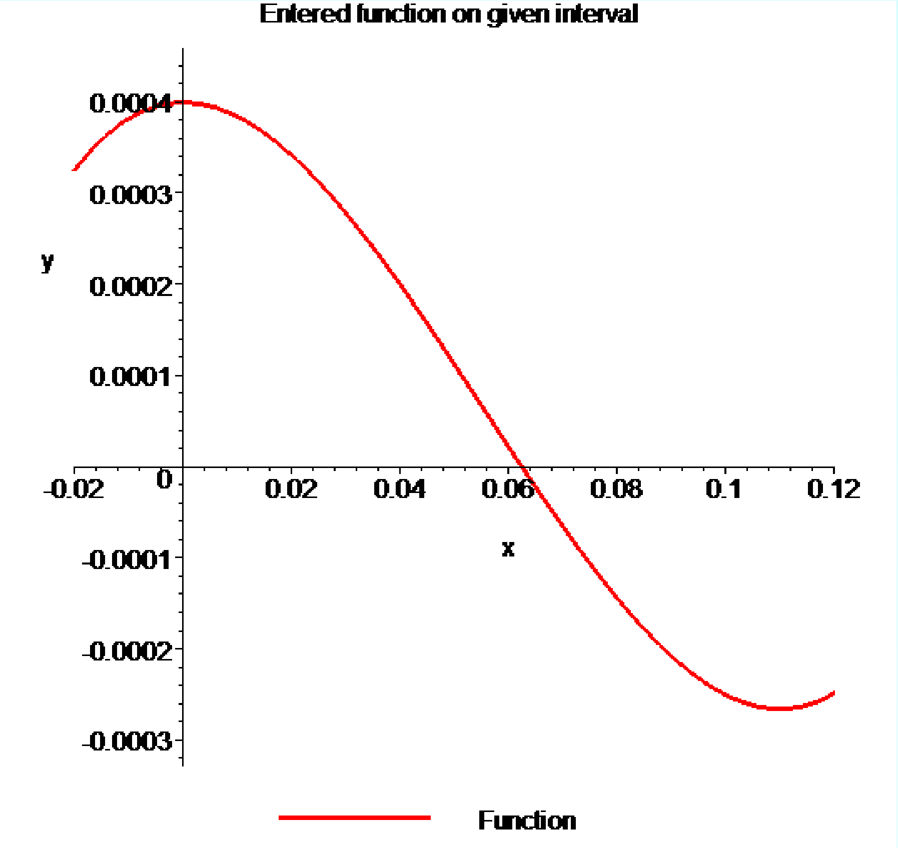


Figure 7 Graph of the function f(x)

Example 1 Cont.

Let us assume

$$x_\lambda = 0.00$$

$$x_u = 0.11$$

Check if the function changes sign between x_λ and x_u .

$$f(x_l) = f(0) = (0)^3 - 0.165(0)^2 + 3.993\times10^{-4} = 3.993\times10^{-4}$$

$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993\times10^{-4} = -2.662\times10^{-4}$$

Hence

$$f(x_l)f(x_u) = f(0)f(0.11) = \left(3.993\times10^{-4}\right)\left(-2.662\times10^{-4}\right) < 0$$

So there is at least on root between x_λ and $x_{u,}$ that is between 0 and 0.11

Example 1 Cont.

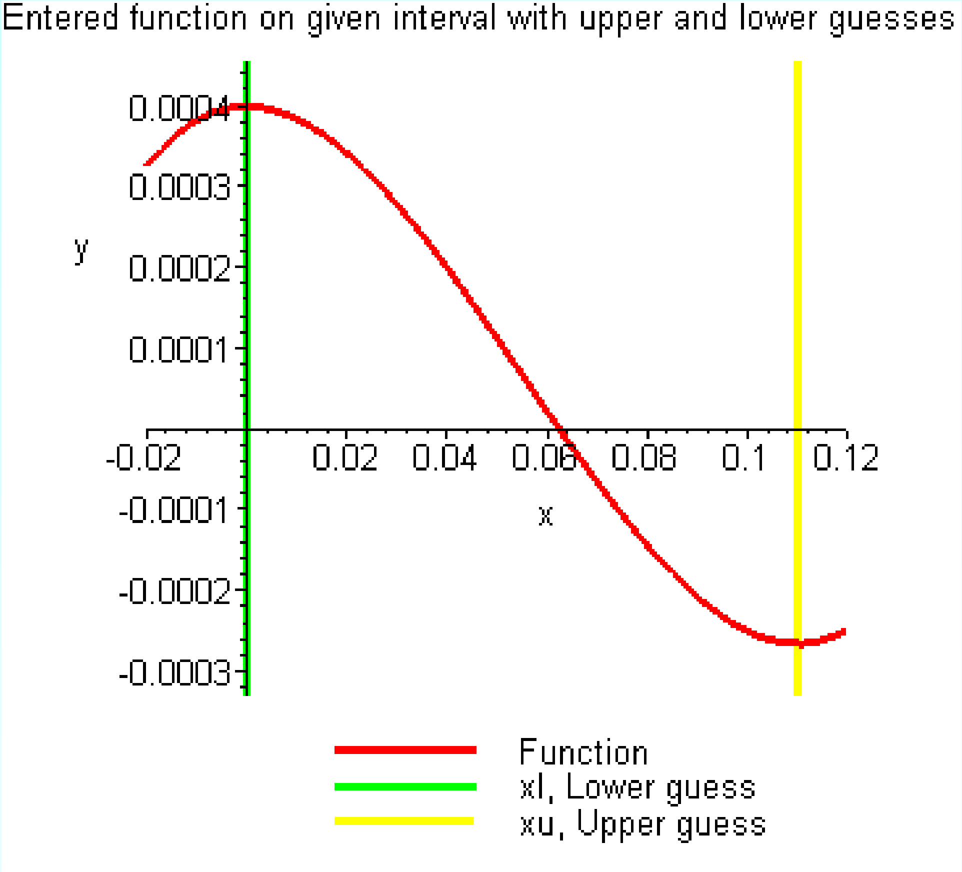


Figure 8 Graph demonstrating sign change between initial limits

Example 1 Cont.

<u>Iteration 1</u>

The estimate of the root is $x_m = \dfrac{x_\lambda + x_u}{2} = \dfrac{0+0.11}{2} = 0.055$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$

$$f(x_l)f(x_m) = f(0)f(0.055) = \left(3.993\times10^{-4}\right)\left(6.655\times10^{-5}\right) > 0$$

According to step 3, slide no. 8

Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Example 1 Cont.

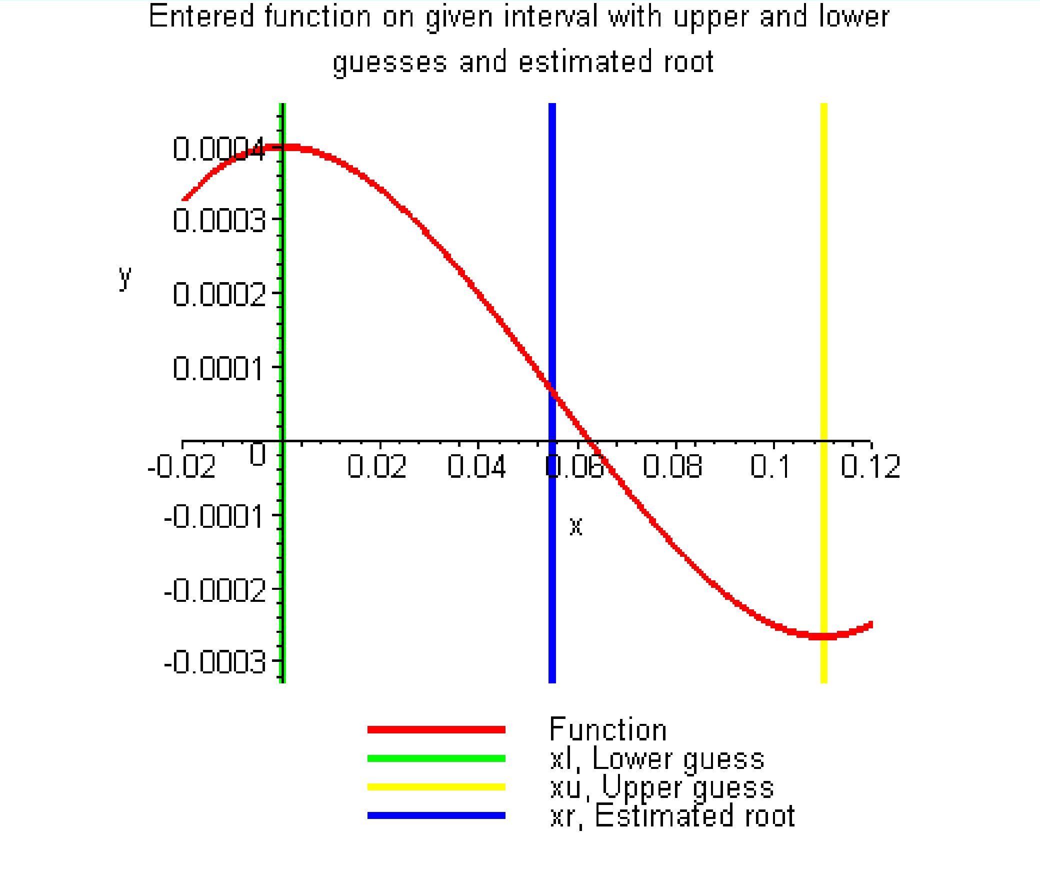


Figure 9 Estimate of the root for Iteration 1

Example 1 Cont.

<u>Iteration 2</u>

The estimate of the root is $x_m = \frac{x_\lambda + x_u}{2} = \frac{0.055 + 0.11}{2} = 0.0825$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993 \times 10^{-4} = -1.622 \times 10^{-4}$$

$$f(x_l)f(x_m) = f(0.055)f(0.0825) = (-1.622 \times 10^{-4})(6.655 \times 10^{-5}) < 0$$

Hence the root is bracketed between x_λ and x_m, that is, between 0.055 and 0.0825. So, the lower and upper limits of the new bracket are

$$x_l = 0.055, \; x_u = 0.0825$$

Example 1 Cont.

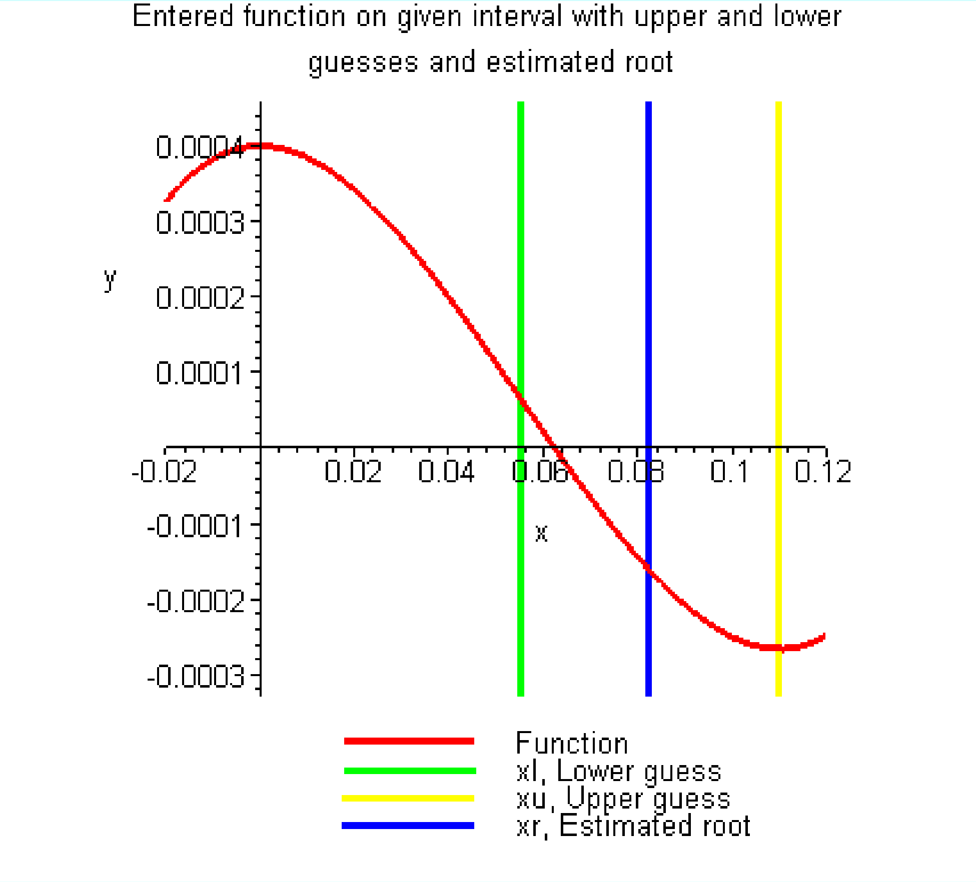


Figure 10 Estimate of the root for Iteration 2

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 2 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100$$

$$= 33.333\%$$

None of the significant digits are at least correct in the estimate root of x_m = 0.0825 because the absolute relative approximate error is greater than 5%.

Example 1 Cont.

<u>Iteration 3</u>
The estimate of the root is $x_m = \dfrac{x_\lambda + x_u}{2} = \dfrac{0.055 + 0.0825}{2} = 0.06875$

$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993\times 10^{-4} = -5.563\times 10^{-5}$$
$$f(x_l)f(x_m) = f(0.055)f(0.06875) = (6.655\times 10^{-5})(-5.563\times 10^{-5}) < 0$$

Hence the root is bracketed between x_λ and x_m, that is, between 0.055 and 0.06875. So, the lower and upper limits of the new bracket are

$$x_l = 0.055,\ \ x_u = 0.06875$$

Example 1 Cont.

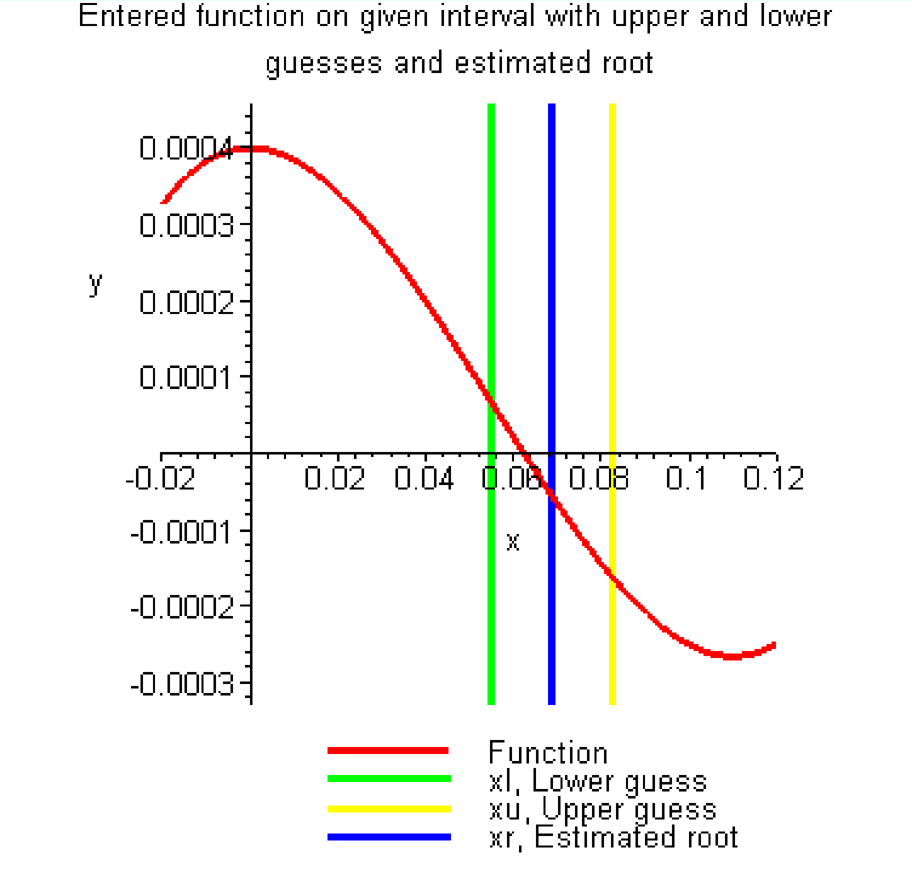


Figure 11 Estimate of the root for Iteration 3

Example 1 Cont.

The absolute relative approximate error $\left|\in_a\right|$ at the end of Iteration 3 is

$$\left|\in_a\right| = \left|\frac{x_m^{new} - x_m^{old}}{x_m^{new}}\right| \times 100$$

$$= \left|\frac{0.06875 - 0.0825}{0.06875}\right| \times 100$$

$$= 20\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.
Seven more iterations were conducted and these iterations are shown in Table 1.

Table 1 Cont.

Table 1 Root of f(x)=0 as function of number of iterations for bisection method.

Iteration	x_λ	x_u	x_m	$\lvert\in_a\rvert$ %	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

Advantages

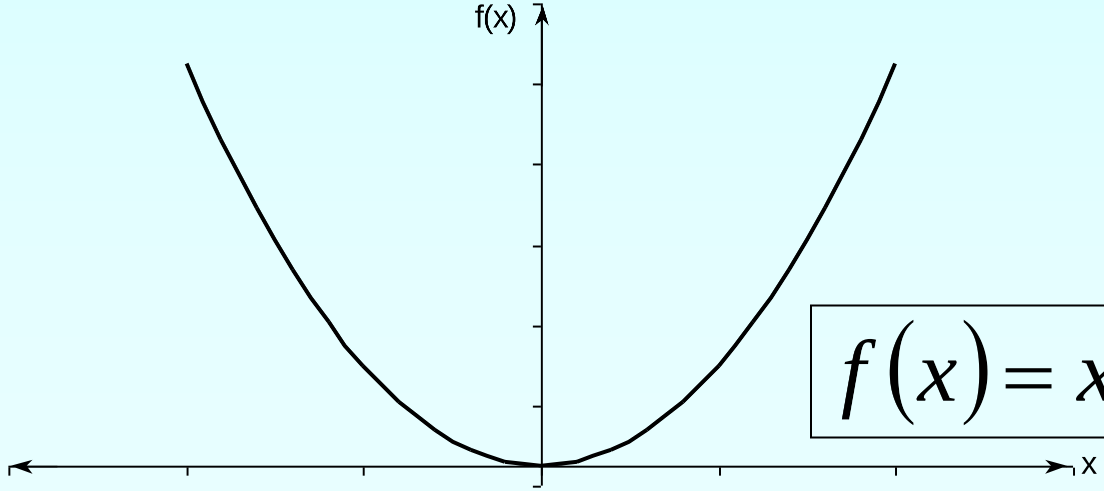
- Always convergent (approaching closer to the real root)
- The root bracket gets halved (divide into two parts of equal) with each iteration - guaranteed.

Drawbacks

- Slow convergence
- If one of the initial guesses is close to the root, the convergence is slower

Drawbacks (continued)

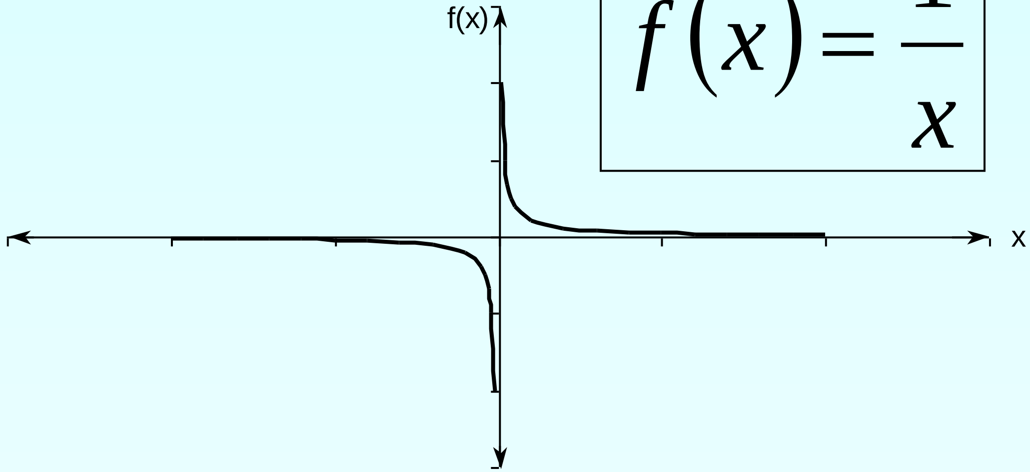
- If a function f(x) is such that it just touches the x-axis it will be unable to find the lower and upper guesses.



$$f(x) = x^2$$

Drawbacks (continued)

- Function changes sign but root does not exist



$$f(x) = \frac{1}{x}$$

THE END