Runge-Kutta 2nd Order Method

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Runge-Kutta 2nd Order Method

For $\frac{dy}{dx} = f(x,y), y(0) = y_0$

$$y_{i+1} = y_i + f(x_i, y_i)(x_{i+1} - x_i) + \frac{1}{2!} f'(x_i, y_i)(x_{i+1} - x_i)^2 + \frac{1}{3!} f''(x_i, y_i)(x_{i+1} - x_i)^3 + \ldots$$

As you can see the first two terms of the Taylor series

$y_{i+1} = y_i + f(x_i, y_i)h$ are the Euler's method.

Runge-Kutta took this particular expression and equated it to the second-order . . . the first three terms of the Taylor series expansion for y

Runge Kutta 2nd order method is given by

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

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what they found out is that they got three equations as follows:

$$a_1+a_2=1 \ldots\ldots(1)$$

$$a_2p_1=1/2 \ldots\ldots(2)$$

$$a_2q_{11}=1/2 \ldots\ldots(3)$$

Heun's Method

<u>Heun's method</u>

Here a_2=1/2 is chosen

$$a_1 = \frac{1}{2}$$

$$p_1 = 1$$

$$q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + h, y_i + k_1 h\right)$$

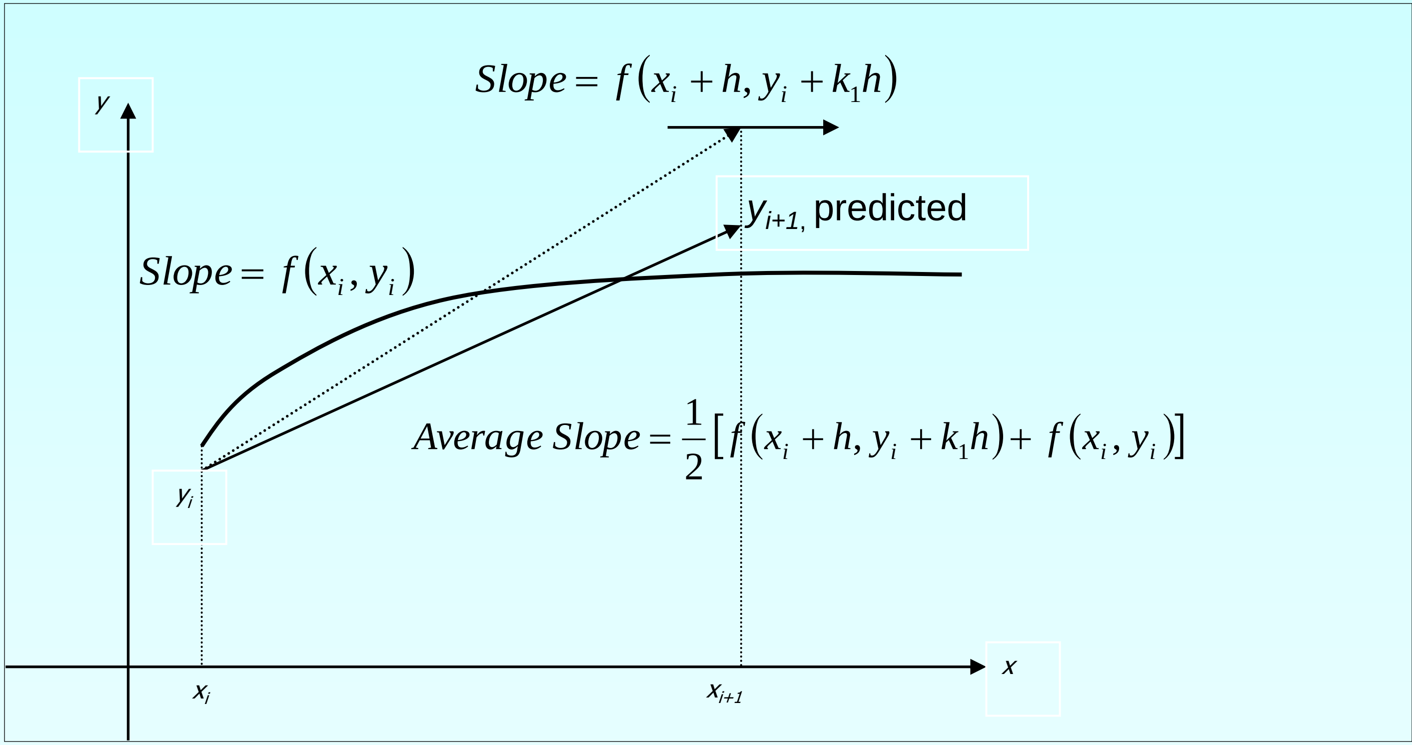

Figure 1 Runge-Kutta 2nd order method (Heun's method)

Midpoint Method

Here $a_2 = 1$ **is chosen, giving**

$$a_1 = 0$$

$$p_1 = \frac{1}{2}$$

$$q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

Here $a_2 = \frac{2}{3}$ is chosen, giving

$$a_1 = \frac{1}{3}$$

$$p_1 = \frac{3}{4}$$

$$q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left(\frac{1}{3} k_1 + \frac{2}{3} k_2 \right) h$$

where

$$k_1 = f\left(x_i, y_i\right)$$

$$k_2 = f\left(x_i + \frac{3}{4} h, y_i + \frac{3}{4} k_1 h \right)$$

How to write Ordinary Differential Equation

How does one write a first order differential equation in the form of

$$\frac{dy}{dx} = f(x, y)$$

Example

$$\frac{dy}{dx} + 2y = 1.3e^{-x}, \; y(0) = 5$$

is rewritten as

$$\frac{dy}{dx} = 1.3e^{-x} - 2y, \; y(0) = 5$$

In this case

$$f(x, y) = 1.3e^{-x} - 2y$$

Example

A ball at 1200K is allowed to cool down in air at an ambient temperature of 300K. Assuming heat is lost only due to radiation, the differential equation for the temperature of the ball is given by

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right), \theta(0) = 1200K$$

Find the temperature at $t = 480$ seconds using Heun's method. Assume a step size of $h = 240$ seconds.

$$\frac{d\theta}{dt} = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$f(t,\theta) = -2.2067\times10^{-12}\left(\theta^4 - 81\times10^8\right)$$

$$\theta_{i+1} = \theta_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$$

Solution

Step 1: $i = 0, t_0 = 0, \theta_0 = \theta(0) = 1200K$

$$
\begin{aligned}
k_1 &= f\left(t_0, \theta_o\right) \\
&= f\left(0,1200\right) \\
&= -2.2067 \times 10^{-12}\left(1200^4 - 81 \times 10^8\right) \\
&= -4.5579
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f\left(t_0 + h, \theta_0 + k_1 h\right) \\
&= f\left(0 + 240, 1200 + \left(-4.5579\right)240\right) \\
&= f\left(240, 106.09\right) \\
&= -2.2067 \times 10^{-12}\left(106.09^4 - 81 \times 10^8\right) \\
&= 0.017595
\end{aligned}
$$

$$
\begin{aligned}
\theta_1 &= \theta_0 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\
&= 1200 + \left(\frac{1}{2}\left(-4.5579\right) + \frac{1}{2}\left(0.017595\right)\right)240 \\
&= 1200 + \left(-2.2702\right)240 \\
&= 655.16K
\end{aligned}
$$

Solution Cont

Step 2: $i=1, t_1 = t_0 + h = 0 + 240 = 240, \theta_1 = 655.16K$

$$
\begin{aligned}
k_1 &= f(t_1, \theta_1) \\
&= f(240, 655.16) \\
&= -2.2067\times 10^{-12}\left(655.16^4 - 81\times 10^8\right) \\
&= -0.38869
\end{aligned}
$$

$$
\begin{aligned}
k_2 &= f(t_1 + h, \theta_1 + k_1 h) \\
&= f(240+240, 655.16 + (-0.38869)240) \\
&= f(480, 561.87) \\
&= -2.2067\times 10^{-12}\left(561.87^4 - 81\times 10^8\right) \\
&= -0.20206
\end{aligned}
$$

$$
\begin{aligned}
\theta_2 &= \theta_1 + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h \\
&= 655.16 + \left(\frac{1}{2}(-0.38869) + \frac{1}{2}(-0.20206)\right)240 \\
&= 655.16 + (-0.29538)240 \\
&= 584.27K
\end{aligned}
$$

Solution Cont

The exact solution of the ordinary differential equation is given by the solution of a non-linear equation as

$$0.92593 \ln \frac{\theta - 300}{\theta + 300} - 1.8519 \tan^{-1}(0.0033333\theta) = -0.22067 \times 10^{-3} t - 2.9282$$

The solution to this nonlinear equation at t=480 seconds is

$$\theta(480) = 647.57K$$

Comparison with exact results

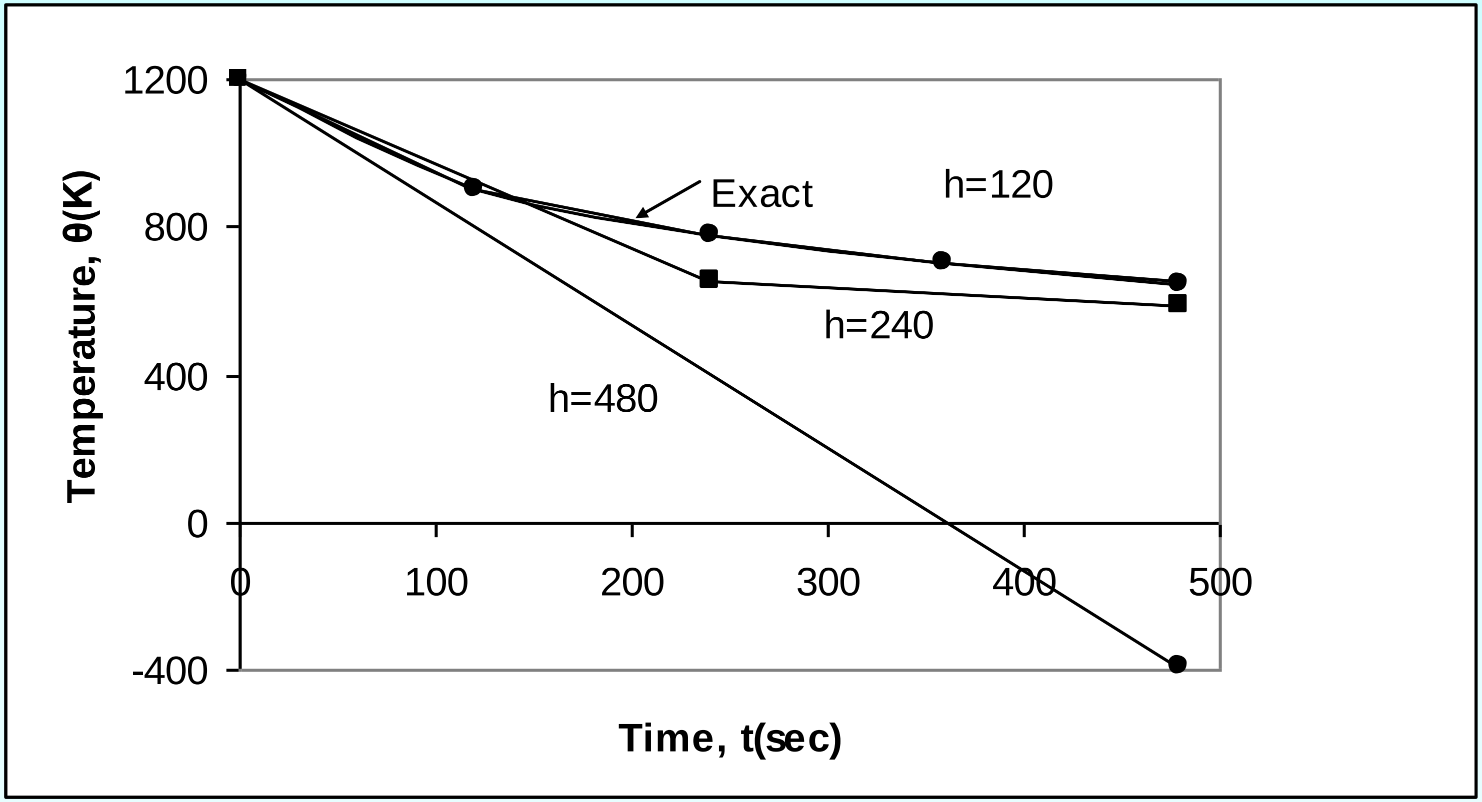

Figure 2. Heun's method results for different step sizes

Effect of step size

Table 1. Temperature at 480 seconds as a function of step size, h

Step size, h	$\theta(480)$	E_t	$\lvert\epsilon_t\rvert\%$
480	−393.87	1041.4	160.82
240	584.27	63.304	9.7756
120	651.35	−3.7762	0.58313
60	649.91	−2.3406	0.36145
30	648.21	−0.63219	0.097625

$$\theta(480) = 647.57K \quad \text{(exact)}$$

Effects of step size on Heun's Method

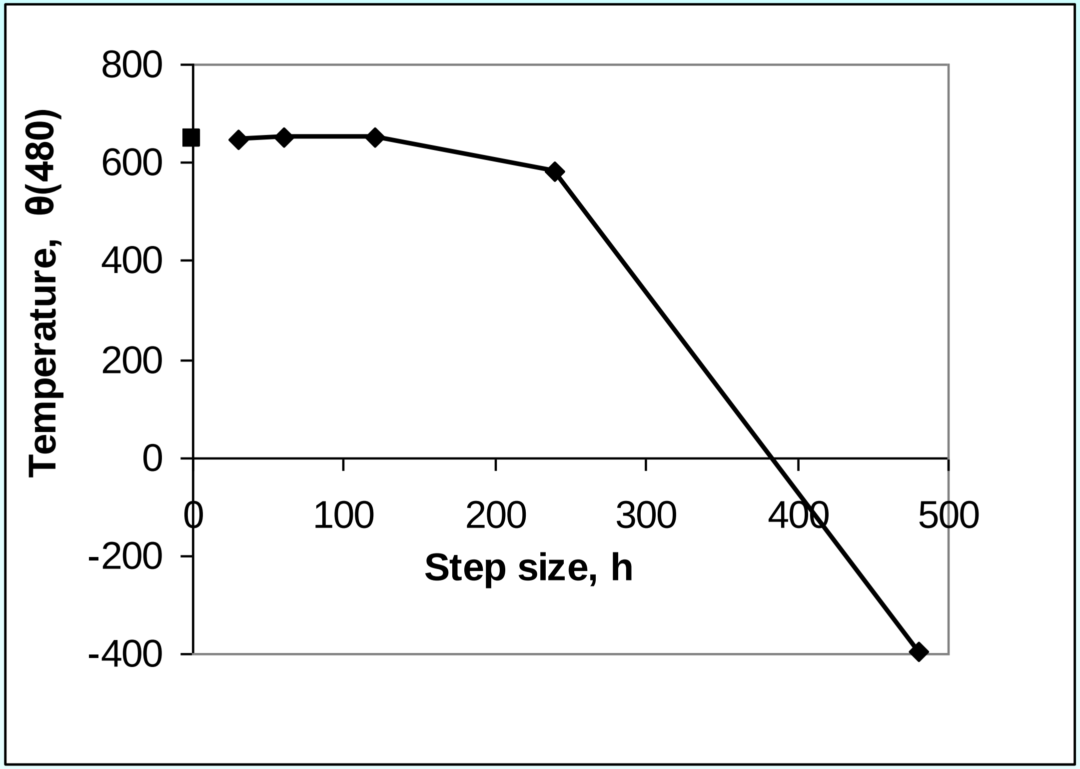

Figure 3. Effect of step size in Heun's method

Comparison of Euler and Runge-Kutta 2^{nd} Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\theta(480)$			
	Euler	Heun	Midpoint	Ralston
480	−987.84	−393.87	1208.4	449.78
240	110.32	584.27	976.87	690.01
120	546.77	651.35	690.20	667.71
60	614.97	649.91	654.85	652.25
30	632.77	648.21	649.02	648.61

$$\theta(480) = 647.57K$$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

Table 2. Comparison of Euler and the Runge-Kutta methods

Step size, h	$\lvert \in_t \rvert \%$			
	Euler	Heun	Midpoint	Ralston
480	252.54	160.82	86.612	30.544
240	82.964	9.7756	50.851	6.5537
120	15.566	0.58313	6.5823	3.1092
60	5.0352	0.36145	1.1239	0.72299
30	2.2864	0.097625	0.22353	0.15940

$\theta(480) = 647.57K$ (exact)

Comparison of Euler and Runge-Kutta 2nd Order Methods

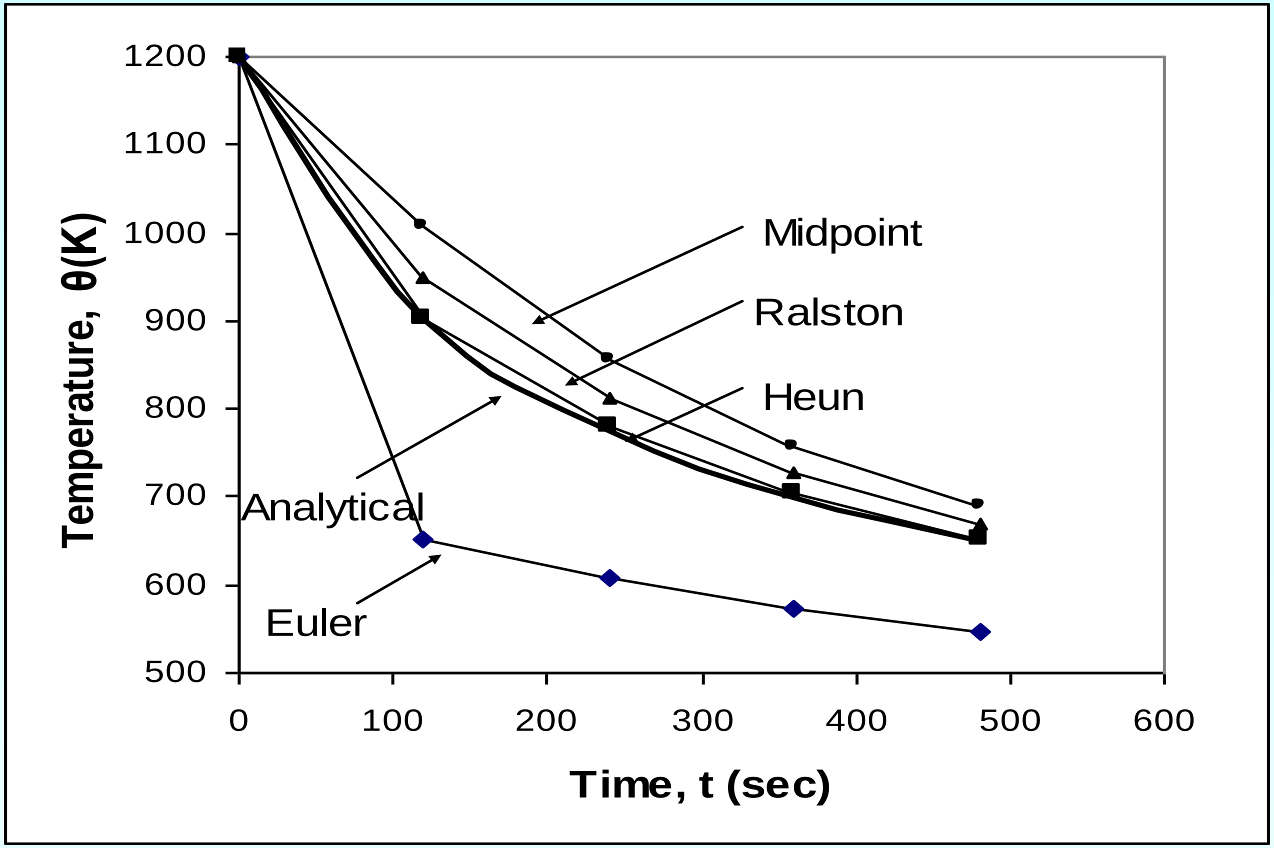

Figure 4. Comparison of Euler and Runge Kutta 2nd order methods with exact results.

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