# Mathematics for Computer Science

# CSE 401

## Poisson Distribution

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# Poisson Distribution

Siméon Denis Poisson (1781-1840)

**Poisson distribution**, in statistics, a distribution function useful for characterizing events with very low probabilities of occurrence within some definite time or space. The French mathematician Siméon-Denis Poisson developed his function in 1830

It gives us the **probability of a given number of events happening in a fixed interval of time**.

# Poisson Distribution

The Poisson distribution is used to model the number of events occurring within a given time interval.

Poisson distribution formula is used to find the probability of an event that happens independently, discretely over a fixed time period, when the mean rate of occurrence is constant over time. The Poisson distribution formula is applied when there is a large number of possible outcomes. For a random discrete variable X that follows the Poisson distribution, and λ is the average rate of value, then the probability of x is given by

$f(x) = P(X=x) = (e^{-\lambda} \lambda^{x})/x!$

Where

- $x = 0, 1, 2, 3\ldots$
- e is the Euler's number($e = 2.718$)
- λ is an average rate of the expected value and λ = variance, also $\lambda>0$

For Poisson distribution, the mean and the variance of the distribution are equal.

For the Poisson distribution, λ is always greater than 0.

# Poisson Distribution

Let $X$ be the discrete random variable that represents the number of events observed over a given time period. Let $\lambda$ be the expected value (average) of $X$. If $X$ follows a Poisson distribution, then the probability of observing $k$ events over the time period is:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!},$$

where $e$ is Euler's number.

A **discrete random variable** is a random variable which takes only finitely many or countably infinitely many different values.
In probability theory, an **expected value** is the theoretical mean value of a numerical experiment over many repetitions of the experiment.

# Poisson Distribution

In the World Cup, an average of 2.5 goals are scored each game. Modeling this situation with a Poisson distribution, what is the probability that *k* goals are scored in a game?

In this instance, $\lambda = 2.5$. The above formula applies directly:

$$P(X = 0) = \frac{2.5^0 e^{-2.5}}{0!} \approx 0.082$$

$$P(X = 1) = \frac{2.5^1 e^{-2.5}}{1!} \approx 0.205$$

$$P(X = 2) = \frac{2.5^2 e^{-2.5}}{2!} \approx 0.257$$

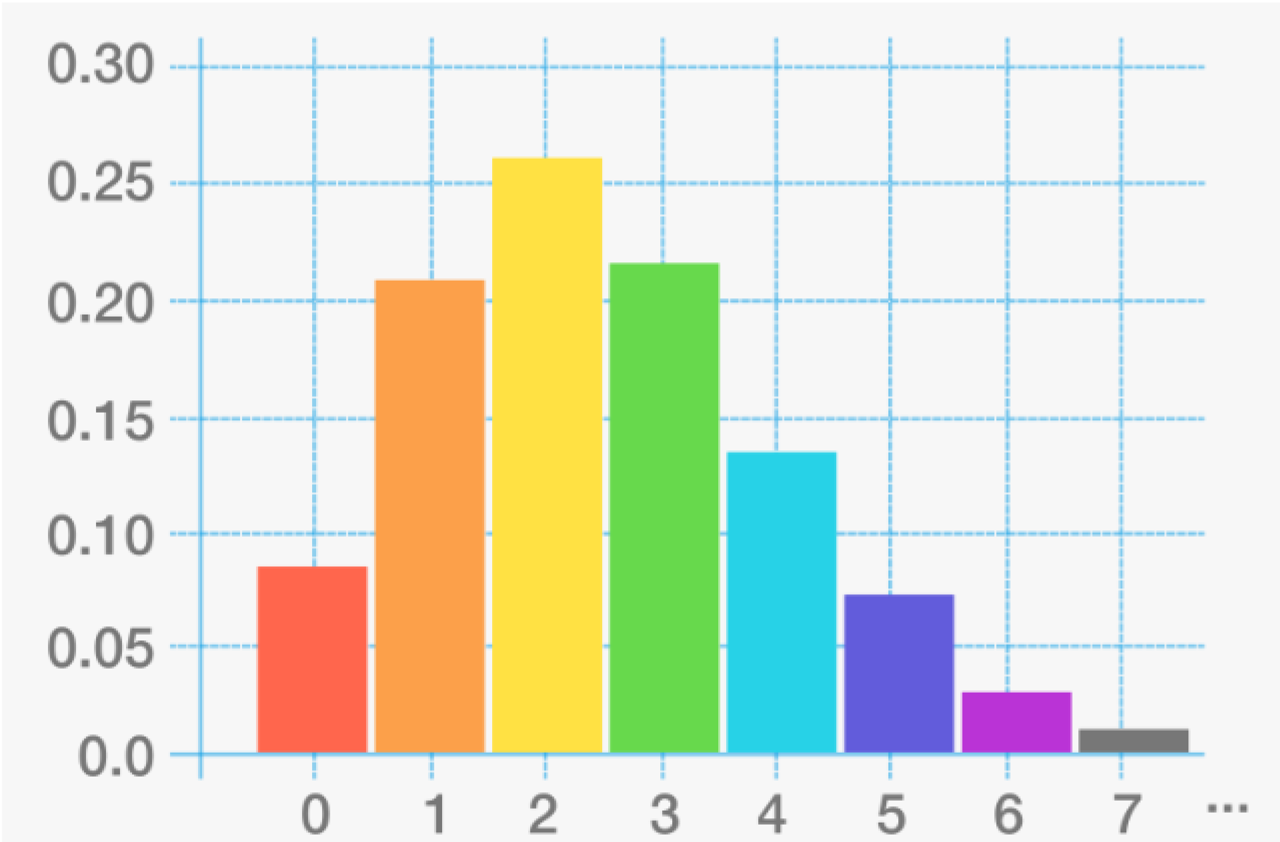
$$P(X = 3) = \frac{2.5^3 e^{-2.5}}{3!} \approx 0.213$$

$$P(X = 4) = \frac{2.5^4 e^{-2.5}}{4!} \approx 0.133$$

And so on up to $X = k$

# Poisson Distribution

There is no upper limit on the value of $k$ for this formula, though the probability rapidly approaches 0 as $k$ increases.




*The Poisson distribution with* $\lambda = 2.5$

# Poisson Distribution

A statistician records the number of cars that approach an intersection. He finds that an average of 1.6 cars approach the intersection every minute.

Assuming the number of cars that approach this intersection follows a Poisson distribution, what is the probability that 3 or more cars will approach the intersection within a minute?

For this problem, $\lambda=1.6$. The goal of this problem is to find $P(X\geq 3)$, the probability that there are 3 or more cars approaching the intersection within a minute. Since there is no upper limit on the value of $k$, this probability cannot be computed directly. However, its complement, $P(X\leq 2)$, can be computed to give $P(X\geq 3)$:

# Poisson Distribution

$$P(X=0)=\frac{1.6^0 e^{-1.6}}{0!}\approx 0.202$$

$$P(X=1)=\frac{1.6^1 e^{-1.6}}{1!}\approx 0.323$$

$$P(X=2)=\frac{1.6^2 e^{-1.6}}{2!}\approx 0.258$$

$$\begin{aligned}\Rightarrow P(X\leq 2) &= P(X=0)+P(X=1)+P(X=2)\\ &\approx 0.783\end{aligned}$$

$$\begin{aligned}\Rightarrow P(X\geq 3) &= 1-P(X\leq 2)\\ &\approx 0.217.\end{aligned}$$

Therefore, the probability that there are 3 or more cars approaching the intersection within a minute is approximately 0.217

# Why we need Poisson Distribution?

Poisson distribution used in cases where the chance of any individual event being a success is very small. The distribution is used to describe the behaviour of rare events.

Examples;

- The number of defective screws per box of 5000 screws.
- The number of printing mistakes in each page of the first proof of book.
- The number of air accidents in India in one year.
- Occurrence of number of scratches on a sheet of glass.

# Example of Poisson Distribution

Q- The average number of accidents at a particular intersection every <u>year</u> is 18.
(a) Calculate the probability that there are <u>exactly 2</u> accidents there this *<u>month</u>*.

There are 12 months in a year, so $\lambda = \dfrac{18}{12}$

$= \mathbf{1.5}$ accidents per **<u>month</u>**

$$P(X = 2) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \frac{e^{-1.5}1.5^{2}}{2!}$$

$$= \mathbf{0.2510}$$

# Example of Poisson Distribution

(b) Calculate the probability that there is <u>at least one accident</u> this month.

$P(X \geq 1) = P(X=1) + P(X=2) + P(X=3) + \ldots$ Infinite.

So… Take the complement: **P(X=0)** $= \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$$= \frac{e^{-1.5}1.5^{0}}{0!}$$

$$= \frac{e^{-1.5}\times 1}{1}$$

$$= \mathbf{0.22313}$$

So $P(X \geq 1) = 1 - P(X=0)$

$= 1 - 0.22313$

$= 0.7769$

# Example of Poisson Distribution

c What is the probability that there are *more than 2* accidents in a particular month

There are an infinite number of cases so instead consider $X \leq 2$

**b** $P(X > 2) = \mathbf{1 - P(X \leq 2)}$

$= 1 - [\,P(X = 0) + P(X = 1) + P(X = 2)]$

$= 1 - [\,\mathbf{0.2231 + 0.3347 + 0.2510}]$

$= 1 - \mathbf{0.8088}$

$= \mathbf{0.1912}$

# Example of Poisson Distribution

**The average number of major storms in your city is 2 per year. What is the probability that exactly 3 storms will hit your city next year?**

Step 1: Figure out the components you need to put into the equation.

- $\mu = 2$ (average number of storms per year, historically)
- $x = 3$ (the number of storms we think might hit next year)
- $e = 2.71828$ (e is Euler's number, a constant)

Step 2: Plug the values from Step 1 into the Poisson distribution formula:

- $P(x; \mu) = (e^{-\mu}) (\mu^{x}) / x!$
- $= (2.71828^{-2}) (2^{3}) / 3!$
- $= (0.13534) (8) / 6$
- $= 0.180$

The probability of 3 storms happening next year is 0.180, or 18%

# Example of Poisson Distribution

**Question:** As only 3 students came to attend the class today, find the probability for exactly 4 students to attend the classes tomorrow.

**Solution:**

Given,

Average rate of value($\lambda$) = 3

Poisson random variable(x) = 4

Poisson distribution = P(X = x) = $\frac{e^{-\lambda}\lambda^{x}}{x!}$

$$P(X = 4) = \frac{e^{-3}.3^{4}}{4!}$$

$$P(X = 4) = 0.16803135574154$$

# Example of Poisson Distribution

A call center receives an average of 4.5 calls every 5 minutes. Each agent can handle one of these calls over the 5 minute period. If a call is received, but no agent is available to take it, then that caller will be placed on hold. Assuming that the calls follow a Poisson distribution, what is the minimum number of agents needed on duty so that calls are placed on hold at most 10% of the time?

In order for all calls to be taken, the number of agents on duty should be greater than or equal to the number of calls received. If $X$ is the number of calls received and $k$ is the number of agents, then $k$ should be set such that $P(X>k) \leq 0.1$, or equivalently, $P(X \leq k) > 0.9$.

# Example of Poisson Distribution

The average number of calls is 4.5, so $\lambda = 4.5$ :

$$
\begin{aligned}
P(X=0) &= \frac{4.5^0 e^{-4.5}}{0!} \approx 0.011 \\
P(X=1) &= \frac{4.5^1 e^{-4.5}}{1!} \approx 0.050 \quad \Longrightarrow \quad P(X \leq 1) \approx 0.061 \\
P(X=2) &= \frac{4.5^2 e^{-4.5}}{2!} \approx 0.112 \quad \Longrightarrow \quad P(X \leq 2) \approx 0.173 \\
P(X=3) &= \frac{4.5^3 e^{-4.5}}{3!} \approx 0.169 \quad \Longrightarrow \quad P(X \leq 3) \approx 0.342 \\
P(X=4) &= \frac{4.5^4 e^{-4.5}}{4!} \approx 0.190 \quad \Longrightarrow \quad P(X \leq 4) \approx 0.532 \\
P(X=5) &= \frac{4.5^5 e^{-4.5}}{5!} \approx 0.171 \quad \Longrightarrow \quad P(X \leq 5) \approx 0.703 \\
P(X=6) &= \frac{4.5^6 e^{-4.5}}{6!} \approx 0.128 \quad \Longrightarrow \quad P(X \leq 6) \approx 0.831 \\
P(X=7) &= \frac{4.5^7 e^{-4.5}}{7!} \approx 0.082 \quad \Longrightarrow \quad P(X \leq 7) \approx 0.913.
\end{aligned}
$$

If the goal is to make sure that less than 10% of calls are placed on hold, then $\boxed{7}$ agents should be on duty. □

# Thank you