Mathematics for Computer Science
CSE 401
Random Variable

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Probability Theory and Random Variable

PROBABILITY AND STATISTICS : RANDOM VARIABLES AND PROBABILITY DISTRIBUTION (1) RANDOM VARIABLE

RANDOM VARIABLE ≡ IS A VARIABLE WHOSE OUTCOME (VALUE) IS SUBJECT TO A RANDOM PROCESS, WHICH MEANS THAT ITS VALUE IS DETERMINED BY CHANCE

EXAMPLE: LET X = # OF HEADS WHEN FLIPPING A COIN 3 TIMES

$S = \{TTT, TTH, THT, THH, HTT, HTH, HHT, HHH\}$

T T T	$X=0$
T T H	$X=1$
T H T	$X=1$
T H H	$X=2$
H T T	$X=1$
H T H	$X=2$
H H T	$X=2$
H H H	$X=3$

$X=0$	(1 OCCURRENCE)	$P(X=0) = \frac{1}{8}$
$X=1$	(3 OCCURRENCES)	$P(X=1) = \frac{3}{8}$
$X=2$	(3 OCCURRENCES)	$P(X=2) = \frac{3}{8}$
$X=3$	(1 OCCURRENCE)	$P(X=3) = \frac{1}{8}$

SAMPLE POINT ≡ OUTCOME OF AN EXPERIMENT

SAMPLE SPACE ≡ SET OF ALL POSSIBLE SAMPLE POINTS (OR OUTCOMES)

Probability Theory and Random Variable

Sample Space and Events

Suppose that we are about to perform an experiment whose outcome is not predictable in advance.

However, while the outcome of the experiment will not be known in advance, let us suppose that **the set of all possible outcomes is known.**

This **set of all possible outcomes** of an experiment is known as the ***sample space*** of the experiment and is denoted by ***S*.**

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Some examples are the following.

1. If the experiment consists of the flipping of a coin, then
$S = \{H, T\}$

where *H means that the outcome of the toss is a head and T that it is a tail.*

2. If the experiment consists of rolling a die, then the sample space is $S = \{1, 2, 3, 4, 5, 6\}$
where the outcome *i means that i appeared on the die*, $i = 1, 2, 3, 4, 5, 6$.

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Some examples are the following.

3. If the experiments consists of flipping two coins, then the **sample space** consists of the following four points:

S = {(H,H), (H, T), (T,H), (T, T)}

The outcome will be *(H,H) if both coins come up heads; it will be (H, T) if the* first coin comes up heads and the second comes up tails; it will be *(T,H) if the* first comes up tails and the second heads; and it will be *(T, T) if both coins come* up tails.

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Some examples are the following.

If the experiment consists of rolling two dice, then the sample space consists of the following **36** points. the number of total possible outcomes is equal to the **sample space** of the first die (6) multiplied by the **sample space** of the second die (6), which is 36.

where the outcome *(i, j) is said to occur if i appears on the first die and j on the second* die.

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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Some examples are the following.

Three or More Dice

The same principle applies if we are working on problems involving three dice. We multiply and see that there are **6 x 6 x 6 = 216 possible outcomes.** As it gets cumbersome to write the repeated multiplication, we can use **exponents** to simplify work. For two dice, there are 6^2 possible outcomes. For three dice, there are 6^3 possible outcomes. In general, if we roll ***n*** dice, then there are a total of $\mathbf{6}^{\boldsymbol{n}}$ possible outcomes.

Probability Theory and Random Variable

Sample Problems

1. Two six-sided dice are rolled. What is the probability that the sum of the two dice is seven? Explain your answer

The easiest way to solve this problem is to consult the **previous table**. You will notice that in each row there is one dice roll where the sum of the two dice is equal to seven. Since there are six rows, there are **six possible outcomes** where the sum of the two dice is equal to **seven.** The number of total possible outcomes remains 36. Again, we find the probability by dividing the event frequency (6) by the size of the **sample space** (36), resulting in a probability of **1/6.**

(6/36 = 1/6)

EVENT = A SUBSET OF A SAMPLE SPACE (OF AN EXPERIMENT)
(EVENT IS 1 OR MORE OUTCOMES OR SAMPLE POINTS)

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Sample Problems

2. Two six-sided dice are rolled. What is the probability that the sum of the two dice is three?

In the previous problem, you may have noticed that the cells where the sum of the two dice is equal to seven form a diagonal. The same is true here, except in this case there are only two cells where the sum of the dice is three. That is because there are only two ways to get this outcome. You must roll a 1 and a 2 or you must roll a 2 and a 1. The combinations for rolling a sum of seven are much greater (1 and 6, 2 and 5, 3 and 4, and so on). To find the probability that the sum of the two dice is three, we can divide the event frequency (2) by the size of the sample space (36), resulting in a probability of 1/18.

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Sample Problems

3. Two six-sided dice are rolled. What is the probability that the numbers on the dice are different?

Again, we can easily solve this problem by consulting the table above. You will notice that the cells where the numbers on the dice are the **same** form a diagonal. There are **only six of them**, and once we cross them out we have **the remaining cells in which the numbers on the dice are different.** We can take the number of combinations (30) and divide it by the size of the sample space (36), resulting in a probability of 5/6.

Probability Theory and Random Variable

Probabilities Defined on Events

Consider an experiment whose sample space is *S. For each event E of the sample* space *S, we assume that a number P(E) is defined and satisfies the following three* conditions:

(i) $0 \leqslant P(E) \leqslant 1$.
(ii) $P(S) = 1$.
(iii) For any sequence of events $E_1, E_2, \ldots$ that are mutually exclusive, that is, events for which $E_n E_m = \varnothing$ when $n \neq m$, then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

We refer to $P(E)$ as the probability of the event E.

Probability Theory and Random Variable

Probabilities Defined on Events

Example 1.1 In the coin tossing example, if we assume that a head is equally likely to appear as a tail, then we would have

$$P(\{H\}) = P(\{T\}) = \tfrac{1}{2}$$

On the other hand, if we had a biased coin and felt that a head was twice as likely to appear as a tail, then we would have

$$P(\{H\}) = \tfrac{2}{3}, \qquad P(\{T\}) = \tfrac{1}{3} \qquad \blacksquare$$

Example 1.2 In the die tossing example, if we supposed that all six numbers were equally likely to appear, then we would have

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \tfrac{1}{6}$$

Probability Theory and Random Variable

Random variables

Example

Flipping three coins, let X count the number of Heads obtained. Then, as a function on Ω,

$$
\begin{aligned}
&X(T, T, T) = 0;\\
&X(T, T, H) = X(T, H, T) = X(H, T, T) = 1;\\
&X(T, H, H) = X(H, T, H) = X(H, H, T) = 2;\\
&X(H, H, H) = 3.
\end{aligned}
$$

Instead, we'll just say that X can take on values $0, 1, 2, 3$ with respective probabilities $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$.

How did we get $\frac{3}{8}$? Well,

$\mathbf{P}\{X = 1\} = \mathbf{P}\{(T, T, H), (T, H, T), (H, T, T)\} = \frac{3}{8}$.

Probability Theory and Random Variable

Since the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of the random variable.

Example 2.1 Letting X denote the random variable that is defined as the sum of two fair dice; then

$$P\{X = 2\} = P\{(1,1)\} = \tfrac{1}{36},$$
$$P\{X = 3\} = P\{(1,2),(2,1)\} = \tfrac{2}{36},$$
$$P\{X = 4\} = P\{(1,3),(2,2),(3,1)\} = \tfrac{3}{36},$$
$$P\{X = 5\} = P\{(1,4),(2,3),(3,2),(4,1)\} = \tfrac{4}{36},$$
$$P\{X = 6\} = P\{(1,5),(2,4),(3,3),(4,2),(5,1)\} = \tfrac{5}{36},$$

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$$
\begin{aligned}
P\{X=7\} &= P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = \tfrac{6}{36}, \\
P\{X=8\} &= P\{(2,6),(3,5),(4,4),(5,3),(6,2)\} = \tfrac{5}{36}, \\
P\{X=9\} &= P\{(3,6),(4,5),(5,4),(6,3)\} = \tfrac{4}{36}, \\
P\{X=10\} &= P\{(4,6),(5,5),(6,4)\} = \tfrac{3}{36}, \\
P\{X=11\} &= P\{(5,6),(6,5)\} = \tfrac{2}{36}, \\
P\{X=12\} &= P\{(6,6)\} = \tfrac{1}{36} \qquad (2.1)
\end{aligned}
$$

In other words, the random variable X can take on any integral value between two and twelve, and the probability that it takes on each value is given by Equation (2.1). Since X must take on one of the values two through twelve, we must have

$$1 = P\left\{\bigcup_{i=2}^{12} \{X=n\}\right\} = \sum_{n=2}^{12} P\{X=n\}$$

which may be checked from Equation (2.1). ■

Probability Theory and Random Variable

Example 2.2 For a second example, suppose that our experiment consists of tossing two fair coins. Letting Y denote the number of heads appearing, then Y is a random variable taking on one of the values 0, 1, 2 with respective probabilities

$$P\{Y=0\}=P\{(T,T)\}=\tfrac{1}{4},$$
$$P\{Y=1\}=P\{(T,H),(H,T)\}=\tfrac{2}{4},$$
$$P\{Y=2\}=P\{(H,H)\}=\tfrac{1}{4}$$

Of course, $P\{Y=0\}+P\{Y=1\}+P\{Y=2\}=1$. ■

Probability Model, By: Sheldon Ross