Mathematics for Computer Science
CSE 401
Big Number
Ackermann Function

Shah Murtaza Rashid Al Masud, PhD
Department of CSE
University of Asia Pacific

Googol, Googolplex

What does **Google** mean?

Google is a misspelling of a real life mathematical term, **Googol**.

The name “Google” actually came from a graduate student at Stanford named Sean Anderson.

Anderson suggested the word “googolplex” during a brainstorming session, and **Larry Page** replied with the shortest “googol.”

The name of Google is derived from the number googol, which is equal to the number 1 followed by 100 zeros (1×10^{100}). A googol is 10 to the 100th power, which is 1 followed by 100 zeros.

And **Backrub** was the original name for the search engine until **Sergey Brin** and **Larry Page** rebranded the company as Google, Inc.

A googol is 10 to the 100th power, which is 1 followed by 100 zeros.

Googol, Googolplex

A *googol* is a 1 followed by 100 zeros (or 10^{100}). It was given its whimsical name in 1937 by mathematician Edward Kasner's young nephew Milton Sirotta, and became famous when an internet search engine, wanting to suggest that it could process a huge amount of data, named itself Google.

A (big) step up from there is *googolplex*, which is 10^{googol} , or 1 with a googol of zeros. This is the last widely accepted name for a really big number. People have tried to get *googolplexian* ($10^{\text{googolplex}}$) and *googolduplex* (same thing) to catch on, but there doesn't seem to be enough occasion to use them.

Googol, Googolplex

Googol is a mathematical term to describe a huge quantity.

A googol has no special significance in mathematics.

But, it is useful when comparing with other very large quantities.

Googol (10^{100})

10, 000, 000, 000, 000,
000, 000, 000, 000, 000,
000, 000, 000, 000, 000,
000, 000, 000, 000, 000,
000, 000, 000, 000, 000,
000, 000, 000, 000, 000,
000, 000, 000, 000

Googol, Googolplex

Big numbers in money some of you didn’t know about …

5 × 10^11 Yugoslav dinar banknotes from 1993.

500,000,000,000 Yugoslav dinar banknote, featuring image of Jovan Jovanovic Zmaj

Googol, Googolplex

1014 Zimbabwean dollars banknote from 2009.

Googol, Googolplex

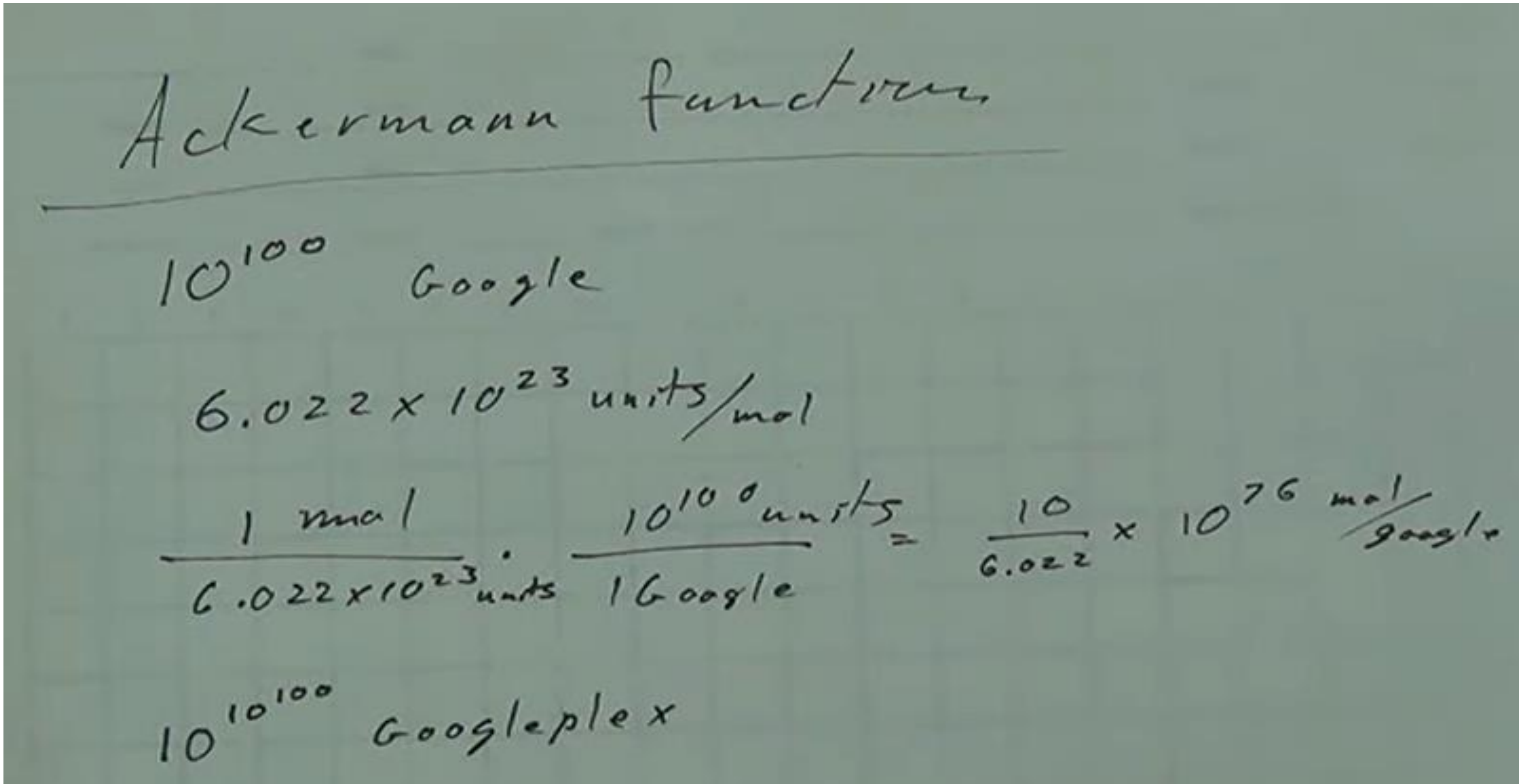
Infinity

Infinity is not a number. But there is nothing as large as **infinity.**
Something that is larger than any real or natural number. It is often denoted by the infinity symbol ∞.

Infinity represents something that is boundless or endless.

Ackermann Function (example of recursion)

Very Big number



Ackermann Function (function)

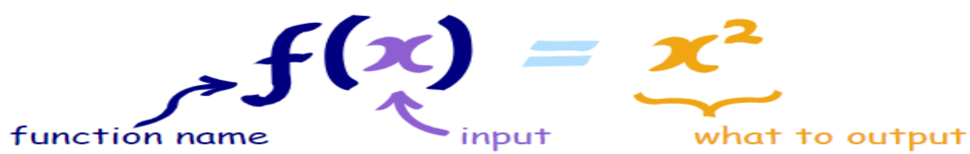
Function, in mathematics, an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable).

We will see many ways to think about functions, but there are always three main parts and a function relates an input to an output:

The input

The relationship

The output



Example: with $\mathbf{f(x) = x^2}$:

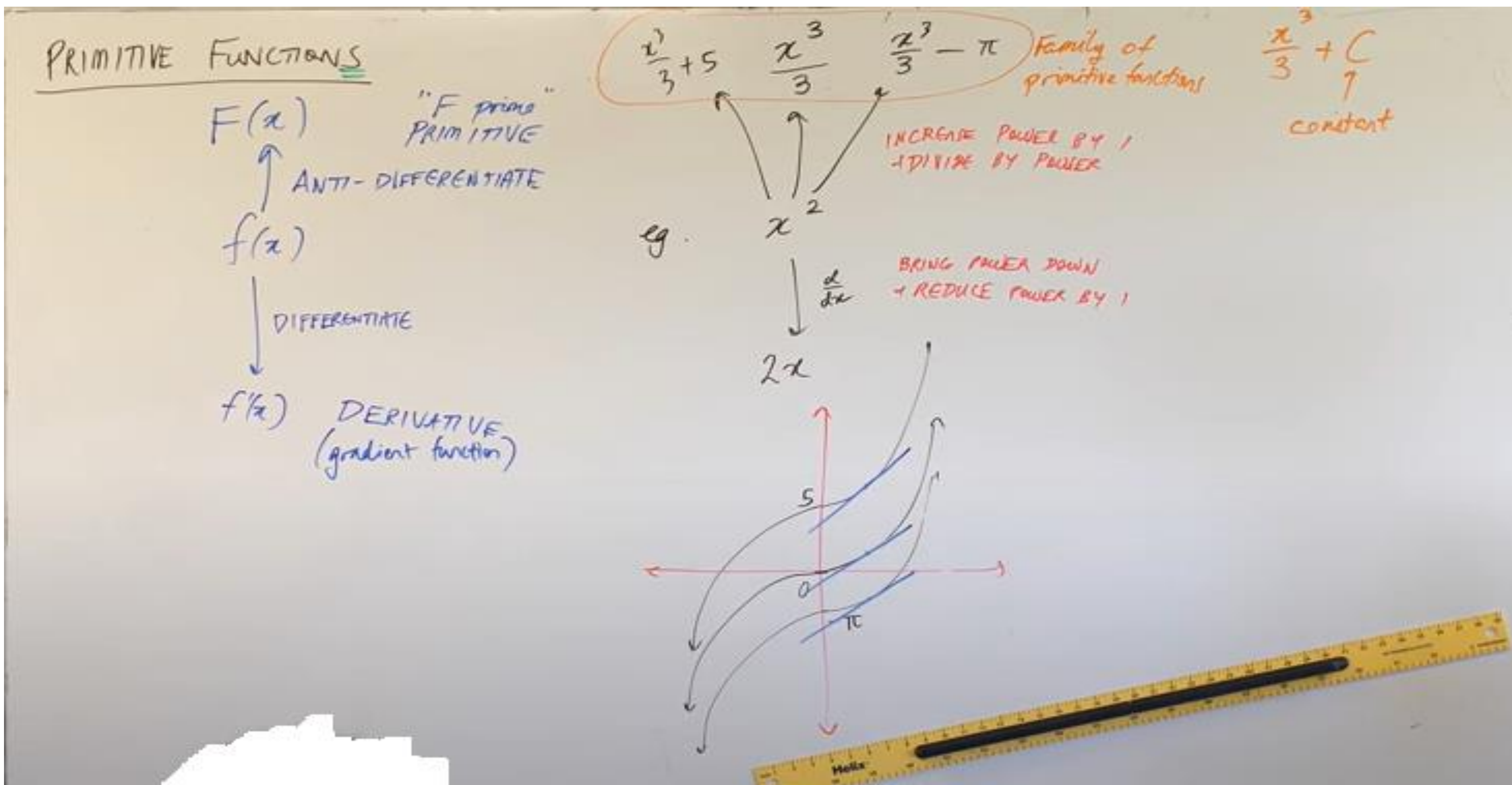
an input of 4

becomes an output of 16.

In fact we can write $\mathbf{f(4) = 16}$.

Ackermann Function (example of recursion)

Ackermann Function is a very famous rapidly-increasing function introduced by **Ackermann** to settle a problem in logic. He wanted to show that there exist “general recursive” functions that are not “primitive recursive” is Ackermann’s function. The primitive recursive functions are a subset of the total/general recursive functions, which are a subset of the partial recursive functions. For example, the Ackermann function can be proven to be total recursive, and to be non-primitive.



Ackermann Function (example of recursion)

a function is primitive recursive if it can be defined using **a finite number of basic operations and recursion, and it can be computed using a finite number of steps.** This concept is used to study the limits of computability and to classify the complexity of functions in the theory of computability.

A general recursive function is called **total recursive function** if it is defined for every input, or, equivalently, if it can be computed by a total Turing machine (decider of undecidable problem).

Ackermann Function (example of recursion)

In computability theory, the Ackermann function, named after **Wilhelm Ackermann**, is one of the simplest and earliest-discovered examples of a total computable function that is not primitive recursive. All primitive recursive functions are total and computable, but the **Ackermann function** illustrates that not all total computable functions are primitive recursive.

Any **computable function** can be incorporated into a program using while-loops (i.e. repeatedly executes a target statement as long as a given condition is true). For-loops (which have a fixed iteration limit) are a special case of while-loops, so computable functions could also be coded using a combination of for- and while-loops.

Ackermann Function (example of recursion)

As first shown by Meyer and Ritchie (1967), for-loops (which have a fixed iteration limit) are a special case of while-loops. A function that can be implemented using only for-loops is called **primitive recursive**. (In contrast, a computable function can be coded using a combination of for- and while-loops, or while-loops).

The **Ackermann function** is the simplest example of a well-defined total function that is computable but not primitive recursive.

Ackermann Function (example of recursion)

The function we wish to calculate is called Ackermann's function, and it is defined as follows:

Ackermann Function: $A(m,n)$, $m, n \in N$

for nonnegative integers *m* and *n*

1) $m=0$: $A(0,n)=n+1$

2) $m\neq 0, n=0$: $A(m,0)=A(m-1,1)$

3) $m\neq 0, n\neq 0$: $A(m,n)=A(m-1, A(m,n-1))$

Example:

$$\mathbf{A(1,1)= A(0, A(1,0)) = A(0, A(0,1)) =A(0,2)=2+1=3}$$

$$A(2,0) =A(1,1)=3$$

Ackermann Function (example of recursion)

- (0,0) (0,1) (0,2) (0,3) (0,4) (0,5) (0,6) (0,7)
  1 2 3 4 5 6 7 8 A(0,n)=n+1
- (1,0) (1,1) (1,2) (1,3) (1,4) (1,5) (1,6) (1,7)
  2 3 4 5 6 7 8 9 A(1,n)=n+2
- (2,0) (2,1) (2,2) (2,3) (2,4)(2,5)
  3 5 7 9 11 13 A(2,n)=2n+3
- (3,0) (3,1) (3,2)
  5 13 29 A(3,n)= $2^{n+3}-3$
- (4,0)(4,1)
  13 65533 **A(4,n)=$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$-3** ($\underbrace{\quad}_{\text{n+3 2's}}$)

$$A(4,1)=2^{2^{2^{2}}}-3=2^{2^{4}}=2^{16}-3=65536-3$$

Ackermann Function (example of recursion)

The function we wish to calculate is called Ackermann's function, and it is defined as follows:

$$A(m,n) = \begin{cases} n+1 & : \text{ if m} = 0 \\ A(m-1,1) & : \text{ if m} > 0, \text{n} = 0 \\ A(m-1, A(m, n-1)) & : \text{ if m} > 0, \text{n} > 0 \end{cases} \tag{1}$$

The easiest way to understand it is to make a table of the values of $A(m,n)$ beginning with the easy ones. In the table below, n increases to the right, and the rows correspond to $m = 0, 1, 2, 3, \ldots$. The "*"s indicate values we have not yet determined. The first row, where $m = 0$, is easy. The general formula is obviously $A(0,n) = n+1$.

$m \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10	11
1	*	*	*	*	*	*	*	*	*	*	*
2	*	*	*	*	*	*	*	*	*	*	*
3	*	*	*	*	*	*	*	*	*	*	*
4	*	*	*	*	*	*	*	*	*	*	*

The next row where $m = 1$ is a bit trickier. If $n = 0$ we can use the second line in formula 1 that defines Ackermann's function to obtain: $A(1,0) = A(0,1) = 2$.

Ackermann Function (example of recursion)

What is $A(1,1)$? We must apply the third row in formula 1 to obtain: $A(1,1) = A(0, A(1,0)) = A(0,2) = 3$. Similarly: $A(1,2) = A(0, A(1,1)) = A(0,3) = 4$. Make sure you understand what is going on by working out a few more, and finally we can fill out the second row of the table as follows.

$m \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	*	*	*	*	*	*	*	*	*	*	*
3	*	*	*	*	*	*	*	*	*	*	*
4	*	*	*	*	*	*	*	*	*	*	*

The general formula for this row is $A(1,n) = n + 2$.

Ackermann Function (example of recursion)

The third row can be approached in the same way:

$$
\begin{aligned}
A(2,0) &= A(1,1) = 3 \\
A(2,1) &= A(1,A(2,0)) = A(1,3) = 5 \\
A(2,2) &= A(1,A(2,1)) = A(1,5) = 7 \\
A(2,3) &= A(1,A(2,2)) = A(1,7) = 9
\end{aligned}
$$

We can continue (do so for a few more) to obtain the third row:

$m \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	3	5	7	9	11	13	15	17	19	21	23
3	*	*	*	*	*	*	*	*	*	*	*
4	*	*	*	*	*	*	*	*	*	*	*

The general formula for this row is A(2, n) = 2n + 3.

Ackermann Function (example of recursion)

Repeat the process for $A(3, n)$:

$$\begin{aligned}
A(3,0) &= A(2,1) = 5 \\
A(3,1) &= A(2, A(3,0)) = A(2,5) = 13 \\
A(3,2) &= A(2, A(3,1)) = A(2,13) = 29 \\
A(3,3) &= A(2, A(3,2)) = A(2,29) = 61
\end{aligned}$$

If you do a few more, you will see that the table now looks like this:

$m \backslash n$	0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10	11
1	2	3	4	5	6	7	8	9	10	11	12
2	3	5	7	9	11	13	15	17	19	21	23
3	5	13	29	61	125	253	509	1021	2045	4093	8189
4	*	*	*	*	*	*	*	*	*	*	*

The general formula for this row is $A(3, n) = 2^{n+3} - 3$.

Ackermann Function (example of recursion)

Beginning with the next line, things begin to get

$$
\begin{aligned}
A(4,0) &= A(3,1) = 13 = 2^{2^{2}} - 3 \\
A(4,1) &= A(3, A(4,0)) = A(3,13) = 65533 = 2^{2^{2^{2}}} - 3 \\
A(4,2) &= A(3, A(4,1)) = A(3,65533) = 2^{65536} - 3 = 2^{2^{2^{2^{2}}}} - 3 \\
A(4,3) &= A(3, A(4,2)) = A(3, 2^{65536} - 3) = 2^{2^{65536}} - 3 = 2^{2^{2^{2^{2^{2}}}}} - 3
\end{aligned}
$$

The general form for $A(4,n)$ is this: $A(4,0) = 2^{2^{2}} - 3$, $A(4,1) = 2^{2^{2^{2}}} - 3$, and in general, each time we increase the value of n, the height of the tower of exponents of 2 increases by 1.

$m \backslash n$	0	1	2	3	4	5	6	7	8
0	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	5	7	9	11	13	15	17	19
3	5	13	29	61	125	253	509	1021	2045
4	$2^{2^{2}} - 3$	$2^{2^{2^{2}}} - 3$	$2^{2^{2^{2^{2}}}} - 3$	$2^{2^{2^{2^{2^{2}}}}} - 3$	$2^{2^{2^{2^{2^{2^{2}}}}}} - 3$	$\cdots$			

Ackermann Function (example of recursion)

- Ackermann Function: grows faster than an exponential function

  $$A(4,2) = 2 \times 10^{19728}$$

- Googol: $10^{100} \approx 70!$
- Visible universe ≈ 10^{81} atoms

Ackermann Function (example of recursion)

Computing the Ackermann function can be restated in terms of an infinite table. First, place the natural numbers along the top row. To determine a number in the table, take the number immediately to the left. Then use that number to look up the required number in the column given by that number and one row up. If there is no number to its left, simply look at the column headed "1" in the previous row. Here is a small upper-left portion of the table:

Values of *A*(*m*, *n*)

m \ *n*	0	1	2	3	4	*n*
0	1	2	3	4	5	$n+1$
1	2	3	4	5	6	$n+2=2+(n+3)-3$
2	3	5	7	9	11	$2n+3=2\cdot(n+3)-3$
3	5	13	29	61	125	$2^{(n+3)}-3$
4	13 $=2^{2^{2}}-3$ $=2\uparrow\uparrow 3-3$	65533 $=2^{2^{2^{2}}}-3$ $=2\uparrow\uparrow 4-3$	$2^{65536}-3$ $=2^{2^{2^{2^{2}}}}-3$ $=2\uparrow\uparrow 5-3$	$2^{2^{65536}}-3$ $=2^{2^{2^{2^{2^{2}}}}}-3$ $=2\uparrow\uparrow 6-3$	$2^{2^{2^{65536}}}-3$ $=2^{2^{2^{2^{2^{2^{2}}}}}}-3$ $=2\uparrow\uparrow 7-3$	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n+3}-3$ $=2\uparrow\uparrow(n+3)-3$
5	65533 $=2\uparrow\uparrow(2\uparrow\uparrow 2)-3$ $=2\uparrow\uparrow\uparrow 3-3$	$2\uparrow\uparrow\uparrow 4-3$	$2\uparrow\uparrow\uparrow 5-3$	$2\uparrow\uparrow\uparrow 6-3$	$2\uparrow\uparrow\uparrow 7-3$	$2\uparrow\uparrow\uparrow(n+3)-3$
6	$2\uparrow\uparrow\uparrow\uparrow 3-3$	$2\uparrow\uparrow\uparrow\uparrow 4-3$	$2\uparrow\uparrow\uparrow\uparrow 5-3$	$2\uparrow\uparrow\uparrow\uparrow 6-3$	$2\uparrow\uparrow\uparrow\uparrow 7-3$	$2\uparrow\uparrow\uparrow\uparrow(n+3)-3$
m	$(2\rightarrow 3\rightarrow(m-2))-3$	$(2\rightarrow 4\rightarrow(m-2))-3$	$(2\rightarrow 5\rightarrow(m-2))-3$	$(2\rightarrow 6\rightarrow(m-2))-3$	$(2\rightarrow 7\rightarrow(m-2))-3$	$(2\rightarrow(n+3)\rightarrow(m-2))-3$

Ackermann Function (example of recursion)

The numbers here which are only expressed with recursive exponentiation or Knuth arrows are very large and would take up too much space to notate in plain decimal digits.

Despite the large values occurring in this early section of the table, some even larger numbers have been defined, such as Graham's number, which cannot be written with any small number of Knuth arrows. This number is constructed with a technique similar to applying the Ackermann function to itself recursively.

This is a repeat of the above table, but with the values replaced by the relevant expression from the function definition to show the pattern clearly:

Ackermann Function (example of recursion)

Values of $A(m, n)$

$m \backslash n$	0	1	2	3	4	n
0	0+1	1+1	2+1	3+1	4+1	$n + 1$
1	$A(0, 1)$	$A(0, A(1, 0))$ $= A(0, 2)$	$A(0, A(1, 1))$ $= A(0, 3)$	$A(0, A(1, 2))$ $= A(0, 4)$	$A(0, A(1, 3))$ $= A(0, 5)$	$A(0, A(1, n-1))$
2	$A(1, 1)$	$A(1, A(2, 0))$ $= A(1, 3)$	$A(1, A(2, 1))$ $= A(1, 5)$	$A(1, A(2, 2))$ $= A(1, 7)$	$A(1, A(2, 3))$ $= A(1, 9)$	$A(1, A(2, n-1))$
3	$A(2, 1)$	$A(2, A(3, 0))$ $= A(2, 5)$	$A(2, A(3, 1))$ $= A(2, 13)$	$A(2, A(3, 2))$ $= A(2, 29)$	$A(2, A(3, 3))$ $= A(2, 61)$	$A(2, A(3, n-1))$
4	$A(3, 1)$	$A(3, A(4, 0))$ $= A(3, 13)$	$A(3, A(4, 1))$ $= A(3, 65533)$	$A(3, A(4, 2))$	$A(3, A(4, 3))$	$A(3, A(4, n-1))$
5	$A(4, 1)$	$A(4, A(5, 0))$	$A(4, A(5, 1))$	$A(4, A(5, 2))$	$A(4, A(5, 3))$	$A(4, A(5, n-1))$
6	$A(5, 1)$	$A(5, A(6, 0))$	$A(5, A(6, 1))$	$A(5, A(6, 2))$	$A(5, A(6, 3))$	$A(5, A(6, n-1))$

Probability Model, By: Knuth