Mathematics for Computer Science
CSE 401
Eigenvalues and Eigenvectors

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Definition of eigenvalues and eigenvectors of a matrix

'Eigen' is a German word that means 'proper' or 'characteristic'. Therefore, the term eigenvalue can be termed as characteristic value, characteristic root, proper values or latent roots as well. In simple words, the eigenvalue is a scalar that is used to transform the eigenvector.

The basic equation: Let **A** be any square matrix. A non-zero vector **x** is an **eigenvector** of **A** if $\mathbf{Ax} = \lambda\mathbf{x}$

for some number λ, called the corresponding **eigenvalue**.

That example demonstrates a very important concept in engineering and science - **eigenvalues and eigenvectors** - which is used widely in many applications, including calculus, search engines, population studies, aeronautics and so on.

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That example demonstrates a very important concept in engineering and science - **eigenvalues and eigenvectors** - which is used widely in many applications, including calculus, search engines, population studies, aeronautics and so on.

It has numerous applications in data science. They provide a way to analyze the structure of linear transformations and matrices, and are used extensively in many areas of **machine learning, including feature extraction, dimensionality reduction, and clustering,** facial recognition, Designing communication systems, Quantum computing, Electrical & mechanical engineering, Determining oil reserves by oil companies, Construction/bridge design, Stability of the system

Definition of eigenvalues and eigenvectors of a matrix

Let's take a look at the following picture (**Fig: Tiger explained using Eigenvectors & Eigenvalues**):

When you look at the above picture (**data**) and identify it as a tiger, what are some of the key information (dimensions / principal components) you use to call it out like a tiger? Is it not the face, body, legs, etc information? These principal components/dimensions can be seen as eigenvectors with each one of them having its own elements. For example, the body will have elements such as color, built, shape, etc. The face will have elements such as nose, eyes, color, etc. The overall data (image) can be seen as a transformation matrix. The data (transformation matrix) when acted on the eigenvectors (principal components) will result in the eigenvectors multiplied by the scale factor (eigenvalue). And, accordingly, you can identify the image as the tiger.

Definition of eigenvalues and eigenvectors of a matrix

The solution to real-world problems often depends upon processing a large volume of data representing different variables or dimensions. For example, take the problem of predicting the stock prices. This is a machine learning / predictive analytics problem. Here the dependent value is stock price and there are a large number of independent variables on which the stock price depends. Using a large number of independent variables (also called features), training one or more machine learning models for predicting the stock price will be **computationally intensive**. Such models turn out to be complex models.

Can we use the information stored in these variables and extract a smaller set of variables (features) to train the models and do the prediction while ensuring that most of the information contained in the original variables is retained/maintained. This will result in **simpler and computationally efficient models**. **This is where eigenvalues and eigenvectors come into the picture.**

Definition of eigenvalues and eigenvectors of a matrix

Eigenvalues and Eigenvectors concepts are key to training computationally efficient and high-performing machine learning models. Data scientists must understand these concepts very well.

The primary goal is to achieve optimal computational efficiency.

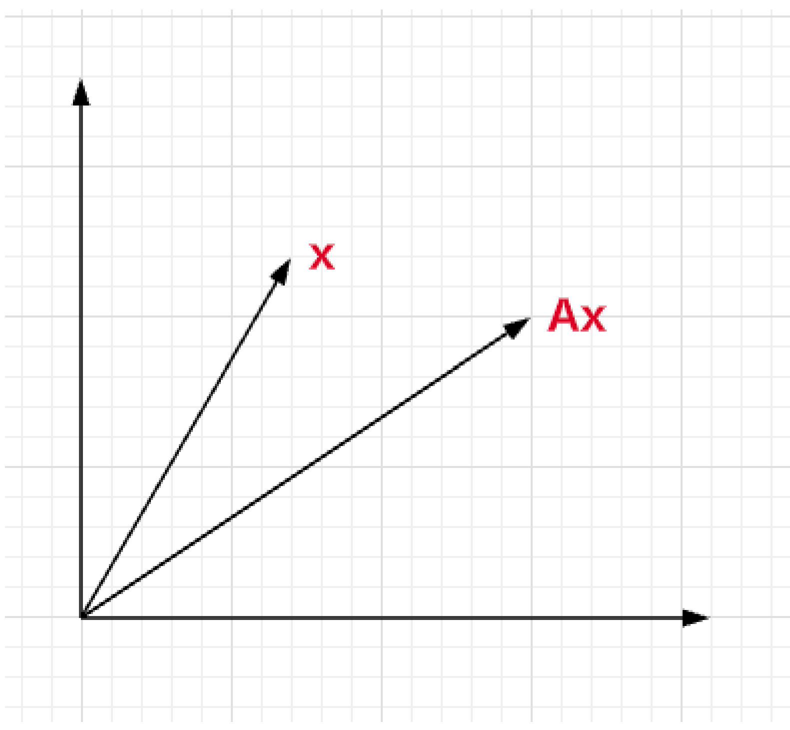
Eigenvectors are the vectors that when multiplied by a matrix (linear combination or transformation) result in another vector having the same direction but scaled (hence scalar multiple) in forward or reverse direction by a magnitude of the scalar multiple which can be termed as **Eigenvalue**. In simpler words, the eigenvalues are scalar values that represent the scaling factor by which a vector is transformed when a linear transformation is applied. In other words, eigenvalues are the values that scale eigenvectors when a linear transformation is applied.

Definition of eigenvalues and eigenvectors of a matrix

Here is the formula for what is called **eigenequation**.

Ax = λx

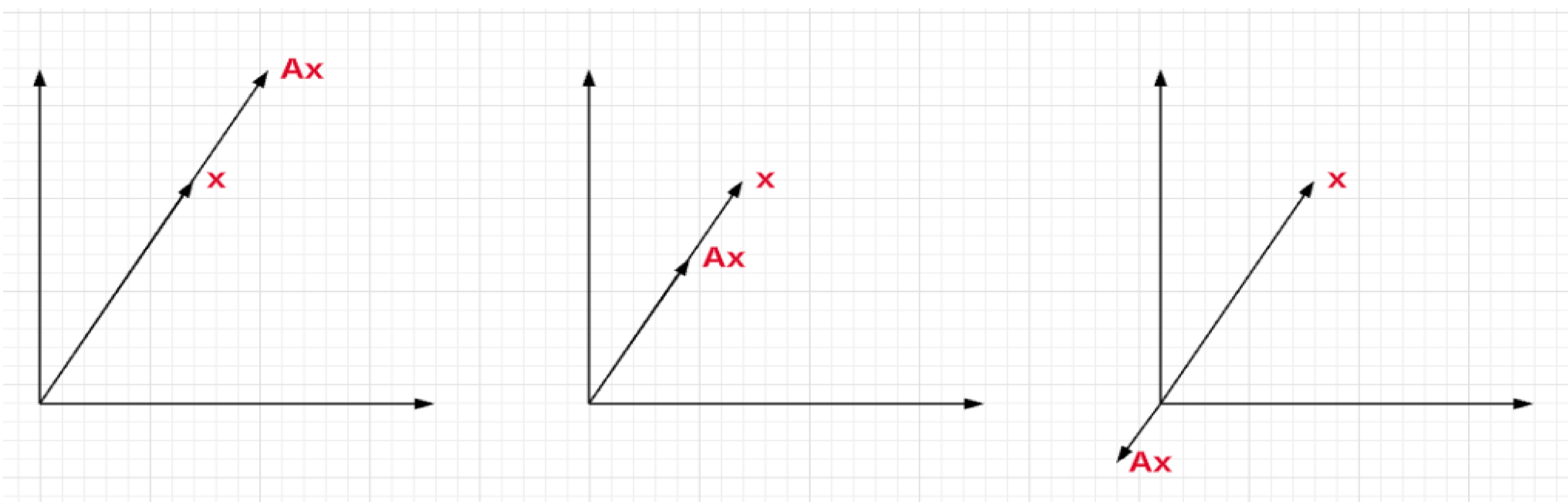
In the above equation, the **matrix A** acts on the **vector x** and the outcome is another **vector Ax** having the same direction as the original vector x but scaled/shrunk in forward or reverse direction by a magnitude of scalar multiple, λ. The vector x is called an the eigenvector of A and λ is called its eigenvalue. Let's understand pictorially what happens when a matrix A acts on a vector x. Note that the new vector Ax has a different direction than vector x.



Matrix A acts on x resulting in another vector Ax

Definition of eigenvalues and eigenvectors of a matrix

When the **matrix multiplication with vector** results in **another vector in the same/opposite** direction but scaled in forward / reverse direction by a magnitude of scalar multiple or eigenvalue λ, then the vector is called the eigenvector of that matrix. Here is the diagram representing the eigenvector x of matrix A because the vector Ax is in the same/opposite direction of x.



x is eigenvector of A

Definition of eigenvalues and eigenvectors of a matrix

Here is further information on the value of eigenvalues:

- $\lambda \in \mathbf{R}$, $\lambda > 0$: v and Av point in same direction
- $\lambda \in \mathbf{R}$, $\lambda < 0$: v and Av point in opposite directions
- $\lambda \in \mathbf{R}$, $|\lambda| < 1$: Av smaller than v
- $\lambda \in \mathbf{R}$, $|\lambda| > 1$: Av larger than v

Characteristic Equation

The characteristic/Eigen equation is the equation which is used to find the Eigenvalues of a matrix.

This is **also called** the characteristic polynomial.

Definition- Let **A** be a square matrix, $\boldsymbol{\lambda}$ be any **scalar value** then $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = 0$ is called the characteristic equation of a matrix **A.**

<u>Note:</u>

Let a be a square matrix and '$\boldsymbol{\lambda}$ ' be any scalar then,

1) $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = 0$ is called characteristic matrix
2) $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = 0$ is called characteristic polynomial.

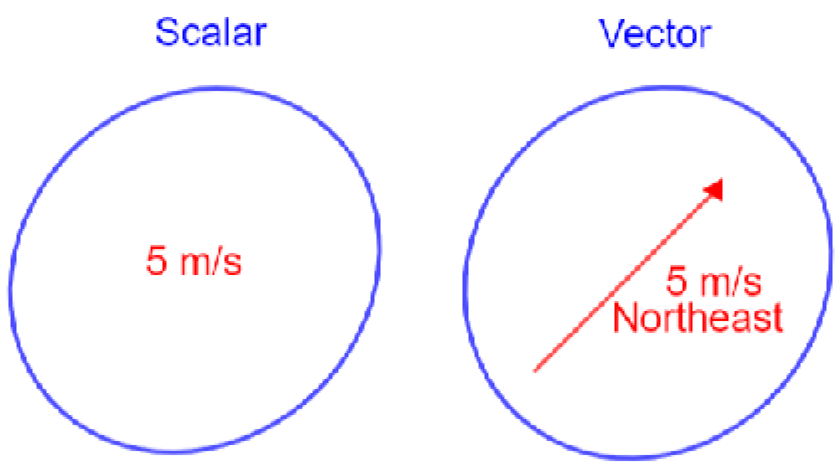
The roots of a characteristic equation are known as characteristic root or Eigenvalues of a matrix **A**.

Scalar Value

A scalar value is simply a value that only has one component to it, the magnitude.

For example, your speed is a scalar value because it only has one component, how fast you are going.

Your height is also a scalar value because the only component is how tall you are.



Determinant of a Matrix

The **determinant of a matrix** is the scalar value computed for a given square matrix.

Symbol

The determinant of a matrix is represented by two vertical lines or simply by writing det and writing the matrix name. eg. $|A|$, $\det(A)$, $\det A$

Eigen Vector

Eigenvectors are the vectors (non-zero) that do not change the direction when any linear transformation is applied. It changes by only a scalar factor. In a brief, we can say, if A is a linear transformation from a vector space V and **x** is a vector in V, which is not a zero vector, then v is an eigenvector of A if A(X) is a scalar multiple of **x**.

An **Eigenspace** of vector **x** consists of a set of all eigenvectors with the equivalent eigenvalue collectively with the zero vector. Though, the zero vector is not an eigenvector.

Let us say A is an "n × n" matrix and λ is an eigenvalue of matrix A, then **x**, a non-zero vector, is called as eigenvector if it satisfies the given below expression;

$A\mathbf{x} = \lambda\mathbf{x}$

x is an eigenvector of A corresponding to eigenvalue, λ.

Note:

There could be infinitely many Eigenvectors, corresponding to one eigenvalue.

For distinct eigenvalues, the eigenvectors are linearly dependent.

Properties of Eigenvalues

1. The sum of the Eigenvalues of a matrix A is equal to the sum of the diagonal elements of a matrix A.

2. The product of all Eigenvalues of a matrix A is equal to the value of the determinant.

How to find the eigenvalues two-dimensional matrix?

1. Taking the determinant to set up the **characteristic equation**, using $|\mathbf{A} - \lambda\mathbf{I}| = 0$

2. Solve the characteristic equation, giving us the **eigenvalues** (2 eigenvalues for a 2x2 system)

Example: Find out the Eigenvalues and Eigenvectors of

Sol. The Characteristics equation is given by

$$\left|\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

I is an identical matrix

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)[(1-\lambda)^2 - 1] - 1[1-\lambda-1] + 1[1-1+\lambda] = 0$$

$$(1-\lambda)[1-2\lambda+\lambda^2-1] - 1[-\lambda] + 1[\lambda] = 0$$

$$(1-\lambda)(-2\lambda+\lambda^2) + \lambda + \lambda = 0$$

$$-2\lambda + \lambda^2 + 2\lambda^2 - \lambda^3 + 2\lambda = 0$$

$$-\lambda^3 + 3\lambda^2 = 0$$

$$-\lambda^2(\lambda - 3) = 0$$

$$\text{Or } \lambda = 0{,}0{,}3$$

Hence the Eigen values are 0, 0 and 3.

The Eigen vector corresponding to Eigen value $\lambda = 0$ is

How to find the eigenvalues two-dimensional matrix?

The Eigen vector corresponding to Eigen value $\lambda = 0$ is

$[A - 0I]X = 0$

Where X is the column matrix of order 3 i.e. $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

This implies that x + y + z = 0

Here the number of unknowns is 3 and the number of equations is 1.

Hence we have (3-1) = 2 linearly independent solutions.

Let z = 0, y = 1 then x = -1 or let z = 1, y = 1 then x = -2

Thus the Eigenvectors corresponding to the Eigenvalue $\lambda = 0$ are (-1,1,0) and (-2,1,1).

How to find the eigenvalues two-dimensional matrix?

The Eigenvector corresponding to Eigenvalue $\lambda = 3$ is

$[A - 3I]X = 0$

Where X is the column matrix of order 3, it means $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$

$$\therefore\ [A - 0I]X = 0 \rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. = 0$$

This implies that -2x + y + z = 0

x – 2y + z = 0

x + y – 2z = 0

Taking last two equations we get

$$\frac{x}{4-1} = -\frac{y}{-2-1} = \frac{z}{1+2}$$

Or

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

Thus the Eigenvectors corresponding to the Eigenvalue $\lambda = 3$ are (3,3,3).

How to find the eigenvalues two-dimensional matrix?

Thus the Eigenvectors corresponding to the Eigenvalue $\lambda = 3$ are (3,3,3).

Hence the three Eigenvectors obtained are (-1,1,0), (-2,1,1) and (3,3,3).

How to find the eigenvalues two-dimensional matrix?

1. Taking the determinant to set up the **characteristic equation**, using $|\mathbf{A} - \lambda\mathbf{I}| = 0$

2. Solve the characteristic equation, giving us the **eigenvalues** (2 eigenvalues for a 2x2 system)

How to find eigen Values

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

$$[A - \lambda I] = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

I is an identical matrix

How to find the eigenvalues

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$$= \begin{bmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix}$$

$$|A - \lambda I| = 0$$

How to find the eigenvalues

$$\begin{vmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{vmatrix} = 0$$

$$-3\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda^2 - 2\lambda - \lambda + 2 = 0$$

$$\lambda(\lambda - 2) - 1(\lambda - 2) = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$$\lambda = 2, 1$$

How to find the eigenvalues....more example

- **Two-dimensional matrix example-**

Ex.1 Find the eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Taking the determinant to find characteristic polynomial A-

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3 - 4\lambda + \lambda^2 = 0$$

It has roots at $\lambda = 1$ and $\lambda = 3$, which are the two eigenvalues of A.

How to find the eigenvectors

Eigenvectors v of this transformation satisfy the equation,

$$Av = \lambda v$$

Rearrange this equation to obtain-

$$(A - \lambda I)v = 0$$

For $\lambda = 1$, Equation becomes,

$$(A - I)v = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution,

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

How to find the eigenvectors

For $\lambda = 3$, Equation becomes,

$$(A - 3I)u = 0$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which has the solution- $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus, the vectors $v_{\lambda=1}$ and $v_{\lambda=3}$ are eigenvectors of A associated with the eigenvalues $\lambda = 1$ and $\lambda = 3$, respectively.

How to find the eigenvalues of three-dimensional matrix?

Prob: Find the eigen values and eigen vectors of the

matrix $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.

Sol:

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{vmatrix} - \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)(5-\lambda) - 0\right] - \cancel{1[0-0]}^{0} + \cancel{4[0-0]}^{0} = 0$$

$$\boxed{(3-\lambda)(2-\lambda)(5-\lambda) = 0} \checkmark$$

$$\boxed{\lambda = 3,\ \lambda = 2,\ \lambda = 5} \checkmark$$

How to find the eigenvectors of three-dimensional matrix?

⇒ eigen Vector for $\lambda = 2$ is

$$(A - \lambda I)\bar{x} = 0$$

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x + y + \cancel{4z}^{\,0} = 0 \checkmark$$

$$\left.\begin{aligned} 0 + 0 + 6z &= 0 \\ 3z &= 0 \end{aligned}\right\} \rightarrow \boxed{z = 0}$$

$$\text{let } \boxed{y = K_1} \Rightarrow \boxed{x = -K_1} \Rightarrow \bar{x} = \begin{bmatrix} -K_1 \\ K_1 \\ 0 \end{bmatrix} = K_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

How to find the eigenvectors of three-dimensional matrix?

$$(A-\lambda I) = \begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

> eigen vector at $\lambda = 3$ is:

$$(A-3I)\bar{X} = 0$$

$$\begin{bmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$y + 4z = 0$$

$$-y + 6z = 0$$

$$2z = 0 \rightarrow \boxed{z = 0}$$

By substituting the value of z, we get

$$\left.\begin{aligned} y + 4z &= 0 \\ -y + 6z &= 0 \end{aligned}\right\} \Rightarrow y = 0$$

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$$2z = 0 \rightarrow \boxed{z = 0}$$

let, $\boxed{x = k_2}$ $\Rightarrow \bar{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} k_2 \\ 0 \\ 0 \end{bmatrix} = k_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ ✓

How to find the eigenvectors of three-dimensional matrix?

> eigen vector for $\lambda = 5$ is:

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 5-\lambda \end{bmatrix}$$

$$(A - 5I)\bar{x} = 0$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$-2x + y + 4z = 0$$

$$-3y + 6z = 0$$

let $\boxed{z = k_3}$

By substituting the value of z, we get

$$-2x + y + 4z = 0 \Rightarrow 2x = 2k_3 + 4k_3 \Rightarrow \boxed{x = 3k_3}$$

$$-3y + 6z = 0 ✓ \Rightarrow 3y = 6k_3$$

$$\Rightarrow \bar{x} = \begin{bmatrix} 3k_3 \\ 2k_3 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} // ✓$$

How to find the eigenvalues of three-dimensional matrix?

more example

- **Three-dimensional matrix example-**

Ex.2 Find the eigenvalue and eigenvector of matrix A.

$$A = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

the matrix has the characteristics equation-

$$|\lambda I - A| = \begin{vmatrix} \lambda+4 & -1 & 0 \\ 0 & \lambda+3 & -1 \\ 0 & 0 & \lambda+2 \end{vmatrix}$$

$$= (\lambda+4)(\lambda+3)(\lambda+2) = 0$$

How to find the eigenvector of three-dimensional matrix?

therefore the eigen values of A are-

$$\lambda_1 = -2, \lambda_2 = -3, \lambda_3 = -4$$

For $\lambda = -2$, Equation becomes,

$$(\lambda_1 I - A)v_1 = 0$$

$$\begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which has the solution-

$$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

How to find the eigenvector of three-dimensional matrix?

Similarly for $\lambda = -3$ and $\lambda = -4$ the corresponding eigenvectors u and x are-

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, x = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$