Number Theory

$1395^4 \cdot 675^3 + 12 \cdot 17 \cdot 22$ by 7

$\text{rem}(1395, 7) = 2$

$\text{rem}(675, 7) = 3$

$\text{rem}(12, 7) = 5$

$\text{rem}(17, 7) = 3$

$\text{rem}(22, 7) = 1$

$2^4 \cdot 3^3 + 5 \cdot 3 \cdot 1$

$= 16 \cdot 27 + 15$

$= 241$

~~rem(241, 7) = 3~~

~~Ans: 3~~

$\text{rem}(16, 7) = 2$

$\text{rem}(27, 7) = 6$

$\text{rem}(15, 7) = 1$

$2 \cdot 6 + 1$

$= 13$

$\text{rem}(13, 7) = 6$

Ans: 6

$\text{rem}((44427^{3456789} + 15555858^{5555})\, 403^{6666666}, 36)$

$\text{rem}(44927, 36) = 3$

$\text{rem}(15555858, 36) = 6$

$\text{rem}(403, 36) = 7$

$\text{rem}((3^{3456789} + 6^{5555})\, 7^{6666666}, 36)$

$rem(3^0, 36) = 1$ $rem(6^0, 36) = 1$ $rem(7^0, 36) = 1$

$rem(3^1, 36) = 3$ $rem(6^1, 36) = 6$ $rem(7^1, 36) = 7$

$rem(3^2, 36) = 9$ $rem(6^2, 36) = 0$ $rem(7^2, 36) = 13$

$rem(3^3, 36) = 27$ $rem(6^3, 36) = 0$ $rem(7^3, 36) = 19$

$rem(3^4, 36) = 9$ $rem(6^4, 36) = 0$ $rem(7^4, 36) = 25$

$rem(3^5, 36) = 27$ $rem(7^5, 36) = 31$

$rem(7^6, 36) = 1$

$rem((27 + 0) 1)$

$rem(27, 36) = 27$ Ans: 27.

Euler's function (ϕ)

$\phi(300)$

$300 = 2 \times 150$

$= 2 \times 2 \times 75$

$= 2 \times 2 \times 3 \times 25$

$= 2 \times 2 \times 3 \times 5 \times 5$

$= 2^2 . 3 . 5^2$

$\phi(300) = \phi(2^2) . \phi(3) . \phi(5^2)$

$= (2^2 - 2^1)(3^1 - 3^0) . (5^2 - 5^1)$

$= 2 . 2 . 20 = 80$ Ans.

$\phi(500) = ??$

$500 = 2 \times 250$

$= 2 \times 2 \times 125$

$= 2 \times 2 \times 5 \times 25$

$= 2 \times 2 \times 5 \times 5 \times 5$

$= 2^2 \times 5^3$

$\phi(500) = \phi(2^2) . \phi(5^3)$

$= (2^2 - 2^1) . (5^3 - 5^2)$

$= 2 . 100$

$= 200$ Ans!

gcd (120.500)

$120 = 2.2.2.3.5$

$= 2^3.3.5$

$500 = 2.2.5.5.5$

$= 2^2.5^3$

we know,

$$gcd(a,b) = P_1^{min(a_1,b_1)} \; P_2^{min(a_2,b_2)} \; P_3^{min(a_3,b_3)}$$

$$gcd(120,500) = 2^{min(3,2)} . 3^{min(1,0)} . 5^{min(1,3)}$$

$$= 2^2 . 3^0 . 5^1$$

$$= 20$$ Ans:

gcd(256, 100)

$256 = 2.2.2.2.2.2.2$

$= 2^7$

$100 = 2.2.5.5$

$= 2^2.5^2$

$$gcd(a,b) = P_1^{min(a_1,b_1)} . P_2^{min(a_2.b_2)}$$

$$= 2^{min(7,2)} . 5^{min(0.2)}$$

$$= 2^2 . 5^0$$

$$= 4$$ Ans:

LCM(1515, 2020)

$1515 = 3 . 5 . 101$

$2020 = 2 . 2 . 5 . 101 = 2^2 . 5 . 101$

$LCM(a, b) = P_1^{max(a_1, b_1)} P_2^{max(a_2, b_2)} P_3^{max(a_3, b_3)} P_4^{max(a_4, b_4)}$

$= 3^{max(1,0)} . 2^{max(0,2)} . 5^{max(1,1)} . 101^{max(1,1)}$

$= 3^1 . 2^2 . 5^1 . 101$

$= 6060$ Ans:

LCM(3500, 625)

$3500 = 2 . 2 . 5 . 5 . 5 . 7$

$= 2^2 . 5^3 . 7$

$625 = 5 . 5 . 5 . 5$

$= 5^4$

$LCM = P_1^{max(a_1, b_1)} P_2^{max(a_2, b_2)} P_3^{max(a_3, b_3)}$

$= 2^{max(2,0)} 5^{max(3,4)} 7^{max(1,0)}$

$= 2^2 . 5^4 . 7^1$

$= 17500$ Ans:

Double Hashing

Table size 11 (0 . . . 10)

Hash function:

$h_1(x) = x \bmod 11$ where t = tablesize − 1

$h_2(x) = 1 + (x \bmod t)$ ($t = 10$ here)

Insert keys : 58, 14, 91, 69, 80, 102, 25, 113, 124

$58 \bmod 11 = 3$ $1 + (14 \bmod 10) = 1 + 4 = 5$

$14 \bmod 11 = 3 \rightarrow$ $3 + (1+4) = 8 \bmod 11 = 8$

$91 \bmod 11 = 3 \rightarrow$ $3 + (1+1) = 5 \bmod 11 = 5$

$69 \bmod 11 = 3 \rightarrow$ $3 + (1+9) = 13 \bmod 11 = 2$

$80 \bmod 11 = 3 \rightarrow$ $3 + (1+0) = 4 \bmod 11 = 4$

$102 \bmod 11 = 3 \rightarrow$ $3 + (1+2) = 6 \bmod 11 = 6$

$25 \bmod 11 = 3 \rightarrow$ $3 + (1+5) = 9 \bmod 11 = 9$

$113 \bmod 11 = 3 \rightarrow$ $3 + (1+3) = 7 \bmod 11 = 7$

$124 \bmod 11 = 3 \rightarrow$ $3 + (1+4) = 8$, $3 + 2(1+4) = 13$,

$3 + 3(1+4) = 18$, $3 + 4(1+4) = 23 \bmod 11 = 1$

	0
124	1
69	2
58	3
80	4
91	5
102	6
113	7
14	8
25	9
	10

Linear Probing

Insert keys : 20, 30, 2, 13, 25, 24, 10, 9

20 mod 11 = 9

30 mod 11 = 8

2 mod 11 = 2

13 mod 11 = 2 $\quad 2+1 = 3$

25 mod 11 = 3 $\quad 3+1 = 4$

24 mod 11 = 2 $\quad 2+1, 2+2, 2+3 = 5$

10 mod 11 = 10

9 mod 11 = 9 $\quad 9+1, 9+2$ mod 11 = 0

9	0
	1
2	2
13	3
25	4
24	5
	6
	7
30	8
20	9
10	10

Quadratic Probing:

Insert keys : 20, 30, 2, 13, 25, 24, 10, 9

20 mod 11 = 9

30 mod 11 = 8

2 mod 11 = 2

13 mod 11 = 2 $\quad 2+1^2 = 3$

25 mod 11 = 3 $\quad 3+1^2 = 4$

24 mod 11 = 2 $\quad 2+1^2, 2+2^2 = 6$

10 mod 11 = 10

9 mod 11 = 9 $\quad 9+1^2, 9+2^2, 9+3^2$ mod 11 = 7

	0
	1
2	2
13	3
25	4
	5
24	6
9	7
30	8
20	9
10	10

Eigenvalues and Eigen vectors

$$\left|\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right|=0$$

$$\Rightarrow \begin{vmatrix}1-\lambda&1&1\\1&1-\lambda&1\\1&1&1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\left((1-\lambda)^2-1\right)-1(1-\lambda-1)+1(1-1+\lambda)=0$$

$$\Rightarrow (1-\lambda)(1-2\lambda+\lambda^2-1)-1(-\lambda)+1(\lambda)=0$$

$$\Rightarrow (1-\lambda)(-2\lambda+\lambda^2)+\lambda+\lambda=0$$

$$\Rightarrow -2\lambda+\lambda^2+2\lambda^2-\lambda^3+2\lambda=0$$

$$\Rightarrow -\lambda^3+3\lambda^2=0$$

$$\Rightarrow \lambda^3-3\lambda^2=0$$

$$\Rightarrow \lambda^2(\lambda-3)=0$$

$$\therefore \lambda=0,0,3$$

For eigenvalues:

$[A-\lambda I]X=0$

$$\begin{vmatrix}1-0&1&1\\1&1-0&1\\1&1&1-0\end{vmatrix}=0$$

$$\begin{vmatrix}1&1&1\\1&1&1\\1&1&1\end{vmatrix}=0$$

$x+y+z=0$
$x+y+z=0$
$x+y+z=0$

$x+y+z=0$

Here the number of unknown is 3 and the number of eqn 1 we have $(3-1)=2$ linearly independent solutions.

Let: $z=0$, $y=1$ then $x=-1$

or, $z=1$, $y=1$ then $x=-2$

so, Eigenvectors are $(-1,1,0)$, $(-2,1,1)$

$[A-\lambda I]X=0$

$$\Rightarrow \begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1-3 & 1 & 1 \\ 1 & 1-3 & 1 \\ 1 & 1 & 1-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} -2x+y+z &= 0 \\ x-2y+z &= 0 \\ x+y-2z &= 0 \end{aligned}\right]$$

For first two equation:

$$\frac{x}{1+2} = -\frac{y}{-2-1} = \frac{z}{4-1}$$

$$\Rightarrow \frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

$|A-\lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 2 \\ 2 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{-\lambda(-1-\lambda)-0\} = 0$$

$$\Rightarrow (1-\lambda)(\lambda+\lambda^2) = 0 \qquad \Rightarrow \lambda+\lambda^2-\lambda^2-\lambda^3 = 0$$

$$\cancel{\Rightarrow \lambda+\lambda^2+\lambda^2-\lambda^3=0} \qquad \Rightarrow \lambda^3-\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2-1) = 0$$

$\lambda = 0$

$\lambda^2 - 1 = 0$

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = +1, -1$

$\Rightarrow -\lambda^3 + 2\lambda^2 - \lambda = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 + \lambda = 0$

$\Rightarrow \lambda(\lambda^2 - 2\lambda + 1) = 0$

$\Rightarrow \lambda(\lambda^2 - \lambda - \lambda + 1) = 0$

$\Rightarrow \lambda(\lambda(\lambda - 1) - 1(\lambda - 1)) = 0$

$\Rightarrow \lambda(\lambda - 1)(\lambda - 1) = 0$

$\therefore \lambda = 0, 1, -1$

$|A - \lambda I| X = 0$

$$\Rightarrow \begin{vmatrix} 1-0 & 0 & 0 \\ 0 & -1-0 & 2 \\ 2 & 0 & -0 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 2 & 0 & 0 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow x = 0$

$-y + 2z = 0$

$2x = 0$

let, $y = 1$

$-1 + 2z = 0$

$\Rightarrow 2z = 1$

$\Rightarrow z = 1/2$

$\left(0, 1, \frac{1}{2}\right)$

$$\begin{vmatrix} 1-1 & 0 & 0 \\ 0 & -1-1 & 2 \\ 2 & 0 & -1 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 2 & 0 & -1 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow -2y + 2z = 0$

$2x - z = 0$

Here, $x = 1$

so, $2 - z = 0$

$\Rightarrow z = 2$

$-2y + 4 = 0$

$\Rightarrow 2y = 4$

$\Rightarrow y = 2$

$\therefore (1, 2, 2)$

For another -1,

$x = 2$

$4 - z = 0$

$\Rightarrow z = 4$

$\Rightarrow -2y + 8 = 0$

$\Rightarrow y = 4$

$(2, 4, 4)$ ans

$300 = 2 \times 150$

$=$

$$\begin{vmatrix} 1+1 & 0 & 0 \\ 0 & -1+1 & 2 \\ 2 & 0 & 1 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 2 & 0 & 1 \end{vmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow 2x = 0$ $\quad x = 0$

$2z = 0$ $\quad z = 0$ $\quad (0, 1, 0)$

$2x + z = 0$ $\quad y = 1$

Ans!

Markov Model

Two states : Rain and Dry

$P(Rain|Rain) = 0.3$, $P(Dry|Rain) = 0.7$, $P(Rain|Dry) = 0.2$

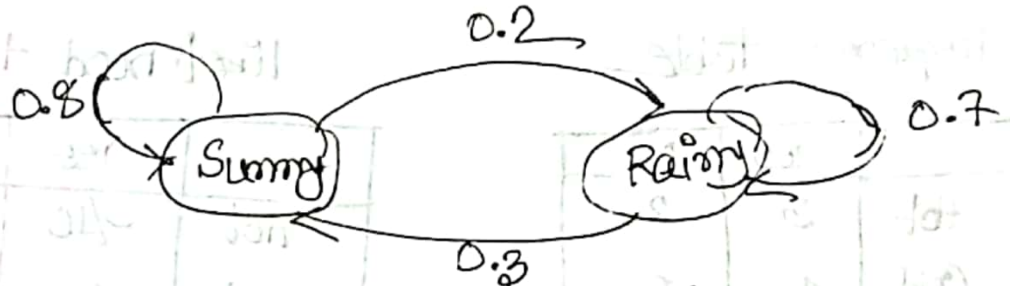
$P(Dry|Dry) = 0.8$

$P(Rain) = 0.4$ $\quad P(Dry) = 0.6$

Sequence of states : Dry , Dry , Rain , Rain

$P(Dry) \, P(Dry|Dry) \, P(Rain|Dry) \, P(Rain|Rain) =$

$0.6 \times 0.8 \times 0.2 \times 0.3 = 0.0288$ Ans:



State space = {Sunny, Rainy}

Initial Prob = {0.4 , 0.6}

Sequence {sunny, Rainy , Sunny , Sunny, Rainy}

$P(sunny) * P(Rainy|Sunny) * P(Sunny|Rainy) * P(Sunny|Sunny) * P(R|S)$

$= 0.4 * 0.2 * 0.3 * 0.8 * 0.2$

$= \cancel{0.00576}$ 3.84×10^{-3}

Ans:

Two states: Hot and Cold

$P(Hot) = 0.4 \quad P(Cold) = 0.6$

$P(Hot|Hot) = 0.5 \quad P(Cold|Hot) = 0.5, \; P(Hot|Cold) = 0.3, \; P(Cold|Cold) = 0.7$

Cold, Cold, Hot, Hot, Hot, Cold, Hot

$P(Cold) * P(Cold|Cold) * P(Hot|Cold) * P(Hot|Hot) * P(Hot|Hot) * P(Cold|Hot) *$

$P(Hot|Cold) = 0.6 * 0.7 * 0.3 * 0.5 * 0.5 * 0.5 * 0.3$

$= 4.725 \times 10^{-3}$

Ans:

Ex: Frequency table

	Yes	No
Hot	3	2
Cold	4	0
Rainy	3	3
Total	10	5

likelihood table

	Yes	No
Hot	3/10	2/5
Cold	4/10	0
Rainy	3/10	3/5
Total	100%	100%

~~P(Yes|Hot, Cold, Rai~~

	Play	P(Yes)/P(No)
Yes	10	10/15
No	5	5/15
Total	15	100%

$$P(Yes \mid Hot, Cold, Rainy) = \frac{P(Hot \mid Yes)\, P(Cold \mid Yes)\, P(Rainy \mid Yes)\, P(Yes)}{P(Hot)\, P(Cold)\, P(Rainy)}$$

$$P(Yes \mid Hot, Cold, Rainy) \propto \frac{3}{10} * \frac{4}{10} * \frac{3}{10} * \frac{10}{15}$$

$$\approx 0.024$$

$$P(No \mid Hot, Cold, Rainy) \propto \frac{P(Hot \mid NO)\, P(Cold \mid NO)\, P(Rainy \mid NO)\, P(NO)}{P(Hot)\, P(Cold)\, P(Rainy)}$$

$$P(No \mid Hot, Cold, Rainy) \propto \frac{2}{5} * 0 * \frac{3}{5} * \frac{5}{15}$$

$$\approx 0$$

$$P(Yes \mid Test) + P(No \mid Test) = 1$$

$$P(Yes \mid Test) = \frac{0.024}{0.024 + 0} = 1 = 100\%$$

$$P(No \mid Test) = \frac{0}{0.024 + 0} = 0 = 0\%$$

$$P(Yes \mid Test) > P(No \mid Test)$$

so, Yes