CHAPTER 7

Mathematics of Projection

Needless to say, there are fundamental differences between the true three-dimensional world and its pictorial description. For centuries, artists, engineers, designers, drafters, and architects have tried to come to terms with the difficulties and constraints imposed by the problem of representing a three-dimensional object or scene in a two-dimensional medium—the problem of *projection*. The implementers of a computer graphics system face the same challenge.

Projection can be defined as a mapping of point $P(x, y, z)$ onto its image $P'(x', y', z')$ in the *projection plane* or *view plane*, which constitutes the display surface (see Fig. 7-1). The mapping is determined by a projection line called the *projector* that passes through P and intersects the view plane. The intersection point is P'.

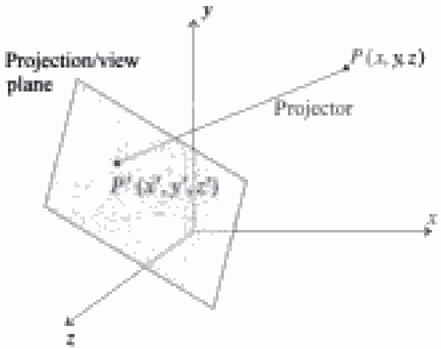

Fig. 7-1 The problem of projection.

The result of projecting an object is dependent on the spatial relationship among the projectors that project the points on the object, and the spatial relationship between the projectors and the view plane (see Sec. 7.1). An important observation is that projection preserves lines. That is, the line joining the projected images of the endpoints of the original line is the same as the projection of that line.

The two basic methods of projection—*perspective* and *parallel*—are designed to solve the basic but mutually exclusive problems of pictorial representation: showing an object as it appears and preserving its true size and shape. We characterize each method and introduce the mathematical description of the projection process in Sec. 7.2 and 7.3, respectively.

7.1 TAXONOMY OF PROJECTION

We can construct different projections according to the view that is desired.

Figure 7-2 provides a taxonomy of the families of perspective and parallel projections. Some projections have names—cavalier, cabinet, isometric, and so on. Other projections qualify the main type of projection—one principal vanishing-point perspective, and so forth.

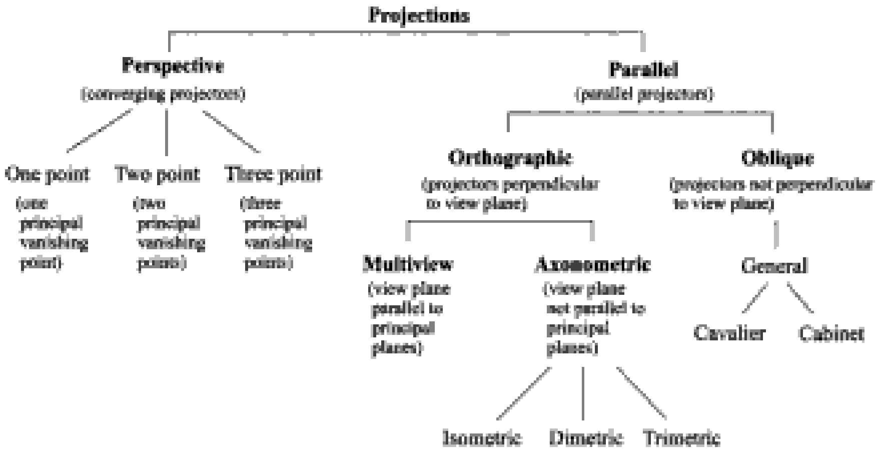

Fig. 7-2 Taxonomy of projection.

7.2 PERSPECTIVE PROJECTION

Basic Principles

The techniques of perspective projection are generalizations of the principles used by artists in preparing perspective drawings of three-dimensional objects and scenes. The eye of the artist is placed at the *center of projection*, and the canvas, or more precisely the plane containing the canvas, becomes the view plane. An image point is determined by a projector that goes from an object point to the center of projection (see Fig. 7-3).

Perspective drawings are characterized by perspective foreshortening and vanishing points. *Perspective foreshortening* is the illusion that objects and lengths appear smaller as their distance from the center of projection increases. The illusion that certain sets of parallel lines appear to meet at a point is another feature of perspective drawings. These points are called *vanishing points*. *Principal vanishing points* are formed by the apparent intersection of lines parallel to one of the three principal x, y, or z axes. The number of principal vanishing points is determined by the number of principal axes intersected by the view plane (Prop. 7.7).

Mathematical Description of a Perspective Projection

A perspective transformation is determined by prescribing a center of projection and a view plane. The view plane is determined by its *view reference point* R_0 and *view plane normal* N. The *object point* P is located in world coordinates at (x, y, z). The problem is to determine the *image point* coordinates $P'(x', y', z')$ (see Fig. 7-3).

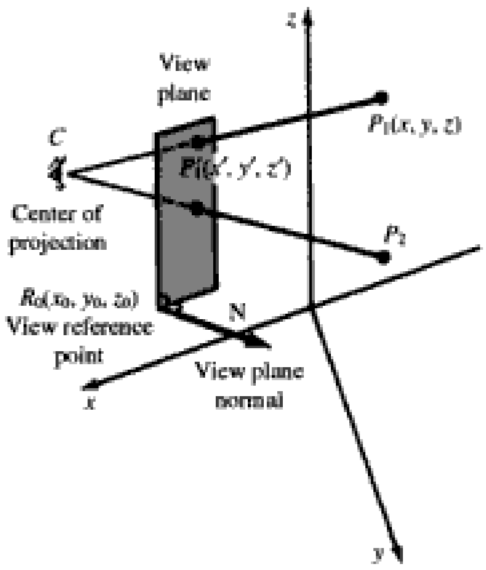

Fig. 7-3

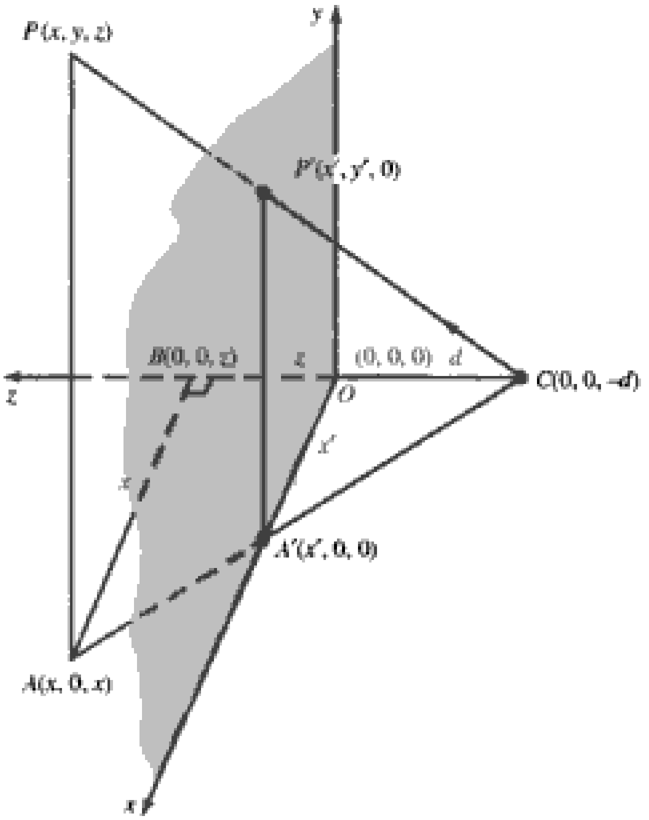

Fig. 7-4

EXAMPLE 1. The standard perspective projection is shown in Fig. 7-4. Here, the view plane is the *xy* plane, and the center of projection is taken as the point $C(0, 0, -d)$ on the negative z axis.

Using similar triangles ABC and $A'OC$, we find

$$x' = \frac{d \cdot x}{z + d} \qquad y' = \frac{d \cdot y}{z + d} \qquad z' = 0$$

The perspective transformation between object and image point is nonlinear and so cannot be represented as a 3×3 matrix transformation. However, if we use homogeneous coordinates, the perspective transformation can be represented as a 4×4 matrix:

$$\begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} = \begin{pmatrix} d \cdot x \\ d \cdot y \\ 0 \\ z + d \end{pmatrix} = \begin{pmatrix} d & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & d \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$$

The general form of a perspective transformation is developed in Prob. 7.5.

Perspective Anomalies

The process of constructing a perspective view introduces certain anomalies which enhance realism in terms of depth cues but also distort actual sizes and shapes.

1. *Perspective foreshortening*. The farther an object is from the center of projection, the smaller it appears (i.e. its projected size becomes smaller). Refer to Fig. 7-5.

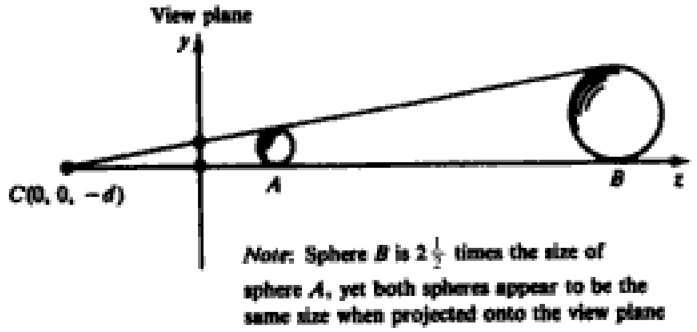

Fig. 7-5

2. *Vanishing points*. Projections of lines that are not parallel to the view plane (i.e. lines that are not perpendicular to the view plane normal) appear to meet at some point on the view plane. A common manifestation of this anomaly is the illusion that railroad tracks meet at a point on the horizon.

EXAMPLE 2. For the standard perspective projection, the projections L_1' and L_2' of parallel lines L_1 and L_2 having the direction of the vector **K** appear to meet at the origin (Prob. 7.8). Refer to Fig. 7-6.

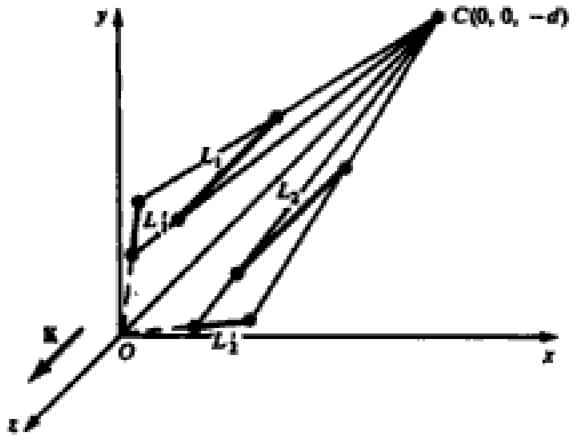

Fig. 7-6

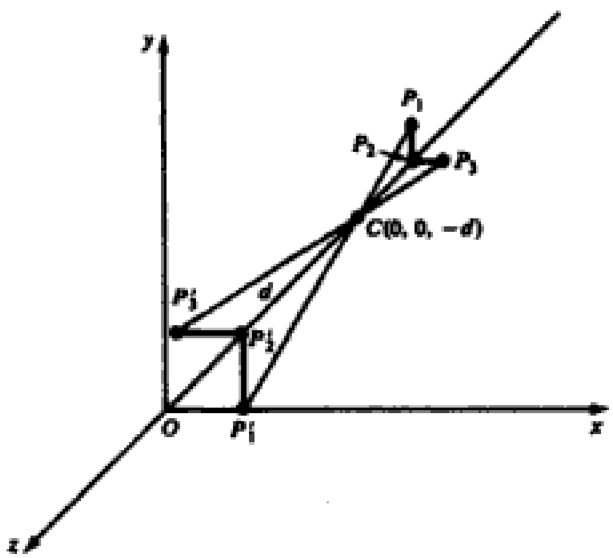

Fig. 7-7

3. *View confusion*. Objects behind the center of projection are projected upside down and backward onto the view plane. Refer to Fig. 7-7.

4. *Topological distortion*. Consider the plane that passes through the center of projection and is parallel to the view plane. The points of this plane are projected to infinity by the perspective transformation. In particular, a finite line segment joining a point which lies in front of the viewer to a point in back of the viewer is actually projected to a broken line of infinite extent (Prob. 7.2) (see Fig. 7-8).

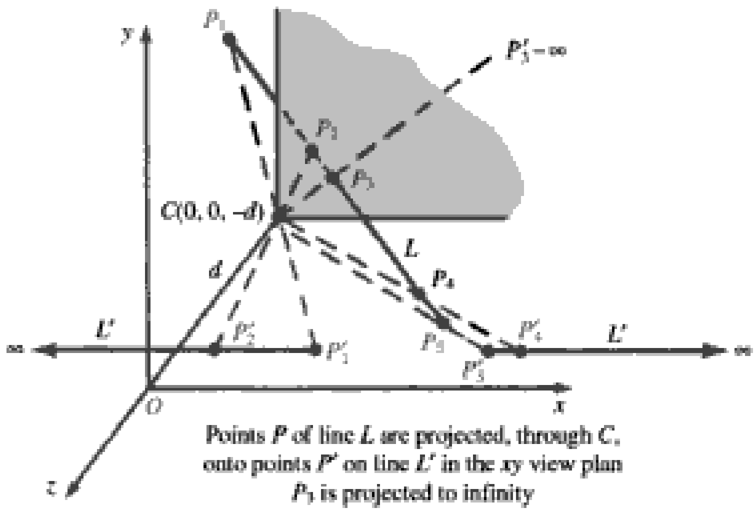

Fig. 7-8

7.3 PARALLEL PROJECTION

Basic Principles

Parallel projection methods are used by drafters and engineers to create working drawings of an object which preserves its scale and shape. The complete representation of these details often requires two or more views (projections) of the object onto different view planes.

In parallel projection, image points are found as the intersection of the view plane with a projector drawn from the object point and having a fixed direction (see Fig. 7-9). The *direction of projection* is the prescribed direction for all projectors. *Orthographic projections* are characterized by the fact that the direction of projection is perpendicular to the view plane. When the direction of projection is parallel to any of the principal axes, this produces the front, top, and side views of mechanical drawings (also referred to as *multiview drawings*). *Axonometric projections* are orthographic projections in which the direction of projection is not parallel to any of the three principal axes. Nonorthograhic parallel projections are called *oblique parallel projections*. Further subcategories of these main types of parallel projection are described in the problems. (See also Fig. 7-10.)

Mathematical Description of a Parallel Projection

A *parallel projective transformation* is determined by prescribing a *direction of projection vector* **V** and a view plane. The view plane is specified by its view plane reference point R_0, and view plane normal **N**. The object point P is located at (x, y, z) in world coordinates. The problem is to determine the image point coordinates $P'(x', y', z')$. See Fig. 7-9.

If the projection vector **V** has the direction of the view plane normal **N**, the projection is said to be *orthographic*. Otherwise it is called *oblique* (see Fig. 7-10).

Some common subcategories of orthographic projections are:

1. *Isometric*—the direction of projection makes equal angles with all of the three principal axes (Prob. 7.14).
2. *Dimetric*—the direction of projection makes equal angles with exactly two of the principal axes (Prob. 7.15).
3. *Trimetric*—the direction of projection makes unequal angles with the three principal axes.

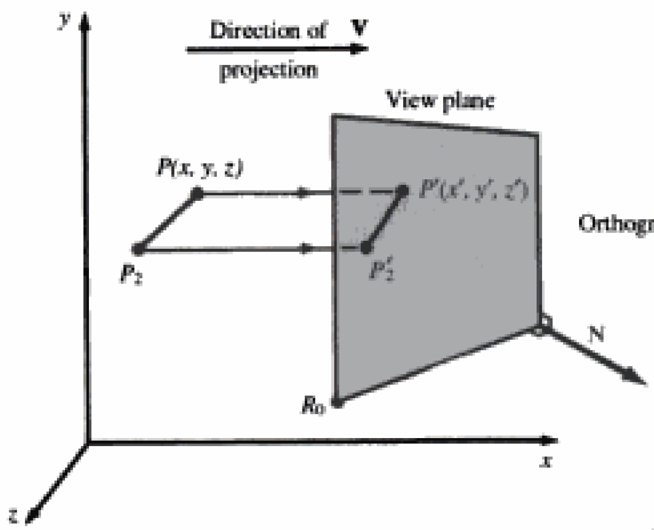

Fig. 7-9

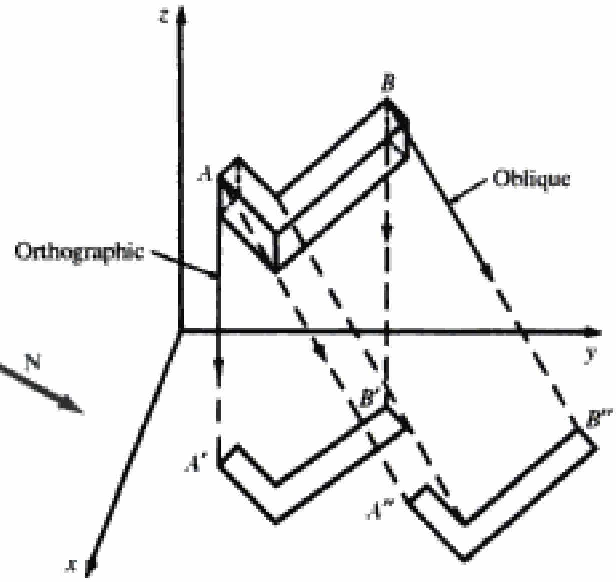

Fig. 7-10

Some common subcategories of oblique projections are:

1. *Cavalier*—the direction of projection is chosen so that there is no foreshortening of lines perpendicular to the xy plane (Prob. 7.13).
2. *Cabinet*—the direction of projection is chosen so that lines perpendicular to the xy planes are foreshortened by half their lengths (Prob. 7.13).

EXAMPLE 3. For orthographic projection onto the xy plane, from Fig. 7-11 it is easy to see that

$$Par_{\mathbf{K}}: \begin{cases} x' = x \\ y' = y \\ z' = 0 \end{cases}$$

The matrix form of $Par_{\mathbf{K}}$ is

$$Par_{\mathbf{K}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The general parallel projective transformation is derived in Prob. 7.11.

Solved Problems

7.1 The unit cube (Fig. 7-12) is projected onto the xy plane. Note the position of the x, y, and z axes. Draw the projected image using the standard perspective transformation with (a) $d = 1$ and (b) $d = 10$, where d is distance from the view plane.