1.
$$\frac{1}{2\lambda}exp\left\{-\frac{|x-\mu|}{\lambda}\right\}\mathbb{1}\{x\in\mathbb{R},\lambda>0\}$$

(a)
$$\int_{-\infty}^{\mu}\frac{1}{2\lambda}exp\left\{\frac{x-\mu}{\lambda}\right\}dx+\int_{\mu}^{\infty}\frac{1}{2\lambda}exp\left\{-\frac{x-\mu}{\lambda}\right\}dx$$

$$\implies \frac{1}{2}exp\left\{\frac{x-\mu}{\lambda}\right\}\Bigg|_{-\infty}^{\mu}+\frac{1}{2}exp\left\{-\frac{x-\mu}{\lambda}\right\}\Bigg|_{\mu}^{\infty}$$

$$\implies \frac{1}{2}exp\left\{\frac{\mu-\mu}{\lambda}\right\}-\frac{1}{2}exp\left\{-\frac{-\infty-\mu}{\lambda}\right\}-\frac{1}{2}exp\left\{-\frac{\infty-\mu}{\lambda}\right\}+\frac{1}{2}exp\left\{\frac{\mu-\mu}{\lambda}\right\}$$

If $\mu=\mathbb{R}$ and $\lambda>0$ we can show that the PDF equals 1. Also note the function is always >0 because it is an exponential function therefore, this is indeed a PDF.

$$\implies \frac{1}{2}-0-0+\frac{1}{2}=1$$

(b)
$$\implies \int_{-\infty}^{x}\frac{1}{2\lambda}exp\left\{\frac{x-\mu}{\lambda}\right\}+\int_{x}^{\infty}\frac{1}{2\lambda}exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

$$F(x<\mu)=\frac{1}{2}exp\left\{\frac{x-\mu}{\lambda}\right\}\Bigg|_{-\infty}^{x}$$

$$\frac{1}{2}exp\left\{\frac{x-\mu}{\lambda}\right\}$$

$$F(x\geq\mu)=\frac{1}{2}exp\left\{-\frac{x-\mu}{\lambda}\right\}\Bigg|_{x}^{\infty}$$

$$\implies -\frac{1}{2}exp\left\{-\frac{\infty-\mu}{\lambda}\right\}+\frac{1}{2}exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

$$=\frac{1}{2}exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

Note that the CDF must be equal to one therefore when F(x $\geq\mu$) the CDF will be 1 - $\frac{1}{2}exp\{-\frac{x-\mu}{\lambda}\}$ Additionally, the function will always be positive because it is exponential.

$$F(x)=\left\{\begin{array}{ll}\frac{1}{2}exp\{\frac{x-\mu}{\lambda}\} & x<\mu\\ 1-\frac{1}{2}exp\{-\frac{x-\mu}{\lambda}\} & x\geq\mu\end{array}\right\}$$

2.

$$f(x,y) = \left\{ \begin{array}{ll} 4xy, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & otherwise \end{array} \right\}$$

(a) Find the joint CDF

$$\int_0^x \int_0^y 4uv \; du \; dv$$

$$\int_0^x 2y^2 v \; dv$$

$$x^2y^2$$

(b) Compute $\mathbb{E}[XY]$

$$\int_0^1 \int_0^1 4x^2y^2 \; dx \; dy$$

$$\int_0^1 \frac{4}{3} y^2 dy = \frac{4}{9}$$

(c) Find the marginal CDF of X and Y

$$F_x(x,1) = x^2 \implies \int_0^x \int_0^1 4xy = x^2$$

$$F_y(1,y) = y^2 \implies \int_0^y \int_0^1 4xy = y^2$$

(d) Are X and Y independent? Yes

$$F(x,y) = F_x(x)F_y(y) = x^2y^2$$

$$\int_0^1 4xy \; dy = 2x$$

$$\int_0^1 4xy \; dx = 2y$$

$$f(x,y) = f_x(x)f_y(y) = 4xy$$

3. Prove $\mathbb{E}[X] = \int_0^\infty 1 - F(x)dx$ to show that $E[X] = \frac{1}{\lambda}$

$$E[X] = \int_0^\infty xf(x)dx \implies \lim_{n\to\infty} \int_0^n xf(x)dx$$

$$\implies \lim_{n\to\infty} [nF(n) - \int_0^n F(x)dx]$$

$$nF(n) - n + n - \int_0^n F(x)dx$$

$$\implies \lim_{n\to\infty} [-n(1 - F(n) + \int_0^n 1 - F(x)dx] = \int_0^n 1 - F(x)dx$$

$$\implies \lim_{n\to\infty}\int_0^n 1-(1-e^{-\lambda x})dx$$

$$\implies \int_0^\infty e^{-\lambda x}dx = -\frac{e^{-\lambda x}}{\lambda}\bigg|_0^\infty = \frac{1}{\lambda}$$

4. Find c where c(2x-x^2), if $0 \leq x \leq 2$ compute $\mathbb{E}[X]$ and $\mathbb{V}[X]$

$$c\int_0^2 2x - x^2 dx = 1$$

$$c\left(x^2 - \frac{x^3}{3}\bigg|_0^2\right)$$

$$c\left(4 - \frac{8}{3}\right) = 1$$

$$c = \frac{3}{4}$$

$$\mathbb{E}[X] = \int_0^2 \frac{3}{2}x^2 - \frac{3}{4}x^3 dx = 1$$

$$\mathbb{V}[X] = \int_0^2 x^2\left(\frac{3}{2}x^2 - \frac{3}{4}x^3\right) - 1 = \frac{3}{5}$$