

1.

$$\frac{1}{2\lambda} \exp\left\{-\frac{|x-\mu|}{\lambda}\right\} \mathbb{1}\{x \in \mathbb{R}, \lambda > 0\}$$

(a)

$$\int_{-\infty}^{\mu} \frac{1}{2\lambda} \exp\left\{\frac{x-\mu}{\lambda}\right\} dx + \int_{\mu}^{\infty} \frac{1}{2\lambda} \exp\left\{-\frac{x-\mu}{\lambda}\right\} dx$$

$$\Rightarrow \frac{1}{2} \exp\left\{\frac{x-\mu}{\lambda}\right\} \Big|_{-\infty}^{\mu} + \frac{1}{2} \exp\left\{-\frac{x-\mu}{\lambda}\right\} \Big|_{\mu}^{\infty}$$

$$\Rightarrow \frac{1}{2} \exp\left\{\frac{\mu-\mu}{\lambda}\right\} - \frac{1}{2} \exp\left\{-\frac{-\infty-\mu}{\lambda}\right\} - \frac{1}{2} \exp\left\{-\frac{\infty-\mu}{\lambda}\right\} + \frac{1}{2} \exp\left\{\frac{\mu-\mu}{\lambda}\right\}$$

If  $\mu = \mathbb{R}$  and  $\lambda > 0$  we can show that the PDF equals 1. Also note the function is always  $> 0$  because it is an exponential function therefore, this is indeed a PDF.

$$\Rightarrow \frac{1}{2} - 0 - 0 + \frac{1}{2} = 1$$

(b)

$$\Rightarrow \int_{-\infty}^x \frac{1}{2\lambda} \exp\left\{\frac{x-\mu}{\lambda}\right\} + \int_x^{\infty} \frac{1}{2\lambda} \exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

$$F(x < \mu) = \frac{1}{2} \exp\left\{\frac{x-\mu}{\lambda}\right\} \Big|_{-\infty}^x$$

$$\frac{1}{2} \exp\left\{\frac{x-\mu}{\lambda}\right\}$$

$$F(x \geq \mu) = \frac{1}{2} \exp\left\{-\frac{x-\mu}{\lambda}\right\} \Big|_x^{\infty}$$

$$\Rightarrow -\frac{1}{2} \exp\left\{-\frac{\infty-\mu}{\lambda}\right\} + \frac{1}{2} \exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

$$= \frac{1}{2} \exp\left\{-\frac{x-\mu}{\lambda}\right\}$$

Note that the CDF must be equal to one therefore when  $F(x \geq \mu)$  the CDF will be  $1 - \frac{1}{2} \exp\left\{-\frac{x-\mu}{\lambda}\right\}$ . Additionally, the function will always be positive because it is exponential.

$$F(x) = \begin{cases} \frac{1}{2} \exp\left\{\frac{x-\mu}{\lambda}\right\} & x < \mu \\ 1 - \frac{1}{2} \exp\left\{-\frac{x-\mu}{\lambda}\right\} & x \geq \mu \end{cases}$$

2.

$$f(x, y) = \begin{cases} 4xy, & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the joint CDF

$$\begin{aligned} \int_0^x \int_0^y 4uv \, du \, dv \\ \int_0^x 2y^2v \, dv \\ x^2y^2 \end{aligned}$$

(b) Compute  $\mathbb{E}[XY]$

$$\begin{aligned} \int_0^1 \int_0^1 4x^2y^2 \, dx \, dy \\ \int_0^1 \frac{4}{3}y^2 dy = \frac{4}{9} \end{aligned}$$

(c) Find the marginal CDF of X and Y

$$\begin{aligned} F_x(x, 1) = x^2 &\implies \int_0^x \int_0^1 4xy \, dy = x^2 \\ F_y(1, y) = y^2 &\implies \int_0^y \int_0^1 4xy \, dx = y^2 \end{aligned}$$

(d) Are X and Y independent? Yes

$$\begin{aligned} F(x, y) &= F_x(x)F_y(y) = x^2y^2 \\ \int_0^1 4xy \, dy &= 2x \\ \int_0^1 4xy \, dx &= 2y \\ f(x, y) &= f_x(x)f_y(y) = 4xy \end{aligned}$$

3. Prove  $\mathbb{E}[X] = \int_0^\infty 1 - F(x)dx$  to show that  $E[X] = \frac{1}{\lambda}$

$$\begin{aligned} E[X] &= \int_0^\infty xf(x)dx \implies \lim_{n \rightarrow \infty} \int_0^n xf(x)dx \\ &\implies \lim_{n \rightarrow \infty} [nF(n) - \int_0^n F(x)dx] \\ &\quad nF(n) - n + n - \int_0^n F(x)dx \\ &\implies \lim_{n \rightarrow \infty} [-n(1 - F(n)) + \int_0^n 1 - F(x)dx] = \int_0^\infty 1 - F(x)dx \end{aligned}$$

$$\begin{aligned} &\implies \lim_{n \rightarrow \infty} \int_0^n 1 - (1 - e^{-\lambda x}) dx \\ &\implies \int_0^\infty e^{-\lambda x} dx = -\frac{e^{-\lambda x}}{\lambda} \Big|_0^\infty = \frac{1}{\lambda} \end{aligned}$$

4. Find  $c$  where  $c(2x-x^2)$ , if  $0 \leq x \leq 2$  compute  $\mathbb{E}[X]$  and  $\mathbb{V}[X]$

$$c \int_0^2 2x - x^2 dx = 1$$

$$c \left( x^2 - \frac{x^3}{3} \Big|_0^2 \right)$$

$$c \left( 4 - \frac{8}{3} \right) = 1$$

$$c = \frac{3}{4}$$

$$\mathbb{E}[X] = \int_0^2 \frac{3}{2}x^2 - \frac{3}{4}x^3 dx = 1$$

$$\mathbb{V}[X] = \int_0^2 x^2 \left( \frac{3}{2}x^2 - \frac{3}{4}x^3 \right) - 1 = \frac{3}{5}$$