16711 HW 4 - Writeup

Saeed Bai, Mohammadreza Mousaei

TOTAL POINTS

18 / 18

QUESTION 1

1 Lagrange 2.5 / 3

✓ - 0.5 pts Partial

QUESTION 2

2 Inertia Matrix 5 / 5

✓ - 0 pts Correct

QUESTION 3

Dynamic Parameter Estimation 10 pts

3.1 1 / 2

✓ - 1 pts Partial

Didn't compute the velocity.

3.2 5 / 5

✓ - 0 pts Correct

3.3 Extra credit 2 / 0

✓ + 2 pts Correct

+ 1 pts Partial

+ 0 pts Incorrect/Not Attempted

3.4 1 / 1

✓ - 0 pts Correct

3.5 0.5 / 1

✓ - 0.5 pts The answer is wrong.

3.6 1 / 1

✓ - 0 pts Correct

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16-711
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Homework 4

Problem 1

Define the length of the string to be l

$$KE = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 + \dot{x}\dot{\theta}l\cos\theta$$

$$PE = mgl(1 - \cos\theta)$$

$$L = KE - PE = \frac{1}{2}(I\dot{\theta}^2 + m\dot{x}^2 + 2\dot{x}\dot{\theta}l\cos\theta) - mgl + mgl\cos\theta$$

$$\longrightarrow \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0$$

$$\text{where} \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} + 2\dot{\theta}l\cos\theta$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) = m(\ddot{x} + \ddot{\theta}l\cos\theta - \dot{\theta}^2 l\sin\theta) = 0$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta - m\dot{x}\dot{\theta}l\sin\theta \qquad \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} + m\dot{x}l\cos\theta = ml^2\dot{\theta} + m\dot{x}l\cos\theta$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = ml^2\ddot{\theta} + m\ddot{x}l\cos theta - m\dot{x}\dot{\theta}l\sin\theta$$

$$ml^2\ddot{\theta} + mgl\sin\theta + m\ddot{x}l\cos\theta = 0$$

$$\longrightarrow \ddot{\theta} + \frac{g}{l}\sin\theta + \frac{\ddot{x}}{l} = 0$$

$$\begin{bmatrix} m & l\cos\theta \\ \frac{1}{l} & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & -ml\sin\dot{\theta} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ g/l\sin\theta \end{bmatrix} = \vec{0}$$

1 Lagrange **2.5 / 3**

✓ **- 0.5 pts** **Partial**

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Homework 4

Problem 2

(a)

Knowing the inertia tensor matrix is

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \longrightarrow \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$let \;\; x = r\cos\theta\sin\phi \;\; y = r\sin\theta\phi \;\; z = c\cos\theta$$

$$x' = \frac{x}{a} \;\; y' = \frac{y}{b} \;\; z' = \frac{z}{c}$$

$$I_{yy} = \rho \int_V (x^2 + y^2) dxdydz$$

$$= abc\rho' \int_0^1 \int_0 2\pi \int 0\pi (a^2 \cos\theta^2 + b^2 \sin\theta^2) r^4 \sin\phi^3 d\phi d\theta dr = \frac{4}{15}\pi abc(a^2 + b^2)$$

$$where \; \rho = \frac{m}{V} \; where \; V = \frac{4}{3}\pi abc$$

$$\longrightarrow \rho = \frac{3m}{4abc\pi}$$

$$plug \; \rho \; into \; I_{yy} \longrightarrow I_{yy} = \frac{m}{5}(a^2 + b^2)$$

Applying the same equation to I_{zz} and I_{xx} will get the following results:

$$I_{xx} = \frac{m}{5}(b^2 + c^2) \quad I_{zz} = \frac{m}{5}(a^2 + c^2)$$

$$I = \frac{m}{5} \begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + b^2 & 0 \\ 0 & 0 & a^2 + c^2 \end{bmatrix}$$

(b)

Define the origin at center of mass where z is along h, x and y is perpendicular to each other along r

$$V = \pi r^2 h \quad \longrightarrow \rho = \frac{m}{v} = \frac{m}{\pi r^2 h}$$

$$I = \rho \int_{-h/2}^{h/2} \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \begin{bmatrix} y^2+z^2 & 0 & 0 \\ 0 & x^2+z^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} dxdydz$$

$$= \begin{bmatrix} \frac{1}{12}mh^2 + \frac{1}{4}mr^2 & 0 & 0 \\ 0 & 12mh^2 + \frac{1}{4}mr^2 & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix}$$

Problem 3

The accompanying code for this problem is provided in $Q3.m$

(a)

To estimate the velocity at the center of mass (cm), we have:

$$p_b = p_{cm} + Rq \longrightarrow p_{cm} = p_b - Rq$$

where p_b is the position of the beacon (which is read from the file), p_{cm} is the position of the center of the mass in the inertial frame , R is the rotation of the beacon w.r.t. body frame and q is the vector from the center of mass to the beacon expressed in body frame. Since the speed is constant (no external force), we can write

$$\left.\begin{aligned} P_{cm}^i &= p_b^i - R^i q \\ P_{cm}^{i+1} &= p_b^{i+1} - R^{i+1} q \\ P_{cm}^{i+2} &= p_b^{i+2} - R^{i+2} q \end{aligned}\right\} \xrightarrow[\text{speed}]{\text{constant}} P_{cm}^{i+2} - P_{cm}^{i+1} = P_{cm}^{i+1} - P_{cm}^i$$

$$\longrightarrow P_{cm}^{i+2} - 2P_{cm}^{i+1} + P_{cm}^i = 0$$

$$\longrightarrow q = (R^i - 2R^{i+1} + R^{i+2})^{-1}(p_b^i - 2p_b^{i+1} + p_b^{i+2})$$

Having q, we can calculate p_{cm} for each p_b. And since $F = 1Hz$, the velocity of the center of mass will be the difference between P_{cm}^i's at two consecutive times.

2 Inertia Matrix 5 / 5

✓ **- 0 pts** Correct

(b)

Define the origin at center of mass where z is along h, x and y is perpendicular to each other along r

$$V = \pi r^2 h \quad \longrightarrow \rho = \frac{m}{v} = \frac{m}{\pi r^2 h}$$

$$\begin{aligned} I &= \rho \int_{-h/2}^{h/2} \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \begin{bmatrix} y^2+z^2 & 0 & 0 \\ 0 & x^2+z^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} dxdydz \\ &= \begin{bmatrix} \frac{1}{12}mh^2 + \frac{1}{4}mr^2 & 0 & 0 \\ 0 & 12mh^2 + \frac{1}{4}mr^2 & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix} \end{aligned}$$

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Having q, we can calculate p_{cm} for each p_b. And since $F = 1Hz$, the velocity of the center of mass will be the difference between P_{cm}^{i}'s at two consecutive times.

3.1 1 / 2

✓ **- 1 pts** Partial

Didn't compute the velocity.

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Homework 4

(b)

Using Newton-Euler equations of motion for the body coordinates we have:

$$\tau = I\dot{w}_b + w_b \times Iw_b \xrightarrow{\tau=0} I\dot{w}_b + \hat{w}_b I w_b = 0$$

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{bmatrix} + \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

rewriting the above equation we have

$$\left(\begin{bmatrix} \dot{w}_1 & \dot{w}_2 & \dot{w}_3 & 0 & 0 & 0 \\ 0 & \dot{w}_1 & 0 & \dot{w}_2 & \dot{w}_3 & 0 \\ 0 & 0 & \dot{w}_1 & 0 & \dot{w}_2 & \dot{w}_3 \end{bmatrix} + \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 & 0 & 0 & 0 \\ 0 & w_1 & 0 & w_2 & w_3 & 0 \\ 0 & 0 & w_1 & 0 & w_2 & w_3 \end{bmatrix} \right) \begin{bmatrix} I_{11} \\ I_{12} \\ I_{13} \\ I_{22} \\ I_{23} \\ I_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the inertial tensor can be calculated up to a scale m, we can fix $I_{33} = 1$, so we get

$$\left(\begin{bmatrix} \dot{w}_1 & \dot{w}_2 & \dot{w}_3 & 0 & 0 \\ 0 & \dot{w}_1 & 0 & \dot{w}_2 & \dot{w}_3 \\ 0 & 0 & \dot{w}_1 & 0 & \dot{w}_2 \end{bmatrix} + \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix} \begin{bmatrix} w_1 & w_2 & w_3 & 0 & 0 \\ 0 & w_1 & 0 & w_2 & w_3 \\ 0 & 0 & w_1 & 0 & w_2 \end{bmatrix} \right) \begin{bmatrix} I_{11} \\ I_{12} \\ I_{13} \\ I_{22} \\ I_{23} \end{bmatrix} = - \left(\begin{bmatrix} 0 \\ 0 \\ \dot{w}_3 \end{bmatrix} + \hat{w} \begin{bmatrix} 0 \\ 0 \\ w_3 \end{bmatrix} \right)$$

Using above equation, I is calculated in the code as

$$I = \begin{bmatrix} 0.4609 & -0.0004 & -0.0009 \\ -0.0004 & 1.1215 & -0.0489 \\ -0.0009 & -0.0489 & 1.0000 \end{bmatrix}$$

(c)

To guess the shape of the asteroid, we first calculate the transform to principal axes (principal moments). To do so, we calculate the eigenvalues

$$I_p = \begin{bmatrix} 0.4609 & 0 & 0 \\ 0 & 0.9828 & 0 \\ 0 & 0 & 1.1387 \end{bmatrix}$$

which matches two well known shapes

1. ellipsoid with axes a, b, c, such that a is the largest axis and c is the smallest.
2. cuboid with width w, height h and depth d

with ratios
$$\frac{b^2+c^2}{a^2+c^2} = \frac{h^2+d^2}{w^2+d^2} = \frac{0.4609}{0.9828}, \quad \frac{b^2+c^2}{a^2+b^2} = \frac{h^2+d^2}{w^2+h^2} = \frac{0.4609}{1.1387}$$

(d)

The vector q is computed in section (a), the distance is computed as $\|q\| = 0.3$

(e)

The angular momentum is constant (there is no external force). Therefore we get

$$L = Iw = \begin{bmatrix} -0.2391 \\ -0.0642 \\ -0.0359 \end{bmatrix}$$

(f)

Having the velocity and angular velocity of the asteroid, we can calculate the pose as

$$Position = \begin{bmatrix} 1.3891 \\ 3.1706 \\ 2.2802 \end{bmatrix}$$
$$Rotation = \begin{bmatrix} -0.2394 & 0.7089 & -0.6634 \\ -0.9706 & -0.1915 & 0.1455 \\ -0.0239 & 0.6788 & 0.7339 \end{bmatrix}$$

Problem 4

Motivation

Delivery drones may become widespread over the next five to ten years, particularly for what is known as the "last-mile" logistics of small, light items. Companies like Amazon, Google, the United Parcel Service, DHL, and Alibaba have been running high-profile experiments testing drone delivery systems, and the development of such systems reached a milestone when the first commercial drone delivery approved by the Federal Aviation Administration took place on July 17, 2015. In the future, drones could augment, or even replace, truck

3.2 5 / 5

✓ **- 0 pts** Correct

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3.3 Extra credit **2 / 0**

✓ + **2 pts** **Correct**

\+ **1 pts** Partial

\+ **0 pts** Incorrect/Not Attempted

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3.4 1 / 1

✓ - 0 pts Correct

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3.5 **0.5 / 1**

✓ **- 0.5 pts** The answer is wrong.

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3.6 1 / 1

✓ **- 0 pts** Correct