16711 HW 2 - Writeup

Saeed Bai

TOTAL POINTS

25.05 / 26

QUESTION 1

Unit Quaternions 6 pts

1.1 **2 / 2**

✓ **- 0 pts** Correct

- 0.5 pts Need to verify that the vector part describes the point to which _x_ is rotated under the rotation associated with _Q_.

- 0.25 pts Need to show the product is a pure quaternion.

1.2 **1.8 / 2**

- 0 pts Correct

- 0.2 Point adjustment

It said to "show" the answer. Just citing something else that proves it is sort of a weak answer. At least give some explanation along with the citation.

1.3 **1.75 / 2**

- 0 pts Correct

- 0.25 pts Mistake on part 1

- 0.25 pts Mistake on part 2

- 0.25 pts Mistake on part 3

✓ **- 0.25 pts** Mistake on part 4

1.4 **2 / 0**

- 0 pts Extra credit not attempted

✓ **+ 2 pts** Correct

+ 1 pts Small mistake

QUESTION 2

Planar Rigid Body Transformations 8 pts

2.1 **1.5 / 2**

Pure Translation

✓ **- 0 pts** Correct

- 0.5 pts Partial

- 1 pts Incorrect

General case

- 0 pts Correct

✓ **- 0.5 pts** Partial

- 1 pts Incorrect

2.2 **0.5 / 1**

- 0 pts Correct

✓ **- 0.5 pts** Partial

- 1 pts Incorrect

Not clear what to conclude from your equations.

2.3 **1 / 2**

- 0 pts Correct

✓ **- 1 pts** Partial

- 2 pts Incorrect

1 Please explain what you did here.

2.4 **1.5 / 2**

- 0 pts Correct

- 1 pts Partial

- 2 pts Incorrect

✓ **- 0.5 pts** Incorrect/Missing interpretation

- 0.5 pts Partial

2.5 **1 / 1**

✓ **- 0 pts** Correct

- 0.5 pts Partial

- 1 pts Incorrect

QUESTION 3

Forward Kinematics 12 pts

3.1 2 / 2

- ✓ **- 0 pts** Correct
- **- 0.5 pts** 1 or 2 values wrong
- **- 1 pts** None of the values is correct

3.2 6 / 6

- ✓ **- 0 pts** Correct
- **- 6 pts** No code provided.
- **- 3 pts** Correct. 50% for resubmission.

3.3 2 / 2

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Surface normal is partially correct
- **- 1 pts** Answers are wrong

3.4 2 / 2

- ✓ **- 0 pts** Correct
- **- 0.5 pts** "kdc" should be smoothly written
- **- 1 pts** The answer is wrong.
- **- 1 pts** Correct. 50% from resubmission.

Please include your figure in the write up next time.

1. a) $\bar{X} = (0, x)$

$$U_Q = \frac{Q}{\|Q\|} = \frac{Q}{\sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}}$$

Q is unit quaternion when $\sqrt{q_0^2+q_1^2+q_2^2+q_3^2} = 1$

$$Q = (q, \bar{q}) = q_0 + q_1 i + q_2 j + q_3 k$$
$$Q^* = (q, -\bar{q}) = q_0 - q_1 i - q_2 j - q_3 k$$
$$X = (x_0, x) = 0 + x_1 i + x_2 j + x_3 k$$

scalar part:

$$QX : \cancel{q_0 x_0}^{0} - q_1 x_1 - q_2 x_2 - q_3 x_3$$

$$\begin{aligned}
QXQ^* : \ & (\cancel{q_0 x_0} + q_1 x_1 + q_2 x_2 + q_3 x_3) q_0 \\
& + (q_0 x_1 + \cancel{x_0 q_1} + q_2 x_3 - q_3 x_2) i \cdot (-q_1) i \\
& + (q_0 x_2 - q_1 x_3 + \cancel{q_2 x_0} + q_3 x_1) j \cdot (-q_2) j \\
& + (q_0 x_3 + q_1 x_2 - q_2 x_1 + \cancel{q_3 x_0}) k \cdot (-q_3) k
\end{aligned}$$

$$\begin{aligned}
= \ & \cancel{q_0^2 x_0}^{0} + \cancel{q_1 x_1 q_0} + \cancel{q_2 x_2 q_0} + \cancel{q_3 x_3 q_0} = 0 \\
& - \cancel{q_0 x_1 q_1} - \cancel{q_2 x_3 q_1} + \cancel{q_3 x_2 q_1} \\
& - \cancel{q_0 x_2 q_2} + \cancel{q_1 x_3 q_2} - \cancel{q_3 x_1 q_2} \\
& - \cancel{q_0 x_3 q_3} - \cancel{q_1 x_2 q_3} + \cancel{q_2 x_1 q_3}
\end{aligned}$$

Hence, $Q\bar{X}Q^*$ is pure quaternion

vector part:

$$QX = (-\bar{q} \cdot \bar{x}, \ \bar{q}\bar{x} + \bar{q} \times \bar{x})$$

$$\begin{aligned}
QXQ^* &= (-\bar{q} \cdot \bar{x})(-\bar{q}) + q_0(q_0 \bar{x} + \bar{q} \times \bar{x}) + (q_0 \bar{x} + \bar{q} \times \bar{x}) \times (-\bar{q}) \\
&= (\bar{x} \cdot \bar{q})\bar{q} + q_0^2 \bar{x} + q_0 \bar{q} \times \bar{x} + \underbrace{q_0(\bar{x} \times (-\bar{q}))}_{q_0(\bar{q} \times \bar{x})} + \underbrace{(\bar{q} \times \bar{x}) \times (-\bar{q})}_{\bar{q} \times \bar{q} \times \bar{x}} \\
&= (\bar{x} \cdot \bar{q})\bar{q} + q_0^2 \bar{x} + 2q_0(q \times \bar{x}) + (\bar{q} \cdot \bar{x})\bar{q} - (q \cdot q)\bar{x} \\
&= (q_0^2 - \bar{q} \cdot \bar{q})\bar{x} + 2(q_0(\bar{q} \times \bar{x}) + (x \cdot \bar{q})\bar{q})
\end{aligned}$$

$$Q = (q_0, \vec{q}) = (\cos\frac{\theta}{2}, \ \bar{\omega} \cdot \sin\frac{\theta}{2})$$

$$(q_0^2 - \bar{q}\cdot\bar{q})\bar{x} + 2(q_0(\bar{q}\times\bar{x}) + (x\cdot\bar{q})\bar{q})$$

$$= \left(\cos^2\frac{\theta}{2} - (\bar{w}\cdot\sin\frac{\theta}{2})\cdot(\bar{w}\cdot\sin\frac{\theta}{2})\right)\vec{x} + 2(\cos\frac{\theta}{2}(\bar{w}\sin\frac{\theta}{2}\times\bar{x}) + (x\cdot\bar{w}\cdot\sin\frac{\theta}{2})\ \bar{w}\cdot\sin\frac{\theta}{2}$$

$$= \cos\theta\,\bar{x} + \sin\theta\,\hat{w}\bar{x} + \bar{w}w^T(1-\cos\theta)\bar{x}$$

$$= \bar{x}(\cos\theta I - \sin\theta\,\hat{w} + (\hat{w}^2 + I)(1-\cos\theta))$$

$$= \dot{x}(I + \sin\theta\,\hat{w} + (1-\cos\theta)\hat{w}^2) \longrightarrow$$ Rodriguez' formula

$$\bar{x}e^{\hat{w}\theta} \Rightarrow R\vec{x}$$

b) "The exponential map is surjective onto SO(3)"

Murray, Li, Shastry, "A Mathematical Introduction to Robotic Manipulation", 1994. pg. 29

c) I. $R_A^B \cdot R_B^C$ in SO(3)

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$$

27 multiplication

18 addition

II. $Q_1 \cdot Q_2$ 1 multiplication

$$Q_1\cdot Q_2 = w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2$$
$$+ (w_1x_1 + x_1w_2 + y_1z_2 - z_1y_2)\,i$$
$$+ (w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2)\,j$$
$$+ (w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2)\,k$$

16 multiplication

12 addition

III.

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}\begin{bmatrix} x \\ x \\ x \end{bmatrix}$$

9 multiplication

6 addition

IV.

$$-x_1x_2 - y_1y_2 - z_1z_2$$
$$+ (w_1x_1 + y_1z_2 - z_1y_2)\,i$$
$$+ (w_1y_2 - x_1z_2 + z_1x_2)\,j$$
$$+ (w_1z_2 + x_2y_2 - y_1x_2)\,k$$

12 multiplication

14 addition

1.1 2 / 2

✓ **- 0 pts** Correct

- 0.5 pts Need to verify that the vector part describes the point to which _x_ is rotated under the rotation associated with _Q_.

- 0.25 pts Need to show the product is a pure quaternion.

$$(q_0^2 - \bar{q}\cdot\bar{q})\bar{x} + 2(q_0(\bar{q}\times\bar{x}) + (x\cdot\bar{q})\bar{q})$$

$$= \left(\cos^2\frac{\theta}{2} - (\bar{\omega}\cdot\sin\frac{\theta}{2})\cdot(\bar{\omega}\cdot\sin\frac{\theta}{2})\right)\vec{x} + 2(\cos\frac{\theta}{2}(\bar{\omega}\sin\frac{\theta}{2}\times\bar{x}) + (x\cdot\bar{\omega}\cdot\sin\frac{\theta}{2})\ \bar{\omega}\cdot\sin\frac{\theta}{2}$$

$$= \cos\theta\,\bar{x} + \sin\theta\,\hat{\omega}\bar{x} + \bar{\omega}\omega^T(1-\cos\theta)\bar{x}$$

$$= \bar{x}(\cos\theta I - \sin\theta\,\hat{\omega} + (\hat{\omega}^2 + I)(1-\cos\theta))$$

$$= \dot{x}(I + \sin\theta\,\hat{\omega} + (1-\cos\theta)\hat{\omega}^2) \longrightarrow$$ Rodriguez' formula

$$\bar{x}e^{\hat{\omega}\theta} \triangleq R\vec{x}$$

b) "The exponential map is surjective onto SO(3)"
Murray, Li, Shastry, "A Mathematical Introduction to Robotic Manipulation", 1994. pg. 29

c) I. $R_A^B \cdot R_B^C$ in SO(3)

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$$

27 multiplication
18 addition

II. $Q_1 \cdot Q_2$ 1 multiplication

$$Q_1\cdot Q_2 = w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2$$
$$+ (w_1x_1 + x_1w_2 + y_1z_2 - z_1y_2)\,i$$
$$+ (w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2)\,j$$
$$+ (w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2)\,k$$

16 multiplication
12 addition

III.

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}\begin{bmatrix} x \\ x \\ x \end{bmatrix}$$

9 multiplication
6 addition

IV.

$$-x_1x_2 - y_1y_2 - z_1z_2$$
$$+ (w_1x_1 + y_1z_2 - z_1y_2)\,i$$
$$+ (w_1y_2 - x_1z_2 + z_1x_2)\,j$$
$$+ (w_1z_2 + x_2y_2 - y_1x_2)\,k$$

12 multiplication
14 addition

1.2 1.8 / 2

- **0 pts** Correct

- 0.2 Point adjustment

It said to "show" the answer. Just citing something else that proves it is sort of a weak answer. At least give some explanation along with the citation.

$$(q_0^2 - \bar{q}\cdot\bar{q})\bar{x} + 2(q_0(\bar{q}\times\bar{x}) + (x\cdot\bar{q})\bar{q})$$

$$= \left(\cos^2\tfrac{\theta}{2} - (\bar{\omega}\cdot\sin\tfrac{\theta}{2})\cdot(\bar{\omega}\cdot\sin\tfrac{\theta}{2})\right)\vec{x} + 2(\cos\tfrac{\theta}{2}(\bar{\omega}\sin\tfrac{\theta}{2}\times\bar{x}) + (x\cdot\bar{\omega}\cdot\sin\tfrac{\theta}{2})\ \bar{\omega}\cdot\sin\tfrac{\theta}{2}$$

$$= \cos\theta\,\bar{x} + \sin\theta\,\hat{\omega}\bar{x} + \bar{\omega}\omega^T(1-\cos\theta)\bar{x}$$

$$= \bar{x}(\cos\theta I - \sin\theta\,\hat{\omega} + (\hat{\omega}^2 + I)(1-\cos\theta))$$

$$= \bar{x}(I + \sin\theta\,\hat{\omega} + (1-\cos\theta)\hat{\omega}^2) \longrightarrow \text{Rodriguez' formula}$$

$$\bar{x}e^{\hat{\omega}\theta} \Rightarrow R\vec{x}$$

b) "The exponential map is surjective onto SO(3)"

Murray, Li, Shastry, "A Mathematical Introduction to Robotic Manipulation", 1994. pg. 29

c) I. $R_A^B \cdot R_B^C$ in SO(3)

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$$

27 multiplication
18 addition

II. $Q_1 \cdot Q_2$ 1 multiplication

$$Q_1 \cdot Q_2 = w_1w_2 - x_1x_2 - y_1y_2 - z_1z_2$$
$$+ (w_1x_1 + x_1w_2 + y_1z_2 - z_1y_2)\, i$$
$$+ (w_1y_2 - x_1z_2 + y_1w_2 + z_1x_2)\, j$$
$$+ (w_1z_2 + x_1y_2 - y_1x_2 + z_1w_2)\, k$$

16 multiplication
12 addition

III.

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}\begin{bmatrix} x \\ x \\ x \end{bmatrix}$$

9 multiplication
6 addition

IV.

$$-x_1x_2 - y_1y_2 - z_1z_2$$
$$+ (w_1x_1 + y_1z_2 - z_1y_2)\, i$$
$$+ (w_1y_2 - x_1z_2 + z_1x_2)\, j$$
$$+ (w_1z_2 + x_2y_2 - y_1x_2)\, k$$

12 multiplication
14 addition

1.3 **1.75 / 2**

- **- 0 pts** Correct
- **- 0.25 pts** Mistake on part 1
- **- 0.25 pts** Mistake on part 2
- **- 0.25 pts** Mistake on part 3
- ✓ **- 0.25 pts Mistake on part 4**

d) let $Q = (\sin\frac{\theta}{2}, \bar{\omega}\cos\frac{\theta}{2})$ $\qquad \dot{Q} = \frac{1}{2}\cos\frac{\theta}{2}, -\frac{\bar{\omega}}{2}\sin\frac{\theta}{2})$

$$\dot{Q}Q^* = \left(-\frac{1}{2}\cos\frac{\theta}{2}, -\frac{\bar{\omega}}{2}\sin\frac{\theta}{2}\right)\left(\sin\frac{\theta}{2}, -\bar{\omega}\cos\frac{\theta}{2}\right)$$

$$= -\frac{1}{2}\cancel{\cos\frac{\theta}{2}\cdot\sin\frac{\theta}{2}} + \frac{\bar{\omega}^2}{2}\cancel{\sin\frac{\theta}{2}\cos\frac{\theta}{2}},\ \frac{\bar{\omega}}{2}\sin^2\frac{\theta}{2} + \frac{\bar{\omega}}{2}\cos^2\frac{\theta}{2}$$

$$= (0, \omega/2)$$

1.4 2 / 0

- **- 0 pts** Extra credit not attempted
- ✓ **+ 2 pts** Correct
- **+ 1 pts** Small mistake

2. a) $e^{\hat{\xi}\theta} \in se2$ gives RB transformation in $SE(2)$

when $w = 0$ $\xi = (v, 0)$

$$\hat{\xi}^2 = \hat{\xi}^3 = \hat{\xi}^4 = \cdots = 0$$

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta \quad \Rightarrow \quad e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \quad w = 0$$

which is in $SE(2)$

when $w \neq 0$, assume $\|w\| = 1$

$$g = \begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix}$$

$$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -w \times v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{w} & hw \\ 0 & 0 \end{bmatrix}$$

$$h = w^T v$$

$$(\hat{\xi}')^2 = \begin{bmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad (\hat{\xi}')^3 = \begin{bmatrix} \hat{w}^3 & 0 \\ 0 & 0 \end{bmatrix}, \quad \cdots$$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) + ww^T v\theta \\ 0 & 1 \end{bmatrix}$$ which belongs to $SE(2)$

b) let $\hat{\xi}' = \begin{bmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $g = \begin{bmatrix} 1 & 0 & q_x \\ 0 & 1 & q_y \\ 0 & 0 & 1 \end{bmatrix}$

$e^{\hat{\xi}\theta} = e^{g\hat{\xi}g^{-1}\theta}$

equal

$\hat{\xi} = \begin{bmatrix} 0 & -w & q_y w \\ w & 0 & -q_x w \\ 0 & 0 & 0 \end{bmatrix}$

which gives the same twist

$\hat{\xi} = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$

$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$

$\xi = \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix}$

c) $H = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$ $\hat{\xi} = \begin{bmatrix} \hat{w} & \bar{v} \\ 0 & 0 \end{bmatrix}$

when $\hat{w} = 0$ $\hat{\xi} = \begin{bmatrix} 0 & \bar{v} \\ 0 & 0 \end{bmatrix}$ $(\hat{\xi})^{\vee} = (\vec{v}, 0)$ Pure translation

$\hat{w} = 1$ $-\hat{w}\vec{q} = \vec{v}$ $-\begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

$wq_y = v_x$ $-wq_x = v_x$ $\Rightarrow$ $v_x = q_y$ $v_x = -q_x$

2.1 1.5 / 2

Pure Translation

✓ **- 0 pts** **Correct**

- 0.5 pts Partial

- 1 pts Incorrect

General case

- 0 pts Correct

✓ **- 0.5 pts** **Partial**

- 1 pts Incorrect

2. a) $e^{\hat{\xi}\theta} \in se2$ gives RB transformation in $SE(2)$

When $w = 0$ $\xi = (v, 0)$

$\hat{\xi}^2 = \hat{\xi}^3 = \hat{\xi}^4 = \cdots = 0$

$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta \Rightarrow e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$ $w=0$

which is in $SE(2)$

when $w \neq 0$, assume $\|w\| = 1$

$g = \begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix}$

$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -w \times v \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{w} & hw \\ 0 & 0 \end{bmatrix}$

$h = w^T v$

$(\hat{\xi}')^2 = \begin{bmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{bmatrix}, (\hat{\xi}')^3 = \begin{bmatrix} \hat{w}^3 & 0 \\ 0 & 0 \end{bmatrix}, \cdots$

$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) + ww^T v\theta \\ 0 & 1 \end{bmatrix}$ which belongs to $SE(2)$

b) let $\hat{\xi}' = \begin{bmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $g = \begin{bmatrix} 1 & 0 & q_x \\ 0 & 1 & q_y \\ 0 & 0 & 1 \end{bmatrix}$

$e^{\hat{\xi}\theta} = e^{g\hat{\xi}g^{-1}\theta}$

equal

$\hat{\xi} = \begin{bmatrix} 0 & -w & q_y w \\ w & 0 & -q_x w \\ 0 & 0 & 0 \end{bmatrix}$

which gives the same twist

$\hat{\xi} = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$

$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$

$\xi = \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix}$

c) $H = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$ $\hat{\xi} = \begin{bmatrix} \hat{w} & \bar{v} \\ 0 & 0 \end{bmatrix}$

when $\hat{w} = 0$ $\hat{\xi} = \begin{bmatrix} 0 & \bar{v} \\ 0 & 0 \end{bmatrix}$ $(\hat{\xi})^{\vee} = (\vec{v}, 0)$ Pure translation

$\hat{w} = 1$ $-\hat{w}\vec{q} = \vec{v}$ $-\begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

$wq_y = v_x$ $-wq_x = v_x$ $\Rightarrow$ $v_x = q_y$ $v_x = -q_x$

2.2 0.5 / 1

- **- 0 pts** Correct
- ✓ **- 0.5 pts Partial**
- **- 1 pts** Incorrect

Not clear what to conclude from your equations.

2. a) $e^{\hat{\xi}\theta} \in se2$ gives RB transformation in $SE(2)$

When $w = 0$ $\xi = (v, 0)$

$$\hat{\xi}^2 = \hat{\xi}^3 = \hat{\xi}^4 = \cdots = 0$$

$$e^{\hat{\xi}\theta} = I + \hat{\xi}\theta \Rightarrow e^{\hat{\xi}\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix} \quad w = 0$$

which is in $SE(2)$

when $w \neq 0$, assume $\|w\| = 1$

$$g = \begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix}$$

$$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} I & -w \times v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{w} & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & w \times v \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \hat{w} & hw \\ 0 & 0 \end{bmatrix}$$

$h = w^T v$

$$(\hat{\xi}')^2 = \begin{bmatrix} \hat{w}^2 & 0 \\ 0 & 0 \end{bmatrix}, \ (\hat{\xi}')^3 = \begin{bmatrix} \hat{w}^3 & 0 \\ 0 & 0 \end{bmatrix}, \cdots$$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{w}\theta} & (I - e^{\hat{w}\theta})(w \times v) + ww^T v\theta \\ 0 & 1 \end{bmatrix}$$ which belongs to $SE(2)$

b) let $\hat{\xi}' = \begin{bmatrix} 0 & -w & 0 \\ w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $g = \begin{bmatrix} 1 & 0 & q_x \\ 0 & 1 & q_y \\ 0 & 0 & 1 \end{bmatrix}$

$e^{\hat{\xi}\theta} = e^{g\hat{\xi}g^{-1}\theta}$

equal $\hat{\xi} = \begin{bmatrix} 0 & -w & q_y w \\ w & 0 & -q_x w \\ 0 & 0 & 0 \end{bmatrix}$

which gives the same twist

$$\hat{\xi} = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{\xi}' = g^{-1}\hat{\xi}g = \begin{bmatrix} 1 & 0 & v_x \\ 0 & 1 & v_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xi = \begin{bmatrix} v_x \\ v_y \\ 1 \end{bmatrix}$$

c) $H = \begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$ $\hat{\xi} = \begin{bmatrix} \hat{w} & \bar{v} \\ 0 & 0 \end{bmatrix}$

when $\hat{w} = 0$ $\hat{\xi} = \begin{bmatrix} 0 & \bar{v} \\ 0 & 0 \end{bmatrix}$ $(\hat{\xi})^{\vee} = (\vec{v}, 0)$ Pure translation

$\hat{w} = 1$ $-\hat{w}\vec{q} = \vec{v}$ $-\begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$

$w q_y = v_x$ $-w q_x = v_x$ $\Rightarrow$ $v_x = q_y$ $v_x = -q_x$

$\hat{\xi} = g\hat{\xi}'g^{-1}$ $\quad g = \begin{bmatrix} I & q \\ 0 & 1 \end{bmatrix}$, $\hat{\xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix}$ Pure rotation around q.

$g^{-1} = \begin{bmatrix} I & -q \\ 0 & 1 \end{bmatrix}$

d) $\hat{V}_s = \dot{g}g^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T \bar{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T\bar{p} + \dot{\bar{p}} \\ 0 & 0 \end{bmatrix}$

$\dot{R}R^T = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{\omega}^s$

where $\dot{R}R^T$ is $\hat{\omega}^s$ and $-\dot{R}R^T\bar{p} + \dot{\bar{p}}$ is $\bar{v}^s$

$\hat{V_b} = g^{-1}\dot{g} = \begin{bmatrix} R^T & -R^T\bar{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$

$R^T\dot{R} = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{\omega}^b$ $\quad R^T\dot{p}$ is $\hat{v}^b$

V^s, the spacial velocity, written as $V^s = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T)^\vee \end{bmatrix}$

is independent of choosing RB.

V^b, the body velocity. $V^b = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix}$ is defining the velocity of a point in the chosen body frame.

e) To map body velocity into spatial velocity

$\omega_{ab}^s = R_{ab}\omega_{ab}^b$ $\quad v_{ab}^s = -\omega_{ab}^s \times p_{ab} + \dot{p}_{ab} = p_{ab} \times R_{ab}\omega_{ab}^b + R_{ab}v_{ab}^b$

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab}R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{Adg} \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix}$$

2.3 1 / 2

- **- 0 pts** Correct
- ✓ **- 1 pts** **Partial**
- **- 2 pts** Incorrect

1 Please explain what you did here.

$\hat{\xi} = g\hat{\xi}'g^{-1}$ $\quad g = \begin{bmatrix} I & q \\ 0 & 1 \end{bmatrix}$, $\hat{\xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix}$ Pure rotation around q.

$g^{-1} = \begin{bmatrix} I & -q \\ 0 & 1 \end{bmatrix}$

d) $$\hat{V}_s = \dot{g}g^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\bar{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T\bar{p} + \dot{\bar{p}} \\ 0 & 0 \end{bmatrix}$$

$$\dot{R}R^T = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{\omega}^s$$

where $\dot{R}R^T$ is $\hat{W}^s$ and $-\dot{R}R^T\bar{p} + \dot{\bar{p}}$ is $\bar{V}^s$

$$\hat{V_b} = g^{-1}\dot{g} = \begin{bmatrix} R^T & -R^T\bar{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$R^T\dot{R} = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{\omega}^b \qquad R^T\dot{p} \text{ is } \hat{v}^b$$

V^s, the spacial velocity, written as $V^s = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T)^\vee \end{bmatrix}$ is independent of choosing RB.

V^b, the body velocity. $V^b = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix}$ is defining the velocity of a point in the chosen body frame.

e) To map body velocity into spatial velocity

$W_{ab}^s = R_{ab}W_{ab}^b \qquad V_{ab}^s = -W_{ab}^s \times P_{ab} + \dot{P}_{ab} = P_{ab} \times R_{ab}W_{ab}^b + R_{ab}V_{ab}^b$

$$V_{ab}^s = \begin{bmatrix} V_{ab}^s \\ W_{ab}^s \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{P}_{ab}R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{Adg} \begin{bmatrix} V_{ab}^b \\ W_{ab}^b \end{bmatrix}$$

2.4 1.5 / 2

- **- 0 pts** Correct
- **- 1 pts** Partial
- **- 2 pts** Incorrect
- ✓ **- 0.5 pts Incorrect/Missing interpretation**
- **- 0.5 pts** Partial

$\hat{\xi} = g\hat{\xi}'g^{-1}$ $\quad g = \begin{bmatrix} I & q \\ 0 & 1 \end{bmatrix}, \quad \hat{\xi}' = \begin{bmatrix} \hat{\omega} & 0 \\ 0 & 0 \end{bmatrix}$ Pure rotation around q.

$g^{-1} = \begin{bmatrix} I & -q \\ 0 & 1 \end{bmatrix}$

d) $$\hat{V}_s = \dot{g}g^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T\bar{p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T\bar{p} + \dot{\bar{p}} \\ 0 & 0 \end{bmatrix}$$

$$\dot{R}R^T = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{\omega}^s$$

where $\dot{R}R^T$ is $\hat{\omega}^s$ and $-\dot{R}R^T\bar{p} + \dot{\bar{p}}$ is $\bar{v}^s$

$$\hat{V_b} = g^{-1}\dot{g} = \begin{bmatrix} R^T & -R^T\bar{p} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix}$$

$$R^T\dot{R} = \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \hat{\omega}^b \qquad R^T\dot{p} \text{ is } \hat{v}^b$$

V^s, the spacial velocity, written as $V^s = \begin{bmatrix} -\dot{R}R^T p + \dot{p} \\ (\dot{R}R^T)^\vee \end{bmatrix}$ is independent of choosing RB.

V^b, the body velocity. $V^b = \begin{bmatrix} R^T\dot{p} \\ (R^T\dot{R})^\vee \end{bmatrix}$ is defining the velocity of a point in the chosen body frame.

e) To map body velocity into spatial velocity

$$\omega_{ab}^s = R_{ab}\omega_{ab}^b \qquad v_{ab}^s = -\omega_{ab}^s \times p_{ab} + \dot{p}_{ab} = p_{ab} \times R_{ab}\omega_{ab}^b + R_{ab}v_{ab}^b$$

$$V_{ab}^s = \begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix} = \underbrace{\begin{bmatrix} R_{ab} & \hat{p}_{ab}R_{ab} \\ 0 & R_{ab} \end{bmatrix}}_{Adg} \begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix}$$

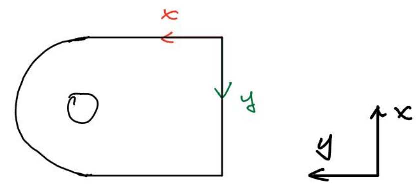
2.5 1 / 1

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Partial
- **- 1 pts** Incorrect

3. a) $x = 0.75\,m = 750\,mm$

$y = 0.5\,m = 500\,mm$

$z = 1\,m = 1000\,mm$



marker tip location to base

12cm = 120mm

$x = 220\,mm$ $\quad y = 140\,mm$ $\quad z = 160\,mm$

respect to world origin

$x = 750 - 140 = 610\,mm$

$y = 500 + 220 = 720\,mm$

$z = 1000 + 346 + 910 + 120 = 2376\,mm$

b) As shown in Code section

c)

A: output (1000,:) $\Rightarrow$ $(x,y,z) = (90.6765,\ 1181.8,\ 1364.4)$

B: output (3000,:) $\Rightarrow$ $(x,y,z) = (162.9133,\ 1076.8,\ 1239.1)$

C: output (5000,:) $\Rightarrow$ $(x,y,z) = (251.197,\ 1246.1,\ 1296.4)$

$\vec{BA} = (-72.237,\ 105.0389,\ 125.3329)$

$\vec{BC} = (88.2837,\ 169.2968,\ 57.2849)$

$\vec{BC} \times \vec{BA} = (15205,\ -15205,\ 21503)$

d) The robot write "kdc". figure attached in the code.

3.1 2 / 2

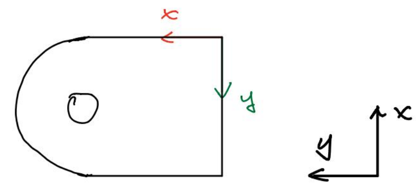
- ✓ **- 0 pts** Correct
- **- 0.5 pts** 1 or 2 values wrong
- **- 1 pts** None of the values is correct

3. a) $x = 0.75\,m = 750\,mm$

$y = 0.5\,m = 500\,mm$

$z = 1\,m = 1000\,mm$

marker tip location to base



12 cm = 120 mm

$x = 220\,mm$ $y = 140\,mm$ $z = 160\,mm$

respect to world origin

$x = 750 - 140 = 610\,mm$

$y = 500 + 220 = 720\,mm$

$z = 1000 + 346 + 910 + 120 = 2376\,mm$

b) As shown in Code section

c)

A: output (1000, :) $\Rightarrow$ $(x, y, z) = (90.6765, 1181.8, 1364.4)$

B: output (3000, :) $\Rightarrow$ $(x, y, z) = (162.9133, 1076.8, 1239.1)$

C: output (5000, :) $\Rightarrow$ $(x, y, z) = (251.197, 1246.1, 1296.4)$

$\vec{BA} = (-72.237, 105.0389, 125.3329)$

$\vec{BC} = (88.2837, 169.2968, 57.2849)$

$\vec{BC} \times \vec{BA} = (15205, -15205, 21503)$

d) The robot write "kdc". figure attached in the code.

3.2 6 / 6

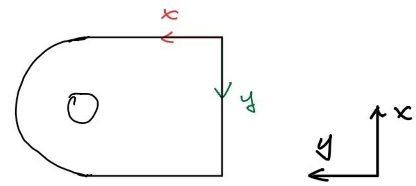
✓ **- 0 pts** Correct

- 6 pts No code provided.

- 3 pts Correct. 50% for resubmission.

3. a) $x = 0.75\,m = 750\,mm$
$y = 0.5\,m = 500\,mm$
$z = 1\,m = 1000\,mm$

marker tip location to base



12cm = 120mm

$x = 220\,mm$ $y = 140\,mm$ $z = 160\,mm$

respect to world origin

$x = 750 - 140 = 610\,mm$
$y = 500 + 220 = 720\,mm$
$z = 1000 + 346 + 910 + 120 = 2376\,mm$

b) As shown in Code section

c)

A: output (1000, :) $\Rightarrow$ $(x, y, z) = (90.6765, 1181.8, 1364.4)$
B: output (3000, :) $\Rightarrow$ $(x, y, z) = (162.9133, 1076.8, 1239.1)$
C: output (5000, :) $\Rightarrow$ $(x, y, z) = (251.197, 1246.1, 1296.4)$

$\vec{BA} = (-72.237, 105.0389, 125.3329)$
$\vec{BC} = (88.2837, 169.2968, 57.2849)$
$\vec{BC} \times \vec{BA} = (15205, -15205, 21503)$

d) The robot write "kdc". figure attached in the code.

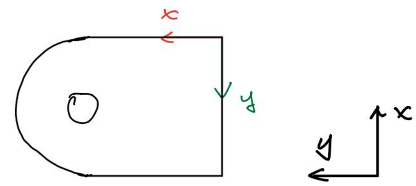
3.3 **2 / 2**

- ✓ **- 0 pts** Correct
- **- 0.5 pts** Surface normal is partially correct
- **- 1 pts** Answers are wrong

3. a) $x = 0.75\,m = 750\,mm$

$y = 0.5\,m = 500\,mm$

$z = 1\,m = 1000\,mm$



marker tip location to base

$12\,cm = 120\,mm$

$x = 220\,mm \quad y = 140\,mm \quad z = 160\,mm$

respect to world origin

$x = 750 - 140 = 610\,mm$

$y = 500 + 220 = 720\,mm$

$z = 1000 + 346 + 910 + 120 = 2376\,mm$

b) As shown in Code section

c)

A: output (1000, :) $\Rightarrow$ $(x, y, z) = (90.6765, 1181.8, 1364.4)$

B: output (3000, :) $\Rightarrow$ $(x, y, z) = (162.9133, 1076.8, 1239.1)$

C: output (5000, :) $\Rightarrow$ $(x, y, z) = (251.197, 1246.1, 1296.4)$

$\vec{BA} = (-72.237, 105.0389, 125.3329)$

$\vec{BC} = (88.2837, 169.2968, 57.2849)$

$\vec{BC} \times \vec{BA} = (15205, -15205, 21503)$

d) The robot write "kdc". figure attached in the code.

3.4 2 / 2

- ✓ **- 0 pts** Correct
- **- 0.5 pts** "kdc" should be smoothly written
- **- 1 pts** The answer is wrong.
- **- 1 pts** Correct. 50% from resubmission.

Please include your figure in the write up next time.