Assignment 6 - Control Fundamentals

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Assignment 6 covers the fundamentals of control. Problem 1 will let you revisit the basics of LTI systems and expose you to feedback design. In problem 2, you will practice filtering methods that cope with noisy signals.

After completing this assignment, you should be able to formulate state-space representations of dynamical system models, analyze the stability of LTI systems, and apply design methods to stabilize feedback control. In addition, you should be able to use Kalman filters to cope with noisy signals in control systems.

1 Feedback Stabilization of LTI Systems (8pts)

Consider a motor with rotational inertia J_1 driving an external load J_2 through a torsional spring, as shown in Figure 1. Assume that the motor delivers a torque that is proportional to the current, $\tau_m = k_I I$. The dynamics of the system can be described by

$$J_1\ddot{\varphi}_1 + c(\dot{\varphi}_1 - \dot{\varphi}_2) + k(\varphi_1 - \varphi_2) = k_1 I \tag{1}$$

$$J_2\ddot{\varphi}_2 + c(\dot{\varphi}_2 - \dot{\varphi}_1) + k(\varphi_2 - \varphi_1) = \tau_d \tag{2}$$

where τ_d is an unknown disturbance torque. Similar equations are obtained when considering robot dynamics with flexible links.

a (2pts) Derive a state space model for the system by introducing the normalized state variables $x_1 = \varphi_1, x_2 = \varphi_2, x_3 = \dot{\varphi}_1/w_0$, and $x_4 = \dot{\varphi}_2/w_0$, where $w_0 = \sqrt{k(J_1+J_2)/(J_1J_2)}$ is the undamped natural frequency of the system when the control signal is zero.

Solution We define our state variables defined as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \frac{\dot{\phi}_1}{\omega_0} \\ \frac{\dot{\phi}_2}{\omega_0} \end{bmatrix} \longrightarrow \dot{\mathbf{x}} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \frac{\ddot{\phi}_1}{\omega_0} \\ \frac{\ddot{\phi}_2}{\omega_0} \end{bmatrix}$$

The input to the system is $\mathbf{u} = I$, so, the state space model of the system can be defined as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \\ \frac{\ddot{\phi}_1}{\omega_0} \\ \frac{\ddot{\phi}_2}{\omega_0} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \frac{\dot{\phi}_1}{\omega_0} \\ \frac{\dot{\phi}_2}{\omega_0} \end{bmatrix} + \mathbf{BI}$$

Expanding the above equation, we get the first 3 equations:

$$\dot{\phi}_1 = a_{11}\phi_1 + a_{12}\phi_2 + \frac{a_{13}\dot{\phi}_1}{\omega_0} + \frac{a_{14}\dot{\phi}_2}{\omega_0} + b_1 \longrightarrow a_{11} = a_{12} = a_{14} = b_1 = 0, \quad a_{13} = \omega_0$$

$$\dot{\phi}_2 = a_{21}\phi_1 + a_{22}\phi_2 + \frac{a_{23}\dot{\phi}_1}{\omega_0} + \frac{a_{24}\dot{\phi}_2}{\omega_0} + b_2 \longrightarrow a_{21} = a_{22} = a_{23} = b_2 = 0, \quad a_{24} = \omega_0$$

$$\ddot{\phi}_1 = a_{31}\phi_1 + a_{32}\phi_2 + \frac{a_{33}\dot{\phi}_1}{\omega_0} + \frac{a_{34}\dot{\phi}_2}{\omega_0} + b_3$$

To find $a_{31}, a_{32}, a_{33}, a_{34}$, we have:

$$J_1\ddot{\phi}_1 = c(\dot{\phi}_1 - \dot{\phi}_2) + k(\phi_1 - \phi_2) = k_I \longrightarrow \frac{\ddot{\phi}_1}{\omega_0} = \frac{-c\dot{\phi}_1 + c\dot{\phi}_2 - k\phi_1 + k\phi_2 + k_I}{J_1\omega_0}$$

$$a_{31} = \frac{-k}{J_1\omega_0}, \quad a_{32} = \frac{k}{J_1\omega_0}, \quad a_{33} = \frac{-c}{J_1}, \quad a_{34} = \frac{c}{J_1}, \quad b_3 = \frac{k_I}{J_1\omega_0}$$

and for the forth equation, assuming torque disturbance $\tau_d = 0$, we get

$$\ddot{\phi}_2 = a_{41}\phi_1 + a_{42}\phi_2 + \frac{a_{43}\dot{\phi}_1}{\omega_0} + \frac{a_{44}\dot{\phi}_2}{\omega_0} + b_4$$

$$J_2\ddot{\phi}_2 = c(\dot{\phi}_2 - \dot{\phi}_1) + k(\phi_2 - \phi_1) = 0 \longrightarrow \frac{\ddot{\phi}_1}{\omega_0} = \frac{-c\dot{\phi}_2 + c\dot{\phi}_1 - k\phi_2 + k\phi_1}{J_2\omega_0}$$

$$a_{41} = \frac{k}{J_2\omega_0}, \quad a_{42} = \frac{-k}{J_2\omega_0}, \quad a_{43} = \frac{c}{J_2}, \quad a_{44} = \frac{-c}{J_2}, \quad b_4 = 0$$

Finally, we get the full state space model as:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & \omega_0 \\ \frac{-k}{J_1\omega_0} & \frac{k}{J_1\omega_0} & \frac{-c}{J_1} & \frac{c}{J_1} \\ \frac{k}{J_2\omega_0} & \frac{-k}{J_2\omega_0} & \frac{c}{J_2} & \frac{-c}{J_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_I}{J_1\omega_0} \\ 0 \end{bmatrix} \mathbf{I}$$

If we include the torque disturbance term, we'll get:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & \omega_0 \\ \frac{-k}{J_1\omega_0} & \frac{k}{J_1\omega_0} & \frac{-c}{J_1} & \frac{c}{J_1} \\ \frac{k}{J_2\omega_0} & \frac{-k}{J_2\omega_0} & \frac{c}{J_2} & \frac{-c}{J_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_I}{J_1\omega_0} \\ 0 \end{bmatrix} \mathbf{I} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_2\omega_0} \end{bmatrix} \tau_d$$

b (2pts) Using the following normalized parameters, $J_1 = 10/9, J_2 = 10, c = 0.1, k = 1$, and $k_I = 1$, verify that the eigenvalues of the open loop system are $0, 0$, and -0.05 $\pm\, i$.

Solution We have:

$$\omega_0 = \sqrt{\frac{k(J_1 + J_2)}{J_1 J_2}} = 1$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & \omega_0 & 0 \\ 0 & 0 & 0 & \omega_0 \\ \frac{-k}{J_1\omega_0} & \frac{k}{J_1\omega_0} & \frac{-c}{J_1} & \frac{c}{J_1} \\ \frac{k}{J_2\omega_0} & \frac{-k}{J_2\omega_0} & \frac{c}{J_2} & \frac{-c}{J_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1.0000 & 0 \\ 0 & 0 & 0 & 1.0000 \\ -0.9000 & 0.9000 & -0.0900 & 0.0900 \\ 0.1000 & -0.1000 & 0.0100 & -0.0100 \end{bmatrix}$$

Eigenvalues of the open loop system are the eigenvalues of $\mathbf{A}$, therefore:

$$eig(system) = eig(\mathbf{A}) = \begin{bmatrix} -0.0500 + 0.9987i \\ -0.0500 - 0.9987i \\ -0.0000 + 0.0000i \\ 0.0000 + 0.0000i \end{bmatrix}$$

c (2pts) Design a state feedback that gives a closed loop system with eigenvalues $-2, -1, -1 \pm i$. This choice implies that the oscillatory eigenvalues will be well damped and that the eigenvalues at the origin are replaced by eigenvalues on the negative real axis.

Solution By adding feedback to the system, the input u of the system will include the current state x of the system and the desired output r:

$$\mathbf{u} = k_r\mathbf{r} - \mathbf{Kx}$$

Then, the new state space model of the system is:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} = \mathbf{Ax} + \mathbf{B}k_r\mathbf{r} - \mathbf{BKx}$$

$$\longrightarrow \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{Bu}'$$

To design a feedback that gives a closed loop system with eigenvalues of -2, -1 and $-1 \pm i$, we should find a matrix K such that the eigenvalues of $(A - BK)$will be these values. The MATLAB code is provided to find the K. The result is:

$$K = \begin{bmatrix} 8.9333 \\ 35.5111 \\ 5.4444 \\ 101.2222 \end{bmatrix}$$

d (2pts) Simulate and plot the responses of the closed loop system to step changes in the reference signal for φ_2 and a step change in the disturbance torque τ_d on the second rotor. Shortly describe the system behavior.

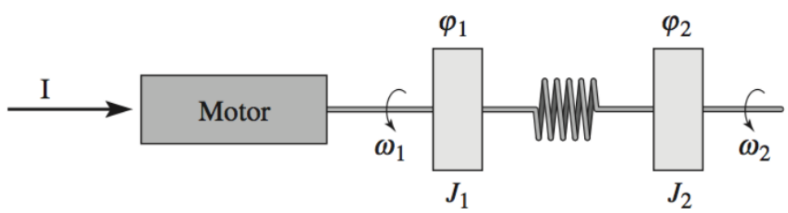

Figure 1: Motor-load coupling through series spring.

Solution For the steady-state solution the state x does not change over time, therefore the state change rate $\dot{x}$ will be zero. Thus, we will have:

$$\dot{x_e} = (\mathbf{A} - \mathbf{BK})\mathbf{x_e} + \mathbf{B}k_r\mathbf{r} = 0 \longrightarrow \mathbf{x_e} = -(\mathbf{A} - \mathbf{B})^{-1}\mathbf{B}k_r\mathbf{r}$$

In the steady-state solution, the output is the desired output (reference value), i.e. $\mathbf{y_e} = \mathbf{r}$. Since it is an LTI system, there is a linear relationship between the state, input and output. If we consider our output as $y = [\phi_1 \quad \phi_2]^T$, the relationship is calculated as:

$$\mathbf{y} = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \mathbf{Cx} + \mathbf{Du} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \dot{\phi}_1/\omega_0 \\ \dot{\phi}_2/\omega_0 \end{bmatrix}$$

$$\longrightarrow \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \mathbf{0}$$

To calculate $y_1 = \phi_1$ we then have:

$$\mathbf{x_e} = -(\mathbf{A} - \mathbf{B})^{-1}\mathbf{B}k_{r1}r_1 = -\mathbf{C_1}(\mathbf{A} - \mathbf{B})^{-1}\mathbf{Bx_e}k_{r1}$$

$$\longrightarrow k_{r1} = \frac{1}{-\mathbf{C_1}(\mathbf{A} - \mathbf{B})^{-1}\mathbf{B}}$$

Where $\mathbf{C}_1$ is the first row of the matrix $\mathbf{C}$. Similarly, to calculate $y_1 = \phi_2$ we then have:

$$\mathbf{x_e} = -(\mathbf{A} - \mathbf{B})^{-\mathbf{1}}\mathbf{B}k_{r2}r_2 = -\mathbf{C_2}(\mathbf{A} - \mathbf{B})^{-\mathbf{1}}\mathbf{B}\mathbf{x_e}k_{r2}$$
$$\longrightarrow k_{r2} = \frac{1}{-\mathbf{C_2}(\mathbf{A} - \mathbf{B})^{-\mathbf{1}}\mathbf{B}}$$

The response of the closed loop system to step change in the reference signal r is shown in Fig. 2.

To calculate the response of the system to the step change in the disturbance torque τ_d , we use the complete model (with noise) derived at the end of the part (a). Since there is no reference output ($\mathbf{r} = 0$), we have:

$$\dot{x_e} = (\mathbf{A} - \mathbf{BK})\mathbf{x_e} + \mathbf{B}k_r\mathbf{r} + \mathbf{E}\tau_d = (\mathbf{A} - \mathbf{BK})\mathbf{x_e} + \mathbf{E}\tau_d$$

The response of the closed loop system to step change in the disturbance torque d on the second rotor is shown in Fig. 3.

As can be seen in the Fig. 2, in the absence of noise (disturbance torque), we can achieve the desired state without a steady-state error. Also, since the two outputs are coupled, the ϕ_1 value also ends up as the same value as the desired ϕ_2 . However, in the presence of the noise (disturbance torque), it can be seen in Fig. 3 that the system outputs are not end up with the same value and has a difference of the torque disturbance value. Also, without enforcement of a desired value to the system, there is a steady-state error. Although the Fig.3 is shown for a reference output of zero, due to the linear behavior of LTI systems, we 5can generalize this result and claim that generally in the presence of noise the steady state output will differ from a reference (desired) value in this system.

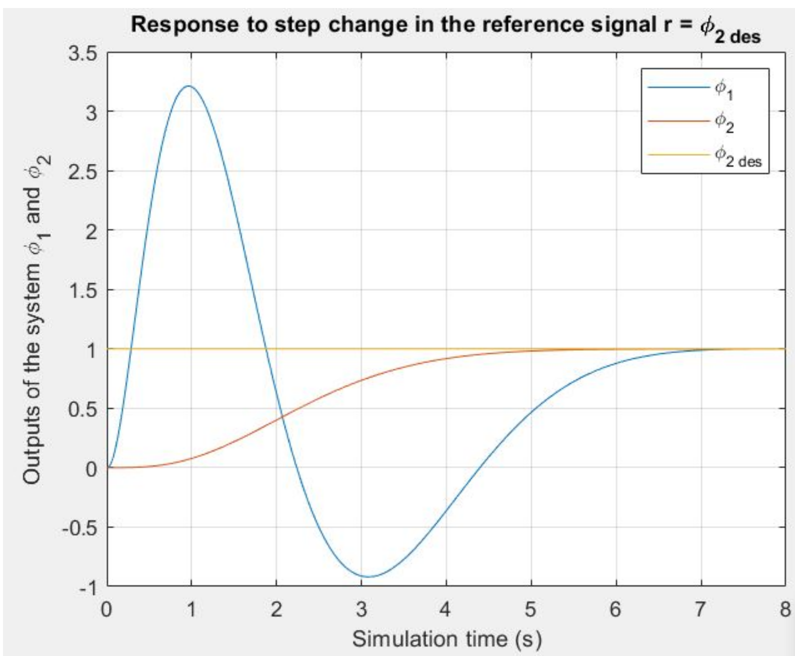

Figure 2: System response respect to reference signal.

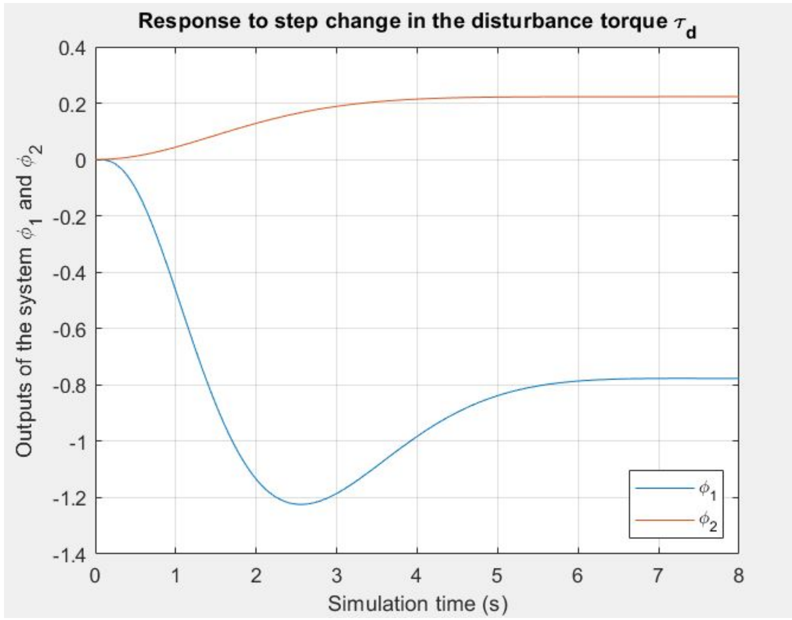

Figure 3: System response respect to disturbance.

2 Kalman Filter (7+2pts)

You are studying an insect as a model of a high-performance Micro-UAV. The insect is flying in an experimental lab environment with controlled airflow and you want to estimate its position and velocity (all translational, neglect rotations). The position and velocity are discretized with time step length Δt, and described by the linear state space:

$$X[k] = \begin{bmatrix} \vec{x}[k] \\ \dot{\vec{x}}[k] \end{bmatrix} \in R^{6\times 1}$$

We assume that the insect can be viewed as a point mass M. At time step k, we can control the airflow such that the net force imposed on the insect is:

$$\vec{f}_{total}[k] = \vec{u} + \vec{v}[k] \in R^{3\times 1}$$

where the control input $\vec{u}$ is a constant force, and $\vec{v}[k]$ is a white noise process:

$$E(\vec{v}[k]) = [0,\ 0,\ 0]^T$$

$$E(\vec{v}[i]\vec{v}[j]^T) = R_v \delta_{ij}$$

where the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

At each time step, a noisy measurement $\vec{y}[k]$ of the true position of the insect is made. Suppose that the measurement is imprecise, with an additive, zero-mean, Gaussian white noise $\vec{w}[k]$:

$$E(\vec{w}[k]) = [0,\ 0,\ 0]^T$$

$$E(\vec{w}[i]\vec{w}[j]^T) = R_w \delta_{ij}$$

The matrices R_v and R_w are the covariance matrices for the process disturbance $\vec{v}$ and measurement noise $\vec{w}$. We assume that the initial condition is also modeled as a Gaussian random variable with $E(X[0]) = \vec{0}$, and $E(X[0]X[0]^T) = P_0$

a (2pts) Compute the expected value and covariance of the insect position and velocity as a function of k.
Solution Expected value of the insect position and velocity is:

$$Position : P_0(1:3, 1:3)$$
$$Velocity : P_0(4:6, 4:6)$$

Expected covariance of the insect position and velocity is

$$cov = P_0(1:3, 1:3) + P_0(4:6, 4:6)dt + R_v(\frac{t^2}{2M})^2$$

b (2pts) Derive the state space representation for the discrete-time linear dynamical system of the insect.
Solution

$$A = \begin{bmatrix} 1 & 0 & 0 & dt & 0 & 0 \\ 0 & 1 & 0 & 0 & dt & 0 \\ 0 & 0 & 1 & 0 & 0 & dt \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} B = \begin{bmatrix} \frac{dt^2}{2M} & 0 & 0 \\ 0 & \frac{dt^2}{2M} & 0 \\ 0 & 0 & \frac{dt^2}{2M} \\ \frac{dt}{M} & 0 & 0 \\ 0 & \frac{dt}{M} & 0 \\ 0 & 0 & \frac{dt}{M} \end{bmatrix} C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Next, you will implement the Kalman filter and RTS smoother. Suppose the parameters are set as follows: $M = 0.01kg, \Delta t = 1s, \vec{u} = [0.01, 0.01, 0.01]^T$ $P_0 = diag[50, 50, 50, 10, 10, 10]$

$$R_v = \begin{bmatrix} 10^{-5} & 0 & 0 \\ 0 & 10^{-5} & 0 \\ 0 & 0 & 10^{-5} \end{bmatrix} N^2, \quad R_w = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 50 \end{bmatrix} m^2$$

and the noisy measurements are stored in the file `ydata.txt`. Each line is a point on the motion trajectory; the file should have 20 points.

c (3pts) Implement a Kalman filter to estimate the position and velocity of the insect at each time k, given the noisy measurement of its position. Compute and output the steady-state expected value and covariance of the error of your estimate. Plot a figure to show your result together with the noisy measurement. What is the the steady-state expected value and error covariance of the last point?
Solution Steady-state expected value of the last point is

$$[166.4196 \quad 158.56 \quad 204.30 \quad 18.36 \quad 17.38 \quad 21.33]$$

Steady state error covariance of the last point is

$$\begin{bmatrix} 12.97 & 0 & 0 & 1.92 & 0 & 0 \\ 0 & 12.97 & 0 & 0 & 1.92 & 0 \\ 0 & 0 & 12.97 & 0 & 0 & 1.92 \\ 1.92 & 0 & 0 & 0.62 & 0 & 0 \\ 0 & 1.92 & 0 & 0 & 0.62 & 0 \\ 0 & 0 & 1.92 & 0 & 0 & 0.62 \end{bmatrix}$$

Plots are showing below

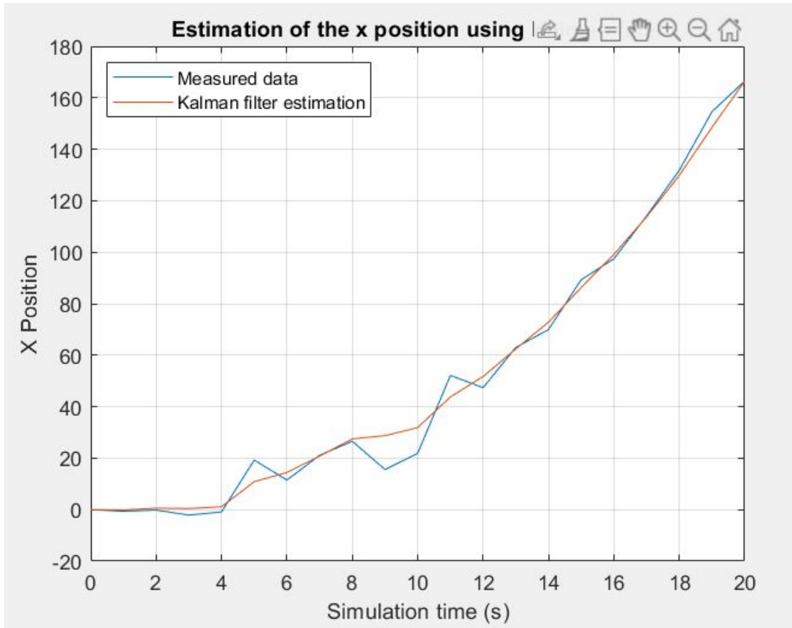

Figure 4: Estimation of the x position using Kalman filter.

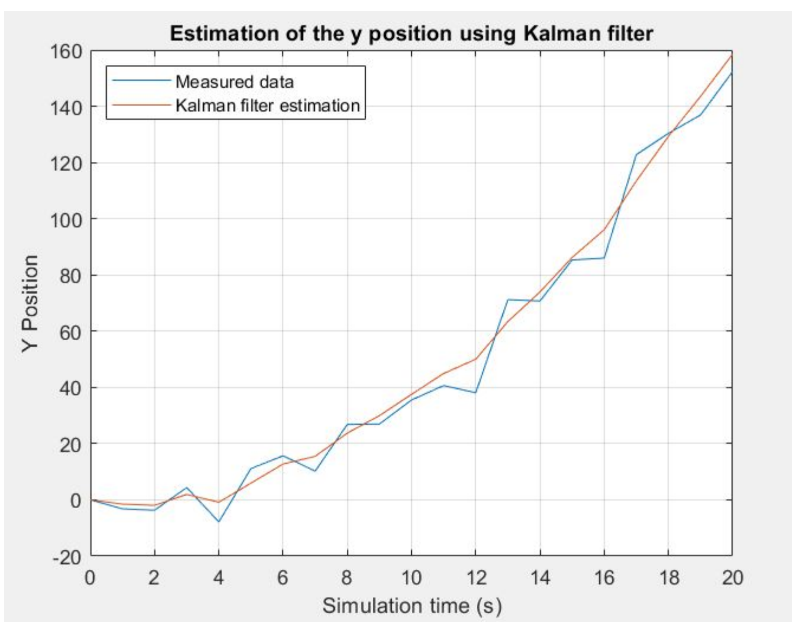

Figure 5: Estimation of the y position using Kalman filter.

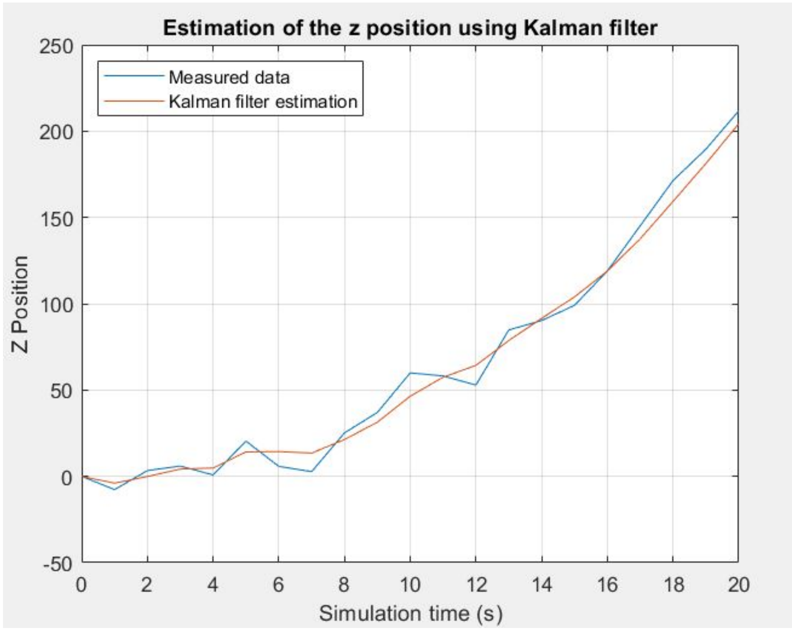

Figure 6: Estimation of the z position using Kalman filter.

d (+2pts) Implement a two-pass Rauch-Tung-Striebel (RTS) smoother which uses regular Kalman filter as a forward pass, and adds a backward pass to refine the previous estimate. Compare your result with (c) in terms of the expected value and covariance of the error of your estimate, and briefly explain the difference in one sentence. What is the the steady-state expected value and error covariance of the first point?
Solution Steady-state expected value of the last point is

$$[1.59 \quad 0.80 \quad -3.53 \quad -1.61 \quad -1.56 \quad -0.30]$$

Steady state error covariance of the last point is

$$\begin{bmatrix} 12.62 & 0 & 0 & -1.81 & 0 & 0 \\ 0 & 12.62 & 0 & 0 & -1.81 & 0 \\ 0 & 0 & 12.62 & 0 & 0 & -1.81 \\ -1.81 & 0 & 0 & 0.59 & 0 & 0 \\ 0 & -1.81 & 0 & 0 & 0.59 & 0 \\ 0 & 0 & -1.81 & 0 & 0 & 0.59 \end{bmatrix}$$

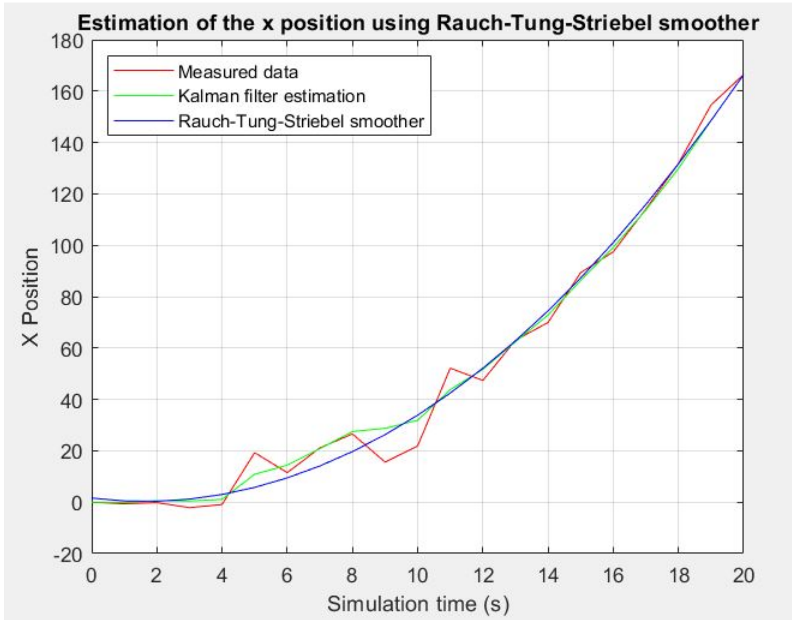

Figure 7: Estimation of the x position using Rauch-Tung-Striebel smoother.

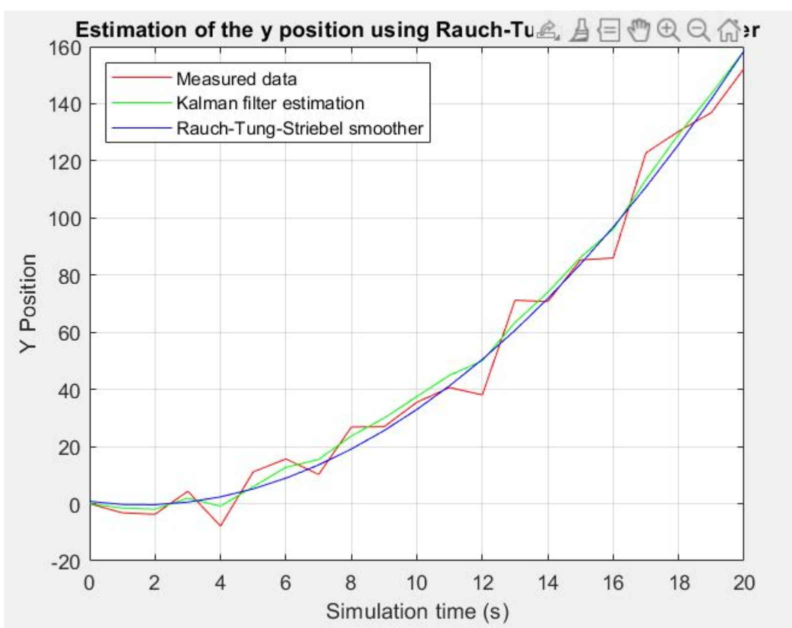

Figure 8: Estimation of the x position using Rauch-Tung-Striebel smoother.

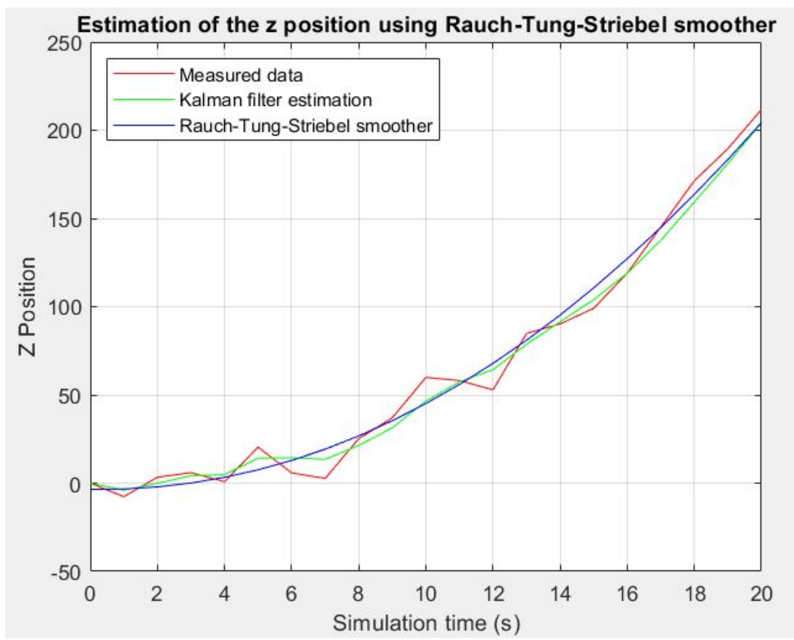

Figure 9: Estimation of the x position using Rauch-Tung-Striebel smoother.