Assignment 4 – Rigid Body Dynamics

Assignment 4 covers the dynamics of constrained systems including rigid bodies. Problem 1 will help you revisit the Lagrange method for deriving the equations of motion of dynamic systems with constraints. Problem 2 will introduce you to calculating the inertia matrix of rigid bodies with known shapes. In problem 3, you will deepen your understanding of dynamic parameter identification by taking advantage of the linearity of the equations of motion with respect to suitable dynamic parameters. Dynamic parameter identification is an essential method for developing high-quality dynamic models of robotic systems. The method also provides insight into how machines and animals can learn their own 'dynamics' over time.

After completing this assignment, you should be able to derive the equations of motion of constrained point-mass systems, to explain what inertia matrices are and how you can calculate them, and to apply the method of dynamic parameter identification to learn models of rigid body systems.

1. Lagrange method for deriving equations of motion (3pts)

Derive the equations of motion for a pendulum on a wire shown in Figure 1. The pendulum is planar, has a massless rod connected to a point mass m. The pendulum's pivot can slide along a horizontal wire without friction.

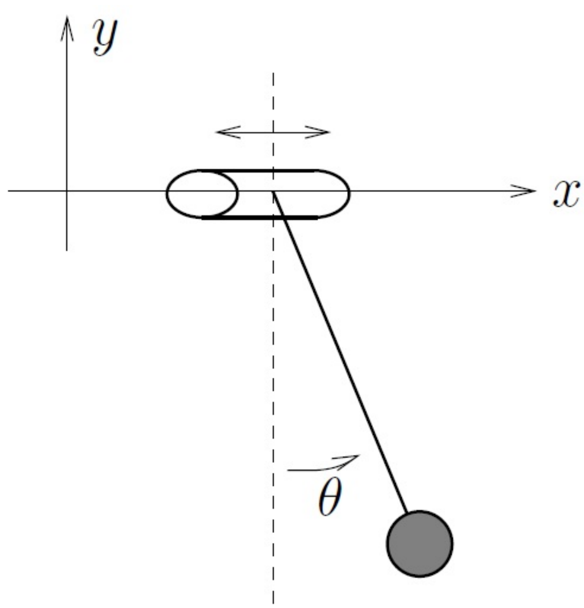
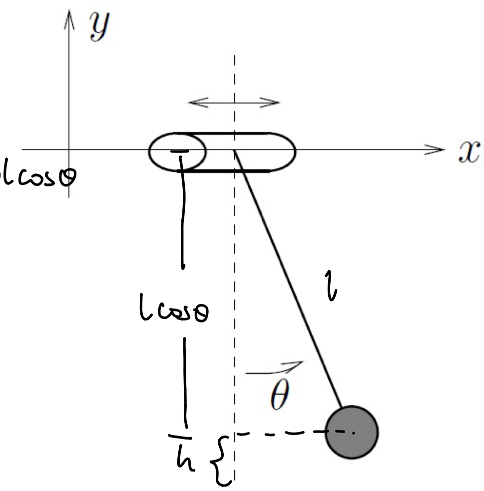


Figure 1: Pendulum on a wire.

1. $KE = \frac{1}{2} I\dot{\theta}^2 + \frac{1}{2} m\dot{x}^2$

$PE = mgl(1-\cos\theta)$

$L = KE - PE = \frac{1}{2}(I\dot{\theta}^2 + m\dot{x}^2) - mgL + mgl\cos\theta$



$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$m\ddot{x} = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \theta} = -mgL\sin\theta$

$\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} = ml^2\dot{\theta}$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$

$ml^2\ddot{\theta} + mgl\sin\theta = 0$

$l\ddot{\theta} + g\sin\theta = 0$

$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$

when $\sin\theta \approx 0 \Rightarrow \sin\theta \approx \theta$

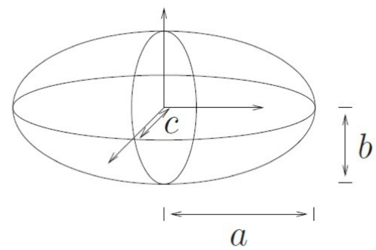
$\ddot{\theta} + \frac{g}{L}\theta = 0$

$$\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{g}{L} \end{bmatrix} = \bar{0}$$

2.

$$I = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

$$x' = \frac{x}{a} \qquad y' = \frac{y}{b} \qquad z' = \frac{z}{c}$$



(a) Ellipsoid

$$x = r\cos\theta\sin\phi \qquad y = r\sin\theta\sin\phi \qquad z = c\cos\phi$$

$$I_{yy} = \rho\int_V (x^2+y^2)\,dx\,dy\,dz$$

$$= abc\rho\int_{V'} (ax')^2 + (by')^2\,dx'\,dy'\,dz'$$

$$= abc\rho \int_0^1\int_0^{2\pi}\int_0^{\pi} a^2\cos^2\theta + b^2\sin^2\theta)\,r^4\sin^3\phi\,d\phi\,d\theta\,dr$$

$$= \frac{4}{15}\pi\rho abc(a^2+b^2) \text{ where } \rho = \frac{m}{V} \text{ where } V = \frac{4}{3}\pi abc$$

$$\Rightarrow \rho = \frac{3m}{4abc\pi}$$

$$\frac{\cancel{4\pi}}{5\,\cancel{15}}\cdot\frac{3m}{\cancel{4}\,\cancel{abc\pi}}\cdot\cancel{abc}\,(a^2+b^2) = \frac{m}{5}(a^2+b^2)$$

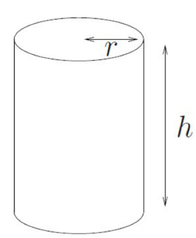
I_{zz} and I_{xx} will be similar to I_{yy} where

$$I_{xx} = \frac{m}{5}(b^2+c^2)$$

$$I_{zz} = \frac{m}{5}(a^2+c^2)$$

$$I = \frac{m}{5}\begin{bmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+b^2 & 0 \\ 0 & 0 & a^2+c^2 \end{bmatrix}$$

b) $V = \pi r^2 h \qquad \rho = \frac{m}{V} = \frac{m}{\pi r^2 h}$



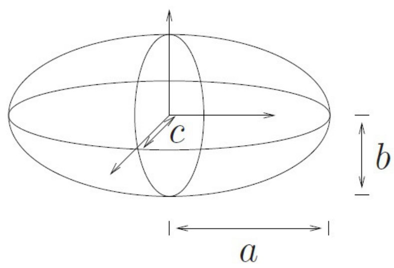
(b) Cylinder

$$I = \rho\int_{-h/2}^{h/2}\int_{-r}^{r}\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \begin{bmatrix} y^2+z^2 & 0 & 0 \\ 0 & x^2+z^2 & 0 \\ 0 & 0 & x^2+y^2 \end{bmatrix} dx\,dy\,dz$$

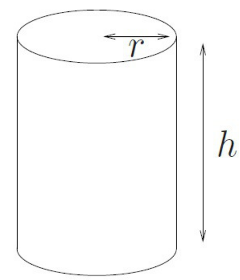
$$= \begin{bmatrix} \frac{1}{12}mh^2 + \frac{1}{4}mr^2 & 0 & 0 \\ 0 & \frac{1}{12}mh^2 + \frac{1}{4}mr^2 & 0 \\ 0 & 0 & \frac{1}{2}mr^2 \end{bmatrix}$$

2. Inertia matrix for known shapes (5pts)

Derive the inertia matrices for the objects shown in Figure 2.



(a) Ellipsoid



(b) Cylinder

Figure 2: Parameterized ellipsoid and cylinder.

3. Dynamic Parameter Estimation (10pts+2pts) 2021.3.23 15 min

A meteorite crashed into earth last week. It was a small meteorite, and the damage was limited. But another rogue asteroid was recently discovered tumbling towards earth. Because it may enter a collision course with earth, it is essential that you generate a precise dynamic model of it so that you can predict its future trajectory, which will be used to determine if interventions are necessary.

The asteroid is not being acted upon by any external forces, and you cannot discern its shape. In an incredible show of technical prowess, your team of rocket scientists and engineers has managed to land a small localization beacon on the surface of the asteroid. The beacon was able to emit a series of messages containing its inertial pose for a minute at a frequency of 1Hz before running out of power.

You've saved the data in a file called `poses.txt`. It contains each timestamped pose; the format of the data file is:

$$t\ x\ y\ z\ q_i\ q_j\ q_k\ q_0.$$

1. (2pts) What is the x, y, z velocity of the asteroid's center of mass?
2. (5pts) Determine the inertia tensor of the asteroid.
3. (+2pts) What is your best guess about the asteroid's shape? Does it match any simple geometries? (you may want to transform to principal axes)
4. (1pts) How far did the beacon land from the center of mass?
5. (1pts) What is the asteroid's angular momentum?
6. (1pts) What will the asteroid's pose be after another minute has passed (e.g. t = 120s)?

4. Project Proposal (5pts)

Describe with no more than two pages (11pt font) the motivation and planned effort of your work as well as its intellectual merit and its potential broader impact, assuming all goes well and you achieve the best possible outcome. All these are elements you typically have to address if you propose a project to funding agencies such as the National Science Foundation.

1. Motivation. Explain your motivation for the proposed project. What is the general background related to your project? What technical or scientific problem do you try to solve? (2pt)
2. Planned effort. Provide a plan of how you will approach your problem. What are the major steps (1 week sections)? What minor steps do they entail? What alternatives can you pursue if things do not work as initially planned? Create a timetable with expected dates for achieving the main steps. (2pt)
3. Intellectual merit. Explain what intellectual merit this project has. How will it advance scientific or your knowledge? In what scientific domains? (0.5pt)
4. Broader impact. Explain how this project can help advance society beyond intellectual merit. How does it contribute to training, education, work force development, or other societal factors? (0.5pt)