1) $\dot{x} = \begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix} x$; use LDM ; range s.t. System AS

a) Determine $V + \dot{V}$

Choosing $V = x_1^2 + x_2^2$, $\dot{V} = 2x_1 \cdot \dot{x}_1 + 2x_2 \cdot \dot{x}_2$

$\dot{V} = 2x_1(ax_1) + 2x_2(x_1 - x_2) = 2ax_1^2 + 2x_1x_2 - 2x_2^2$

b) Set up $\dot{V} < 0$ to get AS

$2ax_1^2 + 2x_1x_2 - 2x_2^2 < 0$

$ax_1^2 + x_1x_2 - x_2^2 < 0 \longrightarrow$ form an eqn. that can be factored to solve for a

$a\left(\frac{x_1^2}{x_2^2}\right) + \frac{x_1x_2}{x_2^2} - 1 < 0$

$a\left(\frac{x_1}{x_2}\right)^2 + \frac{x_1}{x_2} - 1 < 0$; Let $k = \frac{x_1}{x_2}$

$ak^2 + k - 1 < 0$: need to find an a s.t. $ak^2 + k - 1$ has no real roots

roots: $\frac{-1 \pm \sqrt{1+4a}}{2a}$; having $1+4a < 0$ would ensure no real roots

$1 + 4a < 0$ so $a < -0.25$

$a \in (-\infty, -0.25)$

2/3) Det. AS, BIBO, s.i.s.L.

a) $A = \begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}$; DTLTI

i) AS:

eig(A) = 1 + 0.5 so not AS

ii) Lyapunov:

$\exists \lambda_i, r_i = 1 + m = 0$ ✓ so s.i.s.L.

iii) BIBO:

$H(z) = 0$; since tr. func. is 0, output is bounded, so system is BIBO stable

b) $A = \begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}$; CTLTI

i) AS:

$eig(A) = -5, -3, -1$

$\forall \lambda_i, Re(\lambda_i) < 0$, thus AS

ii) Lyapunov:

S.i.S.L. since AS

iii) BIBO Stable:

BIBO stable since AS

c) CTLTI

i) AS:

$eig(A) = 5.583, -3.583$ so not AS

ii) Lyapunov:

$\exists \lambda_i, Re(\lambda_i) > 0$ so unstable

iii) BIBO :

$H(z) = \frac{3}{s^2 - 2s - 20}$, poles @ 5.5826 + -3.5826 so not BIBO Stable

c) DTLTI

i) AS:

$eig(A) = 5.583, -3.583$ so not AS

ii) Lyapunov:

$\exists \lambda_i, r(\lambda_i) > 1$ so unstable

iii) BIBO :

$H(z) = \frac{3}{s^2 - 2s - 20}$, poles @ 5.5826 + -3.5826 so not BIBO Stable

4) $\dot{x}_1 = x_2 - x_1 x_2^2$
$\dot{x}_2 = -x_1^3$

$V(x_1, x_2) = x_1^4 + 2x_2^2$ Need to show $\dot{V}(x_1, x_2)$ negative definite

$\dot{V} = 4x_1^3 \cdot \dot{x}_1 + 4x_2 \cdot \dot{x}_2$

$\dot{V} = 4x_1^3(x_2 - x_1 x_2^2) + 4x_2(-x_1^3)$

$\dot{V} = 4x_1^3 x_2 - 4x_1^4 x_2^2 - 4x_1^3 x_2 = -4x_1^4 x_2^2$

$\dot{V} = -4x_1^4 x_2^2 < 0$, thus energy of system decreasing and so system is stable via Lyapunov's Direct Method.

5) $V(x_1, x_2) = x_1^2 - x_2^2$
$\dot{x}_1 = 3x_1 + x_2^3$
$\dot{x}_2 = -x_2 + x_1^2$

Need to show three things for Instability:

Origin unstable if $\exists V(x,t)$:
1) $V(0,t) = 0$, $\forall t > t_0$
2) $V(x, t_0) > 0$
3) $\dot{V}(x,t) > 0$

1) $V(x_1 = 0, x_2 = 0, t) = 0$, $\forall t > t_0$ based on substitution into expression

2) $V(x_1, x_2, t_0) > 0$ for some point close to Origin?

$V(x_1, x_2) = x_1^2 - x_2^2$, choose a point $z = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}$ close to origin

$V(z, t_0) = \varepsilon^2 > 0$

3) $\dot{V} > 0$?

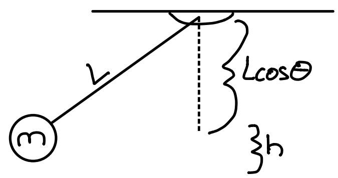
$\dot{V} = 2x_1 \cdot \dot{x}_1 - 2x_2 \cdot \dot{x}_2 = 2x_1(3x_1 + x_2^3) - 2x_2(-x_2 + x_1^2)$

$\dot{V} = 6x_1^2 + 2x_1 x_2^3 + 2x_2^2 - 2x_1^2 x_2$

$\dot{V} = 6x_1^2 + 2x_1 x_2^3 + 2x_2(x_2 - x_1^2) > 0$ for point $x_1, x_2 \neq 0$ close to origin

Since all 3 conditions met, $\exists V$ s.t. origin of system unstable

6) $\ddot{\theta} + \frac{g}{L}\sin\theta = 0$



$KE = \frac{1}{2}mv^2$
$PE = mgh$, $h = L - L\cos\theta$

States θ + $\dot{\theta}$
$\quad\quad\; x_1 \quad x_2$

a) $V = KE + PE = \frac{1}{2}mv^2 + mg(L - L\cos x_1)$

Rewrite in terms of states and show $\dot{V} \le 0$ (neg. semi-def)

$V = \frac{1}{2}m(Lx_2)^2 + mg(L - L\cos x_1) = \frac{1}{2}mL^2x_2^2 + mgL - mgL\cos x_1$

$\dot{V} = \frac{1}{2}mL^2 \cdot 2x_2 \cdot \dot{x}_2 + 0 - [mgL(-\sin x_1)\cdot \dot{x}_1]$

$\dot{V} = mL^2 x_2 \dot{x}_2 + mgL\sin x_1 \cdot \dot{x}_1$

$\dot{V} = mL^2 x_2\left(-\frac{g}{L}\sin x_1\right) + mgL\sin x_1\, x_2$

$\dot{V} = -mLx_2 g\sin x_1 + mLg\sin x_1\, x_2 = 0$

Since $\dot{V} = 0$, system s.i.s.L.

b) Show that system @ $\theta_0 = \pi$ is unstable , $x = \theta - \pi$

$V = \dot{x} - \dot{x}\sin(x)$ $\quad$; $x = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} = \begin{bmatrix} \theta - \pi \\ \dot{\theta} \end{bmatrix}$

Origin unstable if $\exists V(x,t)$:
1) $V(0,t) = 0, \forall t > t_0$
2) $V(x,t_0) > 0$
3) $\dot{V}(x,t) > 0$

$\dot{x} = \begin{bmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \tilde{x}_2 \\ -\frac{g}{L}\sin(\tilde{x}_1 + \pi) \end{bmatrix}$

1) $V(0,t) = 0$?

$V = 0 - 0 \quad = 0 \;\; \forall\; t > t_0$

2) $V(x,t_0) = \dot{x} - \dot{x}\sin(x)$

Choose a pt. $z = [0, \varepsilon]^T$ close to origin

$V(x, t_0) = \dot{\varepsilon} - 0 > 0$

3) $\dot{V}(x, t_0) > 0$?

$$\dot{V}(x,t_0) = \ddot{x} - [\dot{x}^2\cos(x) + \ddot{x}\sin(x)]$$

$$= \ddot{x} - \dot{x}^2\cos(x) - \ddot{x}\sin(x)$$

$$= -\frac{g}{L}\sin(x) - \dot{x}^2\cos(x) - \left(-\frac{g}{L}\sin(x)\right)\sin(x)$$

$$= \frac{g}{L}\sin(x)^2 - \frac{g}{L}\sin(x) - \dot{x}^2\cos(x)$$

$$= \frac{g}{L}\sin(x)\left[\sin(x) - 1\right] - \dot{x}^2\cos(x) > 0 \text{ around origin}$$

thus energy cont. to increase and syst. is unstable at the equilibrium point.