1)

a) Skew-Symmetric $\Rightarrow A^T = -A$

$A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ subspace of $M_{2\times 2}$ w/ real entries

i) $0 \in$

i) $0 \in$ Skew Set$_{(n\times n)}$?

$$\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & & 0 \end{bmatrix}^T = -\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \ddots & \\ \vdots & & \ddots \\ 0 & & 0 \end{bmatrix} \checkmark$$

ii) $A, B \in$ Skew Set$_{(n\times n)}$
$A+B \in$ Skew Set$_{(n\times n)}$?

Set of $n\times n$ skew-symmetric matrices is a subspace.

Need to show $(A+B)^T = -(A+B)$

$(A+B)^T = A^T + B^T = -A + (-B) = -(A+B) \checkmark$

iii) $A \in$ Skew Set$_{(n\times n)}$, scalar c
$cA \in$ Skew Set$_{(n\times n)}$?

Need to show $(cA)^T = -cA$

$(cA)^T = c(A)^T = c(-A) = -cA \checkmark$

b) Diagonal Matrices

i) $0 \in$ Diagonal Set$_{(n\times n)}$?

$$\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

All off-diagonal elements $= 0$ $\checkmark$

Set of $n\times n$ diagonal matrices is a subspace.

ii) $A, B \in$ Diagonal Set$_{(n\times n)}$
$A+B \in$ Diagonal Set$_{(n\times n)}$?

Need to show $A+B$ has all off-diagonal terms$=0$

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & b_n \end{bmatrix} \quad \text{so } A+B = \begin{bmatrix} a_1+b_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2+b_2 & & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & & & \\ 0 & \cdots & & 0 & a_n+b_n \end{bmatrix} \checkmark$$

iii) $A \in$ Diagonal Set$_{(n\times n)}$, scalar c
$cA \in$ Diagonal Set$_{(n\times n)}$?

Need to show cA has all off-diagonal terms$=0$

$$cA = c\begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & a_n \end{bmatrix} = \begin{bmatrix} ca_1 & 0 & \cdots & 0 \\ 0 & ca_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & ca_n \end{bmatrix} \checkmark$$

c) Upper Diagonal Matrices

i) $0 \in$ Upper Diagonal Set $(n \times n)$?

$$\begin{bmatrix} 0 & \cdots & 0 \\ & 0 & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

All elements below diagonal $= 0$ ✓

Set of $n \times n$ upper diagonal matrices is a subspace.

ii) $A, B \in$ Upper Diagonal Set $(n \times n)$
$A + B \in$ Upper Diagonal Set $(n \times n)$?

Need to show $A+B$ has all elements below diagonal $= 0$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ 0 & \ddots & \vdots \\ 0 & \cdots & b_{nn} \end{bmatrix} \quad \text{so } A+B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ 0 & \ddots & \vdots \\ 0 & \cdots & a_{nn}+b_{nn} \end{bmatrix} \checkmark$$

iii) $A \in$ Upper Diagonal Set $(n \times n)$, scalar c
$cA \in$ Upper Diagonal Set $(n \times n)$?

Need to show cA has all elements below diagonal $= 0$

$$cA = c\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ 0 & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ 0 & \ddots & \vdots \\ 0 & \cdots & ca_{nn} \end{bmatrix} \checkmark$$

d) Singular Matrices

i) $0 \in$ Singular Set $(n \times n)$?

$$\left| \begin{bmatrix} 0 & \cdots & 0 \\ & 0 & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \right| = 0 \checkmark$$

Set of $n \times n$ upper diagonal matrices is NOT a subspace.

ii) $A, B \in$ Singular Set $(n \times n)$
$A + B \in$ Singular Set $(n \times n)$?

Need to show A+B has det. = 0

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \times$$

↳ Not Singular

2)

a) Form a basis for Span(W)

Done in Matlab

$$\left\{ \underset{w_1}{\begin{bmatrix} -1 \\ 1 \\ -5 \\ 7 \end{bmatrix}}, \underset{w_2}{\begin{bmatrix} 2 \\ 1 \\ 7 \\ -8 \end{bmatrix}} \right\}$$

b) $z = \begin{bmatrix} 3 \\ -1 \\ 13 \\ -17 \end{bmatrix}$ Done in Matlab

Yes

c) $u = \begin{bmatrix} 4 \\ 9 \\ 12 \\ -8 \end{bmatrix}$ Done in Matlab

No

d) $v = \begin{bmatrix} -1 \\ -1 \\ -3 \\ 3 \end{bmatrix}$ Done in Matlab

Yes

- Need to determine if z, u, v can be written as linear combinations of w_1 + w_2
- Procedure: Set up $[w_1 \ w_2 \ \underset{\downarrow}{\overset{z,u,v}{}}]$ and reduce to rref. If rank > 2, not in W

3)

a)

i) $x_1 = [2 \ -3 \ -1]^T$

$\|x_1\|_1 = 6$

$\|x_1\|_2 = 3.74$

$\|x_1\|_\infty = 3$

ii) $x_2 = [1 \ 1 \ -1]^T$

$\|x_2\|_1 = 3$

$\|x_2\|_2 = 1.73$

$\|x_2\|_\infty = 1$

b) $\left\{ \overset{v_1}{\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}}, \overset{v_2}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}} \right\}$ $\quad \text{proj}_u v = \dfrac{\langle u, v \rangle}{\langle u, u \rangle} u$

$u_1 = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = v_1 \quad , \quad x_1 = \dfrac{u_1}{\|u_1\|} = \dfrac{[2 \ -3 \ -1]^T}{\sqrt{14}} = [0.535, -0.8018, -0.2673]^T$

$u_2 = v_2 - \text{proj}_{u_1} v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - 0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad , \quad x_2 = \dfrac{u_2}{\|u_2\|} = \dfrac{[1 \ 1 \ -1]}{\sqrt{3}} = [0.5774, 0.5774, -0.5774]^T$

$\left\{ \begin{bmatrix} 0.535 \\ -0.8018 \\ -0.2673 \end{bmatrix}, \begin{bmatrix} 0.5774 \\ 0.5774 \\ -0.5774 \end{bmatrix} \right\}$

4)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \ \ 0] x$$

a) $\dot{x}(t) = \dfrac{x[(k+1)T] - x[kT]}{T}$

Discrete System: $\quad x[k+1] = A(T) x[kT] + B(T) u[kT]$
$\qquad\qquad\qquad\qquad y[k] = C x[kT]$

$$\frac{x[(k+1)T] - x[kT]}{T} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x[kT] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[kT]$$

$$x[(k+1)T] = x[kT] + \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} x[kT] + \begin{bmatrix} 0 \\ T \end{bmatrix} u[kT]$$

$$y[kT] = [1 \ \ 0] x[kT]$$

$$x[(k+1)T] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x[kT] + \begin{bmatrix} 0 \\ T \end{bmatrix} u[kT]$$

$$y[kT] = [1 \ \ 0] x[kT]$$

$$x[(k+1)T] = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} x[kT] + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u[kT]$$

$$y[kT] = [1 \ \ 0] x[kT]$$

b) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at $t=0$ $\rightarrow$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ at $t=0.05$ w/ $T_s=0.01$

$$x[(k+1)T] = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} x[kT] + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u[kT]$$
$$y[kT] = \begin{bmatrix} 1 & 0 \end{bmatrix} x[kT]$$

Using Generalized Solns. to DTLTI Systems:

$$x(k) = A^k x(0) + \sum_{m=0}^{k-1} A^{k-m-1} Bu(m)$$

$$y(k) = CA^k x(0) + \sum_{m=0} CA^{k-m-1} Bu(m) + Du(k)$$

Form eqns. using this soln. for $x(k)$

$$x(5) = A^5 x(0) + \sum_{m=0}^{4} A^{k-m-1} Bu(m)$$
$$x(5) = A^4 Bu(0) + A^3 Bu(1) + A^2 Bu(2) + A^1 Bu(3) + A^0 Bu(4)$$
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.0004\,u(0) + 0.0003u(1) + 0.0002u(2) + 0.0001u(3) + 0 \cdot u(4) \\ 0.01u(0) + 0.01u(1) + 0.01u(2) + 0.01u(3) + 0.01u(4) \end{bmatrix}$$

Possible Soln: $u_0 = 1000$, $u_1 = 1000$, $u_2 = 1000$, $u_3 = 1000$, $u_4 = -4000$