1)

a) $\ddot{\theta} + \frac{g}{l}\sin\theta = \frac{\tau}{ml^2}$; $x = \begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$, $\dot{x} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix}$

input

state

$\dot{x} = \begin{bmatrix}\dot{\theta} \\ \frac{\tau}{ml^2} - \frac{g}{l}\sin\theta\end{bmatrix} = 0$, $\theta = \pi/2$, $\dot{\theta} = 0$, $\tau = mgl$

take jacobian wrt state + input var.

$\frac{\partial f}{\partial x} = \begin{bmatrix}0 & 1 \\ -g/l\cos\theta & 0\end{bmatrix}$, $\frac{\partial f}{\partial u} = \begin{bmatrix}0 \\ 1/ml^2\end{bmatrix}$

$\left.\frac{\partial f}{\partial x}\right|_{(\pi/2, 0, mgl)} = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}$, $\frac{\partial f}{\partial u} = \begin{bmatrix}0 \\ 1/ml^2\end{bmatrix}$

$\dot{x} = Ax + Bu$ so $\dot{x} = \begin{bmatrix}\dot{\theta} \\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0 \\ 1/ml^2\end{bmatrix}\tau$

$\dot{x} = \ddot{\theta} = \frac{\tau}{ml^2}$

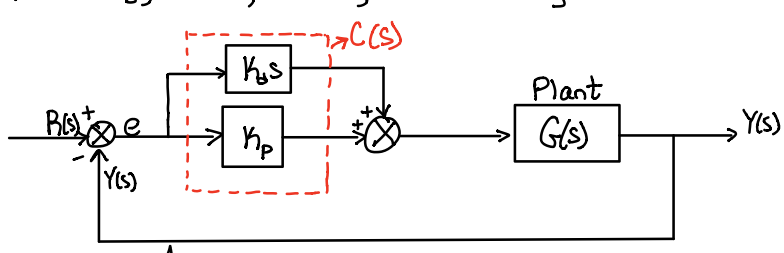
Stability: Analyze evals of $A = \begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}$ *evals = eigenvalues

Upper triangular so evals $= \underset{\#1}{0}, \underset{\#2}{0}$

Marginally / Critically Stable

Unstable also works

b) $m = 1\,kg$, $l = 4m$, 25% OS, 2 sec. 2% settling time



$\ddot{\theta} + \frac{g}{l}\cancel{\sin\theta}^{\,1} = \frac{\tau}{ml^2}$ so $\ddot{\theta} + \frac{g}{l} = \frac{\tau}{ml^2}$, $G(s) = \frac{\theta(s)}{\tau(s)} = C(sI-A)^{-1}B + D = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1/s & 1/s^2 \\ 0 & 1/s\end{bmatrix}\begin{bmatrix}0 \\ 1/ml^2\end{bmatrix}$

$G(s) = \frac{1}{ml^2s^2}$ and $C(s) = K_p + K_d s$

$$H(s) = \frac{C(s)G(s)}{1+C(s)G(s)} = \frac{(K_p+K_d s)\left(\frac{1}{m\ell^2 s^2}\right)}{1+(K_p+K_d s)\left(\frac{1}{m\ell^2 s^2}\right)} = \frac{\frac{(K_p+K_d s)}{m\ell^2 s^2}}{\frac{m\ell^2 s^2 + K_p + K_d s}{m\ell^2 s^2}}$$

$$= \frac{\frac{(K_p+K_d s)}{m\ell^2}}{(s^2+(K_p+K_d s)/m\ell^2)} = \boxed{\frac{\frac{(K_p+K_d s)}{m\ell^2}}{s^2+\frac{K_d}{m\ell^2}s+\frac{K_p}{m\ell^2}}}$$

Second Order Transfer Func. Form

$$\omega_n = \sqrt{\frac{K_p}{m\ell^2}} \qquad \zeta = \frac{K_d}{2m\ell^2\omega_n}$$

$$\%OS: \ \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2+\ln^2(\%OS/100)}} = 0.4037$$

$$T_s: \ T_s = \frac{4}{\zeta\omega_n} \Rightarrow 2 = \frac{4}{\zeta\omega_n} \text{ so } \omega_n = 4.9540$$

$$K_p = 392.674, \ K_d = 63.9978$$

c) $\ddot{\theta} + \frac{g}{\ell}\sin\theta = \frac{\tau}{m\ell^2}$

$$\ddot{\theta} = \frac{\tau}{m\ell^2} - \frac{g}{\ell}\sin\theta$$

Rest of Solution in Simulink

PD controller will stabilize for every angle.

2)

a) $y(t) = a$, $a \neq 0 \ \forall t$, constant value

<u>Linearity</u>

$y_1(t) = a \rightarrow u_1(t)$
$y_2(t) = a \rightarrow u_2(t)$
$u(t) = \alpha u_1(t) + \beta u_2(t)$

1) $y(t) = a$
2) $y(t) = \alpha y_1(t) + \beta y_2(t) = \alpha a + \beta a$

1) $\neq$ 2) so Nonlinear

<u>Time-Variance</u>

No dependence on time so Time-Invariant

b) $y(t) = -3u(t) + 2$

Linearity

$y_1(t) = -3u_1(t) + 2 \rightarrow u_1(t)$
$y_2(t) = -3u_2(t) + 2 \rightarrow u_2(t)$
$u(t) = \alpha u_1(t) + \beta u_2(t)$

1) $y(t) = -3\alpha u_1(t) - 3\beta u_2(t) + 2$

2) $y(t) = \alpha y_1(t) + \beta y_2(t)$
$= \alpha(-3u_1(t)+2) + \beta(-3u_2(t)+2)$
$= -3\alpha u_1(t) + 2\alpha - 3\beta u_2(t) + 2\beta$

1) $\neq$ 2) so Nonlinear

Time-Variance

$u_\tau(t) = u(t-\tau) \Rightarrow$ 1) $y_\tau(t) = -3u(t-\tau) + 2$

2) $y(t-\tau) = -3u(t-\tau) + 2$

1) = 2) so Time-Invariant

c) $y(t) = u^3(t)$

Linearity

$y_1(t) = u_1^3(t) \rightarrow u_1(t)$
$y_2(t) = u_2^3(t) \rightarrow u_2(t)$
$u(t) = \alpha u_1(t) + \beta u_2(t)$

1) $y(t) = [\alpha u_1(t) + \beta u_2(t)]^3$

2) $y(t) = \alpha y_1(t) + \beta y_2(t)$
$= \alpha u_1^3(t) + \beta u_2^3(t)$

1) $\neq$ 2) so Nonlinear

Time-Variance

$u_\tau(t) = u(t-\tau) \Rightarrow$ 1) $y_\tau(t) = u^3(t-\tau)$

2) $y(t-\tau) = u^3(t-\tau)$

1) = 2) so Time-Invariant

d) $y(t) = u(t^3)$

Linearity

$y_1(t) = u_1(t^3) \rightarrow u_1(t)$
$y_2(t) = u_2(t^3) \rightarrow u_2(t)$
$u(t) = \alpha u_1(t) + \beta u_2(t)$

1) $y(t) = \alpha u_1(t^3) + \beta u_2(t^3)$

2) $y(t) = \alpha y_1(t) + \beta y_2(t)$
$= \alpha u_1(t^3) + \beta u_2(t^3)$

1) = 2) so Linear

Time-Variance

$u_\tau(t) = u(t-\tau) \Rightarrow$ 1) $y_\tau(t) = u(t^3-\tau)$

2) $y(t-\tau) = u((t-\tau)^3)$

1) $\neq$ 2) so Time-Variant

e) $y(t) = e^{-t}u(t-T)$

<u>Linearity</u>

$y_1(t) = e^{-t}u_1(t-T) \rightarrow u_1(t)$
$y_2(t) = e^{-t}u_2(t-T) \rightarrow u_2(t)$
$u(t) = \alpha u_1(t) + \beta u_2(t)$

1) $y(t) = e^{-t}[\alpha u_1(t-T) + \beta u_2(t-T)]$

2) $y(t) = \alpha y_1(t) + \beta y_2(t)$
$= \alpha e^{-t}u_1(t-T) + \beta e^{-t}u_2(t-T)$

1) = 2) so Linear

<u>Time-Variance</u>

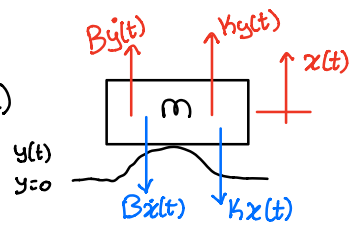
$u_\tau(t) = u(t-\tau) \Rightarrow$ 1) $y_\tau(t) = e^{-t-\tau}u(t-T-\tau)$

2) $y(t-\tau) = e^{-(t-\tau)}u((t-\tau)-T)$

1) ≠ 2) so Time-Variant

3)

a)



<u>Diff. Eqn.</u>

$$M\ddot{x}(t) = -B\dot{x}(t) - kx(t) + B\dot{y}(t) + ky(t)$$

$$\ddot{x}(t) = \frac{-B\dot{x}(t) - kx(t) + B\dot{y}(t) + ky(t)}{M}$$

<u>State Space</u>

State Variables: $\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$

Input : $y(t)$

$\dot{z} = Az + Bu$
$\delta = Cz + Du$

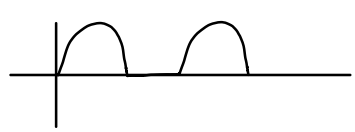
* using z & δ to avoid confusion w/ x, y

$$\dot{z} = \begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -B/m \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y(t)$$

$$\delta = \begin{bmatrix} k/m & B/m \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$$

bi) Provided in Matlab/Simulink

bii) Half sine wave



$y = \sin(x) + |\sin(x)|$

Amplitude = 0.08 , frequency = $\frac{1}{\text{period}}$ = 50 so period = $\frac{1}{50}$

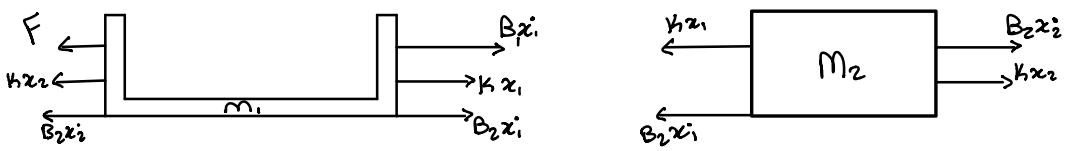
$y = A\sin(Bx)$, period $= \frac{1}{50} = \frac{2\pi}{B}$ so $B = 100\pi$

$\Downarrow$

$y = 0.08\sin(100\pi \cdot x) + |0.08\sin(100\pi \cdot x)|$

Provided in Matlab / Simulink

4)



1) $m_1\ddot{x}_1 = F + kx_2 + B_2\dot{x}_2 - B_1\dot{x}_1 - kx_1 - B_2\dot{x}_1 = F + kx_2 - kx_1 + B_2\dot{x}_2 - (B_1 + B_2)\dot{x}_1$

2) $m_2\ddot{x}_2 = kx_1 + B_2\dot{x}_1 - B_2\dot{x}_2 - kx_2 = kx_1 - kx_2 + B_2\dot{x}_1 - B_2\dot{x}_2$

a) $X = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$, $y = x_2$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k/m_1 & -(B_1+B_2)/m_1 & k/m_1 & B_2/m_1 \\ 0 & 0 & 0 & 1 \\ k/m_2 & B_2/m_2 & -k/m_2 & -B_2/m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{bmatrix} F$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{bmatrix}$$

b) $X = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 - x_1 \\ \dot{x}_2 - \dot{x}_1 \end{bmatrix}$, $y = x_2$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \ddot{x}_1 \\ \dot{x}_2 - \dot{x}_1 \\ \ddot{x}_2 - \ddot{x}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -B_1/m_1 & k/m_1 & B_2/m_1 \\ 0 & 0 & 0 & 1 \\ 0 & B_1/m_1 & \left(-\frac{k}{m_2} - \frac{k}{m_1}\right) & \left(-\frac{B_2}{m_2} - \frac{B_2}{m_1}\right) \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 - x_1 \\ \dot{x}_2 - \dot{x}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ -1/m_1 \end{bmatrix} F$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dot{x}_1 \\ x_2 - x_1 \\ \dot{x}_2 - \dot{x}_1 \end{bmatrix}$$

* Refer to theory below

5) Use a similarity transformation to go from $x \to \hat{x}$

Theory: Given $\dot{x} = Ax + Bu$, $y = Cx + Du$ and new set of independent variables $\hat{x}$

$$\hat{x} = Px$$
$$x = P^{-1}\hat{x}$$

Plug in for x

$$P^{-1}\dot{\hat{x}} = AP^{-1}\hat{x} + Bu$$
$$y = CP^{-1}\hat{x} + Du$$

Multiply top eqn. by P to get $\dot{\hat{x}}$

$$\dot{\hat{x}} = PAP^{-1}\hat{x} + PBu$$
$$y = CP^{-1}\hat{x} + Du$$

$$\boxed{\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u \\ y &= \hat{C}\hat{x} + \hat{D}u \end{aligned} \qquad \begin{aligned} \hat{A} &= PAP^{-1} \\ \hat{B} &= PB \\ \hat{C} &= CP^{-1} \\ \hat{D} &= D \end{aligned}}$$

Solution: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} -4x_1 - 2x_2 - 3x_3 \\ 15x_1 + 7x_2 + 10x_3 \\ -5x_1 - 2x_2 - 3x_3 \end{bmatrix}$

$$\hat{x} = Px$$

$$\begin{bmatrix} -4x_1 - 2x_2 - 3x_3 \\ 15x_1 + 7x_2 + 10x_3 \\ -5x_1 - 2x_2 - 3x_3 \end{bmatrix} = \begin{bmatrix} -4 & -2 & -3 \\ 15 & 7 & 10 \\ -5 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$P = \begin{bmatrix} -4 & -2 & -3 \\ 15 & 7 & 10 \\ -5 & -2 & -3 \end{bmatrix}$; rest of solution provided in Matlab

$$\dot{\hat{x}} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -45 & -25 & -40 \end{bmatrix} \hat{x}$$

6) Inputs τ_1, τ_2 , State $[\theta_1 \ \dot{\theta}_1 \ d_2 \ \dot{d}_2]^T$, Constants m, I, l, g

$$\dot{x} = \begin{bmatrix} \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{d}_2 \\ \ddot{d}_2 \end{bmatrix} \quad ; \quad \ddot{\theta}_1 = \frac{\tau_1 - 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 - (m_1 l_1 + m_2 d_2) g \cos\theta_1}{m_1 l_1^2 + I_1 + I_2 + m_2 d_2^2}$$

$$\ddot{d}_2 = \frac{\tau_2 + m_2 d_2 \dot{\theta}_1^2 - m_2 g \sin\theta_1}{m_2}$$

$$\dot{\theta}_1 = \dot{\theta}_1 \quad \forall \quad \dot{d}_2 = \dot{d}_2$$

Solutions provided in Matlab