1)

$\{a_1, a_2, a_3\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \right\}$, $\{b_1, b_2, b_3\} = \left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix} \right\}$

$A = \begin{bmatrix} 8 & -2 & -1 \\ 4 & -2 & -3 \\ 2 & -3 & -3 \end{bmatrix}$ in the $\{a_1, a_2, a_3\}$ basis

Aside: Calculating inv. of 3x3 by hand

1) Finding Matrix of Minors + Apply Signs

$S = \begin{bmatrix} -4 & 2 & -3 \\ 10 & -5 & 8 \\ 23 & -11 & 18 \end{bmatrix} \rightarrow \begin{bmatrix} -4 & -2 & -3 \\ -10 & -5 & -8 \\ 23 & +11 & 18 \end{bmatrix}$

2) Divide Transpose Matrix by Deter.

$\det S = 1$

$S^{-1} = \begin{bmatrix} -4 & -10 & 23 \\ -2 & -5 & 11 \\ -3 & -8 & 18 \end{bmatrix}$

a) ${}_{B\leftarrow A}P$: Change of Coordinates Matrix

$[a_1]_b = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $[a_2]_b = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, $[a_3]_b = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$[b_1 \ b_2 \ b_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = a_1$

$[b_1 \ b_2 \ b_3] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a_2$

$[b_1 \ b_2 \ b_3] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = a_3$

$[b_1 \ b_2 \ b_3 \ a_1 \ a_2 \ a_3]$, now construct red. row echelon form and take the right half of the resulting matrix.

$${}_{A\leftarrow B}P = \begin{bmatrix} 74 & -38 & -13 \\ 35 & -18 & -7 \\ 58 & -29 & -10 \end{bmatrix}$$

$$\hat{A} = {}_{A\leftarrow B}P \cdot A \cdot {}_{A\leftarrow B}P^{-1} = \begin{bmatrix} -109.86 & -128.29 & 224.71 \\ -52.53 & -62.94 & 108.35 \\ -87.04 & -97.92 & 175.80 \end{bmatrix}$$

b) $x = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix}$ in basis $\{a_1, a_2, a_3\} \rightarrow \{b_1, b_2, b_3\}$

$$\hat{x} = {}_{A\leftarrow B}P \, x = \begin{bmatrix} 238 \\ 116 \\ 185 \end{bmatrix}$$

2) Get all soln. to $Ax=y$; $x = x_p + \alpha x_n$

particular soln.

nullspace

i) get pivot columns to determine free variables

x_3, x_5 not in pivot columns so free variables

ii) determine particular soln. by setting free variables $=0$

$$\begin{aligned} x_1 + \cancel{2x_3} - \cancel{2x_5} &= 3 \\ x_2 \cancel{-x_3} + \cancel{x_5} &= 1 \\ x_3 &= 0 \\ x_4 \cancel{-2x_5} &= 2 \\ x_5 &= 0 \end{aligned} \qquad \text{so} \quad x_p = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

iii) determine additional solns. by finding null space

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_3 - 2x_5 &= 0 \\ x_2 - x_3 + x_5 &= 0 \\ x_4 - 2x_5 &= 0 \end{aligned} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_3 + 2x_5 \\ x_3 - x_5 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = \overset{x_{n_1}}{\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} x_3 + \overset{x_{n_2}}{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}} x_5$$

iv) form entire soln.

$$x = x_p + \alpha x_n = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

3) Optimal Least-Squared Soln.

$$\begin{bmatrix} 1 & -2 \\ 1 & -2 \\ -2 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 1 \\ -3 \end{bmatrix}$$

more eqns. than unknowns so $x = (A^TA)^{-1}A^Tb$

$$x = \begin{bmatrix} 0 \\ 0.222 \end{bmatrix}$$

4)

a) $\begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

* Hand Computed Solution Expected

i) Eigenvalues

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & -2 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} = 0$$

Using Cofactor Expansion: $\det(A-\lambda I) = \sum_{i=1}^{n} a_{i1} C_{i1} M_{i1} = 0$

$$\det(A-\lambda I) = a_{11}C_{11}M_{11} + a_{21}C_{21}M_{21} + a_{31}C_{31}M_{31} = 0$$

$$= (1-\lambda)(-1)^2((2-\lambda)(1-\lambda)-1) + (-1)(-1)^3(-2+2\lambda) + 0 = 0$$

$$= (1-\lambda)(2-3\lambda+\lambda^2-1) + (2\lambda-2) = 0$$

$$= \lambda^2 - 3\lambda + 1 - \lambda^3 + 3\lambda^2 - \lambda + 2\lambda - 2 = 0$$

$$= \lambda^3 - 4\lambda^2 + 2\lambda + 1 = 0$$

↳ possible factors: ± 1 ; Using $\lambda = 1$ gives 0 so $(\lambda - 1)$ is a factor

$$\begin{array}{r|l} & \lambda^2 - 3\lambda - 1 \\ \hline \lambda - 1 & \lambda^3 - 4\lambda^2 + 2\lambda + 1 \\ & -\lambda^3 - \lambda^2 \\ \hline & -3\lambda^2 + 2\lambda + 1 \\ & -(-3\lambda^2 + 3\lambda) \\ \hline & -\lambda + 1 \\ & -(-\lambda + 1) \\ \hline & 0 \end{array}$$

Applying Quad. Formula to get last 2 roots.

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9+4}}{2}$$

$$\lambda_{1,2} = \frac{3 \pm \sqrt{13}}{2} = 3.3028, -0.3028$$

$\lambda_1 = 1$, $\lambda_2 = 3.3028$, $\lambda_3 = -0.3028$

ii) Eigenvectors

λ_1: 1

$$\hookrightarrow \begin{bmatrix} 0 & -2 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad \text{so} \quad \begin{aligned} -2x_2 &= 0 \\ -1x_1 + x_2 - 1x_3 &= 0 \\ -1x_2 &= 0 \end{aligned} \quad \text{so} \quad \begin{aligned} x_2 &= 0 \\ x_1 &= -x_3 + x_2 \end{aligned}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 + x_2 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

λ_2: 3.3028

$$\hookrightarrow \begin{bmatrix} -2.3028 & -2 & 0 \\ -1 & -1.3028 & -1 \\ 0 & -1 & -2.3028 \end{bmatrix} \quad \text{so} \quad \begin{aligned} -2.3028x_1 - 2x_2 &= 0 \\ -x_1 - 1.3028x_2 - x_3 &= 0 \\ -x_2 - 2.3028x_3 &= 0 \end{aligned}$$

Select $x_2 = 1$: $x_1 = -0.8685$, $x_2 = 1$, $x_3 = -0.4343$

Aside: You can do so for any variable

$$x = \begin{bmatrix} -0.8685 \\ 1 \\ -0.4343 \end{bmatrix}, \text{ now normalize to get } x = \begin{bmatrix} -0.6231 \\ 0.7174 \\ -0.3116 \end{bmatrix}$$

$\lambda_3 = -0.3028$

$$\hookrightarrow \begin{bmatrix} 1.3028 & -2 & 0 \\ -1 & 2.3028 & -1 \\ 0 & -1 & 1.3028 \end{bmatrix} \quad \text{so} \quad \begin{aligned} 1.3028x_1 - 2x_2 &= 0 \\ -x_1 + 2.3028x_2 - x_3 &= 0 \\ -x_2 + 1.3028x_3 &= 0 \end{aligned}$$

Select $x_2 = 1$: $x_1 = 1.5352$, $x_2 = 1$, $x_3 = 0.7676$

$$x = \begin{bmatrix} 1.5352 \\ 1 \\ 0.7676 \end{bmatrix}, \text{ now normalize to get } x = \begin{bmatrix} 0.7728 \\ 0.5034 \\ 0.3864 \end{bmatrix}$$

eigenvectors: $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -0.6231 \\ 0.7174 \\ -0.3116 \end{bmatrix}, \begin{bmatrix} 0.7728 \\ 0.5034 \\ 0.3864 \end{bmatrix}$

$\lambda = 1$ $\quad \lambda = 3.3028$ $\quad \lambda = -0.3028$

b) $\begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ -3 & -3 & -5 \end{bmatrix}$ following similar procedure as above

eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = -2$

eigenvectors: $\begin{bmatrix} -0.5774 \\ -0.5774 \\ 0.5774 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.7071 \\ -0.7071 \end{bmatrix}, \begin{bmatrix} -0.8121 \\ 0.3328 \\ 0.4793 \end{bmatrix}$ OR $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix}$

i) Eigenvalues:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -\omega_n^2 & -\lambda \end{bmatrix}\right) = 0$$

$\lambda^2 + \omega_n^2 = 0$ so $\lambda^2 = -\omega_n^2$

$\lambda_{1,2} = i(\pm \omega_n)$

$\lambda_1 = i\omega_n$, $\lambda_2 = -i\omega_n$

ii) Eigenvectors:

$\lambda = i\omega_n$

$\hookrightarrow \begin{bmatrix} -i\omega_n & 1 \\ -\omega_n^2 & -i\omega_n \end{bmatrix}$ so $\begin{aligned} -i\omega_n x_1 + x_2 &= 0 \\ -\omega_n^2 x_1 - i\omega_n x_2 &= 0 \end{aligned}$

Set $x_2 = 1$: $x_1 = \dfrac{-i}{\omega_n}$, $x_2 = 1$

$x = \begin{bmatrix} -i/\omega_n \\ 1 \end{bmatrix}$

$\lambda = -i\omega_n$

$\hookrightarrow \begin{bmatrix} i\omega_n & 1 \\ -\omega_n^2 & i\omega_n \end{bmatrix}$ so $\begin{aligned} i\omega_n x_1 + x_2 &= 0 \\ -\omega_n^2 x_1 + i\omega_n x_2 &= 0 \end{aligned}$

Set $x_2 = 1$: $x_1 = \dfrac{i}{\omega_n}$, $x_2 = 1$

$$x = \begin{bmatrix} i/\omega_n \\ 1 \end{bmatrix}$$

eigenvalues: $i\omega_n$, $-i\omega_n$

eigenvectors: $\begin{bmatrix} -i/\omega_n \\ 1 \end{bmatrix}, \begin{bmatrix} i/\omega_n \\ 1 \end{bmatrix}$

5)

a) $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$

Get eigenvalues of AA^T and $\sigma = \sqrt{\lambda}$

$\sigma_1 = \sqrt{6}$, $\sigma_2 = 1$

b) $A = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$

$\sigma_1 = 4.7016$, $\sigma_2 = 1.7016$

6) $T = 0.01$

$$x_{k+1} = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k$$

a) Optimal feedforward control sequence

$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_{10} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; will get underdetermined system so $x = A^T(AA^T)^{-1}y$

Using Generalized Solns. to DTLTI Systems:

$$x(k) = A^k x(0) + \sum_{m=0}^{k-1} A^{k-m-1} Bu(m)$$

$$x(10) = A^{10} x(0) + \sum_{m=0}^{9} A^{k-m-1} Bu(m)$$

$$x(10) = A^{10}x(0) + A^9Bu(0) + A^8Bu(1) + A^7Bu(2) + A^6Bu(3) + A^5Bu(4) + A^4Bu(5) + A^3Bu(6) + A^2Bu(7) + A^1Bu(8) + A^0Bu(9)$$

$$x(10) - A^{10}x(0) = \sum_{m=0}^{9} A^{10-m-1}Bu(m)$$

$$\underbrace{\begin{bmatrix} B & A^1B & A^2B & \dots & A^9B \end{bmatrix}}_{\hat{A}} \underbrace{\begin{bmatrix} u(9) \\ u(8) \\ u(7) \\ \vdots \\ u(0) \end{bmatrix}}_{x} = \underbrace{x(10) - A^{10}x(0)}_{\hat{b}}$$

b) $u = \underbrace{[k_1 \;\; k_2]}_{k} x = k_1x_1 + k_2x_2$

↳ want k_1, k_2 s.t. $\lambda_1 = \lambda_2 = 0$

General Procedure for Pole Placement:

1) Check if System Controllable
2) Convert SS → TF + form Controller Canonical Form
3) Use last row of CCF to get desired char. poly. in terms of k
4) Find k values and apply similarity transform to get k values in original coordinates.

State char. eqn: $s^2 - 2s - 1$
Desired char. eqn: s^2

Using $(A+Bk)$ form and Controller Canonical Form:

$$s^2 + (-2-k_1)s + (+1-k_0) = s^2$$

$-2-k_1 = 0$ and $1-k_0 = 0$
$k_1 = -2$ and $k_0 = 1$

$$\tilde{k} = [1 \;\; -2]$$

$$k = [-10000 \;\; -200]$$