1) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, 3×3 so $n = 3$

a) $A^{10} = ?$

i) Get eigenvalues:

$$\det(A - \lambda I) = \left| \begin{bmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \right| = (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} - (1) \begin{vmatrix} 0 & 1 \\ 0 & 1-\lambda \end{vmatrix} + 0$$

$$= (1-\lambda)(-\lambda)(1-\lambda) = 0 \text{ so } \lambda = 0, 1, 1$$

ii) Set up $f(\lambda)$ to get β coefficients

$$f(\lambda) = \beta_{n-1}\lambda^{n-1} + \beta_{n-2}\lambda^{n-2} + \ldots + \beta_0 I$$

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 1$$

Aside: For repeated eigenvals, take the deriv. of the eqn. to get a new eqn.

Using $f(\lambda)$ to get β coefficients:

$$\left.\begin{aligned} \lambda_1^{10} &= \beta_2\lambda_1^2 + \beta_1\lambda_1^1 + \beta_0 \\ \lambda_2^{10} &= \beta_2\lambda_2^2 + \beta_1\lambda_2^1 + \beta_0 \\ 10\lambda_2^9 &= 2\beta_2\lambda_2 + \beta_1 \end{aligned}\right\} \quad \left.\begin{aligned} 0 &= \beta_0 \\ 1 &= \beta_2 + \beta_1 \\ 10 &= 2\beta_2 + \beta_1 \end{aligned}\right\} \quad \left.\begin{aligned} \beta_0 &= 0 \\ \beta_1 &= -8 \\ \beta_2 &= 9 \end{aligned}\right\}$$

iii) Apply β coefficients into $f(A)$

$$f(A) = \beta_{n-1}A^{n-1} + \beta_{n-2}A^{n-2} + \ldots + \beta_0 I$$

$$A^{10} = \beta_2 A^2 + \beta_1 A^1 + \beta_0$$

$$A^2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^{10} = 9\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - 8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 9 \\ 0 & 0 & 9 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 8 & 8 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

b) $A^{103} = ?$

i) Get eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$

ii) Set up $f(\lambda)$ to get β coefficients

$$f(\lambda) = \beta_{n-1}\lambda^{n-1} + \beta_{n-2}\lambda^{n-2} + \dots + \beta_0 I$$

$$\left.\begin{aligned} \lambda_1^{103} &= \beta_2\lambda_1^2 + \beta_1\lambda_1^1 + \beta_0 I \\ \lambda_2^{103} &= \beta_2\lambda_2^2 + \beta_1\lambda_2^1 + \beta_0 I \\ 103\lambda_2^{102} &= 2\beta_2\lambda_2 + \beta_1 \end{aligned}\right\} \left.\begin{aligned} 0 &= \beta_0 \\ 1 &= \beta_2 + \beta_1 \\ 103 &= 2\beta_2 + \beta_1 \end{aligned}\right\} \left.\begin{aligned} \beta_0 &= 0 \\ \beta_1 &= -101 \\ \beta_2 &= 102 \end{aligned}\right.$$

iii) Apply β coefficients into $f(A)$

$$f(A) = \beta_{n-1}A^{n-1} + \beta_{n-2}A^{n-2} + \dots + \beta_0 I$$
$$A^{103} = \beta_2 A^2 + \beta_1 A^1 + \beta_0$$

$$A^{103} = 102\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} - 101\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 102 & 102 & 102 \\ 0 & 0 & 102 \\ 0 & 0 & 102 \end{bmatrix} - \begin{bmatrix} 101 & 101 & 0 \\ 0 & 0 & 101 \\ 0 & 0 & 101 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $e^{At} = ?$

i) Get eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 1$

ii) Set up $f(\lambda)$ to get β coefficients

$$\left.\begin{aligned} e^{\lambda_1 t} &= \beta_2\lambda_1^2 + \beta_1\lambda_1^1 + \beta_0 \\ e^{\lambda_2 t} &= \beta_2\lambda_2^2 + \beta_1\lambda_2^1 + \beta_0 \\ te^{\lambda_2 t} &= 2\beta_2\lambda_2 + \beta_1 \end{aligned}\right\} \left.\begin{aligned} 1 &= \beta_0 \\ e^t &= \beta_2 + \beta_1 + 1 \\ te^t &= 2\beta_2 + \beta_1 \end{aligned}\right\} \left.\begin{aligned} \beta_0 &= 1 \\ \beta_1 &= 2e^t - te^t - 2 \\ \beta_2 &= te^t - e^t + 1 \end{aligned}\right.$$

iii) Apply β coefficients into $f(A)$

$$e^{At} = \beta_2 A^2 + \beta_1 A + \beta_0$$

$$= te^t - e^t + 1\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - te^t - 2)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + I$$

$$= \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

2)

Closed form A^k can be found through diagonalization:

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 9 & 23 & 30 \\ -7 & -18 & -235 \end{bmatrix}$$

$$\lambda_1, \lambda_2, \lambda_3 = -232.89,\ 20.89,\ 0$$

$$\text{evecs} = \begin{bmatrix} 0 \\ 0.1164 \\ -0.9932 \end{bmatrix}, \begin{bmatrix} 0 \\ -0.9975 \\ 0.0702 \end{bmatrix}, \begin{bmatrix} 0.9985 \\ 0.0431 \\ -0.0330 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & 0 & 0.9985 \\ 0.1164 & -0.9975 & 0.0431 \\ -0.9932 & 0.0702 & -0.0330 \end{bmatrix}$$

$$V = \begin{bmatrix} -232.89 & 0 & 0 \\ 0 & 20.89 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^k = M V^k M^{-1}$$

$$A^k = \begin{bmatrix} 0 & 0 & 0.9985 \\ 0.1164 & -0.9975 & 0.0431 \\ -0.9932 & 0.0702 & -0.0330 \end{bmatrix} \begin{bmatrix} -232.89^k & 0 & 0 \\ 0 & 20.89^k & 0 \\ 0 & 0 & 0^k \end{bmatrix} \begin{bmatrix} -0.0278 & -0.0714 & -1.0152 \\ -0.3958 & -1.0108 & -0.1185 \\ 1.0739 & 0 & 0 \end{bmatrix}$$

3)

Solution to DTLTI System:

$$x(k) = A^k x(0) + \sum_{m=0}^{k-1} A^{k-m-1} B u(m)$$

$$y(k) = C A^k x(0) + \sum_{m=0}^{k-1} C A^{k-m-1} B u(m) + D u(k)$$

Want expression for $y(n)$ if $u(k) = 1\ \forall k \geq 0$ and $x(0) = [0\ \ 0]^T$

$$y(n) = C A^n x(0) + \sum_{m=0}^{n-1} C A^{n-m-1} B u(m)$$

Need

Known

i) Setting up CH to solve for A^n

$$A = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}^{2\times 2}$$

$\det(A-\lambda I)=0$ so $\left|\begin{bmatrix} -\lambda & 1 \\ -0.5 & -1-\lambda \end{bmatrix}\right| = (-\lambda)(-1-\lambda)+0.5=0$

so $\lambda^2+\lambda+0.5=0$ so $\lambda = \frac{-1\pm\sqrt{1-2}}{2} = \frac{-1\pm i}{2}$

$\lambda_1 = \frac{-1+i}{2}$, $\lambda_2 = \frac{-1-i}{2}$

ii) Set up $f(\lambda)$ to get β coefficients

$f(\lambda) = \beta_{n-1}\lambda^{n-1} + \beta_{n-2}\lambda^{n-2} + \ldots + \beta_0 I$

$\lambda_1^n = \beta_1\lambda_1 + \beta_0$ $\Rightarrow$ $\left(\frac{-1+i}{2}\right)^n = \beta_1\left(\frac{-1+i}{2}\right) + \beta_0$ ①

$\lambda_2^n = \beta_1\lambda_2 + \beta_0$

$\left(\frac{-1-i}{2}\right)^n = \beta_1\left(\frac{-1-i}{2}\right) + \beta_0$ ②

From ①: $\beta_0 = \left(\frac{-1+i}{2}\right)^n - \beta_1\left(\frac{-1+i}{2}\right)$

From ②: $\left(\frac{-1-i}{2}\right)^n = \frac{-\beta_1 - \beta_1 i}{2} + \left(\frac{-1+i}{2}\right)^n + \frac{\beta_1 - \beta_1 i}{2}$

$\left(\frac{-1-i}{2}\right)^n - \left(\frac{-1+i}{2}\right)^n = -\beta_1 i$

$\beta_1 = \frac{1}{i}\left[\left(\frac{-1+i}{2}\right)^n - \left(\frac{-1-i}{2}\right)^n\right]$

From ①: $\beta_0 = \left(\frac{-1+i}{2}\right)^n - \frac{1}{i}\left[\left(\frac{-1+i}{2}\right)^n - \left(\frac{-1-i}{2}\right)^n\right]\left(\frac{-1+i}{2}\right)$

iii) Apply β coefficients into $f(A)$

$f(A) = A^n = \beta_1 A^1 + \beta_0 I$

Expression for A^n in Matlab

iv) Writing out expression for $y(n)$

$$y(n) = \cancel{CA^n x(0)}^{\,0} + \sum_{m=0}^{n-1} CA^{n-m-1}Bu(m) = \sum_{m=0}^{n-1} CA^{n-m-1}Bu(m)$$

$$y(n) = \sum_{m=0}^{n-1} CA^{n-m-1}Bu(m)$$

$$y(n) = \sum_{m=0}^{n-1}\left[\left(-\frac{1}{2}-\frac{1}{2}i\right)^{n-m-1}\left(\frac{1}{2}+\frac{3}{2}i\right)+\left(-\frac{1}{2}+\frac{1}{2}i\right)^{n-m-1}\left(\frac{1}{2}-\frac{3}{2}i\right)\right]$$

4)

Tank 1: $\frac{dx_1}{dt} = -\alpha x_1 + u$

Tank 2: $\frac{dx_2}{dt} = \alpha x_1 - \beta x_2$

Given: $\alpha = 0.1$, $\beta = 0.2$, $u = 1$, $x_1(0) = 2$, $x_2(0) = 1$, water level post 5 sec = ?

SS Form: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 \\ \alpha & -\beta \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$, $x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + 0u$$

a) $y(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$, want $y(5) = \ell[\,'\,]$

Need e^{At}

i) Set up CH and solve for eigenvalues

$$\det\left(\begin{bmatrix} -\alpha-\lambda & 0 \\ \alpha & -\beta-\lambda \end{bmatrix}\right) = 0 \quad \text{so} \quad (-\alpha-\lambda)(-\beta-\lambda) = 0$$

$\lambda = -\alpha, -\beta$

ii) Set up $f(\lambda)$ to get C coefficients

$$\left.\begin{aligned} e^{-\alpha t} &= C_1(-\alpha) + C_0 \\ e^{-\beta t} &= C_1(-\beta) + C_0 \end{aligned}\right\} \quad \begin{aligned} e^{-\alpha t} &= -\alpha C_1 + C_0 \quad (1) \\ e^{-\beta t} &= -\beta C_1 + C_0 \quad (2) \end{aligned}$$

① $e^{-0.1t} = -0.1C_1 + C_0$
② $e^{-0.2t} = -0.2C_1 + C_0$

① $C_0 = e^{-0.1t} + 0.1C_1$

$C_0 = e^{-0.1t} - e^{-0.2t} + e^{-0.1t}$

$C_0 = 2e^{-0.1t} - e^{-0.2t}$

② $e^{-0.2t} = -0.2C_1 + e^{-0.1t} + 0.1C_1$

$\frac{e^{-0.2t} - e^{-0.1t}}{-0.1} = C_1$

iii) Apply C coefficients into F(A)

$e^{At} = C_1 A + C_0 I$

$e^{At} = \left(\frac{e^{-0.2t} - e^{-0.1t}}{-0.1}\right) A + (2e^{-0.1t} - e^{-0.2t}) I$

$= \begin{bmatrix} e^{-0.2t} - e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & 2e^{-0.2t} - 2e^{-0.1t} \end{bmatrix} + \begin{bmatrix} 2e^{-0.1t} - e^{-0.1t} & 0 \\ 0 & 2e^{-0.1t} - e^{-0.2t} \end{bmatrix}$

$e^{At} = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix}$

iv) Plug into solution for CTLTI System

$$\begin{bmatrix} x_1(5) \\ x_2(5) \end{bmatrix} = \begin{bmatrix} e^{-0.1t} & 0 \\ e^{-0.1t} - e^{-0.2t} & e^{-0.2t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \int_0^5 \begin{bmatrix} e^{-0.1(t-\tau)} & 0 \\ e^{-0.1(t-\tau)} - e^{-0.2(t-\tau)} & e^{-0.2(t-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 2e^{-0.1t} \\ 2e^{-0.1t} - 2e^{-0.2t} + e^{-0.2t} \end{bmatrix} + \int_0^5 \begin{bmatrix} e^{-0.1(t-\tau)} \\ e^{-0.1(t-\tau)} - e^{-0.2(t-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 2e^{-0.5} \\ 2e^{-0.5} - e^{-1} \end{bmatrix} + \begin{bmatrix} \int_0^5 e^{-0.1(5-\tau)} \\ \int_0^5 e^{-0.1(5-\tau)} - e^{-0.2(5-\tau)} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 1.2131 \\ 0.8452 \end{bmatrix} + \begin{bmatrix} 3.9347 \\ 0.7741 \end{bmatrix}$$

$$= \begin{bmatrix} 5.1478 \\ 1.6193 \end{bmatrix}$$

b) Get Discretized SS Form w/ $T = 1s$

$A_d = e^{AT}$, $B_d = A^{-1}(A_d - I)B$, $C_d = C$, $D_d = D$

$$A_d = e^{AT} = \begin{bmatrix} e^{-0.1T} & 0 \\ e^{-0.1T} - e^{-0.2T} & e^{-0.2T} \end{bmatrix} = \begin{bmatrix} 0.9048 & 0 \\ 0.0861 & 0.8187 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0.9516 \\ 0.0453 \end{bmatrix}$$

$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x(k+1) = \begin{bmatrix} 0.9048 & 0 \\ 0.0861 & 0.8187 \end{bmatrix} x(k) + \begin{bmatrix} 0.9516 \\ 0.0453 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(k)$$

c) $y(5)$ in DT using Zero Order Hold

$$y(k) = CA^k x(0) + \sum_{m=0}^{k-1} CA^{k-m-1} Bu(m) + Du(k)$$

$$y(5) = CA^5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \sum_{m=0}^{4} CA^{4-m} B$$

Need A^k:

i) Use Similarity Transform

$A = \begin{bmatrix} 0.9048 & 0 \\ 0.0861 & 0.8187 \end{bmatrix}$, eigenvalues $= 0.9048, 0.8187$

eigenvectors $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\hat{A} = \begin{bmatrix} 0.9048 & 0 \\ 0 & 0.8187 \end{bmatrix}$, $M^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$A^k = M \hat{A}^k M^{-1}$$

Solve for $y(5)$:

$y[5] = \begin{bmatrix} 5.1472 \\ 1.6190 \end{bmatrix}$ with Zero Order Hold

$y[5] = \begin{bmatrix} 5.2761 \\ 1.5868 \end{bmatrix}$ with Forward Difference

5) $C(s) = \dfrac{K_d s^2 + K_p s + K_i}{s(\tau s + 1)}$

a) Need controllable canonical form

$$\frac{K_d s^2 + K_p s + K_i}{s(\tau s+1)} = \frac{K_d s^2 + K_p s + K_i}{\tau s^2 + s} = \frac{\frac{K_d}{\tau} s^2 + \frac{K_p}{\tau} s + \frac{K_i}{\tau}}{s^2 + \frac{1}{\tau} s}$$

Cont. Canon. TF: $\dfrac{\beta_n z^n + \beta_{n-1} z^{n-1} + \dots + \beta_0}{z^n + \alpha_{n-1} z^{n-1} + \dots + \alpha_0}$, $\dfrac{\beta_2 s^2 + \beta_1 s + \beta_0}{s^2 + \alpha_1 s}$

Matching Coeff: $\beta_2 = K_d/\tau$, $\beta_1 = K_p/\tau$, $\beta_0 = K_i/\tau$, $\alpha_1 = 1/\tau$

Plugging into Cont. Canon. SS.

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_c = \begin{bmatrix} \frac{K_i}{\tau}, & \frac{K_p}{\tau} - \frac{K_d}{\tau^2} \end{bmatrix}, \quad D_c = \frac{K_d}{\tau}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1/\tau \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{K_i}{\tau}, & \frac{K_p}{\tau} - \frac{K_d}{\tau^2} \end{bmatrix} x + \frac{K_d}{\tau} u$$

b) $u = 1$, $K_d = 10$, $K_p = 100$, $K_i = 1$, $\tau = 0.001$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1000 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1000 & -9900000 \end{bmatrix}, \quad D = 10000$$

$$y(t) = \int_{t_0}^{t} C e^{A(t-\tau)} B u(\tau) d\tau + D u(t)$$

Applying Step Input : $H(s) = \frac{1}{s} \cdot \frac{10s^2 + 100s + 1}{s(0.001s+1)} = \dfrac{10s^2 + 100s + 1}{s^2(0.001s+1)}$

Pole with multiplicity of 2 at origin => unbounded