24677-A Homework 5

Saeed Bai

TOTAL POINTS

110 / 110

QUESTION 1

1 Range of a for stability **20 / 20**

- ✓ **- 0 pts** Correct
- **- 1 pts** Incorrect answer

QUESTION 2

Three LTI systems 30 pts

2.1 Part a **10 / 10**

- ✓ **- 0 pts** Correct
- **- 1 pts** Incorrect Lyapunov stability
- **- 1 pts** Incorrect asymptotic stability
- **- 1 pts** Incorrect BIBO stability

2.2 Part b **10 / 10**

- ✓ **- 0 pts** Correct
- **- 1 pts** Incorrect Lyapunov Stability
- **- 1 pts** Incorrect Asymptomatic Stability
- **- 1 pts** Incorrect BIBO Stability

2.3 Part c **10 / 10**

- ✓ **- 0 pts** Correct
- **- 1 pts** Incorrect Lyapunov Stability
- **- 1 pts** Incorrect Asymptotic Stability
- **- 1 pts** Incorrect BIBO Stability

QUESTION 3

3 Lyapunov stability **20 / 20**

- ✓ **- 0 pts** Correct
- **- 10 pts** Not using Lyapunov's Direct Method

QUESTION 4

4 Lyapunov instability **20 / 20**

- ✓ **- 0 pts** Correct
- **- 10 pts** Not using instability criterion

QUESTION 5

Pendulum analysis 20 pts

5.1 Stability of down position **10 / 10**

- ✓ **- 0 pts** Correct
- **- 5 pts** Not using Lyapunov's Direct Method

5.2 Instability of up position **10 / 10**

- ✓ **- 0 pts** Correct

1、

$$\dot{x} = \underbrace{\begin{bmatrix} a & 0 \\ 1 & -1 \end{bmatrix}}_{A} x. \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^T P + PA = \begin{bmatrix} 2a & 1 \\ 1 & -2 \end{bmatrix} \qquad \det \begin{vmatrix} 2a-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0$$

$$(2a-\lambda)(-2-\lambda) - 1 = 0$$

$$-4a - 2a\lambda + 2\lambda + \lambda^2 = 1$$

$$\lambda(\lambda + 2 + 2a) = 4a + 1$$

$$\lambda < 0 \Rightarrow \alpha < -\frac{1}{4}$$

1 Range of a for stability **20 / 20**

✓ **- 0 pts** Correct

- 1 pts Incorrect answer

2. a)

$$x(k+1) = \underbrace{\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}}_{A} x(k) + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{B} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 5 & 5 \end{bmatrix}}_{C} x(k) \qquad D=0$$

$$\det \begin{vmatrix} 1-\lambda & 0 \\ -0.5 & 0.5-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (1-\lambda)(0.5-\lambda)=0 \qquad \lambda_1 = 1 \quad \lambda_2 = 0.5$$

all eigenvalues of A satisfy $|\lambda_i| \le 1$ & are not repeated Lyapunov stable

am & gm are both 1, AUS

$$G(s) = \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0.5 & s-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

BIBO stable

b)

$$\dot{x} = \underbrace{\begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}}_{A} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

$$\det \begin{vmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda^3 - 9\lambda^2 - 23\lambda - 15 = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -3 \quad \lambda_3 = -5$$

all eigenvalues of A have non-positive real parts & are not repeated. Lyapunov stable & AS & BIBO

2.1 Part a 10 / 10

✓ **- 0 pts** **Correct**

- 1 pts Incorrect Lyapunov stability

- 1 pts Incorrect asymptotic stability

- 1 pts Incorrect BIBO stability

2, a)

$$x(k+1) = \overbrace{\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix}}^{A} x(k) + \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}^{B} u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 5 & 5 \end{bmatrix}}_{C} x(k) \qquad D = 0$$

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all eigenvalues of A satisfy $|\lambda_i| \leq 1$ & are not repeated. Lyapanov stable

am & gm are both 1, AUS

$$G(s) = \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} s-1 & 0 \\ 0.5 & s-0.5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

BIBO stable

b)

$$\dot{x} = \underbrace{\begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix}}_{A} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix} x$$

$$\det \begin{vmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad -\lambda^3 - 9\lambda^2 - 23\lambda - 15 = 0$$

$$\lambda_1 = -1, \quad \lambda_2 = -3 \quad \lambda_3 = -5$$

all eigenvalues of A have non-positive real parts & are not repeated. Lyaponuo stable & AS & BIBO

2.2 Part b **10 / 10**

✓ **- 0 pts** **Correct**

- 1 pts Incorrect Lyapunov Stability

- 1 pts Incorrect Asymptomatic Stability

- 1 pts Incorrect BIBO Stability

c) $$A = \begin{bmatrix} 2 & -5 \\ -4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0$$

$$\det \begin{vmatrix} 2-\lambda & -5 \\ -4 & -\lambda \end{vmatrix} = \lambda(\lambda-2) - 20 = 0 \qquad \lambda^2 - 2\lambda - 20 = 0$$

$$\lambda_1 = 5.58 \quad \lambda_2 = -3.58$$

CT: one eigenvalue of A has positive real part

Lyapunov unstable & AUS

$$G(s) = C(sI - A)^{-1}B + D$$

$$= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s-2 & -5 \\ -4 & s \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0$$

$$= \frac{2}{s^2 - 2s - 20} \qquad s^2 - 2s - 20 = 0$$

$$\Rightarrow s_1 = 5.58 \quad s_2 = -3.58$$

BIBO unstable

DT: not all eigenvalues of A satisfy $|\lambda_i| \leq 1$

Lyapunov unstable & AUS

from above : $s^2 - 2s - 20 = 0 \Rightarrow s_1 = 5.58, \ s_2 = -3.58$

BIBO unstable

2.3 Part c 10 / 10

- ✓ **- 0 pts** **Correct**
- **- 1 pts** Incorrect Lyapunov Stability
- **- 1 pts** Incorrect Asymptotic Stability
- **- 1 pts** Incorrect BIBO Stability

3、

$$\dot{x}_1 = x_2 - x_1 x_2^2$$
$$\dot{x}_2 = -x_1^3$$

$$V(x_1, x_2) = x_1^4 + 2x_2^2.$$

$V(x) > 0 \quad \forall \ x \neq 0$

$$\dot{V} = 4x_1^3 \cdot \dot{x}_1 + 4x_2 \cdot \dot{x}_2$$
$$= 4x_1^3 (x_2 - x_1 x_2^2) + 4x_2 (-x_1)^3$$
$$= \cancel{4x_1^3 x_2} - 4x_1^4 x_2^2 - \cancel{4x_2 x_1^3}$$
$$= -4x_1^4 x_2^2 < 0 \quad \forall \ x \neq 0$$

as $x_1, x_2 \to \infty$, $V(x) \to \infty$

Therefore origin is globally, uniformly AS.

3 Lyapunov stability **20 / 20**

- ✓ **- 0 pts** **Correct**
- **- 10 pts** Not using Lyapunov's Direct Method

4.

n $V(x_1, x_2) = x_1^2 - x_2^2$

$$\dot{x}_1 = 3x_1 + x_2^3$$
$$\dot{x}_2 = -x_2 + x_1^2$$

$V(0) = 0$

$V(x_1, x_2) = x_1^2 - x_2^2 = x^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x$

$\dot{V} = 2x_1\dot{x}_1 - 2x_2\dot{x}_2 = 2x_1(3x_1 + x_2^3) - 2x_2(-x_2 + x_1^2)$

$= 6x_1^2 + 2x_1x_2^3 + 2x_2^2 - 2x_2x_1^2 = 3x_1^2 + x_1x_2^3 - x_2x_1^2 + 2x_2^2$

let $x = \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix}$ $V(x) = \varepsilon^2 > 0$

when $x_1, x_2 = 0$ $V(x) = 0$

let $-1 < x_1 < 1$, $-1 < x_2 < 1$

$3x_1^2 + x_1x_2^3 - x_2x_1^2 + 2x_2^2 > 0$

Therefore system is unstable

4 Lyapunov instability 20 / 20

✓ **- 0 pts** **Correct**

- 10 pts Not using instability criterion

5: 20 points

Use instability criterion with the function $V(x_1, x_2) = x_1^2 - x_2^2$ to prove that the origin of the following system is unstable.

$$\dot{x}_1 = 3x_1 + x_2^3$$
$$\dot{x}_2 = -x_2 + x_1^2$$

6: 20 points

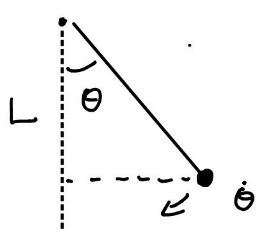
Consider the equation of motion for a simple pendulum

$$\ddot{\theta} + \frac{g}{L}\sin\theta = 0$$

(a) Using the total energy of the system as a Lyapunov function, show that $\theta_0 = 0$ is stable in the sense of Lyapunov.

(b) Using the system energy as a Lyapunov function, show that the equilibrium point $\theta_0 = \pi$ is unstable. For this you will need to use a change of variables $x = \theta - \pi$ to give an equivalent system with $x_0 = 0$ the relevant equilibrium point.

a)



$$V(\theta) = KE + PE = \frac{1}{2}mv^2 + mgL(1-\cos\theta)$$
$$= \frac{1}{2}mL^2\dot{\theta}^2 + mgL(1-\cos\theta)$$
$$\dot{V}(\theta) = mL^2\dot{\theta}\ddot{\theta} + mgL\sin\theta\,\dot{\theta}$$
$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$
$$\dot{V}(\theta) = -mL\dot{\theta}g\sin\theta + mgL\sin\theta\,\dot{\theta}$$
$$= 0 \quad \forall \quad \theta \neq 0$$

$$V(0) = 0$$

$$V(\theta) > 0 \quad \forall \quad x \neq 0$$

Lyapunov stable

5.1 Stability of down position **10 / 10**

✓ **- 0 pts** **Correct**

- 5 pts Not using Lyapunov's Direct Method

b) let $\theta = x_1$ $\quad \dot{\theta} = x_2$

$\dot{x_2} = \frac{g}{L} \sin x_1$ $\quad \dot{\tilde{x}_2} = \frac{g}{L} \sin \tilde{x}_1$

$V(\tilde{x}) = \hat{x}_1 (1 - \sin \tilde{x}_1)$ $\quad V(0) = 0$

$V(x) = \tilde{x_2}(1 + \sin \tilde{x}_1)$

Set $x_1 = \varepsilon > 0$ $\quad V(\varepsilon) > 0$

$\dot{V}(\tilde{x}_1) = \dot{\tilde{x_2}} - \dot{\tilde{x}_2} \sin \tilde{x}_1 + \hat{x}_2 \cos \tilde{x}_1 \dot{\tilde{x}_1}$

$\dot{V} = \dot{\tilde{x}_2} + \dot{\tilde{x}_2} \sin x_1 + \tilde{x}_2^2 \cos \tilde{x}_1 = \frac{g}{L} \sin \tilde{x} + \frac{g}{L} \sin^2 \tilde{x} + \tilde{x}^2 \cos \tilde{x}_1$

close to origin $\quad \dot{V} > 0$

Therefore system unstable

5.2 Instability of up position 10 / 10

✓ - 0 pts Correct