CMPUT 675 - Assignment #3

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Exercise 1: Equality Constraints & Unconstrained Variables

Marks: 3

Recall we noted how an equality constraint can be represented as two inequality constraints (i.e. $x = a$ is equivalent to $x \leq a$ and $-x \leq -a$). Also observe that if some variable x_i does not have a non-negativity constraint then we can introduce two new variables x_i^+, x_i^-, replace x_i throughout the LP with the term $(x_i^+ - x_i^-)$, and add constraints $x_i^+, x_i^- \geq 0$ to get an "equivalent" LP in that a feasible solution for one corresponds naturally to a feasible solution to the other with the same objective function value (take a moment to dwell on this to make sure you understand).

Now consider the following LP.

$$\begin{aligned} \text{maximize}: \quad & 2x_1 + x_2 & & \\ \text{subject to}: \quad & x_1 + 2 \cdot x_2 & = & \quad 5 \\ & 3x_1 + x_2 & \leq & \quad 4 \\ & x_2 & \geq & \quad 0 \end{aligned}$$

Observe the first constraint is actually an *equality* constraint and that the nonnegativity constraint for x_1 is *missing*.

- Use the rules above to rewrite this LP in standard form. That is, **all** constraints should be $\leq$ constraints and **all** variables should be restricted to be nonnegative.

  Solution 1.1: We will have the following LP after introducing the non-negative x_1^+ and x_1^-:

$$\begin{aligned} \text{maximize}: \quad & 2x_1^+ - 2x_1^- + x_2 & & \\ \text{subject to}: \quad & x_1^+ - x_1^- + 2 \cdot x_2 & \leq & \quad 5 \\ & -x_1^+ + x_1^- - 2 \cdot x_2 & \leq & \quad -5 \\ & 3x_1^+ - 3x_1^- + x_2 & \leq & \quad 4 \\ & x_1^+, x_1^-, x_2 & \geq & \quad 0 \end{aligned}$$

  This primal LP has the standard form $\max\{c^T x : Ax \leq b, x \geq 0\}$ if we consider $c = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$,

$$x = \begin{bmatrix} x_1^+ \\ x_1^- \\ x_2 \end{bmatrix},\ b = \begin{bmatrix} 5 \\ -5 \\ 4 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 3 & -3 & 1 \end{bmatrix}.$$

- Then write the dual of the LP you obtained in last part using the simple rule from the lectures:

$$\max\{c^T x : Ax \leq b, x \geq 0\} \Longrightarrow \min\{b^T y : A^T y \geq c, y \geq 0\}.$$

Solution 1.2: The dual will have the following form:

$$\begin{aligned} \text{minimize}: \quad & 5y_1^+ - 5y_1^- + 4y_2 & & \\ \text{subject to}: \quad & y_1^+ - y_1^- + 3 \cdot y_2 & \geq & \ 2 \\ & -y_1^+ + y_1^- - 3 \cdot y_2 & \geq & \ -2 \\ & 2y_1^+ - 2y_1^- + y_2 & \geq & \ 1 \\ & y_1^+, y_1^-, y_2 & \geq & \ 0 \end{aligned}$$

- Observe how the dual constraints for the variables that replaced x_1 can be replaced with a single equality constraint and that the dual variables for the two primal constraints that replaced the equality constraint can be consolidated into one variable that may take negative values (i.e. we don't have the nonnegativity constraint for this variable). Write this new dual LP after performing these simplifications. It should have 2 constraints and 2 variables.

Solution 1.3: By introducing $y_1 = y_1^+ - y_1^-$, we can reach to the following simplified dual form:

$$\begin{aligned} \text{minimize}: \quad & 5y_1 + 4y_2 & & \\ \text{subject to}: \quad & y_1 + 3 \cdot y_2 & = & \ 2 \\ & 2y_1 + y_2 & \geq & \ 1 \\ & y_2 & \geq & \ 0 \end{aligned}$$

- Finally, describe a general rule for handling nonnegativity constraints and unconstrained variables when computing the dual of an LP that allows us to skip reducing it to standard form. More precisely, consider an LP of the form:

$$\max\left\{c^T x + (c')^T x' : A\begin{pmatrix} x \\ x' \end{pmatrix} \leq b, A'\begin{pmatrix} x \\ x' \end{pmatrix} = b', x \geq 0\right\}.$$

Here, $c \in \mathbb{R}^n, c' \in \mathbb{R}^{n'}, A \in \mathbb{R}^{m \times (n+n')}, A' \in \mathbb{R}^{m' \times (n+n')}, b \in \mathbb{R}^m, b' \in \mathbb{R}^{m'}$. Also, x is a collection of n variables and x' is a collection of n' variables. That is, n variables are nonnegative and n' are not constrained to be nonnegative, m constraints are inequalities ($\leq$) and m' are equalities.

Describe the dual of this LP using exactly $m + m'$ variables and $n + n'$ constraints (apart from the nonnegativity constraints that will apply to some dual variables).

You don't need to prove why this is the dual if you are confident you got it right, I just want to see that you know how it is done. If you want to provide an argument, it can be at a high level (i.e. I am not going to grade the proof, just the result but it might help if you explain your approach if there is a mistake).

Solution 1.4: Generally, if x_j can be negative, then the dual will have the equality constraint $A_j^T \cdot y = c_j$. Conversely, if we have the equality constraint $A_i \cdot x = b_i$ in the prime LP, then y_i corresponding to the row i of the constraint matrix can be negative in the dual problem.

If we assume $A = (A_x | A_{x'})$ where $A_x \in \mathbb{R}^{m \times n}$ and $A_{x'} \in \mathbb{R}^{m \times n'}$, and $A' = (A'_x | A'_{x'})$ where $A'_x \in \mathbb{R}^{m' \times n}$ and $A'_{x'} \in \mathbb{R}^{m' \times n'}$, then the dual will have the following form (easily proved using block matrices):

$$\min \left\{ b^T y + (b')^T y' : (\frac{A_x}{A'_x})^T \begin{pmatrix} y \\ y' \end{pmatrix} \geq c, (\frac{A_{x'}}{A'_{x'}})^T \begin{pmatrix} y \\ y' \end{pmatrix} = c', y \geq 0 \right\}.$$

The $y \in \mathbb{R}^m$ is non-negative, but $y' \in \mathbb{R}^{m'}$ can be negative.

Exercise 2: Routing Demands in a Tree

Marks: 4

In the Demand-Routing in a Tree problem, we are given a tree $\mathcal{T} = (V, E)$ and a collection of paths $D = \{P_1, P_2, \ldots, P_k\}$ in $\mathcal{T}$ (think of each as a demand for the edges lying between the endpoints). Each path P_i spans at least one edge. A subset $D' \subseteq D$ is said to be **routable** if no edge $e \in E$ lies on more than one path in D'.

Give a polynomial-time algorithm for finding a routable subset $D' \subseteq D$ of maximum size. For simplicity, you may assume each path P_i has both endpoints being leaves in $\mathcal{T}$ and each leaf is the endpoint of precisely one path (this is without loss of generality, there is a simple reduction from the general problem to this more structured one but you don't have to show it).

I will allow you to use the fact that Maximum-Weight Matching in a general graph can be solved in polynomial time. We will eventually see this.

Hint: Imagine the tree is rooted at some arbitrarily chosen vertex r. Use dynamic programming over subtrees: what do you have to remember about how a solution within a subtree can "interact" with a solution outside of a subtree?

Bonus (0.5 marks)
Devise an algorithm that does not rely on Maximum-Weight Matching. It can still use the algorithm for unweighted matching we saw in the lectures.

Solution 2: N/A

Exercise 3: Routing Demands in a Bi-Coloured Tree

Marks: 5

Consider the problem from Exercise 1 but with the following modifications.

Now there is a root node $r \in V$ and every subtree rooted under r is labelled either **blue** or **red**. Each edge $e \in E$ has a capacity $\mu_e \geq 0$ indicating how many demands can be chosen across this edge. Further, for each path P_i, either P_i only contains nodes from one subtree rooted under r (plus, perhaps, r itself) or the two subtrees that P_i has nodes in are labelled differently (one red, one blue). Finally, suppose each P_i has a value $v_i \geq 0$. It might help if you quickly sketch out an example.

Consider the problem of selecting the maximum value collection of paths D' subject to the constraint that across every edge e, the number of pairs in D' whose paths cross e is at most μ_e. A single path P_i can be selected multiple times (i.e. D' can be a multiset of paths from D).

- Cast problem this as a simple linear program where you have a single variable x_i for each path P_i. Prove the constraint matrix is totally unimodular. You can do this by describing the network matrix.

  Hint: The input sort of already gives you a hint as to what the network matrix could look like. A quick sketch/figure of your ideas could help explain your solution.

  Solution 3.1: Let x_i be the number of times the path P_i appears in the collection D'. We define the constraint matrix $A \in \mathbb{R}^{|E| \times k}$ such that $A_{i,j} = 1$ if $e_i \in P_j$, and zero otherwise. Considering the original graph, A is a network matrix by directing all edges from source to target endpoints given a path. If an edge gets a new alternating direction from a different (source,target) pair, we can imagine a new (target, source) pair which is reversed for that path, then the corresponding entry in the network matrix can still stay $+1$ (All other edges of the graph not part of any path can be directed arbitrarily). Therefore, A is TUM. Let's define the row vectors $\mu^T = (\mu_{e_1}, \mu_{e_2}, ..., \mu_{e_i}, ..., \mu_{e_{|E|}})$ and $xe^T = (x_1, x_2, ..., x_i, ..., x_k)$. The LP becomes $\max\{\sum_{i=1}^{k} v_i * xe_i : A \cdot xe \leq \mu, xe \geq 0\}$.

- Write the dual of this linear programming relaxation.

  Tip: Deriving the dual of a linear program whose constraints are presented symbolically can be challenging if you are not experienced at it. Do it explicitly for a small example and then look for the pattern so you can write the dual of an arbitrary instance symbolically.

  Solution 3.2: For the dual, we can consider the variable y_i over the edge i and the row vector $v^T = (v_1, v_2, ..., v_i, ..., v_k)$. The dual becomes $\min\{\sum_{i=1}^{|E|} \mu_i * y_i : A^T \cdot y \geq v, y_i \geq 0\}$.

- Now suppose $\mu_e = 1$ for each $e \in E$ and $v_i = 1$ for each $P_i \in D$ (i.e. the setting from Problem 2 again, except with the new red/blue structure). Obviously your solution to the first part means the dual constraint matrix is totally unimodular. Using LP duality, write a max/min theorem for this problem in terms of its combinatorial structure relating the maximum number of paths that can be chosen to the minimum of some combinatorial problem you identify using duality.

  That is, write the max/min theorem for the case $\mu_e = 1$ and $v_i = 1$ in a natural way that can be understood by anyone who does not know any linear programming.

  Solution 3.3: The dual LP finds the minimum number of edges that if get congested, can block flows from all sources to all targets.

Food for thought (no marks, just for fun): Describe a linear-time algorithm for the case where each path is entirely contained in a single subtree rooted under r. Here, you may assume the input is the adjacency-list representation of the tree $\mathcal{T}$, the root node r, and just the list of endpoints (s_i, t_i) of the paths P_i.

Exercise 4: Polytopes in $[0,1]^n$

Marks: 4

For this exercise, fix $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$ and suppose $\mathcal{P} \subseteq [0,1]^n$ (i.e. the two bounds $0 \leq x_i \leq 1$ appear among all constraints for each i). Frequently, combinatorial optimization problems that are cast as linear programs will satisfy these $[0,1]$ constraints, eg. when the problem is to decide what subset of items to pick.

- Show every point in $\mathcal{P} \cap \mathbb{Z}^n$ is an extreme point of $\mathcal{P}$.

  Solution 4.1: Assume $y \in \mathcal{P} \cap \mathbb{Z}^n$. Let $1 \leq \alpha \leq n$ be the dimension such that $y_\alpha = 0$ and $1 \leq \beta \leq n$ be the dimension such that $y_\beta = 1$. If y is not an extreme point, then $\exists z \neq 0 \in \mathbb{R}^n$ such that $y + z$ and $y - z$ are both feasible. We would have $0 \leq y_\alpha + z_\alpha \leq 1$ and $0 \leq y_\alpha - z_\alpha \leq 1$. Hence, $0 \leq z_\alpha \leq 1$ and $-1 \leq z_\alpha \leq 0$. Therefore, $z_\alpha = 0$. Similarly, we would have $0 \leq y_\beta + z_\beta \leq 1$ and $0 \leq y_\beta - z_\beta \leq 1$. Hence, $-1 \leq z_\beta \leq 0$ and $0 \leq z_\beta \leq 1$. Therefore, $z_\beta = 0$. We would have $z = 0$ which is a contradiction. Therefore, y is an extreme point.

- Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k} \in \mathbb{R}^n$. A **convex combination** of these points is a point $\mathbf{p}$ of the form

  $$\mathbf{p} = \sum_{i=1}^{n} \lambda_i \cdot \mathbf{x_i}$$

  where the λ_i values are nonnegative and $\sum_{i=1}^{n} \lambda_i = 1$.

  Show that any $\overline{\mathbf{x}} \in \mathcal{P}$ is a convex combination of the extreme points of $\mathcal{P}$. For full credit, show that $\mathbf{x}$ is a convex combination of at most $n+1$ extreme points of $\mathcal{P}$.
  Hint: use induction on the number of constraints defining $\mathcal{P}$ that hold with equality under $\overline{\mathbf{x}}$.

  Note: This will hold for general polytopes that are bounded (i.e. contained in some n-dimensional box) and does not require the specific $[0,1]$ constraints.

  Solution 4.2: Assume $\overline{\mathbf{x_k}} \in \mathcal{P}$ such that the rank of the matrix corresponding to the equality constraints of $\overline{\mathbf{x_k}}$ is k (i.e. $\mathbf{A}^{'} \cdot \overline{\mathbf{x_k}} = \mathbf{b}^{'}$ and rank$(\mathbf{A}^{'}) = k$). We prove the claim by the induction over k where the base case is at $k = n$. The claim holds for the base case $k = n$ since rank$(\mathbf{A}^{'})$ $= n$ and $\overline{\mathbf{x}}_n$ is already an extreme point.

  Assume the claim now holds for $k = l < n$ and we need to prove it for $k = l - 1$. if $\overline{\mathbf{x}}_{l-1} \in \mathcal{P}$, $\mathbf{A}^{'} \cdot \overline{\mathbf{x}}_{l-1} = \mathbf{b}^{'}$ and rank$(\mathbf{A}^{'}) = l - 1$, then $\exists z \neq 0 \in \mathbb{R}^n$ such that $\mathbf{A}^{'} \cdot z = 0$. Consider $u_{\alpha_j} = \frac{1 - \overline{\mathbf{x}}_{l-1,j}}{z_j}$ (u_{α_j} makes the dimension j of $\overline{\mathbf{x}}_{l-1} + u_{\alpha_j} z$ to be 1). Similarly, define $d_{\alpha_j} = \frac{-\overline{\mathbf{x}}_{l-1,j}}{z_j}$ (d_{α_j} makes the dimension j of $\overline{\mathbf{x}}_{l-1} + d_{\alpha_j} z$ to be 0). Let $u = \min_j u_{\alpha_j}$ ($u > 0$ and $z_j \neq 0$) and $d = \max_j d_{\alpha_j}$ ($d < 0$ and $z_j \neq 0$). For every $d \leq \alpha \leq u$, we have $\overline{\mathbf{x}}_{l-1} + \alpha z \in \mathcal{P}$ and $\mathbf{A}^{'} \cdot (\overline{\mathbf{x}}_{l-1} + \alpha z) = \mathbf{b}^{'}$. Now consider $y_d = \overline{\mathbf{x}}_{l-1} + dz$ and $y_u = \overline{\mathbf{x}}_{l-1} + uz$. We have $\overline{\mathbf{x}}_{l-1} = \frac{u}{u-d} y_d + \frac{-d}{u-d} y_u$. Therefore, $\overline{\mathbf{x}}_{l-1}$ is a convex combination of y_d and y_u. By construction, y_d and y_u both have one extra tight dimension and $\mathbf{A}^{'} \cdot y_d = \mathbf{b}^{'}$ and $\mathbf{A}^{'} \cdot y_u = \mathbf{b}^{'}$, therefore it has one more equality constraint compared to $\overline{\mathbf{x}}_{l-1}$. So the rank of the matrix corresponding to the equality constraints for y_d and y_u must be greater than $l - 1$. By the assumption of induction, both y_d and y_u are now convex combinations of the extreme points of $\mathcal{P}$: $y_d = \sum_k \lambda_k x_k^*$ and $y_u = \sum_k \lambda_k^{'} x_k^*$. Hence, $\overline{\mathbf{x}}_{l-1} = \sum_k (\frac{u}{u-d} \lambda_k + \frac{-d}{u-d} \lambda_k^{'}) x_k^*$. It is clear that

$\sum_k \frac{u}{u-d}\lambda_k + \frac{-d}{u-d}\lambda_k^{'} = \sum_k \frac{u}{u-d}\lambda_k + \sum_k \frac{-d}{u-d}\lambda_k^{'} = 1$. Therefore, $\overline{\mathbf{x}}_{l-1}$ is a convex combination of the extreme points and the claim holds for all values of k.

- Show if any $\overline{\mathbf{x}} \in \mathcal{P}$ can be written as a convex combination of points in $\mathcal{P} \cap \mathbb{Z}^n$, then every extreme point of $\mathcal{P}$ is integral.

  Solution 4.3: N/A

Comment: The last part will be used when we discuss perfect graphs.

Exercise 5: The Matching Polytope

Marks: 2

Let $\mathcal{G} = (V; E)$ be a graph. Let $\mathbf{x}_e, e \in E$ be a collection of variables, one per edge. Consider the polytope

$$\mathcal{P} = \{\mathbf{x}_e \in \mathbb{R}^E : \mathbf{x}(\delta(v)) \leq 1 \text{ for each } v \in V, \mathbf{x} \geq 0\}.$$

You already saw in the lectures that if $\mathcal{G}$ is bipartite then this polytope is integral due to total unimodularity.

Show the converse: if $\mathcal{G}$ is not bipartite then there is some extreme point of $\mathcal{P}$ that is not integral. You will get partial credit for simply describing such an extreme point even if you cannot prove it is actually an extreme point.

Solution 5: A counter example is when $\mathcal{G}$ is the triangle graph and (0.5, 0.5, 0.5) is a non-integral extreme point for the triangle.

Let's define the node-edge incident matrix similar to the bipartite case and assume the constraints of LP is in the standard form of $A \cdot X_e \leq 1$. Consider $A^{'}$ with rank $|E|$ corresponding to the equality constraints: $A^{'} \cdot X_e = 1$. Let's construct the square matrix $A^{''}$ with the rank $|E|$ from $A^{'}$. To compute the $det(A^{''})$, we first perform the Laplace (determinant) expansion along the rows with a single 1 entries. In the remaining submatrix of the expansion, every row has at least two 1 entries. The nodes corresponding to these rows in the submatrix form a graph with a degree of at least 2 for every node. Hence, the graph must have an odd cycle since the graph is not bipartite (the nodes with degree 1 in the original graph $\mathcal{G}$ doesn't invalidate the argument).

Consider the columns in the submatrix corresponding to the edges along this odd cycle. Let's name these columns $c_1, c_2, ..., c_{2k+1}$. By changing rows in the submatrix and multiplying the determinant by -1 every time, we can assume that in column c_i, the dimension i and $i+1$ have one entries. We can perform the following column operation to replace $c_1 \leftarrow c_1 - c_2 + c_3 - c_4 + ... + c_{2k+1}$ ($c_1 = c_{2k+1}$). This operation does not change the determinant. The new c_1 will be $2e_1$. We can now continue the expansion along c_1 and this will introduce a factor of 2 in the determinant. We can conclude that $det(A^{''})$ will have a factor of 2 and $det(A^{''}) \neq \pm 1$ (we know that the determinant of an integer matrix is always integer).

Let's now consider the square matrix $A^{''}(c_1, 1)$ where the column c_1 (with the original values) is replaced with a vector of ones. We can still change rows and multiply the determinant by -1 to form c_i $(1 < i)$ to have ones in the dimension i and $i+1$ (same columns corresponding to the edges along the odd cycle). Before starting the Laplace expansion by expanding along the rows with a single one entry, we can perform the following column operation on c_1: $c_1 \leftarrow c_1 - c_2 - c_4 - ... - c_{2k}$. The new c_1 will be e_1. If we start the Laplace expansion similar to the previous part and decide to expand along the new column c_1, then we won't multiply the determinant by 2. We can conclude that $det(A^{''})$ will have an extra factor of 2 compared to $det(A^{''}(c_1, 1))$. According to the Cramer's rule, the following dimension of the extreme point will not be integral: $X_{index(c_1)} = \frac{det(A^{''}(c_1,1))}{det(A^{''})}$.

Exercise 6: Unboundedness Certification

Marks: 2

You have seen how to certify the following for an LP of the form $\max\{c^T x : Ax \leq b, x \geq 0\}$ with n variables and m constraints (apart from nonnegativity).

- **Optimal Solution**: To certify a given feasible solution $\overline{x}$ is an optimal solution, you just need to produce a feasible dual solution $\overline{y}$ with $b^T\overline{y} = c^T\overline{x}$ (one exists by strong duality).
- **Infeasibility**: By Farkas' lemma, $\{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$ is empty if and only if there is some $\overline{y} \in \mathbb{R}^m$ such that $A^T \cdot \overline{y} \geq 0, b^T\overline{y} < 0$ and $\overline{y} \geq 0$.

But what about unboundedness?

Prove that the LP $\max\{c^T x : Ax \leq b, x \geq 0\}$ is unbounded if and only if there exists a feasible solution $\overline{x}$ plus a vector $\overline{z} \in \mathbb{R}^n$ such that: $c^T\overline{z} > 0, A\overline{z} \leq 0$, and $\overline{z} \geq 0$.

In this way, to convince someone the LP is unbounded you simply have to provide a feasible solution $\overline{x}$ and a vector $\overline{z}$ satisfying the conditions above.

Solution 6: Let's first prove that if the LP is unbounded, then there exists a vector $\overline{z} \in \mathbb{R}^n$ such that: $c^T\overline{z} > 0, A\overline{z} \leq 0$, and $\overline{z} \geq 0$. Since the LP is unbounded, then it has a feasible solution by definition (otherwise it is infeasible). We know that the dual will be infeasible. Therefore, $\min\{b^T y : A^T y \geq c, y \geq 0\}$ is infeasible. Therefore, the set $\{y \in \mathbb{R}^m : -A^T y \leq -c, y \geq 0\}$ is empty. According to the Farkas' lemma, there exists $\overline{z} \in \mathbb{R}^n$ such that: $-c^T\overline{z} < 0, (-A^T)^T\overline{z} \geq 0$, and $\overline{z} \geq 0$. Therefore, there exists $\overline{z} \in \mathbb{R}^n$ such that: $c^T\overline{z} > 0, A\overline{z} \leq 0$, and $\overline{z} \geq 0$.

To prove the other direction, assume there exists a feasible solution x and a vector $\overline{z} \in \mathbb{R}^n$ such that: $c^T\overline{z} > 0, A\overline{z} \leq 0$, and $\overline{z} \geq 0$. It is clear that $\forall t > 0 : A \cdot (x + t\overline{z}) = A \cdot x + tA \cdot \overline{z} \leq b$ as $tA \cdot \overline{z} \leq 0$. Hence, $x + t\overline{z}$ is also feasible ($x + t\overline{z}$ is also non-negative). Let's assume the opposite is correct and the LP is bounded, therefore, there exists $\lambda \in \mathbb{R}$ such that $c^T x^{'} < \lambda$ for every feasible $x^{'}$. We can pick $t^* \geq \frac{\lambda - c^T x}{c^T\overline{z}} > 0$ to have $c^T(x + t^*\overline{z}) \geq \lambda$. Therefore, the feasible solution $x + t^*\overline{z}$ contradicts the assumption that the LP is bounded.