CMPUT 675 - Assignment #5
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Pages: 8

Exercise 1 - Filling in the gap

Marks: 2

Recall we left one step as an exercise when proving a rank function with certain properties yields a matroid. Here, you fill in the gap.

Let X be a finite set and let $r : 2^X \to \mathbb{Z}_{\geq 0}$ be a function with the following properties:

- $r(Y) \leq |Y|$ for all $Y \subseteq X$,
- $r(Y) \leq r(Z)$ for all $Y \subseteq Z \subseteq X$,
- $r(Y \cup Z) + r(Y \cap Z) \leq r(Y) + r(Z)$ for all $Y, Z \subseteq X$.

Now consider two sets $A, B \subseteq X$. Suppose $r(A + x) = r(A)$ for each $x \in B - A$. Show $r(B \cup A) = r(A)$.

Solution 1:

Let's assume $B - A = \{x_1, x_2, ..., x_k\}$ for $k \geq 1$. By repeated application of the third feature of the rank functions, we have $k \times r(A) = r(A \cup \{x_1\}) + r(A \cup \{x_2\}) + ... + r(A \cup \{x_k\}) \geq r((A \cup \{x_1\}) \cap (A \cup \{x_2\})) + r((A \cup \{x_1\}) \cup (A \cup \{x_2\})) + ... + r(A \cup \{x_k\}) \geq r(A) + r(A \cup \{x_1, x_2\}) + r(A \cup \{x_3\}) + ... + r(A \cup \{x_k\}) \geq ... \geq (k-1) \times r(A) + r(A \cup \{x_1, x_2, ..., x_k\}) = (k-1) \times r(A) + r(A \cup B)$. Hence, $r(A) \geq r(A \cup B)$. From the second feature of the rank functions, we also know that $r(A) \leq r(A \cup B)$. We can conclude that $r(A \cup B) = r(A)$.

Exercise 2 - A 2-Player Game

Marks: 5

Let $\mathcal{G} = (V, E)$ be a connected graph. Consider the following 2-player game. Intuitively, players take turns "buying" edges: once an edge is bought it is removed from the graph. Player 1 wins if the set of edges they bought includes a spanning tree (i.e. keeps the graph connected), player 1 wins otherwise.

More precisely, players alternate turns where each player $i \in \{1, 2\}$ maintains some $F_i \subseteq E$ where $F_1 \cap F_2 = \emptyset$. Initially both F_1 and F_2 are empty. Player 1 plays first.

A play by player i is simply adding some edge of $E - (F_1 \cup F_2)$ to F_i. Once $F_1 \cup F_2 = E$, the game ends. Player 1 wins if (V, F_2) is disconnected, player 2 wins if (V, F_2) is connected.

Here you will show player 2 has a winning strategy if and only if there are two edge-disjoint spanning trees. To get you started, note that the specialization of the matroid partition min/max theorem proven in class shows $\mathcal{G}$ contains two edge-disjoint spanning trees if and only if

$$2 \cdot (|V| - 1) = \min_{F \subseteq E} |E - F| + 2 \cdot r(F).$$

- Show that if $\mathcal{G}$ has two edge-disjoint spanning trees then player 2 can ensure the following holds after each of their moves (no matter how player 1 played in their turn). There are two spanning trees T_1, T_2 such that $T_1 \cap T_2 = F_2$ and both $T_1 - F_2, T_2 - F_2 \subseteq E - F_1$ (i.e. are not yet grabbed by any player). Conclude player 2 has a winning strategy if and only if there are two edge-disjoint spanning trees in $\mathcal{G}$.

  Tip: One can solve this using generic matroid exchange arguments.

  Solution 2.1: When $F_1 = \emptyset$, we know that $E - F_1$ has two edge-disjoint spanning trees T_1 and T_2. It is clear that $F_2 = \emptyset = T_1 \cap T_2$.

  If player 1 picks an edge e from T_1 for example, we know that e will divide the nodes into two parts of A and B according to the edges in T_1. However, there is another edge $e^{'}$ from T_2 that connects A and B. Therefore, we can define $T_1^{'} = T_1 - e + e^{'}$ and $T_1^{'}$ is a new spanning tree with $T_1^{'} \cap T_2 = \{e^{'}\}$. Therefore, player 2 can pick $e^{'}$ and $F_2 = \{e^{'}\} = T_1^{'} \cap T_2$. ($T_1^{'}$ will become T_1).

  If player 1 picks a node from T_2, a similar process will define a new $T_2^{'}$ and player 2 can pick the new edge added to $T_1 \cap T_2^{'}$.

  If player 1 picks other nodes from $E - F_1 - (T_1 \cup T_2)$, then player 2 can still assume the edge e has been picked from T_1 and can define a new spanning tree $T_1^{'}$ and continue picking from the new edge added to $T_1^{'} \cap T_2$. Player 2 can pick the assumed edge e in future rounds if e has not been picked by the player 1. Picking e does not disconnect the tree in F_2. Therefore, F_2 always stays connected and the player 2 will win the game.

- For a partition $\pi \subseteq 2^V$ of V into $|\pi|$ parts let $\partial(\pi)$ be all edges in E that have endpoints in different parts. Show $\mathcal{G}$ has two edge-disjoint spanning trees if and only if $|\partial(\pi)| \geq 2(|\pi| - 1)$ for all partitions π of V.

  Hint: For the "harder" direction, start by showing the minimum of the expression in the min/max relationship recalled above is achieved at some set F where $E - F$ is of the form $\partial(\pi)$ for some partition π.

  If You Prefer: You can prove a slight generalization instead. For an integer $k \geq 1$, a graph $\mathcal{G}$ has k edge-disjoint spanning trees if and only if $|\partial(\pi)| \geq k(|\pi| - 1)$ for all partitions π of V. It is not any harder than the case $k = 2$ (at least my solution generalizes to arbitrary k immediately).

  Solution 2.2: It is clear that if the graph has $k \geq 1$ edge-disjoint spanning trees, then by contracting the nodes within a part, $|\pi|$ different parts of nodes must also be connected via k edge-disjoint spanning trees. This means the number of edges between the contracted nodes must be at least $k \times (|\pi| - 1)$. Hence, $|\partial(\pi)| \geq k(|\pi| - 1)$.

If for every partition π, we have $|\partial(\pi)| \geq k(|\pi| - 1)$, then we can consider a partition of size $|\pi| = |V|$, and we have $|\partial(\pi)| = |E| \geq k \times (|V| - 1)$. If we consider $F = \emptyset$, then in the RHS of the min-max theorem for matroid partitioning, $RHS = |E| + k \times r(\emptyset) = |E| \geq k \times (|V| - 1)$. Since according to the assumption of the problem, the graph is connected, then it must have a spanning tree, so if we set $F = E$, then we have $RHS = 0 + k \times r(E) = k \times (|V| - 1)$. To further prove that RHS has its minimum value at $F = E$, we must verify that the RHS is a concave function. To minimize RHS, F, as a subset of edges, can form multiple complete graphs of the same size, therefore $r(F)$ equals the size of a spanning tree in each of the complete graphs. So if F forms S complete graphs of the same size, we have $|F| = (S - 1) + S \times \frac{n \times (n-1)}{2}$ and $r(F) = n - 1$. Therefore, we have $RHS \geq J = |E| - (S - 1) - S \times \frac{n \times (n-1)}{2} + k \times (n - 1)$. The second derivative of J with respect to $|F|$ is negative, and this makes J a concave function with respect to $|F|$. Hence, the minimum values of J are achieved at the endpoints of $F = E$ or $F = \emptyset$. As the min-max theorem holds for a subset of edges (LHS = RHS for $F = E$), we know that there are k edge-disjoint spanning trees in the graph.

- Show that if $\mathcal{G}$ does not have two edge-disjoint spanning trees then player 1 can ensure player 2 does not win. The previous part might be helpful.

  Solution 2.3: We know that there is a partition of the nodes π^* such that $|\partial(\pi^*)| < 2(|\pi^*| - 1)$. Player 2 must keep the parts connected, but as the player 1 starts the game and knows about this specific partition, it can start picking as many edges as possible from $|\partial(\pi^*)|$. Hence, player 2 cannot keep the parts connected and F_2 will be disconnected at the end.

Note, players 1 and 2 can determine their winning move (if they have one) in polynomial time. Think about it a bit if you do not see it right away: this is not for marks, I am simply pointing it out if you are curious.

Generalization: For your interest (not assigned work), this generalizes readily to prove that if the game is played in an arbitrary matroid and the goal of player 2 is to ensure their items includes a base, then player 2 has a winning strategy if and only if there are two disjoint bases in the matroid.

A much more challenging variant is to have the goal of player 2 be to construct an $s-t$ path (sometimes called the edge version of Hex). Winning strategies are possible to determine in polynomial time, but it is a bit more challenging than this exercise.

Exercise 3 - Not Quite a Graphic Matroid

Marks: 2
Let $\mathcal{G} = (V, E)$ be an undirected graph. Let $\mathcal{I} = \{F \subseteq E : \text{each component of } (V, F) \text{ has at most one cycle}\}$. Show $\mathcal{M} = (E, \mathcal{I})$ is a matroid.

A direct proof is possible (though tedious). A more elegant proof shows that this supposed matroid is can be viewed as a type of matroid we already discussed in the class.

Thought Exercise (again, not for marks): Consider how to find a minimum-weight base of this matroid when given the graph $\mathcal{G}$ as input in $O(m \log m)$ time where $m = |E|$.

Solution 3: We provide a direct verbal proof. It is clear that in an empty set of edges, each component has at most one cycle. Hence, $\emptyset \in \mathcal{I}$. If $A \in \mathcal{I}$, then for every subset of edges $B \subseteq A$, each component of B still has at most one cycle as we have fewer edges in B. This means B has fewer cycles or more isolated connected components. Therefore, $B \in \mathcal{I}$.

If $A, B \in \mathcal{I}$ and $|A| < |B|$, if B and A have the same number of connected components, then B must have extra edge forming a cycle in some spanning tree compared to A. Therefore, A can also take that extra edge to have a connected component with at most one cycle. If B has fewer connected components compared to A, then similar to the graphic matroid, we can find an edge in B that connects two separate connected components of A. Such edge can be added to A and still be an independent set. We can conclude that there is $x \in B - A$ such that $A + x \in \mathcal{I}$.

Exercise 4 - Representations of Matroids

Marks: 2

A common question asked in matroid theory is whether a matroid is *representable* over a field $\mathbb{F}$. That is, given a matroid $\mathcal{M} = (X, \mathcal{I})$ is there some matrix $\mathbf{A}$ over $\mathbb{F}$ whose columns are indexed by X such that $Y \in \mathcal{I}$ if and only if the columns of $\mathbf{A}$ indexed by Y are linearly independent?

In this question, you focus on the graphic matroid. That is, let $\mathcal{M} = (E, \mathcal{I})$ be a graphic matroid over a graph $\mathcal{G} = (V, E)$. Pick an arbitrary direction for each edge $e \in E$. Let $\mathbf{A} \in \mathbb{R}^{V \times E}$ (rows $\equiv$ vertices, columns $\equiv$ edges) such that for each column $e = (u, v) \in E$ we have $\mathbf{A}_{v,e} = 1$ and $\mathbf{A}_{u,e} = -1$, and $\mathbf{A}_{w,e} = 0$ for $w \notin \{u, v\}$. Show that $\mathcal{M} = (E, \mathcal{I})$ is the same as the vector matroid given by the columns of $\mathbf{A}$.

Tip: Draw a small example (with 5 or 6 edges) and write the matrix explicitly to get some intuition on why acyclic subsets of edges $\equiv$ linearly independent subsets of columns.

Lots of Interesting Notes: You can maybe see that $\mathbf{A}$ is totally unimodular (if not, think about it). The class of matroids representable over $\mathbb{R}$ using totally unmodular matrices is, interestingly, the class of matroids representable over *all* fields. These are called **regular** matroids. One can show the *dual* of the cographic matroid ($F \subseteq E$ is independent iff $E - F$ contains a spanning tree) is also a regular matroid. All regular matroids decompose in some appropriate sense (details omitted) into a product of graphic matroids, cographic matroids, and copies of one particular fixed 10-element matroid that is neither graphic nor cographic. Such a decomposition can be done efficiently (given the TUM matrix representation of the matroid) and leads to a polynomial-time algorithm for deciding if a given matrix is TUM.

Solution 4: We will prove two claims, if there is a subset of edges $Y \subseteq E$ forming a cycle, then the columns corresponding to the edges along this cycle will be linearly dependent. The second claim is to prove that the column vectors corresponding to the acyclic subset of edges will be linearly independent.

To prove the first claim, imagine a cycle and name the nodes along the cycle as $\{v_1, v_2, v_3, ..., v_k, v_1\}$. We can modify the original direction of the edges along the cycle such that we have a new direction from v_i to v_{i+1}. This corresponds to multiplying the original vector columns by -1, if the original direction is from v_{i+1} to v_i. In this new directed cycle $\{e_1 = v_1v_2, e_2 = v_2v_3, ..., e_{k-1} = v_{k-1}v_k, e_k = v_kv_1\}$, the column vectors sum to 0: $A_{v_1v_2} + A_{v_2v_3} + ... + A_{v_kv_1} = 0$, where $A_{u,v}$ is the modified column vector for the edge uv. This means that the original column vectors corresponding to the edges along the cycle has a linear combination with weights +1 or -1 such that they sum to zero. Therefore, that set of column vectors become linearly dependent.

To prove the second claim, if we have a set of acyclic edges $\{e_1, ..., e_k\}$, consider the node v with degree 1. Since we don't have cycles, such node exits. The vector column corresponding to the edge incident to v will have +1 or -1 in the dimension for node v in the space $\mathbb{R}^V$. No linear combination of the vector columns of the edges $\{e_1, ..., e_k\}$ can cancel out that +1 or -1 value as there are no other edges incident to v and the rest of column vectors have zero value in that dimension. This shows that no linear combination of the column vectors for these edges can have non-zero weights summing to the zero vector. Hence, the vector columns are linear independent.

Exercise 5 - Exploring Matroid Duality

Marks: 3

Let $\mathcal{M} = (X, \mathcal{I})$. The **dual** of $\mathcal{M}$ is the matroid $\mathcal{M}^* = (X, \mathcal{I}^*)$ where $\mathcal{I}^* = \{Y \subseteq X : X - Y \text{ contains a base of } \mathcal{M}\}$.

Verify this is indeed a matroid. There is an indirect proof of this fact in the textbook that describes the rank function for $\mathcal{M}^*$ and uses the lemma about rank functions to show it is indeed a matroid (as we did with matroid partitioning). You should use a more direct proof.

That is, prove the following **without ever using the rank function** of $\mathcal{M}$ (or $\mathcal{M}^*$):

- $\emptyset \in \mathcal{I}^*$,
- $A \in \mathcal{I}^*$ and $B \subseteq A$ implies $B \in \mathcal{I}^*$, and
- $A, B \in \mathcal{I}^*$ and $|A| < |B|$ implies there is some $x \in B - A$ such that $A + x \in \mathcal{I}^*$.

Hint: For the last part, one way to start is first to show there is a base $B_1 \subseteq X - A$ and a base $B_2 \subseteq X - B$ such that $B_1 - B = B_2 - A$ (i.e. the bases agree outside of $A \cup B$): you can take this for granted if you want to make progress towards partial credit. For your own benefit, draw a "Venn"-like diagram to help you keep your thoughts straight! It may be helpful to prove if B_1, B_2 are different bases of $\mathcal{M}$, then for every $y \in B_2 - B_1$ there is some $x \in B_1 - B_2$ such that $B_1 + y - x \in \mathcal{I}$ (the proof is very simple).

Very Interesting Fact: Let $\mathcal{G}$ be a planar graph. The dual of the graphic matroid over $\mathcal{G}$ is the same as the graphic matroid itself for the planar dual[1] of $\mathcal{G}$. This turns out to be an if and only if: the dual of the graphic matroid of $\mathcal{G}$ is itself a graphic matroid (for some graph) if and only if $\mathcal{G}$ is a planar graph (in which case, the "for some graph" would be the planar dual).

Solution 5:

To prove the first feature, it is clear that $X - \emptyset$ contains a base of $\mathcal{M}$. Hence, $\emptyset \in \mathcal{I}^*$.

To prove the second feature, $X - A$ contains the base B_1 of $\mathcal{M}$. Since $B \subseteq A$, $X - A \subseteq X - B$. Hence, $X - B$ also contains the base B_1 of $\mathcal{M}$. Therefore, $B \in \mathcal{I}^*$.

To prove the third feature, let's verify the following claim:

- If B_1, B_2 are different bases of $\mathcal{M}$, then for every $y \in B_2 - B_1$ there is some $x \in B_1 - B_2$ such that $B_1 + y - x \in \mathcal{I}$.

  Since B_1 is a base, then for every $y \in B_2 - B_1$, $B_1 + y \notin \mathcal{I}$. Hence, there is a circuit $C(B_1, y) \subseteq B_1 + y$. We must have $C(B_1, y) \nsubseteq B_2$ otherwise $C(B_1, y)$ would be independent. Therefore, there is $x \in B_1 - B_2$ and $x \in C(B_1, y)$ such that $B_1 + y - x \in \mathcal{I}$. Note that $B_1^{'} = B_1 + y - x$ is also a base for $\mathcal{M}$ as $|B_1| = |B_1^{'}|$.

[1] https://en.wikipedia.org/wiki/Dual_graph

By the repeated application of the mentioned claim, if there is a base $B_1 \subseteq X - A$ and a base $B_2 \subseteq X - B$ such that they don't agree outside of $A \cup B$, then we can have $B_1^{'}$ that agrees with B_2 outside of $A \cup B$.

We therefore can assume that there exists a base $B_1 \subseteq X - A$ and a base $B_2 \subseteq X - B$ such that $B_1 - B = B_2 - A$. If $B - A \nsubseteq B_1$, then there exists $x \in B - A - B_1$ such that $B_1 \subseteq X - A - x$. This means $A + x \in \mathcal{I}^*$.

In the case of $B - A \subseteq B_1$, we would reach a contradiction. We know that $|(B - A) \cap B_1| = |(A - B) \cap B_2|$ as the B_1 and B_2 agree outside of $A \cup B$ and $|B_1| = |B_2|$. If $B - A \subseteq B_1$, then we would have $|B - A| = |(A - B) \cap B_2| \leq |A - B|$. Hence, we would have $|B - A| + |A \cap B| \leq |A - B| + |A \cap B|$ which means $|B| \leq |A|$. This contradicts the assumption of $|B| > |A|$.

Exercise 6 - We Need Shortest Paths

Marks: 4

The image associated with this exercise has some colours that are crucial to the understanding of this problem. Let me know if you are having difficulty seeing the different colours in the picture.

Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ be the graphic matroid given by the graph below and let $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be the following partition matroid over E. Each $e \in E$ is coloured either **red**, **green**, or **blue** as depicted in the figure. Then $F \subseteq E$ is in $\mathcal{I}_2$ if F contains at most 2 red edges, at most 1 green edge, and at most 3 blue edges.

Finally, let $F \subseteq E$ be the thick edges in the picture (the thinner dashed edges are $E - F$).

- Construct the bipartite directed graph we used in the matroid intersection algorithm. Include a picture (either a photo or electronically-created figure is ok) that clearly indicates the sets S and T from the lecture (eg. $S = \{x \in E - F : F + x \in \mathcal{I}_1\}$ and $T = \{x \in E - F : F + x \in \mathcal{I}_2\}$).

  Solution 6.1: The figure 1 illustrates the bipartite-graph.

- Find a shortest path from S to T and record the resulting set $F' \in \mathcal{I}_1 \cap \mathcal{I}_2$ obtained by alternating F along this path.

  Solution 6.2: The shortest path $= df \to bc \to ab \to ac \to cf$ will form $F^{'} = \{df, ab, cf, ae, ef, fg\}$. Figure 2 illustrates the resulting edges after augmenting F.

- Describe a set Z such that $|F'| = r_1(E - Z) + r_2(Z)$. You do not have to show how you found this set Z, but you should explain how you calculated the ranks.

  Solution 6.3: If we define $r_1(Z)$ as the number of edges in a spanning tree for the component restricted to Z, and $r_2(Z)$ as the max number of edges that hold the color constraints tight in Z, then if we define $Z = E$, then $r_1(E - E) = 0$ and $r_2(E) = 6$ (3 blue edges, 2 red edges and 1 green edge). Then we have $|F^{'}| = 6 = 0 + 6$. It is clear that $F^{'}$ from the previous part is the maximum independent set for $M_1 \cap M_2$.

- Demonstrate some S to T path such that alternating F along this path does **not** produce a set $F' \in \mathcal{I}_1 \cap \mathcal{I}_2$. Thus, we see it is important to alternate along **shortest** paths.

  Solution 6.4: Consider the path $= \{df, bc, ab, ef, ed, bc, ab, ac, cf\}$ which will result in the new $F^{'} = \{df, ab, ed, cf, ae, fg\}$. However, the resulting subset of edges violates the color constraints as illustrated by the Figure 3 since it uses $3 > 2$ red edges.

Note: Your example may have to use an item of S or T as an intermediate node on the path. It is possible to come up with an example where augmenting along one such path that excludes S or T as intermediate nodes still creates dependent sets, but I couldn't think of a small enough example that would be fair to ask you to work through on an assignment.

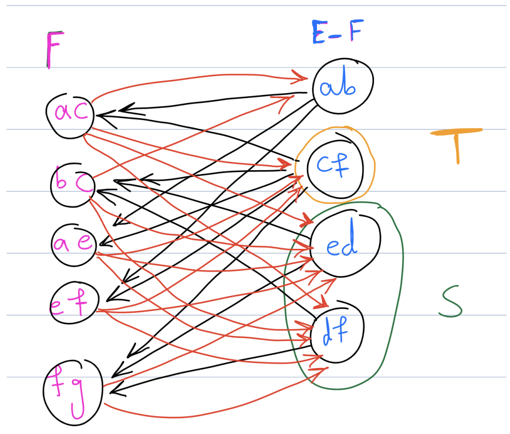

Figure 1: The corresponding bipartite graph built in the maximum intersection algorithm for the edges $F = \{ac, bc, ae, ef, fg\}$. $T = \{cf\}$ and $S = \{ed, df\}$.

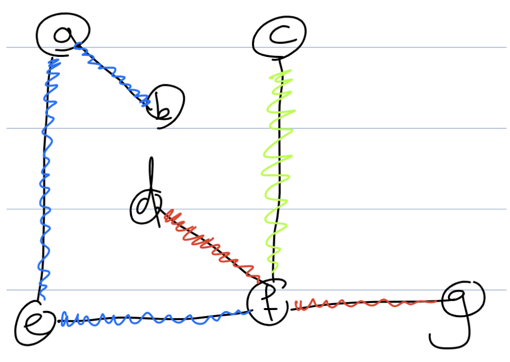

Figure 2: Maximum independent set of $M_1 \cap M_2$ using the shortest path $df \to bc \to ab \to ac \to cf$.

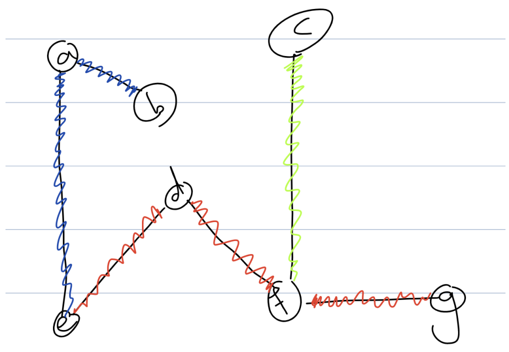

Figure 3: The subset of edges formed by a path which is not shortest. The set of edges is not an independent set for the second matroid as it has 3 red edges.