BIFURCATIONS

Introduction

- A **bifurcation** of a dynamical system is a qualitative change in its dynamics produced by varying parameters.
- A **bifurcation** occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden qualitative or topological change in its behavior.

Types of bifurcations

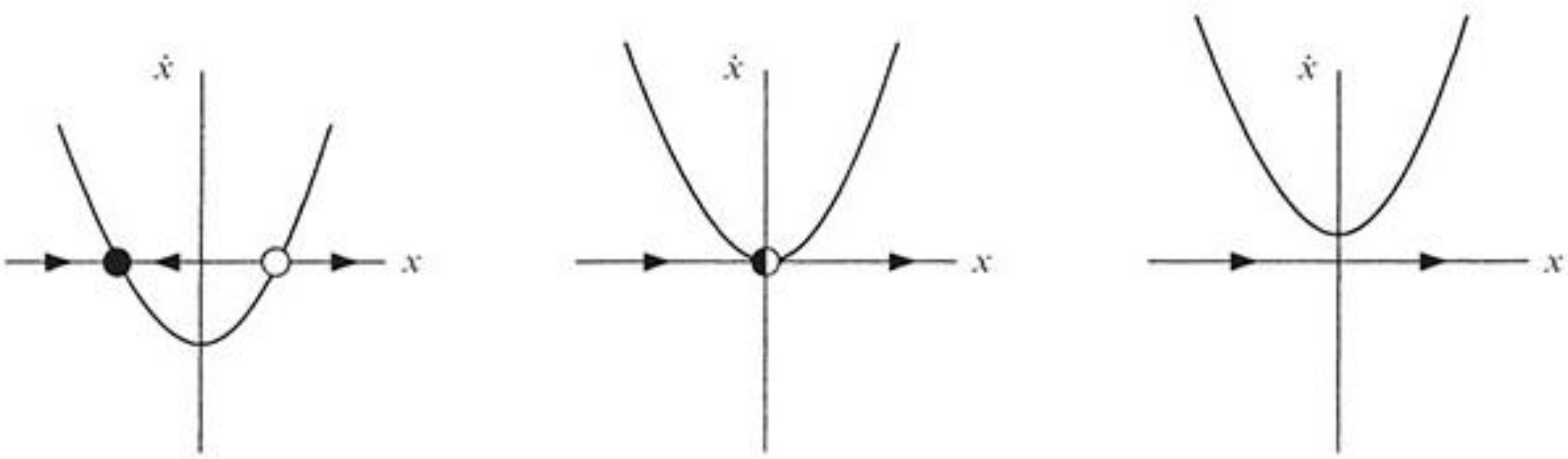
- Local bifurcations, which can be analyzed entirely through changes in the local stability properties of equilibrium, periodic orbits or other invariant sets as parameters cross through critical thresholds.
- Global bifurcations, which often occur when larger invariant sets of the system 'collide' with each other, or with equilibrium of the system. They cannot be detected purely by a stability analysis of the equilibrium (fixed points).

Bifurcation types

Local	Saddle-node
	Transcritical
	Pitchfork
	Period-doubling
	Hopf
	Neimark
Global	Homoclinic
	Heteroclinic
	Infinite-period
	Blue sky catastrophe

Saddle-Node Bifurcation

- The saddle-node bifurcation is the basic mechanism by which fixed points are *created, collided and destroyed*

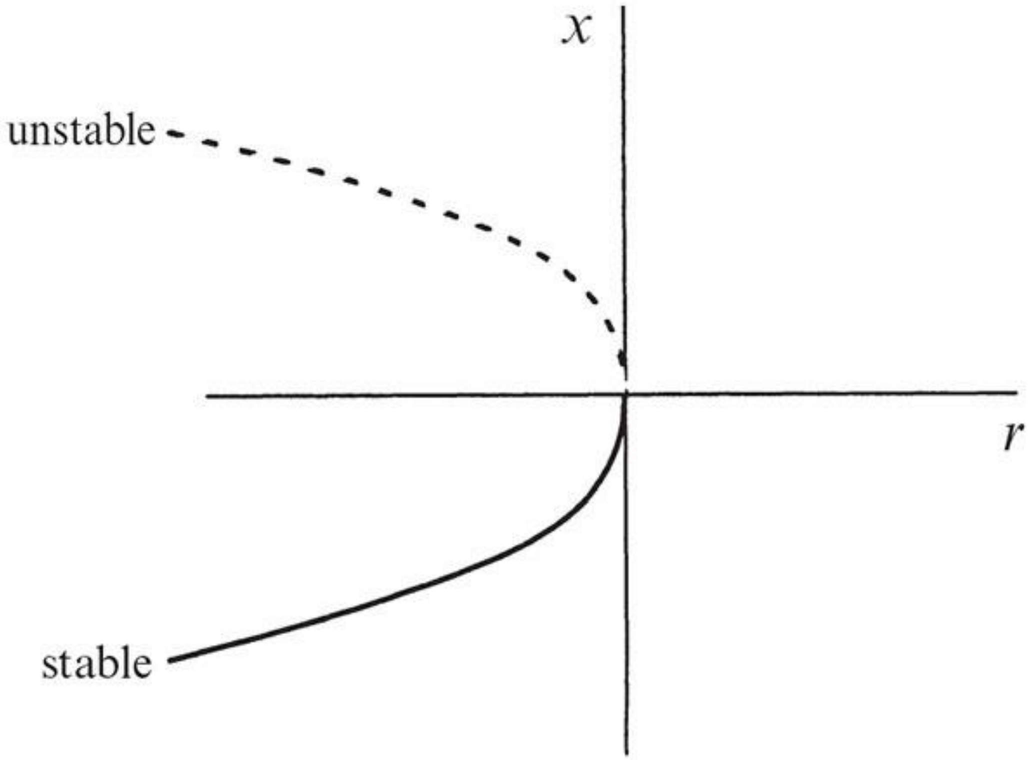


(a) $r<0$ (b) $r=0$ (c) $r>0$

Saddle-Node Bifurcation

- **Bifurcation diagram:**

$$\dot{x} = r + x^2$$



Saddle-Node Bifurcation

- **Normal Forms**

Taylor's expansion:

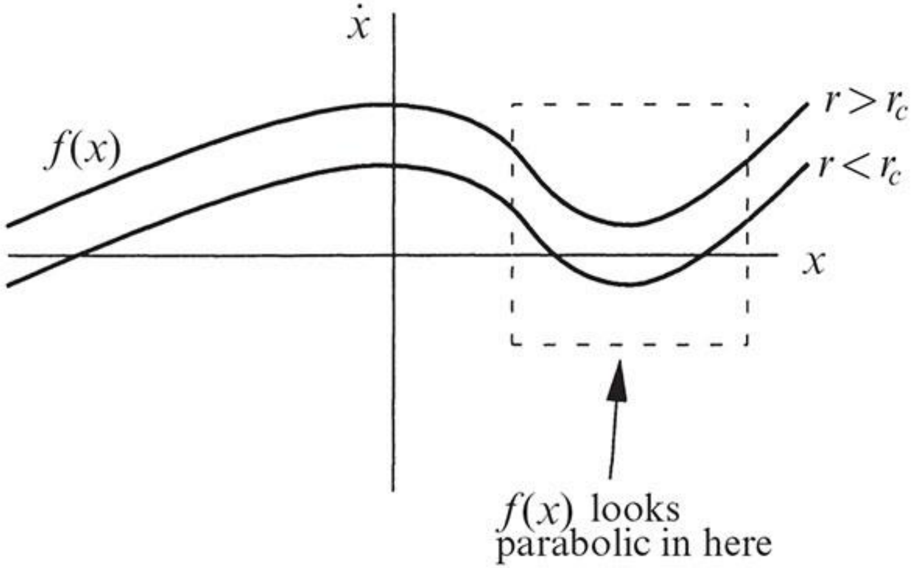
$$\dot{x} = f(x, r)$$
$$= f(x^*, r_c) + (x - x^*)\left.\frac{\partial f}{\partial x}\right|_{(x^*, r_c)} + (r - r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \tfrac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \cdots$$

Which $\begin{cases} f(x^*, r_c) = 0 \\ \partial f/\partial x|_{(x^*, rc)} = 0 \end{cases}$ $\Rightarrow$ $\dot{x} = (r - r_c)\left.\frac{\partial f}{\partial r}\right|_{(x^*, r_c)} + \tfrac{1}{2}(x - x^*)^2 \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x^*, r_c)} + \cdots$

Saddle-Node Bifurcation

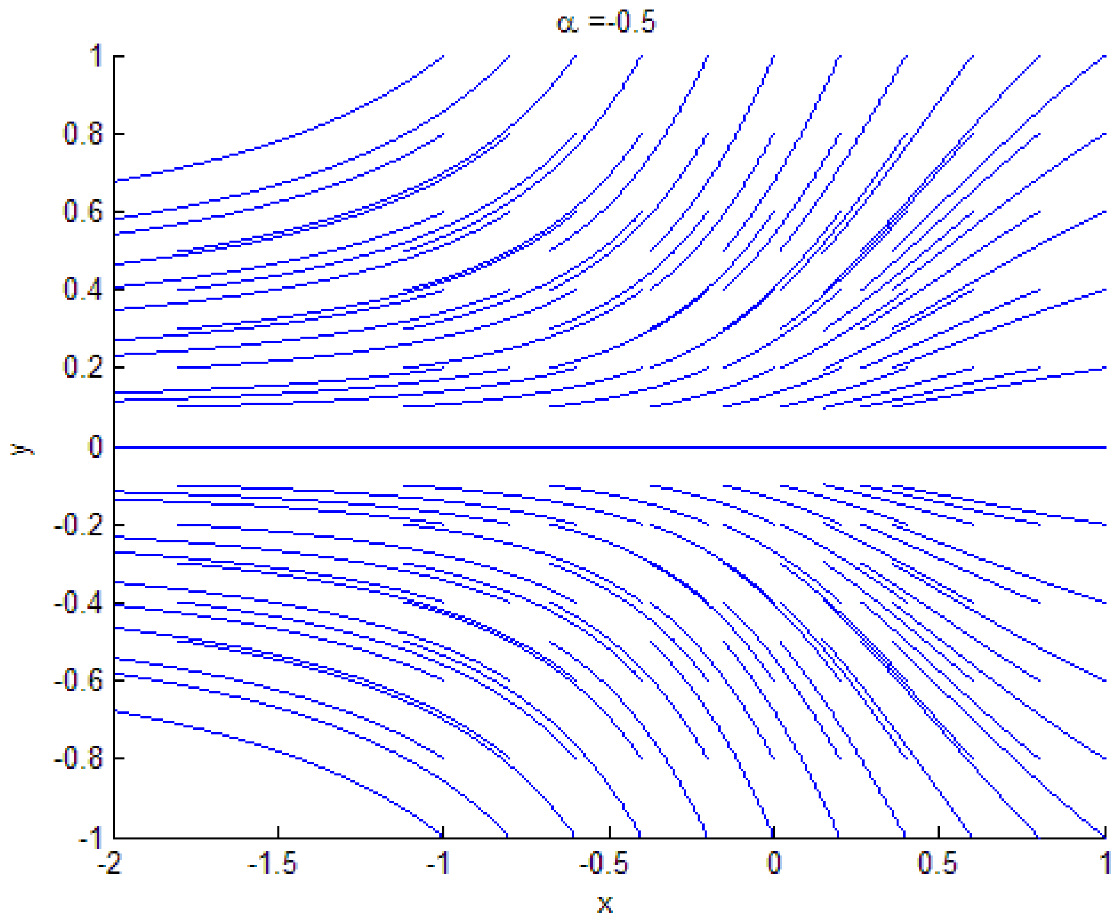
- **Normal Forms**

$$\dot{x} = r - x - e^{-x}$$
$$= r - x - \left[1 - x + \frac{x^2}{2!} + \cdots\right]$$
$$= (r-1) - \frac{x^2}{2} + \cdots$$



$$\dot{x} = a(r - r_c) + b(x - x^*)^2 + \cdots$$

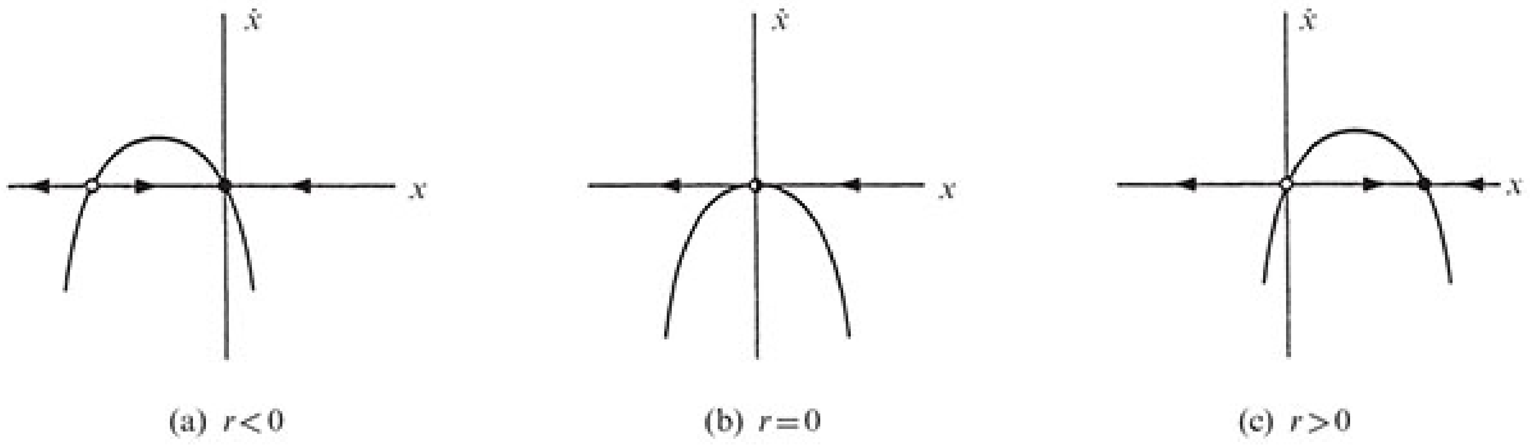
Saddle-Node Bifurcation



Transcritical Bifurcation

- There are certain scientific situations where a fixed point must exist for all values of a parameter and can never be destroyed.

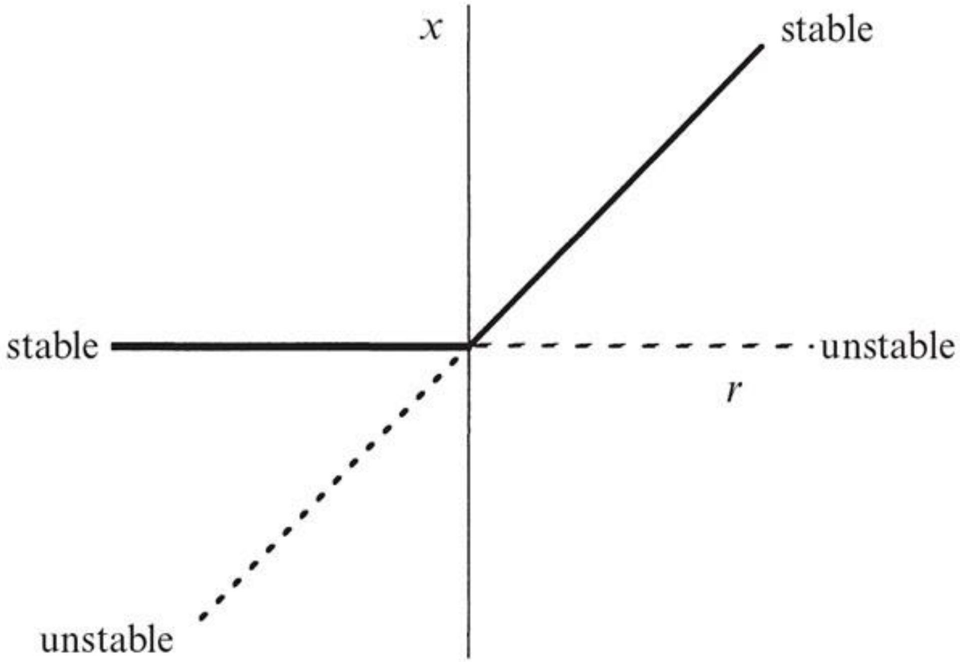
$$\dot{x} = rx - x^2.$$



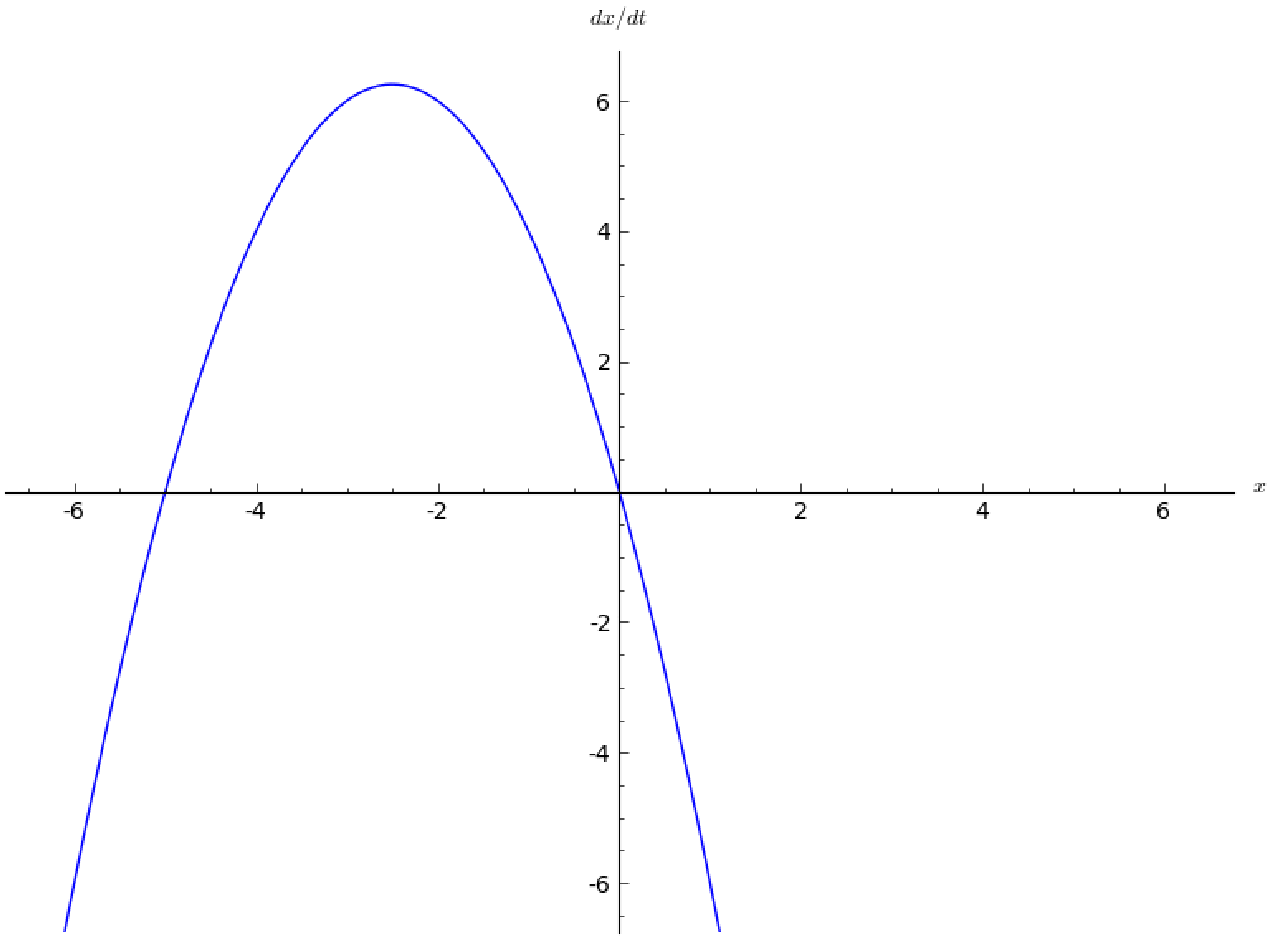
Transcritical Bifurcation

> “ in the transcritical case, the two fixed points don’t disappear after the bifurcation instead they just switch their stability ”

$$\dot{x} = rx - x^2.$$



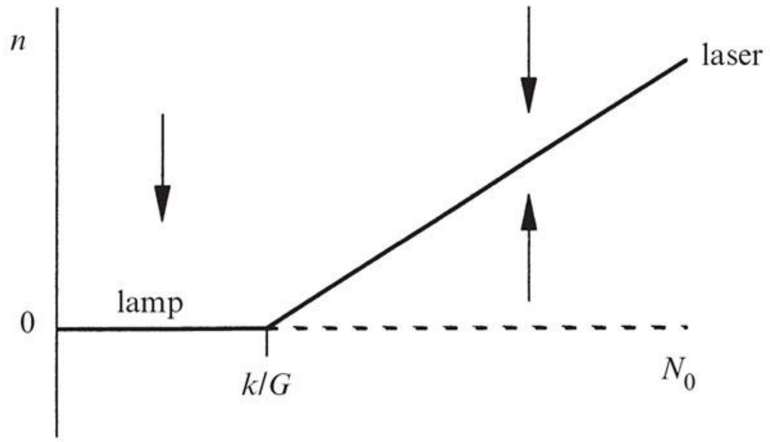
Transcritical Bifurcation

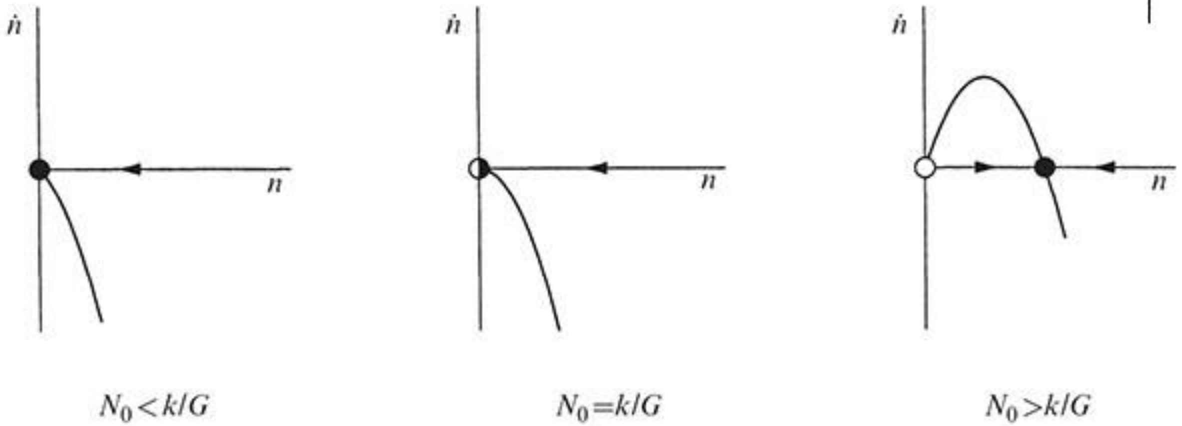


Transcritical Bifurcation

- **Laser Threshold**

$$\dot{n} = Gn(N_0 - \alpha n) - kn$$
$$= (GN_0 - k)n - (\alpha G)n^2.$$





Pitchfork Bifurcation

- This bifurcation is common in physical problems that have a **symmetry**.
- There are two very different types of pitchfork bifurcation:
  - **Supercritical Pitchfork Bifurcation**

    Normal Form:

    $$\dot{x} = rx - x^3$$

  - **Subcritical Pitchfork Bifurcation**

    Normal Form:

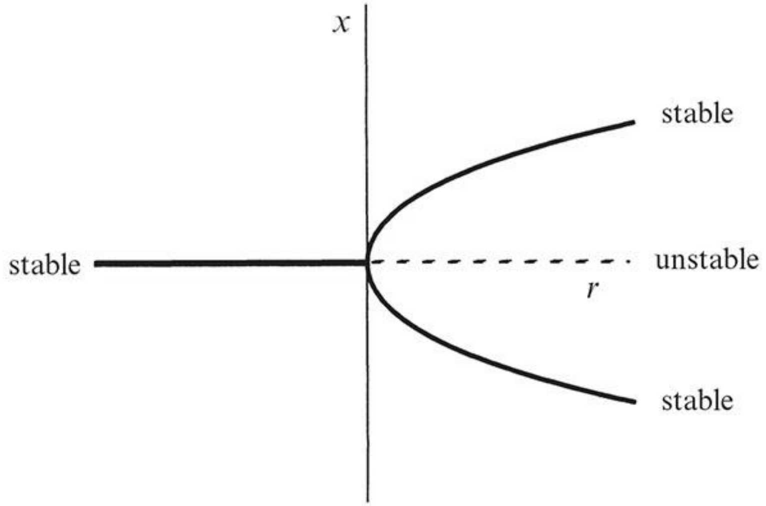
    $$\dot{x} = rx + x^3$$

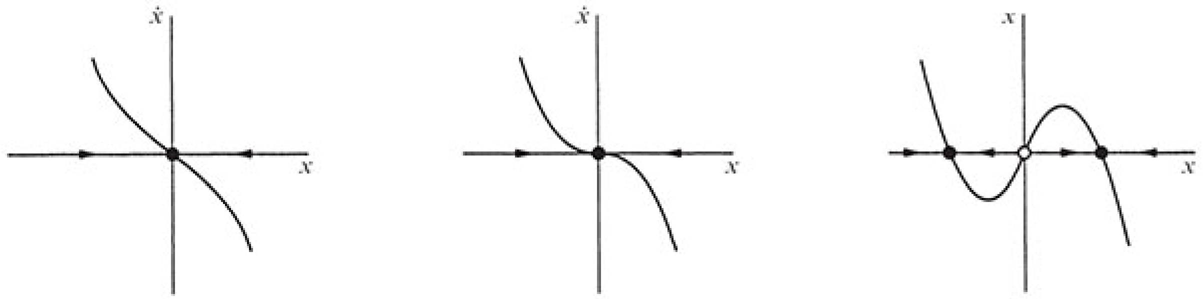
Pitchfork Bifurcation

- **Supercritical Pitchfork Bifurcation**

  Pitchfork Trifurcation!

$$\dot{x} = rx - x^3$$





(a) $r<0$ (b) $r=0$ (c) $r>0$

Pitchfork Bifurcation

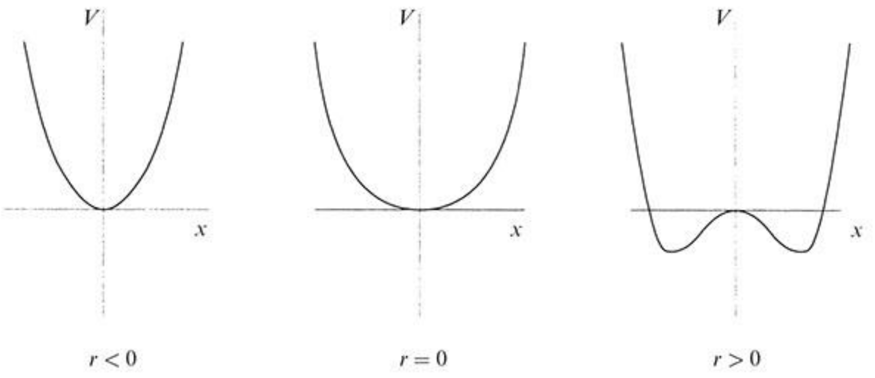
- **The potential of System**

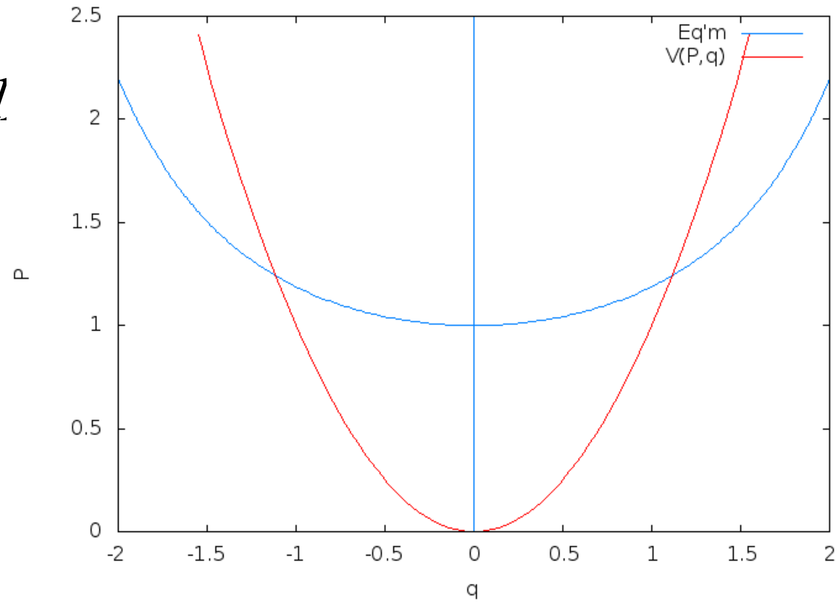
$\dot{x} = rx - x^3$

The potential is defined by $f(x) = -dV/d$

Then

$-dV/dx = rx - x^3 \Rightarrow V(x) = -\frac{1}{2}rx^2 + \frac{1}{4}x^4$

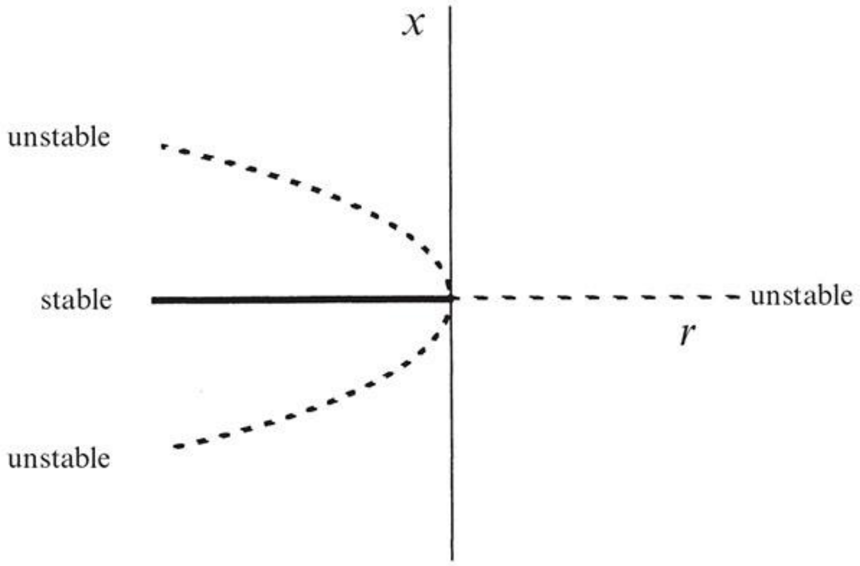




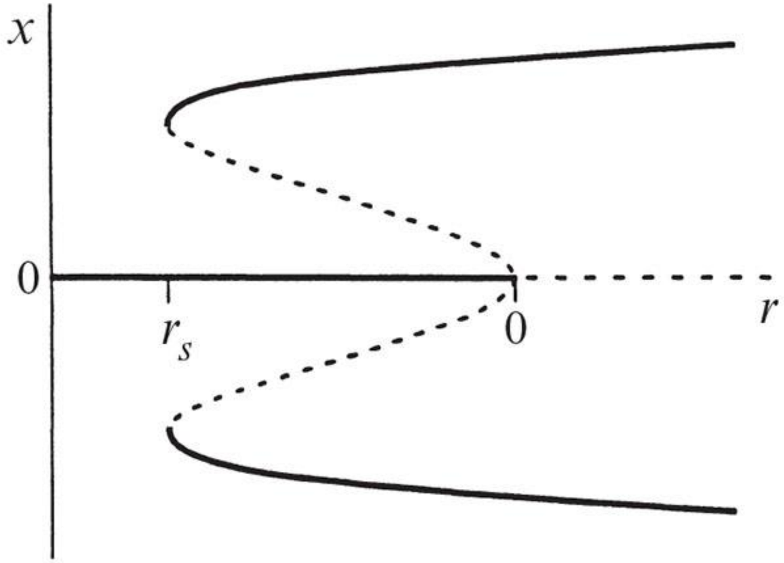
Pitchfork Bifurcation

- **Subcritical Pitchfork Bifurcation**

$$\dot{x} = rx + x^3$$

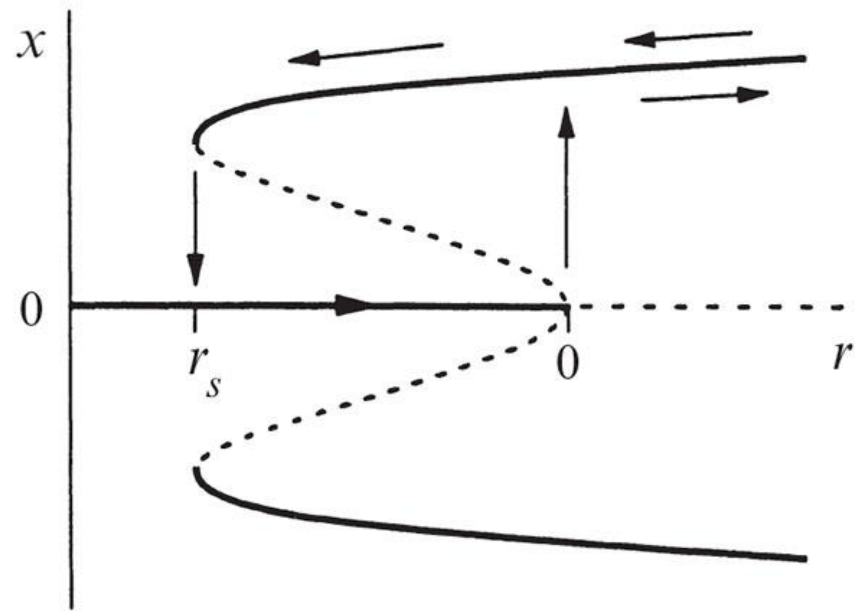


$$\dot{x} = rx + x^3 - x^5$$



Pitchfork Bifurcation

- In the range $r_s < r < 0$, two qualitatively different stable states coexist, namely the origin and the large-amplitude fixed points. The initial condition x_0 determines which fixed point is approached as $t \to \infty$.
- The existence of different stable states allows for the possibility of jumps and hysteresis as r is varied.
- The bifurcation at r_s is a saddle-node bifurcation.



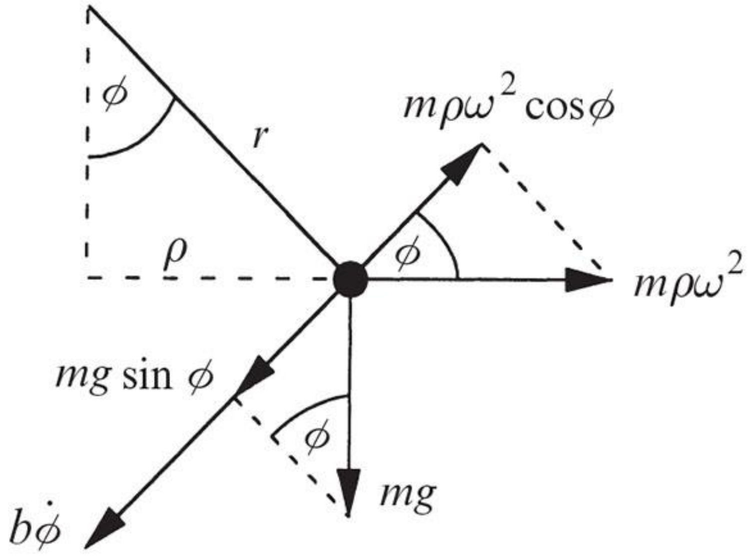
Bifurcation in a Mechanical System

- **Overdamped Bead on a Rotating Hoop**

$$mr\ddot{\phi} = -b\dot{\phi} - mg\sin\phi + mr\omega^2 \sin\phi\cos\phi.$$

we would like to find some conditions under which we can safely neglect $mr\ddot{\phi}$ term.
Then

$$\begin{aligned} b\dot{\phi} &= -mg\sin\phi + mr\omega^2 \sin\phi\cos\phi \\ &= mg\sin\phi\left(\frac{r\omega^2}{g}\cos\phi - 1\right). \end{aligned}$$



Bifurcation in a Mechanical System

- **Overdamped Bead on a Rotating Hoop**

Dimensional Analysis and Scaling

$$\tau = \frac{t}{T}$$

where T is a characteristic time scale

then

$$\ddot{\phi} = \frac{1}{T^2}\frac{d^2\phi}{d\tau^2}.$$

And finally,

Each of the terms in parentheses is a dimensionless group.

$$\left(\frac{r}{gT^2}\right)\frac{d^2\phi}{d\tau^2} = -\left(\frac{b}{mgT}\right)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi.$$

Bifurcation in a Mechanical System

- **Overdamped Bead on a Rotating Hoop**

Dimensional Analysis and Scaling

$$\left(\frac{r}{gT^2}\right)\frac{d^2\phi}{d\tau^2} = -\left(\frac{b}{mgT}\right)\frac{d\phi}{d\tau} - \sin\phi + \left(\frac{r\omega^2}{g}\right)\sin\phi\cos\phi.$$

$$\boxed{\frac{b}{mgT} \approx O(1),} \text{ and } \boxed{\frac{r}{gT^2} << 1.}$$

$$T = \frac{b}{mg}.$$

$$b^2 >> m^2 gr.$$

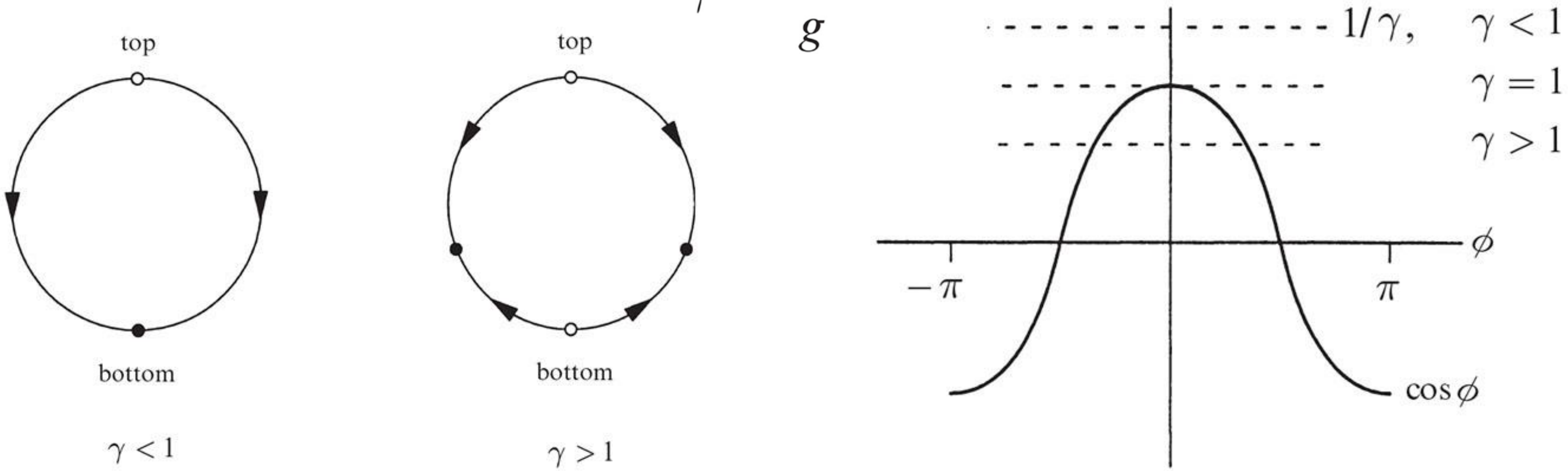
$$\varepsilon\frac{d^2\phi}{d\tau^2} = -\frac{d\phi}{d\tau} - \sin\phi + \gamma\sin\phi\cos\phi.$$

Bifurcation in a Mechanical System

- **Overdamped Bead on a Rotating Hoop**

  Dimensional Analysis and Scaling

$$\gamma = \frac{r\omega^2}{g}$$



Bifurcation in a Mechanical System

Overdamped Bead on a Rotating Hoop

Phase Plane Analysis

First-order system → Vector field

Second-order system → Phase plane

- The plane is spanned by two axes, one for the angle ϕ and one for the angular velocity.

$$\Omega = \phi' \equiv d\phi / d\tau$$

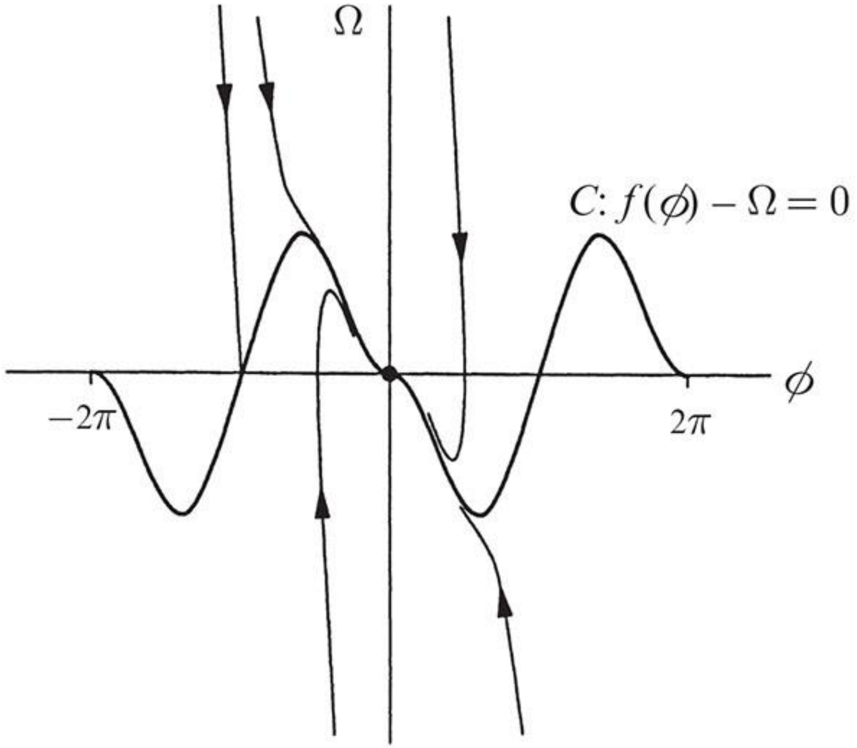
$$\Omega' = \frac{1}{\varepsilon}(f(\phi) - \Omega).$$

Bifurcation in a Mechanical System

- **Overdamped Bead on a Rotating Hoop**

Phase Plane Analysis

All trajectories slam straight up or down onto the curve C defined $f(\phi)=\Omega$ by, and then slowly ooze along this curve until they reach a fixed point.

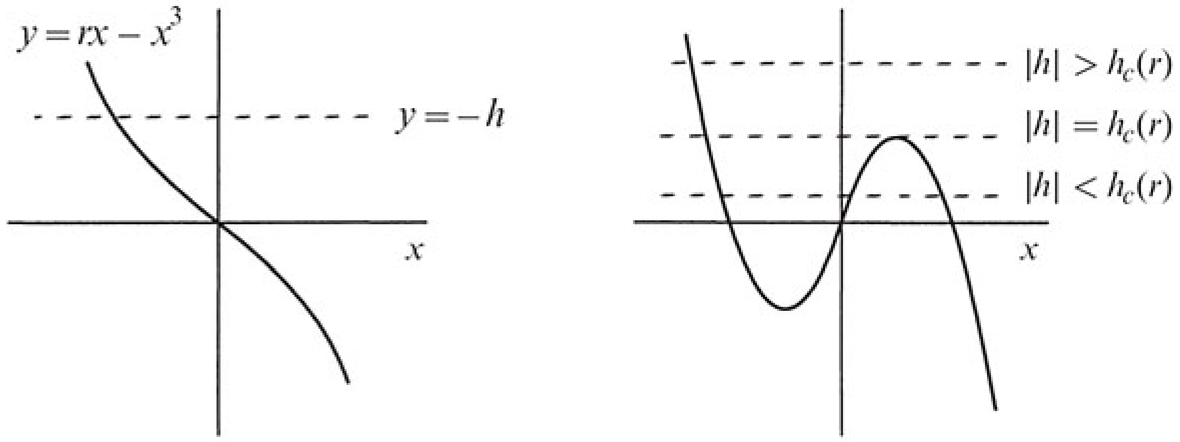


Imperfect Bifurcations

- In many real-world circumstances, the symmetry is only approximate, an imperfection leads to a slight difference between left and right.

$$\dot{x} = h + rx - x^3$$

h as an imperfection parameter.



(a) $r \leq 0$ (b) $r > 0$

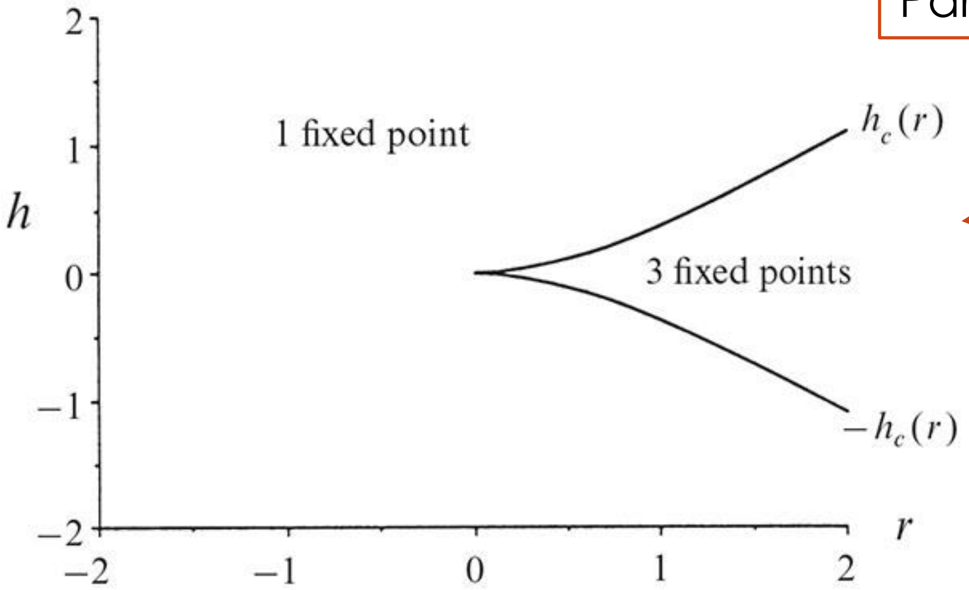
Imperfect Bifurcations

- The critical case occurs when the horizontal line is just tangent to either the local minimum or maximum of the cubic; then we have a saddle-node bifurcation.

  Saddle-node bifurcations occur when $h = \pm h_c(r)$, where

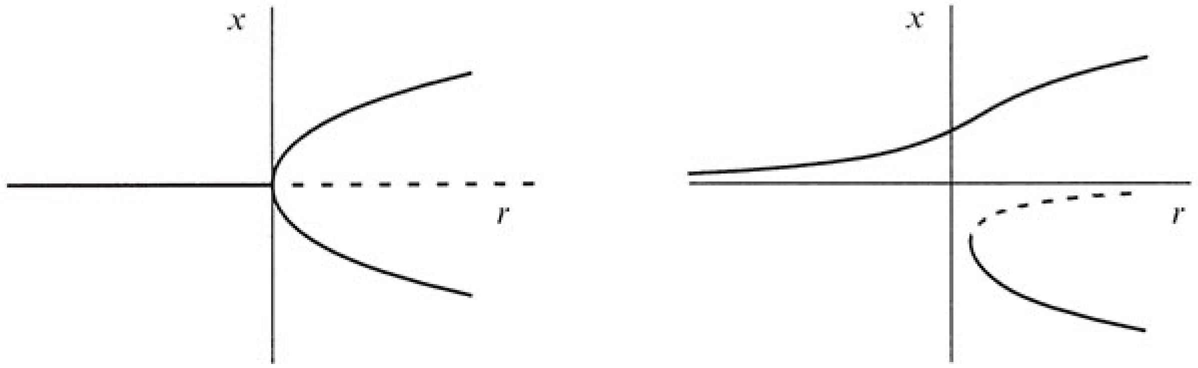
$$h_c(r) = \frac{2r}{2}\sqrt{\frac{r}{3}}.$$

Parameter Space



Imperfect Bifurcations

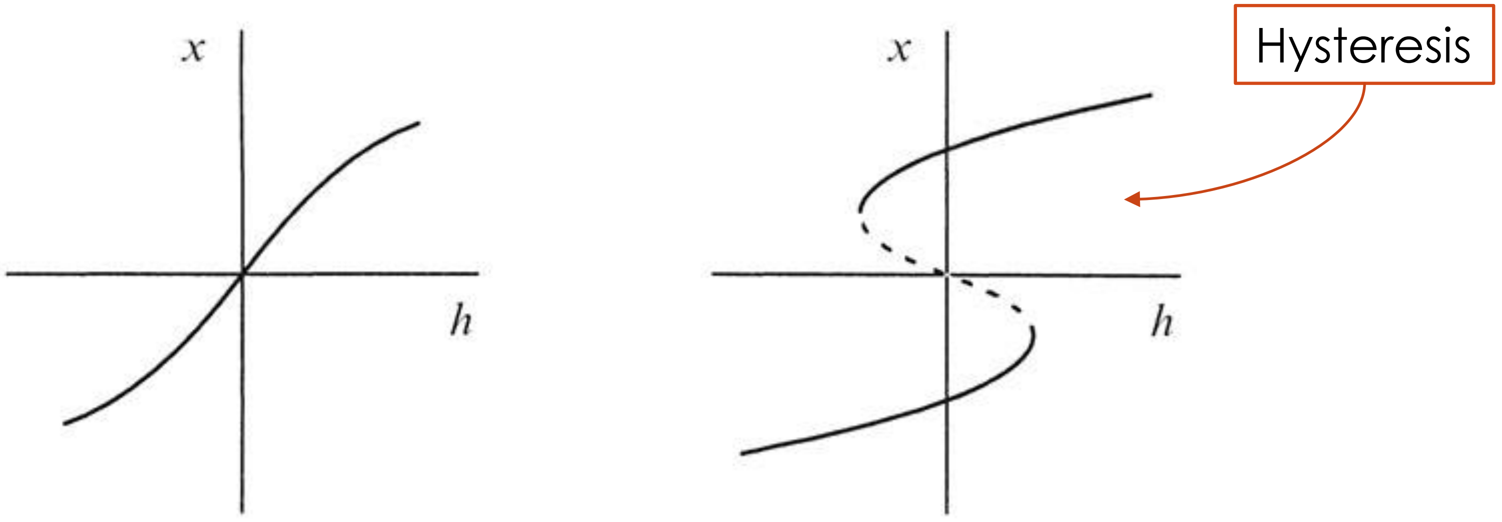
- The bifurcation diagram of x^* vs. r, for fixed h



(a) $h = 0$ (b) $h \neq 0$

Imperfect Bifurcations

- The bifurcation diagram of x^* vs. h, for fixed r

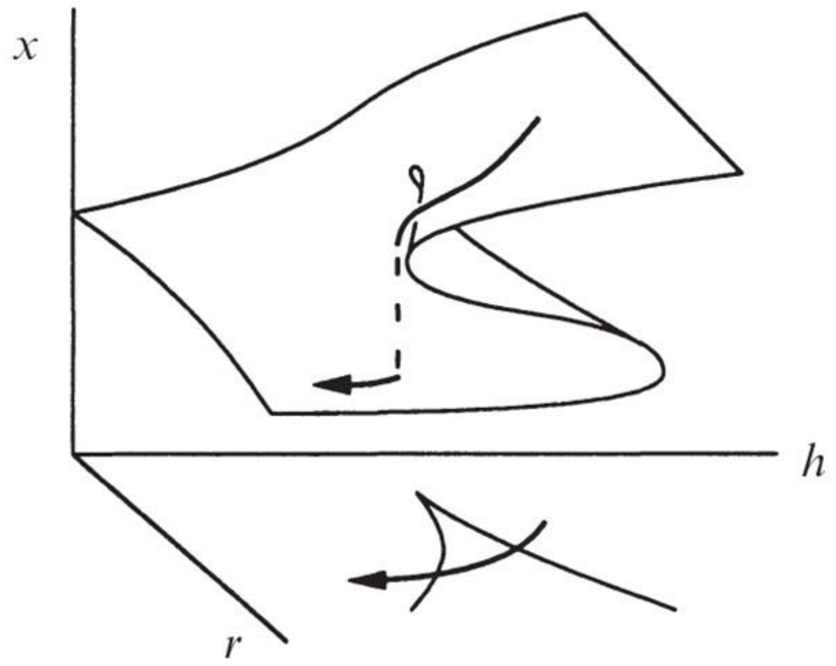


(a) $r \leq 0$ (b) $r > 0$

Imperfect Bifurcations

- If we plot the fixed points x^* above the (r,h) plane, we get the cusp catastrophe surface.

The term catastrophe is motivated by the fact that as parameters change, the state of the system can be carried over the edge of the upper surface, after which it drops discontinuously to the lower surface.

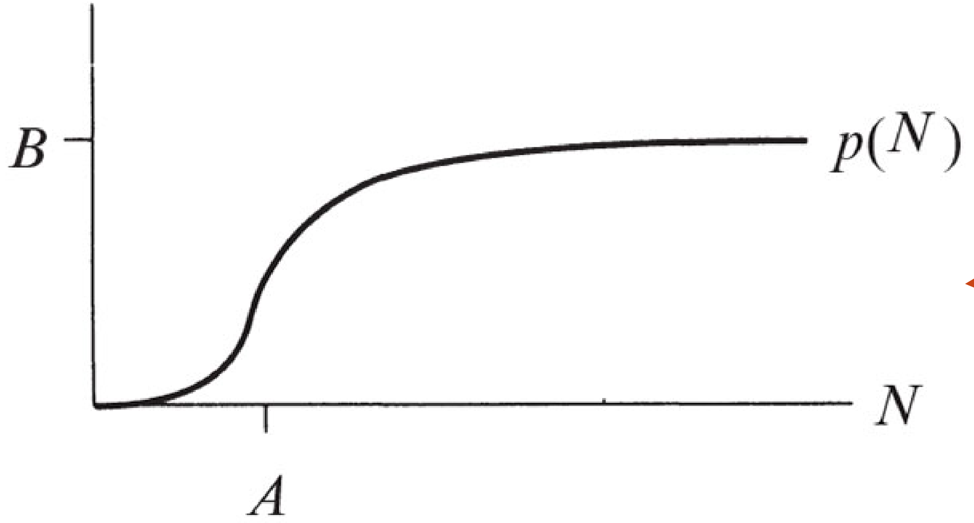


Insect Outbreak

- The proposed model for the budworm population dynamics is

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2 + N^2}.$$

The term $p(N)$ represents the death rate due to predation.



$$p(N) = \frac{BN^2}{A^2 + N^2}$$

Insect Outbreak

- To get rid of the parameters:

$x = N/A,$ $\longrightarrow$ $$\frac{A}{B}\frac{dx}{dt} = \frac{R}{B}Ax\left(1 - \frac{Ax}{K}\right) - \frac{x^2}{1+x^2}.$$

Then

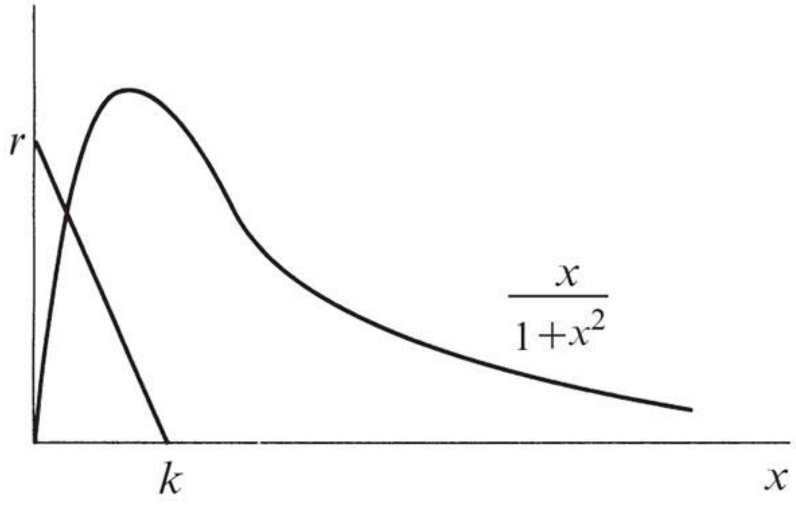
$$\tau = \frac{Bt}{A}, \qquad r = \frac{RA}{B}, \qquad k = \frac{K}{A}.$$

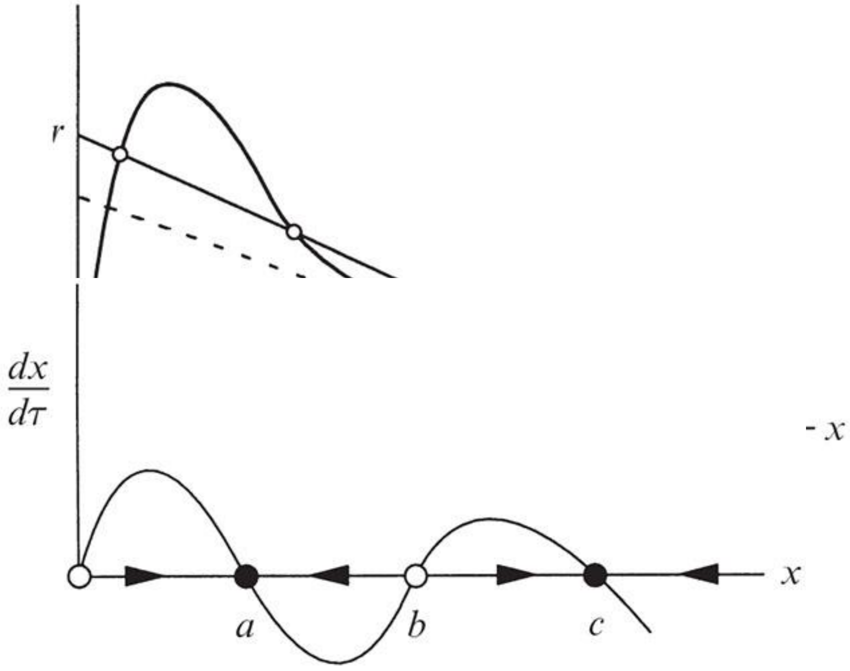
And finally

$$\frac{dx}{d\tau} = rx\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2},$$

Insect Outbreak

$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2}.$$





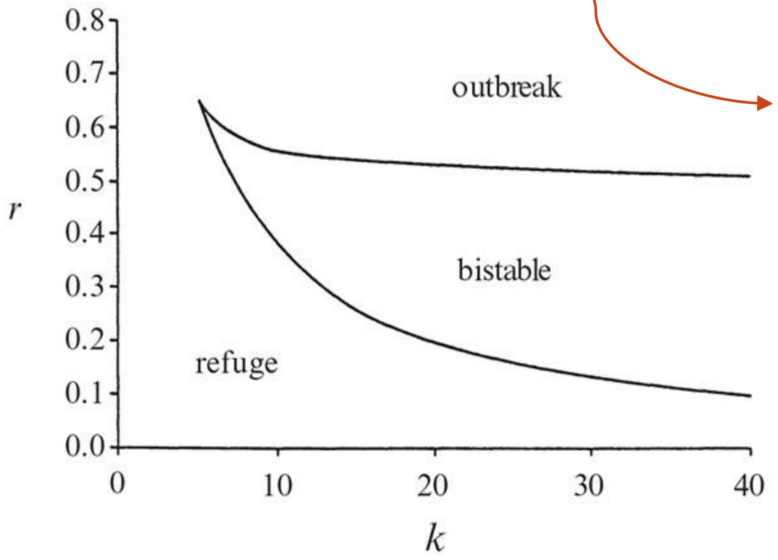
Insect Outbreak

- **Calculating the Bifurcation Curves**

$$r\left(1-\frac{x}{k}\right)=\frac{x}{1+x^2} \Rightarrow \frac{d}{dx}\left[r\left(1-\frac{x}{k}\right)\right]=\frac{d}{dx}\left[\frac{x}{1+x^2}\right] \Rightarrow -\frac{r}{k}=\frac{1-x^2}{(1+x^2)^2}.$$

$$r=\frac{2x^3}{(1+x^2)^2}.$$

$$k=\frac{2x^3}{x^2-1}.$$



Insect Outbreak

- **Calculating the Bifurcation Curves**

The refuge level a is the only stable state for low r, and the outbreak level c is the only stable state for large r. In the bistable region, both stable states exist.