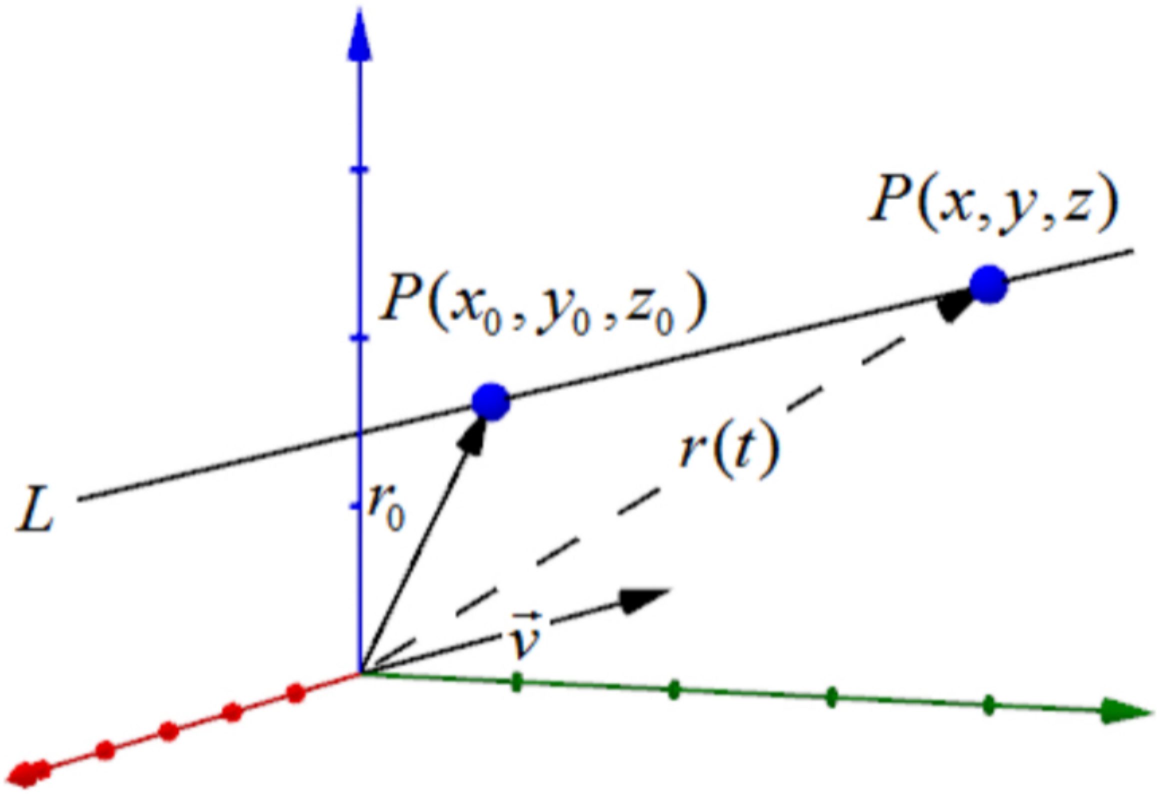
12.5 Equations of Lines and Planes (in $\mathbb{R}^3$)

There are 3 forms for the equation of a 3D line. (Vector, Parametric, and Symmetric)

Def 1 Vector Equation of a line L

Let L be a line in $\mathbb{R}^3$, and let $P(x_0, y_0, z_0)$ be a fixed point on L. $\vec{r}_0$ is the vector that connects to $P(x_0, y_0, z_0)$, and $\vec{r}(t)$ is the vector that connects to an arbitrary point $P(x, y, z)$ on L. $\vec{v}$ is the position vector in the direction of L.



Then the vector equation of L is

$$\vec{r}(t) = \vec{v}t + \vec{r}_0$$

Here t is a parameter, and since $\vec{r}(t)$ connects to point $P(x, y, z)$ we can write $\vec{r}(t) = \langle x, y, z \rangle$.

Letting $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\vec{v} = \langle a, b, c \rangle$ we have,

$$\vec{r}(t) = \vec{v}t + \vec{r}_0$$

$$\begin{aligned} \Rightarrow \langle x, y, z \rangle &= \langle a, b, c \rangle \cdot t + \langle x_0, y_0, z_0 \rangle \\ &= \langle at, bt, ct \rangle + \langle x_0, y_0, z_0 \rangle \\ &= \langle at + x_0, bt + y_0, ct + z_0 \rangle \end{aligned}$$

Setting the components of these vectors equal gives us the parametric equations of a line L.

$$x = at + x_0, \quad y = bt + y_0, \quad z = ct + z_0$$

where $\vec{v} = \langle a, b, c \rangle$ is the direction vector of L and $P(x_0, y_0, z_0)$ is any fixed point on L.

Def 2 (Parametric Eqns of line L)

The parametric equations of a line through the point (x_0, y_0, z_0) and parallel to the position vector $\langle a, b, c \rangle$ are

$$x = at + x_0, \quad y = bt + y_0, \quad z = ct + z_0$$

Note:

The vector and parametric eqns. of a line are not unique. Choosing a different point $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ or a different direction vector $\vec{v}$ will change the eqns.

The last type of line equation can be found by eliminating parameter t in the para. eqns.

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

If $a, b, c \neq 0$ we can solve each eqn. for t,

$$t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

Setting these equal gives us the symmetric equations of L,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Summary of Equations for Lines

Let L be a line in $\mathbb{R}^3$ with a fixed point (x_0, y_0, z_0) and direction vector $\vec{v} = \langle a, b, c \rangle$.

Vector:

$$\vec{r}(t) = \vec{v}t + \vec{r}_0 \quad , \quad \text{where } \vec{r}_0 = \langle x_0, y_0, z_0 \rangle .$$

Parametric:

$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

Symmetric:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Ex 1 Find the vector and parametric equations of a line passing through $(2,4,-5)$ and parallel to $\langle -1,3,8\rangle$.

Sol

Here we have $\vec{r}_0 = \langle x_0, y_0, z_0\rangle = \langle 2,4,-5\rangle$

and $\vec{v} = \langle a,b,c\rangle = \langle -1,3,8\rangle$.

Vector:

$$\vec{r}(t) = \vec{v}t + \vec{r}_0$$

$$= \langle -1,3,8\rangle t + \langle 2,4,-5\rangle$$

Parametric:

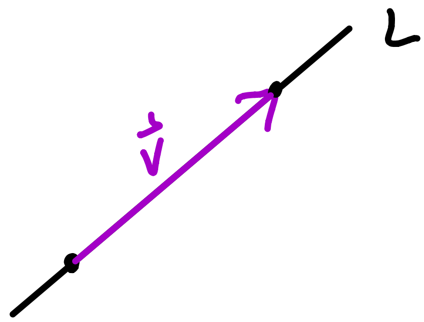
$$x = at + x_0 \qquad y = bt + y_0 \qquad z = ct + z_0$$

$$x = -t + 2 \qquad y = 3t + 4 \qquad z = 8t - 5$$

Ex 2 Find the parametric and symmetric equations of a line passing through the points $(2,-1,7)$ and $(3,2,-4)$.

Sol

First we need to find the direction vector (a vector between the two points will work),



$$\vec{v} = \langle 3-2, 2-(-1), -4-7 \rangle$$
$$= \langle 1, 3, -11 \rangle$$

Then we can choose either of the points to be our fixed point, let $\vec{r}_0 = \langle 3, 2, -4 \rangle$.

So the parametric eqns. are,

$$x = t + 3 \qquad y = 3t + 2 \qquad z = -11t - 4$$

To get the symmetric eqn. we can solve for t in each and then set them equal,

$$t = x + 3 \qquad t = \frac{y-2}{3} \qquad t = \frac{z+4}{-11}$$

<u>Symmetric:</u>

$$x + 3 = \frac{y-2}{3} = \frac{z+4}{-11}$$

<u>Ex 3</u> Find the point where the line from Ex 2 intersects the xz-plane.

<u>Sol</u>

The xz-plane is given by the equation $y=0$. The easiest way to find the intersection would be to substitute $y=0$ into the symmetric eqn. of our line,

$$x-3 = \frac{y-2}{3} = \frac{z+4}{-11}$$

<u>$y=0$:</u> $$x-3 = \frac{-2}{3} = \frac{z+4}{-11}$$

Now we can solve for x and z seperately,

$$x - 3 = \frac{-2}{3} \Rightarrow x = \frac{-2}{3} + 3 = \frac{7}{3}$$

$$\frac{z+4}{-11} = \frac{-2}{3} \Rightarrow z = \frac{22}{3} - 4 = \frac{10}{3}$$

So the line and plane intersect at the point $\left(\frac{7}{3}, 0, \frac{10}{3}\right)$.

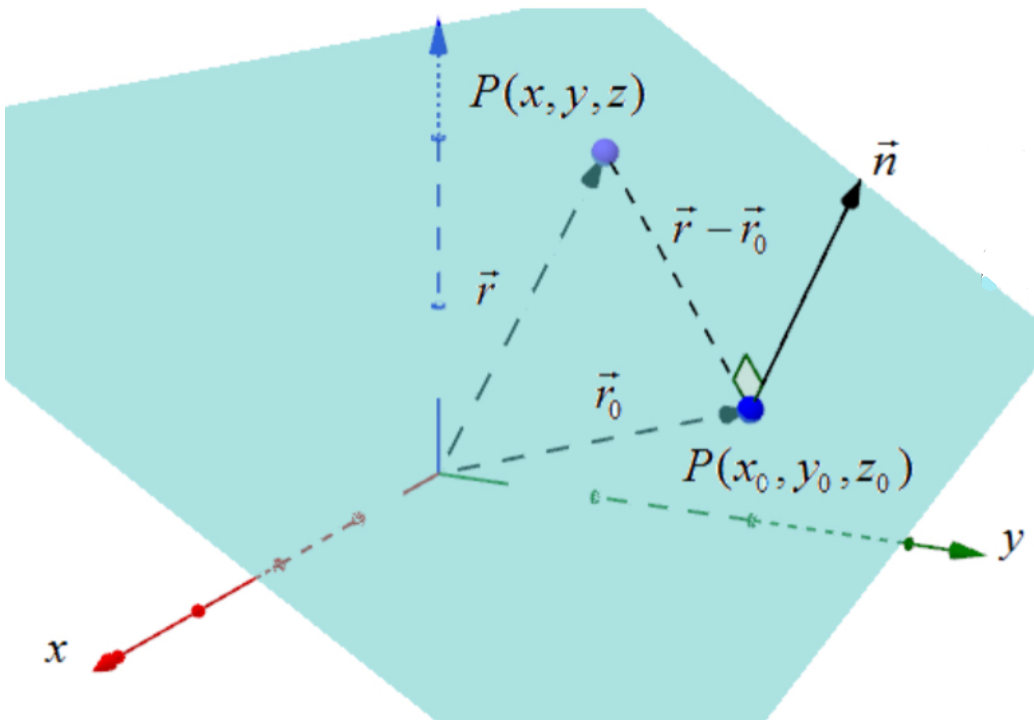
Planes

A line is uniquely determined by a point and a direction vector.

A single vector parallel to a plane is not enough to determine the "direction" of the plane.

To write the eqn. of a plane you need two things, a point on the plane (x_0, y_0, z_0) and a vector $\vec{n}$ which is orthogonal to the plane.

Vector Equation of a Plane



In the graph the only vector that lies on the plane is $\vec{r}-\vec{r}_0$.

Since $\vec{n}$ is orthogonal to the plane we have $(\vec{r}-\vec{r}_0)\cdot\vec{n}=0$.

So the vector equation of a plane is

$$(\vec{r}-\vec{r}_0)\cdot\vec{n}=0 \quad \text{or} \quad \vec{r}\cdot\vec{n}=\vec{r}_0\cdot\vec{n}$$

Let $\vec{n}=\langle a,b,c\rangle$ be the <u>normal</u> (orthogonal) vector to the plane, $\vec{r}=\langle x,y,z\rangle$, and $\vec{r}_0=\langle x_0,y_0,z_0\rangle$. Then

$$(\vec{r}-\vec{r}_0)\cdot\vec{n}=0$$

$$\Rightarrow \langle x-x_0, y-y_0, z-z_0\rangle\cdot\langle a,b,c\rangle=0$$

$$\Rightarrow a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

This is the <u>scalar eqn.</u> of a plane.

Summary of Equations for Planes

Let (x_0, y_0, z_0) be a point on the plane and $\vec{n} = \langle a, b, c \rangle$ be the normal vector.

Vector:

$$(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0 \quad \text{or} \quad \vec{r} \cdot \vec{n} = \vec{r}_0 \cdot \vec{n}$$

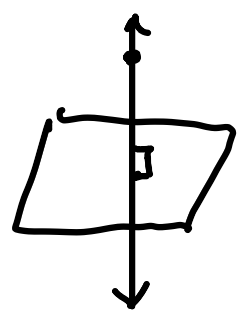
where $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$.

Scalar:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Ex 4 Find the vector and scalar eqns. for a plane that passes through $(4, -2, -1)$ and is perpendicular to the line

$$x = 3t - 4 \quad y = -5t + 1 \quad z = t - 7$$



<u>Sol</u>

We were given a point and a perpendicular line. To find the equations of a plane we need a point and a normal vector.

Any vector which is parallel to the given line will be orthogonal (normal) to the plane. In particular, we can choose $\vec{n}$ to be the direction vector of the line,

$$x = 3t - 4 \qquad y = -5t + 1 \qquad z = t - 7$$

So $\vec{n} = \langle 3, -5, 1 \rangle$ and $\vec{r}_0 = \langle 4, -2, -1 \rangle$.

<u>Vector</u>: $(\vec{r} - \vec{r}_0) \cdot \vec{n} = 0$

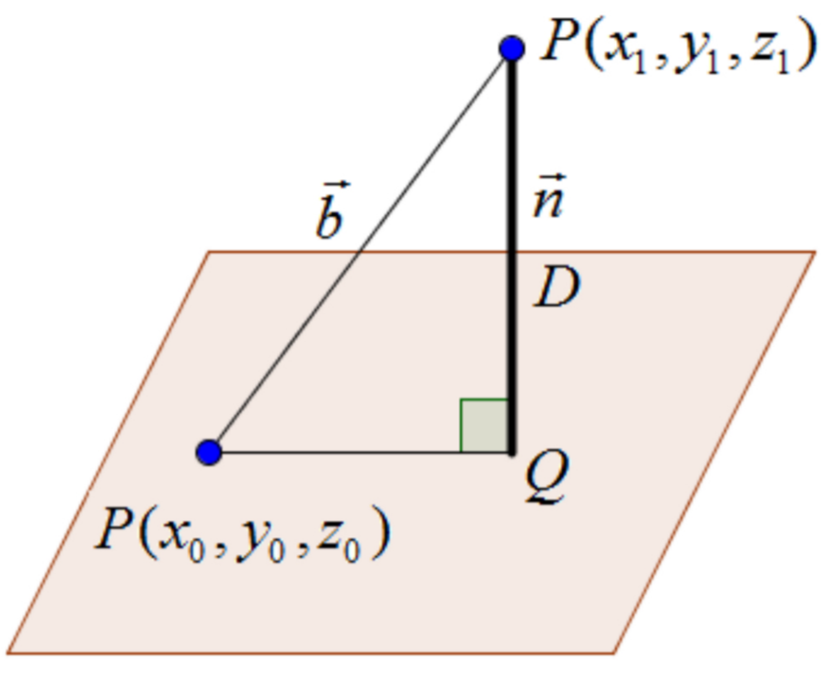
$$\left(\langle x, y, z \rangle - \langle 4, -2, -1 \rangle \right) \cdot \langle 3, -5, 1 \rangle = 0$$

<u>Scalar</u>: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$$3(x-4)-5(y+2)+(z+1)=0$$

<u>Distances</u>

Suppose you want to find the closest distance from a p int $P(x_1, y_1, z_1)$ and a plane.



Letting $\vec{n} = \langle a, b, c \rangle$ and $\vec{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

the length D is given by,

$$D = \left|\underbrace{\text{comp}_{\vec{n}}\vec{b}}_{}\right| = \left|\frac{\vec{n}\cdot\vec{b}}{|\vec{n}|}\right|$$

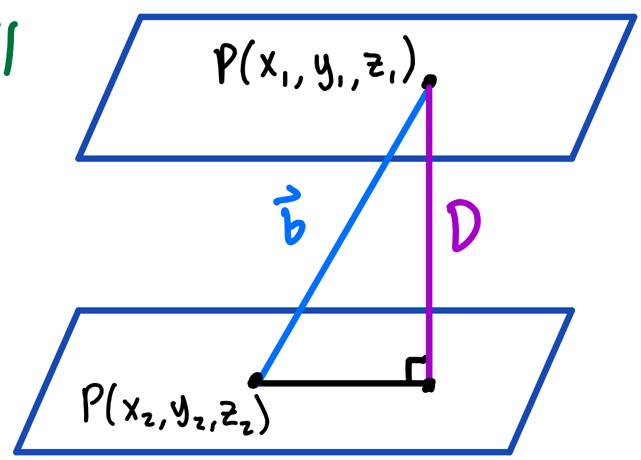
positive scalar
proj. of $\vec{b}$ onto $\vec{n}$

$$= \left|\frac{a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)}{\sqrt{a^2 + b^2 + c^2}}\right|$$

Ex 5 Find the distance between the parallel planes $2(x-1) + 5(y-3) - 4(z-7) = 0$ and $4x + 10y - 8z = 20$.

Sol

For this problem we will need a point on each plane and a normal vector for one of the planes.



For the plane $2(x-1)+5(y-3)-4(z-7)=0$

we can see the normal vector is

$\vec{n}=\langle 2,5,-4\rangle$ and $(1,3,7)$ is a point

on the plane.

We still need a point from plane

$4x+10y-8z=20$. Any point will work

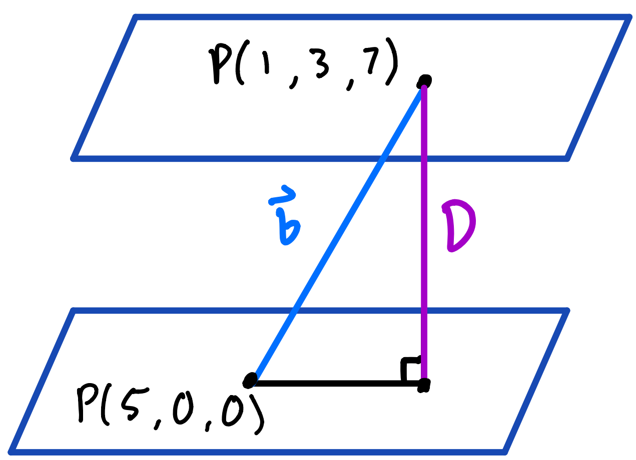
so we can let $y=0$ and $z=0$ to get

$$4x=20 \Rightarrow x=5$$

So the point $(5,0,0)$ lies on this plane.

$\vec{b}=\langle 1-5, 3-0, 7-0\rangle$

$=\langle -4,3,7\rangle$



$$D = |\text{comp}_{\vec{n}}\vec{b}| = \left|\frac{\vec{b}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{\langle -4,3,7\rangle \cdot \langle 2,5,-4\rangle}{\sqrt{2^2+5^2+(-4)^2}}\right|$$

$$= \left|\frac{-8+15-28}{\sqrt{45}}\right| = \left|\frac{-21}{\sqrt{45}}\right| = \frac{21}{\sqrt{45}}$$

Practice Problems

1) Let $P(1,-2,4), Q(0,2,-1), R(5,-3,1) \in \mathbb{R}^3$

a) Find the scalar equation of the plane that passes through these 3 points.

b) Find the vector and parametric eqns. of a line that passes through point $(4,0,-1)$ and is perpendicular to the plane found in part a.

c) Find the symmetric eqn. of a line through points P and Q. Then find where this line crosses the xy-plane.

<u>Solutions</u>

1) $P(1,-2,4), Q(0,2,-1), R(5,-3,1)$

a) We need to find a normal vector, i.e. a vector orthogonal to the plane.

Letting $\vec{n} = \overrightarrow{PQ} \times \overrightarrow{QR}$ will work,

$$\overrightarrow{PQ} = \langle 0-1, 2-(-2), -1-(4) \rangle = \langle -1, 4, -5 \rangle$$

$$\overrightarrow{QR} = \langle 5-0, -3-2, 1-(-1) \rangle = \langle 5, -5, 2 \rangle$$

$$\vec{PQ} \times \vec{QR} = \begin{vmatrix} i & j & k \\ -1 & 4 & -5 \\ 5 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -5 \\ -5 & 2 \end{vmatrix} i - \begin{vmatrix} -1 & -5 \\ 5 & 2 \end{vmatrix} j + \begin{vmatrix} -1 & 4 \\ 5 & -5 \end{vmatrix} k$$

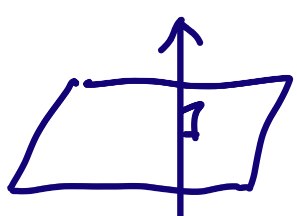
$$= (8-25)i - (-2-(-25))j + (5-20)k$$

$$= \langle -17, -23, -15 \rangle = \vec{n}$$

Choosing $P(1,-2,4)$ as our fixed point the scalar eqn of the plane is,

$$-17(x-1) - 23(y+2) - 15(z-4) = 0$$

b) Find the vector and parametric eqns. of a line that passes through point $(4,0,-1)$ and is perpendicular to the plane found in part a.



We need a direction vector for the line.

Since the line is perpendicular to the plane

$$-17(x-1)-23(x+2)-15(x-4)=0$$

we can use the normal vector $\langle -17, -23, -15 \rangle$.

So $\vec{v} = \langle -17, -23, -15 \rangle$ and $\vec{r}_0 = \langle 4, 0, -1 \rangle$.

<u>Vector</u>: $\vec{r}(t) = \vec{v}t + \vec{r}_0$

$$\vec{r}(t) = \langle -17, -23, -15 \rangle t + \langle 4, 0, -1 \rangle$$

<u>Parametric</u>:

$x = -17t + 4 \qquad y = -23t \qquad z = -15t - 1$

c) Find the symmetric eqn. of a line through points P and Q. Then find where this line crosses the xy-plane.

$$P(1,-2,4), \quad Q(0,2,-1)$$

$\vec{v} = \overrightarrow{PQ} = \langle 0-1, 2-(-2), -1-4 \rangle = \langle -1, 4, -5 \rangle$

$\vec{r}_0 = \langle 1, -2, 4 \rangle$

<u>Symmetric</u>:

$$\frac{x-1}{-1} = \frac{y+2}{4} = \frac{z-4}{-5}$$

Intersection with xy-plane $(z=0)$:

$$\frac{x-1}{-1} = \frac{y+2}{4} = \frac{0-4}{-5} = \frac{4}{5}$$

Solving for x and y,

$$\frac{x-1}{-1} = \frac{4}{5} \Rightarrow x = \frac{1}{5}$$

$$\frac{y+2}{4} = \frac{4}{5} \Rightarrow y = \frac{16}{5} - 2 = \frac{6}{5}$$

So the xy-plane and line intersect at

$\underline{\left(\frac{1}{5}, \frac{6}{5}, 0\right)}$.

<u>Suggested Textbook Exc</u> (12.5)

3, 7, 11, 23, 26, 31, 34, 45, 71