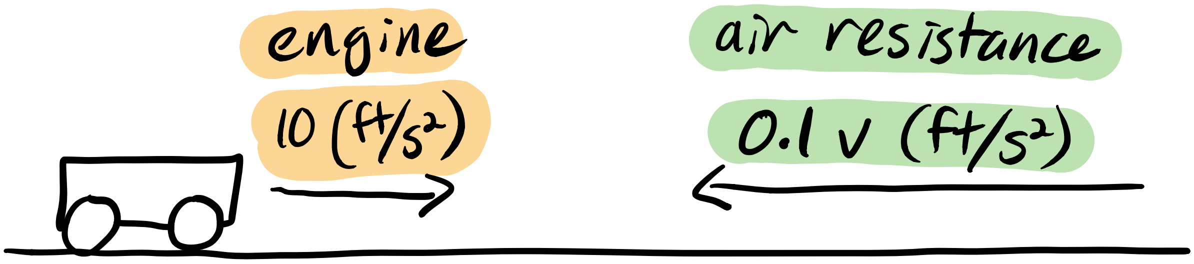
1.8 / Acceleration - Velocity Models

Example: (#7)

Suppose that a car starts from rest, its engine providing an acceleration of 10 ft/s^2, while air resistance provides 0.1 ft/s^2 of deceleration for each foot per second of the car's velocity.



$x_0 = 0$ $\quad a_E = 10$ $\quad a_R = -0.1v$

$v_0 = 0$

(Newton's 2nd Law: $F = ma$)

$$\boxed{\begin{aligned} F_E &= m a_E = m \cdot 10 \\ F_R &= m a_R = m \cdot (-0.1v) \\ F &= F_E + F_R = m(10 - 0.1v) \end{aligned}}$$

$$F = ma \implies m(10 - 0.1v) = ma$$
$$\implies 10 - 0.1v = a$$
$$a = \frac{dv}{dt} \qquad \implies \boxed{\frac{dv}{dt} = 10 - 0.1v}$$

(a) Find the car's maximum possible (limiting) velocity.

$$\frac{dv}{dt} = 0 \implies 10 - 0.1v = 0$$
$$\implies v = 100 \text{ ft/s}$$

$\therefore$ the limiting velocity is 100 ft/s.

(b) Find how long it takes the car to attain 90% of its limiting velocity, and how far it travels while doing so.

$$\frac{dv}{dt} = 10 - 0.1v, \qquad v(0) = 0$$

$$\int \frac{dv}{10-0.1v} = \int dt$$

$$\frac{1}{-0.1} \ln|10-0.1v| = t + C_1$$

$$\ln|10-0.1v| = -0.1t - 0.1C_1$$

$$|10-0.1v| = e^{-t/10} \cdot e^{-C_1/10}$$

$$10-0.1v = \pm e^{-C_1/10} \cdot e^{-t/10}$$

$$-0.1v = C_2 e^{-t/10} - 10 \qquad \left[C_2 = \pm e^{-C_1/10} \right]$$

$$v = \frac{C_2}{-0.1} e^{-t/10} + 100 \qquad \left[C = \frac{C_2}{-0.1} \right]$$

$$\boxed{v(t) = C e^{-t/10} + 100}$$

$$\lim_{t \to \infty} v(t) = C \cdot 0 + 100 = 100$$

$$v(0) = 0 \quad \Rightarrow \quad C \cdot 1 + 100 = 0$$

$$\Rightarrow \quad C = -100$$

$$\therefore \boxed{v(t) = -100e^{-t/10} + 100}$$

Car attains 90% of limiting velocity when $v(t) = 90$ ft/s.

$$-100e^{-t/10} + 100 = 90$$

$$e^{-t/10} = \frac{-10}{-100}$$

$$-t/10 = \ln 10^{-1}$$

$$t = 10 \ln 10 = 23.025\cdots$$

$\therefore$ car attains 90% of limiting velocity after about <u>23 seconds</u>.

$$\boxed{v(t) = -100e^{-t/10} + 100}$$

$$x(t) = \int v(t)\,dt + C, \quad x(0) = 0$$

$$x(t) = -100 \cdot (-10)e^{-t/10} + 100t + C$$

$$x(t) = 1000e^{-t/10} + 100t + C$$

$$x(0) = 0 \implies 1000 \cdot 1 + 100 \cdot 0 + C = 0$$

$$\implies C = -1000$$

$$\therefore \boxed{x(t) = 1000e^{-t/10} + 100t - 1000}$$

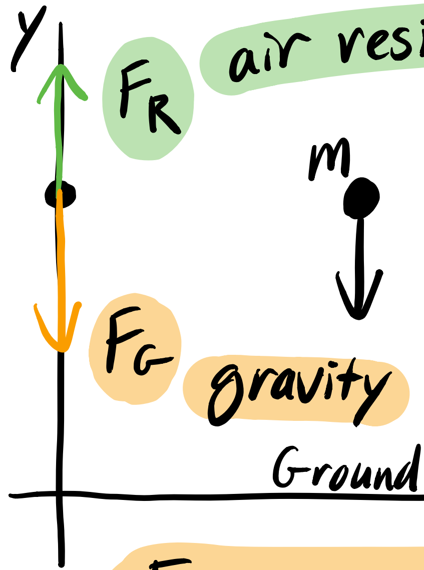
$$x(10\ln 10) = 1000\exp\left(\frac{-10\ln 10}{10}\right) + 100 \cdot (10\ln 10) - 1000$$

$$= 1000 \cdot 10^{-1} + 1000 \cdot \ln 10 - 1000$$

$$= 1000\ln 10 - 900 = 1402.58\ldots$$

$\therefore$ The car travelled about 1403 ft to attain 90% of its limiting velocity.

Vertical Motion



$$F = F_G + F_R$$

Note: If mass is moving up, then F_R is a downward force.

$$\boxed{F_G = -mg}$$

$[g \approx 9.8\ m/s^2 \approx 32\ ft/s^2]$

① Air resistance proportional to velocity

$$a_R = -\rho v$$

[This is a good model at low velocities.]

$$F_R = m a_R$$

$$\left.\begin{aligned} F_R &= -m\rho v \\ F_G &= -mg \end{aligned}\right\}$$

$$F = F_G + F_R$$

$$ma = -mg - m\rho v$$

linear and separable

$$\boxed{\frac{dv}{dt} = -g - \rho v}$$

ρ is called the <u>drag coefficient</u>.

<u>eg</u> Falling w/o parachute: $\rho = 0.15$

Falling w/ parachute: $\rho = 1.5$

The <u>terminal velocity</u> is given by $\frac{dv}{dt} = 0 \Rightarrow -g - \rho v = 0$

$$\Rightarrow v = -g/\rho.$$

$$\boxed{v_\tau = -g/\rho}$$

$$\rho = -\frac{g}{v_\tau}$$

<u>Solving the diff. eq'n:</u>

$$\frac{dv}{dt} = -g - \rho v, \qquad v(0) = v_0$$

$$\int \frac{dv}{-g - \rho v} = \int dt$$

$$\frac{-1}{\rho}\ln|-g-\rho v| = t + C_1$$

$$\ln|-g-\rho v| = -\rho t - \rho C_1$$

$$|-g-\rho v| = e^{-\rho t}\cdot e^{-\rho C_1}$$

$$-g-\rho v = \pm e^{-\rho C_1}\cdot e^{-\rho t}$$

$$-\rho v = C_2 e^{-\rho t} + g \qquad \left[\begin{array}{l} C_2 = \pm e^{-\rho C_1} \\ C = \frac{C_2}{-\rho} \end{array}\right]$$

$$v = Ce^{-\rho t} - g/\rho$$

$$v(0) = v_0 \implies C\cdot 1 - g/\rho = v_0$$

$$\implies C = v_0 + g/\rho$$

Recall That $v_T = -g/\rho$.

$$\therefore \boxed{v(t) = (v_0 - v_T)e^{-\rho t} + v_T}$$

<u>Note</u>: $\lim_{t\to\infty} v(t) = (v_0 - v_T)\cdot 0 + v_T = v_T$.

Example: (#11)

According to a newspaper account, a paratrooper survived a training jump from 1200 ft when his parachute failed to open but provided some resistance by flapping unopened in the wind. Allegedly he hit the ground at 100 mi/h after falling for 8 s. Test the accuracy of this account.

[Use $\frac{dv}{dt} = -g - \rho v$. Find ρ assuming that $v_\tau = 100$ mi/h.]

Sol'n:

$$v_\tau = -100 \text{ mi/h} = -100 \frac{\text{mi}}{\text{h}} \cdot 5280 \frac{\text{ft}}{\text{mi}} \cdot \frac{1}{3600} \frac{\text{h}}{\text{s}}$$

$$= -\frac{440}{3} \text{ ft/s} = -146.66\cdots \text{ ft/s}$$

$$v_\tau = -g/\rho \implies \rho = -g/v_\tau$$

$$\rho = -32/\left(-\tfrac{440}{3}\right) = \frac{12}{55} = 0.218\cdots$$

$$v(t) = (v_0 - v_\tau)e^{-\rho t} + v_\tau$$

$$v(t) = \left(0 + \frac{440}{3}\right)\exp\left(-\frac{12}{55}t\right) - \frac{440}{3}$$

$$v(t) = \frac{440}{3}\left(\exp\left(\frac{-12}{55}t\right) - 1\right)$$

$$y(t) = \int v(t)\,dt, \qquad y(0) = 1200$$

$$y(t) = \frac{440}{3}\left(\frac{-55}{12}\exp\left(\frac{-12}{55}t\right) - t\right) + C$$

$$y(t) = \frac{-6050}{9}\exp\left(\frac{-12}{55}t\right) - \frac{440}{3}t + C$$

$$y(0) = 1200 \Rightarrow \frac{-6050}{9} + C = 1200$$

$$\Rightarrow C = 1200 + \frac{6050}{9}$$

$$\Rightarrow C = \frac{16850}{9} = 1872.22\ldots$$

$$y(t) = -\frac{6050}{9}\exp\left(-\frac{12}{55}t\right) - \frac{440}{3}t + \frac{16850}{9}$$

$y(t) = 0$ ← solve using GeoGebra.

$t = 12.46\cdots$ s

$\therefore$ we calculate that the paratrooper fell for about 12.46 seconds.

② Air resistance proportional to the square of the velocity

Upward motion: $(v > 0)$ $\quad \dfrac{dv}{dt} = -g - \rho v^2$

Downward motion: $(v < 0)$ $\quad \dfrac{dv}{dt} = -g + \rho v^2$

This is a better model for motion at high velocities.

Terminal velocity:

$$-g + \rho v^2 = 0 \implies v = \pm\sqrt{\frac{g}{\rho}}.$$

But since $v_\tau < 0$, we have

$$\boxed{v_\tau = -\sqrt{\frac{g}{\rho}}}.$$

Upward motion:

$$v(t) = \sqrt{\frac{g}{\rho}}\tan\left(C_1 - t\sqrt{\rho g}\right), \quad C_1 = \tan^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)$$

$$y(t) = y_0 + \frac{1}{\rho}\ln\left|\frac{\cos\left(C_1 - t\sqrt{\rho g}\right)}{\cos C_1}\right|$$

Downward motion:

$$v(t) = \sqrt{\frac{g}{\rho}}\tanh\left(C_2 - t\sqrt{\rho g}\right), \quad C_2 = \tanh^{-1}\left(v_0\sqrt{\frac{\rho}{g}}\right)$$

$$y(t) = y_0 - \frac{1}{\rho}\ln\left|\frac{\cosh\left(C_2 - t\sqrt{\rho g}\right)}{\cosh C_2}\right|$$

(<u>Note</u>:

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right)$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

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