12.1 3-D Coordinate System

Definitions

The 3-D coordinate system, denoted $\mathbb{R}^3$, is the set of points

$$\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$$

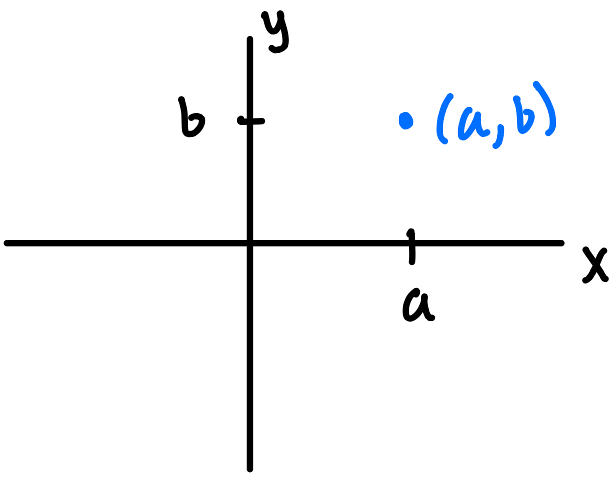
"in the set of real numbers"

The 2-D Cartesian coordinate system is denoted $\mathbb{R}^2$.

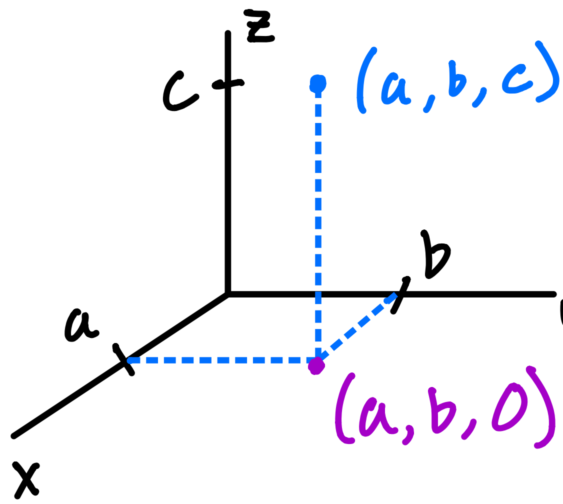
An equation involving x, y in $\mathbb{R}^2$ is called a curve.

An equation involving x, y, z in $\mathbb{R}^3$ is called a surface.

Point (a,b) in $\mathbb{R}^2$:



Point (a,b,c) in $\mathbb{R}^3$:



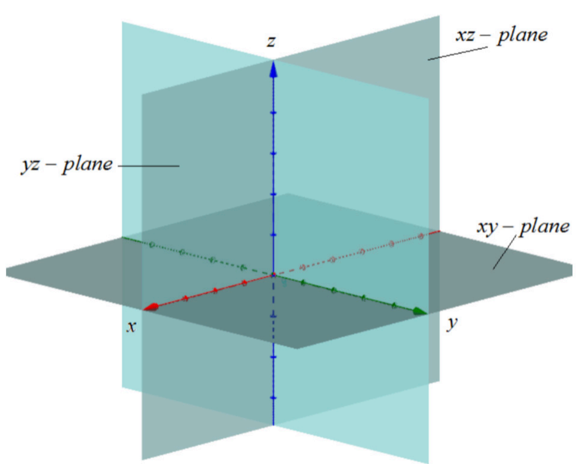
Coordinate Planes

Notice when $z=0$ in $\mathbb{R}^3$ we get the xy-plane.

Similarly, $x=0 \Rightarrow$ yz-plane

$y=0 \Rightarrow$ xz-plane

These are called the coordinate planes.



Distance and Midpoint in $\mathbb{R}^3$

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2) \in \mathbb{R}^3$.

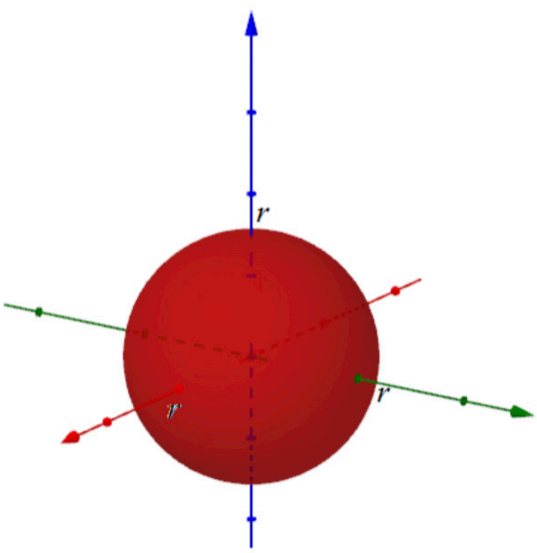
Distance:

$$|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

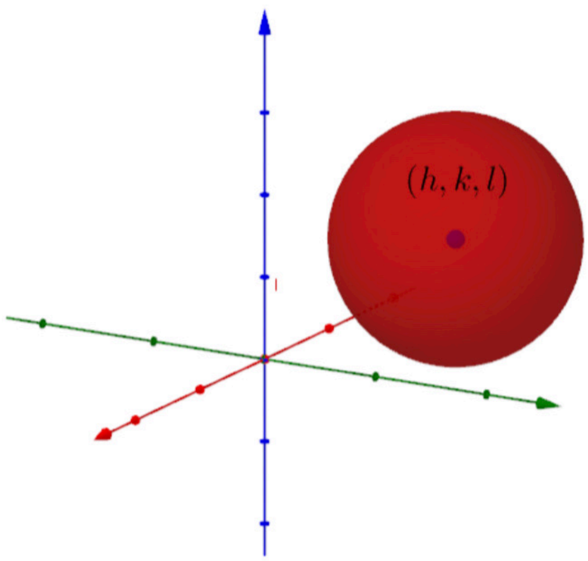
Midpoint:

$$\text{Mid}(PQ) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$$

Equation of a sphere



$x^2 + y^2 + z^2 = r^2$

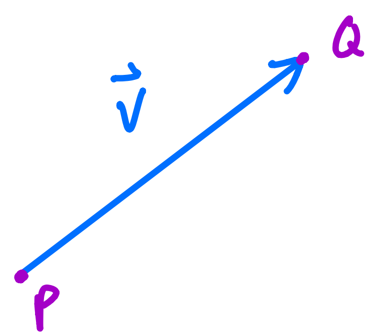


$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$

12.2 Vectors

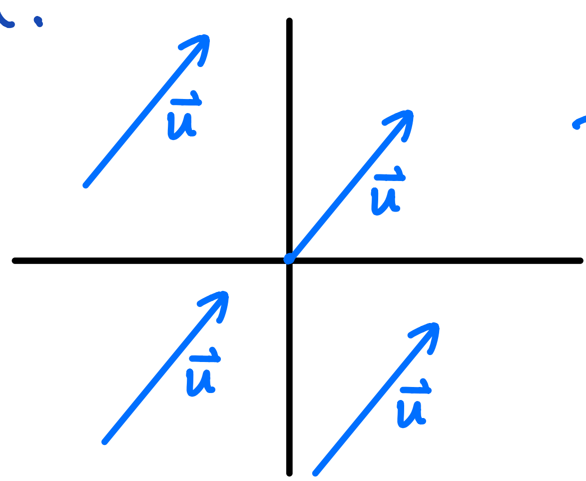
Def (Vector)

A quantity with both a direction and a magnitude (length).



$\vec{v} = \overrightarrow{PQ}$

A vector is defined only by length and direction, so it's position is not determined.

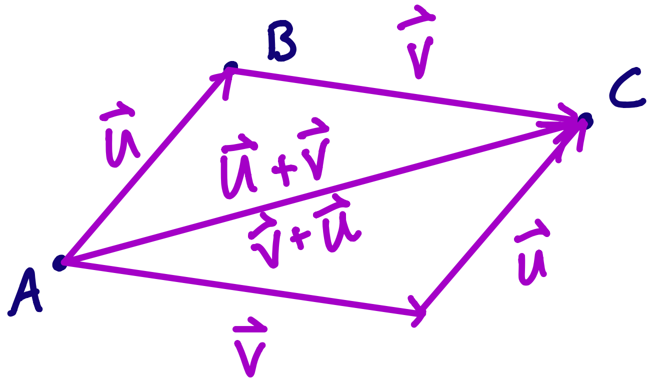


These are all the same vector $\vec{u}$

The vector with initial point at the origin is called the position vector.

Vector Addition

Let $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{BC}$.



Then $\vec{u} + \vec{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

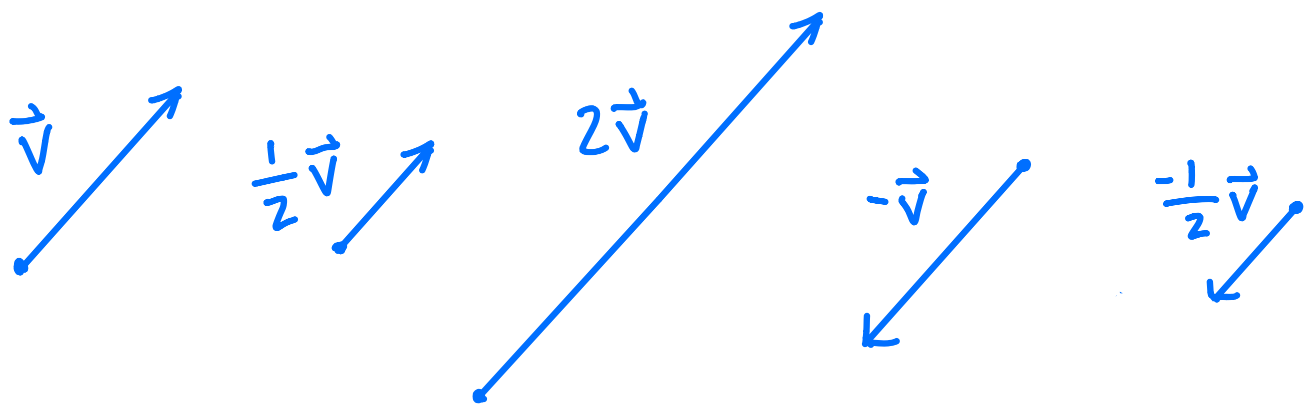
Notice how the picture above forms a parallelogram. This implies that $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

Scalar mult./Subtraction

Multiplying a vector $\vec{v}$ by a scalar c will stretch or compress the vector.

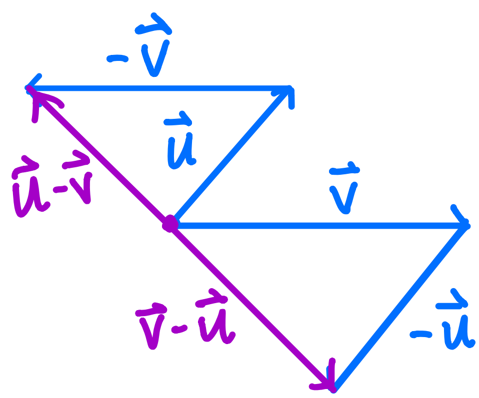
If $c > 0$, then $c\vec{v}$ keeps the same direction.

If $c < 0$, then $c\vec{v}$ points in the opposite direction of $\vec{v}$.



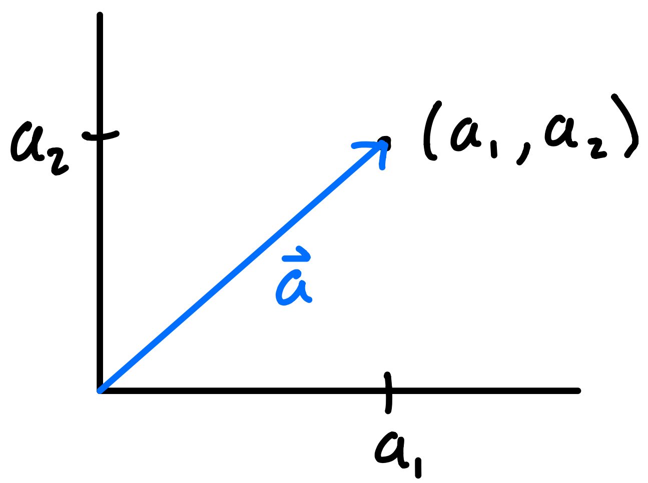
Vector Subtraction

$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$

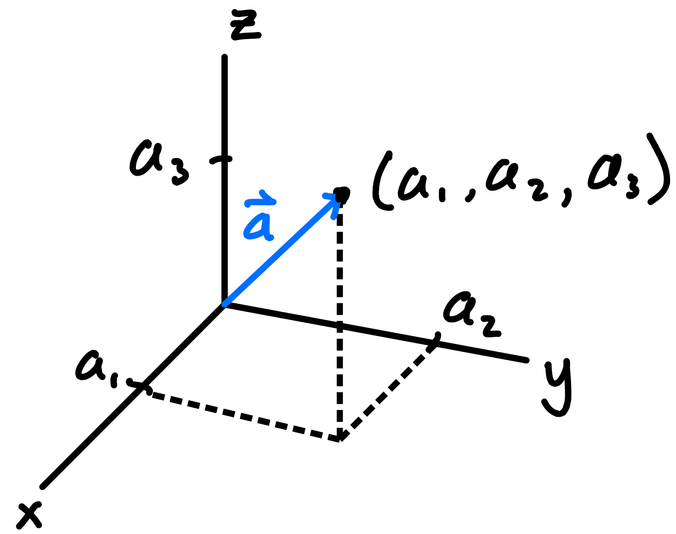


Vector Components

Let $\vec{a}$ be defined by $\vec{a} = \langle a_1, a_2 \rangle$ or $\vec{a} = \langle a_1, a_2, a_3 \rangle$. a_1, a_2, a_3 are called the components of vector $\vec{a}$.

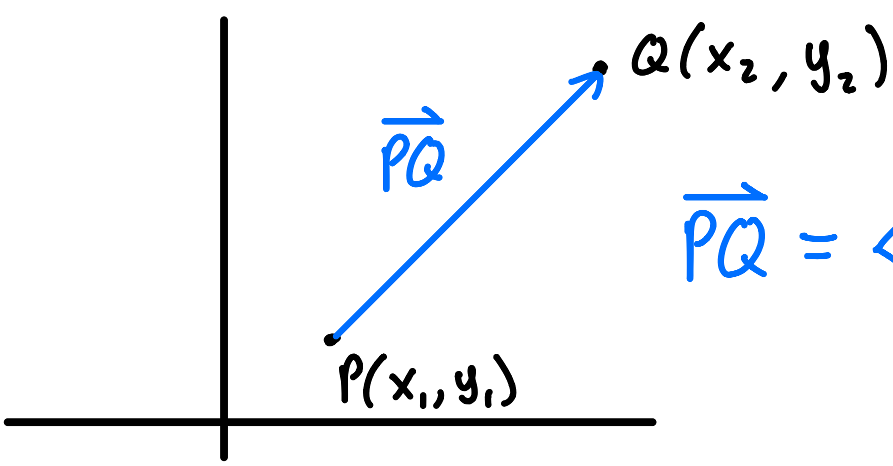


$\vec{a} = \langle a_1, a_2 \rangle$



$\vec{a} = \langle a_1, a_2, a_3 \rangle$

The components are the displacement from the initial point.



$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$

Vector Magnitude (length)

Let $\vec{a} = \langle a_1, a_2 \rangle$, then the magnitude is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2}$$

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, then the magnitude is

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vector Operations

Let $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$, and c be a scalar. Then

i) $\vec{a} + \vec{b} = \langle a_1 + b_1 \, , a_2 + b_2 \rangle$

ii) $\vec{a} - \vec{b} = \langle a_1 - b_1 \, , a_2 - b_2 \rangle$

iii) $c\vec{a} = \langle ca_1 \, , ca_2 \rangle$

Works similarly in 3-D coordinates.

Ex 1 Let $\vec{a} = \langle 4, 0, 3 \rangle$, $\vec{b} = \langle -2, 1, 5 \rangle$.

Find $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $3\vec{b}$, $|\vec{a}|$, $|\vec{b}-\vec{a}|$, and $2\vec{a}+4\vec{b}$.

Sol

- $\vec{a}+\vec{b} = \langle 4,0,3 \rangle + \langle -2,1,5 \rangle = \langle 2,1,8 \rangle$
- $\vec{a}-\vec{b} = \langle 4,0,3 \rangle - \langle -2,1,5 \rangle = \langle 6,-1,-2 \rangle$
- $3\vec{b} = 3\langle -2,1,5 \rangle = \langle -6,3,15 \rangle$
- $|\vec{a}| = |\langle 4,0,3 \rangle| = \sqrt{4^2+0^2+3^2} = \sqrt{25} = 5$
- $|\vec{b}-\vec{a}| = |\langle -2,1,5 \rangle - \langle 4,0,3 \rangle|$

  $= |\langle -6,1,2 \rangle| = \sqrt{(-6)^2+1^2+2^2} = \sqrt{41}$
- $2\vec{a}+4\vec{b} = 2\langle 4,0,3 \rangle + 4\langle -2,1,5 \rangle$

  $= \langle 8,0,6 \rangle + \langle -8,4,20 \rangle = \langle 0,4,26 \rangle$

Standard Base Vectors

$i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, $k = \langle 0, 0, 1 \rangle$

(in 2D they are $i = \langle 1, 0 \rangle$, $j = \langle 0, 1 \rangle$)

Any vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ can be written as $\vec{a} = a_1 i + a_2 j + a_3 k$.

For example,

$$\vec{a} = \langle 2, 1, -3 \rangle = 2i + j - 3k$$

$$\vec{b} = \langle 4, 2 \rangle = 4i + 2j$$

Unit Vector

A unit vector is any vector with length (magnitude) 1. The standard base vectors are unit vectors.

In general, if $\vec{a}$ is any vector then

$\dfrac{\vec{a}}{|\vec{a}|}$ is the unit vector in the direction of $\vec{a}$.

Ex 2 Let $\vec{a} = \langle 2, -1, 2 \rangle$. Find the unit vector in the direction of $\vec{a}$.

Sol

First step is to find the magnitude:

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$$

The unit vector is

$$\vec{u} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{3}\vec{a} = \frac{1}{3}\langle 2, -1, 2 \rangle = \left\langle \frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

Practice Problems

1) Let $P(1,2,3)$, $Q(-3,0,4) \in \mathbb{R}^3$.

Find $\overrightarrow{PQ}$ and $|\overrightarrow{PQ}|$.

2) Let $\vec{a} = \langle 6,-4,1 \rangle$, $\vec{b} = \langle 0,2,-3 \rangle$.

i) Find $\vec{a}+\vec{b}$ and $3\vec{b}-2\vec{a}$.

ii) Find the unit vector in the direction of $\vec{b}$.

Solutions

1) $P(1,2,3)$, $Q(-3,0,4)$

$$\overrightarrow{PQ} = \langle -3-1, 0-2, 4-3 \rangle = \underline{\langle -4, -2, 1 \rangle}$$

$$|\overrightarrow{PQ}| = |\langle -4, -2, 1 \rangle| = \sqrt{(-4)^2 + (-2)^2 + 1^2}$$

$$= \underline{\sqrt{21}}$$

2) $\vec{a} = \langle 6, -4, 1 \rangle$, $\vec{b} = \langle 0, 2, -3 \rangle$

i) $\vec{a} + \vec{b} = \langle 6, -4, 1 \rangle + \langle 0, 2, -3 \rangle$

$$= \underline{\langle 6, -2, -2 \rangle}$$

$$3\vec{b} - 2\vec{a} = 3\langle 0, 2, -3 \rangle - 2\langle 6, -4, 1 \rangle$$

$$= \underline{\langle -12, 14, -11 \rangle}$$

ii) Unit vector in the dir. of $\vec{b} = \langle 0, 2, -3 \rangle$:

$$\frac{\vec{b}}{|\vec{b}|} = \frac{\vec{b}}{\sqrt{0^2+2^2+(-3)^2}} = \frac{1}{\sqrt{13}}\vec{b} = \underline{\left\langle 0, \frac{2}{\sqrt{13}}, \frac{-3}{\sqrt{13}} \right\rangle}$$

Suggested Textbook Exc.

12.1 (5, 8, 13, 26, 31, 36)

12.2 (17, 19, 22, 23, 25, 26, 27)

Quiz 3

(10.5, 12.1, 12.2)