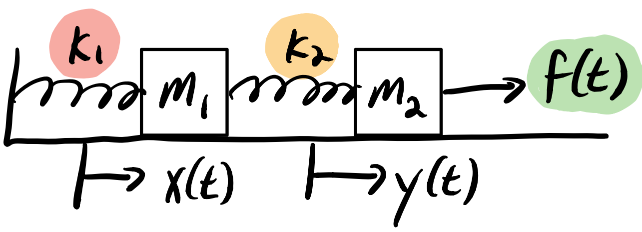
Chapter 5: Linear Systems of Differential Equations

5.1 / First-Order Systems and Applications

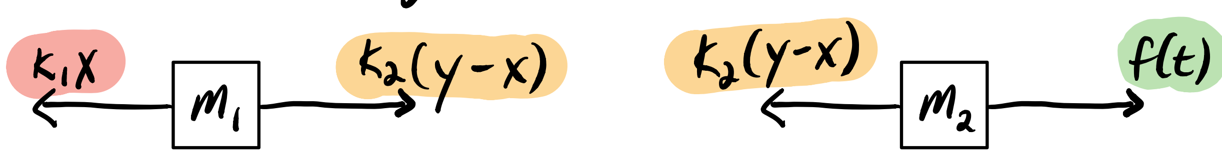
Often in applications, we need to solve a system of differential equations involving more than one function.

Example: Two masses and two springs



- Spring 1 is stretched by $x(t)$ units of length, resulting in a force of $k_1 x(t)$.
- Spring 2 is stretched by $y(t) - x(t)$ units of length, resulting in a force of $k_2\left(y(t) - x(t)\right)$.

The forces acting on m_1 and m_2 are:



Since $F = ma$, we have:

$$\boxed{\begin{aligned} -k_1 x + k_2(y-x) &= m_1 x'' \\ -k_2(y-x) + f(t) &= m_2 y'' \end{aligned}}$$

This is a system of two differential equations involving the two functions $x(t)$ and $y(t)$.

Suppose, for example, that

$$m_1 = 2, \quad m_2 = 1, \quad k_1 = 4, \quad k_2 = 2$$

and $f(t) = 40 \sin 3t$. Then:

$$\boxed{\begin{aligned} 2x'' &= -4x + 2(y-x) \\ y'' &= -2(y-x) + 40 \sin 3t \end{aligned}}$$

Let's solve this numerically using the initial conditions:

$$\boxed{x(0) = x'(0) = y(0) = y'(0) = 0}$$

This is a second-order linear system of differential equations. In order to solve this using standard numerical solvers, we need to convert it into a first-order linear system.

Let $u_1 = x$, $u_2 = x'$, $u_3 = y$, $u_4 = y'$.

Then $u_1' = u_2$, $u_2' = x''$, $u_3' = u_4$, $u_4' = y''$,

so we have:

$$2u_2' = -4u_1 + 2(u_3 - u_1)$$
$$u_4' = -2(u_3 - u_1) + 40\sin 3t$$

In summary:

$$\boxed{\begin{array}{l} u_1' = u_2 \\ u_2' = -3u_1 + u_3 \\ u_3' = u_4 \\ u_4' = 2u_1 - 2u_3 + 40\sin 3t \end{array}}$$

first-order linear system

The initial conditions are:

$$\boxed{u_1(0) = u_2(0) = u_3(0) = u_4(0) = 0}$$

Watch the lecture video to see how to solve this initial value problem in the Julia language with the DifferentialEquations package.

Examples:

Transform the given diff. eq'n into an equivalent system of first-order diff. eq'ns.

(#5) $x^{(3)} = (x')^2 + \cos x$

Let $u_1 = x$, $u_2 = x'$, $u_3 = x''$.

Then $u_1' = u_2$, $u_2' = u_3$, $u_3' = x^{(3)}$.

$$\boxed{\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ u_3' &= (u_2)^2 + \cos u_1 \end{aligned}}$$

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(#9) $x'' = 3x - y + 2z$

$y'' = x + y - 4z$

$z'' = 5x - y - z$

Let $u_1 = x$, $u_2 = x'$, $u_1' = u_2$, $u_2' = x''$

$u_3 = y$, $u_4 = y'$, $u_3' = u_4$, $u_4' = y''$

$u_5 = z$, $u_6 = z'$. $u_5' = u_6$, $u_6' = z''$

$$\boxed{\begin{aligned} u_1' &= u_2 \\ u_2' &= 3u_1 - u_3 + 2u_5 \\ u_3' &= u_4 \\ u_4' &= u_1 + u_3 - 4u_5 \\ u_5' &= u_6 \\ u_6' &= 5u_1 - u_3 - u_5 \end{aligned}}$$

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Solving Linear Systems

Examples:

(#11) $x' = y, \quad y' = -x, \quad x(0) = 1, \quad y(0) = 0$

$$x' = y \implies x'' = y' \qquad x'(0) = 0$$

$$\implies x'' = -x$$

$$\implies \boxed{x'' + x = 0}$$

$$\left[s^2 X(s) - s x(0) - x'(0)\right] + X(s) = 0$$

$$(s^2+1) X(s) - s = 0$$

$$X(s) = \frac{s}{s^2+1} \implies x(t) = \cos t$$

$$y = x' \implies y(t) = -\sin t$$

(plot the direction field and the sol'n in Julia)

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(#17) $x' = y, \quad y' = 6x - y, \quad x(0) = 1, \quad y(0) = 2$

$$x'' = y' \implies x'' = 6x - y \qquad x'(0) = 2$$

$$\implies x'' = 6x - x'$$

$$\implies \boxed{x'' + x' - 6x = 0}$$

$$\left[s^2 X(s) - s x(0) - x'(0)\right] + \left[s X(s) - x(0)\right] - 6X(s) = 0$$

$$(s^2 + s - 6) X(s) - s - 2 - 1 = 0$$

$$X(s) = \frac{s+3}{(s+3)(s-2)} = \frac{1}{s-2}$$

$$\boxed{x(t) = e^{2t}, \qquad y(t) = 2e^{2t}}$$

(plot the direction field and the sol'n in Julia)

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