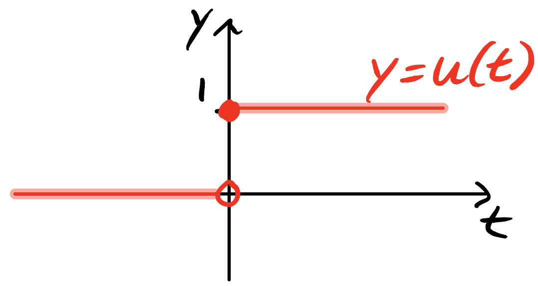
4.5] Piecewise Continuous Functions

The <u>unit step function</u> (a.k.a. the **Heaviside step function**) is given by:

$$\boxed{u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}}$$

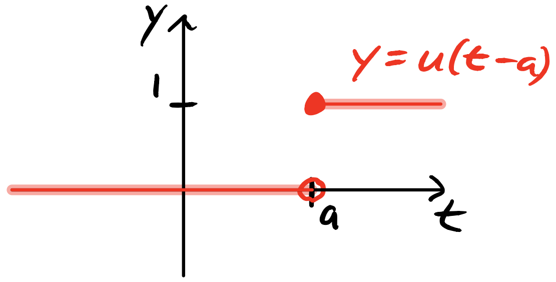


We can use this function to "switch on" or "switch off" a signal at time $t=a$.

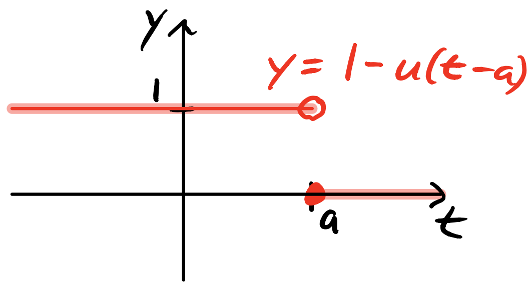
$$\boxed{u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}}$$

$$\boxed{1 - u(t-a) = \begin{cases} 1, & t < a \\ 0, & t \geq a \end{cases}}$$

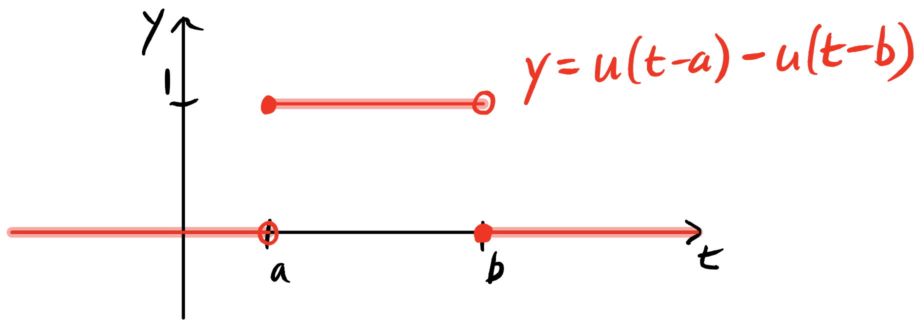
$$\boxed{u(t-a) - u(t-b) = \begin{cases} 0, & t < a \\ 1, & a \leq t < b \\ 0, & t \geq b \end{cases}}$$



"switch on"



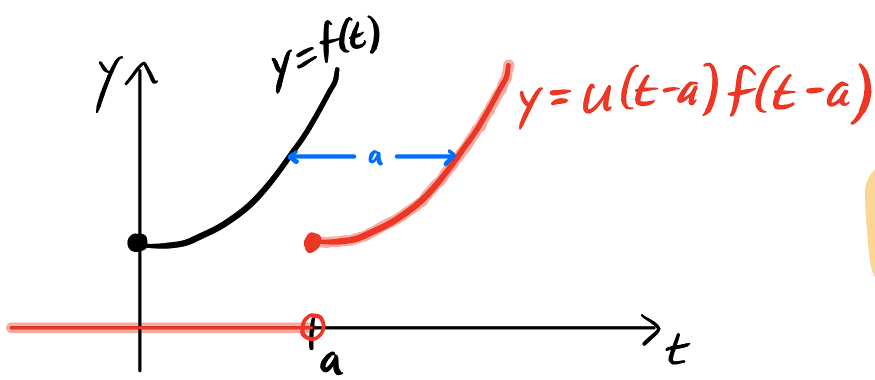
"switch off"



"switch on and off"

We use $u(t-a)$ to model add a "time delay" to a function $f(t)$:

$$\boxed{u(t-a)f(t-a) = \begin{cases} 0, & t < a \\ f(t-a), & t \geq a \end{cases}}$$



"time delay"

$$\boxed{\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)}$$

Examples:

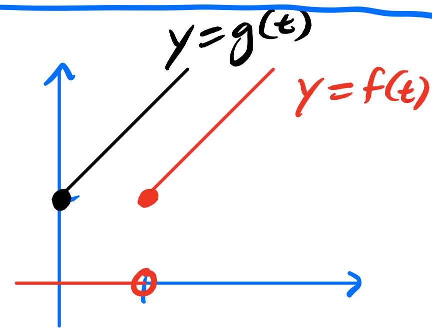
(#11) $f(t) = \begin{cases} 2, & 0 \le t < 3 \\ 0, & t \ge 3 \end{cases}$

$$f(t) = (1 - u(t-3)) \cdot 2$$

$$F(s) = 2\left(\frac{1}{s} - \frac{e^{-3s}}{s}\right)$$

$$\boxed{F(s) = \frac{2(1-e^{-3s})}{s}}$$

(#19) $f(t) = \begin{cases} 0, & t < 1 \\ t, & t \ge 1 \end{cases}$



$$f(t) = u(t-1)\,t$$

$$f(t) = u(t-1)\,g(t-1)$$

$g(t) = t+1$

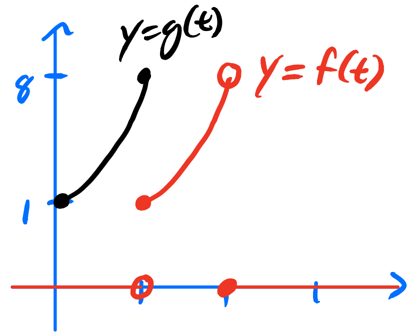
$g(t-1) = t$

$$F(s) = e^{-s}G(s)$$

$$F(s) = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$\boxed{F(s) = \frac{e^{-s}(1+s)}{s^2}}$$

(#22) $$f(t) = \begin{cases} 0, & t<1 \\ t^3, & 1 \le t \le 2 \\ 0, & t>2 \end{cases}$$



$$f(t) = \big(u(t-1) - u(t-2)\big)t^3$$

$$f(t) = u(t-1)t^3 - u(t-2)t^3$$

$$f(t) = u(t-1)g(t-1) - u(t-2)h(t-2)$$

$$g(t) = (t+1)^3$$

$$G(s) = \mathcal{L}\{(t+1)^3\}$$

$$G(s) = \mathcal{L}\{t^3 + 3t^2 + 3t + 1\}$$

$$G(s) = \frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}$$

$$h(t) = (t+2)^3$$

$$H(s) = \mathcal{L}\{(t+2)^3\}$$

$$H(s) = \mathcal{L}\{t^3 + 6t^2 + 12t + 4\}$$

$$H(s) = \frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{4}{s}$$

$$F(s) = e^{-s}G(s) - e^{-2s}H(s)$$

$$F(s) = e^{-s}\left(\frac{6}{s^4} + \frac{6}{s^3} + \frac{3}{s^2} + \frac{1}{s}\right)$$

$$- e^{-2s}\left(\frac{6}{s^4} + \frac{12}{s^3} + \frac{12}{s^2} + \frac{4}{s}\right)$$

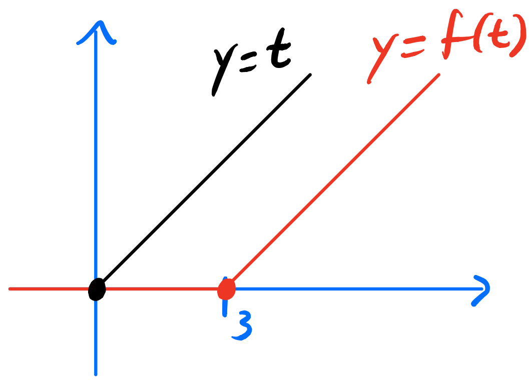
$$\boxed{\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)}$$

<u>Examples</u>:

(#1) $F(s) = \dfrac{e^{-3s}}{s^2}$ $\qquad\qquad$ $\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

$$f(t) = \mathcal{L}^{-1}\{e^{-3s} \cdot \frac{1}{s^2}\} = u(t-3)(t-3)$$

$$f(t) = \begin{cases} 0, & t<3 \\ t-3, & 3 \le t \end{cases}$$

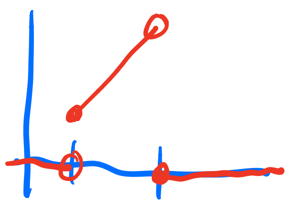


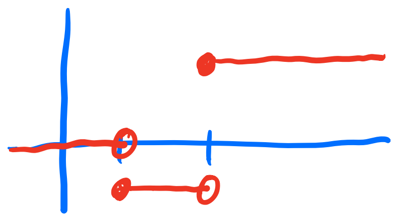
(#2) $F(s) = \dfrac{e^{-s} - e^{-3s}}{s^2}$ $\qquad \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$

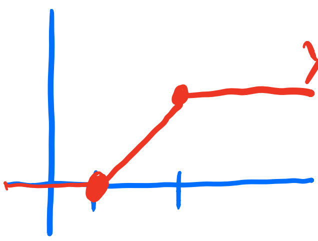
$$f(t) = \mathcal{L}^{-1}\left\{e^{-s} \cdot \frac{1}{s^2}\right\} - \mathcal{L}^{-1}\left\{e^{-3s} \cdot \frac{1}{s^2}\right\}$$

$$f(t) = u(t-1)(t-1) - u(t-3)(t-3)$$

$$f(t) = \left[u(t-1) - u(t-3)\right] t + \left[3u(t-3) - u(t-1)\right]$$







$$\boxed{f(t) = \begin{cases} 0, & t < 1 \\ t-1, & 1 \le t < 3 \\ 2, & t \ge 3 \end{cases}}$$

$$f(t) = u(t-1)(t-1) - u(t-3)(t-3)$$

$t < 1 \Rightarrow f(t) = 0$

$1 \le t < 3 \Rightarrow f(t) = t-1$

$t > 3 \Rightarrow f(t) = (t-1) - (t-3)$

ie $f(t) = 2$. //

Now we can solve differential equations involving piecewise continuous functions.

<u>Example:</u>

(#34) $x'' + x = f(t)$, $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

$x(0) = x'(0) = 0$

$$f(t) = \big(1 - u(t-1)\big)t$$

$g(t) = t+1$

$G(s) = \frac{1}{s^2} + \frac{1}{s}$

$$f(t) = t - u(t-1)g(t-1)$$

$$F(s) = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right)$$

$$F(s) = \frac{1 - e^{-s}(1+s)}{s^2}$$

$$x'' + x = f(t)$$

$$\left(s^2 X(s) - s x(0) - x'(0)\right) + X(s) = \frac{1 - e^{-s}(1+s)}{s^2}$$

$$X(s) = \frac{1 - e^{-s}(1+s)}{s^2(s^2+1)}$$

$$X(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\cdot\frac{1}{s^2(s^2+1)} - e^{-s}\cdot\frac{1}{s(s^2+1)}$$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$1 = A(s^2+1) + (Bs+C)s$$

$s=0$: $\quad 1 = A \qquad A = 1$

$s=i$: $\quad 1 = (Bi+C)i \quad \Big| \quad B = -1$

$\qquad\quad 1 = -B + Ci \quad \Big| \quad C = 0$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau} \leftarrow \text{let's use this!}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos\tau)\,d\tau$$

$$= \Big[\tau - \sin\tau\Big]_0^t$$

$$= t - \sin t$$

$$X(s) = \frac{1}{s^2(s^2+1)} - e^{-s}\cdot\frac{1}{s^2(s^2+1)} - e^{-s}\cdot\frac{1}{s(s^2+1)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t$$

$$x(t) = t - \sin t - u(t-1)\left(t-1-\sin(t-1)\right)$$
$$- u(t-1)\left(1-\cos(t-1)\right)$$

$0 \le t < 1 \Rightarrow x(t) = t - \sin t$

$t \ge 1 \Rightarrow x(t) = t - \sin t - t + 1 + \sin(t-1)$
$- 1 + \cos(t-1)$

$x(t) = \cos(t-1) + \sin(t-1) - \sin t$

$$\boxed{x(t) = \begin{cases} t - \sin t, & 0 \le t < 1 \\ \cos(t-1) + \sin(t-1) - \sin t, & t \ge 1 \end{cases}}$$

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