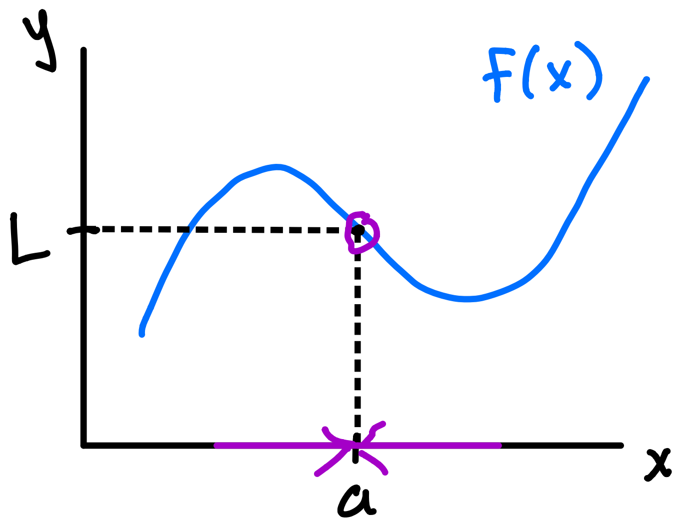
14.2 Limits and Continuity

Our goal in this section is to evaluate limits of the form

$$\lim_{(x,y)\to(a,b)} f(x,y)$$

Let's recall limits in one variable,

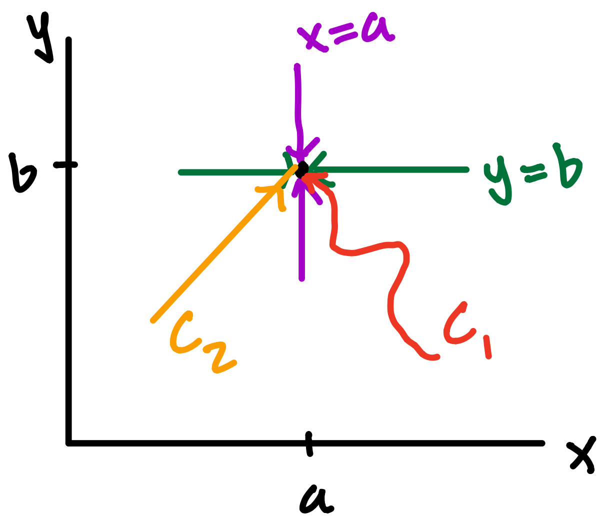


If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$

then $\lim_{x\to a} f(x) = L$

There are only two ways to approach the point $x=a$. That's why we only need to consider the left and right limits.

In the two variable case there are infinitely many ways to approach the point $(x,y)=(a,b)$.



To show $\lim\limits_{(x,y)\to(a,b)} f(x,y)=L$

we would need

$f(x,y)\to L$ as $(x,y)\to(a,b)$

along every path C_i

Note:

Suppose $f(x,y)\to L_1$ as $(x,y)\to(a,b)$ along C_1

and $f(x,y)\to L_2$ as $(x,y)\to(a,b)$ along C_2

If $L_1\neq L_2$ then the limit does not exist.

Thm 1

$f(x,y)$ is continuous at (a,b) iff

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Ex 1

Let $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

a) Find $\displaystyle\lim_{(x,y)\to(0,1)} f(x,y)$.

b) Show $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Sol

a) Since $f(x,y)$ is continuous everywhere except $(0,0)$, by Thm 1 we have,

$$\lim_{(x,y)\to(0,1)} f(x,y) = f(0,1) = \frac{0^2 - 1^2}{0^2 + 1^2} = -1$$

b) To show $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE we need to find two paths to approach $(0,0)$ that give different limits.

Let's try approaching along the x and y-axis first.

<u>Along y-axis</u>: $(x=0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{0^2 - y^2}{0^2 + y^2}$$

$$= \lim_{y\to 0} \frac{-y^2}{y^2} = -1$$

<u>Along x-axis</u>: $(y=0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

Since the limits don't match we know

$\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

<u>Ex 2</u> Let $f(x,y) = \dfrac{xy}{x^2+y^2}$. Determine

if $\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not.

<u>Sol</u>

Notice along x-axis $(y=0)$ we have

$f(x,0)=0$. So $f(x,y)\to 0$ as $(x,y)\to(0,0)$

along the x-axis.

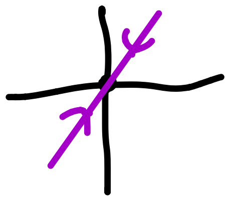
Similarly, along y-axis $(x=0)$ we have

$f(0,y)=0$. So $f(x,y)\to 0$ as $(x,y)\to(0,0)$

along y-axis.

We got the same limit but this does NOT mean the limit exists.

Let's try approaching $(0,0)$ along the line $y=x$,



$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{x\cdot x}{x^2+x^2}$$

$$= \lim_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$$

So $f(x,y) \to \frac{1}{2}$ as $(x,y)\to(0,0)$ along $y=x$.

Therefore the limit does not exist.

Ex 3 Let $f(x,y) = \dfrac{xy^2}{x^2+y^4}$.

Show $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Sol

We can try to save time by considering the limit along any line through the origin, i.e. along $y=mx$.

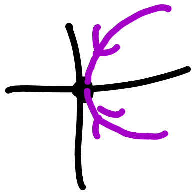
Along $y=mx$:

$$f(x,y) = f(x,mx) = \frac{x(mx)^2}{x^2+(mx)^4} = \frac{m^2x^3}{x^2(1+m^4x^2)}$$

$$= \frac{m^2x}{1+m^4x^2} \longrightarrow 0 \quad \text{as} \quad x\to 0.$$

So $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along $y=mx$.

All of these linear paths gave the same limit so let's try approaching $(0,0)$ along the quadratic path $x=y^2$.

Along $x=y^2$: 

$$f(x,y)=f(y^2,y)=\frac{y^2\cdot y^2}{(y^2)^2+y^4}=\frac{y^4}{2y^4}=\frac{1}{2}$$

So $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along $x=y^2$ and the limit does not exist.

Note: We choose the path $x=y^2$ because this substitution lets us combine the denominator terms of $f(x,y)=\frac{xy^2}{x^2+y^4}$.

This can be an effective strategy if you are not sure which path to try next.

Showing that a limit does exist is usually more difficult. Since there are infinitely many paths we can't show that they all give the same limit directly. Instead we would need to prove it using the following definition.

<u>Def 1</u> (Not Testable)

Let $f(x,y)$ be a function of two variables.

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all $(x,y) \in D$ with $\sqrt{(x-a)^2+(y-b)^2} < \delta$ we have $|f(x,y) - L| < \varepsilon$.

Ex 4 (Not Testable)

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$ if it exists.

Along $y=mx$:

$$f(x,y) = f(x,mx) = \frac{3x^2(mx)}{x^2+(mx)^2} = \frac{3mx}{1+m} \to 0$$

as $x \to 0$.

Along $x=y^2$:

$$f(x,y) = f(y^2,y) = \frac{3(y^2)^2y}{(y^2)^2+y^2} = \frac{3y^5}{y^2(y^2+1)}$$

$$= \frac{3y^3}{y^2+1} \to 0 \text{ as } y \to 0$$

Along $y=x^2$:

$$f(x,y) = f(x,x^2) = \frac{3x^2}{1+x^2} \to 0 \text{ as } x \to 0$$

Starting to suspect limit exists and is 0.

Let's prove it using definition.

Let $\varepsilon > 0$ be given. We need to find $\delta > 0$ such that if $\sqrt{x^2+y^2} < \delta$ then $|f(x,y) - 0| < \varepsilon$.

$$|f(x,y)| = \left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} = 3|y| \underbrace{\left(\frac{x^2}{x^2+y^2}\right)}_{\leq 1}$$

So $|f(x,y)| \leq 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}$

Therefore if we choose $\delta = \frac{\varepsilon}{3}$ then if $\sqrt{x^2+y^2} < \delta$ we have

$$|f(x,y)| \leq 3\sqrt{x^2+y^2} < 3\delta = 3\left(\frac{\varepsilon}{3}\right) = \varepsilon.$$

So by Def 1, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Sometimes it is also useful to convert to polar coordinates. In Ex 4,

$$f(x,y) = \frac{3x^2y}{x^2+y^2} \ominus$$

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

$$\ominus \frac{3r^2\cos^2\theta \cdot r\sin\theta}{r^2} = 3r\cos^2\theta\sin\theta$$

If $(x,y) \to (0,0)$ then $r \to 0$ (If $r \to 0$ it doesn't matter what θ is). So

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{r\to 0} 3r\cos^2\theta\sin\theta = 0$$

Practice Problems

1) Show that the following limits do not exist.

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5+y^4}{x^4+3y^4}$ b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$

Solutions

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^5+y^4}{x^4+3y^4}$, $f(x,y) = \dfrac{x^5+y^4}{x^4+3y^4}$

Along x-axis: $(y=0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x^5}{x^4} = \lim_{x\to 0} x$$

$= 0$. So $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along x-axis

Along y-axis: $(x=0)$

$$f(x,y) = f(0,y) = \frac{y^4}{3y^4} = \frac{1}{3}$$

So $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along y-axis.

Therefore the limit does not exist.

b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$, $f(x,y) = \dfrac{x^3 y}{x^6 + y^2}$

Along x-axis: $(y=0)$

$$f(x,y) = f(x,0) = \frac{0}{x^6 + 0^2} = 0$$

So $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along x-axis.

The y-axis will also give us 0 so let's try something different.

Letting $y=x^3$ will allow us to simplify the denominator x^6+y^2 into a single term.

Along $y=x^3$:

$$f(x,y)=f(x,x^3)=\frac{x^3(x^3)}{x^6+(x^3)^2}=\frac{x^6}{2x^6}=\frac{1}{2}$$

So $f(x,y)\to\frac{1}{2}$ as $(x,y)\to(0,0)$ along $y=x^3$ and therefore the limit does not exist.

Suggested Textbook Exc (14.2)

6, 11, 12, ~~16~~ 15

Midterm

Thurs. 3/4 5:00pm–9:00pm

(Covers Weeks 1–8)