1.6 Substitution Methods and Exact Eq'ns

To solve some integrals, we need to use <u>u-substitution</u>.

---

eg $\displaystyle\int \frac{2x}{\sqrt{1+x^2}}\,dx = \int \frac{du}{\sqrt{u}} = 2u^{1/2}+C$

$\left[\begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \end{array}\right]$

$= 2\sqrt{u}+C$

$= 2\sqrt{1+x^2}+C$

The same is true for some differential equations.

Example: (#17)

$y' = (4x+y)^2$

$v = 4x+y$

$y = v-4x$

$y' = v'-4$

$v'-4 = v^2$

$v' = v^2+4$

$$\frac{dv}{dx} = v^2+4$$

$$\int \frac{dv}{v^2+4} = \int dx$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\left[\begin{array}{l} v = 2\tan\theta \\ dv = 2\sec^2\theta \, d\theta \end{array}\right]$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$4 + 4\tan^2\theta = 4\sec^2\theta$$

$$\int \frac{2\sec^2\theta \, d\theta}{4 + 4\tan^2\theta} = \int dx$$

$$\int \frac{2\sec^2\theta \, d\theta}{4\sec^2\theta} = x + C_1$$

$$\int \frac{1}{2} d\theta = x + C_1$$

$$v = 2\tan\theta$$

$$\frac{1}{2}\theta = x + C_1$$

$$\theta = \tan^{-1}\left(\frac{v}{2}\right)$$

$$v = 4x + y$$

$$\frac{1}{2}\tan^{-1}\left(\frac{v}{2}\right) = x + C_1$$

$$\tan^{-1}\left(\frac{v}{2}\right) = 2x + 2c_1$$

$C = 2c_1$

$$\frac{v}{2} = \tan(2x + C)$$

$$v = 2\tan(2x + C)$$

$$4x + y = 2\tan(2x + C)$$

$$\therefore \boxed{y(x) = 2\tan(2x + C) - 4x}.$$

Homogeneous Equations

A diff. eq'n of the form

$$A x^m y^n \frac{dy}{dx} = B x^p y^q + C x^r y^s$$

is called homogeneous if

$$m+n = p+q = r+s.$$

eg (#2) $2xyy' = x^2 + 2y^2$

(#3) $xy' = y + 2\sqrt{xy}$

(#7) $xy^2 y' = x^3 + y^3$

are homogeneous.

Dividing both sides of

$$A x^m y^n \frac{dy}{dx} = B x^p y^q + C x^r y^s$$

by x^K, where $K = m+n = p+q = r+s$,

we obtain

$$A\left(\frac{y}{x}\right)^n \frac{dy}{dx} = B\left(\frac{y}{x}\right)^q + C\left(\frac{y}{x}\right)^s.$$

For homogeneous eq'ns we use the substitution $\boxed{v = \frac{y}{x}}$.

Example: (#2)

$$2xyy' = x^2 + 2y^2$$

$$\boxed{y' = \frac{1}{2}\frac{x}{y} + \frac{y}{x}}$$

$$\frac{2xyy'}{x^2} = \frac{x^2}{x^2} + 2\frac{y^2}{x^2}$$

$$2\left(\frac{y}{x}\right)y' = 1 + 2\left(\frac{y}{x}\right)^2$$

$$v = \frac{y}{x} \qquad y = xv$$

$$y' = xv' + 1 \cdot v$$

$$y' = xv' + v$$

$$2v(xv' + v) = 1 + 2v^2$$

$$xv' + \cancel{v} = \frac{1}{2v} + \cancel{v}$$

$$\boxed{v' = \frac{1}{2xv}}$$ separable

$$\frac{dv}{dx} = \frac{1}{2xv} \qquad \int 2v\,dv = \int \frac{dx}{x}$$

$$v^2 = \ln|x| + C$$

$$\left(\frac{y}{x}\right)^2 = \ln|x| + C$$

$$\boxed{y^2 = x^2(\ln|x| + C)}$$

is the general implicit sol'n.

Bernoulli Equations

A diff. eq'n of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a <u>Bernoulli equation</u>.

Note:

$n=0$: $\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$ ← linear

$n=1$: $\boxed{\frac{dy}{dx} = \big(Q(x) - P(x)\big)y}$ ← separable

If $n \neq 0, 1$, then we can use the substitution

$$v = y^{1-n}$$

to obtain a <u>linear eq'n</u>.

Example: (#20)

$$y^2 y' + 2xy^3 = 6x$$

Bernoulli with $n = -2$

$$y' + 2xy = 6xy^{-2}$$

$$v = y^{1-n}$$

$$v = y^{1-(-2)}$$

$$\boxed{\begin{aligned} v &= y^3 \\ y &= v^{1/3} \end{aligned}}$$

$$y' = \frac{1}{3} v^{-2/3} \cdot v'$$

$$\frac{1}{3} v^{-2/3} \cdot v' + 2x v^{1/3} = 6x (v^{1/3})^{-2}$$

$$\boxed{v' + 6xv = 18x}$$ linear

$$\rho(x) = \exp\left(\int 6x\,dx\right)$$

$$\rho(x) = \exp(3x^2) = e^{3x^2}$$

$$e^{3x^2} v' + 6x e^{3x^2} v = 18x e^{3x^2}$$

$$\frac{d}{dx}\left[e^{3x^2}v\right] = 18xe^{3x^2}$$

$$e^{3x^2}v = \int 18xe^{3x^2}dx + C_1$$

$$\left[\begin{array}{l} u = 3x^2 \\ du = 6x\,dx \end{array}\right]$$

$$e^{3x^2}v = \int 3e^u du + C_1$$

$$e^{3x^2}v = 3e^u + C_1$$

$$e^{3x^2}v = 3e^{3x^2} + C_1$$

$$v = 3 + C_1e^{-3x^2}$$

$$v = y^3$$

$$y^3 = 3 + Ce^{-3x^2}$$

$$C = C_1$$

$$\therefore \boxed{y(x) = \sqrt[3]{3 + Ce^{-3x^2}}}$$

Exact Equations

Let's start with the implicit sol'n

$$x^2y^2 + x^3 + y^4 = C$$

and find the diff. eq'n.

Then

$$\frac{d}{dx}\left(x^2y^2 + x^3 + y^4\right) = \frac{d}{dx}(C)$$

$$x^2(2y)\frac{dy}{dx} + 2xy^2 + 3x^2 + 4y^3\frac{dy}{dx} = 0$$

$$(2xy^2+3x^2) + (2x^2y+4y^3)\frac{dy}{dx} = 0$$

$$\frac{\partial F}{\partial x} = F_x \qquad \frac{\partial F}{\partial y} = F_y$$

where $F(x,y) = x^2y^2 + x^3 + y^4$.

$$\boxed{\frac{d}{dx}F(x,y) = \frac{\partial F}{\partial x}\cdot\overbrace{\frac{dx}{dx}}^{=1} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx}}$$

Now how do we go from the eq'n

$$(2xy^2+3x^2) + (2x^2y+4y^3)\frac{dy}{dx} = 0$$

or, as it is often written,

$$(2xy^2+3x^2)\,dx + (2x^2y+4y^3)\,dy = 0,$$

to the sol'n $F(x,y) = C$?

<u>Ans</u>: Find $F(x,y)$ such that

$$F_x = 2xy^2+3x^2, \quad F_y = 2x^2y+4y^3.$$

<u>Note</u>: For any twice continuously diff'ble $F(x,y)$, we always have $F_{xy} = F_{yx}$, where

$$F_{xy} = \frac{\partial^2 F}{\partial y \partial x} \text{ and } F_{yx} = \frac{\partial^2 F}{\partial x \partial y}.$$

So we will only be able to find $F(x,y)$ if the diff. eq'n

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

satisfies $M_y = N_x$. ← test for exactness

For the diff. eq'n

$$\underbrace{(2xy^2+3x^2)}_{M(x,y)}dx + \underbrace{(2x^2y+4y^3)}_{N(x,y)}dy = 0,$$

we have

$$M_y = 4xy \quad \text{and} \quad N_x = 4xy.$$

Therefore, this diff. eq'n is <u>exact</u>.

Okay, now we know we can find $F(x,y)$.

Since $F_x = 2xy^2 + 3x^2$, we have

$$F(x,y) = \int (2xy^2 + 3x^2)\,dx + g(y)$$

integration constant could depend on y.

$$F(x,y) = x^2y^2 + x^3 + g(y).$$

Okay, now we just need to find $g(y)$. Differentiating w.r.t. y we have

$$F_y = 2x^2y + g'(y).$$

But recall that $F_y = 2x^2y + 4y^3$, so

$$2x^2y + g'(y) = 2x^2y + 4y^3$$

$$g'(y) = 4y^3$$

$$\therefore \quad g(y) = \int 4y^3\,dy + C_1$$

$$g(y) = y^4 + C_1.$$

$$\therefore \; F(x,y) = x^2y^2 + x^3 + g(y)$$

$$= x^2y^2 + x^3 + y^4 + C_1$$

and the <u>sol'n</u> to the <u>exact diff. eq'n</u> is $F(x,y) = 0$:

$$x^2y^2 + x^3 + y^4 + C_1 = 0$$

or, letting $C = -C_1$,

$$\boxed{x^2y^2 + x^3 + y^4 = C.}$$

<u>Example:</u> (#37)

$$\underbrace{(\cos x + \ln y)}_{M(x,y)}dx + \underbrace{\left(\frac{x}{y} + e^y\right)}_{N(x,y)}dy = 0$$

$$F_x \qquad\qquad\qquad F_y$$

Exactness test: $M_y = N_x$?

$$M_y = \frac{1}{y} \qquad N_x = \frac{1}{y}$$

$\therefore$ diff. eq'n is <u>exact</u>.

$$F_x = \cos x + \ln y$$

$$F(x,y) = \int (\cos x + \ln y)\,dx + g(y)$$

$$F(x,y) = \sin x + x \ln y + g(y)$$

$$F_y = \frac{x}{y} + g'(y)$$

Also, $F_y = N(x,y) = \frac{x}{y} + e^y$

$$\therefore \quad \frac{x}{y} + g'(y) = \frac{x}{y} + e^y$$

$$g'(y) = e^y$$

$$g(y) = e^y + C_1$$

$\therefore \quad F(x,y) = \sin x + x\ln y + e^y + C_1$

$\therefore$ the general implicit sol'n is

$$\boxed{\sin x + x\ln y + e^y = C}$$

//