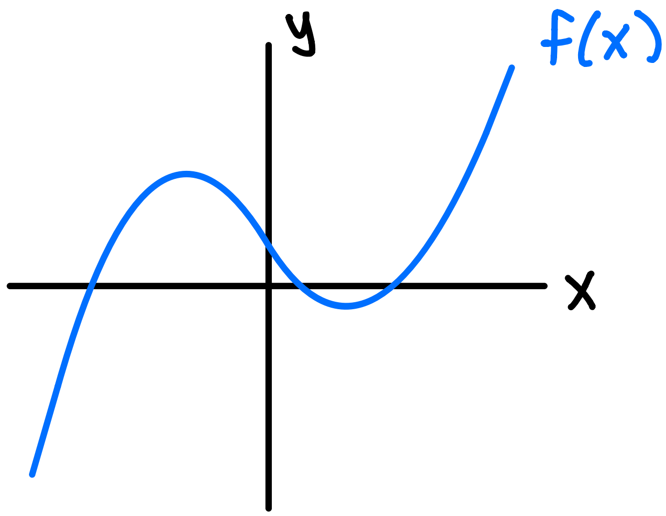
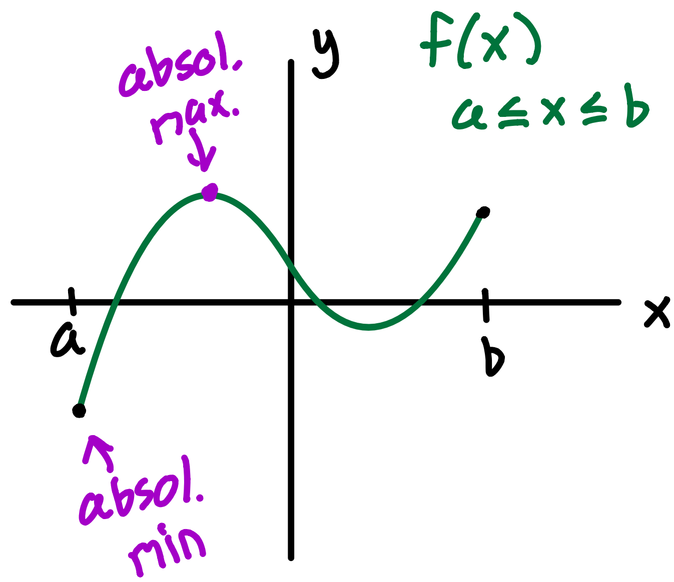
14.8 Lagrange Multipliers

In this section we will be finding absolute (global) extrema.



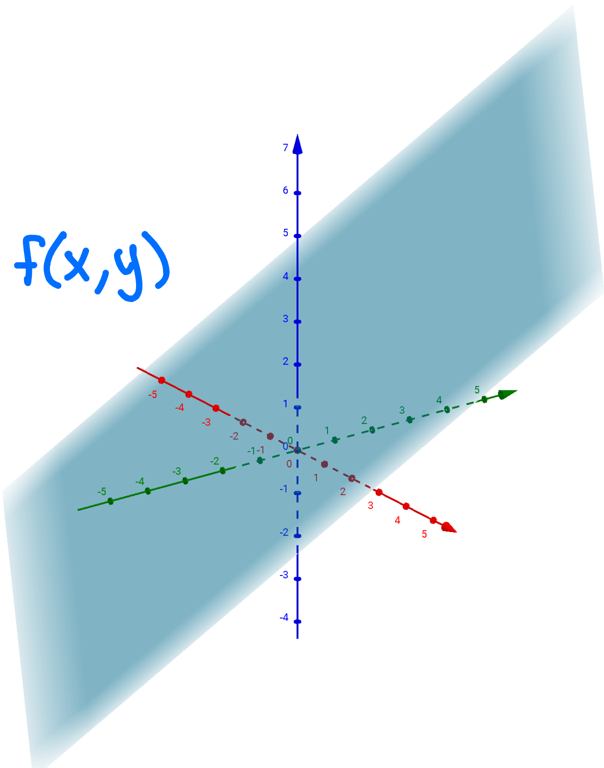


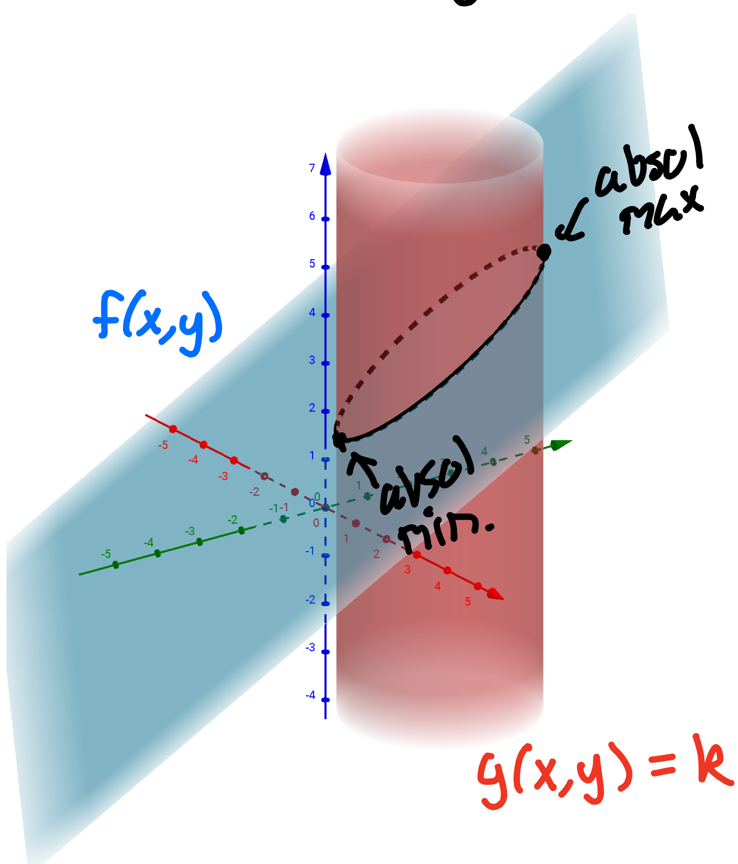
Notice in the unrestricted case on left there is no absolute max/min.

The Extreme Value Thm from Calc 1 says that if f is cont. on a closed interval $[a,b]$, then f must have an absolute max. and absolute min. in $[a,b]$.

These absolute extrema might occur at critical points in $[a,b]$ or on the endpoints $x=a$, $x=b$.

Now let's consider what happens when we restict the domain of $f(x,y)$.





No absolute max/min on left but if we restrict the domain to $g(x,y)=k$ then we do get absolute extrema.

Method of Lagrange Multipliers

Find the absolute max/min. of $f(x,y)$ subject to the constraint $g(x,y)=k$ (with $\nabla g \neq 0$).

Step 1: Solve the system of equations

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = k$$

Step 2: Plug in all the points (x,y) found above into the function $f(x,y)$. The largest value is the absolute max and the smallest is the absolute min.

Note: The scalar λ is called the Lagrange Multiplier.

Notice that the eqn $\nabla f(x,y) = \lambda \nabla g(x,y)$ can be written as

$$\langle f_x(x,y), f_y(x,y) \rangle = \lambda \langle g_x(x,y), g_y(x,y) \rangle$$

$$\Rightarrow \begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \end{cases}$$

So including the constraint eqn $g(x,y) = k$ we are solving a system of 3 eqns with 3 unknowns x, y, λ.

Ex 1 Find the absolute min of $f(x,y) = x^2 + y^2 - 6x - 2y + 1$ subject to the constraint $x + y = 2$.

Sol

We need our constraint to have the form $g(x,y) = k$. So we can let $g(x,y) = x + y$.

To use Langrange Mult. method we must solve,

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$g(x,y) = k$$

1. Find ∇f and ∇g.

$$\nabla f = \langle 2x-6, 2y-2 \rangle$$

$$\nabla g = \langle 1, 1 \rangle$$

2. Solve system.

$$\langle 2x-6, 2y-2 \rangle = \lambda \langle 1, 1 \rangle$$

$$\Rightarrow \begin{cases} 2x-6 = \lambda \\ 2y-2 = \lambda \\ x+y = 2 \end{cases}$$

$$2x-6 = 2y-2$$
$$x-3 = y-1$$

One good strategy is to try to solve an eqn for x and then substitute.

$$\Rightarrow \begin{cases} x-3=y-1 \Rightarrow y=x-2 \\ x+y=2 \end{cases}$$

substitute

Sub. $y=x-2$ into eqn 2,

$$x+x-2=2$$

$$2x=4 \Rightarrow x=2$$

If $x=2$, then $y=x-2=0$.

So $(2,0)$ is a soln to the system.

$$f(2,0)=2^2+0^2-6(2)-2(0)+1=-7$$

To see that $(2,0,-7)$ is an absolute min we can graph $f(x,y)$ and $g(x,y)=k$.

Ex 2 Find the absolute max and min values of $f(x,y) = 4x^2 + 10y^2$ with the constraint $x^2+y^2=4$.

Sol

Let $g(x,y) = x^2+y^2$.

$\nabla f = \langle 8x, 20y \rangle \qquad \nabla g = \langle 2x, 2y \rangle$

Need to solve,

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases} \Rightarrow \begin{cases} \langle 8x, 20y \rangle = \lambda \langle 2x, 2y \rangle \\ x^2+y^2 = 4 \end{cases}$$

$$\Rightarrow \begin{cases} 8x = \lambda 2x \Rightarrow \lambda = \frac{8x}{2x} = 4, \quad x \neq 0 \\ 20y = \lambda 2y \\ x^2+y^2=4 \end{cases}$$

Eqn ① $8x = \lambda 2x \Rightarrow \lambda = 4$ or $x = 0$

If $x=4$, then ② becomes $20y = 8y \Rightarrow y=0$

when $y=0$ the contraints $x^2+y^2=4$ becomes

$x^2=4 \Rightarrow x=\pm 2$

If $x=0$, then ③ $y^2=4 \Rightarrow y=\pm 2$.

The solutions are $(-2,0),(2,0),(0,-2),(0,2)$.

Plugging these into $f(x,y)=4x^2+10y^2$:

$f(\pm 2,0)=16$ ← absol. min

$f(0,\pm 2)=40$ ← absol. max

Ex 3 Find the absolute extrema for

$f(x,y)=xy$ subject to $x^2-y=12$.

Sol

Let $g(x,y) = x^2 - y$.

$\nabla f = \langle y, x \rangle$ $\quad$ $\nabla g = \langle 2x, -1 \rangle$

$$\begin{cases} y = \lambda 2x \\ x = -\lambda \Rightarrow \lambda = -x \\ x^2 - y = 12 \end{cases}$$

Sub. $\lambda = -x$ into ①, $y = -2x^2$. Sub. $y = -2x^2$ into ③, $x^2 + 2x^2 = 12$

$$3x^2 = 12 \Rightarrow x = \pm 2$$

If $x = \pm 2$, then $y = -2x^2 = -8$

So the solutions are $(-2,-8)$ and $(2,-8)$.

Plug into $f(x,y) = xy$:

$f(-2,-8) = 16 \leftarrow$ absol. max

$f(2,-8) = -16 \leftarrow$ absol. min.

Practice Problems

1) Find the absolute max and min for the following problems,

i) $f(x,y) = 3x + y$ subject to $x^2 + y^2 = 10$

ii) $f(x,y) = x^2 + y^2 - 6x - 2y + 1$ subject to $(x-1)^2 + (y-1)^2 = 1$

Solutions

i) $f(x,y) = 3x + y$, $g(x,y) = x^2 + y^2 = 10$

$$\nabla f = \langle 3, 1 \rangle \ , \ \nabla g = \langle 2x, 2y \rangle$$

So we must solve

$$\begin{cases} 3 = 2x\lambda \Rightarrow \lambda = \dfrac{3}{2x} \ , \ \underline{x \neq 0} \\ 1 = 2y\lambda \Rightarrow \lambda = \dfrac{1}{2y} \ , \ \underline{y \neq 0} \\ x^2 + y^2 = 10 \end{cases}$$

We should also consider what happens when $x=0$ and $y=0$ but these won't satisfy the first two eqns so there are no solns with $x=0$ or $y=0$.

So $\dfrac{3}{2x} = \dfrac{1}{2y}$

$\Rightarrow 6y = 2x$

$\Rightarrow 3y = x$

Plugging $3y = x$ into our constraint gives

$$(3y)^2 + y^2 = 10$$

$$\Rightarrow 10y^2 = 10 \Rightarrow y = \pm 1$$

If $y=1$, then $x=3y=3$

If $y=-1$, then $x=3y=-3$

So we have $(3,1)$ and $(-3,-1)$. Plug into

$f(x,y)=3x+y$,

$f(3,1)=10 \leftarrow$ absolute max

$f(-3,-1)=-10 \leftarrow$ absolute min

ii) $f(x,y)=x^2+y^2-6x-2y+1$

$g(x,y)=(x-1)^2+(y-1)^2=1$

So $\nabla f = \langle 2x-6, 2y-2 \rangle$

$\nabla g = \langle 2(x-1), 2(y-1) \rangle$

The system we must solve is

$$\begin{cases} 2x-6 = 2\lambda(x-1) \\ 2y-2 = 2\lambda(y-1) \\ (x-1)^2+(y-1)^2 = 1 \end{cases} \Rightarrow \begin{cases} x-3 = \lambda(x-1) \\ y-1 = \lambda(y-1) \\ (x-1)^2+(y-1)^2 = 1 \end{cases}$$

Eqn ① implies $\lambda = \dfrac{x-3}{x-1}$, $\underline{x \neq 1}$

If $x=1$, eqn ① becomes $-2=0$

Eqn ② implies $\lambda = \dfrac{y-1}{y-1} = 1$, $\underline{y \neq 1}$

Notice how $y=1$ would also satisfy eqn ②

Setting these equal,

$$\frac{x-3}{x-1} = 1 \Rightarrow x-1 = x-3$$
$$\Rightarrow \underline{-1=-3}$$

This is impossible so we won't get any solutions from this approach

If $y=1$, then by eqn ③

$$(x-1)^2 + 0 = 1$$
$$\Rightarrow x-1 = \pm\sqrt{1}$$
$$\Rightarrow x = 0, 2$$

So the only solns are $(0,1)$ and $(2,1)$

$f(0,1) = 0$ $\qquad$ $f(2,1) = -8$

absolute max $\qquad$ absolute min

Suggested Textbook Exc (14.8)

1, 3, 5, 7

Quiz 7

14.7, 14.8