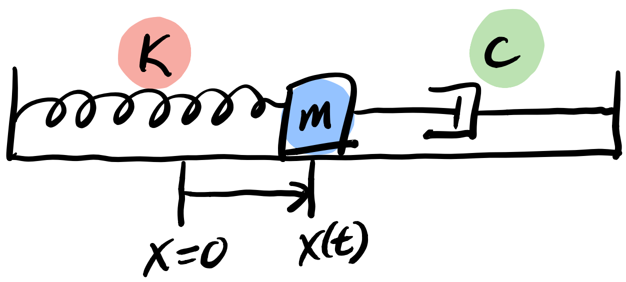
2.4/ Mechanical Vibrations

Recall from Section 2.1:

$$mx'' + cx' + kx = F(t)$$

mass · damping · spring · external force



$x(t)$ = displacement from equilibrium at time t

Note:

$F(t) = 0 \implies$ free motion

$c = 0 \implies$ undamped motion

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① **Free undamped motion**

$$\boxed{mx'' + kx = 0}$$

$mr^2 + k = 0$ ← Characteristic eq'n

$r = \pm i\sqrt{\frac{k}{m}}$ ← roots of char. eq'n

$mr^2 = -k$

$r^2 = \frac{-k}{m}$

Let $\omega_0 = \sqrt{\frac{k}{m}}$. circular frequency (rad/s)

Then $r = \pm i\omega_0$, so the sol'n is:

$$\boxed{x(t) = A\cos\omega_0 t + B\sin\omega_0 t}.$$

We can use the trig. identity

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta-\phi)$$

to write the sol'n as

$$\boxed{x(t) = C\cos(\omega_0 t - \alpha)}.$$

First we factor out C:

$$x(t) = C\left(\frac{A}{C}\cos\omega_0 t + \frac{B}{C}\sin\omega_0 t\right).$$

In order for

$$\boxed{\frac{A}{C} = \cos\alpha, \qquad \frac{B}{C} = \sin\alpha}$$

defines α

we need

defines C

$$\left(\frac{A}{C}\right)^2 + \left(\frac{B}{C}\right)^2 = 1 \implies \boxed{C = \sqrt{A^2+B^2}}.$$

Then

$$x(t) = C\,(\cos\alpha \cos\omega_0 t + \sin\alpha \sin\omega_0 t)$$

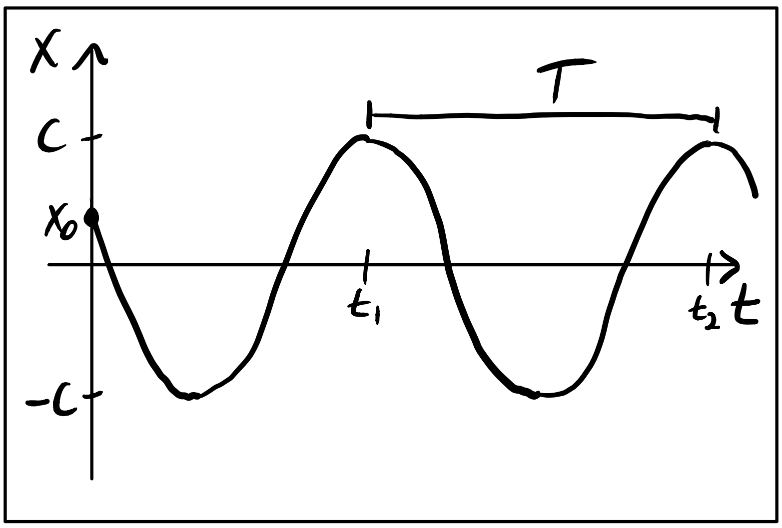
so

$$x(t) = C\cos(\omega_0 t - \alpha).$$

<u>Note:</u> C = amplitude

ω_0 = circular frequency ($^{rad}/_{s}$)

α = phase angle (rad)



T = the period = $t_2 - t_1$

$$x(t_1) = C \qquad \omega_0 t_1 - \alpha = 0 \qquad t_1 = \alpha/\omega_0$$

$$x(t_2) = C \qquad \omega_0 t_2 - \alpha = 2\pi \qquad t_2 = \frac{2\pi + \alpha}{\omega_0}$$

$$\therefore\ T = \frac{2\pi+\alpha}{\omega_0} - \frac{\alpha}{\omega_0} = \frac{2\pi}{\omega_0}.$$

$$\boxed{T = \frac{2\pi}{\omega_0}}$$ $\Rightarrow$ the <u>frequency</u> is

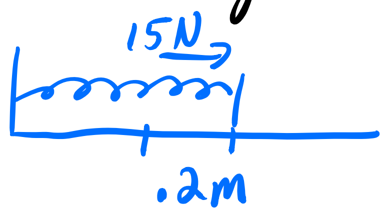
seconds per cycle

$$\boxed{\nu = \frac{1}{T} = \frac{\omega_0}{2\pi} \text{ Hz.}}$$

<u>Example:</u> (#3)

A mass of 3kg is attached to the end of a spring that is stretched 20 cm by a force of 15 N. It is set in motion with initial position $x_0 = 0$ and initial velocity $v_0 = -10$ m/s. Find the amplitude, period, and frequency of the resulting motion.

<u>Sol'n:</u> $m = 3\text{kg}$



$c = 0$

$$F_s = -kx$$

$$-15\text{ N} = -k(.2\text{m}) \rightarrow k = \frac{15\text{N}}{.2\text{m}} = 75\ \text{N/m}$$

$$mx'' + cx' + kx = 0, \quad x(0) = 0, \quad x'(0) = -10$$

$$3x'' + 75x = 0$$

$3r^2 + 75 = 0$

$r^2 + 25 = 0$

$r = \pm i5$

$\therefore \omega_0 = 5$ $\left(\frac{rad}{s}\right)$

$T = \frac{2\pi}{\omega_0} = \frac{2\pi}{5}$ (s)

$\nu = \frac{5}{2\pi}$ (Hz)

$\therefore x(t) = A\cos 5t + B\sin 5t.$

$x'(t) = -5A\sin 5t + 5B\cos 5t$

$x(0) = 0 \; : \; A \cdot 1 + B \cdot 0 = 0$

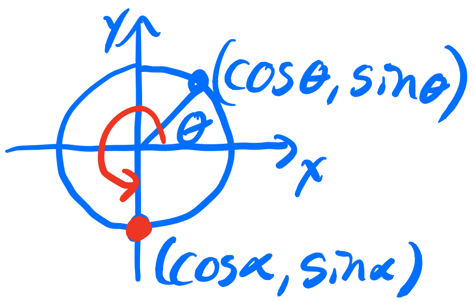
$x'(0) = -10 \; : \; -5A \cdot 0 + 5B \cdot 1 = -10$

$\therefore A = 0, \quad B = -2$

$\therefore x(t) = -2\sin 5t$

$\therefore C = \sqrt{A^2 + B^2} = \sqrt{0^2 + (-2)^2} = 2.$

$\cos\alpha = \frac{A}{C} = 0, \quad \sin\alpha = \frac{B}{C} = -1$



$\alpha = \frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}.$

$\therefore \ x(t) = 2\cos\left(5t - \frac{3\pi}{2}\right).$

$\therefore$ amplitude is $C = 2$ (m),

the period is $T = \frac{2\pi}{5} = 1.26\ldots$ (s),

and the freq. is $\nu = \frac{5}{2\pi} = 0.795\ldots$ (Hz).

② Free damped motion

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\boxed{mx'' + cx' + kx = 0}$$

$mr^2 + cr + k = 0$ ← Characteristic eq'n

$r = \dfrac{-c \pm \sqrt{c^2-4mk}}{2m}$ ← roots of char. eq'n

(i) $c^2 - 4mk > 0 \Rightarrow$ distinct real roots: $r_1 \neq r_2$

$\therefore \boxed{x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$ $(r_1, r_2 < 0)$

This is called <u>overdamping</u> since the damping overpowers the spring and eliminates all oscillations.

(ii) $c^2 - 4mk = 0 \Rightarrow$ one repeated root: $r_1 = r_2 = \dfrac{-c}{2m}$

$\therefore \boxed{x(t) = e^{-pt}(c_1 + c_2 t)}$ $\left(p = \dfrac{c}{2m} > 0\right)$

This is called <u>critical damping</u>.

No oscillations, but small change could lead to underdamping.

(iii) $c^2 - 4mk < 0 \Rightarrow$ complex pair of roots:

$$r = -p \pm i\,\omega_1$$

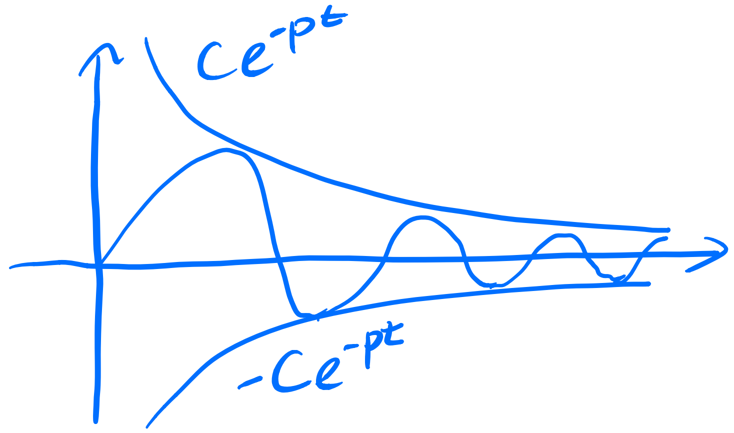
where $\omega_1 = \dfrac{\sqrt{4mk - c^2}}{2m}$.

$$\therefore \boxed{x(t) = e^{-pt}(A\cos\omega_1 t + B\sin\omega_1 t)}$$

This is called <u>underdamping</u> since the damping is not strong enough to prevent oscillations.

Let $C = \sqrt{A^2+B^2}$, $\cos\alpha = \dfrac{A}{C}$, $\sin\alpha = \dfrac{B}{C}$.

Then $\boxed{x(t) = Ce^{-pt}\cos(\omega_1 t - \alpha)}$.



Example: (#21)

$m=1, \quad c=10, \quad k=125; \quad x_0=6, \quad v_0=50$

$$\boxed{x''+10x'+125x=0, \quad x(0)=6, \quad x'(0)=50}$$

Find the damped motion $x(t)$ and the undamped motion $u(t)$:

$$\boxed{u''+125u=0, \quad u(0)=6, \quad u'(0)=50}.$$

Sol'n:

$$x''+10x'+125x=0$$

$$r^2+10r+125=0$$

$$r=\frac{-10\pm\sqrt{10^2-4\cdot1\cdot125}}{2\cdot1}=-5\pm\frac{\sqrt{100-500}}{2}$$

$$=-5\pm i\frac{20}{2}=-5\pm i\cdot 10$$

$$\therefore\ x(t)=e^{-5t}(A\cos 10t+B\sin 10t)$$

$$x'(t)=e^{-5t}(-10A\sin 10t+10B\cos 10t)$$
$$-5e^{-5t}(A\cos 10t+B\sin 10t)$$

$x(0) = 6 \quad : \quad 1 \cdot (A \cdot 1 + B \cdot 0) = 6$

$$x'(0) = 50 \quad : \quad 1 \cdot (-10A \cdot 0 + 10B \cdot 1) - 5 \cdot 1 \cdot (A \cdot 1 + B \cdot 0) = 50$$

$\boxed{A = 6}$

$10B - 5A = 50$

$10B - 5 \cdot 6 = 50$

$10B = 80$

$\boxed{B = 8}$

$C = \sqrt{A^2 + B^2}$

$C = \sqrt{36 + 64}$

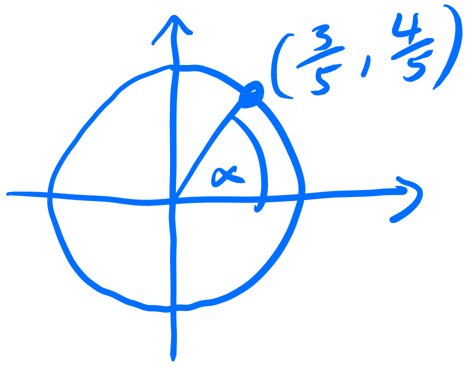
$C = \sqrt{100}$

$\boxed{C = 10}$

$\cos\alpha = \frac{A}{C} = \frac{6}{10} = \frac{3}{5}$

$\sin\alpha = \frac{B}{C} = \frac{8}{10} = \frac{4}{5}$

$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = \frac{9+16}{25} = 1$ ✓



$$\alpha = \begin{cases} \cos^{-1}\left(\frac{A}{C}\right) & \text{if } \frac{B}{C} \geq 0 \\ 2\pi - \cos^{-1}\left(\frac{A}{C}\right) & \text{if } \frac{B}{C} < 0 \end{cases}$$

$\therefore \boxed{\alpha =} \cos^{-1}\left(\frac{3}{5}\right) = \boxed{0.927\ldots}$ (rad)

$$\therefore \boxed{x(t) = 10e^{-5t}\cos(10t - \alpha)}$$

$$u'' + 125u = 0$$

$$r^2 + 125 = 0$$

$$r = \pm i\, 5\sqrt{5}$$

$$u(t) = A\cos 5\sqrt{5}t + B\sin 5\sqrt{5}t$$

$$u'(t) = -5\sqrt{5}A\sin 5\sqrt{5}t + 5\sqrt{5}B\cos 5\sqrt{5}t$$

$$u(0) = 6 \; : \quad A\cdot 1 + B\cdot 0 = 6 \qquad A = 6$$

$$u'(0) = 50 \; : \quad -5\sqrt{5}\cdot A\cdot 0 + 5\sqrt{5}\cdot B\cdot 1 = 50$$

$$B = \frac{50}{5\sqrt{5}} = \frac{10}{\sqrt{5}} = 2\sqrt{5}$$

$$C = \sqrt{A^2 + B^2} = \sqrt{36 + 20} = \sqrt{56} = 2\sqrt{14}$$

$$\cos\alpha_0 = \frac{A}{C} = \frac{6}{2\sqrt{14}} = \frac{3}{\sqrt{14}}$$

$$\sin\alpha_0 = \frac{2\sqrt{5}}{2\sqrt{14}} = \sqrt{\frac{5}{14}} > 0$$

$$\left(\frac{3}{\sqrt{14}}\right)^2 + \left(\sqrt{\frac{5}{14}}\right)^2 = \frac{9+5}{14} = 1 \checkmark$$

$$\therefore \alpha_0 = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right) = 0.6405\ldots \text{ (rad)}$$

$$\therefore \boxed{u(t) = 2\sqrt{14}\cos\left(5\sqrt{5}\,t - \alpha_0\right)}.$$