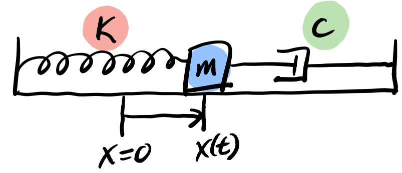
2.6) Forced Oscillations and Resonance

$$\boxed{mx'' + cx' + kx = F(t)}$$



$$F(t) = F_0 \cos \omega t \quad \underline{\text{or}} \quad F(t) = F_0 \sin \omega t$$

① Undamped Forced Oscillations $(c=0)$

$$\boxed{mx'' + kx = F(t)}$$

Example: (#2)

$$x'' + 4x = 5\sin 3t, \quad x(0) = 0, \quad x'(0) = 0$$

Sol'n:

$x'' + 4x = 0$

$r^2 + 4 = 0$

$r = \pm i2$

$$\boxed{x_c = c_1 \cos 2t + c_2 \sin 2t}$$

$f(t) = 5 \sin 3t$

$f'(t) = 15 \cos 3t$

$f''(t) = -45 \sin 3t$

$\phi_1 = \sin 3t$

$\phi_2 = \cos 3t$

$$x_p = A \sin 3t + B \cos 3t$$

$$x_p' = 3A\cos 3t - 3B\sin 3t$$

$$x_p'' = -9A\sin 3t - 9B\cos 3t$$

$$x_p'' + 4x_p = 5\sin 3t$$

$$(-9A\sin 3t - 9B\cos 3t)$$
$$+ 4(A\sin 3t + B\cos 3t) = 5\sin 3t$$

$$-5A\sin 3t - 5B\cos 3t = 5\sin 3t$$

$$\therefore -5A = 5, \quad -5B = 0$$
$$A = -1, \quad B = 0$$

$$\therefore \boxed{x_p = -\sin 3t} \qquad (x = x_c + x_p)$$

$$\therefore \boxed{x(t) = c_1\cos 2t + c_2\sin 2t - \sin 3t}$$

$$x'(t) = -2c_1\sin 2t + 2c_2\cos 2t - 3\cos 3t$$

$$x(0) = 0 \; : \; c_1 \cdot 1 + c_2 \cdot 0 - 0 = 0$$

$x'(0) = 0$: $-2c_1 \cdot 0 + 2c_2 \cdot 1 - 3 \cdot 1 = 0$

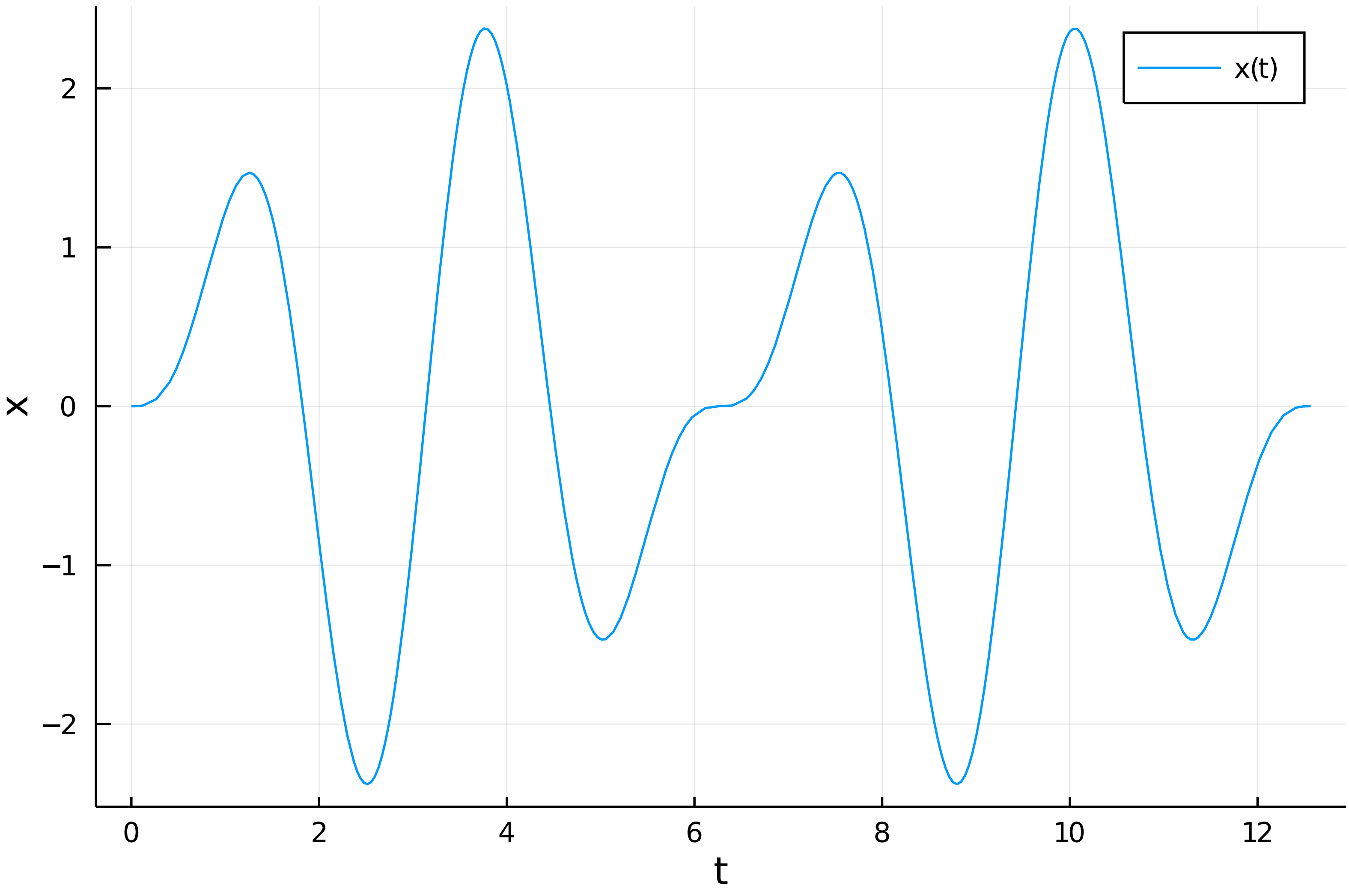
$$\boxed{c_1 = 0, \quad c_2 = 3/2}$$

$$\therefore \boxed{x(t) = \frac{3}{2}\sin 2t - \sin 3t}$$

The period of $\sin 2t$ is π.

The period of $\sin 3t$ is $\frac{2\pi}{3}$.

The period of $x(t)$ is the least common multiple of π and $\frac{2\pi}{3}$, which is $\boxed{2\pi}$ //


x(t)
2
1
0
−1
−2
x
0
2
4
6
8
10
12
t

Resonance

$$mx'' + kx = 0$$
$$mr^2 + k = 0$$
$$r = \pm i\sqrt{\tfrac{k}{m}}$$

$$\boxed{mx'' + kx = F_0 \cos \omega t}$$

The natural frequency is $\omega_0 = \sqrt{\frac{k}{m}}$

and the complementary sol'n is

$$\left[x_c = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t\right].$$

If $\omega \neq \omega_0$, then the particular sol'n has the form

$$\left[x_p = A \cos \omega t + B \sin \omega t\right].$$

$$x_p' = -A\omega \sin \omega t + B\omega \cos \omega t$$

$$x_p'' = -A\omega^2 \cos \omega t - B\omega^2 \sin \omega t$$

Substituting this into the diff. eq'n:

$$m x_p'' + k x_p = F_0 \cos \omega t$$

$$m(-A\omega^2 \cos \omega t - B\omega^2 \sin \omega t)$$
$$+ k(A \cos \omega t + B \sin \omega t) = F_0 \cos \omega t$$

$$A(-m\omega^2+k)\cos\omega t + B(-m\omega^2+k)\sin\omega t = F_0\cos\omega t$$

$\therefore \quad A(k-m\omega^2) = F_0$ and $B(k-m\omega^2)=0$

$$\Rightarrow \quad A = \frac{F_0}{k-m\omega^2}, \quad B = 0.$$

Recall that $\omega_0 = \sqrt{\frac{k}{m}}$, so $m\omega_0^2 = k$.

Thus $A = \dfrac{F_0}{m\omega_0^2 - m\omega^2} = \dfrac{F_0/m}{\omega_0^2-\omega^2}$.

$$\therefore \quad \boxed{x_p = \frac{F_0/m}{\omega_0^2-\omega^2}\cos\omega t}$$

Then $x = x_c + x_p$ implies that

$$\boxed{x(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t + \frac{F_0/m}{\omega_0^2-\omega^2}\cos\omega t}$$

amplitude

As $\omega \to \omega_0$, we have that

$$\frac{F_0/m}{\omega_0^2-\omega^2} \to +\infty.$$

Therefore, the amplitude of the oscillations will increase without bound as $\omega \to \omega_0$.

When $\omega = \omega_0$, we have:

$$\boxed{x_p = At\cos\omega_0 t + Bt\sin\omega_0 t}$$

$$x_p = t(A\cos\omega_0 t + B\sin\omega_0 t)$$

$$x_p' = t(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + 1\cdot(A\cos\omega_0 t + B\sin\omega_0 t)$$

$$x_p'' = t(-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t) + 1\cdot(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t) + (-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t)$$

$$x_p'' = (-A\omega_0^2 t + 2B\omega_0)\cos\omega_0 t + (-B\omega_0^2 t - 2A\omega_0)\sin\omega_0 t$$

Then $\quad m x_p'' + k x_p = F_0 \cos \omega_0 t$

$$\Rightarrow m\Big[(-A\omega_0^2 t + 2B\omega_0)\cos\omega_0 t + (-B\omega_0^2 t - 2A\omega_0)\sin\omega_0 t\Big] + k\Big[At\cos\omega_0 t + Bt\sin\omega_0 t\Big] = F_0\cos\omega_0 t$$

$$\Rightarrow \Big[m(\cancel{-A\omega_0^2 t} + 2B\omega_0) + \cancel{Akt}\Big]\cos\omega_0 t + \Big[m(\cancel{-B\omega_0^2 t} - 2A\omega_0) + \cancel{Bkt}\Big]\sin\omega_0 t = F_0\cos\omega_0 t$$

[<u>Recall</u>: $\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow m\omega_0^2 - k = 0$]

$$\Rightarrow 2B\omega_0 m\cos\omega_0 t - 2A\omega_0 m\sin\omega_0 t = F_0\cos\omega_0 t$$

$\therefore \quad 2B\omega_0 m = F_0 \quad$ <u>and</u> $\quad -2A\omega_0 m = 0$

$\therefore \quad B = \dfrac{F_0}{2m\omega_0} \quad$ <u>and</u> $\quad A = 0$

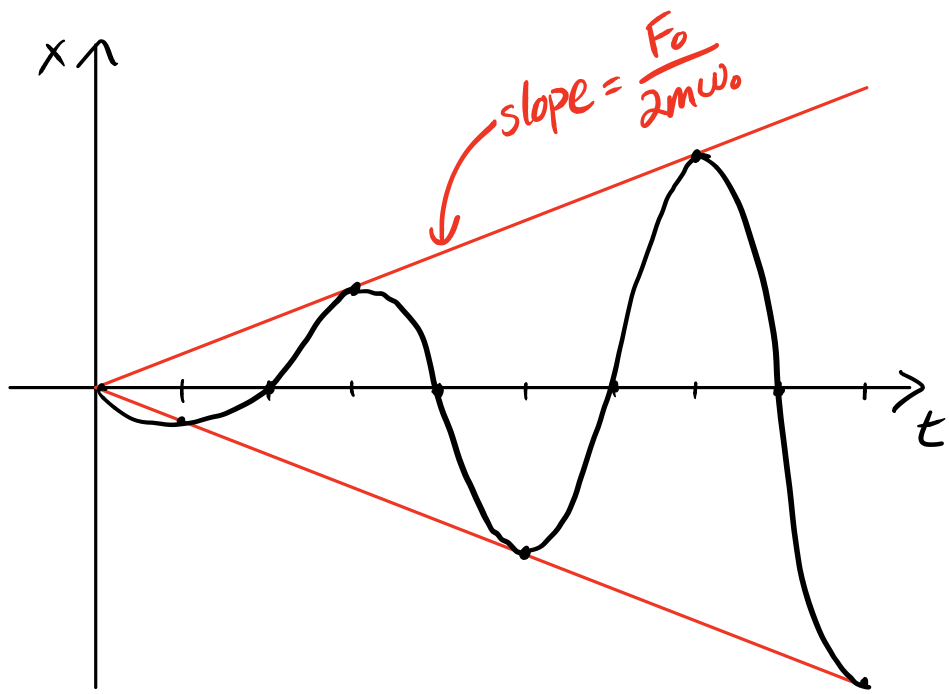
$\therefore$ $$\boxed{x_p = \frac{F_0}{2m\omega_0} t \sin \omega_0 t}$$

Therefore,

$$\boxed{x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \underbrace{\frac{F_0}{2m\omega_0}}_{\text{amplitude}} t \sin \omega_0 t}$$

Now the amplitude of the oscillations is increasing with time.

This phenomenon is known as resonance.



Read: p. 165–166

Example: (#20)

A front-loading washing machine is mounted on a thick rubber pad that acts like a spring, the weight $W = mg$ (with $g = 9.8\, m/s^2$) of the machine depresses the pad exactly 0.5 cm.

When its rotor spins at ω radians per second, the rotor exerts a vertical force $F_0 \cos \omega t$ newtons on the machine.

At what speed (in revolutions per minute) will resonance vibrations occur?

Neglect friction.

Sol'n:

$$F_s = -kx$$

$$mg = -k(-0.005)$$

$$k = \frac{m \cdot (9.8)}{0.005}$$

$$\boxed{k = 1960\, m}$$

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{1960\cancel{m}}{\cancel{m}}} = \sqrt{1960} = 14\sqrt{10}$$

$$\therefore \omega_0 = 44.27\ldots \text{ (rad/s)}$$

$$\omega_0 = 14\sqrt{10} \cdot \frac{1}{2\pi} \text{ (cycles/s} = \text{Hz)}$$

$$\omega_0 = 7.046\ldots \text{ (Hz)}$$

$\therefore$ resonance vibrations will occur at about 7.046 Hz.

② Damped Forced Oscillations $(c \neq 0)$

Example: (#12)

$$x'' + 6x' + 13x = 10\sin 5t, \quad x(0)=0, \quad x'(0)=0$$

Find both the steady periodic sol'n

$$x_{sp}(t) = C_0\cos(\omega_0 t - \alpha_0)$$

and the transient sol'n

$$x_{tr}(t) = C_1 e^{-pt}\cos(\omega_1 t - \alpha_1)$$

where $x(t) = x_{sp}(t) + x_{tr}(t)$.

Sol'n:

$$x'' + 6x' + 13x = 0$$

$$r^2 + 6r + 13 = 0$$

$$r = \frac{-6 \pm \sqrt{6^2 - 4\cdot 1\cdot 13}}{2\cdot 1}$$

$$r = \frac{-6 \pm \sqrt{-16}}{2}$$

$$r = \frac{-6 \pm i4}{2}$$

$$\boxed{r = -3 \pm i2}$$

$$\boxed{x_c = c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t}$$

transient sol'n

$$x'' + 6x' + 13x = 10\sin 5t$$

$$f(t) = 10\sin 5t \qquad \phi_1 = \sin 5t$$

$$f'(t) = 50\cos 5t \qquad \phi_2 = \cos 5t$$

$$x_p = A\sin 5t + B\cos 5t$$

$$x_p' = 5A\cos 5t - 5B\sin 5t$$

$$x_p'' = -25A\sin 5t - 25B\cos 5t$$

$$x_p'' + 6x_p' + 13x_p = 10\sin 5t$$

$$(-25A\sin 5t - 25B\cos 5t)$$
$$+6(5A\cos 5t - 5B\sin 5t)$$
$$+13(A\sin 5t + B\cos 5t) = 10\sin 5t$$

$$(-12A - 30B)\sin 5t$$
$$+(-12B + 30A)\cos 5t = 10\sin 5t$$

$$\therefore \begin{bmatrix} -12A - 30B = 10 \\ 30A - 12B = 0 \end{bmatrix}$$

<u>Cramer's Rule:</u>

$$A = \frac{\begin{vmatrix} 10 & -30 \\ 0 & -12 \end{vmatrix}}{\begin{vmatrix} -12 & -30 \\ 30 & -12 \end{vmatrix}} = \frac{-120}{144+900} = \frac{-10}{87}$$

$$B = \frac{\begin{vmatrix} -12 & 10 \\ 30 & 0 \end{vmatrix}}{\begin{vmatrix} -12 & -30 \\ 30 & -12 \end{vmatrix}} = \frac{-300}{1044} = \frac{-25}{87}$$

$$x_p = A\sin 5t + B\cos 5t$$

$$\therefore \boxed{x_p = \frac{-10}{87}\sin 5t - \frac{25}{87}\cos 5t}$$

steady periodic sol'n

$$x(t) = e^{-3t}(c_1\cos 2t + c_2\sin 2t) - \frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t$$

$$x'(t) = e^{-3t}(-2c_1\sin 2t + 2c_2\cos 2t)$$
$$-3e^{-3t}(c_1\cos 2t + c_2\sin 2t)$$
$$+\frac{125}{87}\sin 5t - \frac{50}{87}\cos 5t$$

$$x(0)=0 : \quad 1\cdot(c_1\cdot 1 + c_2\cdot 0) - \frac{25}{87}\cdot 1 - \frac{10}{87}\cdot 0 = 0$$

$$\therefore \quad c_1 - \frac{25}{87} = 0 \quad \Rightarrow \quad \boxed{c_1 = \frac{25}{87}}$$

$$x'(0)=0 : \quad 1\cdot(-2c_1\cdot 0 + 2c_2\cdot 1)$$
$$-3\cdot 1\cdot(c_1\cdot 1 + c_2\cdot 0)$$
$$+\frac{125}{87}\cdot 0 - \frac{50}{87}\cdot 1 = 0$$

$$2c_2 - 3c_1 - \frac{50}{87} = 0$$

$$c_2 = \frac{1}{2}\left(\frac{50}{87} + 3c_1\right) = \frac{1}{2}\left(\frac{50}{87} + 3\cdot\left(\frac{25}{87}\right)\right)$$

$$\therefore \quad \boxed{c_2 = \frac{125}{174}}$$

$$\therefore\ x(t) = e^{-3t}\left(\frac{25}{87}\cos 2t + \frac{125}{174}\sin 2t\right)$$

$$- \frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t$$

$\therefore$ The steady periodic sol'n is:

$$\boxed{x_{sp}(t) = -\frac{25}{87}\cos 5t - \frac{10}{87}\sin 5t}$$

$$C_0 = \sqrt{A^2+B^2} = \sqrt{\left(\frac{-25}{87}\right)^2 + \left(\frac{-10}{87}\right)^2}$$

$$C_0 = \sqrt{\frac{25}{261}} = \frac{5}{3\sqrt{29}}$$

$$\cos\alpha_0 = \frac{A}{C_0}, \qquad \sin\alpha = \frac{B}{C_0} < 0$$

$$\alpha_0 = 2\pi - \cos^{-1}\left(\frac{A}{C_0}\right)$$

$$\boxed{\alpha_0 = 3.522\ldots}$$

$$\omega_0 = 5$$

$$\therefore \boxed{x_{sp}(t) = \frac{5}{3\sqrt{29}}\cos(5t - \alpha_0)}$$

The transient sol'n is:

$$x_{tr}(t) = e^{-3t}\left(\frac{25}{87}\cos 2t + \frac{125}{174}\sin 2t\right)$$

$$C_1 = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{25}{87}\right)^2 + \left(\frac{125}{174}\right)^2}$$

$$C_1 = \sqrt{\frac{625}{1044}} = \frac{25}{6\sqrt{29}}$$

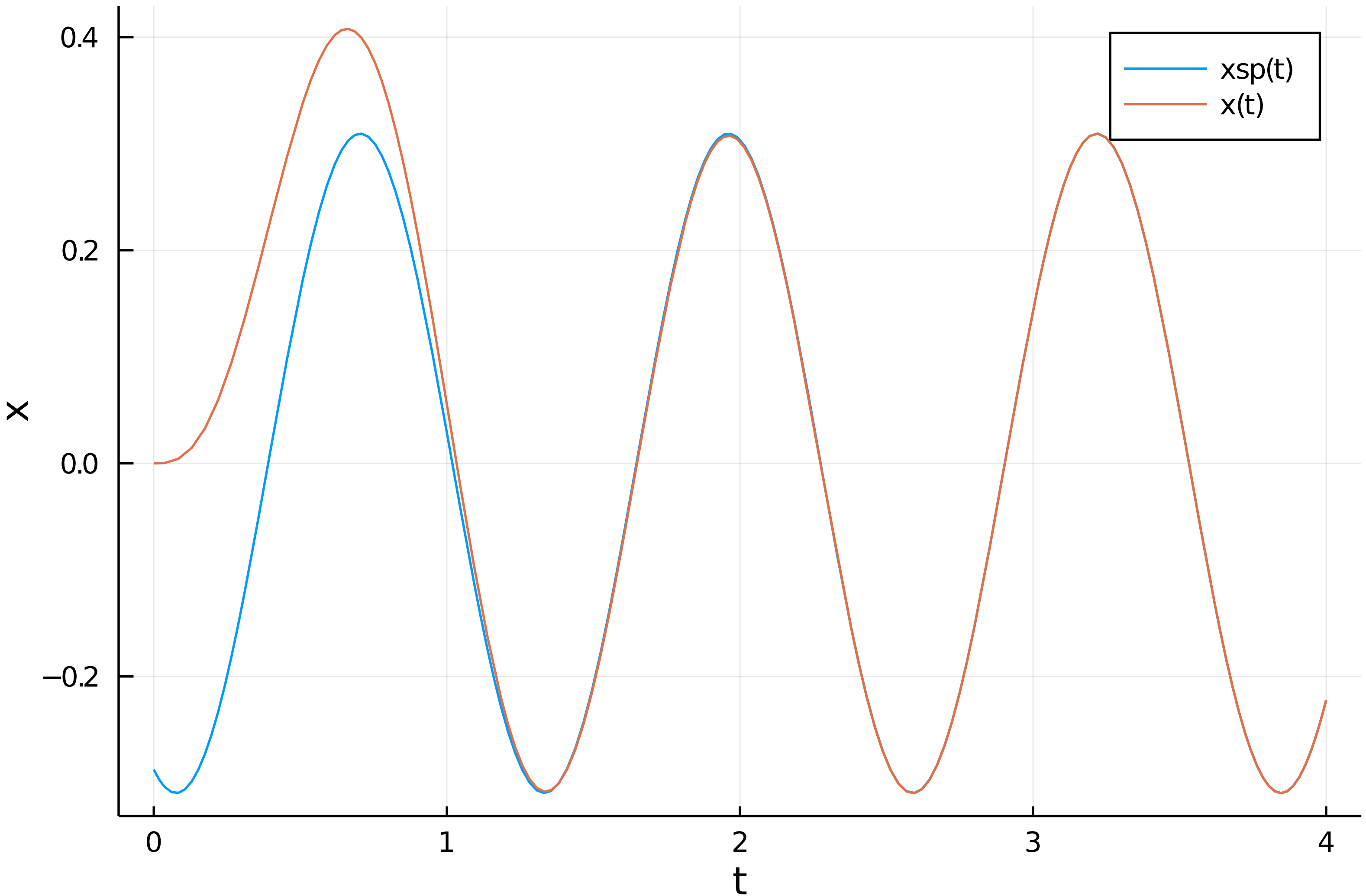
$$\cos\alpha_1 = \frac{A}{C_1}, \quad \sin\alpha_1 = \frac{B}{C_1} > 0$$

$$\alpha_1 = \cos^{-1}\left(\frac{A}{C_1}\right)$$

$$\alpha_1 = 1.19\ldots \qquad \omega_1 = 2$$

$$\therefore \boxed{x_{tr}(t) = \frac{25}{6\sqrt{29}} e^{-3t} \cos\left(2t - \alpha_1\right)}$$

See below for the plot of $x(t)$ and $x_{sp}(t)$.



xsp(t)
x(t)
0.4
0.2
0.0
−0.2
x
0
1
2
3
4
t