12.4 Cross Product

Def 1

If $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$ then the cross product of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$

$$= (a_2 b_3 - a_3 b_2)i + (a_3 b_1 - a_1 b_3)j + (a_1 b_2 - a_2 b_1)k$$

The cross product only applies to 3D vectors.

This formula is difficult to memorize.

An easier method for finding $\vec{a} \times \vec{b}$ is to use matrix determinates.

Determinate of a 2x2 matrix

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinate of a 3x3 matrix

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \underbrace{\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}}_{2\times 2 \text{ determinates}} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

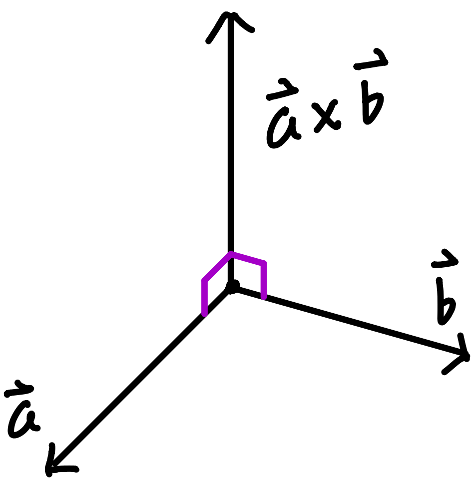
Cross Product Using Determinates

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$

What does the vector $\vec{a} \times \vec{b}$ represent?



$\vec{a} \times \vec{b}$ is the vector which is orthogonal to both $\vec{a}$ and $\vec{b}$.

Note that,

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = 0 \quad \text{and} \quad (\vec{a} \times \vec{b}) \cdot \vec{b} = 0.$$

Ex 1 Find $\vec{a} \times \vec{b}$ with $\vec{a} = \langle 2, -1, 4 \rangle$ and $\vec{b} = \langle 5, -2, 0 \rangle$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ 5 & -2 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 4 \\ -2 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & 4 \\ 5 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} k$$

$$= ((-1)(0) - (-2)(4))i - ((2)(0) - (5)(4))j$$

$$+ ((2)(-2) - (5)(-1))k$$

$$= 8i + 20j + k = \langle 8, 20, 1 \rangle$$

We can verify our answer by making sure it is perpendicular to $\vec{a}$ and $\vec{b}$.

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle 8, 20, 1 \rangle \cdot \langle 2, -1, 4 \rangle$$

$$= 16 - 20 + 4 = 0 \checkmark$$

$$(\vec{a} \times \vec{b}) \cdot \vec{b} = \langle 8, 20, 1 \rangle \cdot \langle 5, -2, 0 \rangle$$

$$= 40 - 40 + 0 = 0 \checkmark$$

Theorem 1

Let θ be the angle between $\vec{a}$ and $\vec{b}$ in $[0, \pi]$, then

$$|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin\theta$$

Def 2 (Parallel)

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if they point in the same or opposite direction (i.e. $\theta = 0$ or $\theta = \pi$).

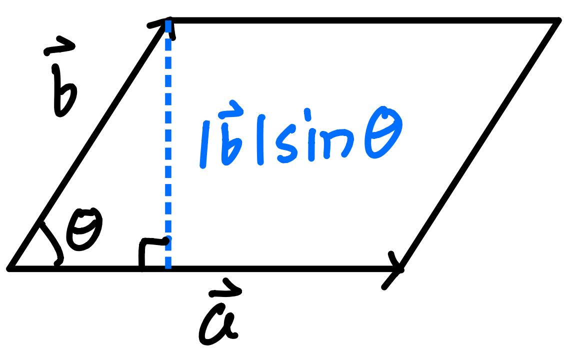
By Thm 1 it follows that

$\vec{a} \times \vec{b} = \vec{0}$ if $\vec{a}$ and $\vec{b}$ are parallel.

An equivalent definition for parallel vectors is that one is a scalar multiple of the other. In other words, we can find a scalar c such that $c\vec{a} = \vec{b}$.

Application of Cross Product

Consider the parallelagram formed by $\vec{a}$ and $\vec{b}$.

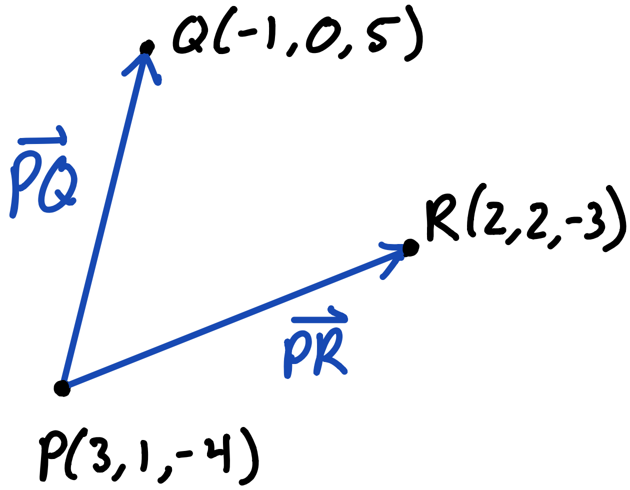


The area of this parallelagram is

$$|\vec{a}||\vec{b}|\sin\theta = |\vec{a} \times \vec{b}|$$

Ex 2 Find a vector which is orthogonal to the plane that passes through the points $P(3,1,-4)$, $Q(-1,0,5)$, $R(2,2,-3)$.

<u>Step 1</u>: First we need to find two vectors that lie on the plane.



$$\vec{PQ} = \langle -1-3, 0-1, 5-(-4) \rangle$$
$$= \langle -4, -1, 9 \rangle$$

$$\vec{PR} = \langle 2-3, 2-1, -3-(-4) \rangle$$
$$= \langle -1, 1, 1 \rangle$$

Since these vectors lie in the plane any vector that is orthogonal to both of these will be orthogonal to the plane.

<u>Step 2</u>: Find $\vec{PQ} \times \vec{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ -4 & -1 & 9 \\ -1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 9 \\ 1 & 1 \end{vmatrix} i - \begin{vmatrix} -4 & 9 \\ -1 & 1 \end{vmatrix} j + \begin{vmatrix} -4 & -1 \\ -1 & 1 \end{vmatrix} k$$

$$= \big((-1)(1) - (9)(1)\big) i - \big((-4)(1) - (-1)(9)\big) j$$

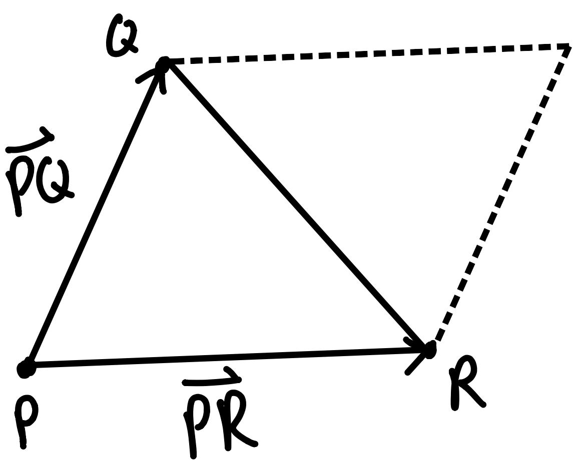
$$+ \big((-4)(1) - (-1)(-1)\big) k$$

$$= -10i - 5j - 5k$$

$$= \langle -10, -5, -5 \rangle = -5\langle 2, 1, 1 \rangle$$

<u>Note</u>: There are many possible answers for this problem.

Ex 3 Find the area of the triangle formed by the points $P(4,1,-1)$, $Q(6,0,3)$ and $R(2,1,-5)$.



Area of parallelagram:

$$|\overrightarrow{PQ} \times \overrightarrow{PR}|$$

Area of triangle PQR:

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$$

Step 1: Find $\overrightarrow{PQ}$, $\overrightarrow{PR}$.

$$\overrightarrow{PQ} = \langle 6-4, 0-1, 3-(-1) \rangle = \langle 2, -1, 4 \rangle$$

$$\overrightarrow{PR} = \langle 2-4, 1-1, -5-(-1) \rangle = \langle -2, 0, -4 \rangle$$

Step 2: Find $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ -2 & 0 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & 4 \\ 0 & -4 \end{vmatrix} i - \begin{vmatrix} 2 & 4 \\ -2 & -4 \end{vmatrix} j + \begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} k$$

$$= (4-0)i - (-8-(-8))j + (0-2)k$$

$$= 4i + 0j - 2k = \langle 4, 0, -2 \rangle$$

Step 3: Compute $\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$.

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}|\langle 4, 0, -2 \rangle|$$

$$= \frac{1}{2}\sqrt{4^2 + 0^2 + (-2)^2}$$

$$= \frac{1}{2}\sqrt{20} = \frac{1}{2}(2\sqrt{5}) = \sqrt{5}$$

In general, the cross product is not commutative or associative.

$$\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$$

Properties of Cross Product

1) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

2) $(c\vec{a}) \times \vec{b} = c(\vec{a} \times \vec{b})$

3) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

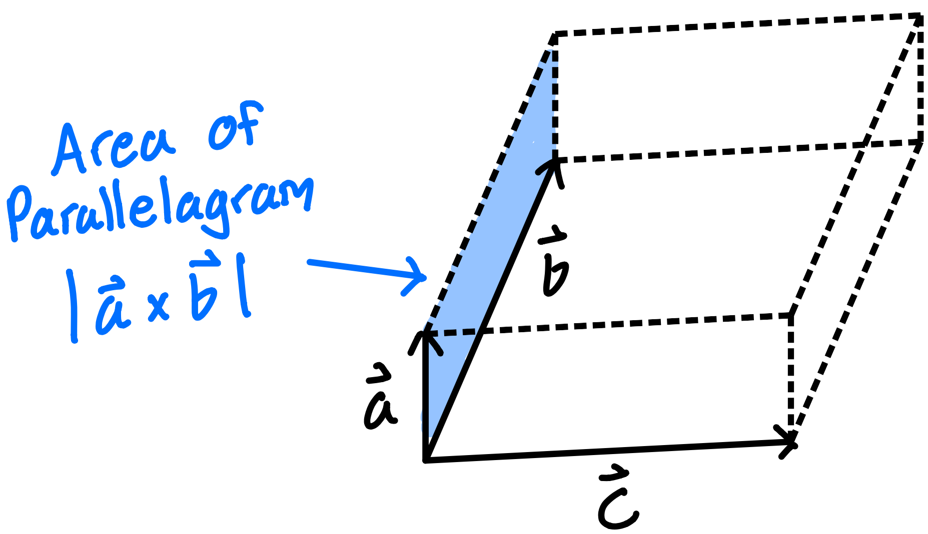
4) $(\vec{b} + \vec{c}) \times \vec{a} = \vec{b} \times \vec{a} + \vec{c} \times \vec{a}$

5) $\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$

6) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

Def 3

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors and consider the 3D solid



This solid is called a parallelepiped and it's volume is given by

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = |(\vec{a} \times \vec{b}) \cdot \vec{c}| = \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|$$

<u>Note</u>: Absolute values are there to avoid getting a negative volume.

Ex 4 Find the volume of the parallelepiped formed by vectors $\vec{a} = \langle 1, 5, -1 \rangle$, $\vec{b} = \langle 0, 3, 4 \rangle$, $\vec{c} = \langle -2, 1, 7 \rangle$.

Sol

$$V = |(\vec{a} \times \vec{b}) \cdot \vec{c}| = \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\|$$

Dropping absolute values for now we have,

$$\begin{vmatrix} 1 & 5 & -1 \\ 0 & 3 & 4 \\ -2 & 1 & 7 \end{vmatrix} = (1)\begin{vmatrix} 3 & 4 \\ 1 & 7 \end{vmatrix} - (5)\begin{vmatrix} 0 & 4 \\ -2 & 7 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= (21-4) - 5(0-(-8)) - (0-(-6))$$

$$= 17 - 40 - 6 = -29$$

So $V = |-29| = 29$.

Practice Problems

1) Let $P(1,2,3), Q(4,-2,0), R(-1,5,3) \in \mathbb{R}^3$

a) Find a vector which is orthogonal to the plane that passes through the points P, Q, and R.

b) Find the area of the triangle PQR.

2) Find the volume of the parallelepiped formed by $\vec{a} = \langle 2,-3,1 \rangle$, $\vec{b} = \langle 0,7,-2 \rangle$, $\vec{c} = \langle 4,1,1 \rangle$.

Solutions

1) $P(1,2,3)$, $Q(4,-2,0)$, $R(-1,5,3)$

a) We can find $\vec{PQ} \times \vec{PR}$

$$\vec{PQ} = \langle 4-1, -2-2, 0-3 \rangle = \langle 3, -4, -3 \rangle$$
$$\vec{PR} = \langle -1-1, 5-2, 3-3 \rangle = \langle -2, 3, 0 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 3 & -4 & -3 \\ -2 & 3 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} -4 & -3 \\ 3 & 0 \end{vmatrix} i - \begin{vmatrix} 3 & -3 \\ -2 & 0 \end{vmatrix} j + \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix} k$$

$$= (0-(-9))i - (0-6)j + (9-8)k$$

$$= \underline{\langle 9, 6, 1 \rangle}$$

(one of many possible answers)

b) $A = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$ so using part (a),

$$A = \frac{1}{2}|\langle 9, 6, 1 \rangle| = \frac{1}{2}\sqrt{81 + 36 + 1}$$

$$= \underline{\frac{1}{2}\sqrt{118}}$$

2) $\vec{a} = \langle 2, -3, 1 \rangle$, $\vec{b} = \langle 0, 7, -2 \rangle$, $\vec{c} = \langle 4, 1, 1 \rangle$.

$$V = \left\| \begin{matrix} 2 & -3 & 1 \\ 0 & 7 & -2 \\ 4 & 1 & 1 \end{matrix} \right\|$$

$$\begin{vmatrix} 2 & -3 & 1 \\ 0 & 7 & -2 \\ 4 & 1 & 1 \end{vmatrix} = 2\begin{vmatrix} 7 & -2 \\ 1 & 1 \end{vmatrix} - (-3)\begin{vmatrix} 0 & -2 \\ 4 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 7 \\ 4 & 1 \end{vmatrix}$$

$$= 2(7 - (-2)) + 3(0 - (-8)) + (0 - 28)$$

$= 18 + 24 - 28 = 14$

So $V = |14| = \underline{14}$

Suggested Textbook Exc. (12.4)

2, 15, 19, 30, 33, 43

Exam 1 (2/5-2/7)

10.1-10.5

12.1-12.4