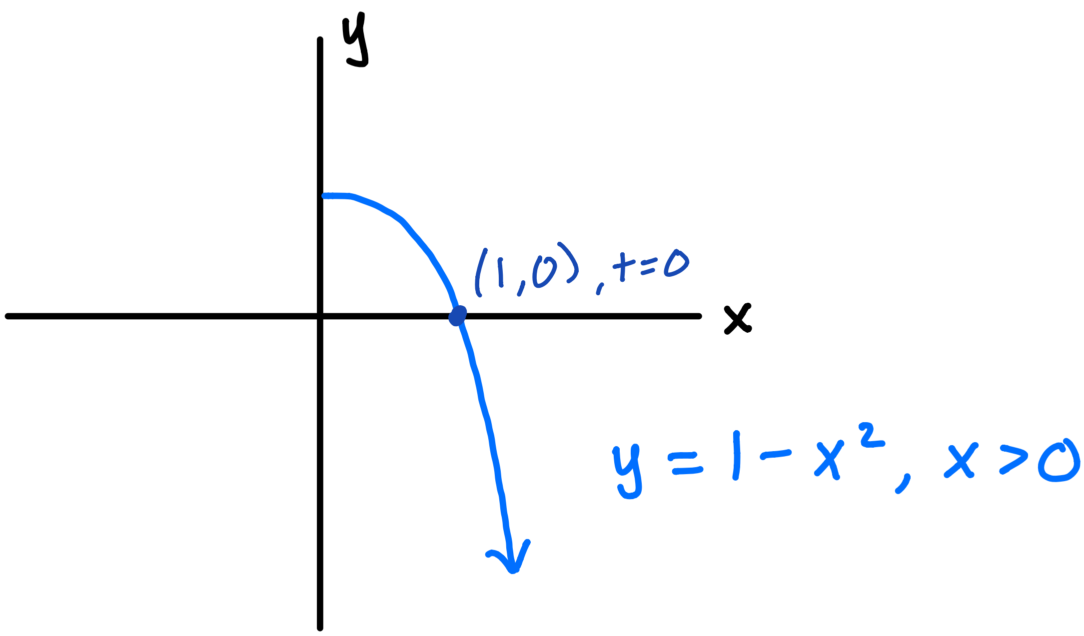
1. (10 points) Consider the parametric curve given by $x = e^{-t}$ , $y = 1 - e^{-2t}$. Eliminate the parameter, sketch the graph of the parametric curve and give any limits that might exist on x and y.

Eliminate parameter:

$$y = 1 - e^{-2t} = 1 - (e^{-t})^2 = 1 - x^2$$

but since $e^{-t} > 0$ for all $t$ we have the restriction $x > 0$.




2. (15 points) Let $x = t^3 - t$ , $y = t^2$, $-2 \le t \le 2$.

(a) (8 points) Show that there are two tangents at the point $(0, 1)$ and find their equations.

First we should find the $t$ values that correspond to the point $(0,1)$.

$$x = t^3 - t \qquad\qquad y = t^2$$

$$0 = t(t^2-1) \qquad\qquad 1 = t^2$$

$$= t(t+1)(t-1) \qquad\qquad \Rightarrow t = \underline{-1}, \underline{1}$$

$$\Rightarrow t = 0, \underline{-1}, \underline{1}$$

this corresponds to point $(0,0)$

<u>Slope of tangent line</u>:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2-1}$$

<u>$t=-1$</u>:

$$\left.\frac{dy}{dx}\right|_{t=-1} = \frac{-2}{3-1} = -1$$

So the equation of the tangent line at $t=-1$ is

$$y-1=-1(x-0)$$
$$y=-x+1$$

<u>$t=1$</u>:

$$\left.\frac{dy}{dx}\right|_{t=1}=\frac{2}{3-1}=1$$

Tangent equation for $t=1$,

$$y-1=1(x-0)$$
$$y=x+1$$

(b) (7 points) Find all the points with vertical or horizontal tangents.

$$y = t^2$$

$$\frac{dy}{dt} = 2t = 0$$

$$\Rightarrow t = 0$$

HT: $(0,0)$

$$x = t^3 - t$$

$$\frac{dx}{dt} = 3t^2 - 1 = 0$$

$$t^2 = \frac{1}{3}$$

$$\Rightarrow t = \pm\frac{1}{\sqrt{3}}$$

VT: $\left(\left(\frac{1}{\sqrt{3}}\right)^3 - \frac{1}{\sqrt{3}}, \frac{1}{3}\right), \left(\left(\frac{-1}{\sqrt{3}}\right)^3 + \frac{1}{\sqrt{3}}, \frac{1}{3}\right)$

or

VT: $\left(\frac{-2}{3\sqrt{3}}, \frac{1}{3}\right), \left(\frac{2}{3\sqrt{3}}, \frac{1}{3}\right)$

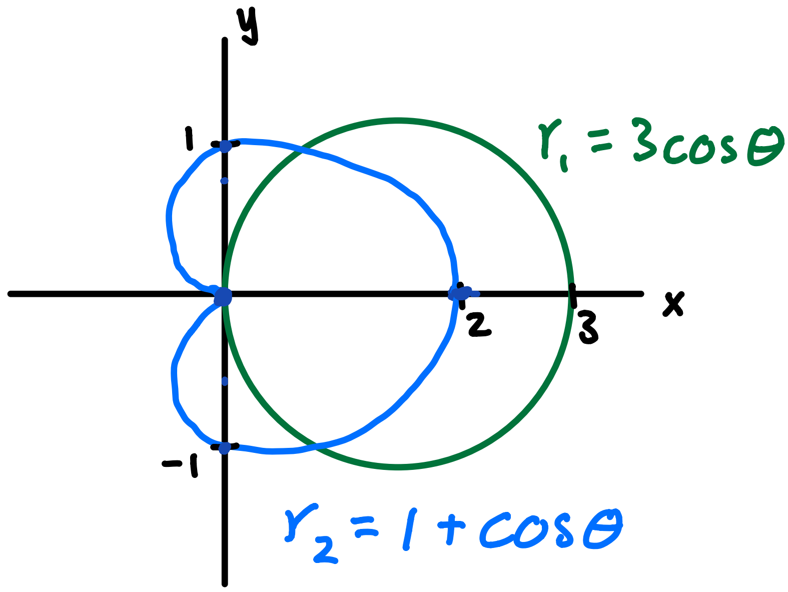
3. (20 points) Let $r_1 = 3\cos\theta$ , $r_2 = 1 + \cos\theta$.

(a) (5 points) Sketch the curves.

| $\theta$ | $r_1$ |
|---|---|
| $0$ | $3$ |
| $\pi/2$ | $0$ |
| $\pi$ | $-3$ |

same point

| $\theta$ | $r_2$ |
|---|---|
| $0$ | $2$ |
| $\pi/2$ | $1$ |
| $\pi$ | $0$ |
| $\frac{3\pi}{2}$ | $1$ |
| $2\pi$ | $2$ |




(b) (5 points) Set up (Do Not Evaluate) the integral which represents the arc length of $r_2 = 1 + \cos\theta$.

The curve $r_2 = 1 + \cos\theta$ is traversed once on the interval $[0, 2\pi]$.

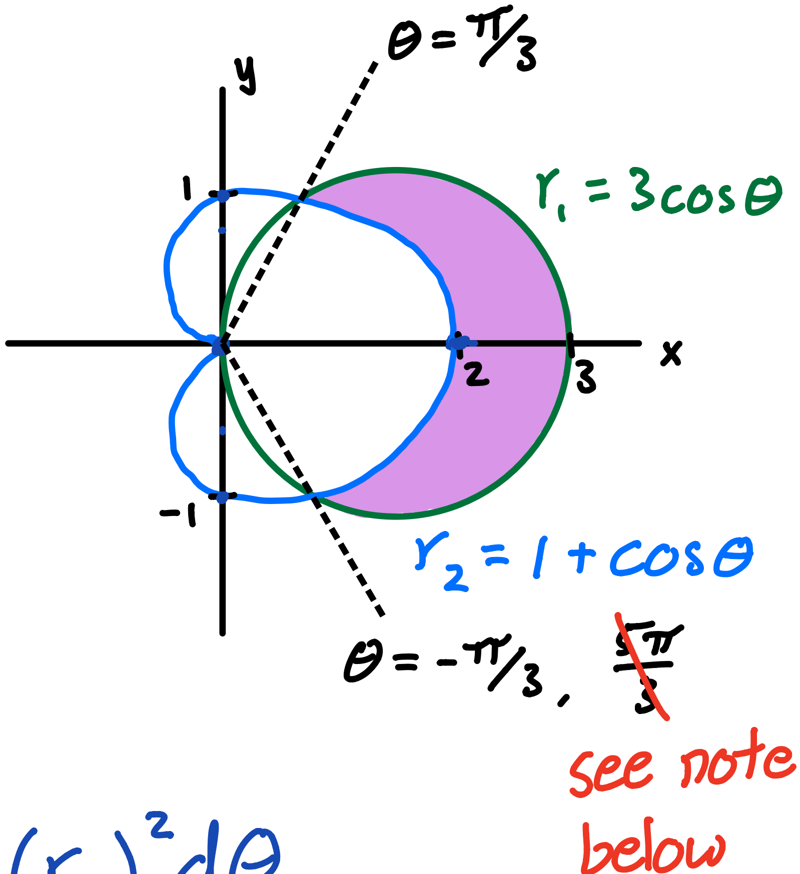
$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = (-\sin\theta)^2 + (1+\cos\theta)^2$$

$$= \underbrace{\sin^2\theta + \cos^2\theta}_{=1} + 2\cos\theta + 1$$

$$= 2\cos\theta + 2$$

$$L = \int_0^{2\pi} \sqrt{2\cos\theta + 2}\, d\theta$$

(c) (10 points) Find the area of the region that lies inside of the circle $r_1 = 3\cos\theta$ but outside the cardiod $r_2 = 1 + \cos\theta$.




Intersections:

$$3\cos\theta = 1 + \cos\theta$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = -\frac{\pi}{3}, \frac{\pi}{3}$$

$$A = \int_{-\pi/3}^{\pi/3} \frac{1}{2}(r_1)^2 - \frac{1}{2}(r_2)^2 d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} (3\cos\theta)^2 - (1+\cos\theta)^2 d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 8\underbrace{\cos^2\theta}_{=\frac{1}{2}(1+\cos 2\theta)} - 2\cos\theta - 1\, d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos 2\theta - 2\cos\theta + 3\, d\theta$$

By symmetry,

$$= \int_{0}^{\pi/3} 4\cos 2\theta - 2\cos\theta + 3\, d\theta$$

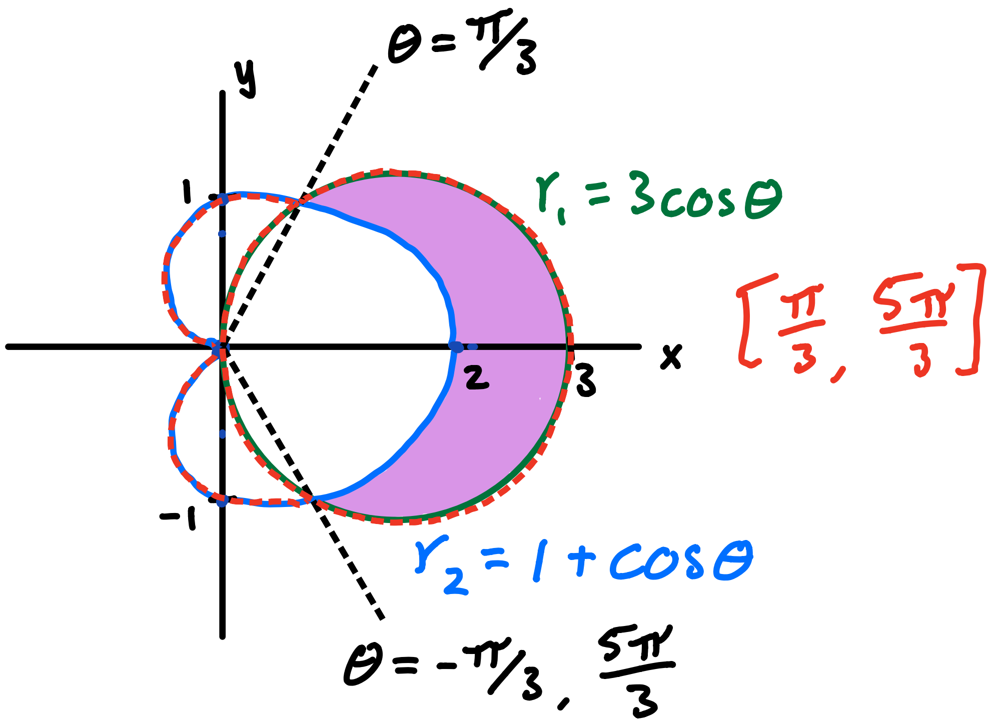
$$= \Big[2\sin 2\theta - 2\sin\theta + 3\theta\Big]_0^{\pi/3}$$

$$= \left(2\sin\tfrac{2\pi}{3} - 2\sin\tfrac{\pi}{3} + \pi\right) - 0$$

$$= \sqrt{3} - \sqrt{3} + \pi = \underline{\pi}$$

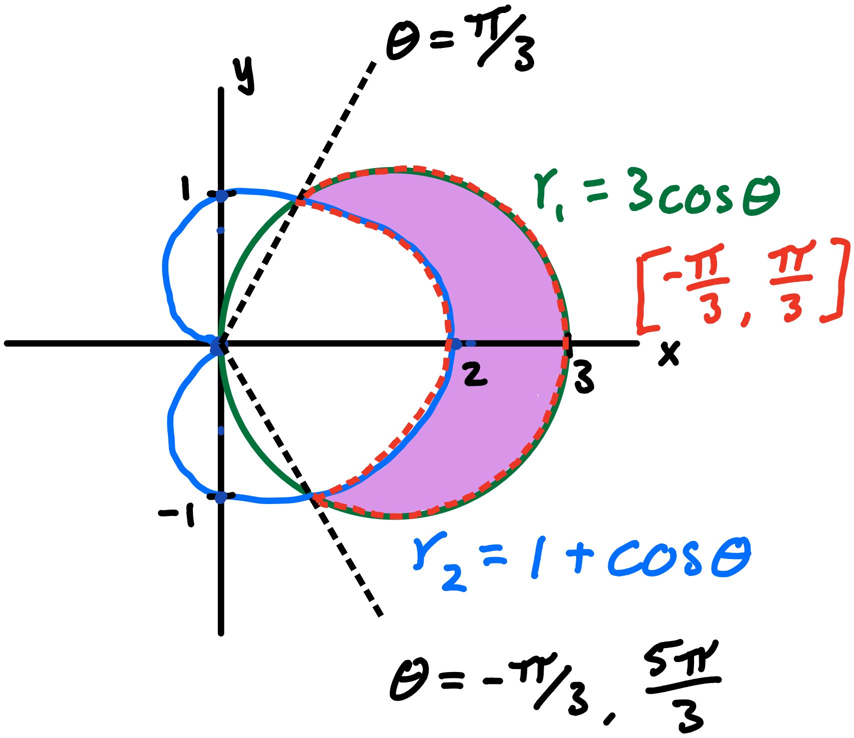
<u>Note</u>: Why is intgrating over $[\frac{\pi}{3}, \frac{5\pi}{3}]$ not correct?

The parts of the curves that correspond to the interval $[\frac{\pi}{3}, \frac{5\pi}{3}]$ are traced out in red.




Notice how this interval gives us the left half of the cardiod $r_2 = 1 + \cos\theta$ and it gives the entire circle $r = 3\cos\theta$. So this won't give us the shaded area.

Compare this to the interval $[-\pi/3, \pi/3]$.




Now we are tracing out the parts of the curve that make up the boundary of the shaded region, which means this is the correct interval to integrate over.

4. (10 points) Find the focus, directrix and vertex of the parabola $(x+2)^2 = 3y - 6$, then sketch the curve.

Ignoring translations we have

$x^2 = 3y$. General form is $x^2 = 4py$.

So $4p = 3 \Rightarrow p = 3/4$

$\underline{x^2 = 3y}$

Vertex: $(0,0)$

Focus: $(0,p) = (0, 3/4)$

Directrix: $y = -p = -3/4$

<u>Translations:</u>

$(x+2)^2$ is a shift left by 2.

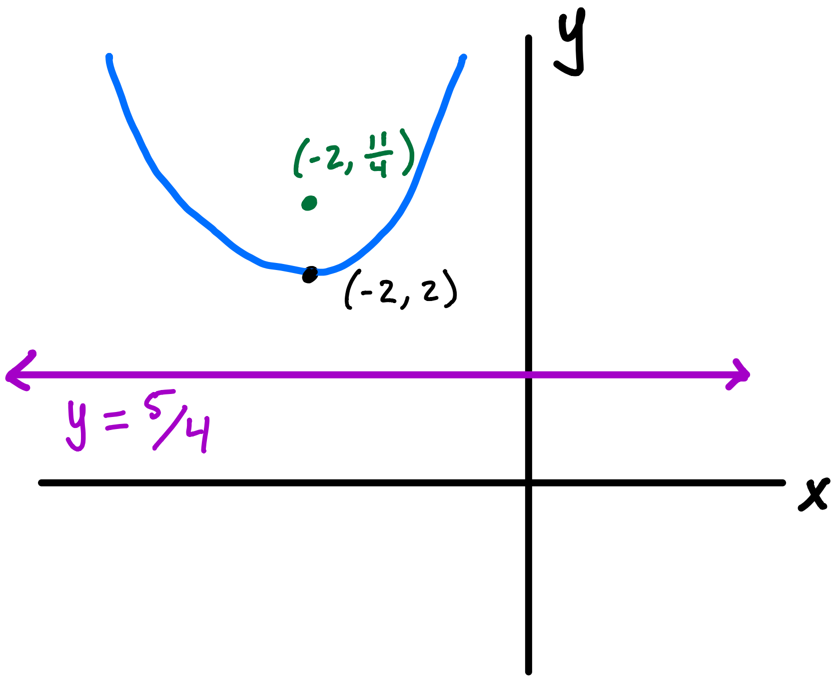
$3y-6 = 3(y-2)$ is a shift up 2.

$$(x+2)^2 = 3y - 6$$

Vertex: $(-2, 2)$

Focus: $\left(-2, \frac{11}{4}\right)$

Directrix: $y = \frac{5}{4}$




5. (30 points) Let $\vec{a} =< 2, 3, -4 >$, $\vec{b} =< -2, 4, 2 >$, and $\vec{c} =< 6, -1, -3 >$ .

(a) (4 points) Find the unit vector in the direction of $\vec{a}$.

$$\frac{\vec{a}}{|\vec{a}|} = \frac{\langle 2, 3, -4 \rangle}{\sqrt{2^2+3^2+4^2}} = \frac{1}{\sqrt{29}} \langle 2, 3, -4 \rangle$$

$$= \left\langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{-4}{\sqrt{29}} \right\rangle$$

(b) (4 points) Show that $\vec{a}$ and $\vec{b}$ are orthogonal.

$$\vec{a} \cdot \vec{b} = \langle 2, 3, -4 \rangle \cdot \langle -2, 4, 2 \rangle$$

$$= 2(-2) + 3(4) - 4(2) = -4 + 12 - 8 = 0 \checkmark$$

So $\vec{a}$ and $\vec{b}$ are orthogonal.

(c) (6 points) Find the angle between $\vec{b}$ and $\vec{c}$.

By Thm 1 in 12.3,

$$\cos\theta = \frac{\vec{b}\cdot\vec{c}}{|\vec{b}||\vec{c}|} = \frac{\langle -2, 4, 2\rangle \cdot \langle 6, -1, -3\rangle}{\left(\sqrt{2^2+4^2+2^2}\right)\left(\sqrt{6^2+1^2+3^2}\right)}$$

$$= \frac{-12-4-6}{\sqrt{24}\sqrt{46}} = \frac{-22}{4\sqrt{69}} = \frac{-11}{2\sqrt{69}} \approx -0.66$$

So $\theta = \cos^{-1}\left(\frac{-11}{2\sqrt{69}}\right) \approx \underline{131.46^\circ}$

(d) (8 points) Find the scalar and vector projections of $\vec{b}$ onto $\vec{c}$.

<u>Scalar</u>:

$$\text{Comp}_{\vec{c}}\vec{b} = \frac{\vec{b}\cdot\vec{c}}{|\vec{c}|} = \underline{\frac{-22}{\sqrt{46}}}$$

Vector:

$$\text{proj}_{\vec{c}}\vec{b} = \text{comp}_{\vec{c}}\vec{b}\left(\frac{\vec{c}}{|\vec{c}|}\right) = \left(\frac{-22}{\sqrt{46}}\right)\left(\frac{1}{\sqrt{46}}\right)\vec{c}$$

$$= \frac{-11}{23}\langle 6, -1, -3\rangle = \left\langle \frac{-66}{23}, \frac{11}{23}, \frac{33}{23} \right\rangle$$

(e) (8 points) Find the volume of the parallelepiped which is formed by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

$$V = \left|\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\right|$$

$$\begin{vmatrix} 2 & 3 & -4 \\ -2 & 4 & 2 \\ 6 & -1 & -3 \end{vmatrix} = 2\begin{vmatrix} 4 & 2 \\ -1 & -3 \end{vmatrix} - 3\begin{vmatrix} -2 & 2 \\ 6 & -3 \end{vmatrix} - 4\begin{vmatrix} -2 & 4 \\ 6 & -1 \end{vmatrix}$$

$$= 2(-12+2) - 3(6-12) - 4(2-24)$$

$$= -20 + 18 + 88 = 86$$

So $V = |86| = \underline{86}$

6. (15 points) Given the points $P(1,2,0)$, $Q(2,-1,3)$, and $R(2,0,-1) \in \mathbb{R}^3$.

   (a) (8 points) Find a vector which is orthogonal to the plane that passes through these points.

$$\overrightarrow{PQ} = \langle 2-1, -1-2, 3-0 \rangle = \langle 1, -3, 3 \rangle$$

$$\overrightarrow{QR} = \langle 2-2, 0+1, -1-3 \rangle = \langle 0, 1, -4 \rangle$$

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & -3 & 3 \\ 0 & 1 & -4 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & 3 \\ 1 & -4 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ 0 & -4 \end{vmatrix} j + \begin{vmatrix} 1 & -3 \\ 0 & 1 \end{vmatrix} k$$

$$= (12-3)i - (-4-0)j + (1-0)k$$

$$= 9i + 4j + k = \underline{\langle 9, 4, 1 \rangle}$$

(b) (7 points) Find the area of the triangle $PQR$.

$$A = \frac{1}{2}\left|\vec{PQ} \times \vec{QR}\right| = \frac{1}{2}\left|\langle 9, 4, 1\rangle\right|$$

$$= \frac{1}{2}\sqrt{9^2+4^2+1^2} = \frac{\sqrt{98}}{2}$$

**Bonus Question**(5 pts): Consider two vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a} \times \vec{b} = \vec{0}$. What can be said about vectors $\vec{a}$ and $\vec{b}$? Moreover, is it possible that $\vec{a} \cdot \vec{b} = 0$ as well? Explain why or why not.

For two nonzero vectors $\vec{a}$ and $\vec{b}$,

i) $\vec{a} \times \vec{b} = 0 \Rightarrow \vec{a}, \vec{b}$ are parallel

ii) $\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a}, \vec{b}$ are perpendicular

The only way they are both 0 is if $\vec{a} = \vec{0}$ or $\vec{b} = 0$.