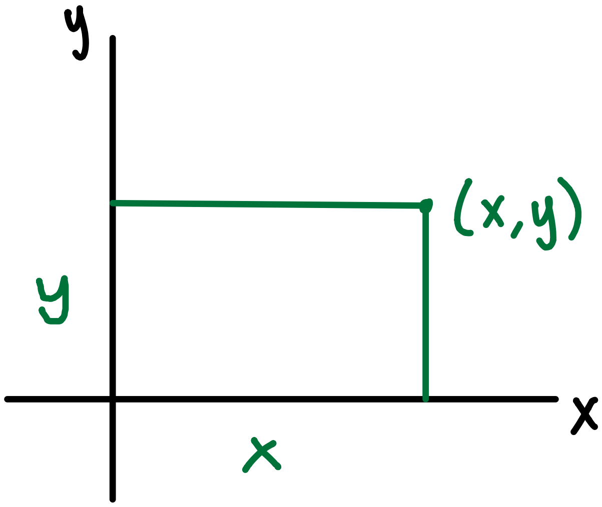
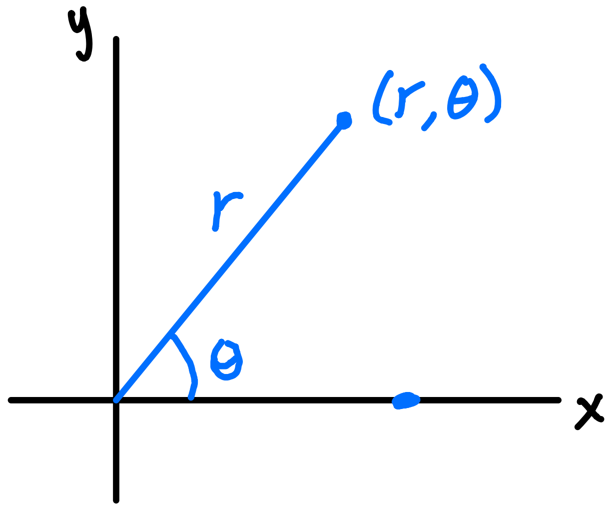
10.3 Polar Coordinates

Cartesian (Rectangular)	Polar
	
$(1, 0)$ $(1, 2\pi)$	r - distance from origin θ - angle made with positive x-axis

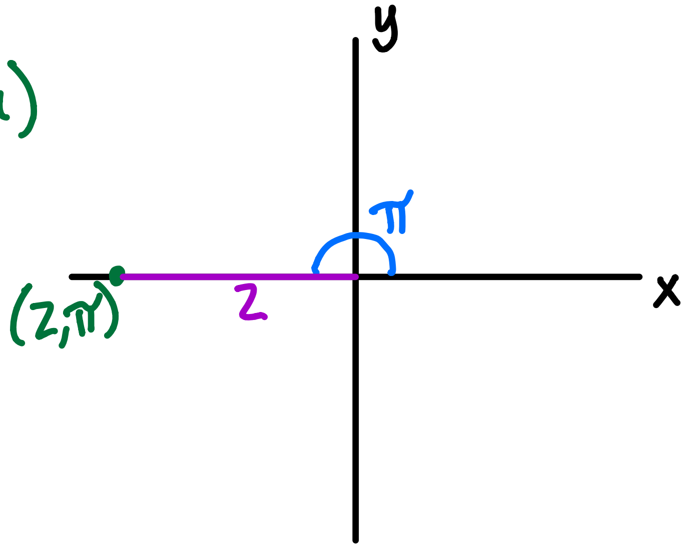
Ex 1 Plot the following polar points.

a) $(2, \pi)$ b) $\left(\frac{1}{2}, -\frac{2\pi}{3}\right)$

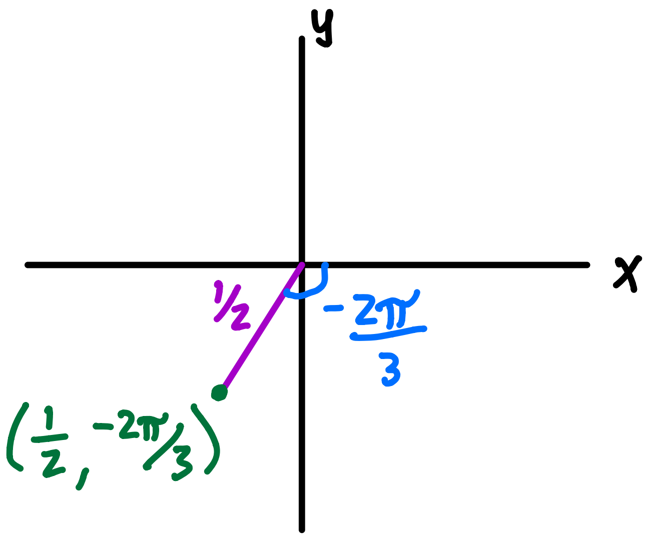
c) $\left(-1, \frac{\pi}{4}\right)$ d) $\left(-4, -\frac{\pi}{2}\right)$

Sol

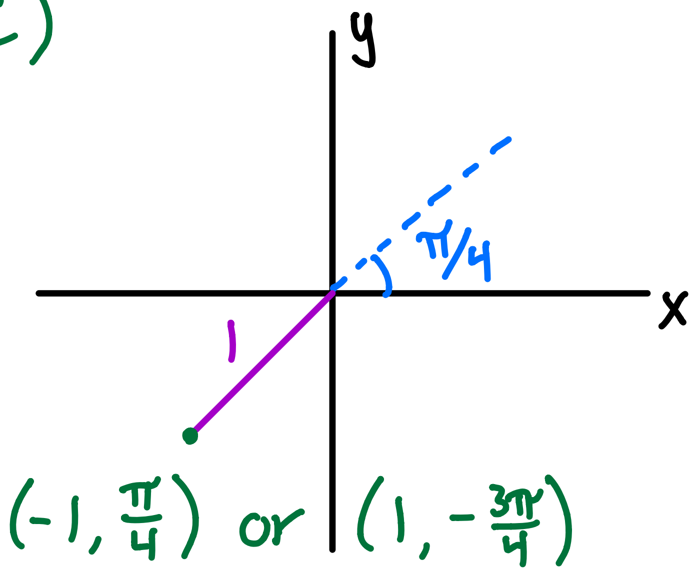
a)



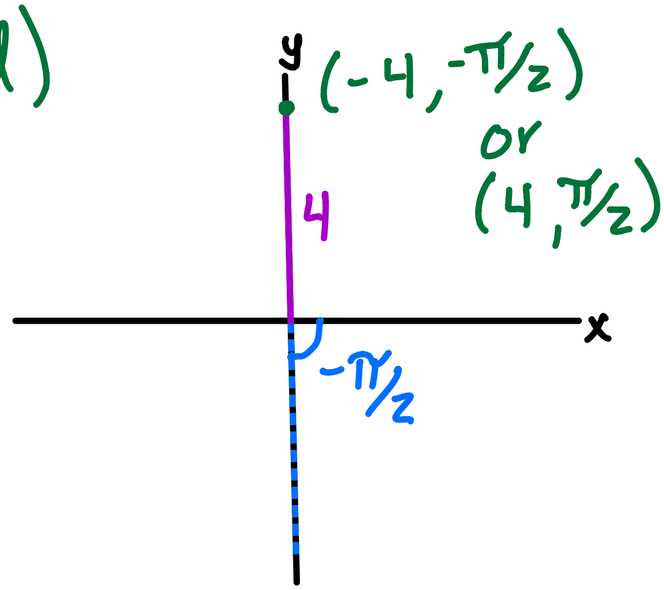
b)



c)



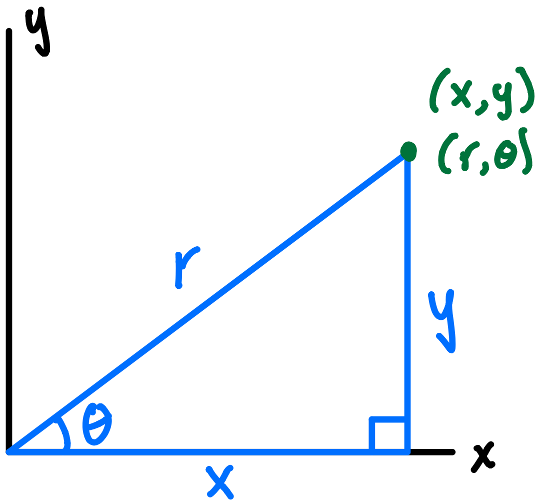
d)



Unlike Cartesian coordinates each point can have many polar representations (r, θ). In fact for any real numbers r, θ and integer k,

$$(r,\theta) = (r, \theta + 2\pi k) \qquad (-r, \theta) = (r, \theta + \pi)$$

Relationship Between Cart. and Polar



$\cos\theta = \frac{x}{r}$

$\Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r}$

$\Rightarrow y = r\sin\theta$

Polar → Cart.

$x = r\cos\theta$

$y = r\sin\theta$

Cart. → Polar

$r^2 = x^2 + y^2$

$\tan\theta = \frac{y}{x}$

Ex 2

a) Convert $(2, \pi/3)$ into Cartesian coor.

b) Convert $(-1, 1)$ into Polar coor.

Sol

a) $(2, \pi/3) \longrightarrow$ Cart.

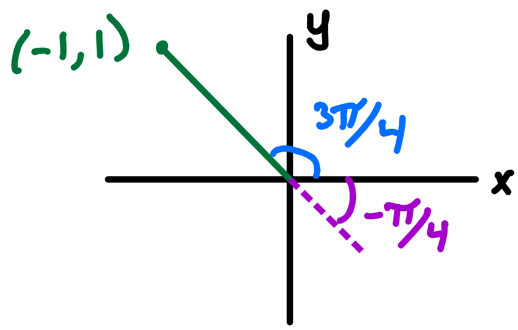
$$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2\left(\frac{1}{2}\right) = 1$$

$$y = r\sin\theta = 2\sin\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

Typo in the video

So the point is $(1, \sqrt{3})$ in Cartesian coor.

b) $(-1, 1) \longrightarrow$ Polar



$$r^2 = x^2 + y^2 = 2 \Rightarrow r = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1 \Rightarrow \theta = \cancel{-\frac{\pi}{4}}, \frac{3\pi}{4} \checkmark$$

Not in the correct quadrant

So the point is $(\sqrt{2}, \frac{3\pi}{4})$ in Polar coor.

Polar Curves

The graph of a polar equation $r=f(\theta)$ (more generally $F(r,\theta)=0$) consists of all the points with at least one representation (r,θ) satisfying the equation.

Ex 3 Sketch the following polar curves.

a) $r=2$

b) $\theta = 3\pi/4$

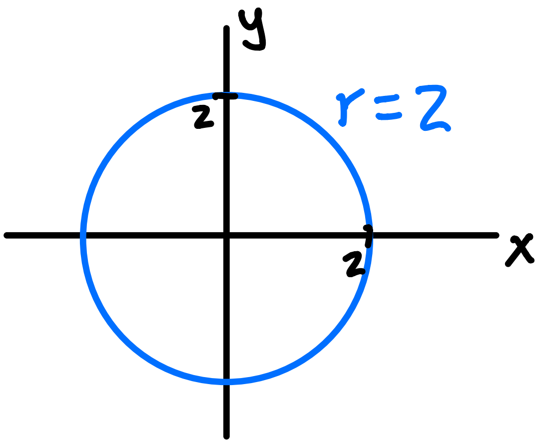
c) $r=1+\sin\theta$

d) $r=\cos 2\theta$

Sol

a) $r=2$ consits of all the points (r,θ) where $r=2$ (θ can be anything).

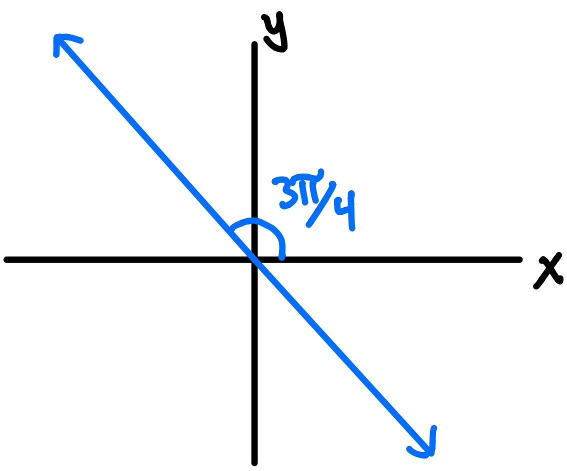
So all the points with a distance of 2 from the origin.



$$r^2 = x^2 + y^2$$
$$\Rightarrow 2^2 = x^2 + y^2$$
$$\Rightarrow x^2 + y^2 = 4$$

b) $\theta = 3\pi/4$

Here r can be anything (including negatives).

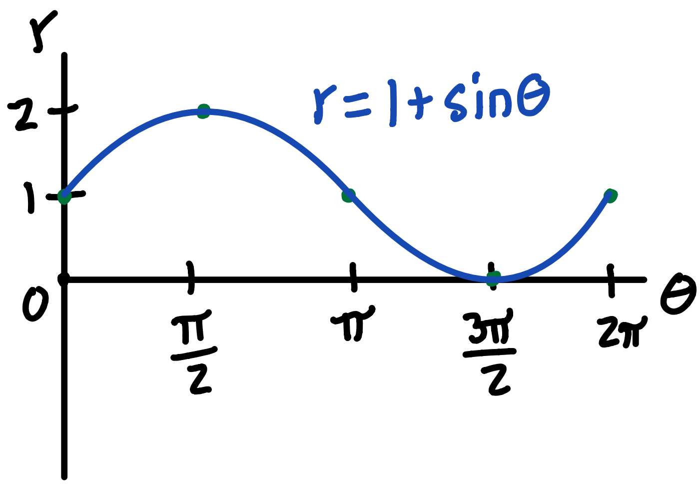


$$\frac{y}{x} = \tan\theta = \tan\left(\frac{3\pi}{4}\right) = -1$$

$$\Rightarrow y = -x$$

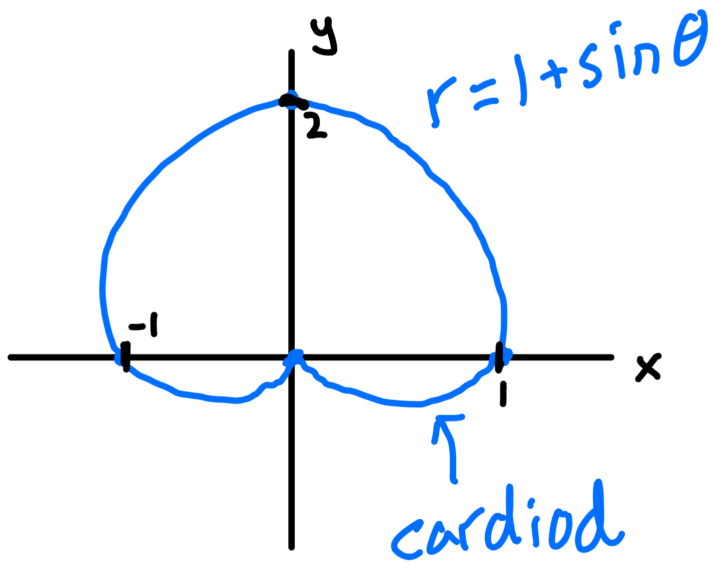
c) $r = 1 + \sin\theta$

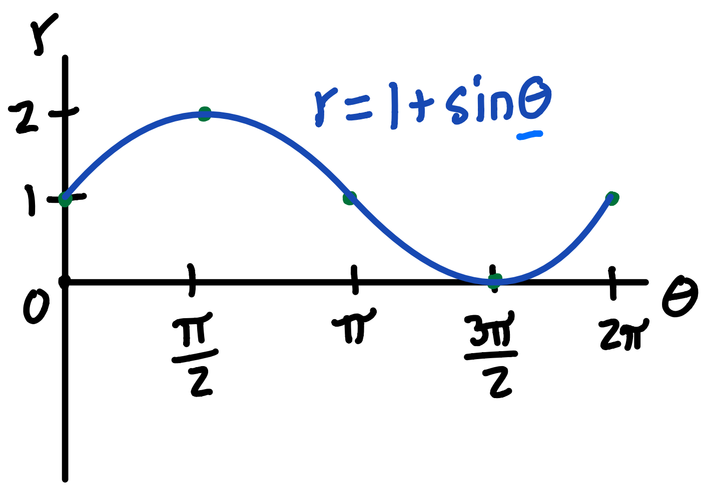
For more complicated polar equations it can be useful to first try graphing it as a Cartesian curve. (Pretend it's $y = 1 + \sin x$).



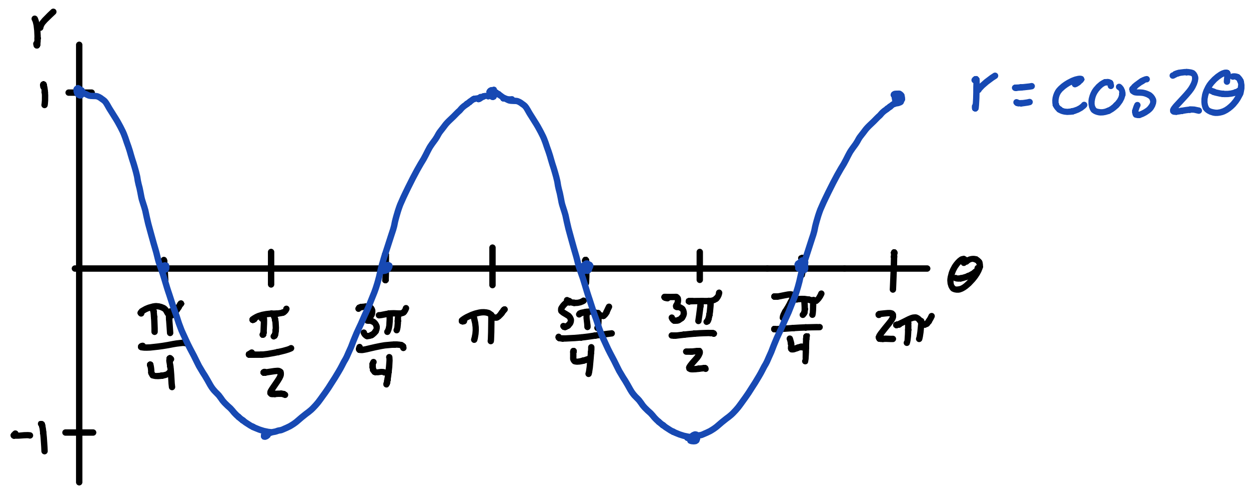
θ	r
0	1
$\pi/2$	2
π	1
$3\pi/2$	0
2π	1

Doing this can help identify some key points to graph like: $(1, 0)$, $(2, \pi/2)$, $(1, \pi)$ $(0, 3\pi/2)$, $(1, 2\pi)$

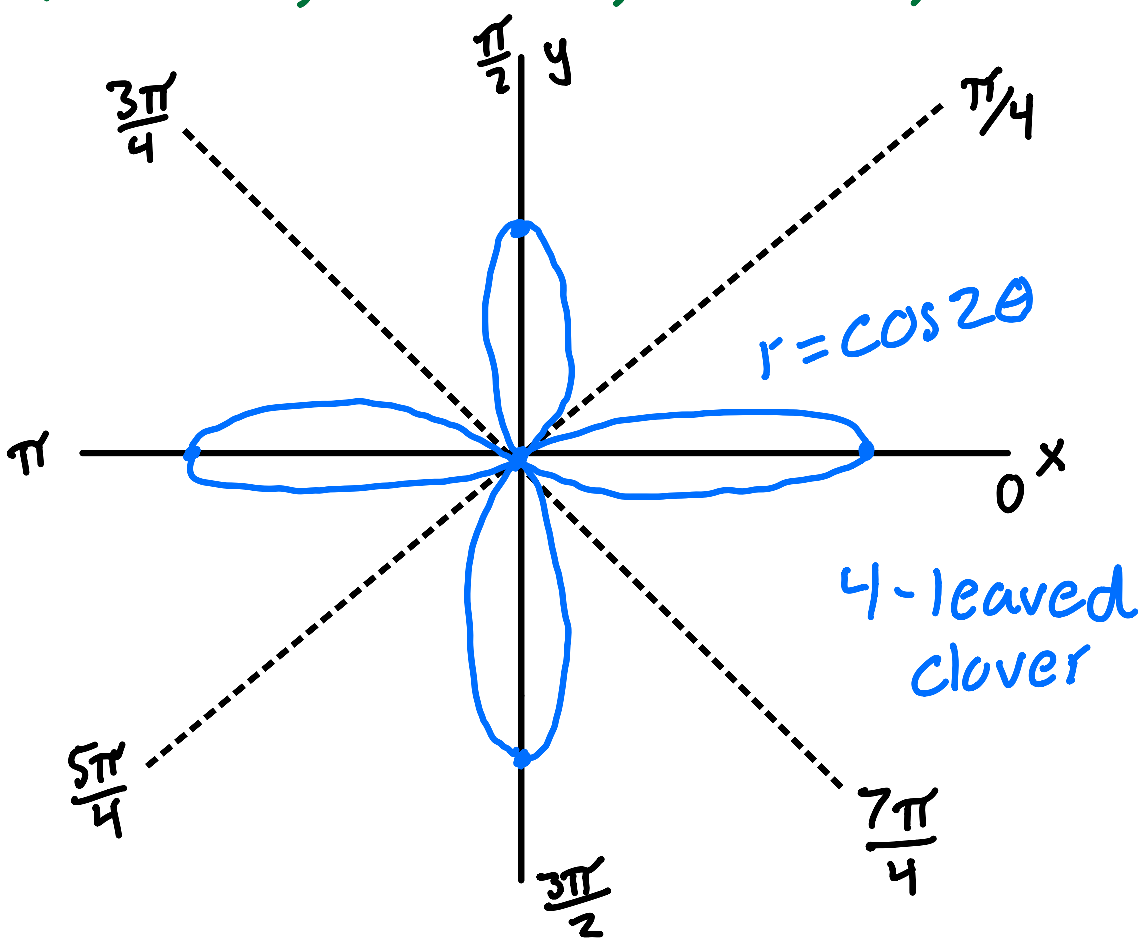




d) $r = \cos 2\theta$



Some key points: $(1, 0)$, $(0, \pi/4)$, $(-1, \pi/2)$, $(0, 3\pi/4)$, $(1, \pi)$, $(0, 5\pi/4)$, $(-1, \frac{3\pi}{2})$, $(0, \frac{7\pi}{4})$, $(1, 2\pi)$



Tangents to Polar Curves

Let $r = f(\theta)$ be a polar curve. Then

$$x = r\cos\theta = f(\theta)\cdot\cos\theta$$
$$y = r\sin\theta = f(\theta)\cdot\sin\theta$$

Parametric Equations!

By product rule,

$$\frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

So the slope of the tangent line is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Ex 4 Let $r = 2 + \sin(3\theta)$. Find the slope of the tangent line at $\theta = \pi/4$.

Sol

$$r = 2 + \sin 3\theta \qquad \frac{dr}{d\theta} = 3\cos 3\theta$$

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$= \frac{3\cos(3\theta)\sin\theta + (2+\sin 3\theta)\cos\theta}{3\cos(3\theta)\cos\theta - (2+\sin 3\theta)\sin\theta}$$

Not worth trying to simplify

For $\theta = \pi/4$ we have

$$r = 2 + \sin\left(\frac{3\pi}{4}\right) = 2 + \frac{\sqrt{2}}{2} \qquad \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\frac{dr}{d\theta} = 3\cos\left(\frac{3\pi}{4}\right) = \frac{-3\sqrt{2}}{2} \qquad \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

So

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\left(\frac{-3\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(2+\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}{\left(\frac{-3\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(2+\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$$

$$= \frac{-\frac{3}{2} + \sqrt{2} + \frac{1}{2}}{-\frac{3}{2} - \sqrt{2} - \frac{1}{2}} = \frac{\sqrt{2} - 1}{-\sqrt{2} - 2} = \frac{1-\sqrt{2}}{1+\sqrt{2}}$$

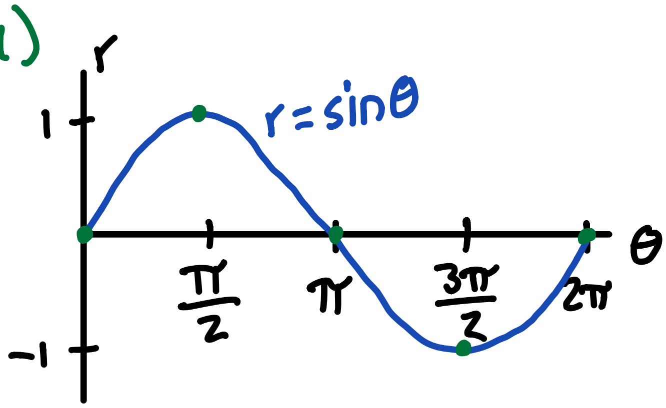
Ex 5 Let $r = \sin\theta$, $0 \le \theta \le 2\pi$.

a) Sketch the curve.

b) Find the vertical and horizontal tangents.

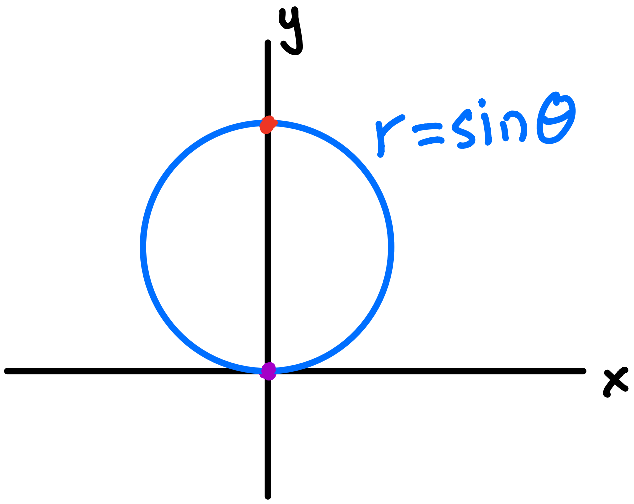
Sol

a)



Key points:

$(0,0)$, $(1,\frac{\pi}{2})$, $(0,\pi)$, $(-1,\frac{3\pi}{2})$, $(0,2\pi)$



This circle is being traversed twice on the interval $0 \le \theta \le 2\pi$.

b) Recall,

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta \qquad \left(HT: \frac{dy}{d\theta} = 0\right)$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \left(VT: \frac{dx}{d\theta} = 0\right)$$

We have $r = \sin\theta$, $\frac{dr}{d\theta} = \cos\theta$ so

$$\frac{dy}{d\theta} = \cos\theta\sin\theta + \sin\theta\cos\theta$$

$$= 2\sin\theta\cos\theta = \sin(2\theta)$$

$$\sin(2\theta) = 0 \Rightarrow 2\theta = 0, \pi, 2\pi$$

$$\Rightarrow \theta = 0, \frac{\pi}{2}, \pi$$

HT (polar): $(0,0)$, $(1, \pi/2)$

HT (rect.): $(0,0)$, $(0,1)$

Similarly,

$$\frac{dx}{d\theta} = \cos^2\theta - \sin^2\theta = 0$$

$$\Rightarrow \cos^2\theta = \sin^2\theta$$

$$\Rightarrow \tan^2\theta = 1$$

$$\Rightarrow \tan\theta = \pm 1$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{3\pi}{4}$$

$r = \sin\theta$

$r = \frac{\sqrt{2}}{2}$ when

$\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$

VT (polar): $\left(\frac{\sqrt{2}}{2}, \frac{\pi}{4}\right)$, $\left(\frac{\sqrt{2}}{2}, \frac{3\pi}{4}\right)$

VT (rect.): $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{-1}{2}, \frac{1}{2}\right)$

Practice Problems

1) Sketch the polar curve $r = \sin 3\theta$
$0 \le \theta \le \pi$.

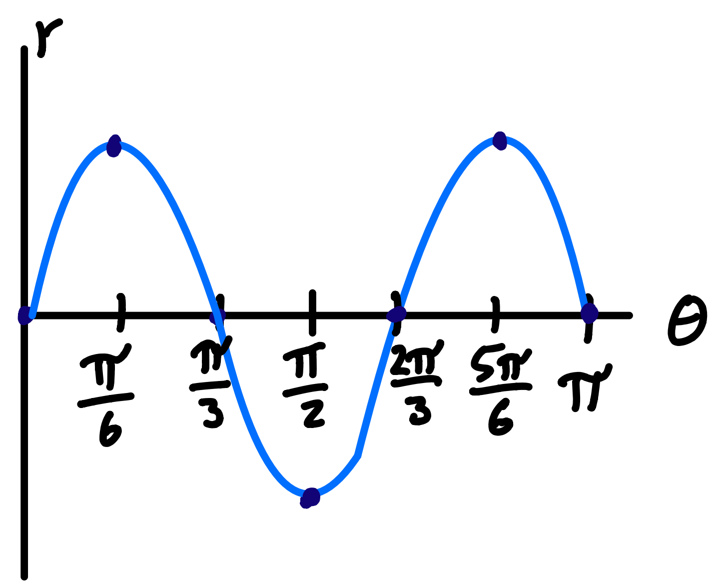
2) Let $r = \cos 2\theta$. Find the equation of the tangent line at $\theta = \pi/6$.

3) Find the vertical tangents of $r = 2 + 3\cos\theta$, $0 \le \theta \le 2\pi$.

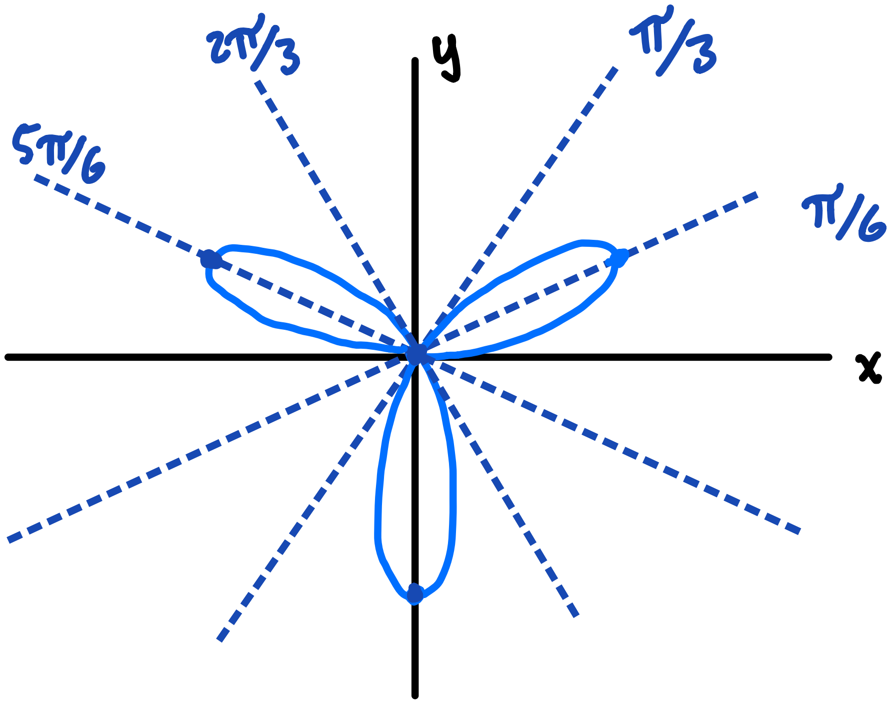
Solutions

1) $r = \sin 3\theta$

θ	r
0	0
$\pi/6$	1
$\pi/3$	0
$\pi/2$	-1
$2\pi/3$	0



$5\pi/6$	1
π	0



2) Let $r = \cos 2\theta$. Find the equation of the tangent line at $\theta = \pi/6$.

We have $\dfrac{dr}{d\theta} = -2\sin 2\theta$. For $\theta = \dfrac{\pi}{6}$,

$r = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, $\quad \dfrac{dr}{d\theta} = -2\sin\left(\frac{\pi}{3}\right) = -\sqrt{3}$

The slope is given by

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\pi/6} = \frac{(-\sqrt{3})\left(\frac{1}{2}\right)+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)}{(-\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)-\frac{1}{2}\left(\frac{1}{2}\right)} = \frac{-\frac{\sqrt{3}}{4}}{-\frac{7}{4}} = \frac{\sqrt{3}}{7}$$

mistake in video

The point (in polar) corresponding to $\theta=\frac{\pi}{6}$ is $\left(\frac{1}{2},\frac{\pi}{6}\right)$. Converting to rectangular,

$$\left.\begin{aligned} x &= r\cos\theta = \tfrac{1}{2}\cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{4} \\ y &= r\sin\theta = \tfrac{1}{2}\sin\tfrac{\pi}{6} = \tfrac{1}{4} \end{aligned}\right\rangle \left(\frac{\sqrt{3}}{4},\frac{1}{4}\right)$$

So equation of tangent line is

$$\underline{y-\frac{1}{4}=\frac{\sqrt{3}}{7}\left(x-\frac{\sqrt{3}}{4}\right)}$$

3) Find the vertical tangents of

$r = 3 + 3\cos\theta, \quad 0 \le \theta \le 2\pi.$

$$\frac{dr}{d\theta} = -3\sin\theta$$

VTs occur when $\frac{dx}{d\theta} = 0$ $\left(\text{assuming } \frac{dy}{d\theta} \neq 0 \right)$.

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$= -3\sin\theta\cos\theta - (3 + 3\cos\theta)\sin\theta$$

$$= -3\sin\theta - 6\sin\theta\cos\theta$$

$$= -3\sin\theta(1 + 2\cos\theta) = 0$$

$\sin\theta = 0$

$\Rightarrow \theta = 0, \pi, 2\pi$

$1 + 2\cos\theta = 0$

$\Rightarrow \cos\theta = -1/2$

$\Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

Need to check that none of these values also make $\frac{dy}{d\theta} = 0$.

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$= -3\sin^2\theta + 3\cos\theta + 3\cos^2\theta$$

$$= 3(\cos^2\theta - \sin^2\theta + \cos\theta)$$

$$= 3(\cos 2\theta + \cos\theta)$$

Notice that $\frac{dy}{d\theta} = 0$ when $\theta = \pi$ so we must use limits to confirm if there is a vertical or horizontal tangent at $\theta = \pi$.

$$\lim_{\theta\to\pi} \frac{dy}{dx} = \lim_{\theta\to\pi} \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \lim_{\theta\to\pi} \frac{3(\cos 2\theta + \cos\theta)}{-3\sin\theta(1+2\cos\theta)}$$

Indet. Form

Type $\frac{0}{0}$

$$\overset{L'H}{=} \lim_{\theta\to\pi} \frac{-2\sin 2\theta - \sin\theta}{-\cos\theta(1+2\cos\theta) + 2\sin^2\theta} = \frac{0}{-1} = 0$$

This implies there is a horizontal tangent at $\theta = \pi$ (slope of 0).

That leaves us with $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$

VT(polar): $\underline{(6,0), \left(\frac{3}{2}, \frac{2\pi}{3}\right), \left(\frac{3}{2}, \frac{4\pi}{3}\right)}$

VT(rect.): $\underline{(6,0), \left(-\frac{3}{4}, \frac{3\sqrt{3}}{4}\right), \left(-\frac{3}{4}, \frac{-3\sqrt{3}}{4}\right)}$

<u>Suggested Textbook Exercises</u> (10.3)

3, 16, 22, 29, 39, 55, 63, 64