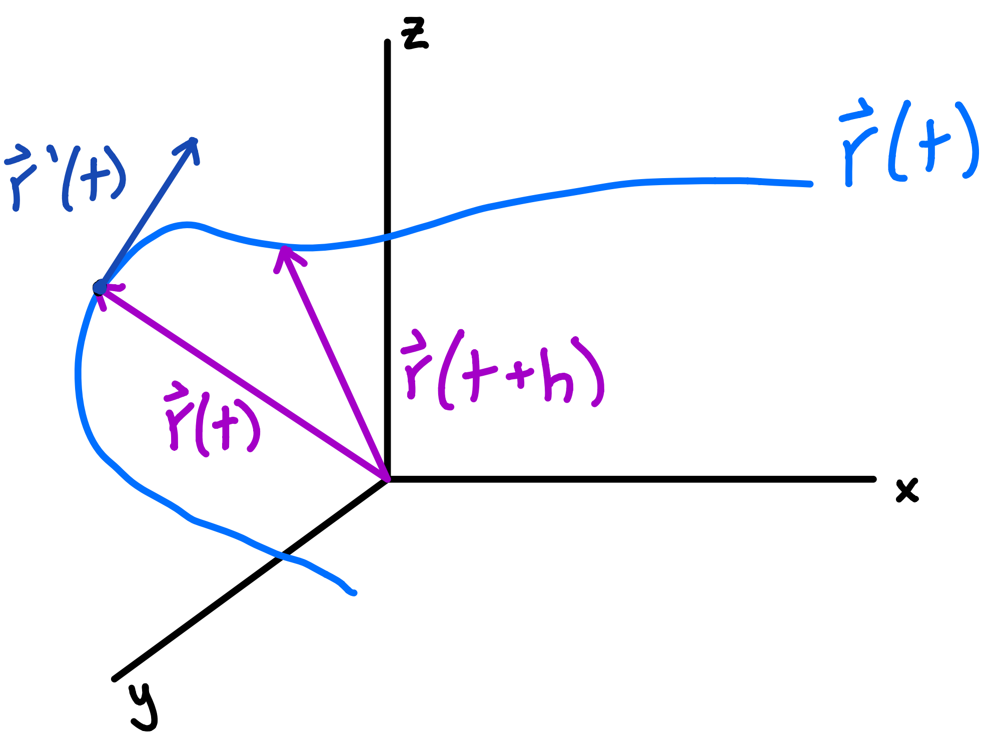
13.2 Derivatives and Integrals of Vector Functions

The derivative of $\vec{r}(t)$ is defined the same way as real-valued functions,

$$\frac{dr}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$



Theorem 1

If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ where f, g, h are all differentiable, then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

If $\vec{r}\,'$ is the tangent vector then $\dfrac{\vec{r}\,'}{|\vec{r}\,'|}$ is the unit tangent vector.

Ex 1

Let $\vec{r}(t) = \langle 1+t^3, te^{-t}, \sin(2t) \rangle$

Find the unit tangent vector and the eqn. of the tangent line at $t=0$.

Sol

$$\vec{r}\,'(t) = \langle 3t^2, e^{-t} - te^{-t}, 2\cos(2t) \rangle$$

$$\vec{r}\,'(0) = \langle 0, 1, 2 \rangle$$ ← tangent vector at $t=0$

$$\frac{\vec{r}\,'(0)}{|\vec{r}\,'(0)|} = \frac{\langle 0,1,2\rangle}{\sqrt{5}} = \langle 0, \tfrac{1}{\sqrt{5}}, \tfrac{2}{\sqrt{5}} \rangle$$

unit tangent vector at $t=0$

$$\vec{r}(0) = \langle 1,0,0 \rangle \quad , \quad \vec{v} = \langle 0,1,2 \rangle$$

Eqn of tangent line at $t=0$:

$$\vec{s}(t) = \vec{v}t + \vec{r}_0 = \langle 0,1,2\rangle \cdot t + \langle 1,0,0 \rangle$$

$$= \langle 1, t, 2t \rangle$$ ← vector eqn.

$$x = 1 \qquad y = t \qquad z = 2t$$ ← para. eqns.

Ex 2 Find the parametric eqns. of the tangent line of $x = \ln(t+1)$ $y = t\cos(2t)$ $z = 2^t$ at point $(0,0,1)$.

Sol

Let $\vec{r}(t) = \langle \ln(t+1), t\cos(2t), 2^t \rangle$

$$\vec{r}\,'(t) = \left\langle \frac{1}{t+1}, \cos(2t) - 2t\sin(2t), 2^t \cdot \ln(2) \right\rangle$$

$(0,0,1) \Rightarrow t = 0$

$$\vec{r}\,'(0) = \langle 1, 1, \ln(2) \rangle$$ ↙ tangent vector

Tangent line at $(0,0,1)$:

$\vec{v} = \langle 1, 1, \ln(2) \rangle$, $\vec{r}_0 = \langle 0, 0, 1 \rangle$

$x = 1 \cdot t + 0$ $\quad$ $y = 1 \cdot t + 0$ $\quad$ $z = \ln(2) \cdot t + 1$

$x = t$ $\quad$ $y = t$ $\quad$ $z = \ln(2) + 1$

Derivative Properties

Let $\vec{u}$ and $\vec{v}$ be diff. vector functions,

c a scalar, and f a real-valued diff. function.

1) $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \vec{u}'(t) \pm \vec{v}'(t)$

2) $\frac{d}{dt}\left[c\vec{u}(t)\right] = cu'(t)$

3) $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$

4) $\frac{d}{dt}\left[\vec{u}(t) \cdot \vec{v}(t)\right] = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$

5) $\frac{d}{dt}\left[\vec{u}(t) \times \vec{v}(t)\right] = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

6) $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$

Integrals

Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then,

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$$

Ex 4

Let $\vec{r}(t) = \langle 2\cos t, \sin t, 2t \rangle$. Find

$$\int \vec{r}(t)\,dt \quad \text{and} \quad \int_0^{\pi/2} \vec{r}(t)\,dt.$$

Sol

$$\int \vec{r}(t)\,dt = \left\langle \int 2\cos t\,dt,\ \int \sin t\,dt,\ \int 2t\,dt \right\rangle$$

$$= \langle 2\sin t, -\cos t, t^2 \rangle + \vec{C}$$

$\vec{C} = \langle C_1, C_2, C_3 \rangle$

So,

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \left\langle 2\sin t, -\cos t, t^2 \right\rangle \Big|_0^{\pi/2}$$

$$= \left\langle 2\sin\tfrac{\pi}{2} - 2\sin 0, -\cos\tfrac{\pi}{2} + \cos 0, \left(\tfrac{\pi}{2}\right)^2 - 0 \right\rangle$$

$$= \left\langle 2, 1, \tfrac{\pi^2}{4} \right\rangle$$

Ex 5

Find $\vec{r}(t)$ where $\vec{r}\,'(t) = \langle t, e^t, -te^{t^2} \rangle$ and $\vec{r}(0) = \langle 1, 2, 1 \rangle$.

Sol

$$\int \vec{r}\,'(t)\,dt = \int \langle t, e^t, -te^{t^2} \rangle\,dt \;\ominus$$

i) $\int t\,dt = \frac{t^2}{2}$ ii) $\int e^t\,dt = e^t$

iii) $\int -te^{t^2}\,dt$ $\quad u = t^2$, $du = 2t\,dt$

$$= -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^{t^2}$$

$$\boxed{=} \left\langle \frac{t^2}{2}, e^t, -\frac{1}{2}e^{t^2} \right\rangle + \vec{c}$$

$$= \left\langle \frac{t^2}{2} + c_1, e^t + c_2, -\frac{1}{2}e^{t^2} + c_3 \right\rangle$$

Use $\vec{r}(0) = \langle 1, 2, 1 \rangle$ to find $\vec{c}$:

$$1 = \frac{0^2}{2} + c_1 \Rightarrow c_1 = 1$$

$$2 = e^0 + c_2 \Rightarrow c_2 = 1$$

$$1 = -\frac{1}{2}e^{0^2} + c_3 \Rightarrow c_3 = \frac{3}{2}$$

Therefore,

$$\vec{r}(t) = \left\langle \frac{t^2}{2} + 1, e^t + 1, -\frac{1}{2}e^{t^2} + \frac{3}{2} \right\rangle$$

13.3 Arc Length

Recall from 10.2, for para. eqns.

$x = f(t)$ $y = g(t)$, $a \le t \le b$, the arc length is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Letting $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \le t \le b$

or $x = f(t)$ $y = g(t)$ $z = h(t)$, $a \le t \le b$, the

arc length of this curve is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Ex 6

Find the arc length of the circular helix with vector eqn. $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$ from $(1,0,0)$ to $(1,0,4\pi)$.

Sol

Need to find bounds in terms of t:

$(1,0,0) \Rightarrow \cos t = 1, \sin t = 0, 2t = 0 \Rightarrow t = 0$

$(1,0,4\pi) \Rightarrow 2t = 4\pi \Rightarrow t = 2\pi$

$$\vec{r}'(t) = \langle \underbrace{-\sin t}_{\frac{dx}{dt}}, \underbrace{\cos t}_{\frac{dy}{dt}}, \underbrace{2}_{\frac{dz}{dt}} \rangle$$

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (2)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{5}\, dt = 2\sqrt{5}\pi$$

Practice Problems

1) Let $\vec{r}(t) = \langle t^3 - 1, e^{2t-2}, \cos(\pi t) \rangle$

a) Find the unit tangent vector at $t = 1$.

b) Write eqn for tangent line at $t = 1$.

c) Evaluate $\int_0^2 \vec{r}(t)\, dt$.

d) Set up integral for the arc length of $\vec{r}(t)$ between $(0, 1, -1)$ and $(7, e^2, 1)$.

Solutions

1) $\vec{r}(t) = \langle t^3 - 1,\ e^{2t-2},\ \cos(\pi t) \rangle$

a) $\vec{r}\,'(t) = \langle 3t^2,\ 2e^{2t-2},\ -\pi \sin(\pi t) \rangle$

$\vec{r}\,'(1) = \langle 3, 2, 0 \rangle$ ← tangent vector

$$\frac{\vec{r}\,'(1)}{|\vec{r}\,'(1)|} = \frac{\langle 3,2,0 \rangle}{\sqrt{3^2+2^2+0^2}} = \underline{\left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}}, 0 \right\rangle}$$

unit tangent vector

b) Tangent line at $t=1$:

$\vec{r}(1) = \langle 0, 1, -1 \rangle$, $\vec{v} = \langle 3, 2, 0 \rangle$

$x = 0 + 3t \qquad y = 1 + 2t \qquad z = -1 + 0t$

$x = 3t \qquad y = 1 + 2t \qquad z = -1$

para. eqns. of tangent line

c) $\int_0^2 \vec{r}(t)\,dt = \int_0^2 \langle t^3-1, e^{2t-2}, \cos(\pi t)\rangle\,dt$

$$\int_0^2 t^3-1\,dt = \left[\frac{t^4}{4}-t\right]_0^2 = (4-2)-0 = 2$$

$u = 2t-2$
$du = 2dt$

$$\int_0^2 e^{2t-2}\,dt = \left[\frac{1}{2}e^{2t-2}\right]_0^2 = \frac{1}{2}(e^2-e^{-2})$$

$$\int_0^2 \cos(\pi t)\,dt = \left[\frac{\sin \pi t}{\pi}\right]_0^2 = (0-0) = 0$$

Therefore,

$$\int_0^2 \vec{r}(t)\,dt = \underline{\langle 2, \tfrac{1}{2}(e^2-e^{-2}), 0\rangle}$$

d) Arc length between $(0,1,-1)$ and $(7, e^2, 1)$.

$\vec{r}(t) = \langle t^3-1, e^{2t-2}, \cos(\pi t) \rangle$

$$\frac{dx}{dt} = 3t^2 \qquad \frac{dy}{dt} = \frac{1}{2}e^{2t-2} \qquad \frac{dz}{dt} = -\pi \sin(\pi t)$$

<u>Finding bounds for t:</u>

$(0, 1, -1) \Rightarrow t = 1$

$(7, e^2, 1) \Rightarrow t = 2$

So the arc length is given by

$$L = \int_1^2 \sqrt{(3t^2)^2 + \left(\frac{1}{2}e^{(2t-2)}\right)^2 + (-\pi \sin(\pi t))^2}\, dt$$

$$= \int_1^2 \sqrt{9t^4 + \frac{1}{4}e^{4t-4} + \pi^2 \sin^2 \pi t}\, dt$$

Suggested Textbook Exc.

13.2

(9, 13, 17, 19, 23, 25, 35, 41)

13.3

(2, 3, 5)