1.4) Separable Equations and Applications

An o.d.e. of the form

$$\frac{dy}{dx} = g(x)\,h(y) \quad \underline{\text{or}} \quad \frac{dy}{dx} = \frac{g(x)}{f(y)}$$

is called <u>separable</u>. $\left[h(y) = \frac{1}{f(y)} \right]$

<u>Sol'n procedure</u>:

$$\frac{dy}{dx} = \frac{g(x)}{f(y)} \implies \boxed{\int f(y)\,dy = \int g(x)\,dx}$$

↑ <u>separate</u> the variables

Also in this lecture, we will discuss:

- Singular sol'ns
- explicit sol'ns
- implicit sol'ns
- heat transfer

Examples:

(#2) $$\frac{dy}{dx} + 2xy^2 = 0$$

$$\frac{dy}{dx} = -2xy^2$$

$$\int \frac{dy}{y^2} = \int -2x\,dx$$

$$-y^{-1} = -x^2 + C_1$$

Note: We divided by y^2, which is only allowed for $y^2 \neq 0$ (ie $y \neq 0$).

Solve for y:

$$y^{-1} = x^2 - C_1$$

$$y = \frac{1}{x^2 - C_1}$$

Let $C = -C_1$.

$$\therefore \boxed{y(x) = \frac{1}{x^2 + C}}$$

Note: $y(x) \equiv 0$ is also a sol'n, but is not of the form given here.

$\therefore$ $y(x) \equiv 0$ is a singular sol'n.

(#3) $\frac{dy}{dx} = y \sin x$

$$\int \frac{dy}{y} = \int \sin x \, dx$$

Note: $y \neq 0$ here

$$\ln|y| = -\cos x + C_1$$

$\boxed{\exp(x) = e^x}$

Solve for y:

$$\exp(\ln|y|) = \exp(-\cos x + C_1)$$

$$|y| = e^{-\cos x + C_1}$$

$$|y| = e^{-\cos x} \cdot e^{C_1}$$

$$y = \pm e^{C_1} \cdot e^{-\cos x}$$

$$\therefore \boxed{y(x) = C \cdot e^{-\cos x}}$$

Let $C = \pm e^{C_1}$. Then C is any nonzero constant.

Note: $y(x) \equiv 0$ is also a sol'n (i.e. a singular sol'n), which we include with letting $C = 0$.

Implicit Sol'ns

In the examples above, we obtained <u>explicit</u> formulas

$$y(x) = \cdots$$

for the sol'ns of the o.d.e.'s.

Sometimes it is not possible to solve for y, and instead sol'ns are <u>implicitly</u> given by an equation of the form

$$K(x,y) = 0.$$

[recall <u>implicit</u> differentiation in your calculus courses]

Example:

(#12) $\quad yy' = x(y^2+1)$

$$y\frac{dy}{dx} = x(y^2+1)$$

$$\int \frac{y\,dy}{y^2+1} = \int x\,dx$$

Note: $y^2+1 \neq 0$ for all y.

$$\left[\begin{aligned} u &= y^2+1 \\ du &= 2y\,dy \end{aligned}\right]$$

$$\int \frac{1}{2}\frac{du}{u} = \int x\,dx$$

$$\frac{1}{2}\ln|u| = \frac{1}{2}x^2 + C_1$$

$$\frac{1}{2}\ln|y^2+1| = \frac{1}{2}x^2 + C_1$$

Note: $y^2+1 \geq 1$, so $|y^2+1| = y^2+1$.

$$\frac{1}{2}\ln(y^2+1) = \frac{1}{2}x^2 + C_1$$

$$\ln(y^2+1) = x^2 + 2C_1$$

$$\exp(\ln(y^2+1)) = \exp(x^2+2c_1)$$

$$y^2+1 = e^{x^2}\cdot e^{2c_1}$$

$$\boxed{C = e^{2c_1}}$$

$C > 0$

$$\boxed{y^2 = Ce^{x^2}-1}$$

implicit sol'n

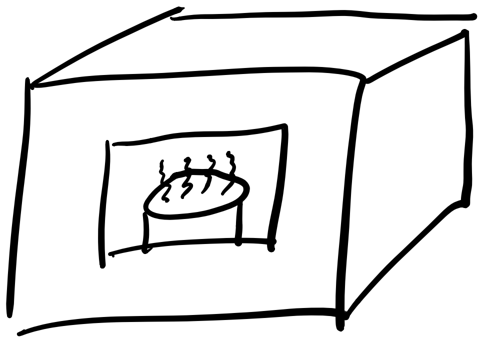
In fact, both

$$y(x) = \sqrt{Ce^{x^2}-1} \quad \text{and} \quad y(x) = -\sqrt{Ce^{x^2}-1}$$

are explicit sol'ns. //

[geogebra.org demo]

Application: Cooling and heating



baking cake in the oven

$T(t)$ = temp. of cake at time t

A = ambient temp. inside the oven

$$\boxed{\frac{dT}{dt} = k(A-T), \quad k>0}$$

Example:

(#49) We remove the cake from the oven and measure its temperature at 210°F. The room temperature is 70°F. After 30 min. the cake is 140°F. When will the cake be 100°F?

Sol'n:

$T(0) = 210 \qquad A = 70$

$T(30) = 140 \qquad T(t) = 100\ ?$

$$\frac{dT}{dt} = k(A-T)$$

$$\frac{dT}{dt} = k(70-T)$$

$$\int \frac{dT}{70-T} = \int k\,dt$$

Note: $70 - T \neq 0$

$$\ln|70-T| = kt + C_1$$

[Note: $T > 70$, so $|70-T| = T-70$.]

$$\ln(T-70) = kt + C_1$$

$$T - 70 = e^{kt} \cdot e^{C_1}$$

$$\boxed{T = 70 + Ce^{kt}}$$

$C = e^{C_1}$

$C > 0$

$$T(0) = 210$$

$$70 + Ce^{k \cdot 0} = 210$$

$$C = 210 - 70$$

$$\boxed{C = 140}$$

$$T(30) = 140$$

$$70 + 140e^{k \cdot 30} = 140$$

$$140e^{30k} = 70$$

$$e^{30k} = \frac{1}{2}$$

$$30k = \ln(2^{-1}) = -\ln 2$$

$$\boxed{k = \frac{-\ln 2}{30}} = -0.0231\ldots$$

$$\therefore T(t) = 70 + 140 \exp\left(\frac{-\ln 2}{30} t\right)$$

$T(t) = 100$?

$$70 + 140 \exp\left(-\frac{\ln 2}{30} t\right) = 100$$

$$\exp\left(-\frac{\ln 2}{30} t\right) = \frac{30}{140}$$

$$-\frac{\ln 2}{30} t = \ln\left(\frac{3}{14}\right)$$

$$t = -\frac{\ln\left(\frac{3}{14}\right)}{\ln 2} \cdot 30$$

$$= 66.67\ldots$$

$\therefore$ After 1 h 6 m 40 s, the cake will be 100°F.