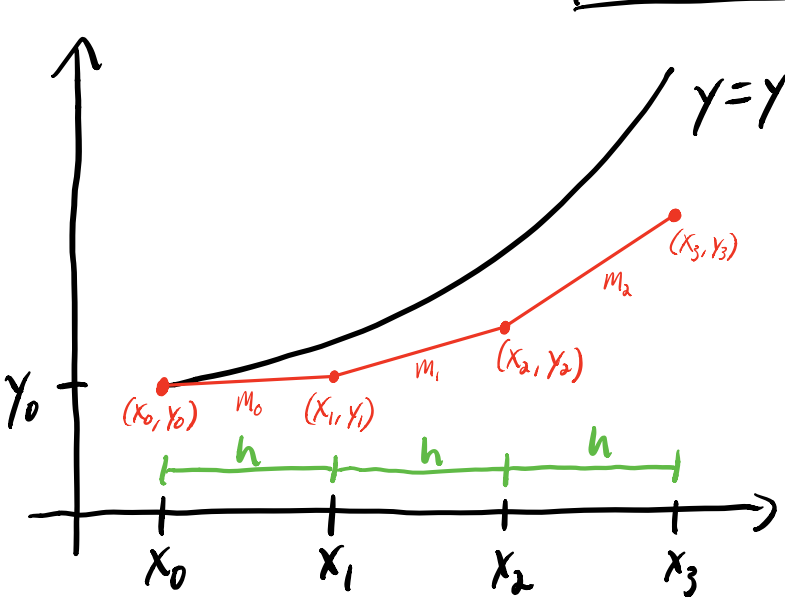
6.1 Numerical Approximation: Euler's Method

Recall:

$$\boxed{\frac{dy}{dx} = f(x,y)}$$ ← Says the slope of the graph $y = y(x)$ at the point (x,y) is $f(x,y)$.

Initial condition: $\boxed{y(x_0) = y_0}$



$\boxed{h = \text{step size}}$

$\boxed{m_i = \text{slope of sol'n at } (x_i, y_i)}$

What is x_1? $\boxed{x_1 = x_0 + h}$

What is y_1?

Slope of line is $m_0 = f(x_0, y_0)$.

Eq'n of line is $y - y_0 = m_0(x - x_0)$.

$\Rightarrow$ $y_1 - y_0 = m_0(x_1 - x_0)$, with $x_1 - x_0 = h$

$$\Rightarrow \quad y_1 = y_0 + h \cdot m_0$$

$$\Rightarrow \quad \boxed{y_1 = y_0 + h \cdot f(x_0, y_0)}$$

Similarly:

$$x_2 = x_1 + h \qquad y_2 = y_1 + h \cdot f(x_1, y_1)$$

$$x_3 = x_2 + h \qquad y_3 = y_2 + h \cdot f(x_2, y_2)$$

That is, for $n = 0, 1, 2, \ldots,$ **Euler's Method** is

$y(x_0) = y_0$

(x_0, y_0)

$$\boxed{\begin{aligned} x_{n+1} &= x_n + h \\ y_{n+1} &= y_n + h \cdot f(x_n, y_n) \end{aligned}}$$

Example: (#4)

$$y' = x - y, \quad y(0) = 1; \quad y(x) = 2e^{-x} + x - 1$$

Use step size $h = 0.1$ to approx. the sol'n $y(x)$ at $x = 0.1, 0.2, 0.3, 0.4, 0.5$ using Euler's method.

Sol'n: Note: $f(x, y) = x - y$

$(x_0, y_0) = (0, 1)$

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$m_0 = f(x_0, y_0) = x_0 - y_0 = 0 - 1 = -1$

$y_1 = y_0 + h \cdot m_0 = 1 + (0.1) \cdot (-1) = 0.9$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$m_1 = f(x_1, y_1) = 0.1 - 0.9 = -0.8$

$y_2 = y_1 + h \cdot m_1 = 0.9 + (0.1) \cdot (-0.8) = 0.82$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$m_2 = f(x_2, y_2) = 0.2 - 0.82 = -0.62$

$y_3 = y_2 + h \cdot m_2 = 0.82 + (0.1) \cdot (-0.62) = 0.758$

$x_4 = 0.4$

$m_3 = f(x_3, y_3) = 0.3 - 0.758 = -0.458$

$y_4 = y_3 + h \cdot m_3 = 0.758 + (0.1) \cdot (-0.458)$

$= 0.7122$

$x_5 = 0.5$; $m_4 = 0.4 - 0.7122 = -0.3122$

$y_5 = 0.7122 + (0.1) \cdot (-0.3122) = 0.68098$

n	x_n	y_n	$y(x_n)$	$y(x_n) - y_n$
0	0	1	1	0
1	0.1	0.9	0.90967	0.00967
2	0.2	0.82	0.83746	0.01746
3	0.3	0.758	0.78164	0.02364
4	0.4	0.7122	0.74064	0.02844
5	0.5	0.68098	0.71306	0.03208

[Julia code demo]