1.1) Diff. Eq'ns and Mathematical Models

Showing that a function $y=f(x)$ satisfies (ie is a <u>solution</u> of) a diff. eq'n is easy.

<u>Examples:</u>

(#5) $y = e^x - e^{-x}$ $\quad [y' = y + 2e^{-x}]$

$$y' = e^x - e^{-x}\cdot(-1)$$
$$= e^x + e^{-x}$$

$$y' - y = (\cancel{e^x} + e^{-x}) - (\cancel{e^x} - e^{-x})$$
$$= 2e^{-x} \quad \Rightarrow \quad y' = y + 2e^{-x}$$

$\therefore$ $\boxed{y = e^x - e^{-x}}$ is <u>a sol'n of</u> the diff. eq'n $\boxed{y' = y + 2e^{-x}}$

(#6) $y_1 = e^{-2x}$ $\quad [y'' + 4y' + 4y = 0]$

$y_2 = xe^{-2x}$

$$y_1' = e^{-2x}\cdot(-2) = -2e^{-2x}$$

$$y_1'' = -2\cdot e^{-2x}\cdot(-2) = 4e^{-2x}$$

$$y_1'' + 4y_1' + 4y_1 = 4e^{-2x} + 4(-2e^{-2x}) + 4(e^{-2x})$$

$$\Rightarrow \quad y_1'' + 4y_1' + 4y_1 = 8e^{-2x} - 8e^{-2x} = 0$$

$\therefore$ $\boxed{y_1 = e^{-2x}}$ is a sol'n of the diff. eq'n $\boxed{y'' + 4y' + 4y = 0}$.

Now, let's look at $y_2 = xe^{-2x}$.

$$y_2' = (1)(e^{-2x}) + (x)(e^{-2x} \cdot (-2))$$
$$= e^{-2x} - 2xe^{-2x}$$

$$y_2'' = e^{-2x} \cdot (-2)$$
$$- 2\left[(1)(e^{-2x}) + (x)(e^{-2x} \cdot (-2))\right]$$
$$= -2e^{-2x} - 2(e^{-2x} - 2xe^{-2x})$$
$$= -4e^{-2x} + 4xe^{-2x}$$

$\therefore$ $y_2'' + 4y_2' + 4y_2$

$$= (\cancel{-4e^{-2x}} + \cancel{4xe^{-2x}})$$
$$+ 4(\cancel{e^{-2x}} - 2\cancel{xe^{-2x}}) + 4(\cancel{xe^{-2x}})$$
$$= 0.$$

$\therefore$ $\boxed{y_2 = xe^{-2x}}$ is also a sol'n of the diff. eq'n $\boxed{y'' + 4y' + 4y = 0.}$

Going the opposite direction (ie the inverse problem) of finding the sol'ns of a diff. eq'n is not as straight forward.

Sometimes we can guess the form of the sol'n.

Example:

(#14) $\boxed{4y'' = y}$ Guess: $y = e^{rx}$.

$$y = e^{rx}$$

$$y' = e^{rx} \cdot (r) = re^{rx}$$

$$y'' = r \cdot e^{rx} \cdot (r) = r^2 e^{rx}$$

$$4y'' = y \Rightarrow 4r^2 \cancel{e^{rx}} = \cancel{e^{rx}}$$

$$\Rightarrow 4r^2 = 1$$

$$\Rightarrow \boxed{r = \pm \frac{1}{2}}.$$

$\therefore$ $\boxed{y_1 = e^{\frac{1}{2}x}}$ and $\boxed{y_2 = e^{-\frac{1}{2}x}}$ are both sol'ns of the diff. eq'n $\boxed{4y'' = y.}$

Note that in the previous example,

$y = 2e^{-x/2}$, $y = \frac{-1}{10}e^{-x/2}$, $y = 100e^{-x/2}$

are all sol'ns of $\boxed{4y'' = y}$.

In fact, any function of the form $y = Ce^{-x/2}$, where C is a constant number, is a sol'n.

The same is true for any function of the form $y = Ce^{x/2}$.

The general form of a sol'n of the diff. eq'n $\boxed{4y'' = y}$ is

$$y = C_1 e^{-x/2} + C_2 e^{x/2}$$

where C_1 and C_2 are any constant numbers.

Such unknown constants can be determined using additional information such as initial position and velocity.

Example:

(#20) Show that $y(x) = Ce^{-x} + x - 1$ satisfies the diff. eq'n

$$\boxed{y' = x - y}$$

and determine the constant C that satisfies the initial condition

$$\boxed{y(0) = 3}$$

$y(x) = Ce^{-x} + x - 1$ $\qquad [y' = x - y]$

$y' = C \cdot e^{-x} \cdot (-1) + 1 - 0$

$= -Ce^{-x} + 1$

$y' + y = (\cancel{-Ce^{-x}} \cancel{+1}) + (\cancel{Ce^{-x}} + x \cancel{-1})$

$= x$

$\therefore$ $y(x) = Ce^{-x} + x - 1$ is a sol'n of the diff. eq'n $y' = x - y$.

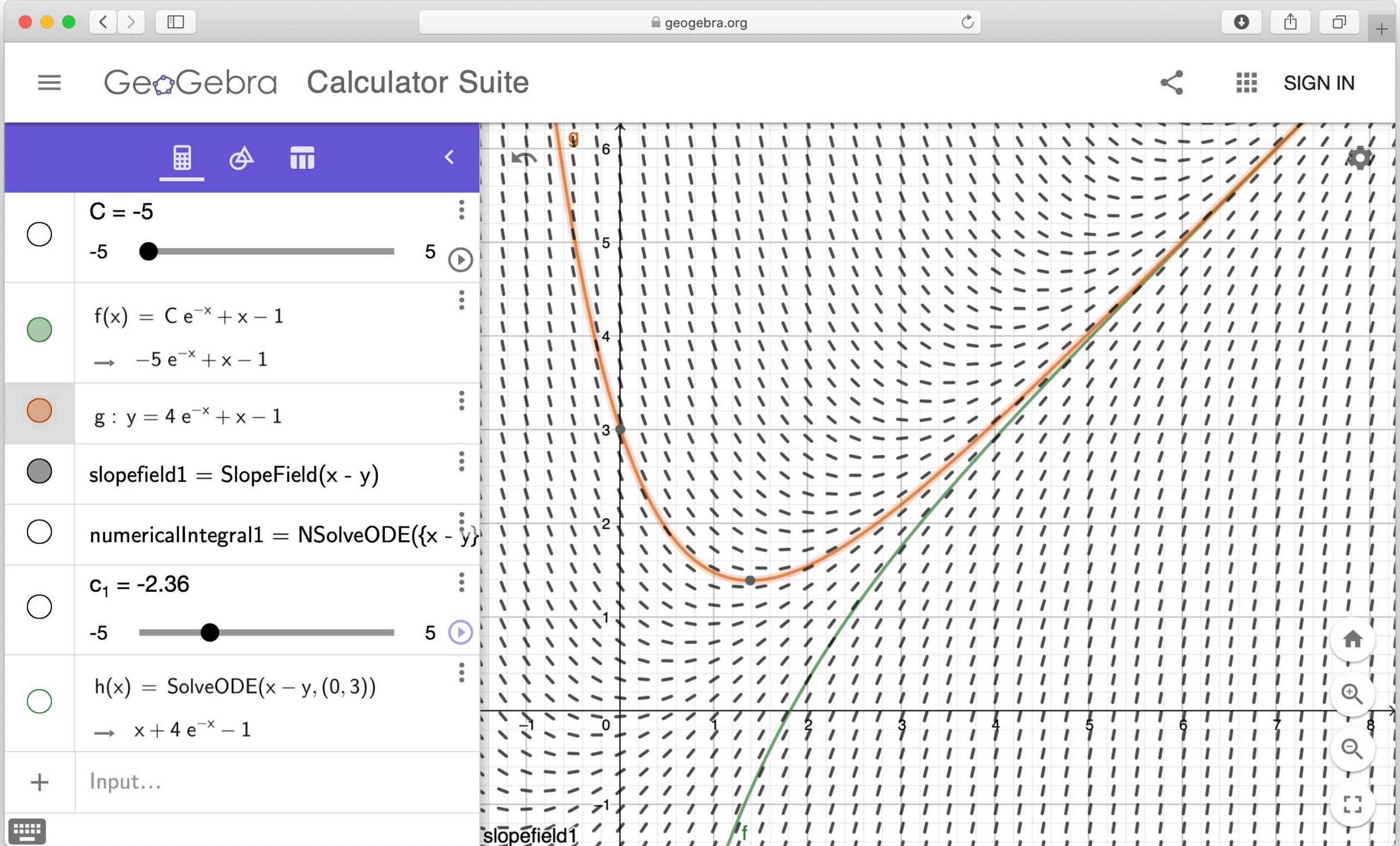
Initial condition: $y(0) = 3$.

$$\Rightarrow Ce^{-0} + 0 - 1 = 3$$
$$\Rightarrow C \cdot 1 - 1 = 3$$
$$\Rightarrow \boxed{C = 4}.$$

$\therefore$ $\boxed{y(x) = 4e^{-x} + x - 1}$ is the sol'n of the diff. eq'n $\boxed{y' = x - y}$ satisfying the initial condition that $\boxed{y(0) = 3.}$

Use <u>geogebra.org</u> to plot several typical sol'ns.

$\downarrow$


geogebra.org
GeoGebra Calculator Suite
SIGN IN
C = -5
-5
5
f(x) = C e^−x + x − 1
→ −5 e^−x + x − 1
g : y = 4 e^−x + x − 1
slopefield1 = SlopeField(x - y)
numericalIntegral1 = NSolveODE({x - y}
c1 = -2.36
-5
5
h(x) = SolveODE(x − y, (0, 3))
→ x + 4 e^−x − 1
Input...
slopefield1
g
f