13.1 Vector Functions

Def 1

A vector function (aka a vector-valued function) is a function whose domain is a set of real numbers and whose range is a set of vectors.

In the 3D case they have the form,

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

where f, g, h are real-valued functions called the component functions of $\vec{r}$.

Ex 1 Find the domain of the vector function $\vec{r}(t) = \langle t^2, \ln(t), \sqrt{4-t} \rangle$.

Sol

The domain of $\vec{r}$ is the set of all t values for which $\vec{r}(t)$ is defined.

We can find this by looking at the domains of the component functions.

$f(t) = t^2$, Dom: $\mathbb{R}$

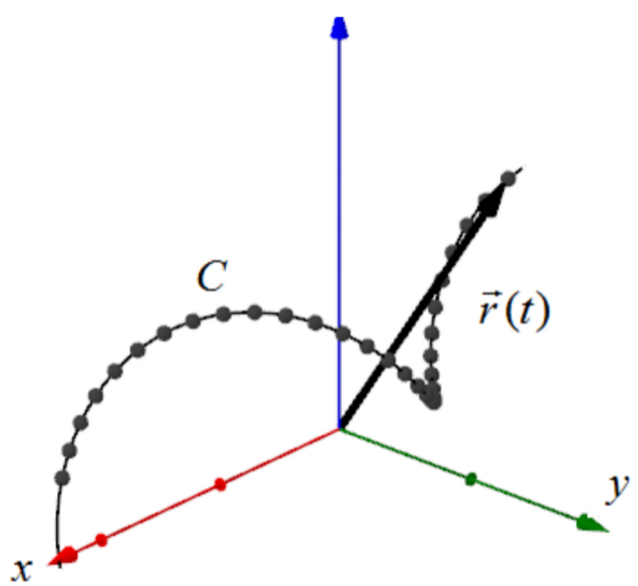
$g(t) = \ln t$, Dom: $t > 0$, $(0, \infty)$

$h(t) = \sqrt{4-t}$, Dom: $4 - t \geq 0 \Rightarrow t \leq 4$ $(-\infty, 4]$

So the domain of $\vec{r}$ is $(0, 4]$.

Def 2 (Space Curve)

Suppose f, g and h are continuous on a domain D. Then the set of all points (x, y, z) where $x = f(t)$, $y = g(t)$ and $z = h(t)$ as t varies throughout D is called a space curve C.



The curve C is traced out by the tip of the vectors from $\vec{r}(t)$.

Def 3 (Limits and Continuity)

The limit of $\vec{r}(t)$ is defined by taking the limit of it's components

$$\lim_{t\to a} \vec{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

Ex 2 Find the limit of

$$\vec{r}(t) = \left\langle \frac{1+t^2}{1-t^2}, te^{-t}, \frac{2-e^{-t}}{t} \right\rangle \quad \text{as } t\to\infty.$$

Sol

Taking the limit of each component,

i) $$\lim_{t\to\infty} \frac{1+t^2}{1-t^2} \overset{L'H}{=} \lim_{t\to\infty} \frac{2t}{-2t} = -1$$

$\uparrow$

Indeterminate Form: $\frac{\infty}{\infty}$

ii) $\lim_{t\to\infty} te^{-t} = \lim_{t\to\infty} \frac{t}{e^t} \overset{L'H}{=} \lim_{t\to\infty} \frac{1}{e^t} = \frac{1}{\infty} = 0$

Indet. Form: $0 \cdot \infty$

iii) $\lim_{t\to\infty} \frac{2-e^{-t}}{t} = \frac{2-0}{\infty} = \frac{2}{\infty} = 0$

not an indet. form

So $\lim_{t\to\infty} \vec{r}(t) = \langle -1, 0, 0 \rangle$

Ex 3 Describe the following space curves.

a) $\vec{r}(t) = \langle 2t-1, t+4, -3t-6 \rangle$

b) $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$

Sol

a) $\vec{r}(t) = \langle 2t-1, t+4, -3t-6 \rangle$

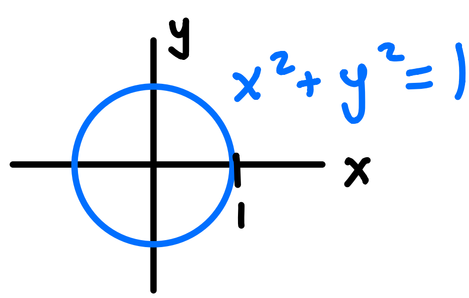
This is the vector equation for a line with direction vector $\vec{v} = \langle 2, 1, -3 \rangle$ and initial point $(-1, 4, -6)$.

b) $\vec{r}(t) = \langle \cos t, \sin t, 2t \rangle$

If we ignore the $2t$ for now we have

$$x = \cos t \qquad y = \sin t$$

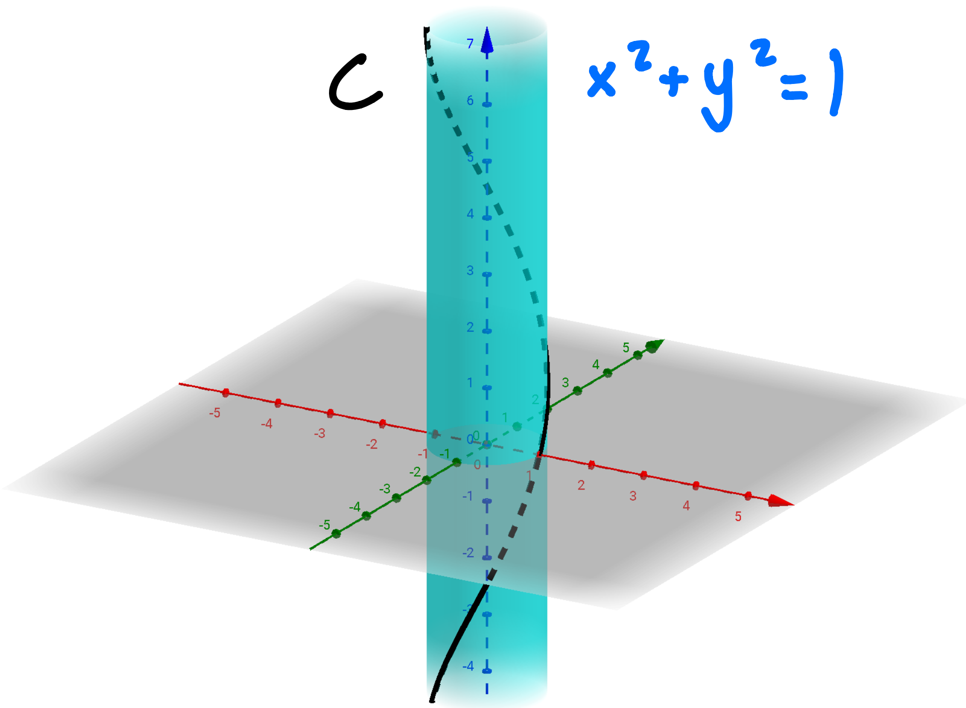
which are the parametric eqns. for a circle in the xy-plane.



In $\mathbb{R}^3$, $x^2+y^2=1$ is a circular cylinder.

The $z=2t$ equation tells us that as t increases so does the z value.

So the space curve C spirals upward around the cylinder as t increases.



Ex 5 Find the vector and parametric equation of a line segment between the points $P(1,-3,3)$ and $Q(5,-2,4)$.

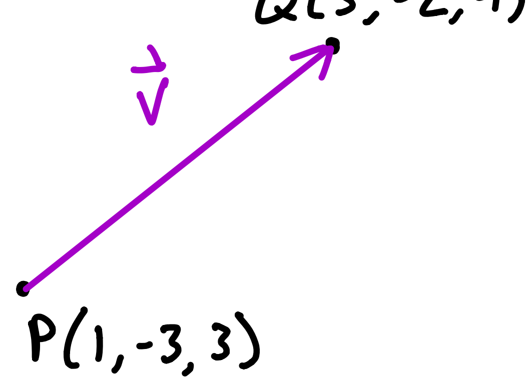
Sol

This won't be very different from what was done in 12.5 to find the eqn. of a line through two points.

We need a point and a direction vector.

Direction vector:

$$\vec{v} = \overrightarrow{PQ} = \langle 5-1, -2-(-3), 4-3 \rangle$$
$$= \langle 4, 1, 1 \rangle$$



Let $\vec{r}_0 = \langle 1, -3, 3 \rangle$.

The vector/parametric equations of the line are

$$\vec{r}(t) = \vec{v}t + \vec{r}_0 = \langle 4,1,1\rangle t + \langle 1,-3,3\rangle$$
$$= \langle 4t+1, t-3, t+3\rangle$$

$x = 4t+1 \qquad y = t-3 \qquad z = t+3$

Notice how $t=0$ corresponds to point $P(1,-3,3)$ and $t=1$ gives $Q(5,-2,4)$.

So to get the line segment we must restrict parameter t to $0 \le t \le 1$,

Vector:

$$\vec{r}(t) = \langle 4,1,1\rangle t + \langle 1,-3,3\rangle \ , \ 0 \le t \le 1$$

Parametric:

$x = 4t + 1 \quad y = t - 3 \quad z = t + 3, \quad 0 \le t \le 1$

Practice Problems

1) Find the domain of vector function

$$\vec{r}(t) = \left\langle \cos t, \ln t, \frac{1}{t-2} \right\rangle$$

2) Find the vector equation of the line segment between $(0,2,1)$ and $(7,-9,2)$.

Solutions

1) $\vec{r}(t) = \langle \cos t, \ln t, \frac{1}{t-2} \rangle$

$f(t) = \cos t$, Dom: $\mathbb{R}$ or $(-\infty, \infty)$

$g(t) = \ln t$, Dom: $t > 0$ or $(0, \infty)$

$h(t) = \frac{1}{t-2}$, Dom: $t \neq 2$ or $(-\infty, 2) \cup (2, \infty)$

So the domain of $\vec{r}$ is

$t > 0, t \neq 2$ or $(0, 2) \cup (2, \infty)$

2) $(0, 2, 1), (7, -9, 2)$

Let $\vec{v} = \langle 7-0, -9-2, 2-1 \rangle = \langle 7, -11, 1 \rangle$

$\vec{r}_0 = \langle 0, 2, 1 \rangle$

The vector equation of the line segment,

$$\vec{r}(t) = \vec{v}t + \vec{r}_0 \quad , 0 \le t \le 1$$

$$\vec{r}(t) = \langle 7, -11, 1 \rangle t + \langle 0, 2, 1 \rangle \quad , 0 \le t \le 1$$

<u>Suggested Textbook Exc.</u> (13.1)

1, 3, 4, 11, 17, 19

<u>Quiz 4</u> (2/15 - 2/17)

12.5, 13.1