2.1 Second-Order Linear Equations

$$\boxed{A(x)y'' + B(x)y' + C(x)y = F(x)}$$

ie linear in the dependent variable y and its derivatives y' and y''.

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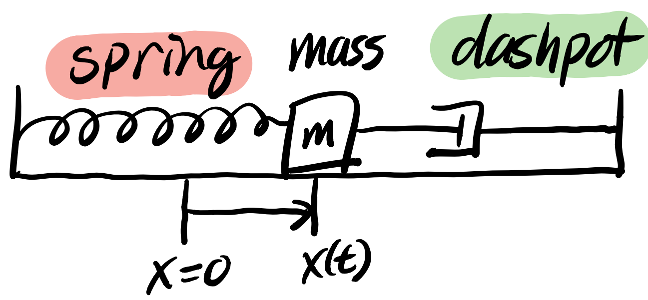
eg



$F_S = -kx$

$F_D = -cx'$

$$F = ma \quad \Rightarrow \quad F_S + F_D = mx''$$

$$\boxed{mx'' + cx' + kx = 0}$$ homogeneous

If there is an external force $F(t)$, then $\boxed{mx'' + cx' + kx = F(t)}$. non-homogeneous

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Homogeneous eq'n:

$$\boxed{A(x)y'' + B(x)y' + C(x)y = 0}$$

If y_1 and y_2 are two sol'ns of the homogeneous eq'n, then any <u>linear combination</u>

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

is also a sol'n.

<u>Example:</u> (#9)

$y'' + 2y' + y = 0$, $y(0) = 2$, $y'(0) = -1$

Show that $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ are both sol'ns. Show that

$$y = c_1 y_1 + c_2 y_2$$

is a sol'n. Then find c_1 and c_2 such that $y = c_1 y_1 + c_2 y_2$ satisfies the initial conditions.

<u>Sol'n:</u> $y'' + 2y' + y = 0$

$$\left.\begin{aligned} y_1 &= e^{-x} \\ y_1' &= -e^{-x} \\ y_1'' &= e^{-x} \end{aligned}\right]$$

$$\begin{aligned} &y_1'' + 2y_1' + y_1 \\ &= e^{-x} + 2(-e^{-x}) + e^{-x} \\ &= 0. \quad \checkmark \end{aligned}$$

$$\left.\begin{aligned} y_2 &= xe^{-x} \\ y_2' &= -xe^{-x} + e^{-x} \\ y_2'' &= xe^{-x} - e^{-x} - e^{-x} \end{aligned}\right]$$

$$\begin{aligned} &y_2'' + 2y_2' + y_2 \\ &(x-2)e^{-x} + 2(1-x)e^{-x} \\ &+ xe^{-x} \\ &= 2xe^{-x} - 2xe^{-x} \\ &= 0 \quad \checkmark \end{aligned}$$

$$\left.\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ y' &= c_1 y_1' + c_2 y_2' \\ y'' &= c_1 y_1'' + c_2 y_2'' \end{aligned}\right]$$

$$\begin{aligned} &y'' + 2y' + y \\ &= (c_1 y_1'' + c_2 y_2'') \\ &\quad + 2(c_1 y_1' + c_2 y_2') \\ &\quad + (c_1 y_1 + c_2 y_2) \end{aligned}$$

$$\Rightarrow \quad y'' + 2y' + y = (c_1 y_1'' + 2c_1 y_1' + c_1 y_1) + (c_2 y_2'' + 2c_2 y_2' + c_2 y_2)$$

$$\Rightarrow \quad y'' + 2y' + y = c_1(y_1'' + 2y_1' + y_1) + c_2(y_2'' + 2y_2' + y_2)$$

$$\Rightarrow \; y'' + 2y' + y = c_1 \cdot 0 + c_2 \cdot 0 = 0 \; \checkmark$$

$$\boxed{y = c_1 e^{-x} + c_2 x e^{-x}}$$

$$y' = -c_1 e^{-x} + c_2\left(-xe^{-x} + e^{-x}\right)$$

$$y(0) = 2 \;\Rightarrow\; c_1 \cdot 1 + c_2 \cdot 0 \cdot 1 = 2$$

$$y'(0) = -1 \;\Rightarrow\; -c_1 \cdot 1 + c_2 \cdot (0 \cdot 1 + 1) = -1$$

$$\therefore \; \boxed{\begin{aligned} c_1 \quad\quad\; &= 2 \\ -c_1 + c_2 &= -1 \end{aligned}} \qquad \begin{aligned} c_1 &= 2 \\ -2 + c_2 &= -1 \\ c_2 &= 1 \end{aligned}$$

$\therefore$ the sol'n of the initial value problem is $\boxed{y = 2e^{-x} + xe^{-x}}$.

Linear Independence of Functions

If $y_2 = Cy_1$ and $y = c_1 y_1 + c_2 y_2$,

then $y = c_1 y_1 + c_2 (Cy_1)$

$$y = (c_1 + c_2 C) y_1$$

$$c_3 = c_1 + c_2 C$$

$$\boxed{y = c_3 y_1}$$

$\therefore$ y_2 is **redundant**, so not needed.

When this happens, we say that y_1 and y_2 are **linearly dependent**:

> there exists c_1 and c_2, not both zero such that $c_1 y_1 + c_2 y_2 = 0$.

We need to have two **linearly independent** functions y_1 and y_2 to **avoid redundancy** and be able to satisfy the initial conditions.

ie $c_1 y_1 + c_2 y_2 = 0 \implies c_1 = c_2 = 0$.

Example:

Determine if the functions are lin. dep. or lin. indep. on the real line.

(#21) $f(x) = x^3$, $g(x) = x^2|x|$

$$c_1 f(x) + c_2 g(x) = 0 \quad \text{for all } x$$

$$c_1 x^3 + c_2 x^2|x| = 0 \quad \text{for all } x$$

$\underline{x=1:}$ $\quad c_1\cdot(1)^3 + c_2\cdot(1)^2|1| = 0$

$$\boxed{c_1 + c_2 = 0}$$

$\underline{x=-1:}$ $\quad c_1\cdot(-1)^3 + c_2\cdot(-1)^2\cdot|-1| = 0$

$$\boxed{-c_1 + c_2 = 0}$$

$$\Rightarrow \begin{array}{l} c_2 = c_1 \\ c_1 + c_1 = 0 \end{array} \Rightarrow \begin{array}{l} c_2 = c_1 \\ 2c_1 = 0 \end{array}$$

$$\Rightarrow \boxed{c_1 = c_2 = 0.}$$

$\therefore$ $f(x)$ and $g(x)$ are <u>lin. indep.</u>

An easier method:

$$\begin{bmatrix} c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \end{bmatrix} \qquad \boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb}$$

The determinant of the matrix of coefficients is

$$\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = 1\cdot 1 - (-1)\cdot 1 = 2 \neq 0$$

$\Rightarrow \boxed{c_1 = c_2 = 0}$. $\therefore$ lin. indep. $f(x)$ and $g(x)$

(#26) $f(x) = 2\cos + 3\sin x, \quad g(x) = 3\cos x - 2\sin x$

$$c_1 f(x) + c_2 g(x) = 0 \quad \text{for all } x$$

$$c_1(2\cos x + 3\sin x) + c_2(3\cos x - 2\sin x) = 0$$

$$(2c_1 + 3c_2)\cos x + (3c_1 - 2c_2)\sin x = 0$$

$\underline{x=0:}$ $(2c_1 + 3c_2)\cdot 1 + (3c_1 - 2c_2)\cdot 0 = 0$

$$\boxed{2c_1 + 3c_2 = 0}$$

$x = \pi/2$: $(2c_1+3c_2)\cdot 0 + (3c_1 - 2c_2)\cdot 1 = 0$

$$\boxed{3c_1 - 2c_2 = 0}$$

$$\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 2\cdot(-2) - 3\cdot 3 = -13 \neq 0$$

$\therefore\ c_1 = c_2 = 0$

$\therefore$ $f(x)$ and $g(x)$ are <u>lin. indep.</u> //

General solutions

If y_1 and y_2 are lin. indep. sol'ns of

$$y'' + p(x)y' + q(x)y = 0 \qquad (1)$$

then

$$\boxed{y = c_1 y_1 + c_2 y_2}$$

is a general solution of (1).

We use the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1$$

to find c_1 and c_2:

$$\begin{aligned} y(a) = b_0 &\implies \\ y'(a) = b_1 &\implies \end{aligned} \boxed{\begin{aligned} c_1 y_1(a) + c_2 y_2(a) &= b_0 \\ c_1 y_1'(a) + c_2 y_2'(a) &= b_1 \end{aligned}}$$

Note that this linear system has a sol'n for c_1 and c_2 if the following determinant is nonzero:

$$\begin{vmatrix} y_1(a) & y_2(a) \\ y_1'(a) & y_2'(a) \end{vmatrix} = y_1(a)y_2'(a) - y_1'(a)y_2(a).$$

$$\left[\text{Recall that } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.\right]$$

<u>Example:</u> (#2) $\boxed{y'' - 9y = 0}$

$y_1 = e^{3x}$, $y_2 = e^{-3x}$; $y(0) = -1$, $y'(0) = 15$

Find c_1 and c_2 such that

$$y = c_1 y_1 + c_2 y_2$$

satisfies the initial conditions.

<u>Sol'n:</u>

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

$$y' = 3c_1 e^{3x} - 3c_2 e^{-3x}$$

$y(0) = -1 \implies c_1 \cdot 1 + c_2 \cdot 1 = -1$

$y'(0) = 15 \implies 3c_1 \cdot 0 - 3c_2 \cdot 0 = 15$

$$\begin{bmatrix} c_1 + c_2 = -1 \\ 3c_1 - 3c_2 = 15 \end{bmatrix} \Rightarrow \begin{array}{r} 3c_1 + 3c_2 = -3 \\ \oplus \; 3c_1 - 3c_2 = 15 \\ \hline 6c_1 + 0 = 12 \end{array}$$

$$\therefore \; c_1 = 2$$

$$2 + c_2 = -1$$

$$c_2 = -3$$

$$\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix} = 1\cdot(-3) - 3\cdot 1 = -6 \neq 0$$

$\therefore$ There is a unique sol'n to the linear system, which is

$$c_1 = 2 \text{ and } c_2 = -3.$$

$$\therefore \; \boxed{y = 2e^{3x} - 3e^{-3x}}$$

Also, $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix}$

$$= e^{3x}(-3e^{-3x}) - (3e^{3x})(e^{-3x})$$

$$= -3e^{3x-3x} - 3e^{3x-3x}$$

$$= -3\cdot 1 - 3\cdot 1 = -6 \neq 0$$

The Wronskian

We call this determinant the **Wronskian** of y_1 and y_2:

$$\boxed{W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2}$$

Wronskian Test of Linear Independence

If y_1 and y_2 are sol'ns of

$$y'' + p(x)y' + q(x)y = 0$$

then:

(a) If $W(y_1, y_2) = 0$ for all x, then y_1 and y_2 are lin. dep.

(b) If $W(y_1, y_2) \neq 0$ for some value of x, then y_1 and y_2 are lin. indep.

Example: (#9)

Recall that $y_1 = e^{-x}$ and $y_2 = xe^{-x}$ are both sol'ns of the diff. eq'n

$$y'' + 2y' + y = 0.$$

Use the Wronskian to show they are lin. indep.

Sol'n:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & (1-x)e^{-x} \end{vmatrix}$$

$$= e^{-x} \cdot (1-x)e^{-x} - (-e^{-x})(xe^{-x})$$

$$= (1-x) \cdot e^{-2x} + x e^{-2x}$$

$$= e^{-2x} \neq 0 \quad \text{for all } x$$

$\therefore$ y_1 and y_2 are lin. indep.

Example: (#9)

Note that $y_1 = e^{-x}$ and $y_2 = 2e^{-x}$ are both sol'ns of the diff. eq'n

$$y'' + 2y' + y = 0.$$

Use the Wronskian to show they are lin. dep.

Sol'n:

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & 2e^{-x} \\ -e^{-x} & -2e^{-x} \end{vmatrix}$$

$$= e^{-x}(-2e^{-x}) - (-e^{-x})(2e^{-x})$$

$$= -2e^{-2x} + 2e^{-2x}$$

$$= 0$$

$\therefore$ y_1 and y_2 are lin. dep.