4.4) Products of Transforms

$$\boxed{\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t)}$$

where $f(t)*g(t)$ is a different way to multiply functions called the **convolution** of $f(t)$ and $g(t)$:

$$\boxed{f(t)*g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau}$$

We can also write $(f*g)(t)$ instead of $f(t)*g(t)$.

Some other properties of convolution:

(1) $f(t)*g(t) = g(t)*f(t)$

(2) $a\cdot\left(f(t)*g(t)\right) = \left(a\cdot f(t)\right)*g(t)$

(3) $f(t)*\left(g(t)+h(t)\right) = f(t)*g(t) + f(t)*h(t)$

(4) $\delta(t)*g(t) = g(t)$ [$\delta(t)$ is the Dirac delta function]

Examples:

(#7) $F(s) = \dfrac{1}{s(s-3)}$

$$\boxed{\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} &= 1 \\ \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} &= e^{3t} \end{aligned}}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{s-3}\right\}$$

$$= 1 * e^{3t}$$

$$= \int_0^t 1 \cdot e^{3(t-\tau)}\, d\tau$$

$$= \int_0^t e^{3\tau} \cdot 1\, d\tau \quad \left(= e^{3t} * 1\right)$$

$$= \frac{1}{3} e^{3\tau} \Big|_0^t$$

$$= \frac{1}{3} e^{3t} - \frac{1}{3} = \boxed{\frac{1}{3}\left(e^{3t} - 1\right)}$$

Check: $\mathcal{L}\left\{\frac{1}{3}\left(e^{3t} - 1\right)\right\} = \frac{1}{3}\left(\frac{1}{s-3} - \frac{1}{s}\right)$

$$= \frac{1}{3}\left(\frac{s - (s-3)}{s(s-3)}\right) = \frac{1}{s(s-3)} \quad \checkmark$$

(#9) $F(s) = \dfrac{1}{(s^2+9)^2}$

$\mathcal{L}^{-1}\left\{\dfrac{3}{s^2+9}\right\} = \sin 3t$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{9}\cdot\frac{3}{s^2+9}\cdot\frac{3}{s^2+9}\right\}$$

$$= \frac{1}{9}(\sin 3t) * (\sin 3t)$$

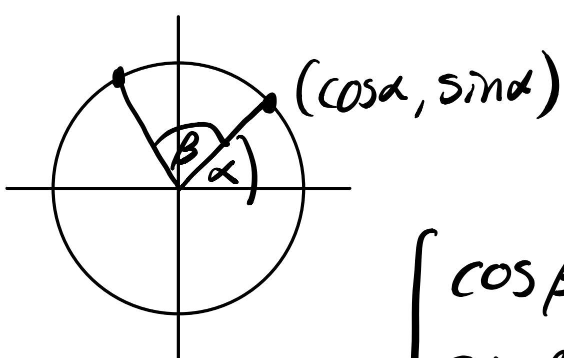
$$= \frac{1}{9}\int_0^t \sin(3\tau)\sin(3(t-\tau))\,d\tau$$

Product-to-sum trig identities:

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$



Rotate by an angle of β:

$$\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}x\cos\beta - y\sin\beta\\ x\sin\beta + y\cos\beta\end{bmatrix}$$

$$\begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix} = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos\alpha\sin\beta + \cos\beta\sin\alpha \end{bmatrix}$$

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ + \quad \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ \hline \cos(\alpha+\beta) + \cos(\alpha-\beta) &= 2\cos\alpha\cos\beta \end{aligned}$$

$$\begin{aligned} \cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ - \quad \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \hline \cos(\alpha-\beta) - \cos(\alpha+\beta) &= 2\sin\alpha\sin\beta \end{aligned}$$

$$\begin{aligned} \sin(\alpha+\beta) &= \cos\alpha\sin\beta + \cos\beta\sin\alpha \\ + \quad \sin(\alpha-\beta) &= -\cos\alpha\sin\beta + \cos\beta\sin\alpha \\ \hline \sin(\alpha+\beta) + \sin(\alpha-\beta) &= 2\cos\beta\sin\alpha. \end{aligned}$$

$$f(t) = \frac{1}{9}\int_0^t \sin(3\tau)\sin(3(t-\tau))\,d\tau$$

$$= \frac{1}{9}\int_0^t \frac{1}{2}\Big(\cos(6\tau - 3t) - \cos(3t)\Big)\,d\tau$$

$$= \frac{1}{18}\left[\frac{1}{6}\sin(6\tau - 3t) - \tau\cos(3t)\right]_0^t$$

$$= \frac{1}{18}\Big[\frac{1}{6}\sin(6t - 3t) - t\cos(3t)$$
$$- \frac{1}{6}\sin(0 - 3t) + 0\Big]$$

$$= \frac{1}{18}\left[\frac{2}{6}\sin(3t) - t\cos(3t)\right]$$

$$= \frac{1}{54}\Big(\sin(3t) - 3t\cos(3t)\Big)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \frac{1}{2k^3}\left(\sin kt - kt\cos kt\right)}$$

(#10) $F(s) = \dfrac{1}{s^2(s^2+k^2)}$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{k}\cdot\frac{1}{s^2}\cdot\frac{k}{s^2+k^2}\right\}$$

$$= \frac{1}{k}\, t * \sin kt$$

$$= \frac{1}{k}(\sin kt) * t$$

$$= \frac{1}{k}\int_0^t \sin k\tau\,(t-\tau)\,d\tau$$

$$= \frac{1}{k}\left(\int_0^t t\sin k\tau\,d\tau - \int_0^t \tau\sin k\tau\,d\tau\right)$$

$$= \frac{1}{k}\left(\left[-\frac{t}{k}\cos k\tau\right]_0^t - \int_0^t \tau\sin k\tau\,d\tau\right)$$

$$= \frac{1}{k}\left(-\frac{t}{k}\cos kt + \frac{t}{k} - \int_0^t \tau\sin k\tau\,d\tau\right)$$

$$\int_0^t \tau\sin k\tau\,d\tau = \left[-\frac{\tau}{k}\cos k\tau\right]_0^t - \int_0^t -\frac{1}{k}\cos k\tau\,d\tau$$

$$\int_0^t \tau \sin k\tau \, d\tau = \left(\frac{-t}{k}\cos kt\right) - \left[\frac{-1}{k^2}\sin k\tau\right]_0^t$$

$$= \frac{-t}{k}\cos kt - \left(\frac{-1}{k^2}\sin kt\right)$$

$$= \frac{1}{k^2}\sin kt - \frac{t}{k}\cos kt$$

$$f(t) = \frac{1}{k}\left(\frac{-t}{k}\cos kt + \frac{t}{k} - \int_0^t \tau \sin k\tau \, d\tau\right)$$

$$= \frac{1}{k}\left(\frac{-t}{k}\cos kt + \frac{t}{k} - \frac{1}{k^2}\sin kt + \frac{t}{k}\cos kt\right)$$

$$= \frac{1}{k}\left(\frac{t}{k} - \frac{1}{k^2}\sin kt\right)$$

$$= \frac{1}{k^3}\left(kt - \sin kt\right)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+k^2)}\right\} = \frac{1}{k^3}\left(kt - \sin kt\right)$$

Check:

$$\mathcal{L}\left\{\frac{1}{k^3}(kt - \sin kt)\right\} = \frac{1}{k^3}\left(\frac{k}{s^2} - \frac{k}{s^2+k^2}\right)$$

$$= \frac{1}{k^2}\left(\frac{1}{s^2} - \frac{1}{s^2+k^2}\right) = \frac{1}{k^2}\left(\frac{s^2+k^2-s^2}{s^2(s^2+k^2)}\right)$$

$$= \frac{1}{s^2(s^2+k^2)} \checkmark$$

$$\left[\begin{array}{c}\dfrac{1}{s^2(s^2+k^2)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+k^2} \\ \left(A=0,\ B=\dfrac{1}{k^2},\ C=0,\ D=\dfrac{-1}{k^2}\right)\end{array}\right]$$

(#13) $F(s) = \dfrac{s}{(s-3)(s^2+1)}$

$$\boxed{\begin{aligned} \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} &= e^{3t} \\ \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} &= \cos t \end{aligned}}$$

$$f(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s-3} \cdot \frac{s}{s^2+1} \right\}$$

$$= e^{3t} * \cos t$$

$$= \int_0^t \cos\tau \, e^{3(t-\tau)} \, d\tau = e^{3t} \int_0^t \cos\tau \, e^{-3\tau} \, d\tau$$

$$\int \cos\tau \, e^{-3\tau} d\tau = -\frac{1}{3}\cos\tau \, e^{-3\tau} - \int \frac{1}{3} \sin\tau \, e^{-3\tau} d\tau$$

$$= -\frac{1}{3}\cos\tau \, e^{-3\tau} - \frac{1}{3}\left[-\frac{1}{3}\sin\tau \, e^{-3\tau} - \int -\frac{1}{3}\cos\tau \, e^{-3\tau} d\tau \right]$$

$$= \left(-\frac{1}{3}\cos\tau + \frac{1}{9}\sin\tau\right) e^{-3\tau} - \frac{1}{9}\int \cos\tau \, e^{-3\tau} d\tau$$

$$\therefore \left(1 + \frac{1}{9}\right) \int \cos\tau \, e^{-3\tau} d\tau = \left(-\frac{1}{3}\cos\tau + \frac{1}{9}\sin\tau\right) e^{-3\tau}$$

$$\frac{10}{9}\int \cos\tau\, e^{-3\tau}\,d\tau = \left(-\frac{1}{3}\cos\tau + \frac{1}{9}\sin\tau\right)e^{-3\tau}$$

$$\int \cos\tau\, e^{-3\tau}\,d\tau = \frac{9}{10}\left(-\frac{1}{3}\cos\tau + \frac{1}{9}\sin\tau\right)e^{-3\tau}$$

$$= \frac{1}{10}\left(-3\cos\tau + \sin\tau\right)e^{-3\tau}$$

$$f(t) = e^{3t}\int_0^t \cos\tau\, e^{-3\tau}\,d\tau$$

$$= e^{3t}\left[\frac{1}{10}\left(-3\cos\tau + \sin\tau\right)e^{-3\tau}\right]_0^t$$

$$= e^{3t}\left(\frac{1}{10}\left(-3\cos t + \sin t\right)e^{-3t} - \frac{1}{10}\left(-3+0\right)\cdot 1\right)$$

$$= \frac{1}{10}\left(-3\cos t + \sin t + 3e^{3t}\right)$$

$$\therefore\ \mathcal{L}^{-1}\left\{\frac{s}{(s-3)(s^2+1)}\right\} = \frac{1}{10}\left(-3\cos t + \sin t + 3e^{3t}\right)$$

Check:

$$\mathcal{L}\left\{\frac{1}{10}\left(-3\cos t+\sin t+3e^{3t}\right)\right\}$$

$$=\frac{1}{10}\left(-3\cdot\frac{s}{s^2+1}+\frac{1}{s^2+1}+3\cdot\frac{1}{s-3}\right)$$

$$=\frac{1}{10}\left(\frac{-3s+1}{s^2+1}+\frac{3}{s-3}\right)$$

$$=\frac{1}{10}\left(\frac{(-3s+1)(s-3)+3(s^2+1)}{(s^2+1)(s-3)}\right)$$

$$=\frac{1}{10}\left(\frac{-3s^2+9s+s-3+3s^2+3}{(s-3)(s^2+1)}\right)$$

$$=\frac{s}{(s-3)(s^2+1)} \quad \checkmark$$

$$\left[\begin{array}{c}\dfrac{s}{(s-3)(s^2+1)}=\dfrac{A}{s-3}+\dfrac{Bs+C}{s^2+1}\\[2ex] A=\dfrac{3}{10},\quad B=\dfrac{-3}{10},\quad C=\dfrac{1}{10}\end{array}\right]$$