1.5 | Linear First-Order Eq'ns

$$\boxed{\frac{dy}{dx} + P(x)\,y = Q(x)}$$

← linear in y and the derivatives of y

↑ first-order

$(fg)' = fg' + f'g$

Idea: Mult. eq'n by a function $\rho(x)$.

[credit: Michael Penn on his YouTube channel]

$$\underbrace{\rho(x)\frac{dy}{dx} + \rho(x)P(x)\,y}_{\text{almost a product rule}} = \rho(x)\,Q(x)$$

ρ: "rho"

We want $\boxed{\rho'(x) = \rho(x)\,P(x)}$.

a separable o.d.e.!

$$\frac{d\rho}{dx} = \rho \cdot P(x) \implies \int \frac{d\rho}{\rho} = \int P(x)\,dx$$

$$\implies \ln|\rho| = \int P(x)\,dx + C_1$$

$$\Rightarrow \quad |\rho| = \exp\left(\int P(x)\,dx\right) \cdot e^{C_1}$$

$$\Rightarrow \quad \rho(x) = \pm e^{C_1} \cdot \exp\left(\int P(x)\,dx\right)$$

$$\Rightarrow \quad \rho(x) = C \exp\left(\int P(x)\,dx\right)$$

<u>Note</u>: Any constant C will work, so choose $C = 1$.

$$\therefore \quad \boxed{\rho(x) = \exp\left(\int P(x)\,dx\right)}$$

Then we have $\rho'(x) = \rho(x)\,P(x)$, so

$$\rho(x)\frac{dy}{dx} + \rho(x)P(x)\,y = \rho(x)\,Q(x)$$

becomes

$$\underbrace{\rho(x)\frac{dy}{dx} + \rho'(x)\,y}_{\text{a product rule!}} = \rho(x)\,Q(x)$$

Then

$$\frac{d}{dx}\left[\rho(x)\, y\right] = \rho(x) Q(x)$$

$$\Longrightarrow \quad \rho(x)\, y = \int \rho(x) Q(x)\, dx + C$$

$$\Longrightarrow \quad \boxed{y(x) = \frac{1}{\rho(x)}\left(\int \rho(x) Q(x)\, dx + C\right)}$$

Example: (#6)

$$x y' + 5y = 7x^2, \quad y(2) = 5$$

$$y' + \left(\frac{5}{x}\right) y = 7x \qquad \left[\begin{array}{l} P(x) = 5/x \\ Q(x) = 7x \end{array}\right]$$

$$\rho(x) = \exp\left(\int P(x)\, dx\right)$$

$$\rho(x) = \exp\left(\int \frac{5}{x}\, dx\right)$$

$$\rho(x) = \exp\left(5 \ln|x|\right)$$

[Note: $x_0 = 2 > 0$, so we will take $x > 0$.]

$$\rho(x) = \exp(5\ln x)$$

$$\rho(x) = \exp(\ln x^5)$$

$$\boxed{\rho(x) = x^5}$$

$$x^5\left(y' + \frac{5}{x}y\right) = x^5(7x)$$

$$x^5 y' + 5x^4 y = 7x^6$$

$$\frac{d}{dx}\left[x^5 y\right] = 7x^6$$

$$x^5 y = \int 7x^6\,dx + C$$

$$x^5 y = x^7 + C$$

$$y = \frac{x^7 + C}{x^5}$$

$$\boxed{y(x) = x^2 + \frac{C}{x^5}}$$

$$y(2)=5 \implies (2)^2+\frac{C}{2^5}=5$$

$$\implies 4+\frac{C}{32}=5$$

$$\implies \boxed{C=32}$$

$$\therefore \boxed{y(x)=x^2+\frac{32}{x^5}}$$

[geogebra.org demo]

Example: (#18)

$$y' + P(x)y = Q(x)$$

$$xy' = 2y + x^3\cos x$$

$$y' = \frac{2}{x}y + x^2\cos x$$

$$y' + \left(\frac{-2}{x}\right)y = x^2\cos x$$

$$\left[\begin{array}{l} P(x) = -2/x \\ Q(x) = x^2\cos x \end{array}\right]$$

$$\rho(x) = \exp\left(\int P(x)\,dx\right)$$

$$\rho(x) = \exp\left(\int \frac{-2}{x}\,dx\right)$$

$$\rho(x) = \exp\left(-2\ln|x|\right)$$

$$\rho(x) = \exp\left(-\ln|x|^2\right)$$

Note:
$$|x|^2 = x^2$$

$$\rho(x) = \exp\left(-\ln x^2\right)$$

$$\rho(x) = \exp\left(\ln (x^2)^{-1}\right)$$

$$\rho(x) = \exp\left(\ln x^{-2}\right)$$

$$\boxed{\rho(x) = x^{-2}}$$

$$y' + \left(\frac{-2}{x}\right) y = x^2 \cos x$$

$$x^{-2}\left(y' - \frac{2}{x} y\right) = x^{-2}(x^2 \cos x)$$

$$x^{-2} y' - 2x^{-3} y = \cos x$$

$$\frac{d}{dx}\left[x^{-2} y\right] = \cos x$$

$$x^{-2} y = \int \cos x \, dx + C$$

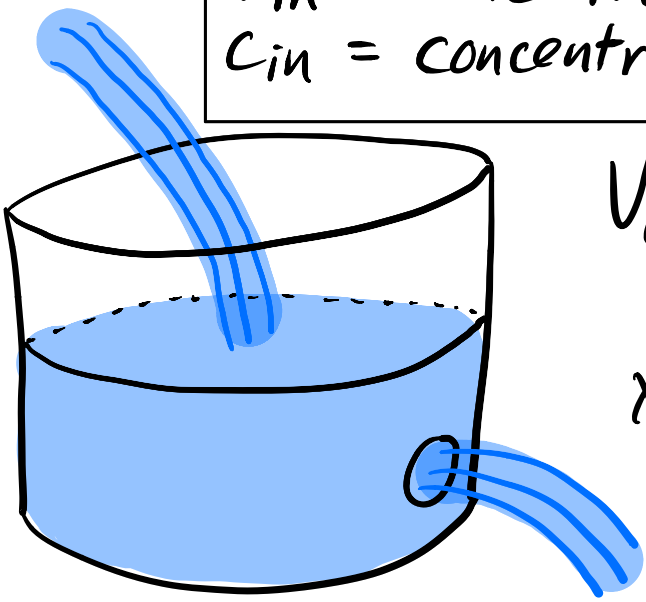
$$x^{-2} y = \sin x + C$$

$$\boxed{y(x) = x^2(\sin x + C)}$$

[geogebra.org demo]

Mixture Problems

r_{in} = rate in (L/min)
c_{in} = concentration in (g/L)



V_0 = initial volume of mixture in tank (L)

x_0 = initial amount of solute in tank (g)

r_{out} = rate out (L/min)
c_{out} = concentration out (g/L)

$V(t)$ = volume of mixture at time t (L)

$x(t)$ = amount of solute at time t (g)

How to formulate the diff. eq'n:

Δt = very small change in time

ΔV = change in volume after Δt min.

Δx = change in amount after Δt min.

Go from time t to time $t + \Delta t$.

① After Δt min:

$r_{in} \cdot \Delta t$ (L) added to volume

$r_{out} \cdot \Delta t$ (L) removed from volume

$\therefore \; \Delta V = r_{in} \cdot \Delta t - r_{out} \cdot \Delta t$

$\therefore \; \dfrac{\Delta V}{\Delta t} = r_{in} - r_{out}$

Letting $\Delta t \to 0$, we get

$$\boxed{\frac{dV}{dt} = r_{in} - r_{out}}$$

The initial condition is $\boxed{V(0) = V_0}$.

This diff. eq'n is separable:

$$\int dV = \int (r_{in} - r_{out})\,dt$$

$$V = (r_{in} - r_{out})t + C$$

$V(0) = V_0 \implies C = V_0$.

$$\therefore \boxed{V(t) = (r_{in} - r_{out})t + V_0}$$

② After Δt min:

$r_{in} \cdot C_{in} \cdot \Delta t$ (g) of solute added

$r_{out} \cdot C_{out} \cdot \Delta t$ (g) of solute removed

$$\therefore \Delta x \approx r_{in} \cdot C_{in} \cdot \Delta t - r_{out} \cdot C_{out} \cdot \Delta t$$

$$\therefore \frac{\Delta x}{\Delta t} \approx r_{in} \cdot C_{in} - r_{out} \cdot C_{out}$$

But C_{out} depends on time: $C_{out}(t) = \dfrac{X(t)}{V(t)}$.

$$\therefore \frac{\Delta x}{\Delta t} \approx r_{in} \cdot C_{in} - r_{out} \cdot \frac{x(t)}{V(t)}$$

Letting $\Delta t \to 0$, we get

$$\boxed{\frac{dx}{dt} = r_{in} \cdot C_{in} - \frac{r_{out}}{V(t)} x}$$

The initial condition is $\boxed{x(0) = X_0}$.

This diff. eq'n is linear.

Example: (#36)

A tank initially contains 60 gal of pure water. Brine containing 1 lb of salt per gallon enters the tank at 2 gal/min, and the (perfectly mixed) solution leaves the tank at 3 gal/min.

(a) Find the amount of salt in the tank after t minutes.

(b) What is the maximum amount of salt ever in the tank?

Sol'n: $V_0 = 60$ (gal), $x_0 = 0$ (lb)

$c_{in} = 1$ (lb/gal)

$r_{in} = 2$ (gal/min)

$r_{out} = 3$ (gal/min)

$$V(t) = (r_{in} - r_{out})t + V_0$$

$$V(t) = (2-3)t + 60$$

$$\boxed{V(t) = 60 - t}$$

Note: tank will be empty after 60 minutes.

$$\frac{dx}{dt} = r_{in} \cdot C_{in} - r_{out} \cdot C_{out}(t)$$

$$C_{out}(t) = \frac{x(t)}{V(t)}$$

$$\frac{dx}{dt} = 2 \cdot 1 - 3 \cdot \frac{x(t)}{V(t)}$$

$$\boxed{\frac{dx}{dt} = 2 - \frac{3}{60-t}x, \quad x(0) = 0}$$

$$\frac{dx}{dt} + \frac{3}{60-t}x = 2 \qquad \left[\begin{array}{l} P(t) = \frac{3}{60-t} \\ Q(t) = 2 \end{array}\right]$$

$$\rho(t) = \exp\left(\int P(t)\,dt\right)$$

$$\rho(t) = \exp\left(\int \frac{3}{60-t}\,dt\right) \qquad [u = 60-t]$$

$$\rho(t) = \exp\left(-3 \ln|60-t|\right)$$

<u>Note:</u> $0 \le t < 60 \Rightarrow |60-t| = 60-t$

$$\rho(t) = \exp\left(-3 \ln(60-t)\right)$$

$$\boxed{\rho(t) = (60-t)^{-3}}$$

$$\frac{dx}{dt} + \frac{3}{60-t}x = 2$$

$$(60-t)^{-3}\left(\frac{dx}{dt} + \frac{3}{60-t}x\right) = (60-t)^{-3} \cdot 2$$

$$(60-t)^{-3}\frac{dx}{dt} + 3(60-t)^{-4}x = \frac{2}{(60-t)^3}$$

$$\frac{d}{dt}\left[(60-t)^{-3}x\right] = \frac{2}{(60-t)^3}$$

$$(60-t)^{-3}x = \int \frac{2}{(60-t)^3}\,dt + C$$

$$\left[\begin{aligned} u &= 60-t \\ du &= -dt \end{aligned}\right]$$

$$(60-t)^{-3}x = \int -2u^{-3}\,du + C$$

$$(60-t)^{-3}x = u^{-2} + C$$

$$(60-t)^{-3}x = (60-t)^{-2} + C$$

$$x = (60-t)^3\left[(60-t)^{-2} + C\right]$$

$$\boxed{x(t) = 60-t + C(60-t)^3}$$

$$x(0) = 0 \implies 60-0 + C(60-0)^3 = 0$$

$$\implies C = \frac{-60}{60^3} = \frac{-1}{3600}$$

(a)

$$\therefore \boxed{x(t) = 60 - t - \frac{1}{3600}(60-t)^3}$$

(b) $$x'(t) = -1 - \frac{1}{3600}\cdot 3(60-t)^2\cdot(-1) = 0$$

$$\frac{1}{1200}(60-t)^2 = 1$$

$$(60-t)^2 = 1200$$

$$60 - t = \pm\sqrt{1200}$$

$$t = 60 \mp \sqrt{1200}$$

Note: $0 < t < 60 \Rightarrow t = 60 - \sqrt{1200}$

$$\boxed{t = 60 - 20\sqrt{3}}$$

$$t = 25.35898\ldots$$

$$x(t) = 60 - t - \frac{1}{3600}(60-t)^3$$

$$x(60-20\sqrt{3}) = 60 - (60-20\sqrt{3}) - \frac{1}{3600}(60-(60-20\sqrt{3}))^3$$

$$x(60-20\sqrt{3}) = 20\sqrt{3} - \frac{1}{3600}(20\sqrt{3})^3$$

$$= 20\sqrt{3} - \frac{1}{3600}8000\cdot 3\sqrt{3}$$

$$= 20\sqrt{3} - \frac{20}{3}\sqrt{3}$$

$$= 20\sqrt{3}\left(1-\frac{1}{3}\right)$$

$$= \boxed{\frac{40\sqrt{3}}{3}} = 23.09\ldots$$

$\therefore$ The maximum salt ever in the tank is about <u>23.09 lb</u> and happens after about <u>25.36 min.</u>