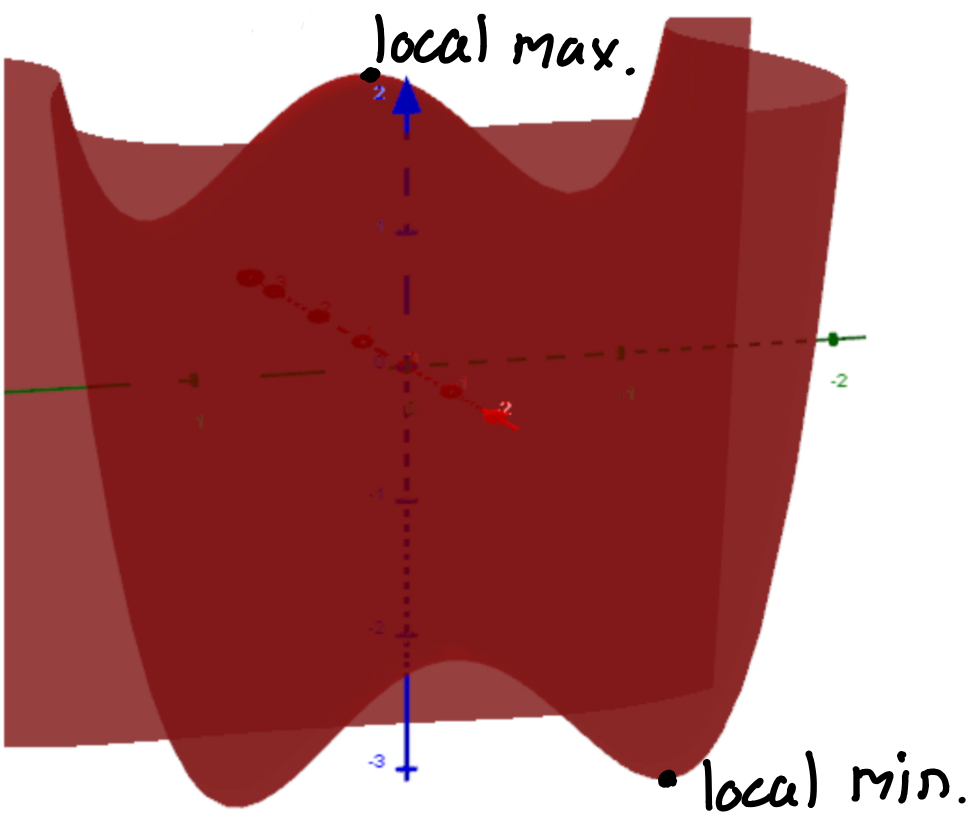
14.7 Maximum and Minimum Values

Def 1 Local Extrema

i) $f(x,y)$ has a local maximum at (a,b)
if $f(x,y) \leq f(a,b)$ for (x,y) near (a,b).

ii) $f(x,y)$ has a local minimum at (a,b)
if $f(x,y) \geq f(a,b)$ for (x,y) near (a,b).



Thm 1

If (a,b) is a local max/min of f, then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

So all the potential min/max values occur when the partial derivatives are all zero. (i.e. when $\nabla f = 0$)

Just like in Calc 1 the points (a,b) with $f_x(a,b) = 0$ and $f_y(a,b) = 0$ are called <u>critical points</u>.

<u>Note:</u>

Not all critical points are local extrema.

Ex 1 Let $f(x,y) = x^2 + y^2 - 6x + 8y + 31$.

Find all the critical points.

Sol

$$f_x = 2x - 6 = 0 \qquad f_y = 2y + 8 = 0$$

$$\left.\begin{array}{l} 2x - 6 = 0 \Rightarrow x = 3 \\ 2y + 8 = 0 \Rightarrow y = -4 \end{array}\right\} \Rightarrow (3, -4)$$

Since there are no contradictions with these solns the only crit. point is at $(3, -4)$.

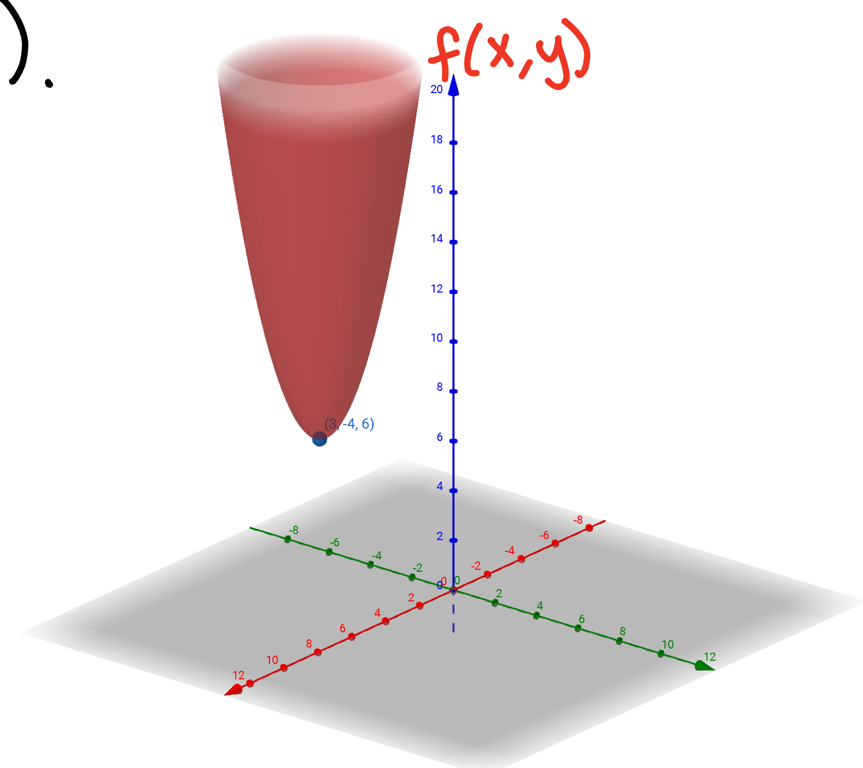
How can we tell if this is a max, min, or neither? By completing the square twice we can write,

$$f(x,y) = x^2 + y^2 - 6x + 8y + 31$$

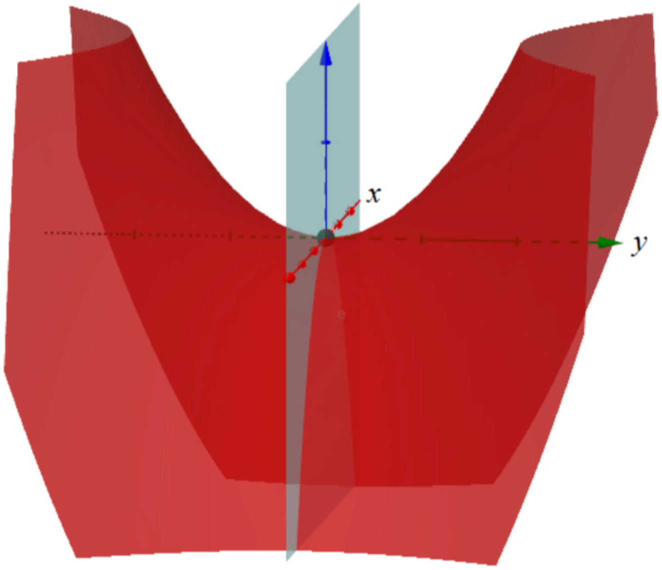
$$= x^2 - 6x + 9 + y^2 + 8y + 16 + 6$$

$$z = (x-3)^2 + (y+4)^2 + 6$$

Notice $f(3,-4) = 6$ and since $(x-3)^2 \geq 0$, $(y+4)^2 \geq 0$ we have $f(x,y) \geq f(3,-4)$ for all (x,y). So f has an absolute minimum at $(3,-4)$.



Def 2 Saddle Point



Consider (a,b) at the origin. If you move along the y-axis the slope is positive.

Along the x-axis the slope is negative.
This critical point is not a local max/min.
We call the point (a,b) a saddle point.

To determine if a crit. point (a,b) is a saddle point or a local max/min we can use the second derivative test.

Def 3 Second Derivative Test

Assume all the second order partial derivatives of f are continuous on a disk with center (a,b). Suppose (a,b) is a critical point (i.e. $f_x(a,b) = f_y(a,b) = 0$) and let

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

i) If $D > 0$ and $f_{xx}(a,b) > 0$, then (a,b) is a local minimum.

ii) If $D > 0$ and $f_{xx}(a,b) < 0$, then (a,b) is a local maximum.

iii) If $D<0$, then (a,b) is a saddle point.

iv) If $D=0$ the test is inconclusive.

Ex 2 Let $f(x,y) = x^4 + y^4 - 4xy + 3$.

Find all of the local extema and saddle points.

Sol

$f_x = 4x^3 - 4y = 0 \qquad f_y = 4y^3 - 4x = 0$

$x^3 - y = 0 \Rightarrow y = x^3$

$y^3 - x = 0$

substitute

$(x^3)^3 - x = 0$

$x^9 - x = 0$

$x(x^8 - 1) = 0$

$x(x^4-1)(x^4+1) = 0$

$x(x^2-1)\underbrace{(x^2+1)(x^4+1)}_{>0} = 0$

$\Rightarrow x = 0$ or $x = \cancel{\pm\sqrt{2}}\ \pm 1$

Using substitution $y = x^3$,

mistake in video

If $x = 0$, then $y = 0^3 = 0$
If $x = -1$, then $y = -1$
If $x = 1$, then $y = 1$

So crit. points are $(0,0)$, $(-1,-1)$, $(1,1)$

2^{nd} Deriv. Test

$f_x = 4x^3 - 4y$ $\quad\quad$ $f_y = 4y^3 - 4x$

$f_{xx} = 12x^2$ $\quad$ $f_{xy} = -4$ $\quad$ $f_{yy} = 12y^2$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 144x^2y^2 - 16$$

$\underline{(0,0)}$:

$D(0,0) = -16 < 0 \Rightarrow (0,0)$ is a saddle pt.

$\underline{(-1,-1)}$:

$D(-1,-1) = 128 > 0$, $f_{xx}(-1,-1) = 12(-1)^2 = 12 > 0$

So $(-1,-1)$ is a local min.

$(1,1)$:

$D(1,1) = 128 > 0$, $f_{xx}(1,1) = 12 > 0$

So $(1,1)$ is a local min.

Ex 3 Find and clasify the critical points

for $f(x,y) = \frac{1}{2}x^4 - 4xy^2 - 2x^2 + 8y^2$.

Sol

$f_x = 2x^3 - 4y^2 - 4x \qquad f_y = -8xy + 16y$

$f_{xx} = 6x^2 - 4 \qquad f_{xy} = -8y \qquad f_{yy} = -8x + 16$

① $2x^3 - 4y^2 - 4x = 0$

② $-xy + 2y = 0 \Rightarrow y(2-x) = 0$

$\Rightarrow y = 0$ or $x = 2$

If $y=0$, then ① becomes $2x^3-4x=0$

$$2x(x^2-2)=0$$
$$x=0, \pm\sqrt{2}$$

If $x=2$, then ① becomes $-4y^2+8=0$

$$y^2=2$$
$$\Rightarrow y=\pm\sqrt{2}$$

So crit. points are $(0,0)$, $(\pm\sqrt{2},0)$, $(2,\pm\sqrt{2})$

$f_{xx}=6x^2-4$ $\quad f_{xy}=-8y \quad f_{yy}=-8x+16$

$\quad 2(3x^2-2) \qquad\qquad\qquad\qquad -8(x-2)$

$$D=-16(3x^2-2)(x-2)-(-8y)^2$$
$$=-16(3x^2-2)(x-2)-64y^2$$

$D(0,0)=-16(-2)(-2)=-64<0 \Rightarrow$ saddle pt.

$D(\sqrt{2},0)=-16(4)(\sqrt{2}-2)>0$, $f_{xx}(\sqrt{2},0)=8>0$

So $(\sqrt{2},0)$ is local min.

$D(-\sqrt{2},0) = -16(4)(-\sqrt{2}-2) > 0$, $f_{xx}(-\sqrt{2},0) = 8 > 0$

So $(-\sqrt{2},0)$ is local min.

$D(2,\pm\sqrt{2}) = -16(10)(0) - 64(2) = -128 < 0$

So $(2,\pm\sqrt{2})$ are both saddle points.

Saddle: $(0,0)$, $(2,\pm\sqrt{2})$

Local Min: $(\pm\sqrt{2},0)$

Practice Problems

1) Find and clasify all the critical points for the following functions

i) $f(x,y) = (y-2)x^2 - y^2$

ii) $g(x,y) = (3x+4x^3)(y^2+2y)$

Solutions

i) $f(x,y) = (y-2)x^2 - y^2$

Let's find all the partial derivatives,

$$f_x = 2x(y-2) \qquad f_y = x^2 - 2y$$

$$f_{xx} = 2(y-2) \qquad f_{xy} = 2x \qquad f_{yy} = -2$$

Critical points:

$f_x = 2x(y-2) = 0 \Rightarrow x = 0$ or $y = 2$

If $x=0$, $f_y = -2y = 0 \Rightarrow y = 0$

If $y=2$, $f_y = x^2 - 4 = 0 \Rightarrow x = \pm 2$

So the critical points are $(0,0), (2,2), (-2,2)$

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = -4(y-2) - (2x)^2 = -4x^2 - 4y + 8$$

$D(0,0) = 8 > 0$, $f_{xx}(0,0) = -4 < 0$

So there is a local max at $(0,0)$

$D(2,2) = -16 - 8 + 8 < 0 \Rightarrow$ saddle pt. at $(2,2)$

$D(-2,2) = -16 - 8 + 8 < 0 \Rightarrow$ saddle pt. at $(-2,2)$

ii) $g(x,y) = (3x + 4x^3)(y^2 + 2y)$

$g_x = (3 + 12x^2)(y^2 + 2y)$ $\quad g_y = (3x + 4x^3)(2y + 2)$

$g_{xx} = 24x(y^2 + 2y)$ $\quad g_{yy} = 2(3x + 4x^3)$

$g_{xy} = (3 + 12x^2)(2y + 2)$

Crit. pts.

$g_x = \underbrace{(3 + 12x^2)}_{>0}(y^2 + 2y) = 0$

$\Rightarrow y(y+2) = 0 \Rightarrow y = 0$ or $y = -2$

If $y=0$, $g_y = 2(3x+4x^3) = 0$

$$2x\underbrace{(3+4x^2)}_{>0} = 0 \Rightarrow x=0$$

If $y=-2$, $g_y = -2(3x+4x^3) = 0 \Rightarrow x=0$

So critical points are $(0,0)$ and $(0,-2)$.

$$g_{xx} = 24x(y^2+2y) \qquad g_{yy} = 2(3x+4x^3)$$

$$g_{xy} = (3+12x^2)(2y+2)$$

$$D = 48x(y^2+2y)(3x+4x^3) - (3+12x^2)^2(2y+2)^2$$

$D(0,0) = 0 - (3)^2(2)^2 = -36 < 0$

$D(0,-2) = 0 - (3)^2(-2)^2 = -36 < 0$

So $(0,0)$ and $(0,-2)$ are both saddle points.

Suggested Textbook Exc. (14.7)

2, 5, 9, 11, 21