Table of Laplace Transforms

This table summarizes the general properties of Laplace transforms and the Laplace transforms of particular functions derived in Chapter 4.

Function	Transform	Function	Transform
$f(t)$	$F(s)$	e^{at}	$\dfrac{1}{s-a}$
$af(t)+bg(t)$	$aF(s)+bG(s)$	$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
$f'(t)$	$sF(s)-f(0)$	$\cos kt$	$\dfrac{s}{s^2+k^2}$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$	$\sin kt$	$\dfrac{k}{s^2+k^2}$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\cdots-f^{(n-1)}(0)$	$\cosh kt$	$\dfrac{s}{s^2-k^2}$
$\displaystyle\int_0^t f(\tau)\,d\tau$	$\dfrac{F(s)}{s}$	$\sinh kt$	$\dfrac{k}{s^2-k^2}$
$e^{at}f(t)$	$F(s-a)$	$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$	$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$
$\displaystyle\int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$	$\dfrac{1}{2k^3}(\sin kt-kt\cos kt)$	$\dfrac{1}{(s^2+k^2)^2}$
$tf(t)$	$-F'(s)$	$\dfrac{t}{2k}\sin kt$	$\dfrac{s}{(s^2+k^2)^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	$\dfrac{1}{2k}(\sin kt+kt\cos kt)$	$\dfrac{s^2}{(s^2+k^2)^2}$
$\dfrac{f(t)}{t}$	$\displaystyle\int_s^\infty F(\sigma)\,d\sigma$	$u(t-a)$	$\dfrac{e^{-as}}{s}$
$f(t)$, period p	$\displaystyle\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$	$\delta(t-a)$	e^{-as}
1	$\dfrac{1}{s}$	$(-1)^{[\![t/a]\!]}$ (square wave)	$\dfrac{1}{s}\tanh\dfrac{as}{2}$
t	$\dfrac{1}{s^2}$	$\left[\!\!\left[\dfrac{t}{a}\right]\!\!\right]$ (staircase)	$\dfrac{e^{-as}}{s(1-e^{-as})}$
t^n	$\dfrac{n!}{s^{n+1}}$		
$\dfrac{1}{\sqrt{\pi t}}$	$\dfrac{1}{\sqrt{s}}$		
t^a	$\dfrac{\Gamma(a+1)}{s^{a+1}}$		

4.2) Initial Value Problems

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

the Laplace transform of $f(t)$

Before we study this formal definition of the Laplace transform in Section 4.1, let's see how we can use the Table of Laplace Transforms (above and in the front cover of your textbook) to solve initial value problems like the ones in Chapter 2.

$$\begin{tikzcd}
\text{time domain } (t) & \boxed{\text{ivp in } x(t)} \arrow[d, "\mathcal{L}"] & \boxed{x(t) = \text{sol'n of ivp}} \\
\text{Laplace domain } (s) & \boxed{\text{alg. eq'n in } \mathcal{L}\{x(t)\}} \arrow[r, "\text{solve}"] & \boxed{\mathcal{L}\{x(t)\} = \cdots} \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}$$

For the following examples, we will use the these properties from the Table of Laplace Transforms:

① $\mathcal{L}\{f(t)\} = F(s)$

② $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$

③ $\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$

④ $\mathcal{L}\{1\} = \frac{1}{s}$

⑤ $\mathcal{L}\{t\} = \frac{1}{s^2}$

⑥ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

⑦ $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

⑧ $\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

⑨ $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

(ie $\mathcal{L}$ is linear) [Note: $\mathcal{L}\{0\} = 0$]

Examples:

(#2) $x'' + 9x = 0, \quad x(0) = 3, \quad x'(0) = 4$

Sol'n: First we will use the Chap. 2 technique.

$r^2 + 9 = 0$	$x = c_1 \cos 3t + c_2 \sin 3t$
$r = \pm 3i$	$x' = -3c_1 \sin 3t + 3c_2 \cos 3t$

$x(0) = 3: \quad c_1 + 0 = 3 \qquad c_1 = 3$

$x'(0) = 4: \quad 0 + 3c_2 = 4 \qquad c_2 = 4/3$

$\therefore \boxed{x(t) = 3\cos 3t + \frac{4}{3}\sin 3t}$

Now we will solve the i.v.p. using Laplace transforms.

$X(s) = \mathcal{L}\{x(t)\}$

$$\mathcal{L}\{x'' + 9x\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{x''\} + 9\mathcal{L}\{x\} = 0$$

$$\left[s^2 X(s) - s\,x(0) - x'(0)\right] + 9X(s) = 0$$

$$(s^2 + 9)X(s) - 3s - 4 = 0$$

$$X(s) = \frac{3s+4}{s^2+9}$$

$$X(s) = 3\cdot\frac{s}{s^2+9} + \frac{4}{3}\cdot\frac{3}{s^2+9}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{3\cdot\frac{s}{s^2+9} + \frac{4}{3}\cdot\frac{3}{s^2+9}\right\}$$

$$x(t) = 3\cdot\mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{4}{3}\cdot\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{x(t) = 3\cos 3t + \frac{4}{3}\sin 3t}$$

(#4) $x'' + 8x' + 15x = 0, \quad x(0) = 2, \quad x'(0) = -3$

Sol'n:

$$r^2 + 8r + 15 = 0$$

$$(r+3)(r+5) = 0$$

$$r = -3, -5$$

$$x = c_1 e^{-3t} + c_2 e^{-5t}$$

$$x' = -3c_1 e^{-3t} - 5c_2 e^{-5t}$$

$x(0) = 2: \quad c_1 + c_2 = 2$

$x'(0) = -3: \quad -3c_1 - 5c_2 = -3$

$$c_1 = \frac{\begin{vmatrix} 2 & 1 \\ -3 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -5 \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} 1 & 2 \\ -3 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ -3 & -5 \end{vmatrix}}$$

$$c_1 = \frac{-10+3}{-5+3} \qquad c_2 = \frac{-3+6}{-2}$$

$$c_1 = \frac{-7}{-2} \qquad c_2 = -\frac{3}{2}$$

$$c_1 = \frac{7}{2}$$

$$\therefore \boxed{x(t) = \frac{7}{2}e^{-3t} - \frac{3}{2}e^{-5t}}$$

$$x'' + 8x' + 15x = 0, \quad x(0) = 2, \quad x'(0) = -3$$

$$[s^2X(s) - sx(0) - x'(0)]$$
$$+ 8[sX(s) - x(0)] + 15X(s) = 0$$

$$(s^2+8s+15)X(s) - 2s + 3 - 16 = 0$$

$$\boxed{X(s) = \frac{2s+13}{s^2+8s+15}}$$

$$X(s) = \frac{2s+13}{(s+3)(s+5)} = \frac{A}{s+3} + \frac{B}{s+5}$$

[ie partial fractions]

$$2s+13 = A(s+5) + B(s+3)$$

$s=-3$: $\quad -6+13 = 2A \qquad A = 7/2$

$s=-5$: $\quad -10+13 = -2B \qquad B = -3/2$

$$X(s) = \frac{7}{2}\cdot\frac{1}{s+3} - \frac{3}{2}\cdot\frac{1}{s+5}$$

$$\boxed{x(t) = \frac{7}{2}e^{-3t} - \frac{3}{2}e^{-5t}}$$

(#9) $x'' + 4x' + 3x = 1, \quad x(0) = 0, \quad x'(0) = 0$

Sol'n: $r^2 + 4r + 3 = 0$

$(r+3)(r+1) = 0$

$r = -3, -1$

$$\boxed{x_c = c_1 e^{-3t} + c_2 e^{-t}}$$

$f(t) = 1 \qquad \phi_1(t) = 1$

$$\boxed{x_p = A}$$

$$x_p' = 0 \qquad x_p'' + 4x_p' + 3x_p = 1$$

$$x_p'' = 0 \qquad 0 + 0 + 3A = 1$$

$$A = 1/3$$

$$\boxed{x = c_1 e^{-3t} + c_2 e^{-t} + \frac{1}{3}}$$

$$x' = -3c_1 e^{-3t} - c_2 e^{-t}$$

$$x(0) = 0 : \quad c_1 + c_2 + \frac{1}{3} = 0$$

$$x'(0) = 0 : \quad -3c_1 - c_2 = 0$$

$$\begin{aligned} 3c_1 + 3c_2 &= -1 \\ (+)\ -3c_1 - c_2 &= 0 \\ \hline 2c_2 &= -1 \end{aligned}$$

$$c_2 = -1/2$$

$$c_1 - \frac{1}{2} = -\frac{1}{3}$$

$$c_1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6}$$

$$c_1 = 1/6$$

$$\therefore \boxed{x(t) = \frac{1}{6} e^{-3t} - \frac{1}{2} e^{-t} + \frac{1}{3}}$$

$$x'' + 4x' + 3x = 1, \quad x(0) = 0, \quad x'(0) = 0$$

$$\left[s^2 X(s) - s\,x(0) - x'(0)\right] + 4\left[s X(s) - x(0)\right] + 3X(s) = \frac{1}{s}$$

$$(s^2 + 4s + 3)X(s) = \frac{1}{s}$$

$$X(s) = \frac{1}{s(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}$$

$$1 = A(s+3)(s+1) + Bs(s+1) + Cs(s+3)$$

$\underline{s=0:}$ $\quad 1 = 3A \qquad A = 1/3$

$\underline{s=-3:}$ $\quad 1 = 6B \qquad B = 1/6$

$\underline{s=-1:}$ $\quad 1 = -2C \qquad C = -1/2$

$$X(s) = \frac{1}{3}\cdot\frac{1}{s} + \frac{1}{6}\cdot\frac{1}{s+3} - \frac{1}{2}\cdot\frac{1}{s+1}$$

$$\boxed{x(t) = \frac{1}{3} + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}}$$

(#10) $x'' + 3x' + 2x = t, \quad x(0) = 0, \quad x'(0) = 2$

Sol'n: $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0$

$r = -2, -1$

$\boxed{x_c = c_1 e^{-2t} + c_2 e^{-t}}$

$f(t) = t \qquad \phi_1(t) = t$

$f'(t) = 1 \qquad \phi_2(t) = 1$

$\boxed{x_p = At + B}$

$x_p' = A$

$x_p'' = 0$

$x_p'' + 3x_p' + 2x_p = t$

$0 + 3A + 2(At+B) = t$

$2At + (3A+2B) = t$

$2A = 1 \qquad 3A + 2B = 0$

$A = 1/2 \qquad \frac{3}{2} + 2B = 0$

$B = -\frac{3}{4}$

$\therefore \boxed{x_p = \frac{1}{2}t - \frac{3}{4}}$

$$\boxed{x = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{2}t - \frac{3}{4}}$$

$x' = -2c_1 e^{-2t} - c_2 e^{-t} + \frac{1}{2}$

$x(0)=0: \quad c_1 + c_2 - \frac{3}{4} = 0$

$x'(0)=2: \quad -2c_1 - c_2 + \frac{1}{2} = 2$

$$\begin{aligned} 4c_1 + 4c_2 &= 3 \\ \oplus\ -4c_1 - 2c_2 &= 3 \\ \hline 2c_2 &= 6 \end{aligned}$$

$c_2 = 3$

$4c_1 + 4\cdot 3 = 3$

$c_1 = \frac{-9}{4}$

$$\therefore \boxed{x(t) = -\frac{9}{4}e^{-2t} + 3e^{-t} + \frac{1}{2}t - \frac{3}{4}}$$

$x'' + 3x' + 2x = t, \quad x(0)=0, \quad x'(0)=2$

$$\left[s^2X(s) - sx(0) - x'(0)\right] + 3\left[sX(s) - x(0)\right] + 2X(s) = \frac{1}{s^2}$$

$$(s^2+3s+2)X(s) - 2 = \frac{1}{s^2}$$

$$X(s) = \frac{1+2s^2}{s^2(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+1}$$

$$1+2s^2 = As(s+2)(s+1) + B(s+2)(s+1) + Cs^2(s+1) + Ds^2(s+2)$$

$\underline{s=0:}$ $\quad 1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

$\underline{s=-2:}$ $\quad 9 = -4C \quad \Rightarrow \quad C = -\frac{9}{4}$

$\underline{s=-1:}$ $\quad 3 = D \quad \Rightarrow \quad D = 3$

$\underline{\text{Coeff. of } s^3:}$ $\quad 0 = A + C + D$

$$0 = A - \frac{9}{4} + 3$$

$$0 = 4A - 9 + 12$$

$$A = -\frac{3}{4}$$

$$X(s) = -\frac{3}{4} \cdot \frac{1}{s} + \frac{1}{2} \cdot \frac{1}{s^2} - \frac{9}{4} \cdot \frac{1}{s+2} + 3 \cdot \frac{1}{s+1}$$

$$\boxed{x(t) = -\frac{3}{4} + \frac{1}{2}t - \frac{9}{4}e^{-2t} + 3e^{-t}}$$

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