1.3 Slope Fields and Solution Curves

$$\boxed{\frac{dy}{dx} = f(x,y)}$$

Says the slope of the graph $y = y(x)$ at the point (x,y) is $f(x,y)$.

Example:

(#9) $\dfrac{dy}{dx} = x^2 - y - 2$

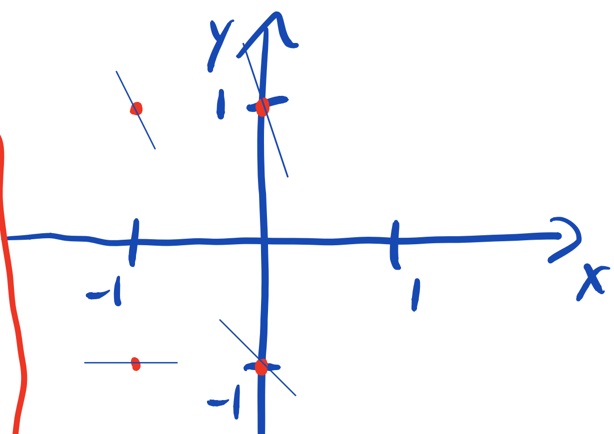
$$f(x,y) = x^2 - y - 2$$

$$f(-1,1) = (-1)^2 - (1) - 2 = -2$$

$$f(-1,-1) = (-1)^2 - (-1) - 2 = 0$$

$$f(0,-1) = (0)^2 - (-1) - 2 = -1$$

$$f(0,1) = (0)^2 - (1) - 2 = -3$$



$y = y(x)$

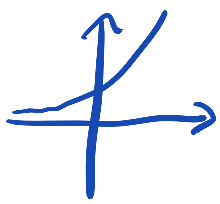
See lecture video for geogebra.org demo

Existence and Uniqueness of Solutions

Do all equations in algebra have exactly one solution? No.

Examples:

(i) $e^x = 0$: has no sol'ns since $e^x > 0$ for all x



(ii) $x^2 - 5x + 6 = 0$: has exactly two sol'ns
$(x-2)(x-3) = 0$ $\quad$ $x = 2$ and $x = 3$.

(iii) $x - y = 0$: has infinitely many sol'ns $x = y = k$ for all numbers k.

The same is true for diff. eq'ns, and it can depend on the initial condition.

<u>Examples</u>:

See lecture video for geogebra.org demo

(#28) $xy' = y, \quad y(a) = b$

$$x\frac{dy}{dx} = y \implies \boxed{\frac{dy}{dx} = y/x}$$

$\boxed{y = mx}$ are all sol'ns of the diff. eq. where m is any scalar.

$$y(a) = b \implies m \cdot a = b$$

(i) if $a \neq 0$, then $m = b/a$, so we get <u>exactly one sol'n</u>.

(ii) if $a = 0$ and $b = 0$, then $m \cdot 0 = 0$, which is satisfied by all m, so we get <u>infinitely many sol'ns</u>.

(iii) if $a = 0$ and $b \neq 0$, then $m \cdot 0 = b \neq 0$ $\implies 0 \neq 0$, so there are <u>no sol'ns</u>.

Sufficient conditions for existence and uniqueness of solutions

Theorem 1 in the textbook gives technical conditions that guarantee an initial value problem

$$\frac{dy}{dx} = f(x,y), \quad y(a) = b$$

has a unique sol'n near the point $x = a$ on the x-axis. These conditions are:

(1) f is continuous near the point (a,b) in the xy-plane

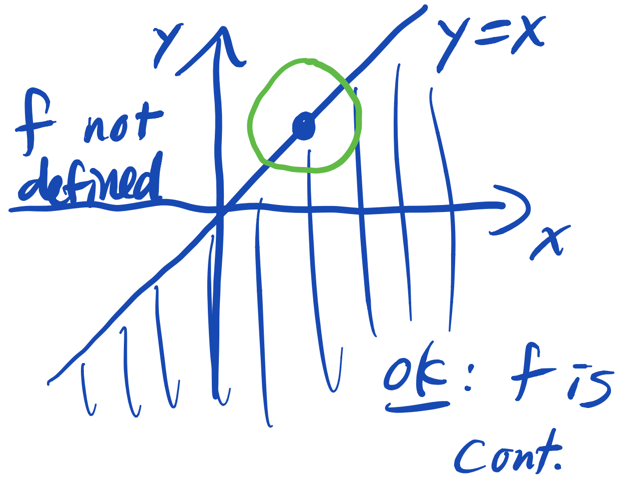
(2) $\frac{\partial f}{\partial y}$, the <u>partial</u> derivative of f w.r.t. y, is continuous near the point (a,b) in the xy plane.

Examples:

(#15) $\dfrac{dy}{dx} = \sqrt{x-y}\,, \quad y(2)=2$ $\qquad (a,b)=(2,2)$

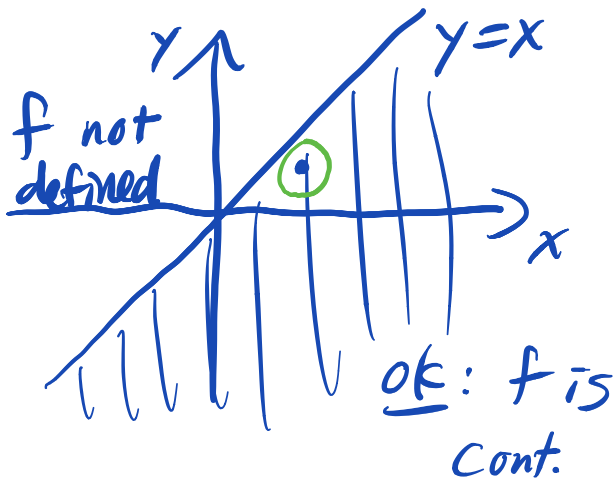
$f(x,y) = \sqrt{x-y}$

$\therefore$ f is not cont. near $(2,2)$.



$\therefore$ the theorem cannot guarantee this initial value problem has a unique sol'n.

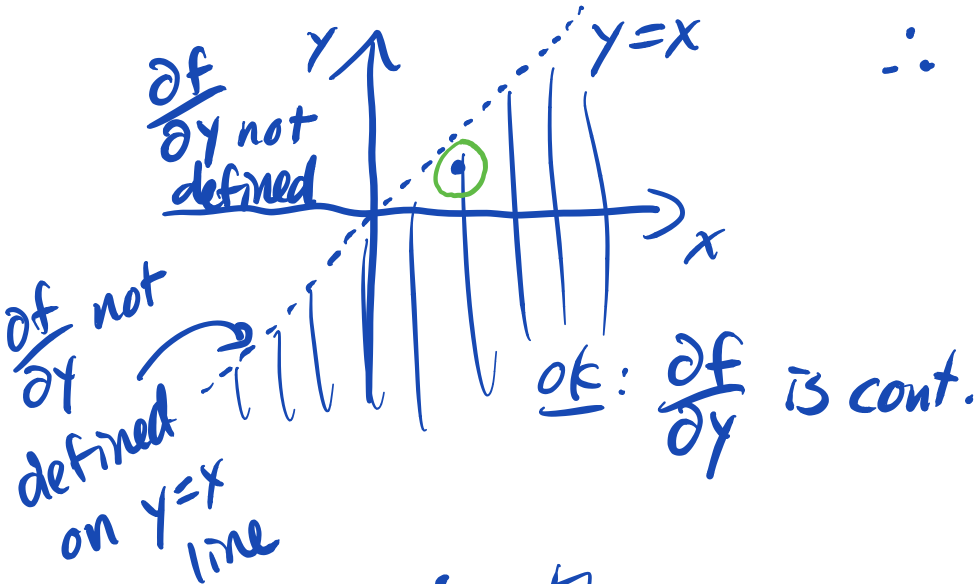
(#16) $\dfrac{dy}{dx} = \sqrt{x-y}\,, \quad y(2)=1$



$\therefore$ f is cont. near the point $(2,1)$.

$$f(x,y) = \sqrt{x-y} \Rightarrow \frac{\partial f}{\partial y} = \tfrac{1}{2}(x-y)^{-1/2}\cdot(-1).$$

$$\therefore \frac{\partial f}{\partial y}(x,y) = \frac{-1}{2\sqrt{x-y}}$$



$\therefore \frac{\partial f}{\partial y}$ is cont. near the point $(2,1)$.

$\therefore$ Theorem guarantees that the i.v.p. has a unique sol'n near $x=2$.