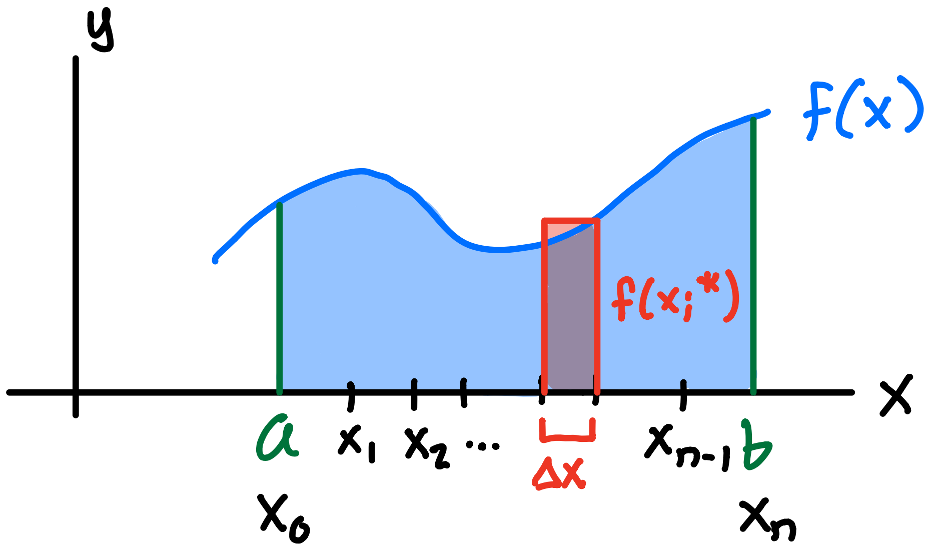
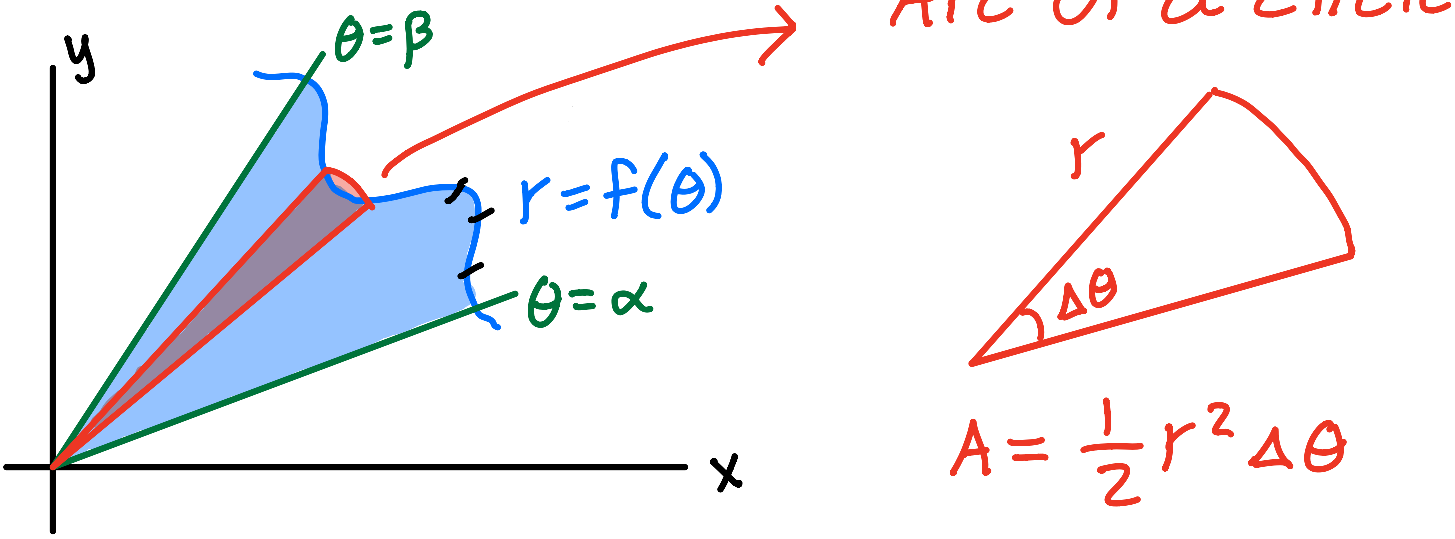
10.4 Areas and Lengths in Polar

Recall from Calc II,



$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{f(x_i^*)\Delta x}_{\text{Area of each rectangle}} = \int_a^b f(x)\,dx$$

For polar curves,



$$A = \frac{1}{2}r^2\Delta\theta$$

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} \underline{\tfrac{1}{2}\left[r(\theta_i^*)\right]^2 \Delta\theta}$$

Area of each arc

$$= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

The area under the polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$ is

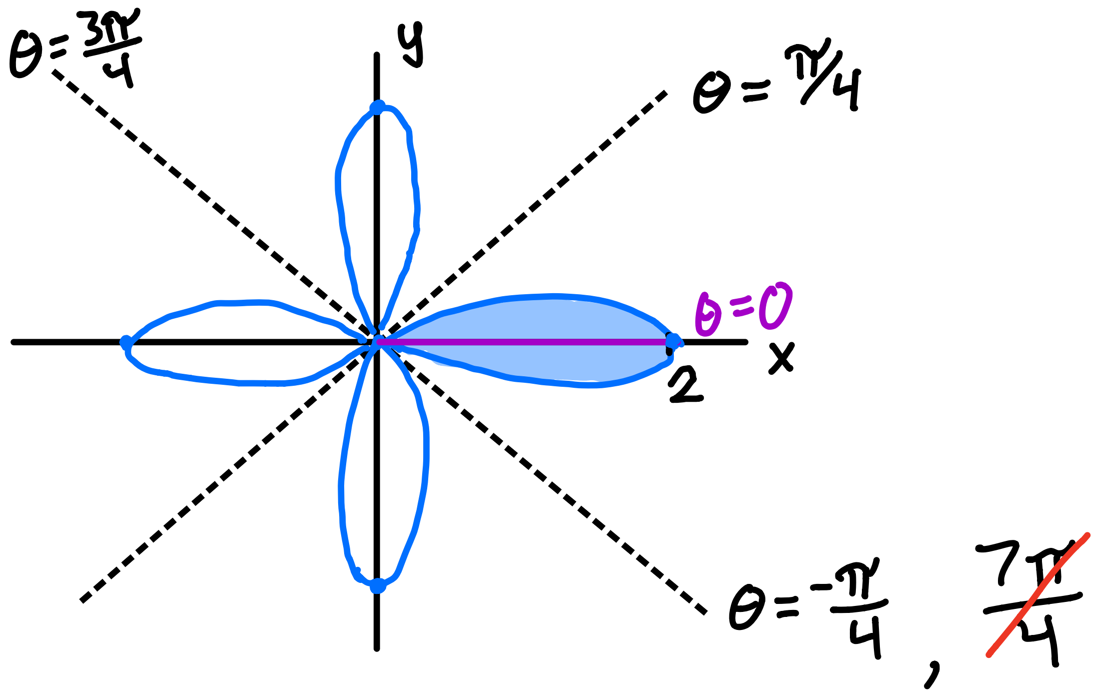
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left[f(\theta)\right]^2 d\theta$$

Evaluating these will sometimes require the trig identities

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

Ex 1 Find the area enclosed by one loop of the curve $r = 2\cos(2\theta)$.



need an interval like $[-\frac{\pi}{4}, \frac{\pi}{4}]$

$$A = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2}(2\cos 2\theta)^2 \, d\theta$$

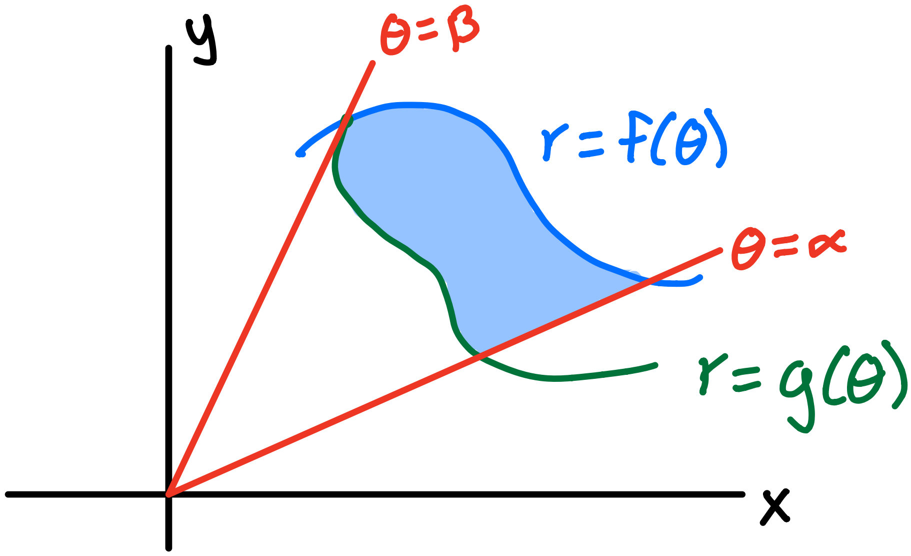
$$= 2\int_{-\pi/4}^{\pi/4} \underline{\cos^2(2\theta)} \, d\theta$$

Using $\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$ we have $\cos^2(2\theta) = \frac{1}{2}(1+\cos 4\theta)$

$$= \int_{-\pi/4}^{\pi/4} 1 + \cos 4\theta \, d\theta = \left[\theta + \frac{\sin 4\theta}{4}\right]_{-\pi/4}^{\pi/4}$$

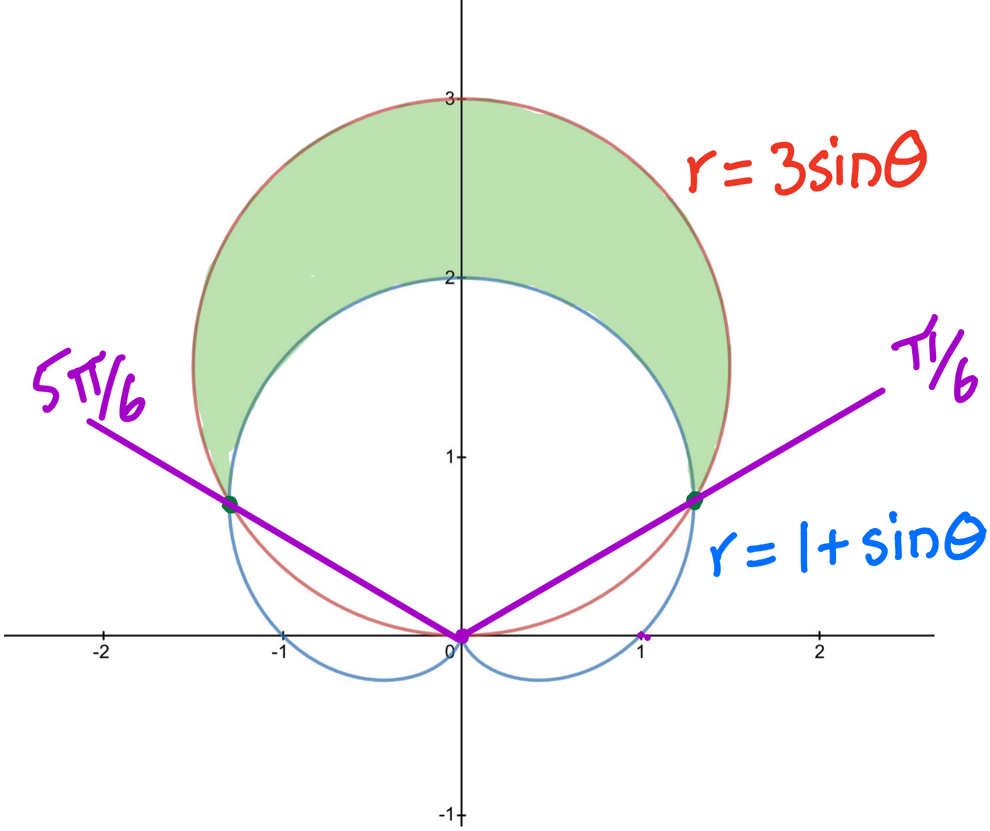
$$= \left(\frac{\pi}{4} + 0\right) - \left(-\frac{\pi}{4} + 0\right) = \frac{\pi}{2}$$

Areas Between Polar Curves



$$A = \int_{\alpha}^{\beta} \frac{1}{2}(f(\theta))^2 - \frac{1}{2}(g(\theta))^2 \, d\theta$$

Ex 2 Find the area of the region inside the circle $r = 3\sin\theta$ and outside the cardiod $r = 1 + \sin\theta$.



We need to find the intersections to know the bounds on θ.

<u>Intersections:</u>

$$3\sin\theta = 1+\sin\theta$$

$$\Rightarrow \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Notice how we couldn't find the intersection at the origin from this method.

$$A = \int_{\pi/6}^{5\pi/6} \frac{1}{2}(3\sin\theta)^2 - \frac{1}{2}(1+\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} 8\underbrace{\sin^2\theta}_{=\frac{1}{2}(1-\cos 2\theta)} - 2\sin\theta - 1 \, d\theta$$

$$= \frac{1}{2}\int_{\pi/6}^{5\pi/6} -4\cos 2\theta - 2\sin\theta + 3 \, d\theta$$

$$= \frac{1}{2}\Big[-2\sin 2\theta + 2\cos\theta + 3\theta\Big]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2}\left(-2\left(-\frac{\sqrt{3}}{2}\right) + 2\left(\frac{-\sqrt{3}}{2}\right) + \frac{15\pi}{6}\right) - \frac{1}{2}\left(-2\left(\frac{\sqrt{3}}{2}\right) + 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{2}\right)$$

$$= \frac{1}{2}\left[\sqrt{3} - \sqrt{3} + \sqrt{3} - \sqrt{3} + \frac{15\pi}{6} - \frac{\pi}{2}\right]$$

$$= \underline{\pi}$$

Arc Length for Polar Curves

Let $r = f(\theta)$, $\alpha \le \theta \le \beta$. We have

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Para. Eqns.

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Consider

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta \cancel{- 2r\frac{dr}{d\theta}\cos\theta\sin\theta} + r^2\sin^2\theta$$

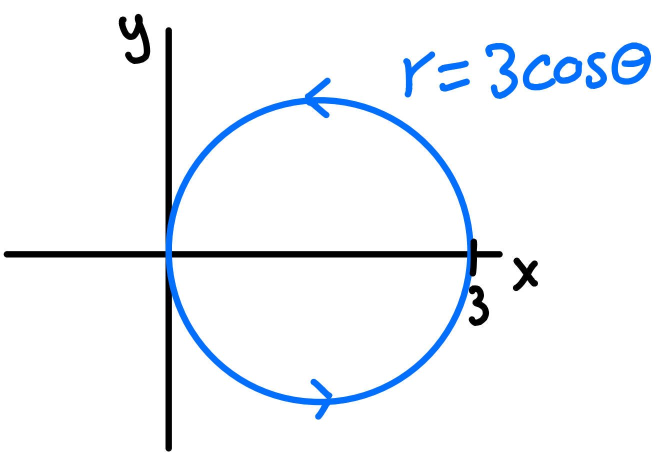
$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta \cancel{+ 2r\frac{dr}{d\theta}\sin\theta\cos\theta} + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

The arc length of $r = f(\theta)$, $\alpha \le \theta \le \beta$ (assuming the curve is only traversed once) is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$$

<u>Ex 3</u> Find the arc length of $r = 3\cos\theta$.



This curve completes a full rotation on the interval $[0, \pi]$. (NOT $[0, 2\pi]$)

$r = 3\cos\theta$ $\qquad$ $\dfrac{dr}{d\theta} = -3\sin\theta$

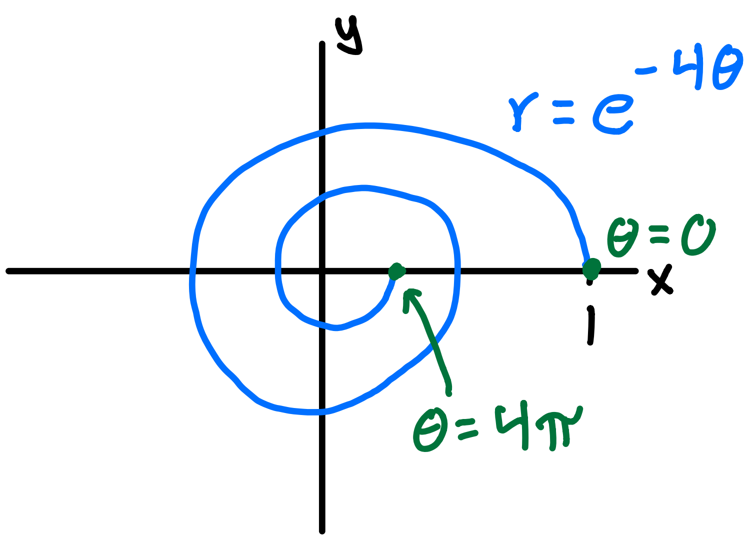
$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = 9\sin^2\theta + 9\cos^2\theta = 9$$

$$L = \int_0^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta = \int_0^{\pi} \sqrt{9}\, d\theta$$

$$= 3\int_0^{\pi} d\theta = 3\pi$$

<u>Ex 4</u> Find the arc length of

$r = e^{-4\theta}$, $0 \le \theta \le 4\pi$.



$$\frac{dr}{d\theta} = -4e^{-4\theta}$$

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \left(-4e^{-4\theta}\right)^2 + \left(e^{-4\theta}\right)^2$$

$$= 17e^{-8\theta}$$

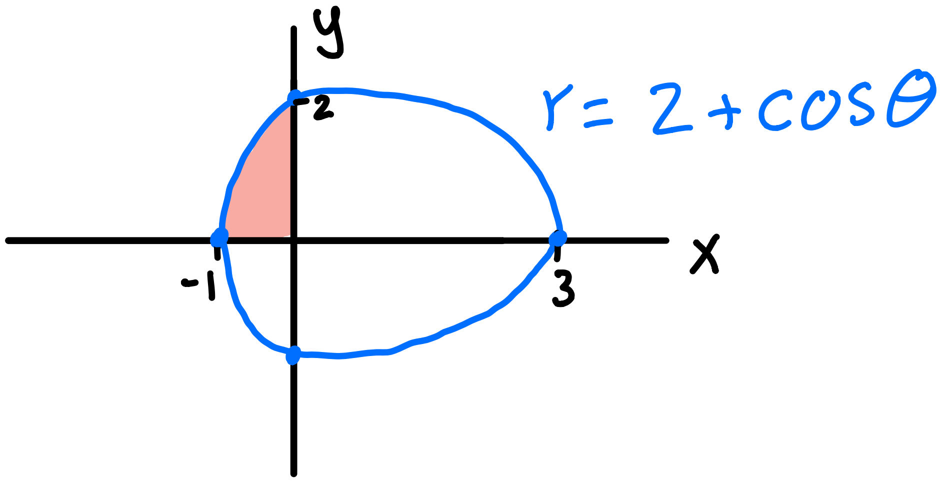
$$L = \int_0^{4\pi} \sqrt{17e^{-8\theta}}\, d\theta = \sqrt{17}\int_0^{4\pi} \left(e^{-8\theta}\right)^{1/2} d\theta$$

$$= \sqrt{17}\int_0^{4\pi} e^{-4\theta}\, d\theta = \frac{-\sqrt{17}}{4}\left[e^{-4\theta}\right]_0^{4\pi}$$

$$= \frac{-\sqrt{17}}{4}\left(e^{-16\pi} - 1\right) \approx \frac{\sqrt{17}}{4} \approx \underline{1.03}$$

Practice Problems

1) Find the area of the shaded region.

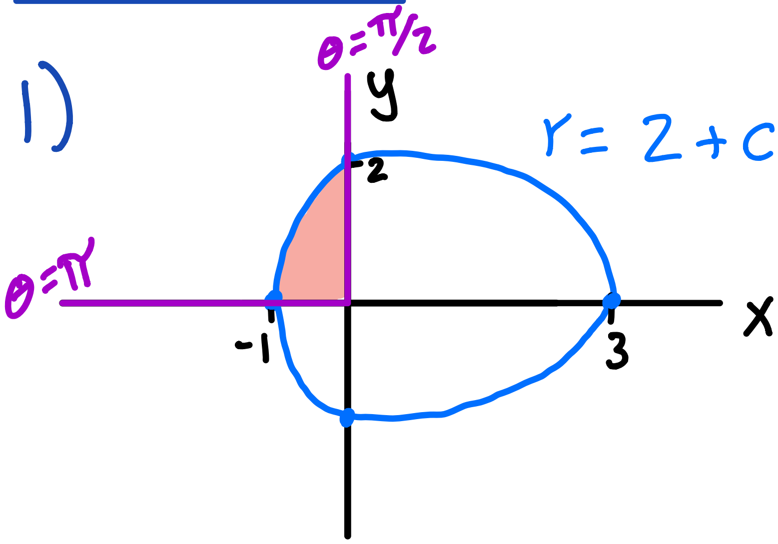


2) Find the area outside the circle $r=1$ but inside the cardiod $r=1-\sin\theta$.

3) Set up the integral for the arc length of $r=1+\sin\theta$. Do not evaluate.

Solutions

1)



$$A = \int_{\pi/2}^{\pi} \frac{1}{2} r^2 \, d\theta = \frac{1}{2}\int_{\pi/2}^{\pi} 4 + 4\cos\theta + \underbrace{\cos^2\theta}_{=\frac{1}{2}(1+\cos 2\theta)} \, d\theta$$

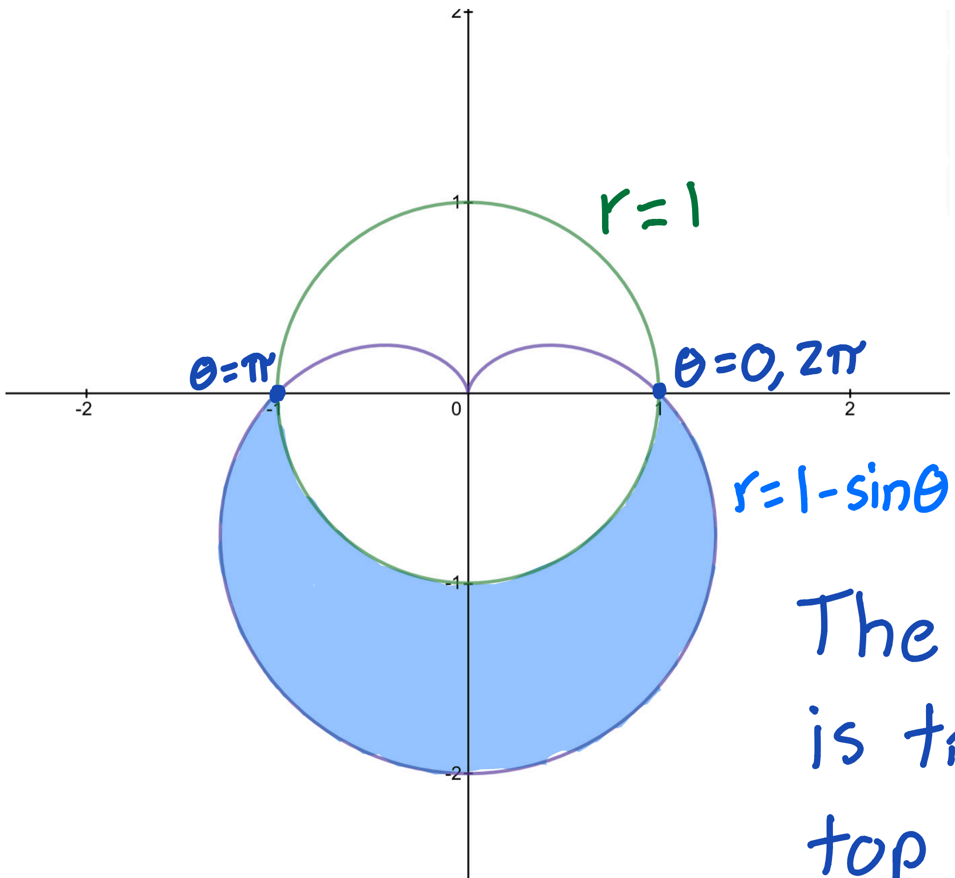
$$= \int_{\pi/2}^{\pi} 2 + 2\cos\theta + \frac{1}{4}(1+\cos 2\theta)\, d\theta$$

$$= \left[\frac{9}{4}\theta + 2\sin\theta + \frac{\sin 2\theta}{8} \right]_{\pi/2}^{\pi}$$

$$= \left(\frac{9\pi}{4} + 0 + 0 \right) - \left(\frac{9\pi}{8} + 2 + 0 \right)$$

$$= \frac{9\pi}{8} - 2 \approx \underline{1.5343}$$

2) Find the area outside the circle $r=1$ but inside the cardiod $r = 1 - \sin\theta$.



Intersections:

$$1 = 1 - \sin\theta$$
$$\sin\theta = 0$$
$$\theta = 0, \pi, 2\pi$$

The interval $[0, \pi]$ is tracing out the top part of each curve.

To get the shaded area we need to look at the interval $[\pi, 2\pi]$ which traces out the borders of the region.

$$A = \int_{\pi}^{2\pi} \frac{1}{2}(1-\sin\theta)^2 - \frac{1}{2}(1)^2 \, d\theta$$

$$= \int_{\pi}^{2\pi} \frac{1}{2}\sin^2\theta - \sin\theta \, d\theta$$

$$= \int_{\pi}^{2\pi} \frac{1}{4} - \frac{1}{4}\cos 2\theta - \sin\theta \, d\theta$$

$$= \left[\frac{1}{4}\theta - \cancel{\frac{\sin 2\theta}{8}} + \cos\theta\right]_{\pi}^{2\pi}$$

$$= \left(\frac{\pi}{2} - 0 + 1\right) - \left(\frac{\pi}{4} - 0 - 1\right)$$

$$= \underline{\frac{\pi}{4} + 2} \approx \underline{2.7854}$$

3) Set up the integral for the arc length of $r = 1 + \sin\theta$. Do not evaluate.

The curve $r = 1 + \sin\theta$ is traversed once on the interval $[0, 2\pi]$.

We have $\frac{dr}{d\theta} = \cos\theta$ so,

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \underbrace{\cos^2\theta + 1 + 2\sin\theta + \sin^2\theta}_{=1}$$

$$= 2 + 2\sin\theta$$

$$L = \int_0^{2\pi} \sqrt{2 + 2\sin\theta}\, d\theta$$

Suggested Textbook Exc. (10.4)

3, 5, 9, 17, 26, 46

Quiz 2
(10.3, 10.4)