2.2) General Sol'ns of Linear Eq'ns

(*) $$\boxed{y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = f(x)}$$

linear n-th order diff. eq'n.

To solve (*):

(1) Find the complementary sol'n:

$$\boxed{y_c = c_1 y_1 + \cdots + c_n y_n}.$$

This is a general sol'n of the homogeneous eq'n:

(**) $$\boxed{y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0}.$$

(2) Find a particular sol'n y_p of (*).

Then a general sol'n of (*) is

$$\boxed{y = c_1 y_1 + \cdots + c_n y_n + y_p}.$$

Example: (#21)

Solve the initial value problem

$$y'' + y = 3x, \quad y(0) = 2, \quad y'(0) = -2,$$

which has the complementary sol'n

$$y_c = c_1 \cos x + c_2 \sin x$$

and particular sol'n

$$y_p = 3x.$$

Sol'n:

A general sol'n of $y'' + y = 3x$ is

$$y = c_1 \cos x + c_2 \sin x + 3x.$$

$$[y' = -c_1 \sin x + c_2 \cos x + 3]$$

$$y(0) = 2 \Rightarrow c_1 \cdot 1 + c_2 \cdot 0 + 3 \cdot 0 = 2$$

$$\Rightarrow \boxed{c_1 = 2}$$

$$y'(0) = -2 \Rightarrow -c_1 \cdot 0 + c_2 \cdot 1 + 3 = -2$$

$$\Rightarrow \boxed{c_2 = -5}$$

$\therefore$ the sol'n of the i.v.p. is

$$\boxed{y = 2\cos x - 5\sin x + 3x}.$$

Linear independence

The functions $y_1, \ldots, y_n$ in the complementary sol'n

$$\boxed{y_c = c_1 y_1 + \cdots + c_n y_n}$$

ie no redundant sol'ns

need to be linearly independent sol'ns of the homogeneous eq'n

$$\boxed{y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_{n-1}(x) y' + p_n(x) y = 0}.$$

We say that functions $f_1, \ldots, f_n$ are linearly dependent if there are scalars $c_1, \ldots, c_n$, not all zero, such that

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0, \quad \text{for all } x.$$

We say that $f_1, \ldots, f_n$ are lin. indep. if

$$c_1 f_1 + \cdots + c_n f_n = 0 \implies c_1 = \cdots = c_n = 0.$$

Example: (#1)

Show that

$$f_1(x) = 2x, \quad f_2(x) = 3x^2, \quad f_3(x) = 5x - 8x^2$$

are **lin. dep.** by finding c_1, c_2, c_3, **not all zero**, such that

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0, \text{ for all } x.$$

Sol'n:

$$c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x) = 0$$

$$c_1 \cdot 2x + c_2 \cdot 3x^2 + c_3 \cdot (5x - 8x^2) = 0$$

$$(2c_1 + 5c_3)x + (3c_2 - 8c_3)x^2 = 0$$

[Note: x and x^2 are lin. indep.:

$$W(x) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2,$$

which is not zero everywhere.]

$$\Rightarrow \quad 2c_1 + 5c_3 = 0, \quad 3c_2 - 8c_3 = 0$$

Take $c_3 = 1$. Then $2c_1 + 5 = 0$, $3c_2 - 8 = 0$.

$\therefore$ $c_1 = -5/2$ and $c_2 = 8/3$.

$$(2c_1 + 5c_3)x + (3c_2 - 8c_3)x^2 = 0$$

$\underline{x=1}$: $(2c_1+5c_3)\cdot 1 + (3c_2 - 8c_3)\cdot 1 = 0$

$$2c_1 + 3c_2 - 3c_3 = 0$$

$\underline{x=-1}$: $(2c_1+5c_3)\cdot(-1) + (3c_2-8c_3)\cdot 1 = 0$

$$-2c_1 + 3c_2 - 13c_3 = 0$$

Take $c_3 = 1$. Then

$$\begin{array}{rrcl} & 2c_1 + 3c_2 &=& 3 \\ \oplus & -2c_1 + 3c_2 &=& 13 \\ \hline & 0 + 6c_2 &=& 16 \end{array} \Rightarrow \boxed{c_2 = 8/3}$$

$\Rightarrow 2c_1 + 3(8/3) = 3$

$\Rightarrow 2c_1 + 8 = 3$

$\Rightarrow 2c_1 = -5 \Rightarrow \boxed{c_1 = -5/2}$.

Using the Wronskian to prove lin. indep.

Suppose $f_1, \dots, f_n$ are lin. dep.
Then there are scalars $c_1, \dots, c_n$, not all zero, such that

$$c_1 f_1(x) + \cdots + c_n f_n(x) = 0, \text{ for all } x.$$

Then:

$$c_1 f_1'(x) + \cdots + c_n f_n'(x) = 0,$$
$$c_1 f_1''(x) + \cdots + c_n f_n''(x) = 0,$$
$$\vdots$$
$$c_1 f_1^{(n-1)}(x) + \cdots + c_n f_n^{(n-1)}(x) = 0, \text{ for all } x.$$

This is a linear system in $c_1, \dots, c_n$ that has the trivial all zero sol'n and the not all zero sol'n.

[This means the determinant of the coefficient matrix is zero!]

That is,

$$W(x) = \begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ & \vdots & \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix} = 0, \quad \text{for all } x.$$

the Wronskian of $f_1, \ldots, f_n$

<u>Linear Independence Test</u>:

[If $W(x) \neq 0$ for some value of x, then $f_1, \ldots, f_n$ are <u>lin. indep.</u>]

Example: (#7)

Show that

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = x^2$$

are lin. indep.

Sol'n: Using cofactor expansion.

$$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= +1 \cdot \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} - x \cdot \begin{vmatrix} 0 & 2x \\ 0 & 2 \end{vmatrix} + x^2 \cdot \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1 \cdot (1 \cdot 2 - 0 \cdot 2x) - x(0 \cdot 2 - 0 \cdot 2x) + x^2(0 \cdot 0 - 0 \cdot 1)$$

$$= 2 \neq 0$$

$\therefore$ $1, x, x^2$ are lin. indep.

$$W(x) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} \qquad \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} 1 & 2x \\ 0 & 2 \end{vmatrix} - 0 + 0$$

$$= 1 \cdot (1 \cdot 2 - 0 \cdot 2x) = 2 \neq 0.$$

<u>Note</u>: Another way to compute 3x3 determinant is:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh$$
$$- ceg - afh - bdi$$

$$\begin{array}{ccccc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

$$\begin{array}{ccccc} 1 & x & x^2 & 1 & x \\ 0 & 1 & 2x & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{array} \qquad \therefore\ W(x) = 2 + 0 + 0$$
$$-0 - 0 - 0$$
$$= 2$$

Example: (#8)

Show that

$$f_1(x) = e^x, \quad f_2(x) = e^{2x}, \quad f_3(x) = e^{3x}$$

are lin. indep.

Sol'n:

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 18e^{6x} + 3e^{6x} + 4e^{6x} - 2e^{6x} - 12e^{6x} - 9e^{6x}$$

$$\begin{matrix} e^x & e^{2x} & e^{3x} & e^x & e^{2x} \\ e^x & 2e^{2x} & 3e^{3x} & e^x & 2e^{2x} \\ e^x & 4e^{2x} & 9e^{3x} & e^x & 4e^{2x} \end{matrix}$$

$$= 2e^{6x} \neq 0$$

for all x.

$\therefore$ e^x, e^{2x}, e^{3x} are lin. indep.

Example: (#14)

Solve the homogeneous i.v.p.:

$$y^{(3)} - 6y'' + 11y' - 6y = 0,$$

$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 3.$$

Three lin. indep. sol'ns are:

$$y_1(x) = e^x, \quad y_2 = e^{2x}, \quad y_3 = e^{3x}.$$

Sol'n:

A general sol'n of the diff. eq'n is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

$$y' = c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x}$$

$$y'' = c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}$$

$y(0) = 0 \Rightarrow c_1 \cdot 1 + c_2 \cdot 1 + c_3 \cdot 1 = 0$

$y'(0) = 0 \Rightarrow c_1 \cdot 1 + 2c_2 \cdot 1 + 3c_3 \cdot 1 = 0$

$y'(0) = 3 \Rightarrow c_1 \cdot 1 + 4c_2 \cdot 1 + 9c_3 \cdot 1 = 3$

$$\begin{bmatrix} c_1 + c_2 + c_3 = 0 \\ c_1 + 2c_2 + 3c_3 = 0 \\ c_1 + 4c_2 + 9c_3 = 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & 4 & 9 & 3 \end{array}\right]$$

[Subtract row 1 from rows 2 and 3.]

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 8 & 3 \end{array}\right]$$

ie $\quad c_2 + 2c_3 = 0$

$\quad 3c_2 + 8c_3 = 3$

$$\downarrow$$

[Add $(-3) \times (\text{row } 2)$ to row 3.]

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right]$$

ie $\quad 2c_3 = 3 \implies c_3 = 3/2$

$c_2 + 2c_3 = 0 \implies c_2 = -3$

$c_1 + c_2 + c_3 = 0 \implies c_1 = 3 - 3/2 = 3/2$

$$\therefore \boxed{y = \frac{3}{2}e^{x} - 3e^{2x} + \frac{3}{2}e^{3x}.}$$