2.5 / Nonhomogeneous Equations + Undetermined Coefficients

$$\boxed{a_n y^{(n)} + \cdots + a_1 y' + a_0 y = f(x)} \qquad (1)$$

$$\boxed{a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0} \qquad (2)$$

Recall that if y_c is the <u>complementary sol'n</u> (ie a general sol'n of (2)) and y_p is a <u>particular sol'n</u> of (1), then $y = y_c + y_p$ is a <u>general sol'n</u> of (1).

Section 2.3 covered how to find y_c.
Now we will see how to find y_p.

Method of Undetermined Coefficients

* $f(x)$ needs to be a sum of products of:

1. a polynomial in x $(c_k x^k + \cdots + c_1 x + c_0)$
2. an exponential function (e^{ax})
3. a trig. function $(\cos bx$ or $\sin bx)$

Determine all the terms that appear in $f(x)$ and all the derivatives of $f(x)$:

$$\phi_1(x),\ \phi_2(x),\ \ldots,\ \phi_m(x).$$

[If some $\phi_i(x)$ is a sol'n of (2), replace it with $x\phi_i(x)$.]

Then

$$y_p = A_1\phi_1(x) + \cdots + A_m\phi_m(x). \qquad (3)$$

Substitute (3) into (1) to determine the coefficients $A_1, \ldots, A_m$.

Examples: Find y_p.

(#1) $y'' + 16y = e^{3x}$ $\qquad f(x) = e^{3x}$

$y'' + 16y = 0$

$r^2 + 16 = 0$

$r = \pm i4$

$y_c = c_1\cos 4x + c_2 \sin 4x$

$f(x) = e^{3x}$

$f'(x) = 3e^{3x}$

$\boxed{\phi_1(x) = e^{3x}}$ ✓

$\therefore \boxed{y_p = A_1 e^{3x}}$

$y_p' = 3A_1 e^{3x}$

$y_p'' = 9A_1 e^{3x}$

$y_p'' + 16y_p = e^{3x}$

$9A_1 e^{3x} + 16A_1 e^{3x} = e^{3x}$

$25A_1 e^{3x} = 1 \cdot e^{3x}$

$\therefore 25A_1 = 1 \Rightarrow A_1 = \frac{1}{25}$

$\therefore \boxed{y_p = \frac{1}{25} e^{3x}}$

Then $y = y_c + y_p$

$\Rightarrow \boxed{y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{25} e^{3x}}$

(#2) $y'' - y' - 2y = 3x + 4$

$y'' - y' - 2y = 0$

$r^2 - r - 2 = 0$

$(r-2)(r+1) = 0$

$r = -1, 2$

$\boxed{y_c = c_1 e^{-x} + c_2 e^{2x}}$

$f(x) = 3x + 4 \cdot 1$

$f'(x) = 3 \cdot 1$

$f''(x) = 0$

$\phi_1(x) = x$ ✓

$\phi_2(x) = 1$ ✓

$\therefore \boxed{y_p = A_1 x + A_2 \cdot 1}$

$y_p' = A_1$

$y_p'' = 0$

$y_p'' - y_p' - 2y_p = 3x + 4$

$0 - A_1 - 2(A_1 x + A_2) = 3x + 4$

$$-2A_1 x + (-A_1 - 2A_2) = 3x + 4$$

$$-2A_1 = 3 \Rightarrow A_1 = -\frac{3}{2}$$

$$-A_1 - 2A_2 = 4 \Rightarrow -\left(-\frac{3}{2}\right) - 2A_2 = 4$$

$$\Rightarrow \frac{3}{4} - A_2 = 2$$

$$\Rightarrow A_2 = \frac{3}{4} - 2$$

$$\Rightarrow A_2 = -\frac{5}{4}$$

$$\therefore \boxed{y_p = -\frac{3}{2}x - \frac{5}{4}}$$

Then $y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 e^{-x} + c_2 e^{2x} - \frac{3}{2}x - \frac{5}{4}}$$

(#10) $y'' + 9y = 2\cos 3x + 3\sin 3x$

$y'' + 9y = 0$

$r^2 + 9 = 0$

$r = \pm i3$

$$\boxed{y_c = c_1 \cos 3x + c_2 \sin 3x}$$

$f(x) = 2\cos 3x + 3\sin 3x$

$f'(x) = -6\sin 3x + 9\cos 3x$

$\cancel{\phi_1(x) = \cos 3x}$

$\cancel{\phi_2(x) = \sin 3x}$

reject since both are sol'ns of the homogeneous eq'n

$$\boxed{\begin{aligned} \phi_1(x) &= x\cos 3x \\ \phi_2(x) &= x\sin 3x \end{aligned}}$$

✓ ✓

$y_p = A_1 x\cos 3x + A_2 x\sin 3x$

$y_p = x(A_1 \cos 3x + A_2 \sin 3x)$

$y_p' = x(-3A_1 \sin 3x + 3A_2 \cos 3x)$
$\quad + 1 \cdot (A_1 \cos 3x + A_2 \sin 3x)$

$$y_p'' = x\left(-9A_1\cos 3x - 9A_2\sin 3x\right)$$
$$+ 1\cdot\left(-3A_1\sin 3x + 3A_2\cos 3x\right)$$
$$+ \left(-3A_1\sin 3x + 3A_2\cos 3x\right)$$

$$y_p'' = \left(-9A_1x + 6A_2\right)\cos 3x$$
$$+ \left(-9A_2x - 6A_1\right)\sin 3x$$

$$\left[y_p'' + 9y_p = 2\cos 3x + 3\sin 3x\right]$$

$$\left(-9A_1x + 6A_2\right)\cos 3x$$
$$+ \left(-9A_2x - 6A_1\right)\sin 3x$$
$$+ 9\left(A_1x\cos 3x + A_2x\sin 3x\right)$$
$$= 2\cos 3x + 3\sin 3x$$

$$\left(\cancel{-9A_1x} + 6A_2 + \cancel{9A_1x}\right)\cos 3x$$
$$+ \left(\cancel{-9A_2x} - 6A_1 + \cancel{9A_2x}\right)\sin 3x$$
$$= 2\cos 3x + 3\sin 3x$$

$$6A_2\cos 3x - 6A_1\sin 3x = 2\cos 3x + 3\sin 3x$$

$$\therefore \quad 6A_2 = 2$$
$$-6A_1 = 3$$
$$\Rightarrow \quad A_2 = \frac{1}{3}, \quad A_1 = \frac{-1}{2}$$

$$\therefore \boxed{y_p = \frac{-1}{2} x\cos 3x + \frac{1}{3} x \sin 3x}$$

Then $y = y_c + y_p$

$$\Rightarrow \boxed{\begin{aligned} y = {} & c_1 \cos 3x + c_2 \sin 3x \\ & -\frac{1}{2} x\cos 3x + \frac{1}{3} x \sin 3x \end{aligned}}$$

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Example: (#24)

Determine the form of y_p.

$$y^{(3)} - y'' - 12y' = x - 2xe^{-3x}$$

Sol'n:

$$y^{(3)} - y'' - 12y' = 0$$

$$r^3 - r^2 - 12r = 0$$

$$r(r^2 - r - 12) = 0$$

$$r(r-4)(r+3) = 0$$

$$\therefore r = -3, 0, 4$$

$$\therefore \boxed{y_c = c_1 e^{-3x} + c_2 + c_3 e^{4x}}$$

$$f(x) = x - 2xe^{-3x}$$

$$f'(x) = 1 - 2x(-3e^{-3x}) - 2e^{-3x}$$

$$f'(x) = 1 + 6xe^{-3x} - 2e^{-3x}$$

$\phi_1(x) = x$ ✓

$\phi_2(x) = xe^{-3x}$ ✓

~~$\phi_3(x) = 1$~~ ~~$\phi_3(x) = x$~~ $\phi_3(x) = x^2$ ✓

~~$\phi_4(x) = e^{-3x}$~~ ~~$\phi_4(x) = xe^{-3x}$~~ $\phi_4(x) = x^2e^{-3x}$ ✓

$$\therefore \boxed{y_p = A_1 x + A_2 xe^{-3x} + A_3 x^2 + A_4 x^2e^{-3x}}$$

Example: (#31)

Solve the i.v.p.

$$y'' + 4y = 2x, \quad y(0) = 1, \quad y'(0) = 2$$

Sol'n:

$y'' + 4y = 0$

$r^2 + 4 = 0$

$r = \pm i2$

$$\boxed{y_c = c_1 \cos 2x + c_2 \sin 2x}$$

$f(x) = 2x$

$f'(x) = 2 \cdot 1$

$$\boxed{\begin{aligned} \phi_1(x) &= x \\ \phi_2(x) &= 1 \end{aligned}}$$

✓ ✓

$\therefore \boxed{y_p = A_1 x + A_2}$

$y_p' = A_1$

$y_p'' = 0$

$y_p'' + 4y_p = 2x$

$0 + 4(A_1 x + A_2) = 2x$

$4A_1 x + 4A_2 = 2x + 0$

$4A_1 = 2 \Rightarrow A_1 = \frac{1}{2}$

$4A_2 = 0 \Rightarrow A_2 = 0$

$\therefore \boxed{y_p = \frac{1}{2}x}$

Then $y = y_c + y_p$

$\Rightarrow \boxed{y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}x}$

$y' = -2c_1 \sin 2x + 2c_2 \cos 2x + \frac{1}{2}$

$y(0) = 1 \quad : \quad c_1 \cdot 1 + c_2 \cdot 0 + \frac{1}{2} \cdot 0 = 1$

$y'(0) = 2 \quad : \quad -2c_1 \cdot 0 + 2c_2 \cdot 1 + \frac{1}{2} = 2$

$c_1 = 1$

$2c_2 + \frac{1}{2} = 2$

$c_2 + \frac{1}{4} = 1$

$c_2 = \frac{3}{4}$

$\therefore \boxed{y(x) = \cos 2x + \frac{3}{4} \sin 2x + \frac{1}{2}x}$

Method of Variation of Parameters

$$\boxed{y'' + P(x)y' + Q(x)y = f(x)} \qquad (3)$$

Suppose y_1 and y_2 satisfy:

$$\boxed{y'' + P(x)y' + Q(x)y = 0} \qquad (4)$$

Then $y_c = c_1 y_1 + c_2 y_2$ is a general sol'n of the homogeneous eq'n (4).

Goal: Find functions u_1 and u_2 such that

$$\boxed{y_p = u_1 y_1 + u_2 y_2}$$

is a particular sol'n of (3).

We need to substitute y_p into (3).

$$y_p' = u_1 y_1' + u_1' y_1 + u_2 y_2' + u_2' y_2$$

$$y_p' = (u_1 y_1' + u_2 y_2') + (u_1' y_1 + u_2' y_2)$$

To simplify, we set:

$$\boxed{u_1' y_1 + u_2' y_2 = 0}. \qquad (5)$$

Then $y_p' = u_1 y_1' + u_2 y_2'$ so

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

and $y_p'' + P(x) y_p' + Q(x) y_p = f(x)$ implies

$$(u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2')$$
$$+ P(x)(u_1 y_1' + u_2 y_2') + Q(x)(u_1 y_1 + u_2 y_2) = f(x)$$

$$u_1 (y_1'' + P(x) y_1' + Q(x) y_1) + u_2 (y_2'' + P(x) y_2' + Q(x) y_2)$$
$$+ (u_1' y_1' + u_2' y_2') = f(x)$$

$$\Rightarrow \quad \boxed{u_1' y_1' + u_2' y_2' = f(x)}. \qquad (6)$$

To solve (5) and (6) we use <u>Cramer's Rule</u>.

Cramer's Rule:

$$\left[\begin{array}{l} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{array}\right]$$

$$\Rightarrow \quad x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} \, , \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{12} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

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eg $$\left[\begin{array}{l} c_1 + c_2 = -1 \\ 3c_1 - 3c_2 = 15 \end{array}\right]$$

$$c_1 = \frac{\begin{vmatrix} -1 & 1 \\ 15 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix}} = \frac{3-15}{-3-3} = \frac{-12}{-6} = 2$$

$$c_2 = \frac{\begin{vmatrix} 1 & -1 \\ 3 & 15 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & -3 \end{vmatrix}} = \frac{15-(-3)}{-6} = \frac{18}{-6} = -3$$

$$\left[\begin{array}{l} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = f(x) \end{array}\right] \quad \begin{array}{l}(5)\\(6)\end{array}$$

$$\Rightarrow \quad u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}, \quad u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}.$$

$$\Rightarrow \quad \boxed{u_1' = \frac{-y_2 f(x)}{W(x)}, \quad u_2' = \frac{y_1 f(x)}{W(x)}}$$

$$\Rightarrow \quad \boxed{u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx, \quad u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx}.$$

<u>Example:</u> (#47)

Use Variation of Parameters to find y_p.

$$y'' + 3y' + 2y = 4e^x$$

Sol'n:

$$y'' + 3y' + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$r = -2, -1$$

$$\boxed{y_c = c_1 e^{-2x} + c_2 e^{-x}}$$

$$\boxed{\begin{aligned} y_1 &= e^{-2x} \\ y_2 &= e^{-x} \end{aligned}}$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(x) = \begin{vmatrix} e^{-2x} & e^{-x} \\ -2e^{-2x} & -e^{-x} \end{vmatrix}$$

$$W(x) = (e^{-2x})(-e^{-x}) - (e^{-x})(-2e^{-2x})$$

$$W(x) = -e^{-3x} + 2e^{-3x}$$

$$\boxed{W(x) = e^{-3x}}$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = \int \frac{-e^{-x}\cdot 4e^{x}}{e^{-3x}}\,dx$$

$$u_1 = \int -4e^{3x}\,dx = \frac{-4}{3}e^{3x}$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{e^{-2x}\cdot 4e^{x}}{e^{-3x}}\,dx$$

$$u_2 = \int 4e^{2x}\,dx = 2e^{2x}$$

Then $y_p = u_1 y_1 + u_2 y_2$

$$\Rightarrow y_p = \frac{-4}{3}e^{3x}e^{-2x} + 2e^{2x}e^{-x}$$

$$\Rightarrow y_p = -\frac{4}{3}e^{x} + 2e^{x}$$

$$\Rightarrow \boxed{y_p = \frac{2}{3}e^{x}}$$

Example: (#57)

$$x^2y'' + xy' - y = 72x^5$$

$$\left[y_c = c_1 x + c_2 x^{-1}\right]$$ Use Variation of Parameters to find y_p.

Sol'n:

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 72x^3$$

$$\therefore \quad f(x) = 72x^3$$

$$\boxed{\begin{aligned} y_1 &= x \\ y_2 &= x^{-1} \end{aligned}}$$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(x) = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1}$$

$$u_1 = \int \frac{-y_2 f(x)}{W(x)}\,dx = \int \frac{-x^{-1} \cdot 72x^3}{-2x^{-1}}\,dx$$

$$u_1 = \int 36x^3\,dx = 9x^4$$

$$u_2 = \int \frac{y_1 f(x)}{W(x)}\,dx = \int \frac{x \cdot 72x^3}{-2x^{-1}}\,dx$$

$$u_2 = \int -36x^5\,dx = -6x^6$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = 9x^4 \cdot x - 6x^6 \cdot x^{-1}$$

$$y_p = 9x^5 - 6x^5$$

$$\boxed{y_p = 3x^5}$$