14.6 Directional Derivatives and the Gradient Vector

In previous sections we discussed the partial derivatives $f_x(x,y)$ and $f_y(x,y)$. These represent the rate of change of f as we vary x (fixing y) and as we vary y (fixing x) respectively.

Now let's consider the rate of change as we vary x and y simultaneously.

Def 1

The rate of change of $f(x,y)$ in the direction of the unit vector $\vec{u} = \langle a, b \rangle$

is called the directional derivative and it is denoted by $D_{\vec{u}} f(x,y)$.

$$D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+ah,\, y+bh) - f(x,y)}{h}$$

Note: f_x and f_y are the directional derivatives in the direction of the x and y axis respectively. So

$$D_i f(x,y) = f_x(x,y) \qquad i = \langle 1,0 \rangle$$

$$D_j f(x,y) = f_y(x,y) \qquad j = \langle 0,1 \rangle$$

Moreover we can use f_x and f_y to find any directional derivative. To see this let

$$g(z) = f(x_0 + az,\ y_0 + bz)$$

for a fixed point (x_0, y_0) and a unit vector $\vec{u} = \langle a, b \rangle$. We have

$$g'(z) = \lim_{h \to 0} \frac{g(z+h) - g(z)}{h}$$

So

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

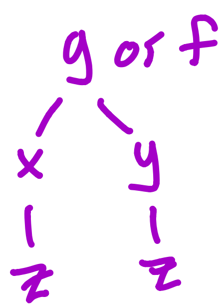
$$= \lim_{h \to 0} \frac{f(x_0 + ah,\ y_0 + bh) - f(x_0, y_0)}{h}$$

$$= D_{\vec{u}} f(x_0, y_0)$$

But we can also write $g(z) = f(x, y)$

where $x = x_0 + az$ and $y = y_0 + bz$. By Chain Rule,

$$g'(z) = \frac{dg}{dz} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dz} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dz}$$



$$= f_x(x,y)\,a + f_y(x,y)\,b$$

Since $z=0 \Rightarrow x = x_0$ and $y = y_0$ we have

$$g'(0) = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\,b$$

So

$$D_{\vec{u}} f(x_0,y_0) = f_x(x_0,y_0)\,a + f_y(x_0,y_0)\,b$$

$$= \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle \cdot \langle a, b\rangle$$

The vector $\langle f_x, f_y\rangle$ will show up a lot in later sections, it is called the gradient of f.

Def 2

The gradient (vector) of $f(x,y)$ is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$

(For $f(x,y,z)$, $\nabla f = \langle f_x, f_y, f_z\rangle$)

Let's summarize what we have found. Let $\vec{u} = \langle a, b \rangle$ be a unit vector, then

$$\begin{aligned} D_{\vec{u}} f(x,y) &= f_x(x,y)\,a + f_y(x,y)\,b \\ &= \nabla f(x,y) \cdot \vec{u} \end{aligned}$$

Ex 1

Let $f(x,y) = \sqrt{4 - x^2 - y^2}$. Find the directional derivative at $P(1,1)$ in the direction of $\vec{v} = \langle 1, 1 \rangle$.

Sol

The def. of $D_{\vec{u}} f$ only works when $\vec{u}$ is a unit vector so the first step is to let

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 1, 1 \rangle}{\sqrt{1^2 + 1^2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle.$$

Next we find $f_x(1,1)$ and $f_y(1,1)$:

$$f(x,y) = \sqrt{4-x^2-y^2}$$

$$f_x = \frac{1}{2}(4-x^2-y^2)^{-1/2}(-2x)$$

$$f_x(1,1) = \frac{1}{2}(2)^{-1/2}(-2) = -\frac{1}{\sqrt{2}}$$

$$f_y = \frac{1}{2}(4-x^2-y^2)^{-1/2}(-2y)$$

$$f_y(1,1) = -\frac{1}{\sqrt{2}}$$

So $\nabla f(1,1) = \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle$ and

$$D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{u}$$

$$= \langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$= -\frac{1}{2} - \frac{1}{2} = -1$$

This value represents the slope of the tangent line in the direction of $\vec{v} = \langle 1, 1 \rangle$.

We could even find an eqn for this tangent line by using direction vector $\vec{v} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1 \rangle$ and point $(1, 1, \sqrt{2})$.

Ex 2

Let $f(x,y) = x^2y^3 - 4y$. Find $\nabla f(2,-1)$ and use this to find directional der. in the direction of $\vec{v} = 2i + 5j$.

Sol

$f_x = 2xy^3$ $\qquad\qquad$ $f_y = 3x^2y^2 - 4$

$f_x(2,-1) = 2(2)(-1)^3$
$= -4$

$f_y(2,-1) = 3(4)(1) - 4$
$= 8$

$$\nabla f(2,-1) = \langle -4, 8 \rangle$$

$$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 2, 5 \rangle}{\sqrt{4+25}} = \langle \frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \rangle$$

So $D_{\vec{u}}f(2,-1) = \langle -4, 8\rangle \cdot \frac{1}{\sqrt{29}}\langle 2,5\rangle$

$$= \frac{1}{\sqrt{29}}(-8+40)$$

$$= \frac{32}{\sqrt{29}}$$

Maximizing Directional Derivative

Of all the directional derivatives which direction gives the fastest (biggest) positive rate of change. What is this rate?

Thm 1

Suppose f is a differentiable function of 2 or 3 variables. Then the max value of $D_{\vec{u}}f$ is $|\nabla f|$ and it occurs in the direction of the gradient vector ∇f.

(i.e. when $\vec{u} = \frac{\nabla f}{|\nabla f|}$)

Ex 3 Let $f(x,y) = xe^y$. At the point $(2,0)$ in what direction does f have the max rate of change? What is it?

Sol

$D_{\vec{u}}f$ is maximized in the direction of ∇f.

$$\nabla f = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2,0) = \langle e^0, 2e^0 \rangle = \langle 1, 2 \rangle$$

So $D_{\vec{u}}f(2,0)$ is maximized when

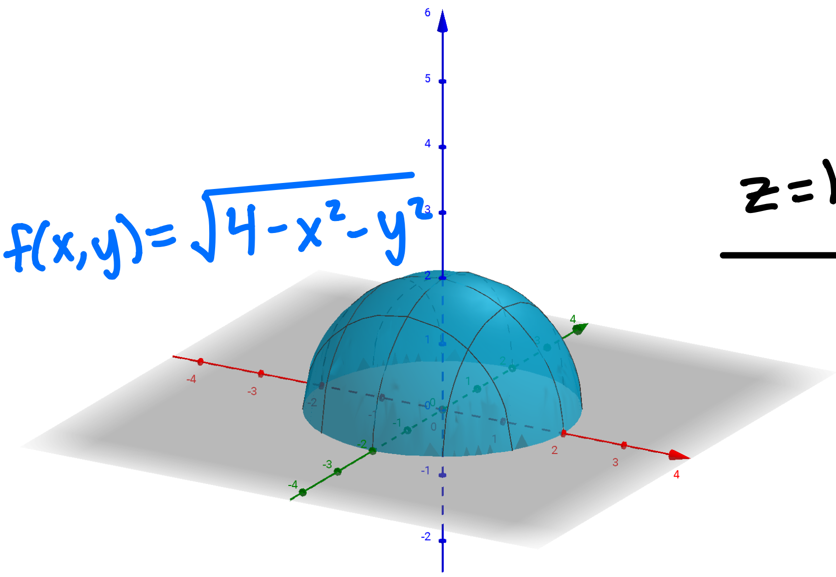
$$\vec{u} = \frac{\langle 1,2 \rangle}{|\langle 1,2 \rangle|} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

and the value is $|\nabla f(2,0)| = |\langle 1,2 \rangle| = \sqrt{5}$

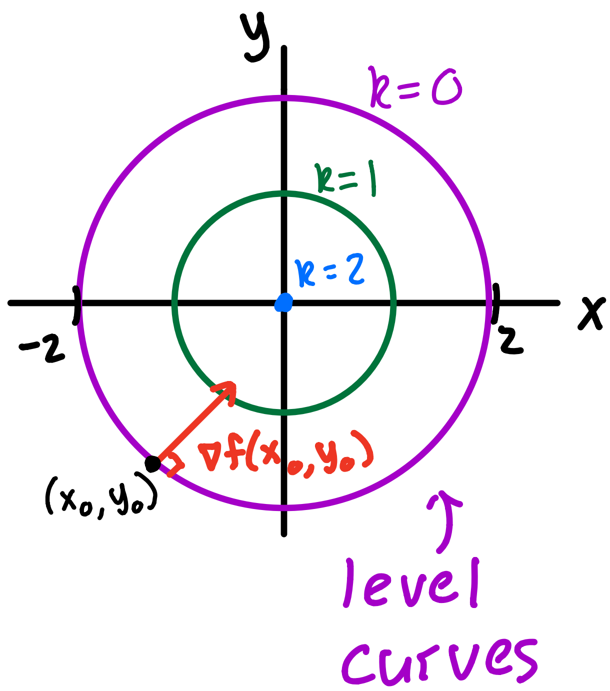
Important Fact

The gradient $\nabla f(x_0, y_0)$ is orthogonal to the <u>level curve</u> $f(x,y) = k$ at the point (x_0, y_0).
$z = k$

Similarly $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface $f(x,y,z) = k$.



$\xrightarrow{z=k}$



Practice Problems

1) Let $f(x,y) = e^{xy} + 2xy^2 - x^3y$.

a) Find the gradient $\nabla f(0,2)$.

b) Let $\vec{u}$ be the unit vector in the direction of $\langle -1,-3 \rangle$. Find $D_{\vec{u}} f(0,2)$.

c) Find a unit vector $\vec{v}$ for which $D_{\vec{v}} f(0,2)$ is maximized.

Solutions

1) $f(x,y) = e^{xy} + 2xy^2 - x^3y$

a) $\nabla f = \langle f_x, f_y \rangle$

$$= \langle ye^{xy} + 2y^2 - 3x^2y,\ xe^{xy} + 4xy - x^3 \rangle$$

$$\nabla f(0,2) = \langle 2+8-0,\ 0+0-0 \rangle$$

$$= \underline{\langle 10, 0 \rangle}$$

b) $$\vec{u} = \frac{\langle -1,-3 \rangle}{|\langle -1,-3 \rangle|} = \frac{1}{\sqrt{10}} \langle -1, -3 \rangle$$

$$D_{\vec{u}} f(0,2) = \nabla f(0,2) \cdot \vec{u}$$

$$= \langle 10, 0 \rangle \cdot \frac{1}{\sqrt{10}} \langle -1, -3 \rangle$$

$$= \frac{1}{\sqrt{10}} (-10 + 0)$$

$$= \frac{-10}{\sqrt{10}} = \underline{-\sqrt{10}}$$

c) Need $\vec{v}$ to be in the direction of

$\nabla f(0,2) = \langle 10,0 \rangle$.

So $\vec{v} = \dfrac{\langle 10,0\rangle}{|\langle 10,0\rangle|} = \underbrace{\langle 1,0\rangle}_{i}$ will

maximize $D_{\vec{v}} f(0,2)$.

Suggested Textbook Exc (14.6)

8, 10, 12, 21, 30

Midterm 3/5

5:00pm - 9:00pm