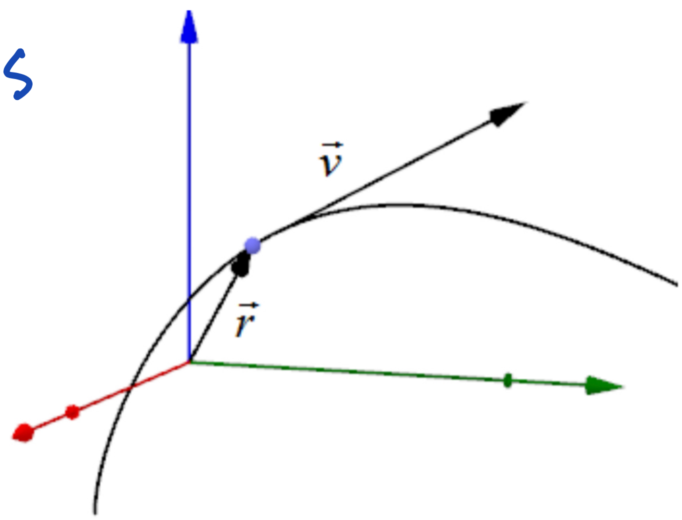
13.4 Motion in Space

Def 1 (Velocity and Acceleration)

Let $\vec{r}(t)$ be a vector function which gives the position of an object in space.



Then $\vec{v}(t) = \vec{r}\,'(t)$ is the velocity vector and points in the direction of the tangent line.

The speed of the object at time t is given by the magnitude of $\vec{v}$

$$s(t) = |\vec{v}(t)| = |\vec{r}\,'(t)|$$

The acceleration at time t is

$$\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$$

Ex 1

The position vector of a moving object is $\vec{r}(t) = \langle t^2+1,\ t-4,\ t^{3/2} \rangle$.

Find its velocity, speed, and acceleration at $t=1$.

$$\vec{v}(t) = \vec{r}\,'(t) = \langle 2t,\ 1,\ \tfrac{3}{2}t^{1/2} \rangle$$

$$s(t) = |\vec{v}(t)| = \sqrt{(2t)^2 + 1^2 + (\tfrac{3}{2}t^{1/2})^2}$$

$$= \sqrt{4t^2 + 1 + \tfrac{9}{4}t}$$

$$\vec{a}(t) = \vec{v}\,'(t) = \langle 2, 0, \tfrac{3}{4}t^{-1/2} \rangle$$

$\underline{t=1:}$

$$\vec{v}(1) = \langle 2, 1, \tfrac{3}{2} \rangle$$

$$s(1) = |\langle 2, 1, \tfrac{3}{2} \rangle| = \sqrt{4 + 1 + \tfrac{9}{4}} = \frac{\sqrt{29}}{2}$$

$$\vec{a}(1) = \langle 2, 0, \tfrac{3}{4} \rangle$$

Ex 2

Find the velocity and position of a particle with $\vec{r}(0) = \langle 0, -1, 4 \rangle$ and $\vec{v}(0) = \langle 0, -1, 0 \rangle$ where $\vec{a}(t) = \sin(t)i + 2\cos(t)j + 6tk$.

Sol

We are given acceleration so we to use integration to work our way back to the position.

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle -\cos t, 2\sin t, 3t^2 \rangle + \vec{c}$$

We have,

$$\vec{v}(0) = \langle -1, 0, 0 \rangle + \vec{c} = \langle 0, -1, 0 \rangle$$

$$\Rightarrow \begin{cases} -1 + c_1 = 0 \\ 0 + c_2 = -1 \\ 0 + c_3 = 0 \end{cases} \Rightarrow \vec{c} = \langle 1, -1, 0 \rangle$$

So

$$\vec{v}(t) = \langle -\cos t + 1, 2\sin t - 1, 3t^2 \rangle$$

Repeating these steps,

$$\vec{r}(t) = \int \vec{v}(t)\,dt$$

$$= \langle -\sin t + t, -2\cos t - t, t^3 \rangle + \vec{D}$$

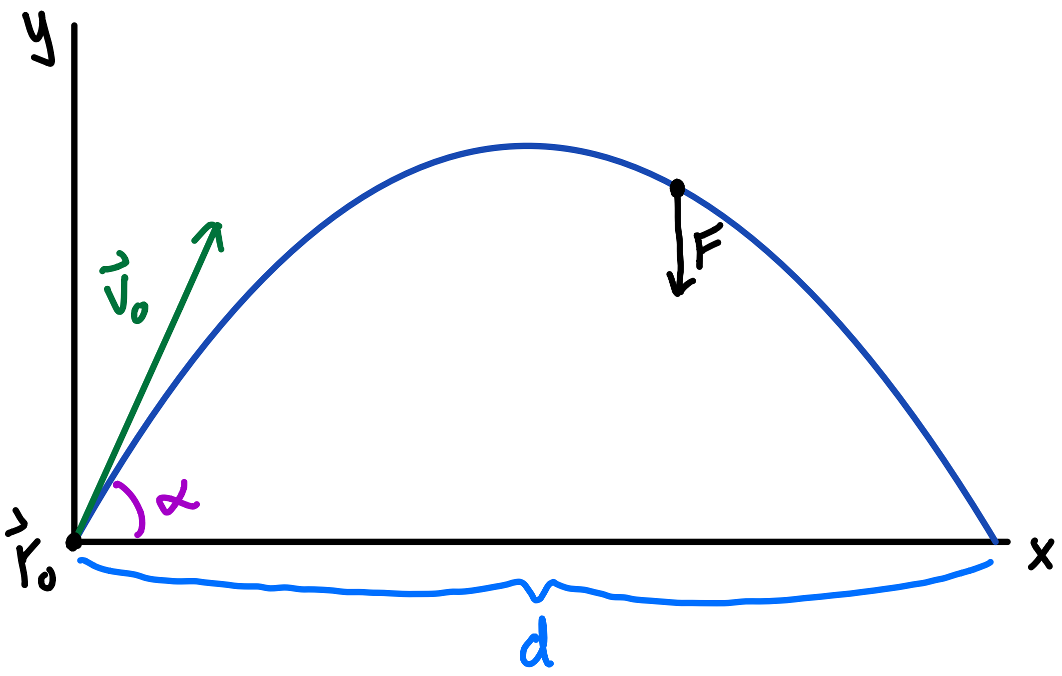
$$\vec{r}(0) = \langle 0, -2, 0 \rangle + \vec{D} = \langle 0, -1, 4 \rangle$$

$$\Rightarrow \vec{D} = \langle 0, 1, 4 \rangle$$

$$\vec{r}(t) = \langle -\sin t + t, -2\cos t - t + 1, t^3 + 4 \rangle$$

Projectile Motion

Consider the situation where a particle is fired with angle of elevation α, initial velocity $\vec{v}_0$, and initial position $\vec{r}_0$.



Assuming air resistance is negligible and gravity is the only outward force, our goal is to find $\vec{r}(t)$.

By Newton Second Law,

$$F = ma = m(-gj)$$, where g is gravity

$(g = 9.8\ m/sec^2)$

Thus $\vec{a}(t) = -gj = \langle 0, -g \rangle$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \int \langle 0, -g \rangle\,dt$$

$$\vec{v}(t) = \langle 0, -gt \rangle + \vec{C}$$

But $\vec{C} = \vec{v}(0) = \vec{v}_0$, so

$$\vec{v}(t) = \langle 0, -gt \rangle + \vec{v}_0$$

Integrating again,

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \int \langle 0, -gt \rangle + \vec{v}_0\,dt$$

$$= \langle 0, -\tfrac{1}{2}gt^2 \rangle + \vec{v}_0 t + \underset{\vec{r}_0}{\underline{\vec{D}}}$$

Now let $v_0 = |\vec{v}_0|$ (initial speed of object).

Then using the direction angle we can write,

$$\vec{v}_0 = \langle v_0 \cos\alpha , v_0 \sin\alpha \rangle$$

So position eqn. becomes,

$$\vec{r}(t) = \langle 0, -\tfrac{1}{2}gt^2 \rangle + t\langle v_0\cos\alpha, v_0\sin\alpha \rangle + \vec{r}_0$$

$$= \langle (v_0\cos\alpha)t, (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 \rangle + \vec{r}_0$$

Suppose $\vec{r}_0 = \langle 0, r_0 \rangle$ then we have,

$$\vec{r}(t) = \langle (v_0\cos\alpha)t, (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 + r_0 \rangle$$

So the para. eqns of the trajectory are,

$$x = (v_0\cos\alpha)t \qquad y = (v_0\sin\alpha)t - \tfrac{1}{2}gt^2 + r_0$$

To find the horizontal distance travelled we must find x when $y = 0$.

Suppose $r_0 = 0$, then solving $y = 0$:

$$(v_0 \sin\alpha)t - \frac{1}{2}gt^2 = 0$$

$$t(v_0 \sin\alpha - \frac{1}{2}gt) = 0$$

$$\Rightarrow t = 0, \quad v_0 \sin\alpha - \frac{1}{2}gt = 0$$

$$\Rightarrow t = \frac{2v_0 \sin\alpha}{g}$$

Therefore

$$d = v_0 \cos\alpha \left(\frac{2v_0 \sin\alpha}{g}\right) = \frac{2\sin\alpha\cos\alpha\, v_0^2}{g}$$

$$= \frac{\sin(2\alpha)\, v_0^2}{g}$$

Notice the value of α that maximizes d is $\alpha = \frac{\pi}{4}$ or $45°$.

Ex 3

A projectile is fired from ground level with initial speed 200 m/sec and angle of elevation 60°. Find

a) the horizontal distance travelled.

b) the maximum height reached.

c) the speed at impact.

Sol

Fired from ground so

$$\vec{r}_0 = \langle 0, 0 \rangle \quad \text{and} \quad v_0 = |\vec{v}_0| = 200 \text{ m/s}$$

Para. eqns for trajectory are

$$x = (v_0 \cos\alpha)t \qquad y = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2 + r_0$$

Since $g = 9.8\ m/s$ we have,

$$x = 200\cos(60°)t \qquad y = 200\sin(60°)t - \tfrac{1}{2}gt^2$$

$$= 100t \qquad\qquad = 100\sqrt{3}\,t - 4.9t^2$$

a) Solving for $y=0$:

$$t(100\sqrt{3} - 4.9t) = 0$$

$$\Rightarrow t = 0, \quad t = \frac{100\sqrt{3}}{4.9} \approx 35.35$$

So $d \approx x(35.35) = 100(35.35) = 3{,}535\ m$

b) To find max height we must maximize

$$y = 100\sqrt{3}\,t - 4.9t^2$$

(use derivative to find critical points)

$$y' = 100\sqrt{3} - 9.8t = 0$$

$$\Rightarrow t = \frac{100\sqrt{3}}{9.8} \approx 17.67$$

Max height:

$$y(17.67) = 100\sqrt{3}(17.67) - 4.9(17.67)^2$$
$$= 1530.61 \text{ m}$$

c) Impact is at $t = 35.35$ so speed at impact is

$$s(35.35) = |\vec{v}(35.35)| = |\vec{r}\,'(35.35)|$$

We have,

$$\vec{r}(t) = \langle 100t,\ 100\sqrt{3}t - 4.9t^2 \rangle$$

$$\vec{r}\,'(t) = \langle 100,\ 100\sqrt{3} - 9.8t \rangle$$

So

$$s(35.35) = |\vec{r}\,'(35.35)| = |\langle 100, -173.22\rangle|$$

$$= \sqrt{100^2 + (173.22)^2} \approx 200.02 \text{ m/s}$$

Practice problem

1) An object's acceleration is $\vec{a}(t) = \langle 3t, -4e^{-t}, 12t^2 \rangle$. Determine the velocity and position function with initial conditions $\vec{v}(0) = \langle 0, 1, -3 \rangle$ and $\vec{r}(0) = \langle -5, 2, -3 \rangle$.

2) A projectile is fired from 5m above ground with initial speed 100 m/s at an angle of 45°. Find where it hits the ground and its speed at impact.

Hint: 2) will require the quadratic formula:

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Solutions

1) $\vec{a}(t) = \langle 3t, -4e^{-t}, 12t^2 \rangle$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle \tfrac{3}{2}t^2, 4e^{-t}, 4t^3 \rangle + \vec{C}$$

$$\vec{v}(0) = \langle 0, 4, 0 \rangle + \vec{C} = \langle 0, 1, -3 \rangle$$

$$\Rightarrow \vec{C} = \langle 0, -3, -3 \rangle$$

So $\vec{v}(t) = \underline{\langle \tfrac{3}{2}t^2, 4e^{-t} - 3, 4t^3 - 3 \rangle}$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle \tfrac{1}{2}t^3, -4e^{-t} - 3t, t^4 - 3t \rangle + \vec{D}$$

$$\vec{r}(0) = \langle 0, -4, 0 \rangle + \vec{D} = \langle -5, 2, -3 \rangle$$

$$\Rightarrow \vec{D} = \langle -5, 6, -3 \rangle$$

Thus

$$\vec{r}(t) = \underline{\left\langle \frac{1}{2}t^3 - 5, -4e^{-t} - 3t + 6, t^4 - 3t - 3 \right\rangle}$$

2) $\vec{r}_0 = \langle 0, 5 \rangle$, $v_0 = |\vec{v}_0| = 100$, $\alpha = 45°$

Para eqns. of trajectory:

$$x = 100\cos(45°)t \qquad y = 100\sin(45°)t - gt^2 + 5$$
$$= 50\sqrt{2}t \qquad = 50\sqrt{2}t - 4.9t^2 + 5$$

<u>Solving $y=0$:</u>

$$0 = -4.9t^2 + 50\sqrt{2}t + 5$$

$$\Rightarrow t = \frac{-50\sqrt{2} \pm \sqrt{(50\sqrt{2})^2 - 4(4.9)(5)}}{2(4.9)}$$

quadratic formula

$$= \frac{-50\sqrt{2} \pm \sqrt{4902}}{9.8} \Rightarrow t = 14.36 \text{ or } \cancel{-0.07}$$

$t \geq 0$

So $d = x(14.36) = 50\sqrt{2}(14.36)$

$\approx \underline{1015.41\ m}$

<u>Speed at impact</u> $(t = 14.36)$

$s(14.36) = |\vec{v}(14.36)| = |\vec{r}\,'(14.36)| \ominus$

$\vec{r}(t) = \langle 50\sqrt{2}t,\ 50\sqrt{2}t - 4.9t^2 + 5 \rangle$

$\vec{r}\,'(t) = \langle 50\sqrt{2},\ 50\sqrt{2} - 9.8t \rangle$

$\ominus\ |\langle 50\sqrt{2}, -70.02 \rangle| = \sqrt{(50\sqrt{2})^2 + (-70.02)^2}$

$\approx \underline{99.51\ m/s}$

<u>Suggested Text Excs.</u> (13.4)

10, 12, 15, 18a, 25, ~~37~~ 31

<u>Quiz 5</u>

13.2, 13.3, 13.4