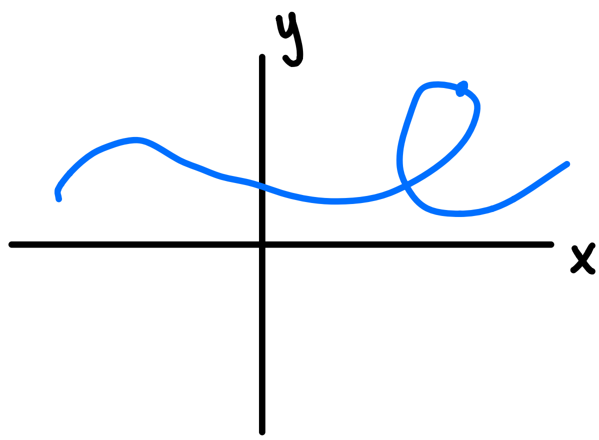
10.1 Curves Defined by Parametric Equations



$y = f(x)$

$x = g(y)$

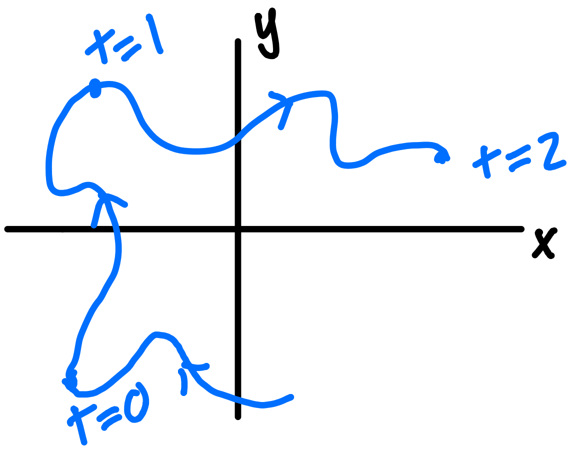
Suppose x and y are functions of a third variable t (called a parameter) given by the equations

$$x = f(t) \qquad y = g(t)$$

called parametric equations.

Each value of t determines a point $(x, y) = (f(t), g(t))$.

As t varies these points trace out a curve C, which is called a parametric curve.



Ex 1 Sketch and identify the curve given by para. eqns.

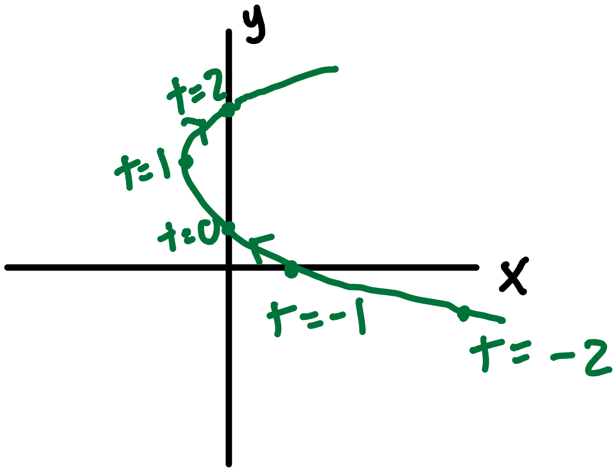
$$x = t^2 - 2t \qquad y = t + 1$$
$$= t(t-2)$$

x,y-int

$t = -1, 0, 2$

Sol

t	x	y
-2	8	-1
-1	3	0
0	0	1
1	-1	2
2	0	3



3 | 3 | 4

Eliminating the parameter:

$x = t^2 - 2t \qquad y = t + 1$

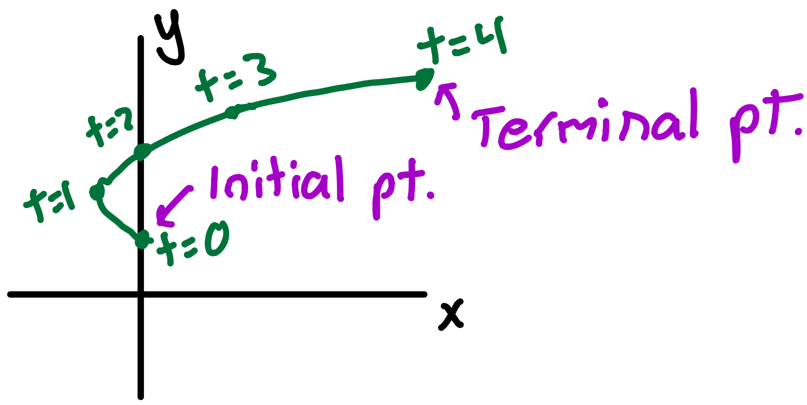
$\Rightarrow t = y - 1$ subsititute

$x = (y-1)^2 - 2(y-1)$

$x = y^2 - 4y + 1$

In Ex 1 no restriction was placed on parameter t, but it can be restricted to an interval.

$x = t^2 - 2t \quad y = t+1 \;, \; 0 \le t \le 4$



In general,

$$x = f(t) \quad y = g(t) \,, \; a \le t \le b$$

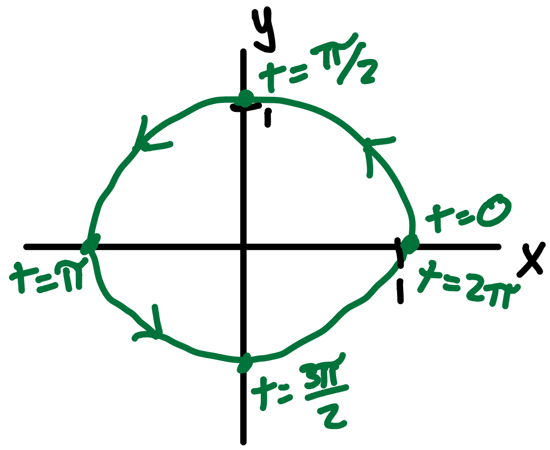
has intitial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

Ex 2 What curve is represented by the following para. equations

$$x = \cos t \quad y = \sin t \quad 0 \le t \le 2\pi$$

Sol

t	x	y
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\vdots$		
2π	1	0



To eliminate the parameter consider the identity

$$\cos^2 t + \sin^2 t = 1$$

$$\Rightarrow \underline{x^2 + y^2 = 1}$$

In general the para. eqns. for a circle centered at (h, k) with radius r are given by

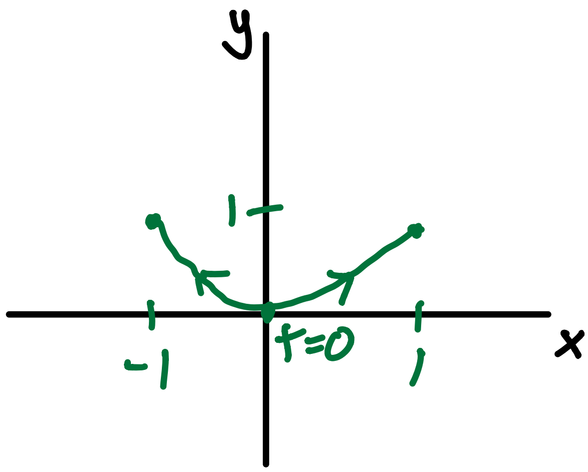
$$x = h + r\cos t \quad y = k + r\sin t, \quad 0 \le t \le 2\pi$$

Ex 3 Sketch the curve with para. eqns.

$$x = \sin t \qquad y = \sin^2 t$$

Sol

$$y = (\sin t)^2 = x^2$$



Be Careful!

Since $-1 \le \sin t \le 1$

$$\Rightarrow -1 \le x \le 1$$

Graphing using Parametric Eqns

Use a graphing device to graph $x = y^4 - 3y^2$.

If we let parameter $t = y$ then we have

$$x = t^4 - 3t^2 \qquad y = t$$

Practice Problems

1) Sketch the curve given by

$$x = t^3 + t \qquad y = t^2 + 2 \quad , \quad -2 \le t \le 2$$

2) $x = e^t \qquad y = e^{-2t}$

a) Eliminate the parameter to find a Cartesian eqn. of the curve

b) Sketch the curve

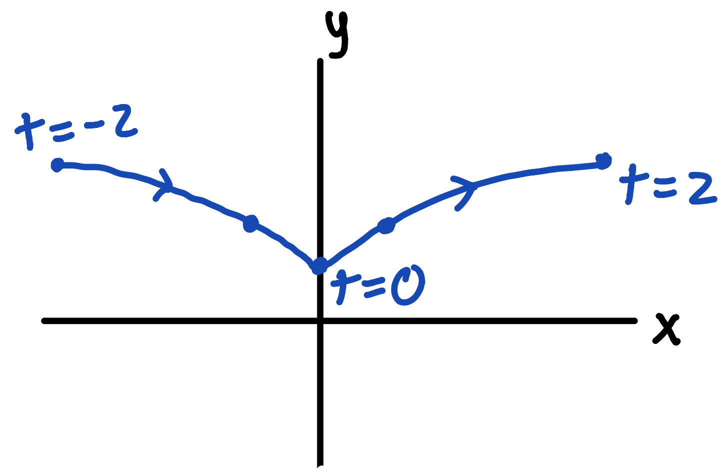
3) Eliminate the parameter

$$x = 3 + 2\sin t \qquad y = -1 + 4\cos t$$

Solutions

1) $x = t^3 + t \quad y = t^2 + 2 \;, \quad -2 \le t \le 2$

t	x	y
-2	-10	6
-1	-2	3
0	0	2
1	2	3
2	10	6



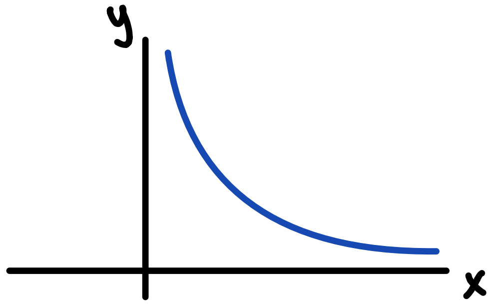
2) $x = e^t \quad y = e^{-2t}$

a) Eliminate parameter:

$$y = e^{-2t} = (e^t)^{-2} = x^{-2} = \frac{1}{x^2}$$

So $y = \frac{1}{x^2}$ but since $e^t > 0$ we must also have $x > 0$.

b)



$y = \frac{1}{x^2}, \; x > 0$

3) $x = 3 + 2\sin t \quad y = -1 + 4\cos t, \quad 0 \le t \le \pi$

We can use identity $\cos^2 t + \sin^2 t = 1$ by solving each eqn $\cos t$ and $\sin t$.

$x = 3 + 2\sin t \qquad\qquad y = -1 + 4\cos t$

$\Rightarrow \sin t = \frac{x-3}{2} \qquad\qquad \Rightarrow \cos t = \frac{y+1}{4}$

So

$$\sin^2 t + \cos^2 t = 1$$

$$\left(\frac{x-3}{2}\right)^2 + \left(\frac{y+1}{4}\right)^2 = 1$$

Additional Exercises (10.1)

7, 9, 11, 20