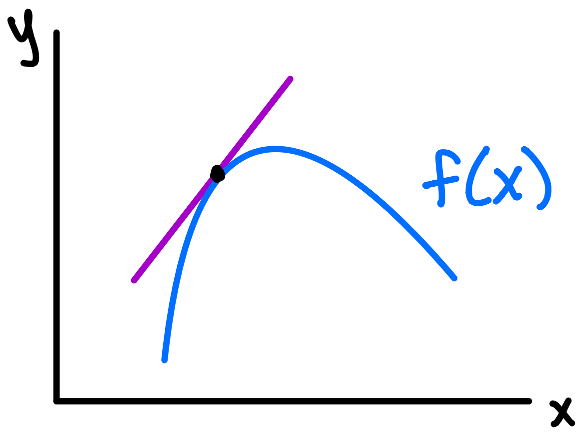
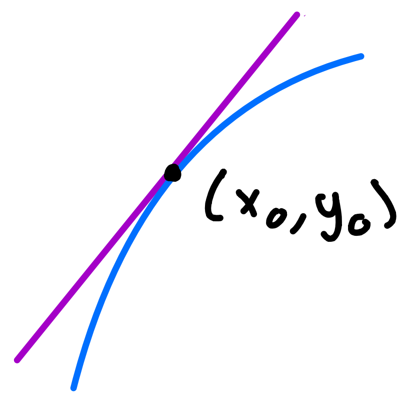
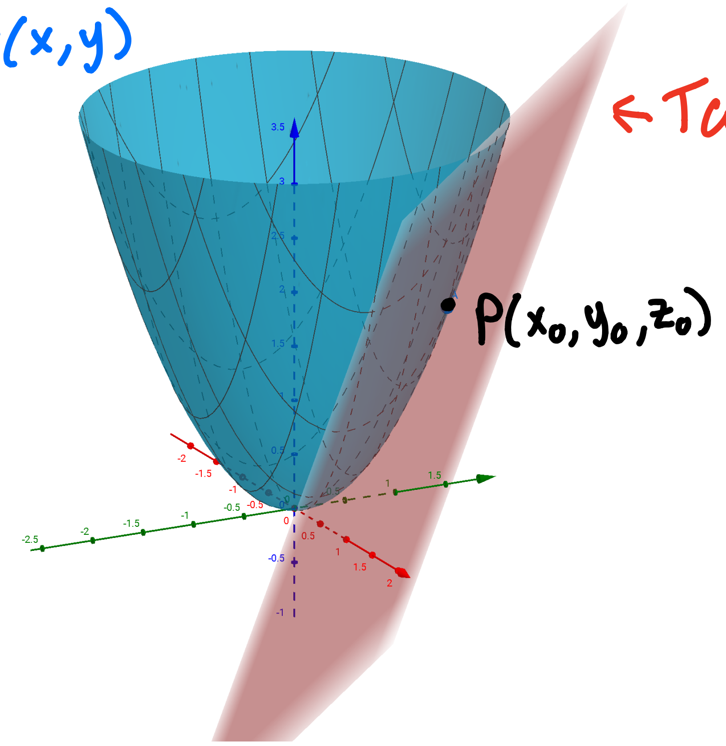
14.4 Tangent Planes and Linearization

Zoom in on a single variable function and it looks like a line. We call this the tangent line.





Functions of two variables are surfaces in $\mathbb{R}^3$ so when you zoom in at a point (x_0, y_0, z_0) it will look like a plane.



Def 1 (Eqn of Tangent Plane)

The tangent plane of the surface $z=f(x,y)$ at the point $P(x_0,y_0,z_0)$ is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Compare this to the tangent line eqn of the curve $y=f(x)$ at point (x_0,y_0),

$$y-y_0=f'(x_0)(x-x_0)$$

Ex 1 Find the tangent plane of $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Sol

Let $f(x,y) = 2x^2 + y^2$.

$f_x = 4x \qquad f_y = 2y$

$f_x(1,1) = 4 \qquad f_y(1,1) = 2$

So eqn of tangent plane is

$$z - 3 = 4(x-1) + 2(y-1)$$

$$z = 4x + 2y - 3$$

Ex 2 Find the tangent plane of $z = x^2 + x\cos(y)$ at point $(\pi, \frac{\pi}{2})$.

Sol

1. Find f_x and $f_x(\pi, \frac{\pi}{2})$.

$$f_x = 2x + \cos(y)$$
$$f_x(\pi, \tfrac{\pi}{2}) = 2\pi + \cos(\tfrac{\pi}{2}) = 2\pi$$

2. Find f_y and $f_y(\pi, \frac{\pi}{2})$.

$$f_y = -x\sin(y)$$
$$f_y(\pi, \tfrac{\pi}{2}) = -\pi\sin(\tfrac{\pi}{2}) = -\pi$$

3. Find z_0.

$$f(\pi, \tfrac{\pi}{2}) = \pi^2 + \pi\cos(\tfrac{\pi}{2}) = \pi^2$$

Eqn of tangent plane:

$$z - \pi^2 = 2\pi(x - \pi) - \pi(y - \tfrac{\pi}{2})$$

14.5 The Chain Rule

For functions of one variable we use chain rule to differentiate composite functions.

Let $y = f(x)$ and $x = g(t)$, here

y is indirectly a function of t

$$y = f(x) = f(g(t))$$

$$\frac{dy}{dt} \quad \begin{array}{c} y \\ | \\ x \\ | \\ t \end{array} \begin{array}{c} \frac{dy}{dx} \\ \\ \frac{dx}{dt} \end{array}$$

By Chain Rule,

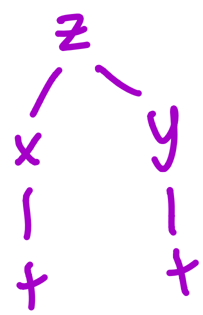
$$\frac{dy}{dt} = f'(g(t)) \cdot g'(t) = f'(x) \cdot g'(t)$$

$$= \frac{dy}{dx} \cdot \frac{dx}{dt}$$

For functions of several variables we have the following cases.

Chain Rule Case 1 (one parameter)

Suppose $z=f(x,y)$ is differentiable in x and y where $x=g(t)$ and $y=h(t)$ are diff. functions of t. Then



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Ex 1

Let $z = x^2y + 3xy^4$, $x = \sin(2t)$, and $y=\cos(t)$. Find $\frac{dz}{dt}$ when $t=0$.

Sol

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} =$$

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[d, dash] & & y \arrow[d, dash] \\ t & & t \end{tikzcd}$$

$$\frac{\partial z}{\partial x} = 2xy + 3y^4 \qquad \frac{\partial z}{\partial y} = x^2 + 12xy^3$$

$$\frac{dx}{dt} = 2\cos(2t) \qquad \frac{dy}{dt} = -\sin(t)$$

$$= (2xy + 3y^4)2\cos(2t) - (x^2 + 12xy^3)\sin t$$

$t=0$

$$x = \sin(0) = 0$$

$$y = \cos(0) = 1$$

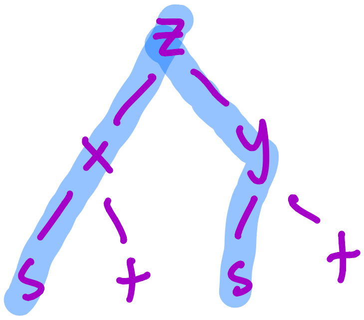
$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)2\cos(0) - (0+0)\sin(0)$$

$$= 6$$

Chain Rule Case 2 (two parameters)

Let $z = f(x,y)$, $x = g(s,t)$, $y = h(s,t)$.

Assuming everthing is differentiable,



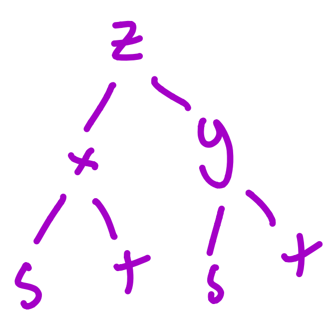
$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

Ex 2

Let $z = e^x \sin y$, $x = st^2$, and $y = s^2 t$.

Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.



Sol

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (e^x \sin y)(t^2) + (e^x \cos y)(2st)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= (e^x \sin y)(2st) + (e^x \cos y)(s^2)$$

General Chain Rule

Suppose z is a diff. function of $x_1, x_2, \cdots, x_n$ with each x_j a diff. function of $t_1, t_2, \cdots, t_m$. Then

$$\frac{\partial z}{\partial t_i} = \frac{\partial z}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_i} + \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial z}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_i}$$

for $i = 1, 2, \cdots, m$.

Implicit Differentiation (2 cases)

Case 1

Suppose $y=f(x)$ is defined implicitly by $F(x,y)=0$. Then by Chain Rule,

$$\frac{\partial F}{\partial x}\cdot\underbrace{\frac{\partial x}{\partial x}}_{=1}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0$$

Solving for $\frac{dy}{dx}$,

$$\frac{dy}{dx}=\frac{-F_x}{F_y}$$

where $F_x=\frac{\partial F}{\partial x}$ and $F_y=\frac{\partial F}{\partial y}$.

Case 2

Suppose $z=f(x,y)$ is given implicitly by $F(x,y,z)=0$. Then

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

Ex 3

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = f(x, y)$ is given by $x^3 + y^3 + z^3 + 6xyz = 1$.

Sol

Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$.

Then

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(3x^2 + 6yz)}{3z^2 + 6xy} = \frac{-x^2 - 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(3y^2 + 6xz)}{3z^2 + 6xy} = \frac{-y^2 - 2xz}{z^2 + 2xy}$$

Practice Problems

1) Find the tangent plane of $f(x,y) = x\sin(x+y)$ at point $(-1,1,0)$.

2) Let $z = x^2 - y^2$ where $x = u^2e^{-v}$ and $y = v\ln(u^2v)$. Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.

3) Let $e^z = xyz$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solutions

1) $f(x,y) = x\sin(x+y)$, $(-1,1,0)$

$$f_x = \sin(x+y) + x\cos(x+y)$$

$$f_x(-1,1) = \sin(0) - 1\cos(0)$$
$$= -1$$

$$f_y = x\cos(x+y)$$

$$f_y(-1,1) = -1\cos(0)$$
$$= -1$$

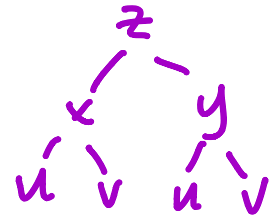
Eqn of tangent plane is

$$z - 0 = -1(x+1) - 1(y-1)$$

or

$$z = -x - y$$

2) $z = x^2 - y^2$, $x = u^2 e^{-v}$, $y = v\ln(u^2 v)$



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$= (2x)(2ue^{-v}) + (-2y)\left(\frac{v}{u^2v}\cdot 2uv\right)$$

$$= 4xue^{-v} - \frac{4yv}{u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= (2x)(-u^2e^{-v}) + (-2y)\left[\ln(u^2v) + \frac{v}{u^2v}\cdot u^2\right]$$

$$= -2xu^2e^{-v} - 2y\ln(u^2v) - 2y$$

3) $e^z = xyz$.

Let $F(x,y,z) = xyz - e^z = 0$.

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = -\frac{yz}{xy-e^z}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = -\frac{xz}{xy-e^z}$$

Suggested Textbook Exc.

14.4

1, 3, 4, 19

14.5

3, 8, 14, 21, 32