14.3 Partial Derivatives

Let $f(x,y)$ be a function of 2 variables.

Suppose we let y be fixed and let x vary. Then

$$g(x) = f(x,y)$$

If $g'(x)$ exists then we call it the partial derivative of f with respect to x and denote it $f_x(x,y)$.

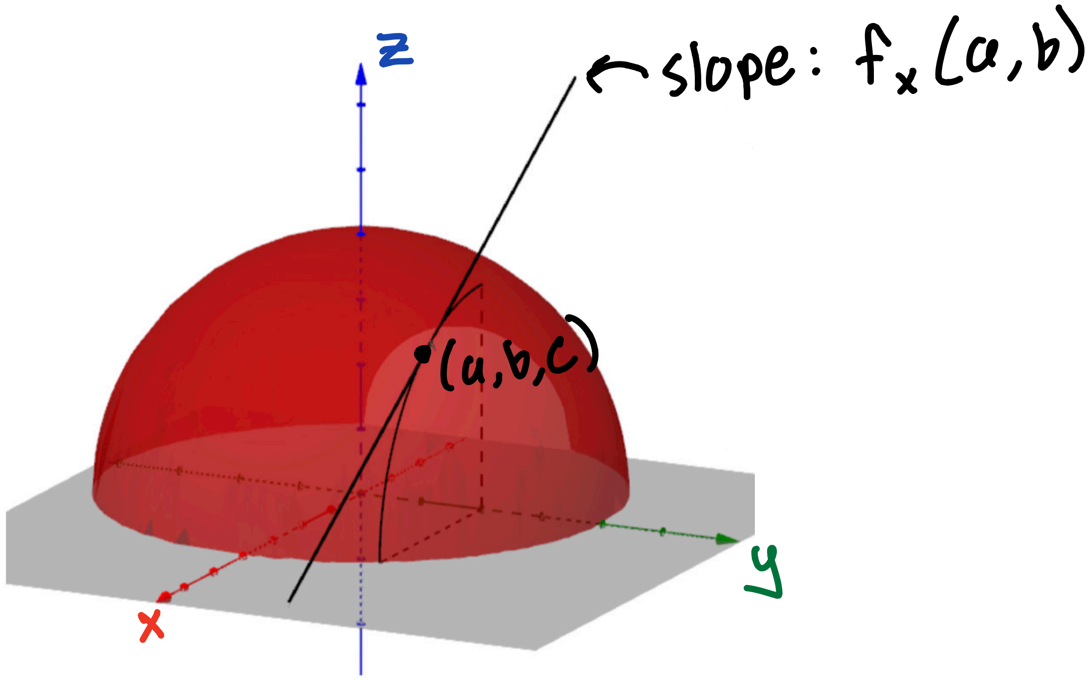
From Calc I,

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

since $g(x) = f(x,y)$ this becomes

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

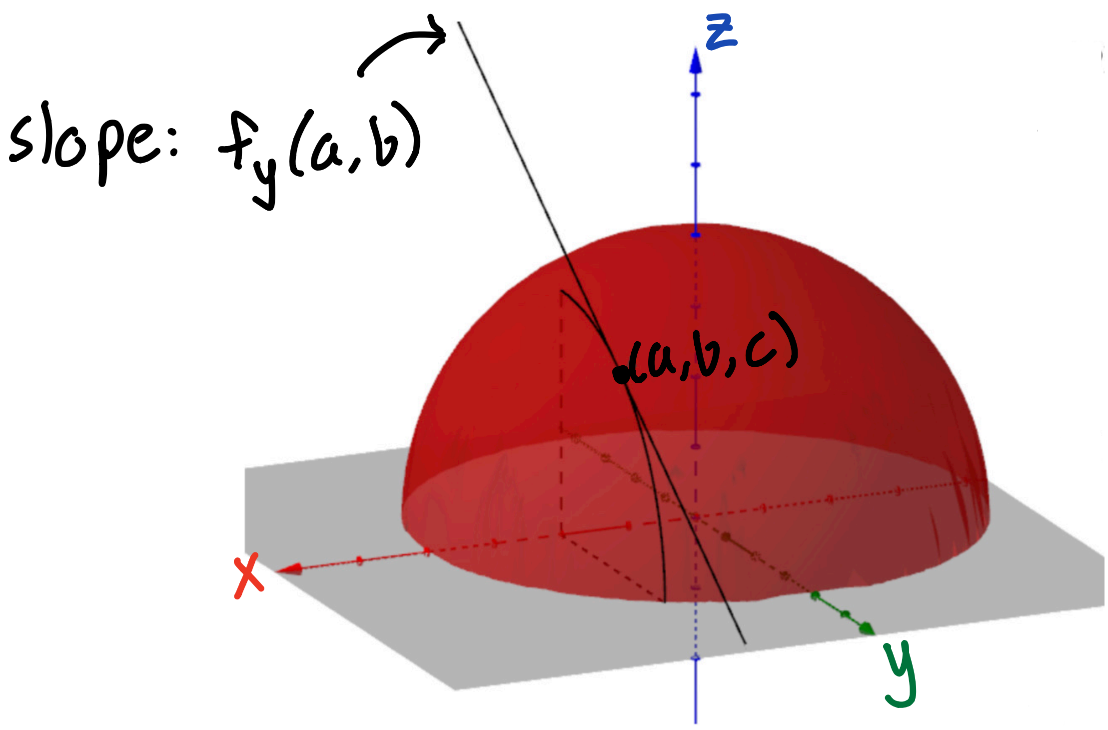
This represents the variation of f in the x-direction.



We can define the partial derivative with respect to y in a similar way.

Treating x as constant and letting y vary we have,

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h)-f(x,y)}{h}$$



All of our usual differentiation rules (chain, product, quotient) will apply to partial dereviatives.

Notations

$$f'(x) = \frac{df}{dx}$$

Let $z = f(x,y)$.

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = D_y f$$

Ex 1 Let $f(x,y) = x^3 + 2x^2y^3 - 4y^2 + 6$.

Find f_x, f_y, $f_x(1,2)$, and $f_y(1,2)$.

Sol

To find f_x we must treat y as a constant and differentiate with respect to x.

$f_x = 3x^2 + 4xy^3$

$f_x(1,2) = 3 + 4(1)(2)^3 = 35$

Treating x as constant,

$f_y = 6x^2y^2 - 8y$

$f_y(1,2) = 6(2)^2 - 8(2) = 8$

<u>Ex 2</u> Let $f(x,y) = \sin\left(\frac{x}{1+y}\right)$.

Find f_x and f_y.

<u>Sol</u>

$$f_x = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left[\frac{x}{1+y}\right]$$

chain rule

$$= \cos\left(\frac{x}{1+y}\right) \cdot \left(\frac{1}{1+y}\right)$$

$$f_y = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left[\frac{x}{1+y}\right]$$

$$= \cos\left(\frac{x}{1+y}\right) x \cdot \frac{\partial}{\partial y}\left[(1+y)^{-1}\right]$$

$$= \cos\left(\frac{x}{1+y}\right) x \cdot \left(-1(1+y)^{-2}\right)$$

$$= \cos\left(\frac{x}{1+y}\right)\left(\frac{-x}{(1+y)^2}\right)$$

Partial derivatives work the same way for functions of more than two variables. The key is to focus on a single variable and treat all the other variables as constants.

<u>Ex 3</u> Let $f(x,y,z) = e^{xyz}\ln(z)$.

Find f_x, f_y and f_z.

Sol

$$f_x = yze^{xyz} \cdot \ln(z)$$

$$f_y = xze^{xyz} \cdot \ln(z)$$

product rule

$$f_z = xye^{xyz} \cdot \ln(z) + e^{xyz} \cdot \frac{1}{z}$$

$$= e^{xyz}\left(xy\ln(z) + \frac{1}{z}\right)$$

Higher Order Partials

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Ex 4 Find all of the second order partial derivatives of

$$f(x,y) = 2x^2y - y^3 + 3x^2y^2 + 7xy.$$

Sol

Start by finding the first order partials,

$$f_x = 4xy + 6xy^2 + 7y$$

$$f_y = 2x^2 - 3y^2 + 6x^2y + 7x$$

$$f_{xx} = 4y + 6y^2$$

$$f_{xy} = 4x + 12xy + 7$$

$$f_{yx} = 4x + 12xy + 7$$

$$f_{yy} = -6y + 6x^2$$

Clairaut's Theorem

Let $f(x,y)$ be a function. If f_{xy} and f_{yx} are both continuous, then

$$f_{xy} = f_{yx}$$

This also applies to higher order partials. For example,

$$f_{xxy} = f_{xyx} = f_{yxx}$$

Practice Problems

1) Let $f(x,y) = y \cdot \ln(3x - y^2)$.

Find f_x and f_y.

2) Find all the second order partials of $f(x,y) = 2xy^4 - 3x^2y^3 + 8\sqrt{xy}$

Solutions

1) $f(x,y) = y\ln(3x - y^2)$

$$f_x = y \cdot \frac{1}{3x - y^2} \cdot \underbrace{\frac{\partial}{\partial x}\left[3x - y^2\right]}_{=3}$$

$$= \frac{3y}{3x - y^2}$$

$$f_y = \ln(3x-y^2) + y \cdot \frac{1}{3x-y^2} \cdot \underbrace{\frac{\partial}{\partial y}(3x-y^2)}_{=-2y}$$

product rule

$$= \ln(3x-y^2) - \frac{2y^2}{3x-y^2}$$

2) $f(x,y) = 2xy^4 - 3x^2y^3 + 8\sqrt{xy}$

$(xy)^{1/2}$

$$f_x = 2y^4 - 6xy^3 + 4y(xy)^{-1/2}$$

$$f_y = 8xy^3 - 9x^2y^2 + 4x(xy)^{-1/2}$$

$$f_{xx} = -6y^3 - 2y^2(xy)^{-3/2}$$

$$f_{xy} = 8y^3 - 18xy^2 + 4(xy)^{-1/2} - 2xy(xy)^{-3/2}$$

$$f_{yx} = 8y^3 - 18xy^2 + 4(xy)^{-1/2} - 2xy(xy)^{-3/2}$$

$$f_{yy} = 24xy^2 - 18x^2y - 2x^2(xy)^{-3/2}$$

Suggested Textbook Exc (14.3)

15, 20, 25, 31, 53, 54

Quiz 6

(14.1-14.3)