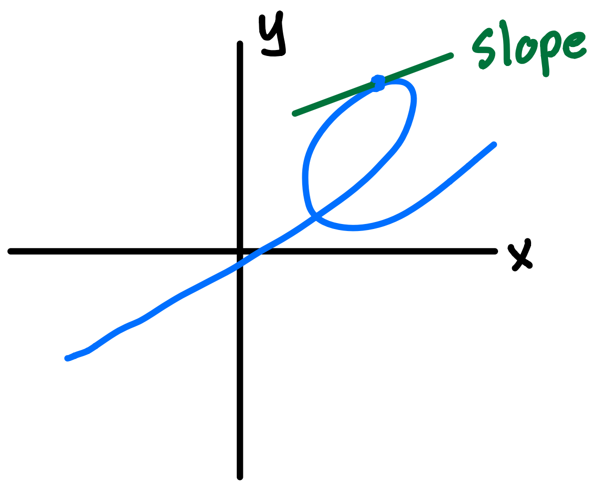
10.2 Calculus with Parametric Curves



$x = f(t) \quad y = g(t)$

$y = h(x)$

$y = h(f(t))$

$$\frac{dy}{dt} = \underline{h'(x)} \cdot f'(t)$$

Suppose f, g are differentiable and we want to find the tangent line at a point on the para. curve $x = f(t)$ $y = g(t)$, where y is also a differentiable function of x. Then by Chain Rule,

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$ we can solve for $\frac{dy}{dx}$,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

This allows us to find the slope $\left(\frac{dy}{dx}\right)$ of the tangent line without having to eliminate the parameter t.

Note 1: When $\frac{dx}{dt} = 0$ we have a vertical tangent, similarly, $\frac{dy}{dt} = 0$ would give a horizontal tangent.

Note 2: $\frac{d^2y}{dx^2} \neq \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}}$

Second Derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)$$

$$= \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Ex 1 Consider the para. curve C given by $x = t^2 \quad y = t^3 - 3t$.

a) Find points on C where the tangent is vertical or horizontal

b) Show that C has two tangents at the point $(3,0)$ and find their equations.

c) Sketch the curve.

Sol

$$x = t^2 \quad y = t^3 - 3t$$

a) Horizontal Tangent (HT): $\frac{dy}{dt} = 0$

$$\frac{dy}{dt} = 3t^2 - 3 = 0$$
$$= 3(t^2 - 1) = 0$$
$$= 3(t+1)(t-1) = 0$$
$$\Rightarrow t = -1, 1$$

$t = -1 \Rightarrow (1, 2)$
$t = 1 \Rightarrow (1, -2)$

> HT

Vertial Tangents (VT): $\frac{dx}{dt} = 0$

$$\frac{dx}{dt} = 2t = 0 \Rightarrow t = 0$$

VT: $t = 0 \Rightarrow (0, 0)$

b) $x = t^2 \quad y = t^3 - 3t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t} = \frac{3}{2}\left(\frac{t^2-1}{t}\right)$$

Notice $t^3 - 3t = 0 \Rightarrow t(t^2 - 3) = 0$

$$\Rightarrow t(t-\sqrt{3})(t+\sqrt{3}) = 0$$

$$\Rightarrow t = 0, \sqrt{3}, -\sqrt{3}$$

So the point $(3,0)$ corresponds to two parameter values $t = -\sqrt{3}, \sqrt{3}$. This means that the curve crosses itself at $(3,0)$.

$t = \sqrt{3}$

$$\left.\frac{dy}{dx}\right|_{t=\sqrt{3}} = \frac{3}{2}\left(\frac{(\sqrt{3})^2 - 1}{\sqrt{3}}\right) = \frac{3}{2}\left(\frac{2}{\sqrt{3}}\right) = \frac{3}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{3\sqrt{3}}{3} = \sqrt{3}$$

Recall pt.-slope

$y - y_0 = m(x - x_0)$

Eqn of tangent line is $y - 0 = \sqrt{3}(x-3)$

$$y = \sqrt{3}(x-3).$$

$t = -\sqrt{3}$

$$\left.\frac{dy}{dx}\right|_{t=-\sqrt{3}} = \frac{3}{2}\left(\frac{(-\sqrt{3})^2 - 1}{-\sqrt{3}}\right) = -\sqrt{3}$$

So the tangent line is

$$y - 0 = -\sqrt{3}(x-3)$$

c) Sketch Curve

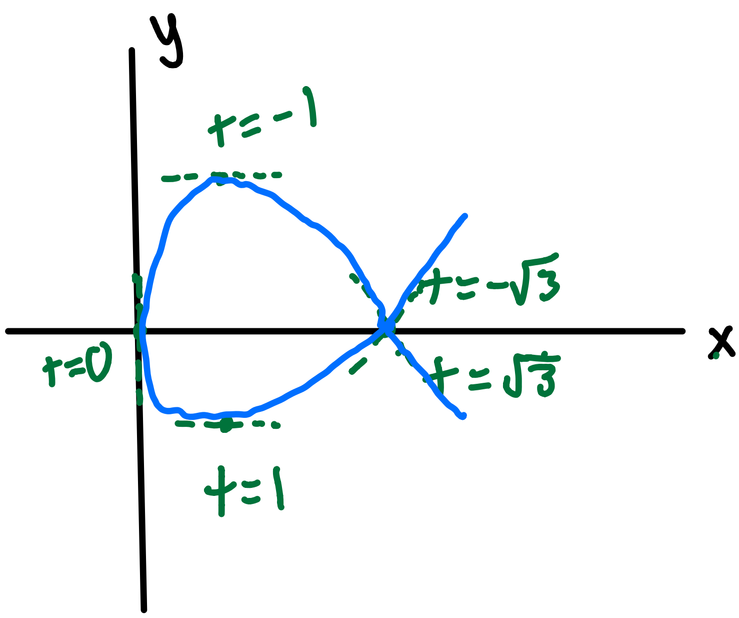
$t=-1$ $t=1$

HT: $(1,2)$, $(1,-2)$

VT: $(0,0) \leftarrow t=0$

$(3,0)$ slope $\sqrt{3}, -\sqrt{3}$

$\uparrow t = \sqrt{3}, -\sqrt{3}$



Ex 2 $\quad x = 1 + \ln t \quad y = t^2 + 2$

a) Find the tangent line at $(1, 3)$ by NOT eliminating the parameter.

b) Find the tangent at $(1, 3)$ by eliminating the parameter.

Sol $\quad x = 1 + \ln t \quad y = t^2 + 2 \qquad (1, 3)$

a)
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{\frac{1}{t}} = 2t^2$$

$3 = t^2 + 2$

$t^2 = 1$

$\Rightarrow t = 1$

$$\left.\frac{dy}{dx}\right|_{t=1} = 2$$

Tangent line: $y - 3 = 2(x - 1)$

b) $x = 1 + \ln t \quad y = t^2 + 2 \quad (1, 3)$

$\ln t = x - 1$

$\Rightarrow \cancel{e}^{\cancel{\ln} t} = e^{x-1}$

$t = e^{x-1}$

$y = (e^{x-1})^2 + 2$

$y = e^{2x-2} + 2$

$$\frac{dy}{dx} = 2e^{2x-2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 2e^{2-2} = 2$$

Tangent line: $y - 3 = 2(x - 1)$

Arc Length

Recall the arc length L of a curve C given by $y = f(x)$, $a \le x \le b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose C can also be described by para. eqns. $x = f(t)$ $y = g(t)$, $\alpha \le t \le \beta$ where $\frac{dx}{dt} = f'(t) > 0$. This means that C is traversed once from left to right as t increases from α to β.

Using $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ and a substitution,

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\, \frac{dx}{dt}\, dt$$

$$= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 \left[1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2\right]}\, dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Arc Length for Para. Equations

Thm If C is a para. curve given by $x = f(t)$ $y = g(t)$, $\alpha \le t \le \beta$, where f', g' are continuous on $[\alpha, \beta]$ and C is traversed only once, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex 3 Find the arc length of

$$x = 1 + 3t^2 \quad y = 4 + 2t^3, \quad 0 \le t \le 1$$

$$\frac{dx}{dt} = 6t \qquad \frac{dy}{dt} = 6t^2$$

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt = \int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$

$$= 6\int_0^1 t\sqrt{1+t^2}\, dt$$

u-sub

$u = 1 + t^2$
$du = 2t\,dt$

$t = 0, \; u = 1$
$t = 1, \; u = 2$

$$= 6\int_1^2 \sqrt{u} \cdot \frac{du}{2}$$

$$= 3\left[\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^2 = \cancel{3}\left[\frac{2}{\cancel{3}} u^{3/2}\right]_1^2 = 2\left(2^{3/2} - 1^{3/2}\right)$$

$$= 2\left(2\sqrt{2} - 1\right) \approx 3.66$$

Practice Problems

1) Find equation of the tangent line corresponding to $t = 4$.

$$x = \sqrt{t} \qquad y = t^2 - 2t$$

2) Find all the points where there is a vertical or horizontal tangent

$$x = 2t^3 - 6t \qquad y = t^3 + 6t^2$$

3) Find the arc length of the para. eqn.

$$x = r\cos\theta \quad y = r\sin\theta, \; 0 \le \theta \le 2\pi, \; (r > 0)$$

to show that the circumference of a circle of radius r is $2\pi r$.

Solutions

1) $x = \sqrt{t}$ $y = t^2 - 2t$ tangent line at $t=4$ or $(2,8)$

Sol 1 (Eliminate para.)

$$x = \sqrt{t} \Rightarrow t = x^2$$

So $y = (x^2)^2 - 2(x^2) = x^4 - 2x^2$

$$\frac{dy}{dx} = 4x^3 - 4x \ , \quad \left.\frac{dy}{dx}\right|_{x=2} = 4\cdot 2^3 - 4\cdot 2 = 24$$

Eqn of tangent line is $y - 8 = 24(x-2)$.

Sol 2

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} \qquad \frac{dy}{dt} = 2t - 2$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-2}{\frac{1}{2\sqrt{t}}} = 4\sqrt{t}(t-1)$$

$$\left.\frac{dy}{dx}\right|_{t=4} = 4\sqrt{4}(4-1) = 24$$

Eqn of tangent line is $\underline{y-8 = 24(x-2)}$.

2) $x = 2t^3 - 6t$ $\qquad$ $y = t^3 + 6t^2$

<u>Vertical</u>

$$\frac{dx}{dt} = 6t^2 - 6$$
$$= 6(t^2-1)$$
$$= 6(t+1)(t-1) = 0$$
$$\Rightarrow t = -1, 1$$

<u>VT</u>: $\underline{(4,-5), (-4,7)}$

<u>Horizontal</u>

$$\frac{dy}{dt} = 3t^2 + 12t$$
$$= 3t(t+4) = 0$$
$$\Rightarrow t = 0, -4$$

<u>HT</u>: $\underline{(0,0), (-104, 32)}$

3) $x = r\cos\theta$ $\quad y = r\sin\theta$, $0 \le \theta \le 2\pi$, $r > 0$

$$\frac{dx}{d\theta} = -r\sin\theta \qquad \frac{dy}{d\theta} = r\cos\theta$$

So

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{r^2(\underbrace{\sin^2\theta + \cos^2\theta}_{=1})}\, d\theta$$

$$= \int_0^{2\pi} r\, d\theta \underset{\substack{\uparrow \\ r \text{ is a} \\ \text{constant}}}{=} r\int_0^{2\pi} d\theta = r\left[\theta\right]_0^{2\pi} = \underline{2\pi r}$$

Suggested Exercises (10.2)

2, 6, 14, 17, 18, 29, 42