6.3 | The Runge-Kutta Method

The <u>RK4</u> method numerically approximates the sol'n $y(x)$ of an initial value problem

$$\boxed{\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0}$$

at evenly spaced points $x_1, \ldots, x_N$, with <u>step size</u> h, yielding values $y_1, \ldots, y_N$ approximating $y(x_1), \ldots, y(x_N)$ and satisfying:

$$|y_n - y(x_n)| \le C h^4.$$

This means that if we divide the <u>step size</u> by 10, the <u>error</u> will also be divided by approximately $10^4 = 10{,}000$.

For $n = 0, 1, 2, \ldots$, the RK4 method is:

$$x_{n+1} = x_n + h$$

$$k_1 = f(x_n, y_n)$$ initial slope estimate

$$k_2 = f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h \cdot k_1\right)$$
$$k_3 = f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h \cdot k_2\right)$$
$\tfrac{1}{2}$-step slope estimates

$$k_4 = f\left(x_{n+1},\ y_n + h \cdot k_3\right)$$ full-step slope estimate

$$k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$ weighted average of slope estimates

$$y_{n+1} = y_n + h \cdot k$$

Example: (#4)

$y' = x - y$, $\quad y(0) = 1$; $\quad y(x) = 2e^{-x} + x - 1$

Use step size $h = 0.25$ to approx. the sol'n $y(x)$ at $x = 0.25, 0.5$ using RK4.

Sol'n: $f(x,y) = x - y$, $(x_0, y_0) = (0,1)$

$x_1 = x_0 + h = 0.25$

$k_1 = f(x_0, y_0) = -1$

$k_2 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}h \cdot k_1) = -0.75$

$k_3 = f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}h \cdot k_2) = -0.78125$

$k_4 = f(x_1, y_0 + h \cdot k_3) = -0.5546875$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.76953125$

$y_1 = y_0 + h \cdot k = 0.8076171875$

$x_2 = x_1 + h = 0.5$

$k_1 = f(x_1, y_1) = -0.5576171875$

$k_2 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}h \cdot k_1) = -0.362915039062 5$

$k_3 = f(x_1 + \frac{1}{2}h, y_1 + \frac{1}{2}h \cdot k_2) = -0.3872528076171875$

$k_4 = f(x_2, y_1 + h \cdot k_3) = -0.2108039855957031 2$

$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = -0.3781261444091797$

$y_2 = y_1 + h \cdot k = 0.7130856513977051$

Improved Euler Method: (10 function evaluations in total)

n	x_n	y_n	$y(x_n)$	$\lvert y(x_n) - y_n \rvert$
0	0	1	1	0
1	0.1	0.91	0.90967	0.00033
2	0.2	0.83805	0.83746	0.00059
3	0.3	0.782435	0.78164	0.00080
4	0.4	0.741604	0.74064	0.00096
5	0.5	0.714152	0.71306	0.00109

RK4: (8 function evaluations in total)

n	x_n	y_n	$y(x_n)$	$\lvert y(x_n) - y_n \rvert$
0	0	1	1	0
1	0.25	0.807617	0.807602	0.0000156
2	0.5	0.713086	0.713061	0.0000243

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[Julia code demo]