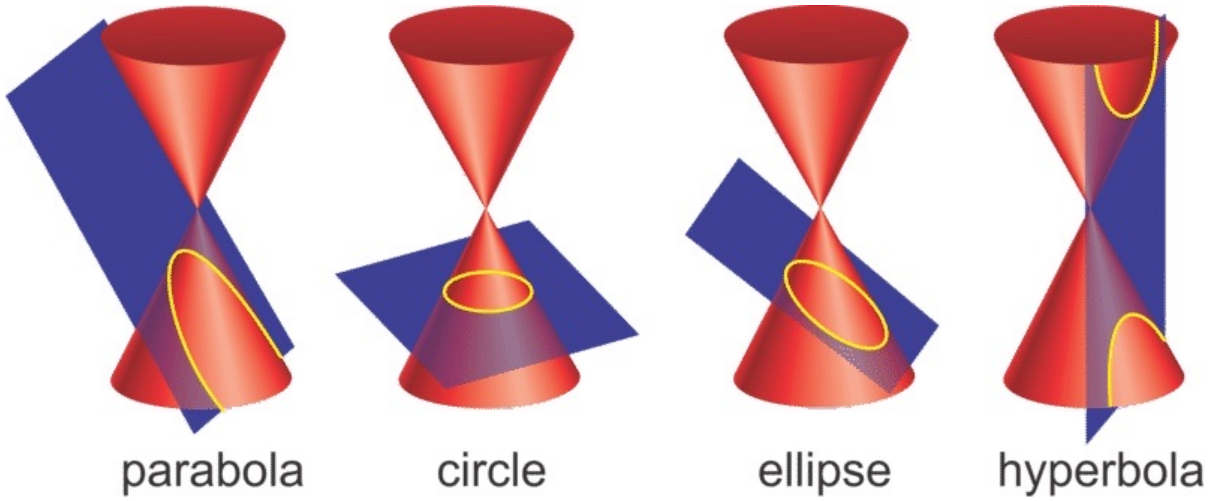
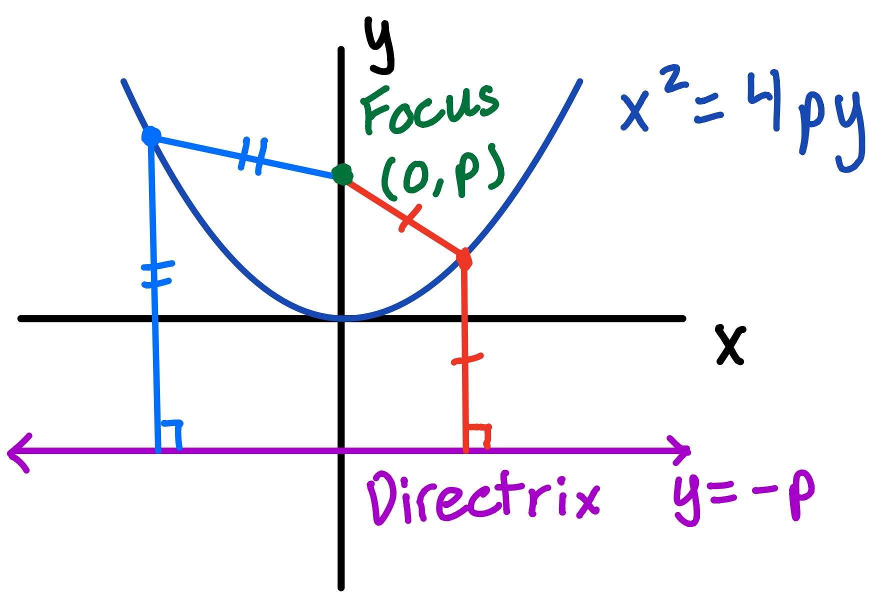
10.5 Conic Sections



Def (Parabola)

A set of points that are equidistant from a fixed point F (called the Focus) and a fixed line (called a directrix).

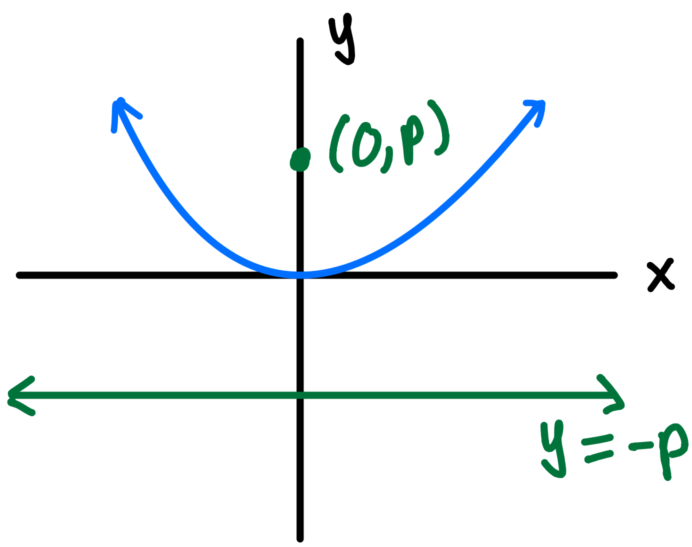


A parabola with focus $(0,p)$ and directrix $y=-p$ has equation $x^2=4py$.

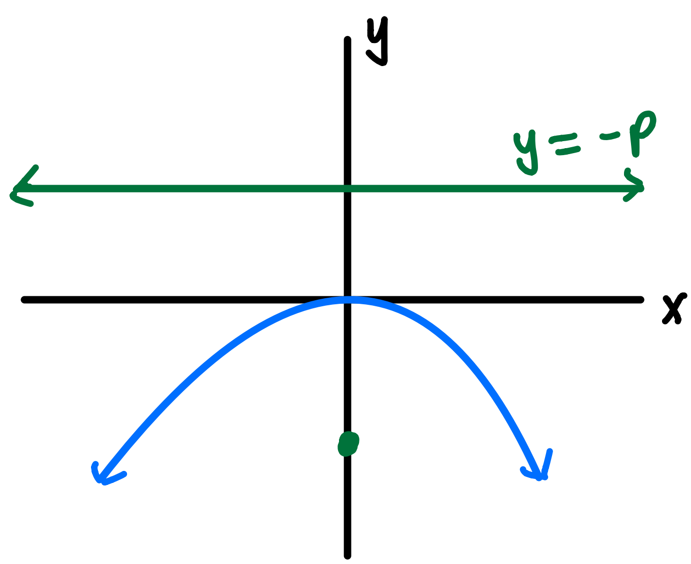
<u>Note</u>: You are used to seeing this as

$$y=ax^2 \quad \text{where} \quad a=\frac{1}{4p}.$$

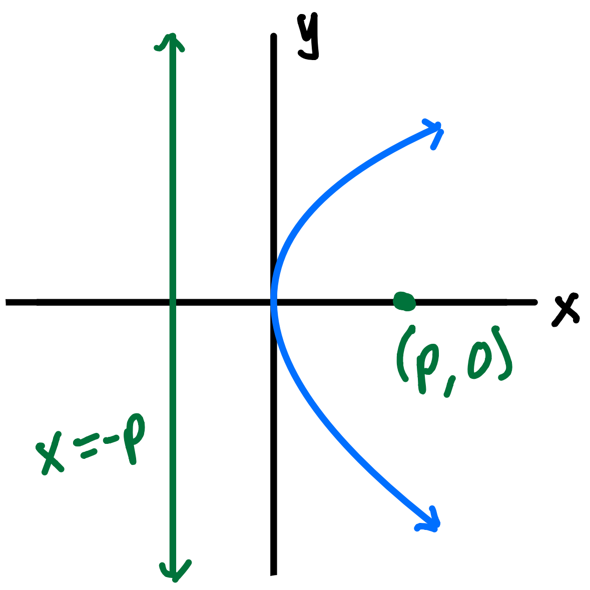
There are 4 basic cases:



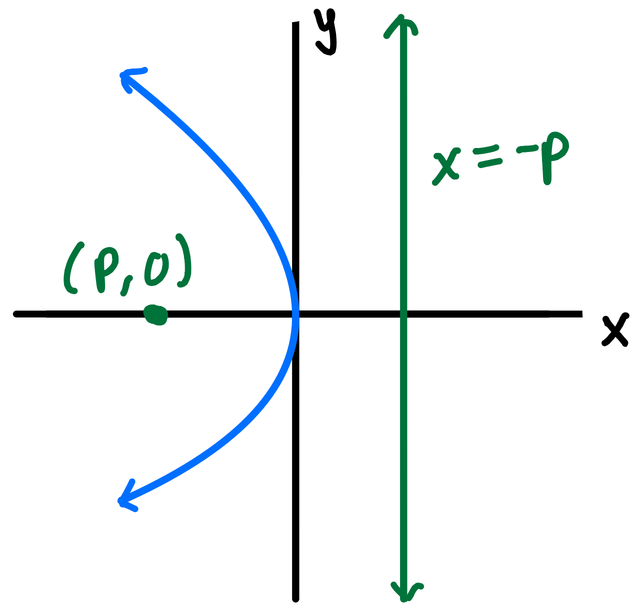
$x^2=4py \ (p>0)$



$x^2=4py \ (p<0)$



$y^2=4px \ (p>0)$



$y^2=4px \ (p<0)$

Ex 1 Let $3x^2=-8y$. Find the vertex, focus, directrix and sketch.

$$3x^2=-8y \Rightarrow x^2=-\frac{8}{3}y$$

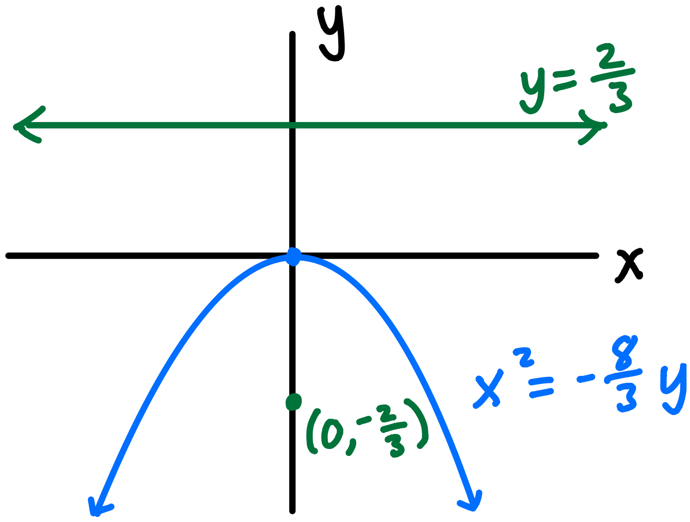
The general form is $x^2=4py$ so to solve for p,

$$4p=-\frac{8}{3} \Rightarrow p=-\frac{2}{3}$$

Focus: $(0, -\frac{2}{3})$

Directrix: $y = \frac{2}{3}$

Vertex: $(0,0)$



<u>Ex 2</u> Sketch $(y-2)^2 = 2x+1$

We can ignore the translations for now and consider the parabola $y^2 = 2x$

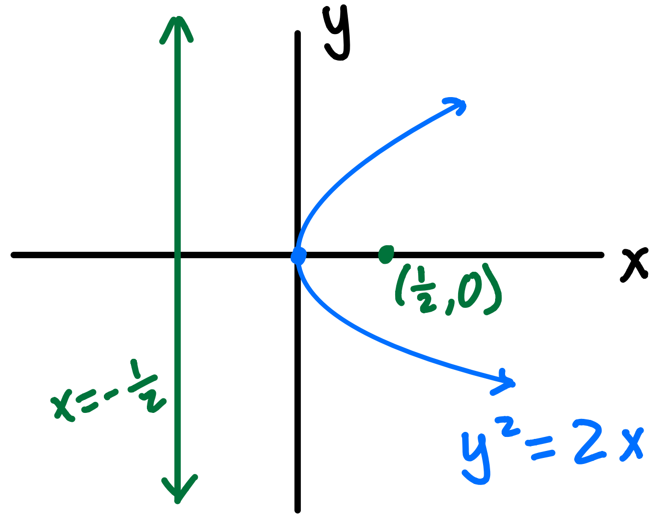
<u>General form</u>: $y^2 = 4px$

So $4p = 2 \Rightarrow p = \frac{1}{2}$

Focus: $(\frac{1}{2}, 0)$

Directrix: $x = -\frac{1}{2}$

Vertex: $(0,0)$



We still need to translate this:

The $(y-2)^2$ tells us to shift up by 2.

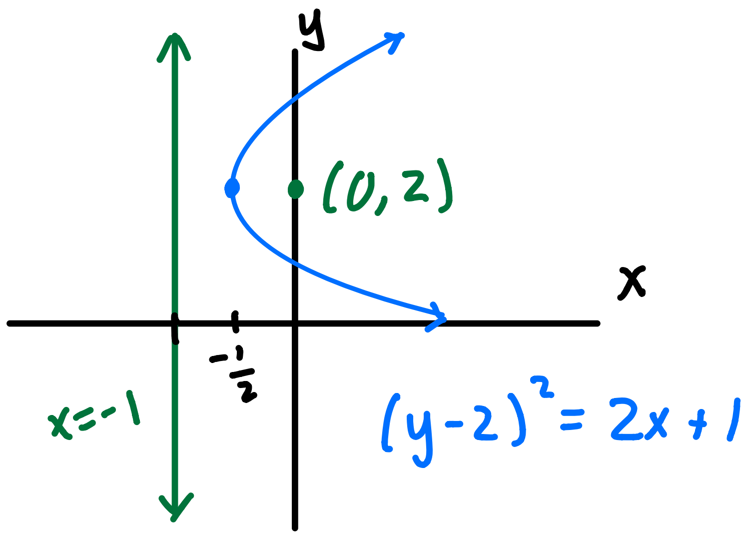
The $2x+1 = 2(x+\frac{1}{2})$ tells us to shift left by $\frac{1}{2}$.

So the new points become

Focus: $(0,2)$

Directrix: $x=-1$

Vertex: $(-\frac{1}{2}, 2)$

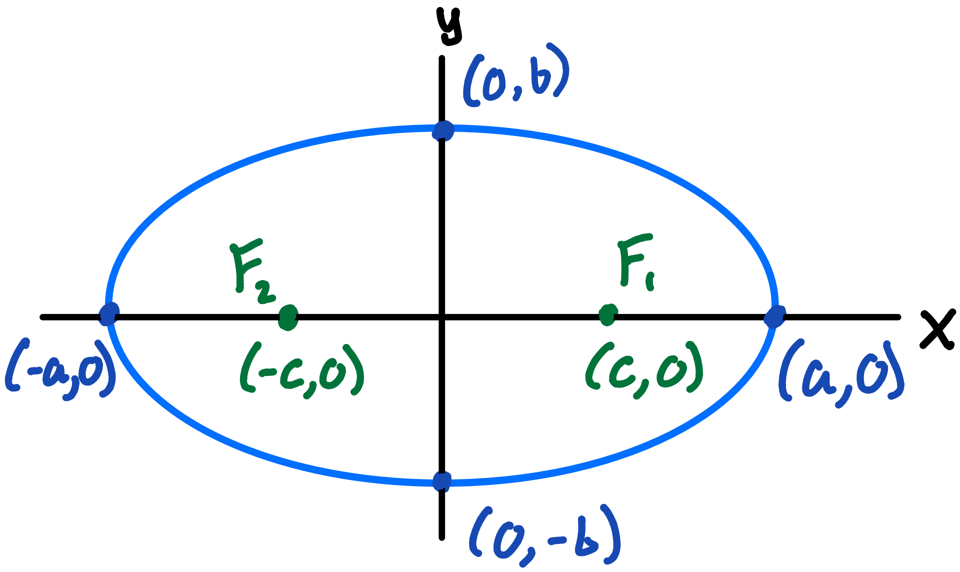


<u>Def</u> (Ellipse)

A set of points such that the sum of the distances between two fixed points (F_1 and F_2) is constant.

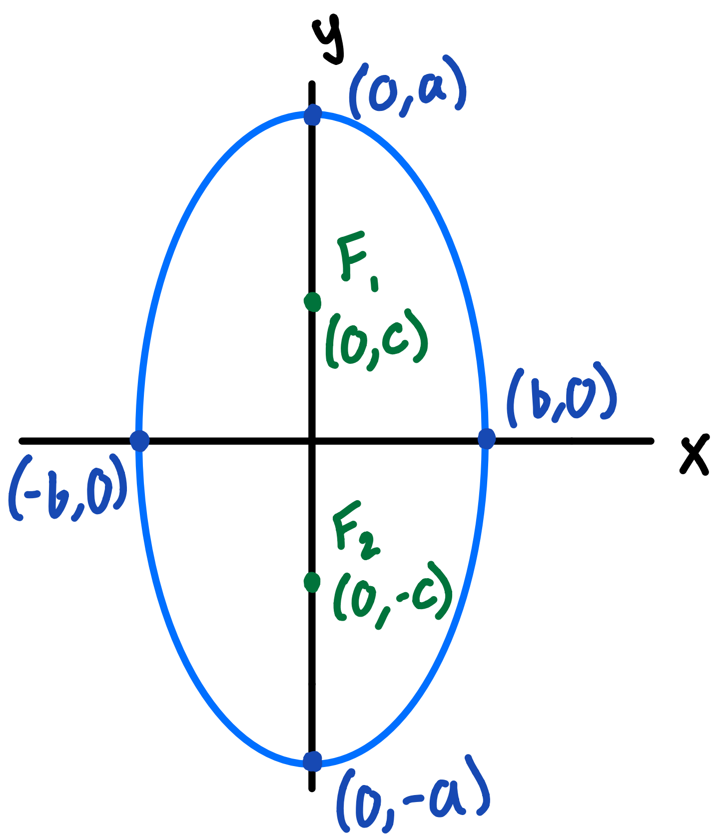
Horizontal Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a \geq b)$

$$c^2 = a^2 - b^2$$



Vertical Ellipse: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a \geq b)$

$$c^2 = a^2 - b^2$$



Ex 3 Sketch $9x^2 + 36y^2 = 324$.

Divide by 324,

$$\frac{x^2}{36} + \frac{y^2}{9} = 1$$

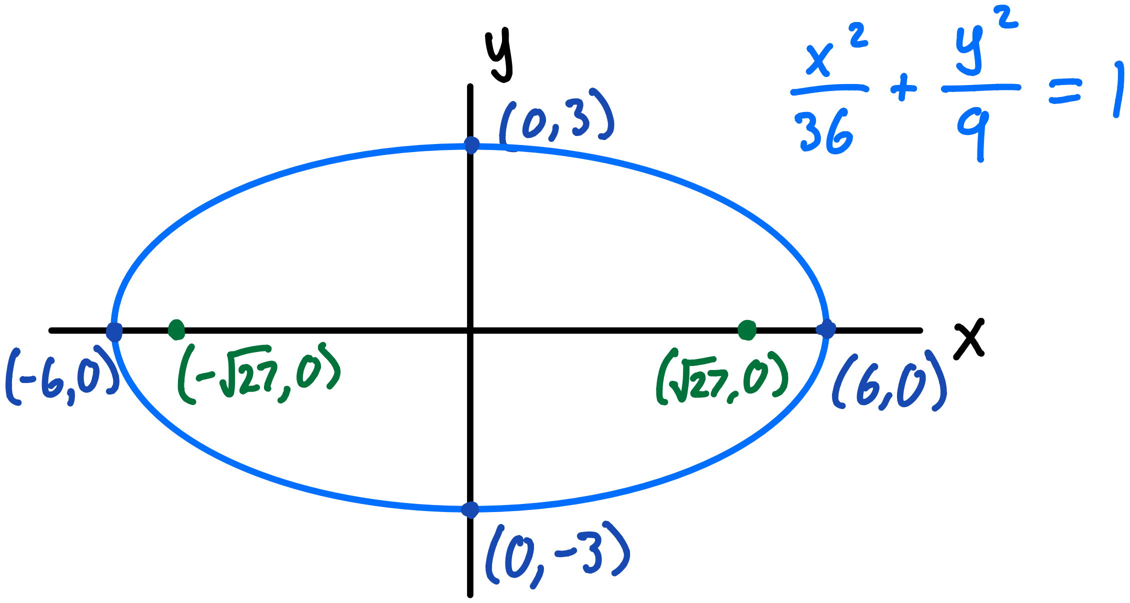
$36 > 9$ so this is a horizontal ellipse

General Form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So $a^2 = 36 \Rightarrow a = 6$

$b^2 = 9 \Rightarrow b = 3$

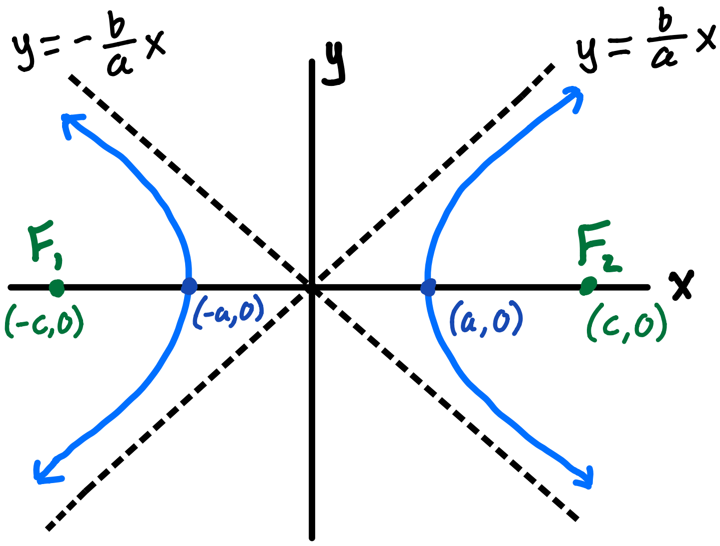
$c^2 = a^2 - b^2 = 27 \Rightarrow c = \sqrt{27}$



Def (Hyperbola)

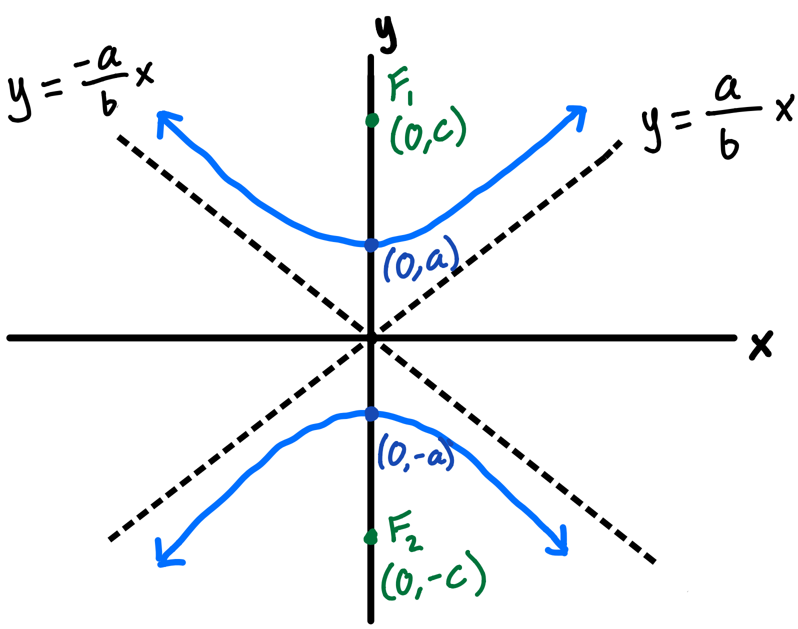
A set of points where the difference of the distances between two fixed points (F_1 and F_2) is constant.

There are two forms:



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$c^2 = a^2 + b^2$$



$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$c^2 = a^2 + b^2$$

Ex 4 Sketch $\frac{y^2}{9} - \frac{x^4}{4} = 1$.

General Form: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

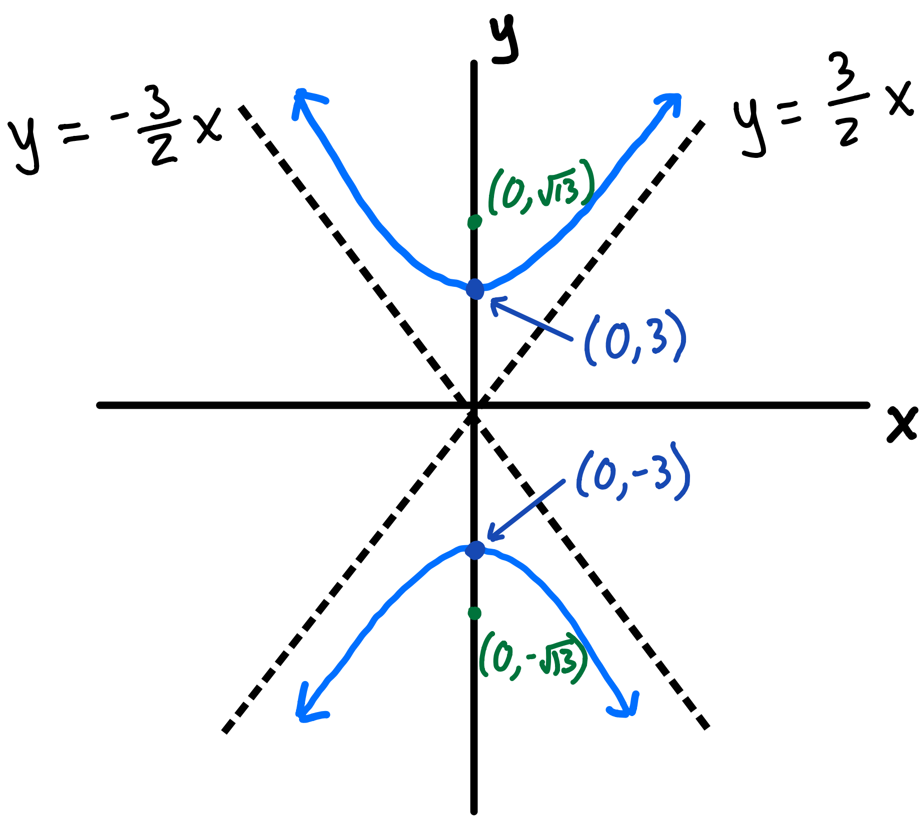
$a^2 = 9 \Rightarrow a = 3$

$b^2 = 4 \Rightarrow b = 2$

$c^2 = a^2 + b^2 = 13 \Rightarrow c = \sqrt{13}$

Asymptotes: $\left(y = \frac{a}{b}x,\ y = -\frac{a}{b}x\right)$

$y = \frac{3}{2}x \quad , \quad y = \frac{-3}{2}x$



Practice Problems

1) Let $(x-1)^2 = -12y - 24$. Find the focus, vertex, directrix, and sketch.

2) Sketch $64x^2 + 4y^2 = 256$. Label the focus points and vertices.

3) Sketch $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$. Label the focus points, vertices, and asymptotes.

Solutions

1) $(x-1)^2 = -12y - 24$

Let's ignore the translations and look at the basic form $x^2 = \underline{-12}y$.

General Form: $x^2 = \underline{4}py$

So $4p = -12 \Rightarrow p = -3$ and we have

Focus: $(0, -3)$
Directrix: $y = 3$
Vertex: $(0, 0)$

Now we can translate,

The $(x-1)^2$ tells us to shift right 1.

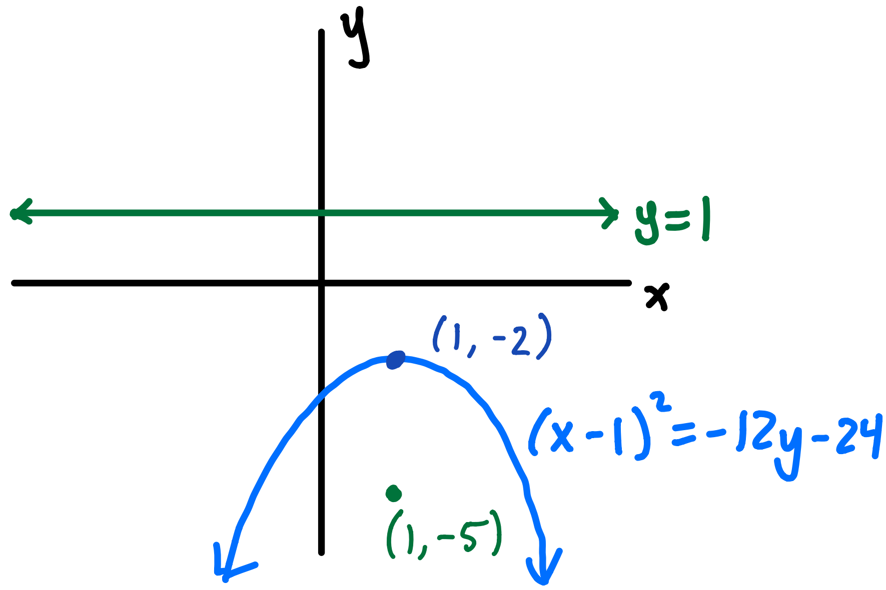
The $-12y - 24 = -12(y+2)$ tells us to shift down 2.

So our new points are

Focus: $(1,-5)$

Directrix: $y=1$

Vertex: $(1,-2)$



2) $64x^2+4y^2=256$

Divide by 256,

$$\frac{x^2}{4}+\frac{y^2}{64}=1$$

$64>4$ so this is a vertical ellipse

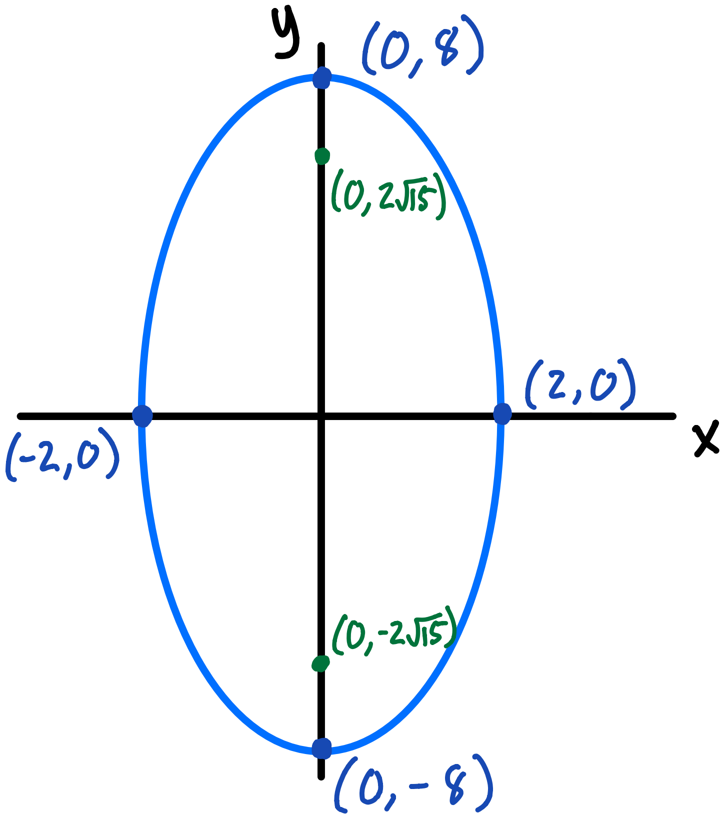
$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

We have,

$b^2=4 \Rightarrow b=2$

$a^2=64 \Rightarrow a=8$

$c^2=a^2-b^2=60 \Rightarrow c=\sqrt{60}=2\sqrt{15}$



3) $\dfrac{x^2}{16} - \dfrac{y^2}{4} = 1$

General Form: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

So we have,

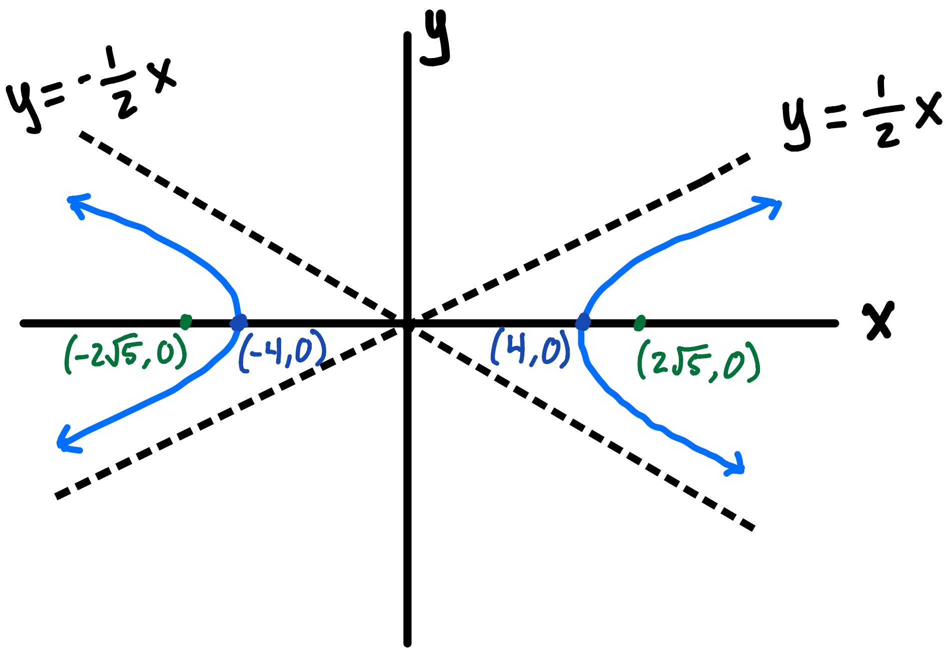
$a^2 = 16 \Rightarrow a = 4$

$b^2 = 4 \Rightarrow b = 2$

$c^2 = a^2 + b^2 = 20 \Rightarrow c = \sqrt{20} = 2\sqrt{5}$

Asymptotes: $\left(y = \frac{b}{a}x,\ y = \frac{-b}{a}x\right)$

$$y = \frac{2}{4}x = \frac{1}{2}x \quad , \quad y = -\frac{1}{2}x$$



Suggested Textbook Exc. (10.5)

2, 14, 20, 33, 43