6.21 A Closer Look at the Euler Method

Euler's method is used to approximate the sol'n $y(x)$ of an **initial value problem**

$$\boxed{\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0}$$

at evenly spaced points $x_1, \ldots, x_n$.

That is, we compute $y_1, \ldots, y_n$ such that

approx. value

$$y_n \approx y(x_n)$$

true value

The error in this approximation satisfies:

$$|y_n - y(x_n)| \leq Ch$$

where h is the **step size**.

This means that if we divide the <u>step size</u> by 10, the <u>error</u> will also be divided by approximately 10.

eg $h = 0.1 \Rightarrow |y_5 - y(x_5)| \approx 3 \times 10^{-2}$
(5 steps)

$h = 0.01 \Rightarrow |y_{50} - y(x_{50})| \approx 3 \times 10^{-3}$
(50 steps)

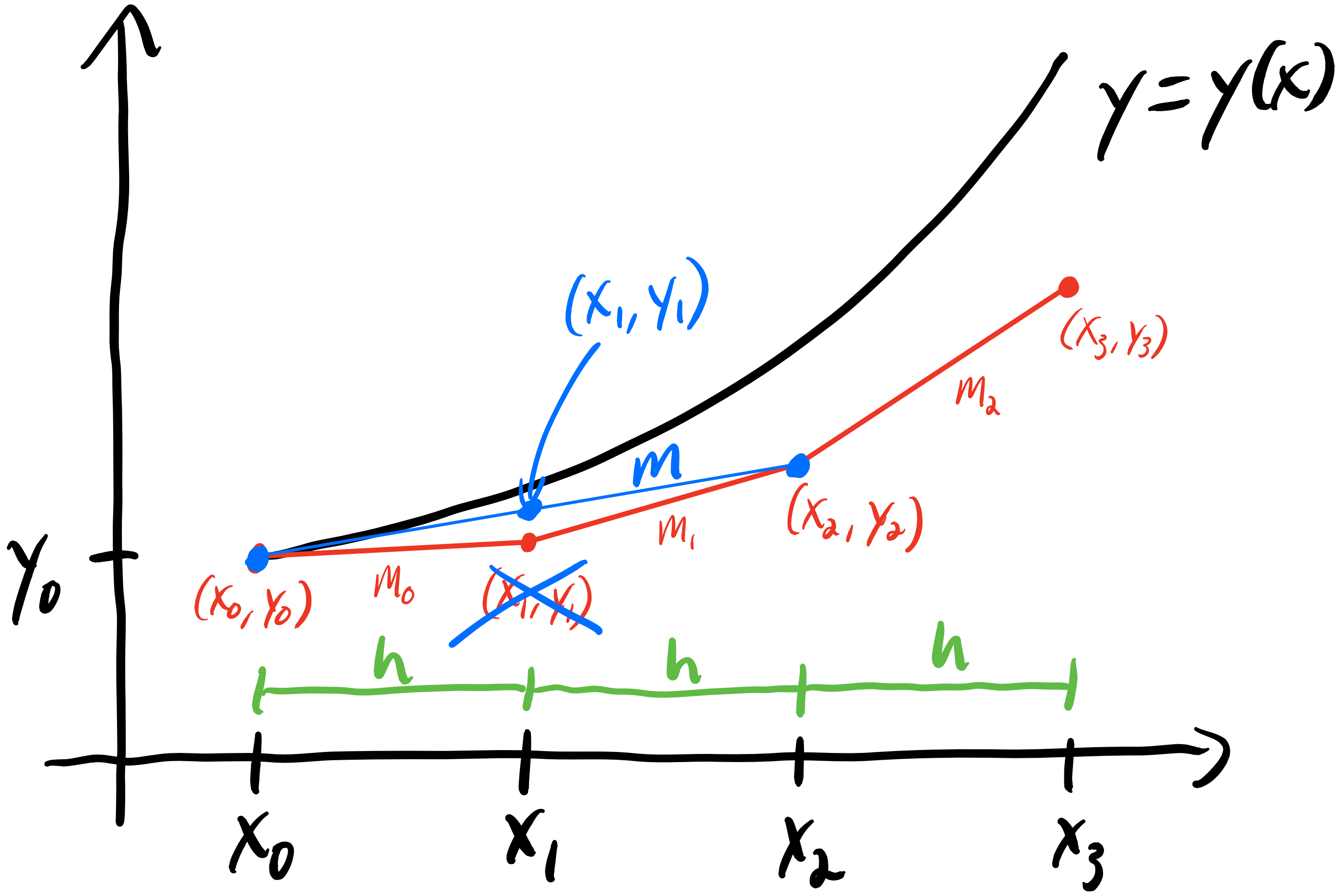
How many steps are needed to have $|y_N - y(x_N)| \approx 3 \times 10^{-8}$?

<u>Ans:</u> $N = 5 \times 10^6 = \underline{5{,}000{,}000}$!

That is not good...

We can do <u>a lot</u> better.

<u>Idea:</u>



What is m? $\boxed{m = \dfrac{m_0 + m_1}{2}}$

Thus, for $n = 0, 1, 2, \ldots,$ the **Improved Euler Method** is:

$$x_{n+1} = x_n + h$$

$$m_0 = f(x_n, y_n)$$

$$m_1 = f(x_{n+1}, y_n + h \cdot m_0)$$

$$m = \tfrac{1}{2}(m_0 + m_1)$$

$$y_{n+1} = y_n + h \cdot m$$

We will see that the error in the Improved Euler Method satisfies:

$$\boxed{|y_n - y(x_n)| \leq Ch^2}$$

Thus, dividing h by 10 divides the error by 100!

Example: (#4)

$y' = x - y, \quad y(0) = 1; \quad y(x) = 2e^{-x} + x - 1$

Use step size $h = 0.1$ to approx. the sol'n $y(x)$ at $x = 0.1, 0.2, 0.3, 0.4, 0.5$ using the Improved Euler Method.

Sol'n: $f(x,y) = x - y, \quad (x_0, y_0) = (0, 1)$

$x_1 = x_0 + h = 0.1$

$m_0 = f(x_0, y_0) = -1$

$m_1 = f(x_1, y_0 + h \cdot m_0) = -0.8$

$m = \frac{1}{2}(m_0 + m_1) = -0.9$

$y_1 = y_0 + h \cdot m = \boxed{0.91}$

$x_2 = x_1 + h = \boxed{0.2}$

$m_0 = f(x_1, y_1) = -0.81$

$m_1 = f(x_2, y_1 + h \cdot m_0) = -0.629$

$m = \frac{1}{2}(m_0 + m_1) = -0.7195$

$y_2 = y_1 + h \cdot m = \boxed{0.83805}$

$x_3 = x_2 + h = \boxed{0.3}$

$m_0 = f(x_2, y_2) = -0.63805$

$m_1 = f(x_3, y_2 + h \cdot m_0) = -0.474245$

$m = \frac{1}{2}(m_0 + m_1) = -0.5561475$

$y_3 = y_2 + h \cdot m = \boxed{0.78243525}$

$x_4 = x_3 + h = \boxed{0.4}$

$m_0 = f(x_3, y_3) = -0.48243525$

$m_1 = f(x_4, y_3 + h \cdot m_0) = -0.334191725$

$m = \frac{1}{2}(m_0 + m_1) = -0.4083134875$

$y_4 = y_3 + h \cdot m = \boxed{0.74160390125}$

$$x_5 = x_4 + h = \boxed{0.5}$$

$$m_0 = f(x_4, y_4) = -0.34160390125$$

$$m_1 = f(x_5, y_4 + h\cdot m_0) = -0.20744351125$$

$$m = \tfrac{1}{2}(m_0 + m_1) = -0.274523706187 45$$

$$y_5 = y_4 + h\cdot m = \boxed{0.7141515306 3125}$$

Euler's method:

n	x_n	y_n	$y(x_n)$	$\lvert y(x_n) - y_n \rvert$
0	0	1	1	0
1	0.1	0.9	0.90967	0.00967
2	0.2	0.82	0.83746	0.01746
3	0.3	0.758	0.78164	0.02364
4	0.4	0.7122	0.74064	0.02844
5	0.5	0.68098	0.71306	0.03208

Improved Euler Method:

n	x_n	y_n	$y(x_n)$	$\lvert y(x_n) - y_n \rvert$
0	0	1	1	0
1	0.1	0.91	0.90967	0.00033
2	0.2	0.83805	0.83746	0.00059
3	0.3	0.782435	0.78164	0.00080
4	0.4	0.741604	0.74064	0.00096
5	0.5	0.714152	0.71306	0.00109

eg $h = 0.1 \Rightarrow |y_5 - y(x_5)| \approx 1 \times 10^{-3}$

(5 steps)

$h = 0.01 \Rightarrow |y_{50} - y(x_{50})| \approx 1 \times 10^{-5}$

(50 steps)

How many steps are needed to have $|y_N - y(x_N)| \approx 1 \times 10^{-8}$?

Ans: $N = 5 \times 10^3 = \underline{5{,}000}$!

[Julia code demo]