12.3 The Dot Product

Def 1 (Dot Product)

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.
Then the dot product is given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Note: The dot product of two vectors is a scalar. (not another vector)

Properties

i) $\vec{a} \cdot \vec{a} = |\vec{a}|^2$

ii) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

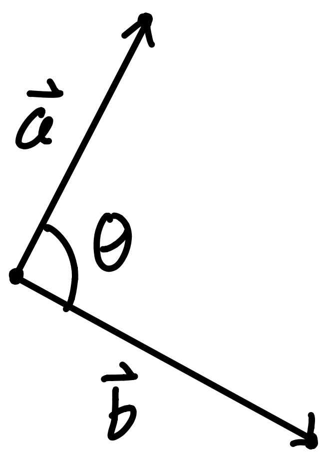
iii) $\vec{a} \cdot (\vec{b} + \vec{v}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{v}$

iv) $\vec{0}\cdot\vec{a} = 0$ $\quad(\vec{0} = \langle 0,0,0\rangle)$

v) $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b})$

Theorem 1

Let θ be the angle between vectors $\vec{a}$ and $\vec{b}$. Then



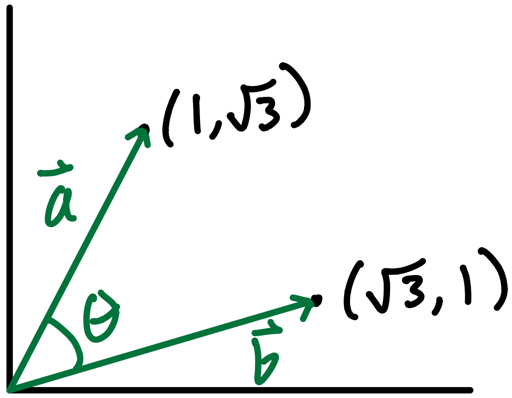
$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$

or

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Note: We can always choose θ to be in the interval $[0,\pi]$.

Ex 1 Find the angle between the vectors $\vec{a} = \langle 1, \sqrt{3} \rangle$ and $\vec{b} = \langle \sqrt{3}, 1 \rangle$.



$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

Step 1: Find $\vec{a}\cdot\vec{b}$

$$\vec{a}\cdot\vec{b} = \langle 1, \sqrt{3} \rangle \cdot \langle \sqrt{3}, 1 \rangle$$
$$= 1\cdot\sqrt{3} + \sqrt{3}\cdot 1$$
$$= 2\sqrt{3}$$

Step 2: Find $|\vec{a}|$ and $|\vec{b}|$.

$$|\vec{a}| = |\langle 1, \sqrt{3} \rangle| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$|\vec{b}| = |\langle \sqrt{3}, 1 \rangle| = 2$$

Step 3: Use formula $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2\sqrt{3}}{2\cdot 2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Def 2 (Orthogonality)

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are orthogonal or perpendicular if $\vec{a}\cdot\vec{b} = 0$.

3 cases

i) If $\vec{a}\cdot\vec{b} > 0$ then θ is acute.

ii) If $\vec{a}\cdot\vec{b} = 0$ then $\theta = \frac{\pi}{2}$ and the vectors are orthogonal.

iii) If $\vec{a}\cdot\vec{b} < 0$ then θ is obtuse.

Ex 2 Show that $2i + 2j - k$ is orthogonal to $5i - 4j + 2k$.

Sol

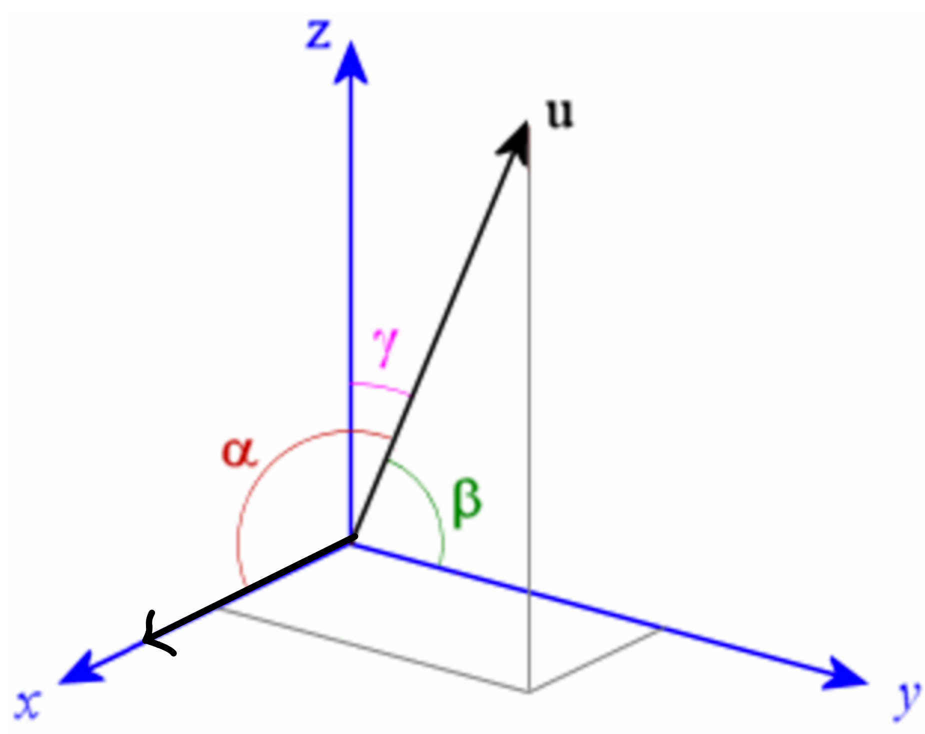
Taking the dot product,

$$\langle 2, 2, -1 \rangle \cdot \langle 5, -4, 2 \rangle$$
$$= 2(5) + 2(-4) + (-1)(2)$$
$$= 10 - 8 - 2$$
$$= 0$$

So the vectors are orthogonal.

Def 3 (Direction Angles and Cosines)

Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$. The direction angles of $\vec{a}$ are the angles α, β, γ in $[0, \pi]$ that $\vec{a}$ makes with the positive x, y and z axes respectively.



The direction cosines are $\cos(\alpha)$, $\cos(\beta)$ and $\cos(\gamma)$.

For $\vec{a} = \langle a_1, a_2, a_3 \rangle$ we have,

$$\cos(\alpha) = \frac{a_1}{|\vec{a}|} \Rightarrow a_1 = |\vec{a}|\cos(\alpha)$$

$$\cos(\beta) = \frac{a_2}{|\vec{a}|} \Rightarrow a_2 = |\vec{a}|\cos(\beta)$$

$$\cos(\gamma) = \frac{a_3}{|\vec{a}|} \Rightarrow a_3 = |\vec{a}|\cos(\gamma)$$

Therefore,

$$\vec{a} = \langle a_1, a_2, a_3 \rangle$$

$$= |\vec{a}|\langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

$$\Rightarrow \underline{\frac{\vec{a}}{|\vec{a}|}} = \langle \cos(\alpha), \cos(\beta), \cos(\gamma) \rangle$$

Unit vector in the direction of $\vec{a}$.

<u>Ex 3</u> Find the direction angles of $\vec{a} = \langle 1, 2, 3 \rangle$.

<u>Sol</u>

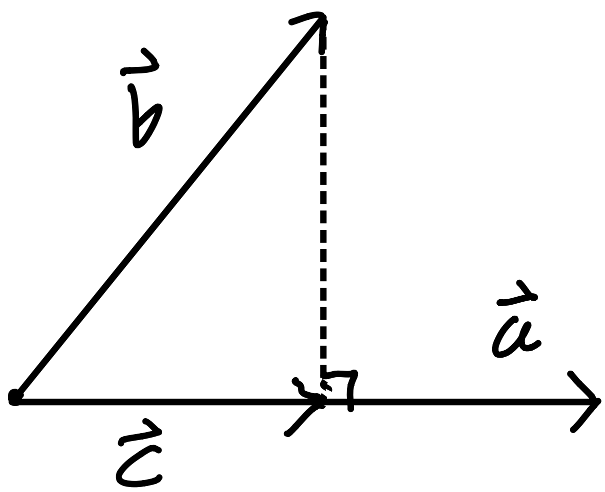
$$|\vec{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\cos(\alpha) = \frac{a_1}{|\vec{a}|} = \frac{1}{\sqrt{14}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{14}}\right)$$
$$\approx 74^\circ$$

$$\cos(\beta) = \frac{a_2}{|\vec{a}|} = \frac{2}{\sqrt{14}} \Rightarrow \beta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right)$$
$$\approx 58^\circ$$

$$\cos(\gamma) = \frac{a_3}{|\vec{a}|} = \frac{3}{\sqrt{14}} \Rightarrow \gamma = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$
$$\approx 37^\circ$$

Def 4 (Projections)



Vector projection of $\vec{b}$ onto $\vec{a}$:

$$\vec{c} = \mathrm{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\right)\vec{a}$$

Scalar projection of $\vec{b}$ onto $\vec{a}$:

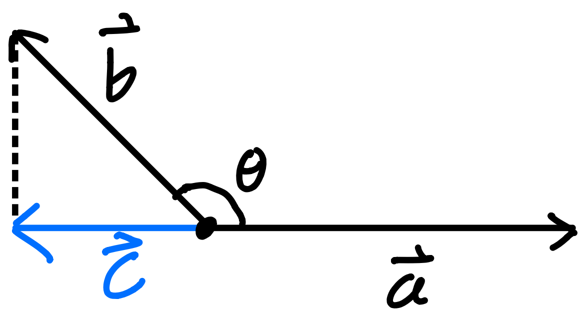
$$\mathrm{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Note: $\mathrm{proj}_{\vec{a}}\vec{b} = \mathrm{comp}_{\vec{a}}\vec{b}\left(\frac{\vec{a}}{|\vec{a}|}\right)$

The scalar projection (aka the component of $\vec{b}$ along $\vec{a}$) can be thought of as the signed magnitude of the projection of $\vec{b}$ onto $\vec{a}$.

Notice that $\text{comp}_{\vec{a}}\vec{b} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = |\vec{b}|\underbrace{\cos\theta}_{<0 \text{ when } \frac{\pi}{2}<\theta<\pi}$

If $\frac{\pi}{2}<\theta<\pi$ then the picture becomes,



Ex 4 Let $\vec{a} = \langle 4,7,-4\rangle$, $\vec{b} = \langle 3,-1,1\rangle$.

Find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$.

Sol

$$\vec{a} = \langle 4, 7, -4 \rangle \qquad \vec{b} = \langle 3, -1, 1 \rangle$$

Scalar:

$$\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|} = \frac{4(3) - 7 - 4}{\sqrt{4^2 + 7^2 + (-4)^2}}$$

$$= \frac{1}{\sqrt{81}} = \frac{1}{9}$$

Vector:

$$\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|}\right) \frac{\vec{a}}{|\vec{a}|} = \left(\frac{1}{9}\right) \frac{\vec{a}}{9}$$

$$= \frac{1}{81} \vec{a} = \left\langle \frac{4}{81}, \frac{7}{81}, \frac{-4}{81} \right\rangle$$

Practice Problems

1) Let $\vec{a} = \langle -5, 4, 1 \rangle$, $\vec{b} = \langle 3, 4, -1 \rangle$.

   a) Show that $\vec{a}$ and $\vec{b}$ are orthogonal.

   b) Find the direction angles of $\vec{a}$.

2) Let $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 5, 0, -1 \rangle$.
Find the scalar and vector projections of $\vec{b}$ onto $\vec{a}$.

Solutions

1) $\vec{a} = \langle -5, 4, 1 \rangle$, $\vec{b} = \langle 3, 4, -1 \rangle$

a) We must show $\vec{a} \cdot \vec{b} = 0$.

$$\vec{a} \cdot \vec{b} = (-5)(3) + (4)(4) + (1)(-1)$$
$$= -15 + 16 - 1 = 0$$

b) $\vec{a} = \langle -5, 4, 1 \rangle$

$$|\vec{a}| = \sqrt{(-5)^2 + 4^2 + 1^2} = \sqrt{42}$$

$$\cos(\alpha) = \frac{a_1}{|\vec{a}|} = \frac{-5}{\sqrt{42}} \Rightarrow \alpha = \cos^{-1}\left(\frac{-5}{\sqrt{42}}\right)$$
$$\approx \underline{140.5^\circ}$$

$$\cos(\beta) = \frac{a_2}{|\vec{a}|} = \frac{4}{\sqrt{42}} \Rightarrow \beta = \cos^{-1}\left(\frac{4}{\sqrt{42}}\right)$$
$$\approx \underline{52^\circ}$$

$$\cos(\gamma) = \frac{a_3}{|\vec{a}|} = \frac{1}{\sqrt{42}} \Rightarrow \gamma = \cos^{-1}\left(\frac{1}{\sqrt{42}}\right)$$
$$\approx \underline{81^\circ}$$

2) $\vec{a} = \langle 1, 2, 3 \rangle$, $\vec{b} = \langle 5, 0, -1 \rangle$

Scalar proj.

$$\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} = \frac{5+0-3}{\sqrt{1^2+2^2+3^2}} = \frac{2}{\sqrt{14}}$$

Vector proj.

$$\text{proj}_{\vec{a}}\vec{b} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\frac{\vec{a}}{|\vec{a}|} = \left(\frac{2}{\sqrt{14}}\right)\cdot\left(\frac{1}{\sqrt{14}}\right)\vec{a}$$

$$= \frac{2}{14}\vec{a} = \frac{1}{7}\langle 1, 2, 3 \rangle = \left\langle \frac{1}{7}, \frac{2}{7}, \frac{3}{7} \right\rangle$$

Suggested Textbook Exc. (12.3)

5, 9, 18, 20, 23, 33, 39, 43