2.3 | Homogeneous Eq'ns w/ Constant Coefficients

$$\boxed{a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0} \quad (1)$$

Consider $y = e^{rx}$. Then

$$y' = re^{rx}, \quad y'' = r^2 e^{rx}, \ldots, \quad y^{(n)} = r^n e^{rx}.$$

Substituting into (1):

$$a_n r^n e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} = 0.$$

Dividing by e^{rx}:

$$\boxed{a_n r^n + \cdots + a_1 r + a_0 = 0}. \quad (2)$$

The <u>characteristic eq'n</u> of (1)

<u>Note</u>: (2) is a degree-n polynomial, so has <u>n roots</u>, counting multiplicities:

$$r_1, r_2, \ldots, r_n.$$

These roots could be <u>real numbers</u> or <u>complex numbers</u>. $x^2+1=0$ $\quad x=-i, i$

Case 1: Distinct real roots

If $r_1, \ldots, r_n$ are distinct real numbers, then $y_1 = e^{r_1 x}, \ldots, y_n = e^{r_n x}$ are n linearly independent sol'ns of (1).

$\therefore$ $y(x) = c_1 e^{r_1 x} + \cdots + c_n e^{r_n x}$

is a general sol'n of (1).

Example: (#3)

$$y'' + 3y' - 10y = 0$$

Sol'n:

Characteristic eq'n: $r^2 + 3r - 10 = 0$

$(r-2)(r+5) = 0$

$r = 2, -5 \implies y_1 = e^{2x}$

$y_2 = e^{-5x}$

$\therefore$ a general sol'n is

$$y(x) = c_1 e^{2x} + c_2 e^{-5x}.$$

Case 2: Repeated real roots

If r is a root of (2) that is repeated k times, then

$$e^{rx},\ xe^{rx},\ \ldots,\ x^{k-1}e^{rx}$$

are k linearly independent sol'ns of (1), so we add

$$c_1 e^{rx} + c_2 x e^{rx} + \cdots + c_k x^{k-1} e^{rx}$$

to the general sol'n of (1).

Example: (#7)

$$4y'' - 12y' + 9y = 0$$

Sol'n:

Char. eq'n: $4r^2 - 12r + 9 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{144 - 4\cdot 4\cdot 9}}{2\cdot 4}$$

$$= \frac{12 \pm \sqrt{0}}{8} = \frac{3}{2}, \frac{3}{2}$$

$\therefore$ $y_1 = e^{\frac{3}{2}x}$, $y_2 = xe^{\frac{3}{2}x}$ are lin. indep. sol'ns.

$\therefore$ a general sol'n is:

$$y(x) = c_1 e^{\frac{3}{2}x} + c_2 x e^{\frac{3}{2}x}.$$

<u>Example:</u> (#11)

$$y^{(4)} - 8y^{(3)} + 16y'' = 0$$

<u>Sol'n:</u>

Char. eq'n: $r^4 - 8y^3 + 16r^2 = 0$

$$r^2(r^2 - 8r + 16) = 0$$

$$r^2(r-4)^2 = 0$$

$$r = 0, 0, 4, 4$$

$\therefore$ $y_1 = e^{0x} = 1$ | $y_3 = e^{4x}$

$y_2 = xe^{0x} = x$ | $y_4 = xe^{4x}$

$\therefore$ a gen. sol'n is:

$$y(x) = c_1 \cdot 1 + c_2 x + c_3 e^{4x} + c_4 x e^{4x}$$

Complex Roots: $r = a + ib, \quad i^2 = -1$

Euler's Formula: $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Proof: Recall the Taylor series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$\therefore e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

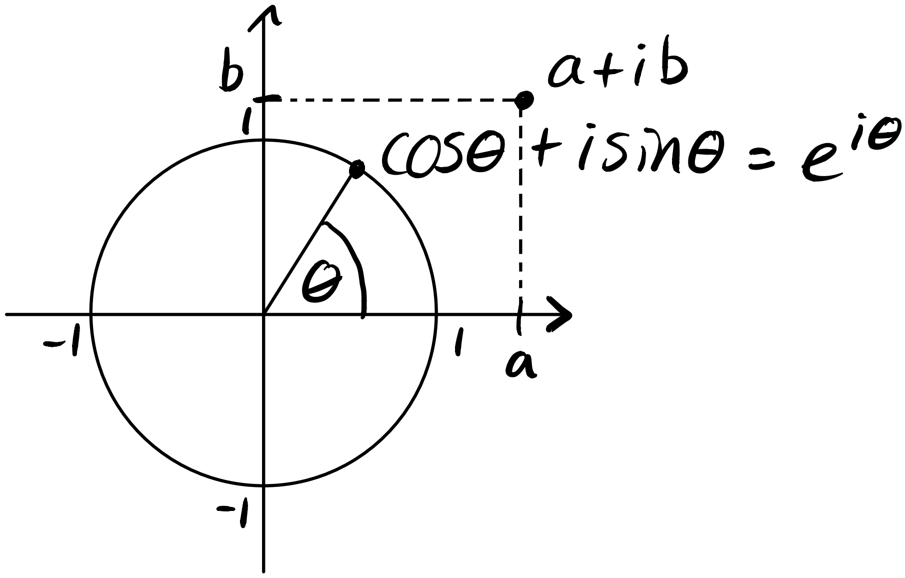
$$= \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$= \cos\theta + i\sin\theta. \quad \blacksquare$$

Note: $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0$

$$\therefore \boxed{e^{i\pi} + 1 = 0}.$$

The Complex Plane: $\mathbb{C}$



Case 3: Unrepeated complex pair of roots

If $r = a \pm ib$ is an unrepeated pair of complex roots of (2), then

$$e^{ax}\cos bx, \quad e^{ax}\sin bx$$

are linearly independent sol'ns of (1), so we add

$$c_1 e^{ax}\cos bx + c_2 e^{ax}\sin bx$$

to the general sol'n of (1).

$$e^{(a\pm ib)x} = e^{ax}e^{\pm ibx} = e^{ax}\big(\cos(\pm bx) + i\sin(\pm bx)\big)$$
$$= e^{ax}(\cos bx \pm i\sin bx).$$

Example: (#31)

$$y^{(3)} + 3y'' + 4y' - 8y = 0$$

Sol'n:

Char. eq'n: $r^3 + 3r^2 + 4r - 8 = 0$

r=1: $(1)^3 + 3(1)^2 + 4(1) - 8 = 0$ ✓

$$\begin{array}{r|l} & r^2 + 4r + 8 \\ \hline r-1 & r^3 + 3r^2 + 4r - 8 \\ & \underline{r^3 - r^2} \\ & \quad 4r^2 + 4r \\ & \quad \underline{4r^2 - 4r} \\ & \qquad 8r - 8 \\ & \qquad \underline{8r - 8} \\ & \qquad\quad 0 \end{array}$$

$\therefore\ r^3 + 3r^2 + 4r - 8 = (r-1)(r^2 + 4r + 8) = 0$

$r - 1 = 0$ or $\boxed{r^2 + 4r + 8 = 0}$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16 - 32}}{2}$$

$$= -2 \pm \frac{\sqrt{-16}}{2} = -2 \pm \frac{4i}{2} = -2 \pm 2i$$

$\therefore\ r = 1,\ -2 \pm 2i$

$\therefore\ y_1 = e^{x},\quad y_2 = e^{-2x}\cos 2x,$

$y_3 = e^{-2x}\sin 2x$

are lin. indep. sol'ns of the diff. eq'n.

$\therefore$ a gen. sol'n is

$$y(x) = c_1 e^{x} + c_2 e^{-2x}\cos 2x + c_3 e^{-2x}\sin 2x.$$

Case 4: Repeated complex pair of roots

If $r = a \pm ib$ is a pair of complex roots of (2) that are repeated k times, then

$$e^{ax}\cos bx,\quad xe^{ax}\cos bx,\ \ldots,\ x^{k-1}e^{ax}\cos bx,$$

$$e^{ax}\sin bx,\quad xe^{ax}\sin bx,\ \ldots,\ x^{k-1}e^{ax}\sin bx,$$

are linearly independent sol'ns of (1), so we add

$$c_1 e^{ax}\cos bx + \cdots + c_k x^{k-1} e^{ax}\cos bx + c_{k+1} e^{ax}\sin bx + \cdots + c_{2k} x^{k-1} e^{ax}\sin bx$$

to the general sol'n of (1).

Example: (#20)

$$y^{(4)} + 2y^{(3)} + 3y'' + 2y' + y = 0$$

Sol'n: Char. eq'n: $r^4 + 2r^3 + 3r^2 + 2r + 1 = 0$

Hint: expand $(r^2+r+1)^2$

$$
\begin{aligned}
(r^2+r+1)^2 &= (r^2+r+1)(r^2+r+1) \\
&= r^4+r^3+r^2 \\
&\quad + r^3+r^2+r \\
&\quad\quad + r^2+r+1 \\
&= r^4+2r^3+3r^2+2r+1
\end{aligned}
$$

$$r^4+2r^3+3r^2+2r+1=0$$

$$(r^2+r+1)^2=0$$

$r^2+r+1=0$ or $r^2+r+1=0$

$$r=\frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2\cdot1}$$

$$=\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}.$$

$$\therefore\ r=\frac{-1}{2}\pm i\frac{\sqrt{3}}{2},\ \frac{-1}{2}\pm i\frac{\sqrt{3}}{2}.$$

$$\therefore\ y_1=e^{-x/2}\cos\tfrac{\sqrt{3}}{2}x,\quad y_2=xe^{-x/2}\cos\tfrac{\sqrt{3}}{2}x,$$

$$y_3=e^{-x/2}\sin\tfrac{\sqrt{3}}{2}x,\quad y_4=xe^{-x/2}\sin\tfrac{\sqrt{3}}{2}x.$$

$\therefore$ a gen. sol'n is:

$$y(x) = c_1 e^{-x/2} \cos \tfrac{\sqrt{3}}{2}x + c_2 x e^{-x/2} \cos \tfrac{\sqrt{3}}{2}x + c_3 e^{-x/2} \sin \tfrac{\sqrt{3}}{2}x + c_4 x e^{-x/2} \sin \tfrac{\sqrt{3}}{2}x.$$

<u>ie</u>

$$y(x) = e^{-x/2}\left[(c_1 + c_2 x) \cos \tfrac{\sqrt{3}}{2}x + (c_3 + c_4 x) \sin \tfrac{\sqrt{3}}{2}x \right]$$