4.3 Translation + Partial Fractions

$$\boxed{\mathcal{L}\{e^{at}f(t)\} = F(s-a)}$$

translation rule

Examples:

(a) $\mathcal{L}\{te^{2t}\}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$$\mathcal{L}\{te^{2t}\} = \frac{1}{(s-2)^2}$$

(b) $\mathcal{L}\{e^{-2t}\cos 3t\}$

$\mathcal{L}\{\cos 3t\} = \frac{s}{s^2+9}$

$$\mathcal{L}\{e^{-2t}\cos 3t\} = \frac{(s+2)}{(s+2)^2+9}$$

A good way to use the translation rule is:

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}}$$

ie factor out e^{at}

Examples:

(#7) $F(s) = \dfrac{1}{s^2+4s+4}$

$$F(s) = \frac{1}{(s+2)^2}$$

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$f(t) = te^{-2t}$$

(#9) $F(s) = \dfrac{3s+5}{s^2-6s+25}$

$$F(s) = \frac{3s+5}{(s-3)^2+16} = \frac{3(s-3)+14}{(s-3)^2+16}$$

$$f(t) = e^{3t}\mathcal{L}^{-1}\left\{\frac{3s+14}{s^2+16}\right\}$$

$$f(t) = e^{3t}\mathcal{L}^{-1}\left\{3\cdot\frac{s}{s^2+16} + \frac{14}{4}\cdot\frac{4}{s^2+16}\right\}$$

$$f(t) = e^{3t}\left(3\cos 4t + \frac{7}{2}\sin 4t\right)$$

Partial Fractions

Repeated real roots: $(s-a)^n$ in denominator

$$\boxed{\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^n}}$$

add to partial fractions

Repeated complex roots: $\left((s-a)^2+b^2\right)^n$ in denominator

$$\boxed{\frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{\left((s-a)^2+b^2\right)^2} + \cdots + \frac{A_n s + B_n}{\left((s-a)^2+b^2\right)^n}}$$

add to partial fractions

Examples:

(#15) $F(s) = \dfrac{1}{s^3 - 5s^2}$

$$F(s) = \frac{1}{s^2(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-5}$$

$$1 = As(s-5) + B(s-5) + Cs^2$$

$\underline{s=0}$: $1 = -5B$ $\quad B = -1/5$

$\underline{s=5}$: $1 = 25C \qquad C = 1/25$

$\underline{\text{coeff. of } s^2}$: $0 = A + C \qquad A = -1/25$

$$F(s) = -\frac{1}{25}\cdot\frac{1}{s} - \frac{1}{5}\cdot\frac{1}{s^2} + \frac{1}{25}\cdot\frac{1}{s-5}$$

$$f(t) = \frac{-1}{25} - \frac{1}{5}t + \frac{1}{25}e^{5t}$$

(#19) $F(s) = \dfrac{s^2 - 2s}{s^4 + 5s^2 + 4}$

$$F(s) = \frac{s^2-2s}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$s^2 - 2s = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$\underline{s=i}$: $-1-2i = (Ai+B)(3)$ | $A = -2/3$

$-2 = 3A \qquad -1 = 3B$ | $B = -1/3$

$\underline{s=2i}$: $-4-4i = (2Ci+D)(-3)$ | $C = 2/3$

$-4 = -6C \qquad -4 = -3D$ | $D = 4/3$

$$F(s) = \frac{-\frac{2}{3}s - \frac{1}{3}}{s^2+1} + \frac{\frac{2}{3}s + \frac{4}{3}}{s^2+4}$$

$$F(s) = -\frac{2}{3}\cdot\frac{s}{s^2+1} - \frac{1}{3}\cdot\frac{1}{s^2+1} + \frac{2}{3}\cdot\frac{s}{s^2+4} + \frac{2}{3}\frac{2}{s^2+4}$$

$$f(t) = -\frac{2}{3}\cos t - \frac{1}{3}\sin t + \frac{2}{3}\cos 2t + \frac{2}{3}\sin 2t$$

Now let's apply these techniques to solve initial value problems.

$$\boxed{\begin{aligned} \mathcal{L}\{x''\} &= s^2X(s) - s\,x(0) - x'(0) \\ \mathcal{L}\{x'\} &= sX(s) - x(0) \end{aligned}}$$

<u>Examples:</u>

(#30) $x'' + 4x' + 8x = e^{-t}, \quad x(0) = x'(0) = 0$

$$\left[s^2X(s) - s\,x(0) - x'(0)\right] + 4\left[sX(s) - x(0)\right] + 8X(s) = \frac{1}{s+1}$$

$$(s^2+4s+8)X(s) = \frac{1}{s+1}$$

$$X(s) = \frac{1}{(s+1)\left[(s+2)^2+4\right]} = \frac{A}{s+1} + \frac{Bs+C}{(s+2)^2+4}$$

$$1 = A\left[(s+2)^2+4\right] + (Bs+C)(s+1)$$

$\underline{s=-1:}$ $1 = A(1+4)$ $\quad$ $A = 1/5$

$\underline{s=-2+2i:}$ $1 = (B(-2+2i)+C)(-1+2i)$

$1 = (-2B+2Bi+C)(-1+2i)$

$1 = 2B - 2Bi - C - 4Bi - 4B + 2Ci$

$1 = -2B - 6Bi - C + 2Ci$

$$1 = -2B - C \qquad \left[\begin{array}{cc|c} -2 & -1 & 1 \\ -6 & 2 & 0 \end{array}\right]$$
$$0 = -6B + 2C$$

$$B = \frac{\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}}{\begin{vmatrix} -2 & -1 \\ -6 & 2 \end{vmatrix}} \qquad C = \frac{\begin{vmatrix} -2 & 1 \\ -6 & 0 \end{vmatrix}}{\begin{vmatrix} -2 & -1 \\ -6 & 2 \end{vmatrix}}$$

$$B = \frac{2}{-10} \qquad C = \frac{6}{-10}$$

$$B = -1/5 \qquad C = -3/5$$

$$X(s) = \frac{1}{5}\cdot\frac{1}{s+1} + \frac{-\frac{1}{5}s - \frac{3}{5}}{(s+2)^2+4}$$

$$X(s) = \frac{1}{5}\cdot\frac{1}{s+1} - \frac{1}{5}\,\frac{(s+2)+1}{(s+2)^2+4}$$

$$x(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+4}\right\}$$

$$x(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4} + \frac{1}{2}\cdot\frac{2}{s^2+4}\right\}$$

$$x(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

(#34) $x^{(4)} + 13x'' + 36x = 0$

$x(0) = 0, \quad x'(0) = 2, \quad x''(0) = 0, \quad x'''(0) = -13$

$$\left[s^4X(s) - s^3\cancel{x(0)} - s^2x'(0) - s\cancel{x''(0)} - x'''(0)\right]$$

$$+ 13\left[s^2X(s) - s\cancel{x(0)} - x'(0)\right] + 36X(s) = 0$$

$$(s^4 + 13s^2 + 36)X(s) - 2s^2 + 13 - 26 = 0$$

$$X(s) = \frac{2s^2 + 13}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$$

$$2s^2 + 13 = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

$\underline{s = 2i:}$ $\quad -8+13 = (2Ai+B)(-4+9)$

$$5 = 10Ai + 5B$$

$$0 = 10A \qquad 5 = 5B \qquad \Bigg| \quad A = 0, \quad B = 1$$

$\underline{s = 3i:}$ $\quad -18+13 = (3Ci+D)(-9+4)$

$$-5 = -15Ci - 5D$$

$$0 = -15C \qquad -5 = -5D \qquad \Bigg| \quad C = 0, \quad D = 1$$

$$X(s) = \frac{1}{s^2+4} + \frac{1}{s^2+9} = \frac{1}{2} \cdot \frac{2}{s^2+4} + \frac{1}{3} \cdot \frac{3}{s^2+9}$$

$$x(t) = \frac{1}{2}\sin 2t + \frac{1}{3}\sin 3t$$