4.1) Laplace Transforms

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt}$$

the Laplace transform of $f(t)$

Examples: Use the above definition to directly find the Laplace transform.

(#1) $f(t) = t$ $\qquad F(s) = \frac{1}{s^2}$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot t\,dt$$

$$= \lim_{b\to\infty} \int_0^b t e^{-st}\,dt$$

$$\boxed{\begin{aligned} (fg)' &= fg' + f'g \\ fg &= \int fg' + \int f'g \\ \int fg' &= fg - \int f'g \end{aligned}}$$

$$= \lim_{b\to\infty} \left[-\frac{t}{s} e^{-st} \Big|_0^b - \int_0^b \frac{-1}{s} e^{-st}\,dt \right]$$

$$= \lim_{b\to\infty} \left[-\frac{b}{s} e^{-sb} - 0 - \left[\frac{1}{s^2} e^{-st} \right]_0^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{b}{s} e^{-sb} - \left(\frac{1}{s^2} e^{-sb} - \frac{1}{s^2} \right) \right]$$

$$= \lim_{b\to\infty}\left[-\frac{b}{s}e^{-sb} - \frac{1}{s^2}e^{-sb} + \frac{1}{s^2}\right] \quad (s>0)$$

$$= 0 - 0 + \frac{1}{s^2} = \frac{1}{s^2}.$$

(#3) $f(t) = e^{3t+1}$

$f(t) = e\cdot e^{3t}$ $\qquad$ $F(s) = \dfrac{e}{s-3}$

$$F(s) = \int_0^\infty e^{-st}e^{3t+1}\,dt$$

$$= \lim_{b\to\infty}\int_0^b e^{(3-s)t}\cdot e\,dt$$

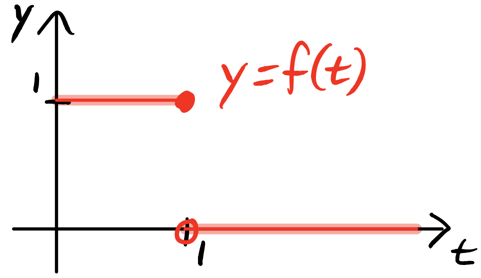
$$= \lim_{b\to\infty}\left[\frac{1}{3-s}e^{(3-s)t}\cdot e\right]_0^b$$

$$= \lim_{b\to\infty}\left[\frac{1}{3-s}e^{(3-s)b}\cdot e - \frac{1}{3-s}\cdot 1\cdot e\right]$$

$$= 0 + \frac{1}{s-3}\cdot e \qquad (s>3)$$

$$= \frac{e}{s-3}$$

(#7) $f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ 0, & t > 1. \end{cases}$



$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st} \cdot 1\,dt + \int_1^\infty e^{-st} \cdot 0\,dt$$

$$= \left[-\frac{1}{s} e^{-st} \right]_0^1$$

$$= -\frac{1}{s} e^{-s} + \frac{1}{s} = \boxed{\frac{1-e^{-s}}{s}}$$

$$\boxed{u(t-a) = \begin{cases} 0, & t < a \\ 1, & t \ge a \end{cases}}$$

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$F(s) = \frac{1}{s} - \frac{e^{-s}}{s} \Rightarrow \boxed{f(t) = 1 - u(t-1)}$$

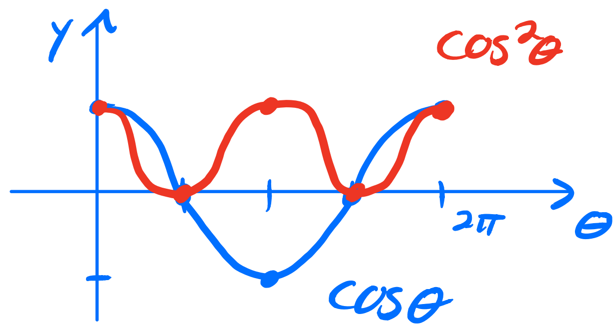
Examples: Use the Table of Laplace Transforms to find the Laplace transform.

(#15) $f(t) = 1 + \cosh 5t$

$$F(s) = \frac{1}{s} + \frac{s}{s^2-25} = \frac{s^2-25+s^2}{s(s^2-25)}$$

$$F(s) = \frac{2s^2-25}{s(s^2-25)}$$

(#17) $f(t) = \cos^2 2t$



$$f(t) = \frac{1}{2}(1+\cos 4t)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$F(s) = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+16}\right)$$

$$F(s) = \frac{1}{2}\left(\frac{s^2+16+s^2}{s(s^2+16)}\right)$$

$$F(s) = \frac{s^2+8}{s(s^2+16)}$$

(#21) $f(t) = t\cos 2t$

$$\boxed{\mathcal{L}\{t f(t)\} = -F'(s)}$$

$$\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -\left(\frac{s^2+4-s(2s)}{(s^2+4)^2}\right)$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

$$F(s) = \int_0^\infty e^{-st}\cdot t\cos 2t\,dt$$

$$\boxed{\int fg' = fg - \int f'g}$$

$$\int e^{-st}\cos 2t\,dt = -\frac{1}{s}e^{-st}\cos 2t - \int \frac{2}{s}e^{-st}\sin 2t\,dt$$

$$= -\frac{1}{s}e^{-st}\cos 2t - \int \frac{2}{s}e^{-st}\sin 2t\,dt$$

$$= -\frac{1}{s}e^{-st}\cos 2t - \left[\frac{-2}{s^2}e^{-st}\sin 2t - \int \frac{-4}{s^2}e^{-st}\cos 2t\,dt\right]$$

$$= -\frac{1}{s}e^{-st}\cos 2t + \frac{2}{s^2}e^{-st}\sin 2t - \frac{4}{s^2}\int e^{-st}\cos 2t\,dt$$

$$\Rightarrow \left(1+\frac{4}{s^2}\right)\int e^{-st}\cos 2t\,dt$$

$$= -\frac{1}{s}e^{-st}\cos 2t + \frac{2}{s^2}e^{-st}\sin 2t$$

$$\Rightarrow \left(\frac{s^2+4}{s^2}\right)\int e^{-st}\cos 2t\,dt$$

$$= e^{-st}\left(-\frac{1}{s}\cos 2t + \frac{2}{s^2}\sin 2t\right)$$

$$\Rightarrow \int e^{-st}\cos 2t\,dt = \frac{s^2}{s^2+4}e^{-st}\left(-\frac{1}{s}\cos 2t + \frac{2}{s^2}\sin 2t\right)$$

$$\boxed{\int e^{-st}\cos 2t\,dt = \frac{1}{s^2+4}e^{-st}(-s\cos 2t + 2\sin 2t)}$$

$$F(s) = \int_0^\infty t e^{-st}\cos 2t\,dt \qquad \boxed{\int fg' = fg - \int f'g}$$

$$= \left[t\frac{1}{s^2+4}e^{-st}(-s\cos 2t + 2\sin 2t)\right]_0^\infty$$

$$- \int_0^\infty \frac{1}{s^2+4}e^{-st}(-s\cos 2t + 2\sin 2t)\,dt$$

$$= \lim_{b\to\infty}\left[\frac{b}{s^2+4}e^{-sb}(-s\cos 2b + 2\sin 2b) - 0\right]$$

$$+ \frac{s}{s^2+4}\int_0^\infty e^{-st}\cos 2t\,dt - \frac{2}{s^2+4}\int_0^\infty e^{-st}\sin 2t\,dt$$

$$= 0 + \frac{s}{s^2+4}\cdot\frac{s}{s^2+4} - \frac{2}{s^2+4}\cdot\frac{2}{s^2+4}$$

$$= \frac{s^2-4}{(s^2+4)^2}$$

Examples: Use the Table of Laplace Transforms to find the inverse Laplace transform.

(#23) $F(s) = \frac{3}{s^4}$

$(n=3)$

$$\boxed{\begin{aligned} \mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}} \\ \mathcal{L}\{t^3\} &= \frac{6}{s^4} \end{aligned}}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s^4}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{6}{s^4}\right\} = \frac{1}{2}t^3$$

(#27) $F(s) = \frac{3}{s-4}$

$$\boxed{\begin{aligned} \mathcal{L}\{e^{at}\} &= \frac{1}{s-a} \\ \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} &= e^{4t} \end{aligned}}$$

$$\mathcal{L}^{-1}\left\{\frac{3}{s-4}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} = 3e^{4t}$$

(#29) $F(s) = \dfrac{5-3s}{s^2+9}$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$$

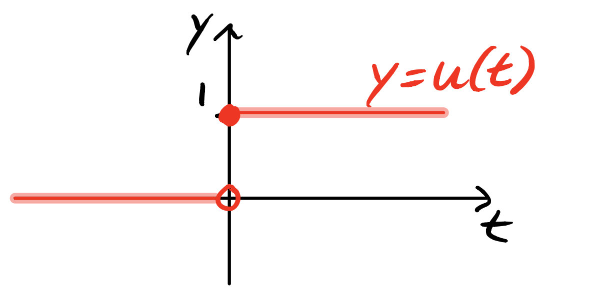
$$\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$$

$$F(s) = \frac{5-3s}{s^2+9} = \frac{5}{3}\cdot\frac{3}{s^2+9} - 3\cdot\frac{s}{s^2+9}$$

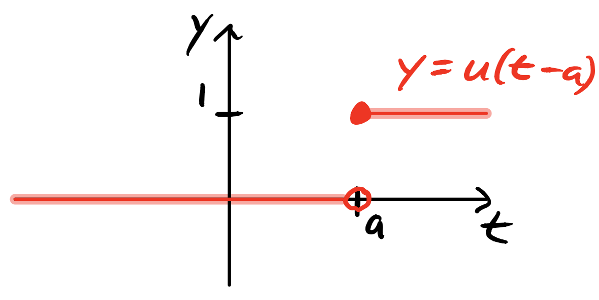
$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{5}{3}\sin 3t - 3\cos 3t.$$

Unit step function

$$u(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$$



$$u(t-a) = \begin{cases} 0, & t<a \\ 1, & t \geq a \end{cases}$$



Example: Find the Laplace transform.

$f(t) = u(t-a)$

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} \cdot 1\,dt$$

$$= \lim_{b \to \infty} \left[-\frac{1}{s} e^{-st} \right]_a^b$$

$$= \lim_{b \to \infty} \left[-\frac{1}{s} e^{-sb} + \frac{1}{s} e^{-sa} \right] \qquad (s > 0)$$

$$= 0 + \frac{e^{-as}}{s} = \frac{e^{-as}}{s}.$$

<u>Transforms of Integrals</u> (from Section 4.2)

$$\boxed{\mathcal{L}\left\{ \int_0^t f(\tau)\,d\tau \right\} = \frac{F(s)}{s}}$$

$$\boxed{\mathcal{L}^{-1}\left\{ \frac{F(s)}{s} \right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}\,d\tau}$$

Examples: (Section 4.2)

(#17) $F(s) = \dfrac{1}{s(s-3)}$ $\qquad \mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{3t}$

$$f(t) = \int_0^t e^{3\tau}\,d\tau = \frac{1}{3}e^{3\tau}\Big|_0^t$$

$$f(t) = \frac{1}{3}e^{3t} - \frac{1}{3}$$

(#21) $F(s) = \dfrac{1}{s^2(s^2+1)}$ $\qquad \mathcal{L}^{-1}\left\{\dfrac{1}{s^2+1}\right\} = \sin t$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin\tau\,d\tau = -\cos\tau\Big|_0^t$$

$$= 1 - \cos t$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1-\cos\tau)\,d\tau$$

$$= \Big[\tau - \sin\tau\Big]_0^t$$

$$= t - \sin t$$

Check: $\dfrac{1}{s^2} - \dfrac{1}{s^2+1} = \dfrac{s^2+1-s^2}{s^2(s^2+1)} = \dfrac{1}{s^2(s^2+1)}$ ✓