MTL783: Theory of Computation	1st Semester, 2025-2026

Quiz 1 Solutions - 18/08/2025

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Q1.

Problem: Write regular expressions for the following cases:

1.1) $L = \{w \in \{0,1\}^* : w \text{ starts and ends with } 1\}$

1.2) $L = \{w \in \{a,b\}^* : w \text{ has exactly one } b\}$

Solution:

1.1) Regular expression:

$$w = 1(0+1)^*1 + 1$$

Explanation: The string must start and end with 1. Between them, there can be any string over $\{0,1\}$. The case $w =$ "1" (a single symbol) is not captured by $1(0+1)^*1$, so we include it separately.

1.2) Regular expression:

$$w = a^*ba^*$$

Explanation: The string must contain exactly one b. It can be preceded by zero or more a's and followed by zero or more a's. (This expression also covers edge cases like $w = b$, $w = ab$, and $w = ba$).

Q2. Design an NFA that accepts $L = \{a\} \cup \{b^n : n \geq 1\}$, over $\Sigma = \{a,b\}$, using exactly one final state and *no* λ-transitions.

Solution: Define $M = (Q, \Sigma, \delta, q_0, F)$ where

$$Q = \{q_0, q_b, q_f\}, \qquad \Sigma = \{a,b\}, \qquad q_0 \text{ is the start state}, \qquad F = \{q_f\}.$$

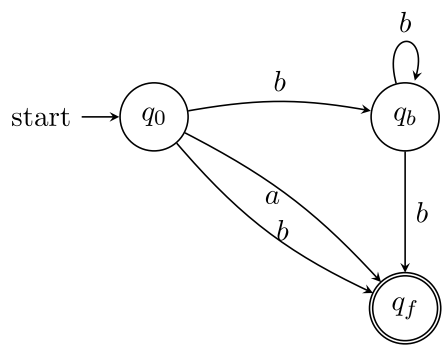
The transition function δ is

$$\begin{aligned} \delta(q_0, a) &= \{q_f\}, & \delta(q_0, b) &= \{q_b, q_f\}, \\ \delta(q_b, a) &= \emptyset, & \delta(q_b, b) &= \{q_b, q_f\}, \\ \delta(q_f, a) &= \emptyset, & \delta(q_f, b) &= \emptyset. \end{aligned}$$

Transition table.

	a	b
q_0	$\{q_f\}$	$\{q_b, q_f\}$
q_b	$\emptyset$	$\{q_b, q_f\}$
q_f	$\emptyset$	$\emptyset$

Diagram (no λ-moves, one final state).



Why it works (proof).

- The string a is accepted by $q_0 \xrightarrow{a} q_f$; since q_f has no outgoing edges, any extra symbol would be impossible to read, so only the single a is accepted from that branch.
- For b^n with $n \geq 1$:
  - If $n = 1$, take $q_0 \xrightarrow{b} q_f$.
  - If $n \geq 2$, take $q_0 \xrightarrow{b} q_b$, then loop on q_b with b for the first $n-1$ symbols and finally take $q_b \xrightarrow{b} q_f$ on the last b.
- No other strings are accepted:
  - ε is rejected since $q_0 \notin F$ and there are no λ-moves.
  - Any string containing an a after the first symbol is rejected because q_b and q_f have no a-transitions.
  - Any string longer than one symbol that begins with a is rejected since after $q_0 \xrightarrow{a} q_f$ there are no outgoing transitions to consume more input.

Hence $L(M) = \{a\} \cup b^+$, with exactly one final state (q_f) and no λ-transitions.

Q3. Prove that if L is a regular language, then its reversal L^R is also regular.

Solution: The reversal of a string $w = a_1 a_2 \dots a_n$ is defined as

$$w^R = a_n \dots a_2 a_1.$$

For a language $L \subseteq \Sigma^*$, we define

$$L^R = \{ w^R \mid w \in L \}.$$

We aim to show that L^R is regular whenever L is regular.

Step 1: Construction of the New NFA

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA such that $L = L(M)$. We construct a new NFA $M^R = (Q', \Sigma, \Delta, q'_{\text{new}}, F')$ as follows:

- $Q' = Q \cup \{q'_{\text{new}}\}$, where q'_{new} is a fresh start state.
- $F' = \{q_0\}$ (the old start state becomes the only final state).
- For every $f \in F$, add a λ-transition from q'_{new} to f.
- For every $p \in Q$, $a \in \Sigma$, and every $q \in \delta(p, a)$, add p to $\Delta(q, a)$. Equivalently,

$$\Delta(q, a) \supseteq \{ p \in Q \mid q \in \delta(p, a) \}.$$

Intuitively: - We reverse all the edges, - Swap initial and final roles, - Add a new start state with λ-moves into the old final states.

Step 2: Correctness Proof

We need to show that $L(M^R) = L^R$.

(i) If $w \in L$, then $w^R \in L(M^R)$. Suppose $w \in L$. Then there exists an accepting run in M:

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} q_2 \cdots \xrightarrow{a_n} q_n, \quad \text{with } q_n \in F \text{ and } w = a_1 a_2 \cdots a_n.$$

By construction, M^R has the reversed edges:

$$q_n \xrightarrow{a_n} q_{n-1} \xrightarrow{a_{n-1}} \cdots \xrightarrow{a_1} q_0.$$

Moreover, M^R starts at q'_{new}, takes a λ-transition to q_n, and then follows this reversed path on $a_n a_{n-1} \dots a_1 = w^R$. It ends in q_0, which is final in M^R. Thus M^R accepts w^R, so $w^R \in L(M^R)$.

(ii) If $x \notin L^R$, then $x \notin L(M^R)$. Let $x \in \Sigma^*$ and suppose $x \notin L^R$. Since reversal is a bijection on Σ^*, there exists a unique $w \in \Sigma^*$ with $x = w^R$. The assumption $x \notin L^R$ is equivalent to $w \notin L$.

Assume for contradiction that $x \in L(M^R)$. Then there is an accepting run of M^R on x:

$$q'_{\text{new}} \xrightarrow{\lambda} f \in F \xrightarrow{a_k} q_{k-1} \xrightarrow{a_{k-1}} \cdots \xrightarrow{a_2} q_1 \xrightarrow{a_1} q_0,$$

where $x = a_k a_{k-1} \cdots a_1$ and q_0 is the (only) final state of M^R. By construction of M^R, each transition $q_i \xrightarrow{a_{i+1}} q_{i+1}$ in M^R corresponds to a reversed transition in M, hence in M we have the forward path

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} f \in F,$$

which is an accepting computation of M on the word $a_1 a_2 \cdots a_k = w$. Therefore $w \in L$, contradicting $w \notin L$. Hence $x \notin L(M^R)$, and thus $L(M^R) \subseteq L^R$.

(iii) Conclusion. We have shown both inclusions:

$$L^R \subseteq L(M^R) \quad \text{and} \quad L(M^R) \subseteq L^R.$$

Therefore $L(M^R) = L^R$. Since M^R is an NFA, L^R is regular.

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