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Recall

Weak law of large numbers (WLLN) $p=1$

$X_1, X_2, \dots, X_n$ random sample

then $\bar{X}_n \xrightarrow{p} \underbrace{\mu}_{\text{population mean}}$ $\left[\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} x_i\right] \leftarrow$ sample mean

that is $\boxed{\lim_{n\to\infty} P(|\bar{X}_n - \mu| > \varepsilon) = 0}$

Strong law of large numbers

$$P\left(\lim_{n\to\infty} \bar{X}_n = \mu\right) = 1 \Rightarrow \bar{X}_n \xrightarrow[\text{almost surely}]{p} \mu$$

for $p > 1$

$\underset{\sim}{X}_{p\times 1}$ population with mean $\underset{\sim}{\mu}_{p\times 1} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix}$

let $\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_n$ be a random sample (iid)

$$\underset{\sim}{\bar{X}}_{(n)\,p\times 1} = \frac{1}{n}\sum_{i=1}^{n} X_i = \begin{bmatrix} \bar{X}_{(n)1} \\ \bar{X}_{(n)2} \\ \vdots \\ \bar{X}_{(n)p} \end{bmatrix}_{p\times 1}$$

use WLLN on $\underset{\sim}{x}_1$: $\underset{\sim}{x}_{(n)1} \xrightarrow{p} \mu$

component wise convergence $\forall\ 1 \le i \le p$

Weak law of large number in p-dim

$$\underset{\sim}{\bar{X}}_{(n)} \xrightarrow{p} \mu \quad \text{component wise}$$

$\therefore \underset{\sim}{\bar{X}}_{(n)}$ is a Consistent Estimator of μ

$S_{p\times p} \rightarrow$ Sample var-cov-var matrix

$$s_{ij} = \frac{1}{n-1}\sum_{k=1}^{n}(x_{ki} - \bar{x}_i)(x_{kj} - \bar{x}_j)$$

$$= \frac{1}{n-1} \sum_{k=1}^{n} (x_{ki} - \mu_i + \mu_i - \overline{x}_i)(x_{kj} - \mu_j + \mu_j - \overline{x}_j)$$

$$= \frac{1}{n-1} \left\{ \sum_{k=1}^{n} (x_{ki} - \mu_i)(x_{kj} - \mu_j) + (\mu_i - \overline{x}_i) \sum_{k=1}^{n} (x_{kj} - \mu_j) + (\mu_j - \overline{x}_j) \sum_{k=1}^{n} (x_{ki} - \mu_i) + n(\mu_i - \overline{x}_i)(\mu_j - \overline{x}_j) \right\}$$

$$(n-1) s_{ij} = \sum_{k=1}^{n} (x_{ki} - \mu_i)(x_{kj} - \mu_j) + (\mu_i - \overline{x}_i) n (\overline{x}_j - \mu_j) + (\mu_j - \overline{x}_j) n (\overline{x}_i - \mu_i) + n(\overline{x}_i - \mu_i)(\overline{x}_j - \mu_j)$$

$$= \sum_{k=1}^{n} (x_{ki} - \mu_i)(x_{kj} - \mu_j) - n(\overline{x}_i - \mu_i)(\overline{x}_j - \mu_j)$$

by WLLN argument

$$\overline{x}_i \xrightarrow{p} \mu_i$$

$$\overline{x}_j \xrightarrow{p} \mu_j$$

$$\Rightarrow (\overline{x}_i - \mu_i)(\overline{x}_j - \mu_j) \xrightarrow{p} 0$$

for the 1st term

$$E(x_{ki} - \mu_i)(x_{kj} - \mu_j) = \sigma_{ij}$$

$$\Sigma = [\sigma_{ij}]_{p \times p} \quad \text{pop var cov matrix}$$

by WLLN:

$$\frac{1}{n} \sum_{k=1}^{n} (x_{ki} - \mu_i)(x_{kj} - \mu_j) \xrightarrow{p} \sigma_{ij}$$

$$\frac{n-1}{n} \times \frac{1}{n-1} \sum_{k=1}^{n} (x_{ki} - \mu_i)(x_{kj} - \mu_j) \xrightarrow{p} \sigma_{ij}$$

$$s_{ij} \xrightarrow{p} \sigma_{ij} \quad \forall i, j$$

$\therefore$ Componentwise

$$S = [s_{ij}] \xrightarrow{p} \Sigma = [\sigma_{ij}]$$

under WLLN

$\therefore$ ★ S is a Consistent Estimator of Σ

$$\lim_{n \to \infty} P(|s_{ij} - \sigma_{ij}| > \varepsilon) = 0 \quad \forall i, j$$

$n \to \infty$

Central limit theorem

Recal $p = 1$

let X be a random variable representing the random variable representing the population with $E(X) = \mu < \infty$ and $Var(X) = \sigma^2 <$

then $Z = \frac{\bar{X}_n - \mu}{\sigma / \sqrt{n}} \xrightarrow{D} N(0,1)$ $\left(\bar{X}_n = \frac{1}{n} \sum X_i \right)$

convergence in distribution $X_1, \dots, X_n$ are iid

→ If we have the population RV $X \sim N(\mu, \sigma^2)$

then $Z \sim N(0,1)$ $\forall$ $n \geq 1$