$S = \cancel{\frac{1}{n}} X^T H X$ $\qquad H = I - \frac{1}{n} \mathbf{1} . \mathbf{1}^T$

$\underbrace{H}$ → Centering matrix

$H^T H \qquad H^2 = H$

~~LT~~ $n \leq p \Rightarrow |S| = 0 \Rightarrow$ The sample indicates that features have degeneracy or dependency

~~Proof~~ $\Delta = \begin{bmatrix} d_1, \ldots, d_p \end{bmatrix}_{n \times p}$ $n \leq p \Rightarrow \text{rank}\, \Delta \leq n$

$$\sum_{i=1}^{n} d_{ij} = 0 \quad \forall \; 1 \leq j \leq p$$

$\therefore$ By row transformation, last row of Δ can be reduced to zero row $\Rightarrow \text{rank}\, \Delta \leq n-1 < p$

$S = \frac{1}{(n-1)} \Delta^T \Delta$ $\quad$ rank S = rank Δ $\quad$ [rank M = rank $M^T M$ derived using rank-nullity Thm]

$\Rightarrow$ rank $S_{p\times p} < p$

$\Rightarrow |S| = 0$

Limitation of $|S| \rightarrow$ Doesn't capture orientation

$p=2$ & we have n-sample data pts.

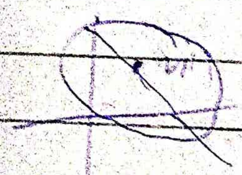
Suppose Sample mean $\bar{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ & let $S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$

$R = [r_{ij}]$ $\quad r_{ij} = \frac{s_{ij}}{\sqrt{s_{ii}}\sqrt{s_{jj}}}$ $\quad R = \begin{bmatrix} 1 & 4/5 \\ 4/5 & 1 \end{bmatrix}$

Sample I

Take same features but second Sample to Compute $\bar{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

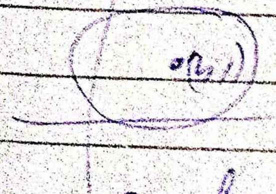
Var-Cov matrix $\rightarrow \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$ $\quad R = \begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix}$



Sample II

$|S| = 9$ in both cases.

Take IIIrd sample $\quad \bar{X} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad S = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$



Sample III

$D = \begin{bmatrix} \cancel{\frac{1}{\sqrt{s_{11}}}}\ \frac{1}{s_{11}} & 0 \\ 0 & \frac{1}{s_{22}} \end{bmatrix}$

$D^{1/2} R D^{1/2} = S \longrightarrow$ Sure ??

$D^{-1/2} S D^{-1/2} = R$

$$\sum d_{ik} s_{kl} d_{lj} = d_{ij} s_{ij} d_{jj}$$

Multivaliate Normal Distribution

$X \sim N(\mu, \sigma^2)$ $p=1$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$p=2$ $X = [x_1, x_2]_{1\times 2}$ $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ $\sigma_{11} = \sigma_1^2$, $\sigma_{22} = \sigma_2^2$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_{2\times 1}$$

$$d(x,\mu) = d((x_1,x_2),(\mu_1,\mu_2)) = (x-\mu)^T \Sigma^{-1} (x-\mu)$$

Mahlanobanis distance

$p=2$

$$(x_1-\mu_1 \quad x_2-\mu_2) \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}^{-1} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$

$$= \frac{(x_1-\mu_1 \quad x_2-\mu_2)}{(\sigma_1^2\sigma_2^2 - (\sigma_{12})^2)} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix} \begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}$$

$$= \frac{\sigma_2^2}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}(x_1-\mu_1)^2 + \frac{\sigma_1^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}(x_2-\mu_2)^2 - \frac{2\sigma_{12}(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1^2\sigma_2^2-(\sigma_{12})^2}$$

$$\frac{\sigma_2^2}{\sigma_1^2\sigma_2^2-\sigma_{12}^2} = \frac{1}{\sigma_1^2 - \frac{(\sigma_{12})^2}{\sigma_2^2}} = \frac{1}{\sigma_1^2\left(1-\frac{\sigma_{12}^2}{\sigma_1^2\sigma_2^2}\right)} = \frac{1}{\sigma_1^2(1-\rho^2)}$$

$$\frac{\sigma_{12}}{\sigma_1^2\sigma_2^2-(\sigma_{12})^2} = \frac{1}{(\sigma_{12})\left(\frac{\sigma_1^2\sigma_2^2}{(\sigma_{12})^2}-1\right)} = \frac{1}{\sigma_{12}\left(\frac{1}{\rho^2}-1\right)}$$

$$= \frac{\sigma_{12}}{\sigma_1^2\sigma_2^2}\left(\frac{1}{1-\rho^2}\right) = \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}$$

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$$\therefore d(\vec{x}, \vec{\mu}) = \frac{1}{(1-\rho^2)} \left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - \frac{2\rho}{\sigma_1\sigma_2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) \right)$$

pdf of $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ $\quad f_X(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2} d(\vec{x}, \vec{\mu})\right)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_X(x_1, x_2)\, dx_1\, dx_2 = 1$$

In general

$$f_X(\vec{x} = x_1, \dots, x_p) = \frac{1}{(2\pi)^{p/2} \underbrace{|\Sigma|^{1/2}}_{\det(\Sigma)}} \exp\left(-\frac{(x-\mu)^T \Sigma^{-1} (x-\mu)}{2}\right)$$

Result $\rightarrow X \sim N_p(\mu, \Sigma) \iff \forall\ \vec{a} \in \mathbb{R}^p$ $\quad \underbrace{\vec{a}^T X}_{1\times 1} \sim N(\vec{a}^T\mu, \vec{a}^T\Sigma\vec{a})$

① Choosing $\vec{a} = \vec{e_i}$ (standard basis)

$X \sim N_p(\mu, \Sigma) \Rightarrow$ each $X_i \sim N(\mu_i, \underbrace{\sigma_{ii}}_{\sigma_i^2})$ $\quad i = 1, \dots, p$

② Let $A \rightarrow q \times p$ (deterministic)

$b \rightarrow q \times 1$ $\rightarrow$ deterministic

$Y = A\underset{\sim}{X} + b$

$Y \sim N_q(A\mu + b, A^T\Sigma A^T)$ $\rightarrow$ ~~Check if $A\Sigma A^T$ or A^T~~ ΣA