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$p=1$ Sample $X_1, \dots, X_n$ random iid

$\bar{X} \sim N(\mu, \sigma^2)$

$(n-1)\dfrac{S^2}{\sigma^2} \sim \chi^2_{(n-1)}$

Consider $p>1$

$\bar{X} \sim N_p\left(\mu_{p\times 1}, \Sigma_{p\times p}\right)$

If you have a χ^2 distributed RV, then it is square of a Nor

$y \sim \chi^2_\nu$

$y = z_1^2 + \dots + z_\nu^2$ $\quad z_i \sim N(0,1) \ \forall i$

independent rv

$\therefore \quad (n-1)\dfrac{S^2}{\sigma^2} = z_1^2 + \dots + z_{n-1}^2$

$(n-1)S^2 = (\sigma z_1)^2 + \dots + (\sigma z_{n-1})^2 = \underset{1\times 1}{y_1^2} + \dots + \underset{1\times 1}{y_{n-1}^2}$ $\quad y_i = \sigma z_i$ $\sim N(0,$

independent

Getting motivation from this expression, for general $p>1$,

$\underset{p\times p}{(n-1)S} = \underset{p\times 1}{z_1}\underset{1\times p}{z_1^T} + z_2 z_2^T + \dots + z_{n-1}z_{n-1}^T$

$\boxed{\text{all rank 1 matrix}}$

$\underset{p\times 1}{z_i} \sim N_p(0, \Sigma)$ and each z_i are independent random vector

decomposition of into rank 1 random mat

n: sample size
S: sample variance covariance matrix

Wishart Distribution

let A be a $p\times p$ symetric random matrix

if we can write

$A = z_1 z_1^T + \dots + z_m z_m^T$

$z_i \sim N_p(0, \Sigma) \quad 1 \le i \le m$

and $z_1, \dots, z_m$ are independent random vectors

Then we say that $A \sim W_p(m, \Sigma)$ or $W_p(\Sigma, m)$

★ A is p-dim Wishart distributed and m is degree of
↓
number of rank 1 ma

In case $\Sigma = I_{p \times p}$

↳ then $W_p(m, I)$ is called standard form of Wishart distribution

$$\boxed{(n-1)\, S \sim W_p(n-1, \Sigma)}$$

let $p = 1$

$A \sim W_1(m, \Sigma)$

$\Rightarrow A = z_1^2 + z_2^2 + \dots + z_m^2$, $z_i \sim N(0, \sigma^2)$ are independent rv

$\dfrac{A}{\sigma^2} \sim \chi^2_m$

we can take wishart dist. as an extension of χ^2 dist in p-dim

pdf of $W_p(m, \Sigma)$ is

$$f_{W_p}(A) = \begin{cases} \dfrac{|A|^{\frac{1}{2}(m-p-1)} \exp\left(-\frac{1}{2} Tr(\Sigma^{-1} A)\right)}{2^{mp/2}\, \pi^{p(p-1)/4}\, |\Sigma|^{m/2} \prod_{i=1}^{p} \Gamma\left(\frac{m+1-i}{2}\right)} \\ 0 \end{cases}$$

↳ γ-function

① if $A_1 \sim W_p(m_1, \Sigma)$

$A_2 \sim W_p(m_2, \Sigma)$

then $A_1 + A_2 \sim W_p(m_1 + m_2, \Sigma)$

② if $A \sim W_p(m, \Sigma)$ $\vec{a} \in \mathbb{R}^p$

$a^T A a = a^T (z_1 z_1^T + z_2 z_2^T + \dots + z_m z_m^T)\, \vec{a}$

$$a^T A a = a^T(z_1 z_1^T + z_2 z_2^T + \dots + z_m z_m^T)a$$
$$= (a^T z_1)(z_1^T a) + (a^T z_2)(z_2^T a) + \dots + (a^T z_m)(z_m^T a)$$
$$= y_1^2 + y_2^2 + \dots + y_m^2$$

$z_i \sim N_p(0, \Sigma)$

$a^T z_i \sim N_1(0, a^T \Sigma a)$

$y_i \sim N_1(0, a^T \Sigma a)$

$$\frac{y_i}{a^T \Sigma a} \sim N_1(0,1) \quad (a \neq 0)$$

$$\frac{\vec{a}^T A \vec{a}}{\vec{a}^T \Sigma \vec{a}} = \underbrace{\left(\frac{y_1}{\sqrt{a^T \Sigma a}}\right)^2}_{N(0,1)} + \dots + \left(\frac{y_m}{\sqrt{a^T \Sigma a}}\right)^2$$

all are independent

$$\sim \chi^2_{(m)}$$

Converse is Not true

i.e. if $\frac{a^T A a}{a^T \Sigma a} \sim \chi^2_{(m)} \quad \forall \vec{a} \in \mathbb{R}^p$

then we can Not say that $A \sim W_p(\ )$

Counter example

let $\underset{p\times p}{A} = \underset{p\times p}{W} + \underset{p\times p}{S} = z_1 z_1^T + \dots + z_m z_m^T + S$ random

where $W \sim W_p(m, \Sigma)$

$\underset{p\times p}{S} = -S^T$ (skew sym)

S can not be written as bb^T as it is skew sym

$a \in \mathbb{R}^p \quad a^T A a = a^T W a + \cancel{a^T S a}^{\,0}$

$$\frac{a^T A a}{a^T \Sigma a} = \frac{a^T W a}{a^T \Sigma a} \sim \chi^2_{(m)}$$