Lecture 19

Wednesday, 3 September 2025 10:10 AM

$p=1$

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$\frac{\bar{X}-\mu}{s/\sqrt{n}} \sim t_{(n-1)}$$

$$t^2 \sim F_{1,n-1} \Rightarrow \frac{n(\bar{X}-\mu)^2}{s^2} \sim {}_{1,n-1}$$

$$\Rightarrow n(\bar{X}-\mu)(s^2)^{-1}(\bar{X}-\mu) \sim F_{p,n-p}$$

H_0

$p=1$

Population $X \sim N_p(\mu, \Sigma)$

$H_0 : \mu = \mu_0 \quad p\times 1$

$H_1 : \mu \neq \mu_0 \quad p\times 1$

Hotelling T^2 Test
named after Harold Hotelling.

Define a statistics under H_0,

$$T^2 = n(\bar{X}-\mu_0)^T_{1\times p}\,\Sigma^{-1}_{p\times p}\,(\bar{X}-\mu_0)_{p\times 1} \geq 0$$

$T^2 : 1\times 1$

Claim:

$$\frac{n-p}{p(n-1)} T^2 \sim F_{p,n-p\alpha}$$

$\alpha \in (0,1)$
$\hookrightarrow$ level of significance

$1-\alpha \to$ level of confidence

Recall: Under H_0 (H_0 is true)

$$\sqrt{n}(\bar{X}-\mu_0) \sim N_p(0, \Sigma)$$

(CLT in p-dim)

$(n-1)S \sim W_p(n-1, \Sigma) \to$ Wishart

$$\sqrt{n}\,\Sigma^{-1/2}(\bar{x}-\mu_0) \sim N_p(0, \Sigma^{-1}\Sigma)$$
$$= N_p(0, I)$$

Recall if A is Wishart $C^T A C$ is also wishart.

$$(n-1)(\Sigma^{-1/2})^T S\, \Sigma^{-1/2} \sim W_p(n-1, \Sigma^{-1/2} I \Sigma^{-1/2})$$
$$\sim W_p(n-1, \Sigma^{-1})$$

$(\because (\Sigma^{-1/2})^T = \Sigma^{-1/2}$ geometrically?)

Let $A = \sqrt{n}\,\Sigma^{-1/2}(\bar{X}-\mu_0) \sim N_p(0, I)$
$B = (n-1)\Sigma^{-1/2} S\, \Sigma^{-1/2} \sim W_p(n-1, \Sigma)$

$$B^{-1} = (n-1)^{-1}(\Sigma^{-1/2})^{-1} S^{-1} (\Sigma^{-1/2})^{-1}$$
$$\Sigma^{-1/2} S^{-1} \Sigma^{-1/2} = S^{-1}/n-1$$

(We will use 'Bartlett decomposition')

$\Rightarrow S^{-1} = (n-1)\, \Sigma^{-1/2} S^{-1} \Sigma^{-1/2}$

$$T^2 = n(\bar{X} - \mu_0)^T \Sigma^{-1} (\bar{X} - \mu_0) > 0$$

$$= n(\bar{X} - \mu_0)^T (n-1)\, \Sigma^{-1/2} S^{-1} \Sigma^{-1/2} (\bar{X} - \mu_0)$$

$$= (n-1) \left(\sqrt{n}(\bar{X} - \mu_0)^T \Sigma^{-1/2}\right) S^{-1}$$

$$* \left(\sqrt{n}\, \Sigma^{-1/2} (\bar{X} - \mu_0)\right)$$

$$T^2 = (n-1) \underset{1\times p}{A^T} \underset{p\times p}{\Sigma^{-1}} \underset{p\times 1}{A}$$

Bartlett's decomposition

If $A \sim N_p(0, I)$ & $B = W_p(v, I)$

(Random vector & matrix)

then $A^T \Sigma^{-1} A \overset{D}{\sim} \dfrac{p}{v-p+1} F_{p, n-p+1}$

Set $v = n-1$

$\Rightarrow \dfrac{T^2}{(n-1)} \sim \dfrac{p}{n-p} F_{p, n-p}$

$\Rightarrow T^2 \sim \dfrac{(n-1)p}{n-p} F_{p, n-p}$

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$\therefore\ \Sigma_X = Z_1 Z_1^T + Z_2 Z_2^T + \dots + Z_p Z_p^T$

$Z_p : p \times 1$, $Z_i \sim N(0, I)$,

Z_i are iid

$= LL^T$

$B^{-1} = (LL^T)^{-1}$

$= (L^{-1})^{-1} L^{-1}$

$A^T \Sigma^{-1} A = A^T (L^{-1})^T (L^{-1}) A$

$= (L^{-1} A)^T (L^{-1} A)$

$= \| L^{-1} A \|^2$

$= \sum_j \cancel{v_j}^2$

$$L^{-1} A = \begin{bmatrix} V_1 \\ \vdots \\ V_p \end{bmatrix}$$

P{Critical region} $= \alpha$

P{Reject H_0 | H_0 is true}

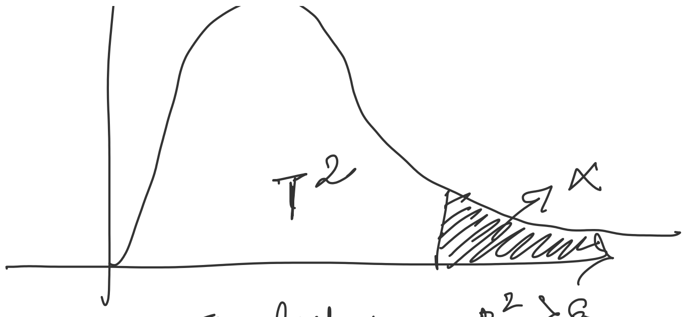
= Type $\underline{I}$ error $= \alpha$

$P(T^2 > c) = \alpha$ under H_0

reject

called

$$\frac{(n-1)p}{n-p} F_{p,n-p,\alpha}$$



T^2 distr
is related to
F distr

reject H_0 with $1-\alpha$ confidence.

In rejection 1 sample is sufficient
to reject. (simpler)
But 1 sample acceptance doesn't mean
that we will be accepting the model.

eg: $X = \begin{bmatrix} 6 & 9 \\ 10 & 6 \\ 1 & 3 \\ 12 & 10 \end{bmatrix} \rightarrow$ sample matrix
$n=4, p=2$

$H_0: \mu = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ $\quad \Sigma = \begin{bmatrix} & \\ & \end{bmatrix}$

$H_1: \mu \neq \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ under H_0

So $T^2 = 4 \begin{pmatrix} 4-6 \\ 7-4 \end{pmatrix} S^{-1} \begin{bmatrix} 4-6 \\ 7-4 \end{bmatrix}$

≈ 7.373

$\alpha = 5\% \longrightarrow 95\%$ conf

then,

$$\frac{(n-1)p}{n-p} F_{p, n-1}$$

Approximately $c = 57$.

Sample does not support rejecting H_0.

- Neyman-Pearson test
- Likelihood Ratio Test.