Lec 5

Tuesday, 5 August 2025   10:05 AM

$X_{n \times p}$. $[X_{ij}]$ $i = 1 \dots n$, $j = 1 \dots p$.

$E(X) \rightarrow$ Component wise expp
$\hookrightarrow [E(x_{ij})]_{n \times p}$.

$\underset{m \times n}{A}, \underset{p \times k}{B} \longrightarrow$ deterministic matrices.

$\underset{n \times p}{X} \longrightarrow$ random variable matrix.

AXB $m \times k$ matrix.
$\hookrightarrow$ random matrix.

$E(AXB) = A\,E(X)\,B$.
$E(X+Y) = E(X) + E(Y)$.

$E(\alpha X) = \alpha\, E(X)$.
$E(\cdot)$ is a linear map.

Suppose we take any arb. random vect.
(vector comprising r.v.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \longrightarrow \text{population features.}$$

pop. mean $= \mu_{p \times 1}$ pop. variance covariance $= \Sigma_{p \times p}$.

$$E(X) = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_p) \end{bmatrix} = \mu$$

let $\vec{a} \in \mathbb{R}^p$ deterministic constant vector

$Y = a^T X = a_1 x_1 + \dots + a_p x_p$

1×1 random variable.
$E(Y) = \vec{a}^T \mu$.
$Var(Y)$ Y is a random variable.

$Var(Y) = E[Y - E(Y)]^2$.

$= E(\vec{a}^T X - \vec{a}^T \mu)^2$
$= E(a^T(X - \mu))^2$
$= E[\underset{1 \times p}{a^T}\underset{p \times 1}{(X-\mu)}\ \underset{1 \times p}{a^T}\underset{p \times 1}{(X-\mu)}]$.

$= E(\vec{a}^T \underbrace{(X-\mu)(X-\mu^T)}_{\text{random part.}} \vec{a})$. $\leftarrow$ number

$= \vec{a}^T E[\underbrace{\underset{p \times 1}{(X-\mu)}\underset{1 \times p}{(X-\mu)^T}}_{p \times p}]\vec{a}$

$= \vec{a}^T \Sigma \vec{a}$.

This whole calc for population.
Now we talk about sample $\Rightarrow$ μ replaced by $\bar{x}$ & Σ replaced by S. something like $n \to \infty$; collecting data of each pssbl pt.

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix}_{p \times 1}$$

$\vec{a_1}, \vec{a_2}, \dots, \vec{a_q} \in \mathbb{R}^p$ const vector in $\mathbb{R}^p$.

$y_1 = \vec{a_1}^T X$.
$y_2 = \vec{a_2} X$.
$\vdots$
$y_q = \vec{a_q} X$.

$$\underset{\text{random vector}}{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_q \end{bmatrix}_{q \times 1} = \begin{bmatrix} a_1^T & 1 \times p \\ a_2^T & 1 \times p \\ \vdots & \\ a_q^T & 1 \times p \end{bmatrix}_{q \times p} X_{p \times 1}.$$

$= C_{q \times p} X$.
deterministic constant matrix.

$\underset{q \times 1}{E(Y)} = \underset{q \times p}{C}\, \underset{p \times 1}{E(X)} = C\mu$.

$Cov(Y) = E(Y - E(Y))^2$
$= E((\underset{q \times 1}{Y - E(Y)})(\underset{1 \times q}{Y - E(Y))^T})$.

$= E[(CX - C\mu)(CX - C\mu)^T]$.

$= E[\underbrace{\underset{q \times p}{C}\underset{p \times 1}{(X-\mu)}}_{q \times 1}\ \underbrace{\underset{1 \times p}{(X-\mu)^T}\underset{p \times q}{C^T}}_{1 \times q}]$.
$q \times q$.

$= C\, E(X-\mu)(x-\mu)^T]\, C^T$

$$Y = \begin{bmatrix} X_1 \\ \hline X_2 \end{bmatrix} \begin{matrix} p \times 1 \\ q \times 1. \end{matrix}$$

$(p+q) \times 1$
$\uparrow$
random vector.

two different blocks corresponding to diff sets of questions in the blocks.
eg. 1st block has general ques.
2nd " specific " .

$$E(Y) = \begin{bmatrix} E(X_1) \\ \hdashline E(X_2) \end{bmatrix} \begin{matrix} \rightarrow \mu_1 \ p \times 1 \\ \rightarrow \mu_2 \ q \times 1. \end{matrix}$$

These may come from same population or different.

Variance covariance matrix of Y is $(p+q) \times (p+q)$.

$$\left[\begin{array}{c|c} \underset{p \times p}{cov(X_1, X_1)} & \underset{p \times q}{cov(X_1, X_2)} \\ \hline \underset{q \times p}{cov(X_2, X_1)} & \underset{q \times q}{cov(X_2, X_2)} \end{array}\right]$$

diagonal x_{11} x_{22} ... variance of p features
cross diagonal x_{12} x_{31} ... covariance of p features.

$$= \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \underset{q \times p}{\Sigma_{21}} & \Sigma_{22} \end{array}\right] \quad \Sigma_{12} \rightarrow p \times q.$$

$\Sigma_{12} = \Sigma_{21}^T$.

$cov(Z, T) = cov(T, Z)$.

eg: $p \times 3$.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1} \qquad Y = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 - x_2 \\ x_1 - x_3 \end{bmatrix}_{5 \times 1}$$

computing
stat math
comp - stat math.
can compare b/w comp & (comp-stat math)

$$Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}_{5 \times 3} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}_{3 \times 1}$$

$\underset{5 \times 1}{Y} = \underset{5 \times 3}{C}\, \underset{3 \times 1}{X}$

$$E(Y) = C\,E(X) = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_1 - \mu_2 \\ \mu_1 - \mu_3 \end{bmatrix}.$$

$Cov(Y) = C \Sigma C^T$.

identity. identity.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{array}\right).$$

$\sigma_{ij} = E(X_i - \mu_i)(X_j - \mu_j)$.

because of both the identity matrices.

$\Sigma_{3 \times 3}$ | $[cov(X, X_1 - X_2)]$. 3×2.

2×3 | $(\sigma_{11} - \sigma_{12} - \sigma_{21} + \sigma_{22})$ $(\sigma_{11} - \sigma_{13} - \sigma_{21} + \sigma_{23})$

$cov(X_1 - X_2, X_1 - X_3)_{2 \times 2}$.