Lec 18

Tuesday, 2 September 2025 10:10 AM

CLT:

$p=1$ $X \to$ pop. random variable. No. distribution assumption.

$E(X)=\mu<\infty$

$Var(X)=\sigma^2<\infty$,

$\bar{X}_n$ – sample mean $\to$ random variable.

$n\to\infty$.

$$Z=\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \xrightarrow[n\to\infty]{D} N(0,1).$$

$\sqrt{n}(\bar{X}-\mu)\sigma^{-1} \sim N(0,1)$.

$\sqrt{n}(\bar{X}-\mu) \sim N(0,\sigma^2)$.

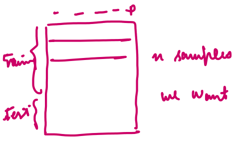
In case if $X\sim N(\mu,\sigma^2)$ then we can drop condition of n.

$\sqrt{n}(\bar{x}-\mu) \sim N(0,\sigma^2)$ $\forall n\in\mathbb{N}$.

$p>1$. under similar conditions :–

$\sqrt{n}(\bar{\underset{\sim}{x}}-\mu) \sim N_p(0,\Sigma)$ as $n\to\infty$.

i.e. distribution of $\bar{\underset{\sim}{X}}$ is normal / gaussian if $n\to\infty$.



n samples $\longrightarrow$ model $\longrightarrow$ θ parameters $\longrightarrow$ θ_t estimated. $\longrightarrow \bar{\theta}$ $E(\bar{\theta})=\mu$, $Cov(\bar{\theta})=\frac{\Sigma}{n}$.

we want $n>>p$.

(100 iterations; take mean, value very close to original)

crude ways of creating samples :– attach prob to samples & take combination.
Simulated data. Mix up, shuffle the original & simulated & then divide 70–30.

Recall from notes :–

$(\underset{\sim}{X}-\mu)^T_{p\times 1}\ \Sigma^{-1}_{p\times p}(\underset{\sim}{X}-\mu) \sim \chi^2_{(p)}$.

here $\mu=E[\underset{\sim}{X}]$ & $\Sigma=Cov(\underset{\sim}{X})$.

If we take $\bar{\underset{\sim}{X}}$ instead of X.

$(\bar{\underset{\sim}{X}}-\mu)^T\left(\frac{\Sigma}{n}\right)^{-1}(\bar{\underset{\sim}{X}}-\mu) \sim \chi^2(p)$. $\mu=E(\bar{\underset{\sim}{X}})$ & $\frac{\Sigma}{n}=Cov(\bar{\underset{\sim}{X}})$.

$n(\bar{\underset{\sim}{X}}-\mu)^T\Sigma^{-1}(\bar{\underset{\sim}{X}}-\mu) \sim \chi^2_{(p)}$

when $n\to\infty$ then we have by WLLN

$S \xrightarrow{P} \Sigma$ component wise.

$\lim_{n\to\infty} P(|s_{ij}-\sigma_{ij}|>\varepsilon)=0$ $\forall (i,j)$.

\# If n is suff. large then Σ (if unknown) can be estimated by S.
Then $n(\bar{X}-\mu)^T S^{-1}(\bar{X}-\mu) \sim \chi^2_{(p)}$ when $n\to\infty$.

$S_n = \frac{n-1}{n} S \longrightarrow$ consistent & unbiased estimator of Σ.

$\uparrow$ MLE of Σ.

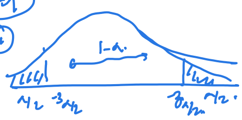
$\boxed{S=\frac{n}{n-1}S_n} \Rightarrow S^{-1}=\frac{n-1}{n}S_n^{-1} \Rightarrow (n-1)(\bar{x}-\mu)^T S_n^{-1}(\bar{x}-\mu) \sim \chi^2_{(p)}$.

\# Hypothesis testing for population mean :– Assume population $X\sim N_p(\mu,\Sigma)$.

Null hypothesis $H_0: \mu=\mu_0$.
Alternative " $H_1: \mu\neq\mu_0$. } Two-tailed test.

$p=1$

(1)



$\boxed{(\sigma \text{ known})}$

$z=\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$

$\left|\frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}}\right| > z_{\alpha/2}$: reject H_0. Z test.

If this not satisfied; we don't have enough info to reject H_0.

(2) σ unknown $p=1$.

Result 1: $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$.

Result 2: $\bar{x}$ & s are indep r.v.

Result 3: $\frac{\sqrt{n}(\bar{x}-\mu)}{\sigma} \sim N(0,1)$.

Recall: $p=1$.

$Z\sim N(0,1)$

$Y\sim\chi^2_{(n)} \iff Y=Z_1^2+Z_2^2+\cdots+Z_n^2$ $Z_i\to$ independent.

Z & Y are indep r.v.

$\Rightarrow T=\frac{Z}{\sqrt{Y/n}}=\frac{\sqrt{n}Z}{\sqrt{Y}} \sim t_{(n)}$. student t-distribution w n-degree of freedom

$\sqrt{\frac{Y}{n}}$ $\sqrt{y}$

mean z (CLT) t : ratio $z/\sqrt{Y_n}$
varia f → related to σ.
goodness of fit χ^2 test

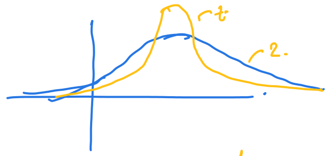
If σ unknown can't use z test so we try t test.

It follows:

$$\frac{\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}}{\sqrt{\frac{(n-1)s^2/\sigma^2}{(n-1)}}} \sim t_{(n-1)}. \Rightarrow \sqrt{n}\left(\frac{\bar{X}-\mu}{s/\sqrt{n}}\right) \sim t_{(n-1)}.$$

$H_0 : \mu = \mu_0$.
vs. $H_1 : \mu \neq \mu_0$

$\left|\frac{\bar{X}-\mu}{s/\sqrt{n}}\right| > t_{(n-1),\alpha/2}$ reject H_0.



For larger value of $n > 40/50$. t dist. as good as normal table dist

For f you need 2 chi square dist.

$p=1$. $Y_1 \sim \chi^2_{(n_1)}$ $Y_2 \sim \chi^2_{(n_2)}$
Y_1, Y_2 are independent r.v.
$F = \frac{Y_1/n_1}{Y_2/n_2}$.
F is a r.v. & distribution of F is called F-distribution w (n_1, n_2) degree of freedom.

suppose T is a r.v. which is t-dist. w n degree of freedom
$T = \sqrt{n}\frac{Z}{\sqrt{Y}}$ $Z \sim N(0,1)$, $Y \sim \chi^2_{(n)}$.
$T^2 = n\frac{Z^2}{Y} = \frac{Z^2/1}{Y/n}$ if $Z \sim N(0,1) \Rightarrow Z^2 \sim \chi^2_{(1)}$
$\sim F_{1,n}$. f distributed w.