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$$\max_{\mu, \Sigma} \quad L(\mu, \Sigma)$$

$\hat{\mu} = \bar{X}$: optimal sol^n

$\hookrightarrow$ MLE of μ

$$\text{Trace}(\Sigma^{-1} S_n) = \text{Tr}(\Sigma^{-1} S_n^{1/2} S_n^{1/2})$$

$$\max_{(\Sigma)} L(\hat{\mu}, \Sigma) = \text{Tr}(S_n^{1/2} \Sigma^{-1} S_n^{1/2}) = \sum_{i=1}^{p} \eta_i$$

Σ: sym pd $p \times p$

η_i : eigenvalues of $S_n^{1/2} \Sigma^{-1}$

$\eta_i > 0 \quad \forall i$

$$|S_n| |\Sigma^{-1}| = |S_n^{1/2} \Sigma^{-1} S_n^{1/2}| = \prod_{i=1}^{p} \eta_i$$

$$\Rightarrow |\Sigma^{-1}| = \frac{\prod \eta_i > 0}{|S_n| > 0} \Rightarrow |\Sigma| = \frac{|S_n|}{\prod \eta_i}$$

$$\max_{\Sigma} L(\hat{\mu}, \Sigma) = \max_{\Sigma} \frac{1}{(2\pi)^{np/2} |\Sigma|^{n/2}} \exp\left(-\frac{n}{2} \text{Tr}(\Sigma^{-1} S_n)\right)$$

$$= \max_{\Sigma} \frac{\prod_{i=1}^{p} \eta_i^{n/2}}{(2\pi)^{np/2} |S_n|^{n/2}} \exp\left(\frac{-n}{2} \sum_{i=1}^{p} \eta_p\right)$$

$$= \frac{1}{(2\pi)^{np/2} |S_n|^{n/2}} \times \max_{\eta} \prod_{i=1}^{p} \eta_i^{n/2} \exp\left(\frac{-n}{2} \eta_i\right)$$

$$= \frac{1}{(2\pi)^{np/2} |S_n|^{n/2}} \times \prod_{i=1}^{p} \max\left(\eta_i^{n/2} \exp\left(\frac{-n}{2} \eta_i\right)\right)$$

$$h(x) = x^{n/2} \exp\left(\frac{-nx}{2}\right) \qquad x > 0$$

want to $\max_{x > 0}$imize $h(x)$

$$h'(x) = x^{n/2} \cancel{\exp\left(\frac{-nx}{2}\right)} \left(\frac{-n}{2}\right) + \frac{n}{2} x^{\frac{n}{2}-1} \cancel{\exp\left(\frac{-nx}{2}\right)} = 0$$

$$\Rightarrow x^{n/2} \left(-1 + x^{-1}\right) = 0 \qquad x > 0$$

$x = 1$

$\Rightarrow$ each $\eta_i = 1$

$$\therefore \quad L(\hat{\mu}, \hat{\Sigma}) = \frac{1}{(2\pi)^{np/2} |S_n|^{n/2}} \times \exp\left(-\frac{np}{2}\right)$$

max

upper Boun

Attained iff

$\Sigma^{-1} S_n = I$

i.e. $\boxed{\Sigma = S_n}$

$\hat{\Sigma} = S_n$

MLE of Σ

$S_n = \frac{n-1}{n} S$

	unbiased estimator	MLE estimator
μ:	$\bar{x}$	$\bar{x}$
Σ:	S	$S_n = \frac{n-1}{n} S$

Note:

max value of $L(\mu, \Sigma) \propto \frac{1}{(S_n)^{n/2}} = \frac{1}{\left(\frac{n-1}{n} S\right)^{n/2}}$

$$\Rightarrow \boxed{L(\hat{\mu}, \hat{\Sigma}) \propto \frac{1}{|S|^{n/2}}}$$

$|S|$ = generalized variance of sample data.

- The smaller the generalized variance the more accurate is the estimation

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_n$ let the population be p-dimentional Normal $N_p(\underset{p\times 1}{\mu},$

$\hookrightarrow p\times 1$ $\hookrightarrow p\times 1$

$$\underset{\sim}{X}_i \sim N_p(\mu, \Sigma) \quad 1 \le i \le n$$

$$\underset{\sim}{V} = \sum_{i=1}^{n} c_i \underset{\sim}{X}_i$$

$$\sim N_p\left(\sum_{i=1}^{n} c_i \mu, \sum_{i=1}^{n} c_i^2 \Sigma\right)$$

for mean: $c_i = 1/n \quad \forall i$

$$\bar{X} \sim N_p\left(\mu, \frac{\Sigma}{n}\right)$$