1. Let $X \sim \mathbb{N}(0,1)$ and $Y = \begin{cases} X & -1 \le X \le 1 \\ -X & \text{otherwise} \end{cases}$
Prove that $Y \sim N(0,1)$. Show that (X,Y) do not follow a bivariate normal distribution.

2. Let $\underset{\sim}{X} \sim \mathbb{N}_3(\underset{\sim}{0}, \Sigma)$, where $\Sigma = \begin{pmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and $\underset{\sim}{X} = (X_1, X_2, X_3)^T$.
Show that X_1 and X_2 are not independently distributed; (X_1, X_2) and X_3 are independently distributed; $2.5X_1 + X_2X_3$ and X_2 are not independently distributed. Determine a vector $\tilde{a} \in \mathbb{R}^3$ such that $a_1X_1 + a_2X_2 + a_3X_3$ and (X_2, X_3) are independent.

3. Let $\underset{\sim}{X}_i$, $i = 1, \dots, 4$, be independent $\mathbb{N}_p(\underset{\sim}{\mu}, \Sigma)$ random vectors. Define
$$\underset{\sim}{V}_1 = 0.25\underset{\sim}{X}_1 - 0.25\underset{\sim}{X}_2 + 0.25\underset{\sim}{X}_3 - 0.25\underset{\sim}{X}_4$$
$$\underset{\sim}{V}_2 = 0.25\underset{\sim}{X}_1 + 0.25\underset{\sim}{X}_2 - 0.25\underset{\sim}{X}_3 - 0.25\underset{\sim}{X}_4.$$
Find the marginal distributions of $\underset{\sim}{V}_1$ and $\underset{\sim}{V}_2$. Find the joint density of vectors $\underset{\sim}{V}_1$ and $\underset{\sim}{V}_2$.

4. Let X and Y have the joint pdf
$$f(x,y) = \exp(c + 4x + 4y - 0.5x^2 - 0.5y^2 - 0.5x^2y^2), \ -\infty < x, y < \infty,$$
where c is a constant. Determine the marginal pdfs of X and Y. Using them find the conditional pdfs of X given $Y = y$ and Y given $X = x$. Notice that the joint pdf is not normal distributed but the two conditional pdfs are normally distributed.

5. Let $X \sim \mathbb{N}_2\left(\left(\begin{array}{c} 1 \\ 1 \end{array}\right), \left(\begin{array}{cc} 3 & 1 \\ 1 & 2 \end{array}\right)\right)$. Compute the conditional distribution of $(X_1 + X_2)|(X_1X_2)$.

6. Let $\underset{\sim}{X}_i$, $i = 1, \dots, 50$, be $\mathbb{N}_4(\underset{\sim}{\mu}, \Sigma)$. Specify the distributions of the following:
$\bar{\underset{\sim}{X}}$; $\quad (\underset{\sim}{X}_1 - \underset{\sim}{\mu})^T\Sigma^{-1}(\underset{\sim}{X}_1 - \underset{\sim}{\mu})$; $\quad 50\,(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu})^T\Sigma^{-1}(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu})$.

7. Let $\underset{\sim}{X}_i$, $i = 1, \dots, 25$, be $\mathbb{N}_6(\underset{\sim}{\mu}, \Sigma)$. Specify the distributions of the following:
$(\underset{\sim}{X}_1 - \underset{\sim}{\mu})^T\Sigma^{-1}(\underset{\sim}{X}_1 - \underset{\sim}{\mu})$; $\quad 5(\bar{\underset{\sim}{X}} - \underset{\sim}{\mu})$, $\quad 24\,S$.

8. Let D be a positive definite of order p. Show that maximum of the function $f(G) = n\log|G| - trace(G^{-1}D)$ with respect to $p \times p$ positive definite matrices G exists. Find the maximum value.

9. Let x_1 be body weight (in Kg) and x_2 be heart weight (in gm) of cats. In a sample of 47 cats, it is observed that
$$\sum x_i == \begin{pmatrix} 110.9 \\ 432.5 \end{pmatrix}, \qquad \sum x_i x_i^T = \begin{pmatrix} 265.13 & 1029.62 \\ 1029.62 & 4064.71 \end{pmatrix}.$$
Find $\hat{\underset{\sim}{\mu}}$, S, $\hat{\Sigma}, r\hat{h}o$, where hat symbol means MLE.

10. Prove that $(1/n)\sum_{i=1}^n(\underset{\sim}{X}_i - \underset{\sim}{\mu})(\underset{\sim}{X}_1 - \underset{\sim}{\mu})^T$ is an unbiased estimator of σ when $\underset{\sim}{\mu}$ is known.