Hausdorff Measure

For $s \geq 0$, we define

$$\mathcal{H}^s(A) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(A),$$

where $\mathcal{H}^s_\delta(A) = \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : A \subseteq \bigcup_{i=1}^{\infty} U_i, \ \operatorname{diam}(U_i) \leq \delta \right\}$

- $\mathcal{H}^0(A) =$ number of points in A.
- $\mathcal{H}^s$ is an outer measure.
- All Borel sets are $\mathcal{H}^s$-measurable.
- For $n \in \mathbb{N}$, $\mathcal{H}^n(A) = c_n \mathcal{L}^n(A)$, where $\mathcal{L}^n$ is the Lebesgue meas. on $\mathbb{R}^n$,

$$c_n = \frac{1}{\text{volume of ball of dia 1}}$$

For $n=1$, $c_1 = 1 \Rightarrow \mathcal{H}^1(A) = \mathcal{L}^1(A)$

For $n=2$, $c_2 = \frac{1}{\pi(\frac{1}{2})^2} = \frac{4}{\pi} \Rightarrow \mathcal{H}^2(A) = \frac{4}{\pi}\mathcal{L}^2(A)$

For $n=3$, $c_2 = \frac{1}{\frac{4}{3}\pi(\frac{1}{2})^3} = \frac{6}{\pi}$

In general, $c_n = \begin{cases} \dfrac{2^n (\frac{n}{2})!}{\pi^{n/2}}, & \text{if } n \text{ is even} \\ \dfrac{n!}{\pi^{(n-1)/2} (\frac{n-1}{2})!}, & \text{if } n \text{ is odd.} \end{cases}$

<u>Recall</u>: $f: D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is said to Hölder continuous with Hölder exponent $\alpha > 0$ if

$$d(f(x), f(y)) \leq c\,(d(x,y))^\alpha \quad \forall x, y \in D$$

for some constant $c > 0$.

A map is said to Lipschitz continuous if $d(f(x), f(y)) \leq c\, d(x,y) \quad \forall x, y \in D$.

$\therefore$ Lipschitz cont. maps are Hölder cont. with Hölder exponent $\alpha = 1$.

<u>Proposition</u>: Let $F \subseteq \mathbb{R}^n$ and $f: F \to \mathbb{R}^m$ be Hölder cont. with exponent $\alpha > 0$.

Then $\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s(F)$ for any $s \geq 0$.

<u>Proof</u>: If $\{U_i\}_{i=1}^\infty$ is a δ-cover of F, then

$$\begin{aligned} \operatorname{diam}(f(F \cap U_i)) &= \sup\{d(f(x), f(y)) : x, y \in F \cap U_i\} \\ &\leq c \sup\{(d(x,y))^\alpha : x, y \in F \cap U_i\} \\ &= c\,(\operatorname{diam}(F \cap U_i))^\alpha \leq c\,\delta^\alpha \end{aligned}$$

$\Rightarrow \{f(F \cap U_i)\}_{i=1}^\infty$ is a $c\delta^\alpha$-cover of $f(F)$.

$$\Rightarrow \mathcal{H}^{s/\alpha}_{c\delta^\alpha}(f(F)) \leq c^{s/\alpha} \mathcal{H}^s_\delta(F).$$

(because $\mathcal{H}^{s/\alpha}_{c\delta^\alpha}(f(F)) \le \inf\left\{ \sum_{i=1}^{\infty} \left(\operatorname{diam} f(F \cap U_i)\right)^{s/\alpha} : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$

Now, $\sum_{i=1}^{\infty} \left(\operatorname{diam} f(F\cap U_i)\right)^{s/\alpha} \le \sum_{i=1}^{\infty}\left(c\,(\operatorname{diam} U_i)^{\alpha}\right)^{s/\alpha}$

$$= c^{s/\alpha} \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s$$

$$\Rightarrow \mathcal{H}^{s/\alpha}_{c\delta^\alpha}(f(F)) \le c^{s/\alpha}\mathcal{H}^s_\delta(F) \quad)$$

$$\therefore \mathcal{H}^{s/\alpha}(f(F)) = \lim_{\delta\to 0+} \mathcal{H}^{s/\alpha}_{c\delta^\alpha}(f(F)) \le c^{s/\alpha}\mathcal{H}^s(F)$$

Corollary: If f is a Lipschitz map, then
$$\mathcal{H}^s(f(F)) \le c^s \mathcal{H}^s(F).$$

Corollary: If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a similarity transformation of scale factor $\lambda > 0$, i.e.,
$$\|f(x) - f(y)\| = \lambda\|x-y\|,$$
then $\mathcal{H}^s(f(F)) = \lambda^s \mathcal{H}^s(F)$.

Pf: Apply the prev. corollary to f & f^{-1}.

Hausdorff Dimension

Let $0 \leq s < t$. (Notation: $|U| = \text{diam}(U)$)

$$\sum_{i=1}^{\infty} |U_i|^t = \sum_{i=1}^{\infty} |U_i|^s |U_i|^{t-s}$$

If $\{U_i\}_{i=1}^{\infty}$ is a δ-cover of F (assume $0<\delta<1$),

$$|U_i|^{t-s} \leq \delta^{t-s}$$

$$\sum_{i=1}^{\infty} |U_i|^t \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s$$

Taking the infimum over all δ-covers of F,

$$\mathcal{H}^t_\delta(F) \leq \delta^{t-s} \mathcal{H}^s_\delta(F)$$

As $\delta \to 0+$, $\delta^{t-s} \to 0$

If $\mathcal{H}^s(F) < \infty$, i.e., $\lim_{\delta \to 0+} \mathcal{H}^s_\delta(F) < \infty$, then

$$\mathcal{H}^t(F) = 0.$$

Hence, $\mathcal{H}^s(F) < \infty \Rightarrow \mathcal{H}^t(F) = 0 \quad \forall t > s.$

This implies, $\mathcal{H}^t(F) > 0 \Rightarrow \mathcal{H}^s(F) = \infty \quad \forall s < t$

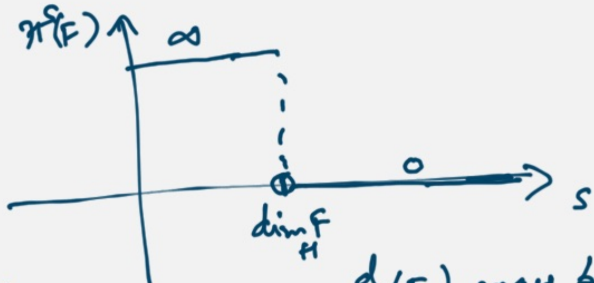
$\therefore \exists!\, s_0 \geq 0$ s.t. $\mathcal{H}^s(F) = \begin{cases} \infty, & \text{if } s < s_0 \\ 0, & \text{if } s > s_0 \end{cases}$

$$\left(\begin{aligned} s_0 &= \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\} \\ &= \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} \end{aligned} \right)$$

Defn: (Hausdorff dimension)

For any $F \subseteq \mathbb{R}^n$ (or any metric space) we define the Hausdorff dimension of F as

$$\dim_H F = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}$$
$$= \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\}$$



If $d = \dim_H F$, then $\mathcal{H}^d(F)$ may be $0, \infty$ or any positive real number.

Properties of Hausdorff dimension:

(i) Monotonicity: $E \subseteq F \Rightarrow \dim_H E \leq \dim_H F$.

(ii) Range of values: $F \subseteq \mathbb{R}^n \Rightarrow 0 \leq \dim_H F \leq n$.

(iii) If F is a singleton set, $\dim_H F = 0$.

(iv) If F is an open set in $\mathbb{R}^n$, $\dim_H F = n$.

(v) Countable stability:

$$\dim_H\left(\bigcup_{i=1}^{\infty} F_i\right) = \sup_i \dim_H F_i$$