Lecture 7

Recall

- X - nLs $M \subseteq X$, closed $\Rightarrow$ $\|\cdot\|$ is a norm in X/M

$$\|x+M\| = \text{dist}(x, M)$$
$$= \inf\{\|x-m\|_X \mid m \in M\} \leq \|x\|$$

- $\text{dist}(x, M) = 0 \iff x \in \overline{M}$ $\ast$ $0 \in M$

- Theo: X nLs, M closed
  X Banach $\Rightarrow$ X/M Banach.

- Converse: X nLs, M closed
  X/M Banach + M Banach $\Rightarrow$ X Banach.

Convergent of series.

$(X, \|\cdot\|)$ $\qquad \sum_{n\in\mathbb{N}} a_n$, $a_n \in X$

$\sum_{n\in\mathbb{N}} \|a_n\| < \infty \underset{\text{defn.}}{\Longrightarrow} \sum a_n$ is absolutely convergent.

Theo. Let X be a nls. Then X is Banach space.
$\Longleftrightarrow$ Every absolutely convergent series is convergent.

Proof: Suppose X is a Banach space.

Let $\sum x_n$ be abs. convergent

ie $\sum \|x_n\| < \infty$.

$$S_n = \sum_{i=1}^{n} x_i$$

$$\|S_n - S_m\| \leq \sum_{i=n+1}^{m} \|x_i\| \longrightarrow 0$$

as $n, m \to \infty$

$$\left(\because \sum \|x_n\| < \infty \right)$$

$\Rightarrow \{S_n\}$ is C.S and X is a BS so $S_n \to x$ as $n \to \infty$

$\therefore \quad \sum x_n < \infty$.

Converse.

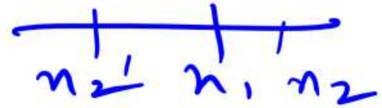
Let (x_n) be a c.s in X.

$\exists \; n_1 \in \mathbb{N}$ s.t

$$\|x_n - x_m\| < 1/2 \quad \forall m,n \geq n_1$$

$\exists \; n_2 > n_1$ s.t

$$\|x_n - x_m\| < 1/2^2 \quad \forall m,n \geq n_2$$



$\exists \; n_k$ s.t $\quad n_1 < n_2 < n_3 \cdots < n_k$

$$\|x_n - x_m\| < \frac{1}{2^k} \quad \forall n,m \geq n_k .$$

$n = n_{k+1}$ & $m = n_k$

$$\boxed{\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k} \quad \forall\ k \in \mathbb{N}}.$$

$$x_{n_1} + (x_{n_2} - x_{n_1}) + (x_{n_3} - x_{n_2}) + \cdots$$

$$\|x_{n_1}\| + \|x_{n_2} - x_{n_1}\| + \|x_{n_3} - x_{n_2}\| + \cdots$$

$$\leq \|x_{n_1}\| + \sum_{k=1}^{\infty} \frac{1}{2^k}.$$

$$< \infty$$

Then $x_{n_1} + (x_{n_2} - x_{n_1}) + \cdots$

is convergent. ($\because$ abs conv $\Rightarrow$ conv)

Then the seq of partial sum of the above series is convergent.

However the seq of partial sum is x_{n_k}

$\therefore \; x_{n_k} \to x \,, \quad x \in X$.

And $\{x_{n_k}\}$ is a s. seq of the C.S $\{x_n\}$

$\therefore \; \{x_n\}$ is convergent in X.

$\Rightarrow$ X is Banach.

- Recall.

X – finite dimensional NLS.

$\{u_1, \dots u_k\}$ basis of X

Then $x \in X$

$$x = \alpha_1 u_1 + \dots + \alpha_k u_k$$

α_i are unique and ordered.

$j = 1, 2 \dots k$

$$f_j : X \to \mathbb{K}$$

$$f_j(x) = f_j(\alpha_1 u_1 + \dots \alpha_k u_k)$$
$$= \alpha_j$$

Each f_j, $j = 1, 2 \dots k$ is a linear fnal on X

$\longrightarrow$ coordinate functionals.

- Note $f_j(u_i) = \delta_{ij}$

$$= \begin{cases} 1 & , i = j \\ 0 & i \neq j \end{cases}$$

- $x \in X$

$$x = \sum_{j=1}^{k} f_j(x)\, u_j$$

- $\{f_1, f_2, \dots f_k\}$ forms a basis of $L(X, \mathbb{K})$ = the space of all linear fnals defined on X.

$$\sum_{i=1}^{k} c_i f_j = 0$$

$$\Rightarrow \sum_{i=1}^{k} c_i f_i(u_j) = 0$$

$$\Rightarrow c_j = 0 \quad \forall j$$

$\{f_1, \dots f_k\}$ is lin ind.

Let $f \in L(X, \mathbb{K})$.

$$f(x) = f\left(\sum \underline{f_j(x)}\, u_j\right)$$

$$= \sum f_j(x)\, f(u_j)$$

$$= \left(\sum f(u_j)\, f_j\right)(x)$$

$$\boxed{f = \sum f(u_j)\, f_j}$$

$\{f_1, \ldots f_n\} \rightarrow$ dual basis for $L(X, \mathbb{K})$.

<u>Goal</u> : Any finite dimensional space is a Banach space.

<u>Theo</u>: Let X be a finite dimensional vector space

of dim $= k$.

Let $E = \{u_1 \ldots u_k\}$ be an ordered basis.

Consider

$$\|x\|_E = \max \{|f_1(x)|, |f_2(x)|, \ldots |f_k(x)|\}$$

for each $x \in X$, where (Check)

f_i's are coordinate fnals wrt the basis, ~~of~~ E.

Then X is a B.S wrt $\|\cdot\|_E$.

Proof: Let (x_n) be a Cauchy seq in X wrt $\|\cdot\|_E$

Then for any $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ $\ni$

$$\|x_n - x_m\|_E < \epsilon \quad \forall\, n, m \geq N$$

$$\max\left\{ |f_1(x_n) - f_1(x_m)|, \ldots, |f_k(x_n) - f_k(x_n)| \right\} < \epsilon \quad \forall\, n, m \geq N.$$

$$\Rightarrow \ast \; |\underline{f_j(x_n)} - f_j(x_m)| < \epsilon \quad \forall\, n, m \geq N, \; j = 1, 2 \ldots k.$$

$\Rightarrow \{f_j(x_n)\}$ is a C.S in $\mathbb{K}$, $j = 1, 2 \ldots k$.

Since $\mathbb{K}$ is complete

$\Rightarrow \{f_j(x_n)\}$ converges.

$f_j(x_n) \longrightarrow \alpha_j$, as $n \to \infty$

for $j = 1, 2, \ldots k$.

Define

$$x = \sum_{j=1}^{k} \alpha_j u_j, \quad x \in X$$

By defn $\alpha_j = f_j(x)$

$\|x_n - x\|_E$

$= \max \{ |f_1(x_n - x)|, \ldots,$
$|f_k(x_n - x)| \}$

Since

$$x_n - x = \sum f_j(x_n) u_j - \sum f_j(x) u_j$$

$$= \sum (f_j(x_n) - f_j(x)) u_j$$

$$= \sum (f_j(x_n) - \alpha_j)) u_j$$

$\longrightarrow 0$ as $n \to \infty$

$\therefore \quad x_n \xrightarrow[\|\cdot\|_E]{} x$ as $n \to \infty$ $x \in X$

$\therefore (X, \|\cdot\|_E)$ is a Banach space.

Proposition Let Y be a closed subspace of a nls X. & Z be a finite dimensional subspace of X.

Then $Y+Z$ is a closed subspace of X.

Proof: By induction on dim of Z.

$\dim Z = 1$ (say).

$Z = \text{span}\{v_1\}$.

If $v_1 \in Y \Rightarrow Y + Z = Y$

$\Rightarrow Y + Z$ is closed.

Hence assume $v_1 \notin Y = \overline{Y}$

$\Rightarrow \text{dist}(v_1, Y) > 0$

Let (x_n) be a seq in $Y + Z$ s.t $x_n \to x$ as $n \to \infty$

Claim $x \in Y + Z$

$x_n \in Y + Z$

$x_n = y_n + \alpha_n e_1 \quad \forall n$

$\operatorname{dist}(x_n - x_m, Y) =$

$\leq \|x_n - x_m\| \longrightarrow 0$ as $n, m \to \infty$.

— (1)

$\operatorname{dist}(x_n - x_m, Y)$

$= \operatorname{dist}(y_n + \alpha_n e_1 - y_m - \alpha_m e_1, Y)$

$$= \mathrm{dist}\left(\underline{y_n - y_m} + (\alpha_n - \alpha_m) e_1 , Y \right)$$

$$= \mathrm{dist}\left((\alpha_n - \alpha_m) e_1 , Y \right)$$

$$= |\alpha_n - \alpha_m| \, \underline{\mathrm{dist}\,(e_1, Y)}$$

$\therefore$ $|\alpha_n - \alpha_m| \to 0$ as $n, m \to \infty$
by (1)

$\therefore$ (α_n) is a C.S in $\mathbb{K}$.

$\therefore$ $\alpha_n \to \underline{\alpha}$ in $\mathbb{K}$ as $n \to \infty$

Now $\quad x_n = y_n + \alpha_n e_1$

$\therefore \quad \underline{y_n} = x_n - \alpha_n e_1$

$\longrightarrow x - \alpha e_1$ in X

as $n \to \infty$

Since Y is closed

$\Rightarrow \quad x - \alpha e_1 \in Y$

$y = x - \alpha e_1.$

$\Rightarrow \quad x = y + \alpha e_1 \in Y + Z.$

$\therefore \quad Y + Z$ is closed $(\dim Z = 1)$

Next let $\dim Z = k$. , $k>1$.

Let $\{v_1, v_2, \ldots v_k\}$ be a basis of Z.

Let $$X_j = Y + Z_j$$

$$Z_j = \text{span}\{v_1 \ldots v_j\}.$$

Then, X_1 is a closed subspace of ~~$\&$~~ X (Proved above).

Assume X_{j-1} is a closed subspace of X.

$$X_j = \underbrace{X_{j-1}}_{Y} + \underbrace{\mathrm{span}\{e_j\}}_{= Z}.$$

$\therefore$ X_j is closed.

By induction,

$X_k = \underline{Y} + Z_k$. is closed $\forall k$.

Cor. In particular if Z is finite dimensional subspace of X then Z is closed.

(Take $Y = \{0\}$)

Theo: Any two norms on a finite dimensional linear space are equivalent.

Proof: Let $\|\cdot\|$, $\|\cdot\|_*$ are two norms in X.

We will show $\|\cdot\|$ & $\| \|_E$ are equivalent.

ie $\exists \; a, b > 0$

s. t

$$a \|x\|_E \leq \|x\| \leq b \|x\|_E$$

Let $x \in X$

$$x = f_1(x) u_1 + \cdots + f_k(x) u_k$$

$$\|x\| = \| \cdots \|$$

$$\leq |f_1(x)| \, \|u_1\| + \cdots + |f_k(x)| \, \|u_k\|$$

$$\leq \|x\|_E \|u_1\| + \cdots + \|x\|_E \|u_k\|$$

$$\leq \|x\|_E \big(\underbrace{\|u_1\| + \|u_2\| + \cdots + \|u_k\|} \big)$$

$$\| \quad b.$$

$$\leq \quad b \, \|x\|_E$$

$$\therefore \quad \boxed{\|x\| \leq b \, \|x\|_E}$$

$$X_j = \operatorname{span}\{u_i\}_{i \neq j}$$

$$\|x\| = \| f_1(x) u_1 + \cdots + f_k(x) u_k \|$$

$$= \| f_j(x) u_j + \big(\underline{f_1(x) u_1 + \cdots + f_{j-1} u_{j-1}}$$

$$\underbrace{+ f_{j+1} u_{j+1} \cdots + f_k u_k}_{\in X_j} \big) \|$$

$$\geq \operatorname{dist}\left(f_j(x)\, u_j\,,\ X_j\right)$$

$$= |f_j(x)|\ \underbrace{\operatorname{dist}\left(u_j\,,\ X_j\right)}_{a_j}.$$

X_j is a finite dim subspace

$\Rightarrow X_j$ is closed

& $u_j \notin X_j$

$a > 0$.

$$\|x\| \geq a_j |f_j(x)| \quad \forall\ j$$

$$\Rightarrow \quad \|x\| \geq a\, \|x\|_E \qquad a = \min_{j=1,\dots,n} a_j$$

$$\underline{\|\cdot\|} \sim \|\cdot\|_E \sim \|\ \|_*$$

Theo. Every finite dimensional n. linear space is complete. wrt any norm. Banach.