Prop: If $F \subseteq \mathbb{R}^n$, then $\dim_H F \leq n$.

Proof. We'll show that for $s > n$, $\mathcal{H}^s(F) = 0$.
(This implies $\dim_H F \leq n$)
Since, $\mathcal{H}^s(F) \leq \mathcal{H}^s(\mathbb{R}^n)$, it is enough to show that $\mathcal{H}^s(\mathbb{R}^n) = 0$.
For this it is enough to show that $\mathcal{H}^s(C) = 0$, where C is a unit cube.
(because $\mathbb{R}^n$ is a countable union of such cubes)

$$\mathcal{H}^s(C) = \sup_{\delta > 0} \mathcal{H}^s_\delta(C).$$

Let $\delta > 0$. We'll show that $\mathcal{H}^s_\delta(C) = 0$.

We can cover C by k^n subcubes each of side length $\frac{1}{k}$ (i.e. diameter $\sqrt{n} \cdot \frac{1}{k}$)

Choose $k \in \mathbb{N}$ s.t. $\frac{\sqrt{n}}{k} < \delta$.
Then these k^n cubes form a δ-cover of C.

$$\therefore \quad \mathcal{H}^s_\delta(C) \leq k^n \left(\frac{\sqrt{n}}{k}\right)^s = \underbrace{(\sqrt{n})^s \cdot \left(\frac{1}{k}\right)^{s-n}}_{\to\, 0 \text{ as } k \to \infty}.$$

$$\therefore \quad \mathcal{H}^s(C) = 0$$

Prop: Let $F \subseteq \mathbb{R}^n$ and suppose $f : F \to \mathbb{R}^m$ satisfies the Hölder condition

$$\|f(x) - f(y)\| \le c\,\|x-y\|^{\alpha} \quad \forall x,y \in F$$

Then $\dim_H f(F) \le \frac{1}{\alpha} \dim_H F$

Proof: We know that $\mathcal{H}^{s/\alpha}(f(F)) \le c^{s/\alpha}\, \mathcal{H}^{s}(F)$ for any $s \ge 0$. — (i)

If $s > \frac{1}{\alpha} \dim_H F$ i.e. $s\alpha > \dim_H F$,

$$\mathcal{H}^{s\alpha}(F) = 0$$

$\therefore$ From (i), $\mathcal{H}^{s}(f(F)) \le c^{s}\, \mathcal{H}^{s\alpha}(F) = 0$

$\Rightarrow \dim_H (f(F)) \le s \quad \forall s > \frac{1}{\alpha} \dim_H F$

$\Rightarrow \dim_H f(F) \le \frac{1}{\alpha} \dim_H F$.

Corollary: If $f : F \to \mathbb{R}^m$ is a Lipschitz map, then $\dim_H f(F) \le \dim_H F$

Pf: Take $\alpha = 1$ in the prev. proposition.

Corollary: If $f : F \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is a bi-Lipschitz map, i.e., $c_1 \|x-y\| \le \|f(x) - f(y)\| \le c_2 \|x-y\|$ $\forall x,y \in F$,

where $0 < c_1 \le c_2 < \infty$, then

$$\dim_H f(F) = \dim_H F.$$

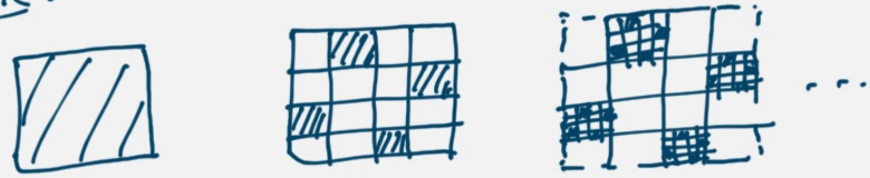
Pf: Since f is Lipschitz, $\dim_H f(F) \le \dim_H F$.

Also, $f^{-1} : f(F) \to F$ is Lipschitz.

$\Rightarrow \dim_H F = \dim_H f^{-1}(f(F)) \le \dim_H f(F)$

Calculation of Hausdorff dimension

Example: Cantor dust $= \bigcap_{n=0}^{\infty} E_n = E$



E_0 $\quad$ E_1 $\quad$ E_2

E_n consists of 4^n squares of side length $\left(\frac{1}{4}\right)^n$, i.e. diameter $4^{-n}\cdot\sqrt{2}$.

For $\delta > 0$, choose $n \in \mathbb{N}$ s.t. $4^{-n}\cdot\sqrt{2} < \delta$.

$$\mathcal{H}^s_\delta(E) \leq 4^n\left(4^{-n}\sqrt{2}\right)^s = (\sqrt{2})^s\cdot\left(\frac{1}{4^{s-1}}\right)^n \to 0 \text{ as } n\to\infty \text{ if } s>1$$

$$\Rightarrow \mathcal{H}^s(E) = 0 \quad \forall s > 1 \Rightarrow \boxed{\dim_H E \leq 1}$$

If we take f to be the projection of $\mathbb{R}^2$ onto $\mathbb{R}$, then $f(E) = [0,1]$.

Since $\|f(x)-f(y)\| \leq \|x-y\|$, f is a Lipschitz map.

$$\therefore\ 1 = \dim_H[0,1] = \dim_H(f(E)) \leq \dim_H E.$$

$$\Rightarrow \boxed{\dim_H E \geq 1}$$

$$\therefore \boxed{\dim_H E = 1}$$

Example: Hausdorff dimension of the middle third Cantor set, C.

$$\dim_H C = \frac{\ln 2}{\ln 3}.$$

Heuristic calculation:

$f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = \frac{x}{3}$

$f_2 : \mathbb{R} \to \mathbb{R}$, $f_2(x) = \frac{2}{3} + \frac{x}{3}$.

Then $C = f_1(C) \cup f_2(C)$

$\left(f_1(C) = C \cap [0, \frac{1}{3}] \; ; \; f_2(C) = C \cap [\frac{2}{3}, 1] \right)$

Note that f_1 and f_2 are similarity transformations with similarity ratios $\frac{1}{3}$ each.

$\therefore \; \mathcal{H}^s(f_i(C)) = \left(\frac{1}{3}\right)^s \mathcal{H}^s(C)$ for $i = 1, 2$

Since, $\mathcal{H}^s(C) = \mathcal{H}^s(f_1(C)) + \mathcal{H}^s(f_2(C))$

$= \left(\frac{1}{3}\right)^s \mathcal{H}^s(C) + \left(\frac{1}{3}\right)^s \mathcal{H}^s(C)$

$\Rightarrow \quad \mathcal{H}^s(C) = \frac{2}{3^s} \mathcal{H}^s(C)$ —— (i)

Let $s = \dim_H C$. If we know that $0 < \mathcal{H}^s(C) < \infty$, then (i) $\Leftrightarrow \frac{2}{3^s} = 1$

$\Leftrightarrow \boxed{s = \frac{\ln 2}{\ln 3}}$

Rigorous calculation

- $\dim_H C \le \frac{\ln 2}{\ln 3}$

At the kth level, C_k consists of 2^k intervals each of length 3^{-k}.

$$\mathcal{H}^s_{3^{-k}}(C) \le 2^k (3^{-k})^s = \left(\frac{2}{3^s}\right)^k$$

If $s > \frac{\ln 2}{\ln 3}$, i.e., $\frac{2}{3^s} < 1$, $\left(\frac{3}{3^s}\right)^k \to 0$ as $k \to \infty$.

$$\Rightarrow \quad \mathcal{H}^s(C) = 0$$

$$\therefore \quad \dim_H C \le \frac{\ln 2}{\ln 3}$$