Lec 6

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$X(\mathbb{K}) \to M$ · subspace
$x, y \in X$
$x \sim y \iff x - y \in M$
$\sim$ is a equivalence relation

$[x] = x + M$

$X/M = \{ [x] \mid x \in X \} = \{ x + M \mid x \in X \}$

$+ , \cdot$ well defined

$(x+M) + (y+M) = (x+y) + M$

$\|x+M\| = dis(x, M)$ $\quad \left(dis(x, M) = 0 \iff x \in \overline{M} \right)$

$M \underset{\text{closed}}{\subseteq} \overline{X}$.

'$\|\cdot\|$' on X/M

Theorem

Let X be a banach space and M be a closed subspace of X. Then X/M is a Banach space.

lemma Let (y_n) is a CS in NLS Y

then $\exists$ a subseq (y_{n_k}) of (y_n) st

$$\|y_{n_{k+1}} - y_{n_k}\| < 2^{-k}$$

proof

$\|x_m - x_n\| < \frac{1}{2^{-k}} \quad \forall\ m, n \geq n_k$

$n_1 < n_2 < n_3 < \cdots$
$\hookrightarrow$ subseq

Given (y_n) is CS
by def. $\forall\ \varepsilon > 0\ \exists\ N$ such that

$$\|y_m - y_n\| < \varepsilon \quad \forall\ n, m \geq N$$

choose $\varepsilon = 1/2$ then
$\varepsilon = 1/2^2$
$\vdots$
get sub seq

Claim: X/M is a BS

M: closed

let $x_n + M$ be a CS in X/M

when a CS have a convergent subseq then the CS is also convergent

to show $\exists\ x_{n_k}$ st $x_{n_k} + M$ is convergent

X/M is a nls so for $(x_n + M)$ $\exists$ a subseq $(x_{n_k} + M)$ such that

$$\|(x_{n_{k+1}} + M) - (x_{n_k} + M)\| < 1/2^{k} \quad \text{—(1)}$$

$\|(x_{n_{k+1}} \dots) - (x_{n_k} + M)\| < 1/2^{k}$ — (1)

consider $y_1 = 0$

$$\inf_{y \in M} \|(x_{n_1} - y_1) - (x_{n_2} - y)\| = \inf_{y \in M} \|(x_{n_1} - x_{n_2}) + y\|$$

$$= \|x_{n_1} - x_{n_2} + M\| < \frac{1}{2}$$

$\exists\, y_2$ such that

$$\|(x_{n_1} - y_1) - (x_{n_2} - y_2)\| < 2 \cdot \frac{1}{2}$$

(using defn of infimum)

since $y_2 \in M$

$$\inf_{y \in M} \|(x_{n_2} - y_2) - (x_{n_3} - y)\| = \inf \|(x_{n_2} - x_{n_3}) + y\|$$

$(\ \text{dist}(x, M) = \text{dist}(x+y, M) \quad y \in M\ )$

$$= \|x_{n_2} - x_{n_3} + M\|$$

$$< 1/2^2$$

By def. of inf. $\exists\, y_3 \in M$ such that

$$\|(x_{n_2} - y_2) - (x_{n_3} - y_3)\| < 2 \cdot \frac{1}{2^2}$$

continuing like this $\exists\, y_k$ such that

$$\|(x_{n_k} - y_k) - (x_{n_{k+1}} - y_{k+1})\| < 2 \cdot \frac{1}{2^k}$$

$z_k = x_{n_k} - y_k$

we can say $\|z_k - z_{k+1}\| < \frac{1}{2^{k-1}}$

if a seq satisfy this
then that is a cauchy convergent seq

$z_k \to z \ (\in X)$ as X is BS

$$\|(x_{n_k} + M) - (z + M)\| = \|(x_{n_k} - z) + M\|$$

$$= \|x_{n_k} - y_k - z + M\|$$

$$= \|z_k - z + M\|$$

$$(= \inf \|z_k - z + u\|, \quad u \in M)$$

$$\leq \|z_k - z\| \to 0 \quad \text{as } k \to \infty$$

$$\therefore (x_{n_k} + M) \longrightarrow (z + M) \quad \text{as } k \to \infty$$

Converse

theorem let X be a NLS and M a closed subspace of X

If X/M is BS and M is also BS then X is BS

proof: let (x_n) be a CS in X

then $\|(x_n + M) - (x_m + M)\| = \|x_n - x_m + M\|$

$= \inf \{ \|x_n - x_m + u\| : u \in M \}$

as $0 \in M$

$\leq \|x_n - x_m\| \longrightarrow 0$ as $m, n \to \infty$

$\therefore (x_n + M)$ is CS in X/M

since X/M is BS

then $x_n + M \to x + M$ $x \in X$

$\|(x_n + M) - (x + M)\| < \varepsilon$ $\forall$ $n \geq N_0$

$\Rightarrow \|x_n - x + M\| < \varepsilon$ $\forall$ $n \geq N_0$

LHS $= \inf \{ \|x_n - x + y\| : y \in M \}$

choose $y_n \in M$ st

$\|x_n - x + y_n\| < \|(x_n - x) + M\| + \frac{1}{n}$

$\to 0$ as $n \to \infty$

$\therefore x_n + y_n \to x$ in X

now to show (y_n) is a CS (H.W.)

$\|y_n - y_m\| = \|y_n + x_n - x_n + x_m - x_m - y_m\|$

Convergent Series in NLS

let (x_n) be a seq in a NLS X.

consider $\sum_{n \in N} x_n$

this series convergent to x

$S_N = \sum_{n=1}^{N} x_n \to x$ in X

means $\|S_N - x\|_X \to 0$ as $N \to \infty$

Absolutely convergent series

If $\sum \|x_n\| < \infty$ then $\sum x_n$ is absolute convergent

Ex in $\mathbb{R}$ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$ → convergent

but $\sum_{n=1}^{\infty} \left|(-1)^{n+1} \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent

Remark

[absolute convergent in a NLS is NOT stronger that convergence]

Ex. $X = P([0,1])$, $\|\cdot\|_\infty$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

here $\sum_{n=1}^{\infty} \left\|\frac{x^n}{n!}\right\|$ is convergent

$\therefore \sum \frac{x^n}{n!}$ is absolutely convergent

however $\sum \frac{x^n}{n!} \to e^x \notin P[0,1]$

→ Not convergent

theorem

for a NLS X

X is BS $\Longleftrightarrow$ every abs. convergent series in X is convergent

proof: Suppose X is a BS

let $\sum x_n$ be abs convergent

(i.e. $\sum \|x_n\|$ is convergent)

let $S_N = \sum_{n=1}^{N} x_n$ $\quad N > M$

$$\|S_N - S_M\| = \left\|\sum_{n=1}^{N} x_n - \sum_{n=1}^{M} x_n\right\| \leq \sum_{n=M+1}^{N} \|x_n\|$$

Since X is BS

so S_N is convergent and that implies $\sum x_n$ is convergent.