Lec 4

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2 norms are equivalent if $\exists\ c_1, c_2 > 0$ such that

$$c_1 \|x\| \leq \|x\|_* \leq \|x\|$$

<u>remark</u>: Two norms are equivalent $\Leftrightarrow$

$E \subseteq X$, E is open wrt $\|\cdot\|$
iff E is open wrt $\|\cdot\|_*$

$\Leftrightarrow$ (x_n) is a cauchy seq in $\|\cdot\|$
iff ——— " ——— $\|\cdot\|_*$

$\Leftrightarrow$ (x_n) is convergent wrt $\|\cdot\|$
iff ——— " ——— $\|\cdot\|_*$

then $(X, \|\cdot\|)$ is a BS iff $(X, \|\cdot\|_*)$ is a BS

$\sup_{t \in (a,b)} |x(t)|$

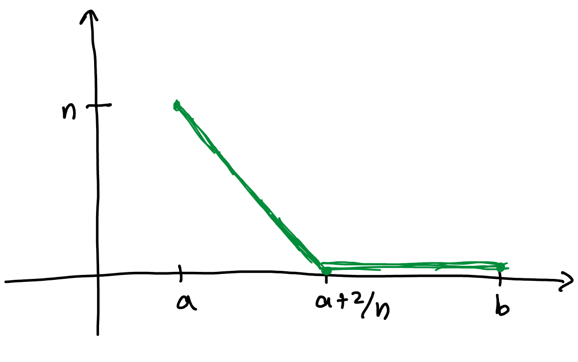
$(C[a,b], \|\cdot\|_\infty)$ is a normed linear space

$(C[a,b], \|\cdot\|_1)$ is also NLS

$$\|x\|_1 = \int_a^b |x(t)|\, dt$$

$$\leq \|x\|_\infty \cdot (b-a)$$

we show $\nexists\ c > 0$ such that $\|x\|_\infty < c\|x\|_1$



choose n such that $a + \frac{2}{n} < b$ ($\exists$ such an n by archemedian principle)

i.e. $n > \frac{2}{b-a}$

consider $x_n(t) = \begin{cases} n - \frac{n^2}{2}(t-a) & a < t < a + 2/n \\ 0 & a + \frac{2}{n} \leq t \leq b \end{cases}$

$\|x_n\|_\infty = n$

$\|x_n\|_1 = 1$

if $\exists$ $c > 0$ st $\|x\|_\infty < c\|x\|_1$

$\Leftrightarrow$ $n \leq c \quad \forall n$

$\therefore \nexists$ any such c.

Infinite dim BS

$l_p = \{ x \in \mathcal{F}(\mathbb{N}, K) \mid \sum_{i=1}^{\infty} |x(i)|^p < \infty \}$

$(l^p, \|\cdot\|_p)$ is BS.

let (x_n) be a CS (cauchy seq) in l_p

for any $\varepsilon > 0$ $\exists$ $N_\varepsilon \in \mathbb{N}$ st

$$\|x_n - x_m\|_p < \varepsilon \quad \forall \ n, m \geq N_\varepsilon$$

i.e. $\left(\sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^p \right)^{1/p} < \varepsilon \quad \forall \ n, m \geq N_\varepsilon$

for each $j \in \mathbb{N}$

$$|x_n(j) - x_m(j)|^p \leq \left(\sum_{j=1}^{\infty} |x_n(j) - x_m(j)|^p \right) < \varepsilon^p \quad \forall \ n, m \geq N_\varepsilon$$

i.e. $|x_n(j) - x_m(j)| < \varepsilon \quad \forall \ n, m \geq N_\varepsilon$

i.e. $(x_n(j))_{n \in \mathbb{N}}$ is a CS in $(K, |\cdot|)$

since $(K, |\cdot|)$ is complete

$\therefore$ for each j, $x_n(j) \longrightarrow \alpha_j$ as $n \to \infty$

let $x = (x(1), x(2), \ldots)$
$= (\alpha_1, \alpha_2, \ldots)$

TP
i) $x \in l^p$
ii) $\|x_n - x\|_p \to 0$ as $n \to \infty$

if $x \in l^p$:
Consider $\sum_{j=1}^{k} |x_n(j) - x(j)|^p = \sum_{j=1}^{k} \left| x_n(j) - \lim_{m \to \infty} x_m(j) \right|^p$

$$= \lim_{m \to \infty} \sum_{j=1}^{k} |x_n(j) - x_m(j)|^p < \varepsilon^p \quad \forall \ n \geq N_\varepsilon \quad \forall k$$

$$\therefore \sum_{j=1}^{\infty} |x_n(j) - x(j)|^p < \varepsilon^p \quad \forall \ n \geq N_\varepsilon$$

$$\|x_n - x\|_p < \varepsilon \quad \forall \ n \geq N_\varepsilon$$

$$\therefore x_n \longrightarrow x$$

TP $x \in l^p$

$$\left(\sum_{j=1}^{k} |x(j)|^p \right)^{1/p} \leq \left(\sum_{j=1}^{k} |x(j) - x_{N_\varepsilon}(j)|^p \right)^{1/p} + \left(\sum_{j=1}^{k} |x_{N_\varepsilon}(j)|^p \right)^{1/p}$$

(Minkowski)

$$\leq \left(\sum_{j=1}^{\infty} |x(j) - x_{N_\varepsilon}(j)|^p \right)^{1/p} + \|x_{N_\varepsilon}\|_p$$

$$< \varepsilon + \|x_{N_\varepsilon}\|_p < \infty$$

$$\Rightarrow x \in l_p$$

Q $(l^\infty, \|\cdot\|_\infty)$ is BS

Ex $c_0 \subseteq c \subseteq l^\infty$

$c_0, c \to \|\cdot\|_\infty$

are c_0, c B.S. wrt $\|\cdot\|_\infty$?

Show that c_0 forms a closed subspace of $(l^\infty, \|\cdot\|_\infty)$

$x \in c_0 \Rightarrow x$ seq converge to zero

$x_n \longrightarrow x$ in $(l^\infty, \|\cdot\|)_\infty$

$C_b(\Omega) \to$ cont. func^n on Ω $\mathbb{K}$-valued

(x_n) CS $\Rightarrow \varepsilon > 0 \; \exists N_\varepsilon$

$\|x_m - x_n\|_\infty < \varepsilon \quad \forall n, m \geq N_\varepsilon$

$\Rightarrow (x_n(t))_{(n \in \mathbb{N})}$ is a CS in $\mathbb{K}$

using $(\mathbb{K}, |\cdot|)$ complete

$x_n(t) \to \alpha_t$ as $n \to \infty$ for each t

Define $x(t) = \alpha_t \qquad x: \Omega \to \mathbb{K}$

show $x \in C_b(\Omega)$ ie x is cont. bdd $\mathbb{K}$ valued

to prove x is cont. let $t_0 \in \Omega$

$$|x(t) - x(t_0)| = |x(t) - x_N(t) + x_{N_\varepsilon}(t) - x_{N_\varepsilon}(t_0) + x_{N_\varepsilon}(t_0) - x(t_0)|$$

$$\leq \underbrace{|x(t) - x_{N_\varepsilon}(t)|}_{< \varepsilon/3} + |x_{N_\varepsilon}(t) - x_{N_\varepsilon}(t_0)| + \underbrace{|x_{N_\varepsilon}(t_0) - x(t_0)|}_{< \varepsilon/3}$$

$\left(|x(z) - x_N(z)| \leq \|x - x_N\|_\infty < \varepsilon/3 \quad \forall N \geq N_\varepsilon \right)$

$$\leq \frac{2\varepsilon}{3} + \underbrace{|x_{N_\varepsilon}(t) - x_{N_\varepsilon}(t_0)|}_{< \varepsilon/3}$$

cont func

$\forall \; d(t, t_0) < \delta$

$< \varepsilon$

$\therefore x$ is cont. func^n

Non Banach Space

$c_{00} \subseteq l^p \subseteq l^\infty$

$(c_{00}, \|\cdot\|_p)$ BS?

(c_{00}, l_∞) BS?

C_{00} is dense subspace of l^p

let $x \in l^p$ $1 \le p < \infty$

consider a seq x_n

$$x_n(j) = \begin{cases} x(j) & j \le n \\ 0 & j > n \end{cases}$$

$x_n \in C_{00}$

$x_n \to x$ in $\|\cdot\|_p$ and $\|\cdot\|_\infty$

$x \notin C_{00}$

C_{00} is Not BS

$\overline{C_{00}} = l^p$ with norm $\|\cdot\|_p$

$P[a,b]$ = space of all poly $\subseteq C[a,b]$

$a_0 + a_1 t + \dots + a_n t^n$

$(P[a,b], \|\cdot\|_\infty) \to$ Weierstrass Approx th

$\overline{P[a,b]} = C[a,b]$

$\underline{\underline{Q}}$ $(P[a,b], \|\cdot\|_p)$ is BS ?