Assignment 1 :

We try to prove that $C[a,b]$ is dense in $L^1[a,b]$.

First, we show that $C[a,b] \subseteq L^1[a,b]$

Let $f \in C[a,b]$

$\Rightarrow$ f is bounded, let's say by M,
ie $|f(x)| \leq M \quad \forall \ x \in [a,b]$

$$\Rightarrow \int_{[a,b]} |f| \, d\mu \leq M \mu([a,b])$$
$$= M(b-a) < \infty$$

$$\Rightarrow f \in L^1[a,b]$$

Now, we try to show that every function in $L^1[a,b]$ can be very well-approximated by a continuous function.

That is, $\forall \ \varepsilon > 0 \ \forall \ f \in L^1[a,b] \ \exists \ g \in C[a,b]$ such that $\|f-g\|_{L^1[a,b]} < \varepsilon$

We get a sequence of bounded functions converging pointwise to f, defined by

$$f_n(x) := \begin{cases} n & f(x) > n \\ f(x) & |f(x)| \le n \\ -n & f(x) < -n \end{cases}$$

Then, $f_n \in L^1 \cap L^\infty[a,b]$

because $\displaystyle\int_{[a,b]} |f_n| \, d\mu \le n(b-a) < \infty$

Now,

$$(f_n - f)(x) = \begin{cases} n - f(x) & f(x) > n \\ 0 & |f(x)| \le n \\ -n - f(x) & f(x) < -n \end{cases}$$

$$\Rightarrow \quad |f_n - f|(x) = \begin{cases} |f(x)| - n & |f(x)| > n \\ 0 & |f(x)| \le n \end{cases}$$

$$\| f_n - f \|_{L^1} = \int_{[a,b]} |f_n - f| \, d\mu$$

Now, $|f_n - f|(x) \le |f(x)| - n < |f(x)|$

$\Rightarrow \quad |f_n - f| \le |f|$ with $f \in L^1$

$\Rightarrow |f| \in L^1$

For almost every $x \in [a,b]$, $f(x) \in \mathbb{R}$
so for $n \geq |f(x)|$, $f_n(x) = f(x)$

$$\Rightarrow \quad f_n(x) - f(x) \to 0$$
$$\Rightarrow \quad |f_n - f| \to 0 \quad \text{a.e. on } [a,b]$$

Using DCT,

$$\| f_n - f \|_{L^1} \to \int_{[a,b]} 0 \, d\mu = 0$$

Now, we use the fact that every bounded absolutely integrable function can be approximated by a seq. of simple functions.

So we first choose an n such that
$$\| f_n - f \|_{L^1} < \frac{\varepsilon}{3}$$

and then a simple function s such that
$$\| s - f_n \| < \frac{\varepsilon}{3}$$

where s looks like
$$s = \sum_{k=1}^{n} c_k \, \chi(E_k) \, , \quad c_k \in \mathbb{C}$$
for measurable sets E_k, $1 \leq k \leq n$

Now, we use the regularity property of Lebesgue measure, for each of the sets E_k, we have an open set $U_k \supseteq E_k$ in $[a,b]$ such that $\mu(U_k \setminus E_k) < \varepsilon'$ where ε' will be chosen suitably.

By using the regularity property again, for each E_k we have a closed set $V_k \subseteq E_k$ in $[a,b]$ such that $\mu(E_k \setminus V_k) < \varepsilon'$

Now, since $V_k \subseteq E_k \subseteq U_k \Rightarrow V_k \subseteq E_k$ and $E_k \cap U_k^c = \emptyset$

with both V_k and U_k^c closed, $V_k \cap U_k^c = \emptyset$
We use Urysohn's lemma to get a continuous $\varphi_k : [a,b] \to [0,1]$ such that

$$\varphi_k\big|_{V_k} \equiv 1 \quad \text{and} \quad \varphi_k\big|_{U_k^c} \equiv 0$$

Then, define $g := \sum_{k=1}^{n} c_k \varphi_k$

Clearly, $g \in C[a,b]$

We observe that

$$\int_{[a,b]} |g - s| \, d\mu = \int_{[a,b]} \left| \sum_{k=1}^{n} c_k (\varphi_k - \chi(E_k)) \right| d\mu$$

$$\leq \int_{[a,b]} \sum_{k=1}^{n} |c_k| |\varphi_k - \chi(E_k)| \, d\mu$$

It follows from the definition of φ_k that

$$\chi(V_k) \leq \varphi_k \leq \chi(U_k)$$

On E_k, $|\varphi_k - \chi(E_k)| = 1 - \varphi_k \leq 1 - \chi(V_k)$
$= \chi(E_k \setminus V_k)$

On E_k^c, $|\varphi_k - \chi(E_k)| = \varphi_k \leq \chi(U_k)$
$= \chi(U_k \setminus E_k)$

$$\Rightarrow \int_{[a,b]} |\varphi_k - \chi(E_k)| \, d\mu = \int_{E_k} + \int_{E_k^c}$$

$$\leq \int_{E_k} \chi(E_k \setminus V_k) \, d\mu + \int_{E_k^c} \chi(U_k \setminus E_k) \, d\mu$$

$$= \mu(E_k \setminus V_k) + \mu(U_k \setminus E_k)$$

$$< \varepsilon' + \varepsilon' = 2\varepsilon'$$ for all $1 \leq k \leq n$

So $\int_{[a,b]} |g - s| \, d\mu \leq M \int_{[a,b]} \sum_{k=1}^{n} |\varphi_k - \chi(E_k)| \, d\mu$

where $M := \sup\{|c_k| : 1 \leq k \leq n\}$

$$< 2Mn\varepsilon'$$

$$= \frac{\varepsilon}{3}$$ given $\varepsilon' = \frac{\varepsilon}{6Mn}$

$$\Rightarrow \| g - s \|_{L^1} < \frac{\varepsilon}{3}$$

Now,

$$\| g - f \|_{L^1} \le \| g - s \|_{L^1} + \| s - f_n \|_{L^1} + \| f_n - f \|_{L^1}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

with $g \in C[a,b]$

Hence under L^1-norm, $\overline{C[a,b]} = L^1[a,b]$