Lecture – 2/2/26.

IFS: (X, d) : complete metric space

$f_1, f_2, \dots, f_N$ contraction maps on (X, d)

Then $\exists$ a unique compact set $K \subseteq X$ such that
$$K = \bigcup_{i=1}^{N} f_i(K)$$

- K is called an attractor or invariant set for the IFS.

$F : \mathcal{K}(X) \to \mathcal{K}(X)$ (space of all nonempty compact subsets of X with the Hausdroff metric)

$$F(K) = \bigcup_{i=1}^{N} f_i(K)$$

Also, $D(F^n(K), A) \to 0$ as $n \to \infty$ for any $K \in \mathcal{K}(X)$

In fact, $D(F^n(K), A) = D(F(K), F(A))$
$\leq c\, D(K, A)$,

where $c = \max\{c_i : 1 \leq i \leq N\} < 1$

$\therefore D(F^n(K), A) \leq c^n D(K, A)$
$\to 0$ as $n \to \infty$.

→ code Space and Code map :

Let $\{(X, d) ; f_1, f_2, \dots, f_N\}$ be an I.F.S.

Let $\Lambda = \{1, 2, \dots, N\}$ → alphabet set

For $(i_1, i_2, \dots, i_n) \in \Lambda^n$, we define

$$f_w = f_{i_1} \circ f_{i_2} \circ f_{i_3} \cdots \circ f_{i_n}$$

Since, $K = \bigcup_{i=1}^{N} f_i(K)$, $K = \bigcup_{w \in \Lambda^n} f_w(K)$

Let $w = (i_1, i_2, \dots, i_n, \dots) \in \Lambda^{\mathbb{N}}$

For $x \in X$, and $w = (i_1, i_2, \dots) \in \Lambda^{\mathbb{N}}$

define, $x_n = f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_n}(x)$

Then, $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence

(If $n \geq m$, $d(x_n, x_m) = d(f_{i_1} \circ \cdots \circ f_{i_m}(f_{i_{m+1}} \circ \cdots f_{i_n}(x)), f_{i_1} \circ \cdots \circ f_{i_m}(x))$

$\leq c^m d(f_{i_{m+1}} \circ \cdots f_{i_n}(x), x)$

$\to 0$ as $m \to \infty$)

Since (X, d) is complete, $\{x_n\}$ converges to some point in X,

say $\pi(x, w)$

Suppose $x, y \in X$

Claim: $\pi(x, w) = \pi(y, w)$ for any $w \in \Lambda^{\mathbb{N}}$

$$\pi(x, w) = \lim_{n \to \infty} f_{i_1} \circ \cdots \circ f_{i_n}(x)$$

$$\pi(y, w) = \lim_{n \to \infty} f_{i_1} \circ \cdots \circ f_{i_n}(y)$$

$$d(f_{i_1} \circ \cdots \circ f_{i_n}(x), f_{i_1} \circ \cdots \circ f_{i_n}(y))$$

$$\leq c^n d(x, y) \longrightarrow 0 \text{ as } n \to \infty.$$

$\Rightarrow \pi(x, w) = \pi(y, w)$

Thus, we can define a map $\pi : \Lambda^{\mathbb{N}} \to X$
by $\pi(i_1, i_2, \ldots, i_n, \ldots) = \lim_{n \to \infty} f_{i_1} \circ \cdots \circ f_{i_n}(x)$
for any $x \in X$

Let $J = \{\pi(w) : w \in \Lambda^{\mathbb{N}}\}$

$$J = \bigcup_{i=1}^{N} f_i(J)$$

Proof: $f_i(\pi(w)) = f_i(\lim_{n \to \infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(x))$

$= \lim_{n \to \infty} f_i \circ f_{i_1} \circ \cdots \circ f_{i_n}(x)$ ($\because f_i$ is continuous)

$= \pi(iw)$

where if $w = (i_1, i_2, \ldots, i_n, \ldots)$,
$iw = (i, i_1, i_2, \ldots, i_n, \ldots)$

$\therefore f_i(J) \subset J$ for each i.

Conversely, let $x \in J$

Then, $x = \pi(\omega)$ for some $\omega = (i_1, i_2, \dots) \in \Lambda^{\mathbb{N}}$

$= f_{i_1}(\pi((i_2, i_3, \dots)))$

$\in \bigcup_{i=1}^{N} f_i(J)$

Claim: J is nonempty compact set

$\pi : \Lambda^{\mathbb{N}} \to X$

If we take the product topology on $\Lambda^{\mathbb{N}}$
(discrete topology on $\Lambda = \{1, 2, \dots, N\}$)

then the map π is continuous.

Since Λ is compact, $\Lambda^{\mathbb{N}}$ is compact by the Tychonoff theorem.

Hence, $J = \pi(\Lambda^{\mathbb{N}})$ is compact.

By the uniqueness of attractor,
J is the attractor for the IFS.

- $\Lambda^{\mathbb{N}}$ is called the code space
- $\pi : \Lambda^{\mathbb{N}} \to X$ is called the code map

→ Self-similar sets

(X, d) — complete metric space.

Assume $f_1, f_2, \dots f_N$ are similarity maps,

i.e. $d(f_i(x), f_i(y)) = c_i d(x, y) \quad \forall x, y \in X$

Also, assume that each $c_i \in (0, 1)$

Then, $\exists$ a unique attractor $A \subseteq X$ satisfying $A = \bigcup_{i=1}^{N} f_i(A)$

- A is called a self-similar set.

e.g. Cantor set

$f_1(x) = \frac{x}{3} \qquad f_2(x) = \frac{2}{3} + \frac{x}{3} \qquad c_1 = c_2 = \frac{1}{3}$

similarity dimension: There exists a unique $s \in (0,\infty)$ s.t.

$$\sum_{i=1}^{N} c_i^s = 1$$

This s is called the similarity dimension of the self similar set A.

(Let $\phi : [0,\infty) \to \mathbb{R}$ be defined by,

$$\phi(s) = \sum_{i=1}^{N} c_i^s$$

Then,
- ϕ is continuous
- ϕ is strictly decreasing
- $\phi(0) = N \geq 1$
- $\lim_{s\to\infty} \phi(s) = 0$)

e.g. For the Cantor set, C, the similarity dimension is given by

$$\left(\frac{1}{3}\right)^s + \left(\frac{1}{3}\right)^s = 1$$

$$\Rightarrow \frac{2}{3^s} = 1 \Rightarrow s = \frac{\ln 2}{\ln 3}$$

- For Sierpinski Gasket, $c_1 = c_2 = c_3 = \frac{1}{2}$

$$3\left(\frac{1}{2}\right)^s = 1 \Rightarrow 2^s = 3 \Rightarrow s = \frac{\ln 3}{\ln 2}$$

For the Koch curve, $c_1 = c_2 = c_3 = c_4 = 1/3$

$$4\left(\frac{1}{3}\right)^s = 1 \Rightarrow s = \frac{\ln 4}{\ln 3}$$

* Separation Conditions:

Let $\{(X,d) : f_1, f_2, \dots, f_N\}$ be an IFS.

Let A be the attractor for this IFS.

* Strong separation condition

$$f_i(A) \cap f_j(A) = \emptyset \text{ for } i \neq j$$

★ Open Set Condition

$\exists$ a nonempty open set $V \subseteq X$

s.t. $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$

and $\bigcup_{i=1}^{N} f_i(V) \subseteq V$

★ Strong open set condition:

$\exists$ a nonempty set $V \subseteq X$ s.t.

$$f_i(V) \cap f_j(V) = \emptyset, \quad \bigcup_{i=1}^{N} f_i(V) \subseteq V$$

and $V \cap A = \emptyset$, where A is the attractor.