Lec 5

Recap:

- Example $\rightarrow \ell^p$ (Banach space)
- Non Banach space $\left((c_{00}, \|\cdot\|_\infty, \|\cdot\|_p \right)$
- $(C[a,b], \|\cdot\|_\infty) \rightarrow$ BS. $(P[a,b], \|\cdot\|_\infty)$

  $(B([a,b]), \|\cdot\|_\infty) \rightarrow$ BS.

Qn. $(P[a,b], \|\cdot\|_p)$ is a B.S ? Not. $1 \le p < \infty$

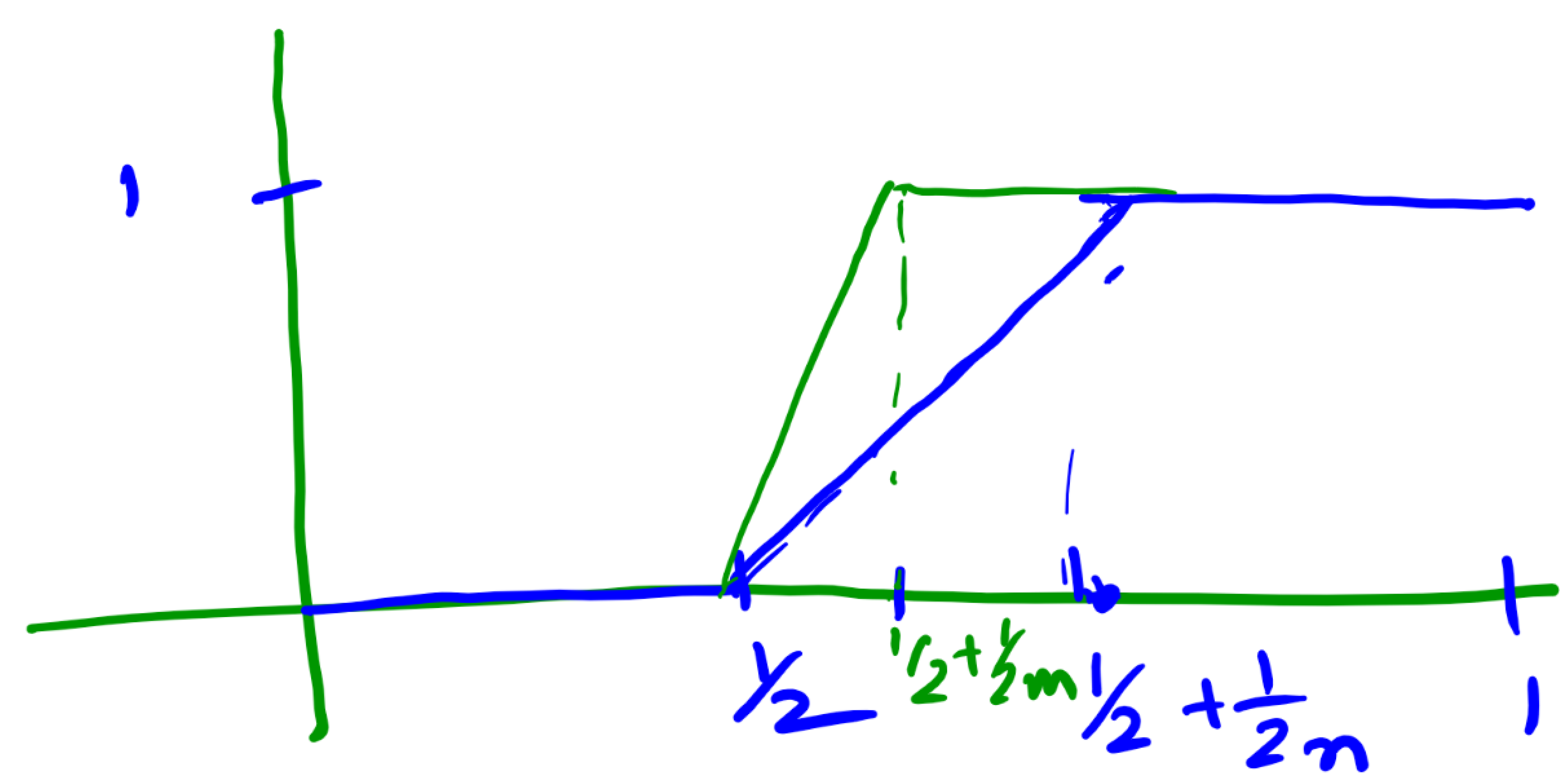
Today

- $(C[a,b], \|\cdot\|_1)$ is not a BS.

$$\|x\|_1 = \int_a^b |x(t)|\, dt$$

$|x_n(t)| \le 1$

$[0,1]$



For each $n \in \mathbb{N}$.

$$x_n(t) = 0, \quad 0 \le t \le \tfrac{1}{2}$$
$$= \cancel{a}^{2n}t + \cancel{b}^{-n}, \quad \tfrac{1}{2} \le t \le \tfrac{1}{2}+\tfrac{1}{2n}$$
$$= 1 \quad \tfrac{1}{2}+\tfrac{1}{2n} \le t \le 1.$$

$m > n$ $\quad \|x_m - x_n\|_1.$

$$\int_0^1 |x_m(t) - x_n(t)|\, dt.$$

$$= \int_{1/2}^{1/2+\frac{1}{2n}} |x_m(t) - x_n(t) dt)| \le \int_{1/2}^{1/2+\frac{1}{2n}} |x_m(t)| + |x_n(t)|\, dt$$

$$\le 2 \int_{1/2}^{1/2+1/2n} dt \le 2 \cdot \frac{1}{2n} \longrightarrow 0$$

as $n \to \infty$.

Thus $\|x_m - x_n\|_1 \to 0$ as $m, n \to \infty$.

ie (x_n) is a C.S in $C[0,1]$.

Let $\|x_n - x\|_1 \to 0$ as $n \to \infty$

$$\int_0^{1/2} |x(t)|\, dt.$$

$$= \int_0^{1/2} |x_n(t) - x(t)|\, dt$$

$$\leq \|x_n - x\|_1 \to 0 \text{ as } n \to \infty$$

$$\int_0^{1/2} |x(t)|\, dt = 0$$

$$x(t) = 0, \quad 0 \leq t \leq \tfrac{1}{2}$$

$$\int_{1/2+1/2n}^{1} |1 - x(t)|\, dt$$

$$= \int_{1/2+1/2n}^{1} |x_n(t) - x(t)|\, dt$$

$$\leq \quad \|x_n - x\|_1$$

$$\int_{1/2}^{1} |1 - x(t)|\, dt = 0$$

$$x(t) = 1, \quad 1/2 \leq t \leq 1$$

$$x(t) \notin C[0, 1]$$

• Ex. $(C[a,b], \|\cdot\|_p)$, $1 \leq p < \infty$ is NOT a BS.

Infact it is a dense subspace of $L^p[a,b]$.

$$\int_a^b |f(x)|^p \, dx < \infty .$$

Q Does for an nLS which are not Banach space

∃ a B.S s.t $nLS \subseteq BS$.

Yes!!

$X \to$ n l s. $\|\cdot\|_X$

Let (x_n) be a C.S in X.

- $(\|x_n\|)$ is a C S in $\mathbb{K}$.

  So $\lim_{n\to\infty} \|x_n\|$ exists.

$(x_n), (y_n) \in X$ both C. Seq.

- $(\|x_n - y_n\|)$ is C.S. in $\mathbb{K}$.

  ie $\lim_{n\to\infty} \|x_n - y_n\|$ exists.

$\mathcal{X}$: collection of all Cauchy seq in X.

'$\sim$' $\quad x = (x_n) \in \mathcal{X}$, $y = (y_n) \in \mathcal{X}$

$x \sim y$ if $\lim_{n\to\infty} \|x_n - y_n\| = 0$

'$\sim$' is an equivalence reln.

$\tilde{X}$ be the equivalence classes

for an $(x_n) \in X$ — $[(x_n)] \in \tilde{X}$

For $[(x_n)]$, $[(y_n)] \in \tilde{X}$, $\alpha \in \mathbb{K}$

$$[(x_n)] \underline{+} [(y_n)] = [(x_n + y_n)]$$

$$\alpha \underline{\cdot} [(x_n)] = [(\alpha \cdot x_n)]$$

$(+, \cdot)$ are well defined.

$\tilde{X}(\mathbb{K}) \rightarrow$ linear space.

$\| \ \|_*$ defined by $[(\theta_n)]$

$\theta_n = 0 \ \forall n.$

$$\| [(x_n)] \|_* = \lim_{n \to \infty} \| x_n \|_X$$

$\| \cdot \|_*$ is a norm (verify).

Then $(\tilde{X}, \| \cdot \|_*)$ is a nls.

Theo. Let X be nls and $\tilde{X}$ be a nls as above. Then $\tilde{X}$ is a completion of X.

Completion: $X \to$ nls.

$Y \to$ BS.

Y is a completion of X if $\exists$

isometry, $T : X \to Y$ s.t $R(T)$ is dense in Y.

(range of T)

Claim

- $\tilde{X}$ is a BS.

Let $(\tilde{x}_k)$ be a C.S where

$\tilde{x}_k \in \tilde{X}$ that is,

$\tilde{x}_k = [(x_{k,n})]$, $(x_{k,n})$ is a CS in X.

$\rightarrow \tilde{x}_1 = [(x_{1,n})]$, $\underline{(x_{1,n})}$ is a C.S in X

$\tilde{x}_2 = [(x_{2,n})]$, $(x_{2,n})$ is a C.S in X.

$\epsilon = 1 \quad \exists \; N_1 \quad$ s.t

$$\| x_{1,N_1} - x_{1,p} \| < 1, \quad \forall \; p \geq N_1$$

$\epsilon = 1/2 \quad \exists \; N_2 \quad$ s.t

$$\| x_{2,N_2} - x_{2,p} \| < 1/2, \quad \forall \; p \geq N_2$$

Then continuing we have

$$\| x_{k,\underline{N_k}} - x_{k,p} \| < \frac{1}{k}, \quad \forall \; p \geq N_k. \quad \text{—— (1)}$$

Now consider

$$\| x_{n,N_n} - x_{m,N_m} \|$$

$$= \| x_{r,N_r} - x_{r,n} + x_{r,n} - x_{m,n} + x_{m,n} - x_{m,N_m} \|$$

$$\le \| x_{r,N_r} - x_{r,n} \| + \| x_{r,n} - x_{m,n} \| + \| x_{m,n} - x_{m,N_m} \|$$

$$\le \frac{1}{r} + \frac{1}{m} + \| x_{r,n} - x_{m,n} \|$$

$\forall n \geq \max\{N_r, N_m\}$

$$< \frac{1}{r} + \frac{1}{m} + \epsilon \qquad r > m \quad m > N_\epsilon \qquad 2$$

$(\tilde{x}_k)$ is a CS in $\tilde{X}$

$\forall \epsilon > 0$

$$\| \tilde{x}_r - \tilde{x}_m \|_{\tilde{X}} < \epsilon \qquad r > m$$

$$m > N_\epsilon$$

$$\therefore \lim_{n \to \infty} \| x_{r,n} - x_{m,n} \| < \epsilon$$

$\forall \; r > m$

$m > N_\epsilon$.

$\| x_{r,n} - x_{m,n} \| < \epsilon$ for arbitrary n. — (3)

So from (2) & (3) we get

$(x_{n,N_n})_{n \in \mathbb{N}}$ is a C.S in X. — (4)

Let $a_n = x_{n,N_n}$

(a_n) is a C.S in X.

Then
$$[(a_n)] \in \tilde{\mathcal{X}}$$
$$\tilde{a} = [(a_n)] \in \tilde{\mathcal{X}}$$

Claim

$$\tilde{x}_k \xrightarrow[\|\cdot\|_*]{} \tilde{a} \quad \text{as } k \to \infty$$

ie, $\| \tilde{x}_k - \tilde{a} \|_* \longrightarrow 0$ as $k \to \infty$.

$$\| \tilde{x}_k - \tilde{a} \|_* = \lim_{n\to\infty} \| x_{k,n} - a_n \|_X$$
$$= \lim_{n\to\infty} \| x_{k,n} - x_{n,N_n} \|_X$$

$\underline{\underline{k > n}}$

$$\| x_{k,n} - x_{n,N_n} \|_X$$

$$= \| x_{k,n} - x_{k,N_k} + x_{k,N_k} - x_{n,N_n} \|$$

$$\leq \| x_{k,n} - x_{k,N_k} \| + \| x_{k,N_k} - x_{n,N_n} \|$$

$$\leq \frac{1}{k} + \| x_{k,N_k} - \underline{\underline{x_{n,N_n}}} \| \quad \forall n \geq N_k$$

Then taking $n \to \infty$

$$\lim_{n\to\infty} \| x_{k,n} - x_{n,N_n} \|_X \qquad \underline{\underline{k > n}}$$

$$\leq 1/k + \epsilon$$

$$\|\tilde{x}_k - \tilde{x}\|_* \leq \frac{1}{k} + \epsilon$$

$$\leq \epsilon' \text{ for all sufficiently large } k.$$

$$\lim_{k \to \infty} \tilde{x}_k = \tilde{x} \quad \text{in } \|\cdot\|_*.$$

$\therefore$ $\tilde{X}$ is a B.S.

- $\tilde{X}$ is a completion.

$T : X \longrightarrow \tilde{X}$

$x \in X$

$\tilde{y}_x = [(x)]$ — equivalence class of const seq

$(x, x, \ldots\ldots)$

$$Tx = \tilde{y}_x$$

$$\|Tx\|_* = \|y_x\|_*$$

$$= \lim_{n\to\infty} \|x\|$$

$$= \|x\|$$

T is an isometry.

- $R(T)$ is dense in $\hat{\tilde{X}}$.

Let $\tilde{x} = [(x_n)] \in \tilde{X}$.

Let $\epsilon > 0$ $\exists$ $N \in \mathbb{N}$ s.t

$$\|x_m - x_n\| < \epsilon \quad \forall\, m, n \geq N.$$

$$u_n = x_N \quad \forall\, n.$$

$$\therefore \quad \tilde{u} = [(u_n)] \in R(T)$$

$$\|\tilde{x} - \tilde{u}\|_* = \lim_{n\to\infty} \|x_n - u_n\|$$

$$= \lim_{n\to\infty} \|x_n - x_N\|$$

$$< \epsilon$$

$\therefore$ $R(T)$ is dense in $\tilde{X}$.

Hence $\tilde{X}$ is a completion of X.

Ex:

- $(c_{00}, \|\cdot\|_p)$ $1 \le p < \infty$.

  completion $(\ell^p, \|\cdot\|_p)$

- $(P[a,b], \|\cdot\|_\infty)$

  $(C[a,b], \|\cdot\|_\infty)$

$C^k[a,b]$ Sobolev norm.

Recall

$X \longrightarrow$ v. sp.

$M \subseteq X$ subspace. $0 \in M$.

Define $x, y \in X$ '$\sim$'

$x \sim y$, $x - y \in M$.

'$\sim$' eq relⁿ.

$$[x] = \{ y \in X \mid x - y \in M \}$$

$$= x + M \quad \text{cosets}.$$

$$X/M = \{ x + M \mid x \in X \}.$$

$$= \{ [x] \mid x \in X \}$$

Define:

$$(x+M) + (y+M)$$

$$= x + y + M.$$

$$\alpha(x+M) = \alpha x + M.$$

$$\alpha[x] = [\alpha x]$$

Check these are well defined!!

X/M is a linear space.

Define

$$\underline{\|x+M\|_*} = \operatorname{dist}(x, M) = \inf\{\|x-m\|_X \mid m \in M\}.$$

$$\left(\operatorname{dist}(x, M) = 0 \iff x \in \overline{M}\right)$$

$\| \ \|_*$ is a seminorm

but if M is a closed subspace then $\|\cdot\|_*$ is a norm.

- If M is a closed subspace of X then X/M is a nls with $\|\cdot\|_*$.