Lec 2

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Normed Linear Space

$X \neq \phi$ linear space over $\mathbb{K}$ $(\mathbb{R}/\mathbb{C})$

$\|\cdot\| : X \to \mathbb{R}$

N1) $\|x\| \geq 0 \quad \forall x \in X$
$\|x\| = 0 \iff x = 0$ — positivity

N2) $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall x \in X, \alpha \in \mathbb{K}$ — Homogenity

N3) $\|x+y\| \leq \|x\| + \|y\|$ (Triangle ineq)

then $\|\cdot\|$ is called a norm and $(X, \|\cdot\|)$ is called a normed linear space

remark

Given a NLS $(X, \|\cdot\|)$ define $d(x,y) = \|x-y\| \quad \forall x,y \in X$

forms a metric induced from the norm.

Can we induce a norm from a metric?

Semi Norm

N2 automatically gives suppose $x=0$
$\|x\| = \|0\| = \|0 \cdot 0\| = |0| \, \|0\| = 0$

That gives
$x = 0 \Rightarrow \|x\| = 0$

- N1 satisfy partially $\|x\| \geq 0$ — allow non zero vector to have zero norm
- N2 ✓
- N3 ✓ — like normal norm

$\|\cdot\| \to$ seminorm

Ex $(X, \|\cdot\|)$ is a NLS
$d(x,y) = \|x-y\|$

- Translation invariance
$d(x+u, y+u) = d(x,y)$
- Homogenity $d(\lambda x, \lambda y) = |\lambda| \, d(x,y)$

if d does NOT satisfy any of these property then it can NOT induce a NORM

Discrete Metric does Not satisfy Homogenity
$\therefore$ it is NOT induced from a NORM.

Q if in def. of metric we also add translation invariance and Homogenity then there exist corresponding NORM?

$\|x\| = d(0,x)$ $\boxed{\text{Yes}}$

$\|x\| = 0 \iff x = 0$ ✓
$\|x\| \geq 0$ ✓

$\|\alpha x\| = d(0, \alpha x) = |\alpha| d(0,x)$ (Homogenity)
$= |\alpha| \cdot \|x\|$

$\|x+y\| = d(0, x+y) = d(-y, x) \leq d(-y, 0) + d(0,x) = d(y,0) + d(0,x)$
$= \|y\| + \|x\|$

$\boxed{\text{If } \rho \text{ is a metric induced by } \|\cdot\| \text{ then } \rho = d}$

proof

$$\rho(x,y) = \|x-y\| = d(0, x-y) = d(0+y, x-y+y) \quad \text{[translation]}$$
$$= d(y,x) = d(x,y)$$

Convergence

let $(X, \|\cdot\|)$ be a NLS and (x_n) converges to $x \in X$,
if $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that

$$\|x_n - x\| < \varepsilon \quad \forall\ n \geq N_\varepsilon$$

Cauchy Seq

A seq (x_n) in a NLS, $(X, \|\cdot\|)$ is said to be cauchy seq
if $\forall\ \varepsilon > 0\ \exists\ N_\varepsilon \in \mathbb{N}$ such that

$$\|x_n - x_m\| < \varepsilon \quad \forall\ n, m \geq N_\varepsilon$$

A complete NLS is called **Banach Space**
↳ every cauchy seq is convergent

Lipschitz

$f : (X, d) \to (Y, \rho)$ metric

f is lipschitz iff $\exists\ K > 0$ such that

$$\rho(f(x), f(y)) \leq K \cdot d(x,y)$$

if f is lipschitz then it is <u>uniformly Continuous</u>

continuity @ a pt x_0
$\forall\ \varepsilon > 0\ \exists\ \delta > 0$ st
$\rho(f(x), f(x_0)) < \varepsilon$
$\forall\ d(x, x_0) < \delta$

uniform continuity
$\forall\ \varepsilon > 0\ \exists\ \delta > 0$ st
$\rho(f(x), f(y)) < \varepsilon$
$\forall\ d(x,y) < \delta$

proposition: in a NLS $(X, \|\cdot\|)$ we have

$$\big|\|x\| - \|y\|\big| \leq \|x-y\| \quad \forall\ x, y \in X$$

This shows $f(x) = \|x\|$ is lip. $\Rightarrow$ $\|\cdot\|$ is <u>uniform Continous Function</u>

define metric $d(x,y) = \|x-y\|$
and $f(x) = \|x\| \in \mathbb{R}$
then
$$|f(x) - f(y)| \leq 1 \cdot d(x,y)$$
metric in $\mathbb{R}$ metric in X

$$\|x\| = \|x-y+y\| \leq \|x-y\| + \|y\|$$

$\Rightarrow \|x\| - \|y\| \leq \|x-y\|$ — (1)

similarly $\|y\| - \|x\| \leq \|y-x\| = \|x-y\|$ — (2)

from (1) and (2)

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x-y\|$$

proposition

let $(X, \|\cdot\|)$ is a NLS

let $(x_n), (y_n)$ in X be such that $x_n \to x \in X$, $y_n \to y \in X$

let (λ_n) in $\mathbb{K}$ such that $\lambda_n \to \lambda \in \mathbb{K}$

then $\lambda_n x_n \to \lambda x$

$x_n + y_n \to x + y$

proof $\|\lambda x - \lambda_n x_n\| = \|\lambda x - \lambda x_n + \lambda x_n - \lambda_n x_n\|$

$\leq \lambda \|x - x_n\| + x_n \|\lambda - \lambda_n\|$

$\leq \lambda \cdot \frac{\varepsilon}{2\lambda} + x_n \frac{\varepsilon}{M}$

$\leq \varepsilon$

every converging seq is bdd.

$\|x_n\| \leq M$

$\|x_n + y_n - (x+y)\| \leq \|x_n - x\| + \|y_n - y\|$

$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$

Remark

$(X, \|\cdot\|)$

$+ : X \times X \to X$

$\cdot : \mathbb{K} \times X \to X$

recall

a function $f : X \to Y$ (metric spaces) is continuous iff

$\forall\ x_n \to x$

$f(x_n) \to f(x)$

$f : X \times X \to X$

Def $(x_n, y_n) \to (x, y) \iff x_n \to x$ and $y_n \to y$

we know $(x_n, y_n) \to (x, y)$

then $x_n + y_n \to x + y$ } from above proposition

hence $+$ is continuous function.

Similarly $\bullet$ is also continuous function.

Lemma $(X, \|\cdot\|)$ a NLS

X_0 be a subspace of X

then $\overline{X_0}$ is a closed subspace of X

proof by def of $\overline{X_0}$ is subspace of X

if $x, y \in \overline{X_0}$ and $\lambda \in \mathbb{K}$

then they are limit pts

$x_n \to x \quad y_n \to y$

then $\lambda x + y$ is also limit pt

$\{z_n = \lambda x_n + y_n\}$

$\therefore \lambda x + y \in \overline{X_0}$ (limit pt)

Ex ① $\mathbb{K} = \mathbb{R} / \mathbb{C}$

$\forall\ x \in \mathbb{K} \qquad \|x\| = |x|$

remark : X is a vec. space over $\mathbb{K}$

$\|x\|_* = C\|x\| \quad \forall\ x \in X \quad c > 0$

In particular

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$

Any norm will be a const multiple of $\|\cdot\|$

$\|x\| = \|x \cdot 1\| = |x|\ \|1\| = c\,|x|$

② $\mathbb{K}^n \cong \mathcal{F}(S, \mathbb{K})$

$S = \{1, 2, 3, \dots, n\}$

for each $x = (x(1), x(2), \dots, x(n)) \in \mathbb{K}^n$

Define $\|x\| = \sum_{i=1}^{n} |x(i)|$

use Δ-ineq check this is a norm

componentwise

$|x(i) + y(i)| \le |x(i)| + |y(i)|$

$x = (x(1), \dots, x(n))$

$y = (y(1), \dots, y(n))$

③ for $x \in \mathbb{K}^n$

$\|x\|_\infty = \max_{1 \le i \le n} |x(i)|$

$\|\cdot\|_\infty$ is a norm

Cauchy Schwartz
Hölder's
Minkowski

$\Bigg\}$ inequalities