<u>Sigma algebra</u> : Let $X$ be any set.
A sigma algebra $\Sigma$ is a subset of $\mathcal{P}(X)$ satisfying

(i) $\phi \in \Sigma$

(ii) $A \in \Sigma \Rightarrow A^c (= X \setminus A) \in \Sigma$

(iii) $A_n \in \Sigma, n \in \mathbb{N} \Rightarrow \bigcup_{n \in \mathbb{N}} A_n \in \Sigma$

<u>Examples</u> :

- $\Sigma = \{\phi, X\}$
- $\Sigma = \mathcal{P}(X)$
- $\Sigma = \{\phi, A, A^c, X\}$, where $A \neq \phi, X$.

- If $\mathcal{A} \subseteq \mathcal{P}(X)$, then the <u>$\sigma$-algebra generated by $\mathcal{A}$</u> is the smallest $\sigma$-alg. containing $\mathcal{A}$



<u>Measure</u>

Let $X$ be any set and $\Sigma$ be a $\sigma$-algebra on $X$. A measure $\mu : \Sigma \to [0, \infty]$ s.t.

(i) $\mu(\phi) = 0$

(ii) $A \subseteq B$, $A, B \in \Sigma \Rightarrow \mu(A) \leq \mu(B)$

(iii) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$, if $\{A_n\}_{n=1}^{\infty}$ is a collection of pairwise disjoint sets in $\Sigma$.

<u>Example</u> : $\Sigma = \mathcal{P}(X)$, $\mu$ = counting measure.

## Measurable sets

Let $\mu$ be an outer measure on a set $X$. A subset $E$ of $X$ is said to be a measurable set (w.r.t. $\mu$) if for any $A \subseteq X$,

$$\mu(A) = \mu(A \cap E) + \mu(A \setminus E)$$

- $\Sigma = \{E \subseteq X : E \text{ is measurable}\}$ is a $\sigma$-algebra on $X$ (Verify)
- $\mu|_\Sigma$ is a measure on $(X, \Sigma)$. (Verify)
- $\mu(E) = 0 \Rightarrow E$ is measurable.

Pf: $\mu(E) = 0 \Rightarrow \mu(A \cap E) = 0$

$\therefore$ $\mu(A \cap E) + \mu(A \cap E^c) = \mu(A \cap E^c) \leq \mu(A)$



Also, $\mu(A) = \mu((A \cap E) \cup (A \cap E^c))$

$$\leq \mu(A \cap E) + \mu(A \cap E^c)$$

$\therefore$ $\mu(A) = \mu(A \cap E) + \mu(A \cap E^c)$

$\Rightarrow E$ is measurable

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If we take $\mu$ to be the Lebesgue outer measure on $\mathbb{R}$, and let $\mathfrak{M}$ to be the $\sigma$-algebra of Lebesgue measurable sets, we denote by $\mathcal{L}^1$ the Lebesgue measure on $\mathbb{R}$, which is the restriction of $\mu$ on $\mathfrak{M}$.

Fact: Open sets are Lebesgue measurable.

Borel $\sigma$-algebra on $\mathbb{R}$, $\mathcal{B}$ is the $\sigma$-algebra generated by the open sets in $\mathbb{R}$.

- Since open sets are Lebesgue measurable, $\mathcal{B} \subseteq \mathcal{M}$.
- In fact, $\mathcal{B}$ is a proper subset of $\mathcal{M}$, i.e., $\exists$ a Lebesgue measurable set $E$ which is not Borel measurable ($E \notin \mathcal{B}$).
- There exists a non-measurable set (i.e. $A \subseteq \mathbb{R}$ s.t. $A \notin \mathcal{M}$).



## Hausdorff Measure & Hausdorff Dimension

s-dimensional Hausdorff measure

Let $s \geq 0$ be any real number. For any subset $F$ of $\mathbb{R}^n$, we'll define $\mathcal{H}^s(F)$ to be the s-dimensional Hausdorff measure as follows.

First, for $\delta > 0$, we define

$$\mathcal{H}^s_\delta(F) = \inf\left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : \{U_i\}_{i=1}^{\infty} \text{ is a } \delta\text{-cover of } F \right\}$$

( $\{U_i\}_{i=1}^{\infty}$ is a $\delta$-cover of $F$ means $F \subseteq \bigcup_{i=1}^{\infty} U_i$ and $\operatorname{diam} U_i \leq \delta$ )

$\operatorname{diam} U = \sup\{d(x,y) : x, y \in U\}$

Note that if $0 < \delta_1 < \delta_2$, then $\mathcal{H}^s_{\delta_2}(F) \le \mathcal{H}^s_{\delta_1}(F)$

$\therefore$ As $\delta$ decreases, $\mathcal{H}^s_\delta(F)$ increases.

So, we define

$$\mathcal{H}^s(F) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(F) = \sup_{\delta > 0} \mathcal{H}^s_\delta(F).$$

(The limit may be $+\infty$).

<u>Exercise:</u> Prove that $\mathcal{H}^s$ is an outer measure on $\mathbb{R}^n$.