

## Iterated Function System (IFS)

Let  $(X, d)$  be a complete metric space.  
(for example,  $X = \mathbb{R}^n$  with the Euclidean metric)

Recall: A metric space  $(X, d)$  is complete if every Cauchy sequence  $\{x_n\}$  in  $X$  converges to some point in  $X$ .

- $\{x_n\}_{n=1}^{\infty}$  is a Cauchy seq. if given  $\varepsilon > 0$ ,  
 $\exists N \in \mathbb{N}$  st.  $d(x_n, x_m) < \varepsilon \quad \forall n, m \geq N$ .
- A map  $f: (X, d) \rightarrow (X, d)$  is called a contraction if  $\exists c < 1$  such that  
 $d(f(x), f(y)) \leq c d(x, y) \quad \forall x, y \in X$

## Contraction mapping principle (Banach fixed point theorem)

If  $f: X \rightarrow X$  is a contraction map on a complete metric space  $(X, d)$ , then  $f$  has a unique fixed point, i.e.,

$\exists$  a unique  $x^* \in X$  st.  $f(x^*) = x^*$ .

Furthermore, for any  $x \in X$ , the sequence  $\{f^n(x)\}_{n=1}^{\infty}$  converges to  $x^*$ .

Defn: (IFS) Let  $(X, d)$  be a complete metric space and let  $f_1, f_2, \dots, f_k$  be contraction maps on  $(X, d)$ . Then  $(X; f_1, f_2, \dots, f_k)$  is called an IFS.

Defn (Attractor or invariant set)  
 A nonempty compact set  $A \subseteq X$  is called an attractor or an invariant set for an IFS  $(X, d; f_1, f_2, \dots, f_k)$  if

$$A = \bigcup_{i=1}^k f_i(A)$$

Theorem: Any IFS on a complete metric space has a unique attractor.

Example: Cantor set,  $C$  is the attractor for the IFS  $\{\mathbb{R}; f_1, f_2\}$ , where  
 $f_1(x) = \frac{x}{3}$ ,  $f_2(x) = \frac{2}{3} + \frac{x}{3}$ .

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### Hausdorff metric

Let  $(X, d)$  be any metric space.

Let  $H(X) = \{K \subseteq X : K \neq \emptyset, K \text{ is compact}\}$

For  $A, B \in H(X)$ , define

For  $A, B \in H(X)$ , define  
 $D(A, B) = \inf \{ \delta > 0 : A \subseteq B_\delta \text{ and } B \subseteq A_\delta \}$ ,

where  $A_\delta = \{x \in X : d(x, A) < \delta\}$ ,  
 where  $d(x, A) = \inf \{d(x, a) : a \in A\}$

Properties of  $D$ : (Verify these)

- $0 \leq D(A, B) < \infty$
- $D(A, B) = 0$  iff  $A = B$ .
- $D(A, C) \leq D(A, B) + D(B, C)$  (Triangle inequality)

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$\therefore (H(X), D)$  is a metric space.  
 $D$  is called the Hausdorff metric.

Theorem: If  $(X, d)$  is a complete metric space,  
 then  $(H(X), D)$  is a complete metric space.

Proof: Exercise.

Theorem: Let  $(X, d)$  be a complete metric space and  $f_1, f_2, \dots, f_k$  be contraction maps on  $(X, d)$ . Then  $\exists$  a unique attractor  $A$  for the IFS  $\{(X, d); f_1, f_2, \dots, f_k\}$ , i.e.  
 $\exists$  a unique nonempty compact set  $A \subseteq X$  st.  
 $A = \bigcup_{i=1}^k f_i(A)$

Proof: Consider the space  $H(X)$  with the Hausdorff metric  $D$ .

Then  $(H(X), D)$  is a complete metric space

define  $F: H(X) \rightarrow H(X)$  by

$$F(K) = \bigcup_{i=1}^k f_i(K).$$

(Note that  $\bigcup_{i=1}^k f_i(K)$  is compact since each  $f_i(K)$  is compact being continuous image of compact set  $K$ )

Suppose  $c_1, c_2, \dots, c_k < 1$  be st.

$$d(f_i(x), f_i(y)) \leq c_i d(x, y) \quad \forall x, y.$$

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Then  $D(F(K_1), F(K_2))$   
 $= D\left(\bigcup_{i=1}^k f_i(K_1), \bigcup_{i=1}^k f_i(K_2)\right)$   
 $\leq c D(K_1, K_2), \quad (\text{Prove this})$   
 where  $c = \max\{c_i : 1 \leq i \leq k\}$

$\Rightarrow F$  is a contraction map on  $(\mathcal{H}(X), D)$   
 $\therefore$  By Banach fixed point theorem,  
 $\exists$  a unique  $A \in \mathcal{H}(X)$  st.  
 $A = F(A) = \bigcup_{i=1}^k f_i(A)$ .

Also,  $A = \lim_{n \rightarrow \infty} F^n(K)$  for any nonempty compact set  $K$ .  
re.  $D(F^n(K), A) \rightarrow 0$  as  $n \rightarrow \infty$