Hausdorff dimension of the Cantor set

Let C be the Cantor set.

<u>To show</u>: $\dim_H C = s = \frac{\ln 2}{\ln 3}$.

We showed that $\dim_H C \leq \frac{\ln 2}{\ln 3}$

(In fact, $\mathcal{H}^s(C) \leq 1$ for $s = \frac{\ln 2}{\ln 3}$)

<u>Claim</u>: $\mathcal{H}^s(C) \geq \frac{1}{2}$ for $s = \frac{\ln 2}{\ln 3}$

(This will imply that $\dim_H C \geq s$)

To prove $\mathcal{H}^s(C) \geq \frac{1}{2}$, we'll show that

$$\sum_{i=1}^{\infty} |U_i|^s \geq \frac{1}{2} \quad \text{for any cover } \{U_i\}_{i=1}^{\infty} \text{ of } C. \qquad \text{—— } (*)$$

To prove $(*)$ it is enough to show that $(*)$ is true for every <u>finite</u> collection of <u>closed intervals</u> which covers C.

[First, replacing U_i with its closed convex hull (which does not change the diameter) we can assume each U_i is a closed interval. Then by expanding U_i "slightly" we can assume U_i to be open intervals. Now, using compactness of C, $\{U_i\}_{i=1}^{\infty}$ will have a finite subcover of C.]

Let $\{U_i\}$ be a finite collection of closed intervals covering C.

To show: $\sum |U_i|^s \geq \frac{1}{2}$.

For each U_i, let k be the integer such that

$$3^{-(k+1)} \le |U_i| < 3^{-k} \quad \text{—— } (**)$$

Then U_i can intersect at most one level-k interval since the distance between any two such intervals is at least 3^{-k}.

$\therefore$ If $j \geq k$, then U_i can intersect at most 2^{j-k} level-j intervals.

Since $s = \frac{\ln 2}{\ln 3}$, $3^s = 2$

$$2^{j-k} = 2^j \left(3^{-sk}\right) = 2^j \, 3^{-s(k+1)} \cdot 3^s \le 2^j 3^s |U_i|^s$$

[by $(**)$]

If we choose j large enough so that

$$3^{-(j+1)} \le |U_i|,$$

then $\{U_i\}$ must intersect all 2^j intervals of length 3^{-j}.

$$\therefore \quad \sum_i 2^j 3^s |U_i|^s \geq 2^j$$

$$\Rightarrow \quad \sum_i |U_i|^s \geq 3^{-s} = \frac{1}{2} \qquad \left(\because s = \frac{\ln 2}{\ln 3}\right)$$

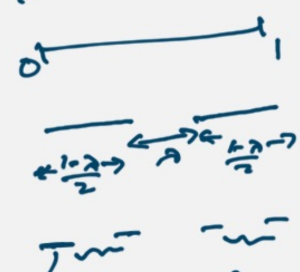
Hence, we are done.

λ-Cantor set :

Let $0 < \lambda < 1$.

Instead of removing the middle one-third, we can remove the middle λ proportion of the intervals to get C_λ.

What is $\mathcal{L}^1(C_\lambda)$?



$$\mathcal{L}^1([0,1]\setminus C_\lambda) = \lambda + 2\lambda\left(\frac{1-\lambda}{2}\right) + 2^2\lambda\left(\frac{1-\lambda}{2}\right)^2 + \cdots$$

$$= \lambda\left[1 + (1-\lambda) + (1-\lambda)^2 + \cdots\right]$$

$$= \lambda \cdot \frac{1}{1-(1-\lambda)} = 1$$

$$\Rightarrow \mathcal{L}^1(C_\lambda) = 0$$

length : $\lambda\left(\frac{1-\lambda}{2}\right)$

$$\frac{1}{2}\left[\left(\frac{1-\lambda}{2}\right) - \lambda\left(\frac{1-\lambda}{2}\right)\right] = \left(\frac{1-\lambda}{2}\right)^2$$

$f_1 : [0,1] \to [0,1]$

$$f_1(x) = \left(\frac{1-\lambda}{2}\right)x$$

$f_2 : [0,1] \to [0,1]$

$$f_2(x) = \left(\frac{1-\lambda}{2}\right)x + \left(\frac{1+\lambda}{2}\right)$$

$$C_\lambda = f_1(C_\lambda) \cup f_2(C_\lambda).$$

By Heuristic calculation, we get the Hausdorff dimension of C_λ is given by

$$2\cdot\left(\frac{1-\lambda}{2}\right)^s = 1 \quad \text{ie. } \left(\frac{2}{1-\lambda}\right)^s = 2$$

$$\text{ie. } \boxed{s = \frac{\ln 2}{\ln\left(\frac{2}{1-\lambda}\right)}}$$

By varying $\lambda \in (0,1)$, we get the Haus. dim. to vary from 0 to 1.

Theorem: Every set $F \subseteq \mathbb{R}^n$ with $\dim_H F < 1$ is totally disconnected.

Proof: Let $x, y \in F$, $x \neq y$.
We will show, $\exists\, U, V$ open sets in F s.t. $x \in U$, $y \in V$ and $U \cup V = F$.
(This implies that F is totally disconnected)

Define $f : \mathbb{R}^n \to [0, \infty)$ by
$$f(z) = \|z - x\|$$

Then
$$|f(z) - f(w)| = \big|\, \|z-x\| - \|w-x\| \,\big|$$
$$\leq \|(z-x) - (w-x)\| \quad \text{(reverse triangle inequality)}$$
$$= \|z - w\|$$

$\Rightarrow$ f is a Lipschitz map.

$\therefore$ $\dim_H f(F) \leq \dim_H F < 1$

$\Rightarrow$ $\mathcal{H}^1(f(F)) = 0$

$\Rightarrow$ $\mathcal{L}^1(f(F)) = 0$.

$\therefore$ $\exists\, r \in (0, f(y))$ s.t. $r \notin f(F)$.
(because $\mathcal{L}^1(f(F)) = 0$)

Now,
$$F = \underbrace{\{z \in F : \underbrace{\|z-x\|}_{f(z)} < r\}}_{U} \cup \underbrace{\{z \in F : \underbrace{\|z-x\|}_{f(z)} > r\}}_{V}$$

U & V are open subsets of F, $U \cap V = \phi$, $x \in U$, $y \in V$.

$\therefore$ F is totally disconnected.