PDE Quiz 2 Revision: Method of Characteristics & Solutions

If you find error, please report in group

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Quick Revision: Method of Characteristics for 1st-Order Non-Linear PDEs

The Method of Characteristics is a powerful technique for solving first-order partial differential equations (PDEs), especially non-linear ones. Here are the key steps and important caveats:

Steps

1. **Define** F: Rewrite the PDE in the form $F(x, y, u, p, q) = 0$ (for 2D) or $F(x, y, z, u, p, q, r) = 0$ (for 3D), where $p = u_x$, $q = u_y$, $r = u_z$.
2. **Derive Characteristic ODEs (Charpit's Equations)**: Calculate the partial derivatives of F with respect to all its arguments (F_x, F_y, F_u, F_p, F_q, etc.). Then write down the system of ODEs:
   - $\frac{dx}{dt} = F_p$
   - $\frac{dy}{dt} = F_q$
   - $\frac{du}{dt} = pF_p + qF_q$
   - $\frac{dp}{dt} = -F_x - pF_u$
   - $\frac{dq}{dt} = -F_y - qF_u$
   - (For 3D, analogous equations for z and r apply: $\frac{dz}{dt} = F_r$, $\frac{dr}{dt} = -F_z - rF_u$)
3. **Parameterize Initial Curve/Surface**: Represent the initial condition $u(x_0, y_0, \dots) = G(x_0, y_0, \dots)$ by introducing parameter(s) (s for 2D, s_1, s_2 for 3D) at $t = 0$.
4. **Determine Initial Values for** p, q, r: Use two conditions to find p_0, q_0, r_0 as functions of the initial parameters:
   - **Compatibility Condition(s)**: Total differential of u_0 along the initial curve/surface. E.g., $\frac{du_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}$.
   - **PDE on Initial Surface**: $F(x_0, y_0, u_0, p_0, q_0, \dots) = 0$.
5. **Integrate Characteristic ODEs**: Solve the system of ODEs to find $x(s,t), y(s,t), u(s,t), p(s,t), q(s,t)$ (and z, r for 3D). Use the initial values from step 4 to determine integration constants.
6. **Check Jacobian Condition**: Compute the Jacobian $J = \det \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{pmatrix}$ (for 2D) or its 3D equivalent. If $J \neq 0$, a local unique solution exists.
7. **Eliminate Parameters**: Solve the equations for x, y (and z) for s, t (and s_1, s_2, t) in terms of x, y (and z). Substitute these back into the expression for $u(s,t)$ to obtain the final explicit solution $u(x, y)$ (or $u(x, y, z)$).

Important Caveats & Things to Remember

- **Signs**: Be extremely careful with signs when calculating partial derivatives and applying the characteristic equations, especially for dp/dt and dq/dt.
- **Multiple Solutions ($\pm$)**: Non-linear PDEs often yield multiple initial values for p, q, r (e.g., from square roots). Each choice can lead to a valid solution branch.
- **Algebraic Complexity**: The parameter elimination step can be algebraically intensive. Look for clever substitutions or relationships between the characteristic solutions.
- **Jacobian $J = 0$**: If $J = 0$, the solution may not be unique, or the method might describe an envelope or a singular solution. This often corresponds to physical phenomena like shocks or caustics.
- **Conceptual Understanding**: Always try to relate the mathematical solution back to any physical interpretation (e.g., wave fronts).

Problems & Complete Solutions

Problem 1

PDE: $u_x^2 + yu_y - u = 0$ **Initial Condition**: $u(x,1) = \frac{x^2}{4} + 1$
Solution:

1. **Define** F: $F(x,y,u,p,q) = p^2 + yq - u = 0$
2. **Derive Characteristic ODEs**: Partial derivatives of F: $F_x = 0$, $F_y = q$, $F_u = -1$, $F_p = 2p$, $F_q = y$. Characteristic ODEs:
$$\begin{aligned}\frac{dx}{dt} &= F_p = 2p\\ \frac{dy}{dt} &= F_q = y\\ \frac{du}{dt} &= pF_p + qF_q = p(2p) + q(y) = 2p^2 + qy\\ \frac{dp}{dt} &= -F_x - pF_u = -0 - p(-1) = p\\ \frac{dq}{dt} &= -F_y - qF_u = -q - q(-1) = 0\end{aligned}$$
3. **Parameterize Initial Curve** (at $t = 0$): $x_0(s) = s$ $y_0(s) = 1$ $u_0(s) = \frac{s^2}{4} + 1$
4. **Determine Initial Values for** p, q: **Compatibility Condition**: $\frac{du_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds}$ $\frac{d}{ds}\left(\frac{s^2}{4} + 1\right) = p_0(1) + q_0(0) \implies \frac{s}{2} = p_0 \implies p_0(s) = \frac{s}{2}$. **PDE on Initial Curve**: $F(s, 1, u_0(s), p_0(s), q_0(s)) = 0$ $p_0^2 + (1)q_0 - u_0 = 0 \implies \left(\frac{s}{2}\right)^2 + q_0 - \left(\frac{s^2}{4} + 1\right) = 0 \implies \frac{s^2}{4} + q_0 - \frac{s^2}{4} - 1 = 0 \implies q_0(s) = 1$.
5. **Integrate Characteristic ODEs**:
   - $\frac{dq}{dt} = 0 \implies q(t) = q_0(s) = 1$.
   - $\frac{dp}{dt} = p \implies p(t) = p_0(s)e^t = \frac{s}{2}e^t$.
   - $\frac{dy}{dt} = y \implies \ln|y| = t + C_y(s) \implies y(t) = K(s)e^t$. Using $y(0,s) = 1$: $1 = K(s)e^0 \implies K(s) = 1$. So, $y(s,t) = e^t$.
   - $\frac{dx}{dt} = 2p = 2\left(\frac{s}{2}e^t\right) = se^t$. $x(t) = \int se^t dt = se^t + C_x(s)$. Using $x(0,s) = s$: $s = se^0 + C_x(s) \implies s = s + C_x(s) \implies C_x(s) = 0$. So, $x(s,t) = se^t$.
   - $\frac{du}{dt} = 2p^2 + qy = 2\left(\frac{s}{2}e^t\right)^2 + (1)(e^t) = \frac{s^2}{2}e^{2t} + e^t$. $u(t) = \int \left(\frac{s^2}{2}e^{2t} + e^t\right) dt = \frac{s^2}{4}e^{2t} + e^t + C_u(s)$. Using $u(0,s) = u_0(s) = \frac{s^2}{4} + 1$: $\frac{s^2}{4} + 1 = \frac{s^2}{4}e^0 + e^0 + C_u(s) \implies \frac{s^2}{4} + 1 = \frac{s^2}{4} + 1 + C_u(s) \implies C_u(s) = 0$. So, $u(s,t) = \frac{s^2}{4}e^{2t} + e^t$.
6. **Check Jacobian Condition**: $x(s,t) = se^t$ and $y(s,t) = e^t$. $\frac{\partial x}{\partial s} = e^t$, $\frac{\partial x}{\partial t} = se^t$ $\frac{\partial y}{\partial s} = 0$, $\frac{\partial y}{\partial t} = e^t$ $J = \det\begin{pmatrix} e^t & se^t \\ 0 & e^t\end{pmatrix} = e^t \cdot e^t - se^t \cdot 0 = e^{2t}$. Since $e^{2t} \neq 0$, a unique solution exists locally.
7. **Eliminate Parameters**: From $y = e^t$, we get $e^t = y$. From $x = se^t$, substitute $e^t = y$: $x = sy \implies s = \frac{x}{y}$. Substitute s and e^t into $u(s,t)$: $u(x,y) = \frac{(x/y)^2}{4}(y^2) + y = \frac{x^2/y^2}{4}y^2 + y = \frac{x^2}{4} + y$.
   Final Solution:
$$u(x,y) = \frac{x^2}{4} + y$$

Problem 2 (Clairaut's Equation)

PDE: $u = xu_x + yu_y + \frac{1}{2}(u_x^2 + u_y^2)$ **Initial Condition**: $u(x,0) = \frac{1}{2}(1-x^2)$
Solution:

1. **Define** F: $F(x,y,u,p,q) = xp + yq + \frac{1}{2}(p^2+q^2) - u = 0$

2. **Derive Characteristic ODEs**: Partial derivatives of F: $F_x = p$, $F_y = q$, $F_u = -1$, $F_p = x+p$, $F_q = y+q$. Characteristic ODEs:

$$\frac{dx}{dt} = F_p = x + p$$
$$\frac{dy}{dt} = F_q = y + q$$
$$\frac{du}{dt} = pF_p + qF_q = p(x+p) + q(y+q) = px + p^2 + qy + q^2$$
$$\frac{dp}{dt} = -F_x - pF_u = -p - p(-1) = 0$$
$$\frac{dq}{dt} = -F_y - qF_u = -q - q(-1) = 0$$

3. **Parameterize Initial Curve** (at $t=0$): $x_0(s) = s$ $y_0(s) = 0$ $u_0(s) = \frac{1}{2}(1-s^2)$

4. **Determine Initial Values for** p,q: **Compatibility Condition**: $\frac{du_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds}$ $\frac{d}{ds}\left(\frac{1}{2}(1-s^2)\right) = p_0(1) + q_0(0) \implies -s = p_0 \implies p_0(s) = -s$. **PDE on Initial Curve**: $F(s,0,u_0(s),p_0(s),q_0(s)) = 0$ $sp_0 + 0q_0 + \frac{1}{2}(p_0^2+q_0^2) - u_0 = 0$ $s(-s) + \frac{1}{2}((-s)^2+q_0^2) - \frac{1}{2}(1-s^2) = 0$ $-s^2 + \frac{1}{2}s^2 + \frac{1}{2}q_0^2 - \frac{1}{2} + \frac{1}{2}s^2 = 0$ $\frac{1}{2}q_0^2 - \frac{1}{2} = 0 \implies q_0^2 = 1 \implies q_0(s) = \pm 1$. We choose the branch $q_0(s) = 1$.

5. **Integrate Characteristic ODEs**:
   - $\frac{dp}{dt} = 0 \implies p(t) = p_0(s) = -s$.
   - $\frac{dq}{dt} = 0 \implies q(t) = q_0(s) = 1$.
   - $\frac{dx}{dt} = x + p = x - s$. This is a linear ODE: $x' - x = -s$. Multiplying by integrating factor e^{-t}: $\frac{d}{dt}(xe^{-t}) = -se^{-t}$. Integrating: $xe^{-t} = se^{-t} + C_x(s) \implies x(t) = s + C_x(s)e^t$. Using $x(0,s) = s \implies s = s + C_x(s) \implies C_x(s) = 0$. So, $x(s,t) = s$.
   - $\frac{dy}{dt} = y + q = y + 1$. This is a linear ODE: $y' - y = 1$. Multiplying by integrating factor e^{-t}: $\frac{d}{dt}(ye^{-t}) = e^{-t}$. Integrating: $ye^{-t} = -e^{-t} + C_y(s) \implies y(t) = -1 + C_y(s)e^t$. Using $y(0,s) = 0 \implies 0 = -1 + C_y(s) \implies C_y(s) = 1$. So, $y(s,t) = e^t - 1$.
   - $\frac{du}{dt} = px + p^2 + qy + q^2$. Substitute $p = -s, q = 1, x = s, y = e^t - 1$: $\frac{du}{dt} = (-s)(s) + (-s)^2 + (1)(e^t - 1) + (1)^2 = -s^2 + s^2 + e^t - 1 + 1 = e^t$. $u(t) = \int e^t dt = e^t + C_u(s)$. Using $u(0,s) = u_0(s) = \frac{1}{2}(1-s^2)$: $\frac{1}{2}(1-s^2) = e^0 + C_u(s) \implies \frac{1}{2}(1-s^2) = 1 + C_u(s) \implies C_u(s) = \frac{1}{2}(1-s^2) - 1 = -\frac{1}{2}(1+s^2)$. So, $u(s,t) = e^t - \frac{1}{2}(1+s^2)$.

6. **Check Jacobian Condition**: $x(s,t) = s$ and $y(s,t) = e^t - 1$. $\frac{\partial x}{\partial s} = 1$, $\frac{\partial x}{\partial t} = 0$ $\frac{\partial y}{\partial s} = 0$, $\frac{\partial y}{\partial t} = e^t$ $J = \det\begin{pmatrix}1 & 0\\ 0 & e^t\end{pmatrix} = 1\cdot e^t - 0\cdot 0 = e^t$. Since $e^t \neq 0$, a unique solution exists locally.

7. **Eliminate Parameters**: From $x = s$, we get $s = x$. From $y = e^t - 1$, we get $e^t = y + 1$. Substitute s and e^t into $u(s,t)$: $u(x,y) = (y+1) - \frac{1}{2}(1+x^2) = y + 1 - \frac{1}{2} - \frac{x^2}{2} = y + \frac{1}{2} - \frac{x^2}{2}$.
   Final Solution (for $q_0 = 1$ branch):

$$u(x,y) = y + \frac{1}{2} - \frac{x^2}{2}$$

Problem 3

PDE: $u = u_x^2 + u_y^2$ **Initial Condition**: $u(x,0) = ax^2$ **Task**: Determine for what positive constants a a solution exists, whether it is unique, and find all such solutions.

Solution:

1. **Define** F: $F(x,y,u,p,q) = p^2 + q^2 - u = 0$

2. **Derive Characteristic ODEs**: Partial derivatives of F: $F_x = 0$, $F_y = 0$, $F_u = -1$, $F_p = 2p$, $F_q = 2q$. Characteristic ODEs:

$$\begin{aligned}
\frac{dx}{dt} &= F_p = 2p \\
\frac{dy}{dt} &= F_q = 2q \\
\frac{du}{dt} &= pF_p + qF_q = p(2p) + q(2q) = 2(p^2+q^2) \\
\frac{dp}{dt} &= -F_x - pF_u = -0 - p(-1) = p \\
\frac{dq}{dt} &= -F_y - qF_u = -0 - q(-1) = q
\end{aligned}$$

3. **Parameterize Initial Curve** (at $t=0$): $x_0(s) = s$ $y_0(s) = 0$ $u_0(s) = as^2$

4. **Determine Initial Values for** p, q: **Compatibility Condition**: $\frac{du_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}$ $\frac{d}{ds}(as^2) = p_0(1) + q_0(0) \implies 2as = p_0 \implies p_0(s) = 2as$. **PDE on Initial Curve**: $F(s,0,u_0(s),p_0(s),q_0(s)) = 0$ $p_0^2 + q_0^2 - u_0 = 0 \implies (2as)^2 + q_0^2 - as^2 = 0$ $4a^2s^2 + q_0^2 - as^2 = 0 \implies q_0^2 = as^2 - 4a^2s^2 = s^2(a-4a^2)$. For q_0 to be real, $a - 4a^2 \geq 0$. Since $a > 0$ (given), we need $1 - 4a \geq 0 \implies a \leq 1/4$. So, $q_0(s) = \pm s\sqrt{a-4a^2}$. We choose the positive branch for $q_0(s) = s\sqrt{a-4a^2}$ for now.

5. **Integrate Characteristic ODEs**:
   - $\frac{dp}{dt} = p \implies p(t) = p_0(s)e^t = 2ase^t$.
   - $\frac{dq}{dt} = q \implies q(t) = q_0(s)e^t = s\sqrt{a-4a^2}e^t$.
   - $\frac{dx}{dt} = 2p = 2(2ase^t) = 4ase^t$. $x(t) = \int 4ase^t dt = 4ase^t + C_x(s)$. Using $x(0,s) = s \implies s = 4as + C_x(s) \implies C_x(s) = s(1-4a)$. So, $x(s,t) = 4ase^t + s(1-4a)$.
   - $\frac{dy}{dt} = 2q = 2(s\sqrt{a-4a^2}e^t)$. $y(t) = \int 2s\sqrt{a-4a^2}e^t dt = 2s\sqrt{a-4a^2}e^t + C_y(s)$. Using $y(0,s) = 0 \implies 0 = 2s\sqrt{a-4a^2} + C_y(s) \implies C_y(s) = -2s\sqrt{a-4a^2}$. So, $y(s,t) = 2s\sqrt{a-4a^2}(e^t-1)$.
   - $\frac{du}{dt} = 2(p^2+q^2)$. From $F=0$, $p^2+q^2 = u$. So $\frac{du}{dt} = 2u$. Integrating: $\frac{du}{u} = 2dt \implies \ln|u| = 2t + K(s) \implies u(t) = C_u(s)e^{2t}$. Using $u(0,s) = u_0(s) = as^2 \implies as^2 = C_u(s)e^0 \implies C_u(s) = as^2$. So, $u(s,t) = as^2e^{2t}$.

6. **Check Jacobian Condition**: $x(s,t) = s(4ae^t + 1 - 4a)$ and $y(s,t) = 2s\sqrt{a-4a^2}(e^t-1)$. $\frac{\partial x}{\partial s} = 4ae^t + 1 - 4a$ $\frac{\partial x}{\partial t} = 4ase^t$ $\frac{\partial y}{\partial s} = 2\sqrt{a-4a^2}(e^t-1)$ $\frac{\partial y}{\partial t} = 2s\sqrt{a-4a^2}e^t$ $J = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}$ $J = (4ae^t+1-4a)(2s\sqrt{a-4a^2}e^t) - (4ase^t)(2\sqrt{a-4a^2}(e^t-1))$ $J = 2s\sqrt{a-4a^2}e^t[(4ae^t+1-4a) - 4a(e^t-1)]$ $J = 2s\sqrt{a-4a^2}e^t[4ae^t+1-4a-4ae^t+4a] = 2se^t\sqrt{a-4a^2}$. For $J \neq 0$, we need $s \neq 0$ and $\sqrt{a-4a^2} \neq 0$. This implies $a(1-4a) \neq 0$. Since $a > 0$, we must have $1 - 4a \neq 0 \implies a \neq 1/4$. Thus, for $0 < a < 1/4$, a unique solution exists. If $a = 1/4$, $J = 0$.

7. **Eliminate Parameters**: We have $u = as^2e^{2t} = a(se^t)^2$. From $x = s(4ae^t + 1 - 4a)$ and $y = 2s\sqrt{a-4a^2}(e^t-1)$. Let's use the relation $s = x - \frac{2ay}{\sqrt{a-4a^2}}$ derived during our earlier discussion. From $x = s + \frac{2ay}{\sqrt{a-4a^2}}$, substitute into the second equation: $x = s(4ae^t + 1 - 4a) \implies \frac{x}{s} = 4ae^t + 1 - 4a \implies e^t = \frac{1}{4a}\left(\frac{x}{s} - 1 + 4a\right)$. Substitute e^t into $u = a(se^t)^2$: $u = a\left(s \cdot \frac{1}{4a}\left(\frac{x}{s} - 1 + 4a\right)\right)^2 = a\left(\frac{1}{4a}(x - s(1-4a))\right)^2$ $u = \frac{1}{16a}(x - s(1-4a))^2$. Now substitute $s = x - \frac{2ay}{\sqrt{a-4a^2}}$: $u = \frac{1}{16a}\left(x - \left(x - \frac{2ay}{\sqrt{a-4a^2}}\right)(1-4a)\right)^2$

$u = \frac{1}{16a}\left(x - x(1-4a) + \frac{2ay}{\sqrt{a-4a^2}}(1-4a)\right)^2$ $u = \frac{1}{16a}\left(4ax + \frac{2ay(1-4a)}{\sqrt{a-4a^2}}\right)^2$. Since $1-4a > 0$, we have $\frac{1-4a}{\sqrt{a-4a^2}} = \frac{(\sqrt{1-4a})^2}{\sqrt{a}\sqrt{1-4a}} = \frac{\sqrt{1-4a}}{\sqrt{a}}$. $u = \frac{1}{16a}\left(4ax + 2ay\frac{\sqrt{1-4a}}{\sqrt{a}}\right)^2 = \frac{1}{16a}\left(4ax + 2y\sqrt{a}\sqrt{1-4a}\right)^2$ $u = \frac{1}{16a}(2\sqrt{a})^2\left(2\sqrt{a}x + y\sqrt{1-4a}\right)^2 = \frac{4a}{16a}\left(2\sqrt{a}x + y\sqrt{1-4a}\right)^2$. $u = \frac{1}{4}\left(2\sqrt{a}x + y\sqrt{1-4a}\right)^2$.

Case for $a = 1/4$: If $a = 1/4$, then $\sqrt{1-4a} = 0$. The solution becomes: $u = \frac{1}{4}(2\sqrt{1/4}x + y \cdot 0)^2 = \frac{1}{4}(2 \cdot \frac{1}{2}x)^2 = \frac{1}{4}x^2$.

Final Solution:

- **Existence and Uniqueness**: A real solution exists for $0 < a \leq 1/4$.
  - For $0 < a < 1/4$, the solution is locally unique ($J \neq 0$).
  - For $a = 1/4$, the Jacobian $J = 0$, indicating a special case.
- **Solution(s)**:
  - For $0 < a < 1/4$:
$$u(x, y) = \frac{1}{4}(2x\sqrt{a} + y\sqrt{1-4a})^2$$
  - For $a = 1/4$:
$$u(x, y) = \frac{x^2}{4}$$

Problem 8

Task: Describe the wave front produced by an initial disturbance at a point. This involves considering (36) with Γ being given by $f = g = h = 0$.

Solution:

(a) **Context from McOwen's text**: Equation (36) is the Eikonal equation, $c^2(u_x^2 + u_y^2) = 1$, and the problem implies its 3D analogue. The text states c is the constant propagation speed.

(b) **Initial Disturbance at a Point**: The condition "$f = g = h = 0$" for Γ signifies that the initial data is concentrated at a single point, typically the origin $(0, 0, 0)$.

(c) **Wave Propagation in Homogeneous Isotropic Medium**: Since the propagation speed c is constant (homogeneous medium) and the Eikonal equation is isotropic (same form in all directions), the wave will spread out uniformly from the point source.

(d) **Description of Wave Front**: The wave front represents the locus of points that the disturbance reaches at a given time. Due to the symmetrical propagation from a point source in a homogeneous, isotropic 3D medium, these loci will form spheres.

(e) **Mathematical Representation**: If the disturbance originates at the origin $(0, 0, 0)$ at time $t = 0$, and propagates with speed c, then at any time T, the wave front is a sphere of radius cT.

Final Description: The wave front produced by an initial disturbance at a single point in a homogeneous and isotropic medium will be an **expanding spherical surface**. Its radius will grow linearly with time, at the constant propagation speed c. Mathematically, if the disturbance starts at the origin at $t = 0$, the wave front at time T is given by:

$$x^2 + y^2 + z^2 = (cT)^2$$

Problem 9

PDE: Eikonal equation $c(x,y)^2(u_x^2 + u_y^2) = 1$. **Special Case**: $c = |x|$ (for $x > 0$, so $c = x$). **Initial Condition**: $u(x,0) = 0$. **Task**: Derive characteristic equations for general $c(x,y)$ and then find the solution for the special case, confirming it matches $u(x,y) = -\log \frac{\sqrt{x^2+y^2}+y}{x}$ for $x > 0$.

Solution:

(a) **Define** F: $F(x,y,u,p,q) = c(x,y)^2(p^2+q^2) - 1 = 0$

(b) **Derive General Characteristic ODEs**: Partial derivatives of F: $F_x = 2cc_x(p^2+q^2)$, $F_y = 2cc_y(p^2+q^2)$, $F_u = 0$, $F_p = 2c^2p$, $F_q = 2c^2q$. Characteristic ODEs:

$$\frac{dx}{dt} = F_p = 2c^2p$$
$$\frac{dy}{dt} = F_q = 2c^2q$$
$$\frac{du}{dt} = pF_p + qF_q = 2c^2(p^2+q^2)$$
$$\frac{dp}{dt} = -F_x - pF_u = -2cc_x(p^2+q^2)$$
$$\frac{dq}{dt} = -F_y - qF_u = -2cc_y(p^2+q^2)$$

(c) **Solve for Special Case** $c = x$ **(**$x > 0$**)**: Here $c(x,y) = x \implies c_x = 1, c_y = 0$. Also, from the PDE $x^2(p^2+q^2) = 1 \implies p^2+q^2 = 1/x^2$. The ODEs become:

$$\frac{dx}{dt} = 2x^2p$$
$$\frac{dy}{dt} = 2x^2q$$
$$\frac{du}{dt} = 2x^2(p^2+q^2) = 2x^2(1/x^2) = 2$$
$$\frac{dp}{dt} = -2x(1)(1/x^2) = -2/x$$
$$\frac{dq}{dt} = -2x(0)(1/x^2) = 0$$

(d) **Parameterize Initial Curve** (at $t = 0$): $x_0(s) = s$ $y_0(s) = 0$ $u_0(s) = 0$

(e) **Determine Initial Values for** p, q: **Compatibility Condition**: $\frac{du_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds}$ $0 = p_0(1) + q_0(0) \implies p_0(s) = 0$. **PDE on Initial Curve**: $x_0^2(p_0^2+q_0^2) = 1$ $s^2(0^2+q_0^2) = 1 \implies q_0^2 = 1/s^2 \implies q_0(s) = \pm 1/s$. To match the given final solution, we will find that $q_0(s) = -1/s$ is the consistent choice.

(f) **Integrate Characteristic ODEs**:

- $\frac{du}{dt} = 2 \implies u(t) = 2t + C_u(s)$. With $u(0,s) = 0 \implies C_u(s) = 0$. So, $\boxed{u(s,t) = 2t}$.
- $\frac{dq}{dt} = 0 \implies q(t) = q_0(s) = -1/s$.
- From PDE: $p^2 = 1/x^2 - q^2 = 1/x^2 - (-1/s)^2 = \frac{s^2-x^2}{s^2x^2}$. $p(t) = \pm\frac{\sqrt{s^2-x^2}}{sx}$. To proceed towards the given solution, we select $p(t) = \frac{\sqrt{s^2-x^2}}{sx}$.
- We use $\frac{dy}{dx} = \frac{q}{p}$. $\frac{dy}{dx} = \frac{-1/s}{\sqrt{s^2-x^2}/(sx)} = -\frac{x}{\sqrt{s^2-x^2}}$. Integrating: $y = \int -\frac{x}{\sqrt{s^2-x^2}}dx = \sqrt{s^2-x^2} + C_y(s)$. Using $y(0,s) = 0$ and $x(0,s) = s$: $0 = \sqrt{s^2-s^2} + C_y(s) \implies C_y(s) = 0$. So, $y(s,t) = \sqrt{s^2-x^2}$. This implies $y^2 = s^2 - x^2 \implies s^2 = x^2+y^2 \implies s = \sqrt{x^2+y^2}$ (for $s > 0$).

- To find t: From $\frac{dx}{dt} = 2x^2 p = 2x^2 \frac{\sqrt{s^2-x^2}}{sx} = \frac{2x}{s}\sqrt{s^2-x^2}$. $dt = \frac{s}{2x\sqrt{s^2-x^2}}dx$. $t = \int_s^x \frac{s}{2\xi\sqrt{s^2-\xi^2}}d\xi$. Using the substitution $\xi = s\cosh\eta$, this integral evaluates to: $t = -\frac{1}{2}\ln\left(\frac{s+\sqrt{s^2-x^2}}{x}\right)$.

(g) **Eliminate Parameters**: We have $u = 2t$. So, $u(x,y) = -\ln\left(\frac{s+\sqrt{s^2-x^2}}{x}\right)$. Substitute $s = \sqrt{x^2+y^2}$ and $\sqrt{s^2-x^2} = y$ (from our integration step where $y = \sqrt{s^2-x^2}$ and $y > 0$ is assumed).

$$u(x,y) = -\ln\left(\frac{\sqrt{x^2+y^2}+y}{x}\right)$$

This matches the given solution for $x > 0$.

Final Solution:

- **General Characteristic Equations for** $c(x,y)^2(p^2+q^2) - 1 = 0$:

$$\frac{dx}{dt} = 2c^2 p$$
$$\frac{dy}{dt} = 2c^2 q$$
$$\frac{du}{dt} = 2c^2(p^2+q^2)$$
$$\frac{dp}{dt} = -2cc_x(p^2+q^2)$$
$$\frac{dq}{dt} = -2cc_y(p^2+q^2)$$

- **Solution for** $c = x$ **and** $u(x,0) = 0$ **(**$x > 0$**)**:

$$u(x,y) = -\log\frac{\sqrt{x^2+y^2}+y}{x}$$

Problem 10

PDE: $u_x^2+u_y^2+u_z^2 = 1$ (3D Eikonal Equation) **Tasks**: (a) Solve IVP with $u = k$ on plane $\alpha x+\beta y+z = 0$. (b) Find a complete integral.

Solution:

(a) **Define** F: $F(x, y, z, u, p, q, r) = p^2 + q^2 + r^2 - 1 = 0$

(b) **Derive Characteristic ODEs**: Partial derivatives of F: $F_x = 0$, $F_y = 0$, $F_z = 0$, $F_u = 0$, $F_p = 2p$, $F_q = 2q$, $F_r = 2r$. Characteristic ODEs:

$$\frac{dx}{dt} = 2p$$
$$\frac{dy}{dt} = 2q$$
$$\frac{dz}{dt} = 2r$$
$$\frac{du}{dt} = p(2p) + q(2q) + r(2r) = 2(p^2 + q^2 + r^2)$$
$$\frac{dp}{dt} = -F_x - pF_u = 0$$
$$\frac{dq}{dt} = -F_y - qF_u = 0$$
$$\frac{dr}{dt} = -F_z - rF_u = 0$$

(c) **(a) Solving the Initial Value Problem Initial Condition**: $u = k$ on the plane $\alpha x+\beta y+z = 0$. **Parameterize Initial Surface** (at $t = 0$): $x_0(s_1, s_2) = s_1$ $y_0(s_1, s_2) = s_2$ $z_0(s_1, s_2) = -\alpha s_1 - \beta s_2$ $u_0(s_1, s_2) = k$
Determine Initial Values for p, q, r: **Compatibility Conditions**: $\frac{\partial u_0}{\partial s_1} = p_0\frac{\partial x_0}{\partial s_1} + q_0\frac{\partial y_0}{\partial s_1} + r_0\frac{\partial z_0}{\partial s_1} \implies 0 = p_0(1) + q_0(0) + r_0(-\alpha) \implies p_0 = \alpha r_0$. $\frac{\partial u_0}{\partial s_2} = p_0\frac{\partial x_0}{\partial s_2} + q_0\frac{\partial y_0}{\partial s_2} + r_0\frac{\partial z_0}{\partial s_2} \implies 0 = p_0(0) + q_0(1) + r_0(-\beta) \implies q_0 = \beta r_0$. **PDE on Initial Surface**: $p_0^2 + q_0^2 + r_0^2 = 1$ $(\alpha r_0)^2 + (\beta r_0)^2 + r_0^2 = 1 \implies r_0^2(\alpha^2 + \beta^2 + 1) = 1 \implies r_0 = \pm\frac{1}{\sqrt{\alpha^2+\beta^2+1}}$. We choose the positive branch for r_0. So, $p_0 = \frac{\alpha}{\sqrt{1+\alpha^2+\beta^2}}$, $q_0 = \frac{\beta}{\sqrt{1+\alpha^2+\beta^2}}$, $r_0 = \frac{1}{\sqrt{1+\alpha^2+\beta^2}}$.
Integrate Characteristic ODEs:

- $\frac{dp}{dt} = 0 \implies p(t) = p_0$.
- $\frac{dq}{dt} = 0 \implies q(t) = q_0$.
- $\frac{dr}{dt} = 0 \implies r(t) = r_0$.
- $\frac{du}{dt} = 2(p^2 + q^2 + r^2) = 2(1) = 2$. $u(t) = 2t + C_u(s_1, s_2)$. With $u(0, s_1, s_2) = k \implies C_u = k$. So, $u(s_1, s_2, t) = 2t + k$.
- $\frac{dx}{dt} = 2p \implies x(t) = 2p_0t + C_x(s_1, s_2)$. With $x(0, s_1, s_2) = s_1 \implies C_x = s_1$. So, $x(s_1, s_2, t) = 2p_0t + s_1$.
- $\frac{dy}{dt} = 2q \implies y(t) = 2q_0t + C_y(s_1, s_2)$. With $y(0, s_1, s_2) = s_2 \implies C_y = s_2$. So, $y(s_1, s_2, t) = 2q_0t + s_2$.
- $\frac{dz}{dt} = 2r \implies z(t) = 2r_0t + C_z(s_1, s_2)$. With $z(0, s_1, s_2) = -\alpha s_1 - \beta s_2 \implies C_z = -\alpha s_1 - \beta s_2$. So, $z(s_1, s_2, t) = 2r_0t - \alpha s_1 - \beta s_2$.

Eliminate Parameters: From $u = 2t+k \implies t = \frac{u-k}{2}$. From $x = 2p_0t + s_1 \implies s_1 = x - 2p_0t$. From $y = 2q_0t+s_2 \implies s_2 = y-2q_0t$. Substitute s_1, s_2 into $z = 2r_0t-\alpha s_1-\beta s_2$: $z = 2r_0t-\alpha(x-2p_0t)-\beta(y-2q_0t)$ $z = 2r_0t-\alpha x+2\alpha p_0t-\beta y+2\beta q_0t$ $z+\alpha x+\beta y = 2t(r_0+\alpha p_0+\beta q_0)$. Substitute $p_0 = \alpha r_0$ and $q_0 = \beta r_0$: $z + \alpha x + \beta y = 2t(r_0 + \alpha(\alpha r_0) + \beta(\beta r_0))$ $z + \alpha x + \beta y = 2tr_0(1 + \alpha^2 + \beta^2)$. Substitute $r_0 = \frac{1}{\sqrt{1+\alpha^2+\beta^2}}$: $z+\alpha x+\beta y = 2t\frac{1}{\sqrt{1+\alpha^2+\beta^2}}(1+\alpha^2+\beta^2) = 2t\sqrt{1 + \alpha^2 + \beta^2}$. Substitute

$t = \frac{u-k}{2}$: $z + \alpha x + \beta y = 2\left(\frac{u-k}{2}\right)\sqrt{1+\alpha^2+\beta^2}$ $z + \alpha x + \beta y = (u-k)\sqrt{1+\alpha^2+\beta^2}$. Solving for u: $u(x,y,z) = k + \frac{\alpha x+\beta y+z}{\sqrt{1+\alpha^2+\beta^2}}$.

Final Solution for (a):

$$u(x,y,z) = k + \frac{\alpha x + \beta y + z}{\sqrt{1+\alpha^2+\beta^2}}$$

(d) **(b) Finding a Complete Integral** The PDE is $u_x^2+u_y^2+u_z^2 = 1$. This is of the form $F(p,q,r) = 0$. For such PDEs, we can seek a complete integral of the form $u(x,y,z) = a_1x + a_2y + a_3z + C$, where a_1, a_2, a_3, C are arbitrary constants. Substituting into the PDE: $a_1^2 + a_2^2 + a_3^2 = 1$. We can express one constant in terms of the others, say $a_3 = \pm\sqrt{1-a_1^2-a_2^2}$. So, a complete integral is:

$$u(x,y,z;a_1,a_2,C) = a_1x + a_2y \pm \sqrt{1-a_1^2-a_2^2}z + C$$

where a_1, a_2, and C are arbitrary constants.

Final Solution for (b):

$$u(x,y,z;a_1,a_2,C) = a_1x + a_2y \pm \sqrt{1-a_1^2-a_2^2}z + C$$