

Lec 3

15 January 2026 08:06

Cauchy Schwartz inequality :

$$\left| \sum_{i=1}^n x(i) y(i) \right| \leq \left(\sum_{i=1}^n |x(i)|^2 \right)^{1/2} \left(\sum_{i=1}^n |y(i)|^2 \right)^{1/2}$$

Hölder's Inequality :

$$\left| \sum_{i=1}^n x(i) y(i) \right| \leq \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p} \left(\sum_{i=1}^n |y(i)|^q \right)^{1/q} \quad p > 1$$

Minkowski Inequality :

$$\left(\sum_{i=1}^n |x(i) + y(i)|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^n |y(i)|^p \right)^{1/p}$$

Example :

 \mathbb{K}^n Define $x \rightarrow \|x\|_p$

$$\|x\|_p = \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p}, \quad x(i) \in \mathbb{K} \quad (p \geq 1)$$

$$\|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Exercise : $\|\cdot\|_p, \|\cdot\|_\infty$ check these are norms
($p \geq 1$)
($p \in \mathbb{R}$)what happens when $p \notin [1, \infty)$ $p=0 \rightarrow$ Definition is NOT valid $p \in \mathbb{R}^- \rightarrow$ Not define always $0 < p < 1$?? \hookrightarrow inequality does not holds

$$e_1 = (1, 0, 0, \dots, 0)$$

$$e_2 = (0, 1, 0, \dots, 0)$$

$$\|e_1\|_p = \|e_2\|_p = 1$$

$$\|e_1 + e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p + \|e_2\|_p$$

we know $\mathbb{K}^n \sim \mathcal{F}(S, \mathbb{K})$

$$S = \{1, 2, \dots, n\}$$

$$\underline{\underline{\mathcal{F}(\mathbb{N}, \mathbb{K})}} = \text{set of seq } (S \neq \emptyset)$$

linear space

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p}$$

$$\left(\sum_{i=1}^n |x(i) + y(i)|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^n |y(i)|^p \right)^{1/p}$$

observe that if $(\sum_{i=1}^{\infty} |x(i)|^p)^{1/p} < \infty$ and $(\sum_{i=1}^{\infty} |y(i)|^p)^{1/p} < \infty$
 then a priori we cannot say

$$(\sum_{i=1}^{\infty} |x(i) + y(i)|^p)^{1/p} < \infty$$

consider $S_N = (\sum_{i=1}^N |x(i) + y(i)|^p)^{1/p}$

note that S_N is monotonically increasing

$$S_N \leq \left(\sum_{i=1}^N |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^N |y(i)|^p \right)^{1/p} \quad [\text{Minkowski ineq.}]$$

↑
Bounded

↑
Bounded

$$\leq \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y(i)|^p \right)^{1/p}$$

$$\therefore \left(\sum_{i=1}^{\infty} |x(i) + y(i)|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y(i)|^p \right)^{1/p} < \infty$$

★

l^p -space

$$l^p = \left\{ x \in F(N, \mathbb{K}) : \sum |x(i)|^p < \infty \right\}$$

PT: $l^p \subseteq F(N, \mathbb{K})$
 ↑
 subspace \hookrightarrow linear space

PT: $x, y \in l^p \Rightarrow x + y \in l^p$ (true from ★)

$x \in l^p \Rightarrow \lambda x \in l^p$ (trivial)

hence l^p is a subspace of $F(N, \mathbb{K})$

we define $\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p}$ $1 \leq p < \infty$

Consider

$$l^\infty = \left\{ x \in F(N, \mathbb{K}) \text{ such that } (x(n)) \text{ is bdd.} \right\}$$

clearly l^∞ is a subspace of $F(N, \mathbb{K})$

define $\|x\|_\infty = \sup_{n \in \mathbb{N}} |x(n)|$

$$C_{00} = \left\{ x \in F(N, \mathbb{K}) \mid \exists N_0 \in \mathbb{N} \text{ with } x(n) = 0 \forall n \geq N_0 \right\}$$

$$C = \left\{ x \in F(N, \mathbb{K}) \mid x(n) \text{ is convergent} \right\}$$

$$C_0 = \left\{ x \in F(N, \mathbb{K}) \mid x(n) \rightarrow 0 \right\}$$

$$C_{00} \subseteq C_0 \subseteq C \subseteq l^\infty$$

let $x \in l^p \Rightarrow \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p} < \infty \Rightarrow x(n) \rightarrow 0$
 $\Rightarrow x \in C_0$

$$\Rightarrow l^p \subseteq C_0$$

$$C_{00} \subseteq l^p \subseteq C_0 \subseteq C \subseteq l^\infty$$

$$1 \leq p < q < \infty$$

$$\text{PT: } l^p \subsetneq l^q$$

take $x \in l^p$

$$(\sum |x(i)|^p)^{1/p} < \infty$$

$$\Rightarrow \sum |x(i)|^p < \infty$$

$$\Rightarrow |x(i)|^p \rightarrow 0$$

$$\Rightarrow |x(i)| < 1 \quad \forall i > N_0$$

we can split seq $x(i)$ into >1 and <1

$$\sum_{i=1}^{N_0} |x(i)|^q \text{ is finite } (\leq M_1)$$

$$|x(i)|^q < |x(i)|^p \quad \text{for } i > N_0 \quad \text{because } |x(i)| < 1 \text{ and } p < q$$

$$\therefore \sum_{i=N_0+1}^{\infty} |x(i)|^q < \sum_{i=N_0+1}^{\infty} |x(i)|^p (\leq M_2)$$

$$\therefore x \in l^q$$

★

$$1 \leq p < q < \infty$$

$$x \in l^p \text{ then } \|x\|_q < \|x\|_p$$

use

$$\sum a_n^\alpha \leq (\sum a_n)^\alpha \quad \text{complete}$$

$$\text{where } a_n = |x(n)|^p \quad \alpha = q/p > 1$$

$$\text{Ex } x \in \bigcup_{1 \leq p < \infty} l^p \text{ then}$$

$$\|x\|_p \rightarrow \|x\|_\infty \text{ as } p \rightarrow \infty$$

Example of NLS:

$$\Omega \neq \emptyset$$

$B(\Omega, \mathbb{K})$ = set of all \mathbb{K} -valued funcⁿ define on Ω that are bdd.

if $x, y \in B(\Omega) \rightarrow$ linear space

$$\Rightarrow x+y \in B(\Omega) \quad \lambda x \in B(\Omega)$$

$$x \in B(\Omega) \Leftrightarrow \sup_{t \in \Omega} |x(t)| < \infty$$

Define $x \rightarrow \|x\|_p$

$$\|x\|_\infty = \sup_{t \in \Omega} |x(t)|$$

note that $\|\cdot\|_\infty$ defines a norm.

then $(B(\Omega), \|\cdot\|_\infty)$ is a NLS

In particular $\Omega = [a, b]$

$$C(\Omega, \mathbb{K}) = C(\Omega) = C[a, b]$$

If $x \in C[a, b]$ then x is ~~odd~~ and $x \in B[a, b]$

$$\therefore C[a, b] \subseteq B[a, b]$$

we define $\|\cdot\|_\infty$ in $C[a, b]$

\cdot $R[a, b]$ = space of all rieman integral funcⁿ

$$\left\{ \begin{array}{l} \subseteq B[a, b] \\ \|\cdot\|_\infty \text{ norm} \end{array} \right.$$

$$\begin{array}{l} \cdot \quad C[a, b] \xrightarrow{\quad} \|f\|_\infty = \sup_{t \in [a, b]} |f(t)| \\ \quad \quad \quad \searrow \quad \|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p} \quad 1 \leq p < \infty \end{array}$$

Δ inequality can be proved by minkowski's integral ineq.

Banach Space

\hookrightarrow complete NLS is Banach Space

Ex 1. $(\mathbb{K}, \|\cdot\|)$ is a Banach space

2. $(\mathbb{K}^n, \|\cdot\|_p)$ ————— ?

Q. is $(\mathbb{K}^n, \|\cdot\|_p)$ is Banach space

for $p = \infty$:

$(\mathbb{K}^n, \|\cdot\|_\infty)$ is Banach :

\rightarrow 1st show the norm is defined (we showed earlier)

now let $(x_m)_{m \in \mathbb{N}}$ be a cauchy seq in $(\mathbb{K}^n, \|\cdot\|_\infty)$

i.e. $\forall \varepsilon > 0 \exists N_\varepsilon$ such that

$$\|x_m - x_k\|_\infty < \varepsilon \quad \forall m, k \geq N_\varepsilon$$

$$\text{i.e. } \max_{1 \leq i \leq n} |x_m(i) - x_k(i)| < \varepsilon \quad \forall m, k \geq N_\varepsilon$$

$$\text{i.e. } |x_m(i) - x_k(i)| < \varepsilon \quad \forall m, k \geq N_\varepsilon \quad i \in \{1, \dots, n\}$$

$$x_1 = (x_1(1), x_1(2), \dots)$$

$$x_2 = (x_2(1), x_2(2), \dots)$$

\vdots

cauchy cauchy

i.e. $(x_m(i))_{m \in \mathbb{N}}$ is a Cauchy seq in \mathbb{K} for each $i = 1, 2, \dots, n$

as \mathbb{K} is Banach space

$\exists \alpha_i$ such that $x_m(i) \rightarrow \alpha_i \quad \forall 1 \leq i \leq n$

define $x = (\alpha_1, \alpha_2, \dots, \alpha_n)$

as $\alpha_i \in \mathbb{K} \quad \forall 1 \leq i \leq n$

$\Rightarrow x \in \mathbb{K}^n$

now to show $x_n \rightarrow x$

$$\begin{aligned} \|x_m - x\|_\infty &= \max_{1 \leq i \leq n} |x_m(i) - x(i)| \\ &= \max_{1 \leq i \leq n} |x_m(i) - \alpha_i| \end{aligned}$$

we know $x_m(i) \rightarrow \alpha_i$

$\therefore \langle \epsilon \rangle \Rightarrow x_n \rightarrow x$

$\therefore (\mathbb{K}^n, \|\cdot\|_\infty)$ is B-S.

$\forall x \in \mathbb{K}^n$ then $|x(i)| \leq \left(\sum_{j=1}^n |x(j)|^p \right)^{1/p} \quad \forall 1 \leq i \leq n$

$$\Rightarrow \max_{1 \leq i \leq n} |x(i)| = \|x\|_\infty \leq \|x\|_p$$

$$\begin{aligned} \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p} &\leq \left(\sum_{i=1}^n M^p \right)^{1/p} \quad (M = \|x\|_\infty) \\ &= n^{1/p} M \end{aligned}$$

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty$$