Fractal Geometry

Evaluation Policy :

1. Midterm Exam — 30%
2. End-term Exam — 40%
3. Quizzes (2) — 15% each

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Audit Pass : Minimum 40%

Reference Books:

1. K. J. Falconer, Fractal Geometry: Mathematical Foundations & Applications, Wiley
2. Gerald Edgar, Measure, Topology and Fractal Geometry, Springer
3. M. F. Barnsley, Fractals Everywhere, Dover.

- Basic Measure Theory

What are fractals?

1. Cantor Set

$E_0 = [0,1]$

$E_1 = [0,\tfrac{1}{3}] \cup [\tfrac{2}{3},1]$

$E_2 = [0,\tfrac{1}{9}] \cup [\tfrac{2}{9},\tfrac{1}{3}]$
$\quad [\tfrac{2}{3},\tfrac{7}{9}] \cup [\tfrac{8}{9},1]$

$\vdots$

E_n = union of 2^n closed intervals each of length 3^{-n}.

Cantor set, $E = \bigcap_{n=0}^{\infty} E_n$

Some properties:

1. E is an uncountable set.
2. "Lebesgue measure" of E is zero.

$$m([0,1]\setminus E) = \frac{1}{3} + 2\times\frac{1}{3^2} + 2^2\times\frac{1}{3^3} + \cdots$$
$$= \frac{1}{3}\left(1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right)$$
$$= \frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}} = 1$$

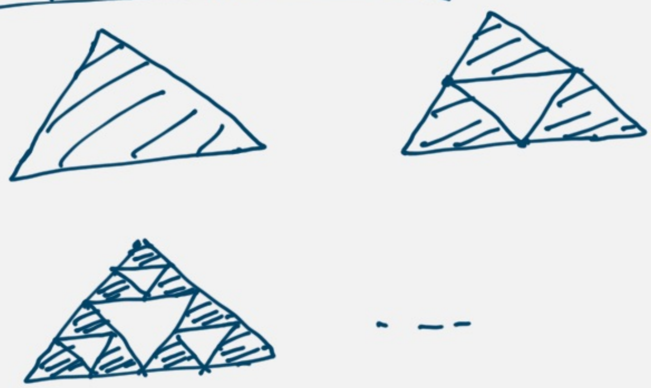
$\Rightarrow m(E) = 1 - 1 = 0$.

3. "Self-similarity"

$$E = \underbrace{\left(E \cap [0,\tfrac{1}{3}]\right)}_{\text{scaling factor} = \frac{1}{3}} \cup \underbrace{\left(E \cap [\tfrac{2}{3},1]\right)}$$

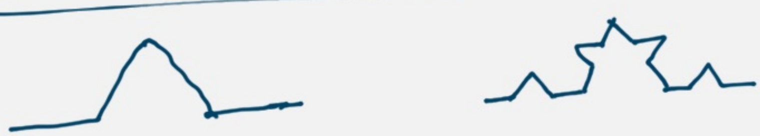
are similar to E.

- Sierpinski gasket



- Uncountable
- Zero Lebesgue measure
- Self-similar with scaling factor $= \frac{1}{2}$

- Koch Snowflake curve



and so on.

This is also self-similar with scaling factor $= \frac{1}{3}$

Basic Measure Theory

- Outer measure

Let X be any set.

An outer measure μ on X is

$$\mu : \mathcal{P}(X) \to [0,\infty]$$

($\mathcal{P}(X)$ = the power set of X
= set of all subsets of X)

satisfying

(i) $\mu(\phi) = 0$

(ii) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ (Monotonicity)

(iii) $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$ (Countable subadditivity)

Examples:

1. Counting measure

$$\mu : \mathcal{P}(X) \to [0,\infty]$$

$$\mu(A) = \begin{cases} \text{no. of elements in } A & \text{, if } A \text{ is finite} \\ \infty & \text{, if } A \text{ is infinite} \end{cases}$$

2. Lebesgue outer measure on $\mathbb{R}$

$$\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$$

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} \ell(I_j) : A \subset \bigcup_{j=1}^{\infty} I_j, \ I_j \text{ is an interval} \right\}$$