

Lec 3

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Cauchy Schwartz inequality :

$$\left| \sum_{i=1}^n x_{(i)} y_{(i)} \right| \leq \left(\sum_{i=1}^n |x_{(i)}|^2 \right)^{1/2} \left(\sum_{i=1}^n |y_{(i)}|^2 \right)^{1/2}$$

Hölder's Inequality :

$$\left| \sum_{i=1}^n x_{(i)} y_{(i)} \right| \leq \left(\sum_{i=1}^n |x_{(i)}|^p \right)^{1/p} \left(\sum_{i=1}^n |y_{(i)}|^q \right)^{1/q} \quad p > 1$$

Minkowski Inequality :

$$\left(\sum_{i=1}^n |x_{(i)} + y_{(i)}|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |x_{(i)}|^p \right)^{1/p} + \left(\sum_{i=1}^n |y_{(i)}|^p \right)^{1/p}$$

Example :

\mathbb{K}^n define $x \rightarrow \|x\|_p$

$$\|x\|_p = \left(\sum_{i=1}^n |x_{(i)}|^p \right)^{1/p}, \quad x_{(i)} \in \mathbb{K} \quad (p > 1)$$

$$\cdot \|x+y\|_p \leq \|x\|_p + \|y\|_p$$

Exercise : $\|\cdot\|_p, \|\cdot\|_\infty$ check these are norms
 $(p \geq 1)$
 $(p \in \mathbb{R})$

what happens when $p \notin [1, \infty)$

$p=0 \rightarrow$ definition is NOT valid

$p \in \mathbb{R}^- \rightarrow$ Not define always

$0 \leq p < 1$??

\hookrightarrow Δ inequality does not holds

$$e_1 = (1, 0, 0, \dots, 0)$$

$$e_2 = (0, 1, 0, \dots, 0)$$

$$\|e_1\|_p = \|e_2\|_p = 1$$

$$\|e_1 + e_2\|_p = 2^{1/p} > 2 = \|e_1\|_p + \|e_2\|_p$$

we know $\mathbb{K}^n \sim \mathcal{F}(S, \mathbb{K})$

$$S = \{1, 2, \dots, n\}$$

$\mathcal{F}(S, \mathbb{K})$ = set of seq ($S \neq \emptyset$)
 linear space

$$\|x\|_p = \left(\sum_{i=1}^n |x_{(i)}|^p \right)^{1/p}$$

$$\left(\sum_{i=1}^n |x_{(i)} + y_{(i)}|^p \right)^{1/p} \leq \left(\sum_{i=1}^n |x_{(i)}|^p \right)^{1/p} + \left(\sum_{i=1}^n |y_{(i)}|^p \right)^{1/p}$$

Observe that if $(\sum_{i=1}^{\infty} |x(i)|^p)^{1/p} < \infty$ and $(\sum_{i=1}^{\infty} |y(i)|^p)^{1/p} < \infty$
then a priori we cannot say

$$\left(\sum_{i=1}^{\infty} |x(i) + y(i)|^p \right)^{1/p} < \infty$$

consider $s_n = \left(\sum_{i=1}^n |x(i) + y(i)|^p \right)^{1/p}$

Note that s_n is monotonically increasing

$$s_n \leq \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^n |y(i)|^p \right)^{1/p} \quad [\text{Minkowski ineq.}]$$

↑ ↑
Bounded Bounded

$$\leq \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y(i)|^p \right)^{1/p}$$

$$\therefore \left(\sum_{i=1}^{\infty} |x(i) + y(i)|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p} + \left(\sum_{i=1}^{\infty} |y(i)|^p \right)^{1/p} < \infty$$

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ℓ^p - space

$$\ell^p = \{ x \in \mathcal{F}(N, \mathbb{K}) : \sum |x(i)|^p < \infty \}$$

PT: $\ell^p \subseteq \mathcal{F}(N, \mathbb{K})$
 \cap linear space
 subspace

PT: $x, y \in \ell^p \Rightarrow x + p \in \ell^p$ (true from *)

$$x \in \ell^p \rightarrow \lambda x \in \ell^p \quad (\text{trivial})$$

hence ℓ^p is a subspace of $\mathcal{F}(N, \mathbb{K})$

we define $\|x\|_p = \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p}$ $1 \leq p < \infty$

Consider

$$\ell^\infty = \{ x \in \mathcal{F}(N, \mathbb{K}) \text{ such that } (x(n)) \text{ is bdd.} \}$$

clearly ℓ^∞ is a subspace of $\mathcal{F}(N, \mathbb{K})$

$$\text{define } \|x\|_\infty = \sup_{n \in N} |x(n)|$$

$$C_\infty = \{ x \in \mathcal{F}(N, \mathbb{K}) \mid \exists n_0 \in N \text{ with } x(n)=0 \text{ for } n \geq n_0 \}$$

$$C = \{ x \in \mathcal{F}(N, \mathbb{K}) \mid x(n) \text{ is convergent} \}$$

$$C_0 = \{ x \in \mathcal{F}(N, \mathbb{K}) \mid x(n) \rightarrow 0 \}$$

$$C_\infty \subseteq C_0 \subseteq C \subseteq \ell^\infty$$

let $x \in \ell^p \Rightarrow \left(\sum_{i=1}^{\infty} |x(i)|^p \right)^{1/p} < \infty \Rightarrow x(n) \rightarrow 0$
 $\Rightarrow x \in C_0$

$$\Rightarrow \ell^p \subseteq C_0$$

$$C_{00} \subseteq \ell^p \subseteq C_0 \subseteq C \subseteq \ell^\infty$$

$$1 \leq p < q < \infty$$

PT: $\ell^p \subsetneq \ell^q$

take $x \in \ell^p$

$$(\sum |x_{(i)}|^p)^{1/p} < \infty$$

$$\Rightarrow \sum |x_{(i)}|^p < \infty$$

$$\Rightarrow |x_{(i)}|^p \rightarrow 0$$

$$\Rightarrow |x_{(i)}| < 1 \quad \forall i > N_0$$

we can split seq $x_{(i)}$ into > 1 and < 1

$$\sum_{i=1}^{N_0} |x_{(i)}|^q \text{ is finite } (\leq M_1)$$

$$|x_{(i)}|^q < |x_{(i)}|^p \quad \text{for } i > N_0 \quad \text{because } |x_{(i)}| < 1 \text{ and } p < q$$

$$\therefore \sum_{i=N_0+1}^{\infty} |x_{(i)}|^q < \sum_{i=N_0+1}^{\infty} |x_{(i)}|^p \quad (\leq M_2)$$

$$\therefore x \in \ell^q$$

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$$1 \leq p < q < \infty$$

$$x \in \ell_p \text{ then } \|x\|_q < \|x\|_p$$

use

$$\sum a_n^\alpha \leq (\sum a_n)^\alpha \quad \text{complete}$$

$$\text{where } a_n = |x(n)|^p \quad \alpha = n/p > 1$$

Ex $x \in \bigcup_{1 \leq p < \infty} \ell^p$ then

$$\|x\|_p \rightarrow \|x\|_\infty \text{ as } p \rightarrow \infty$$

Example of NLS:

$$\mathcal{S} \neq \emptyset$$

$B(S, \mathbb{K}) = \text{set of all } K\text{-valued funct define on } S$
that are bdd.

if $x, y \in B(S) \rightarrow \text{linear space}$

$$\Rightarrow x+y \in B(S) \quad \lambda x \in B(S)$$

$$x \in B(S) \Leftrightarrow \sup_{t \in S} |x(t)| < \infty$$

Define $x \mapsto \|x\|_p$

$$\|x\|_\infty = \sup_{t \in S} |x(t)|$$

note that $\|\cdot\|_\infty$ defines a norm.

then $(B(\mathbb{R}), \|\cdot\|_\infty)$ is a NLS

In particular $\mathcal{S} = [a, b]$

$$C(\mathcal{S}, \mathbb{K}) = C(\mathbb{R}) = C[a, b]$$

If $x \in C[a, b]$ then x is bounded and $x \in B[a, b]$

$$\therefore C[a, b] \subseteq B[a, b]$$

we define $\|\cdot\|_\infty$ in $C[a, b]$

$\cdot R[a, b] = \text{space of all riemann integral func'}$

$$\subseteq B[a, b]$$

$\|\cdot\|_\infty$ norm

$$\cdot C[a, b] \xrightarrow{\quad} \|f\|_\infty = \sup_{t \in [a, b]} |f(t)|$$

$$\|f\|_p = \left(\int_a^b |f(t)|^p dt \right)^{1/p} \quad 1 \leq p < \infty$$

A inequality can be proved by minkowski's integral ineq.

Banach Space

\hookrightarrow complete NLS is Banach Space

Ex 1. $(\mathbb{K}, \|\cdot\|)$ is a Banach space

2. $(\mathbb{K}^n, \|\cdot\|_p)$ _____ ?

Q. is $(\mathbb{K}^n, \|\cdot\|_p)$ is Banach space

for $p = \infty$:

$(\mathbb{K}^n, \|\cdot\|_\infty)$ is Banach:

\rightarrow 1st show the norm is defined (we showed earlier)

now let $(x_m)_{m \in \mathbb{N}}$ be a cauchy seq, in $(\mathbb{K}^n, \|\cdot\|_\infty)$

i.e. $\forall \varepsilon > 0 \exists N_\varepsilon$ such that

$$\|x_m - x_k\|_\infty < \varepsilon \quad \forall m, k \geq N_\varepsilon$$

$$\text{i.e. } \max_{1 \leq i \leq n} |x_m(i) - x_k(i)| < \varepsilon \quad \forall m, k \geq N_\varepsilon$$

$$\text{i.e. } |x_m(i) - x_k(i)| < \varepsilon \quad \forall m, k \geq N_\varepsilon$$

$$x_1 = (x_1(1), x_1(2), \dots) \quad i \in \{1, \dots, n\}$$

$$x_2 = (x_2(1), x_2(2), \dots)$$

$\vdots \quad \vdots$

cauchy cauchy

i.e. $(x_m(i))_{m \in \mathbb{N}}$ is a cauchy seq/ in \mathbb{K} for each $i = 1, 2, \dots, n$

as \mathbb{K} is Banach space

$\exists \alpha_i$ such that $x_m(i) \rightarrow \alpha_i \quad \forall i \leq n$

define $x = (\alpha_1, \alpha_2, \dots, \alpha_n)$

as $\alpha_i \in \mathbb{K} \quad \forall 1 \leq i \leq n$

$\Rightarrow x \in \mathbb{K}^n$

now to show $x_n \rightarrow x$

$$\begin{aligned}\|x_m - x\|_\infty &= \max_{1 \leq i \leq n} |x_m(i) - x(i)| \\ &= \max_{1 \leq i \leq n} |x_m(i) - \alpha_i|\end{aligned}$$

we know $x_m(i) \rightarrow \alpha_i$

$\therefore \lim_{m \rightarrow \infty} \|x_m - x\|_\infty = 0 \Rightarrow x_n \rightarrow x$

$\therefore (\mathbb{K}^n, \|\cdot\|_\infty)$ is B-S.

$$\begin{aligned}\text{If } x \in \mathbb{K}^n \text{ then } |x(i)| &\leq \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p} \quad \forall i \leq n \\ \Rightarrow \max_{1 \leq i \leq n} |x(i)| &= \boxed{\|x\|_\infty \leq \|x\|_p} \\ \left(\sum_{i=1}^n |x(i)|^p \right)^{1/p} &\leq \left(\sum_{i=1}^n M^p \right)^{1/p} \quad (M = \|x\|_\infty) \\ &= n^{1/p} M\end{aligned}$$

$$\boxed{\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty}$$