Simplex Method for Standard Maximization Problem

Previously, we learned the method of corners to solve linear programming problems. However, to solve problems with the method of corners, it is necessary that we know specific information about the feasible solution set. We must know the coordinate points of the corners of the feasible solution set. In 2 variables, this is no problem. However, as the number of variables increases, it becomes difficult, if not impossible, to plot the feasible solution set. With the simplex method, we will be able to solve linear programming problems by only knowing one corner of the feasible solution set.

In this section, we give the formal procedure for the simplex method for the standard maximization problem.

def: A **standard maximization problem** is one in which

1. the objective function is to be maximized,
2. all variables involved in the problem are nonnegative, and
3. all other linear constraints may be written so that the expression involving the variables is less than or equal to a nonnegative constant.

First, we will discuss the algorithm for the simplex method for standard maximization problems, and then we will apply the method to a problem.

Simplex Method: (for standard maximization problems)

Step 1: Transform the system of linear inequalities into a system of linear equations by introducing slack variables.

Consider the inequality $2x + 3y \leq 12$ with x and y nonnegative. For (x, y)-coordinates in a select set, we have that the left hand side is always less than or equal to the right hand side. By adding a nonnegative variable s_1 to the left-hand side of the inequality, we can change the inequality to equality. For example, if $(x, y) = (1, 1)$, then $2(1) + 3(1) + s_1 = 12$ when $s_1 = 7$. Of course the variable s_1 depends on the values of x and y. The variable s_1 is referred to as a slack variable.

Step 2: Given the objective function $P = c_1x_1 + c_2x_2 + ... + c_nx_n$, rewrite the function as $-c_1x_1 - c_2x_2 - ... - c_nx_n + P = 0$ so that all the variables are on the left of the equality sign and so that the coefficient of P is 1.

Step 3: Write the system of constraints (with slack variables inserted) above the objective function (as it is written in step 2). Convert the system of equations into an augmented matrix.

Step 4: Determine whether the optimal solution has been reached by examining all entries in the last row to the left of the vertical bar. If all entries in the last row are nonnegative, then the solution has been reached, and we should skip to step 7. If not, proceed to step 5.

Step 5: Locate the most negative entry to the left of the vertical line in the last row. In the column containing this element, find the element corresponding to the smallest nonegative ratio obtained by dividing the element in the column into the element to the right of the veritcal bar in the same row. Circle this element.

Step 6: If needed, multiply the row containing the circled element by a constant so that the circled element is a 1. Then, using row operations, make every other entry in the column 0. Return to step 4.

Step 7: The value of each variable heading a column of 0's and 1 is given by the entry lying in the column to the right of the vertical bar and in the row containing the 1. The variables heading columns not filled with 0's and a 1 are assigned the value 0.

ex.) Maximize $P = 2x + 4y$ subject to the constraints $x + 4y \leq 12$, $x + 3y \leq 10$, $x \geq 0$, and $y \geq 0$. Notice that this is a standard maximization problem.

Step 1:
$$\begin{aligned} x + 4y \leq 12 &\quad\rightarrow\quad x + 4y + s_1 = 12 \\ x + 3y \leq 10 &\quad\quad x + 3y + s_2 = 10 \end{aligned}$$

Step 2:
$$P = 2x + 4y \quad\rightarrow\quad -2x - 4y + P = 0$$

Step 3:
$$\begin{aligned} x + 4y + s_1 + 0s_2 + 0P &= 12 \\ x + 3y + 0s_2 + s_2 + 0P &= 10 \\ -2x - 4y + 0s_1 + 0s_2 + P &= 0 \end{aligned} \quad\rightarrow\quad \left[\begin{array}{ccccc|c} 1 & 4 & 1 & 0 & 0 & 12 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

Step 4: Not all entries in the last row are nonnegative, and so the optimal solution has not been reached.

Step 5:
$$\left[\begin{array}{ccccc|c} 1 & 4 & 1 & 0 & 0 & 12 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right]$$

Step 6:
$$\left[\begin{array}{ccccc|c} 1 & 4 & 1 & 0 & 0 & 12 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{4}R_1} \left[\begin{array}{ccccc|c} \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 3 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{-3R_1 + R_2 \rightarrow R_2}$$
$$\left[\begin{array}{ccccc|c} \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 3 \\ \frac{1}{4} & 0 & -\frac{3}{4} & 1 & 0 & 1 \\ -2 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{4R_1 + R_3 \rightarrow R_3} \left[\begin{array}{ccccc|c} \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 3 \\ \frac{1}{4} & 0 & -\frac{3}{4} & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 12 \end{array}\right]$$

Step 4: Not all entries in the last row are nonnegative, and so the optimal solution has not been reached.

Step 5:
$$\left[\begin{array}{ccccc|c} \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 3 \\ \frac{1}{4} & 0 & -\frac{3}{4} & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 12 \end{array}\right]$$

Step 6:
$$\left[\begin{array}{ccccc|c} \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 3 \\ \frac{1}{4} & 0 & -\frac{3}{4} & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 & 12 \end{array}\right] \xrightarrow{4R_2 \rightarrow R_2} \left[\begin{array}{ccccc|c} \frac{1}{4} & 1 & \frac{1}{4} & 0 & 0 & 3 \\ 1 & 0 & -3 & 4 & 0 & 4 \\ -1 & 0 & 1 & 0 & 1 & 12 \end{array}\right] \xrightarrow{-\frac{1}{4}R_2 + R_1 \rightarrow R_1}$$
$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -3 & 4 & 0 & 4 \\ -1 & 0 & 1 & 0 & 1 & 12 \end{array}\right] \xrightarrow{R_2 + R_3 \rightarrow R_3} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -3 & 4 & 0 & 4 \\ 0 & 0 & -2 & 4 & 1 & 16 \end{array}\right]$$

Step 4: Not all entries in the last row are nonnegative, and so the optimal solution has not been reached.

Step 5:
$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -3 & 4 & 0 & 4 \\ 0 & 0 & -2 & 4 & 1 & 16 \end{array}\right]$$

ex.) (cont.)

Step 6:

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 0 & -3 & 4 & 0 & 4 \\ 0 & 0 & -2 & 4 & 1 & 16 \end{array}\right] \xrightarrow{3R_1 + R_2 \to R_2} \left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ 0 & 0 & -2 & 4 & 1 & 16 \end{array}\right] \xrightarrow{2R_1 + R_3 \to R_3}$$

$$\left[\begin{array}{ccccc|c} 0 & 1 & 1 & -1 & 0 & 2 \\ 1 & 3 & 0 & 1 & 0 & 10 \\ 0 & 2 & 0 & 2 & 1 & 20 \end{array}\right]$$

Step 4: All entries of the last row are positive, therefore the optimal solution has been reached.

Step 7: $P = 20$ is the maximum value of P, and it occurs when $x = 10$ and $y = 2$.