PART 7
Ordinary Differential Equations
ODEs

Ordinary Differential Equations

Part 7

- Equations which are composed of an unknown function and its derivatives are called ***differential equations*.**

$$\frac{dv}{dt} = g - \frac{c}{m}v$$

v - dependent variable

t - independent variable

- Differential equations play a fundamental role in engineering because many physical phenomena are best formulated mathematically in terms of their rate of change.

Ordinary Differential Equations

- When a function involves **one dependent variable**, the equation is called an **ordinary differential equation (ODE).**

- **A partial differential equation (PDE)** involves **two or more independent variables**.

Ordinary Differential Equations

Differential equations are also classified as to their order:

1. **A first order equation** includes a first derivative as its highest derivative.

- Linear 1st order ODE $$\frac{dy}{dx} + \alpha \cdot y = f(x)$$

- Non-Linear 1st order ODE $$\frac{dy}{dx} = f(x, y)$$

Where f(x,y) is nonlinear

Ordinary Differential Equations

2. **A second order equation** includes a second derivative.

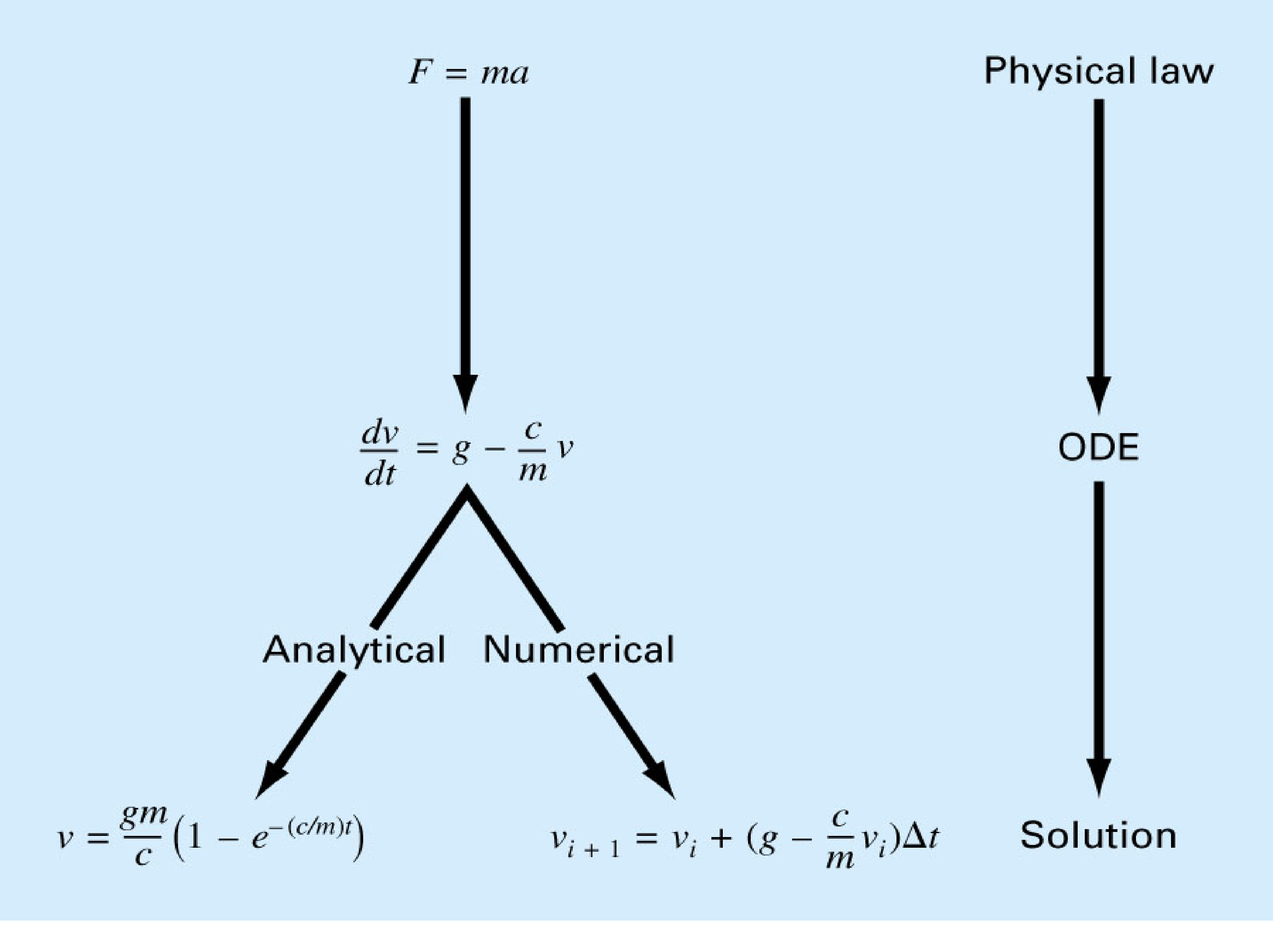
- Linear 2nd order ODE $$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + Qy = f(x)$$

- Non-Linear 2nd order ODE

$$\frac{d^2y}{dx^2} + p(x,y)\frac{dy}{dx} + Q(x,y)\cdot y = f(x)$$

- Higher order equations can be reduced to a system of first order equations, by redefining a variable.

Ordinary Differential Equations



Runge-Kutta Methods

This chapter is devoted to solving ODE of the form:

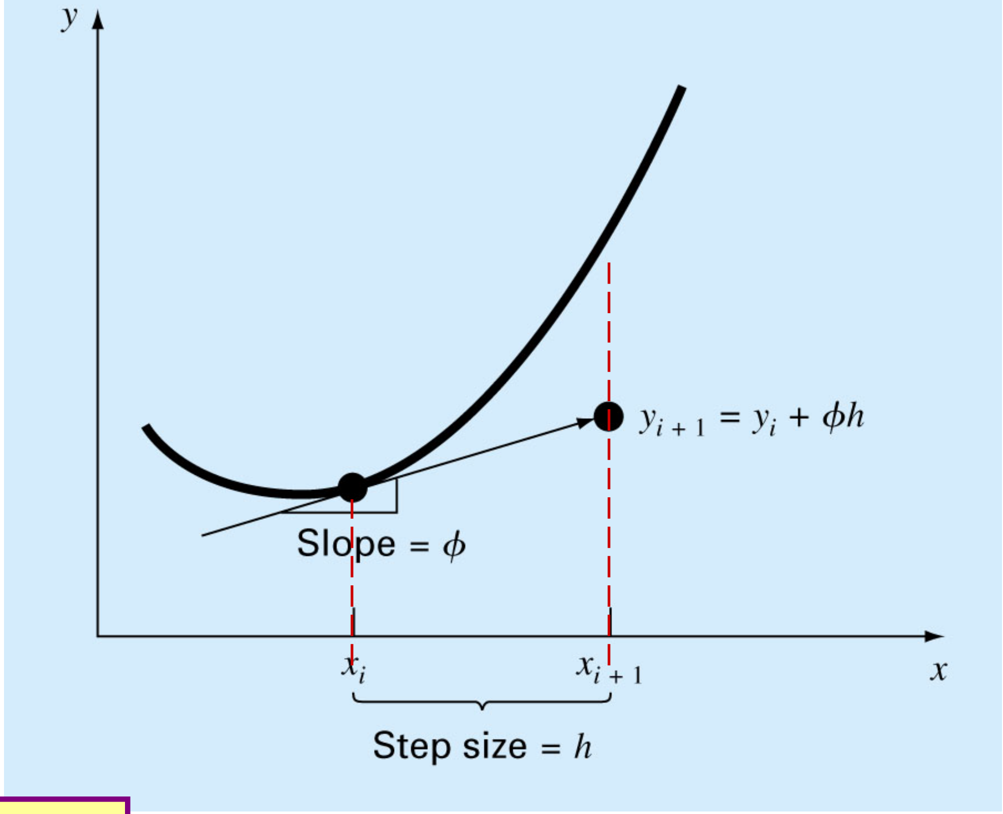
$$\frac{dy}{dx} = f(x, y)$$

- **Euler's Method**

$$\frac{dy}{dx} = f(x, y)$$

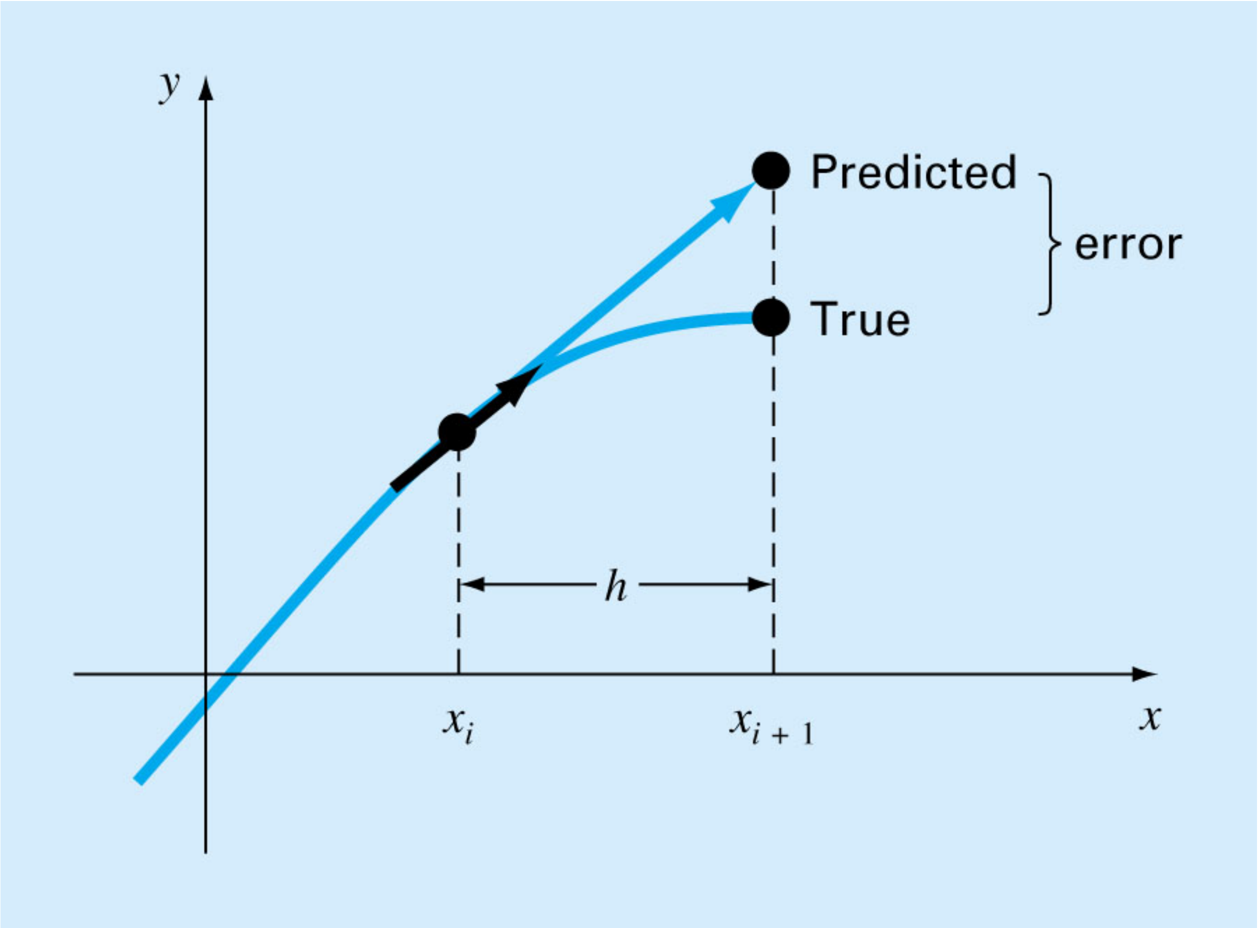
solution

$$y_{i+1} = y_i + f(x_i, y_i) \cdot h$$



Runge-Kutta Methods

$$\frac{dy}{dx} = f(x, y)$$



$$\boxed{y_{i+1} = y_i + f(x_i, y_i) \cdot h}$$

Euler's Method: Example

Obtain a solution between x = 0 to x = 4

with a step size of 0.5 for: $$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

Initial conditions are: x = 0 to y = 1

Solution: $$y_{i+1} = y_i + f(x_i, y_i) \cdot h$$

$$y(0.5) = y(0) + f(0,1).(0.5)$$

$$= 1.0 + 8.5x0.5 = 5.25$$

$$y(1.0) = y(0.5) + f(0.5,5.25).(0.5)$$

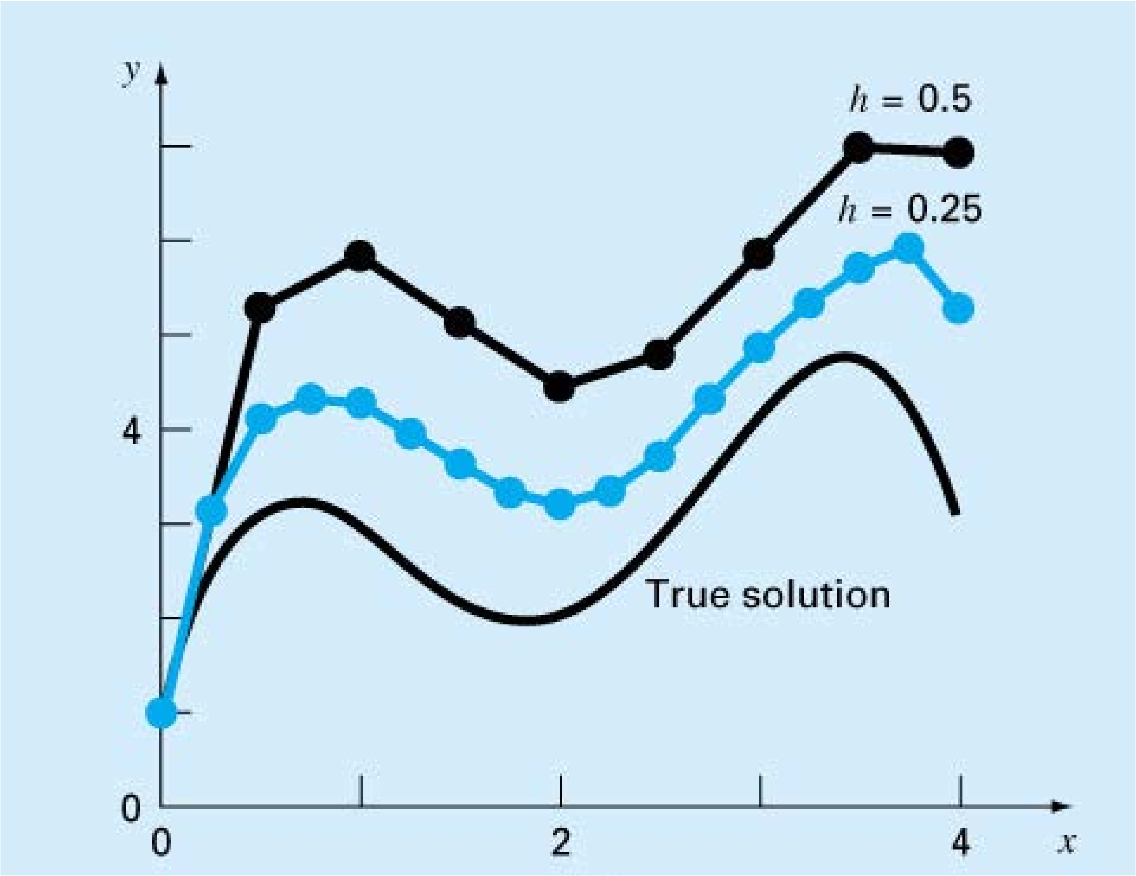
$$= 5.25 + (-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5).(0.5) = 5.875$$

$$y(2.0) = y(1.0) + f(1.0,5.875).(0.5)$$

$$= 5.25 + (-2(1.0)^3 + 12(1.0)^2 - 20(1.0) + 8.5).(0.5) = 5.125$$

Euler's Method: Example

- Although the computation captures the general trend solution, the error is considerable.
- This error can be reduced by using a smaller step size.



Improvements of Euler's method

- A fundamental source of error in Euler's method is that the derivative at the beginning of the interval is assumed to apply across the entire interval.

- Simple modifications are available:
  - **Heun's Method**
  - **The Midpoint Method**
  - **Ralston's Method**

Runge-Kutta Methods

- Runge-Kutta methods achieve the accuracy of a Taylor series approach without requiring the calculation of higher derivatives.

$$y_{i+1} = y_i + \phi(x_i, y_i, h)h$$

$$\boxed{\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n}$$

Increment function (representative slope over the interval)

$$a's = \text{constants}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

$$k_3 = f(x_i + p_3 h, y_i + q_{21} k_1 h + q_{22} k_2 h)$$

$$\vdots$$

$$k_n = f(x_i + p_{n-1} h, y_i + q_{n-1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h)$$

p's and q's are constants

Runge-Kutta Methods

- Various types of RK methods can be devised by employing different number of terms in the increment function as specified by *n*.

1. **First order RK** method with *n=1* is **Euler's method**.
   - **Error is proportional to O(h)**
2. **Second order RK methods:**
   - **Error is proportional to $O(h^2)$**

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

- Values of a_1, a_2, p_1, and q_{11} are evaluated by setting the second order equation to Taylor series expansion to the second order term.

Runge-Kutta Methods

- Three equations to evaluate the four unknown constants are derived:

$$a_1 + a_2 = 1$$

$$a_2 p_1 = \frac{1}{2}$$

$$a_2 q_{11} = \frac{1}{2}$$

A value is assumed for one of the unknowns to solve for the other three.

Runge-Kutta Methods

$$a_1 + a_2 = 1, \quad a_2\, p_1 = \frac{1}{2}, \quad a_2\, q_{11} = \frac{1}{2}$$

- We can choose an infinite number of values for a_2,there are an infinite number of second-order RK methods.
- Every version would yield exactly **the same results** if the solution to ODE were **quadratic**, **linear**, or **a constant**.
- However, they yield different results if the solution is more complicated (typically the case).

Runge-Kutta Methods

Three of the most commonly used methods are:

$$\boxed{y_{i+1} = y_i + (a_1 k_1 + a_2 k_2)h}$$

$$k_1 = f(x_i, y_i)$$

- **Huen Method with a Single Corrector (*a_2=1/2*)**

$$\boxed{y_{i+1} = y_i + (\frac{1}{2}k_1 + \frac{1}{2}k_2) \cdot h}$$

$$k_2 = f\big((x_i + h), (y_i + k_1 h)\big)$$

- **The Midpoint Method (*a_2= 1*)**

$$\boxed{y_{i+1} = y_i + k_2 \cdot h}$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

- **Raltson's Method (*a_2= 2/3*)**

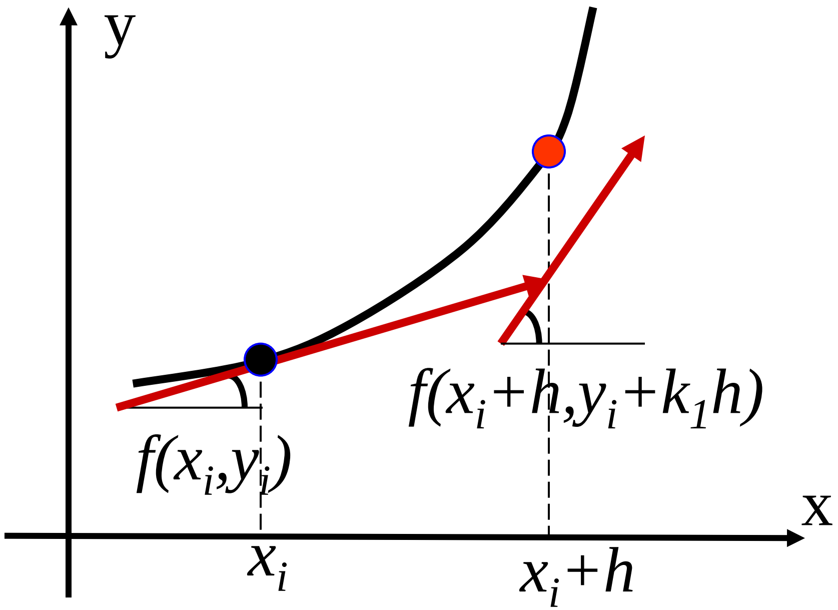
$$\boxed{y_{i+1} = y_i + (\frac{1}{3}k_1 + \frac{2}{3}k_2) \cdot h}$$

$$k_2 = f(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h)$$

Runge-Kutta Methods

- **Heun's Method:**

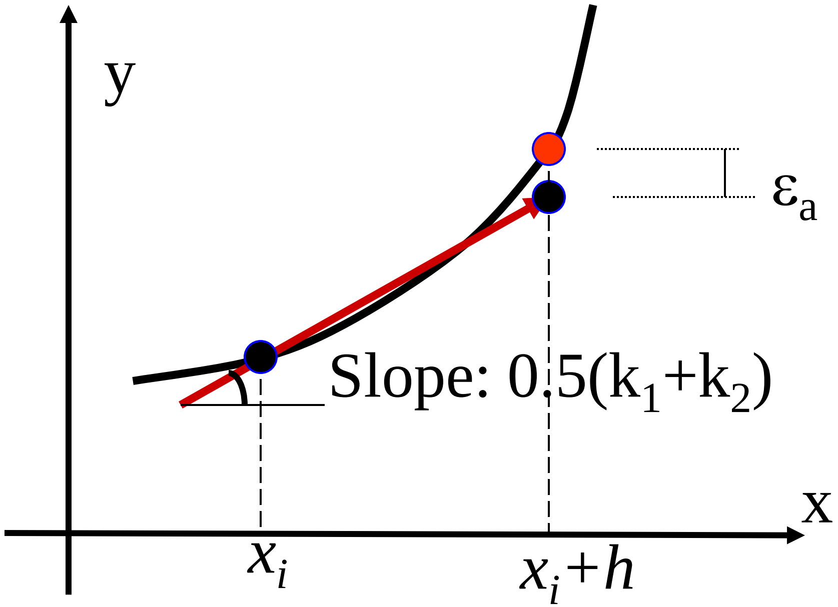
Involves the determination of two derivatives for the interval at the initial point and the end point.



$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)\cdot h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\big((x_i + h), (y_i + k_1 h)\big)$$



Runge-Kutta Methods

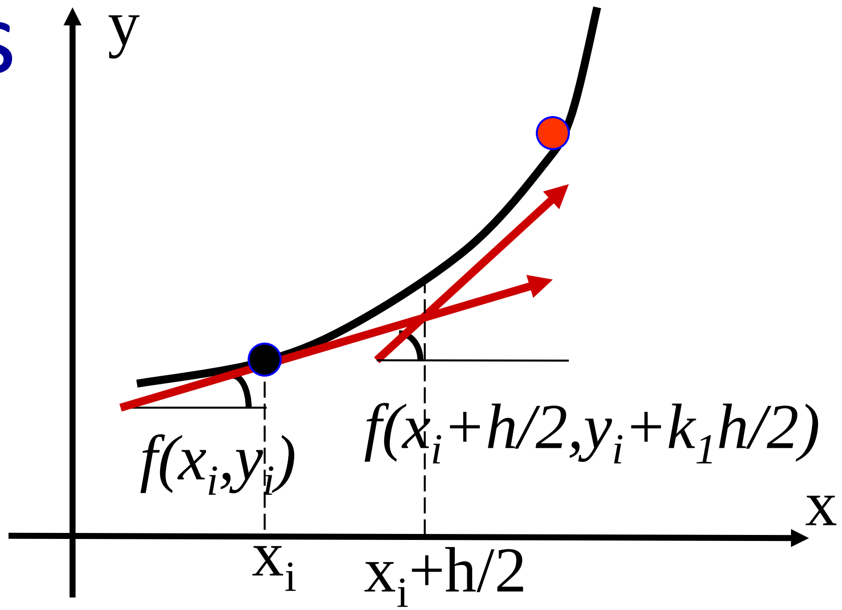
- **Midpoint Method:**

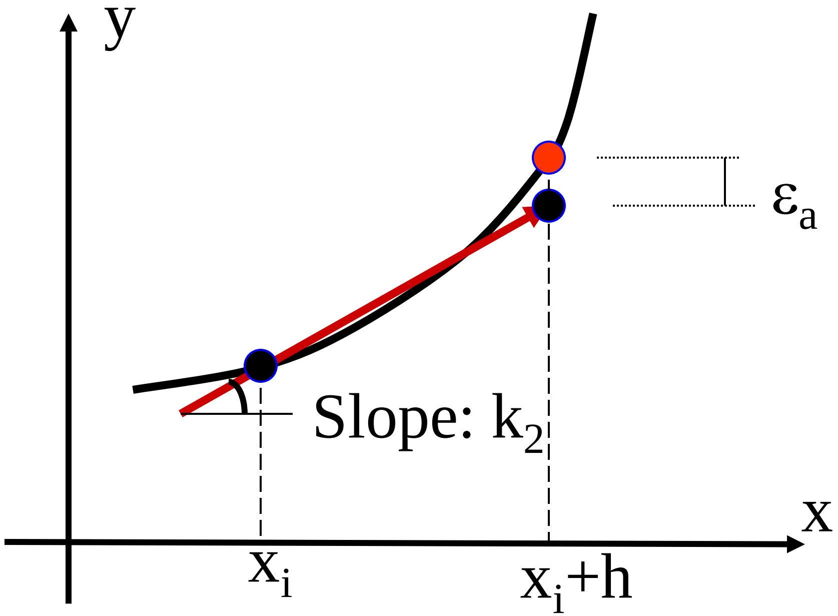
Uses Euler's method to predict a value of *y* at the midpoint of the interval:

$$\boxed{y_{i+1} = y_i + k_2 \cdot h}$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$





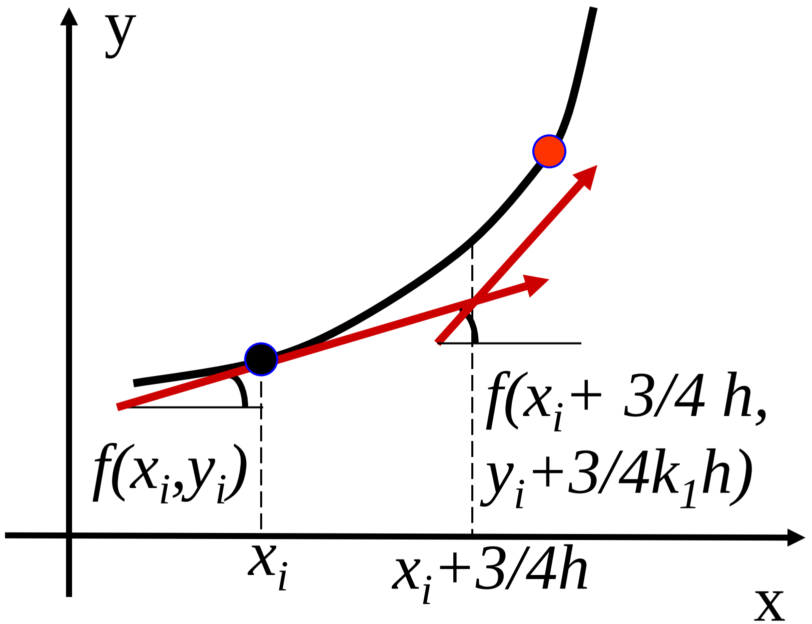
Runge-Kutta Methods

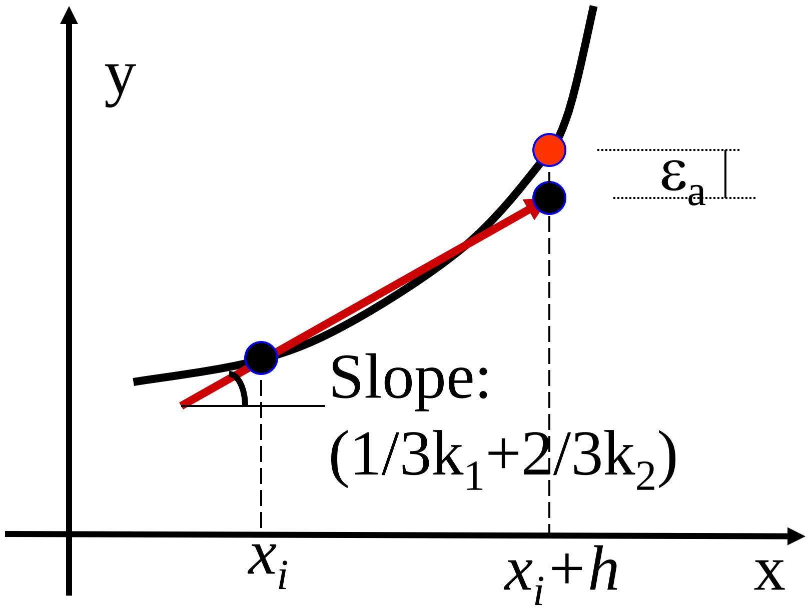
- **Ralston's Method:**

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right) \cdot h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$





EXAMPLE 25.6 Comparison of Various Second-Order RK Schemes

Problem Statement. Use the midpoint method [Eq. (25.37)] and Ralston's method [Eq. (25.38)] to numerically integrate Eq. (PT7.13)

$$f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

from $x = 0$ to $x = 4$ using a step size of 0.5. The initial condition at $x = 0$ is $y = 1$. Compare the results with the values obtained using another second-order RK algorithm, that is, the Heun method

Solution. The first step in the midpoint method is to use Eq. (25.37*a*) to compute

$$k_1 = -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$

However, because the ODE is a function of x only, this result has no bearing on the second step—the use of Eq. (25.37*b*) to compute

$$k_2 = -2(0.25)^3 + 12(0.25)^2 - 20(0.25) + 8.5 = 4.21875$$

Notice that this estimate of the slope is much closer to the average value for the interval (4.4375) than the slope at the beginning of the interval (8.5) that would have been used for Euler's approach. The slope at the midpoint can then be substituted into Eq. (25.37) to predict

$$y(0.5) = 1 + 4.21875(0.5) = 3.109375 \qquad \varepsilon_t = 3.4\%$$

The computation is repeated, and the results are summarized in Fig. 25.14 and Table 25.3.

For Ralston's method, k_1 for the first interval also equals 8.5 and [Eq. (25.38*b*)]

$$k_2 = -2(0.375)^3 + 12(0.375)^2 - 20(0.375) + 8.5 = 2.58203125$$

The average slope is computed by

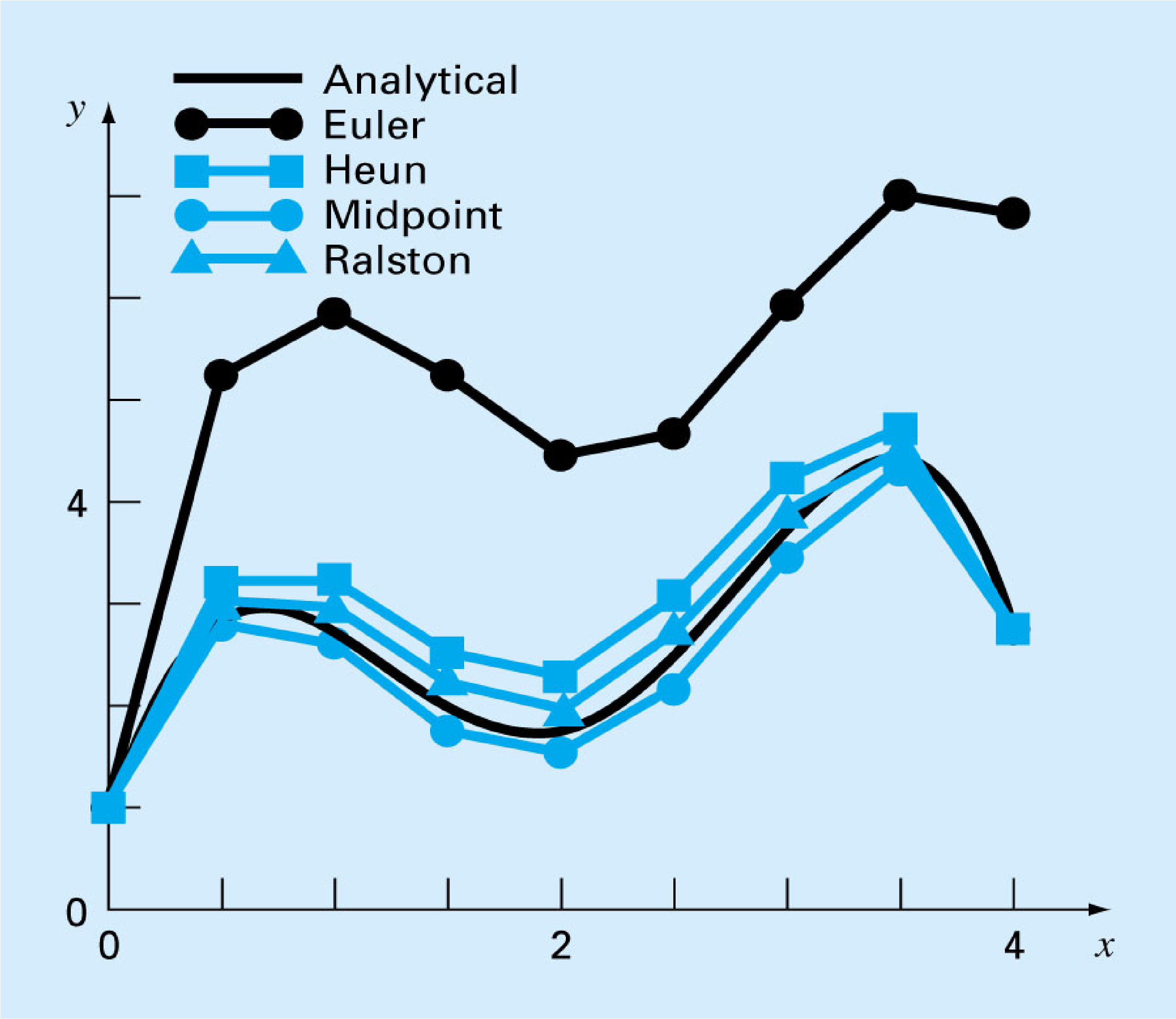
$$\phi = \frac{1}{3}(8.5) + \frac{2}{3}(2.58203125) = 4.5546875$$

which can be used to predict

$$y(0.5) = 1 + 4.5546875(0.5) = 3.27734375 \qquad \varepsilon_t = -1.82\%$$

The computation is repeated, and the results are summarized in Fig. 25.14 and Table 25.3. Notice how all the second-order RK methods are superior to Euler's method.

		Heun		Midpoint		Second-Order Ralston RK	
x	**y_{true}**	**y**	**$\lvert\varepsilon_t\rvert$ (%)**	**y**	**$\lvert\varepsilon_t\rvert$ (%)**	**y**	**$\lvert\varepsilon_t\rvert$ (%)**
0.0	1.00000	1.00000	0	1.00000	0	1.00000	0
0.5	3.21875	3.43750	6.8	3.109375	3.4	3.277344	1.8
1.0	3.00000	3.37500	12.5	2.81250	6.3	3.101563	3.4
1.5	2.21875	2.68750	21.1	1.984375	10.6	2.347656	5.8
2.0	2.00000	2.50000	25.0	1.75	12.5	2.140625	7.0
2.5	2.71875	3.18750	17.2	2.484375	8.6	2.855469	5.0
3.0	4.00000	4.37500	9.4	3.81250	4.7	4.117188	2.9
3.5	4.71875	4.93750	4.6	4.609375	2.3	4.800781	1.7
4.0	3.00000	3.00000	0	3	0	3.031250	1.0



Analytical
Euler
Heun
Midpoint
Ralston
y
4
0
0
2
4
x

Runge-Kutta Methods

3. Third order RK methods

$$\boxed{y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)h}$$

$$where$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$$

Runge-Kutta Methods

4. Fourth order RK methods

$$\boxed{y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h}$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h)$$

$$k_3 = f(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_2 h)$$

$$k_3 = f(x_i + h,\ y_i + k_3 h)$$

Comparison of Runge-Kutta Methods

Use first to fourth order RK methods to solve the equation from x = 0 to x = 4

Initial condition y(0) = 2, exact answer of y(4) = 75.33896

$$f(x, y) = 4e^{0.8x} - 0.5y$$