A Probability - Refresher Training I

Exercise A.1 (easy)
In a communication channel, a zero or a one is transmitted. The probability that a zero is transmitted is 0.1. Due to the noise in the channel, a zero can be received as one with probability 0.01 and a one can be received as a zero with probability 0.05. If you receive a zero, what is the probability p_0 that a zero was transmitted ? If you receive a one, what is the probability p_1 that a one was transmitted ?

Sketch of the solution:
Let X be the transmitted symbol and Y be the received symbol. We know that $\Pr(X = 0) = 0.1$, so $\Pr(X = 1) = 0.9$. We also know that $\Pr(Y = 1|X = 0) = 0.01$ and $\Pr(Y = 0|X = 1) = 0.05$. It follows that $\Pr(Y = 0|X = 0) = 0.99$ and $\Pr(Y = 1|X = 1) = 0.95$ Using the Bayes rule, we get that

$$\begin{aligned} p_0 &= \Pr(X = 0|Y = 0) = \frac{\Pr(Y = 0|X = 0)\Pr(X = 0)}{\Pr(Y = 0)} \\ &= \frac{\Pr(Y = 0|X = 0)\Pr(X = 0)}{\Pr(Y = 0|X = 0)\Pr(X = 0) + \Pr(Y = 0|X = 1)\Pr(X = 1)} = 0.6875. \end{aligned}$$

With the same approach, we get that

$$p_1 = \Pr(X = 1|Y = 1) = 0.9988.$$

Exercise A.2 (easy)
Four suppliers provide 10%, 20%, 30% and 40% of the bolts sold by a hardware shop and the rate of defects in their products are 1%, 1.5%, 2% and 3% respectively. Calculate the probability p of a given defective bolt coming from supplier 1.

Sketch of the solution:
Let X be the random variable indicating the presence of default in the bolt ($X = 0$: no default, $X = 1$: defective bolt). Let S be the random variable indicating the supplier : $S = k$ for supplier k with $k = 1, 2, 3, 4$.

We know that $\Pr(S = 1) = 0.1$, $\Pr(S = 2) = 0.2$, $\Pr(S = 3) = 0.3$ and $\Pr(S = 4) = 0.4$. We also know that $\Pr(X = 1|S = 1) = 0.01$, $\Pr(X = 1|S = 2) = 0.015$, $\Pr(X = 1|S = 3) = 0.02$ and $\Pr(X = 1|S = 4) = 0.03$.

Using the Bayes rule, we get that

$$\begin{aligned} p &= \Pr(S = 1|X = 1) = \frac{\Pr(X = 1|S = 1)\Pr(S = 1)}{\Pr(X = 1)} \\ &= \frac{\Pr(X = 1|S = 1)\Pr(S = 1)}{\sum_{k=1}^{4}\Pr(X = 1|S = k)\Pr(S = k)} = 0.0455. \end{aligned}$$

Exercise A.3 (easy)
The two events A and B have $\Pr(A) = 1/2$, $\Pr(B) = 1/3$, $\Pr(A \cup B) = 2/3$. Are the events A and B independent ? Are they mutually exclusive ?

Sketch of the solution:
Remember that

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Hence, $\Pr(A \cap B) = 1/2 + 1/3 - 2/3 = 1/6 = 1/2 \times 1/3$: the events are independent. They are not mutually exlusive since $A \cap B \neq \emptyset$.

Exercise A.4 (easy)
What is the probability P of having at least six heads when tossing a coin ten times ?

Sketch of the solution:
Counting the heads when tossing a coin is equivalent to observe a random variable X which follows a binomial distribution with parameter $p = 1/2$ (probability of the head) and $n = 10$. The result is

$$P = \Pr(X \geq 6) = 1 - \Pr(X \leq 5) = 1 - \sum_{i=0}^{5} \binom{n}{i} p^i (1-p)^{n-i}.$$

Exercise A.5 (easy)
A die is thrown six times. What is the probability of having at two 4s, two 5s and two 6s ?
Hints : think about the multinomial distribution.

Sketch of the solution:
The analyzed result (two 4s, two 5s and two 6s) is a realization of a multinomial distribution with $n = 6$ trials and the event probabilities $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = 1/6$. Just look for the multinomial distribution in a book or on the web.

Exercise A.6 (easy)
Two boxes containing marbles are placed on a table. The boxes are labeled B_1 and B_2. Box B_1 contains 7 green marbles and 4 white marbles. Box B_2 contains 3 green marbles and 10 yellow marbles. The boxes are arranged so that the probability of selecting box B_1 is $1/3$ and the probability of selecting box B_2 is $2/3$. Kathy is blindfolded and asked to select a marble. She will win a color TV if she selects a green marble.

1. What is the probability p that Kathy will win the TV (that is, she will select a green marble) ?
2. If Kathy wins the color TV, what is the probability q that the green marble was selected from the first box ?

Sketch of the solution:
Let X be the random variable indicating the selection of a green marble ($X = 0$: not a green marble, $X = 1$: a green marble). Let B be the random variable indicating the selected box : $B = k$ for box B_k with $k = 1, 2$.

We know that $\Pr(B = 1) = 1/3$ and $\Pr(B = 2) = 2/3$. We also know that $\Pr(X = 1|B = 1) = 7/11$ and $\Pr(X = 1|B = 2) = 3/13$.

We get that

$$\begin{aligned} p &= \Pr(X = 1) = \Pr(X = 1|B = 1)\Pr(B = 1) + \Pr(X = 1|B = 2)\Pr(B = 2) \\ &= 0.3660. \end{aligned}$$

Using the Bayes rule, we get that

$$q = \Pr(B = 1|X = 1) = \frac{\Pr(X = 1|B = 1)\Pr(B = 1)}{\Pr(X = 1)} = 0.5796.$$

Exercise A.7 (easy)
Let X a random variable following the uniform distribution over the interval (a, b).

1. Recall the probability density function $f_X(x)$ of X.
2. Calculate the cumulative distribution function $F_X(x)$ of X.
3. Calculate $\mathbb{E}[X]$.
4. Calculate var$[X]$.

Sketch of the solution:
Just look for the uniform distribution in a book or on the web.

Exercise A.8 (easy)
Let the random variable X have the density function

$$f(x) = \begin{cases} k\,x & \text{for } 0 \le x \le \sqrt{\frac{2}{k}}, \\ 0 & \text{elsewhere,} \end{cases}$$

where k is a positive real value. If the mode is at $x = \frac{\sqrt{2}}{4}$, then what is the median of X ?

Sketch of the solution:
The mode is given by $x^* = \sqrt{\frac{2}{k}}$ which corresponds to the maximum of $f(x) : f(x^*) \ge f(x)$ for all $x \in \mathbb{R}$. Hence, $k = 16$. The median m satisfy

$$\int_{-\infty}^{m} f(x)dx = \frac{1}{2}.$$

Hence, $m = \frac{1}{\sqrt{k}} = \frac{1}{4}$ (the other solution $m = -\frac{1}{\sqrt{k}} < 0$ is not acceptable).

Exercise A.9 (easy)
What is the probability density function of the random variable whose cumulative distribution function is

$$F(x) = \frac{1}{1 + e^{-x}}, \quad -\infty < x < \infty?$$

Sketch of the solution:
We get

$$f(x) = F'(x) = \frac{e^{-x}}{(1 + e^{-x})^2}.$$

Exercise A.10 (easy)
The length of time required by students to complete a 1-hour exam is a random variable with a pdf given by

$$f(x) = \begin{cases} c\,x^2 + x & \text{if } 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

What the probability a student finishes in less than a half hour ?

Sketch of the solution:
We get

$$\int_0^1 f(x)dx = \frac{c}{3} + \frac{1}{2} = 1.$$

Hence, $c = \frac{3}{2}$. The probability is

$$\int_0^{0.5} f(x)dx = 0.1875.$$

Exercise A.11 (easy)
Show that the variance of an exponential random variable with parameter λ is $\frac{1}{\lambda^2}$.

Sketch of the solution:
Let X be the exponential random variable. We get

$$\mathbb{E}[X] = \int_0^{+\infty} x\lambda e^{-\lambda x} dx = \frac{1}{\lambda}.$$

and

$$\mathbb{E}[X^2] = \int_0^{+\infty} x^2\lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}.$$

It follows that

$$\text{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{1}{\lambda^2}.$$

Exercise A.12 (easy)
Let X be a random variable with the cumulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x} & \text{for } x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

What is $\Pr(0 \leq e^X \leq 4)$?

Sketch of the solution:
We get

$$\Pr(0 \leq e^X \leq 4) = \Pr(X \leq \ln 4) = 1 - e^{-\ln 4} = \frac{3}{4}.$$

Exercise A.13 (easy)
Let X be a random variable with the probability density function

$$f(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

Calculate $\mathbb{E}[X]$ and var$[X]$.

Sketch of the solution:
We get

$$\mathbb{E}[X] = \int_{-1}^{1} x(1-|x|)dx = \int_{-1}^{0} x(1+x)dx + \int_{0}^{1} x(1-x)dx = 0.$$

Alternatively, we can note that $x \mapsto x(1-|x|)$ is an odd function. We can calculate $\mathbb{E}[X^2]$ in the same manner and we deduce var$[X]$.

Exercise A.14 (medium)
Let $W = e^X$ where $X \sim \mathcal{N}(\mu, \sigma^2)$. What is the probability density function of W for $w \in (0, +\infty)$?

Sketch of the solution:
For $w > 0$, we get

$$F_W(w) = \Pr(W \leq w) = \Pr(e^X \leq w) = \Pr(X \leq \ln w) = F_X(\ln w),$$

where $F_X(x)$ is the cdf of X. Let $f_X(x)$ be the pdf of X. Then,

$$f_W(w) = \frac{dF_W(w)}{dw} = \frac{1}{w} f_X(\ln w) = \frac{1}{w\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln w - \mu)^2}{2\sigma^2}}.$$

Exercise A.15 (medium)
If the random variable X is normal with mean 1 and standard deviation 2, then what is

$$\Pr(X^2 - 2X \leq 8)?$$

Sketch of the solution:
We get

$$\begin{aligned}\Pr(X^2 - 2X \leq 8) &= \Pr(X^2 - 2X - 8 \leq 0) = \Pr((X-4)(X+2) \leq 0)\\ &= \Pr(-2 \leq X \leq 4) = F_X(4) - F_X(-2),\end{aligned}$$

where $F_X(\cdot)$ is the cdf of the the normal distribution with mean 1 and standard deviation 2. It follows that

$$\Pr(X^2 - 2X \leq 8) = \Phi\left(\frac{3}{2}\right) - \Phi\left(\frac{-3}{2}\right) = 0.8664,$$

where $\Phi(\cdot)$ is the cdf of the the normal distribution with mean 0 and standard deviation 1.

B Random Variables - Refresher Training II

Exercise B.1 (medium)
Let X an univariate normal random variable with mean 0 and variance σ^2. Let $Y = X^2$.

1. Calculate the cumulative distribution function $F_Y(y)$ of Y.
2. Calculate the probability density function $f_Y(y)$ of Y.
3. Calculate $\mathbb{E}[Y]$.

Sketch of the solution:
For $y \geq 0$, we get

$$F_Y(y) = \Pr(Y \leq y) = \Pr(X^2 \leq y) = \Pr(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

where $F_X(\cdot)$ is the cdf of the the variable X. It follows that

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(\sqrt{y})}{dy} - \frac{dF_X(-\sqrt{y})}{dy} = \frac{f_X(\sqrt{y})}{2\sqrt{y}} + \frac{f_X(\sqrt{y})}{2\sqrt{y}} = \frac{f_X(\sqrt{y})}{\sqrt{y}},$$

where $f_X(y)$ is the pdf of the univariate normal variable with mean 0 and variance σ^2. Finally, we get

$$\mathbb{E}[Y] = \int_0^{+\infty} y f_Y(y) dy = \sigma^2 = \mathbb{E}[X^2] = \text{var}[X].$$

Exercise B.2 (easy)
Let us suppose that X is a continuous random variable with pdf $f_X(x)$. Let $Y = aX + b$ where $a \neq 0$ and b are two real numbers. Show that

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

Sketch of the solution:
Suppose $a > 0$, we get

$$F_Y(y) = \Pr(Y \leq y) = \Pr(aX + b \leq y) = \Pr\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right),$$

where $F_X(\cdot)$ is the cdf of the the variable X. It follows that

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{F_X\left(\frac{y-b}{a}\right)}{dy} = \frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

Suppose $a < 0$, we get

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{1 - F_X\left(\frac{y-b}{a}\right)}{dy} = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right).$$

So, we can summarize both cases with

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

Exercise B.3 (easy)
Let X be normally distributed with mean μ and standard deviation σ. Determine $\Pr(|X-\mu| \geq 2\sigma)$. Compare with Chebyshev's inequality.

Sketch of the solution:
We get

$$\begin{aligned}\Pr(|X-\mu| \geq 2\sigma) &= \Pr(X-\mu \geq 2\sigma) + \Pr(X-\mu \leq -2\sigma) \\ &= 1 - \Phi(2) + \Phi(-2) = 2\Phi(-2) = 0.0455.\end{aligned}$$

The Chebyshev's inequality gives

$$\Pr(|X-\mu| \geq 2\sigma) \leq \frac{\sigma^2}{4\sigma^2} = \frac{1}{4} = 0.25.$$

The exact calculation is more acccurate than the Chebyshev's bound which is too large.

Exercise B.4 (medium)
Let X and Y be any two random variables and let a and b be any two real numbers. Then show that

$$\text{var}[aX+bY] = a^2\text{var}[X] + b^2\text{var}[Y] + 2ab\,\text{cov}(X,Y).$$

Sketch of the solution:
You can use the equality $\text{var}[aX+bY] = \text{cov}(aX+bY, aX+bY)$ and develop the covariance.

Exercise B.5 (difficult)
If the random variable $X \sim Exp(\lambda)$ with $\lambda > 0$, then what is the probability density function of the random variable $Y = X\sqrt{X}$?

Sketch of the solution:
Following the approach in Exercise B.2, we get

$$f_Y(y) = \begin{cases} \frac{2\lambda}{3y^{\frac{1}{3}}} e^{-\lambda y^{\frac{2}{3}}} & \text{for } y > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Exercise B.6 (easy)
Let X and Y be discrete random variables with joint probability density function

$$f(x,y) = \begin{cases} \frac{1}{21}(x+y) & \text{for } x = 1,2,3; y = 1,2, \\ 0 & \text{otherwise.} \end{cases}$$

What are the marginals of X and Y?

Sketch of the solution:
The marginal of X is $\Pr(X=1) = f(1,1) + f(1,2) = \frac{5}{21}$, $\Pr(X=2) = f(2,1) + f(2,2) = \frac{7}{21}$ and $\Pr(X=3) = f(3,1) + f(3,2) = \frac{9}{21}$.

The marginal of Y is $\Pr(Y=1) = f(1,1) + f(2,1) + f(3,1) = \frac{9}{21}$ and $\Pr(Y=2) = f(1,2) + f(2,2) + f(3,2) = \frac{12}{21}$.

Exercise B.7 (difficult)
Let X and Y have the joint probability density function

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{for } 0 \le x, y < +\infty, \\ 0 & \text{otherwise.} \end{cases}$$

What is $\Pr(X \ge Y \ge 2)$?

Sketch of the solution:
By definition, we get

$$\begin{aligned} \Pr(X \ge Y \ge 2) &= \int_{x=2}^{+\infty} \int_{y=2}^{x} f(x,y) dx dy = \int_{x=2}^{+\infty} e^{-x} \left(\int_{y=2}^{x} e^{-y} dy \right) dx \\ &= \int_{x=2}^{+\infty} e^{-x} \left(e^{-2} - e^{-x} \right) dx = \int_{x=2}^{+\infty} e^{-2} e^{-x} - e^{-2x} dx \\ &= \frac{1}{2} e^{-4}. \end{aligned}$$

Exercise B.8 (medium)
Let X and Y be two independent random variables having the joint pdf $f(x,y) = g(x)h(y)$. Then show that

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

Show that

$$\text{var}[XY] = \mathbb{E}[X]^2 \text{var}[Y] + \mathbb{E}[Y]^2 \text{var}[Y] + \text{var}[X]\text{var}[Y].$$

Sketch of the solution:
We get

$$\begin{aligned} \mathbb{E}[XY] &= \int_{(x,y)\in\mathbb{R}^2} xy f(x,y) dx dy = \int_{\mathbb{R}^2} x g(x) y h(y) dx dy \\ &= \left(\int_{\mathbb{R}} x g(x) dx \right) \left(\int_{\mathbb{R}} y g(y) dy \right) = \mathbb{E}[X]\mathbb{E}[Y]. \end{aligned}$$

The second proof is obtained by developing the equality

$$\text{var}[XY] = \mathbb{E}[(XY - \mathbb{E}[XY])^2].$$

and using the well-known equalities $\text{var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ and $\text{var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$.

Exercise B.9 (easy)
Suppose the random variables X and Y are independent and identically distributed. Let $Z = aX + Y$. If the correlation coefficient between X and Z is $\frac{1}{3}$, then what is the value of the constant a ?

Sketch of the solution:
We get

$$\text{cov}(X,Z) = a\text{cov}(X,X) + \text{cov}(X,Y) = a\,\text{var}[X].$$

Futhermore,

$$\text{var}[Z] = a^2 \text{var}[X] + \text{var}[Y] = (a^2+1)\text{var}[X].$$

Hence,

$$\mathrm{corr}(X,Z) = \frac{a}{\sqrt{a^2+1}} = \frac{1}{3}.$$

It follows that $a = \frac{1}{\sqrt{8}}$.

Exercise B.10 (difficult)
Let X and Y be two independent random variables. If $X \sim BIN(n,p)$ and $Y \sim BIN(m,p)$, then what is the distribution of $X+Y$?

Sketch of the solution:

Remember that

$$\Pr(X=x) = \binom{n}{x} p^x (1-p)^{n-x},\ \Pr(Y=y) = \binom{m}{y} p^y (1-p)^{m-y}.$$

It follows that

$$\Pr(Z=z) = \sum_{\substack{0\le x\le n\\ 0\le y\le m\\ x+y=z}} \Pr(X=x \text{ and } Y=z-x) = p^z(1-p)^{n+m-z} \sum_{\substack{0\le x\le n\\ 0\le y\le m\\ x+y=z}} \binom{n}{x}\binom{m}{z-x}.$$

We can show that

$$\sum_{\substack{0\le x\le n\\ 0\le y\le m\\ x+y=z}} \binom{n}{x}\binom{m}{z-x} = \binom{n+m}{z}. \tag{3}$$

In fact, for $u \in \mathbb{R}$, we get

$$(1+u)^n(1+u)^m = (1+u)^{n+m}.$$

We know that

$$(1+u)^n = \sum_{k=0}^{n} \binom{n}{k} u^k 1^{n-k}.$$

It follows that

$$(1+u)^n(1+u)^m = \sum_{x=0}^{n}\sum_{y=0}^{m} \binom{n}{x}\binom{m}{y} u^{x+y} = \sum_{k=0}^{n+m} \sum_{\substack{0\le x\le n\\ 0\le y\le m\\ x+y=k}} \binom{n}{x}\binom{m}{y} u^k. \tag{4}$$

Note that

$$\sum_{\substack{0\le x\le n\\ 0\le y\le m\\ x+y=k}} \binom{n}{x}\binom{m}{y} = \sum_{\substack{0\le x\le n\\ 0\le y\le m\\ x+y=k}} \binom{n}{x}\binom{m}{k-x}.$$

Let z between 0 and $n+m$. Identifying the coefficient of u^z in (4) with the coefficient of u^z in

$$(1+u)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} u^k$$

leads to (3).
Finally, we can conclude that the distribution of $X+Y$ is the binomial distribution $BIN(n+m,p)$.

Exercise B.11 (medium)
Let X and Y be two independent random variables with identical probability density function given by

$$f(x) = \begin{cases} e^{-x} & \text{for } x > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

What is the probability density function $g(w)$ of $W = \min\{X, Y\}$?

Sketch of the solution:
Let $G(w)$ be the cdf of W. A short calculation shows that $G(w) = 1 - e^{-2w}$ for $w > 0$. The derivation of $G(w)$ yields

$$g(w) = \begin{cases} 2e^{-2w} & \text{for } w > 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Exercise B.12 (easy)
Let X_1, X_2, $\ldots$, X_n be a random sample of size n from a Bernoulli distribution with probability of success $p = \frac{1}{2}$. What is the limiting distribution of the sample mean $\bar{X}$?

Sketch of the solution:
Using the Central limit theorem, the limiting distribution of $Z = \frac{\bar{X}-p}{\sqrt{\frac{1}{n}p(1-p)}}$ is $\mathcal{N}(0, 1)$. Hence, when n is sufficiently large, $\bar{X}$ follows approximately an univariate normal distribution with mean p and variance $\frac{1}{n}p(1-p)$.