Digital Signal Processing

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Part I

Inverse Z-transform (continued)

- Inverse z-transform is used to recover a sequence $x[n]$ from its z-transform $X(z)$
- It is essential for many z-transform related tasks, such as difference equation solving, analytic computation of the convolution etc.
- There are three possible approaches:
  1. Partial fraction expansion
  2. Conversion to power series (polynomial long division)
  3. (Contour integration via Cauchy's integral theorem)

Partial fraction expansion (repetition)

- Utilized for z-transforms given in the form of rational function

$$X(z) = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})} \tag{1}$$

- If $p > q$ and all poles are simple ($\alpha_i \neq \alpha_k$for $i \neq k$)

$$X(z) = \sum_{k=1}^{p}\frac{A_k}{1-\alpha_k z^{-1}} \tag{2}$$

- $A_k \in \mathcal{R}$ are constants computed via

$$A_k = [(1-\alpha_k z^{-1})X(z)]_{z=\alpha_k} \tag{3}$$

- If $p > q$, then long polynomial division of numerator and denominator is performed
- DETAILS: How to proceed, when poles are of a higher order?
- EXAMPLE: Inverse z-transform of a rational $X(z)$
- MATLAB: `residuez(B,A)`

Polynomial long division

- Z-transform is defined by a power series

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z^{1} + x[0] + x[1]z^{-1} + \ldots \tag{4}$$

- If the series is *finite*, the samples of $x[n]$ are simply selected
- If the series is *infinite*, it is usually given by a rational function

$$X(z) = \frac{\sum_{k=0}^{q} b[k]z^{-k}}{1 + \sum_{k=1}^{p} a[k]z^{-k}} \tag{5}$$

- By polynomial division of numerator and denominator, the samples of $x[n]$ are obtained
- Suitable for computer-based computation
- EXAMPLE: Inverse z-transform using polynomial division

Solving of difference equations with initial conditions

- Solving of difference equations in time-domain requires experience (we skipped it due to this in our lectures)
- DTFT introduces a rather simple algorithm for this purpose, but requires zero initial conditions
- **Unilateral z-transform:** introduces similar procedure as DTFT, generalized for arbitrary initial conditions
- Unilateral z-transform is a variant of z-transform defined for right-sided sequences

$$X_1(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots \quad (6)$$

- **Shift theorem:** differs from its bi-lateral variant and is given by

$$x[n-1] \overset{Z}{\Longleftrightarrow} z^{-1}X_1(z) + x[-1] \quad (7)$$

- EXAMPLE: Solving of difference equation with non-zero initial conditions using unilateral z-transform

Part II

Transform analysis of LTI systems

Transfer/system function (repetition)

- **Transfer function** $H(z)$ is a z-transform of the impulse response $h[n]$:

$$y[n] = h[n] * x[n] \xrightarrow{Z} Y(z) = H(z)X(z) \tag{8}$$

- Because impulse response is a unique description of an LTI system, so is the transfer function
- For a general LTI system (IIR) given by a difference equation

$$y[n] + \sum_{k=1}^{p} a[k]y[n-k] = \sum_{k=0}^{q} b[k]x[n-k] \tag{9}$$

the transfer function is a rational function of the form

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b[k]z^{-k}}{1 + \sum_{k=1}^{p} a[k]z^{-k}} = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}, \tag{10}$$

where roots of the numerator are *zeros* (β_k) a roots of the denominator are *poles* (α_k).

Tranfer function: causality, stability

- **Stability:** LTI system is stable if ROC of its transfer function contains unit circle
- DETAILS: Why is it so?

- **Causality:** Impulse response of a causal system is a right-sided sequence, ROC of the corresponding transfer function is thus of the form $|z| > \alpha$. The poles of the transfer function cannot lie within the ROC
- Consequently, all poles of a causal system must lie inside or on a circle $|z| \leq \alpha$

- **Realizable system:** is both *stable* and *causal*
- Transfer function has ROC of the form $|z| > \alpha, \quad 0 \leq \alpha < 1$, poles then *must* lie inside a unit circle

Inverse system

- If an LTI system has transfer function $H(z)$, then its **inverse system** $G(z)$ is given by

$$G(z) = \frac{1}{H(z)} \tag{11}$$

- ROC of the inverse system $G(z)$ must have an overlap with the ROC of $H(z)$
- EXAMPLE: Inverse system computation and its properties

- Lets have an LTI system with rational transfer function

$$H(z) = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}, \tag{12}$$

- Let us assume only first-order poles. If $p > q$, then $H(z)$ can be expanded into

$$H(z) = \sum_{k=1}^{p} \frac{A_k}{1-\alpha_k z^{-1}}. \tag{13}$$

If the system is causal, then the *impulse response* is of the form

$$h[n] = \sum_{k=1}^{p} A_k(\alpha_k)^n u[n] \tag{14}$$

- EXAMPLE: Transfer function and impulse response

- If $p \leq q$, then $H(z)$ is of the form

$$H(z) = \sum_{k=0}^{q-p} B_k z^{-k} + \sum_{k=1}^{p} \frac{A_k}{1 - \alpha_k z^{-1}}, \tag{15}$$

and (if the system is causal) the *impulse response* is of the form

$$h[n] = \sum_{k=0}^{q-p} B_k \delta[n-k] + \sum_{k=1}^{p} A_k (\alpha_k)^n u[n]. \tag{16}$$

- If $H(z)$ has only zeros

$$H(z) = \prod_{k=1}^{q} (1 - \beta_k z^{-1}), \tag{17}$$

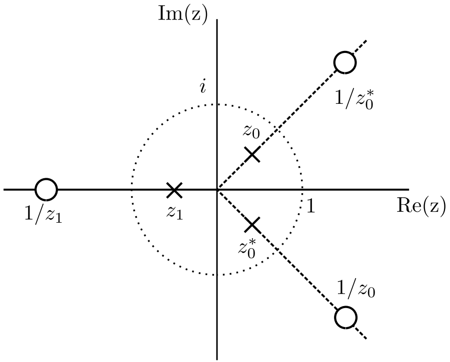
the impulse response is of a finite length and is given by

$$h[n] = \sum_{k=0}^{q} B_k \delta[n-k]. \tag{18}$$

Allpass filters I

- **Allpass filter** is a system with constant magnitude response
$$|H(e^{j\omega})| = 1. \tag{19}$$
- Allpass filters can be used for *equalization of a group delay*. This is a compensation of phase nonlinearities of other filters, such that the magnitude response of the original filter is not changed.
- The constraint placed on the magnitude response determines the location of *zeros and poles* in the z-plane. These occur in complex conjugate pairs of the form
$$H(z) = \prod_{k=1}^{N} \frac{z^{-1} - a_k^*}{1 - a_k z^{-1}}. \tag{20}$$
- Stable/causal allpass filter has *non-negative* group delay
- Stable/causal allpass filter has all poles inside and zeros outside the unit circle
- An inverse system to an allpass filter is also an allpass filter
- EXAMPLE: Equalization of the phase response of a filter

Allpass filters II



Constraints to a location of zeros and poles of a realizable allpass filter

Minimum phase filters I

- Let us have a system with a transfer function

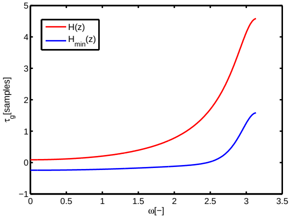
$$H(z) = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}. \tag{21}$$

- This system is realizable, if the poles α_k are located inside a unit circle
- The location of zeros in the z-plane may be arbitrary
- System $H_{min}(z)$ is said to have a *minimum phase*, if it has a realizable inverse system (i.e., all its *zeros* lie inside the unit circle)
- Any realizable system, which does not have zeros on the unit circle, can be transformed into a system with minimum phase
- This transformation is useful when:
  1. the *existence of an inverse filter* must be ensured
  2. the system $H(z)$ is required to have a *minimum group delay* τ_g (given a specific magnitude response)

Minimum phase filters II

- The transfer function $H(z)$ of any realizable system can be written in the form
$$H(z) = H_{min}(z) \cdot H_{apr}(z), \tag{22}$$
where $H_{min}(z)$ is a minimum phase system and $H_{apr}(z)$ is a realizable allpass filter.
- Due to this factorization, $H_{min}(z)$ exhibits a minimum group delay because it holds that
$$\tau(\omega) = \tau_{min}(\omega) + \tau_{apr}(\omega). \tag{23}$$

- To obtain $H_{min}(z)$, it is necessary to "suitably move" zeros located outside the unit circle into the unit circle
- This can be done using a cascade of $H(z)$ and an inverse filter to $H_{apr}(z)$ (it is a non-causal allpass filter with zeros inside and poles outside the unit circle)
- The poles of this non-causal allpass filter cancel with zeros of $H(z)$, its zeros (located inside the unit circle) remain unchanged



Thank you for attention!