Digital Signal Processing

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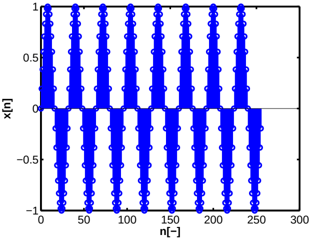
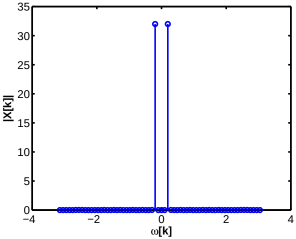
Part I

Practical spectral analysis, windowing

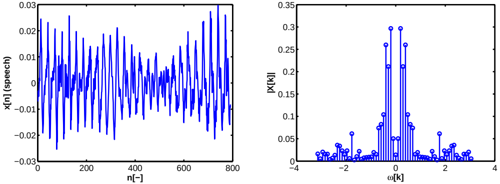
- Spectral analysis: search for a regular inner structure / periodicity in a general signal
- Complications: finite signal length, potential non-stationarity, noise..
- **Signals with harmonic structure** (energy focused into narrow bands, ideally a sparse spectrum)
- EXAMPLE: Signals originated by rotating machinery, musical signals, alternating current ...
- *Harmonic analysis* or frequency estimation, aims at "'accurate"' determination of several frequency components



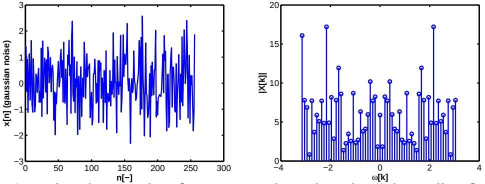


Practical spectral analysis II

- (Locally) **(quasi)-periodic signals** (they have "certain" harmonic structure, spectrum is not sparse, several bands with significant energy)
- EXAMPLE: Speech signal (vowel - a,e,i,o,u)



- **Non-periodic signals** (no harmonic structure, energy spread throughout the spectrum, wide-band signals)
- EXAMPLE: White noise, speech signal (fricatives - s,z)



- *Spectral analysis* - Analyzes the frequency bands, the "shape" of the spectrum, the distribution of energy with respect to frequency (e.g., computation of features for speech recognition, detection of formants etc.).

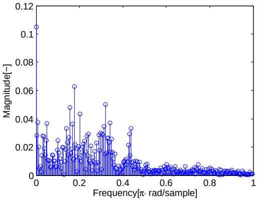
Practical spectral analysis III

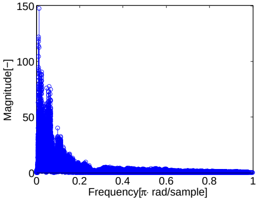
Factors complicating practical spectral analysis:

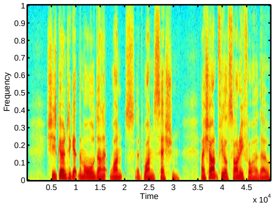
- Finite (short) signal length - prevents accurate detection of frequency components.
- (Potential) non-stationarity - spectrum of the signal evolves in time. Signal should not be analyzed as a whole over the changes. Instead, the signal is analyzed in short intervals, where it is approximately stationary - **short-time spectral analysis**.
- Presence of various unwanted noise components (quantization, sensor, environment noise).

Short-time spectral analysis

- Variant of spectral analysis for non-stationary signals (whose spectrum changes in time)
- In this case, the spectrum should not be computed using the whole signal:
  - The computed spectrum has extraordinary spectral resolution ($\Delta\omega = 2\pi/N$)...
  - ...but practically no time resolution
  - Computational burden is unnecessarily large
- More useful is a sequence of short DFTs, which provides a compromise between time and spectral resolution
- *Short Time Spectral Analysis + Windowing*
- Computation using *Short Time Fourier Transform - STFT*
- EXAMPLE: Analysis of music recording







(a) Short-time DFT spectrum ($N = 512$, $\Delta\omega \approx 3 \cdot 10^{-3}\pi$, $\Delta f \approx 40$Hz), (b) DFT spectrum ($N = 10^5$, $\Delta\omega = 2 \cdot 10^{-5}\pi$, $\Delta f = 0.2$Hz), (c) Spectrogram

- Time-resolution of spectrogram can be improved by *overlapping* of segments for DFT computation.

Spectral leakage - multiplication by rectangular window

- Let us have infinite signal $y[n]$, from which we select short data segment $x[n]$ such, that

$$x[n] = \begin{cases} y[n], & 0 \leq n \leq N-1 \\ 0, & \text{elsewhere} \end{cases} \quad (1)$$

- This operation corresponds to multiplication of $y[n]$ with rectangular window $w_r[n]$
- Multiplication in the time-domain corresponds to the convolution in the frequency domain
- The relationship between DTFT spectrum of long signal $y[n]$ and signal $x[n]$ weighted by the rectangular window is therefore

$$X(e^{j\omega}) = \frac{1}{2\pi}\{Y(e^{j\omega}) * W_r(e^{j\omega})\} \quad (2)$$

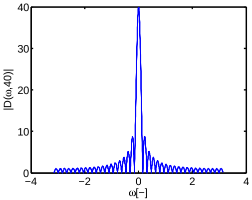
- Function $W_r(e^{j\omega})$ is DTFT$(w_r[n])$ given by

$$W_r(e^{j\omega}) = \frac{sin(0.5\omega N)}{sin(0.5\omega)} e^{-j0.5\omega(N-1)} = D(\omega, N)e^{-j0.5\omega(N-1)} \quad (3)$$

Multiplication by rectangular window II

Dirichlet kernel - Magnitude of DTFT($w_r[n]$) - $D(\omega, N)$

- Maximum value N occurs at frequency $\omega = 0$
- Closest zeros occur at frequencies $\pm 2\pi/N$
  Frequency interval between zeros is denoted as *main lobe*
- Another zeros occur at frequencies
  $\omega = 2m\pi/N, m = \pm 2, \pm 3, \ldots$
  Frequency intervals between these zeros are denotes as *side lobes*
- A lobe with largest magnitude occurs at frequency $\omega = \pm 3\pi/N$ and the ratio between its magnitude and the magnitude of the main lobe is -13.5dB.



Multiplication by rectangular window III

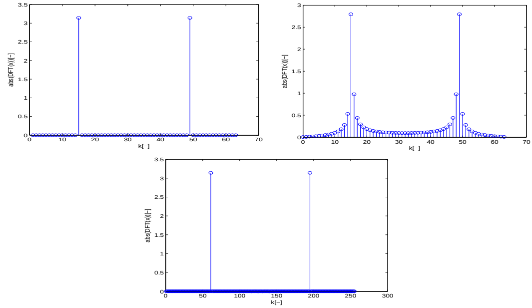
What is the difference between spectrum $X(e^{j\omega})$ of signal $x[n]$ (signal multiplied by rectangular window) from spectrum $Y(e^{j\omega})$ of the original signal $y[n]$?

- There appears **smearing of spectrum**
  - The loss of frequency resolution - if two frequency components in $Y(e^{j\omega})$ are distant less than $2\pi/N$ (width of the side-lobe), then they merge.
- There appears **masking of weak frequency components**
  - If there is one dominant component within the spectrum and some weak components, then the side-lobe of the dominant component masks the main-lobes of the weak ones.
  - This effect is the most significant, when the components differby and odd multiple of π/N

In other words: Selection of signal segments via rectangular window has undesirable side effects on the spectrum of the original signal and may significantly distort results of short-time spectral analysis in some cases.

Spectral leakage II

- Spectral leakage (smearing) occurs **always**, when the signal $y[n]$ is windowed and the (DTFT/DFT) spectrum is computed from shortened sequence $x[n]$
- EXAMPLE: How is it possible that if exactly one period of harmonic signal is selected then its DFT spectrum appears free of leakage?



$$x_1[n] = sin(\frac{2\pi 15}{64} n),\ N_1 = 64;\ x_2[n] = sin(\frac{2\pi 15{,}25}{64} n), N_1 = 64;$$
$$x_2[n] = sin(\frac{2\pi 15{,}25}{64} n),\ N_2 = 256;$$

Windowing

Undesired effects of windowing by rectangular window $w_r[n]$ can be partly mitigated via selection of a more suitable window $w[n]$

Windowing - $x[n] = y[n]w[n]$

The desired sequence $w[n]$ is not arbitrary, it must fulfill the following criteria:

- Sequence $w[n]$ has final duration
- Window length N_w is the same as the length of the segment to be analyzed
- Sequence $w[n]$ should be non-negative

Moreover, following properties are of importance in the frequency domain:

- The main-lobe should have minimal width
- Side-lobes should have minimal magnitude

DTFT sequence $w[n]$ denoted as $W(e^{j\omega})$ is called *kernel function*

Windowing II

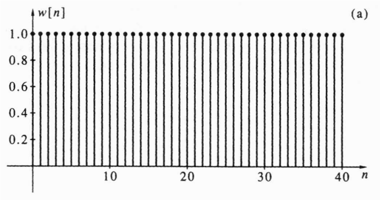
- **Ideal kernel function** $W(e^{j\omega})$**:** approaches $\delta(\omega)$, then convolution

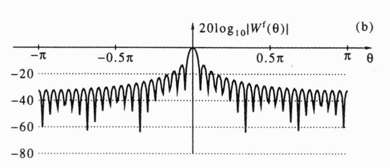
$$X(e^{j\omega}) = \frac{1}{2\pi}\{Y(e^{j\omega}) * W(e^{j\omega})\} \qquad (4)$$

  does not smear spectrum $Y(e^{j\omega})$ too significantly.
- Unfortunately, the window corresponding to kernel function $W(e^{j\omega}) = 2\pi\delta(\omega)$ is $w[n] = 1$, i.e., it is infinite (no windowing occurs).

- **Selection of suitable window:** is a compromise between ...
- ... narrow main-lobe ...
- ... and side-lobes with low magnitude
- The narrower the main-lobe, the higher the magnitude of the side-lobes

- **Rectangular window:** Has the narrowest *main-lobe* from all windows: $4\pi/N$
- *Magnitude of side-lobes* is however the largest: $-13.5dB$, which is highly impractical for spectral analysis - weak frequency components are masked





(a) Rectangular window $w_r[n]$, (b) DTFT spectrum $W_r(e^{j\omega})$ magnitude

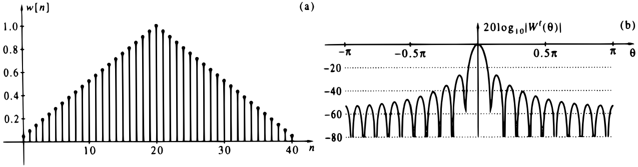
SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Bartlett/triangular window

- Derived from squaring of the kernel function $W_r(e^{j\omega})$
- Result in two-times lower side-lobes (in dB)
- Squaring $W_r(e^j\omega)$ in frequency-domain corresponds to $w_r[n] * w_r[n]$ in time-domain
  (length $w_r[n]$ is $(N+1)/2$ for $w_t[n]$ of length N)

$$w_t[n] = \frac{2}{N+1}\{w_r[n] * w_r[n]\} = 1 - \frac{|2n - N + 1|}{N+1} \qquad (5)$$

- *Main-lobe width*: $8\pi/(N+1)$
- *Side-lobe magnitude*: $-27dB$



(a) Triangular window $w_t[n]$, (b) DTFT spectrum $W_t(e^{j\omega})$ magnitude

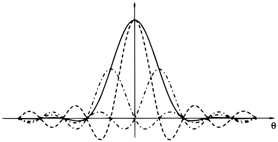
SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Hann window

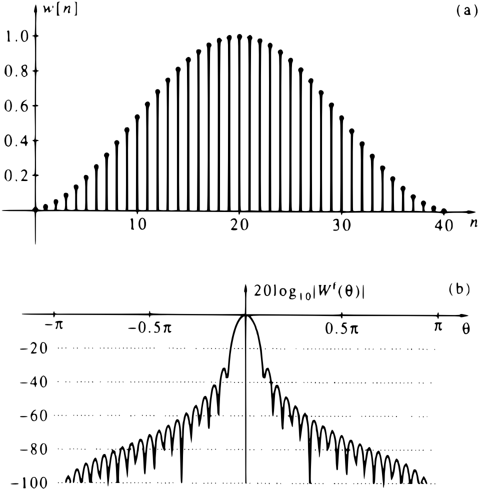
- Derived from superposition of three Dirichlet kenrels shifted in frequency ($\Delta\omega = \pm 2\pi/(N-1)$), which partly cancels its side-lobes
- Magnitude of central kernel is 0.5, magnitude of the two-shifted kernels are 0.25

$$w_{hn}[n] = 0.5\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right], \quad 0 \leq n \leq N-1 \qquad (6)$$

- By mistake denoted as Hanning
- *Main-lobe width*: $8\pi/(N)$
- *Side-lobe magnitude*: $-32dB$
- The boundary samples are equal to 0 (deletes samples $y[0]$ a $y[N-1]$)



SOURCE: BOAZ PORAT, A Course in Digital Signal Processing



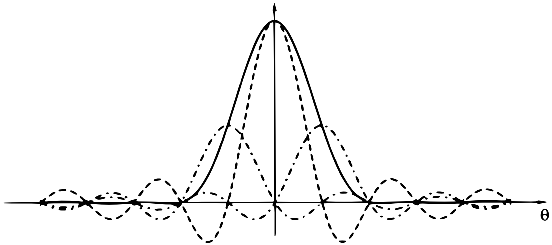
(a) Hann window $w_{hn}[n]$, (b) DTFT spectrum $W_{hn}(e^{j\omega})$ magnitude

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

- **Hamming window** is obtained by modification of magnitudes of Dirichlet kernels summed to obtained the Hann window

$$w_{hm}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \tag{7}$$

- Largest side-lobe is not the closest to the main lobe
- *Main-lobe width*: $8\pi/(N)$
- *Side-lobe magnitude*: $-43dB$



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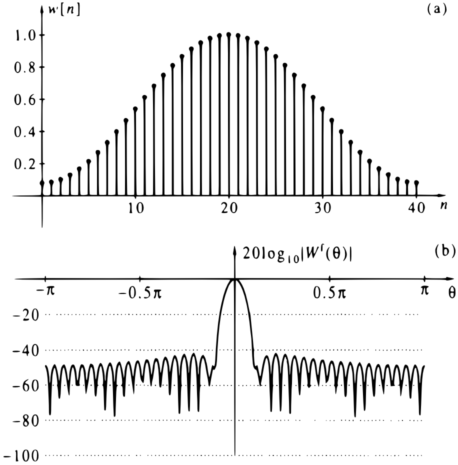


(a) Hamming window $w_{hm}[n]$, (b) DTFT spectrum $W_{hm}(e^{j\omega})$ magnitude

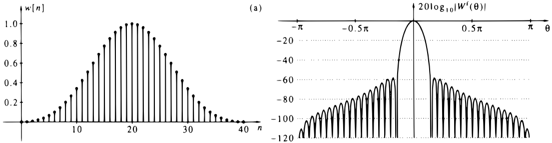
SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Blackmann window

- **Blackmann window** stems from superposition of five Dirichlet kernels shifted in frequency ($\Delta\omega = \pm 2\pi/(N-1)$)

$$w_b[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right) \quad (8)$$

- *Main-lobe width*: $12\pi/(N)$
- *Side-lobe magnitude*: $-57dB$
- The boundary samples of Blackmann window are equal to 0 (deletes samples $y[0]$ a $y[N-1]$)



(a) Blackmann window $w_b[n]$, (b) DTFT spectrum $W_b(e^{j\omega})$ magnitude

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Kaiser window

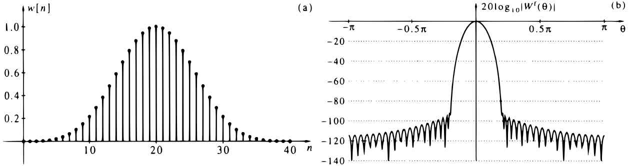
- Previous windows are considered classic - these are based on intuition and qualified guesses
- Kaiser window is an example of modern window, which is based on optimality criterion
- **Kaiser criterion:** Minimize the width of the main-lobe, provided that the length of the window is fixed and the energy of side-lobes does not exceed given percentage of total energy
- **Kaiser window** is given by

$$w_k[n] = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{|2n-N+1|}{N-1}\right)^2}\right]}{I_0[\alpha]}, \quad 0 \leq n \leq N-1 \qquad (9)$$

where $I_0(x) = \sum_{k=0}^{\infty}\left(\frac{x^k}{2^k k!}\right)^2$ is modified Bessel function of order 0, $\alpha \in R$ - a free parameter influencing the main-lobe/side-lobe compromise

Kaiser window II

- **Parameter** α of the Kaiser window influences the width of the main-lobe and magnitude of the side-lobes
- For growing α, the main-lobe width is growing and the magnitude of the side-lobe diminishes
- *Example of Kaiser window:* $N = 41, \alpha = 12$
- *Main-lobe width*: $16\pi/(N)$
- *Side-lobe magnitude*: $-90dB$



(a) Kaiser window $w_k[n]$, (b) DTFT spectrum $W_k(e^{j\omega})$ magnitude

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Part II

Harmonic analysis

Measuring frequency of periodic signals

- Measuring frequency of periodic signals, especially harmonic ones, is a very important taks in digital signal processing
- Fourier analysis is a natural tool for this task
- In practice, signals are measured only within some finite time interval
- Spectrum of such signals can then be evaluated on some discrete finite set of frequencies (using DFT)

Measuring frequency for a set of harmonic functions

We seek argument of maximum of magnitude spectrum computed via:

1. We select a window, which reflects (by side-lobe magnitude) the expected ratio of the weakest and the largest frequency components
2. We multiply signal $y[n]$ by a window of selected length - the length is selected according to the stationarity of the analyzed signal as a compromise between frequency and time resolution
3. We compute $Y(e^{j\omega})$, in practice its sampled variant (DFT spectrum) $Y[k]$ (using FFT)
4. If the window is suitable and the *conditions of distinguish-ability* hold, the the sought frequencies are *close* to the local maxima of $Y[k]$.
   - Conditions of distinguish-ability determine, when two frequency components in a short signal can be distinguished from each other

   The inaccuracy of the detected maximum is caused by
   - limited frequency resolution (mitigated by concatenation of zeros)
   - frequency bias - summation of side lobes of various components; it shifts the local maximum of the magnitude spectrum

Suitable a priory information in this task is:

1. Frequency distance between the frequency components
2. Ratio of magnitudes of the respective frequency components
3. Distance of the frequencies ω_k from 0 and π

Measuring frequency for a single complex exponential

- Let us have continuous signal $y(t) = Ae^{j(\Omega t+\phi_0)}$ and let us measure the frequency Ω_0
- Let us sample the signal with sample period T_s such that $-\pi < \Omega_0 T_s < \pi$
- We obtain a signal $y[n] = Ae^{j(\omega_0 n+\phi_0)}, \quad 0 \leq n \leq N-1$ and $\Omega_0 T_s = \omega_0$
- Dirichlet kernel has a single maximum in point $\omega = 0$, therefore it is in theory possible to find ω_0 exactly as a frequency, where magnitude spectrum $|Y(e^{j\omega})|$ is maximal
- CAREFUL - in practice it is not possible to find the global maximum exactly, we evaluate $|Y(e^{j\omega})|$ only for some finite number of points using DFT
- If it is necessary, it is possible (for enhancement of the frequency resolution) to concatenate the original sequence $y[n]$ with a vector of zeros

Measuring frequency for a two complex exponentials

- Let us have continuous signal given by $y(t) = A_1 e^{j(\Omega_1 t + \phi_1)} + A_2 e^{j(\Omega_2 t + \phi_2)}$ with the task to measure Ω_1, Ω_2
- Let us sample the signal with sampling period T_s such that $-\pi < \Omega_{1,2} T_s < \pi$
- We obtain signal $(\Omega_1 T_s = \omega_1, \Omega_2 T_s = \omega_2)$ $y[n] = A_1 e^{j(\omega_1 n + \phi_1)} + A_2 e^{j(\omega_2 n + \phi_2)}, \quad 0 \leq n \leq N-1$
- Let us search first for ω_1, for $Y(e^{j\omega})$ in point $\omega = \omega_1$ holds

$$Y(e^{j\omega})|_{\omega=\omega_1} = NA_1 e^{j\phi_1} + A_2 e^{-j(0.5(\omega_1-\omega_2)(N-1)-\phi_2)} D(\omega_1 - \omega_2, N)$$

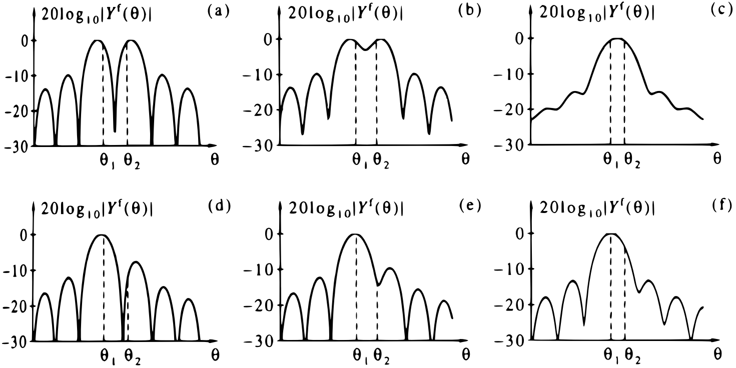
- If $A_2 \neq 0$ and

$$|A_2 D(\omega_1 - \omega_2, N)| << NA_1 \tag{10}$$

  than the *local* maximum $Y(e^{j\omega})$ approaching ω_1 is well distinguishable.
- The condition (10) holds, if $|\omega_2 - \omega_1| \geq 2\pi/N$ and if A_2 "not much larger" than A_1

Similar conditions hold symmetrically for ω_2

EXAMPLE: Measuring frequency for a two complex exponentials: (a,b,c): $A_1 = A_2$, (d,e,f): $A_2 = 0.25A_1$



(a) $\omega_1 - \omega_2 = 2\pi/N$, (b) $\omega_1 - \omega_2 = 1.5\pi/N$, (c) $\omega_1 - \omega_2 = \pi/N$
(d) $\omega_1 - \omega_2 = 2\pi/N$, (e) $\omega_1 - \omega_2 = 1.5\pi/N$, (f) $\omega_1 - \omega_2 = \pi/N$

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

- Condition (10) can be hard to fulfill using rectangular window (large magnitude of side-lobes)
- This problem can be partly mitigated using windowing
- For the value of spectrum $Y(e^{j\omega})$ on frequency ω_1 using window $w[n]$ of length N (with DTFT denoted as $W(e^{j\omega})$) it holds

$$Y(e^{j\omega})|_{\omega=\omega_1} = A_1 e^{j\phi_1} W(e^{j0}) + A_2 e^{j\phi_2} W(e^{j(\omega_1-\omega_2)})$$

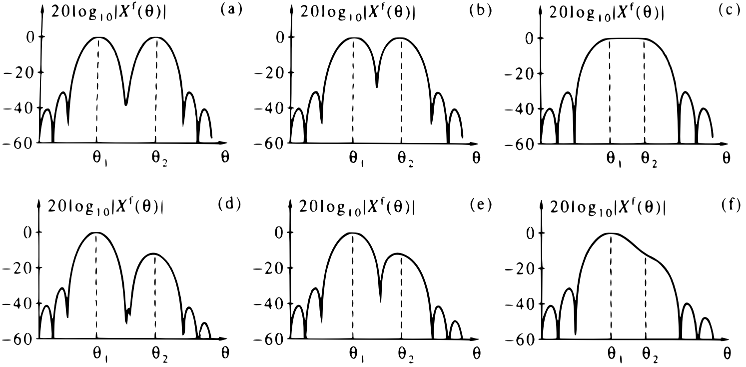
- The condition (10) using window $w[n]$ evolves into

$$|A_2 W(e^{j(\omega_1-\omega_2)})| << A_1 \sum_{n=0}^{N-1} w[n] \qquad (11)$$

where $W(e^{j0}) = \sum_{n=0}^{N-1} w[n]$

- Condition (11) holds:
  1. if $|\omega_1 - \omega_2|$ is larger than $\frac{1}{2}$ of the main-lobe of $W(e^{j\omega})$
  2. if $20log_{10}(A1/A2)$ larger than the magnitude of the side-lobes

EXAMPLE: Measuring frequency for a two complex exponentials: (windowing by Hann window)
(a,b,c): $A_1 = A_2$, (d,e,f): $A_2 = 0.25A_1$



(a) $\omega_1 - \omega_2 = 8\pi/N$, (b) $\omega_1 - \omega_2 = 6\pi/N$, (c) $\omega_1 - \omega_2 = 4\pi/N$
(d) $\omega_1 - \omega_2 = 8\pi/N$, (e) $\omega_1 - \omega_2 = 6\pi/N$, (f) $\omega_1 - \omega_2 = 4\pi/N$

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Measuring frequency for a set of harmonic functions

TASK: We want to learn frequencies of M real-valued harmonic functions

- Real-valued harmonic functions exhibit both-sided symmetric spectrum

For well distinguishable components it must hold:

1. All components $\omega_k, k = 1 \ldots M$ must be distant in the spectrum at least $2\pi/N$
2. No component ω_k is lower than π/N and larger than $\pi(1 - 1/N)$
3. All amplitudes $A_k, k \neq m, k = 1 \ldots M$ are lower or "not much larger" than A_m

Again: side-lobes of the Dirichlet kernel will mask weak frequency components - to mitigate, windowing can be utilized

Measuring frequency for a set of harmonic functions II

Usage of window $w[n]$ changes the conditions of distinguish-ability for two frequency components as follows:

1. All components $\omega_k, k = 1 \dots M$ are mutually distant at least half of the main-lobe $W(e^{j\omega})$
2. No-frequency component ω_k is lower than half of the main-lobe $W(e^{j\omega})$ and larger than π minus half of main-lobe $W(e^{j\omega})$
3. The ratio of logarithmic magnitudes $20 \log_{10} A_k$ cannot be larger than the magnite of the side-lobe of $W(e^{j\omega})$

Influence of noise on harmonic analysis

- Some sort of noise is up to some extend present in all measured signals
- Harmonic analysis in the noisy case is performed as in the noiseless case, up to following differences:
- The noise further **masks weak frequency components** (alongside masking due to window side-lobes)
- *Signal detection:* Distinguishing of the weak harmonic components in the presence of many spurious noise peaks
- The noise **shifts maxims** of the DTFT/DFt spectrum
- *Frequency estimation:* Found maxims are identified with an error corresponding to the random nature of the noise

Influence of noise on harmonic analysis II

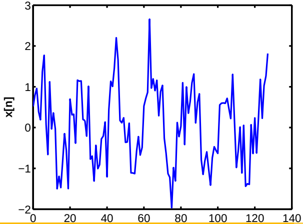
- The influence of noise on signal detection and frequency estimation can be quantified for white noise to some extend
- The analysis is accurate only for signals containing only "a few" harmonic components
- For approximative quantification the following "‘rule-of-thumb"’ is used
- Frequency component can be detected in the presence of noise, if:
  1. The *conditions of distinguishability* (see slide 32)
  2. The following *enequality* holds

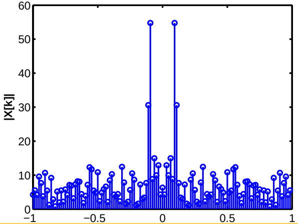
$$\frac{N \cdot A^2 \cdot \mathrm{PG}}{P_v} \geq 100 \tag{12}$$

  - N - Window length
  - A - Harmonig component magnitude
  - PG - *Processing gain* - Parameter characteristic for a specific window, determines amplification of harmonic signal with respect to noise during windowing (the higher the better)
  - P_v - Energy of white noise (zero value for $\omega = 0$)

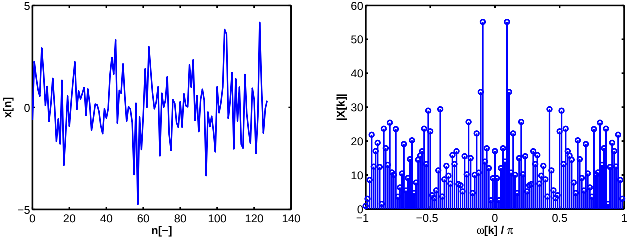
Window	Recangular	Bartlett	Hann	Hamming	Blackman	Kaiser($\alpha = 12$)
PG	1	0.74	0.67	0.73	0.58	0.50

- DETAILS: Boaz Porat, *A course in digital signal processing*, 185 / chapter 6.5
- EXAMPLE: Detection of harmonic signal in the presence of noise
- *Signal*: $x[n] = \sin(0.1\pi \cdot n) + v[n]$, $v[n]$ - white noise
- P_{lim}: Limit to noise energy, which allows to detect the harmonic component as in (12)
- **Scenario 1**: Noise energy $P_v = 0.25 P_{lim}$

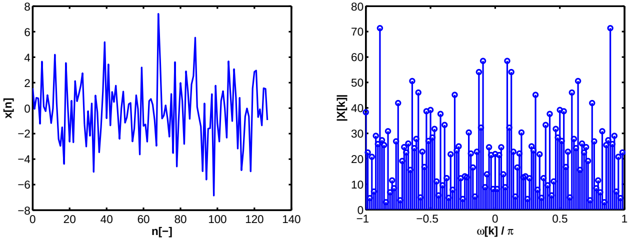




- **Scenario 2**: Noise energy $P_v = P_{lim}$



- **Scenario 3**: Noise energy $P_v = 4P_{lim}$



Thank you for attention!