Digital Signal Processing

Jiří Málek

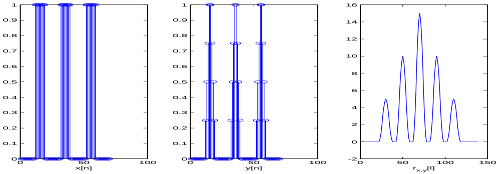
Part I

Correlation of deterministic signals, energy

Cross-correlation of deterministic signals

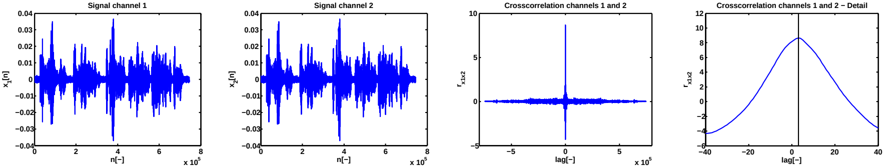
- Operator, which computes mutual similarity of two signals $x[n]$ and $y[n]$ with respect to their mutual shift ℓ (lag)
- For $x[n], y[n] \in \mathcal{R}$ is the cross-correlation $r_{xy}[\ell]$ defined by:

$$r_{xy}[\ell] = \sum_{n=-\infty}^{\infty} x[n+\ell]y[n] \tag{1}$$



- Cross-correlation of real-valued signals is a real-valued function, migth be positive or negative and is not symetric in general
- **Time delay analysis:**
- EXAMPLE: Determination of time-delay and direction of arival of an acoustic signal using binaural microphone array

Time difference of Arrival: (TDOA)



1. Value $\ell_{\max}$ is determined, for which $|r_{xy}[\ell]|$ yields the highest value
2. TDOA $= \ell_{\max} T_s$, (T_s - sampling period)
3. Angle between the axis of the microphone array and the direction of sound arrival is given by

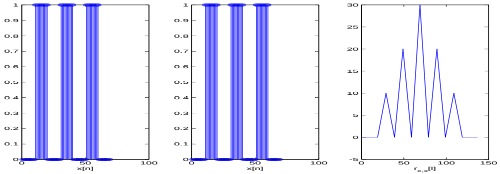
$$\mathrm{DOA} = \arcsin\left(\frac{\mathrm{TDOA} \cdot c}{d}\right) \tag{2}$$

c - speed of sound (343m/s in the air), d - microphone distance

Autocorrelation of deterministic signals

- Operator, which quantifies the similarity of the signal to itself, with respect to a shift ℓ
- For $x[n] \in \mathcal{R}$ is the autocorrelation $r_{xx}[\ell]$ defined by:

$$r_{xx}[\ell] = \sum_{n=-\infty}^{\infty} x[n+\ell]x[n] \tag{3}$$



- Autocorrelation function is real, might be positive or negative and has always even symetry around $\ell = 0$.
- **Periodicity detection:**
- Autocorrelation is often used as a simple tool to determine a periodicity of a general noisy signal.

- **Cross/Auto-correlation** can also be computed using convolution by
$$r_{x,y}[n] = x[n] * y[-n] \tag{4}$$
- (This feature is exploited, when the correlation is computed using the Fast Fourier Transform (FFT))
- Cross-correlation is **not commutative**
$$r_{x,y}[n] = r_{y,x}[-n] \tag{5}$$
- The interchange of variables results into time-reversed correlation function.

Signal energy, signal power

Signal energy:

- Energy is a value of auto-correlation $r_{xx}[0]$, i.e. a quantity given by

$$E_x = \sum_{n=-\infty}^{\infty} x[n]^2 \qquad (6)$$

- Scalar variable measuring "activity"/"size" of the signal
- Finite-energy signals, $E_x < \infty$
- Energy of the harmonic periodic signal is infinite

Signal power:

- Average energy of a signal within the time-interval of length N

$$P_x = \frac{1}{N} \sum_{n=0}^{N-1} x[n]^2 \qquad (7)$$

- The power of periodic signal is computed for a single period
- The power of harmonic signals is therefore finite
- Instantaneous power is computed for $N = 1$

Part II

Spectral analysis - introduction and motivation

Discrete signal $x[n]$:

- Real/complex function defined on a set of integers
- In other words: Indexed infinite sequence of numbers from $\mathcal{R}$ or $\mathcal{C}$
- This type of signal description is often (and in a slightly misleading way) denoted as description **in time-domain**, because n has often the meaning of a time index
- For many applications, this description does not explicitly reflect the most important information stored in the signal
- EXAMPLE: In the speech signal, the value of $x[n]$ has the meaning of instantaneous loudness
  - However, the speech signal carries much more important information in a hidden form (what was said, who speaks, emotion of the speaker)
  - **Spectral analysis** attempts to reveal such information

Spectral analysis - motivation II

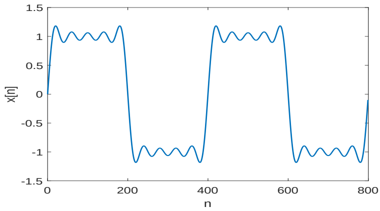
- EXAMPLE: In a musical audio signal, $x[n]$ has also the meaning of the instantaneous loudness
  - The information about tone pitch or used musical instrument is encoded as a speed of (periodical) changes (oscillations) of $x[n]$
  - Fast changes of $x[n]$ correspond to high-pitched tones, slow ones to low-pitched tones
- Similar to physics, oscillations are described using harmonic functions (cosine) and their **frequencies**
- For continuous signals, an analog frequency F[Hz] is used (as in physics, number of repetitions per unit of time), or its scaled version $\Omega = 2\pi F$
- For discrete signals, similar variable is defined - **digital frequency** ω[-]

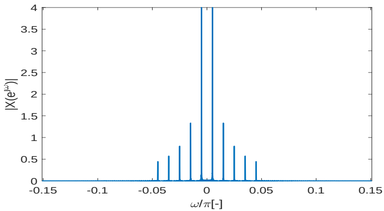
$$\omega = \frac{2\pi F}{F_s}, \tag{8}$$

F_s - sampling frequency

Spectral analysis - motivation III

- One of the results of the spectral analysis is the **magnitude spectrum**, which has several forms (will be discussed lated)
- Magnitude spectrum quantifies, how much is the signal similar to a harmonic function (cosine) on a given frequency
  - In other words: it quantifies an extend, to which are the individual harmonic functions on various frequencies present in the signal
- Spectrum is another view of a signal $x[n]$, it is a description in the **frequency domain**
- Spectrum carries the same information as the sequence $x[n]$, it just explicitly states another its part
- Basic tools of the spectral analysis are various forms of the Fourier transform (DTFT, DFT, STFT)





Part III

Discrete Time Fourier Transform (DTFT)

Discrete Time Fourier Transform (DTFT)

- Mapping from a set of sequences (signal, impulse response) into a set of continuous complex functions of real variable (*DTFT spectrum*, *frequency response*)
- Mapping from *time-domain* into *frequency-domain*, where many properties of signals and systems can be more easily studied
- It is given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \tag{9}$$

  $\omega \in \mathcal{R}$ - digital frequency, $-\infty < n < \infty$
- $X(e^{j\omega})$ is continuous complex function of real variable ω.
- The value of $X(e^{j\omega})$ at ω_0 is *cross-correlation* of sequence $x[n]$ and complex exponential $e^{jn\omega_0}$ (for $\ell = 0$)
- Meaning of $X(e^{j\omega_0})$:
  - *Magnitude* $\left|X(e^{j\omega_0})\right|$ - the level of correlation between $x[n]$ and $e^{jn\omega_0}$
  - *Phase* $\phi(\omega_0)$ - shift between $x[n]$ and $e^{jn\omega_0}$

$$X(e^{j\omega_0}) = \left|X(e^{j\omega_0})\right| e^{j\phi(\omega_0)} \tag{10}$$

Discrete Time Fourier Transform II

- EXAMPLE: Computation of $X(e^{j\omega})$ using the DTFT definition formula
- **Inverse discrete time Fourier transform:**

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{jn\omega}d\omega \qquad (11)$$

- List of commonly used DTFT pairs:

Sequence	DTFT image
$\delta[n]$	1
$\delta[n-n_0]$	$e^{-jn_0\omega}$
1	$2\pi\delta(\omega)$
$e^{jn\omega_0}$	$2\pi\delta(\omega-\omega_0)$
$\alpha^n u[n],\ \lvert\alpha\rvert<1$	$\frac{1}{1-\alpha e^{-j\omega}}$
$-\alpha^n u[-n-1],\ \lvert\alpha\rvert>1$	$\frac{1}{1-\alpha e^{-j\omega}}$
$(n+1)\alpha^n u[n],\ \lvert\alpha\rvert<1$	$\frac{1}{(1-\alpha e^{-j\omega})^2}$
$\cos([n\omega_0])$	$\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

- **Periodicity:**
- DTFT is periodic with period 2π, i.e.,
- $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$
- QUESTION: Why is it so?

- **Symmetry:**

$x[n]$	$X(e^{j\omega})$
Real, even	Real, even
Real, odd	Imaginary, odd
Imaginary, even	Imaginary, even
Imaginary, odd	Real, odd

DTFT properties II

- **Linearity:**
- If $X_1(e^{j\omega})$ is DTFT of $x_1[n]$ and $X_2(e^{j\omega})$ is DTFT of $x_2[n]$, then:
- $ax_1[n] + bx_2[n] \overset{\text{DTFT}}{\Longleftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
- **Time-reversal:**
- Time-reversal $x[n]$ leads to reversal of $X(e^{j\omega})$ in the frequency domain, i.e.:
- $x[-n] \overset{\text{DTFT}}{\Longleftrightarrow} X(e^{-j\omega})$
- **Shifting:**
- Shift of the sequence $x[n]$ leads multiplication of $X(e^{j\omega})$ by a complex exponential, i.e.,:
- $x[n-n_0] \overset{\text{DTFT}}{\Longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$

DTFT properties III

- **Modulation:**
- Multiplication of a sequence by a complex exponential leads to a shift in the frequency domain, i.e.,
- $e^{jn\omega_0} x[n] \overset{\text{DTFT}}{\Longleftrightarrow} X(e^{j\omega-\omega_0})$
- Multiplication of a sequence by a signal $\cos(\omega_0 n)$ leads to two shifted copies of $X(e^{j\omega})$ in the frequency domain, i.e.,
- $\cos(\omega_0 n) x[n] \overset{\text{DTFT}}{\Longleftrightarrow} \frac{1}{2} X(e^{j\omega-\omega_0}) + \frac{1}{2} X(e^{j\omega+\omega_0})$
- **Convolutional theorem:**
- Convolution of two signals in time-domain equals the multiplication of the DTFTs of these signals in the frequency-domain
- $h[n] * x[n] \overset{\text{DTFT}}{\Longleftrightarrow} H(e^{j\omega}) X(e^{j\omega})$

DTFT properties IV

- **Multiplication theorem:**
- Multiplication in the time-domain corresponds to a (periodic) convolution in the frequency-domain
- $x[n]y[n] \overset{\text{DTFT}}{\Longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
- **Parseval theorem:**
- DTFT preserves the energy of a signal when transitioned from the time- into the frequency-domain
- $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$

Part IV

DTFT spectrum

DTFT spectrum $X(e^{j\omega})$

- Complex-value function of independent variable $\omega \in \mathcal{R}$, expresses correlation of the signal $x[n]$ and a complex exponential on a specific frequency ω_0.
- *Computation:* application of DTFT on the signal $x[n]$
- Meaning for *real-valued* $x[n]$: it is a decomposition of $x[n]$ into the sum (of an infinite number) of harmonic functions $\cos(\omega n + \phi(\omega))$ (called also frequency components)
- Euler formula:

$$\cos(\omega) = \frac{1}{2} \cdot (e^{j\omega} + e^{-j\omega}) \qquad (12)$$

- EXAMPLE: Spectrum of a real-valued signal $x[n]$
- Notation: $x[n] \stackrel{DTFT}{\rightarrow} X(e^{j\omega})$
- $X(e^{j\omega})$ can be decomposed into two real-valued functions via

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\phi(\omega)} \qquad (13)$$

  - *magnitude spectrum* $\left|X(e^{j\omega})\right|$
  - *phase spectrum* $\phi(\omega)$

DTFT spectrum $X(e^{j\omega})$ II

- DTFT spectrum is independent of time (it does not have any time resolution, analyzes the signal as a whole)
- This analysis is suitable for **stationary signals:**
  - expected statistical properties (and consequently the spectrum) does not change in time
  - e.g. hum of a fan, constant vibrations of rotating machines, white noise, constant tone/accord
- This type of analysis is also suitable for study of LTI systems, which do not change their properties in time

Thank you for attention!