## Solved examples on topics presented in Lecture 2

**Numerical solution of difference equation:** See also corresponding `Matlab script`

*Example:* Find the response of a filter given by the difference equation

$$y[n] = \frac{3}{4} \cdot y[n-1] - \frac{1}{8} \cdot y[n-2] + x[n] - x[n-1] \tag{1}$$

corresponding to an input $x[n] = \delta[n]$ (unit impulse) and zero initial conditions (i.e., $y[-1] = y[-2] = 0$).

Let us show, how to compute the first several samples of the output signal $y[n]$ using recursive substitution. Recursive substitution is just a gradual substitution of increasing numbers for $n$. The substitution must proceed gradually, because at time $n_0$, we need values $y[n_0 - 1]$ and $y[n_0 - 2]$. Thus, we start with $n = 0$ and $x[n] = \delta[n]$

$$\begin{aligned}
y[0] &= \frac{3}{4} \cdot y[-1] - \frac{1}{8} \cdot y[-2] + \delta[0] - \delta[-1] \\
y[0] &= \frac{3}{4} \cdot 0 - \frac{1}{8} \cdot 0 + 1 - 0 \\
y[0] &= 1.
\end{aligned}$$

We continue with substitution $n = 1$ and $x[n] = \delta[n]$, when we exploit the value $y[0]$ computed in the previous step:

$$\begin{aligned}
y[1] &= \frac{3}{4} \cdot y[0] - \frac{1}{8} \cdot y[-1] + \delta[1] - \delta[0] \\
y[1] &= \frac{3}{4} \cdot 1 - \frac{1}{8} \cdot 0 + 0 - 1 \\
y[1] &= -\frac{1}{4}.
\end{aligned}$$

And so forth, until the required number of samples is obtained.

Check the first two sample in the enclosed Matlab script. In the script, there is a general implementation of this task. Moreover, it contains also a solution specific for Matlab: the function `filter(b,a,x)` performs numerical solution of difference equation, i.e., filtering via a digital filter. Vectors `b,a` are coefficients of the difference equation and `x` is the input sequence/signal.

**Cross-correlation:** See also corresponding `Matlab script`

This example shows the estimation of the direction of arrival. An audio signal originated by some source is recorded using a microphone array. The task is to estimate the angle of the source location with respect to the axis of the microphone array.
Load a stereo recording corresponding to the microphone array from the tt TwoChannelMicrophoneArrayRecording.wav file. Estimate the direction of arrival See the script for more information.

**DTFT spectrum computation using the list of DTFT pairs - Example 1:**
Compute DTFT spectrum for a signal

$$x[n] = \cos(\pi/2n) + \sin(\pi/3n) \tag{2}$$

and plot a graph of magnitude a phase spectrum.
Signal is given as a sum of two harmonic functions. Due to linearity, each function can be transformed separately. Output for the $\cos(\pi/2n)$ function is given in the list of the DTFT pairs (Lecture 2) as

$$\cos(\pi/2n) \xrightarrow{\text{DTFT}} \pi(\delta(\omega - \pi/2) + \delta(\omega + \pi/2)), \tag{3}$$

where $\delta(\omega - \omega_0)$ is the Dirac pulse, which is (slightly inaccurately) a function having a nonzero value at point $\omega_0$ and is equal to zero at all other points. For our purposes (DTFT spectrum calculation) we can simply (and again slightly inaccurately) imagine that it has a value of $1$ at the point $\omega_0$ and is zero elsewhere. More accurately, it has an infinite value at the point $\omega_0$ and the area under its curve (definite integral) is $1$. In the graphs, the Dirac pulse is indicated by an arrow to distinguish it from common functions.
The function $\sin(\pi/3n)$ is not not contained in the list, but we can use the Euler formula and rewrite it in the form of the sum of exponential functions. Subsequently, we use the DTFT linearity, i.e., we apply the transformation in the list to each summand separately. Euler's formula states that

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \tag{4}$$

where $j$ is the imaginary unit. For the function $\sin(\pi/3n)$ we obtain

$$\sin(\pi/3n) = \frac{e^{j\pi/3n} - e^{-j\pi/3n}}{2j}. \tag{5}$$

The two exponential functions give, according to the DTFT list, an output (we factorize the constant $\frac{1}{2j}$)

$$\sin(\pi/3n) = \frac{e^{j\pi/3n} - e^{-j\pi/3n}}{2j} \xrightarrow{\text{DTFT}} \frac{1}{2j}(2\pi\delta(\omega - \pi/3) - 2\pi\delta(\omega + \pi/3)). \tag{6}$$

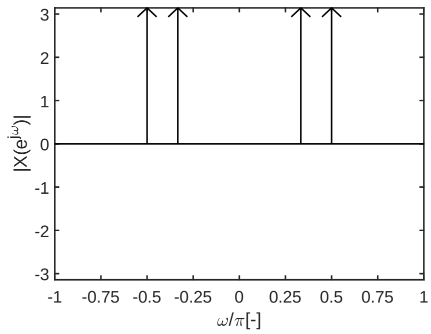
The final DTFT spectrum of $x[n]$ is the sum of the partial ones, that is

$$X(e^{j\omega}) = \pi(\delta(\omega - \pi/2) + \delta(\omega + \pi/2)) + \frac{\pi}{j}(\delta(\omega - \pi/3) - \delta(\omega + \pi/3)). \tag{7}$$

The DTFT spectrum is periodical with period $2\pi$, the interval $\omega \in \langle -\pi, \pi \rangle$ is often stated as the basic period. In the basic period, the magnitude spectrum is non-zero only at four points ($\pi/2, -\pi/2$, $\pi/3, -\pi/3$) and its values are given by the absolute value of $|X(e^{j\omega})|$ (as we stated, $\delta(\omega)$ is assumed to have an absolute value of $1$):

$$\begin{aligned}
\omega_0 = \pi/2 \quad &: \quad |\pi| = \pi \\
\omega_0 = -\pi/2 \quad &: \quad |\pi| = \pi \\
\omega_0 = \pi/3 \quad &: \quad \left|\frac{\pi}{j}\right| = \pi \\
\omega_0 = -\pi/3 \quad &: \quad \left|-\frac{\pi}{j}\right| = \pi.
\end{aligned} \tag{8}$$

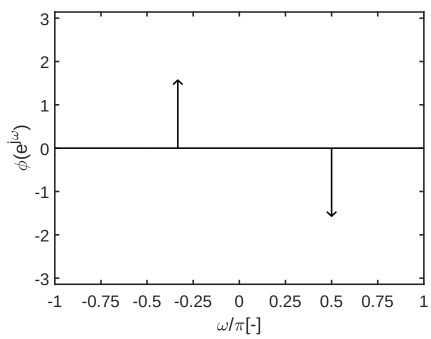
We plot the values into a graph and see an agreement with the theory. The magnitude spectrum of a




real-valued signal $x[n]$ features even symmetry.
Phase spectrum is obtained as an argument (an angle of the complex number w.r.t. the real axis) of the complex spectrum $X(e^{j\omega})$, that is

$$\begin{aligned}
\omega_0 = \pi/2 &: \quad \arg \pi = \arg \pi - 0j = 0 \\
\omega_0 = -\pi/2 &: \quad \arg \pi = \arg \pi - 0j = 0 \\
\omega_0 = \pi/3 &: \quad \arg \frac{\pi}{j} = \arg 0 - \pi j = -\pi/2 \\
\omega_0 = -\pi/3 &: \quad \arg -\frac{\pi}{j} = \arg 0 + \pi j = +\pi/2. \qquad (9)
\end{aligned}$$

We plot the values into a graph and again see an agreement with the theory. The phase spectrum of a real-valued signal $x[n]$ features odd symmetry.




We can see that for the real signal $x[n]$, only the frequency range $\omega \in \langle 0, \pi \rangle$ is of interest (due to symmetry and periodicity). In accordance with the assignment, it consists of only two harmonic functions (the magnitude spectrum contains only two pulses at the frequencies $\omega = \pi/2$ and $\pi/3$).

**DTFT spectrum computation using the list of DTFT pairs - Example 2:**
Compute DTFT spectrum for a signal

$$x[n] = \cos(\pi/2n) \cdot \sin(\pi/3n) \qquad (10)$$

and plot hte graph of magnitude an phase spectrum.
The signal is given as a *product* of two harmonic functions. Therefore, it is not directly listed in the list nor can we use DTFT linearity as in the previous example. However, we can use the Euler formula, rewrite the functions sin and cos in the form of the sum of exponential functions, and then use the DTFT linearity, i.e., apply the transformation in the list to each summand separately. Euler's formula states that

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}, \qquad (11)$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}, \qquad (12)$$

where $j$ is the imaginary unit. By substitution into equation (10), the signal $x[n]$ is obtained as a product

$$x[n] = \left(\frac{e^{j\pi/2n} + e^{-j\pi/2n}}{2}\right) \cdot \left(\frac{e^{j\pi/3n} - e^{-j\pi/3n}}{2j}\right). \tag{13}$$

After multiplication of both parentheses we obtain $x[n]$ (still in the time-domain) in the desired form, as a sum of exponentials

$$x[n] = \frac{1}{4j}\left(e^{j5\pi/6n} - e^{j1\pi/6n} + e^{-j1\pi/6n} - e^{-j5\pi/6n}\right). \tag{14}$$

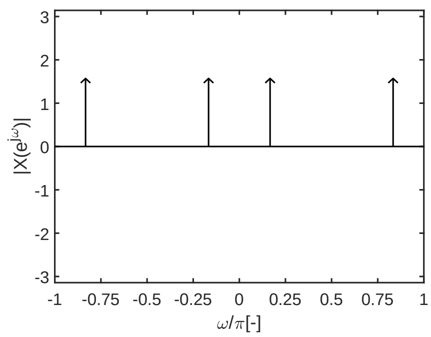
Now we exploit the linearity of the DTFT, i.e., we apply DTFT to each summand separately and factor out the constant. We obtain the DTFT spectrum in the form

$$X(e^{j\omega}) = \frac{2\pi}{4j}\left(\delta(\omega - 5\pi/6) - \delta(\omega - \pi/6) + \delta(\omega + \pi/6) - \delta(\omega + 5\pi/6)\right), \tag{15}$$

where $\delta(\cdot)$ is the Dirac pulse.
Magnitude spectrum in the interval $\omega \in \langle -\pi, \pi \rangle$ is nonzero only at four points ($5\pi/6, \pi/6, -\pi/6, -5\pi/6$) and its values are given by the absolute value of $|X(e^{j\omega})|$ (as we stated, we consider $\delta(\omega)$ to have an absolute value of 1):

$$\begin{aligned}
\omega_0 = 5\pi/6 \quad &: \quad \left|\frac{2\pi}{4j}\right| = \frac{\pi}{2} \\
\omega_0 = \pi/6 \quad &: \quad \left|-\frac{2\pi}{4j}\right| = \frac{\pi}{2} \\
\omega_0 = -\pi/6 \quad &: \quad \left|\frac{2\pi}{4j}\right| = \frac{\pi}{2} \\
\omega_0 = -5\pi/6 \quad &: \quad \left|-\frac{2\pi}{4j}\right| = \frac{\pi}{2}.
\end{aligned} \tag{16}$$

We plot the values and see an agreement with the theory. For the real-valued signal $x[n]$, the magnitude spectrum is even.




Phase spectrum is obtained as an argument of the complex spectrum $X(e^{j\omega})$, that is

$$\begin{aligned}
\omega_0 = 5\pi/6 \quad &: \quad \arg\frac{2\pi}{4j} = \arg 0 - \frac{\pi}{2}j = -\pi/2 \\
\omega_0 = \pi/6 \quad &: \quad \arg -\frac{2\pi}{4j} = \arg 0 + \frac{\pi}{2}j = +\pi/2 \\
\omega_0 = -\pi/6 \quad &: \quad \arg\frac{2\pi}{4j} = \arg 0 - \frac{\pi}{2}j = -\pi/2 \\
\omega_0 = -5\pi/6 \quad &: \quad \arg -\frac{2\pi}{4j} = \arg 0 + \frac{\pi}{2}j = +\pi/2.
\end{aligned} \tag{17}$$

We plot the values and again see the agreement with the theory. For the real-valued signal $x[n]$, the phase spectrum is odd.