Solved examples on topics presented in Lecture 3

Frequency response of the FIR system:

Frequency response $H(e^{j\omega})$ is DTFT of the impulse response $h[n]$.
Generally, the computation of $H(e^{j\omega})$ is very simple, due to linearity and the shift theorem of the DFTF:

$$\begin{aligned} h[n] &= \sum_{k=0}^{q} b[k]\delta[n-k] \\ H(e^{j\omega}) &= \sum_{k=0}^{q} b[k]e^{-j\omega k} \end{aligned}$$

Frequency response can also be obtained from the difference equation (see Schaum 2.7.1/64). This is simple for the FIR as well, due to linearity and the shift theorem of the DTFT:

$$\begin{aligned} y[n] &= \sum_{k=0}^{q} b[k]x[n-k] \\ Y(e^{j\omega}) &= \sum_{k=0}^{q} b[k]X(e^{j\omega})e^{-j\omega k} \\ H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \sum_{k=0}^{q} b[k]e^{-j\omega k} \end{aligned}$$

Note that the coefficients $b[k]$ appear repeatedly in the impulse response, frequency response and the difference equation of the FIR system.

Example: Compute the frequency response of a FIR system (second order moving average filter), determine the magnitude and the phase delay for $\omega = \pi/4$. Using the impulse response and the shift theorem:

$$\begin{aligned} h[n] &= (1/3)(\delta[n] + \delta[n-1] + \delta[n-2]) \\ H(e^{j\omega}) &= (1/3)(1 + e^{-j\omega} + e^{-j\omega 2}) \end{aligned}$$

Alternatively, using the difference equation and the shift theorem:

$$\begin{aligned} y[n] &= (1/3)(x[n] + x[n-1] + x[n-2]) \\ Y(e^{j\omega}) &= (1/3)(X(e^{j\omega}) + X(e^{j\omega})e^{-j\omega} + X(e^{j\omega})e^{-j\omega 2}) \\ H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = (1/3)(1 + e^{-j\omega} + e^{-j\omega 2}) \end{aligned}$$

The complex value of $H(e^{j\omega})$ for frequency $\omega = \pi/4$ is obtained by substitution:

$$\begin{aligned} H(e^{j\omega})|_{w=\pi/4} &= (1/3)(1 + e^{-j\pi/4} + e^{-j\pi/2}) \\ &= (1/3)(1 + \frac{\sqrt{2}}{2}(1-j) - j) \\ &\approx 0.57 - 0.57j \end{aligned}$$

Magnitude and phase is obtained as the absolute value and the argument:

$$
\begin{aligned}
|H(e^{j\omega})||_{w=\pi/4} &\approx 0.8 \\
\phi(e^{j\omega})|_{w=\pi/4} &= -\pi/4
\end{aligned}
$$

Phase delay is computed using its definition formula as

$$
\begin{aligned}
\tau_p(\omega) &= -\frac{\phi(e^{j\omega})}{w} \\
\tau_p(\omega)|_{w=\pi/4} &= -\frac{-\pi/4}{\pi/4} = 1
\end{aligned}
$$

The result can be interpreted: the input $x[n] = \cos(\pi/4n)$ is amplified by approximately 0.8 and delayed by 1 sample (see the related `Matlab script` for details).

Frequency response of the IIR system:

The frequency response $H(e^{j\omega})$ is DTFT of the impulse response $h[n]$. We will discuss the general form of the impulse response for the IIR system later (lecture about Z-transform).
For the IIR case, the frequency response is usually obtained using the difference equation (see Schaum chapter 2.7.1/64 and **example 2.7.1/65**.

$$
\begin{aligned}
y[n] &= \sum_{k=0}^{q} b[k]x[n-k] - \sum_{k=1}^{p} a[k]y[n-k] \\
Y(e^{j\omega}) &= \sum_{k=0}^{q} b[k]X(e^{j\omega})e^{-j\omega k} - \sum_{k=1}^{p} a[k]Y(e^{j\omega})e^{-j\omega k} \\
H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{q} b[k]e^{-j\omega k}}{1 + \sum_{k=1}^{p} a[k]e^{-j\omega k}}
\end{aligned}
$$

Note that the coefficients $b[k], a[k]$ appear repeatedly in the difference equation and the frequency response. Impulse response has different coefficients.

Example: Compute the frequency response of the following IIR system given by its impulse response. Use the list of known DTFT pairs from Lecture 2.

$$
\begin{aligned}
h[n] &= \left(\frac{1}{2}\right)^n u[n], \\
H(e^{j\omega}) &= \frac{1}{1-(1/2)e^{-j\omega}}.
\end{aligned}
$$

Alternatively from the difference equation (using the shift theorem)

$$
\begin{aligned}
y[n] &= x[n] + (1/2)y[n-1] \\
Y(e^{j\omega}) &= X(e^{j\omega}) + (1/2)Y(e^{j\omega})e^{-j\omega} \\
H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1-(1/2)e^{-j\omega}}
\end{aligned}
$$

Convolution of infinite sequences using DTFT:
Analytical approach for computation of convolution of potentially infinite sequences. **Schaum - example 2.7.2/65**

Solving of differences equations using DTFT:
Analytical approach for computation of the output of a system/filter. Suitable for tasks containing infinite signals or filters of the IIR type. Limited for cases with zero initial conditions.
Schaum - example 2.7.3/66

Inverse system:
Inverse system with impulse response $g[n]$ to a system with response $h[n]$ is a filter, which cancells all effects of the filter $h[n]$ applied to the input signal. Exact impulse response does not exist, if filter $h[n]$ deletes any of the frequency components (magnitude $|H(e^{j\omega})| = 0$ for any ω). This is due to the fact that

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})}$$

would be infinite for this frequency (and the inverse system would thus be unstable).
Schaum - example 2.7.4/67