Digital Signal Processing

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Part I

Sampling

Introduction to sampling

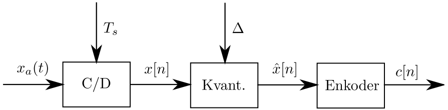
- Most of the discrete signals arises via *sampling* of some analog quantity
  EXAMPLE: audio recording, measurement of biomedical signals etc.
- Transformation of analog signals into discrete sequences is denoted as *A/D conversion* (Analog to Digital Conversion)
- A reverse process is the *D/A conversion* - Digital to Analog Conversion
- *Sampling theorem* states, when is the analog signal uniquely determined by its samples

A/D conversion

- A transformation of the analog signal $x_a(t), t \in \mathcal{R}$ to a discrete sequence $\hat{x}[n]$ and subsequent encoding
- Amplitude $x_a(t)$ is an arbitrary real number, $\hat{x}[n]$ is quantized - has finite number of amplitude levels
- **A/D conversion** typically consists of three parts:
  1. *(Ideal) sampling* - selection of values $x_a(t)$ at times equal to integer multiples of *sampling period* T_s

$$x[n] = x_a(nT_s) \tag{1}$$

  2. *Quantization* of continuous amplitudes in $x[n]$ into a discrete set of amplitude values, gives $\hat{x}[n]$
  3. *Encoding* of the discrete values $\hat{x}[n]$ into a sequence of binary code-words $c[n]$



Sampling I

- Sampling period $T_s[s, s/sample]$, sampling frequency $F_s[Hz, sample/s]$, frequency $F[Hz]$
- *Equidistant sampling*- multiplication of $x_a(t)$ with periodic sequence of Dirac pulses
- DETAILS around equidistant sampling
- Sampling maps frequencies of the analog signal $-\infty < \Omega < \infty$ to digital frequencies $-\pi < \omega < \pi$

$$\omega = \Omega T_s = \frac{\Omega}{F_s} = \frac{2\pi F}{F_s} \tag{2}$$

- The frequency components corresponding to ω repeat periodically with period 2π
- DETAILS to transition from analog to digital frequencies

Sampling II

- **Sampling theorem** - The signal $x_a(t)$ is *band-limited*, if it contains only components with frequency lower than Ω_0 (analog spectrum $X_a(\Omega) = 0; |\Omega| \geq \Omega_0$)
- The band-limited signal $x_a(t)$ can be reconstructed from its samples $x_a(nT_s)$ if

$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_0 \qquad (3)$$

  Ω_0 - Nyquist frequency
- QUESTION: What happens when the sampling theorem does not hold?
- *Aliasing* - Distortion of the digital signal arising during sampling with a low sampling frequency.
  In frequency domain, the periods of DTFT spectrum $X(e^{j\omega})$ overlap and sum together.
  In the time-domain, the signal reconstructed from the samples of $x(nT_s)$ is different than the original $x_a(t)$.
- Most signals are not bend-limited, an anti-aliasing analog filter (low-pass) must be applied, in order to avoid aliasing.

Quantization I

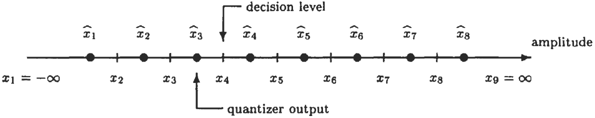
- **Quantization** - Transformation of a continuous amplitude of $x[n]$ to discrete finite set of amplitudes
- May be interpreted as a form of rounding

$$\hat{x}[n] = Q(x[n]) \tag{4}$$

- Quantization splits the continuous amplitude of $x[n]$ to L non-overlapping intervals I_k using $L+1$ decision levels $x_1, x_2, \dots, x_{L+1}$

$$I_k = [x_k, x_{k+1}], \quad k = 1, 2, \dots, L \tag{5}$$

- If $x[n]$ belongs to the interval I_k, quantization assigns to $\hat{x}[n]$ the value $\hat{x}_k$



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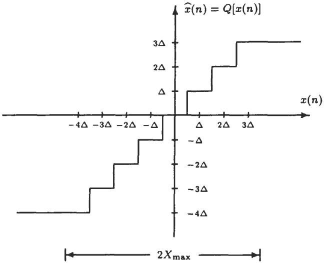
Quantization II

- **Quantization step/resolution** - the width of the interval I_k, often constant for all intervals - *linear/equidistant quantization*
- Number of decision levels is often $L = 2^B + 1$, due to subsequent binary coding, B-Number of bits
- **Quantization error** - $e[n] = x[n] - Q(x[n])$
- **SQNR[dB]** - *Signal to Quantization Noise Ratio*

$$\mathrm{SQNR} = 10 \log \frac{\sigma_x^2}{\sigma_e^2} \approx 6.02B[dB] \tag{6}$$

  SQNR thus grows by about 6dB with every added bit (doubles the number of encoding levels)
  ASSUMPTION: signal is amplified such that it covers all quantization level equally
- **In practice:** Provided that the assumption holds, SQNR is very high, due to high number of quantization levels (approx. 96dB for 16-bit quantizer)
  Thus, $x[n]$ usually can be considered equal to $\hat{x}[n]$

Quantization III, Encoding



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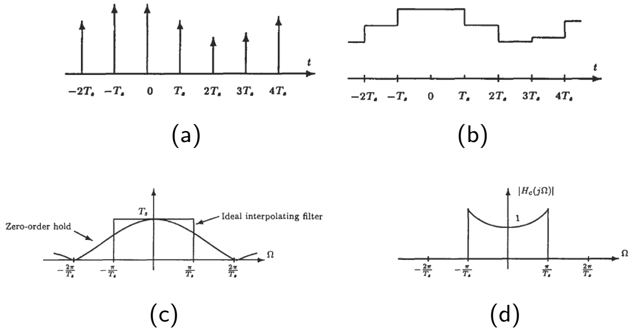
- **Encoder** - An algorithm/device assigning a binary word to each quantization level
- *Many numerical types:* integers, signed/unsigned, fixed/floating point etc.
- *Many encoding schemes:* e.g., *two's complement* for signed integers

D/A conversion I

- **D/A conversion** - Transformation of discrete sequence into analog signal
- If the *sampling theorem* is fulfilled, then the analog signal can be uniquely reconstructed from its samples
- The exact reconstruction is however unavailable, due to quantization errors
- In practice: if quantization is sufficiently fine (SQNR is high), these errors can be neglected
- Ideal (theoretical) D/A conversion proceeds in two steps:
  1. Continuous sequence of pulses $x_s(t)$ is generated using the samples of $x[n]$
  2. Ideal analog low-pass filter - *a reconstruction filter* is applied to $x_s(t)$
- DETAILS - Ideal D/A conversion
- In practice: ideal reconstruction filter is not realizable
- Instead: *zero-order hold* and *compensation filters* are used
- DETAILS - Real-world D/A conversion



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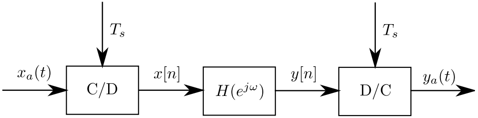
(a) Continuous sequence of pulses $x_s(t)$
(b) Zero-Order Hold signal
(c) Analog frequency response: Ideal low-pass, zero-oder hold
(d) Analog frequency response: Ideal compensation filter

D/A conversion III

- A/D and D/A transducers are often used when an analog signal should be processed by a discrete system (system control, music processing etc.)
- This scenario assumes:
  - In theory: signal is not quantized, in practice: quantization levels are sufficiently fine
  - in theory: ideal low-pass reconstruction filter is used, in practice: zero-order hold and compensation filter instead
- When all assumptions (or their practical approximations) hold, the overall cascade can be considered as continuous system, and any influence of sampling can be neglected



Part II

Sampling - Sample rate conversion

Sample rate conversion

- A frequent task in signal processing
- Two possible implementations:
  1. In theory: D/A conversion and sampling via a different sampling frequency F_s (infeasible)
  2. In practice: *resampling* directly in the time-domain
- The types:
  1. Decrease of F_s by an integer factor
  2. Increase of F_s by an integer factor
  3. Change of F_s by a rational factor

Decrease of sample rate by an integer factor

- **Down-sampling**
- When down-sampling M-times, the down-sampled signal $x_d[n]$ contains every Mth sample of the original $x[n]$
- BEWARE - down-sampling generally leads to aliasing!
- DETAILS - Aliasing by down-sampling
- *Aliasing prevention:* Filtration of $x[n]$ with a low-pass filter with cutt-off frequency $\omega_c = \pi/M$
- Cascade of low-pass filtering and down-sampling is called *decimation*

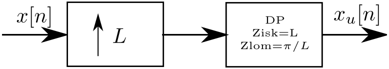
$$x[n] \longrightarrow \boxed{\begin{matrix}\text{DP}\\ \text{Zisk}=1\\ \text{Zlom}=\pi/M\end{matrix}} \longrightarrow \boxed{\downarrow M} \longrightarrow x_d[n]$$

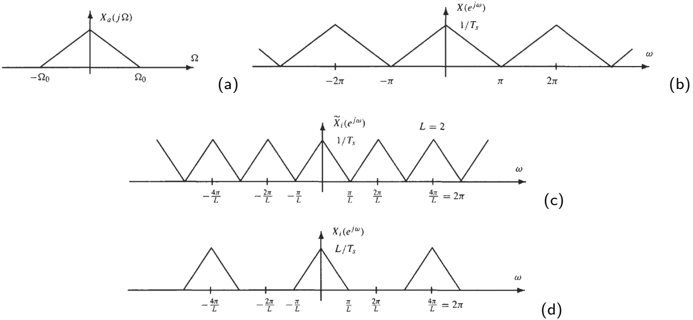
Increase of sample rate by an integer factor

- **Up-sampling**
- When up-sampling L-times, the up-sampled signal $x_u[n]$ contains the samples of $x[n]$ with $L-1$ zeros between every two samples

$$x_u[n] = \begin{cases} x[n/L] & n = 0, \pm L, \pm 2L, \ldots \\ 0 & \textit{elsewhere} \end{cases} \tag{7}$$

- Approximation of zero samples is performed by low-pass filtering with cut-off frequency π/L and gain L.
- Cascade of up-sampling and low-pass filtering is called interpolation
- DETAILS - Interpolation and spectrum





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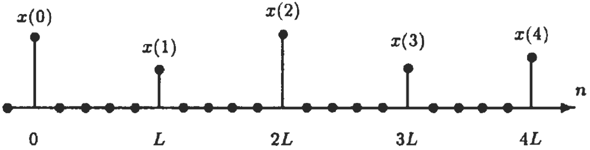
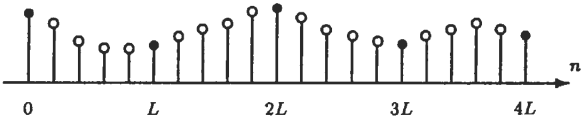
(a) Spectrum of continuous band-limited signal
(b) Spectrum of sampled signal
(c) Spectrum of upsampled signal
(d) Spectrum of interpolated signal (up-sampled and low-pass filtered)

Interpolation in the time-domain



(a)



(b)

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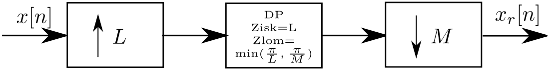
(a) Up-sampled signal $x_u[n]$
(b) Interpolated signal (up-sample and low-pass filtered)

- The sampling rate conversion by rational-factor $\frac{L}{M}$ is performed by interpolation L-times followed by decimation M-times.
- The cascade can be replaced by a single low-pass filter with cut-off frequency

$$\omega_c = \min\left\{\frac{\pi}{M}, \frac{\pi}{L}\right\} \tag{8}$$

and gain L.



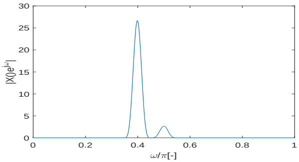
- EXAMPLE: Change of the sampling frequency by a rational factor
- MATLAB: `resample(x,L,M);`

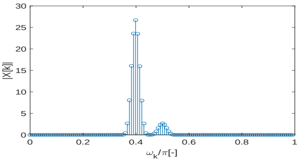
Part III

Spectral analysis - signals with finite duration

Spectral analysis - motivation

- We have discussed the analysis of the signals in the frequency domain via DTFT
- Here, the digital frequency $\omega \in \mathcal{R}$; even when the analyzed signal is of finite duration, its DTFT spectrum is comprised (in theory) of infinitely many frequency components
  - This is advantageous for system analysis (the response to any frequency is known)
  - Unsuitable for analysis of common signals with short duration
- **Discrete Fourier Transform (DFT)**
  - Form of Fourier transform suitable for finite of periodical signals
  - Its output is a "more compact" spectrum (sequence of the same length as the original signal)
  - It retains most (but not all) advantageous properties of the DTFT (invertibility, linearity)





Part IV

Discrete Fourier Transform (DFT)

Discrete Fourier Transform (DFT)

- DTFT allows to transform the discrete sequence $x[n]$ to continuous function of digital frequency ω, i.e., $X(e^{j\omega})$.
- Considering discrete signal $x[n]$, the function $X(e^{j\omega})$ is DTFT spectrum
- Unique inverse transform is possible, provided that values of $X(e^{j\omega})$ are known for all frequencies $\omega \in [0, 2\pi)$
- For finite-length $x[n]$ ($x[n] \neq 0, 0 < n < N-1$), $x[n]$ can be reconstructed using N suitably selected frequency points.
- DTFT is defined as:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \tag{9}$$

- Provided that $x[n]$ has finite duration then only N elements of the sum in non-zero

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n]e^{-jn\omega} \tag{10}$$

Discrete Fourier Transform II

- By uniformly sampling one period of $X(e^{j\omega}), \omega \in [0, 2\pi)$ (frequency resolution being $2\pi/N$) we use frequencies:

$$\omega[k] = \frac{2\pi k}{N}, \quad 0 \leq k \leq N-1, \qquad (11)$$

  and the original $x[n]$ can be uniquely reconstructed using these points.
- **Discrete Fourier Transform** is thus defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j2\pi nk/N}, \quad 0 \leq k \leq N-1 \qquad (12)$$

- Sequence of complex-valued signals $X[k]$ is called N-point DFT of discrete sequence $x[n]$
- If the formula (12) is evaluated for all k (not just $0 \leq k < N$), an infinite periodic DFT image with period N is obtained (denoted by $\tilde{X}[k]$).

Discrete Fourier Transform III

- **Inverse Discrete Fourier Transform** is defined by:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j2\pi nk/N}, \quad 0 \le n \le N-1 \qquad (13)$$

- Again, if the formula (13) is evaluated for all n, an infinite periodic extension of $x[n]$ with period N is obtained (denoted by $\tilde{x}[n]$)
- Equations (12) and (13) form the DFT pair

$$x[n] \overset{DFT}{\Longleftrightarrow} X[k] \qquad (14)$$

- By application of the N-point DFT to a digital signal, a N-point **discrete complex spectrum** $X[k]$ is obtained, which corresponds to sampling of the DTFT spectrum at frequencies $\frac{2\pi k}{N}, \ k = 0 \dots N-1$
- DFT is advantageous due to simplicity of the computation on digital computers
- EXAMPLE: Computation of 4-point DFT.

Thank you for attention!