Digital Signal Processing

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Part I

Application of DTFT to impulse response $h[n]$, frequency response

Frequency response - motivation

- Both difference equation (LCCDE) and impulse response describe the LTI system in the time-domain
- These system models do not provide any information, how the LTI system influences the DTFT spectrum of a signal
- The influence of system on an input signal in the frequency domain is given by the **frequency response**

What is frequency response:

- The response of a system given by impulse response $h[n]$ to signal $x[n]$ is given in the time-domain via

$$y[n] = h[n] * x[n]. \tag{1}$$

- Applying the convolution theorem of the DTFT gives us the response in the frequency domain

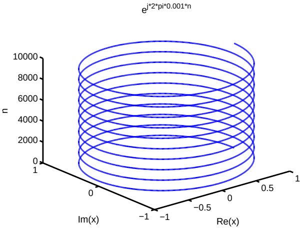
$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}), \tag{2}$$

$Y(e^{j\omega})$ - DTFT spectrum $y[n]$, $X(e^{j\omega})$ - DTFT spectrum $x[n]$

- Thus, the frequency response is a ratio between the output spectrum $Y(e^{j\omega})$ and the input spectrum $X(e^{j\omega})$.

Eigenfunctions of the LTI system:

- Sequences that pass through the LTI system unchanged, up to a change in *complex amplitude*
- If input into the system is $x[n]$, then output $y[n] = \lambda x[n]$, where $\lambda \in \mathcal{C}$ is the *eigenvalue* corresponding to the eigenfunction $x[n]$.
- The complex eigenvalue has absolute value and argument, which can be interpreted as *amplification* and *delay* of the eigenfunction $x[n]$, respectively.
- Eigenfunction of the LTI systems have the form $x[n] = e^{jn\omega_0}$ $\omega_0 \in \mathcal{R}$, $-\infty < n < \infty$
- QUESTION: Why complex exponentials?



Frequency response III

- Eigenvalue corresponding to the complex exponential $e^{j\omega_0 n}$ is denoted $H(e^{j\omega_0})$
- The function, which describes the dependency of the eigenvalues on frequency ω is denoted by $H(e^{j\omega})$ and called the **frequency response** (FR) of the LTI system
- FR states, how the complex exponential on frequency ω_0 is **amplified** and **delayed** when passed through LTI system
- FR is computed by application of DTFT to impulse response $h[n]$ of the system

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-jn\omega} \tag{3}$$

- FR is a complex-valued function of a real-valued independent variable ω
- FR is the third description of the LTI system (along with impulse response and difference equation)

- Discrete signal can be (using DTFT) decomposed into a spectrum of complex exponentials (or harmonic functions for $x[n] \in \mathcal{R}$)
- If the input into the LTI system is in the form

$$x[n] = \sum_{k=1}^{K} \alpha_k e^{jn\omega_k} \tag{4}$$

then the output is

$$y[n] = \sum_{k=1}^{K} \alpha_k H(e^{j\omega_k}) e^{jn\omega_k} \tag{5}$$

- The response of a system with impulse response $h[n]$ to signal $x[n]$ is given by

$$y[n] = h[n] * x[n], \tag{6}$$

then (due to convolutional theorem of DTFT) this results in the frequency domain into

$$Y(e^{j\omega}) = H(e^{j\omega}) X(e^{j\omega}), \tag{7}$$

$Y(e^{j\omega})$ - DTFT spectrum of $y[n]$, $X(e^{j\omega})$ - DTFT spectrum of $x[n]$

Properties of the frequency response

- **Periodicity**
- Frequency response is *periodical* with period 2π
- QUESTION: Why is it so?
- **Symmetry** for systems with real-valued impulse response $h[n]$
- For such a system, FR is *conjugate symmetric* function of ω
  $H(e^{-j\omega}) = H^*(e^{j\omega})$
- This stems from the symmetry properties of the DTFT
- Due to this, the cosine and sine functions are eigenfunctions of such systems along with the exponentials

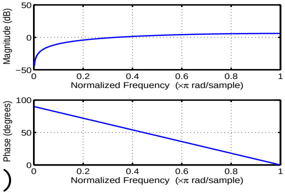
- FR is usually formulated as a pair of functions:
  - *magnitude response* $\left|H(e^{j\omega})\right|$
  - *phase response* $\phi(e^{j\omega})$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\phi(e^{j\omega})} \tag{8}$$

- A more practical substitute of phase characteristic is the **phase delay**

$$\tau_p(\omega) = -\frac{\phi(e^{j\omega})}{\omega} \tag{9}$$

- It states the delay of signal $e^{j\omega n}$ after a pass through LTI system in samples (in contrast to the angle given by phase characteristic)
- MATLAB: `[PHI,W]=phasedelay(b,a,N)`



- Another measure of the delay when signal passes through the LTI systems is the **group delay**

$$\tau_g(\omega) = -\frac{d\phi(e^{j\omega})}{d\omega} \tag{10}$$

- It states a delay (in samples) of a narrow-band signal consisting of a "group" of harmonic components with frequencies close to ω_0
- Let us consider a signal $a[n]$ modulated by a carrier harmonic wave $\cos(\omega_0 n)$, i.e., $x[n] = a[n] \cdot \cos(\omega_0 n)$. The group delay gives a shift of the amplitude envelope $a[n]$ when $x[n]$ passes through the LTI system with phase response $\phi(e^{j\omega})$.
- Phase and group delay are equal at systems with linear phase:

$$\phi(e^{j\omega}) = -\alpha\omega, \tau_p(\omega) = -\frac{-\alpha\omega}{\omega}, \tau_g(\omega) = -\frac{-d\alpha\omega}{d\omega} \tag{11}$$

- MATLAB: `[Gd,W]=grpdelay(b,a,N)`

Part II

Other applications of DTFT

DTFT applications I

- **Frequency response** of a system given by a difference equation
- Frequency response $H(e^{j\omega})$ is the DTFT of the impulse resp. $h[n]$
- FIR systems are often given by $h[n]$, the computation of $H(e^{j\omega})$ is straightforward there
- IIR systems are usually given by their difference equation:

$$y[n] = \sum_{k=0}^{q} b[k]x[n-k] - \sum_{k=1}^{p} a[k]y[n-k] \tag{12}$$

- QUESTION AND EXAMPLE: How do we obtain frequency response from the difference equation?
- Due to linearity and the shift theorem, $H(e^{j\omega})$ is given by

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{q} b[k]e^{-jk\omega}}{1+\sum_{k=1}^{p} a[k]e^{-jk\omega}} \tag{13}$$

- MATLAB: `[H,W]=freqz(b,a,N)`

DTFT applications II

- **Computation of convolutions**: (for infinite sequences)
- DTFT maps convolution in the time-domain into multiplication in the frequency domain
- This fact is used to evaluate convolutions of infinite sequences
- EXAMPLE: Computation of convolution using DTFT

DTFT applications III

- **Analytical solution of difference equations** with zero initial conditions
- *In other words:* analytical computation of filtering
  1. Substitution of the input signal and transformation of the equation into the frequency-domain
  2. Expression of the output
  3. Inverse DTFT
- EXAMPLE: Analytical solution of a difference equation using DTFT

DTFT applications IV

- **Inverse system** to a system given by impulse response $h[n]$ is such a filter, whose impulse response $g[n]$ fulfills

$$h[n] * g[n] = \delta[n]. \tag{14}$$

- Frequency response $G(e^{j\omega})$ therefore fulfills

$$G(e^{j\omega}) = \frac{1}{H(e^{j\omega})} \tag{15}$$

- Not every system is practically invertible (the inverse system may be unstable)
- Inverse system to a causal system may be non-causal
- *In other words:* the existence of a causal and stable inverse system is not always guaranteed
- EXAMPLE: Inverse system and its causality

Thank you for attention!