Digital Signal Processing

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Part I

Introduction to digital filtering

Introduction to digital filtering

Digital filtering:

- Change of magnitude/phase of the spectrum
- **Magnitude filtration:** (more common) Change of the magnitude spectrum, while retaining the phase spectrum
- Delay is the only allowed distortion, (in the ideal case) does not deform the signal in the time-domain (if the signal spectrum lies in the pass-band of the filter)
- **Phase filtration:** Equalization of the phase spectrum, while maintaining the magnitude spectrum, performed by all-pass filters
- **Application:**
  - *Noise reduction:* bio-signals - ECG/EEG, digitization of analog recordings
  - *Frequency band emphasizing:* Equalization of audio signals, line detectors (high frequencies) in image data
  - *Band limitation:* Sampling (aliasing prevention), radio/tv transmission
  - *Zeroing of specific frequency components:* Suppression of DC component, suppression of power-line frequency
  - Special operations: Differentiations, Integrations, Hilbert transform

Analog vs digital filtering

Analog filtering:

- Works on continuous signals, in electronic circuits from amplifiers, resistors, capacitors ...
- In theory, unlimited frequency band, in practice limited by utilized technology
- Sensitive to noise, quality of components, limited flexibility and repeatability

Digital filtering:

- Operates with discrete signals, implemented in computers or specialized hardware (signal processors)
- Limited frequency band, equal to half of the sampling frequency
- Arbitrary level of accuracy, highly linear and flexible (adaptive filtering), perfectly repeatable.
- Allows operations, which cannot be (or with high difficulty) performed in analog domain (pure delay, adaptive filtering)
- Requires pre/post-processing (A/D-D/A conversion), anti-aliasing.

Part II

Design of frequency selective filters

Design of frequency selective filters

- Stems from spectral analysis and analysis of LTI systems via Z-transform
- Digital LTI filters are often described by the system function in the form

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b[k]z^{-k}}{\sum_{k=0}^{p} a[k]z^{-k}} = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}. \quad (1)$$

- Filter having $p \geq 1$ are denoted as *IIR*, these have infinite impulse response $h[n]$ because of the feedback
- Filters having $p = 0$ are denoted as *FIR*, these have finite impulse response $h[n]$, the non-zero samples of $h[n]$ are equal to coefficients $b[k]$
- RCSR - Real, causal, stable, rational

Filter specifications (of magnitude)

Filter specifications:

- Define requirements on magnitude properties of the filter
- *Low-pass, high-pass, band-pass, band-stop, multi-band filters*
- Real-valued filter coefficients desired, magnitude of $H(e^{j\omega})$ is thus even and phase odd
- Specifications are thus necessary only in the band $0 < \omega < \pi$

Filter specifications - parameters:

- Pass band / Stop band / Transition band
- Cutoff frequency for pass-band ω_p / stop-band ω_s
- Positive δ^+/ negative δ^- tolerance in the pass-band
- Desired magnitude in the pass-band is 1
- Tolerance in the stop-band δ_s
- Desired magnitude in the stop-band is 0

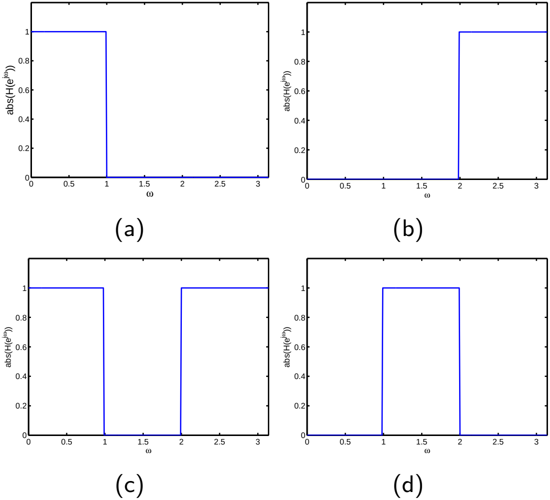


Figure: (a) Low-pass (b) High-pass (c) Band-pass (d) Band-stop

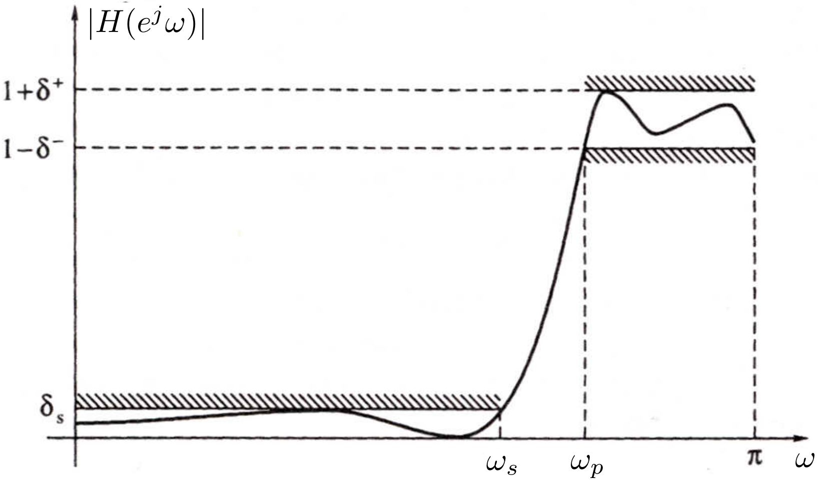
Filter specifications (of magnitude) II

Filter specifications - parameters: (continued)

- Pass band ripple: $\max(\delta^+, \delta^-)$
- Pass band ripple in dB: $A_p = max\{20\log_{10}(1+\delta^+), -20\log_{10}(1-\delta^-)\}$
- Tolerances δ are usually stated differently for FIR and IIR filters
- **IIR:** $\delta^+ = 0, \delta^- = \delta_p$ - Maximal magnitude of IIR is 1
- **FIR:** $\delta^+ = \delta^- = \delta_p$ - Middle magnitude of FIR is 1
- Filter specifications for IIR filters are sometimes (for historical reasons from analog filters) given as magnitude squared
- Attenuation in the stop-band in dB: $A_s = -20\log_{10}(\delta_s)$

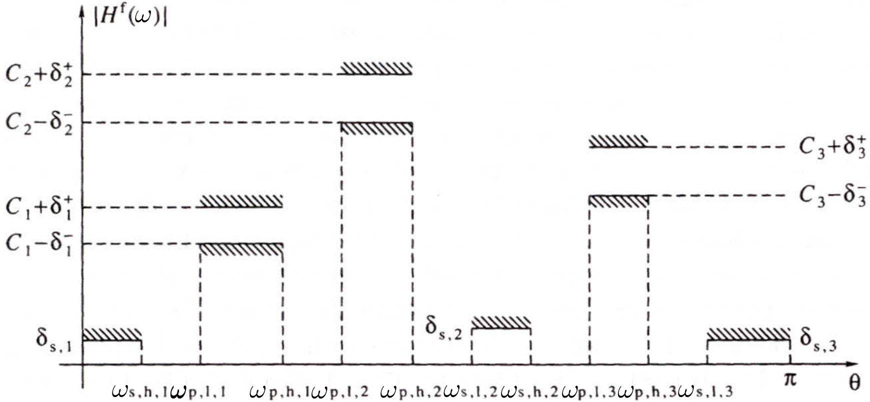
Multi-band filters:

- Allow various gain or attenuation for several frequency bands
- **Piece-wise constant multi-band filter:**
- Splits the frequency interval $[0, \pi]$ into a finite number of bands; some of these are pass-/stop-bands, the rest are transitional bands
- Each pass-/stop-band has its own tolerances δ and amplifications A_p, A_s



Filter specifications for high-pass filter

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing



Filter specifications for multi-band filter

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Phase response of a filter

- **Phase characteristic** states the change in phase of a given frequency component at the output of the filter
- **Phase distortion** appears, if the components present in the signal are shifted by varying phase delay
- This distortion influences the shape of the signal in the time-domain
- We will focus on filters, which (almost) do not distort the signal in the time-domain. These are denoted as **filters with linear phase**
- Digital filter exhibits linear phase, if

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega}, \quad \alpha \in \mathcal{R} \tag{2}$$

  $A(e^{j\omega}) \in \mathcal{R}$ - Amplitude (can be positive or negative)
- The frequency response described in this manner (using $A(e^{j\omega})$ instead of $|H(e^{j\omega})|$) exhibits **continuous phase**
- DETAILS: Continuous representation of the phase response
- Other type of a filter, which does not distort *envelope of modulated signals*, is called **generalized linear phase** filter
- Digital filter exhibits generalized linear phase, if

$$H(e^{j\omega}) = A(e^{j\omega})e^{-j\alpha\omega-\beta}, \quad \alpha, \beta \in \mathcal{R} \tag{3}$$

Phase response of a filter II

- **Group/ phase delay** is defined as

$$\tau_g(\omega) = -\frac{d\phi(e^{j\omega})}{d\omega}, \ \tau_p(\omega) = -\frac{\phi(e^{j\omega})}{\omega} \tag{4}$$

  where $\phi(e^{j\omega})$ is the phase response of the filter
- Unit of τ_g, τ_p is sample
- Systems with linear phase exhibit constant group/phase delay

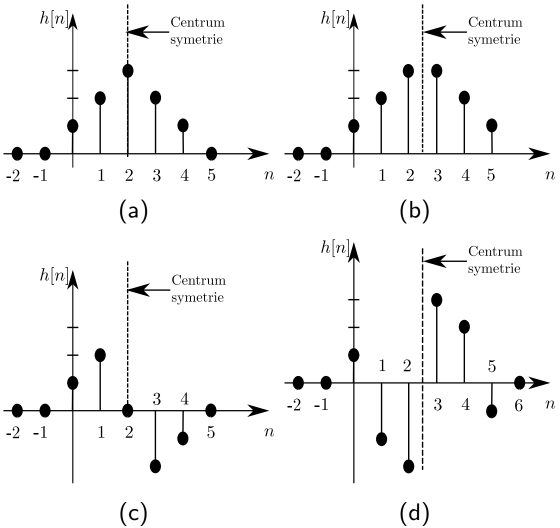
$$\tau_g(\omega) = \tau_p(\omega) = \alpha, \tag{5}$$

  that is, they only delay the signal and do not deform it in the time-domain (if the signal spectrum lies in the pass-band of the filter).
- Group/ phase delay *need not* be integer
- DETAILS: Non-integer group delay

Phase response of a filter III

- As stated in prior lectures, the filter with linear phase need to be a FIR system
- Based on a type of impulse response $h[n]$ with length $N+1$ there exist four types of the filter with (generalized) linear phase:
    1. 1.typ - symmetric $h[n]$, N is even, linear phase
    2. 2.typ - symmetric $h[n]$, N is odd, linear phase
    3. 3.typ - anti-symmetric $h[n]$, N is even, generalized linear phase
    4. 4.typ - anti-symmetric $h[n]$, N is odd, generalized linear phase
- N - order of FIR filter
- Group delay of these FIR filters is equal $\tau_g = N/2$
- Phase delay is for filters of type 1. and 2. equal $\tau_p = N/2$

Phase response of a filter IV



(a) Impulse response of a linear-phase FIR filter - type I, (b) Impulse response of a linear-phase FIR filter - type II
(c) Imp. response of a gen. linear-phase filter - type III, (d) Imp. response of a gen. linear-phase filter - type IV

Part III

Finite impulse response filters

Finite Impulse Response:

- Finite number of non-zero samples within the impulse response $h[n]$
- Described by non-recursive LCCDE

$$y[n] = \sum_{k=0}^{q} b[k]x[n-k] \tag{6}$$

- Always stable, with a system function given by

$$X(z) = \sum_{k=0}^{q} b[k]z^{-k} = C\sum_{k=1}^{q}(1-\beta_k z^{-1}), \tag{7}$$

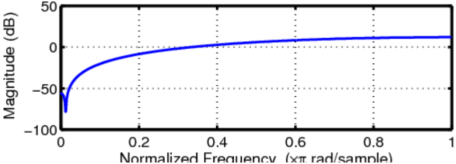
  thus with q zeros and q-fold pole in the origin of the z-plane.
- FIR filters can be designed with linear phase, i.e., they do not distort signal (if its spectrum lies in the pass-band of the filter).

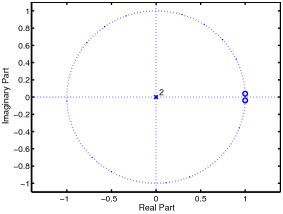
Part IV

Notch filter design

Notch filter - FIR

- Completely suppresses single frequency component
- Designed by setting a null in z-plane on a specific frequency ω_0
- EXAMPLE: Suppression of the power-hum



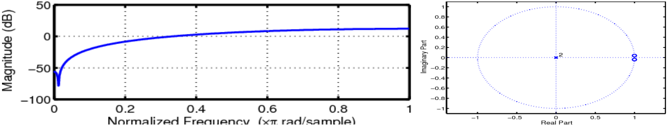


Notch filter - Magnitude response, Z-plane

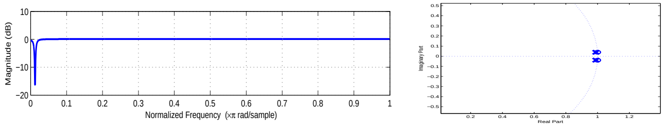
Notch filter - IIR

- Completely suppresses single frequency component
- Designed by setting a null in z-plane on a specific frequency ω_0
- A pole is set "close" to the zero to suppress distortion of neighbor frequencies
- EXAMPLE: Suppression of the power-hum
- $H(z) = \frac{z^2 - 1.9985z^1 + 1}{z^2 - 1.9796z^1 + 0.9801} = \frac{(z - e^{\frac{j\cdot 2\cdot\pi\cdot 50}{8000}})(z - e^{-\frac{j\cdot 2\cdot\pi\cdot 50}{8000}})}{(z - 0.99\cdot e^{\frac{j\cdot 2\cdot\pi\cdot 50}{8000}})(z - 0.99\cdot e^{-\frac{j\cdot 2\cdot\pi\cdot 50}{8000}})}$

FIR:



IIR:



Part V

Impulse response truncation, filter design using windows

Impulse Response Truncation I

Impulse Response Truncation: (IRT)

- Impulse response $h[n]$ of an ideal filter is infinite and non-causal
- However, it has finite energy, it is thus possible to "truncate" it
- Truncated and shifted impulse response the corresponds to a filter, which approximates the ideal filter
- The longer is the part of the impulse response, which is preserved, the more the truncated filter approaches to the ideal filter
- The ideal band-pass filter has the amplitude given by

$$A(e^{j\omega}) = \begin{cases} 1, & \omega_1 \leq |\omega| \leq \omega_2 \\ 0, & \text{else} \end{cases} \quad (8)$$

- Low-pass filter is obtained if $\omega_1 = 0$, the high-pass if $\omega_2 = \pi$
- The linear phase is desired, then the phase $e^{j\phi(e^{j\omega})}$ is given by (FIR filter of type I or II)

$$\phi(e^{j\omega}) = \begin{cases} -\omega N/2, & \omega_1 \leq |\omega| \leq \omega_2 \\ 0, & \text{else} \end{cases} \quad (9)$$

Impulse Response Truncation II

- Impulse response of the desired band-pass is obtained by

$$\begin{aligned} h_{BP}[n] = & \quad \frac{1}{2\pi}\int_{-\omega_2}^{-\omega_1} e^{j\omega(n-0.5N)}d\omega + \frac{1}{2\pi}\int_{\omega_1}^{\omega_2} e^{j\omega(n-0.5N)}d\omega \\ = & \quad \frac{\omega_2}{\pi}\mathrm{sinc}\left[\frac{\omega_2(n-0.5N)}{\pi}\right] - \frac{\omega_1}{\pi}\mathrm{sinc}\left[\frac{\omega_1(n-0.5N)}{\pi}\right] \end{aligned} \tag{10}$$

- Low-pass filter is achieved by substitution $\omega_1 = 0$

$$h_{LP}[n] = \frac{\omega_2}{\pi}\mathrm{sinc}\left[\frac{\omega_2(n-0.5N)}{\pi}\right] \tag{11}$$

- If N is even, high-pass is achieved by substitution $\omega_2 = \pi$ (cannot be designed for N odd, FIR of type II is not suitable for high-pass):

$$h_{HP}[n] = \delta[n-0.5N] - \frac{\omega_1}{\pi}\mathrm{sinc}\left[\frac{\omega_1(n-0.5N)}{\pi}\right] \tag{12}$$

Multi-pass filters

- Amplitude response of an ideal multi-pass filter is obtained by superposition of K suitable band-pass filters

$$A(e^{j\omega}) = \sum_{k=1}^{K} A_k(e^{j\omega}), \tag{13}$$

kde

$$A_k(e^{j\omega}) = \begin{cases} C_k, & \omega_{1,k} \le |\omega| \le \omega_{2,k} \\ 0, & \text{else} \end{cases} \tag{14}$$

- The impulse response of the ideal multi-band filter is thus given by

$$h_{MB}[n] = \sum_{k=1}^{K} \frac{C_k}{\pi} \left\{ \omega_{2,k} \mathrm{sinc} \left[\frac{\omega_{2,k}(n-0.5N)}{\pi} \right] - \omega_{1,k} \mathrm{sinc} \left[\frac{\omega_{1,k}(n-0.5N)}{\pi} \right] \right\} \tag{15}$$

- If N is even, (FIR type II is not suitable for na HP), then band-stop filter for $K = 2$ is obtained by substitution $C_1 = C_2 = 1, \omega_{1,1} = 0, \omega_{2,2} = \pi$

Impulse Response Truncation IV

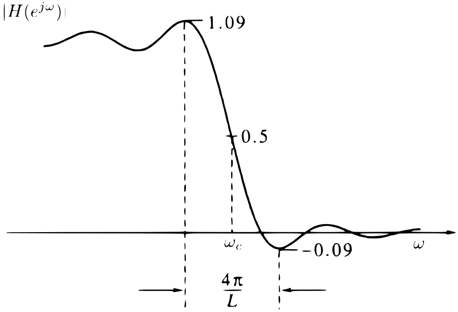
Filter order selection

- Impulse response $h_{ID}[n]$ of an ideal filter is infinite
- By truncation of $h_{ID}[n]$ we obtain an impulse response $h[n]$ of a filter, which differs from the ideal one by ripples δ_p/δ_s in the pass-/stop-band and existence of a transitional band
- With increasing order of the filter N, the transitional band becomes narrower; the ripples remain practically the same - *Gibbs phenomenon*
- The truncation of the impulse response can be described by

$$h[n] = h_{ID}[n]w_r[n] \tag{16}$$

  where $w_r[n]$ is rectangular window of length $L = N + 1$ (N - filter order).
- **Transitional band width** depends on the width of the main-lobe of a Dirichlet kernel and is equal $4\pi/L$
- **Ripples** δ_p, δ_s depend on the magnitude of a side-lobe of Dirichlet kernel and is practically independent of window $w_r[n]$ length
- Ripples $\delta_p \approx \delta_s \approx 0.09$, that is $A_p = 0.75$dB, $A_s = 21$dB.

- Ripples δ achieved using $w_r[n]$ are unsuitable for many real-world applications. These can only be suppressed by utilization of other windows (with lower magnitude of the side-lobes) when truncating the impulse response - Filter design using windows



Transitional band of a filter designed by the IRT method

SOURCE: BOAZ PORAT, A Course in Digital Signal Processing

Filter design using windows:

- Generalization of the Impulse Response Truncation method
- Truncation of the impulse response can be described by

$$h[n] = h_{ID}[n]w[n], \tag{17}$$

  $w[n]$ is a window of length $L = N + 1$ (N - desired filter order)
- The type of the window influences the properties of the filter, because

$$H(e^{j\omega}) = \frac{1}{2\pi}\left\{H_{ID}(e^{j\omega}) * W(e^{j\omega})\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{ID}(e^{j\lambda})W(e^{j\omega-\lambda})d\lambda \tag{18}$$

- This is smoothing of $H_{ID}(e^{j\omega})$ using the spectrum of the window $W(e^{j\omega})$
- **Main-lobe width:** influences the width of the transitional band
- **Side-lobe magnitude:** influences the size of the pass-/stop-band ripples

Filter design using windows II

Design algorithm:

1. According to allowed ripple size, the type of window is selected
2. According to desired transitional band width, the order/length of $h[n]$ is selected ($N = L - 1$)
3. According to equations stated for method IRT, L coefficients of the impulse response are computed, which are multiplied by the selected window

MATLAB: `B=FIR1(n,Wn)`

Parameters of frequently used windows:

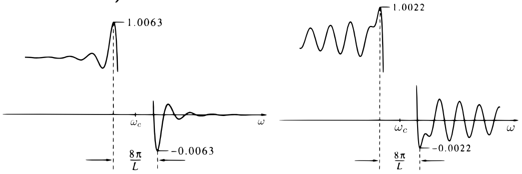
Window	Main-lobe width	Side-lobe magnitude [dB]	δ_p, δ_s
Rectangular	$4\pi/L$	-13,5	0.09
Bartlett	$8\pi/L$	-27	0.05
Hann	$8\pi/L$	-32	0.0063
Hamming	$8\pi/L$	-43	0.0022
Blackman	$12\pi/L$	-57	0.0002
Kaiser	Depends on α		

- $A_p = \max\{20\log_{10}(1+\delta^+), -20\log_{10}(1-\delta^-)\} \approx 8.6859\max(\delta^+, \delta^-)$
- $A_s = -20\log_{10}(\delta_s)$

Filter design using windows III

Properties of the commonly used windows

- **Bartlett window:** Exhibits monotonous magnitude (without ripples) in the vicinity of the transitional band
- Magnitude tolerance is therefore not well defined, it is usually stated as $\delta_p = \delta_s = 0.05$
- The decrease of ripples with respect to rectangular window is rather small, but the transitional band is twice as wide ($8\pi/L$)
- **Hann a Hamming window:** Equally wide transition band (slightly less than $8\pi/L$)
- *Hann:* Larger magnitude of ripples, more distinct ripple attenuation than Hamming
- *Hamming:* Lower magnitude of ripples, gradual ripple attenuation (compared to Hann)



Ripples of filters designed using Hann and Hamming windows

Filter design using windows IV

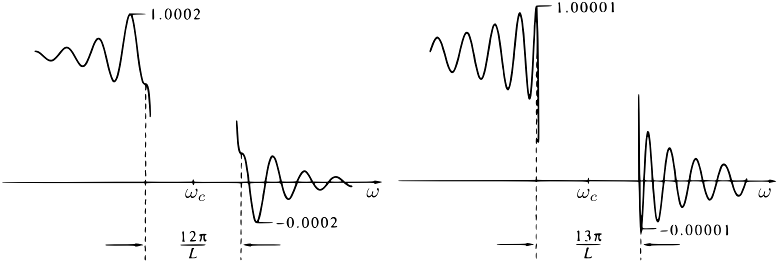
Properties of the commonly used windows (continuation)

- **Blackman window:** Wide transition band ($12\pi/L$), very small ripples $\delta_p = \delta_s = 0.0002$
- **Kaiser window:** Frequently used window for filter design
- Its shape and properties can be altered using free parameter α
- Empiric formulas have been designed to relate α and specifications of the designed filter
- The order of the filter is selected as a nearest integer above the order N computed via (21)
- The formulas are empiric, the designed filter is not guaranteed to fulfill the desired specifications (usually, a higher order needs to be selected)

$$A = -20\log_{10}(\min\{\delta_p, \delta_s\}) \tag{19}$$

$$\alpha = \begin{cases} 0.1102(A-8.7), & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21), & 21 < A \leq 50 \\ 0, & A \leq 21 \end{cases} \tag{20}$$

$$N = \frac{A - 7.95}{2.285|\omega_p - \omega_s|} \tag{21}$$



Ripples of filters designed using Blackman a Kaiser window ($\alpha = 10$)

Source: BOAZ PORAT, A Course in Digital Signal Processing

Thank you for attention!