Digital Signal Processing

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Part I

Z-transform

Z-transform - motivation

- The utilization of DTFT and DFT is very useful for analysis of signals (spectral analysis)
- However, it cannot analyze easily some important properties of systems/filters
  - Stability cannot be simply determined
  - Causality cannot be easily work with (important for realizable filters, inverse systems computation)
- For analysis of these properties, a generalization of DTFT denoted by **Z-transform** is used

Z-transform I

- A transform generalizing DTFT for signals, which are not absolutely summable
- Applications in signal processing:
  1. LTI system analysis (stability, causality)
  2. Filter design
  3. Solving of difference equations with initial conditions
- DTFT is defined by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega} \tag{1}$$

for sequences $x[n]$, which are absolutely summable, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| = S < \infty \tag{2}$$

- For many common sequences, the DTFT does not exist/converge, e.g.,

$$x[n] = u[n] \tag{3}$$

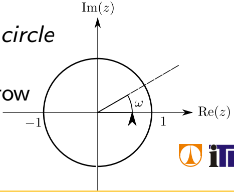
Z-transform II

- **Z-transform** of discrete sequence $x[n]$ is defined by:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{4}$$

  $z = re^{j\omega}, z \in \mathcal{C}, r \in \mathcal{R},\ r > 0$
- The z-transform output is a complex function of a complex variable. Its properties are commonly described in *the z-plane*.
- The transform is denoted by $x[n] \overset{Z}{\Longleftrightarrow} X(z)$
  $X(z)$ - Output of the z-transform applied to $x[n]$
- QUESTION: What is the relation of Z-transform and DTFT?

- **Unit circle:**
- DTFT output can be obtained from the Z-transform output by substitution $z = e^{j\omega}$
- DTFT output thus consists of point located on *the unit circle* in the z-plane
- Point $z = 1$ corresponds to $\omega = 0$ and the frequencies grow counter-clockwise (e.g., $z = j$ corresponds to $\omega = \pi/2$)



Region of Convergence - ROC:

- the area in the z-plane, where the z-transform converges (the series (4) has a finite sum)
- EXAMPLE: Computation of z-transform and the ROC
- *Beware:* Z-transform output is uniquely determined, only when the ROC is known
- *ROC properties:*
    - Region of convergence is an annulus in the form $\alpha < |z| < \beta$
    - When $x[n]$ has *finite duration* the ROC consists of the whole z-plane except $z = 0$ and $z = \infty$.
      $z = \infty$ belongs to the ROC if $x[n]$ is right-sided
      $z = 0$ belongs to the ROC if $x[n]$ is left-sided
    - *Right-sided sequence:* ROC in the form $|z| > \alpha$
    - *Left-sided sequence:* ROC in the form $|z| < \beta$

Z-transform IV

- Z-transform output is a complex function of a complex independent variable
- It is visualized as a magnitude and a phase part (both have 3D graph - a real function of a complex variable)

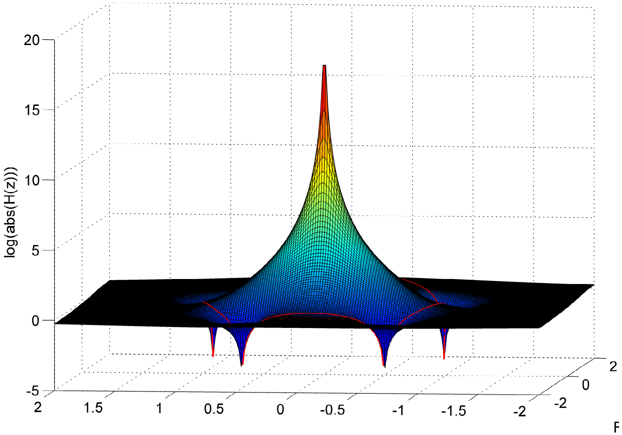
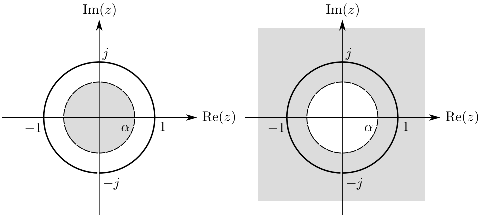


Table of common z-transform pairs



Sequence	Z-transform	ROC
$\delta[n]$	1	z-plane
$\alpha^n u[n]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| > \alpha$
$-\alpha^n u[-n-1]$	$\frac{1}{1-\alpha z^{-1}}$	$\|z\| < \alpha$
$n\alpha^n u[n]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \alpha$
$-n\alpha^n u[-n-1]$	$\frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \alpha$
$\cos[n\omega_0]u[n]$	$\frac{1-(\cos(\omega_0)z^{-1})}{1-2(\cos(\omega_0)z^{-1})+z^{-2}}$	$\|z\| > 1$
$\sin[n\omega_0]u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2(\cos(\omega_0)z^{-1})+z^{-2}}$	$\|z\| > 1$

Z-transform properties I

- **Linearity:**
- Let $X_1(z)$ with ROC R_1 be z-transform output of $x_1[n]$ and $X_2(z)$ with ROC R_2 be z-transform output of $x_2[n]$, then:
- $y[n] = ax_1[n] + bx_2[n] \overset{Z}{\Longleftrightarrow} Y(z) = aX_1(z) + bX_2(z)$
- R_z contains intersection $R_1 \cap R_2$, but can be even larger
- EXAMPLE: ROC of a sum of two sequences

- **Shifting property:**
- Shifting of a sequence by n_0 samples leads to
- $x[n - n_0] \overset{Z}{\Longleftrightarrow} z^{-n_0} X(z)$
- ROC remains unchanged
  (with the exception of points 0 or ∞)

Z-transform properties II

- **Time-reversal:** of sequence $x[n]$ leads to
- $x[-n] \overset{Z}{\Longleftrightarrow} X(z^{-1})$
- When the original ROC is $R_x = (\alpha < |z| < \beta)$, then ROC corresponding to the shifted sequence is $1/R_x$, i.e., $(1/\beta < |z| < 1/\alpha)$

- **Multiplication by an exponential:** leads to scaling of the z-plane
- $\gamma^n x[n] \overset{Z}{\Longleftrightarrow} X(\gamma^{-1} z)$
- The time-domain multiplication scales also the ROC
- Let ROC of the original sequence be $R_x = (\alpha < |z| < \beta)$, then ROC of the multiplied sequence R_y takes the form $(|\gamma|\,\alpha < |z| < |\gamma|\,\beta)$

Z-transform properties III

- **Convolutional theorem:**
- Important property, convolution of two signals in the time-domain maps as multiplication of z-transform outputs in the z-domain
- $y[n] = h[n] * x[n] \overset{Z}{\Longleftrightarrow} Y(z) = H(z)X(z)$
- R_z contains intersection $R_h \cap R_x$, but can be even larger
- EXAMPLE: Convolution of two sequences using z-transform

- **Initial value theorem:**
- If $x[n]$ is a right-sided sequence then $x[0]$ can be found using:
- $x[0] = \lim_{z \to \infty} X(z)$
- QUESTION: Why is it so?

Part II

Description of the LTI systems using z-transform

Transfer (system) function

- Response of the LTI system given by *impulse response* $h[n]$ to the input $x[n]$ is given by convolution

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \tag{5}$$

- Application od DTFT to (5) gives the relation of spectra of the input and output signals via the *frequency response*

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \tag{6}$$

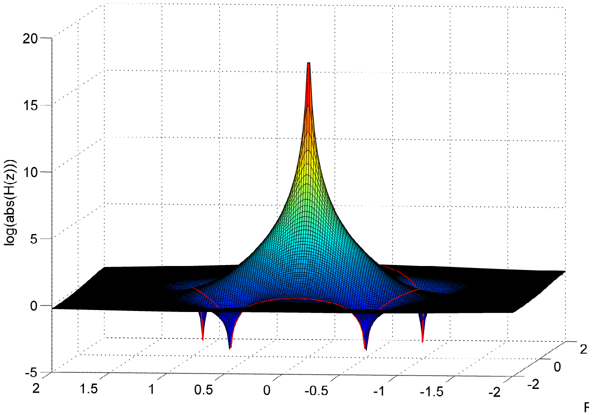
- Similarly, the application of z-transform to (5) gives the relation between z-transform outputs of the input/output signals via a *transfer function* $H(z)$

$$Y(z) = H(z)X(z) \tag{7}$$

- Transfer function is very important from the perspective of LTI system analysis (stability, causality, inverse systems etc.)

Transfer (system) function II

- **Transfer function** is a complex-valued function of a complex independent variable
- It is visualized as a magnitude part and a phase part (both by a 3D graph - a real-valued function of a complex independent variable)



Transfer (system) function III

- **Transfer function** is thus given as a z-transform if the *impulse response* of the LTI system

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \tag{8}$$

- *Frequency response* is obtained by evaluation of $H(z)$ on the unit circle

$$H(e^{j\omega}) = H(z)|_{z=e^{j\omega}} \tag{9}$$

- For the LTI system described via a recursive *difference equation* (i.e., an IIR system)

$$y[n] + \sum_{k=1}^{p} a[k]y[n-k] = \sum_{k=0}^{q} b[k]x[n-k] \tag{10}$$

the transfer function is a rational function in the (normalized) form

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b[k]z^{-k}}{1 + \sum_{k=1}^{p} a[k]z^{-k}}. \tag{11}$$

Transfer (system) function IV

- **Zeros and poles:**
- LTI system with an infinite impulse response can be described using the transfer function in the form

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{q} b[k]z^{-k}}{\sum_{k=0}^{p} a[k]z^{-k}} \tag{12}$$

- *Alternatively:* The numerator and the denominator can be factorized into the product form

$$H(z) = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})} \tag{13}$$

- Roots of the numerator - *zeros* (β_k), roots of the denominator - *poles* (α_k)

- **Pole-zero plot** - a diagram depicting the location of all zeros (o) and poles (x) in the z-plane
- MATLAB: `zplane(B,A)`
- **Region of converge** is also often shown in the pole-zero plot
- For a rational $H(z)$ the ROC does not contain any poles

Transfer (system) function V

- In the z-domain, the LTI system is uniquely given (up to a gain C) by its zeros and poles
- The term $(1 - \beta_k z^{-1}) = \frac{z-\beta_k}{z}$ contributes to the transfer function the zero β_k and the pole in 0
- The term $\frac{1}{(1-\alpha_k z^{-1})} = \frac{z}{z-\alpha_k}$ contributes to the transfer function the pole α_k and the zero in 0
- Zeros and poles located at the same point cancel each other
- In the case of a real-valued impulse response, the complex zeros/poles form conjugate pairs (e.g., $\beta_1 = z_0$ and $\beta_2 = z_0^*$)

Part III

Inverse z-transform

- Inverse z-transform is used to recover a sequence $x[n]$ from its z-transform $X(z)$
- It is essential for many z-transform related tasks, such as difference equation solving, analytic computation of the convolution etc.
- There are three possible approaches:
    1. Partial fraction expansion
    2. Conversion to power series (polynomial long division)
    3. (Contour integration via Cauchy's integral theorem)

Partial fraction expansion

- Utilized for z-transforms given in the form of rational function

$$X(z) = C\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})} \tag{14}$$

- If $p > q$ and all poles are simple ($\alpha_i \neq \alpha_k$for $i \neq k$)

$$X(z) = \sum_{k=1}^{p}\frac{A_k}{1-\alpha_k z^{-1}} \tag{15}$$

- $A_k \in \mathcal{R}$ are constants computed via

$$A_k = [(1-\alpha_k z^{-1})X(z)]_{z=\alpha_k} \tag{16}$$

- If $p > q$, then long polynomial division of numerator and denominator is performed
- DETAILS: How to proceed, when poles are of a higher order?
- EXAMPLE: Inverse z-transform of a rational $X(z)$
- MATLAB: `residuez(B,A)`

Thank you for attention!