IT-24667
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① Let G be a group order pq, where p and q are distinct primes. prove that G is abelian.

Ans: Claim as stated is "false".
S_3 has order $6 = 2.3$ and is non-abelian.
Correct statement/classification: Let, $|G| = pq$ with p, q primes and $p < q$. Then by sylow theory one of the sylow subgroups is normal. hence G is a semidirect product of the sylow subgroups. In particular:

(i) If $p \nmid (q-1)$ then every homomorphism from a sylow-q subgroup to Aut(sylow-p) is trivial. So, the semidirect product is direct and G is cyclic (hence abelian).

(ii) If $p \mid (q-1)$ a nontrivial semidirect product may exist (and then G can be nonabelian, S_3, when $p = 2$, $q = 3$).

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2. Prove that, if G is a group of order p^2, where p is prime, then G is abelian if and only if it has $p+1$ subgroups of order p.

Ans: If $|G| = p^2$ then G is abelian iff it has $p+1$ subgroups of order p. claimed as "False".
All groups of order p^2 are abelian, but there are two abelian types: C_{p^2} (has 1 subgroup of order p) and $C_p \times C_p$ (has $p+1$ subgroups). So, "abelian" does not force $p+1$ subgroups; the correct equivalence is: G has $p+1$ subgroups of order $p \Leftrightarrow G \cong C_p \times C_p$.

3. Let G be a finite group, of and H be a proper subgroup of G. Prove that the union of all conjugates of H can't be equal to G.

Ans: Union of all conjugates of a proper subgroup H can equal G is a "False" statement (so the statement "can't equal G" is true).

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Proof sketch: let the distinct conjugates be m sets each of size $|H|$. Distinct conjugates intersect in proper subsets, so counting nonidentity elements gives $|G|-1 \leq m(|H|-1)$.
But $m = [G : N_G(H)]$ and
$N_G(H) \geq H$ (so $m < |G|/|H|$), which yield to contradiction. Hence union can't be all of G.

4. Let G be a group and N be a Normal subgroup of G. If G/N is cyclic and N is cyclic, prove that G is abelian.

Ans: "If $N \trianglelefteq G$ and both N and G/N are cyclic, then G is abelian" — False.

Counterexample: S_3 has normal cyclic C_3 and quotient C_2, yet S_3 is nonabelian.

5. Prove that in any group G, the set of elements of finite orders form a subgroup of G.

Ans: In any group the set of elements of finite order is a subgroup is a False.

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Infinite dihedral group D_∞: reflection have order 2, but product of two reflections can be a translation of infinite order, so torsion elements are not closed under product.

6. Let G be a finite group and p be the smallest prime dividing $|G|$. prove that any subgroup of index p in G is normal.

Ans: If p is the smallest prime dividing $|G|$ and $[G:H]=p$ then H is normal is true.

proof sketch: action on cosets $\phi: G \to S_p$: image has order a power of p and transitive $\Rightarrow$ order p.

So, $|\ker\phi| = |G|/p = |H|$, hence $\ker\phi = H$ is normal.

~~7. Let G be a group and H be a subgroup of G. prove that $[G:H]=n$, if~~

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7. Let G be a group and $a, b \in G$. prove that if $a^4 = b^2$ and $ab = ba$, then $(ab)^6 = e$.

Ans: If $a^4 = b^2$ and $ab = ba$ then $(ab)^6 = e$ is false.

counterexample in an infinite cyclic group: take $a = t$, $b = t^2$.

Then $a^4 = t^4 = b^2$ and they commute, but $(ab)^6 = (t^3)^6 = t^{18} \neq e$

[No order info is given. So, we can't force the identity]

8. Let G be a group and H be a subgroup of G, prove that if $[G:H] = n$, then for any $x \in G, x^n \in H$

Ans: If $[G:H] = n$ then for any $x \in G, x^n \in H$ is true.

Reason the orbit of coset H under $\langle x \rangle$ has size $k | n$, so $x^k H = H$ hence $x^k \in H$, then $x^n = (x^k)^{n/k} \in H$.