Answer 1: Given vectors:

$$\mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

$$\mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{a_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) Find the projection p of b onto the subspace spanned by $\mathbf{a_1}$ and $\mathbf{a_2}$.

Orthogonalize the basis vectors using the Gram-Schmidt process:

$$\begin{aligned}
\mathbf{v_1} &= \mathbf{a_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
\mathbf{v_2} &= \mathbf{a_2} - \frac{\mathbf{a_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}} \mathbf{v_1} \\
&= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

Calculate the projection:

$$\begin{aligned}
\mathbf{p} &= (\mathbf{b} \cdot \mathbf{v_1})\mathbf{v_1} + (\mathbf{b} \cdot \mathbf{v_2})\mathbf{v_2} \\
&= 6 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}
\end{aligned}$$

b) Find the error vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$ and show that it is orthogonal to both $\mathbf{a_1}$ and $\mathbf{a_2}$.

$$
\begin{aligned}
\mathbf{e} &= \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} \\
&= \begin{bmatrix} -4 \\ -2 \\ 0 \end{bmatrix}
\end{aligned}
$$

Checking the orthogonality:

$$
\begin{aligned}
\mathbf{e} \cdot \mathbf{a_1} &= 0 \\
\mathbf{e} \cdot \mathbf{a_2} &= 0
\end{aligned}
$$

Answer 2: Given the data points:

$$
(x_1, y_1) = (0, 1), \quad (x_2, y_2) = (1, 8), \quad (x_3, y_3) = (2, 8), \quad (x_4, y_4) = (3, 20)
$$

$$
\begin{aligned}
\sum x &= x_1 + x_2 + x_3 + x_4 = 0 + 1 + 2 + 3 = 6 \\
\sum y &= y_1 + y_2 + y_3 + y_4 = 1 + 8 + 8 + 20 = 37 \\
\sum xy &= x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 = 0 + 8 + 16 + 60 = 84 \\
\sum x^2 &= x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0 + 1 + 4 + 9 = 14
\end{aligned}
$$

Let the Slope be: D

$$
D = \frac{n \sum xy - \sum x \sum y}{n \sum x^2 - (\sum x)^2} = \frac{4(84) - 6(37)}{4(14) - 6^2} = \frac{57}{10}
$$

Let the y-intercept be C

$$
C = \frac{\sum y - D \sum x}{n} = \frac{37 - \frac{57}{10} \times 6}{4} = \frac{7}{10}
$$

Therefore, the line of best fit is:

$$
y = \frac{7}{10} + \frac{57}{10} x
$$

ie:

$$
10y = 57x + 7
$$

Answer 3:

Given the projection matrix:

$$P = A(A^T A)^{-1} A^T$$

a) Show that $P^2 = P$
Multiplying P by itself:

$$\begin{aligned} P^2 &= A(A^T A)^{-1} A^T A (A^T A)^{-1} A^T \\ &= A(A^T A)^{-1} I (A^T A)^{-1} A^T \\ &= A(A^T A)^{-1} A^T \\ &= P \end{aligned}$$

b) Geometric Proof

For any vector b, the projection Pb is the vector in the column space of A closest to b. Thus, the error vector $e = b - Pb$ is orthogonal to the column space of A.

Now, projecting b again using P:

$$P(Pb) = P^2 b$$

Since Pb is already in the column space of A, projecting it again will not change it. Thus:

$$P(Pb) = Pb$$

This proves that $P^2 b = Pb$ for any vector b.

c) Is $I - P$ a Projection Matrix?

The matrix $I - P$ projects onto the orthogonal complement of the column space of A. To show it's a projection matrix, we need to prove that:

$$(I - P)^2 = I - P$$

Expanding:

$$\begin{aligned} (I - P)^2 &= I^2 - 2IP + P^2 \\ &= I - 2P + P \quad \text{(since } P^2 = P \text{ from part a)} \\ &= I - P \end{aligned}$$

Thus, $I - P$ is indeed a projection matrix, and it projects onto the orthogonal complement of the column space of A.

Answer 4:

Given vectors:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Step 1: Start with vector **a**:

$$\mathbf{u_1} = \mathbf{a} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Step 2: Subtract the projection of **b** onto $\mathbf{u_1}$ from **b**:

$$\begin{aligned} \mathbf{u_2} &= \mathbf{b} - \frac{\mathbf{b} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \mathbf{u_1} \\ &= \mathbf{b} - \frac{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} \end{aligned}$$

Step 3: Subtract the projections of **c** onto $\mathbf{u_1}$ and $\mathbf{u_2}$ from **c**:

$$\begin{aligned} \mathbf{u_3} &= \mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \mathbf{u_1} - \frac{\mathbf{c} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \mathbf{u_2} \\ &= \mathbf{c} - \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}}{\begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}} \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \end{aligned}$$

Thus, we have calculated the orthogonal basis vectors, u1, u2 and u3

Answer 5

a) Orthogonality of Q_1Q_2
Given that Q_1 and Q_2 are orthogonal matrices, we have:

$$Q_1^TQ_1 = I \quad \text{and} \quad Q_2^TQ_2 = I$$

To show that Q_1Q_2 is orthogonal, we need to prove:

$$(Q_1Q_2)^T(Q_1Q_2) = I$$

Expanding:

$$\begin{aligned}(Q_1Q_2)^T(Q_1Q_2) &= Q_2^TQ_1^TQ_1Q_2 \\ &= Q_2^TIQ_2 \\ &= Q_2^TQ_2 \\ &= I\end{aligned}$$

b) Orthogonal Vectors and Dot Product
Given orthogonal vectors q_1, q_2, q_3 and the equation:

$$x_1q_1 + x_2q_2 + x_3q_3 = b$$

Taking the dot product with q_i:

$$x_1(q_1 \cdot q_i) + x_2(q_2 \cdot q_i) + x_3(q_3 \cdot q_i) = b \cdot q_i$$

Using the orthonormality conditions:

$$x_i = q_i \cdot b$$

c) Using Orthogonal Vectors to Solve the Equation
Given vectors:

$$q_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}, \quad q_3 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Using part (b) to solve:

$$x_1q_1 + x_2q_2 + x_3q_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}$$

We get:

$$x_1 = q_1 \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad x_2 = q_2 \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}, \quad x_3 = q_3 \cdot \begin{bmatrix}1\\2\\1\end{bmatrix}$$

Thus,The computed values are:

$$x_1 = -\frac{1}{\sqrt{2}}$$

$$x_2 = \frac{1}{\sqrt{6}}$$

$$x_3 = \frac{4}{\sqrt{3}}$$