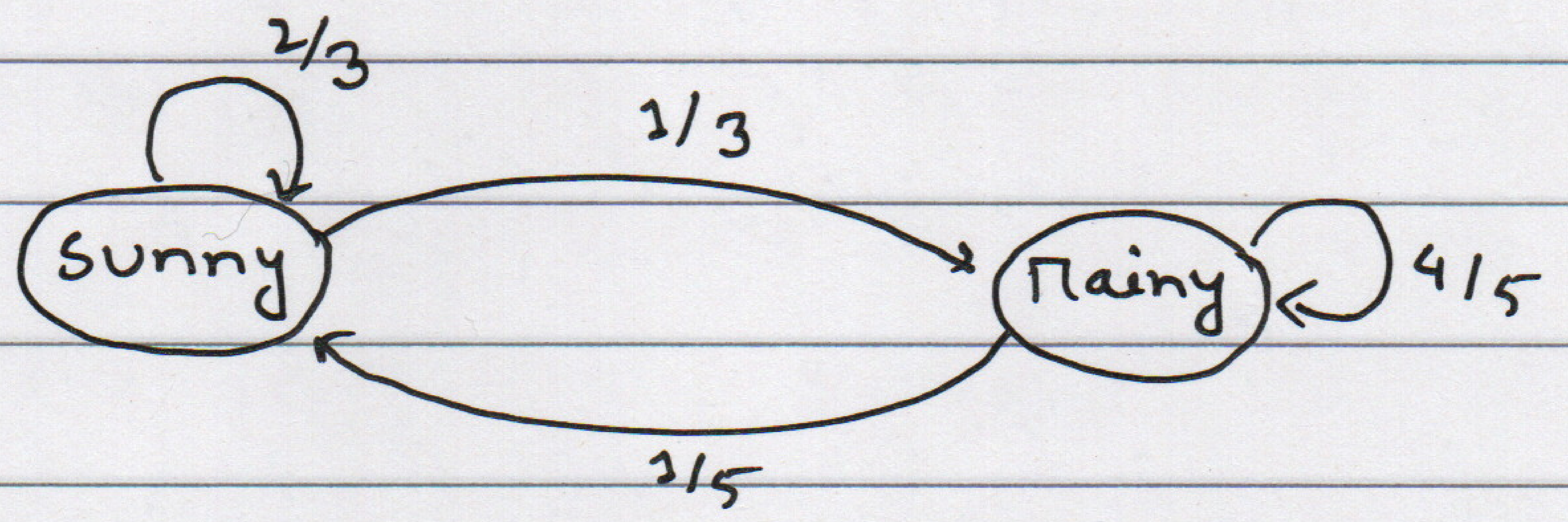
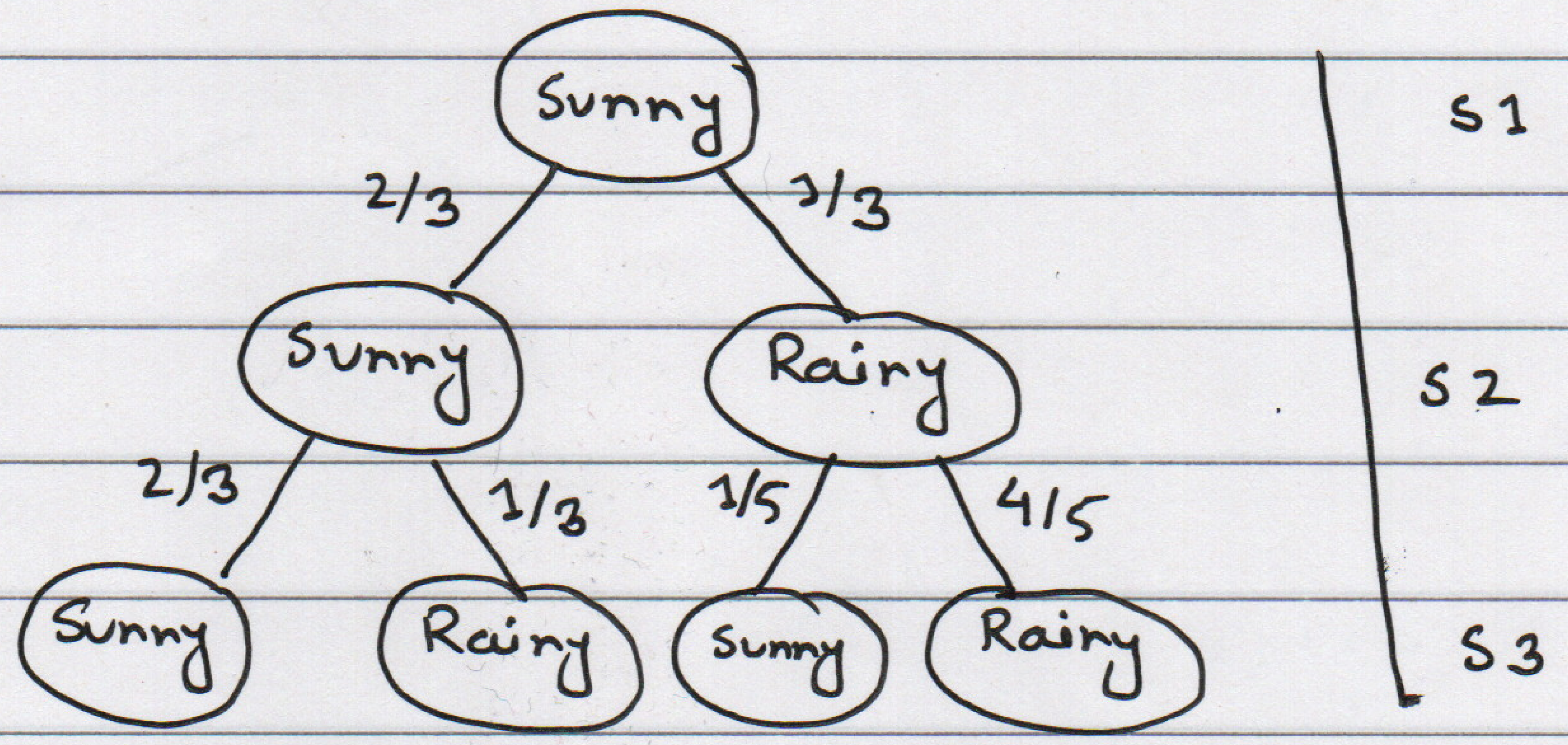
Exercise 1:

(i)





$\mathbb{P}(S_3 = \text{Sunny} \mid S_1 = \text{Sunny}) = \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{5} = \frac{23}{45}$

$\mathbb{P}(S_3 = \text{Rainy} \mid S_1 = \text{Sunny}) = \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{4}{5} = \frac{22}{45}$

$\therefore$ The probability of sunny the day after tomorrow $= \frac{23}{45}$ and the probability of rainy the day

(ii) $P = \begin{bmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{bmatrix}$

The day is sunny, so, $P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Probability in next day, $P_1 = PP_0 = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$

Probability of n days, $P_n = P^n P_0$

Now P, can be written as,

$P = UVU^{-1}$ [here V is the diagonal matrix and U is the matrix whose column correspond to eigen values of P)

Now,

$$\det\begin{pmatrix} 2/3-\lambda & 1/5 \\ 1/3 & 4/5-\lambda \end{pmatrix} = 0$$

$$\Rightarrow \left(\frac{2}{3}-\lambda\right)\left(\frac{4}{5}-\lambda\right) - \frac{1}{15} = 0$$

$$\Rightarrow \frac{8}{15} - \frac{2\lambda}{3} - \frac{4\lambda}{5} + \lambda^2 - \frac{1}{15} = 0$$

$$\Rightarrow \lambda^2 - \frac{22}{15}\lambda - \frac{7}{15} = 0$$

$$\Rightarrow \frac{1}{15}\left(15\lambda^2 - 22\lambda - 7\right) = 0$$

$$\Rightarrow \frac{1}{15}\cdot\frac{1}{15}\left(\lambda - \frac{7}{15}\right)(\lambda - 1) = 0$$

$$\therefore \lambda_1 = \frac{7}{15} \text{ and } \lambda_2 = 1.$$

For, $\lambda_1 = \frac{7}{15}$,

$|A - \lambda I| = 0$

$$\therefore \left(\begin{array}{cc|c} 1/5 & 1/5 & 0 \\ 1/3 & 1/3 & 0 \end{array}\right) = 0$$

$$R_1/\tfrac{1}{5} \to R_1 \quad \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1/3 & 1/3 & 0 \end{array}\right)$$

$$R_2 - \left(\tfrac{1}{3}\right)R_1 \to R_2 \quad \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2$

Let. $x_2 = 1 \quad \therefore v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For, $\lambda_2 = 1$

$|P - \lambda_2 I| = 0$

$$\Rightarrow \left(\begin{array}{cc|c} -1/3 & 1/5 & 0 \\ 1/3 & -1/5 & 0 \end{array}\right) = 0$$

$R_1 / -1/3 \rightarrow R_1$
$$\sim \left(\begin{array}{cc|c} 1 & -3/5 & 0 \\ 1/3 & -1/5 & 0 \end{array}\right)$$

$R_2 - \frac{1}{3}R_1 \rightarrow R_2$
$$\sim \left(\begin{array}{cc|c} 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\therefore x_1 - \frac{3}{5}x_2 = 0$$

$$\Rightarrow x_1 = \frac{3}{5}x_2$$

Let, $x_2 = 1$, $v_2 = \begin{pmatrix} 3/5 \\ 1 \end{pmatrix}$

$$U = \begin{pmatrix} -1 & 3/5 \\ 1 & 1 \end{pmatrix}$$

$$\therefore U^{-1} = \begin{pmatrix} -\frac{5}{8} & \frac{3}{8} \\ \frac{5}{8} & \frac{5}{8} \end{pmatrix}$$

Now, $P^n = (UVU^{-1})^n \cdot P_0$

$$= \begin{pmatrix} -1 & \frac{3}{5} \\ 1 & 1 \end{pmatrix} \begin{bmatrix} (7/15)^n & 0 \\ 0 & 1^n \end{bmatrix} \begin{bmatrix} -5/8 & 3/8 \\ 5/8 & 5/8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-7/15)^n & 3/5 \\ (7/15)^n & 1 \end{bmatrix} \begin{bmatrix} -5/8 \\ 5/8 \end{bmatrix}$$

$$= \begin{bmatrix} (-7/15)^n (-5/8) + 3/5 \cdot 5/8 \\ (7/15)^n \cdot (-5/8) + 5/8 \end{bmatrix}$$

$$= 5/8 \begin{bmatrix} (7/15)^n + 3/5 \\ (-7/15)^n + 1 \end{bmatrix}$$

$$\Rightarrow \lim_{n \to \alpha} p^n = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

$\therefore$ The relative frequency of sunny days is $3/8$.

(iii) The invariant measure is $\pi = (3/8 \quad 5/8)^T$

Now, $P\pi$

$$= \begin{bmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{bmatrix} \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 & 5/8 \end{bmatrix}^T$$

So, the chain converges to it's invariant measures.