Sheet-13

Exercise - 1.

Given,

$$Z^0 \sim N(1,1)$$

$$E[Z^0] = 1 \, . \; var[Z^0] = 1$$

$$\Xi \; \cancel{\xi}, \; \Sigma \sim N(0,1)$$

$$E[\Xi] \; \cancel{E[\xi]} = E[\Sigma] = 0$$

$$var[\Xi] \; \cancel{var[\xi]} = var[\Sigma] = 1.$$

(i)

$$Z^1 = Z^0 + \sqrt{2}\,\Sigma.$$

$$\therefore E[Z^1] = E[Z^0 + \sqrt{2}\,\Sigma]$$

$$= E[Z^0] + \sqrt{2}\,E[\Sigma]$$

$$= 1 + 0$$

$$= 1.$$

$$var[Z^1] = var[Z^0 + \sqrt{2}\,\Sigma]$$

$$= var[Z^0] + (\sqrt{2})^2 \, var[\Sigma]$$

$$= 1 + 2.1$$

$$= 3.$$

$$\therefore Z^1 \sim N(1,3)$$

Now, $R = var(\sqrt{3}, \Xi) = 3. var[\Xi] = 3.\,1 = 3.$

~~Now,~~ $\cancel{R = var(\sqrt{3}.\xi) = (\sqrt{3})^2 \, var[\xi] = 3.1 = 3}$

$$K = \frac{P_M{}^f}{P_M{}^f + R} = \frac{3}{3+3} = 1/2$$

Now, $\bar{z}^a = \bar{z}^f_M - K(\bar{z}^f_M - y_{obs})$

$$= 1 - 1/2\,(1-3)$$

$$= 2$$

$$P^a = P^f_M - K P_M^f$$
$$= 3 - 1/2 \cdot 3$$
$$= 3/2$$

So, mean $= 2$, and variance $= 3/2$

$$Z^a \sim N(2, 3/2)$$

(ii)

From Definition, 7.6 H will be 1.

From Definition. 7.5 variables $\{\xi_i\}$ are realisations of M i.i.d. Gaussian random variables with PDF $N(0, R)$.

$$R = 3$$

So, $N(0,3)$.

(iii) $\hat{Z}^a = Z^1 - K(HZ^1 + \xi - y_{obs})$

Mean, $\mathbb{E}[\hat{Z}^a] = \mathbb{E}[Z^f - K(Z^f + \xi - y_{obs})]$
$$= \mathbb{E}[1 - 1/2(1 + \xi - 3)]$$
$$= 1 - 1/2 - \mathbb{E}[\xi] + 3/2$$
$$= 1 - 1/2 - 0 + 3/2$$
$$= 2$$

$\xi \sim N(0,3)$. So, $\mathbb{E}[\xi] = 0$ and $\mathrm{var}[\xi] = 3$

variance, $\mathrm{var}[\hat{Z}^a] = \mathrm{var}[Z^f] - \frac{1}{2^2}\mathrm{var}[Z^f] - \frac{1}{2^2}\mathrm{var}[\xi]$
$$= 3 - \frac{1}{4} \times 3 - \frac{1}{4} \times 3$$
$$= 3/2$$

So, $\hat{z}^a \sim N(2, 3/2)$.

from $z^a \sim N(2, 3/2)$

So, the mean and variance of z^a and $\hat{z}^a$ are same.

(v) Both $\hat{z}^a$ and z^a are normally distributed and there mean and variance are equal so we can say that they are same distribution.