Exercise 1.

Forward map:

$$Z^{n+1} = \frac{1}{2} Z^n + 1 + E^n$$

Where

$$E^n \sim N(0,1)$$

Observation Operator:

$$Y^n = Z^n + \sqrt{2}\, \Sigma^n$$

where $\Sigma^n \sim N(0,1)$

Assume $Z_0 \sim N(-1, 2)$

(i)

Prediction: What is the distribution of Z_1?

We have following rules for distributions.

$$X \sim N(a, b)$$

$$\alpha X \sim N(\alpha a, \alpha^2 b)$$

$$\alpha + X \sim N(\alpha + a, b)$$

$$X + Y \sim N(a+c, b+d)$$

↳ when cov(b,d) = 0

$$Z_1 = \frac{1}{2} Z^0 + 1 + E^0$$

$$Z_0 \sim \mathcal{N}(-1, 2)$$

$$\frac{1}{2} Z_0 \sim \mathcal{N}\left(\frac{-1}{2}, \frac{1}{2}\right)$$

$$\frac{1}{2} Z_0 + 1 \sim \mathcal{N}\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\frac{1}{2} Z_0 + 1 + \mathcal{E}^0 \sim \mathcal{N}\left(\frac{1}{2}, \frac{3}{2}\right)$$

$$Z_1 \sim \mathcal{N}\left(\frac{1}{2}, \frac{3}{2}\right)$$

(ii)

Filtering: What is the distribution of Z_1 conditioned on $Y_1 = 2$?

forward mean $= m^f = \frac{1}{2}$
forward variance $= V^f = \frac{3}{2}$

We have to find Kalman Gain first:

$$\text{Kalman Gain} = K = \frac{Var[Z_1]}{Var[Z_1] + Var[\sqrt{2}\,\mathcal{E}']}$$

$\sqrt{2}\,\mathcal{E}'$ is observation noise

$$\mathcal{E}'' \sim \mathcal{N}(0, 1)$$
$$\sqrt{2}\,\mathcal{E}'' \sim \mathcal{N}(0, 2)$$

$$K = \frac{3/2}{3/2 + 2} = \frac{3}{7}$$

Now, we need to update our analysis mean and variance.

$$m^a = m^f - \frac{3}{7}\left(m^f - y_{obs}\right)$$

$$= \frac{1}{2} - \frac{3}{7}\left(\frac{1}{2} - 2\right)$$

$$= \frac{8}{7}$$

$$V^a = V^f(1 - K)$$

$$= \frac{3}{2}\left(1 - \frac{3}{7}\right)$$

$$= \frac{6}{7}$$

$$Z' \mid Y' = 2 \sim \mathcal{N}\left(\frac{8}{7}, \frac{6}{7}\right)$$

(iii)

Smoothing : What is the distribution of Z_0 conditioned on $Y_1 = 2$?

We can use Bayes theorm to find $P(Z^0 / Y^1)$

$$P(Z^0 / Y_1 = 2) = \frac{P(\cancel{Z} Y_1 = 2 / Z^0) \, P(Z^0)}{P(Y_1 = 2)}$$

$$Y_1 = Z_1 + \sqrt{2} \, \Sigma^1$$

$$= \frac{1}{2} Z_0 + 1 + E^0 + \sqrt{2} \Sigma^1$$

let $\mathcal{E}_1 = E^0 + \sqrt{2} \Sigma^1$ as they both are constants. and have Guassian distribution.

$$\mathcal{E} \sim \mathcal{N}(0, 3)$$

$$Y_1 = \frac{1}{2} Z_0 + 1 + \mathcal{E}_1$$

As Y_1 is Guassian so Y_1 / Z_0 is also Guassian distributed.

$$Y_1 | Z_0 = z = \frac{1}{2} z + 1 + \varepsilon_1 \sim \mathcal{N}\left(\frac{1}{2} z + 1, 3\right)$$

$$P(Y_1 = 2 | Z_0 = z) \propto \exp\left(-\frac{1}{2} \frac{(1/2 + 1 - 2)^2}{3}\right)$$

$$\cancel{P(Y_1 = 2 | Z_0 = z) \propto \exp\left(\frac{-(\frac{1}{2} z - 1)^2}{6}\right)}$$

$$P(Y_1 = 2 | Z_0 = z) P(Z_0 = z) \propto \exp\left(\cancel{-\tfrac{1}{2}} - \frac{(1/2\, z - 1)^2}{6} - \frac{(z+1)^2}{4}\right)$$

let $$\phi(z) = \frac{(1/2\, z - 1)^2}{6} + \frac{(z+1)^2}{4} \quad \text{— ①}$$

$$P(Y_1 = 2 | Z_0 = z) P(Z_0 = z) \propto \exp(-\phi(z))$$

As $\exp(-\phi(z))$ ~~follows~~ is guassian distribution so,

$$\phi(z) = \frac{(z - m)^2}{2V} + C$$

From above equation we can see that

$$\phi'(m) = 0$$

$$\phi''(z) = V^{-1} \quad \text{for any } z.$$

Equation ① after derivation

$$\phi'(z) = \frac{7z}{12} + \frac{1}{3} \quad \cancel{= 0}$$

$$\phi'(m) = 0$$

$$m = \frac{-4}{7}$$

$$\phi''(z) = \frac{7}{12}$$

$$v = \frac{12}{7}$$

$$Z_0 \mid Y_1 = 2 \sim \mathcal{N}\left(\frac{-4}{7}, \frac{12}{7}\right)$$

(iv)

$$Z^{n+1} = \alpha Z^n + \beta \Xi^n$$

$$Z_0 \sim \mathcal{N}(m, 1) \qquad , Y_1 = y$$

<u>Pseudocode</u> :

```
For distribution of Z1
    mean_Z1 = α * m
    var_Z1 = α^2 + β^2
    return (mean_Z1, var_Z1)
For distribution of Z1 | Y1 = y
    var-obs-noise = 2
    K = (var_Z1 / (var_Z1 + var-obs-noise))
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$mean\_z_1\_y_1 = mean\_z_1 - K\,(mean\_z_1 - y)$

$var\_z_1\_y_1 = var\_z_1\,(1-K)$

return $(mean\_z_1\_y_1,\ var\_z_1\_y_1)$