Exercise 2:

(i) Uncorrelated means that the correlation is 0, or equivalently, that the covariance between them is 0. Therefore, we want to show that for two given (but unknown) random variables that are independent, then the covariance between them is 0.

The formula for covariance,

$$\mathrm{Cov}(X,Z) = E[XZ] - E[X]\,E[Z]$$

If the covariance is 0, then $E[XZ] = E[X]E[Z]$, therefore we say mathematically that it is sufficient to show that $E[XZ] = E[X]E[Z]$. This is because uncorrelated is equivalent to a 0 covariance, which is equivalent to $E[XZ] = E[X]E[Z]$

$$\begin{aligned}
E[XZ] &= \int\int xz\, P_{X,Z}(x,z)\, dx\, dz \\
&= \int\int xz\, P_X(x)\, P_Z(z)\, dx\, dz \\
&= \int\int x\, P_X(x) \left(\int z P_Z(z)\, dz\right) dx \\
&= \left(\int x\, P_X(x)\, dx\right)\left(\int z P_Z(z)\, dz\right) \\
&= E[X]\,E[Z].
\end{aligned}$$

Independent $P(X,Z) = P(X)\cdot P(Z)$

Example: $P(X=1, Z=1) = \frac{1}{4}$, $P(X=1, Z=-1) = \frac{1}{4}$, $P(X=0, Z=0) = 1/2$.

Now, $E[XZ] = 0$, $E[X] = 1/2$, $E[Z] = 0$

and $P(X=1) = 1/2$, $P(X=0) = 1/2$, $P(Z=1) = 1/4$

$P(Z=-1) = 1/4$, $P(Z=0) = 1/2$, so, $E(XZ) - E(X)E(Z) = 0$

uncorrelated.

Now,

$1/4 = P(X=1, Z=1) \neq P(X=1)\,P(Z=1) = 1/8$, So, it is uncorrelated but dependent.

So, the statement is False.

(ii) The statement is False.

Let x denotes the points after n coin flipped where id is heads and z for tails.

if heads $= +1$ and tails $= -1$

$$E[x^2] = n^2 \cdot \frac{1}{2} \cdot 1 = \frac{n^2}{2}$$

$$E[z^2] = n^2 \, \frac{1}{2}(-1) = -\frac{n^2}{2}$$

$$\therefore E[x^2] \neq E[z^2].$$

(iii) From (i) we know that if X and Z are independent then $E[XZ] = E[X] \cdot E[Z]$.

Now, $Cov(X, Z) = E[XZ] - E[X] \cdot E[Z]$

$= E[X] \cdot E[Z] - E[X] \cdot E[Z]$

$= 0$.

So, the statement is True.