10. EXERCISE SHEET *Bayesian inference and data assimilation*

Exercise 1. Assume that X is sampled from a prior distribution $X \sim \mathcal{N}(\bar{x}, P)$. We then observe $h(x) = x_1 = Hx$ with observation operator

$$H = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T .$$

We observe $Y = HX + \Xi \in \mathbb{R}$ with $\Xi \sim \mathcal{N}(0, R)$ being independent of X. Calculate the conditional distribution of X given $Y = y_{\text{obs}}$. How do the two components of X behave when the measurement error R tends to ∞?

Exercise 2. In Exercise 1 we performed inference for a linear forward operator. We now assume that the forward operator $h(x)$ is not linear any more. We assume that the prior is given as $X \sim \mathcal{N}(0, 2)$. The forward operator is

$$h(x) = x^4 + x^2$$

and we observe $Y = h(X) + \Xi$ with $\Xi \sim \mathcal{N}(0, 1)$. Assume we observe $y_{\text{obs}} = 4$.

(i) Write down the density $\pi(x|y_{\text{obs}} = 4)$. You do not need to calculate the normalizing constant explicitly.

(ii) Find a MAP estimator m for the posterior distribution using gradient descent.

(iii) Let $\pi(x) = \exp(-g(x))$ be a density. The Laplace approximation to π is defined as $\tilde{\pi} = \mathcal{N}(m, g''(m)^{-1})$. Write down the Laplace approximation to the posterior of X.

(iv) Use the Langevin SDE to generate $N = 10\ 000$ samples from the posterior. How long do you need to run it to reach its invariant distribution? Overlay the histogram from obtained by running the Langevin SDE for a sufficiently long time with the density you got in (iii).

(v) Which of the approximates in (iv) is the better approximation to the true distribution? Are there any reasons/cases where one would still use the other approximation?

Exercise 3. We prove the statements from Definition 5.8. Let X ne a random variable in one dimension.

(i) Show that the mean $c = \mathbb{E}[X]$ minimises $l(c) = \mathbb{E}[(X - c)^2]$.

(ii) Show that the median minimizes $l(c) = \mathbb{E}[|X - c|]$. The median is defined as a number l s.t. $\mathbb{P}[X < l] = \mathbb{P}[X > l] = 0.5$.