(1) It is given that

$$\hat{\pi}_i(z') = \int_R \pi(z'|z)\, \pi_i(z)\, dz \quad \text{where } i = 1, 2$$

$\therefore$ $i = 1$

$$\hat{\pi}_1(z') = \int_R \pi(z'|z)\, \pi_1(z)\, dz$$

and $i = 2$

$$\hat{\pi}_2(z') = \int_R \pi(z'|z)\, \pi_2(z)\, dz$$

we need to show that

$$d_{TV}(\hat{\pi}_1, \hat{\pi}_2) \leq d_{TV}(\pi_1, \pi_2)$$ and it

is mentioned in the question.

$$d_{TV}(v, u) = \frac{1}{2}_{|f|_\infty \leq 1} \left| E_u[f] - E_v[f] \right|$$

$$\therefore \quad d_{TV}(\pi_1, \pi_2) = \frac{1}{2} |f|_\infty \leq 1 \sup \left| E_{\pi_2}[z] - E_{\pi_1}[z] \right|$$

$$\Rightarrow d_{TV}(\pi_1, \pi_2) = \frac{1}{2} \int_R \left| E_{\pi_2}[z] - E_{\pi_1}[z] \right|$$

$$d_{TV}(\hat{\pi}_1, \hat{\pi}_2) = \frac{1}{2} \int_R \left| E_{\hat{\pi}_2}[z] - E_{\hat{\pi}_1}[z] \right|$$

$$= \frac{1}{2} \int_R \left| f(z)\, \hat{\pi}_2[z'] - f(z)\, \hat{\pi}_1[z'] \right| dz'$$

$$= \frac{1}{2} \int_R \left| f(z) \int_R \pi(z'|z)\, \pi_2(z)\, dz - f(z) \int_R \pi(z'|z)\, \pi_1(z)\, dz \right| dz'$$

$$\leq \frac{1}{2} \int_R \int_R \left| f(z)\, \pi(z'|z) \left(\pi_2(z) - \pi_1(z) \right) \right| dz\, dz'$$

$$\left[\text{As, } \left|\int_x^y f(x)\,dx\right| \le \int_x^y |f(x)|\,dx\right]$$

$$\le \int_R \pi(z'|z)\,dz' \cdot \frac{1}{2}\int_R \left|f(z)\pi_2(z) - f(z)\pi_1(z)\right| dz$$

$$\le 1 \cdot \frac{1}{2}\int_R \left| E_{\pi_2}[z] - E_{\pi_1}[z] \right|$$

$$\le \frac{1}{2}\int_R \left|E_{\pi_2}[z] - E_{\pi_1}[z]\right|$$

$$\le d_{TV}(\pi_1, \pi_2)$$

$\therefore$ Therefore, we can say

$$d_{TV}(\hat{\pi}_1, \hat{\pi}_2) \le d_{TV}(\pi_1, \pi_2)$$

$\square$