11. Exercise sheet *Bayesian inference and data assimilation*

Exercise 1. In this exercise we will do one full filtering and smoothing step by hand in a simple case. The forward map is given as

$$Z^{n+1} = \frac{1}{2}Z^n + 1 + \Xi^n,$$

where $\Xi^n \sim \mathcal{N}(0,1)$. The observation operator is given as

$$Y^n = Z^n + \sqrt{2}\Sigma^n,$$

with $\Sigma^n \sim \mathcal{N}(0,1)$. Assume $Z_0 \sim \mathcal{N}(-1,2)$. All noise processes are independent. Calculate the following exercises by hand.

(i) Prediction: What is the distribution of Z_1?

(ii) Filtering: What is the distribution of Z_1 conditioned on $Y_1 = 2$?

(iii) Smoothing: What is the distribution of Z_0 conditioned on $Y_1 = 2$?

Now we want to implement this in pseudo code. Assume you are given the model

$$Z^{n+1} = \alpha Z^n + \beta \Xi^n.$$

Assume that $Z_0 \sim \mathcal{N}(m,1)$. You also observe that $Y_1 = y$.

(iv) Write pseudocode, that given the inputs α, β, m and y will output the distribution of Z_1 and of Z_1 conditioned on $Y_1 = y$.

Exercise 2. The model in this exercise is

$$Z^{n+1} = Z^n + \delta t d Z^n + \delta t b + \sqrt{2\delta t}\Xi^n$$

with $d = -2$, $b = 1$, $\delta t = 0.01$ and $\Xi^n \sim \mathcal{N}(0,1)$. Generate a reference trajectory $\{Z^i\}_{i=1}^N$ starting at $Z_0 = 10$ with $N = 1000$. Generate observations $\{Y^n\}_{i=1}^N$ by

$$Y^n = Z^n + \Sigma^n$$

with $\Sigma^n \sim \mathcal{N}(0,1)$. N_{out} is set to 1. All noise processes are independent.

(i) Run the Kalman filter with initial distribution $Z_0 \sim \mathcal{N}(1,1)$. Plot the analysis mean and the observations into one plot.

(ii) Run the Kalman filter but with a misspecified model where you set $d = -0.4$. The observations $\{Y_n\}_{i=1}^N$ are still the same ones as before, i.e. they are generated with $d = -2$. Make the same plots as in (i).

(iii) What is the difference between the plots in (i) and (ii)? Create a graphical visualization (a plot of any kind) from which one can see that the model in (ii) was misspecified while the model in (i) was not.

Exercise 3. Write a code to implement the Kalman filter for the fully discrete time system with $Z_k \in \mathbb{R}^{2\times 1}$ and forecast model

$$Z_k = \begin{bmatrix} 0.2 & 0.3 \\ 0 & 0.7 \end{bmatrix} Z_{k-1} + \Gamma_k, \quad \Gamma_k \sim N(0, Q)$$

with $Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $Z_0 \sim \mathcal{N}(0, I)$ and $Y_k \in \mathbb{R}$ with observing model

$$Y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} Z_k + \Xi_k, \quad \Xi_k \sim N(0, 3) \tag{1}$$

with y_k being a solution of (1). All noise processes are independent. Run the model up to $k = 200$. N_{out} is set to 1.

(i) Produce plots of the analysis means of each component for the k time steps. Plot the obervations into the same plot.

(ii) Plot the variances of both components in different plots.

(iii) Assume $Z_k \sim \mathcal{N}(m, v)$ and $Y_{k+1} = y$. Write pseudocode that computes the distribution of Z_{k+1} conditioned on $Y_{k+1} = y$.