Exercise 4

Exercise 4. Let X_1, X_2 and X_3 be three discrete random variables with

$$\mathbb{P}[X_1 = 0] = \mathbb{P}[X_1 = 1] = \mathbb{P}[X_2 = 0] = \mathbb{P}[X_2 = 1] = \frac{1}{2}$$

and

$$\mathbb{P}[X_3 = 0] = 1.$$

(i) Characterize all possible coupling between X_1 and X_2.

$X_2 \backslash X_1$	0	1	
0	a	b	1/2
1	c	d	1/2
	1/2	1/2	

$\}\ \mathbb{P}[X_2 = x]$

$\underbrace{\qquad\qquad}_{\mathbb{P}[X_1 = x]}$

with $a := \mathbb{P}[X_1 = 0, X_2 = 0]$; $b := \mathbb{P}[X_1 = 1, X_2 = 0]$
$c := \mathbb{P}[X_1 = 0, X_2 = 1]$; $d := \mathbb{P}[X_1 = 1, X_2 = 1]$

The parameters a, b, c, d have to fulfill some properties such that the define a valid probability measure:

(I) $a + b = c + d = a + c = b + d = \frac{1}{2}$

which also implies: $a + b + c + d = 1$

(II) $a, b, c, d \geq 0$

which combined with (I) implies $a, b, c, d \in [0, \frac{1}{2}]$

Examples:

1) $a = 0 \Rightarrow b = c = 1 \Rightarrow d = 0$

2) $a = 1/4 \Rightarrow b = c = 1/4 \Rightarrow d = 1/4$

These examples show that it is sufficient to fix one parameter to describe the coupling:

For instance: pick $a = \mathbb{P}[X_1=0, X_2=0] \in [0,\frac{1}{2}]$, then you can compute:

$$b = 1/2 - a$$
$$c = 1/2 - a$$
$$d = 1/2 - b = 1/2 - c = \frac{1}{2} - (1/2 - a) = a$$

$\rightarrow$ In the following we will express a specific coupling by parameter a.

(ii) Which coupling maximizes the correlation? Which coupling minimizes the correlation? Do you have an intuitive explanation why these couplings are the ones that minimize/maximize the correlation?

$$\operatorname{corr}(X_1,X_2) = \frac{\operatorname{cov}(X_1,X_2)}{\sqrt{\operatorname{Var}(X_1)\operatorname{Var}(X_2)}}$$

$$a^* = \underset{a\in[0,\frac{1}{2}]}{\operatorname{argmax}} \frac{\operatorname{cov}(X_1,X_2)}{\sqrt{\operatorname{Var}(X_1)\operatorname{Var}(X_2)}} = \underset{a\in[0,\frac{1}{2}]}{\operatorname{argmax}} \operatorname{cov}(X_1,X_2)$$

$$= \underset{a\in[0,\frac{1}{2}]}{\operatorname{argmax}} \left(\mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]\right)$$

$$= \underset{a\in[0,\frac{1}{2}]}{\operatorname{argmax}} \mathbb{E}[X_1X_2]$$

get rid of terms not depending on a

$$\mathbb{E}[X_1X_2] = 0\cdot\left(\mathbb{P}[X_1=0,X_2=0] + \mathbb{P}[X_1=0,X_2=1] + \mathbb{P}[X_1=1,X_2=0]\right) + 1\cdot\mathbb{P}[X_1=1,X_2=1]$$

$$= 0 + d = d$$

In (i) we have seen that for a valid coupling $a=d$.

$$\rightarrow \quad a^* = \underset{a\in[0,\frac{1}{2}]}{\operatorname{argmax}} \mathbb{E}[X_1X_2] = \underset{a\in[0,\frac{1}{2}]}{\operatorname{argmax}} d = \underset{a}{\operatorname{argmax}}\, a = \frac{1}{2}$$

If $a = d = 1/2$, we have $b = c = 0$.

$$\operatorname{corr}(X_1,X_2) = \frac{\mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]}{\sqrt{\operatorname{Var}(X_1)\operatorname{Var}(X_2)}} = \frac{1/2 - 1/2\cdot 1/2}{\sqrt{1/4\cdot 1/4}} = 1$$

$$\left[\begin{array}{l} \mathbb{E}[X_1] = \mathbb{E}[X_2] = 1/2 \\ \operatorname{Var}(X_1) = \operatorname{Var}(X_2) = \mathbb{E}[(X_2 - 1/2)^2] = (0-1/2)^2\cdot 1/2 + (1-1/2)^2\cdot 1/2 \\ \qquad = 1/4 \end{array}\right.$$

$$a^* = \underset{a \in [0,\frac{1}{2}]}{\operatorname{argmin}} \frac{\operatorname{cov}(X_1, X_2)}{\sqrt{\operatorname{Var}(X_1)\operatorname{Var}(X_2)}} = \ldots \overset{\text{same as before}}{=} \underset{a \in [0,\frac{1}{2}]}{\operatorname{arg\,min}} \mathbb{E}[X_1 X_2]$$

$$= \underset{a \in [0,1/2]}{\operatorname{argmin}} \; d = \underset{a \in [0,1/2]}{\operatorname{argmin}} \; a = 0$$

If $a = d = 0$, we have $b = c = 1/2$.

$$\operatorname{corr}(X_1, X_2) = \frac{\mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]}{\sqrt{\operatorname{Var}(X_1)\ \operatorname{Var}(X_2)}} = \frac{0 - 1/2 \cdot 1/2}{\sqrt{1/4 \cdot 1/4}} = -1$$

In these two scenarios ($a = 1/2$ and $a = 0$) we have $|\operatorname{corr}(X_1, X_2)| = 1$. In both cases it is sufficient to know the value of X_1 or X_2 to infer the value of the respective other RV.

In the case $a = 1/2$, X_1 and X_2 <u>always</u> take on the same value, in the case $a = 0$, they <u>always</u> take on a different value.

This means we have perfect correlation, with the difference that when $a = 0$, we have perfect <u>negative</u> correlation.

(iii) Which coupling makes the two random variables uncorrelated?

$$\operatorname{corr}(X_1, X_2) \overset{!}{=} 0$$

$$\Rightarrow \quad \operatorname{cov}(X_1, X_2) = 0$$

$$\Leftrightarrow \quad \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = 0$$

$$\Leftrightarrow \quad \mathbb{E}[X_1 X_2] = \mathbb{E}[X_1]\,\mathbb{E}[X_2]$$

$$d = 1/2 \cdot 1/2 = 1/4$$

$$\Rightarrow \quad a = d = 1/4, \quad b = c = 1/4$$

Here, we have the opposite to the previous cases. If we know X_1 (or X_2), we cannot make any assumptions about X_2 (or respectively X_1). The two RVs are independent:

$$\mathbb{P}[X_1 = x_1, X_2 = x_2] = 1/4 = \mathbb{P}[X_1 = x_1] \cdot [X_2 = x_2] \quad \forall \; x_1, x_2 \in \{0,1\}$$

(iv) Do the tasks $(i) - (iii)$ but for X_1 and X_3.

$X_3 \backslash X_1$	0	1	$P[X_3 = x]$
0	1/2	1/2	1
1	0	0	0
$P[X_1 = x]$	1/2	1/2	

→ There is only one possible coupling

$$\mathrm{Corr}(X_1, X_3) = \frac{\mathbb{E}[X_1 X_3] - \mathbb{E}[X_1]\mathbb{E}[X_3]}{\sqrt{\underbrace{\mathrm{Var}(X_1)\mathrm{Var}(X_3)}_{=0}}} = 0$$

$$\mathbb{E}[X_1 X_3] = 1 \cdot P[X_1 = 1, X_3 = 1] = 0$$

$$\mathbb{E}[X_1] = 1/2 \quad , \quad \mathbb{E}[X_3] = 1$$

$$\mathrm{Var}(X_1) = 1/4 \quad , \quad \mathrm{Var}(X_3) = 0$$

→ X_1 and X_3 are uncorrelated