Exercise-1: It is given that x is a ~~ran~~ uniform random variable $X \sim U[-1,1]$.

$\therefore$ PDF π_X is

$$\pi_X(x) = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

from Book, we know for a particular PDF π_X, a numerical quadrature rule for an integral is

$$\bar{f} = \int_R f(x)\, \pi_X(x)\, dx$$

and $$\bar{f}_M := \sum_{i=1}^{M} b_i f(c_i)$$

Here $c_i \in R$, $i = 1 \ldots M$ express the quadrature points and b_i are their weights and it is greater than 0. If we consider $K+1$ dimensional linear space of all polynomials of K or less, we can say

$$f(x) = a_0 + a_1 x + \cdots + a_K x^K$$

Moreover, quadrature rule is of order p if $\bar{f} = \bar{f}_M$ for all integrands.

(i) It is given quadrature order of $P = 2$

i.e. $f(x) = a_0 + a_1(x)$

quadrature rule of $M = 1$

i.e. $$\bar{f}_1 = \sum_{i=1}^{1} b_i f(c_i)$$
$$= b_1 f(c_1) \quad \because M = 1$$

From the aforemention definition we can write that

$$\bar{f} = \int_{R} f(x)\, \pi_x(x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2}(a_0 + a_1 x)\, dx$$

$\because$ ~~a~~ PDF ~~of~~ of X
$\pi_x(x) = \frac{1}{2}$ for $x \in [-1, 1]$
$f(x) = a_0 + a_1(x)$

$$= \frac{1}{2}\int_{-1}^{1} \cancel{a_0 dx} + \frac{1}{2}\int_{-1}^{1} a_1 x\, dx$$

$$= \frac{1}{2}\left[a_0 x + \frac{a_1 x^2}{2}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(a_0 + \cancel{a_1}\frac{1}{2}a_1\right) - \left(-a_0 + a_1\frac{(-1)^2}{2}\right)\right]$$

$$= \frac{1}{2}\left[a_0 + \cancel{\tfrac{1}{2}a_1} + a_0 - \cancel{\tfrac{1}{2}a_1}\right]$$

$$= \frac{1}{2}\, 2a_0$$

$$\therefore \bar{f} = a_0$$

Again,

$$\bar{f}_1 = b_1 f(c_1)$$

$$= b_1(a_0 + a_1 c_1)$$

from the book page (66) we know that a quadrature is of order p if $\bar{f} = \bar{f}_M$ for all integrands $f(x) \in \Pi_{p-1}(R)$

To satisfy this

$$\bar{f} = a_0$$

$$\bar{f}_1 = b_1 (a_0 + a_1 c_1)$$

$\therefore$ $\bar{f} = \bar{f}_1$ would be only possible when we have

$$b_1 = 1$$

$$c_1 = 0$$

(ii) case $M = 2$, and $P = 3$

$$\bar{f} = \int_{R} f(x)\,\pi_x(x)\,dx$$

$$= \int_{-1}^{1} \frac{1}{2}\left(a_0 + a_1 x + a_2 x^2\right) dx \qquad \Bigg|\ \because \pi_x(x) = \frac{1}{2}$$

$$f(x) = a_0 + a_1 x + x^2 \text{ as } P = 3$$

$$= \frac{1}{2}\left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right) - \left(-a_0 + a_1 \frac{(-1)^2}{2} + a_2 \frac{(-1)^3}{3}\right)\right]$$

$$= \frac{1}{2}\left[\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3}\right) - \left(-a_0 + \frac{a_1}{2} - \frac{a_2}{3}\right)\right]$$

$$= \frac{1}{2}\left[a_0 + \cancel{\frac{a_1}{2}} + \frac{a_2}{3} + a_0 - \cancel{\frac{a_1}{2}} + \frac{a_2}{3}\right]$$

$$= \frac{1}{2}\left[2a_0 + 2 \cdot \frac{a_2}{3}\right]$$

$$= \frac{1}{\cancel{2}} \cdot \cancel{2}\left(a_0 + \frac{a_2}{3}\right)$$

$$\bar{f} = a_0 + \frac{1}{3} a_2$$

$$\bar{f}_2 = \sum_{i=1}^{2} b_i f(c_i)$$

$$= b_1 f(c_1) + b_2 f(c_2)$$

$$= b_1 a_0 + b_2 a_0 + b_1 c_1 a_1 + b_2 c_2 a_1 + b_1 c_1^2 a_2 + b_2 c_2^2 a_2$$

$$= (b_1 + b_2) a_0 + (b_1 c_1 + b_2 c_2) a_1 + (b_1 c_1^2 + b_2 c_2^2) a_2$$

$\bar{f} = \bar{f}_2$ to satisfy this we should have

$\therefore (b_1 + b_2) a_0 = a_0$ | i.e. $b_1 + b_2 = 1$

$\therefore (b_1 c_1^2 + b_2 c_2^2) a_2 = a_2$ | i.e. $b_1 c_1^2 + b_2 c_2^2 = \frac{1}{3}$

and

$$b_1 c_1 + b_2 c_2 = 0$$

We need to find the value of b_1, c_1, b_2, c_2.

In order to get the values we need to solve all the earnation we get.

$$b_1 + b_2 = 1$$

$$\therefore \boxed{b_1 = 1 - b_2} \text{ and we know}$$

from the book, weights $b_i > 0$

$$b_1 + b_2 = 1$$

$$\Rightarrow \quad b_2 = 1 - b_1$$

$$b_1 c_1 + b_2 c_2 = 0$$

$$\Rightarrow b_2 c_2 = - b_1 c_1 . \Rightarrow (1-b_1) c_2 = - b_1 c_1 \Rightarrow \boxed{c_2 = - \frac{b_1 c_1}{1-b_1}}$$

$$b_1 c_1^2 + b_2 c_2^2 = 1/3$$

$$\Rightarrow 3 b_1 c_1^2 + 3 b_2 c_2^2 = 1 .$$

$$\Rightarrow \quad b_2 c_2^2 = \frac{1}{3} - b_1 c_1^2$$

$$\Rightarrow (1-b_1) \frac{b_1^2 c_1^2}{(1-b_1)^2} = \frac{1}{3} - b_1 c_1^2$$

$$\Rightarrow \quad \frac{b_1^2 c_1^2}{(1-b_1)} = \frac{1 - 3 b_1 c_1^2}{3}$$

$$\Rightarrow 3 b_1^2 c_1^2 = 1 - 3 b_1 c_1^2 - b_1 + 3 b_1^2 c_1^2$$

$$\Rightarrow \quad b_1 = 1 - 3 b_1 c_1^2$$

$$\Rightarrow c_1^2 = \frac{1}{3 b_1} (1 - b_1)$$

$$\therefore c_1 = \pm \frac{1}{\sqrt{3}\sqrt{b_1}} \sqrt{(1-b_1)}$$

$$\therefore c_2 = \pm \frac{b_1}{1-b_2} \left(\frac{1}{\sqrt{3}\sqrt{b_1}} \sqrt{(1-b_1)} \right)$$

$$= \pm \frac{\sqrt{b_1}}{\sqrt{3} \cdot \sqrt{(1-b_2)}} .$$

3(iii) $P=4$ which means

$$\bar{f} = \int_{R} f(x)\, \pi_x(x)\, dx$$

$$= \int_{-1}^{1} \frac{1}{2}\left(a_0 + a_1 x + a_2 x^2 + a_3 x_3^3\right) dx$$

$\because \pi_x(x) = \frac{1}{2}$

$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$[\because P=4]$

$$= \frac{1}{2}\left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}\right) - \left(-a_0 + a_1 \frac{(-1)^2}{2} + a_2 \frac{(-1)^3}{3} + a_3 \frac{(-1)^4}{4}\right)\right]$$

$$= \frac{1}{2}\left[\left(a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4}\right) - \left(-a_0 + \frac{a_1}{2} - \frac{a_2}{3} + \frac{a_3}{4}\right)\right]$$

$$= \frac{1}{2}\left[\left(a_0 + \cancel{\frac{a_1}{2}} + \frac{a_2}{3} + \cancel{\frac{a_3}{4}} + a_0 - \cancel{\frac{a_1}{2}} + \frac{a_2}{3} - \cancel{\frac{a_3}{4}}\right)\right]$$

$$= \frac{1}{2}\left(2a_0 + 2 \cdot \frac{a_2}{3}\right)$$

$$= \cancel{\frac{1}{2}} \cdot \cancel{2}\left(a_0 + \frac{a_2}{3}\right)$$

$$\therefore \bar{f} = a_0 + \frac{a_2}{3}$$

Moreover,

$$\bar{f}_2 = \sum_{i=1}^{2} b_i f(c_i)$$

$$\bar{f}_2 = b_1 f(c_1) + b_2 f(c_2)$$

$$\therefore \bar{f}_2 = (b_1 + b_2) a_0 + (b_1 c_1 + b_2 c_2) a_1 + (b_1 c_1^2 + b_2 c_2^2) a_2 + \underline{(b_1 c_1^3 + b_2 c_2^3) a_3}$$

like the previous question we can see, ~~too~~ to set the $\bar{f} = \bar{f}_2$ equal we need to have

$$b_1 + b_2 = 1 \Rightarrow b_1 = 1 - b_2$$
$$b_1 c_1 + b_2 c_2 = 0$$
$$b_1 c_1^2 + b_2 c_2^2 = \frac{1}{3}$$

and in this case $b_1 c_1^3 + b_2 c_2^3 = 0$

$$\therefore \cancel{b_1 c_1^3} + b_2 c_2^2 c_1 = \frac{1}{3} c_1$$
$$\underset{(-)}{\cancel{b_1 c_1^3}} \underset{(-)}{+} b_2 c_2^3 = 0$$

$$b_2 c_2^2 c_1 - b_2 c_2^3 = \frac{1}{3} c_1$$
$$\Rightarrow b_2 c_2^2 (c_1 - c_2) = \frac{1}{3} c_1$$
$$\Rightarrow \cancel{b_2} c_2^2 \left(\frac{c_1}{\cancel{b_2}}\right) = \frac{1}{3} c_1$$
$$\Rightarrow c_2^2 = \frac{1}{3}$$
$$\boxed{\therefore c_2 = \pm\sqrt{\frac{1}{3}}}$$

$$b_1 c_1 + b_2 c_2 = 0$$
$$(1 - b_2) c_1 + b_2 c_2 = 0$$
$$c_1 - b_2 c_1 + b_2 c_2 = 0$$
$$-b_2 c_1 + b_2 c_2 = -c_1$$
$$-b_2 (c_1 - c_2) = -c_1$$
$$(c_1 - c_2) = \frac{c_1}{b_2}$$