5. Exercise sheet *Bayesian inference and data assimilation*

Exercise 1. Let $X \sim U[-1, 1]$ be a uniform random variable.

(i) Find a quadrature rule for $M = 1$ of order $p = 2$.

(ii) Classify all quadrature rules for $M = 2$ that have order $p = 3$.

(iii) Do you see which of the quadrature rules from (ii) has order $p = 4$?

Exercise 2. Consider the state space $\mathcal{X} = \{a_1 = 0, a_2 = 1\}$ and the two random variables

$$\mathbb{P}[X_1 = a_1] = \frac{1}{4}, \qquad \mathbb{P}[X_1 = a_2] = \frac{3}{4}$$

and

$$\mathbb{P}[X_2 = a_1] = \frac{1}{2}, \qquad \mathbb{P}[X_2 = a_2] = \frac{1}{2}$$

A coupling can be described as a matrix $T \in \mathbb{R}^{3\times 3}$ with $t_{ij} \geq 0$,

$$\sum_{j=1}^{2} t_{ij} = \mathbb{P}[X_1 = a_i],$$

and

$$\sum_{i=1}^{2} t_{ij} = \mathbb{P}[X_2 = a_j].$$

Use the procedure described in Example 2.29 in the book to find the coupling that minimizes

$$J(T) = \sum_{i,j=1}^{2} t_{ij} |a_i - a_j|^2.$$

The resulting coupling is the optimal coupling in optimal transport and the value of $J(T)$ is the squared Wasserstein-2 distance. Calculate $J(T)$. Do the calculations by hand, not using a computer.

Exercise 3. Determine the ANOVA decomposition for

$$f(x_1, x_2) = -12x_1 - 6x_2 + 6x_1x_2$$

under the uniform measure on $[0, 1] \times [0, 1]$ and compute the associated variances σ_1^2, σ_2^2, and σ_{12}^2. Which terms in the ANOVA decomposition contribute most significantly to the total variance σ^2?