Exercise 2:

Given, $X \sim N(\mu, \sigma^2)$ and $\varepsilon \sim N(0, \tau^2)$.

So, $E[X] = \mu$ and $Var[X] = \sigma^2$

and $E[X^2] = var[X] + (E[X])^2$
$= \sigma^2 + \mu^2$

and. $E[\varepsilon] = 0$ and $var[\varepsilon] = \tau^2$

and $E[\varepsilon^2] = var[\varepsilon] + (E[\varepsilon])^2$
$= \tau^2$

and . $X$ and $\varepsilon$ are independent.

so, $E[\varepsilon X] = E[\varepsilon] . E[X] = 0$

Given, $\hat{X} = cY + d$ and $Y = gX + \varepsilon$

$E[(X - \hat{X})^2] = E[(X - cY - d)^2]$ [Replace with the value of $\hat{X}$]

$= E[(X - cgX - c\varepsilon - d)^2]$ [Replace with value of $Y$]

$= E[((1-cg)X - (c\varepsilon + d))^2]$

$= E[(1-cg)^2 X^2 - 2(1-cg)X(c\varepsilon + d) + (c\varepsilon + d)^2]$

$= (1-cg)^2 E[X^2] - 2(1-cg)(E[c\varepsilon X] + E[Xd]) + E[(c\varepsilon + d)^2]$

$= (1-cg)^2(\sigma^2 + \mu^2) - 2(1-cg)[c. E[\varepsilon X] + d E[X]] + E[c^2\varepsilon^2 + 2c.\varepsilon.d + d^2]$

$= (1-cg)^2(\sigma^2 + \mu^2) - 2(1-cg). d\mu + c^2\tau^2 + d^2$

[$\because E[\varepsilon X] = 0$ and $E[\varepsilon] = 0$]

To find the minimize of $E[(X-\hat{X})^2]$, we will derivate $E[(X-\hat{X})^2]$ with respect to c and d and set the derivative result zero.

Derivate $E[(X-\hat{X})^2]$ with respect to d.

$$\frac{\partial}{\partial d} E[(X-\hat{X})^2] = 0 - 2(1-cg)M + 2d$$

Now, $2d = 2(1-cg) \cdot M$

$$\therefore d = (1-cg) \cdot M \quad \text{---- (i)}$$

Derivate $E[(X-\hat{X})^2]$ with respect to c.

$$\frac{\partial}{\partial c} E[(X-\hat{X})^2] = -2g(1-cg)[\sigma^2+M^2] + 2gdM + 2c\pi^2$$

Now, $-2g(1-cg)[\sigma^2+M^2] + 2gdM + 2c\pi^2 = 0$

$$\Rightarrow -g(1-cg)[\sigma^2+M^2] + g(1-cg)M^2 + 2c\pi^2 = 0$$

[values of d in (i)]

$$\Rightarrow -g\sigma^2 - \cancel{gM^2} + cg^2\sigma^2 + \cancel{cg^2M^2} + \cancel{gM^2} - \cancel{cg^2M^2} + c\pi^2 = 0$$

$$\Rightarrow -g\sigma^2 + cg^2\sigma^2 + c\pi^2 = 0$$

$$\therefore \boxed{c = \frac{g\sigma^2}{g^2\sigma^2+\pi^2}} \quad \text{.... (ii)}$$

Put the value of (ii) in (i)

$$d = \left(1 - \frac{g^2\sigma^2}{g^2\sigma^2+\pi^2}\right) \cdot M = \frac{Mg^2\sigma^2 + M\pi^2 - Mg^2\pi^2}{g^2\sigma^2+\pi^2}$$

$$\therefore \boxed{d = \frac{M\pi^2}{g^2\sigma^2+\pi^2}}$$

If the second order derivative is positive definite ( >0) then we can say it a global minimum.

$$\frac{\partial}{\partial d^2}\left[E(x-\hat{x})^2\right] = 2 > 0$$

and $\frac{\partial}{\partial c^2}\left[E(x-\hat{x})^2\right] = 2(\sigma^2+\mu^2)g^2 + 2\tau^2 > 0$

[because $g \neq 0$ and $\tau^2$ can't be negative]

So, both of them are positive definite. and we can say that c and d are ~~also~~ global minimum.