Exercise 2:

Given,

$$T_x(y) = \begin{cases} 1, & \text{if } x \leq y \\ 0, & \text{if } x > y \end{cases}$$

Now, $$T_y(x) = \begin{cases} 1, & \text{if } x \geq y \\ 0, & \text{if } x < y \end{cases}$$

$$\int_{-\alpha}^{\alpha} F(x) \cdot T_y(x) \, \Pi_x(y) \, dy$$

$$= F(x) \int_{-\alpha}^{\alpha} T_y(x) \cdot \Pi_x(y) \cdot dy$$

$$= F(x) \int_{-\alpha}^{\alpha} 1_{x \geq y} \, y \cdot \Pi_x(y) \cdot dy$$

$$= F(x) \int_{-\alpha}^{\alpha} \Pi_x(y) \cdot dy$$

$$= F(x) \cdot F_x(x).$$

Now,

$$\mathbb{E}[S_{CRPS}(F, X)] = \int_{-\alpha}^{\alpha} S_{CRPS}(F, x) \, \Pi_x(x) \cdot dx$$

$$= \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} (F(x) - F_x(x))^2 \, dx \; \Pi_x(x) \cdot dx.$$

$$= \int_{-\alpha}^{\alpha} \int_{-\alpha}^{\alpha} \left[F(x)^2 + F_x(x)^2 - 2F(x) \cdot F_x(x) \right] dx \cdot \Pi_x(x) \cdot dx$$

$$= \int_{-\alpha}^{\alpha} \Big[F(x)^2 \int_{-\alpha}^{\alpha} \Pi_x(x) \cdot dx + \int_{-\alpha}^{\alpha} F_x(x)^2 \cdot \Pi_x(x)$$

$$= \int_{-\alpha}^{\alpha} \Big[F(x)^2 \int_{-\alpha}^{\alpha} \Pi_x(x) \cdot dx + \int_{-\alpha}^{\alpha} F_x(x)^2 \cdot \Pi_x(x) \cdot dx$$

$$- 2 \int_{-\alpha}^{\alpha} F(x) \cdot F_x(x) \, \Pi_x(x) \cdot dx \Big] \, dx$$

$$= \int_{-\infty}^{\infty} \left[F(x)^2 + \int_{-\infty}^{\infty} \mathbb{1}_{x \geq y} \cdot y \, \pi_X(x)\, dx - 2F(x) F_X(x) \right] dx$$

$$= \int_{-\infty}^{\infty} \left[F(x)^2 + \int_{-\infty}^{\infty} \pi_X(x) \cdot dx - 2F(x) \cdot F_X(x) \right] dx$$

$$= \int_{-\infty}^{\infty} \left[F(x)^2 + F_X(x) - 2F(x) F_X(x) \right] dx$$

$$= \int_{-\infty}^{\infty} \left[F(x)^2 + F_X(x)^2 - 2F(x) \cdot F_X(x) - F_X(x)^2 + F_X(x) \right] dx$$

$$= \int_{-\infty}^{\infty} (F(x) - F_X(x))^2 \, dx + \int_{-\infty}^{\infty} (F_X(x) - F_X(x)^2) \, dx$$

$$= \int_{-\infty}^{\infty} (F(x) - F_X(x))^2 \, dx + \int_{-\infty}^{\infty} F_X(x) (1 - F_X(x)) \, dx.$$

[Proved]