Exercise 3. We want to do inference for a non-linear forward operator in a Bayesian inverse problem. We assume that the prior is given as $X \sim \mathcal{N}(2, 2)$. The forward operator is

$$h(x) = x^2$$

and we observe $Y = h(X) + \Xi$ with $\Xi \sim \mathcal{N}(0, 1)$. Assume we observe $y_{\text{obs}} = 2$.

(i) Write down the density $\pi(x|y_{\text{obs}} = 2)$. You do not need to calculate the normalizing constant explicitly.

We know that

$$\pi(x|y_{obs} = 2) \propto \pi_Y(y_{obs} = 2 \,|\, x)\, \pi_X(x) = \pi_\Xi(y_{obs} - h(x))\, \pi_X(x)$$

$$= \pi_\Xi(2 - x^2) \cdot \pi_X(x) \qquad (1)$$

Both Ξ and π_X have Gaussian distr. with density:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}$$

so,

$$\pi_\Xi(2 - x^2) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \cdot (2 - x^2)^2\right)$$

$$\pi_X(x) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{1}{2} \cdot \left(\frac{x-2}{2}\right)^2\right)$$

Plug it into (1):

$$\pi(x|y_{obs} = 2) \propto \exp\left(-\frac{1}{2} \cdot (2 - x^2)^2\right) \cdot \exp\left(-\frac{1}{2} \cdot \left(\frac{x-2}{2}\right)^2\right)$$

$$= \exp\left(-\frac{1}{2} \cdot (2 - x^2)^2 - \frac{1}{2} \cdot \left(\frac{x-2}{2}\right)^2\right)$$

$$= \exp\left(-\frac{1}{2} \cdot (x^4 - 4x^2 + 4) - \frac{1}{8}(x^2 - 4x + 4)\right)$$

$$= \exp\left(-\frac{1}{8}(4x^4 - 16x^2 + 16 + x^2 - 4x + 4)\right)$$

$$= \exp\Big(\underbrace{-\frac{1}{8}(4x^4 - 15x^2 - 4x + 20)}_{g(x)}\Big)$$