Exercise: 4

From Example 2.18 (Page 42), we found that,

conditional mean, $\bar{x}_c = \bar{x}_1 + \sigma_{12}^2 \sigma_{22}^{-2}(x_2 - \bar{x}_2)$

conditional variance, $\sigma_c^2 = \sigma_{11}^2 - \sigma_{12}^2 \sigma_{22}^{-2} \sigma_{21}^2$

and $\sigma_{21} = \sigma_{12}$

From problem sheet,

$$z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \bar{z} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}, \quad P = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 \end{bmatrix}$$

So, $|P| = \sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2$

and $$P^{-1} = \frac{1}{\sigma_{11}^2 \sigma_{22}^2 - \sigma_{12}^2 \sigma_{21}^2} \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12}^2 \\ -\sigma_{21}^2 & \sigma_{11}^2 \end{bmatrix}$$

Now,

$$\frac{1}{2\pi |P|^{1/2}} \exp\left(-\frac{1}{2}(z-\bar{z})^T P^{-1}(z-\bar{z})\right)$$

$$= \frac{1}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}} \exp\left(-\frac{1}{2}\begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}^T \frac{1}{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}\begin{bmatrix} \sigma_{22}^2 & -\sigma_{12}^2 \\ -\sigma_{21}^2 & \sigma_{11}^2 \end{bmatrix} \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}} \exp\left(-\frac{1}{2(\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2)}\left[(x_1-\bar{x}_1)\sigma_{22}^2 - (x_2-\bar{x}_2)\sigma_{21}^2 \quad -(x_1-\bar{x}_1)\sigma_{12}^2 + (x_2-\bar{x}_2)\sigma_{11}^2\right] \begin{bmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{bmatrix}\right)$$

$$= \frac{1}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}} \exp\left(- \frac{(x_1-\bar{x}_1)^2\sigma_{22}^2 - (x_2-\bar{x}_2)(x_1-\bar{x}_1)\sigma_{21}^2 - (x_1-\bar{x}_1)(x_2-\bar{x}_2)\sigma_{12}^2 + (x_2-\bar{x}_2)\sigma_{11}^2}{2(\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2)}\right)$$

$$= \frac{1}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}} \exp\left(- \frac{(x_1-\bar{x}_2)^2\sigma_{22}^2 + (x_2-\bar{x}_2)\sigma_{11}^2 - 2(x_1-\bar{x}_1)(x_2-\bar{x}_2)\sigma_{12}^2}{2(\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2)}\right)$$

$[\because \sigma_{12} = \sigma_{21}]$ ——(i)

For,

$$\frac{1}{\sqrt{2\pi}\,\sigma_c} \exp\left(-\frac{1}{2\sigma_c^2}(x_1-\bar{x}_c)^2\right) \frac{1}{\sqrt{2\pi}\,\sigma_{22}} \exp\left(-\frac{1}{2\sigma_{22}^2}(x_2-\bar{x}_2)^2\right)$$

$$= \frac{1 \times \cancel{\sigma_{22}}}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2 - \sigma_{12}^2\sigma_{21}^2}} \times \frac{1}{\cancel{\sigma_{22}}} \exp\left[-\frac{1}{2\sigma_c^2}(x_1-\bar{x}_c)^2 - \frac{1}{2\sigma_{22}^2}(x_2-\bar{x}_2)^2\right]$$

——(ii)

[put the value of σ_c and $\exp(x) * \exp(y) = \exp(x+y)$]

Now,

$$\exp\left[\frac{-\sigma_{22}^2}{2[\sigma_{22}^2\sigma_{11}^2 - \sigma_{12}^2\sigma_{21}^2]}\left(x_1 - \bar{x}_1 - \sigma_{12}^2\frac{1}{\sigma_{22}^2}(x_2-\bar{x}_2)\right)^2 - \frac{1}{2\sigma_{22}^2}(x_2-\bar{x}_2)^2\right]$$

[put the value of σ_c and $\bar{x}_c$]

$$= \exp\left[\frac{-1}{2\sigma_{22}^2[\sigma_{22}^2\sigma_{11}^2 - \sigma_{12}^2\sigma_{21}^2]}\left(\sigma_{22}^2(x_1-\bar{x}_1) - \sigma_{12}^2(x_2-\bar{x}_2)\right)^2 - \frac{1}{2\sigma_{22}^2}(x_2-\bar{x}_2)^2\right]$$

$$= \exp\left[\frac{-[\sigma_{22}^4(x_1-\bar{x}_1)^2 - 2\sigma_{22}^2(x_1-\bar{x}_1)\sigma_{12}^2(x_2-\bar{x}_2) + \sigma_{12}^4(x_2-\bar{x}_2)^2] - (x_2-\bar{x}_2)^2\sigma_{22}^2\sigma_{11}^2 + (x_2-\bar{x}_2)^2\sigma_{12}^2\sigma_{21}^2}{2\sigma_{22}^2(\sigma_{22}^2\sigma_{11}^2 - \sigma_{12}^2\sigma_{21}^2)}\right]$$

$$= \exp\left[\frac{-\sigma_{22}^4(x_1-\bar{x}_1)^2 - 2\sigma_{22}^2(x_1-\bar{x}_1)\sigma_{12}^2(x_2-\bar{x}_2) - \cancel{\sigma_{12}^4(x_2-\bar{x}_2)^2} - (x_2-\bar{x}_2)^2\sigma_{22}^2\sigma_{11}^2 \cancel{+ (x_2-\bar{x}_2)^2\sigma_{12}^4}}{2\sigma_{22}^2(\sigma_{22}^2\sigma_{11}^2 - \sigma_{12}^2\sigma_{21}^2)}\right]$$

$[\because \sigma_{12} = \sigma_{21}$ so, $\sigma_{12}^2 = \sigma_{21}^2$
$\therefore \sigma_{12}^2\sigma_{21}^2 = \sigma_{12}^2 \cdot \sigma_{12}^2 = \sigma_{12}^4]$

$$= \exp\left[- \frac{\cancel{\sigma_{22}^2}\left(\sigma_{22}^2 (x_1-\bar{x}_1)^2 - 2\sigma_{12}^2 (x_1-\bar{x}_1)(x_2-\bar{x}_2) + \sigma_{11}^2 (x_2-\bar{x}_2)^2\right)}{2\,\cancel{\sigma_{22}^2}\left(\sigma_{22}^2\sigma_{11}^2 - \sigma_{12}^2\sigma_{21}^2\right)} \right]$$

eqvation (ii) will be

$$\frac{1}{2\pi\sqrt{\sigma_{11}^2\sigma_{22}^2-\sigma_{12}^2\sigma_{21}^2}} * \exp\left(- \frac{(x_1-\bar{x}_1)^2\sigma_{22}^2 + (x_2-\bar{x}_2)\sigma_{11}^2 - 2(x_1-\bar{x}_1)(x_2-\bar{x}_2)\sigma_{12}^2}{2(\sigma_{22}^2\sigma_{11}^2 - \sigma_{12}^2\sigma_{21}^2)} \right) \quad \text{——— (iii)}$$

eqvation (i) and (iii) are same so, we can say that.

$$\frac{1}{2\pi|P|^{1/2}} \exp\left(-\frac{1}{2}(z-\bar{z})^T P^{-1}(z-\bar{z})\right) = \frac{1}{\sqrt{2\pi}\,\sigma_c}\exp\left(-\frac{1}{2\sigma_c^2}(x_1-\bar{x}_c)^2\right) \times$$
$$\frac{1}{\sqrt{2\pi}\,\sigma_{22}}\exp\left(-\frac{1}{2\sigma_{22}^2}(x_2-\bar{x}_2)^2\right)$$

Conditional PDFs. [Example 2.18 Page 42 and Page 41]

$$\pi_{X_2}(x_2|x_1) = \frac{\pi_{X_1,X_2}(x_1,x_2)}{\pi_{X_1}(x_1)} = \frac{1}{\sqrt{2\pi}\,\sigma_c} e^{-(x_2-\bar{x}_c)^2/(2\sigma_c^2)}$$

where, conditional mean,

$$\bar{x}_c = \bar{x}_2 + \sigma_{12}^2\sigma_{22}^{-2}(x_1-\bar{x}_1)$$

and conditional variance,

$$\sigma_c^2 = \sigma_{11}^2 - \sigma_{12}^2\sigma_{22}^{-2}\sigma_{21}^2$$

marginal $\pi_{X_1}(x_1)$: [Lemma 2.16 Page 41]

Let X_1 and X_2 be two random variables with joint PDF ~~x~~ π_{X_1,X_2}. then

$$\pi_{X_1}(x_1) = \int_X \pi_{X_1,X_2}(x_1,x_2)\,dx_2$$

$$= \int \pi_{X_1}(x_1 | x_2)\,\pi_{X_2}(x_2)\,dx_2$$

$$= \mathbb{E}\left[\pi_{X_1}(x_1 | X_2)\right].$$