4. Exercise sheet *Bayesian inference and data assimilation*

Exercise 1. Let f be the pdf on a continuous random variable Z. The variance of Z is given by σ_Z and the pdf is symmetric ($f(x) = f(-x)$) and everywhere positive. Define another random variable X as $X = \alpha_3 Z^3 + \alpha_2 Z^2 + \alpha_1 Z + \alpha_0$.

(i) For which values of α_i are X and Z uncorrelated?

(ii) For which values of α_i are X and Z independent?

Exercise 2. Which of the following statements are true and false? Prove the true ones and give counterexamples for the false ones. Let X and Z be random variables.

(i) If X and Z are uncorrelated, then they are independent.

(ii) If X and Z are independent, then $\mathbb{E}[X^2] = \mathbb{E}[Z^2]$.

(iii) If X and Z are correlated, they are also dependent.

Exercise 3.

(i) Verify that the Kullback-Leibler divergence of two univariate Gaussians $X_i \sim \mathrm{N}(\bar{x}_i, \sigma_i^2)$, $i = 1, 2$, is given by

$$\begin{aligned} D_{\mathrm{KL}}(\pi_{X_1} || \pi_{X_2}) &= \int_{\mathbb{R}} \ln \frac{\pi_{X_1}(x)}{\pi_{X_2}(x)} \pi_{X_1}(x) \mathrm{d}x \\ &= \frac{1}{2} \left(\sigma_2^{-2} \sigma_1^2 + \sigma_2^{-2} (\bar{x}_2 - \bar{x}_1)^2 - 1 - 2 \log \frac{\sigma_1}{\sigma_2} \right). \end{aligned}$$

(ii) Verify that the Wasserstein distance between π_{X_1} and π_{X_2} is given by

$$W(\pi_{X_1}, \pi_{X_2})^2 = (\bar{x}_1 - \bar{x}_2)^2 + (\sigma_1 - \sigma_2)^2.$$

Exercise 4. Let X_1, X_2 and X_3 be three discrete random variables with

$$\mathbb{P}[X_1 = 0] = \mathbb{P}[X_1 = 1] = \mathbb{P}[X_2 = 0] = \mathbb{P}[X_2 = 1] = \frac{1}{2}$$

and

$$\mathbb{P}[X_3 = 0] = 1.$$

(i) Characterize all possible coupling between X_1 and X_2.

(ii) Which coupling maximizes the correlation? Which coupling minimizes the correlation? Do you have an intuitive explanation why these couplings are the ones that minimize/maximize the correlation?

(iii) Which coupling makes the two random variables uncorrelated?

(iv) Do the tasks $(i) - (iii)$ but for X_1 and X_3.