9. Exercise sheet *Bayesian inference and data assimilation*

Exercise 1. We define a Markov chain on $X = \{1, 2, 3\}$ by

$$\mathrm{P} = \begin{pmatrix} \frac{1}{3} & \frac{1}{5} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{5} & \frac{1}{4} \\ \frac{1}{3} & \frac{2}{5} & \frac{1}{4} \end{pmatrix},$$

where P_{ij} is the probability to move from state j to state i.

(i) What is the invariant measure of this chain? Does the chain converge to its invariant measure?

(ii) Start with $X_0 = 1$. Then use the above transition probabilities to simulate a path $\{X_t\}_{t=1}^{T}$. Plot the trajectories of the Markov chain with $t = 1, \ldots, 100$ on the x-axis.

(iii) Start in $i = 1$ and do $N = 1000$ simulations. Plot histograms at times $T = 1$, $T = 100$, $T = 1000$ and $T = 1000$. Also plot the invariant measure as a comparison.

We now want to sample $\pi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. The Metropolis-Hastings scheme gives a way to modify any Markov chain into another Markov chain. It works by first proposing a new state X_{i+1}^* using the original transition kernels. But then you only accept that new proposal with probability $a = \min(1, \pi(X_{i+1}^*)/\pi(X_i))$. This means, that you set $X_{i+1} = X_{i+1}^*$ with probability a, and otherwise just stay at $X_{i+1} = X_i$.

(iv) Start with $X_0 = 1$. Then use the above transition probabilities to simulate a path of the adjusted markov chain $\{\tilde{X}_t\}_{t=1}^{T}$. Plot the trajectories of the Markov chain with $t = 1, \ldots, 100$ on the x-axis.

(v) Start in $i = 1$ and do $N = 1000$ simulations. Plot histograms of the MH-adjusted Markov chain at times $T = 1$, $T = 100$, $T = 1000$ and $T = 1000$. Also plot the invariant measure as a comparison.

Exercise 2. We define a Markov chain on $X = \{1, 2, 3\}$ by

$$\mathrm{P} = \begin{pmatrix} \frac{2}{3} & \frac{2}{5} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where P_{ij} is the probability to move from state j to state i.

(i) Assume the Markov chain is started in $X_0 = 1$. What distribution will X_t tend towards?

(ii) Assume the Markov chain is started in $X_0 = 3$. What distribution will X_t tend towards?

(iii) Does this Markov chain have an invariant distribution? Does it converge towards it?

This exercise is meant to be done by hand.

Exercise 3. We want to do inference for a non-linear forward operator in a Bayesian inverse problem. We assume that the prior is given as $X \sim \mathcal{N}(2, 2)$. The forward operator is

$$h(x) = x^2$$

and we observe $Y = h(X) + \Xi$ with $\Xi \sim \mathcal{N}(0, 1)$. Assume we observe $y_{\text{obs}} = 2$.

(i) Write down the density $\pi(x|y_{\text{obs}} = 2)$. You do not need to calculate the normalizing constant explicitly.

(ii) Find a MAP estimator m for the posterior distribution using gradient descent.

(iii) Let $\pi(x) = \exp(-g(x))$ be a density. The Laplace approximation to π is defined as $\tilde{\pi} = \mathcal{N}(m, g''(m)^{-1})$. Write down the Laplace approximation to the posterior of X.

(iv) Generate $N = 10000$ samples from $\tilde{\pi}$ and plot a histogram. Also plot a histogram of $N = 10000$ samples of π that you generate through a Langevin SDE. How do these histograms differ?