Exercise 1. Assume that $Z^0 \sim \mathcal{N}(1,1)$ and Z^1 is given as

$$Z^1 = Z^0 + \sqrt{2}\Sigma.$$

The observation Y is generated according to

$$Y = Z^1 + \sqrt{3}\Xi.$$

The noise variables Ξ and Σ are independent of everything else and have distribution $\mathcal{N}(0,1)$. Assume that $y_{\text{obs}} = 3$. The analysis random variable Z^a is defined as

$$Z^a = Z^1 | Y = y_{\text{obs}}.$$

(i) We already know that under these assumptions Z^a is Gaussian and can calculate the mean and variance using the Kalman Filter. Calculate the mean and variance.

(ii) For the EnKF with perturbed observations the analysis is obtained using the non-deterministc coupling

$$\hat{Z}^a = Z^1 - K(HZ^1 + \xi - y_{\text{obs}}), \tag{1}$$

see Definition 7.5 in the book. What is the value of H in this case? How should ξ be distributed?

(iii) Calculate the mean and the variance of $\hat{Z}^a$ using Equation (1). They point of this exercise is to do the calculation yourself, do not just apply the formula from the book (which is not proven). Are they equal to the mean and variance of Z^a?

(iv) Do $\hat{Z}^a$ and Z^a have the same distribution?

Exercise 2. In Exercise 1 we studied the coupling between Z^f and $\hat{Z}^a$. Especially this gives us a way to transform samples from Z^f to samples from $\hat{Z}^a$ (If we only have the distribution of Z^f and Z^a as in the Kalman Filter we do not know how to transform). We now assume that z_j^f are samples from Z^f. Write the EnKF update step

$$z_j^{\text{a}} = z_j^{\text{f}} - K(z_j^{\text{f}} + \xi_j - y_{\text{obs}}), \quad K = \frac{(\sigma_M^{\text{f}})^2}{(\sigma_M^{\text{f}})^2 + R}, \quad \xi_j \sim \text{N}(0, R)\,,$$

for scalar state variable $z \in \mathbb{R}$ in the form of a LETF

$$z_j^{\text{a}} = \sum_{i=1}^{M} z_i^{\text{f}} d_{ij}.$$

Assume that the forecast ensemble $\{z_i^f\}$ consists of independent samples from a random variable Z^f with PDF π_{Z^f}, $H = 1$, and that a set of realisations $\xi_i \in \mathbb{R}$, $i = 1, \ldots, M$ of Ξ is given.

(i) Find the coefficients d_{ij}. Are they random for a fixed y_{obs}?

(ii) Show that

$$\sum_{i=1}^{M} d_{ij} = 1.$$

(iii) Can the d_{ij} get negative? If yes, give an example.