Exercise 3:

$$\int_{\mathbb{R}} \ln \frac{\pi_{x_1}(x)}{\pi_{x_2}(x)} \pi_{x_1}(x) \cdot dx .$$

$$= - \int_{\mathbb{R}} \pi_{x_1}(x) \ln \pi_{x_2}(x) \cdot dx + \int_{\mathbb{R}} \pi_{x_1}(x) \ln \pi_{x_1}(x) \cdot dx \quad \text{——— (i)}$$

Now,

$$- \int_{\mathbb{R}} \pi_{x_1}(x) \ln \pi_{x_2}(x) \cdot dx$$

$$= - \int_{\mathbb{R}} \pi_{x_1}(x) \ln \frac{1}{(2\pi\sigma_2^2)^{1/2}} e^{-\frac{(x-\bar{x}_2)^2}{2\sigma_2^2}} \cdot dx$$

Which can be separated in,

$$\frac{1}{2} \ln (2\pi\sigma_2^2) - \int_{\mathbb{R}} \pi_{x_1}(x) \ln e^{-\frac{(x-\bar{x}_2)^2}{2\sigma_2^2}} \cdot dx .$$

Taking the Log,

$$\frac{1}{2} \ln (2\pi\sigma_2^2) - \int_{\mathbb{R}} \pi_{x_1}(x) \left(- \frac{(x-\bar{x}_2)^2}{2\sigma_2^2} \right) . dx$$

$$= \frac{1}{2} \ln (2\pi\sigma_2^2) + \frac{\int_{\mathbb{R}} \pi_{x_1}(x) x^2 dx - \int_{\mathbb{R}} \pi_{x_1}(x) 2x\bar{x}_2 dx + \int_{\mathbb{R}} \pi_{x_1}(x) \cdot \bar{x}_2^2 dx}{2\sigma_2^2}$$

Letting $\langle \rangle$ denote the expectation operation under x,

$$\frac{1}{2} \ln (2\pi\sigma_2^2) + \frac{\langle x \rangle^2 - 2\langle x \rangle \bar{x}_2 + \bar{x}_2^2}{2\sigma_2^2}$$

We know $\text{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$. Thus.

$$\langle x^2 \rangle = \sigma_1^2 + \bar{x}_1^2$$

So,

$$\frac{1}{2} \ln (2\pi\sigma_2^2) + \frac{\sigma_1^2 + \bar{x}_1^2 - 2\bar{x}_1\bar{x}_2 + \bar{x}_2^2}{2\sigma_2^2}$$

$$= \frac{1}{2}\ln(2\pi\sigma_2^2) + \frac{\sigma_1^2 + (\bar{x}_1 - \bar{x}_2)^2}{2\sigma_2^2}$$

For,

$\int_R \pi_{x_1}(x) \ln \pi_{x_1}(x) \cdot dx$. we found that.

$-\frac{1}{2}(1+\ln(2\pi\sigma_1^2))$; Similar as above.

Putting everything together, in equation (i)

$$D_{KL}(\pi_{x_1} \| \pi_{x_2}) = \frac{1}{2}\ln(2\pi\sigma_2^2) + \frac{\sigma_1^2 + (\bar{x}_1 - \bar{x}_2)^2}{2\sigma_2^2} - \frac{1}{2}(1+\ln(2\pi\sigma_1^2))$$

$$= \ln\frac{\sigma_2}{\sigma_1} + \frac{\sigma_1^2}{2\sigma_2^2} + \frac{(\bar{x}_1 - \bar{x}_2)^2}{2\sigma_2^2} - \frac{1}{2}$$

$$= \frac{1}{2}\left(\sigma_2^{-2}\sigma_1^2 + \sigma_2^{-2}(\bar{x}_2 - \bar{x}_1)^2 - 1 + 2\ln\frac{\sigma_2}{\sigma_1}\right)$$

$$= \frac{1}{2}\left(\sigma_2^{-2}\sigma_1^2 + \sigma_2^{-2}(\bar{x}_2 - \bar{x}_1)^2 - 1 - 2\ln\frac{\sigma_1}{\sigma_2}\right).$$

(ii)

$$W(\pi_{x_1}, \pi_{x_2})^2 = \mathbb{E}_Z[\|x_1 - x_2\|^2], \quad \text{Law}(Z) = \pi_Z^*$$

$$\pi_Z^* = \arg\inf_{\pi_Z \in \Pi(\pi_{x_1}, \pi_{x_2})} \mathbb{E}_Z[\|x_1 - x_2\|]^2 \quad \text{—— (i)}$$

Now,

$$\mathbb{E}_Z[|x_1 - x_2|^2] = \mathbb{E}_{x_1}[|x_1|^2] + \mathbb{E}_{x_2}[|x_2|^2] - 2\,\mathbb{E}_Z[x_1 x_2]$$

$$= \mathbb{E}_{x_1}[|x_1|^2] + \mathbb{E}_{x_2}[|x_2|^2] - 2\,\mathbb{E}_Z[(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)] - 2\bar{x}_1\bar{x}_2$$

$$= \mathbb{E}_{x_1}[|x_1|^2] + \mathbb{E}_{x_2}[|x_2|^2] - 2\bar{x}_1\bar{x}_2 - 2\,Cov(X_1, X_2) \quad \text{—— (ii)}$$

Here,

$$\mathbb{E}_{x_1}[|x_1|^2] = \sigma_1^2 + \bar{x}_1^2$$

and $\mathbb{E}_{x_2}[|x_2|^2] = \sigma_2^2 + \bar{x}_2^2$

The value of π_Z^* will be maximum when $cov(X_1, X_2)$ is maximum.

Now,

$$\frac{Cov(X_1, X_2)}{\sigma_1\sigma_2} = 1.$$

$$\therefore Cov(X_1, X_2) = \sigma_1\sigma_2$$

equation (ii) will be

$$\sigma_1^2 + \bar{x}_1^2 + \sigma_2^2 + \bar{x}_2^2 - 2\bar{x}_1\bar{x}_2 - 2\sigma_1\sigma_2$$

$$= (\bar{x}_1 - \bar{x}_2)^2 + (\sigma_1 - \sigma_2)^2$$

$$\therefore W(\pi_{x_1}, \pi_{x_2})^2 = (\bar{x}_1 - \bar{x}_2)^2 + (\sigma_1 - \sigma_2)^2.$$