4(ii)

	0	1	$P_{x_2}(x_2)$
0	a	b	0.5
1	c	d	0.5
$P_{x_1}(x_1)$	0.5	0.5	

Here,

$$a = d = P$$
$$b = c = 0.5 - P$$

where $P \in [0, 0.5]$

We can list all possible coupling below,

$$P_{x_1, x_2}(0,0) = P$$
$$P_{x_1, x_2}(0,1) = b = 0.5 - P$$
$$P_{x_1, x_2}(1,0) = c = 0.5 - P$$
$$P_{x_1, x_2}(1,1) = d = P$$

4(ii) $E[x_1] = E[x_2] = 0.5$

$V(x_1) = V(x_2) = 0.25$

$\therefore Corr = \frac{1}{\sigma_{x_1}\sigma_{x_2}} \sum_{x_1,x_2} (x_1 - 0.5)(x_2 - 0.5) P_{x_1,x_2}(x_1, x_2)$

$= 2(P - (0.5 - P))$

$= 4P - 1$

We know maximum correlation is 1. If we set the $P = \frac{1}{2}$ we get the maximum correlation $corr(x_1, x_2) = 1$. And If we set the $P = \frac{1}{4} = 0.25$ we get the minimum correlation.

$\therefore \boxed{P = 0.5}$ for maximized

$\boxed{P = 0.25}$ for minimized

~~4(iii)~~

4(iii) Two independent random variables are uncorrelated.

for all x_1, x_2 $P_{x_1}(x_1) P_{x_2}(x_2) = 0.5 \times 0.5$

$= 0.25$

Proof

$P_{x_1}(0)\,P_{x_2}(0) = 0.25$ $\quad P_{x_1}(0)\,P_{x_2}(1) = 0.25$

$P_{x_1}(1)\,P_{x_2}(0) = 0.25$ $\quad P_{x_1}(1)\,P_{x_2}(1) = 0.25$

4(iv)

		x_3		
x_1	0	0	1	$P_{x_1}(x_1)$
	1	$\frac{1}{2}$	0	$\frac{1}{2}$
		$\frac{1}{2}$	0	$\frac{1}{2}$
$P_{x_3}(3)$		1	0	

Since two variables are independent, they are uncorrelated.

$\therefore\ P_{x}(0)\,P_{x_3}(0) = \frac{1}{2}\times 1 = \frac{1}{2}$ $\quad P_{x_1}(0)\,P_{x_3}(1) = \frac{1}{2}\times 0 = 0$

$P_{x_1}(1)\,P_{x_3}(0) = \frac{1}{2}\times 1 = \frac{1}{2}$ $\quad P_{x_1}(1)\,P_{x_3}(1) = \frac{1}{2}\times 0 = 0$

So, coupling with $\frac{1}{2}$ implies two variables are independent.