Exercise 1

Setting:

We have a discrete time Markov process $\{Z^t\}_{t \in T}$ with $T = \{0, 1, 2, \dots\}$
and $Z^t : \Omega \to S$ with $S = \{\text{sunny, rainy}\}$.

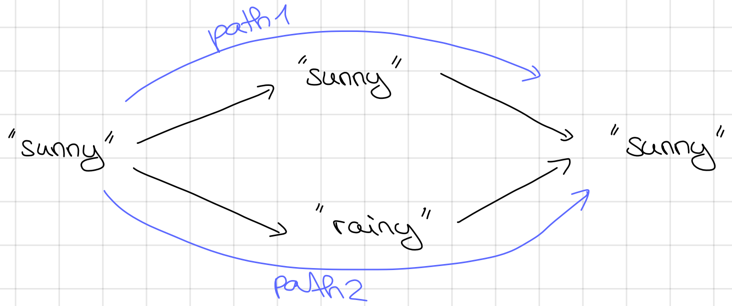
Let's say $t = 0$ represents "today".

Our transition matrix P is defined as $P = \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$

$\longrightarrow \quad p_{ij} = P(Z^{t+1} = a_i \mid Z^t = a_j)$ with $a_1 \hat{=}$ 'sunny'
$a_2 \hat{=}$ 'rainy'

(i) Assume the weather today is sunny. What is the probability that it will be sunny or rainy on the day after tomorrow?

$Z^0 \qquad\qquad Z^1 \qquad\qquad Z^2$



there are two possible paths from Z^0 to Z^2
$\to$ two disjoint events

path 1:

$$\begin{aligned}
&P(Z^1 = a_1, Z^2 = a_1 \mid Z^0 = a_1) \\
&= P(Z^2 = a_1 \mid Z^1 = a_1, Z^0 = a_1)\, P(Z^1 = a_1 \mid Z^0 = a_1) \\
&= P(Z^2 = a_1 \mid Z^1 = a_1)\, P(Z^1 = a_1 \mid Z^0 = a_1) \\
&= p_{11}\, p_{11} \\
&= \left(\tfrac{2}{3}\right)^2 = \tfrac{4}{9}
\end{aligned}$$

path 2:

$$\begin{aligned}
&P(Z^1 = a_2, Z^2 = a_1 \mid Z^0 = a_1) \\
&= P(Z^2 = a_1 \mid Z^1 = a_2, Z^0 = a_1)\, P(Z^1 = a_2 \mid Z^0 = a_1) \\
&= P(Z^2 = a_1 \mid Z^1 = a_2)\, P(Z^1 = a_2 \mid Z^0 = a_1) \\
&= p_{12}\, p_{21} \\
&= \tfrac{1}{5} \cdot \tfrac{1}{3} = \tfrac{1}{15}
\end{aligned}$$

$$\Rightarrow \quad P(Z^2 = a_1 \mid Z^0 = a_1) = \frac{4}{9} + \frac{1}{15} = \frac{23}{45}$$

Because $\mathbb{P}([Z^2 = a_1] \uplus [Z^2 = a_2] \mid Z^0 = a_1) = 1$

$$\mathbb{P}(Z^2 = a_2 \mid Z^0 = a_1) = 1 - \mathbb{P}(Z^2 = a_1 \mid Z^0 = a_1) = 1 - \frac{23}{45} = \frac{22}{45}$$

Alternative strategy:

$$P = \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix}$$

$$\rightarrow \quad P^2 = PP = \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix} \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix} = \begin{pmatrix} \frac{23}{45} & \frac{22}{75} \\ \frac{22}{45} & \frac{53}{75} \end{pmatrix}$$

We can directly derive the results from the matrix P^2:

$$\mathbb{P}(Z^2 = a_1 \mid Z^0 = a_1) = \frac{23}{45}$$

$$\mathbb{P}(Z^2 = a_2 \mid Z^0 = a_1) = \frac{22}{45}$$

$$\mathbb{P}([Z^2 = a_1] \uplus [Z^2 = a_2] \mid Z^0 = a_1) = \frac{23}{45} + \frac{22}{45} = 1$$

(ii) Assume the weather today is sunny. Like a true meteorologist you wait for an infinitely long time and write down the weather every day. What is the relative frequency of sunny days?

- The given Markov chain is irreducible.

  A Markov chain $(X_t), t \in \mathbb{N}_0$ is irreducible if for all possible states s_i, s_j, there exists an $n \in \mathbb{N}$, such that
  $\mathbb{P}(X_n = s_i \mid X_0 = s_j) > 0$

  Since all entries in P are strictly positive, the given Markov chain is irreducible.

- The given Markov chain is positive recurrent.

  Because the Markov chain is irreducible and the state space $S = \{\text{"sunny"}, \text{"rainy"}\}$ is finite, the Markov chain is positive recurrent.

- Because the Markov chain is irreducible and positive recurrent, it has a stationary distribution.

- This unique stationary distribution is given by the eigenvector of P corresponding to the eigenvalue 1 and normalized such that its entries sum up to 1.

$$\det(P-\lambda I) = \det\begin{pmatrix} \frac{2}{3}-\lambda & \frac{1}{5} \\ \frac{1}{3} & \frac{4}{5}-\lambda \end{pmatrix} = \left(\frac{2}{3}-\lambda\right)\left(\frac{4}{5}-\lambda\right) - \frac{1}{15}$$

$$= \frac{8}{15} - \frac{2}{3}\lambda - \frac{4}{5}\lambda + \lambda^2 - \frac{1}{15} = \lambda^2 - \frac{22}{15}\lambda + \frac{7}{15} \overset{!}{=} 0$$

$$\lambda_{1/2} = \frac{\frac{22}{15} \pm \sqrt{\left(\frac{22}{15}\right)^2 - 4\cdot\frac{7}{15}}}{2} \longrightarrow \left.\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = \frac{7}{15} \end{matrix}\right\} \text{eigenvalues}$$

eigenvector for $\lambda_1 = 1$:

$$\begin{pmatrix} \frac{2}{3}-1 & \frac{1}{5} \\ 1/3 & 4/5-1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & 1/5 \\ 1/3 & -1/5 \end{pmatrix} \xrightarrow{\text{Gauß}} \begin{pmatrix} -1/3 & 1/5 \\ 0 & 0 \end{pmatrix}$$

$$-1/3\, x_1 + \frac{1}{5} x_2 = 0$$

$$1/3\, x_1 = 1/5\, x_2$$

$$x_2 = \frac{5}{3} x_1 \longrightarrow \mathbb{L} = \left\{ \begin{pmatrix} 1 \\ 5/3 \end{pmatrix} x \;\middle|\; x \in \mathbb{R} \right\}$$

eigenvector: $\vec{u} = \begin{pmatrix} 1 \\ 5/3 \end{pmatrix} \xrightarrow{\text{normalized}} \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix} = \pi_Z^*$

That means: $\lim_{n\to\infty} P^n = \begin{pmatrix} 3/8 & 3/8 \\ 5/8 & 5/8 \end{pmatrix}$

The relative frequency of sunny days is $3/8$.

(iii) What is the invariant measure for the given Markov chain? Does the chain converge to its invariant measure?

These questions were actually already answered by part (ii).

The invariant measure is the stationary distribution $\pi_Z^* = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix}$.

We have shown why this stationary distribution has to exist, which means that the chain converges to this stationary distribution.

Additionally, we can use the following proof to show that π_Z^* is indeed a stationary distribution:

If $\pi^* = P\pi^* \rightarrow \pi^*$ is a stationary distribution

$$P\pi^* = \begin{pmatrix} 2/3 & 1/5 \\ 1/3 & 4/5 \end{pmatrix} \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix} = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix} = \pi^*$$