Exercise - 3

(i) Given,

$$l(c) = E[(X-c)^2]$$
$$= E[X^2 - 2Xc + c^2]$$
$$= E[X^2] - 2c\,E[X] + c^2. \quad \text{— (i)}$$

Minimizing (i) we get,

$$\frac{d\,l(c)}{dc} = 0$$

$$\Rightarrow \frac{d}{dc}\left(E[X^2] - 2c\,E[X] + c^2\right) = 0$$
$$\Rightarrow -2\,E[X] + 2c = 0$$
$$\therefore\ c = E[X]$$

Again, $$\frac{d^2\,l(c)}{dc^2} = \frac{d^2}{dc^2}\left(-2\,E[X] + 2c\right)$$
$$= 2$$

which is greater than zero, so, it is the minimum value.

(ii) $$l(c) = E[|x-c|] = \int_{-\alpha}^{\alpha} |x-c| f(x) \cdot dx$$

$$= \int_{-\alpha}^{c} (c-x) f(x)\, dx + \int_{c}^{\alpha} (x-c) f(x)\, dx$$

$$= \int_{-\alpha}^{c} (c-x) f(x)\, dx + \int_{c}^{l} (x-c) f(x)\, dx$$

$$+ \int_{l}^{\alpha} (x-c) f(x) \cdot dx$$

$$= \int_{-\alpha}^{l} (c-x) f(x)\, dx + \int_{c}^{l} (x-c) f(x) \cdot dx + \int_{c}^{l} (x-c) f(x)\, dx$$

$$+ \int_{l}^{\alpha} (x-c) f(x) \cdot dx$$

$$= \int_{-\alpha}^{l} (c-l+l-x) f(x) \cdot dx + 2\int_{c}^{l} (x-c) \cdot f(x) \cdot dx$$

$$+ \int_{l}^{\alpha} (x-l+l-c) f(x)\, dx$$

$$= E(|x-l|) + 2\int_{c}^{l} (x-c) f(x) \cdot dx + \int_{-\alpha}^{l} (c-l) f(x)\, dx$$

$$+ \int_{l}^{\alpha} (l-c) f(x) \cdot dx.$$

$$= E(|x-l|) + 2\int_{c}^{l} (x-c) f(x)\, dx + (c-l)\, 1/2$$

$$+ (l-c)\, 1/2$$

[integrate density $-\alpha$ to l (l is median) then $f(x)$ will be $1/2$]

$$= E(|x-l|) + 2\int_{c}^{l} (x-c) f(x) \cdot dx$$

$E(|X-c|)$ is minimized when $2\int_c^{l}(x-c)f(x)\,.dx=0$
which happens when $c=l$.

Now, $\frac{d}{dl}(E|X-l|) = E\left(\frac{d}{dl}|X-l|\right)$

$$= E\left(\frac{(-1)(X-l)}{|X-l|}\right)$$

$$= E\left[1\{X<l\} - 1\{X>l\}\right]$$

$$= P(X<l) - P(X>l) \stackrel{set}{=} 0$$

$$\Leftrightarrow P(X<l) = P(X>l)$$

If they both are $1/2$ then only they are equal.

$$\therefore P(X<l) = P(X>l) = 1/2$$

If $P(X<l) = 1/2$ and $P(X>l) = 1/2$, then
l by defination is the Median.