Exercise 1:

Given,

$$X \sim N(\bar{x}, P)$$

$$H = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and $H^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\Xi \sim \mathcal{N}(0, R)$$

From Example 5.10 we know that if $X \sim N(\bar{x}, P)$ and $h(x) = Hx$, then the posterior distribution of X given an observed $y = y_{obs}$ is also ~~gussian~~ gaussian with mean.

$$\bar{x}^a = \bar{x} - P^a H^T R^{-1} (H\bar{x} - y_{obs})$$

$$= \bar{x} - \underbrace{PH^T (HPH^T + R)^{-1} (H\bar{x} - y_{obs})}_{(a)}$$

$H = \begin{bmatrix} 0 & 1 \end{bmatrix}$ and $H^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Part (a) will be,

$$P \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \end{bmatrix} P \begin{bmatrix} 0 \\ 1 \end{bmatrix} + R \right)^{-1} \left(\begin{bmatrix} 0 & 1 \end{bmatrix} \bar{x} - y_{obs} \right)$$

$$= \begin{bmatrix} 0 \\ P \end{bmatrix} (P + R)^{-1} \left(\begin{bmatrix} 0 & \bar{x} \end{bmatrix} - y_{obs} \right)$$

$$= \begin{bmatrix} 0 \\ P \end{bmatrix} \frac{1}{P+R} \left(\begin{bmatrix} 0 & \bar{x} \end{bmatrix} - y_{obs} \right)$$

Given that R tends to ∞ then $\frac{1}{P+R}$ will be come 0 so part (a) will be 0.

$$\therefore \bar{x}^a = \bar{x}$$

Example 5.11 we can know that,

$$\overline{x}_j^a = \overline{x}_j - P_j H^T (H P_j H^T + R)^{-1} (H \overline{x}_j - y_{obs})$$

we already proof the second part is 0

so, for 1'st component $\overline{x}_1^a = \overline{x}_1$

and for 2'nd component $\overline{x}_2^a = \overline{x}_2$

Now, For convariance matrix,

$$P^a = (P^{-1} + H^T R^{-1} H)^{-1}$$
$$= P - PH^T (HPH^T + R)^{-1} HP \quad \text{— [Example 5.10]}$$

$H = [0 \ 1]$ and $H^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$P^a = P - P\begin{bmatrix} 0 \\ 1 \end{bmatrix} \left([0\,1] P \begin{bmatrix} 0 \\ 1 \end{bmatrix} + R\right)^{-1} [0\,1] P$$
$$= P - \begin{bmatrix} 0 \\ P \end{bmatrix} (P+R)^{-1} [0 \ P]$$
$$= P - \begin{bmatrix} 0 \\ P \end{bmatrix} \frac{1}{P+R} [0 \ R]$$

Given that R tends to α then $\frac{1}{P+R}$ will become 0

$$\therefore P^a = P$$

From Example 5.11 we know.

$$P_j^a = P_j - P_j H^T (H P_j H^T + R)^{-1} H P_j$$

The second part is 0,

So, for 1'st component $P_1^a = P_1$

and 2nd component $P_2^a = P_2$

If we take a column vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

then $\bar{x}_1 = \begin{pmatrix} \bar{x}_1 x_1 \\ \bar{x}_1 x_2 \end{pmatrix}$ and $\bar{x}_2 = \begin{pmatrix} \bar{x}_2 x_1 \\ \bar{x}_2 x_2 \end{pmatrix}$

and the covariance matrix,

$$P_1^a = P_2^a = \begin{pmatrix} V[x_1] & \mathrm{cov}(x_1, x_2) \\ \mathrm{Cov}[x_1, x_2] & V[x_2] \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{x_1}^2 & \mathrm{cov}(x_1, x_2) \\ \mathrm{Cov}(x_1, x_2) & \sigma_{x_2}^2 \end{pmatrix}$$