Exercise 3:

Given, $f(x_1, x_2) = -12x_1 - 6x_2 + 6x_1x_2$

we know, $f(x_1, x_2) = f_0 + f_1(x_1) + f_2(x_2) + f_{12}(x_1, x_2)$ —— (i)

$$
\begin{aligned}
f_0 = E[f(x)] &= \int_0^1 \int_0^1 f(x_1, x_2) \cdot dx_1\, dx_2 \\
&= \int_0^1 \left( \int_0^1 -12x_1 - 6x_2 + 6x_1x_2 \, dx_1 \right) dx_2 \\
&= \int_0^1 \left[ -6x_1^2 - 6x_1x_2 + 3x_1^2x_2 \right]_0^1 dx_2 \\
&= \int_0^1 (-6 - 3x_2)\, dx_2 \\
&= \left[ -6x_2 - \frac{3}{2}x_2^2 \right]_0^1 \\
&= -\frac{15}{2}
\end{aligned}
$$

$$
\begin{aligned}
f_1(x_1) &= \int_0^1 f(x_1, x_2) \cdot dx_2 - f_0 \\
&= \int_0^1 (-12x_1 - 6x_2 + 6x_1x_2)\, dx_2 + \frac{15}{2} \\
&= \left[ -12x_1x_2 - 3x_2^2 + 3x_1x_2^2 \right]_0^1 + \frac{15}{2} \\
&= \left[ -12x_1 - 3 + 3x_1 \right] + \frac{15}{2} \\
&= -9x_1 + \frac{9}{2}
\end{aligned}
$$

$$
\begin{aligned}
f_2(x_2) &= \int_0^1 f(x_1, x_2) \cdot dx_1 - f_0 \\
&= \int_0^1 (-12x_1 - 6x_2 + 6x_1x_2)\, dx_1 + \frac{15}{2} \\
&= \left( -6x_1^2 - 6x_1x_2 + 3x_1^2x_2 \right]_0^1 + \frac{15}{2} \\
&= -6 - 6x_2 + 3x_2 + \frac{15}{2} \\
&= -3x_2 + \frac{15}{2}
\end{aligned}
$$

$$
\begin{aligned}
f_{12}(x_1, x_2) &= f(x_1, x_2) - f_0 - f_1(x_1) - f_2(x_2) \\
&= -12x_1 - 6x_2 + 6x_1x_2 + \frac{15}{2} + 9x_1 - \frac{9}{2} + 3x_2 - \frac{3}{2} \\
&= -3x_1 - 3x_2 + 6x_1x_2 + \frac{3}{2}.
\end{aligned}
$$

For uniform distribution,
$E(x) = \frac{a+b}{2}$ and $var(x) = \frac{1}{2}(b-a)^2$

$$\sigma_1^2 = var(f_1(x_1))$$
$$= var(-9x_1 + \frac{9}{2})$$
$$= var(-9x_1)$$
$$= 81 \cdot \frac{1}{12} \quad [x_1 \sim U[0,1],\ var(x_1) = \frac{(1-0)^2}{12} = \frac{1}{12}]$$
$$= \frac{27}{4}$$

$$\sigma_2^2 = var(f_2(x_2))$$
$$= var(-3x_2 + \frac{3}{2})$$
$$= var(-3x_2)$$
$$= 9 \cdot 1/12$$
$$= 3/4$$

$$\sigma_{12}^2 = var(f_{12}(x_1, x_2))$$
$$= var(-3x_1 - 3x_2 + 6x_1x_2 + \frac{3}{2})$$
$$= var(-3x_1 - 3x_2) + var(6x_1x_2) - 2\,cov(-3x_1 - 3x_2, 6x_1x_2)$$
$$= var(-3x_1) + var(-3x_2) + 2\,cov(-3x_1, -3x_2) + var(6x_1x_2) - 2\,cov(-3x_1 - 3x_2, 6x_1x_2)$$
$$= 9\,var(x_1) + 9\,var(x_2) + 36\,var(x_1x_2) - 36\,cov(x_1 + x_2, x_1x_2)$$
[$x_1$ and $x_2$ are independent so $cov(-3x_1, -3x_2) = 0$]

$$= \cancel{9\,var(x_1) + 9\,var(x_2) + 36\,var(x_1x_2)}\ 3$$
$$= 9\,var(x_1) + 9\,var(x_2) + 36\,var(x_1, x_2) - 36\,cov(x_1, x_1x_2) - 36\,cov(x_2, x_1x_2)$$

—— (ii)

$$\mathrm{var}(x_1, x_2) = E[x_1^2 x_2^2] - (E[x_1 x_2])^2$$

$$= \int_0^1\int_0^1 x_1^2 x_2^2 \, dx_1 dx_2 - (E[x_1]E[x_2])^2$$

[$x_1$ and $x_2$ are independent]

$$= \int_0^1 \left[\frac{x_1^3}{3} x_2^2\right]_0^1 dx_2 - \left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$$

$$= \int_0^1 \frac{x_2^2}{3} \cdot dx_2 - \left(\frac{1}{4}\right)^2$$

$$= \frac{1}{9}\left[x_2^3\right]_0^1 - \frac{1}{16}$$

$$= \frac{1}{9} - \frac{1}{16}$$

$$= 7/144$$

$$\mathrm{cov}(x_1, x_1 x_2) = E[x_1 x_1 x_2] - E[x_1]E[x_1 x_2]$$

$$= \int_0^1\int_0^1 x_1^2 x_2 \, dx_1 dx_2 - \frac{1}{2} E[x_1]E[x_2]$$

[$x_1$ & $x_2$ are independent]

$$= \int_0^1 \left[\frac{x_1^3}{3} x_2\right]_0^1 dx_2 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$$

$$= \int_0^1 \frac{1}{3} x_2 \, dx_2 - \frac{1}{8}$$

$$= \left[\frac{1}{6} x_2^2\right]_0^1 - \frac{1}{8}$$

$$= \frac{1}{6} - \frac{1}{8}$$

$$= \frac{1}{24}$$

(ii) will be.

$$\sigma_{12}^2 = 9\cdot\frac{1}{12} + 9\cdot\frac{1}{12} + 36\cdot\frac{7}{144} - 36\cdot\frac{1}{24} - 36\cdot\frac{1}{24}$$

$$= \frac{1}{4}$$

So, $\sigma_1^2 = \frac{27}{4}$, $\sigma_2^2 = \frac{3}{4}$, $\sigma_{12}^2 = \frac{1}{4}$

6. $\sigma_1^2 > \sigma_2^2 > \sigma_{12}^2$

So, $\sigma_1^2$ contribute most significantly to the total variance $\sigma^2$.