Exercise 2. We define a Markov chain on $X = \{1, 2, 3\}$ by

$$P = \begin{pmatrix} \frac{2}{3} & \frac{2}{5} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where P_{ij} is the probability to move from state j to state i.

I will use a slightly different notation: n as iteration index (instead of t)
Define: $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ (possible states)

The transition matrix shows that there are two "separate systems". If we start in state $a_3 = 3$, we cannot leave this state. If we start in either state $a_1 = 1$ or $a_2 = 2$, we can transition to the respective other state, but never to a_3.

Explanation:

- Let the current state $X^n(\omega)$ be $a_3 = 3$: $\mathbb{P}(X^{n+1} = a_3 \mid X^n = a_3) = 1$
  - $\rightarrow$ it is determined that $X^{n+1}(\omega) = a_3 = 3$
  - $\rightarrow$ by induction: there is no possibility to leave state a_3

- Let the current state $X^n(\omega)$ be $a_1 = 1$: $\mathbb{P}(X^{n+1} = a_3 \mid X^n = a_1) = 0$
  - $\rightarrow$ we cannot reach a_3 from a_1 directly

  Let the current state $X^n(\omega)$ be $a_2 = 2$: $\mathbb{P}(X^{n+1} = a_3 \mid X^n = a_2) = 0$
  - $\rightarrow$ we cannot reach a_3 from a_2 directly

  Because we cannot reach a_3 from either a_1 or a_2 directly, we can conclude that we will never reach a_3 from a_1 or a_2.

Formally, we refer to this as communicating classes. $\{a_1, a_2\}$ and $\{a_3\}$ are <u>two communicating classes</u>. Because there is not exactly one communicating class, the Markov chain is <u>reducible</u>, i.e. there is <u>no unique</u> stationary distribution.

In the following exercises we investigate what happens within each of these two "separate systems".

(i) Assume the Markov chain is started in $X_0 = 1$. What distribution will X_t tend towards?

Let π^n be the distribution of the RV X^n, then $\pi^0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Since, we know that we can never leave the set $\{a_1, a_2\}$, we can just investigate sub-transition matrix P_{12}:

$$P_{12} = \begin{pmatrix} 2/3 & 2/5 \\ 1/3 & 3/5 \end{pmatrix}$$

Because each entry is strictly positive, we know that this "sub-chain" is contracting and there is a stationary distribution $\pi^*_{(12)}$, such that $\pi^*_{(12)} = P_{12}\pi^*_{(12)}$.

Compute eigenvector of P_{12} that corresponds to eigenvalue 1:

$$\det(P_{12} - \lambda I) = \left(\tfrac{2}{3}-\lambda\right)\left(\tfrac{3}{5}-\lambda\right) - \tfrac{1}{3}\cdot\tfrac{2}{5} = \tfrac{2}{5} - \tfrac{2}{3}\lambda - \tfrac{3}{5}\lambda + \lambda^2 - \tfrac{2}{15}$$
$$= \lambda^2 - \tfrac{19}{15}\lambda + \tfrac{4}{15}$$
$$\overset{!}{=} 0$$

$$\lambda_{1/2} = \frac{\frac{19}{15} \pm \sqrt{\left(\frac{19}{15}\right)^2 - 4\cdot\frac{4}{15}}}{2} \rightarrow \begin{array}{l}\lambda_1 = 1\\ \lambda_2 = \frac{4}{15}\end{array}$$

can be skipped

$$P_{12} - I = \begin{pmatrix} \frac{2}{3}-1 & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{5}-1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{5} \\ \frac{1}{3} & -\frac{2}{5} \end{pmatrix} \xrightarrow{\text{Gauß}} \begin{pmatrix} -1 & \frac{6}{5} \\ 0 & 0 \end{pmatrix}$$

$$\rightarrow \quad x_1 = \tfrac{6}{5} x_2$$

eigenvector: $\vec{u} = \begin{pmatrix} \frac{6}{5} \\ 1 \end{pmatrix} \rightarrow \pi^*_{(12)} = \begin{pmatrix} \frac{6}{11} \\ \frac{5}{11} \end{pmatrix}$, $\|\pi^*_{(12)}\|_1 = 1$

Test if $\pi^*_{(12)} = P_{12}\pi^*_{(12)}$ is true:

$$\begin{pmatrix} \frac{2}{3} & \frac{2}{5} \\ \frac{1}{3} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} \frac{6}{11} \\ \frac{5}{11} \end{pmatrix} = \begin{pmatrix} \frac{6}{11} \\ \frac{5}{11} \end{pmatrix} \quad \checkmark$$

To return to the initial question:
If the Markov chain is started in $X_0 = 1$, X_n will tend towards $\pi^*_{(12)} = \begin{pmatrix} \frac{6}{11} \\ \frac{5}{11} \\ 0 \end{pmatrix} \leftarrow$ $a_3 = 3$ cannot be reached

(ii) Assume the Markov chain is started in $X_0 = 3$. What distribution will X_t tend towards?

As we have noted before, if the Markov chain is in state $a_3 = 3$ it cannot leave this state since $P(X_{n+1} = a_3 \mid X_n = a_3) = 1$!
Therefore, the distribution will be constant throught the process with $\pi^*_{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Test: $$P\pi^*_{(3)} = \begin{pmatrix} \frac{2}{3} & \frac{2}{5} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \pi^*_{(3)}$$

(iii) Does this Markov chain have an invariant distribution? Does it converge towards it?

The discussion above has revealed that there are two invariant distributions $\pi^*_{(12)}$ and $\pi^*_{(3)}$.
The initial state decides to which of these two distributions the Markov chain will converge to. It is guaranteed that the chain will converge to one of these.

X^0 is $a_1 = 1$ or $a_2 = 2$ $\rightarrow$ convergence towards $\pi^*_{(12)}$
X^0 is $a_3 = 3$ $\rightarrow$ convergence towards $\pi^*_{(3)}$