Exercise 3. We want to study discretizations of the ODE with noise (also called SDE) for $X_t \in \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t}X_t = -X_t + \sqrt{2}\frac{\mathrm{d}B_t}{\mathrm{d}t}.$$

A straightforward generalization of the Euler method to the stochastic case is the so-called Euler-Maruyama method. It is the same as the Euler-Method, just that the noise needs to be multiplied by $\sqrt{dt}$ instead of dt. The discretization of the above would be

$$x_{n+1} = x_n - \delta t x_n + \sqrt{2\delta t}\xi_n.$$

x_n is an approximation of $X_{n\delta t}$. Assume x_0 is set to $x_0 = 1$.

Define $\Xi_i \sim N(0,1)$ for $i = 0, 1, \dots, n, \dots$
We assume that the ξ_i's are realisations of the Ξ_i RVs.

(i) What is the distribution of x_1 and x_2?

$x_1 = x_0 - \delta t\, x_0 + \sqrt{2\delta t}\,\xi_0$ with ξ_0 being sampled from $\Xi_0 \sim N(0,1)$ and $x_0 = 1$

$x_1 = 1 - \delta t + \sqrt{2\delta t}\,\xi_0$

Therefore, we can see this as a simple linear transformation, i.e. we know the pdf π_{X_1} belongs to a normal distribution. We simply have to find the mean and variance of X_1.

$$\bar{x}_1 = \mathbb{E}\left[\sqrt{2\delta t}\,\Xi_0 + 1 - \delta t\right] = \sqrt{2\delta t}\,\underbrace{\mathbb{E}[\Xi_0]}_{=0} + 1 - \delta t = 1 - \delta t$$

$$\sigma_1^2 = \operatorname{Var}\left(\sqrt{2\delta t}\,\Xi_0 + 1 - \delta t\right) = 2\delta t \operatorname{Var}(\Xi_0) = 2\delta t$$

$$\rightarrow \quad \pi_{X_1}(x) = n(x; \bar{x}_1, \sigma_1^2) \quad \text{with} \quad \begin{aligned} \bar{x}_1 &= 1 - \delta t \\ \sigma_1^2 &= 2\delta t \end{aligned}$$

$$x_2 = x_1 - \delta t\, x_1 + \sqrt{2\delta t}\,\xi_1$$

This can be seen as the sum of two normal distributions. Therefore, also the pdf π_{X_2} belongs to a normal distribution.

$$\begin{aligned} \bar{x}_2 &= \mathbb{E}\left[X_1 - \delta t X_1 + \sqrt{2\delta t}\,\Xi_1\right] = \mathbb{E}[X_1] - \delta t\, \mathbb{E}[X_1] + \sqrt{2\delta t}\,\mathbb{E}[\Xi_1] \\ &= (1-\delta t)^2 \end{aligned}$$

X_1 and Ξ_1 are independent

$$\begin{aligned} \sigma_2^2 &= \operatorname{Var}\left((1-\delta t)X_1 + \sqrt{2\delta t}\,\Xi_1\right) = (1-\delta t)^2 \operatorname{Var}(X_1) + 2\delta t \operatorname{Var}(\Xi_1) \\ &= (1-\delta t)^2\, 2\delta t + 2\delta t \\ &= 2\delta t\left((1-\delta t)^2 + 1\right) \\ &= 2\delta t\left((\delta t)^2 - 2\delta t + 2\right) = 2(\delta t)^3 - 4(\delta t)^2 + 4\delta t \end{aligned}$$

$$\rightarrow \quad \pi_{X_2}(x) = n(x; \bar{x}_2, \sigma_2^2) \quad \text{with} \quad \begin{aligned} \bar{x}_2 &= (1-\delta t)^2 \\ \sigma_2^2 &= 2\delta t\left((1-\delta t)^2 + 1\right) \end{aligned}$$