Exercise 1.

(i) All odd central moments ~~t~~ of a symmetric distribution equal to 0.

i.e. $E(x) = E(x^3) = E(x^5) = \cdots = 0$

$X = \alpha_3 Z^3 + \alpha_2 Z^2 + \alpha_1 Z + \alpha_0$

Now,

$\mathrm{Cov}(X, Z) = 0$

$\Rightarrow \mathrm{Cov}(\alpha_3 Z^3 + \alpha_2 Z^2 + \alpha_1 Z + \alpha_0,\ Z) = 0$

$\Rightarrow \alpha_3 \mathrm{Cov}(Z^3, Z) + \alpha_2 \mathrm{Cov}(Z^2, Z) + \alpha_1 \mathrm{Cov}(Z, Z) + \mathrm{Cov}(\alpha_0, Z) = 0$

$\Rightarrow \alpha_3 \left[E[Z^4] - E[Z^3]E[Z] \right] + \alpha_2 \left[E[Z^3] - E[Z^2]E[Z] \right]$
$+ \alpha_1 E[Z^2] + 0 = 0$

$\Rightarrow \alpha_3 \sigma_Z^4 + \alpha_2 \times 0 + \alpha_1 \sigma_Z^2 + 0 = 0$

$\Rightarrow \sigma_Z^2 (\alpha_3 \sigma_Z^2 + \alpha_1) = 0$

$\Rightarrow \alpha_3 \sigma_Z^2 + \alpha_1 = 0$

$\therefore \alpha_3 = -\frac{\alpha_1}{\sigma_Z^2}$ and $\alpha_1 = -\alpha_3 \sigma_Z^2$.

and $\alpha_2 \in \mathbb{R}$, $\alpha_4 \in \mathbb{R}$

So, X and Z are uncorrelated if $\alpha_3 = -\frac{\alpha_1}{\sigma_Z^2}$, and
$\alpha_1 = -\alpha_3 \sigma_Z^2$

(ii) Correlation = 0 is necessary condition for independence.

i.e. if $\mathrm{corr}(X, Z) \neq 0$, then X, Z are not independent.

For, X, Z independent, corr = 0, $\alpha_1 = -\alpha_3 \sigma_Z^2$, and $\alpha_3 = -\frac{\sigma_1}{\sigma_Z^2}$

So, X will be, $X = -\frac{\alpha_1}{\sigma_Z^2} \cdot Z^3 + \alpha_2 Z^2 + (-\alpha_3 \sigma_Z^2) Z + \alpha_0$

Since X and Z ~~Z, Z^2 and Z^3~~ are not independent, it will be independent only when, $-\frac{\alpha_1}{\sigma_Z^2} = 0$, $\alpha_2 = 0$ and $-\alpha_3 \sigma_Z^2 = 0$

$\therefore$ for $X = \alpha_0$, X and Z are independent.

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\alpha_0 \in \mathbb{R}$, 'α_0' can be anything.