Exercise 1. Assume that X is sampled from a prior distribution $X \sim \mathcal{N}(\bar{x}, P)$. We then observe $h(x) = \cancel{x_1}\ x_2 = Hx$ with observation operator

$$H = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T. \quad \leftarrow \quad 1 \times 2 \quad \text{row vector}$$

We observe $Y = HX + \Xi \in \mathbb{R}$ with $\Xi \sim \mathcal{N}(0, R)$ being independent of X. Calculate the conditional distribution of X given $Y = y_{\text{obs}}$. How do the two components of X behave when the measurement error R tends to ∞?

Define: $\quad \bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}, \quad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$

$$\pi_X(x \mid y_{obs}) = n(x; \bar{x}^a, P^a)$$

with
$$\bar{x}^a = \bar{x} - PH^T(HPH^T + R)^{-1}(H\bar{x} - y_{obs})$$
$$P^a = P - PH^T(HPH^T + R)^{-1} HP$$

according to our lecture.

$$\bar{x}^a = \bar{x} - PH^T(HPH^T + R)^{-1}(H\bar{x} - y_{obs})$$

$$\overset{(*1)}{=} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} - \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} (P_{22} + R)^{-1} (\bar{x}_2 - y_{obs})$$

$$= \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} - \frac{\bar{x}_2 - y_{obs}}{P_{22} + R} \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} \bar{x}_1^a &= \bar{x}_1 - (\bar{x}_2 - y_{obs}) \frac{P_{12}}{P_{22} + R} \\ \bar{x}_2^a &= \bar{x}_2 - (\bar{x}_2 - y_{obs}) \frac{P_{22}}{P_{22} + R} \end{aligned}$$

$$(*1) \quad HPH^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = P_{22}$$

$$P^a = P - PH^T(HPH^T + R)^{-1} HP$$

$$\overset{(*1)}{=} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} - \begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} (P_{22} + R)^{-1} \begin{bmatrix} P_{21} & P_{22} \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} - \frac{1}{P_{22} + R} \begin{bmatrix} P_{12}P_{21} & P_{12}P_{22} \\ P_{21}P_{22} & P_{22}^2 \end{bmatrix}$$

$$= \begin{bmatrix} P_{11} - \frac{P_{12}P_{21}}{P_{22}+R} & P_{12} - \frac{P_{12}P_{22}}{P_{22}+R} \\ P_{21} - \frac{P_{21}P_{22}}{P_{22}+R} & P_{22} - \frac{P_{22}^2}{P_{22}+R} \end{bmatrix}$$

What happens when $R \to \infty$?

$$\lim_{R\to\infty} \bar{x}_1^a = \bar{x}_1 - (\bar{x}_2 - y_{obs}) \underbrace{\lim_{R\to\infty} \frac{p_{12}}{p_{22}+R}}_{\to 0} = \bar{x}_1$$

$$\lim_{R\to\infty} \bar{x}_2^a = \bar{x}_2 - (\bar{x}_2 - y_{obs}) \underbrace{\lim_{R\to\infty} \frac{p_{22}}{p_{22}+R}}_{\to 0} = \bar{x}_2$$

$$\lim_{R\to\infty} P^a = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = P$$

because $\lim_{R\to\infty} \frac{a}{b+R} = 0$ for any constant real numbers a and b

This means, that the posterior converges towards the prior when the measurment error variance R tends to ∞.