Exercise 5:

Given, $d_{Hell}(P,Q) = \left(\frac{1}{2}\int(\sqrt{P(x)} - \sqrt{Q(x)})^2 dx\right)^{1/2}$

$\Rightarrow d^2_{Hell}(P,Q) = \frac{1}{2}\int(\sqrt{P(x)} - \sqrt{Q(x)})^2 dx$

$= \frac{1}{2}\int[P(x) - 2\sqrt{P(x)}\sqrt{Q(x)} + Q(x)]dx$

$= \frac{1}{2}\int P(x)\cdot dx - \int\sqrt{P(x)}\sqrt{Q(x)}\cdot dx + \frac{1}{2}\int Q(x)\cdot dx$

$= \frac{1}{2} - \int\sqrt{P(x).Q(x)}\cdot dx + \frac{1}{2}$

$= 1 - \int\sqrt{P(x)Q(x)}\cdot dx$ ——(i)

$D_{KL}(P||Q) = -\int \log\frac{Q(x)}{P(x)}\cdot P(x)\cdot dx$

$\geq 2\int\left(1 - \sqrt{\frac{Q(x)}{P(x)}}\right).P(x).dx$ [Given, $2(1-\sqrt{x}) \leq -\log x$]

$\geq 2\int\frac{\sqrt{P(x)} - \sqrt{Q(x)}}{\sqrt{P(x)}}\cdot P(x)\cdot dx$

$\geq 2\int(\sqrt{P(x)} - \sqrt{Q(x)})\sqrt{P(x)}\cdot dx$

$\geq 2\int P(x)\cdot dx - 2\int\sqrt{P(x).Q(x)}\cdot dx$

$\geq 2 - 2\int\sqrt{P(x).Q(x)}\cdot dx$

$\geq 2\left(1 - \int\sqrt{P(x).Q(x)}\,dx\right)$

$\geq 2\, d^2_{Hell}(P,Q)$ [From (i)]

So, $d_{Hell}(P,Q)^2 \leq \frac{1}{2} D_{KL}(P||Q)$