7. Exercise sheet *Bayesian inference and data assimilation*

Exercise 1. A very precise model of the weather models it as either sunny or rainy, i.e. $S = \{\text{sunny}, \text{rainy}\}$. The probability of having a sunny day after another sunny day is $\frac{2}{3}$. With a probability of $\frac{1}{3}$ it will rain on the next day. The probability that a rainy day is followed by another rainy day is $\frac{4}{5}$. We can express this with the following matrix

$$\mathrm{P} = \begin{pmatrix} \frac{2}{3} & \frac{1}{5} \\ \frac{1}{3} & \frac{4}{5}. \end{pmatrix}$$

(i) Assume the weather today is sunny. What is the probability that it will be sunny or rainy on the day after tomorrow?

(ii) Assume the weather today is sunny. Like a true meteorologist you wait for an infinitely long time and write down the weather every day. What is the relative frequency of sunny days?

(iii) What is the invariant measure for the given Markov chain? Does the chain converge to its invariant measure?

Exercise 2. We treat the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = -x(x-2).$$

(i) Implement the Euler Method for this scheme. Plot the solutions for $x(0) = -2, 0.5, 2$ and describe their behaviour.

(ii) Plot the function $f(x)$. What does it mean for the system if $f(x) > 0$, $f(x) < 0$ or $f(x) = 0$?

(iii) Find the invariant points of the dynamical system. A point a is invariant if $x(t) = a$ implies that $x(t+s) = a$ for all $s > 0$.

(iv) How does the dynamical system $x(t)$ behave in the long term as $t \to \infty$? Note that this will depend on $x(0)$!

Exercise 3. We want to study discretizations of the ODE with noise (also called SDE) for $X_t \in \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}t} X_t = -X_t + \sqrt{2}\frac{\mathrm{d}B_t}{\mathrm{d}t}.$$

A straightforward generalization of the Euler method to the stochastic case is the so-called Euler-Maruyama method. It is the same as the Euler-Method, just that the noise needs to be multiplied by $\sqrt{dt}$ instead of dt. The discretization of the above would be

$$x_{n+1} = x_n - \delta t x_n + \sqrt{2\delta t}\xi_n.$$

x_n is an approximation of $X_{n\delta t}$. Assume x_0 is set to $x_0 = 1$.

(i) What is the distribution of x_1 and x_2?

(ii) Run the above scheme $N = 10\ 000$ times. If you implement this right (using matrices and numpy), this should not take a long time to run! Create histograms of the distributions at time $t = 0$, $t = 0.5$, $t = 1$ and $t = 10$. Do this once for $\delta t = 0.01$ and $\delta t = 0.5$.

(iii) How do the distributions at the different times look? Do you see if it converges to something?