Exercise 3:

$$E[x_1 x_2^2 \mid x_1 = a] = a\,E[x_2^2 \mid x_1 = a] \quad [\because x_1 = a]$$

$$= a \int_{-\infty}^{\infty} x_2^2\, \pi_{x_2}(x_2 \mid x_1 = a)\, dx_2$$

$$= a \int_{-\infty}^{\infty} \frac{x_2^2\, \pi_{x_1,x_2}(x_2, x_1 = a)}{\pi_{x_1}(x_1 = a)}\, dx_2$$

$$= a\, \frac{\int_{-\infty}^{\infty} x_2^2\, \pi_{x_1,x_2}(x_2, x_1 = a)\, dx_2}{\int_{-\infty}^{\infty} \pi_{x_1}(x_1 = a)\, dx_2}$$

$$= a\, \frac{\int_{-\infty}^{\infty} x_2^2\, \pi_{x_1,x_2}(x_2, x_1 = a)\, dx_2}{\int_{-\infty}^{\infty} \pi_{x_1,x_2}(x_2, x_1 = a)\, dx_2}$$

$$= a\, \frac{\cancel{\frac{1}{Z}} \int_{-\infty}^{\infty} x_2^2\, e^{-x_2^2 - a^2 - a^2 x_2^2}\, dx_2}{\cancel{\frac{1}{Z}} \int_{-\infty}^{\infty} e^{-x_2^2 - a^2 - a^2 x_2^2}\, dx_2}$$

$$= a \cdot \frac{\cancel{e^{-a^2}} \cdot \int_{-\infty}^{\infty} x_2^2\, e^{-x_2^2(1+a^2)}\, dx_2}{\cancel{e^{-a^2}} \int_{-\infty}^{\infty} e^{-x_2^2(1+a^2)}\, dx_2} \quad \text{——(i)}$$

Now,

For, $\int_{-\infty}^{\infty} e^{-x_2^2(1+a^2)}\, dx_2$

$$u = i\sqrt{a^2+1}\; x_2 \rightarrow \frac{du}{dx_2} = i\sqrt{a^2+1} \rightarrow dx_2 = -\frac{i}{\sqrt{a^2+1}} \cdot du$$

$$\int e^{-x_2^2(1+a^2)}\, dx_2 = -\frac{\sqrt{\pi}\, i}{2\sqrt{a^2+1}} \underbrace{\int \frac{2e^{u^2}}{\sqrt{\pi}} \cdot du}$$

It is gauss error function, which is denoted by erfi

$$= -\frac{\sqrt{\pi}\, i}{2\sqrt{a^2+1}} \operatorname{erfi}(u)$$

$$= -\frac{\sqrt{\pi}\, i}{2\sqrt{a^2+1}} \operatorname{erfi}(i\sqrt{a^2+1}\, x_2) \quad [\because u = i\sqrt{a^2+1}\; x_2]$$

$$\int_{-\alpha}^{\alpha} \frac{-\sqrt{\pi}\, i}{2\sqrt{a^2+1}} \operatorname{erfi}(i\sqrt{a^2+1}\, x_2)$$

$$= \frac{\sqrt{\pi}}{\sqrt{a^2+1}} \text{ ——— (ii)}$$

For. $\int_{-\alpha}^{\alpha} x_2^2 e^{-x_2^2(1+a^2)}\, dx_2 = \int_{-\alpha}^{\alpha} x_2^2 e^{(-a^2-1)x_2^2}\, dx_2$

Integrate by parts: $\int fg' = fg - \int f'g$

$$f = x_2 \rightarrow f' = 1$$

$$g' = x_2 e^{(-a^2-1)x_2^2} \rightarrow g = -\frac{e^{(-a^2-1)x_2^2}}{2a^2+2} \quad *$$

$$\int x_2^2 e^{(-a^2-1)x_2^2} \cdot dx_2 = -\frac{x_2 e^{(-a^2-1)x_2^2}}{2a^2+2} - \int -\frac{e^{(-a^2-1)x_2^2}}{2a^2+2}\, dx_2 \text{ ——— (iii)}$$

For. $\int -\frac{e^{(-a^2-1)x_2^2}}{2a^2+2}\, dx_2$

$$\boxed{u = i\sqrt{a^2+1}\; x_2 \rightarrow \frac{du}{dx_2} = i\sqrt{a^2+1} \rightarrow dx_2 = -\frac{i}{\sqrt{a^2+1}} \cdot du}$$

$$= \frac{\sqrt{\pi}\, i}{2\sqrt{a^2+1}\,(2a^2+2)} \underbrace{\int \frac{2e^{u^2}}{\sqrt{\pi}} \cdot du}$$

It is gauss error function, which is denoted by erfi

$$= \frac{\sqrt{\pi}\, i \operatorname{erfi}(u)}{2\sqrt{a^2+1}\,(2a^2+2)}$$

$$= \frac{\sqrt{\pi}\, i \operatorname{erfi}(i\sqrt{a^2+1}\, x_2)}{2\sqrt{a^2+1}\,(2a^2+2)}$$

(iii) will be

$$\int x_2^2 e^{(-a^2-1)x_2^2} dx_2 = -\frac{x_2 e^{(-a^2-1)x_2^2}}{2a^2+2} - \frac{\sqrt{\pi}\, i\, \mathrm{erfi}(i\sqrt{a^2+1}\, x_2)}{2\sqrt{a^2+1}\,(2a^2+2)}$$

$$= -\frac{2\sqrt{a^2+1}\, x_2 e^{-(a^2+1)x_2^2} + \sqrt{\pi}\, i\, \mathrm{erfi}(i\sqrt{a^2+1}\, x_2)}{4(a^2+1)^{3/2}}$$

$$\int_{-\infty}^{\infty} x_2^2 e^{(-a^2-1)x_2^2} . dx_2 = \frac{\sqrt{\pi}}{2(1+a^2)^{3/2}} \quad \text{—— (iv)}$$

Put the value of (ii) and (iv) in (i)

$$E[X_1 X_2^2 \mid X_1 = a] = a \cdot \frac{\sqrt{\pi}}{2(1+a^2)^{3/2}} \times \frac{(1+a^2)^{1/2}}{\sqrt{\pi}}$$

$$\Rightarrow \boxed{E[X_1 X_2^2 \mid X_1 = a] = \frac{a}{2(1+a^2)}}$$

(*)

$$\int x_2 e^{(-a^2-1)x_2^2} . dx_2$$

$$u = (-a^2-1)x_2^2$$

$$\frac{du}{dx_2} = 2(-a^2-1)x_2$$

$$\Rightarrow dx_2 = \frac{1}{2(-a^2-1)x_2} du$$

$$= -\frac{1}{2a^2+2} \int e^u du$$

$$= -\frac{1}{2a^2+2} \cdot \frac{e^u}{\ln(e)}$$

$$= -\frac{e^u}{2a^2+2} = \boxed{-\frac{e^{(-a^2-1)x_2^2}}{2a^2+2}}$$