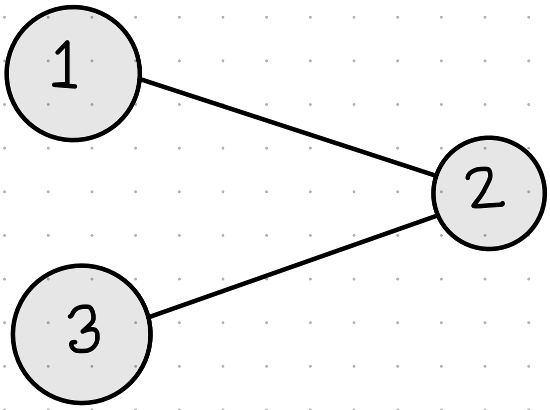
Exercise 1:



- Laplacian matrix: $L(G) = \underbrace{D(G)}_{\text{Degree Matrix}} - \underbrace{A(G)}_{\text{Adjacency matrix}}$

$$D(G) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L(G) = D(G) - A(G)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

From the definition of the eigenvector v corresponding to the eigenvalue λ we have

$$Av = \lambda v$$

Then, $Av - \lambda v = (A - \lambda I) v = 0$

Equation has a non zero solution if and only if:

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$$\begin{array}{ccc:cc} 1-\lambda & -1 & 0 & 1-\lambda & -1 \\ -1 & 2-\lambda & -1 & -1 & 2-\lambda \\ 0 & -1 & 1-\lambda & 0 & -1 \end{array}$$

$$= (1-\lambda)(2-\lambda)(1-\lambda) + (-1)(-1)\cdot 0 + 0\cdot(-1)(-1)$$
$$- 0\cdot(2-\lambda)\cdot 0 - (-1)(-1)(1-\lambda)$$
$$- (1-\lambda)(-1)(-1)$$

$$= -\lambda^3 + 4\lambda^2 - 3\lambda$$
$$= -\lambda(\lambda^2 - 4\lambda + 3)$$
$$= -\lambda(\lambda-1)(\lambda-3) \overset{!}{=} 0$$

$\therefore \lambda_1 = 0, \quad \lambda_2 = 1 \quad \& \quad \lambda_3 = 3$

For $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Solve it by gaussian elimination

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - (-1) R_1 \to R_2} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1) \cdot R_2 \to R_3} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-1) R_2 \to R_1} \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$

$$x = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix}, \quad \text{Let } x_3 = 1, \; v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1$

$$A - \lambda_2 I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \underset{\sim}{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 \times (-1) \to R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 \times (-1) \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1) R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-1) \cdot R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + x_3 = 0$$
$$x_2 = 0$$

$$X = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} ; \text{ Let } x_3 = 1, \; V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For, $\lambda_3 = 3$

$$A - \lambda_3 I = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right) \underset{\sim}{R_1 / (-2) \to R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - (-1)\cdot R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & -1/2 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 / (-1/2) \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1) R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

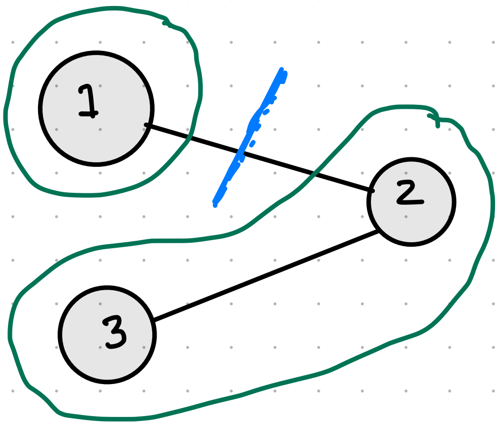
$$\underset{\sim}{R_1 - \left(\tfrac{1}{2}\right) R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$

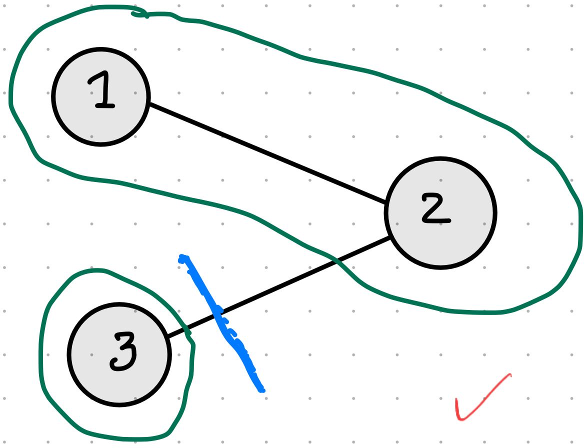
$$\therefore \; X = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} \; ; \text{ let } x_3 = 1, \quad v_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{Ratio Cut} = \sum_{i=1}^{k} \frac{cut(A_i, \bar{A}_i)}{|A_i|}$$

Case 1:

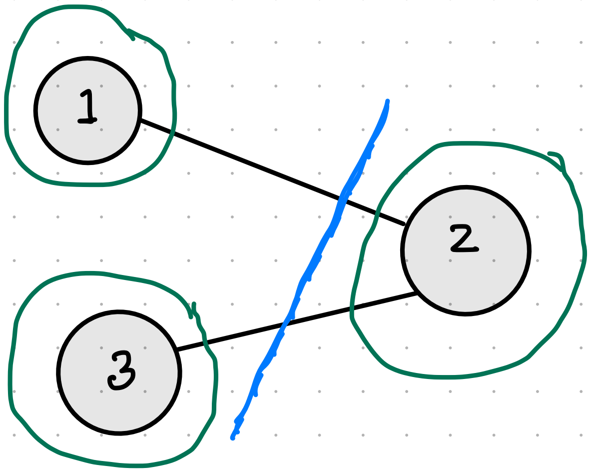


or



$$\text{Ratio cut} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

Case 2:



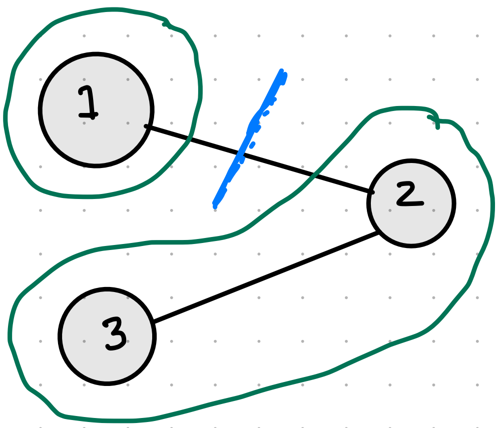
$$\text{Ratio cut} = \frac{1}{1} + \frac{2}{1} + \frac{1}{1}$$

$$= 4$$

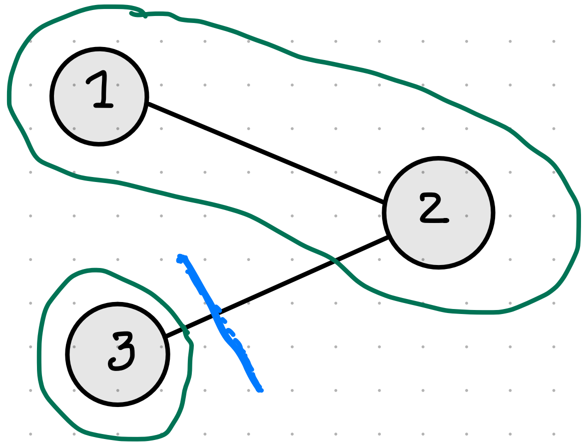
So, case 1 Ratio cut is minimal.

$$Ncut = \sum_{i=1}^{k} \frac{cut(A_i, \bar{A}_i)}{Vol(A_i)}$$

Case 1:

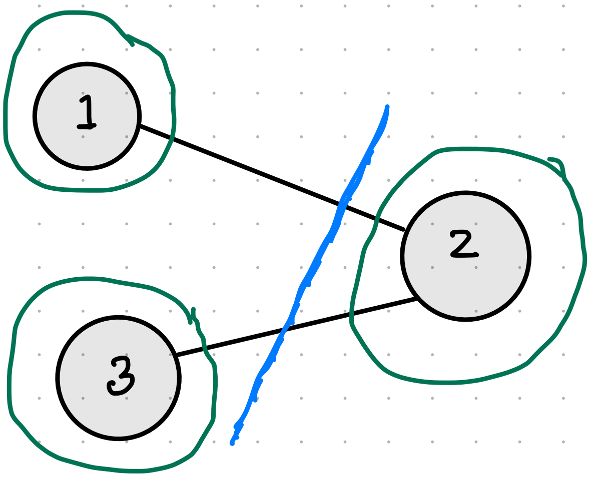


or



$$Ncut = \frac{1}{1} + \frac{1}{3} = \frac{4}{3}$$

Case 2:



$$Ncut = \frac{1}{1} + \frac{2}{2} + \frac{1}{1}$$

$$= 3$$

So, case 1 Ncut is minimul.

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