Exercise 1:

(a)

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

From the definition of the eigenvector v corresponding to the eigen value λ we have,

$$Av = \lambda v$$
$$\Rightarrow (A - \lambda I) v = 0$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) - (-1) \cdot 1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$$\therefore \lambda_1 = 1 - i \ \& \ \lambda_2 = 1 + i$$

For $\lambda_1 = 1 - i$

$$A - \lambda_1 I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

Now,

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \underset{\sim}{R_1/(i) \to R_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 1 & i & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - 1 \cdot R_1 \to R_2} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + i \cdot x_2 = 0$

$\therefore x_1 = -ix_2$ & $x_2 = x_2$

Let $x_2 = 1$, $v_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

For, $\lambda_2 = 1 + i$

$$A - \lambda_2 I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

Now,

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \underset{\sim}{R_1/(-i) \to R_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - 1.R_1 \to R_2} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 - i \cdot x_2 = 0$

$\therefore x_1 = i \cdot x_2$ & $x_2 = x_2$

Let $x_2 = 1$, $v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

A matrix "A" can be diagonized if there exists an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$

$$D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

$$P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \text{ and } P^{-1} = \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}$$

Now, $PDP^{-1} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}$

$$= \begin{pmatrix} -1+i & -1-i \\ 1+i & 1-i \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = A$$

So, A is diagonized.

(b)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -4-\lambda & -2 \\ 3 & -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 - 2\lambda^2 + 24\lambda = 0$$

$$\therefore \lambda_1 = 0, \quad \lambda_2 = -6 \quad \& \quad \lambda_3 = 4$$

For, $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

Now, $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -4 & -2 & 0 \\ 3 & -2 & 1 & 0 \end{array}\right) \underset{\sim}{R_2 - 2.R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -8 & -8 & 0 \\ 3 & -2 & 1 & 0 \end{array}\right)$

$\underset{\sim}{R_3 - 3.R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -8 & -8 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right)$

$\underset{\sim}{R_2/(-8) \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right)$

$\underset{\sim}{R_3 - (-8).R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\underset{\sim}{R_1 - 2.R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\therefore x_1 + x_3 = 0$ & $x_2 + x_3 = 0$

$\Rightarrow x_1 = -x_3$ $\Rightarrow x_2 = -x_3$

Let $x_3 = 1$, $v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

For $\lambda_2 = -6$

$A - \lambda_2 I = \begin{pmatrix} 7 & 2 & 3 \\ 2 & 2 & -2 \\ 3 & -2 & 7 \end{pmatrix}$

Now, $\left(\begin{array}{ccc|c} 7 & 2 & 3 & 0 \\ 2 & 2 & -2 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right) \underset{\sim}{R_1/(7) \to R_1} \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 2 & 2 & -2 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right)$

$$R_2 - 2 \cdot R_1 \to R_2 \sim \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 10/7 & -\frac{20}{7} & 0 \\ 3 & -2 & 7 & 0 \end{array}\right)$$

$$R_3 - 3 \cdot R_1 \to R_3 \sim \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 10/7 & -20/7 & 0 \\ 0 & -20/7 & 40/7 & 0 \end{array}\right)$$

$$R_2 / (10/7) \to R_2 \sim \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -20/7 & 40/7 & 0 \end{array}\right)$$

$$R_3 - \left(\frac{-20}{7}\right) \cdot R_2 \to R_3 \sim \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1 - \frac{2}{7} \cdot R_2 \to R_1 \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + x_3 = 0$ & $x_2 - 2x_3 = 0$

$\Rightarrow x_1 = -x_3$ $\quad \Rightarrow x_2 = 2x_3$

Let, $x_3 = 1$, $v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

For, $\lambda_3 = 4$

$$A - \lambda_3 I = \begin{pmatrix} -3 & 2 & 3 \\ 2 & -8 & -2 \\ 3 & -2 & -3 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 2 & -8 & -2 & 0 \\ 3 & -2 & -3 & 0 \end{array}\right) \underset{\sim}{R_1/(-3) \to R_1} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 2 & -8 & -2 & 0 \\ 3 & -2 & -3 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - 2 \cdot R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 0 & -\frac{20}{3} & 0 & 0 \\ 3 & -2 & -3 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 3 \cdot R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 0 & -20/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 / (-20/3) \to R_2} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-2/3) \cdot R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - x_3 = 0$ & $x_2 = 0$

$\Rightarrow x_1 = x_3$

Let $x_3 = 1$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ -1/6 & 1/3 & 1/6 \end{pmatrix}$$

$$PDP^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ -1/6 & 1/3 & 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

so, A is diagonized.

(c)

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\therefore \quad \lambda_1 = 1, \quad \lambda_2 = 2 \quad \& \quad \lambda_3 = 3$$

For, $\lambda_1 = 1$

$$A - \lambda_1 I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \underset{\sim}{R_2 - 1.R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 1.R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right)$$

$$\overset{R_3 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\overset{R_2/(-2) \to R_2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\overset{R_1 - 1 \cdot R_2 \to R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + x_3 = 0$ & $x_2 = 0$

$\Rightarrow x_1 = -x_3$

Let. $x_3 = 1$, $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For, $\lambda_2 = 2$

$$A - \lambda_2 I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \overset{R_2 \leftrightarrow R_1}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right)$$

$$\overset{R_3 - 1 \cdot R_1 \to R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right)$$

$$\overset{R_3 - (-1) \cdot R_2 \to R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + x_3 = 0$ & $x_2 + x_3 = 0$

$\Rightarrow x_1 = -x_3$ $\quad\Rightarrow x_2 = -x_3$

Let. $x_3 = 1$, $v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

For, $\lambda_3 = 3$

$$A - \lambda_3 I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) \underset{\sim}{R_1/(-1) \to R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - 1 \cdot R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 1 \cdot R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2/(2) \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-1) R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - x_2 = 0$ & $x_3 = 0$

$\Rightarrow x_1 = x_2$

Let $x_2 = 1$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad P = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$PDP^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} = A$$

So, A is diagonized.

Extra:

* If a matrix has distinct eigen values, then it is diagonalisable matrix.

* If the algebraic and geometric multiplicity of each eigen values are equal, then it is diagonalisable matrix.

* Geometric multiplicity $\leqslant$ Algebraic multiplicity.

* If A.M. = 1 then G.M. = 1.

Example:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\lambda = 2, 2, 3$$

A.M. := number of times λ appears as a root of eigen value.

A.M. = 2 $(\lambda = 2)$

$$(A - \overset{\lambda=2}{2}I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\overset{R_2 \Leftrightarrow R_1}{\sim} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore x_2 = 0$ & $x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Dimension $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1$

$x_1 = x_1$

$x_2 = 0$

$x_3 = 0$

$\therefore$ G.M $(2) = 1$

$\therefore$ A.M.$(2) \neq$ G.M.(2), Not diagonalizable

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \lambda = 2, 3, 3$$

A.M. $= 2$ (For $\lambda = 3$)

$$(A - \overset{\lambda=3}{3} I) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = x_2$

$x_2 = x_2$

$x_3 = x_3$

$$-x_1 + x_2 = 0$$

$$\Rightarrow x_1 = x_2$$

Null space $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

dimension of null space $= 2$

G.M. $(3) = 2$

$\therefore$ A.M.$(3) =$ G.M.(3)