Example solution

(i) The likelihood function of σ^3:

$$L(\sigma^3) = \prod_{i=1}^{n} \left[\frac{X_i}{\sigma^3} \exp\left(-\frac{X_i^2}{2\sigma^3} \right) \right]$$

$$= \frac{1}{\sigma^{3n}} \left(\prod_{i=1}^{n} X_i \right) \exp\left(-\frac{1}{2\sigma^3} \sum_{i=1}^{n} X_i^2 \right)$$

(ii) The log likehood function of σ^3:

$$\ln L(\sigma^3) = -n \ln \sigma^3 + \sum_{i=1}^{n} \ln X_i - \frac{1}{2\sigma^3} \sum_{i=1}^{n} X_i^2$$

The derivative with respect to σ:

$$\frac{d \ln L(\sigma^3)}{d\, \sigma^3} = -\frac{n}{\sigma^3} + \frac{1}{2(\sigma^3)^2} \sum_{i=1}^{n} X_i^2 \overset{!}{=} 0$$

$$\Rightarrow \frac{n}{\sigma^3} = \frac{1}{2(\sigma^3)^2} \sum_{i=1}^{n} X_i^2$$

$$\Rightarrow \hat{\sigma}^3 = \frac{1}{2n} \sum_{i=1}^{n} X_i^2$$

Check if $\hat{\sigma}^3$ is maximum:

$$\frac{d^2 \ln L(\sigma^3)}{d(\sigma^3)^2} = \frac{n}{(\sigma^3)^2} - \frac{1}{(\sigma^3)^3} \sum_{i=1}^{n} X_i^2$$

$$= \frac{1}{(\sigma^3)^2} \left(n - \frac{1}{\sigma^3} \sum_{i=1}^{n} X_i^2 \right) \quad \text{(by replacing } \sigma^3 \text{ by } \hat{\sigma}^3\text{)}$$

$$= \frac{1}{(\sigma^3)^2} \left(n - \frac{2n}{\sum X_i^2} \sum X_i^2 \right)$$

$$= \underbrace{\underbrace{\frac{1}{(\sigma^3)^2}}_{>0} \underbrace{(-n)}_{<0}}_{<0} \Rightarrow \hat{\sigma}^3 \text{ is indeed maximum}$$

(iii)

$$\hat{\sigma} = \left(\frac{1}{2n} \sum_{i=1}^{n} X_i^2 \right)^{1/3}$$

(iv) $E[\hat{\sigma}^3] = E\left[\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right]$

$$= \frac{1}{2n}\sum_{i=1}^{n} E[X_i^2]$$

$$= \frac{1}{2n}\sum_{i=1}^{n}\int_0^\infty x^2 \frac{x}{\sigma^3}\exp\left(-\frac{x^2}{2\sigma^3}\right) dx$$

$$= \frac{1}{2n}\sum_{i=1}^{n}\int_0^\infty 2\sigma^3 y \exp(-y)\, dy$$

$$= \frac{2\sigma^3}{2n}\sum_{i=1}^{n}\underbrace{\int_0^\infty y^{2-1}\exp(-y)\, dy}_{\Gamma(2)=1}$$

using change of variable

$y = \frac{x^2}{2\sigma^3}$

$dy = \frac{x}{\sigma^3}dx$

$x = \sqrt{2\sigma^3 y}$

$x = [0,\infty]$

$y = [0,\infty]$

$\Rightarrow E[X_i^2] = 2\sigma^3$

$$dx = \frac{\sigma^3}{n}\sum_{i=1}^{n} 1$$

$$= \frac{\sigma^3}{n} n$$

$$= \sigma^3$$

$\Rightarrow$ $\hat{\sigma}^3$ is unbiased for σ^3.

(v) $MSE(\hat{\sigma}^3) = Var(\hat{\sigma}^3) + \underbrace{Bias(\hat{\sigma}^3)}_{=0}$

$$= var\left(\frac{1}{2n}\sum_{i=1}^{1} X_i^2\right)$$

$$= \frac{1}{4n^2} var(X_1^2 + \dots + X_n^2)$$ (since X_i independent)

$$= \frac{1}{4n^2} n\, var(X_i^2)$$

$$= \frac{1}{4n}\left(E[X_i^4] - (E[X_i^2])^2\right)$$

$$E[X_i^4] = \int_0^\infty x^4 \frac{x}{\sigma^3}\exp\left(\frac{-x^2}{2\sigma^3}\right) dx$$

$$= \int_0^\infty (2\sigma^3 y)^2 \exp(-y)\, dy$$

$$= 4(\sigma^3)^2 \underbrace{\int_0^\infty y^{3-1}\exp(-y)\, dy}_{\Gamma(3)=2\Gamma(2)=2}$$

$$= 8(\sigma^3)^2$$

$$= \frac{1}{4n}\left(8(\sigma^3)^2 - 4(\sigma^3)^2\right)$$

$$= \frac{(\sigma^3)^2}{n} \xrightarrow{n\to\infty} 0$$ $\Rightarrow$ $\hat{\sigma}^3$ is consistent for σ^3.