In-class Exercise for Sheet 07

Show that $\hat{\theta}_{ML}$ of θ for $U_{[0,\theta]}$ is asymptotically unbiased & consistent for θ.

Solution

In the tutorial for sheet 05, we showed that the density of $X_{(n)}$ is $f_{X_{(n)}}(t) = \frac{n t^{n-1}}{\theta^n}$ for $0<t<\theta$ and $E[\hat{\theta}_{ML}] = \frac{n}{n+1}\theta$.

Since $\lim_{n\to\infty} E[\hat{\theta}_{ML}] = \lim_{n\to\infty} \frac{n}{n+1}\theta = \theta$,

$\hat{\theta}_{ML}$ is asymptotically unbiased.

Next we examine $\lim_{n\to\infty} Var(\hat{\theta}_{ML})$.

Since

$$
\begin{aligned}
E[\hat{\theta}_{ML}^2] &= \int_0^\theta t^2 \frac{n t^{n-1}}{\theta^n}\, dt \\
&= \frac{n}{\theta^n} \int_0^\theta t^{n+1}\, dt \\
&= \frac{n}{\theta^n} \frac{1}{n+2} \theta^{n+2} \\
&= \frac{n}{n+2}\theta^2
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} Var(\hat{\theta}_{ML}) &= \lim_{n\to\infty} \left(E[\hat{\theta}_{ML}^2] - E[\hat{\theta}_{ML}]^2 \right) \\
&= \lim_{n\to\infty} \left(\frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\theta\right)^2 \right) \\
&= \lim_{n\to\infty} \frac{n}{(n+2)(n+1)^2}\theta^2 \\
&= 0,
\end{aligned}
$$

it implies that $\hat{\theta}_{ML}$ is consistent.