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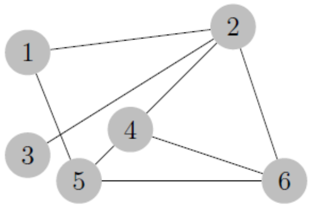
10. Problem sheet for ***Statistical Data Analysis***

Exercise 2

(i)



L(G): Laplacian Matrix

D(G): Degree Matrix

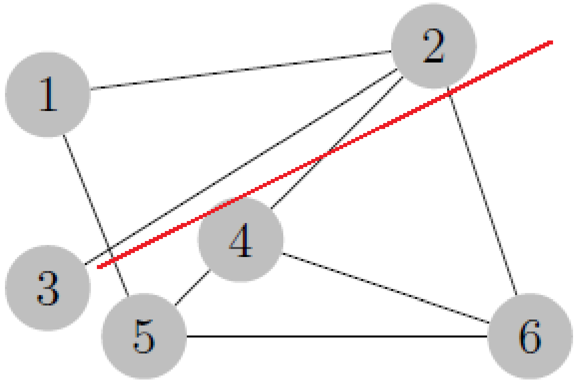
W(G): Affinity Matrix which is A(G): Adjacency Matrix in this exercise since $\omega_{ij} = 1$ *for all ij.*

$$D = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix}$$

(ii)



$A = \{1,2,3\}$

$\bar{A} = \{4,5,6\}$

$$f_i = \begin{cases} \sqrt{|\bar{A}|/|A|}, & if\ v_i \in A, \\ -\sqrt{|A|/|\bar{A}|}, & if\ v_i \in \bar{A}. \end{cases}$$

$f_1 = \sqrt{3/3} = 1, f_2 = \sqrt{3/3} = 1, f_3 = \sqrt{3/3} = 1,$

$f_4 = -\sqrt{3/3} = -1, f_5 = -\sqrt{3/3} = -1, f_3 = -\sqrt{3/3} = -1.$

$f = (f_1, f_2, f_3, f_4, f_5, f_6) = (1,1,1,-1,-1,-1)$

- $f^T Lf = |V| \cdot RatioCut(A, \bar{A})$

$$f^T Lf = (1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1) \cdot \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$= (2 \quad 4 \quad 0 \quad -2 \quad -2 \quad -2) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = 2 + 4 + 2 + 2 + 2 = 12$$

$$RatioCut(A_1, A_2) = \sum_{i=1}^{2} \frac{cut(A_i, \overline{A_i})}{|A_i|} = \frac{cut(A_1, \overline{A_1})}{|A_1|} + \frac{cut(A_2, \overline{A_2})}{|A_2|} = \frac{3}{3} + \frac{3}{3} = 2$$

$|V| = 6$

$\Longrightarrow f^T Lf = 12 = |V| \cdot RatioCut(A, \bar{A})$

- *Let e denotes all-one-vector and we know n=6.*

$$f \cdot e = (1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = 1 + 1 + 1 - 1 - 1 - 1 = 0$$

$\Rightarrow f$ *is orthogonal to the* e.

$$\|f\|^2 = f \cdot f^T = (1 \quad 1 \quad 1 \quad -1 \quad -1 \quad -1) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = 1 + 1 + 1 + 1 + 1 + 1 = 6$$

$\Rightarrow \|f\|^2 = n$