Exercise 2: Determine whether Sn is an unbiased estimator of σ. In case it is not an unbiased estimator, which one is larger E[Sn] or σ?

Solution:

Consider the sample variance:
$\overline{S_n}^2$ = 1/n * $\sum_{i=1}^{n}(Xi - Mn)^{\wedge}$ 2 [here, Mn = Sample Mean]

Now,

E [$\overline{S_n}^2$] = 1/n * E [$\sum_{i=1}^{n} Xi$ ^ 2 – 2 * Mn * $\sum_{i=1}^{n} Xi$ ^ 2 + n * Mn ^ 2]

= E [(1/n) * $\sum_{i=1}^{n} Xi \wedge 2 - 2 * Mn \wedge 2 + Mn \wedge 2$]

= E [(1/n) * $\sum_{i=1}^{n} Xi \wedge 2 - Mn \wedge 2$]

= μ ^ 2 + σ ^ 2 – (μ ^ 2 + $\frac{\sigma \wedge 2}{n}$)

[E(X) ^ 2 = μ ^ 2 + σ ^ 2 and E [Mn ^ 2] = μ ^ 2 + $\frac{\sigma \wedge 2}{n}$]

= $\frac{n-1}{n} * \sigma \wedge 2$

Conclude that $\overline{S_n}^2$ is a biased estimator of the variance. Nevertheless, note that if *n* is relatively large, the bias is very small [1]. Since *E* [$\overline{S_n}^2$] = $\frac{n-1}{n} * \sigma \wedge 2$, so obtain an unbiased estimator of *σ ^ 2* by multiplying $\overline{S_n}^2$ by $\frac{n-1}{n}$. Thus, define

$$S_n{}^2 = \frac{1}{1-n} * \sum_{i=1}^{n}(Xi - Mn)^{\wedge}\ 2$$

$$= \frac{1}{1-n} * (\sum_{i=1}^{n}(X_i^2 - n * M_n^2)$$

By the above discussion, $S_n{}^2$ is an unbiased estimator of the variance. Note that if n is large, the difference between $S_n{}^2$ and $\overline{S_n}^2$ is very small. Also define the sample standard deviation as

$$S_n = \sqrt{S_n 2}$$

Although the sample standard deviation is usually used as an estimator for the standard deviation, it is a biased estimator. To see this, note that *S* is random, so *Var(S)* > 0. Thus,

0 < *Var(S)* = E [$\overline{S_n}^2$] − (E [S_n])^2 = σ^2 – (E [S_n])^2.

Therefore, E [S_n] < σ, which means that *S* is a biased estimator of *σ*.

Reference:

1. https://www.probabilitycourse.com/chapter8/8_2_2_point_estimators_for_mean_and_var.php