

Possion Distribution, $f_{\theta}(x) = \frac{\theta^x}{x!} e^{-\theta}$

The likelihood function,

$$L(\theta; x_1, \dots, x_n) = \prod_{i=1}^n \frac{\theta^{x_i}}{x_i!} e^{-\theta}$$

Taking the log likelihood function

$$\begin{aligned} L(\theta; x_1, \dots, x_n) &= \log \left(\prod_{i=1}^n \frac{\theta^{x_i}}{x_i!} e^{-\theta} \right) \\ &= \sum_{i=1}^n \log \left(\frac{\theta^{x_i}}{x_i!} e^{-\theta} \right) \\ &= \sum_{i=1}^n \left[\log \theta^{x_i} + \log (e^{-\theta}) - \log (x_i!) \right] \\ &= -\theta n + \log \theta \sum_{i=1}^n x_i - \sum_{i=1}^n \log x_i! \end{aligned}$$

Derivative w.r.t. θ .

$$\begin{aligned} \frac{\partial}{\partial \theta} [L(\theta; x_1, \dots, x_n)] &= \frac{\partial}{\partial \theta} \left[-\theta n + \log \theta \sum_{i=1}^n x_i - \sum_{i=1}^n \log x_i! \right] \\ &= -n + \frac{1}{\theta} \sum_{i=1}^n x_i - 0 \end{aligned}$$

Set the derivative equal to 0 and solve for θ .

$$-n + \frac{1}{\theta} \sum_{i=1}^n x_i = 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{1}{n} \sum_{i=1}^n x_i$$

why can we get ML estimator
from $\log L$? (0.5)

Conform $\hat{\theta}_{MLE}$ is maximum
by using second derivative (1)

$$\begin{aligned}
 \mathbb{E}[x] &= \sum_{x=0}^{\infty} x p(x) \\
 &= \sum_{x=0}^{\infty} x \cdot e^{-\theta} \cdot \frac{\theta^x}{x!} \\
 &= e^{-\theta} \sum_{x=0}^{\infty} \frac{x \theta^x}{x!} \\
 &= \theta \cdot e^{-\theta} \sum_{x=1}^{\infty} \frac{\theta^{x-1}}{(x-1)!} \\
 &= \theta \cdot e^{-\theta} \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \quad [x-1=j] \\
 &= \theta \cdot e^{-\theta} \cdot e^{\theta} \quad \left[\sum_{k=0}^{\infty} \frac{\theta^k}{k!} = e^{\theta} \right] \\
 &= \theta.
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \mathbb{E}[\hat{\theta}_{MLE}] &= \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^n x_i\right] \\
 &= \frac{1}{n} \mathbb{E}\left[\sum_{i=1}^n x_i\right] \\
 &= \frac{1}{n} \cdot n \mathbb{E}[x_i] \\
 &= \frac{1}{n} \cdot n \theta \quad [\mathbb{E}[x] = \theta] \\
 &= \theta. \quad \checkmark
 \end{aligned}$$

So, estimate is unbiased.

Therefore, $\hat{\theta} \xrightarrow{P} \theta$

i.e. $\hat{\theta}_n$ is consistent.

Q1

A estimator is consistent if $\lim_{n \rightarrow \infty} \hat{\theta} = \theta$

we know, $E[(x-\mu)^2] = \text{var}(x)$

$$\text{So, } E[(\hat{\theta} - \theta)^2] = \text{var}(\hat{\theta})$$

$$= \text{var}\left(\frac{1}{n} \sum_{i=1}^n x_i\right)$$

$$= \frac{1}{n^2} \text{var}\left(\sum_{i=1}^n x_i\right)$$

$$= \frac{1}{n^2} \sum_{i=1}^n [\text{var}(x_i)]$$

$$= \frac{1}{n^2} \sum_{i=1}^n [E(x_i)^2 - (E(x_i))^2]$$

$$= \frac{1}{n^2} \sum_{i=1}^n [\theta^2 + \theta - \theta^2]$$

$$= \frac{1}{n^2} \sum_{i=1}^n \theta$$

$$= \frac{1}{n^2} \cdot n \theta$$

$$= \theta/n$$

Now, if $n \rightarrow \infty$ then,

$$E[(\hat{\theta} - \theta)^2] \rightarrow 0$$

Therefore, $\hat{\theta} \xrightarrow{P} \theta$

i.e. $\hat{\theta}$ is consistent.

2.5/4

