Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$ independent, where

- σ^2 is known and μ € R is unknown
- μ € R known and σ^2 > 0 unknown

Estimate the respective unknown parameters via the Maximum Likelihood Method.

Solution:

Let uppercase $X_1, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$ random variables, and let lower case x_i be the value X_i takes [1]. The density for each X_i is:

$$f_{X_i}(xi) = \frac{1}{\sqrt{2\pi} * \sigma} * e^{-\frac{(xi-\mu)^2}{2*\sigma^2}}$$

Because the X_i are independent, their joint pdf equals the sum of the separate pdf's

$$f(x_1, \ldots\ldots, x_n | \mu, \sigma) = \left(\frac{1}{\sqrt{2\pi} * \sigma}\right)^n * e^{-\sum_{i=1}^{n} \frac{(xi-\mu)^2}{2*\sigma^2}}$$

The log likelihood for the given $x_1, \ldots., x_n$ is:

Log $(f(x_1, \ldots\ldots, x_n | \mu, \sigma)) = -\frac{n}{2} * log(2 * \pi * \sigma^2) - \frac{1}{2*\sigma^2} * \sum_{i=1}^{n}(xi - \mu)^2$

Since Log $(f(x_1, \ldots\ldots, x_n | \mu, \sigma))$ is a function of the 2 variables μ *and* σ use partial derivatives with respect to μ and σ^2 [2]:

$$\frac{\partial f(x1, \ldots xn|\mu, \sigma^2)}{\partial \mu} = \sum_{i=1}^{n} \frac{(xi-\mu)}{\sigma^2} = 0$$

$$\Rightarrow \sum_{i=0}^{n} xi = n * \mu$$

$$\Rightarrow \hat{\mu} = \frac{\sum_{i=1}^{n} xi}{n} = \bar{x} \text{ [the mean of the data] ....…. (1)}$$

And

$$\frac{\partial f(x1, \ldots xn|\mu, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2} * \frac{1}{\sigma^2} + \frac{\sum_{i=1}^{n}(xi - \mu)^2}{2 * \sigma^4} = 0$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(xi-\mu)^2}{n}$$

$$\Rightarrow \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(xi-\bar{x})^2}{n} \text{ [From (1) we get, } \hat{\mu} = \bar{x}\text{]}$$

$$\Rightarrow \hat{\sigma}^2 = S^2 \text{ [the variance of the data]}$$

Reference:

[1]. https://math.mit.edu/~dav/05.dir/class10-prep.pdf
[2]. https://bookdown.org/egarpor/inference/est-methods.html#est-methods-ml