Reject H_0 if $\lambda(x) \le c$; where c is chosen such that $\boxed{P(\lambda < c) = \alpha}$ or $\boxed{P(\lambda < c \mid H_0) = \alpha}$

Probability of type I error.

Example: Suppose $X_1, X_2, \dots X_n$ are iid $N(\mu, \sigma_0^2)$ where $-\infty < \mu < \infty$ and $\sigma_0^2 = 1$. Consider testing $H_0: \mu = \mu_0$ vs $H_1: \mu \ne \mu_0$

The likelihood function is

$$L(\mu \mid x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-(x_i - \mu)^2/2} = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu)^2}$$

$$\sup_{\mu \in \Theta_0} L(\mu \mid x) = L(\mu_0 \mid x)$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu_0)^2}$$

MLE of $\hat{\mu} = \bar{x}$.

$$\sup_{\mu \in \Theta} L(\mu \mid x) = L(\bar{x} \mid x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$\lambda(x) = \frac{L(\mu_0 \mid x)}{L(\mu \mid x)} = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \mu_0)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

$$= e^{-\frac{1}{2}\left[\underbrace{\sum_{i=1}^{n}(x_i - \mu_0)^2}_{\sum_{i=1}^{n}(x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2} - \sum_{i=1}^{n}(x_i - \bar{x})^2\right]}$$

$$= e^{-\frac{n}{2}(\bar{x} - \mu_0)^2}$$

LRT rejects H_0 when $\lambda(x)$ is too small.

Suppose $X_1, X_2, \dots X_n$ are iid exponential (θ), where $\theta > 0$,

Consider testing $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

$$L(\lambda, x) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_{i=1}^{n} x_i}$$

$$L(\lambda_0 | x) = \theta_0^n e^{-\theta_0 \sum_{i=1}^{n} x_i}$$

$$\text{MLE of } \hat{\theta} = \frac{1}{\bar{x}}$$

$$L\left(\frac{1}{\bar{x}}, x\right) = \left(\frac{1}{\bar{x}}\right)^n e^{-\left(\frac{1}{\bar{x}}\right) \sum_{i=1}^{n} x_i}$$

$$\frac{L(\lambda_0 | x)}{L\left(\frac{1}{\bar{x}}, x\right)} = \frac{\theta_0^n e^{-\theta_0 \sum_{i=1}^{n} x_i}}{\left(\frac{1}{\bar{x}}\right)^n e^{-\left(\frac{1}{\bar{x}}\right) \sum_{i=1}^{n} x_i}}$$

$$= \frac{\theta_0^n e^{-\theta_0 \sum_{i=1}^{n} x_i}}{\left(\frac{1}{\bar{x}}\right)^n e^{-n}}$$

$$= (\theta_0 e)^n (\bar{x})^n e^{-\theta_0 \sum x_i}$$

Suppose $X_1, X_2, \ldots X_n$ are iid $N(\mu, \sigma^2)$ where $-\infty < \mu < \infty$ and $\sigma^2 > 0$
i.e. both parameter are unknown. Set $\theta = (\mu, \sigma^2)$ Consider testing $H_0 = \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$

$$L(\theta|x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\mu)^2/2\sigma^2}$$

$$= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

MLE of N.D.

$$\hat{\mu} = \bar{x}$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

$$L(\hat{\theta}|x) = \begin{pmatrix} \bar{x} \\ \frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2 \end{pmatrix}$$

$$L(\hat{\theta}_0|x) = \begin{pmatrix} \mu_0 \\ \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2 \end{pmatrix}$$

$$\frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)} = \frac{\left(\frac{1}{\sqrt{2\pi\sigma_0^2}}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n}(x_i-\mu_0)^2}}{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$= \left(\frac{\sigma}{\sigma_0}\right)^n e^{-\frac{1}{2\sigma_0^2}\sum(x_i-\mu_0)^2 + \frac{1}{2\sigma^2}(x_i-\bar{x})^2}$$

$$= \left(\frac{\sigma}{\sigma_0}\right)^n e^{-\frac{n}{2}+\frac{n}{2}}$$

$$= \left(\frac{\sigma^2}{\sigma_0}\right)^{n/2} e^0$$

$$= \left[\frac{\Sigma (x_i - \bar{x})^2}{\Sigma (x_i - \mu_0)^2}\right]^{n/2}$$

$\blacksquare$

Uniform Distribution$(0, \theta)$

$$f(\theta) = \frac{1}{\theta}$$

$$L(\theta; x) = \prod_{i=1}^{n} \frac{1}{\theta}$$

$$= \left(\frac{1}{\theta}\right)^n$$

$$\lambda(x) = \frac{L(\hat{\theta}_0 | x)}{L(\hat{\theta} | x)}$$

$$= \frac{(1/\theta_0)^n}{(1/\hat{\theta})^n}$$

$$= \frac{(1/\theta_0)^n}{(1/x_{(n)})^n}$$

$$= \left(\frac{x_n}{\theta_0}\right)^n \quad \text{if } x_{(n)} \leq \theta_0$$

Let $X_1, X_2, \ldots X_n \sim$ IID Poisson(θ). $H_0: \theta = \theta_0$ vs $H_1: \theta \neq \theta_0$

$$\lambda(x) = \frac{L(\theta_0|x)}{L(\hat{\theta}|x)} = \frac{\prod_{i=1}^{n} e^{-\theta_0} \frac{\theta_0^{x_i}}{x_i!}}{\prod_{i=1}^{n} e^{-\bar{x}} \frac{\bar{x}^{x_i}}{x_i!}}$$

[Possion Distribution mle $= \bar{x}$]

$$= e^{-n\theta_0 + n\bar{x}} \left(\frac{\theta_0}{\bar{x}}\right)^{\sum_{i=1}^{n} x_i}$$

$$= e^{n(\bar{x} - \theta_0)} \left(\frac{\theta_0}{\bar{x}}\right)^{\sum_{i=1}^{n} x_i}$$

LRT is the best that rejects H_0 for $\{x : \lambda(x) < c\}$ where $0 \leq c \leq 1$.

Let $x_1, \ldots x_n \sim$ iid Poisson(θ). Find a hypothesis test for $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$

$$\hat{\theta}_0 = \begin{cases} \theta_0 , & \theta_0 < \hat{\theta} \\ \hat{\theta} , & \theta_0 \geq \hat{\theta} \end{cases}$$

$$\lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)}$$

$$= \frac{\prod_{i=1}^{n} \left\{ e^{-\hat{\theta}_0} \frac{\hat{\theta}_0^{x_i}}{x_i!} \right\}}{\prod_{i=1}^{n} \left\{ e^{-\bar{x}} \frac{\bar{x}^{x_i}}{x_i!} \right\}}$$

For $\theta_0 < \hat{\theta}$, $\hat{\theta}_0 = \theta_0$

$$\frac{\prod_{i=1}^{n} e^{-\theta_0} \frac{\theta_0^{x_i}}{x_i!}}{\prod_{i=1}^{n} e^{-\bar{x}} \frac{\bar{x}^{x_i}}{x_i!}}$$

$$= e^{n(\bar{x}-\theta_0)} \left(\frac{\theta_0}{\bar{x}}\right)^{\sum_{i=1}^{n} x_i}$$

for $\theta_0 \geq \hat{\theta}$, $\hat{\theta}_0 = \hat{\theta} = \bar{x}$

$$\frac{\prod_{i=1}^{n} e^{-\bar{x}} \frac{\bar{x}^{x_i}}{x_i!}}{\prod_{i=1}^{n} e^{-\bar{x}} \frac{\bar{x}^{x_i}}{x_i!}} = 1$$

$$\therefore \lambda(x) = \begin{cases} \left(\frac{\theta_0}{x}\right)^{\sum_{i=1}^{n} x_i} e^{n(\bar{x}-\theta_0)} & , \theta_0 < \hat{\theta} \\ 1 & , \theta_0 \geq \hat{\theta} \end{cases}$$

LRT is the best that rejects H_0 for $\{x : \lambda(x) < c\}$ where $0 \leq c \leq 1$.

Normal distribution. $N(10, 2)$, $H_0 = \mu = 10$

$$\lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)}$$

$$= \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi 2}} \exp\left\{-\frac{(x_i-10)^2}{2.2}\right\}}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi.2}} \exp\left\{-\frac{(x_i-\bar{x})^2}{2.2}\right\}}$$

$$= \frac{\exp\left\{-\frac{\sum(x_i-10)^2}{4}\right\}}{\exp\left\{-\frac{\sum(x_i-\bar{x})^2}{4}\right\}}$$

$$\sum(x_i-10)^2 = \sum(x_i-\bar{x}+\bar{x}-10)^2$$

$$= \sum(x_i-\bar{x})^2 + 2(\bar{x}-10)\underbrace{\sum(x_i-\bar{x})}_{=0} + \sum(\bar{x}-10)^2$$

$$= \sum(x_i-\bar{x})^2 + n(\bar{x}-10)^2$$

$$\therefore \lambda(x) = \frac{\cancel{\exp\left\{-\frac{\sum(x_i-\bar{x})^2}{4}\right\}}\exp\left\{-\frac{n(\bar{x}-10)^2}{4}\right\}}{\cancel{\exp\left\{-\frac{\sum(x_i-\bar{x})^2}{4}\right\}}}$$

$$= \exp\left\{-\frac{n(\bar{x}-10)^2}{4}\right\}$$

Normal Distribution: $N(\mu, \sigma^2)$ $H_0: \mu = \mu_0$ and $H_A: \mu \neq \mu_0$

$$\hat{\mu}_{mle} = \bar{x}, \quad \hat{\sigma}^2_{mle} = \frac{1}{n}\sum(x_i-\mu)^2$$

$$\lambda(x) = \frac{L(\hat{\mu}_0 | x)}{L(\hat{\mu} | x)}$$

$$L(\hat{\mu}_0 \mid x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x_i - \mu_0)^2}{2\sigma^2} \right\}$$

$$= \left(\frac{1}{\sqrt{2\pi \frac{1}{n} \Sigma (x_i - \mu_0)^2}} \right)^n \exp\left\{ -\frac{\cancel{\Sigma (x_i - \mu_0)^2}}{2 \cdot \frac{1}{n} \cancel{\Sigma (x_i - \mu_0)^2}} \right\}$$

$$= \left(\frac{\sqrt{n}}{\sqrt{2\pi \, \Sigma (x_i - \mu_0)^2}} \right)^n \exp\left\{ -\frac{n}{2} \right\}$$

$$L(\hat{\mu} \mid x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(x_i - \hat{\mu})^2}{2\sigma^2} \right\}$$

$$= \left(\frac{\sqrt{n}}{\sqrt{2\pi \, \Sigma (x_i - \bar{x})^2}} \right)^n \exp\left\{ -\frac{\cancel{\Sigma (x_i - \bar{x})^2}}{2 \cdot \frac{1}{n} \underbrace{\cancel{\Sigma (x_i - \bar{x})^2}}_{\hat{\mu} = \bar{x}}} \right\}$$

$$= \left(\frac{\sqrt{n}}{\sqrt{2\pi \, \Sigma (x_i - \bar{x})^2}} \right)^n \exp\left\{ -\frac{n}{2} \right\}$$

$$\lambda(x) = \frac{L(\hat{\mu}_0 \mid x)}{L(\hat{\mu} \mid x)}$$

$$= \left(\frac{\Sigma (x_i - \bar{x})^2}{\Sigma (x_i - \mu_0)^2} \right)^{n/2}$$

$$\sum (x_i - \mu_0)^2 = \sum (x_i - \bar{x} + \bar{x} - \mu_0)^2$$

$$= \sum (x_i - \bar{x})^2 + 2(\bar{x} - \mu_0) \underbrace{\sum (x_i - \bar{x})}_{=0} + n(\bar{x} - \mu_0)^2$$

$$= \sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2$$

$$\lambda(x) = \left(\frac{\sum (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2} \right)^{n/2}$$

$$= \left(\frac{1}{1 + \frac{n(\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}} \right)^{n/2}$$

$$f(x \mid \theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad x > 0 \ \Big| \ H_0 : \theta = \theta_0 \ \text{vs} \ H_a : \theta = \theta_1.$$

$$\theta$$

$$\lambda(x) = \frac{L(\hat{\theta}_0 \mid x)}{L(\hat{\theta} \mid x)} = \frac{\prod_{i=1}^{n} \frac{1}{\theta_0} e^{-\frac{x_i}{\theta_0}}}{\prod_{i=1}^{n} \frac{1}{\theta_1} e^{-\frac{x_i}{\theta_1}}}$$

$$= \left(\frac{\theta_1}{\theta_0}\right)^n \exp\left\{ \left(\frac{1}{\theta_1} - \frac{1}{\theta_0} \right) \sum x_i \right\}$$

$N(\mu, \sigma^2)$; $H_0: \sigma^2 = \sigma_0^2$ vs $H_a: \sigma^2 \neq \sigma_0^2$

$$\hat{\mu}_{mle} = \bar{x} \quad , \quad \hat{\sigma}^2_{mle} = \frac{1}{n} \sum (x_i - \mu)^2$$

$$\lambda(x) = \frac{L(\hat{\sigma}_0 | x)}{L(\hat{\sigma} | x)}$$

$$L(\hat{\sigma}_0 | x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left\{ -\frac{(x_i - \mu)^2}{2\sigma_0^2} \right\}$$

$$= \left(\frac{1}{\sqrt{2\pi\sigma_0^2}} \right)^n \exp\left\{ -\frac{\sum (x_i - \bar{x})^2}{2\sigma_0^2} \right\}$$

$$L(\hat{\sigma} | x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \exp\left\{ -\frac{(x_i - \mu)^2}{2\hat{\sigma}^2} \right\}$$

$$= \left(\frac{1}{\sqrt{2\pi\hat{\sigma}^2}} \right)^n \exp\left\{ -\frac{\sum (x_i - \bar{x})^2}{\frac{2}{n} \sum (x_i - \bar{x})^2} \right\}$$

$$= \left(\frac{\sqrt{n}}{\sqrt{2\pi \sum (x_i - \bar{x})^2}} \right)^n \exp\left\{ -\frac{n}{2} \right\}$$

$$\lambda(x) = \left(\frac{\sum (x_i - \bar{x})^2}{\sigma_0^2 \cdot n} \right)^{n/2} \exp\left\{ \frac{n}{2} \right\} \exp\left\{ -\frac{\sum (x_i - \bar{x})^2}{2\sigma_0^2} \right\}$$

Bernoulli Distribution

$$\hat{p}_{mle} = \bar{x}$$

$$\lambda(x) = \frac{L(\hat{p}_0 | x)}{L(\hat{p} | x)} = \frac{\prod_{i=1}^{n} p_0^{x_i} (1-p_0)^{1-x_i}}{\prod_{i=1}^{n} \hat{p}^{x_i} (1-\hat{p})^{1-x_i}}$$

$$= \left(\frac{p_0}{\bar{x}}\right)^{\sum x_i} \left(\frac{1-p_0}{1-\bar{x}}\right)^{n-\sum x_i}$$

$\blacksquare$

Normal Distribution. $H_0: \mu \leq \mu_0$ vs $H_1: \mu > \mu_0$

$$\hat{\mu}_{mle} = \bar{x}$$

$$\hat{\sigma}^2_{mle} = \frac{1}{n} \sum (x_i - \mu)^2$$

$$\lambda(x) = \frac{L(\hat{\mu}_0 | x)}{L(\hat{\mu} | x)}$$

$$L(\hat{\mu}_0 | x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu_0)^2}{2\sigma^2}}$$

$$L(\hat{\mu} | x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\bar{x})^2}{2\sigma^2}}$$

$$\hat{\mu}_{mle} = \bar{x}$$

$$\hat{\sigma}^2_{mle} = \frac{1}{n}\left[\sum (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2\right]$$

$$= (\bar{x} - \mu_0)^2$$

$$\Sigma(x_i - \mu_0)^2 = \Sigma(x_i - \bar{x} + \bar{x} - \mu_0)^2$$

$$= \Sigma(x_i - \bar{x})^2 + 2(\bar{x} - \mu_0)\overset{0}{\overset{\shortparallel}{\Sigma(x_i - \bar{x})}}$$
$$+ n(\bar{x} - \mu_0)^2$$

$$= \Sigma(x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2$$

Now, when $\bar{x} \leq \mu_0$, then $\hat{\mu}_0 = \bar{x}$

and $\bar{x} > \mu_0$ then $\hat{\mu}_0 = \mu_0$

$$\lambda(x) = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\Sigma(x_i - \hat{\mu}_0)/2\sigma^2}}{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\Sigma(x_i - \hat{\mu})^2/2\sigma^2}}$$

$$= e^{-\frac{\Sigma(x_i - \bar{x})^2 + n(\bar{x} - \hat{\mu}_0)^2}{2\sigma^2} + \frac{\Sigma(x_i - \hat{\mu})^2}{2\sigma^2}}$$

$$= e^{-\frac{n(\bar{x} - \hat{\mu}_0)^2}{2\sigma^2}}$$

Now, when $\hat{\mu}_0 = \bar{x}$ then $\lambda(x) = e^0 = 1$

and $\hat{\mu}_0 = \mu_0$, then $\lambda(x) = e^{-\frac{n(\bar{x} - \mu_0)^2}{2\sigma^2}}$

$\blacksquare$

Geometric Distribution:

$$\hat{p}_{mle} = \frac{1}{\bar{x}}$$

when, $\bar{x} = 1$ then $\hat{p} = 1$

and when $\bar{x} > 1$ then $\hat{p} = \frac{1}{\bar{x}}$

$$\hat{p}_0 = \begin{cases} 1/\bar{x} , & \text{if } 1/\bar{x} \leq p_0 \\ p_0 , & \text{if } 1/\bar{x} > p_0 \end{cases}$$

$$\lambda(x) = \frac{L(\hat{p}_0 | x)}{L(\hat{p} | x)}$$

$$= \frac{\prod_{i=1}^{n} (1-\hat{p}_0)^{x_i - 1} \hat{p}_0}{\prod_{i=1}^{n} (1-\hat{p})^{x_i - 1} \hat{p}}$$

$$= \frac{(1-\hat{p}_0)^{\Sigma(x_i - 1)} \cdot \hat{p}_0}{\left(1 - \frac{1}{\bar{x}}\right)^{\Sigma(x_i - 1)} \cdot \frac{1}{\bar{x}}}$$

$$\text{so, } \lambda(x) = \begin{cases} \dfrac{(1-\hat{p}_0)^{\Sigma(x_i - 1)} \hat{p}_0}{\left(1 - \frac{1}{\bar{x}}\right)^{\Sigma(x_i - 1)} \cdot \frac{1}{\bar{x}}} , & \frac{1}{\bar{x}} > p_0 \\ 1 , & \frac{1}{\bar{x}} \leq p_0 \end{cases}$$

$N(\mu, \sigma^2)$, where $\mathbb{E}(Y_i) = \beta x_i$ and $V(Y_i) = 1$

$$f(y, \mu ; \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-\mu)^2}{2\sigma^2} \right)$$

1)

$$L(y ; \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \cdot 1}} \exp\left(\frac{-(y_i - \beta x_i)^2}{2 \cdot 1} \right)$$

Take log likelihood.

$$L(y ; \mu, \sigma^2) = -n \log \sqrt{2\pi} + \frac{-\sum (y_i - \beta x_i)^2}{2}$$

Diff w.r.t. to β

$$\frac{\partial L(y ; \mu, \sigma^2)}{\partial \beta} = 0 - \frac{\sum \cancel{2} (y_i - \beta x_i)(-x_i)}{\cancel{2}}$$

$$= -\left[\sum (-y_i x_i + \beta x_i^2) \right] \stackrel{\text{set}}{=} 0$$

$$\Rightarrow \sum x_i y_i - \sum \beta x_i^2 = 0$$

$$\Rightarrow \hat{\beta}_{mle} = \frac{\sum x_i y_i}{\sum x_i^2}$$

2)

$$\mathbb{E}[\hat{\beta}] = \mathbb{E}\left[\frac{\sum x_i y_i}{\sum x_i^2} \right]$$

$$= \frac{1}{\sum x_i^2} \sum x_i \mathbb{E}[y_i]$$

$$= \frac{1}{\Sigma x_i^2} \Sigma x_i^2 \beta \quad [\because \text{mean} = \beta x_i]$$

$$= \beta \frac{1}{\cancel{\Sigma x_i^2}} \cancel{\Sigma x_i^2}$$

$$= \beta$$

So, $\hat{\beta}$ is unbiased.

③ $$\text{var}[\hat{\beta}] = \text{var}\left[\frac{\Sigma x_i y_i}{\Sigma x_i^2}\right]$$

$$= \frac{1}{(\Sigma x_i^2)^2} (\Sigma x_i)^2 \cdot \Sigma \underbrace{\text{var}(y_i)}_{=1}$$

$$= \frac{1}{\Sigma x_i^2}$$

So, distribution of $\hat{\beta} = \left(\beta, \frac{1}{\Sigma x_i^2}\right)$

④ Likelihood ratio test $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$

$$\lambda(\beta) = \frac{L(\hat{\theta}_0 | \hat{\beta}_0)}{L(\hat{\theta} | \hat{\beta})}$$

$$= \frac{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi}}} \exp\left\{\frac{-(y_i - \hat{\beta}_0 x_i)^2}{2}\right\}}{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi}}} \exp\left\{-\frac{(y_i - \hat{\beta} x_i)^2}{2}\right\}}$$

$$= \exp\left\{ \frac{-\sum(y_i-\hat{\beta}_0 x_i)^2+\sum(y_i-\hat{\beta} x_i)^2}{2} \right\}$$

$$= \exp\left\{ \frac{-\cancel{\sum y_i^2}+2\hat{\beta}_0\sum x_i y_i-\hat{\beta}_0\sum x_i^2 + \cancel{\sum y_i^2}-2\hat{\beta}\sum x_i y_i+\hat{\beta}\sum x_i^2}{2} \right\}$$

$$= \exp\left\{ \frac{2\sum x_i y_i(\hat{\beta}_0-\hat{\beta})+\sum x_i^2(\hat{\beta}-\hat{\beta}_0)}{2} \right\}$$

$$= \exp\left\{ \frac{(\hat{\beta}_0-\hat{\beta})(2\sum x_i y_i-\sum x_i^2)}{2} \right\}$$

$f_X(x|\theta) = \frac{\theta}{x^2} I(x \geq \theta)$; where $\theta > 0$

$$f_X(x|\theta) = \prod_{i=1}^{n} \frac{\theta}{x_i^2} I$$

$$= \theta^n \frac{1}{\sum x_i^2}$$

Take Log likelihood

$$L(x;\theta) = n\log\theta - \log\sum x_i$$

here, $L(x;\theta)$ will be maximum

if $n\log\theta$ is maximum.

$n\log\theta$ will be maximum when θ is maximum

So, $\boxed{\hat{\theta}_{mle} = \max(x_1, \dots x_n)}$

$$\lambda(x) = \frac{L(\hat{\theta}_0 | x)}{L(\hat{\theta} | x)}$$

$$= \frac{\hat{\theta}_0^{\,n} \frac{1}{\Sigma x_i^2}}{\hat{\theta}^{\,n} \frac{1}{\Sigma x_i^2}} = \left(\frac{\hat{\theta}_0}{\max(x_1, \dots, x_n)} \right)^n$$

Binomial Distribution:

$$\lambda(x) = \frac{L(\hat{\theta}_0 | x)}{L(\hat{\theta} | x)}$$

$$= \frac{{}^n c_K \, \theta_0^{K} \, (1-\theta_0)^{n-K}}{{}^n c_K \, \hat{\theta}^{K} \, (1-\hat{\theta})^{n-K}}$$

$$= \left(\frac{\theta_0}{\hat{\theta}} \right)^K \left(\frac{1-\theta_0}{1-\hat{\theta}} \right)^{n-K}$$

Case: 1

$H_0: \theta = \theta_0$ vs $H_a: \theta = \theta_a$

$$\lambda = \frac{L(\theta_0)}{L(\hat{\theta})} = \begin{cases} \frac{L(\theta_0)}{L(\theta_0)} = 1 & \text{if } L(\theta_0) > L(\hat{\theta}) \\ \frac{L(\theta_0)}{L(\hat{\theta})} & \text{if } L(\theta_0) < L(\hat{\theta}) \end{cases}$$

$\lambda = 1$, can't reject H_0

Reject H_0 if

$$\lambda \leq c$$

$$\Rightarrow \frac{L(\theta_0)}{L(\hat{\theta})} \leq c$$

Case: 2

$H_0: \theta \leq a$ vs $H_a: \theta > a$

Reject H_0 when $T > c$

we chose c so that

$$P_{\theta = a}(T > c) \leq \alpha$$