1.

$\mathbb{E}[a+bx] = a + b\,\mathbb{E}[x]$.

Suppose, x is discrete random variable with pmf $P(x)$.

Then by definition,

$$\mathbb{E}[a+bx] = \sum_x (a+bx)\,P(x)$$

$$= \sum_x (a \cdot P(x) + b \cdot x \cdot P(x))$$

$$= \sum_x a \cdot P(x) + b \sum_x x\,P(x)$$ using properties of sum

$$= a\underbrace{\left(\sum_x P(x)\right)}_{1} + b\underbrace{\left(\sum_x x \cdot P(x)\right)}_{\mathbb{E}[x]}.$$

$$= a + b \cdot \mathbb{E}[x]$$

2.

$$\mathrm{Var}(x) = \mathbb{E}[(x - \mathbb{E}[x])^2]$$

$$= \mathbb{E}[x^2 - 2x\,\mathbb{E}[x] + \mathbb{E}[x]^2]$$

why is $\mathbb{E}[\mathbb{E}[x]] = \mathbb{E}[x]$?

$$= \mathbb{E}[x^2] + \mathbb{E}[-2x\,\mathbb{E}[x]] + \mathbb{E}[\mathbb{E}[x^2]]$$

-0.5

$$= \mathbb{E}[x^2] - 2\,\mathbb{E}[x]\,\mathbb{E}[x] + \mathbb{E}[x^2]$$ using 1.

$$= \mathbb{E}[x^2] - 2\,\mathbb{E}[x]^2 + \mathbb{E}[x^2]$$

$$= \mathbb{E}[x^2] - \mathbb{E}[x]^2$$

3. $\text{var}(a+bx) = \mathbb{E}((a+bx)^2 - (\mathbb{E}(a+bx))^2$

$= \mathbb{E}[a^2 + 2abx + b^2x^2] - (a+b\,\mathbb{E}(x))$

$(a+b\,\mathbb{E}(x))$ using 1. -0.5

$= a^2 + 2ab\,\mathbb{E}(x) + b^2\,\mathbb{E}[x^2] - a^2 - 2ab\,\mathbb{E}[x]$

using 1 $- b^2\,\mathbb{E}(x)^2$

$= b^2\,\mathbb{E}[x^2] - b^2\,\mathbb{E}[x]^2$

$= b^2(\mathbb{E}[x^2] - \mathbb{E}[x]^2)$

$= b^2\,\text{var}(x)$

4. The expected value of a constant is equal to itself.

$\mathbb{E}(a) = a$ (using 1., $a \in \mathbb{R}$ constant)

$\text{var}(a) = \mathbb{E}[(a - \mathbb{E}(a))^2]$

$= \mathbb{E}[(a-a)^2]$ ✓

$= \mathbb{E}(0)$

$\mathbb{E}(0) = \sum_x x.f(x) = 0.1 = 0.$ ✓

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2.

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2$$

$$\mathbb{E}[S_n^2] = \frac{1}{n-1} \mathbb{E}\left[\sum_{i=1}^{n} (x_i - \bar{x}_n)^2\right]$$

We know,

$$Var(x_i) = \mathbb{E}(x_i^2) - [\mathbb{E}(x_i)]^2$$

$$\Rightarrow \sigma^2 = \mathbb{E}(x_i^2) - \mu^2$$

$$\Rightarrow \mathbb{E}(x_i^2) = \sigma^2 + \mu^2 \checkmark$$

Also,

$$Var(\bar{x}_n) = \mathbb{E}[\bar{x}_n^2] - (\mathbb{E}[\bar{x}_n])^2$$

$$\Rightarrow \frac{\sigma^2}{n} = \mathbb{E}[\bar{x}_n^2] - \mu^2$$

$$\Rightarrow \mathbb{E}[\bar{x}_n^2] = \frac{\sigma^2}{n} + \mu^2 \checkmark$$

$$\mathbb{E}[S_n^2] = \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E}(x_i^2 - 2x_i\bar{x}_n + \bar{x}_n^2)$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} \mathbb{E}[x_i^2] - 2\mathbb{E}\left[\bar{x}_n \sum_{i=1}^{n} x_i\right] + \mathbb{E}\left(\sum_{i=1}^{n} \bar{x}_n^2\right)\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} \mathbb{E}[x_i^2] - 2n\mathbb{E}[\bar{x}_n^2] + n\mathbb{E}[\bar{x}_n^2]\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} \mathbb{E}[x_i^2] - n\mathbb{E}[\bar{x}_n^2]\right]$$

$$= \frac{1}{n-1}\left[n\mathbb{E}[x_i^2] - n\mathbb{E}[\bar{x}_n^2]\right]$$

$$= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right]$$

$$= \frac{1}{n-1}\left[n(\sigma^2+\mu^2) - \sigma^2 - n\mu^2\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2\right]$$

$$= \sigma^2 \checkmark$$

$$\therefore \mathbb{E}[S_n^2] = \sigma^2.$$

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i^2 - 2x_i\bar{x}_n + \bar{x}_n^2\right)$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}x_i^2 - 2\sum_{i=1}^{n}x_i\bar{x}_n + \sum_{i=1}^{n}\bar{x}_n^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}x_i^2 - 2n\bar{x}_n^2 + n\bar{x}_n^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n}x_i^2 - n\bar{x}_n^2\right] \checkmark$$

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