Exercise 2:

$$\hat{\beta} = (X^T X)^{-1} X^T y \quad \text{and} \quad \sigma^2_{ad} = \frac{1}{n-p-1} \hat{\varepsilon}^T \hat{\varepsilon}.$$

1.
$$\hat{\beta} = (X^T X)^{-1} X^T y$$
$$= (X^T X)^{-1} X^T (X\beta + \varepsilon) \quad [\because y_i = \beta_0 + x_i \beta_1 + \varepsilon_i]$$
$$= (X^T X)^{-1} (X^T X) \beta + (X^T X)^{-1} X^T \varepsilon$$
$$= \beta + (X^T X)^{-1} X^T \varepsilon$$

Now
$$\mathbb{E}[\hat{\beta}] = \beta + (X^T X)^{-1} X^T \mathbb{E}[\varepsilon]$$
$$= \beta \quad [\because \mathbb{E}[\varepsilon] = 0] \checkmark$$

We know,
$$\operatorname{cov}(\hat{\beta}) = \mathbb{E}\left[(\hat{\beta} - \mathbb{E}[\hat{\beta}])(\hat{\beta} - \mathbb{E}[\hat{\beta}])^T\right]$$
$$= \mathbb{E}\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)^T\right]$$

Now,
$$\hat{\beta} - \beta = (X^T X)^{-1} X^T y - \beta$$
$$= (X^T X)^{-1} X^T (X\beta + \varepsilon) - \beta$$
$$= \beta + (X^T X)^{-1} X^T \varepsilon - \beta$$
$$= (X^T X)^{-1} X^T \varepsilon.$$

So,
$$\operatorname{cov}(\hat{\beta}) = \mathbb{E}\left[((X^T X)^{-1} X^T \varepsilon)((X^T X)^{-1} X^T \varepsilon)^T\right]$$
$$= \mathbb{E}\left[(X^T X)^{-1} (X^T X) \varepsilon \varepsilon^T (X^T X)^{-1}\right]$$
$$= \mathbb{E}\left[\varepsilon \varepsilon^T (X^T X)^{-1}\right]$$

$$= (X^TX)^{-1} \underbrace{\mathbb{E}[\varepsilon\varepsilon^T]}$$

$$= (X^TX)^{-1} \cdot \boxed{\sigma^2 I_n}$$

$$= \sigma^2 (X^TX)^{-1}.$$

So, $\text{cov}(\hat{\beta}) = \sigma^2 (X^TX)^{-1}$.

2.

$$\hat{\varepsilon} = y - X\hat{\beta}$$

so,

$$\mathbb{E}[\hat{\varepsilon}^T\hat{\varepsilon}] = \mathbb{E}[(y - X\hat{\beta})^T (y - X\hat{\beta})]$$

$$= \mathbb{E}[(y - X(X^TX)^{-1}X^Ty)^T (y - X(X^TX)^{-1}X^Ty)]$$

$$= \mathbb{E}[(y - Hy)^T (y - Hy)]$$

[Take $X(X^TX)^{-1}X^T = H$]

$$= \mathbb{E}[y^T (I_n - H)^T (I_n - H) y]$$

$$= \mathbb{E}[y^T (I_n - H) y]$$

$$= tr((I_n - H)\sigma^2 I_n) + \beta^T X^T (I_n - H) X\beta$$

[$\mathbb{E}[x^T A x] = tr(A\Sigma) + \mu^T A \mu$]

$$= \sigma^2 (n - p - 1) + \beta^T X^T (I_n - X(X^TX)^{-1}X^T) X\beta$$

[The matrix $I_n - H$ is also symmetric and Idempotent with rank$(I_n - H) = n-p-1$.]

$$= \sigma^2(n-p-1) + \beta^T X^T X \beta - \beta^T X^T X (X^T X)^{-1} X^T X \beta$$

$$= \sigma^2(n-p-1) + \beta^T X^T X \beta - \beta^T X^T X \beta$$

$$= \sigma^2(n-p-1)$$

Now, $\mathbb{E}[\sigma^2_{ad}] = \frac{1}{n-p-1} \mathbb{E}[\hat{\varepsilon}^T \hat{\varepsilon}]$

$$\Rightarrow \mathbb{E}[\sigma^2_{ad}] = \frac{1}{n-p-1} \cdot \sigma^2 (n-p-1)$$

$$\Rightarrow \mathbb{E}[\sigma^2_{ad}] = \sigma^2. \checkmark$$

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