Exercise 2:

Given, $$\beta^*_N = \left(\sum_{i=1}^{N} x_i^2\right)^{-1} \left(\sum_{i=1}^{N} x_i y_i\right)$$

$$= \frac{\sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N} x_i^2}$$

If a new data point (x_{N+1}, y_{N+1}) will add then.

$$\beta^*_{N+1} = \frac{\sum_{i=1}^{N} x_i y_i + x_{N+1} y_{N+1}}{\sum_{i=1}^{N} x_i^2 + x_{N+1}^2}$$ [add (N+1)th term]

$$= \frac{\frac{\sum_{i=1}^{N} x_i y_i}{\sum_{i=1}^{N} x_i^2} \sum_{i=1}^{N} x_i^2 + x_{N+1} y_{N+1}}{\sum_{i=1}^{N} x_i^2 + x_{N+1}^2}$$

$$= \frac{\beta^*_N \sum_{i=1}^{N} x_i^2 + x_{N+1} y_{N+1}}{\sum_{i=1}^{N} x_i^2 + x_{N+1}^2}$$

Point

$x = [1, 2, 3, 4, 5]$

$y = [-4.2947, -2.3880, -1.0445, -1.1596, -0.8999]$

$$\beta_N^* = \left(\sum_{i=1}^{n} x_i^2\right)^{-1} \left(\sum_{i=1}^{n} x_i y_i\right)$$

$$= (0.01818182)(-21.3421)$$

$$= -0.38803818$$

$x_{N+1} = 6 \qquad y_{N+1} = -0.568$

$$\beta_{N+1}^* = \left(\sum_{i=1}^{n} x_i^2 + x_{N+1} x_{N+1}\right)^{-1} \left(\sum_{i=1}^{n} x_i y_i + x_{N+1} y_{N+1}\right)$$

$$= (0.01098901)(-24.7501)$$

$$= -0.27197912$$

$$\beta_{N+1}^* = \frac{\beta_N^* \sum_{i=1}^{n} x_i^2 + x_{N+1} y_{N+1}}{\sum_{i=1}^{n} x_i^2 + x_{N+1}^2}$$

$$= -0.27197912$$

Array

$X = [1, 2, 3, 4, 5]$

$Y = [6.6, 7.9, 9.4, 11.1, 12.4]$

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \\ 1 & 5 \end{bmatrix} \qquad Y = \begin{bmatrix} 6.6 \\ 7.9 \\ 9.4 \\ 11.1 \\ 12.4 \end{bmatrix}$$

$$\hat{\beta}_N = (X^T X)^{-1} X^T y$$

$$= \begin{bmatrix} 5.04 \\ 1.48 \end{bmatrix}$$

New data point: $\underset{X}{6}, \ \underset{Y}{13.8}$

$X\_new = [1, 2, 3, 4, 5, 6]$

$Y\_new = [6.6, 7.9, 9.4, 11.1, 12.4, 13.8]$

$$\hat{\beta}_{N+1} = (X\_new^T X\_new)^{-1} X\_new^T y\_new$$

$$= \begin{bmatrix} 5.08 \\ 1.46 \end{bmatrix}$$

$X\_add = [1 \ 6] \quad Y\_add = [13.8]$

$$\hat{\beta}_{N+1} = (X^T X + X\_add^T X\_add)^{-1}$$

$$(X^T Y + X\_add^T Y\_add)$$

$$= \begin{bmatrix} 5.08 \\ 1.46 \end{bmatrix}$$

$$X^T X = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

$$= \begin{bmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{bmatrix}$$

$$X^T Y = \begin{bmatrix} \sum y_i \\ \sum x_i y_i \end{bmatrix}$$

$$\beta^*_{N+1} = \frac{X^T y + X\_add^T Y\_add}{X^T X + X\_add^T X\_add}$$

$$= \frac{\beta^*_N \; X^T X + X\_add^T Y\_add}{X^T X + X\_add^T X\_add}$$