

```
In [1]: import numpy as np
from sklearn.cluster import KMeans
from matplotlib import pyplot as plt
```

### Part (i)

Adjacency Matrix

```
In [2]: A = np.array([
[0, 1, 0, 0, 1, 0],
[1, 0, 1, 1, 0, 1],
[0, 1, 0, 0, 0, 0],
[0, 1, 0, 0, 1, 1],
[1, 0, 0, 1, 0, 1],
[0, 1, 0, 1, 1, 0]])
```

Degree Matrix

```
In [3]: D = np.diag(A.sum(axis=1))
print(D)

[[2 0 0 0 0 0]
 [0 4 0 0 0 0]
 [0 0 1 0 0 0]
 [0 0 0 3 0 0]
 [0 0 0 3 0 0]
 [0 0 0 0 3 1]]
```

Laplacian matrix

```
In [4]: L = D-A
print(L)

[[ 2 -1  0  0 -1  0]
 [-1  4 -1 -1  0 -1]
 [ 0 -1  1  0  0  0]
 [ 0 -1  0  3 -1 -1]
 [-1  0  0 -1  3 -1]
 [ 0 -1  0 -1 -1  3]]
```

### Part (ii)

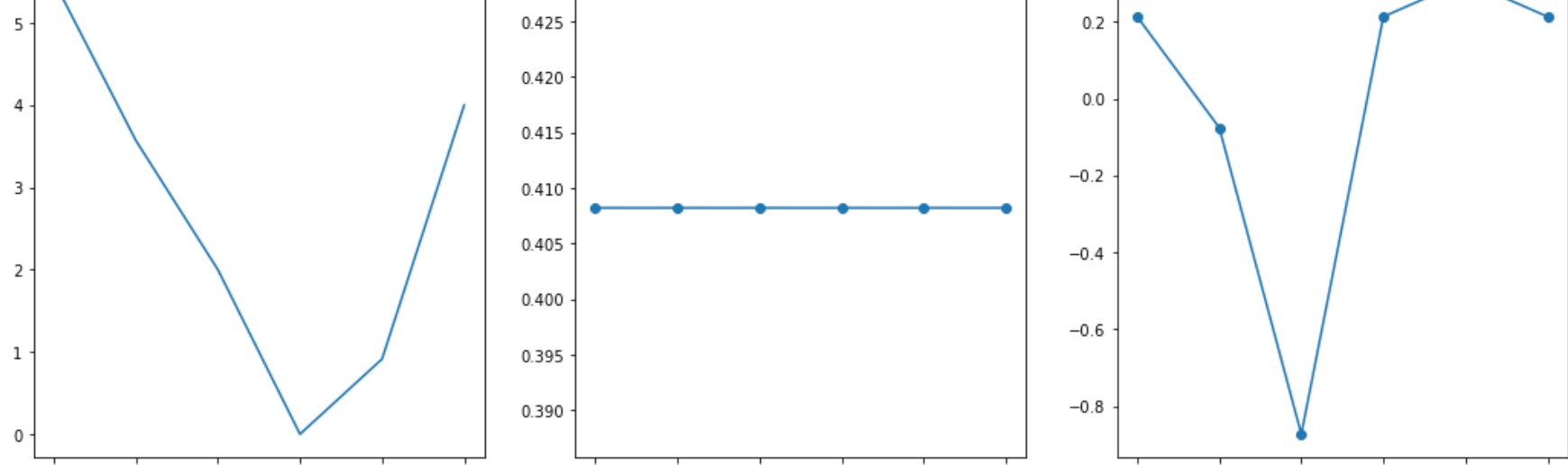
eigenvalues and eigenvectors

```
In [5]: vals, vecs = np.linalg.eig(L)
```

```
In [6]: fig = plt.figure(figsize=[18, 6])
ax1 = plt.subplot(131)
plt.plot(vals)
ax1.title.set_text('eigenvalues')

i = np.where(vals < 1)[0]
ax2 = plt.subplot(132)
plt.plot(vecs[:, i[0]], marker='o')
ax2.title.set_text('first eigenvector with eigenvalue')

ax3 = plt.subplot(133)
plt.plot(vecs[:, i[1]], marker='o')
ax3.title.set_text('second eigenvector with eigenvalue')
```



Because we have a single component, only 1 eigenvalue will be equal to 0. However, if we look at the second smallest eigenvalue, we can still observe a distinction between the two classes. If we drew a horizontal line across, we would correctly classify the nodes. The nodes will be **{1, 2, 4, 5, 6}** and **{3}**

```
In [7]: # sort these based on the eigenvalues
vecs = vecs[:,np.argsort(vals)]
vals = vals[np.argsort(vals)]

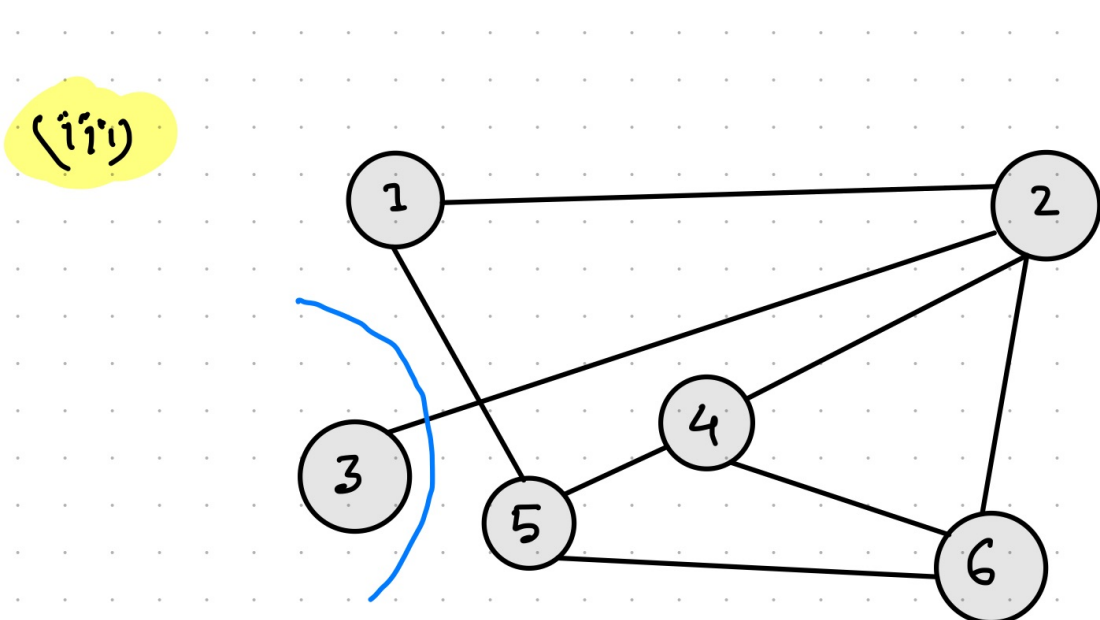
#kmeans on first two vectors with nonzero eigenvalues
kmeans = KMeans(n_clusters=2)
kmeans.fit(vecs[:,0:2])
label = kmeans.labels_
```

```
In [8]: print("Clusters:", label)
```

Clusters: [0 0 1 0 0 0]

Here also we see that node **{3}** will be different label comapre to other nodes **{1, 2, 4, 5, 6}**

### Part (iii)



$$A = \{1, 2, 4, 5, 6\}$$

$$\bar{A} = \{3\}$$

$$f_i = \begin{cases} \sqrt{|\bar{A}|/|A|} & \text{if } v_i \in A \\ -\sqrt{|\bar{A}|/|A|} & \text{if } v_i \in \bar{A} \end{cases}$$

$$f_1 = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$f_2 = \frac{1}{\sqrt{5}}$$

$$f_3 = -\sqrt{\frac{5}{1}} = -\sqrt{5}$$

$$f_4 = \frac{1}{\sqrt{5}}$$

$$f_5 = \frac{1}{\sqrt{5}}$$

$$f_6 = \frac{1}{\sqrt{5}}$$

$$\therefore f = (f_1, f_2, f_3, f_4, f_5, f_6) = \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\sqrt{5}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

### Part (iv)

$$(iv) \quad f^T L f = |V|. \text{RatioCut}(A, \bar{A})$$

$$f^T L f = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -\sqrt{5} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{bmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ -\sqrt{5} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= 7.2 \quad [\text{Calculate using Jupyter Notebook}]$$

$$|V| = 6$$

$$\text{RatioCut} = \frac{1}{1} + \frac{1}{5}$$

$$= \frac{6}{5}$$

$$\therefore |V| \cdot \text{RatioCut}(A, \bar{A}) = 6 \cdot \frac{6}{5}$$

$$= \frac{36}{5}$$

$$= 7.2$$

$$\text{So, } f^T L f = |V|. \text{RatioCut}(A, \bar{A})$$

```
In [9]: import math
x = 1/math.sqrt(5)
y = -math.sqrt(5)
```

```
In [10]: f_t = np.array([x, x, y, x, x, x])
```

```
In [11]: f = np.array([x], [x], [y], [x], [x], [x]])
```

```
In [12]: mul = f_t @ L @ f
print(mul)
```

[7.2]

### Part (v)

$$(v) \quad \text{All one vector, } v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$f \cdot v = \left( \frac{1}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \quad -\sqrt{5} \quad \frac{1}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \right) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= 0$$

$$\text{So, } f \text{ and } v \text{ are orthogonal.}$$

$$\|f\|^2 = f \cdot f^T$$

$$= \left( \frac{1}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \quad -\sqrt{5} \quad \frac{1}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \quad \frac{1}{\sqrt{5}} \right) \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ -\sqrt{5} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= 6 = n$$

