Exercise 1:

1) $\operatorname{Ker}(A) = \langle v_{r+1}, \dots, v_n \rangle$

For $(\subseteq)$ the singular value decomposition (SVD) can easily write $Av_i = 0$ for $i = r+1, \dots, n$. This proves immediately that $v_i \in \operatorname{Ker}(A)$ for $i = r+1, \dots, n$. Since $\operatorname{Ker}(A)$ is vector subspace of $\mathbb{R}^m$, any linear combination of $v_{r+1}, \dots, v_n$ is in $\operatorname{Ker}(A)$. Hence $\langle v_{r+1}, \dots, v_n \rangle \subseteq \operatorname{Ker}(A)$

For $(\supseteq)$

$$x \in \operatorname{Ker}(A) \Leftrightarrow \|Ax\|_2 = 0$$

$$\Leftrightarrow \|U \Sigma V^T x\|_2 = 0$$

$$\Leftrightarrow \|\Sigma V^T x\|_2 = 0$$

$$\Leftrightarrow \|\Sigma y\|_2 = 0; \text{ where } y = V^T x$$

$$\Leftrightarrow y = (0, \dots, 0, y_{r+1}, \dots, y_n)^T$$

$$\Leftrightarrow x = Vy$$

$$[y = (0, \dots, 0, y_{r+1}, \dots, y_n)^T]$$

$$\Leftrightarrow x = \sum_{i=r+1}^{n} y_i v_i$$

$$\Leftrightarrow x \in \langle v_{r+1}, \dots, v_n \rangle$$

This proves that $\langle v_{r+1}, \dots, v_n \rangle \supseteq \operatorname{Ker}(A)$

2. $\mathrm{Im}(A) = \langle u_1, \dots, u_r \rangle$

For $(\subseteq)$ the SVD can easily write $Av_i = \sigma_i v_i$, for $i = 1, \dots r$. This proves immediately that $u_i \in \mathrm{Im}(A)$ for $i = 1, \dots, r$. Since $\mathrm{Im}(A)$ is a vector subspace of $\mathbb{R}^m$, any Linear combination of $u_1, \dots, u_n$ is in $\mathrm{Im}(A)$. Hence $\langle u_1, \dots, u_r \rangle \subseteq \mathrm{Im}(A)$

For $(\supseteq)$

$$
\begin{aligned}
y \in \mathrm{Im}(A) &\iff \exists x \text{ such that } y = Ax \\
&\iff y = U \Sigma V^T x \\
&\iff y = U \Sigma z; \quad \text{where } z = V^T x \\
&\iff y = U (\sigma_1 z_1, \dots, \sigma_r z_r, 0, \dots 0)^T \\
&\iff y = \sum_{i=1}^{r} (\sigma_i z_i) u_i \\
&\iff y = \langle u_1, \dots, u_r \rangle
\end{aligned}
$$

This proves that $\langle u_1, \dots, u_r \rangle \supseteq \mathrm{Im}(A)$