

1.

$$\mathbb{E}[a+bx] = a+b\mathbb{E}[x],$$

Suppose, X is discrete random variable with pmf $P(x)$,

Then by definition,

$$\begin{aligned}\mathbb{E}[a+bx] &= \sum_x (a+bx)P(x) \\ &= \sum_x (a \cdot P(x) + b \cdot x \cdot P(x)) \\ &= \sum_x a \cdot P(x) + b \sum_x x \cdot P(x) \\ &= \underbrace{a \left(\sum_x P(x) \right)}_1 + b \underbrace{\left(\sum_x x \cdot P(x) \right)}_{\mathbb{E}[x]} \\ &= a + b \cdot \mathbb{E}[x]\end{aligned}$$

2.

$$\begin{aligned}\text{Var}(x) &= \mathbb{E}[(x - \mathbb{E}[x])^2] \\ &= \mathbb{E}[x^2 - 2x\mathbb{E}[x] + \mathbb{E}[x]^2] \\ &= \mathbb{E}[x^2] + \mathbb{E}[-2x\mathbb{E}[x]] + \mathbb{E}[\mathbb{E}[x^2]] \\ &= \mathbb{E}[x^2] - 2\mathbb{E}[x]\mathbb{E}[x] + \mathbb{E}[x^2] \\ &= \mathbb{E}[x^2] - 2\mathbb{E}[x]^2 + \mathbb{E}[x^2] \\ &= \mathbb{E}[x^2] - \mathbb{E}[x]^2\end{aligned}$$

$$\begin{aligned}
 3. \quad \text{var}(a + bx) &= E((a + bx)^2 - (E(a + bx))^2 \\
 &= E[a^2 + 2abx + b^2x^2] - (a + bE(x)) \\
 &\quad (a + bE(x)) \\
 &= a^2 + 2abE(x) + b^2E[x^2] - a^2 - 2abE[x] \\
 &\quad - b^2E(x)^2 \\
 &= b^2E[x^2] - b^2E[x]^2 \\
 &= b^2(E[x^2] - E[x]^2) \\
 &= b^2 \text{var}(x)
 \end{aligned}$$

4. The expected value of a constant is equal to itself.

$$E(a) = a$$

$$\begin{aligned}
 \text{var}(a) &= E[(a - E(a))^2] \\
 &= E[(a - a)^2] \\
 &= E(0)
 \end{aligned}$$

$$E(0) = \sum_n n \cdot f(n) = 0 \cdot 1 = 0.$$

2.

$$S_n^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x}_n)^2$$

$$\mathbb{E}[S_n^2] = \frac{1}{n-1} \mathbb{E}\left[\sum_{i=1}^n (x_i - \bar{x}_n)^2\right]$$

We know,

$$\text{Var}(x_i) = \mathbb{E}(x_i^2) - [\mathbb{E}(x_i)]^2$$

$$\Rightarrow \sigma^2 = \mathbb{E}(x_i^2) - M^2$$

$$\Rightarrow \mathbb{E}(x_i^2) = \sigma^2 + M^2$$

Also,

$$\text{Var}(\bar{x}_n) = \mathbb{E}(\bar{x}_n^2) - (\mathbb{E}(\bar{x}_n))^2$$

$$\Rightarrow \frac{\sigma^2}{n} = \mathbb{E}(\bar{x}_n^2) - M^2$$

$$\Rightarrow \mathbb{E}(\bar{x}_n^2) = \frac{\sigma^2}{n} + M^2$$

$$\mathbb{E}[S_n^2] = \frac{1}{n-1} \sum_{i=1}^n \mathbb{E}(x_i^2 - 2x_i\bar{x}_n + \bar{x}_n^2)$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^n \mathbb{E}[x_i^2] - 2 \mathbb{E}[\bar{x}_n] \sum_{i=1}^n x_i + \mathbb{E}\left(\sum_{i=1}^n \bar{x}_n^2\right) \right]$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^n \mathbb{E}[x_i^2] - 2n \mathbb{E}[\bar{x}_n^2] + n \mathbb{E}[\bar{x}_n^2] \right]$$

$$= \frac{1}{n-1} \left[\sum_{i=1}^n \mathbb{E}[x_i^2] - n \mathbb{E}[\bar{x}_n^2] \right]$$

$$= \frac{1}{n-1} \left[n \mathbb{E}[x_i^2] - n \mathbb{E}[\bar{x}_n^2] \right]$$

$$= \frac{1}{n-1} \left[n(\sigma^2 + M^2) - n\left(\frac{\sigma^2}{n} + M^2\right) \right]$$

$$= \frac{1}{n-1} [n(\sigma^2 + \mu^2) - \sigma^2 - n\mu^2]$$

$$= \frac{1}{n-1} [n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2]$$

$$= \sigma^2$$

$$\therefore \text{IE}[S_n^2] = \sigma^2.$$

$$\begin{aligned}
 S_n^2 &= \frac{1}{n-1} \sum_{i=1}^n (x_i^2 - 2x_i\bar{x}_n + \bar{x}_n^2) \\
 &= \frac{1}{n-1} \left[\sum_{i=1}^n x_i^2 - 2 \sum_{i=1}^n x_i \bar{x}_n + \sum_{i=1}^n \bar{x}_n^2 \right] \\
 &= \frac{1}{n-1} \left[\sum_{i=1}^n x_i^2 - 2n \bar{x}_n^2 + n \bar{x}_n^2 \right] \\
 &= \frac{1}{n-1} \left[\sum_{i=1}^n x_i^2 - n \bar{x}_n^2 \right]
 \end{aligned}$$