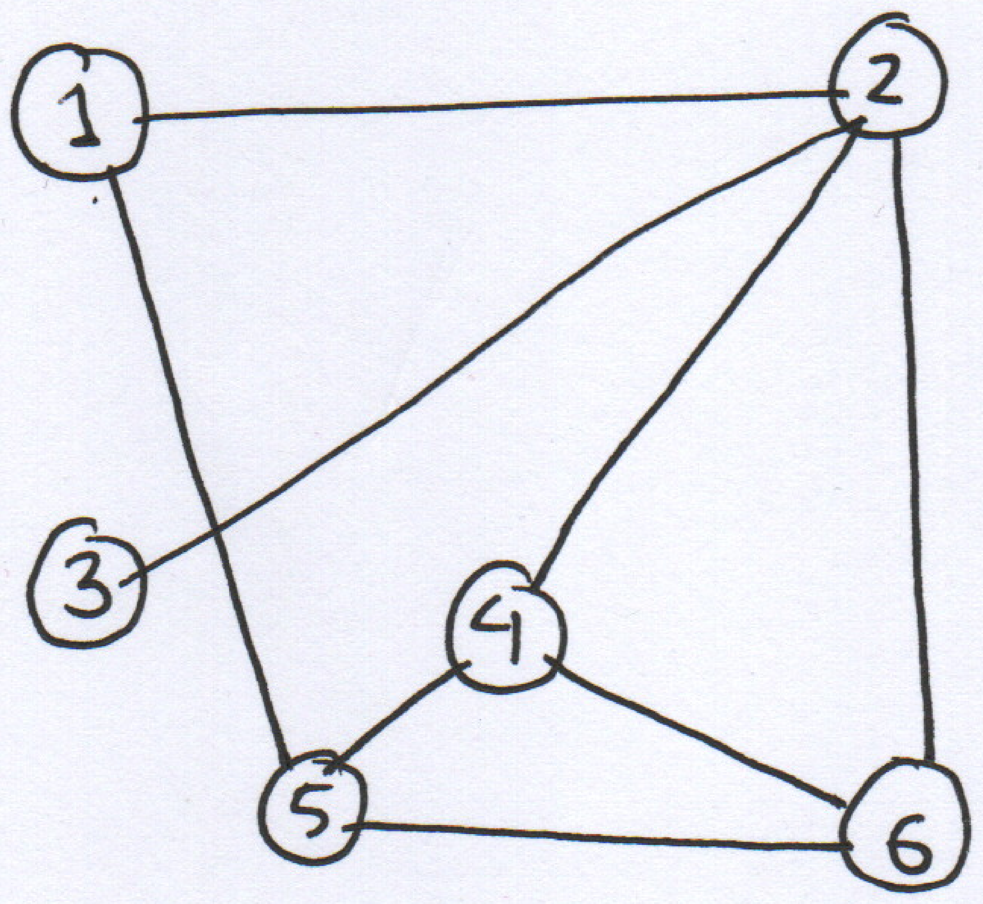


(i) Laplacian Matrix: $L(G) = D(G) - W(G)$

$\uparrow$ Degree Matrix $\uparrow$ Adjacency Matrix.

$D(G)$:

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

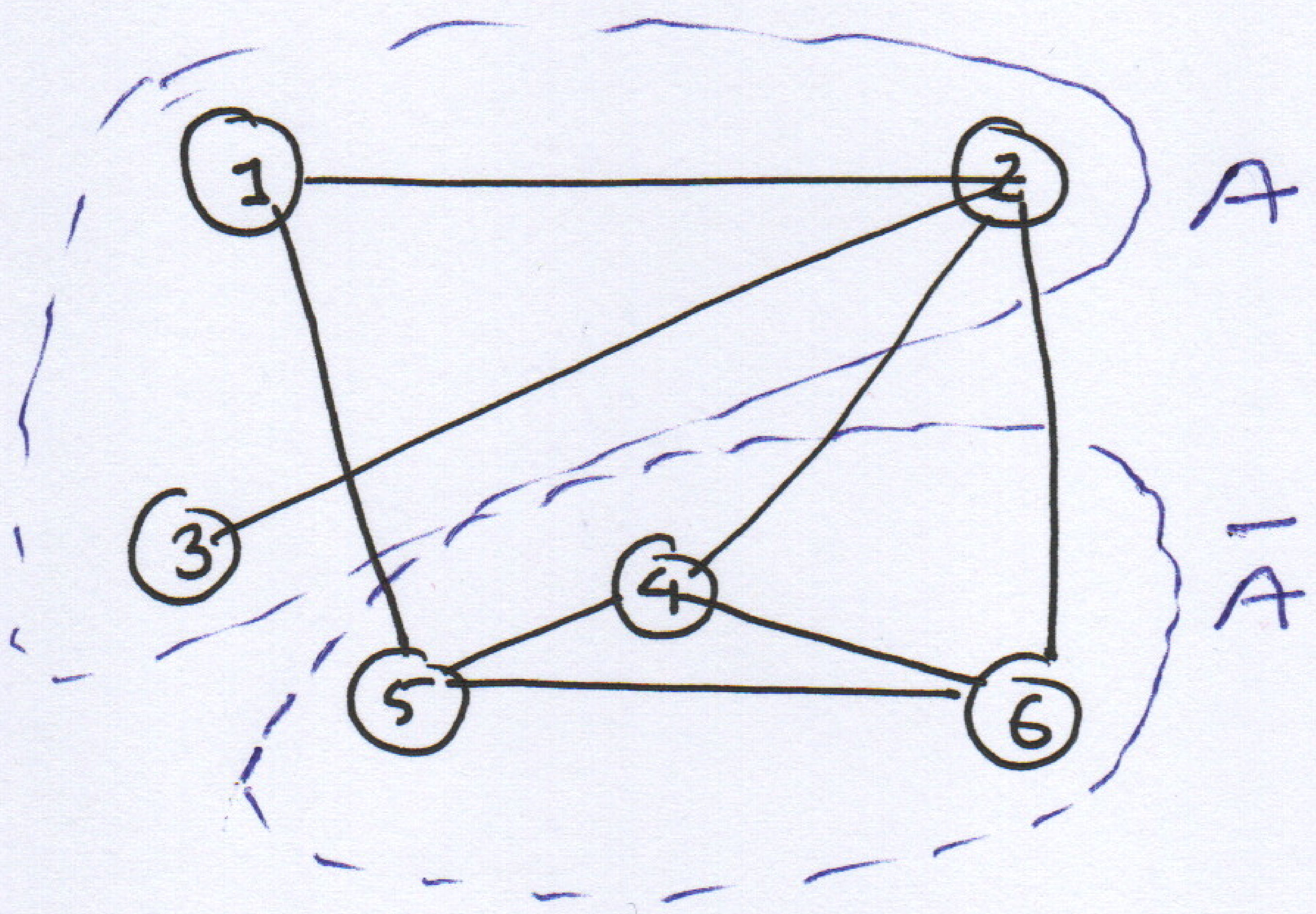
$W(G)$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$L(G) = D(G) - W(G)$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{bmatrix}$$

(ii)



$\therefore |A| = 3$ & $|\bar{A}| = 3$

$$f_i = \begin{cases} \sqrt{|\bar{A}|/|A|} & \text{if } v_i \in A \\ -\sqrt{|A|/|\bar{A}|} & \text{if } v_i \in \bar{A} \end{cases}$$

So the corresponding vector f will be

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

Now, $f = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$, $f^T = (1 \; 1 \; 1 \; -1 \; -1 \; -1)$

and $L = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix}$

$$\therefore f^T L f = (1\ 1\ 1\ {-1}\ {-1}\ {-1}) \begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$= (2\ 4\ 0\ {-2}\ {-2}\ {-2}) \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$$

$$= 12$$

$$|V|.\ \text{Ratio Cut}\ (A, \bar{A}) = |6|.\left(\frac{3}{3} + \frac{3}{3}\right)$$

$$= 12$$

So, we can say that $f^T L f = |V|.\ \text{Ratio Cut}\ (A, \bar{A})$

If 2 vectors are orthogonal if they are ~~part~~ perpendicular to each other, i.e. the dot product of the two vectors is zero.

All one vector: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Now.

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} . \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = 1.1 + 1.1 + 1.1 + 1.(-1) + 1.(-1) + 1.(-1)$$

$$= 1 + 1 + 1 - 1 - 1 - 1$$

$$= 0$$

So, f is orthogonal to all one vector.

$$\|f\|^2 = \sum_i f_i^2$$

$$= 1^2 + 1^2 + 1^2 + (-1)^2 + (-1)^2 + (-1)^2$$

$$= 6$$

$$= n$$

$\therefore \|f\|^2 = n.$