Problem 2:

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ 4 & 6 \end{bmatrix}$$

Now,

$$\begin{vmatrix} 6-\lambda & 4 \\ 4 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (6-\lambda)(6-\lambda) - 16 = 0$$

$$\Rightarrow 36 - 12\lambda + \lambda^2 - 16 = 0$$

$$\Rightarrow \lambda^2 - 12\lambda + 20 = 0$$

$$\Rightarrow (\lambda - 10)(\lambda - 2) = 0$$

$$\therefore \lambda_1 = 10, \quad \sigma_1 = \sqrt{10}$$

$$\lambda_2 = 2, \quad \sigma_2 = \sqrt{2}$$

Now, For $\lambda_1 = 10$

$$\left(\begin{array}{cc|c} -4 & 4 & 0 \\ 4 & -4 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 \to R_1 + R_2} \left(\begin{array}{cc|c} -4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$R_1 \to \frac{R_1}{4} \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So, $-x_1 + x_2 = 0$

$\Rightarrow x_1 = x_2$

Let. $x_2 = 1 \quad \therefore w_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For, $\lambda_2 = 2$

$$\left(\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 \to R_1 - R_2} \left(\begin{array}{cc|c} 4 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 \to \frac{R_1}{4}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

So, $x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

Let $x_2 = 1$, $w_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Now, $v_1 = \frac{1}{\sqrt{1^2+1^2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$u_1 = \frac{1}{\sqrt{\lambda_1}} A v_1$$

$$= \frac{1}{\sqrt{10}} \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\sqrt{5}}{10} \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

Using Gram-Schmidt:

$$u_3^* = e_1 - \langle e_1 \cdot u_1 \rangle u_1 - \langle e_1 \cdot u_2 \rangle \cdot u_2$$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So,

$$U_3^* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(-\frac{2\sqrt{5}}{10}\right)\begin{pmatrix} -\frac{2\sqrt{5}}{10} \\ \frac{4\sqrt{5}}{10} \\ 0 \end{pmatrix} - 0$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/5 \\ -2/5 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4/5 \\ 2/5 \\ 0 \end{pmatrix}$$

$$U_3 = \begin{pmatrix} \frac{4/5}{\sqrt{(4/5)^2 + (2/5)^2}} \\ \frac{2/5}{\sqrt{(4/5)^2 + (2/5)^2}} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2\sqrt{5}}{5} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}$$

Now for Singular value Decompsition.

$$U \Sigma V^T$$

$$= \begin{bmatrix} -\frac{2\sqrt{5}}{10} & 0 & \frac{2\sqrt{5}}{5} \\ \frac{4\sqrt{5}}{10} & 0 & \frac{1}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= A.$$