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11. Problem Sheet for Statistical Data Analysis

Exercise 1

a matrix $A \epsilon \mathbb{R}^{n \times n}$ *is diagonalizable if* $P^{-1}AP = D$ *where*

$D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$ *and* λ_i *are eigenvalues of A,* *i=1,…,n,*

$P = (X_1 \quad \ldots \quad X_n)$ *and* X_i *are eigenvectors of A,* *i=1,…,n.*

If A is $n \times n$ *matrix with n distinct values implies that eeigenvectors of A are linearly independent then A is diagonalizable.*

(for further check: https://www2.math.uconn.edu/~khlee/Teaching/LAlg/MAT223-8.pdf)

a) $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$A - \lambda \cdot I = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{pmatrix}$$

$$det(A - \lambda \cdot I) = \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda) - (-1) = \lambda^2 - 2\lambda + 2$$

$$\Rightarrow \lambda_1 = 1 - i,\ \lambda_2 = 1 + i$$

$$(A - \lambda \cdot I) \cdot v = 0$$

- *for* $\lambda_1 = 1 - i$:

$$A - \lambda_1 \cdot I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

$$(A - \lambda_1 \cdot I) \cdot v = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$ix_1 = x_2, \qquad -ix_2 = x_1, \qquad \begin{cases} ix_1 - x_2 = 0, \\ x_1 + ix_2 = 0, \end{cases}$$

$$\Longrightarrow X = \left(x_2 \begin{pmatrix} -i \\ 1 \end{pmatrix} \right) \Longrightarrow v_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

- *for* $\lambda_2 = 1 + i$:

$$A - \lambda_2 \cdot I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

$$(A - \lambda_2 \cdot I) \cdot v = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$-ix_1 = x_2, \qquad ix_2 = x_1, \qquad \begin{cases} -ix_1 - x_2 = 0, \\ x_1 - ix_2 = 0, \end{cases}$

$\Longrightarrow X = \left(x_2 \begin{pmatrix} i \\ 1 \end{pmatrix} \right) \Longrightarrow v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Since there exists 2 distinct eigenvalues and so linearly independent eigenvectors just over $\mathbb{C}$, A is diagonalizable over $\mathbb{C}$ but not diagonalizable over $\mathbb{R}$.

Check:

$P = (X_1 \quad \dots \quad X_n) = (v_1 \quad v_2) = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$

$P^{-1} = \begin{pmatrix} 1/2i & 1/2 \\ -1/2i & 1/2 \end{pmatrix}$

$P^{-1}AP = \begin{pmatrix} 1/2i & 1/2 \\ -1/2i & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1/2i & 1/2 \\ -1/2i & 1/2 \end{pmatrix} \begin{pmatrix} -i-1 & i-1 \\ -i+1 & i+1 \end{pmatrix} = \begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix} = D$

$D_{11} = \lambda_1 = 1 - i$

$D_{22} = \lambda_2 = 1 + i$ ■

b) $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$

$B - \lambda \cdot I = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 & 3 \\ 2 & -4-\lambda & -2 \\ 3 & -2 & 1-\lambda \end{pmatrix}$

$det(B - \lambda \cdot I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -4-\lambda & -2 \\ 3 & -2 & 1-\lambda \end{vmatrix} = \left| \begin{matrix} 1-\lambda & 2 & 3 \\ 2 & -4-\lambda & -2 \\ 3 & -2 & 1-\lambda \end{matrix} \right| \begin{matrix} 1-\lambda & 2 \\ 2 & -4-\lambda \\ 3 & -2 \end{matrix}$

$= (1-\lambda)(-4-\lambda)(1-\lambda) + (-12) + (-12) - (3)(-4-\lambda)(3) - (4)(1-\lambda)$
$\quad - (4)(1-\lambda)$

$= -\lambda^3 - 2\lambda^2 + 7\lambda - 4 - 24 + 9\lambda + 36 + 4\lambda - 4 + 4\lambda - 4 = -\lambda^3 - 2\lambda^2 + 24\lambda$

$= -\lambda(\lambda^2 + 2\lambda - 24) = -\lambda(\lambda + 6)(\lambda - 4)$

$\Rightarrow \lambda_1 = 0, \lambda_2 = -6, \lambda_3 = 4.$

$(B - \lambda \cdot I) \cdot v = 0$

- *for* $\lambda_1 = 0$:

$$B - \lambda_1 \cdot I = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

$$(B - \lambda_1 \cdot I) \cdot v_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_2 = -x_3, \qquad x_1 = -x_3, \quad \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 - 4x_2 - 2x_3 = 0 \\ 3x_1 - 2x_2 + x_3 = 0 \end{cases}$$

$$\Longrightarrow X = \left(x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right) \Longrightarrow v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

- $for\ \lambda_2 = -6$:

$$B - \lambda_2 \cdot I = \begin{pmatrix} 7 & 2 & 3 \\ 2 & 2 & -2 \\ 3 & -2 & 7 \end{pmatrix}$$

$$(B - \lambda_2 \cdot I) \cdot v_2 = \begin{pmatrix} 7 & 2 & 3 \\ 2 & 2 & -2 \\ 3 & -2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = -x_3, \qquad x_2 = 2x_3, \quad \begin{cases} 7x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 2x_2 - 2x_3 = 0 \\ 3x_1 - 2x_2 + 7x_3 = 0 \end{cases}$$

$$\Longrightarrow X = \left(x_3 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right) \Longrightarrow v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

- $for\ \lambda_3 = 4$:

$$B - \lambda_3 \cdot I = \begin{pmatrix} -3 & 2 & 3 \\ 2 & -8 & -2 \\ 3 & -2 & -3 \end{pmatrix}$$

$$(B - \lambda_3 \cdot I) \cdot v_3 = \begin{pmatrix} -3 & 2 & 3 \\ 2 & -8 & -2 \\ 3 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = x_3, \qquad x_2 = 0, \quad \begin{cases} -3x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 - 8x_2 - 2x_3 = 0 \\ 3x_1 - 2x_2 - 3x_3 = 0 \end{cases}$$

$$\Longrightarrow X = \left(x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \Longrightarrow v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Since there 3 distinct eigenvalues and so linearly independent eigenvectors, B is diagonalizable over $\mathbb{R}$.

Check:

$$P = (X_1 \quad \dots \quad X_n) = (v_1 \quad v_2 \quad v_3) = \begin{pmatrix} -1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ -1/6 & 1/3 & 1/6 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$P^{-1}BP = \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ -1/6 & 1/3 & 1/6 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ -1/6 & 1/3 & 1/6 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 6 & 4 \\ 0 & -12 & 0 \\ 0 & -6 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$D_{11} = \lambda_1 = 0$

$D_{22} = \lambda_2 = -6$

$D_{33} = \lambda_3 = 4$ ■

c) $C = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

$$C - \lambda \cdot I = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{pmatrix}$$

$$det(B - \lambda \cdot I) = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = \left| \begin{array}{ccc|cc} 2-\lambda & 1 & 1 & 2-\lambda & 1 \\ 1 & 2-\lambda & 1 & 1 & 2-\lambda \\ 1 & -1 & 2-\lambda & 1 & -1 \end{array} \right.$$

$= (2-\lambda)(2-\lambda)(2-\lambda) + 1 + (-1) - (2-\lambda) - (-1)(2-\lambda) - (2-\lambda)$

$= -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = -(\lambda - 1)(\lambda^2 - 5 + 6) = -(\lambda - 1)(\lambda - 3)(\lambda - 2)$

$\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3.$

$(C - \lambda \cdot I) \cdot v = 0$

- *for* $\lambda_1 = 1$:

$$C - \lambda_1 \cdot I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(C-\lambda_1 \cdot I)\cdot v_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = -x_3, \qquad x_2 = 0, \quad \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases}$$

$$\Longrightarrow X = \left(x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right) \Longrightarrow v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

- *for* $\lambda_2 = 2$:

$$C-\lambda_1 \cdot I = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$(C-\lambda_1 \cdot I)\cdot v_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = -x_3, \qquad x_2 = -x_3, \quad \begin{cases} x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

$$\Longrightarrow X = \left(x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right) \Longrightarrow v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

- *for* $\lambda_2 = 2$:

$$C-\lambda_1 \cdot I = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$(C-\lambda_1 \cdot I)\cdot v_1 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$x_1 = x_2, \qquad x_3 = 0, \quad \begin{cases} -x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases}$$

$$\Longrightarrow X = \left(x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right) \Longrightarrow v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Since there 3 distinct eigenvalues and so linearly independent eigenvectors, C is diagonalizable over $\mathbb{R}$.

Check:

$$P = (X_1 \quad \dots \quad X_n) = (v_1 \quad v_2 \quad v_3) = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$P^{-1}CP = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 & 3 \\ 0 & -2 & 3 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$D_{11} = \lambda_1 = 1$

$D_{22} = \lambda_2 = 2$

$D_{33} = \lambda_3 = 3$ ■