Theory Part

Exercise 3.

The likelihood function is:

$$L(\theta|x) = \prod_{i=1}^{n} e^{-(x_i-\theta)}\, I(x_i \geq \theta)$$

$$= e^{-\sum_{i=1}^{n} x_i + n\theta}\, I(x_{1} \geq \theta) \prod_{i=1}^{n} I(x_i \in \mathbb{R})$$

$$= \underbrace{e^{n\theta} I(x_{1} \geq \theta)}_{g(x_1|\theta)}\; \underbrace{e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} I(x_i \in \mathbb{R})}_{h(x)}$$

Here, $x_1 = \min(x_1, x_2, \dots x_n)$

Here x_1 is a sufficient statistic by factorization theorem.

Likelihood Ratio Test Statistic is

$$\lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)} \quad \text{———} \quad (1)$$

Here, $\hat{\theta} = \underset{\theta \in \Theta}{\arg\max}\, L(\theta|x)$; $\Theta = \{\theta : -\alpha < \theta < \alpha\}$

and $\hat{\theta}_0 = \underset{\theta \in \Theta_0}{\arg\max}\, L(\theta|x)$; $\Theta_0 = \{\theta : -\alpha < \theta \leq \theta_0\}$

Now for 1st case:

□ when $\theta \leq x_1$

then $L(\theta|x) = e^{-\sum_{i=1}^{n} x_i + n\theta}$,

which increases as θ increases.

□ when $\theta > x_1$, then $L(\theta|x) = 0$

So, $L(\theta|x)$ is an increasing function when θ is less than or equal to the minimum order statistic x_1; when θ is larger than x_1 the likelihood functions drops to zero.

So, $\hat{\theta} = x_1$ or $\min(x_1, \dots, x_n)$

and $\sup_{\theta \in \Theta} L(\theta|x) = L(\hat{\theta}|x)$

$= L(x_1|x)$

For second case:

□ when $\theta_0 < x_1$, then the largest $L(\theta|x)$ can be is $L(\theta_0|x)$.

so, $\hat{\theta}_0 = \theta_0$

□ when $\theta_0 \geq x_1$,

then, $\hat{\theta}_0 = x_1$ or $\min(x_1, \dots x_n)$

Therefore,

$$\hat{\theta}_0 = \begin{cases} \theta_0, & \theta_0 < x_1 \\ x_1, & \theta_0 \geq x_1 \end{cases}$$

Now eqn (1) become

$$\lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)} = \begin{cases} \dfrac{L(\theta_0|x)}{L(x_1|x)}, & \theta_0 < x_1 \\[2ex] \dfrac{L(x_1|x)}{L(x_1|x)} = 1, & \theta_0 \geq x_1 \end{cases}$$

we have $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$

□ If $x_1 \leq \theta_0$, we certainly don't want to reject H_0 and conclude that $\theta > \theta_0$.

□ It is only when $x_1 > \theta_0$ do we have evidence that θ might be larger than θ_0. So, the larger the x_1, the smaller the $\lambda(x)$, the more evidence against H_0.

$$\text{Now, } \lambda(x) = \frac{L(\theta_0|x)}{L(x_1|x)} = \frac{e^{-\sum_{i=1}^{n} x_i + n\theta_0}}{e^{-\sum_{i=1}^{n} x_i + n x_1}}$$

$$= e^{-n(x_1 - \theta_0)}$$

$[x_1 = \min(x_1 \dots x_n)]$

Programming Part

In [1]:
```python
import random
import math
import matplotlib.pyplot as plt
```

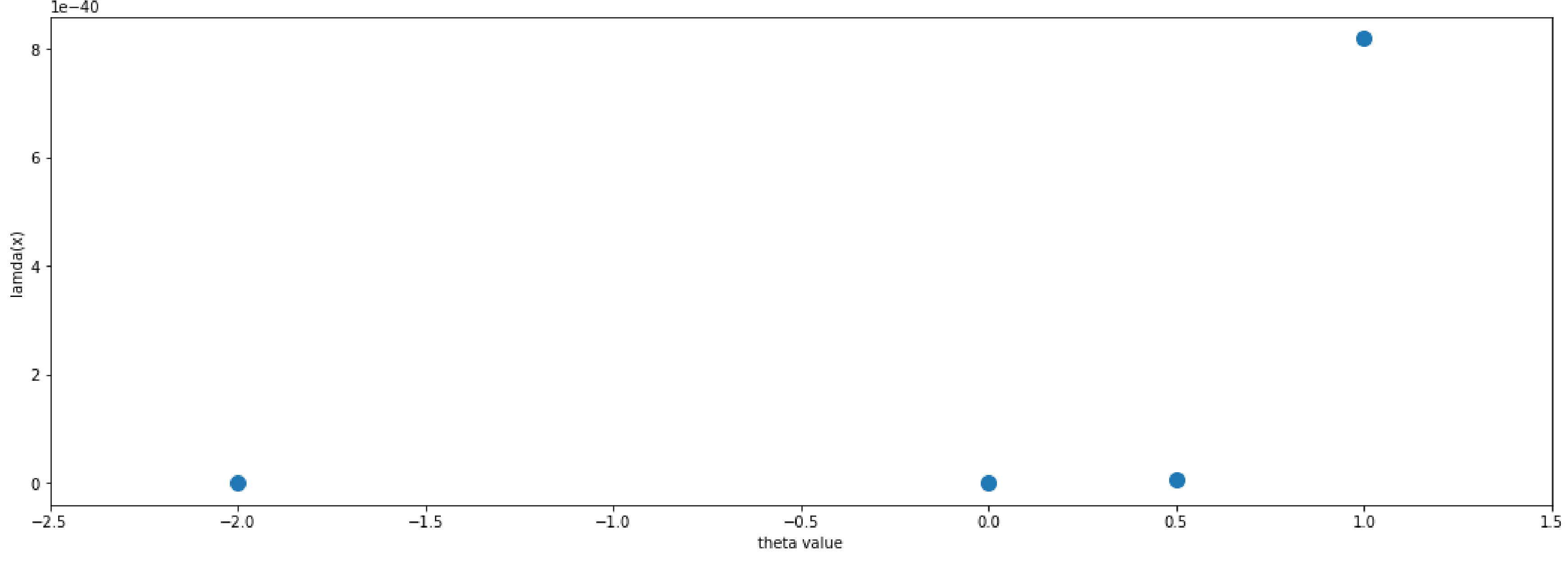
In [2]:
```python
# create 10 random sample. i.e. x > theta, so minimum range is 1
x_list = random.sample(range(1, 100), 10)

# the given theta list
theta_list = [-2, 0, 0.5, 1]
```

In [3]:
```python
# calculate the n (number of samples)
n = len(x_list)

# find the minimum of the list
min_x = min(x_list)
```

In [4]:
```python
lamda_list = []

for theta in theta_list:
    lamda = math.exp(-n * (min_x - theta)) # lamda(x) = e^(-n(min(x) - theta))
    lamda_list.append(lamda) # append all the theta into a list
```

Plot the result

In [5]:
```python
plt.figure(figsize=(18, 6))
plt.scatter(theta_list, lamda_list, marker='o', s=100)
plt.xlabel("theta value")
plt.ylabel("lamda(x)")
plt.xlim(-2.5, 1.5)
plt.show()
```



In the plot we can see that, when the theta values are increasing the lamda(x) values are also increasing.