Exercise 2 (7 Points)
Let $X_1, ..., X_n$ be independent and identically $\mathcal{U}[0,\theta]$−distributed random variables. Show that

$$\left(\prod_{i=1}^{n} X_i\right)^{1/n} \qquad (2)$$

is asymptotically unbiased and and consistent for $\gamma(\theta) = \theta e^{-1}$.

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1) pdf: $f(x|\theta) = \begin{cases} \frac{1}{\theta}, & x \in [0,\theta] \\ 0, & \text{otherwise} \end{cases}$

Asymptotically unbiased, if $\lim_{n\to\infty} E(\hat{\theta}_n) = \theta e^{-1}$

2) Computing the expectation:

since x_i are independent

$$E[\hat{\theta}] = E\left[\left(\prod_{i=1}^{n} X_i\right)^{1/n}\right] = \prod_{i=1}^{n} E(X_i^{1/n}) = \prod_{i=1}^{n} \int_0^\theta \frac{1}{\theta}(x_i)^{\frac{1}{n}}dx = \left(\frac{\theta^{\frac{1}{n}}}{1+\frac{1}{n}}\right)^n = \frac{\theta}{(1+\frac{1}{n})^n}$$

$\lim_{n\to\infty} \frac{\theta}{(1+\frac{1}{n})^n} = \frac{\theta}{e} = \theta \cdot e^{-1}$, therefore for $\gamma(\theta) = \theta e^{-1}$ it is assymptoticaly unbiased

3) • The estimator $T_n = T_n(X_1, ... X_n)$, where $X_1, ..., X_n$ are iid, of θ is consistent if for any $\varepsilon > 0$, $P(|T_n - \theta| > \varepsilon) \to 0$ as $n \to \infty$

• But note from Chebyshev's inequality (for any rand. variable W: $P(|W-\theta| > a) \leq \frac{E((W-\theta)^2)}{a^2}$) the estimator will be consistent if $E((T_n - \theta)^2) \to 0$ and $n \to \infty$

Note also, MSE of T_n is $(b_{T_n}(\theta))^2 + Var_\theta(T_n)$

• So, the estimator will be consistent if it is asymptotically unbiased and its variance $\to 0$ as $n \to \infty$

• $Var(X) = E(X^2) - E^2(X) = \left(\prod_{i=1}^{n} \int_0^\theta \frac{1}{\theta}(x_i)^{\frac{2}{n}}dx\right) - \frac{\theta^2}{(1+\frac{1}{n})^{2n}} =$

$$= \prod_{i=1}^{n} \frac{1}{\theta}\frac{n}{n+2}\theta^{\frac{n+2}{n}} - \frac{\theta^2}{(1+\frac{1}{n})^{2n}} = \frac{1}{\theta^n} \cdot \frac{1}{(1+\frac{2}{n})^n}\theta^{\frac{n+2}{n}} - \frac{\theta^2}{(1+\frac{1}{n})^{2n}} =$$

$$= \frac{\theta^2}{(1+\frac{2}{n})^n} - \frac{\theta^2}{(1+\frac{1}{n})^{2n}}$$

$\lim_{n\to\infty}(1+\frac{1}{n})^{2n} \to \frac{1}{e^2}$ $\quad \lim_{n\to\infty}(1+\frac{2}{n})^n \to \frac{1}{e^2}$ $\Rightarrow \boxed{Var(X) \to \frac{\theta^2}{e^2} - \frac{\theta^2}{e^2} \to 0 \Rightarrow \text{consistent}}$