Exercise 2

$$A = \begin{pmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix} \qquad \text{Goal:} \quad A = U \Sigma V^T \quad \text{with} \quad \begin{matrix} U \in \mathbb{R}^{3\times 3} \\ \Sigma \in \mathbb{R}^{3\times 2} \\ V \in \mathbb{R}^{2\times 2} \end{matrix}$$

$$B := A^T A = \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 6 \end{pmatrix}$$

Compute eigenvalues and -vectors of A^TA:

$$\det(B - \lambda I_2) = \det \begin{pmatrix} 6-\lambda & 4 \\ 4 & 6-\lambda \end{pmatrix} = (6-\lambda)^2 - 16 \overset{!}{=} 0$$

$$\Leftarrow \quad 6 - \lambda = 4 \quad \Rightarrow \quad \lambda_1 = 10, \; \lambda_2 = 2$$

$\underline{\lambda_1 = 10}$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} \to \begin{pmatrix} -4 & 4 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathrm{Eig}(A, \lambda_2) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle$$

$\underline{\lambda_2 = 2}$

$$\begin{pmatrix} 6-2 & 4 \\ 4 & 6-2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \xrightarrow{\text{Gauß}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathrm{Eig}(A, \lambda_1) = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$$

Because AA^T is symmetric, the eigenvectors for different eigenvalues are orthogonal. We just have to transform them to be orthonormal:

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Compute the matrix U:

$$u_1^* = \frac{1}{\sqrt{\lambda_1}} A v_1 = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{\sqrt{5}}{10} \begin{pmatrix} -2 \\ 4 \\ 0 \end{pmatrix}$$

$$u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$u_2^* = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

$u_3 = ?$ $\qquad u_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} ; \; u_2 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$

Gram-Schmidt:

- use $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

- $u_3^* = e_1 - \langle e_1, u_1 \rangle u_1 - \langle e_1, u_2 \rangle u_2$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \left(-\frac{1}{\sqrt{5}}\right) \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} - 0 = \begin{pmatrix} 4/5 \\ 2/5 \\ 0 \end{pmatrix}$$

- $u_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

<u>Put everything together:</u>

$$A = U \Sigma V^T$$

$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{10}$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$$