Exercise 3:

The probability mass function of the multinomial distribution is:

$$P(X_1 = x_1, X_2 = x_2 \dots X_k = x_k) = \frac{n!}{x_1! x_2! \dots x_k!} \times P_1^{x_1} \times P_2^{x_2} \dots \times P_k^{x_k}$$

Here, $n = \sum_{i=1}^{k} x_i$

and $\sum_{i=1}^{k} P_i = 1.$

The likelihood function is:

$$P(x, y, z \mid p) = \binom{x+y+z}{x} \times (1-p)^{2x} \times \binom{y+z}{y} \times (2p(1-p))^{y} \times \binom{z}{z} \times p^{2z}.$$

Taking log-likelihood is given by

$$\log(P(x,y,z \mid p)) = \log\left[\binom{x+y+z}{x} \times (1-p)^{2x} \times \binom{y+z}{y} \times (2p(1-p))^{y} \times \binom{z}{z} \times p^{2z}\right]$$

$$= 2x \log(1-p) + \log\binom{x+y+z}{x} + y \log 2p + y \log(1-p) + \log\binom{y+z}{y} + 2z \log(p) + \log\binom{z}{z}$$

Taking the derivative w.r.t. p

$$\frac{\partial(\log(P(x,y,z \mid p)))}{\partial p} = -\frac{2x}{1-p} + \frac{y}{2p} \cdot 2 - \frac{y}{1-p} + \frac{2z}{p}$$

$$= \frac{y+2z}{p} - \frac{y+2x}{1-p}$$

Set the derivative equal to 0.

$$\frac{y+2z}{p} - \frac{y+2x}{1-p} = 0$$

$$\Rightarrow \frac{y+2z-yp-2pz-yp-2xp}{p(1-p)} = 0$$

$$\Rightarrow y+2z-p(2y+2x+2z) = 0$$

$$\therefore \hat{p}_{MLE} = \frac{y+2z}{2x+2y+2z}$$

Taking 2nd derivative,

$$-\frac{y+2z}{p^2} - \frac{y+2x}{(1-p)^2}$$

which is <0, because here x, y, z are positive and the value of p is $0<p<1$.

So, $\hat{p}_{MLE}$ maximizes the likelihood.

Extension:

Data for Haptoglobin Type in a sample of 190 people.

Haptoglobin Type		
Hp 1-1	Hp 1-2	Hp2-2
10	68	112

$$\hat{\theta} = \frac{y + 2z}{2x + 2y + 2z}$$

$$= \frac{68 + 2 \cdot 112}{2 \cdot 10 + 2 \cdot 68 + 2 \cdot 112}$$

$$= 0.76842 \quad \blacksquare$$

* Find the asymptotic variance of the mle.

$$\mathrm{var}(\hat{\theta}) = \frac{1}{\mathbb{E}[-\ell''(\theta)]}$$

$$\ell''(\theta) = -\frac{y + 2z}{\theta^2} - \frac{y + 2x}{(1-\theta)^2}$$

mean of Binomial $= np$

So, $\mathbb{E}[x] = n(1-\theta)^2$

$\mathbb{E}[y] = n\,2\theta(1-\theta)$

$\mathbb{E}[z] = n\theta^2$

$$\mathbb{E}[-\ell''(\theta)] = \frac{2n\theta(1-\theta) + 2n\theta^2}{\theta^2} + \frac{2n\theta(1-\theta) + 2n(1-\theta)^2}{(1-\theta)^2}$$

$$= 2n\left[\frac{\theta(1-\theta) + \theta^2}{\theta^2} + \frac{\theta(1-\theta) + (1-\theta)^2}{(1-\theta)^2}\right]$$

$$= \frac{2n}{\theta^2(1-\theta)^2}\Big[\theta - 2\theta^2 + \cancel{\theta^3} - \cancel{\theta^2} + 2\cancel{\theta^3} - \cancel{\theta^4} + \theta^2 - 2\cancel{\theta^3} + \cancel{\theta^4} + \cancel{\theta^3} - \cancel{\theta^4} + \cancel{\theta^2} - 2\cancel{\theta^3} + \cancel{\theta^4}\Big]$$

$$= \frac{2n}{\theta^2(1-\theta)^2}(\theta - \theta^2)$$

$$= \frac{2n}{\theta(1-\theta)}$$

$$\therefore \operatorname{var}(\hat{\theta}) = \frac{1}{\mathbb{E}[-\ell''(\theta)]}$$

$$= \frac{\hat{\theta}(1-\hat{\theta})}{2n}$$

$$= \frac{0.76842(1-0.76842)}{2\times 190}$$

$$= 0.0004682898 \quad \blacksquare$$