Exercise 2:

Given, $A = \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$

$\therefore A^T = \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix}$

$\therefore A^T A = \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Now,

$\det(A - \lambda I) = 0$

$\therefore \begin{vmatrix} 2-\lambda & -4 & 0 \\ -4 & 8-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$

$= (2-\lambda)(8-\lambda)(2-\lambda) + (-4).0.0 + 0.(-4).0 - 0.(8-\lambda).0$

$- 0.0(2-\lambda) - (2-\lambda).(-4).(-4)$

$= -\lambda^3 + 12\lambda^2 - 20\lambda^2$

$\therefore \lambda_1 = 10, \lambda_2 = 2, \lambda_3 = 0$

Here, $\sigma_1 = \sqrt{10}$ and $\sigma_2 = \sqrt{2}$

For, $\lambda_1 = 10$

$$(A - \lambda_1 I) = \begin{pmatrix} -8 & -4 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & -8 \end{pmatrix}$$

Now $\left(\begin{array}{ccc|c} -8 & -4 & 0 & 0 \\ -4 & -2 & 0 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right) \underset{\sim}{R_1/(-8) \to R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ -4 & -2 & 0 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right)$

$$\underset{\sim}{R_2 - (-4).R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2/(-8) \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore\ x_1 + \frac{1}{2} x_2 = 0$

and $x_3 = 0$

$\therefore\ x_1 = -\frac{1}{2} x_2 ;\ x_2 = x_2$ and $x_3 = 0$

Let, $x_2 = 1$,

$$v_1 = \begin{pmatrix} -1/2 \\ 1 \\ 0 \end{pmatrix}$$

For, $\lambda_2 = 2$

$$(A - \lambda_2 I) = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 6 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} 0 & -4 & 0 & 0 \\ -4 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \underset{\sim}{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} -4 & 6 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1/(-4) \to R_1} \left(\begin{array}{ccc|c} 1 & -3/2 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2/(-4) \to R_2} \left(\begin{array}{ccc|c} 1 & -3/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-3/2) \cdot R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 = 0$

and $x_2 = 0$

$\therefore \quad x_1 = 0 \,;\ x_2 = 0 \ ;\ x_3 = x_3$

Let, $x_3 = 1$,

$$\therefore v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For, $\lambda_3 = 0$

$$A - \lambda_3 I = \begin{pmatrix} 2 & -4 & 0 \\ -4 & 8 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} 2 & -4 & 0 & 0 \\ -4 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right) \underset{R_1/2 \to R_1}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ -4 & 8 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

$$\underset{R_2 - (-4).R_1 \to R_2}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

$$\underset{R_3 \leftrightarrow R_2}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{R_2/2 \to R_2}{\sim} \left(\begin{array}{ccc|c} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 - 2 \cdot x_2 = 0$

$x_3 = 0$

$\therefore x_1 = 2x_2$; $x_2 = x_2$ & $x_3 = 0$

Let, $x_2 = 1$.

$$v_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

The Σ matrix is a zero matrix with σ_1, on its diagonal.

$$\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$u_1 = \begin{pmatrix} \frac{-1/2}{\sqrt{-\frac{1}{2}^2+1^2}} \\ \frac{1}{\sqrt{-\frac{1}{2}^2+1^2}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} \frac{2}{\sqrt{2^2+1^2}} \\ \frac{1}{\sqrt{2^2+1^2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{2\sqrt{5}}{5} \\ \frac{\sqrt{5}}{5} \\ 0 \end{pmatrix}$$

$$\therefore U = \begin{pmatrix} -\frac{\sqrt{5}}{5} & 0 & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & 0 & \frac{\sqrt{5}}{5} \\ 0 & 1 & 0 \end{pmatrix}$$

Now, $v_i = \frac{1}{\sigma_i} \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}^T u_i$

$$v_1 = \frac{1}{\sigma_1} \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}^T \cdot u_1$$

$$= \frac{1}{\sqrt{10}} \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$v_2 = \frac{1}{\sigma_2} \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}^T u_2$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$$

$$\therefore \; V = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

So, we have,

$$U = \begin{bmatrix} -\frac{\sqrt{5}}{5} & 0 & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & 0 & \sqrt{5}/5 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

Now, $U \Sigma V^T$

$$= \begin{bmatrix} -\sqrt{5}/5 & 0 & \frac{2\sqrt{5}}{5} \\ \frac{2\sqrt{5}}{5} & 0 & \sqrt{5}/5 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{10} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 2 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= A.$$