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In [2]:
```
import numpy as np
import pandas as pd
from matplotlib import pyplot as plt
%matplotlib inline
```

Exercise 1. Write a code to implement the Kalman filter for the fully discrete time system with $Z_k \in \mathbb{R}^{2\times 1}$

$$Z_k = \begin{bmatrix} 0.5 & 0 \\ 0.2 & 4 \end{bmatrix} Z_{k-1} + \Gamma_k, \quad \Gamma_k \sim N(0, Q) \qquad \text{with } Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

- A : state transition matrix of dynamic system (2 X 2 constant matrix)
- Q : noise covariance of dynamic system (2 x 2 constant matrix)
- H : forward map of observing model (1 X 2 constant vector)
- R : noice covariance of observing model (constant)
- E_init : mean of initial distribution of Z (2 X 1 vector)
- P_init : covariance of initial distribution of Z (2 X 2 matrix)
- Z_init : initial state of dynamic system (2 X 1 vector)

In [29]:
```
#A = np.array([[0.5, 0],[0.2, 4]])
A = np.array([[-0.5, 0.6],[-0.1, 0.1]])
Q = np.array([[2, 0],[0, 2]])
H = np.array([[1, 0]])
R = 3
E_init = np.array([[0],[0]])
P_init = np.diag([100, 100]) # please refer to the note below
Z_init = np.array([[np.random.normal(0,1)],[np.random.normal(0,1)]])
Z_init = np.array([[1],[1]])

n_steps = 200 # number of time steps
```

I adjusted the initial covariance because the given 1 seemed too small for the stage where we are very unsure about this initial value.

In [30]:
```
Z_ref_list = np.zeros(shape=(n_steps,2)) # list that stores true states at all time steps
Y_obs_list = np.zeros(shape=(n_steps)) # list that stores observation at all time steps
error_list = np.zeros(shape=(n_steps)) # list that stores residual between estimated state and tr
E_analysis_list = np.zeros(shape=(n_steps,2)) # list that stores E_analysis
P_analysis_list = np.zeros(shape=(n_steps,2)) # list that stores E_analysis
```

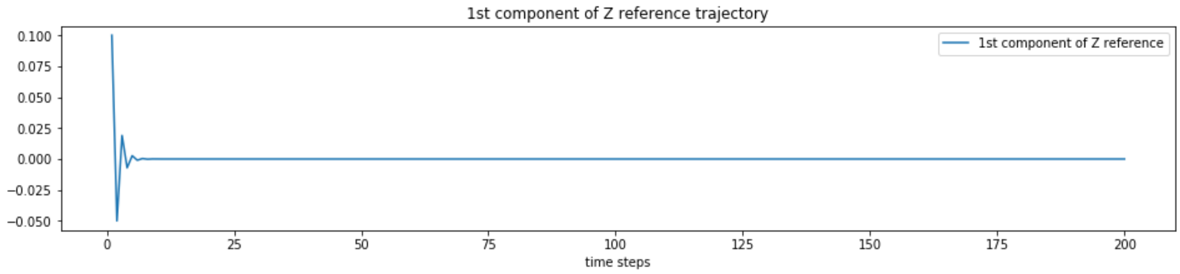
Produce a reference trajectory of the dynamic system.

In [34]:
```
def Forecast_Model (Z_previous):
    Z_current = np.dot(A, Z_previous) #+ np.reshape(np.random.multivariate_normal(np.array([0,0])
    return Z_current

def get_Z_ref_list (Z_init):
    Z_previous = Z_init
    for i in range(n_steps):
        Z_current = Forecast_Model(Z_previous)
        Z_ref_list[i] = np.reshape(Z_current, (1,2))
        Z_previous = Z_current
    return Z_ref_list

Z_ref_list = get_Z_ref_list(Z_init)
```

In [35]:

```python
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), Z_ref_list[:,0])
ax.set(xlabel='time steps', title='1st component of Z reference trajectory')
ax.legend(['1st component of Z reference'])
```

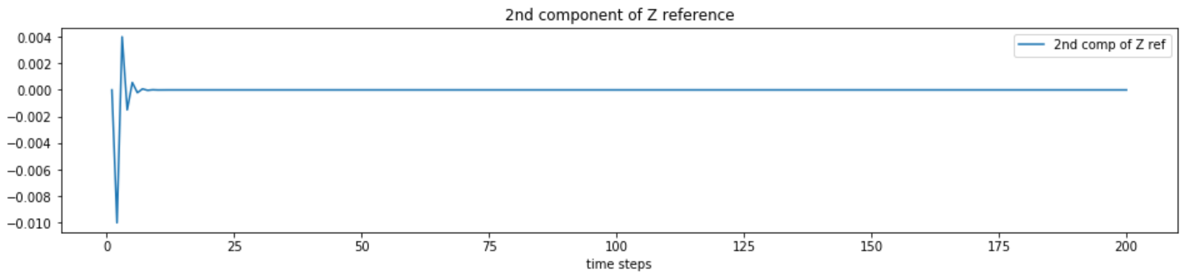
Out[35]: <matplotlib.legend.Legend at 0x194d3bf6710>



In [36]:

```python
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), Z_ref_list[:,1])
ax.set(xlabel='time steps', title='2nd component of Z reference')
ax.legend(['2nd comp of Z ref'])
```

Out[36]: <matplotlib.legend.Legend at 0x194d3c62da0>

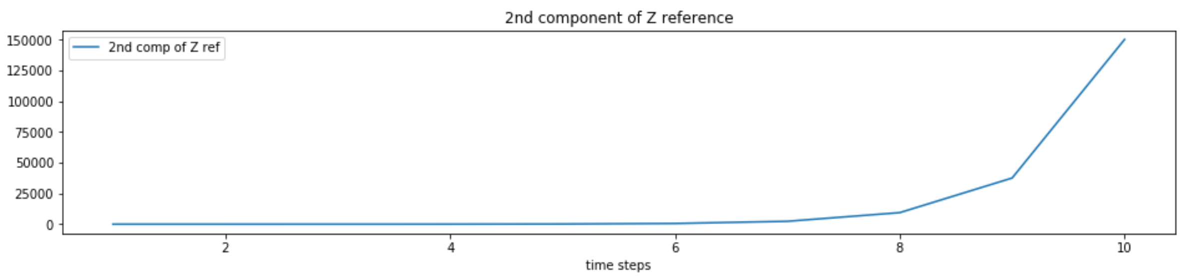


The 2nd component blows up!

In [26]:

```python
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,11), Z_ref_list[:10,1])
ax.set(xlabel='time steps', title='2nd component of Z reference')
ax.legend(['2nd comp of Z ref'])
```

Out[26]: <matplotlib.legend.Legend at 0x203c4b7ae80>



Even after 10 times of recursions, it blows up abruptly!

Input:

$$\begin{pmatrix} 0.5 & 0 \\ 0.2 & 4 \end{pmatrix}^{10}$$

Result:

$$\begin{pmatrix} 0.000976563 & 0 \\ 59918.6 & 1.04858 \times 10^{6} \end{pmatrix}$$

Since the two eigenvalues of A are both positive, this linear state transition matrix is **unstable**.

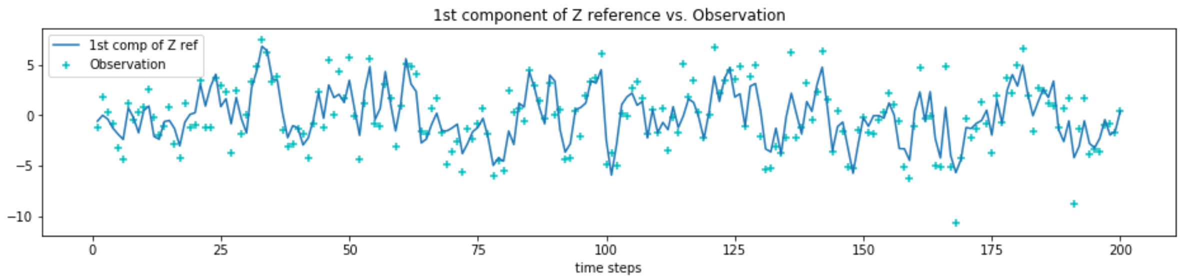
Obstain a obseravation (only for the 1st component of Z ref) at each time step

$$Y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} Z_k + \Xi_k, \quad \Xi_k \sim N(0,3) \tag{1}$$

In [8]:
```
def Observing_Model (Z_ref, R):
    r = np.random.normal(0, np.sqrt(R)) # measurement noise
    Y_obs = np.dot(H, Z_ref) + r
    return Y_obs

def get_Obeservation (Z_ref_list, R):
    for i in range(n_steps):
        Y_obs_list[i] = Observing_Model(Z_ref_list[i], R)
    return Y_obs_list

Y_obs_list = get_Obeservation(Z_ref_list, R)
```

In [9]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), Z_ref_list[:,0])
ax.scatter(np.arange(1,201), Y_obs_list, marker='+', color='c')
ax.set(xlabel='time steps', title='1st component of Z reference vs. Observation')
ax.legend(['1st comp of Z ref','Observation'])
```

Out[9]: `<matplotlib.legend.Legend at 0x293cb2830b8>`



Implement the Prediction step

From the text book:

(i) Set $\overline{z}^0 := \overline{z}^{\mathrm{a}}_{k-1}$, $P^0 := P^{\mathrm{a}}_{k-1}$ and iteratively determine $\overline{z}^{n+1}$ and P^{n+1} for $n = 0, \ldots, N_{\mathrm{out}} - 1$ *via*

$$\overline{z}^{n+1} = [I + \delta t D]\overline{z}^n + \delta t b, \tag{6.10}$$

$$P^{n+1} = [I + \delta t D] P^n [I + \delta t D]^{\mathrm{T}} + 2\delta t Q. \tag{6.11}$$

Set $\overline{z}^{\mathrm{f}}_k := \overline{z}^{N_{\mathrm{out}}}$ and $P^{\mathrm{f}}_k := P^{N_{\mathrm{out}}}$.

- E_previous : mean of the system state at the previous step k-1 (2 X 1 vector)
- P_previous : covariance of the system state at the previous step k-1 (2 x 2 matrix)
- E_forecast : predicted mean of the system state at the current step k (2 X 1 vector)
- P_forecast : predicted covariance of the system state at the current step k (2 x 2 matrix)
- A : transition matrix (2 x 2 matrix)
- Q : noise covariance matrix (2 x 2 matrix)

In [10]:
```
def Prediction (E_previous, P_previous):
    E_forecast = np.dot(A, E_previous)
    P_forecast = np.dot(np.dot(A, P_previous), A.T) + Q
    return E_forecast, P_forecast
```

Implement the Update step

(ii) Compute the Kalman gain matrix

$$K = P_k^{\mathrm{f}} H^{\mathrm{T}} (R + H P_k^{\mathrm{f}} H^{\mathrm{T}})^{-1},$$

and update the mean and covariance matrix according to

$$\bar{z}_k^{\mathrm{a}} := \bar{z}_k^{\mathrm{f}} - K(H\bar{z}_k^{\mathrm{f}} - y_{\mathrm{obs}}(t_k)), \tag{6.12}$$

$$P_k^{\mathrm{a}} := P_k^{\mathrm{f}} - K H P_k^{\mathrm{f}}. \tag{6.13}$$

- E_forecast : mean of prior distribution of Z (2 x 1 vector)
- P_forecast : covariance of prior distribution of Z (2 x 2 matrix)
- K : Kalman gain matrix (2 X 1 vector)
- E_analysis : mean of the postrior distribution of Z (2 x 1 vector)
- P_analysis : covariance of the postrior distribution of Z (2 x 2 matrix)

In [11]:
```
def Assimilation (E_forecast, P_forecast, Y_obs, R):
    K = np.dot((np.dot(P_forecast, H.T)), np.linalg.inv(R + np.dot((np.dot(H, P_forecast)), H.T))
    E_analysis = E_forecast - np.dot(K, (np.dot(H, E_forecast) - Y_obs))
    P_analysis = P_forecast - np.dot(np.dot(K, H), P_forecast)
    return E_analysis, P_analysis
```

Kalman filter

We will do the **update step first** since we have an observation at the first time step (i==0).

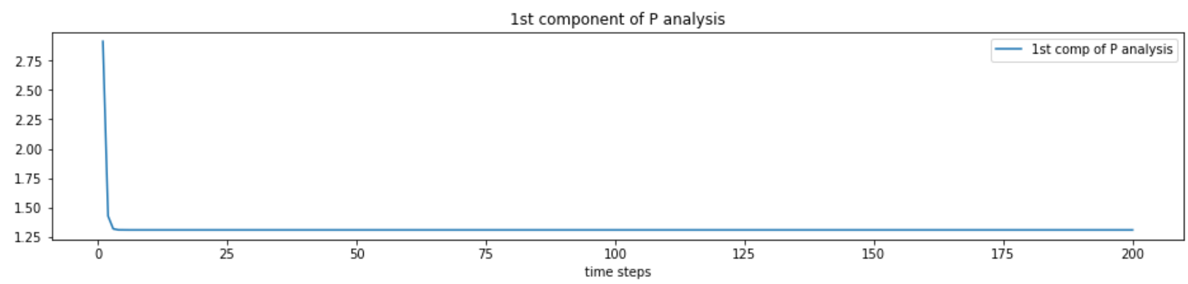
In [12]:
```
def KalmanFilter(E_init, P_init):
    E_forecast, P_forecast = E_init, P_init
    for i in range(n_steps):
        Y_obs = Y_obs_list[i]
        # Update/Assimilation step
        E_analysis, P_analysis = Assimilation (E_forecast, P_forecast, Y_obs, R)
        E_analysis_list[i] = np.reshape(E_analysis, (2,))
        P_analysis_list[i] = np.reshape([P_analysis[0,0], P_analysis[1,1]], (2,))
        E, P = E_analysis, P_analysis
        # Prediction step
        E_forecast, P_forecast = Prediction(E, P)
    return E_analysis_list, P_analysis_list
```

In [13]:
```
E_analysis_list, P_analysis_list  = KalmanFilter(E_init, P_init)
```

Brief view on how the Kalman filter worked

In [14]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), P_analysis_list[:,0])
ax.set(xlabel='time steps', title='1st component of P analysis')
ax.legend(['1st comp of P analysis'])
```

Out[14]: <matplotlib.legend.Legend at 0x293cb847048>

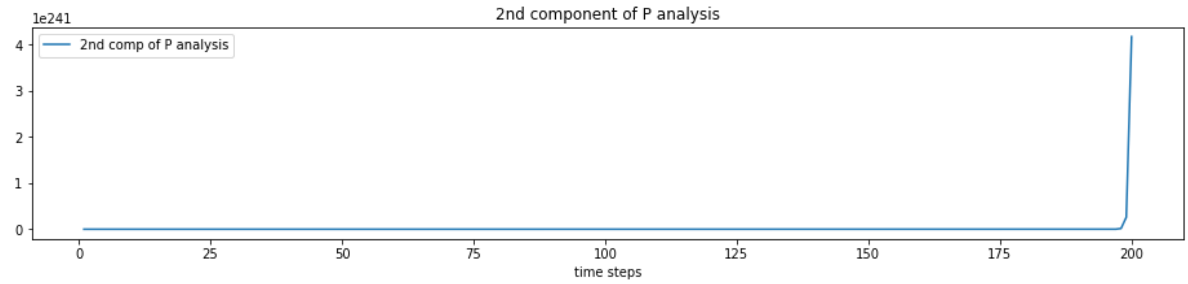


The variance of the first component converges very quickly.

In [15]:

```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), P_analysis_list[:,1])
ax.set(xlabel='time steps', title='2nd component of P analysis')
ax.legend(['2nd comp of P analysis'])
```

Out[15]: `<matplotlib.legend.Legend at 0x293cc8881d0>`

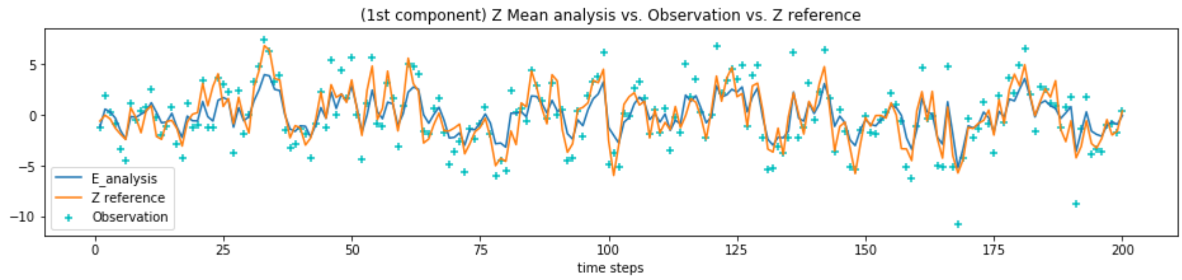


The variance of the second component exponentially increase.

In [16]:

```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), E_analysis_list[:,0], label='E_analysis')
ax.scatter(np.arange(1,201), Y_obs_list, marker='+', color='c', label='Observation')
ax.plot(np.arange(1,201), Z_ref_list[:,0], label='Z reference')
ax.set(xlabel='time steps', title='(1st component) Z Mean analysis vs. Observation vs. Z referenc
ax.legend()
```

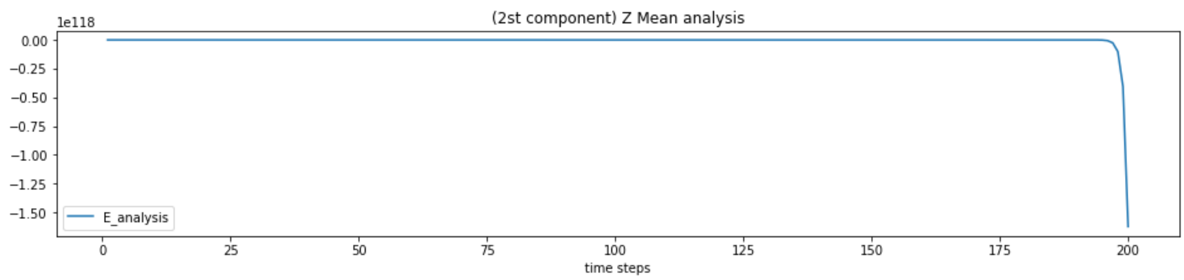
Out[16]: `<matplotlib.legend.Legend at 0x293cc905f60>`



In [17]:

```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), E_analysis_list[:,1], label='E_analysis')
ax.set(xlabel='time steps', title='(2st component) Z Mean analysis') # No observation for the 2nd
ax.legend()
```

Out[17]: `<matplotlib.legend.Legend at 0x293cc96d208>`



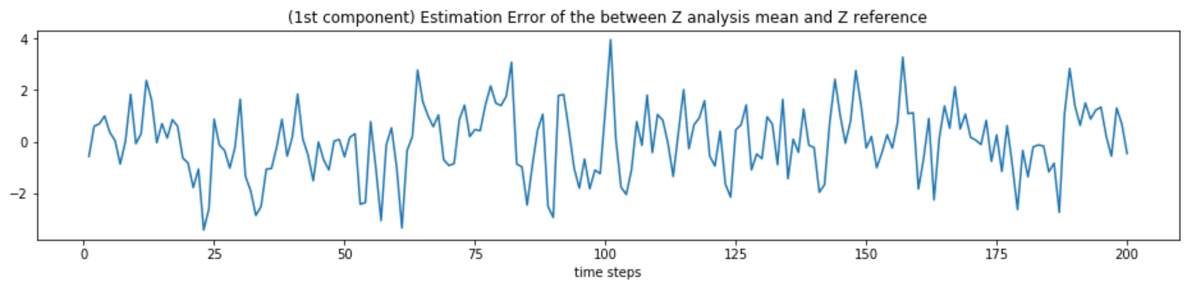
Now let's go through the exercise questions!

i) Produce plots of the error trajectory $\epsilon_k = \bar{z}_k^a - z_k^{ref}$ for at least 200 time steps for both components of the state vector.

In [18]:
```
error_list = E_analysis_list - Z_ref_list
```

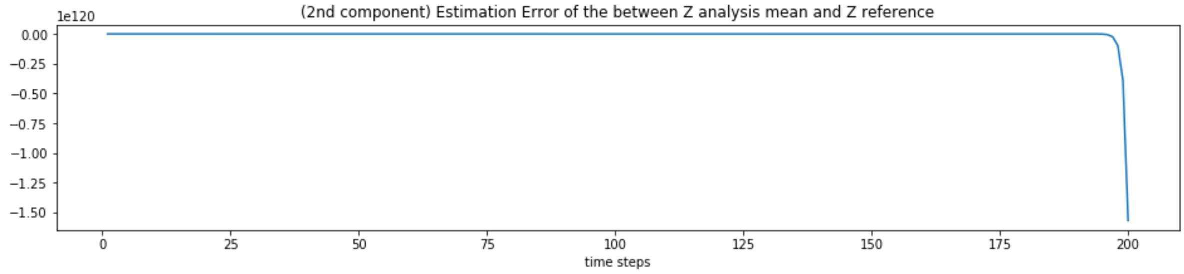
In [19]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), error_list[:,0])
ax.set(xlabel='time steps', title='(1st component) Estimation Error of the between Z analysis mea
print('RMSE : {}'.format(round(np.sqrt(sum(error_list[:,0]**2))/n_steps, 6)))
```

```
RMSE : 0.094675
```



In [20]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), error_list[:,1])
ax.set(xlabel='time steps', title='(2nd component) Estimation Error of the between Z analysis mea
print('RMSE : {}'.format(round(np.sqrt(sum(error_list[:,1]**2))/n_steps, 6)))
```

```
RMSE : 8.095979573969 32e+117
```



ii) What do you notice about ϵ_k for each component? Briefly explain why ϵ_k behaves this way.

- **Estimate error of the first component: the error trajectory stays within certain boundaries.**
- **Estimate error of the second component: exponential growth of errors**
  (As it can be seen previously, the variance of the second component exponentially increases because of the **unstable linear transition matrix**.

iii) Modify the observation setup to improve the performance of the Kalman filter.

Case 1: when R decreases ; we are more confident about the observation.

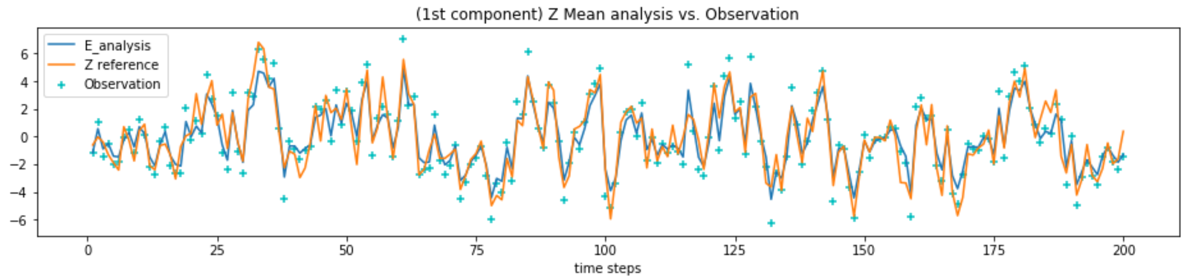
In [21]:
```
R = 1 # previously R=3
Y_obs_list = get_Obeservation(Z_ref_list, R)
E_analysis_list, P_analysis_list  = KalmanFilter(E_init, P_init)
error_list = E_analysis_list - Z_ref_list
```

In [22]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), error_list[:,0])
ax.set(xlabel='time steps', title='(1st component) Error of the between Z analysis mean and Z ref
print('RMSE : {}'.format(round(np.sqrt(sum(error_list[:,0]**2))/n_steps, 6)))
```

```
RMSE : 0.070387
```

In [23]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), E_analysis_list[:,0], label='E_analysis')
ax.scatter(np.arange(1,201), Y_obs_list, marker='+', color='c', label='Observation')
ax.plot(np.arange(1,201), Z_ref_list[:,0], label='Z reference')
ax.set(xlabel='time steps', title='(1st component) Z Mean analysis vs. Observation')
ax.legend()
```

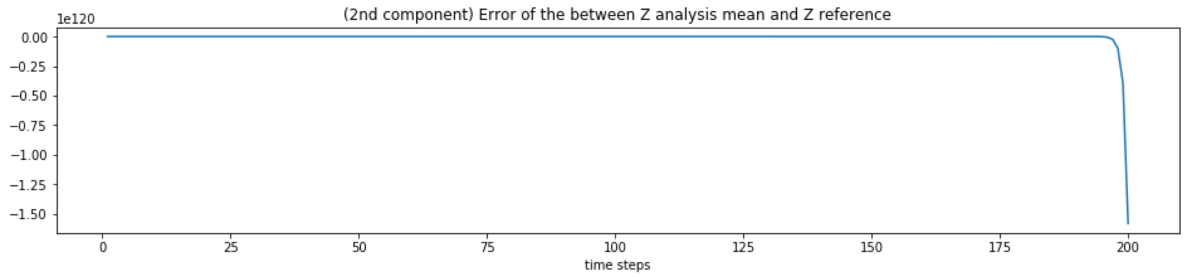
Out[23]: <matplotlib.legend.Legend at 0x293ccb20710>



The estimation errors becomes smaller with a smaller R

In [24]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), error_list[:,1])
ax.set(xlabel='time steps', title='(2nd component) Error of the between Z analysis mean and Z ref
print('RMSE : {}'.format(round(np.sqrt(sum(error_list[:,1]**2))/n_steps, 6)))
```

```
RMSE : 8.166984814708657e+117
```



However, changing R does not affect the second component because we only observe the first component. We need to find another remedy.

Let's inspect further varing R values.

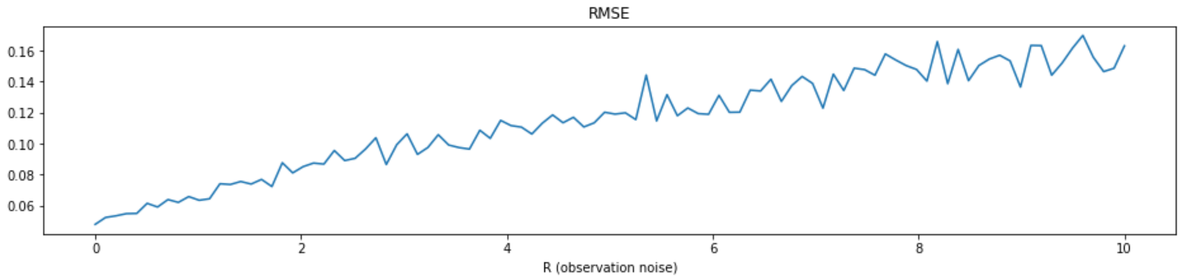
In [25]:

```
def get_RMSE(R_value):
    R = R_value
    Y_obs_list = get_Obeservation(Z_ref_list, R)
    E_analysis_list, P_analysis_list  = KalmanFilter(E_init, P_init)
    error_list = E_analysis_list - Z_ref_list
    return np.sqrt(sum(error_list[:,0]**2))/n_steps

RMSE_list = []
for i in np.linspace(0,10,100):
    RMSE_list.append(get_RMSE(i))
```

In [26]:

```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.linspace(0,10,100), RMSE_list)
ax.set(xlabel='R (observation noise)', title='RMSE')
```

Out[26]: [Text(0.5, 0, 'R (observation noise)'), Text(0.5, 1.0, 'RMSE')]



Decreasing R leads to smaller estimation errors (based on RMSE)

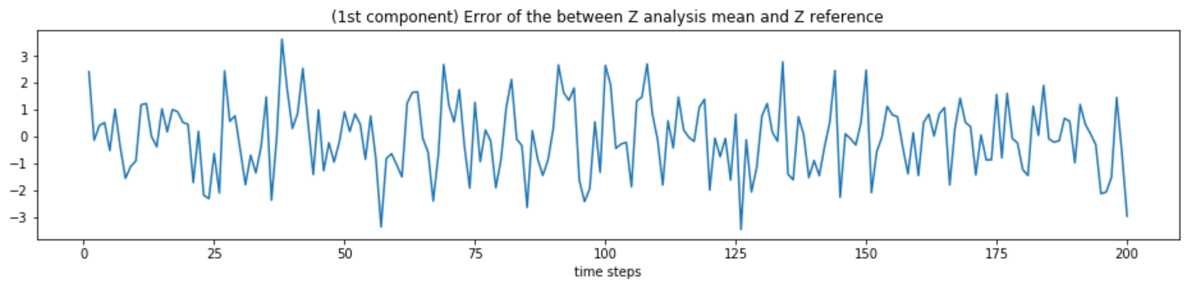
Case 2: When both of the two components are observed (R remained)

In [27]:

```
H = np.eye(2) # identity matrix: np.array([[1,0],[0,1]])
Y_obs_list2 = np.zeros(shape=(n_steps,2))

def Observing_Model2 (Z_ref):
    r1 = np.random.normal(0, np.sqrt(3)) # measurement noise
    r2 = np.random.normal(0, np.sqrt(3))
    Y_obs = np.dot(H, Z_ref) + np.array([[r1,r2]])
    return Y_obs

def get_Obeservation2 (Z_ref_list):
    for i in range(n_steps):
        Y_obs_list2[i,:] = Observing_Model2(Z_ref_list[i])
    return Y_obs_list2

def KalmanFilter2(E_init, P_init):
    E, P = E_init, P_init
    for i in range(n_steps):
        E_forecast, P_forecast = Prediction(E, P)
        Y_obs = Y_obs_list2[i,:]
        E_analysis, P_analysis = Assimilation (E_forecast, P_forecast, Y_obs, R)
        E_analysis_list[i] = np.reshape(np.array([E_analysis[1,0],E_analysis[0,1]]), (2,))
        P_analysis_list[i] = np.reshape([P_analysis[0,0], P_analysis[1,1]], (2,))
        E, P = E_analysis, P_analysis
    return E_analysis_list, P_analysis_list

Y_obs_list2 = get_Obeservation2(Z_ref_list)
E_analysis_list, P_analysis_list  = KalmanFilter2(E_init, P_init)
error_list = E_analysis_list - Z_ref_list
```

In [28]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), error_list[:,0])
ax.set(xlabel='time steps', title='(1st component) Error of the between Z analysis mean and Z ref
print('RMSE : {}'.format(round(np.sqrt(sum(error_list[:,0]**2))/n_steps, 6)))
```

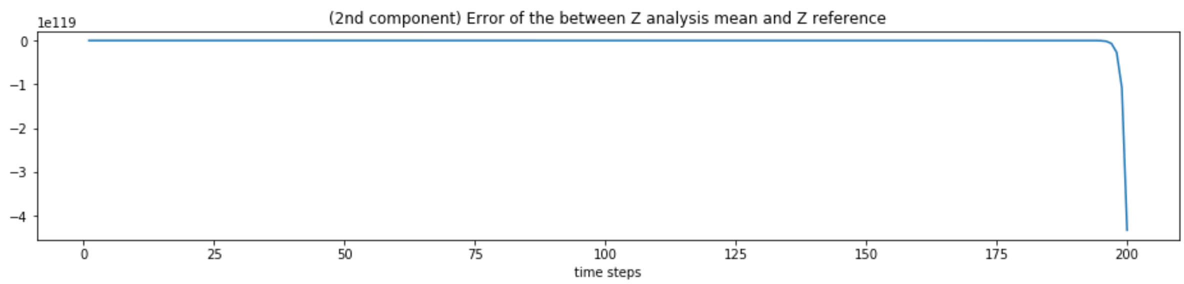
```
RMSE : 0.093722
```



It is obvious that observing the additional component (the second component) is not helpful for estimating the first component.

In [29]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), error_list[:,1])
ax.set(xlabel='time steps', title='(2nd component) Error of the between Z analysis mean and Z ref
print('RMSE : {}'.format(round(np.sqrt(sum(error_list[:,1]**2))/n_steps, 6)))
```

```
RMSE : 2.234858817807371e+117
```



On the other hand, observing the second component, to some extent, reduces the estimation error for the second component.

In [30]:
```
fig, ax = plt.subplots(figsize=(16,3))
ax.plot(np.arange(1,201), E_analysis_list[:,1], label='E_analysis')
ax.plot(np.arange(1,201), Y_obs_list2[:,1], label='Observation')
ax.set(xlabel='time steps', title='(2st component) Z Mean analysis vs. Observation')
ax.legend()
```

Out[30]: `<matplotlib.legend.Legend at 0x293ccb07b00>`