Let X1, ..., Xn be independent and identically U[0, θ] distributed random variables. Show that

$$\prod_{i=1}^{n} E(X_1^{1/n})$$

is asymptotically unbiased and consistent for γ(θ) = θe-1.

Solution:

Since *X1*, *X2*, ..., *Xn* $\overset{i.i.d.}{\sim} U[0,1]$, which density functionis:

$$f_{X_i}(x) \;=\; \frac{1}{\theta} \prod_{x\,\in[0,\theta]} .$$

$\prod_{x\,\in[0,\theta]}$. Is the Dirac function, whose value is 1 if $x \in [0,\;\theta]$ and otherwise *0*. Then we have jointly density function:

$f_{X1,......,Xn}(x1, \dots\dots, xn) \;=\; \left(\frac{1}{\theta}\right)^n \prod_{\{x\,\in[0,\theta],\forall 1\le i\le n\}}$.

Compute the expectation of $(\prod_i^n X_i)^{\frac{1}{n}}$:

$$E\left[\left(\prod_i^n X_i\right)^{\frac{1}{n}}\right] = \int_0^{\theta^n} \left(\prod_i^n X_i\right)^{\frac{1}{n}} \left(\frac{1}{\theta}\right)^n \prod_{\{x\,\in[0,\theta],\forall 1\le i\le n\}} dx$$

$$= \int_0^{\theta^n} \prod_i^n \frac{1}{\theta} (X_i)^{\frac{1}{n}} dx$$

$$\int_0^{\theta} \frac{(x)^{\frac{1}{n}}}{\theta} dx = \frac{1}{\theta}\int_0^{\theta} (x)^{\frac{1}{n}}\, dx = \frac{1}{\theta}.\left[\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}\right]_0^{\theta} = \frac{1}{\theta}.\left[\frac{x^{\frac{1}{\theta}+1}}{\frac{1}{n}+1} - 0\right] = \frac{1}{\theta}.\frac{\theta^{\frac{1+n}{n}}}{\frac{1+n}{n}} = \frac{n\theta^{\frac{1+n}{n}}}{\theta(1+n)}$$

$$= \frac{n\theta}{(1+n)} = \frac{\theta^{\frac{1}{n}}}{1+\frac{1}{n}}$$

$$= \frac{\theta}{\left(1+\frac{1}{n}\right)^n} \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (1)$$

Asymptotic unbiasedness means that the estimator’s bias decrease until it reaches zero, as n → ∞, implying that the estimator’s expected value converges the right value of the parameter [1].

$(\prod_i^n X_i)^{\frac{1}{n}}$is said to asymptically unbiased if bias $\left((\prod_i^n X_i)^{\frac{1}{n}}\right) = E(\prod_i^n X_i)^{\frac{1}{n}} - \theta \to 0$ and $n \to \infty$.

Now, from (1)

$$bias\left(\left(\prod_i^n X_i\right)^{\frac{1}{n}}\right) = \frac{\theta}{\left(1+\frac{1}{n}\right)^n} - \theta = \frac{\theta}{(1+n)} \to 0 \text{ and } n \to \infty$$

So, $(\prod_i^n X_i)^{\frac{1}{n}}$ is asymptotically unbiased.

Take the limit (Both numerator and denominator have limits and $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n \neq 0$:

$$\lim_{n\to\infty}\frac{\theta}{\left(1+\frac{1}{n}\right)^n} = \frac{\theta}{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n} = \frac{\theta}{\lim\limits_{n\to\infty}\left(exp\left(n.log\left(1+\frac{1}{n}\right)\right)\right)}$$

$$= \frac{\theta}{exp\left(\lim\limits_{n\to\infty}\left(n\,log\left(1+\frac{1}{n}\right)\right)\right)} = \frac{\theta}{exp\left(\lim\limits_{n\to\infty}\left(\frac{log\left(1+\frac{1}{n}\right)}{1/n}\right)\right)}$$

$$= \frac{\theta}{exp\left(\frac{1}{\lim\limits_{n\to\infty}\left(1+\frac{1}{x}\right)}\right)} = \frac{\theta}{exp\left(\lim\limits_{n\to\infty}\left(\frac{1}{n}\right)+1\right)} = \frac{\theta}{exp\frac{1}{\frac{1}{\lim\limits_{n\to\infty} n}+1}} = \frac{\theta}{e^{\frac{1}{\frac{1}{\infty}+1}}} = \frac{\theta}{e}$$

$$= \theta e^{-1}$$

Consistent means that for high sample sizes as n → ∞, the likelihood that $\gamma(\theta)$ deviates from the real, but unknown, value becomes negligible [1].

So, $(\prod_i^n X_i)^{\frac{1}{n}}$ is a consistent for $\gamma(\theta) = \theta e^{-1}$

Reference:

[1]. https://stats.stackexchange.com/questions/280684/intuitive-understanding-of-the-difference-between-consistent-and-asymptotically