Exercise 1:

1. $Ker(A) = \langle v_{r+1}, \dots, v_n \rangle$

For $(\subseteq)$ the Singular value decomposition (SVD) can easily write $Av_i = 0$ for $i = r+1, \dots n$. This proves immediately that $v_i \in ker(A)$, for $i = r+1, \dots, n$, Since $Ker(A)$ is vector subspace of $\mathbb{R}^m$, any linear combination of $v_{r+1}, \dots, v_n$ is in $Ker(A)$. Hence $\langle v_{r+1}, \dots v_n \rangle \subseteq Ker(A)$.

For $(\supseteq)$,

$$x \in Ker(A) \Leftrightarrow \|Ax\|_2 = 0 \Leftrightarrow \|U\Sigma V^T x\|_2 = 0$$

$$\Leftrightarrow \|\Sigma V^T x\|_2 = 0 \Leftrightarrow \|\Sigma y\|_2 = 0$$

where $y = V^T x$

$$\Leftrightarrow y = (0, \dots, 0, y_{r+1}, \dots, y_n)^T.$$

where, $y = V^T x$

$$\Leftrightarrow x = Vy, \; y = (0, \dots, 0, y_{r+1}, \dots y_n)^T$$

$$\Leftrightarrow x = \sum_{i=r+1}^{n} y_i v_i$$

$$\Leftrightarrow x \in \langle v_{r+1}, \dots, v_n \rangle$$

This proves that $\langle v_{r+1}, \dots, v_n \rangle \supseteq Ker(A)$

2. $Im(A) = \langle u_1, \ldots, u_r \rangle$

For ($\subseteq$) the SVD can easily write $Av_i = \sigma_i u_i$ for $i = 1 \ldots r$

This proves immediately that $u_i \in Im(A)$ for, $i = 1, \ldots, r$.

Since $Im(A)$ is a vector subspace of $\mathbb{R}^m$, any linear combination of $u_1, \ldots, u_r$ is in $Im(A)$. Hence $\langle u_1, \ldots, u_r \rangle \subseteq Im(A)$

For ($\supseteq$),

$$
\begin{aligned}
y \in Im(A) &\Leftrightarrow \exists x \text{ such that } y = Ax \\
&\Leftrightarrow y = U \Sigma V^* x \\
&\Leftrightarrow y = U \Sigma z, \text{ where, } z = V^* x \\
&\Leftrightarrow y = U(\sigma_1 z_1, \ldots, \sigma_r z_r, 0, \ldots 0)^* \\
&\Leftrightarrow y = \sum_{i=1}^{r} (\sigma_i z_i) u_i \\
&\Leftrightarrow y = \langle u_1, \ldots, u_r \rangle
\end{aligned}
$$

This proves that $\langle u_1, \ldots u_r \rangle \supseteq Im(A)$.