Exercise 3:

The probability mass function of the multinomial distribution is:

$$P(X_1 = x_1, X_2 = x_2 \cdots X_k = x_k) = \frac{n!}{x_1! x_2! \cdots x_k!} \times p_1^{x_1} \times p_2^{x_2} \cdots \times p_k^{x_k}$$

Here, $n = \sum_{i=1}^{k} x_i$

and $\sum_{i=1}^{k} p_i = 1$.

The likelihood function is:

$$P(x, y, z \mid p) = \binom{x+y+z}{x} \times (1-p)^{2x} \times \binom{y+z}{y} \times (2p(1-p))^{y} \times \binom{z}{z} \times p^{2z}.$$

Taking log-likelihood is given by

$$\log(P(x, y, z \mid p)) = \log\left[\binom{x+y+z}{x} \times (1-p)^{2x} \times \binom{y+z}{y} \times (2p(1-p))^{y} \times \binom{z}{z} \times p^{2z}\right]$$

$$= 2x \log(1-p) + \log\binom{x+y+z}{x} + y \log 2p + y \log(1-p) + \log\binom{y+z}{y} + 2z \log(p) + \log\binom{z}{z}$$

Taking the derivative w.r.t. p

$$\frac{\partial(\log(P(x, y, z \mid p)))}{\partial p} = -\frac{2x}{1-p} + \frac{y}{2p} \cdot 2 - \frac{y}{1-p} + \frac{2z}{p}$$

$$= \frac{y+2z}{p} - \frac{y+2x}{1-p}$$

Set the derivative equal to 0.

$$\frac{y+2z}{p} - \frac{y+2x}{1-p} = 0$$

$$\Rightarrow \quad \frac{y+2z-yp-2pz-yp-2xp}{p(1-p)} = 0$$

$$\Rightarrow \quad y+2z - p(2y+2x+2z) = 0$$

$$\therefore \quad \hat{p}_{MLE} = \frac{y+2z}{2x+2y+2z}$$

Taking 2nd derivative,

$$-\frac{y+2z}{p^2} - \frac{y+2x}{(1-p)^2}$$

which is <0, because here x, y, z are positive and the value of p is $0<p<1$.

So, $\hat{p}_{MLE}$ maximizes the likelihood.