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Possion Distribution, $f_\theta(x) = \frac{\theta^x}{x!} e^{-\theta}$

The likelihood function,

$$L(\theta; x_1 \ldots x_n) = \prod_{i=1}^{n} \frac{\theta^{x_i}}{x_i!} e^{-\theta}$$

Taking the log likelihood function

$$L(\theta; x_1 \ldots x_n) = \log\left(\prod_{i=1}^{n} \frac{\theta^{x_i}}{x_i!} e^{-\theta}\right)$$

$$= \sum_{i=1}^{n} \text{Log}\left(\frac{\theta^{x_i}}{x_i!} e^{-\theta}\right)$$

$$= \sum_{i=1}^{n} \left[\log \theta^{x_i} + \log(e^{-\theta}) - \log(x_i!)\right]$$

$$= -\theta n + \log\theta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log x_i!$$

Derivative w.r.to. θ.

$$\frac{\partial}{\partial\theta}\left[L(\theta; x_1 \ldots x_n)\right] = \frac{\partial}{\partial\theta}\left[-\theta n + \log\theta \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \log x_i!\right]$$

$$= -n + \frac{1}{\theta}\sum_{i=1}^{n} x_i - 0$$

Set the derivative equal to 0 and solve for θ.

$$-n + \frac{1}{\theta}\sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

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$$\mathbb{E}[x] = \sum_{x=0}^{\alpha} x\, p(x)$$

$$= \sum_{x=0}^{\alpha} x \cdot e^{-\theta} \cdot \frac{\theta^x}{x!}$$

$$= e^{-\theta} \sum_{x=0}^{\alpha} \frac{x\,\theta^x}{x!}$$

$$= \theta \cdot e^{-\theta} \sum_{x=1}^{\alpha} \frac{\theta^{x-1}}{(x-1)!}$$

$$= \theta \cdot e^{-\theta} \sum_{j=0}^{\alpha} \frac{\theta^j}{j!} \quad [\, x-1 = j \,]$$

$$= \theta \cdot e^{-\theta} \cdot e^{\theta} \quad \left[\sum_{j=1}^{\alpha} \frac{x^k}{k!} = e^x \right]$$

$$= \theta .$$

Now, $$\mathbb{E}[\hat{\theta}_{MLE}] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} x_i \right]$$

$$= \frac{1}{n}\, \mathbb{E}\left[\sum_{i=1}^{n} x_i \right]$$

$$= \frac{1}{n} \cdot \sum_{i=1}^{n} \mathbb{E}[x_i]$$

$$= \frac{1}{n} \cdot n\theta \quad [\, \mathbb{E}[x] = \theta \,]$$

$$= \theta .$$

So, estimate is unbiased.

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A estimator is consistent if $\lim\limits_{n\to\alpha} \hat{\theta} = \theta$

we know, $\mathbb{E}[(x-\mu)^2] = var(x)$

So, $\mathbb{E}[(\hat{\theta}-\theta)^2] = var(\hat{\theta})$

$$= var\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n^2} var\left(\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}[var(x_i)]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left[\mathbb{E}(x_i)^2 - (\mathbb{E}(x_i))^2\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\left[\theta^2+\theta-\theta^2\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\theta$$

$$= \frac{1}{n^2}\cdot n\theta$$

$$= \theta/n$$

Now, if $n\to\alpha$ then,

$\mathbb{E}[(\hat{\theta}-\theta)^2] \to 0$

Therefore, $\hat{\theta} \xrightarrow{P} \theta$

i.e. $\hat{\theta}_n$ is consistent.