Example Solution.

In order for $\hat{m}$ to be unbiased,

$$\begin{aligned} E[\hat{m}] &= E[\alpha X + \beta Y] \\ &= \alpha E[X] + \beta E[Y] \\ &= \alpha \cdot 0.8m + \beta \cdot m \\ &= (0.8\alpha + \beta)\, m \end{aligned}$$

$\Rightarrow$ It must hold : $0.8\alpha + \beta = 1$

In order for $Var(\hat{m})$ to be minimum,

$$\begin{aligned} Var(\hat{m}) &= Var(\alpha X + \beta Y) \\ &= \alpha^2 Var(X) + \beta^2 Var(Y) \\ &= \alpha^2 m^2 + (1-0.8\alpha)^2 (1.5m^2) \\ &= (\alpha^2 + (1-0.8\alpha)^2 (1.5))\, m^2 \\ &= \underbrace{(1.96\alpha^2 - 2.4\alpha + 1.5)}_{:= q}\, m^2 \end{aligned}$$

Since q is a convex funtion in the quadratic form, q is minimized at $\alpha = \left(\frac{2.4}{1.96}\right)\frac{1}{2} = 0.6122$

$$\Rightarrow \quad \beta = 1 - 0.8(0.6122) = 0.5102$$