

Exercise 1:

For general case:

T is sufficient for θ if $P(X=x | T=t)$ is independent of θ .

An estimator T is sufficient for θ iff there exist functions $g(t|\theta)$ and $h(x)$ such that

$$f(x|\theta) = g(T(x)|\theta) \cdot h(x)$$

$$\begin{aligned} P(X=x|\theta) &= P(X=x \text{ and } T=t|\theta) \\ &= P(X=x | T=t, \theta) \cdot P(T=t|\theta) \\ &= \underbrace{P(X=x | T=t)}_{h(x)} \cdot \underbrace{P(T=t|\theta)}_{g(t|\theta)} \end{aligned}$$

Here $h(x)$ and $g(t|\theta)$ are non negative, the function h may depend on x but doesn't depend on θ , and the function g depends on θ but will depend on the observed value x only through the value of statistics $T(x)$

$$\text{Now, } P(X=x | T=t, \theta) = \frac{P(X=x \text{ and } T=t|\theta)}{P(T=t|\theta)}$$

$$\begin{aligned} P(X=x \text{ and } T=t|\theta) &= P(X=x|\theta) \\ &= g(T|\theta) \cdot h(x) \end{aligned}$$

$$\begin{aligned}
P(T=t|\theta) &= \sum_{x: T(x)=t} P(X=x \text{ and } T=t|\theta) \\
&= \sum_{x: T(x)=t} g(T=t|\theta) \cdot h(x) \\
&= g(T=t|\theta) \cdot \sum_{x: T(x)=t} h(x)
\end{aligned}$$

$$\begin{aligned}
\text{So, } P(X=x|T=t, \theta) &= \frac{g(T=t|\theta) \cdot h(x)}{g(T=t|\theta) \left[\sum_{x: T(x)=t} h(x) \right]} \\
&= \frac{h(x)}{\sum_{x: T(x)=t} h(x)}
\end{aligned}$$

Which doesn't depend on θ and therefore T is a sufficient statistic.

Now for our case,

$x_i > i\theta$ is equivalent to $x_i/i \geq \theta$.

we can rewrite the density of the i -th observation as

$$f(x_i; \theta) = \exp(i\theta - x_i) I_{[\theta, \infty)}(x_i/i)$$

The joint density of the random sample
then is

$$\begin{aligned}
 f(x; \theta) &= \prod_{i=1}^n f(x_i) \\
 &= \exp \left\{ \theta \left(\sum_{i=1}^n i \right) - n\bar{x} \right\} \prod_{i=1}^n I_{[\theta, \infty)}(x_i/i) \\
 &= \underbrace{\exp \left\{ \theta \cdot \frac{n(n+1)}{2} \right\} I_{[\theta, \infty)}(\min_i(x_i/i))}_{g(t; \theta)} \cdot \underbrace{\exp \{-n\bar{x}\}}_{h(x)} \quad \checkmark
 \end{aligned}$$

Here $x_i \geq t, i = 1, \dots, n$

$$\begin{aligned}
 \text{Now, } P(X=x | T=t, \theta) &= \frac{h(x)}{\sum_{x: T(x)=t} h(x)} \\
 &= \frac{\exp \{-n\bar{x}\}}{\sum_{x: T(x)=t} \exp \{-n\bar{x}\}}
 \end{aligned}$$

The result does not depend on θ for
each fixed $t = \min(\frac{x_i}{t})$.

So, $T((X_1, X_2, \dots, X_n) = \min_i \left(\frac{x_i}{t} \right)$ is
sufficient statistics for θ . ✓

