Example Solution

By Neyman-Pearson theorem,

$$\frac{L(x;\mu=1)}{L(x;\mu=0)} = \frac{\cancel{(2\pi)^{-n/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-1)^2\right\}}{\cancel{(2\pi)^{-n/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right\}}$$

$$= \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(x_i-1)^2 + \frac{1}{2}\sum_{i=1}^{n} x_i^2\right\}$$

$$= \exp\left\{\cancel{\left(-\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)} + \left(\sum_{i=1}^{n} x_i\right) + \left(-\frac{1}{2}n\right) + \cancel{\left(\frac{1}{2}\sum_{i=1}^{n} x_i^2\right)}\right\}$$

$$= \exp\left\{\left(\sum_{i=1}^{n} x_i\right) + \left(-\frac{1}{2}n\right)\right\} > K$$

$$\alpha = \mathbb{P}\left(\exp\left\{\left(\sum_{i=1}^{n} x_i\right) + \left(-\frac{1}{2}n\right)\right\} > K \;\middle|\; \mu=0\right)$$

$$= \mathbb{P}\left(n\bar{X} - \frac{1}{2}n > \ln K \;\middle|\; \mu=0\right)$$

$$= \mathbb{P}\left(\bar{X} - \frac{1}{2} > \frac{\ln K}{n} \;\middle|\; \mu=0\right)$$

$$= \mathbb{P}\left(\bar{X} > \frac{\ln K}{n} + \frac{1}{2} \;\middle|\; \mu=0\right)$$

(using CLT)

$$= \mathbb{P}\left(\frac{\bar{X}-0}{\sqrt{\frac{1}{n}}} > \frac{\left(\frac{\ln K}{n}+\frac{1}{2}\right)-0}{\sqrt{\frac{1}{n}}} \;\middle|\; \mu=0\right)$$

$$= \mathbb{P}\left(Z > \underbrace{\left(\ln K + \frac{1}{2}\right)\sqrt{n}}_{:=C} \;\middle|\; Z\sim(0,1)\right)$$

$$= 1 - \Phi(C) = 0.05$$

$\Rightarrow \Phi(C) = 0.95$

using Z-table

$\Rightarrow C = \Phi^{-1}(0.95) \approx 1.64 = \left(\frac{\ln K}{n} + \frac{1}{2}\right)\sqrt{n}$

$\Rightarrow \left(\frac{\ln K}{n} + \frac{1}{2}\right) \approx \frac{1.64}{\sqrt{n}}$

$\Rightarrow$ The most powerful test : reject H_0 if

$\mathbb{P}\left(\bar{X} > \frac{1.64}{\sqrt{n}}\right)$