In-Class Exercise Sheet

- Do not write with red pen or pencil.
- Round to 4 decimal places in all computation.

Exercice 1: K-means clustering

Let $x^{(1)}, x^{(2)}, \dots, x^{(N)} \in \mathbb{R}^p$ be a data point and $\theta_1, \theta_2, \dots, \theta_K \in \mathbb{R}^p$ a centroid of a cluster $1, 2, \dots, K$. The objective function of K-means can be defined as follow:

$$\mathcal{L}(\mathbf{\Gamma}, \mathbf{\Theta}) = \sum_{n=1}^{N} \sum_{k=1}^{K} \gamma_{kn} ||x^{(n)} - \theta_k||$$

$$\text{where } \gamma_{kn} = \begin{cases} 1 & \text{if} \quad x^{(n)} \text{ is assigned to the cluster } k \\ 0 & \text{otherwise} \end{cases}$$

Solve the optimization problem in K-means for $\mathbf{\Theta} \in \mathbb{R}^{K \times P}$ given fixed $\mathbf{\Gamma} \in \mathbb{R}^{K \times N}$ and comment on the results you have obtained.

Exercice 2: Neyman-Pearson theorem

Let $x_1, x_2, \dots, x_n$ is an i.i.d. sample from $\mathcal{N}(\mu, 1)$ and consider a test with hypothesis

$$\mathsf{H}_0 : \mu = 0 \text{ vs. } \mathsf{H}_1 : \mu = 1$$

at confidence level $\alpha = 0.05$. Find the test that minimizes type 2 error based on Neyman-Pearson lemma. Explicitly compute using Z table provided.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

Exercice 3: Maximum Likelihood Estimation

Suppose $X_1, X_2, \dots, X_n$ is an i.i.d. sample from a distribution with PDF

$$f_X(x) = \frac{x}{\sigma^3} exp\left(-\frac{x^2}{2\sigma^3}\right) \quad \text{for } x > 0$$

(i) Construct the likelihood function of σ^3.
(ii) Determine the ML estimator of σ^3.
(iii) Determine the ML estimator of σ.
(iv) Examine the unbiasedness of the ML estimator of σ^3.
(v) Examine the consistency of the ML estimator of σ^3.

Exercice 4: Bayesian statistics

Hani is interested in whether the students are doing well or lost in her tutorial. Based on her first impression:

- 75% of the students are doing well in her tutorial
- 25% of the students are lost in her tutorial.

Suppose that the student who is doing well is likely to ask:

- no question with probability of 0.7
- only one question with probability of 0.2
- multiple questions with probability of 0.1

in her tutorial. On the other hand, the student who is lost is likely to ask:

- no question with probability of 0.5
- only one question with probability of 0.3
- multiple questions with probability of 0.2

in her tutorial. Determine the updated belief on whether the students are doing well or lost in Hani's tutorial after Hani has observed:

- the first student asked no question
- the second student asked only 1 question

Exercice 5: Minimum-Variance Unbiased Estimator (MVUE)

Suppose there are two instruments for measuring a distance. The independent random variables X and Y represent:

- X := the measurement with the instrument 1
- Y := the measurement with the instrument 2

and also,

- $E[\mathsf{X}] = 0.8m$, $Var(\mathsf{X}) = m^2$
- $E[\mathsf{Y}] = m$, $Var(\mathsf{Y}) = 1.5m^2$

where m is the true distance. Consider the estimator of m within the class of $\hat{m} = \alpha\mathsf{X} + \beta\mathsf{Y}$. Determine α and β that make $\hat{m}$ the MVUE.

Exercice 6: Singular Value Decomposition

Find singular value of the matrix X = $\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$.