Exercise 1:

For general case:

T is sufficient for θ if $P(X=x \mid T=t)$ is independent of θ.

An estimator T is sufficient for θ iff there exist functions $g(t \mid \theta)$ and $h(x)$ such that

$$f(X \mid \theta) = g(T(x) \mid \theta) \cdot h(x)$$

$$P(X=x \mid \theta) = P(X=x \text{ and } T=t \mid \theta)$$
$$= P(X=x \mid T=t, \theta) \cdot P(T=t \mid \theta)$$
$$= \underbrace{P(X=x \mid T=t)}_{h(x)} \cdot \underbrace{P(T=t \mid \theta)}_{g(t \mid \theta)}$$

Here $h(x)$ and $g(t \mid \theta)$ are non negative, the function h may depend on x but doesn't depend on θ, and the function g depends on θ but will depend on the observed value x only through the value of statistics $T(x)$

Now, $P(X=x \mid T=t, \theta) = \dfrac{P(X=x \text{ and } T=t \mid \theta)}{P(T=t \mid \theta)}$

$$P(X=x \text{ and } T=t \mid \theta) = P(X=x \mid \theta)$$
$$= g(T \mid \theta) \cdot h(x)$$

$$P(T=t|\theta) = \sum_{x:T(x)=t} P(X=x \text{ and } T=t|\theta)$$

$$= \sum_{x:T(x)=t} g(T=t|\theta) \cdot h(x)$$

$$= g(T=t|\theta) \cdot \sum_{x:T(x)=t} h(x)$$

So,

$$P(X=x|T=t,\theta) = \frac{g(T|\cancel{\theta}) \cdot h(x)}{g(T=\cancel{t}|\theta)\left[\sum_{x:T(x)=t} h(x)\right]}$$

$$= \frac{h(x)}{\sum_{x:T(x)=t} h(x)}$$

which doesn't depend on θ and therefore T is a sufficient statistic.

Now for our case,

$x_i \geq i\theta$ is equivalent to $x_i/i \geq \theta$.

we can rewrite the density of the i-th observation as

$$f(x_i;\theta) = \exp(i\theta - x_i)\, I_{[\theta,\infty)}(x_i/i)$$

The Joint density of the random sample then is

$$f(x_i|\theta) = \prod_{i=1}^{n} f(x_i)$$

$$= \exp\left\{\theta\left(\sum_{i=1}^{n} i\right) - n\bar{x}\right\} \prod_{i=1}^{n} I_{[\theta,\infty)}(x_i/i)$$

$$= \underbrace{\exp\left\{\theta \cdot \frac{n(n+1)}{2}\right\} I_{[\theta,\infty)}\left(\min_i (x_i/i)\right)}_{g(t|\theta)} \cdot \underbrace{\exp\{-n\bar{x}\}}_{h(x)}.$$

Here $x_i \geq t$, $i = 1, \ldots, n$

Now, $$P(X = x | T = t, \theta) = \frac{h(x)}{\sum_{x: T(x) = t} h(x)}$$

$$= \frac{\exp\{-n\bar{x}\}}{\sum_{x: T(x) = t} \exp\{-n\bar{x}\}}$$

The result doesnot depend on θ for each fixed $t = \min(\frac{x_i}{t})$.

So, $T((X_1, X_2 \ldots, X_n) = \min_i \left(\frac{x_i}{t}\right)$ is sufficient statistics for θ. ∎