Exercise 3: Let X_1, X_2, X_3, X_4 be a sample from U(0, 1), and let $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}$ be the order statistic. Determine the density of $X_{(1)}, X_{(2)}, X_{(3)}, X_{(4)}$.

Solution:

For $X_1, X_2, \ldots, X_n$ iid continuous random variables with pdf f and cdf F the density of the k^{th} order statistic is

$f_k(x)\, dx = P(X_{(k)} \in dx)$

$= P$ (One of the X's $\in$ dx, k -1 of the others < x)

$= n * P\,(X_1 \in dx$, (k-1) others (exactly) < x)

$$= n * P\,(X_1 \in dx)\left(\binom{n-1}{k-1} (F(x))^{(k-1)} * (1 - F(x))^{(n-k)}\right)$$

$$= n * f(x)\, dx * \left(\binom{n-1}{k-1} * (F(x))^{(k-1)} * (1 - F(x))^{(n-k)}\right)$$

$$= \frac{n!}{(n-k)! * (k-1)!} * f(x)\, dx * (F(x))^{(k-1)} * (1 - F(x))^{(n-k)})$$

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim} U(0,1)$ then the density of X(n) is given by [1]:

$$f_k(x) = \frac{n!}{(n-k)! * (k-1)!} * f(x) * (F(x))^{(k-1)} * (1 - F(x))^{(n-k)})$$

$$= \begin{cases} \frac{n!}{(n-k)! * (k-1)!} * x^{k-1} * (1-x)^{n-k}; & if\ 0 < x < 1 \\ 0; & otherwise \end{cases}$$

Density of $X_{(1)} = n * (1 - x)^{(n-1)}$

Density of $X_{(2)} = n * (n - 1) * x * (1 - x)^{(n-2)}$

Density of $X_{(3)} = \frac{n * (n-1) * (n-2)}{2} * x^2 * (1 - x)^{(n-3)}$

Density of $X_{(4)} = \frac{n * (n-1) * (n-2) * (n-3)}{6} * x^3 * (1 - x)^{(n-4)}$

You were supposed to plug n=3 and derive the exact PDFs.

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Reference:

[1] https://www2.stat.duke.edu/courses/Spring12/sta104.1/Lectures/Lec15.pdf

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