Q. Which estimator is better?

① $\bar{X}$ ② $\dfrac{2X_1 - X_2 + X_3}{2}$

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$n > 3$

Solution.

We know that $\bar{X}$ is an unbiased estimator for μ.

(On the exam, you need to prove this as well)

Let's check if ② is also an unbiased estimator.

$$E\left[\frac{2X_1 - X_2 + X_3}{2}\right] = \mu$$

(by simple calculation)

(Again)

Both ① and ② are unbiased estimators for μ.

Although, unbiasness is not the only measure to evalutate estimators.

Let's check the variance of the estimators.

$$\mathrm{Var}(\bar{X}) = \mathrm{Var}\left(\frac{\sum X_i}{n}\right) =$$

$$= \frac{1}{n^2}\mathrm{Var}(\sum X_i)$$

$$= \frac{1}{n^2} n \,\mathrm{Var}(X_1)$$ (since X_i are independent)

$$= \frac{1}{n}\mathrm{Var}(X_1)$$

$$= \frac{1}{n}\sigma^2$$

$$\mathrm{Var}\left(\frac{2X_1 - X_2 + X_3}{2}\right) = \frac{1}{2^2}\mathrm{Var}(2X_1 - X_2 + X_3)$$

$$= \frac{1}{4}\left\{\mathrm{Var}(2X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3)\right\}$$

$$= \frac{1}{4}\left\{4\sigma^2 + \sigma^2 + \sigma^2\right\}$$

$$= \frac{3}{2}\sigma^2$$

Since $\frac{\sigma^2}{n} < \frac{3}{2}\sigma^2$

$$\mathrm{var}(\bar{X}) < \mathrm{var}\left(\frac{2X_1 - X_2 + X_3}{2}\right),$$

$\bar{X}$ is a better estimator than $\frac{2X_1 - X_2 + X_3}{2}$ ∎

Side note:

In the later lecters, we did (or will) discuss Mean squared error (MSE) of an estimator and see the trade-offs between bias and variance of the estimator, and also consistency of an estimator. Although dicussing variance of an estimator was sufficient for full 3 point because the exercise only requires the knowledge from Problem Sheet 02 & Leture 1 ~ 4.

(+1) noting that the both estimators are unbiased.

(+1) Considering variance of an estimator

(+1) Complete proof to derive the variances of the estimators correctly.

marking scheme