1 Bernouli Distribution:

$$P(x;p) = p^x (1-p)^{1-x}$$

$$L(p) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i}$$

Take the log

$$L(p) = \log p \sum_{i=1}^{n} x_i + \log(1-p) \sum_{i=1}^{n} (1-x_i)$$

Diff. w.r.to p

$$\frac{\partial L(p)}{\partial p} = \frac{1}{p} \sum_{i=1}^{n} x_i - \frac{1}{1-p} \sum_{i=1}^{n} (1-x_i) \stackrel{!}{=} 0$$

$$\Rightarrow \frac{(1-p) \sum_{i=1}^{n} x_i - p \sum_{i=1}^{n} (1-x_i)}{p(1-p)} = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - \cancel{p \sum_{i=1}^{n} x_i} - p \cdot n + \cancel{p \sum_{i=1}^{n} x_i} = 0$$

$$\Rightarrow \boxed{\hat{p} = \frac{1}{n} \sum_{i=1}^{n} x_i}$$

$$\frac{\partial^2 L}{\partial p^2} = -\frac{1}{p^2} \sum_{i=1}^{n} x_i - \frac{1}{(1-p)^2} \sum_{i=1}^{n} (1-x_i)$$

$x \in [0,1]$ & $p > 0$, so $\frac{\partial^2 L}{\partial p^2} < 0$

$$\mathbb{E}[\hat{p}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[x_i]$$

$$= \frac{1}{n}\sum_{i=1}^{n} p$$

$$= \frac{1}{n}\cdot np$$

$$= p$$

(unbiased)

Mean = p

Variance = p(1-p)

$$\mathbb{E}[(\hat{p}-p)] = \mathrm{var}(\hat{p})$$

$$= \mathrm{var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{var}(x_i)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} p(1-p)$$

$$= \frac{1}{n^2}\cdot np(1-p)$$

$$= \frac{p(1-p)}{n}$$

if $n \to \infty$ then $\mathbb{E}[(\hat{p}-p)] \to 0$

So, $\hat{p} \longrightarrow p$

$\therefore$ $\hat{p}$ is consistent.

2. Possion Distribution:

$$P(\lambda, K) = \frac{\lambda^K e^{-\lambda}}{K!}$$

$$L(\lambda, k) = \prod_{i=1}^{n} \frac{\lambda^{k_i} e^{-\lambda}}{k_i!}$$

Take the Log

$$L(\lambda, K) = \sum_{i=1}^{n} \left[k_i \log \lambda - \lambda - \log k_i! \right]$$

$$= \log \lambda \sum_{i=1}^{n} K_i - n\lambda - \sum_{i=1}^{n} \log k_i!$$

Diff. w.r. to λ

$$\frac{\partial L(\lambda, k)}{\partial \lambda} = \frac{1}{\lambda} \sum_{i=1}^{n} k_i - n \stackrel{!}{=} 0$$

$$\Rightarrow \boxed{\hat{\lambda} = \frac{1}{n} \sum_{i=1}^{n} k_i}$$

$$\mathbb{E}[\hat{\lambda}] = \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} k_i\right]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[K_i]$$

$$= \frac{1}{n} \sum_{i=1}^{n} \lambda$$

$$= \frac{1}{n} \cdot n\lambda$$

$= \lambda$ unbiased.

Mean $= \lambda$

variance $= \lambda$

$$\mathbb{E}[(\hat{\lambda}-\lambda)] = var(\hat{\lambda})$$

$$= var\left(\frac{1}{n}\sum_{i=1}^{n} K_i\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n} \lambda$$

$$= \frac{1}{n^2} \cdot n\lambda = \frac{\lambda}{n}$$

if $n \to 0$ $\mathbb{E}[(\hat{\lambda}-\lambda)] \to 0$ so, consistent.

$$MSE(\hat{\theta}) = \mathbb{E}[(\hat{\theta}-\theta)^2]$$

$$= var(\hat{\theta}) + (\underbrace{B(\hat{\theta})}_{\text{Unbiased}=0})^2$$

$$= var(\hat{\theta})$$

$$\sum_{i=1}^{n}(Y_i-\bar{Y})^2 = \sum_{i=1}^{n} Y_i^2 - n\bar{Y}^2$$

$$Bias(\hat{\theta}) = E[\hat{\theta}] - \theta$$

3. Geometric Distribution:

$f(x,p) = (1-p)^{x} p$
$x = 0, 1, 2 \ldots$

mean $= \dfrac{1-p}{p}$

variance $= \dfrac{1-p}{p^2}$

$$f(x,p) = (1-p)^{x-1} p \quad , \quad x = 1, 2, 3, \ldots$$

$$L(x,p) = \prod_{i=1}^{n} (1-p)^{x_i - 1} p$$

Take the Log

$$L(x,p) = \sum_{i=1}^{n} \left[\log (1-p)^{x_i - 1} + \log p \right]$$

$$= \log(1-p) \sum_{i=1}^{n} (x_i - 1) + n \log p$$

Diff. w.r.t. to p

$$\frac{\partial L(x,p)}{\partial p} = \frac{(-1)}{1-p} \sum_{i=1}^{n} (x_i - 1) + \frac{n}{p} \stackrel{!}{=} 0$$

$$\Rightarrow -\frac{1}{1-p} \sum_{i=1}^{n} x_i + \frac{n}{1-p} + \frac{n}{p} = 0$$

$$\Rightarrow \frac{-p \sum_{i=1}^{n} x_i + \cancel{np} + n - \cancel{np}}{p(1-p)} = 0$$

$$\Rightarrow -p \sum_{i=1}^{n} x_i + n = 0$$

$$\Rightarrow \boxed{\hat{p} = \frac{n}{\sum_{i=1}^{n} x_i}}$$

mean $= \dfrac{1}{p}$

variance $= \dfrac{1-p}{p^2}$

$$\frac{\partial^2 L(x,p)}{\partial p^2} = -\frac{1}{(1-p)^2} \sum_{i=1}^{n} (x_i - 1) - \frac{n}{p^2}$$

Here p and $x_i > 0$ so

$$\frac{\partial^2 L(x,p)}{\partial p^2} < 0$$

$$\mathbb{E}[\hat{p}] = \mathbb{E}\left[\frac{n}{\sum_{i=1}^{n} x_i}\right] = n \frac{1}{\sum_{i=1}^{n} \mathbb{E}[x_i]}$$

$$= n \cdot \frac{1}{\frac{n}{p}}$$

$$= n \cdot \frac{p}{n}$$

$$= p \text{ (unbiased)}$$

$$\mathbb{E}[(\hat{p} - p)] = \text{var}(\hat{p})$$

$$= \text{var}\left[\frac{n}{\sum_{i=1}^{n} x_i}\right]$$

$$= \frac{\text{var}[n]}{\mathbb{E}[\Sigma x_i]^2} - \frac{2 \cdot \mathbb{E}[n]}{\mathbb{E}[\Sigma x_i]^3} \text{cov}(n, \Sigma x_i)$$

$$+ \frac{\mathbb{E}[n]^2}{\mathbb{E}[\Sigma x_i]^4} \text{var}[\Sigma x_i]$$

$$= 0 - 0 + \frac{n^2}{n^4 \cdot \frac{1}{p^4}} \cdot n \frac{(1-p)}{p^2}$$

$$= \frac{p^4}{n} \cdot \frac{(1-p)}{p^2}$$

$$= \frac{p^2(1-p)}{n}$$

$n \to \infty$ then $\mathbb{E}[(\hat{p} - p)] \to 0$

so, consistent.

Note:

$$\mathbb{E}\left[\frac{x}{y}\right] = \frac{\mathbb{E}[x]}{\mathbb{E}[y]} - \frac{\mathrm{cov}(x,y)}{\mathbb{E}[y]^2} + \frac{\mathbb{E}[x]}{\mathbb{E}[Y]^3} \cdot \mathrm{var}(y)$$

$$\mathrm{var}\left[\frac{x}{y}\right] = \frac{\mathrm{var}[x]}{\mathbb{E}[y]^2} - \frac{2\,\mathbb{E}[x]}{\mathbb{E}[Y]^3}\mathrm{cov}(x,y) + \frac{\mathbb{E}[x]^2}{\mathbb{E}[Y]^4}\mathrm{var}[Y]$$

https://en.wikipedia.org/wiki/Taylor_expansions_for_the_moments_of_functions_of_random_variables

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4. Binomial Distribution:

$$P(p; n, k) = {}^{n}C_{k}\, p^{k} (1-p)^{n-k}$$

$$L(P) = \prod_{i=1}^{n} {}^{n}C_{k_i}\, p^{k_i} (1-p)^{n-k_i}$$

Take the Log

$$L(P) = \sum_{i=1}^{n}\left(\log {}^{n}C_{k_i}\right) + \log p \sum_{i=1}^{n} k_i + \log(1-p)\sum_{i=1}^{n}(n-k_i)$$

Diff. w.r.to P

$$\frac{\partial L(P)}{\partial p} = 0 + \frac{1}{P}\sum_{i=1}^{n} k_i - \frac{1}{1-P}\sum_{i=1}^{n}(n-k_i) \overset{!}{=} 0$$

$$\Rightarrow \frac{(1-P)\sum_{i=1}^{n} k_i - P\sum_{i=1}^{n}(n-k_i)}{P(1-P)} = 0$$

$$\Rightarrow \sum_{i=1}^{n} k_i - p\cancel{\sum_{i=1}^{n} k_i} - p\sum_{i=1}^{n} n + p\cancel{\sum_{i=1}^{n} k_i} = 0$$

$$\Rightarrow p \cdot n^2 = \sum_{i=1}^{n} k_i$$

$$\Rightarrow \boxed{\hat{p}_{MLE} = \frac{1}{n^2}\sum_{i=1}^{n} k_i}$$

mean $= np$

variance $= np(1-p)$

$$\frac{\partial^2 L(p)}{\partial p^2} = -\frac{1}{p^2}\sum_{i=1}^{n} k_i - \frac{1}{(1-p)^2}\sum_{i=1}^{n}(n-k_i)$$

here $p>0$, $n>0$ & $k_i>0$

so $\frac{\partial^2 L(p)}{\partial p^2} < 0$

$$\mathbb{E}[\hat{p}] = \mathbb{E}\left[\frac{1}{n^2}\sum_{i=1}^{n} k_i\right]$$

$$= \frac{1}{n^2} \cdot \sum \mathbb{E}[k_i] = \frac{1}{n^2} \cdot \sum np = \frac{1}{n^2} \cdot n^2 p$$

$$= p$$ (unbiased)

$$\mathbb{E}[(\hat{p}-p)] = var(\hat{p})$$

$$= var\left[\frac{1}{n^2}\sum_{i=1}^{n} k_i\right]$$

$$= \frac{1}{n^4} \cdot n^2 p(1-p)$$

$$= \frac{p(1-p)}{n^2}$$

if $n \to \alpha$ then $\mathbb{E}[(\hat{p}-p)] \to 0$

so, consistent.

5. Exponential Distribution:

$$f(x,\lambda) = \lambda e^{-\lambda x}$$

$$L(P(x,\lambda)) = \prod_{i=1}^{n} \lambda e^{-\lambda x_i}$$

Take the Log

$$L(P) = \sum_{i=1}^{n} \log \lambda - \lambda \sum_{i=1}^{n} x_i$$

mean $= \frac{1}{\lambda}$

variance $= \frac{1}{\lambda^2}$

Diff. w.r.to λ

$$\frac{\partial L(P)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i \overset{!}{=} 0$$

$$\Rightarrow \boxed{\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} x_i}}$$

$$\frac{\partial^2 L(P)}{\partial \lambda^2} = -\frac{n}{\lambda^2}$$

here, λ and $n > 0$

so, $\frac{\partial^2 L(P)}{\partial \lambda^2} < 0$

$$\mathbb{E}[\hat{\lambda}] = \mathbb{E}\left[\frac{n}{\sum_{i=1}^{n} x_i}\right]$$

$$= n \cdot \frac{1}{n/\lambda}$$

$$= \lambda \text{ (unbiased)}$$

$$\mathbb{E}[(\hat{\lambda}-\lambda)] = var(\hat{\lambda})$$

$$= var\left(\frac{n}{\sum_{i=1}^{n} x_i}\right)$$

$$= \frac{var(n)}{\left[\mathbb{E}\left(\sum_{i=1}^{n} x_i\right)\right]^2} - 0 + \frac{\mathbb{E}[n]^2}{\mathbb{E}\left[\sum_{i=1}^{n} x_i\right]^4} var\left(\sum_{i=1}^{n} x_i\right)$$

$$= \frac{0}{\frac{n^2}{\lambda^2}} + \frac{n^2}{\frac{n^4}{\lambda^4}} \cdot \frac{n}{\lambda^2}$$

$$= 0 + \frac{n^3}{n^4} \cdot \frac{\lambda^4}{\lambda^2}$$

$$= \frac{\lambda^2}{n}$$

if $n \to \alpha$ then $\mathbb{E}[(\hat{\lambda}-\lambda)] \to 0$

consistent.

Suppose we have two light bulbs whose lifetimes follow an exponential(λ) distribution. Suppose also that we independently measure their lifetimes and get data x1 = 2 years and x2 = 3 years. Find the value of λ that maximizes the probability of this data.

$f(x_1, \lambda) = \lambda e^{-2\lambda}$ $\qquad f(x_2, \lambda) = \lambda e^{-3\lambda}$

$L(P) = \lambda e^{-2\lambda} \cdot \lambda e^{-3\lambda} = \lambda^2 e^{-5\lambda}$

Log likelihood: $L(P) = 2 \log \lambda - 5\lambda$

Diff. w.r.t. to λ: $\frac{\partial L(P)}{\partial \lambda} = \frac{2}{\lambda} - 5 = \frac{2-5\lambda}{\lambda} \overset{Set}{=} 0$

$\Rightarrow \hat{\lambda}_{MLE} = 2/5$ (Ans)

6. Normal Distribution:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$L(P) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

Take the Log.

$$L(P) = \sum_{i=1}^{n} -\log\left(\sqrt{2\pi\sigma^2}\right) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

$$= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}$$

Diff. w.r.to σ^2

$$\frac{\partial L(P)}{\partial \sigma^2} = -\frac{n}{2}\cdot\frac{1}{\cancel{2\pi}\sigma^2}\cdot\cancel{2\pi} + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 \stackrel{!}{=} 0$$

$$\Rightarrow -n\sigma^2 + \sum_{i=1}^{n}(x_i-\mu)^2 = 0$$

$$\Rightarrow \boxed{\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\frac{\partial^2 L(P)}{\partial(\sigma^2)^2} = \frac{n}{2(\sigma^2)^2} - \frac{2}{2(\sigma^2)^3}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$= \frac{n}{2\,\frac{1}{n^2}\sum_{i=1}^{n}(x_i-\mu)^4} - \frac{\cancel{2}\cdot\sum_{i=1}^{n}(x_i-\mu)^2}{\frac{1}{n^3}\sum_{i=1}^{n}(x_i-\mu)^6}$$

$$= \frac{n^3}{2\sum_{i=1}^{n}(x_i-\mu)^4} - \frac{n^3}{\sum_{i=1}^{n}(x_i-\mu)^4}$$

$$= \frac{-n^3}{2\sum_{i=1}^{n}(x_i-\mu)^4} < 0$$

Again,

Diff. w.r.to μ.

$$\frac{\partial L(p)}{\partial \mu} = 0 + \frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i-\mu) \stackrel{!}{=} 0$$

$$\Rightarrow \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i - n\mu = 0$$

$$\Rightarrow \boxed{\hat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i}$$

$$\frac{\partial^2 L(p)}{\partial^2 \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(0-1)$$

$$= -\frac{n}{\sigma^2} < 0.$$

MLE and method of moments are same.

Method of moments:

$\mu_1 = \mathbb{E}(X) = \mu$

$\mu_2 = \mathbb{E}(X^2) = \mu^2 + \sigma^2$

First and second sample moments are

$m_1 = \bar{x}$ and $m_2 = \frac{1}{n}\sum x_i^2$

Solving the equations:

$\mu = \bar{x}$ and

$\mu^2 + \sigma^2 = \frac{1}{n}\sum x_i$

Method of moments,

$\hat{\mu} = \bar{x}$

$\hat{\sigma}^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2 = \frac{1}{n}\sum(x_i-\bar{x})^2$

7. Uniform Distribution:

$$f(x) = \frac{1}{b-a}$$

mean $= \frac{a+b}{2}$

variance

$= \frac{1}{12}(b-a)^2$

$$L(P) = \prod_{i=1}^{n} \frac{1}{(b-a)} = \frac{1}{(b-a)^n}$$

Take the Log.

$$L(P) = -\log(b-a)^n$$

$$= -n \log(b-a)$$

Diff. w.r.to a

$$\frac{\partial L(P)}{\partial a} = -\frac{n}{b-a}(-1)$$

$$= \frac{n}{b-a} \rightarrow \text{maximum,}$$

so $b-a \rightarrow \min$

then

$b \rightarrow \min$

$a \rightarrow \max$

$$\frac{\partial^2 L(P)}{\partial a^2} = -\frac{n}{(b-a)^2}(-1)$$

$$= \frac{n}{(b-a)^2}$$

$a < x_1, \cdots x_n < b$

$a < x_1 < x_2 < \cdots < x_n < b$

$a < x_1 \qquad x_n < b$

Diff. w.r.to b

$$\frac{\partial L(P)}{\partial b} = -\frac{n}{b-a}$$

$$\frac{\partial^2 L(P)}{\partial b^2} = \frac{n}{(b-a)^2}$$

The mle for a would be the Largest a possible

$$\hat{a} = \min(x_1, x_2 \cdots x_n)$$

The mle for b would be the Smallest b possible.

$$\hat{b} = \max(x_1, x_2, \cdots, x_n)$$

Case 1: Unif $[0, \theta]$

$$f(\theta) = \frac{1}{\theta}$$

$$L(\theta; x) = \prod_{i=1}^{n} \frac{1}{\theta}$$

$$= \left(\frac{1}{\theta}\right)^n$$

Take Log.

$$L(\theta; x) = -n \operatorname{Log}(\theta)$$

Diff. w.r.to θ

$$\frac{\partial L}{\partial \theta} = -\frac{n}{\theta}.$$

Here $-\frac{n}{\theta}$ is a decreasing function of θ.

So max(L) is at min(θ). The constraint on θ is $\theta \geq x_{max}$

So, $\min(\theta) = x_{max}$

$$\therefore \hat{\theta}_{MLE} = x_{max}.$$

Method of moments:

$$\mathbb{E}[X] = \frac{\theta}{2} \qquad \left[\begin{array}{l} \text{mean} = \frac{a+b}{2} \\ a=0 \\ b=\theta \end{array}\right]$$

$$\Rightarrow \bar{x} = \frac{\theta}{2}$$

$$\Rightarrow \hat{\theta}_{MOM} = 2\bar{x}$$

Gamma Distribution:

$$X_i \sim \text{Gamma}(a, \lambda)$$

$$L(a, \lambda) = \prod_{i=1}^{n} \left[\frac{\lambda^a}{\Gamma(a)} x_i^{a-1} e^{-\lambda x_i} \right]$$

Log likelihood.

$$l(a, \lambda) = \sum_{i=1}^{n} \text{Log}\left[\frac{\lambda^a}{\Gamma(a)} x_i^{a-1} e^{-\lambda x_i} \right]$$

$$= \sum_{i=1}^{n} \text{Log}\ \lambda^a - \sum_{i=1}^{n} \text{Log}\left(\Gamma(a)\right)$$

$$+ (a-1) \sum_{i=1}^{n} \text{Log}(x_i) - \lambda \sum_{i=1}^{n} x_i$$

$$= na\ \text{Log}(\lambda) - n\ \text{Log}\left(\Gamma(a)\right) + (a-1) \sum_{i=1}^{n} \text{Log}(x_i)$$

$$- \lambda \sum_{i=1}^{n} x_i$$

$$\frac{\partial l(a, \lambda)}{\partial a} = n \log(\lambda) - n \frac{\Gamma'(a)}{\Gamma(a)} + \sum_{i=1}^{n} \log(x_i)$$

$$\frac{\partial l(a, \lambda)}{\partial \lambda} = \frac{na}{\lambda} - \sum_{i=1}^{n} x_i \stackrel{!}{=} 0$$

$$\Rightarrow \quad na - \lambda \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \quad \hat{\lambda} = \frac{na}{\sum_{i=1}^{n} x_i} = \frac{a}{\frac{1}{n} \sum_{i=1}^{n} x_i} = \frac{a}{\bar{x}}$$

$$n \log(\lambda) - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} \log(x_i) = 0$$

$$\Rightarrow \quad n(\log(\alpha) - \log(\bar{x})) - n \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} \log(x_i) = 0$$

$$\Rightarrow \quad \log \alpha - \log \bar{x} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \frac{1}{n} \sum_{i=1}^{n} \log(x_i) = 0$$

It is non-linear and a closed form solution doesn't exist.

$$f_Y(y|\theta) = \begin{cases} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} y^{\alpha-1} e^{-y/\theta} & ; y > 0 \\ 0 & , \text{elsewhere} \end{cases}$$

$$L(y|\theta) = \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\theta^{\alpha}} y_i^{\alpha-1} e^{-y_i/\theta}$$

$$= \left(\frac{1}{\Gamma(\alpha)\theta^{\alpha}}\right)^n \Sigma y_i^{\alpha-1} e^{-\Sigma y_i/\theta}$$

Log likelihood.

$$L = -n \log(\Gamma(\alpha)) - n \log \theta^{\alpha} + (\alpha-1) \log \Sigma y_i - \Sigma y_i/\theta$$

$$\frac{\partial L}{\partial \theta} = -\frac{n\alpha}{\theta} + \frac{1}{\theta^2} \Sigma y_i = 0$$

$$\Rightarrow \frac{-n\alpha\theta + \Sigma y_i}{\theta^2} = 0$$

$$\therefore \quad \hat{\theta}_{mle} = \frac{1}{n\alpha} \sum y_i$$

$$E[Y] = \alpha\theta,$$

$$var[Y] = \alpha\theta^2$$

$$\mathbb{E}[\hat{\theta}] = \frac{1}{n\alpha} \sum \mathbb{E}[y_i]$$

$$= \frac{1}{n\alpha} \sum \alpha\theta$$

$$= \frac{1}{n\alpha} \cdot n\alpha\theta$$

$$= \theta \quad \text{(unbiased)}$$

$$Var[\hat{\theta}] = \frac{1}{n^2\alpha^2} \sum V[y_i]$$

$$= \frac{1}{n^2\alpha^2} \sum \alpha\theta^2$$

$$= \frac{1}{n^2\alpha^2} \cdot n\alpha\,\theta^2$$

$$= \frac{\theta^2}{n\alpha}$$

if $n \to \alpha$, $V[\hat{\theta}] = 0$

So, consistent.

Suppose that X is a discrete random variable with the following probability mass function: where $0 \le \theta \le 1$ is a parameter. The following 10 independent observation

X	0	1	2	3
P(X)	$2\theta/3$	$\theta/3$	$2(1-\theta)/3$	$(1-\theta)/3$

were taken from such a distribution: (3,0,2,1,3,2,1,0,2,1). What is the maximum likelihood estimate of θ.

$$L(\theta) = 2 \times P(X=3) \cdot 3 \times P(X=2) \cdot 3 \times P(X=1)$$
$$2 \times P(X=0)$$

$$= 2 \cdot \frac{1-\theta}{3} \cdot \cancel{3} \cdot \frac{2(1-\theta)}{\cancel{3}} \cdot \cancel{3} \cdot \frac{\theta}{\cancel{3}} \cdot 2 \times \frac{2\theta}{3}$$

$$= \frac{32(1-\theta)^2 \cdot \theta^2}{9}$$

Log likelihood.

$$L(P) = \frac{32}{9} \times \left[2\log(1-\theta) + 2\log\theta\right]$$

Diff w.r.to θ

$$\frac{\partial L(P)}{\partial \theta} = \frac{32}{9} \cdot \left[\frac{2(-1)}{1-\theta} + \frac{2}{\theta}\right]$$

$$= -\frac{64}{9(1-\theta)} + \frac{64}{9\theta} \overset{\text{set}}{=} 0$$

$$\Rightarrow \frac{-576\theta + 64 \cdot 9(1-\theta)}{9(1-\theta)\,9\theta} = 0$$

$$\Rightarrow -576\theta + 576 - 576\theta = 0$$

$$\Rightarrow -1152\theta = -576$$

$$\therefore \boxed{\hat{\theta}_{MLE} = 1/2}$$

Method of Moments:

$$E[X] = \sum_{i=1}^{n} x \cdot P(x)$$

$$= 0 \cdot \frac{2\theta}{3} + 1 \cdot \frac{\theta}{3} + 2 \cdot \frac{2(1-\theta)}{3} + \frac{1-\theta}{3}$$

$$= \frac{7}{3} - 2\theta.$$

$$\bar{X} = 3/2$$

$$\frac{7}{3} - 2\theta = \frac{3}{2}$$

$$\Rightarrow \hat{\theta} = 5/12$$

Let $X_1, X_2 \ldots X_n$ be a sample from the inverse Gaussian pdf

$$f(x|\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\lambda(x-\mu)^2/2\mu^2 x\right\},\ x>0$$

Find MLE of μ and λ.

The likelihood is:

$$L(\mu,\lambda|x) = \frac{\lambda^{n/2}}{(2\pi)^n \prod_{i=1}^{n} x_i} \exp\left\{-\frac{\lambda}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\mu^2 x_i}\right\}$$

Log likelihood.

$$L(\mu,\lambda|x) = \frac{n}{2}\log(\lambda) - n\log 2\pi - \log\prod_{i=1}^{n} x_i - \frac{\lambda}{2}\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{\mu^2 x_i}$$

Diff. w. r. to μ

$$\frac{\partial L(\mu,\lambda|x)}{\partial\mu} = -\frac{\lambda}{2}\sum_{i=1}^{n}\frac{2(x_i-\mu)(-1)\mu^2 x_i - (x_i-\mu)^2 2\mu x_i}{(\mu^2 x_i)^2}$$

$$= -\frac{\lambda}{2}\sum_{i=1}^{n}\frac{-2\mu^2 x_i(x_i-\mu) - (x_i-\mu)^2 2\mu x_i}{\mu^4 x_i^2}$$

$$= -\frac{\lambda}{2}\sum_{i=1}^{n}\frac{-2\mu^2 x_i^2 + \cancel{2\mu^3 x_i} - 2\mu x_i^3 + 4x_i^2\mu^2 - \cancel{2\mu^3 x_i}}{\mu^4 x_i^2}$$

$$= -\frac{\lambda}{2}\sum_{i=1}^{n} \frac{-2\mu x_i^3 + 2\mu^2 x_i^2}{\mu^4 x_i^2}$$

$$= -\lambda \sum_{i=1}^{n} \frac{-x_i + \mu}{\mu^3}$$

$$= -\lambda \frac{-\sum_{i=1}^{n} x_i + n\mu}{n\mu^3} \overset{set}{=} 0$$

$$\Rightarrow \lambda \sum_{i=1}^{n} x_i - n\mu\lambda = 0$$

$$\Rightarrow n\mu = \sum_{i=1}^{n} x_i$$

$$\therefore \hat{\mu}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i \,/\, \bar{x}$$

$$\frac{\partial L(\mu, \lambda | x)}{\partial \lambda} = \frac{n}{2}\cdot\frac{1}{\lambda} - \frac{1}{2}\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\mu^2 x_i} \overset{set}{=} 0$$

$$\Rightarrow \frac{n}{2\lambda} - \frac{1}{2}\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\mu^2 x_i} = 0$$

$$\Rightarrow \frac{n}{2\lambda} - \frac{1}{2}\sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{\bar{x}^2 x_i} = 0$$

$$\Rightarrow \boxed{\hat{\lambda}_{MLE} = \frac{n}{\sum_{i=1}^{n} \frac{(x_i-\bar{x})^2}{\bar{x}^2 \cdot x_i}}}$$

$$f(x) = \frac{1}{\theta^2} x e^{-\frac{x}{\theta}}, \quad 0 \le x < \infty, \; 0 < \theta < \infty$$

$$L(x;\theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} x_i e^{-\frac{x_i}{\theta}}$$

$$= \frac{1}{\theta^{2n}} \left(\sum_{i=1}^{n} x_i\right) e^{-\sum_{i=1}^{n} x_i / \theta}$$

Log likelihood,

$$L(x;\theta) = -2n \log(\theta) + \log\left(\sum_{i=1}^{n} x_i\right) - \frac{\sum_{i=1}^{n} x_i}{\theta}$$

Diff. w. r. to θ.

$$\frac{\partial L(x;\theta)}{\partial \theta} = -\frac{2n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} \overset{!}{=} 0$$

$$\Rightarrow -2n\theta + \sum_{i=1}^{n} x_i = 0$$

$$\Rightarrow \hat{\theta} = \frac{1}{2n} \sum_{i=1}^{n} x_i$$

$$= \frac{\bar{x}}{2}$$

$$f_X(x|\lambda) = \frac{3}{\lambda}\left(\frac{x^2}{\lambda^2}\right)\exp\left(-\frac{x^3}{\lambda^3}\right), \quad y, \lambda > 0$$

$$f(x;\lambda) = \prod_{i=1}^{n} \frac{3^n}{\lambda}\left(\frac{x_i^2}{\lambda^2}\right)\exp\left(-\frac{x_i^3}{\lambda^3}\right)$$

$$= \frac{3^n}{\lambda^n}\,\frac{\sum_{i=1}^{n} x_i^2}{\lambda^{2n}}\exp\left(-\frac{\sum_{i=1}^{n} x_i^3}{\lambda^3}\right)$$

$$= \underbrace{\sum_{i=1}^{n} x_i^2}_{h(x)}\;\underbrace{\frac{3^n}{\lambda^{3n}}\exp\left(-\frac{\sum_{i=1}^{n} x_i^3}{\lambda^3}\right)}_{g(\lambda|x)}$$

$$T(x) = \sum_{i=1}^{n} x_i^3$$

Take log likelihood.

$$L(x;\lambda) = \log\left(\frac{3}{\lambda^3}\right)^n + \log\left(\sum_{i=1}^{n} x_i^2\right) - \frac{\sum_{i=1}^{n} x_i^3}{\lambda^3}$$

$$= n\log 3 - 3n\log\lambda + \log\sum_{i=1}^{n} x_i^2 - \frac{\sum_{i=1}^{n} x_i^3}{\lambda^3}$$

Diff. w.r. to λ

$$\frac{\partial L(x;\lambda)}{\partial\lambda} = 0 - \frac{3n}{\lambda} + 0 + \frac{3\cdot\sum_{i=1}^{n} x_i^3}{\lambda^4} \overset{!}{=} 0$$

$$\Rightarrow -3n\lambda^3 + 3\sum_{i=1}^{n} x_i^3 = 0$$

$$\Rightarrow \quad n\lambda^3 = \sum_{i=1}^{n} x_i^3$$

$$\Rightarrow \quad \hat{\lambda}_{mle} = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)^{1/3}$$

$$= \left(\frac{T(x)}{n}\right)^{1/3}$$

which is a function of sufficient statistics.

Raylicgh Distribution:

$$f(x,\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \; ; \quad 0 < x < \infty$$

$$L(x|\sigma) = \prod_{i=1}^{n} \frac{x_i}{\sigma^2} e^{-\frac{x_i^2}{2\sigma^2}}$$

Mean $= \lambda\sqrt{\frac{\pi}{2}}$

$$= \frac{\Sigma x_i}{\sigma^{2n}} e^{-\frac{1}{2\sigma^2}\Sigma x_i^2}$$

Variance $= \frac{4-\pi}{2}\lambda^2$

Take Loglikelihood,

$$L(x|\sigma) = \log \Sigma x_i - 2n \operatorname{Log} \sigma - \frac{\Sigma x_i^2}{2\sigma^2}$$

$$\frac{\partial L}{\partial \sigma} = 0 - \frac{2n}{\sigma} + \frac{2}{2\sigma^3}\Sigma x_i^2 \overset{!}{=} 0$$

$$= \frac{4n\sigma^2 + 2\Sigma x_i^2}{2\sigma^3} = 0$$

$$\Rightarrow \quad \hat{\sigma}^2 = \frac{1}{2n} \sum x_i^2$$

$$\frac{\partial^2 L}{\partial \sigma^2} = \frac{2n}{\sigma^2} - \frac{6}{2\sigma^4} \sum x_i^2$$

$$= \frac{2n \times 2n}{\sum x_i^2} - \frac{6}{\cancel{2} \cdot \underset{2}{\frac{1}{4n^2}} \underset{\sum x_i^2}{\cancel{(\sum x_i^2)^2}}} \times \sum \cancel{x_i^2}$$

$$= \frac{4n^2}{\sum x_i^2} - \frac{12n^2}{\sum x_i^2}$$

$$= \frac{-8n^2}{\sum x_i^2} < 0$$

$$\mathbb{E}[X^2] = \operatorname{var}(X) + (\mathbb{E}[X])^2$$

$$= \frac{4-\pi}{2} \cdot \sigma^2 + \sigma^2 \cdot \frac{\pi}{2}$$

$$= \frac{4\sigma^2 - \pi\sigma^2 + \pi\sigma^2}{2}$$

$$= 2\sigma^2$$

$$\mathbb{E}[\hat{\sigma}^2] = \frac{1}{2n} \sum \mathbb{E}[X_i^2]$$

$$= \frac{1}{2n} \sum (2\sigma^2)$$

$$= \frac{1}{2\cancel{n}} \cdot 2\cancel{n}\sigma^2$$

$$= \sigma^2 \qquad \text{(unbiased)}$$

$$\mathbb{E}[\hat{\sigma}^2 - \sigma^2] = \operatorname{Var}(\hat{\sigma}^2)$$

$$= \frac{1}{\mathbb{E}\left[-\frac{\partial^2}{\partial \sigma^2} L\right]}$$

$$= \frac{1}{\mathbb{E}\left[\frac{8n^2}{\sum x_i^2}\right]}$$

$$= \frac{1 \cancel{\times 2}\sigma^2}{4 \cancel{8} n^2}$$

$$= \frac{\sigma^2}{4n^2}$$

if $n \to \infty$, $\mathbb{E}[\hat{\sigma}^2 - \sigma^2] \to 0$

So, consistent.

$$f_X(x|\theta) = \frac{\theta x^{\theta-1}}{3^\theta}$$

$$L(x|\theta) = \prod_{i=1}^{n} \frac{\theta x_i^{\theta-1}}{3^\theta} = \frac{\theta^n \sum x_i^{\theta-1}}{3^{\theta n}}$$

Log likelihood. $L(x, \theta) = n\log\theta + (\theta-1)\sum \log x_i - \theta n \log 3$

$$\frac{\partial L}{\partial \theta} = \frac{n}{\theta} + \sum \log x_i - n\log 3 \overset{!}{=} 0$$

$$\Rightarrow n + \theta \sum \log x_i - n\theta \log 3 = 0$$

$$\therefore \hat{\theta}_{ml} = \frac{n}{n\log 3 - \sum \log x_i}$$

Linear Regression:

$$y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

$$\varepsilon_i \sim N(0, \sigma^2)$$

$$P(y \mid \beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} P(y_i \mid \beta_0, \beta_1, \sigma^2)$$

$$= \prod_{i=1}^{n} \mathcal{N}(y_i ; \beta_0 + \beta_1 x_i , \sigma^2)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2} \right\}$$

$$= \frac{1}{(\sqrt{2\pi\sigma^2})^n} \exp\left\{ - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2 \right\}$$

Take Log likelihood function:

$$L(\beta_0, \beta_1, \sigma^2 \mid y, x) = -\frac{n}{2} \log 2\pi\sigma^2 - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

Diff. w.r.to σ^2

$$\frac{\partial L}{\partial \sigma^2} = -\frac{n}{2} \cdot \frac{1}{\cancel{2\pi}\sigma^2} \cdot \cancel{2\pi} + \frac{1}{2\sigma^4} \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i^2)$$

$$= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} \Sigma (y_i - \beta_0 - \beta_1 x_i^2) \overset{!}{=} 0$$

$$\Rightarrow \quad \widehat{\sigma^2}_{mle} = \frac{1}{n} \Sigma (y_i - \beta_0 - \beta_1 x_i^2)$$

Diff. w.r.to β_0

$$\frac{\partial L}{\partial \beta_0} = 0 - \frac{1}{2\sigma^2} \Sigma\, 2(y_i - \beta_0 - \beta_1 x_i)\,(-1)$$

$$= \frac{1}{\sigma^2}\left[\Sigma(y_i) - \Sigma(\beta_0) - \Sigma(\beta_1 x_i)\right] \overset{!}{=} 0$$

$$\Rightarrow \Sigma y_i - n\beta_0 - \beta_1 \Sigma x_i = 0$$

$$\Rightarrow \hat{\beta}_0 = -\frac{1}{n}\beta_1 \Sigma x_i + \frac{1}{n}\Sigma y_i$$

$$\Rightarrow \hat{\beta}_{0\,mle} = \bar{y} - \hat{\beta}_1 \bar{x}$$

Diff. w.r.to. β_1.

$$\frac{\partial L}{\partial \beta_1} = 0 - \frac{1}{2\sigma^2} \Sigma\, 2(y_i - \beta_0 - \beta_1 x_i)\,(-x_i)$$

$$\Rightarrow \Sigma y_i x_i - \beta_0 \Sigma x_i - \beta_1 \Sigma x_i^2 \overset{!}{=} 0$$

$$\Rightarrow \Sigma y_i x_i - (\bar{y} - \beta_1 \bar{x})\Sigma x_i - \beta_1 \Sigma x_i^2 = 0$$

$$\Rightarrow \Sigma y_i x_i - \bar{y} n \bar{x} + \beta_1 n \bar{x}^2 - \beta_1 \Sigma x_i^2 = 0$$

$$\Rightarrow \beta_1 (n\bar{x}^2 - \Sigma x_i^2) = n\bar{x}\bar{y} - \Sigma x_i y_i$$

$$\Rightarrow \hat{\beta}_1 = \frac{\Sigma x_i y_i - \Sigma \bar{x}\bar{y}}{\Sigma x_i^2 - \Sigma \bar{x}^2}$$

$$= \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2}$$

$$* \quad \Sigma x_i y_i - \Sigma \bar{x} \bar{y}$$

$$= \Sigma x_i y_i - n\bar{x}\bar{y} + n\bar{x}\bar{y} - n\bar{x}\bar{y}$$

$$= \Sigma x_i y_i - \bar{x} \Sigma y_i + \bar{y} \Sigma x_i - n\bar{x}\bar{y}$$

$$= \Sigma (x_i - \bar{x})(y_i - \bar{y})$$

and. $\Sigma x_i^2 - \Sigma \bar{x}^2$

$$= \Sigma x_i^2 - n\bar{x}^2$$

$$= \Sigma x_i^2 - 2n\bar{x}^2 + n\bar{x}^2$$

$$= \Sigma (x_i - \bar{x})^2$$

$$\hat{\beta}_1 = \frac{\Sigma (x_i - \bar{x})(y_i - \bar{y})}{\Sigma (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

$$\hat{\sigma}^2 = \frac{1}{n} \Sigma (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

$$\mathbb{E}[\hat{\beta}_1] = \frac{\sum\left((x_i-\bar{x})y_i - (x_i-\bar{x})\bar{y}\right)}{\sum (x_i-\bar{x})(x_i-\bar{x})}$$

$$= \frac{\sum (x_i-\bar{x})y_i - 0}{\sum\left((x_i-\bar{x})x_i - \underbrace{(x_i-\bar{x})\cdot\bar{x}}_{=0}\right)}$$

$$\sum (x_i-\bar{x}) = \sum x_i - \sum \bar{x} = n\bar{x} - n\bar{x} = 0$$

$$= \frac{\sum (x_i-\bar{x})y_i}{\sum (x_i-\bar{x})x_i}$$

$$= \frac{1}{\sum (x_i-\bar{x})x_i} \sum (x_i-\bar{x})\,\mathbb{E}[y_i]$$

$$= \frac{\sum (x_i-\bar{x})}{\sum (x_i-\bar{x})x_i}\left[\beta_0 + \beta_1 x_i\right]$$

$$[\mathbb{E}[\varepsilon]=0]$$

$$= \frac{\beta_0 \overbrace{\sum (x_i-\bar{x})}^{=0}}{\sum (x_i-\bar{x})x_i} + \beta_1 \frac{\sum (x_i-\bar{x})\cancel{x_i}}{\sum (x_i-\bar{x})\cancel{x_i}}$$

$$= \beta_1. \text{ (unbiased)}$$

$$\mathbb{E}[\hat{\beta}_0] = \mathbb{E}[\bar{y} - \hat{\beta}_1\bar{x}]$$

$$= \mathbb{E}[\bar{y}] - \mathbb{E}[\hat{\beta}_1\bar{x}]$$

$$= \beta_0 + \beta_1\bar{x} - \beta_1\bar{x}$$

$$= \beta_0 \text{ (unbiased)}$$

$$v(\hat{\beta}_1) = v\left[\frac{\Sigma(x_i-\bar{x})y_i}{\Sigma(x_i-\bar{x})^2}\right]$$

$$= \left[\frac{\Sigma(x_i-\bar{x})}{\Sigma(x_i-\bar{x})^2}\right]^2 v[y]$$

$$= \left[\frac{\Sigma(x_i-\bar{x})}{\Sigma(x_i-\bar{x})^2}\right]^2 \sigma^2 \quad \Big| \; [v(\varepsilon) = \sigma^2]$$

$$= \left[\frac{1}{\Sigma(x_i-\bar{x})^2}\right] \sigma^2$$

$$= \frac{1}{n\,\overline{(x_i-\bar{x})^2}} \cdot \sigma^2$$

$n \to \infty$, $v(\hat{\beta}_1) \to 0$, consistent.

$$v(\hat{\beta}_0) = v\left[\bar{y} - \beta_1 \bar{x}\right]$$

$$= v(\bar{y}) + v(-\beta_1\bar{x}) + 2\underbrace{\mathrm{cov}(\bar{y}, \beta_1\bar{x})}_{=0}$$

$$= v\left(\frac{1}{n}\Sigma y_i\right) + \bar{x}^2 v(\beta_1)$$

$$= \frac{1}{n^2} \cdot n\sigma^2 + \bar{x}^2 \cdot \frac{\sigma^2}{\Sigma(x_i-\bar{x})^2}$$

$$= \frac{\sigma^2}{n} + \frac{\bar{x}^2 \cdot \sigma^2}{n\,\overline{(x_i-\bar{x})^2}}$$

$n \to \infty$, $v(\hat{\beta}_0) \to 0$, consistent.

Pareto Distribution:

$$f(x;\alpha) = \frac{\alpha k^{\alpha}}{x^{\alpha+1}}$$

mean $= \dfrac{\alpha K}{\alpha - 1}$

variance

$$\frac{k^2 \alpha}{(\alpha-1)^2(\alpha-2)}$$

$$L(\alpha \mid x_n) = \prod_{i=1}^{n} \frac{\alpha k^{\alpha}}{x^{\alpha+1}} = (\alpha^n k^{n\alpha}) \prod_{i=1}^{n} x_i^{-(\alpha+1)}$$

log likelihood,

$$L(\alpha \mid x_n) = n \ln \alpha + n\alpha \ln k - (\alpha+1) \ln(\Sigma x_i)$$

$$\frac{\partial L}{\partial \alpha} = \frac{n}{\alpha} + n \ln k - \ln(\Sigma x_i) = 0$$

$$\Rightarrow n + n\alpha \ln k - \alpha \ln(\Sigma x_i) = 0$$

$$\Rightarrow n + \alpha(n \ln k - \ln \Sigma x_i) = 0$$

$$\Rightarrow \hat{\alpha}_{mle} = \frac{-n}{n \ln k - \ln \Sigma x_i}$$

$$= \frac{n}{\ln \Sigma x_i - n \ln k}$$

Laplace Distribution:

$$\frac{1}{2b} \exp\left(-\frac{|x-\mu|}{b}\right)$$

mean $= \mu$

variance $= 2b^2$

$$f(x;b) = \prod_{i=1}^{n} \frac{1}{2b} \exp\left(-\frac{|x_i-\mu|}{b}\right)$$

$$= \left(\frac{1}{2b}\right)^n \exp\left(-\frac{\sum(x_i-\mu)}{b}\right)$$

Log-likelihood:

$$L(x;b) = -n\log 2b - \frac{\sum(x_i-\mu)}{b}$$

Diff. w. r. to b

$$\frac{\partial L}{\partial b} = -\frac{n}{2b}\cdot \cancel{2} + \frac{\sum(x_i-\mu)}{b^2} = 0$$

$$\Rightarrow -\frac{n}{b} + \frac{\sum(x_i-\mu)}{b^2} = 0$$

$$\Rightarrow \frac{-nb + \sum(x_i-\mu)}{b^2} = 0$$

$$\Rightarrow \hat{b}_{mle} = \frac{1}{n}\sum(x_i-\mu)$$

Maximum Spacing Estimator:

$x_{(1)} = 2, \quad x_{(2)} = 4$

exponential distribution

$$F(x;\lambda) = 1 - e^{-x\lambda}, \quad x \geq 0$$

and $\lambda > 0$

	$F(x_{(i)})$	$F(x_{(i-1)})$	$D_i = F(x_{(i)}) - F(x_{(i-1)})$
D_1	$1 - e^{-2\lambda}$	0	$1 - e^{-2\lambda}$
D_2	$1 - e^{-4\lambda}$	$1 - e^{-2\lambda}$	$e^{-2\lambda} - e^{-4\lambda}$
D_3	1	$1 - e^{-4\lambda}$	$e^{-4\lambda}$

Geometric Mean: $\sqrt[3]{(1 - e^{-2\lambda})(e^{-2\lambda} - e^{-4\lambda})\, e^{-4\lambda}}$

$$= \sqrt[3]{(1-\mu)(\mu - \mu^2)\mu^2} \quad [e^{-2\lambda} = \mu]$$

$$= \sqrt[3]{(\mu - \mu^2 - \mu^2 + \mu^3)\mu^2}$$

$$= (\mu^5 - 2\mu^4 + \mu^3)^{1/3}$$

Log likelihood.

$$\frac{1}{3}\log(\mu^5 - 2\mu^4 + \mu^3)$$

Diff w.r. to μ,

$$\frac{\partial L}{\partial \mu} = \frac{1}{3} \cdot \frac{1}{\mu^5 - 2\mu^4 + \mu^3} \cdot (5\mu^4 - 8\mu^3 + 3\mu^2)$$

$$\Rightarrow 5\mu^4 - 8\mu^3 + 3\mu^2 = 0$$

$$\Rightarrow 5\mu^3 - 8\mu^2 + 3\mu = 0$$

$$\therefore \mu = 0, \quad 0.6 \text{ and } 1$$

$\mu = e^{-2\lambda}$; it has to greater than zero but less than one.

so, only acceptable solution:

$$\mu = 0.6$$

$$\Rightarrow e^{-2\lambda} = 0.6$$

$$\Rightarrow -2\lambda = \ln 0.6$$

$$\Rightarrow \hat{\lambda}_{mse} = \frac{\ln 0.6}{-2} = 0.255$$

$$\therefore \text{Mean} = \frac{1}{\lambda} = 3.915$$

$$\hat{\lambda}_{MLE} = \frac{1}{\bar{x}} = \frac{1}{\left(\frac{2+4}{2}\right)} = 0.333$$

let $x_1, \dots, x_n$ be an i.i.d. sample from $U[a,b]$ estimate unknown a and b using maximum spacing estimator.

$$D_1 = \frac{x_1 - a}{b - a}$$

$$D_i = \sum_{i=2}^{n} \frac{x_i - x_{i-1}}{b - a}$$

$$D_{n+1} = \frac{x_{n+1} - x_{n+1-1}}{b - a}$$

$$= \frac{b - x_n}{b - a}$$

$$L = -\log(b-a) + \frac{1}{n+1} \log(x_1 - a) + \frac{1}{n+1} \sum_{i=2}^{n} (x_i - x_{i-1}) + \frac{1}{n+1} (b - x_n)$$

$$\frac{\partial L}{\partial a}, \quad a = \frac{nx_1 + x_1 - b}{n}$$

$$\frac{\partial L}{\partial b}, \quad b = \frac{nx_n + x_n - a}{n}$$

$$\hat{a}_{ms} = \frac{nx_1 - x_n}{n-1}$$

$$\hat{b}_{ms} = \frac{nx_n - x_1}{n-1}$$