Example Solution

Since $\text{rank}(X) = 2$, X has two positive singular values σ_1 and σ_2.
Using the fact that σ_1^2 and σ_2^2 are eigenvalues of $X^T X$, i.e., λ_1 and λ_2,
we first find λ_1 and λ_2:

$$X^T X = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}.$$

We can also use the fact that

$$\text{tr}(X^T X) = \lambda_1 + \lambda_2 \quad \text{and} \quad \det(X^T X) = \lambda_1 \lambda_2$$

that is, $\qquad \lambda_1 + \lambda_2 = 50 \quad$ and $\quad \lambda_1 \lambda_2 = 225$

$$\Rightarrow \quad \lambda_1 = 45, \quad \lambda_2 = 5$$
$$\Rightarrow \quad \sigma_1 = \sqrt{45}, \quad \sigma_2 = \sqrt{5}$$