Exercise 3: Let X_1, X_2, X_3, X_4 be a sample from U(0, 1), and let $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, $X_{(4)}$ be the order statistic. Determine the density of $X_{(1)}$, $X_{(2)}$, $X_{(3)}$, $X_{(4)}$.

Solution:

For $X_1, X_2, \ldots, X_n$ iid continuous random variables with pdf f and cdf F the density of the k^{th} order statistic is

$f_k(x)$ dx = $P(X_{(k)}$ € dx)

= P (One of the X's € dx, k -1 of the others < x)

= n * P (X_1 € dx, (k-1) others (exactly) < x)

= n * P (X_1 € dx) $((\frac{n-1}{k-1})$ (F(x)) $^{(k-1)} * (1 – F(x))^{(n-k)})$

= n * $f(x)$ dx $* ((\frac{n-1}{k-1}) * (F(x))^{(k-1)} * (1 – F(x))^{(n-k)})$

$= \frac{n!}{(n-k)!*(k-1)!} * f(x)\,dx * (F(x))^{(k-1)} * (1 – F(x))^{(n-k)})$

Let $X_1, X_2, \ldots, X_n \overset{iid}{\sim}$ U(0,1) then the density of X(n) is given by [1]:

$$f_k(x) = \frac{n!}{(n-k)!*(k-1)!} * f(x) * (F(x))^{(k-1)} * (1 – F(x))^{(n-k)})$$

$$= \begin{cases} \frac{n!}{(n-k)!*(k-1)!} * x^{k-1} * (1-x)^{n-k}; \; if\ 0 < x < 1 \\ 0; \; otherwise \end{cases}$$

Density of $X_{(1)} = n * (1 – x)^{(n-1)}$

Density of $X_{(2)} = n * (n – 1) * x * (1 – x)^{(n-2)}$

Density of $X_{(3)} = \frac{n*(n-1)*(n-2)}{2} * x^2 * (1 – x)^{(n-3)}$

Density of $X_{(4)} = \frac{n*(n-1)*(n-2)*(n-3)}{6} * x^3 * (1 – x)^{(n-4)}$

Reference:

[1] https://www2.stat.duke.edu/courses/Spring12/sta104.1/Lectures/Lec15.pdf