New exercise 4 (2):

Kalman filter:

Observed data	Reference data
0.39894	0.15127
0.83747	0.7171
−0.16078	0.18423
	−0.6281

$$z_n = \underbrace{0.99}_{A} z_{n-1} + \varepsilon_{n-1} \rightarrow N(0, \underbrace{0.5}_{B})$$

$$z_0 \sim N(\underbrace{0, 0.5}_{m_0,\, C_0})$$

$$y_n = z_n + \eta_n \rightarrow N(0, \underbrace{0.5}_{R})$$

$m_0 = 0, \; C_0 = 0.5, \; R = 0.5, \; A = 0.99, \; B = 0.5$

1st Iteration:

$$\hat{m}_1 = A m_0 = 0.99 \times 0 = 0$$

$$\hat{C}_1 = A^2 C_0 + B = 0.99^2 \times 0.5 + 0.5$$

$$= 0.99005$$

$$K = \frac{0.99005}{0.5 + 0.99005} = 0.6644$$

$$m_1 = \hat{m}_1 - K(\hat{m}_1 - \text{1st obs data})$$

$$= 0 - 0.6644(0 - 0.39894)$$

$$= 0.26507$$

$$C_1 = \hat{C}_1 - K\hat{C}_1 = 0.99005 - 0.6644 \times 0.99005$$

$$= 0.33226$$

2nd Iteration:

$$\hat{m}_2 = Am_1 = 0.99 \times 0.26507 = 0.262421$$

$$\hat{c}_2 = A^2 c_1 + B = 0.825609$$

$$K = \frac{\hat{c}_2}{R + \hat{c}_2} = 0.6228149$$

$$m_2 = \hat{m}_2 - K(\hat{m}_2 - \text{2nd obs data})$$

$$= 0.6205702$$

$$c_2 = \hat{c}_2 - K\hat{c}_2$$

$$= 0.3114074659$$

3rd Iteration:

$$\hat{m}_3 = Am_2 = 0.6143645$$

$$\hat{c}_3 = A^2 c_2 + B = 0.80521045 7$$

$$K = \frac{\hat{c}_3}{R + \hat{c}_3} = 0.61692001$$

$$m_3 = \hat{m}_3 - K(\hat{m}_3 - \text{3rd obs-data})$$

$$= 0.136162346$$

$$c_3 = \hat{c}_3 - K\hat{c}_3$$

$$= 0.30846$$

$m = 0,\ 0.265072,\ 0.6205702127,\ 0.136162346$

Ref-value $= 0.15127,\ 0.7171,\ 0.18423,\ -0.6281.$

$$MSE = (0.15127 - 0)^2 + (0.7171 - 0.265072)^2$$
$$+ (0.18423 - 0.6205702127)^2 +$$
$$(-0.6281 - 0.136162346)^2$$

$= 1.00170164$

$$\text{Mean MSE} = \frac{1.00170164}{4} = 0.250425408 0 9.$$

For MSE, the lower the value the better and 0 means the model is perfect.

Proof (Extra):

$$\mathbb{E}[\hat{\epsilon}] = 0$$

$$y = X\beta + \epsilon \qquad\qquad \mathbb{E}[y] = \mathbb{E}[X\beta + \epsilon]$$

$$= X\beta + \mathbb{E}[\epsilon]$$

$$= X\beta$$

$$\hat{\epsilon} = y - X\hat{\beta}$$

$$= y - \underbrace{X(X^TX)^{-1}X^T}y$$

$$= y - Hy$$

$$\mathbb{E}[\hat{\epsilon}] = \mathbb{E}[y - Hy]$$

$$= X\beta - HX\beta$$

$$= X\beta - X(X^TX)^{-1}X^TX\beta$$

$$= X\beta - X\beta$$

$$= 0$$

$$\mathrm{Cov}(\hat{\epsilon}) = \mathrm{Cov}(y - Hy)$$

$$= \mathrm{Cov}((I_n - H)y)$$

$$= \sigma^2((I_n - H)y)(I_n - H)y)^T$$

$$= \sigma^2((I_n - H)yy^T(I_n - H)^T)$$

$$= \sigma^2 (I_n - H) I_n (I_n - H)^T$$

$$= \sigma^2 (I_n - H)$$

$$\left[(I_n - H)(I_n - H)^T = (I_n - H)(I_n - H) \right.$$

$$\left. = (I_n - H) \right]$$