Exercise 1:

Let X (random variable) is σ- subgaussian if there exists $\sigma > 0$, such as:

$$\forall t \in \mathbb{R}, \mathbb{E}[\exp(tX)] \leq \exp\left(\frac{\sigma^2 t^2}{2}\right) \quad \cdots\cdots (1)$$

$$\mathbb{E}(\exp(tX)) = \sum_{n \geq 0} t^n \frac{\mathbb{E}(X^n)}{n!} \leq \exp\left(\frac{t^2 \sigma^2}{2}\right)$$

$$= \sum_{n \geq 0} \left(\frac{\sigma^2 t^2}{2}\right)^n \frac{1}{n!}$$

Up to order 2 and rearranging terms of order greater than 2 on the left hand side:

$$1 + t\mathbb{E}(X) + \frac{t^2}{2}\mathbb{E}(X^2) \leq 1 + \frac{\sigma^2 t^2}{2} + g(t) \quad \cdots\cdots (2)$$

Where, $\frac{g(t)}{t^2} \xrightarrow[t \to 0]{} 0$. So, by dividing both side by t and taking the limit when $t \to 0_+$ we show that $\mathbb{E}[X] \leq 0$. with $t \to 0_-$ we prove that $\mathbb{E}(X) \geq 0$. So, $\mathbb{E}(X) = 0$.

By dividing both side of (2) by t^2 and taking the limit we obtain

$$\mathbb{E}[X^2] \leq \sigma^2$$

$$\text{or,} \quad \mathbb{V}[X] \leq \sigma^2$$

$$\mathbb{E}[e^{t(X_1+X_2)}]$$

$$= \mathbb{E}[e^{t(X_1)}]\ \mathbb{E}[e^{t(X_2)}]$$

$$= e^{\frac{\sigma_1^2 t^2}{2}} + e^{\frac{\sigma_2^2 t^2}{2}}$$

$$= \exp\left(\frac{t^2\sqrt{\sigma_1^2+\sigma_2^2}}{2}\right)$$

So, X_1+X_2 is $\sqrt{\sigma_1^2+\sigma_2^2}$ Subgaussian.

$$\mathbb{E}[\exp(tcX)] \leq \exp\left(t^2 c^2 \frac{t}{2}\right)$$

$$\leq \exp\left(|c^2| \cdot \frac{\sigma}{2} \cdot t^2\right)$$

So, cX is $|c|\sigma$- Subgaussian.

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