In [1]:
```python
import numpy as np
from sklearn.cluster import KMeans
from matplotlib import pyplot as plt
```

Part (i)

Adjacency Matrix

In [2]:
```python
A = np.array([
  [0, 1, 0, 0, 1, 0],
  [1, 0, 1, 1, 0, 1],
  [0, 1, 0, 0, 0, 0],
  [0, 1, 0, 0, 1, 1],
  [1, 0, 0, 1, 0, 1],
  [0, 1, 0, 1, 1, 0]])
```

Degree Matrix

In [3]:
```python
D = np.diag(A.sum(axis=1))
print(D)
```

```
[[2 0 0 0 0 0]
 [0 4 0 0 0 0]
 [0 0 1 0 0 0]
 [0 0 0 3 0 0]
 [0 0 0 0 3 0]
 [0 0 0 0 0 3]]
```

Laplacian matrix

In [4]:
```python
L = D-A
print(L)
```

```
[[ 2 -1  0  0 -1  0]
 [-1  4 -1 -1  0 -1]
 [ 0 -1  1  0  0  0]
 [ 0 -1  0  3 -1 -1]
 [-1  0  0 -1  3 -1]
 [ 0 -1  0 -1 -1  3]]
```

✓

Part (ii)

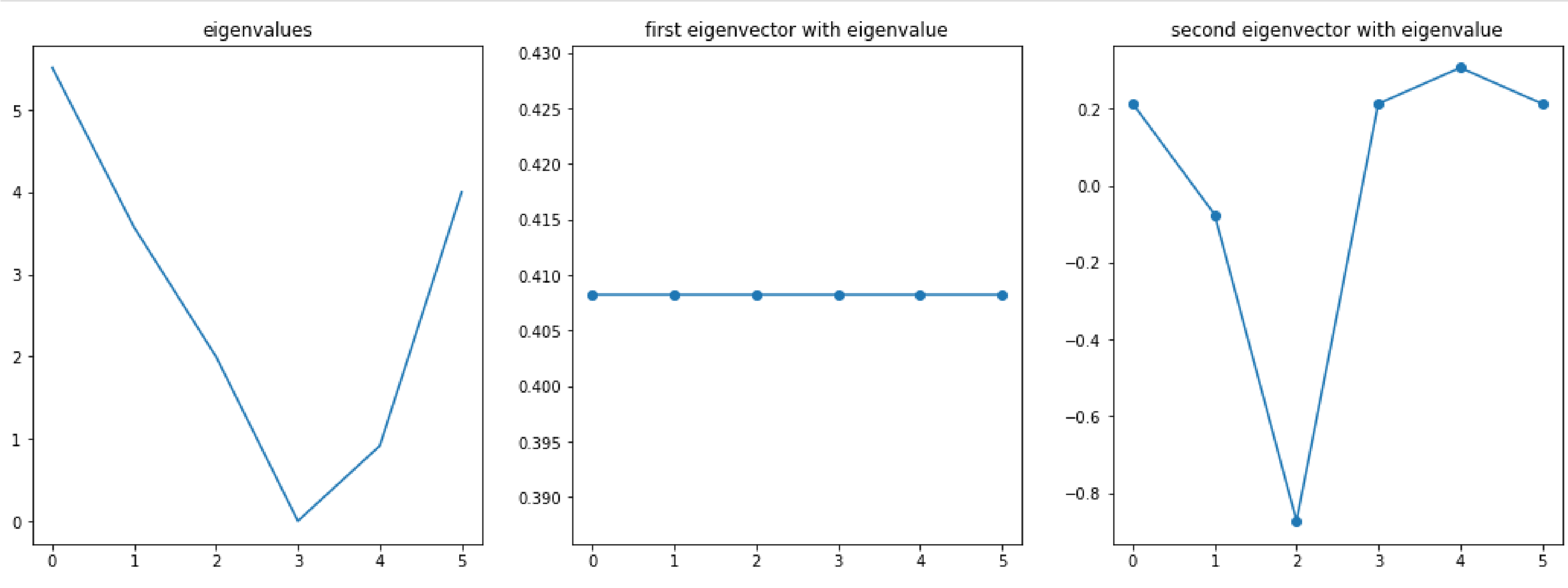
eigenvalues and eigenvectors

In [5]:
```python
vals, vecs = np.linalg.eig(L)
```

In [6]:
```python
fig = plt.figure(figsize=[18, 6])
ax1 = plt.subplot(131)
plt.plot(vals)
ax1.title.set_text('eigenvalues')

i = np.where(vals < 1)[0]
ax2 = plt.subplot(132)
plt.plot(vecs[:, i[0]], marker='o')
ax2.title.set_text('first eigenvector with eigenvalue')

ax3 = plt.subplot(133)
plt.plot(vecs[:, i[1]], marker='o')
ax3.title.set_text('second eigenvector with eigenvalue')
```



Because we have a single component, only 1 eigenvalue will be equal to 0. However, if we look at the second smallest eigenvalue, we can still observe a distinction between the two classes. If we drew a horizontal line across, we would correctly classify the nodes. The nodes will be **{1, 2, 4, 5, 6}** and **{3}**

In [7]:
```python
# sort these based on the eigenvalues
vecs = vecs[:,np.argsort(vals)]
vals = vals[np.argsort(vals)]

#kmeans on first two vectors with nonzero eigenvalues
kmeans = KMeans(n_clusters=2)
kmeans.fit(vecs[:,0:2])
label = kmeans.labels_
```

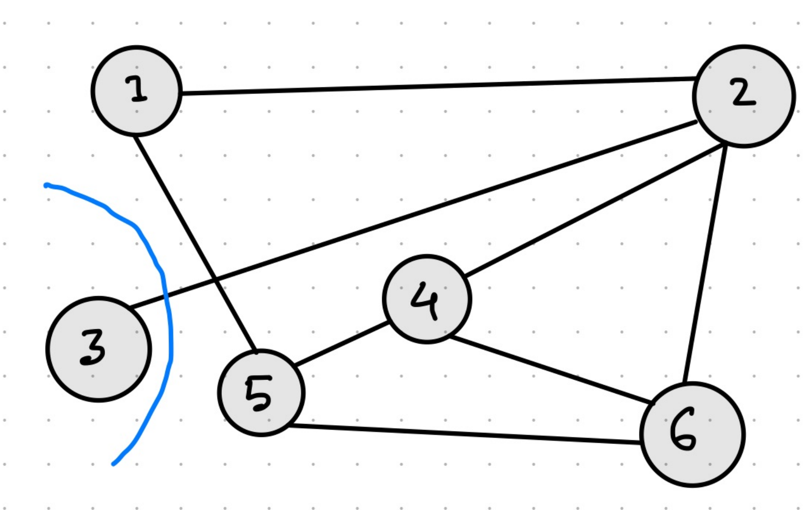
In [8]:
```python
print("Clusters:", label)
```

```
Clusters: [0 0 1 0 0 0]
```

Here also we see that node **{3}** will be different label comapre to other nodes **{1, 2, 4, 5, 6}** ✓

Part (iii)

(iii)



$A = \{1, 2, 4, 5, 6\}$

$\bar{A} = \{3\}$

$$f_i = \begin{cases} \sqrt{|\bar{A}|/|A|} & \text{if } v_i \in A \\ -\sqrt{|A|/|\bar{A}|} & \text{if } v_i \in \bar{A} \end{cases}$$

$$f_1 = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}}$$

$$f_2 = \frac{1}{\sqrt{5}}$$

$$f_3 = -\sqrt{\frac{5}{1}} = -\sqrt{5}$$

$$f_4 = \frac{1}{\sqrt{5}}$$

$$f_5 = \frac{1}{\sqrt{5}}$$

$$f_6 = \frac{1}{\sqrt{5}}$$

$$\therefore f = (f_1, f_2, f_3, f_4, f_5, f_6) = \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -\sqrt{5}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$$

Part (iv)

(iv)

$$f^T L f = |V| . \text{RatioCut}(A, \bar{A})$$

$$f^T L f = \left(\frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}} \ -\sqrt{5} \ \frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}}\right) \begin{bmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 4 & -1 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & -1 & 0 & -1 & -1 & 3 \end{bmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 1/\sqrt{5} \\ -\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

$= 7.2$ [Calculate using jupyter Notebook]

$|V| = 6$

$$\text{RatioCut} = \frac{1}{1} + \frac{1}{5} = \frac{6}{5}$$

$$\therefore |V| . \text{Rationcut}(A, \bar{A}) = 6 . \frac{6}{5} = \frac{36}{5} = 7.2$$

So, $f^T L f = |V| . \text{Ratiocut}(A, \bar{A})$ ✓

In [9]:
```python
import math
x = 1/math.sqrt(5)
y = -math.sqrt(5)
```

In [10]:
```python
f_t = np.array([x, x, y, x, x, x])
```

In [11]:
```python
f = np.array([[x], [x], [y], [x], [x], [x]])
```

In [12]:
```python
mul = f_t @ L @ f
print(mul)
```

```
[7.2]
```

Part (v)

(v)

All one vector, $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$f . v = \left(\frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}} \ -\sqrt{5} \ \frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}}\right) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$= 0$ ✓

So, f and v are orthogonal.

$$\|f\|^2 = f . f^T$$

$$= \left(\frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}} \ -\sqrt{5} \ \frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}} \ \frac{1}{\sqrt{5}}\right) \begin{pmatrix} 1/\sqrt{5} \\ 1/\sqrt{5} \\ -\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$ ✓

$= 6 = n$

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