Exercise 3.

The likelihood function is:

$$L(\theta|x) = \prod_{i=1}^{n} e^{-(x_i-\theta)} I(x_i \geq \theta)$$

$$= e^{-\sum_{i=1}^{n} x_i + n\theta} I(x_1 \geq \theta) \prod_{i=1}^{n} I(x_i \in \mathbb{R})$$

$$= \underbrace{e^{n\theta} I(x_1 \geq \theta)}_{g(x_1|\theta)} \underbrace{e^{-\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} I(x_i \in \mathbb{R})}_{h(x)}$$

Here, $x_1 = \min(x_1, x_2, \dots x_n)$

Here x_1 is a sufficient statistic by Factorization theorem.

Likelihood Ratio Test Statistic is

$$\lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)} \quad \text{———} \quad (1)$$

Here, $\hat{\theta} = \arg\max_{\theta \in \Theta} L(\theta|x)$; $\Theta = \{\theta: -\infty < \theta < \infty\}$

and $\hat{\theta}_0 = \arg\max_{\theta \in \Theta_0} L(\theta|x)$; $\Theta_0 = \{\theta: -\infty < \theta \leq \theta_0\}$

Now for 1st case:

- When $\theta \leq x_1$

  then $L(\theta|x) = e^{-\sum_{i=1}^{n} x_i + n\theta}$,

  which increases as θ increases.

□ when $\theta > x_1$, then $L(\theta|x) = 0$

So, $L(\theta|x)$ is an increasing function when θ is less than or equal to the minimum order statistic x_1; when θ is larger than x_1 the likelihood functions drops to zero.

So, $\hat{\theta} = x_1$ or $\min(x_1, \dots, x_n)$

and $$\sup_{\theta \in \Theta} L(\theta|x) = L(\hat{\theta}|x)$$
$$= L(x_1|x)$$

For second case:

□ when $\theta_0 < x_1$, then the largest $L(\theta|x)$ can be is $L(\theta_0|x)$.

So, $\hat{\theta}_0 = \theta_0$

□ when $\theta_0 \geq x_1$,

then, $\hat{\theta}_0 = x_1$ or $\min(x_1, \dots x_n)$

Therefore,
$$\hat{\theta}_0 = \begin{cases} \theta_0, & \theta_0 < x_1 \\ x_1, & \theta_0 \geq x_1 \end{cases}$$

Now eqn (1) become

$$\lambda(x) = \frac{L(\hat{\theta}_0|x)}{L(\hat{\theta}|x)} = \begin{cases} \dfrac{L(\theta_0|x)}{L(x_1|x)}, & \theta_0 < x_1 \\[2ex] \dfrac{L(x_1|x)}{L(x_1|x)} = 1, & \theta_0 \geq x_1 \end{cases}$$

we have $H_0: \theta \leqslant \theta_0$ vs $H_1: \theta > \theta_0$

- If $x_1 \leqslant \theta_0$, we certainly don't want to reject H_0 and conclude that $\theta > \theta_0$
- It is only when $x_1 > \theta_0$ do we have evidence that θ might be Larger than θ_0. So, the larger the x_1, the smaller the $\lambda(x)$, the more evidence against H_0.

$$\text{Now, } \lambda(x) = \frac{L(\theta_0|x)}{L(x_1|x)} = \frac{e^{-\sum_{i=1}^{n} x_i + n\theta_0}}{e^{-\sum_{i=1}^{n} x_i + n x_1}}$$

$$= e^{-n(x_1 - \theta_0)}$$

$$[x_1 = \min(x_1 \dots x_n)]$$