

Exercise 1:

Let X (random variable) is σ -subgaussian if there exists $\sigma > 0$, such as:

$$\forall t \in \mathbb{R}, \mathbb{E}[\exp(tX)] \leq \exp\left(\frac{\sigma^2 t^2}{2}\right) \quad \dots (1)$$

$$\begin{aligned} \mathbb{E}(\exp(tX)) &= \sum_{n \geq 0} t^n \frac{\mathbb{E}(X^n)}{n!} \leq \exp\left(\frac{t^2 \sigma^2}{2}\right) \\ &= \sum_{n \geq 0} \left(\frac{\sigma^2 t^2}{2}\right)^n \frac{1}{n!} \end{aligned}$$

Up to order 2 and rearranging terms of order greater than 2 on the left hand side:

$$1 + t\mathbb{E}(X) + \frac{t^2}{2} \mathbb{E}(X^2) \leq 1 + \frac{\sigma^2 t^2}{2} + g(t)$$

$$\text{where, } \frac{g(t)}{t^2} \xrightarrow{t \rightarrow 0} 0. \text{ So, by dividing both} \quad \dots (2)$$

side by t and taking the limit when $t \rightarrow 0_+$

We show that $\mathbb{E}[X] \leq 0$. with $t \rightarrow 0_-$ we

Prove that $\mathbb{E}(X) \geq 0$. So, $\mathbb{E}(X) = 0$.

By dividing both side of (2) by t^2 and taking the limit we obtain

$$\mathbb{E}[X^2] \leq \sigma^2$$
$$\text{or, } \text{Var}[X] \leq \sigma^2$$



$$\begin{aligned} & \mathbb{E}[e^{t(X_1+X_2)}] \\ &= \mathbb{E}[e^{tX_1}] \mathbb{E}[e^{tX_2}] \\ &= e^{\frac{\sigma_1^2 t^2}{2}} + e^{\frac{\sigma_2^2 t^2}{2}} \\ &= \exp\left(\frac{t^2 \sqrt{\sigma_1^2 + \sigma_2^2}}{2}\right) \end{aligned}$$

So, $X_1 + X_2$ is $\sqrt{\sigma_1^2 + \sigma_2^2}$ Subgaussian.



$$\begin{aligned} \mathbb{E}[\exp(tcX)] &\leq \exp\left(t^2 c^2 \frac{\sigma}{2}\right) \\ &\leq \exp\left(|c|^2 \cdot \frac{\sigma}{2} \cdot t^2\right) \end{aligned}$$

So, cX is $|c|\sigma$ -Subgaussian.