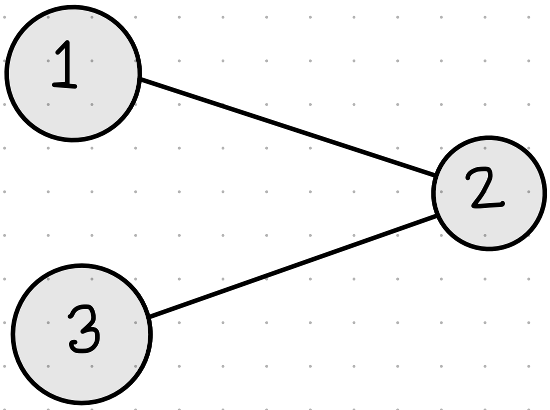
Exercise 1:



- Laplacian matrix: $L(G) = D(G) - A(G)$

$$L(G) = \underset{\substack{\uparrow \\ \text{Degree} \\ \text{Matrix}}}{D(G)} - \underset{\substack{\uparrow \\ \text{Adjacency} \\ \text{matrix}}}{A(G)}$$

$$D(G) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L(G) = D(G) - A(G)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

From the definition of the eigenvector v corresponding to the eigenvalue λ we have

$$Av = \lambda v$$

Then, $Av - \lambda v = (A - \lambda I)v = 0$

Equation has a non zero solution if and only if:

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$$\begin{array}{ccc:cc} 1-\lambda & -1 & 0 & 1-\lambda & -1 \\ -1 & 2-\lambda & -1 & -1 & 2-\lambda \\ 0 & -1 & 1-\lambda & 0 & -1 \end{array}$$

$$\begin{aligned} &= (1-\lambda)(2-\lambda)(1-\lambda) + (-1)(-1)\cdot 0 + 0\cdot(-1)(-1) \\ &\qquad - 0\cdot(2-\lambda)\cdot 0 - (-1)(-1)(1-\lambda) \\ &\qquad - (1-\lambda)(-1)(-1) \end{aligned}$$

$$\begin{aligned} &= -\lambda^3 + 4\lambda^2 - 3\lambda \\ &= -\lambda(\lambda^2 - 4\lambda + 3) \\ &= -\lambda(\lambda - 1)(\lambda - 3) \overset{!}{=} 0 \end{aligned}$$

$$\therefore \lambda_1 = 0, \quad \lambda_2 = 1 \quad \& \quad \lambda_3 = 3$$

For $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Solve it by gaussian elimination

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - (-1) R_1 \to R_2} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1) \cdot R_2 \to R_3} \quad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-1) R_2 \to R_1} \quad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$

$$x = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix}, \text{ Let } x_3 = 1, \; v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 1$

$$A - \lambda_2 I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \underset{\sim}{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 \times (-1) \to R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 \times (-1) \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1) R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-1) \cdot R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + x_3 = 0$$
$$x_2 = 0$$

$$X = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} ; \text{ Let } x_3 = 1, \; V_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For, $\lambda_3 = 3$

$$A - \lambda_3 I = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right) \underset{\sim}{R_1 / (-2) \to R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - (-1) \cdot R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & -1/2 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 / (-1/2) \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1) R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

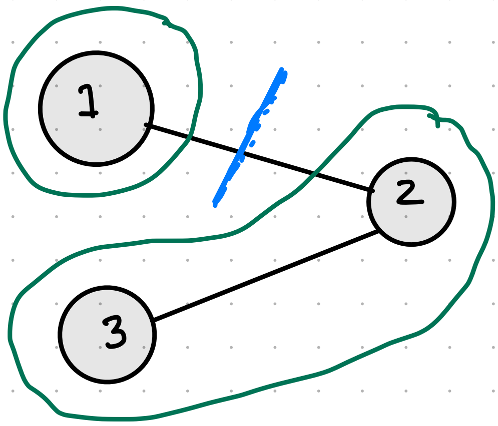
$$\underset{\sim}{R_1 - \left(\tfrac{1}{2}\right) R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$

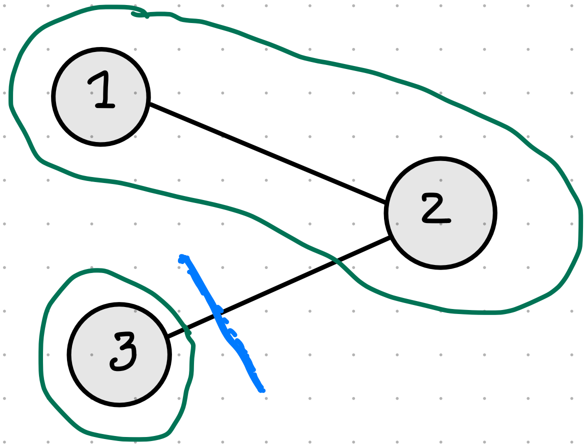
$$\therefore \; X = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} \; ; \text{ let } x_3 = 1, \quad v_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{Ratio Cut} = \sum_{i=1}^{K} \frac{cut(A_i, \bar{A}_i)}{|A_i|}$$

Case 1:

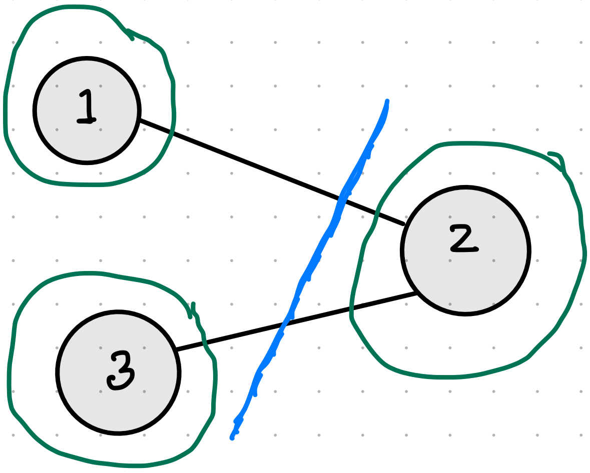


or



$$\text{Ratio cut} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}$$

Case 2:



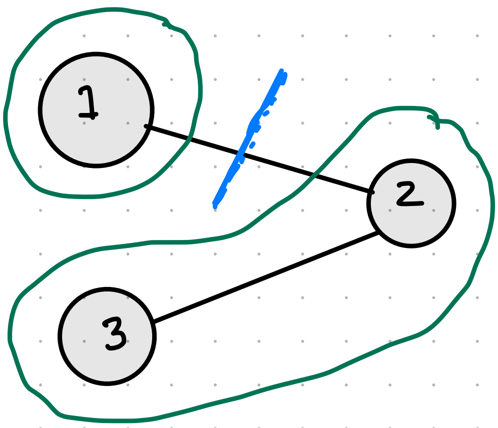
$$\text{Ratio cut} = \frac{1}{1} + \frac{2}{1} + \frac{1}{1}$$

$$= 4$$

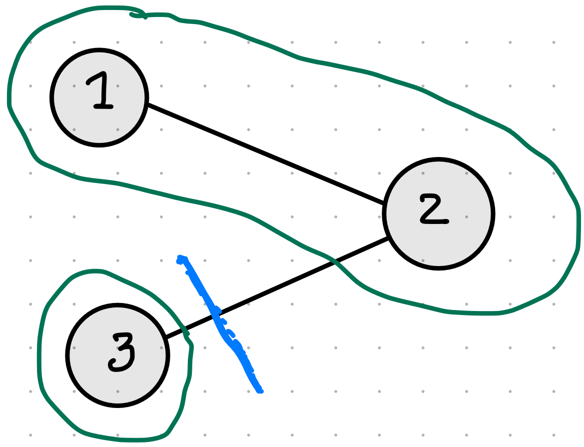
So, case 1 Ratio cut is minimal.

$$\text{Ncut} = \sum_{i=1}^{k} \frac{\text{cut}(A_i, \bar{A}_i)}{\text{Vol}(A_i)}$$

Case 1:

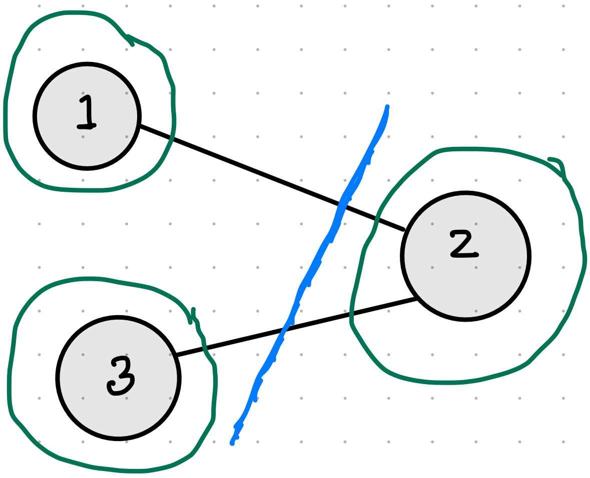


or



$$\text{Ncut} = \frac{1}{1} + \frac{1}{3} = \frac{4}{3}$$

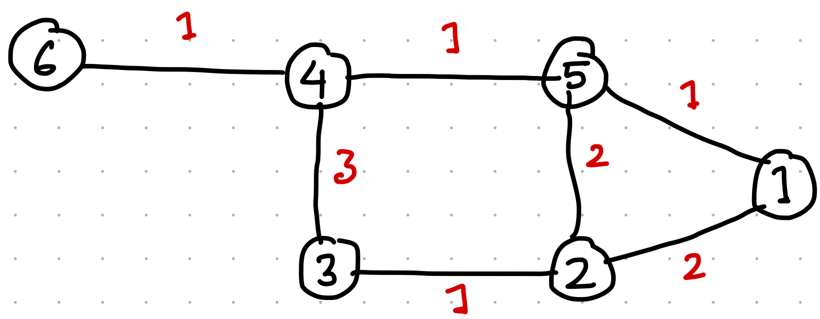
Case 2:



$$\text{Ncut} = \frac{1}{1} + \frac{2}{2} + \frac{1}{1}$$
$$= 3$$

So, case 1 Ncut is minimul

Extra:



$$D(G) = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

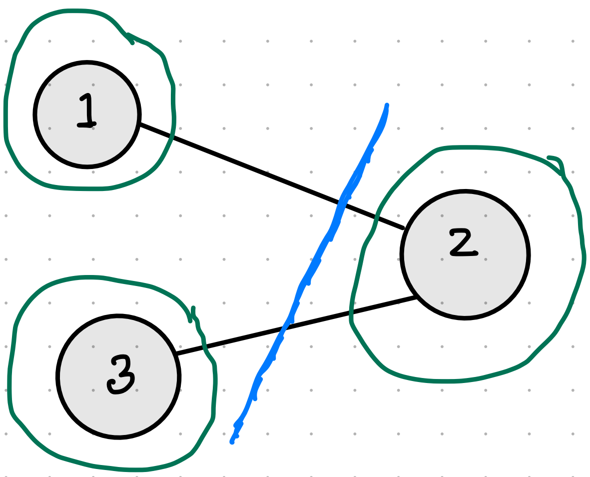
Always Diagonal হবে। Weight দিয়া থাকলে weight এর sum otherwise weight 1 ধরে করতে হবে।

Same as Degree matrix

$$A(G) = \begin{array}{c} \\ \begin{bmatrix} 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} \text{sum} \\ 3 \\ 5 \\ 4 \\ 5 \\ 4 \\ 1 \end{array}$$

যার সাথে যে connected থাকবে সেই weight বসে যাবে, weight না থাকলে 1 ধরে করতে হবে।

$$L(G) = \begin{pmatrix} 3 & -2 & 0 & 0 & -1 & 0 \\ -2 & 5 & -1 & 0 & -2 & 0 \\ 0 & -1 & 4 & -3 & 0 & 0 \\ 0 & 0 & -3 & 5 & -1 & -1 \\ -1 & -2 & 0 & -1 & 4 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$



$$\text{Ratio Cut} = \frac{\text{Node এর যতগুলো edge এর উপর দিয়ে যায়}}{\text{সেই পাশে কতগুলো Node.}}$$

Node 1 এর 1টা edge cut করছে

$$\frac{1}{1} + \frac{2}{1} + \frac{1}{1}$$

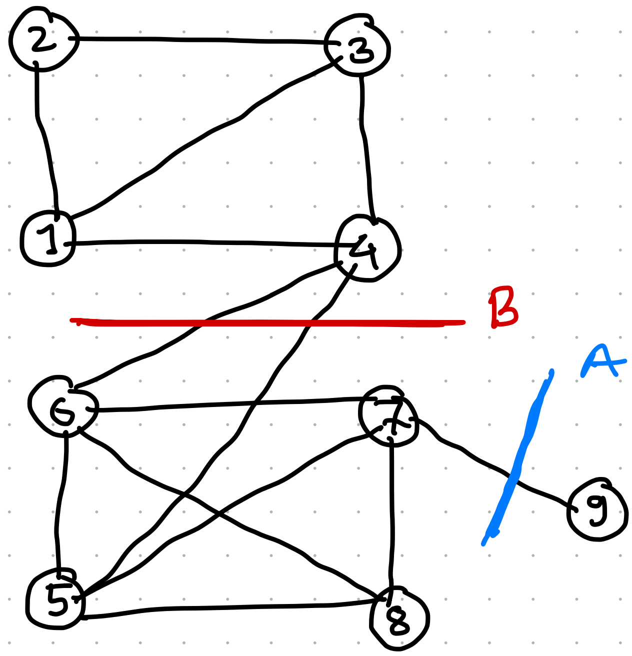
cut এর পাশে 1টাই Node

$$N\,Cut = \frac{\text{Node এর যতগুলো edge এর উপর দিয়ে যাবে}}{\text{ওই Node এর edge সংখ্যা.}}$$

$$\frac{1}{1} + \frac{2}{2} + \frac{1}{1}$$

Node 2 এর edge 2 টাকে cut করেছে

Node 2 এর edge 2 টা.



For Cut A:

$$\text{Ratio Cut} = \frac{1}{1} + \frac{1}{8}$$

$$N\,Cut = \frac{1}{1} + \frac{1}{27}$$

For cut B:

$$\text{Ratio Cut} = \frac{2}{4} + \frac{2}{5}$$

$$\text{Ncut} = \frac{2}{12} + \frac{2}{16}$$