Exercise 1:

i) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

From the definition of the eigenvector v corresponding to the eigen value λ we have

$$Av = \lambda v$$

Then, $Av - \lambda v = (A - \lambda I)v = 0$

$\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(1-\lambda) - (-1).1 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 2 = 0$$

$\therefore \lambda_1 = 1 - i$ & $\lambda_2 = 1 + i$

For $\lambda_1 = 1 - i$

$$A - \lambda_1 I = \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$$

Now

$$\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right) \underset{\sim}{R_1/(i) \to R_1} \left(\begin{array}{cc|c} 1 & i & 0 \\ 1 & i & 0 \end{array}\right) \underset{\sim}{R_2 - 1.R_1 \to R_2} \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + i.x_2 = 0$

$\therefore x_1 = -ix_2$ & $x_2 = x_2$

Let $x_2 = 1$, $v_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

For $\lambda_2 = 1 + i$

$$A - \lambda_2 I = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix}$$

Now,

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \underset{\sim}{R_1/(-i) \to R_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right) \underset{\sim}{R_2 - 1.R_1 \to R_2} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 - i.x_2 = 0$

$\therefore x_1 = i.x_2$ & $x_2 = x_2$

Let $x_2 = 1$, $v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

A matrix "A" can be diagonized if there exists an invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$

The diagonal matrix D is composed of the Eigenvalues:

$$D = \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}$$

The eigenvectors corresponding to the eigenvalues in D compose the ~~compos.~~ columns of P:

$$P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \text{ \& } P^{-1} = \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}$$

Now, $PDP^{-1} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}$

$$= \begin{pmatrix} -1+i & -1-i \\ 1+i & 1-i \end{pmatrix} \begin{pmatrix} -i/2 & 1/2 \\ i/2 & 1/2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= A.$$

~~No.~~ So $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ is diagonized.

(b) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -4-\lambda & -2 \\ 3 & -2-\lambda & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow -\lambda^3 - 2\lambda^2 + 24\lambda = 0$

$\therefore \lambda_1 = 0, \ \lambda_2 = -6 \ \& \ \lambda_3 = 4$

For $\lambda_1 = 0$,

$A - \lambda_1 I = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$

Now,

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -4 & -2 & 0 \\ 3 & -2 & 1 & 0 \end{array}\right) \underset{\sim}{R_2 - 2\cdot R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -8 & -8 & 0 \\ 3 & -2 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 3\cdot R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -8 & -8 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2/(-8) \to R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -8 & -8 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-8)\cdot R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - 2\cdot R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + x_3 = 0 \ \& \ x_2 + x_3 = 0$

$\therefore x_1 = -x_3, \ x_2 = -x_3 \ \& \ x_3 = x_3$

Let $x_3 = 1$. $v_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

For, $\lambda_2 = -6$

$$A - \lambda_2 I = \begin{pmatrix} 7 & 2 & 3 \\ 2 & 2 & -2 \\ 3 & -2 & 7 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} 7 & 2 & 3 & 0 \\ 2 & 2 & -2 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right) \underset{\sim}{R_1/(7) \to R_1} \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 2 & 2 & -2 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - 2.R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 10/7 & -20/7 & 0 \\ 3 & -2 & 7 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 3.R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 10/7 & -20/7 & 0 \\ 0 & -20/7 & 40/7 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2/(10/7) \to R_2} \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & \frac{-20}{7} & \frac{40}{7} & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - \left(\frac{-20}{7}\right).R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 2/7 & 3/7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - \frac{2}{7}.R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore x_1 + x_3 = 0$ & $x_2 - 2x_3 = 0$

$\therefore x_1 = -x_3$, $x_2 = 2x_3$, $x_3 = x_3$

Let $x_3 = 1$, $v_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

For $\lambda_3 = 4$

$$A - \lambda_3 I = \begin{pmatrix} -3 & 2 & 3 \\ 2 & -8 & -2 \\ 3 & -2 & -3 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} -3 & 2 & 3 & 0 \\ 2 & -8 & -2 & 0 \\ 3 & -2 & -3 & 0 \end{array}\right) \underset{\sim}{R_1/(-3) \to R_1} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 2 & -8 & -2 & 0 \\ 3 & -2 & -3 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - 2 \cdot R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 0 & \frac{-20}{3} & 0 & 0 \\ 3 & -2 & -3 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 3 . R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 0 & \frac{-20}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 / \left(\frac{-20}{3}\right) \to R_2} \left(\begin{array}{ccc|c} 1 & -2/3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (-2/3) \cdot R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore \; x_1 - x_3 = 0$

$x_2 = 0$

$\therefore \; x_1 = x_3, \; x_2 = 0 \;\&\; x_3 = x_3$

Let $x_3 = 1$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

The diagonal matrix D is composed of the eigenvalues.

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

The eigenvectors corresponding to the eigenvalues in D compose the columns of P:

$$P = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ -1/6 & 1/3 & 1/6 \end{pmatrix}$$

Now, $PDP^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -6 \end{pmatrix} \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ -1/6 & 1/3 & 1/6 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 4 & 6 \\ 0 & 0 & -12 \\ 0 & 4 & -6 \end{pmatrix} \begin{pmatrix} -1/3 & -1/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ -1/6 & 1/3 & 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 2 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$$

$$= A.$$

$\therefore \begin{pmatrix} 1 & 2 & 3 \\ 3 & -4 & -2 \\ 3 & -2 & 1 \end{pmatrix}$ is diagonized.

(c) $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$

$\det(A-\lambda I)=0$

$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$

$\Rightarrow \quad -\lambda^3+6\lambda^2-11\lambda+6=0$

$\therefore \lambda_1=1, \ \lambda_2=2 \ \& \ \lambda_3=3$

For $\lambda_1=1$,

$A-\lambda_1 I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

Now, $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \underset{\sim}{R_2-1.R_1\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right)$

$\underset{\sim}{R_3-1.R_1\to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right)$

$\underset{\sim}{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\underset{\sim}{R_2/(-2)\to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\underset{\sim}{R_1-1.R_2\to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x_1+x_3=0 \ \& \ x_2=0$

$\therefore x_1=-x_3, \ x_2=0, \ x_3=x_3$

Let $x_3=1$, $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

For $\lambda_2 = 2$

$$(A-\lambda_2 I) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Now,

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right) \underset{\sim}{R_2 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 1.R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1).R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore \; x_1 + x_3 = 0 \;\&\; x_2 + x_3 = 0$

$\therefore \; x_1 = -x_3, \; x_2 = -x_3 \;\&\; x_3 = x_3$

Let $x_3 = 1$. $v_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

For $\lambda_3 = 3$

$$(A-\lambda_3 I) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right) \underset{\sim}{R_1/(-1) \to R_1} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - 1.R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - 1.R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2/(2) \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1-(-1)\cdot R_2 \to R_1 \quad \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore\ x_1 - x_2 = 0 \ \&\ x_3 = 0$

$\therefore\ x_1 = x_2,\ x_2 = x_2 \ \&\ x_3 = 0$

Let, $x_2 = 1$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

The diagonal matrix D is composed of the eigen values:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The eigenvectors corresponding to the eigenvalues in D compose the columns of P:

$$P = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad P^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Now, $$PDP^{-1} = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -2 & 3 \\ 0 & -2 & 3 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

$$= A.$$

$\therefore \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix}$ is diagonized.