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SDA - Sheet 08 - Exercise 1

Problem Setting:

- $n = 8$ multiple-choice questions
- probability to pick randomly the correct answer: $\frac{1}{4}$
- Define θ to be probability of the mathematician to pick the correct answer for a question (independent of the question and the order of the questions.)

  E.g. If $\theta = \frac{3}{4}$, the probability to answer each question correctly is: $\theta^n \approx 10\,\%$
- Define the RV X to identify the number of correct answers: $X \sim Bin(n, \theta)$

1.1) Null Hypothesis H_0: $\theta = 0.25$

Alternative Hypothesis H_1: $0.25 < \theta \leq 1$

1.2) If at least $c = 6$ questions are answered correctly, H_0 is rejected and the person is hired.

$X \geq c \Rightarrow$ rejection of H_0

$$P_{H_0}(X \geq c) = P_{H_0}(X \geq 6) =$$

$$P_{H_0}(X=6) + P_{H_0}(X=7) + P_{H_0}(X=8) =$$

$$\sum_{k \in \{6,7,8\}} \binom{n}{k} \theta^k (1-\theta)^{n-k} =$$

$$\binom{8}{6} \theta^6 (1-\theta)^2 + \binom{8}{7} \theta^7 (1-\theta) + \theta^8 \overset{\theta = 0.25}{=}$$

$$28 \cdot 0.25^6 \cdot 0.75^2 + 8 \cdot 0.25^7 \cdot 0.75 + 0.25^8 \approx 0.004$$

$$\left(\Rightarrow P_{H_0}(X < c) = 1 - P_{H_0}(X \geq c) \approx 0.996 \right)$$

If H_0 is true, the probability that at least 6 questions were answered correctly is approximately 0.4 %.
→ type 1 error: 0.4 %

1.3) sensitivity $\alpha = 0.05$

c is the smallest value in $\{0, \dots, 8\}$ for which it holds $P_{H_0}(X \geq c) \leq \alpha$.

$c = 6$ $\quad P_{H_0}(X \geq 6) \approx 0.004$

$c = 5$ $\quad P_{H_0}(X \geq 5)$

$$\approx \binom{8}{5} \theta^5 (1-\theta)^3 + 0.004$$

$$\overset{0.25}{=} 56 \cdot 0.25^5 \cdot 0.75^3 + 0.004$$

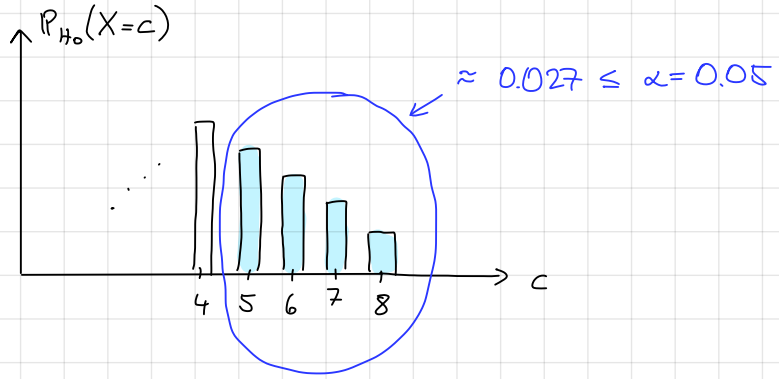
$$\approx 0.027$$

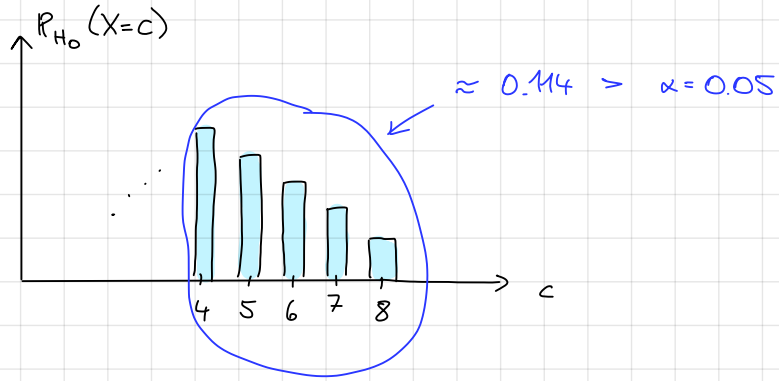
$c = 4$ $\quad P_{H_0}(X \geq 4)$

$$\approx \binom{8}{4} \theta^4 (1-\theta)^4 + 0.027$$

$$\approx 0.114$$

Sketch:





Test: If at least 5 questions were answered correctly ($X \geq 5$), reject H_0.

In the given scenario, where the person has answered 4 questions correctly, he would not be hired.

1.4) $\Theta = 0.85$

Test with $c = 6$:

$$P_\Theta(X < 6) = 1 - \sum_{i=c}^{n} P_\Theta(X = i) = 1 - \sum_{k \in \{6,7,8\}} \binom{n}{k} \Theta^k (1-\Theta)^{n-k}$$

$$= 1 - \sum_{k \in \{6,7,8\}} \binom{8}{k} 0.85^k \, 0.15^{(n-k)} \approx 0.105 = \underline{\underline{10.5\%}}$$

$\Rightarrow$ type 2 error $\approx$ 10.5%

Test with $c = 5$:

$$P_\Theta(X < 5) = \sum_{k \in \{0,1,2,3,4\}} \binom{8}{k} 0.85^k \, 0.15^{(n-k)} \approx 0.021 = 2.1\%$$

$\Rightarrow$ type 2 error $\approx$ 2.1%