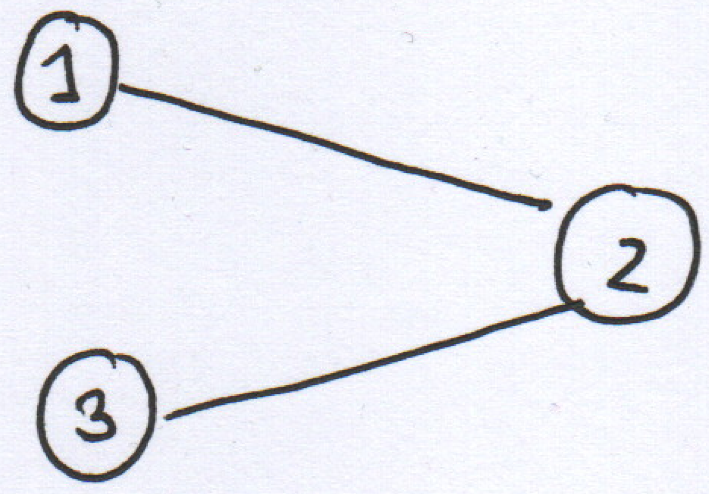
Exercise 1:



Laplacian Matrix: $L(G) = D(G) - W(G)$

Degree Matrix

Adjacency Matrix

$D(G)$:
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$W(G)$:
$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$L(G) = D(G) - W(G)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

From the definition of the eigenvector v corresponding to the eigenvalue λ we have $Av = \lambda v$

Then $Av - \lambda v = (A - \lambda I) \cdot v = 0$

Equation has a non zero solution if and only if.

$$\det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$

$$\begin{array}{ccc:cc} 1-\lambda & -1 & 0 & 1-\lambda & -1 \\ -1 & 2-\lambda & -1 & -1 & 2-\lambda \\ 0 & -1 & 1-\lambda & 0 & -1 \end{array}$$

$$= (1-\lambda)\cdot(2-\lambda)\cdot(1-\lambda) + (-1)\cdot(-1)\cdot 0 + 0\cdot(-1)\cdot(-1)$$

$$- 0\cdot(2-\lambda)\cdot 0 - (-1)\cdot(-1)\cdot(1-\lambda) - (1-\lambda)\cdot(-1)\cdot(-1)$$

$$= -\lambda^3 + 4\lambda^2 - 3\lambda.$$

$$= -\lambda(\lambda^2 - 4\lambda + 3)$$

$$= -\lambda(\lambda - 1)(\lambda - 3) = 0$$

$\therefore \lambda_1 = 0, \ \lambda_2 = 1 \ \& \ \lambda_3 = 3$

For every λ we find its own vectors:

1. $\lambda_1 = 0$

$$A - \lambda_1 I = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$(A - \lambda I)v = 0$$

We solve it by Gaussian Elimination:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - (-1)\cdot R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right)$$

$$R_3 - (-1).R_2 \to R_3 \quad \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1 - (-1).R_2 \to R_1 \quad \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{array}{l} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{array}\right\} (1)$$

$$\therefore X = \begin{pmatrix} x_3 \\ x_3 \\ x_3 \end{pmatrix} ; \text{ Let } x_3 = 1. \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2. $\lambda_2 = 1$.

$$A - \lambda_2 I = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$(A - \lambda I)v = 0$$

$$\left(\begin{array}{ccc|c} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right) \quad R_2 \leftrightarrow R_1 \quad \sim \left(\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$R_1/(-1) \to R_1 \quad \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$R_2/(-1) \to R_2 \quad \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$R_3 - (-1).R_2 \to R_3 \quad \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$R_1 - (-1).R_2 \to R_1 \quad \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left.\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \end{aligned}\right\} (1)$$

$$X = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix} ; \text{ Let } x_3 = 1,\ v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

3. $\lambda_2 = 3$

$$A - \lambda_3 I = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix}$$

$$(A - \lambda I) . v = 0$$

$$\left(\begin{array}{ccc|c} -2 & -1 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right) \underset{\sim}{R_1/(-2) \to R_1} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 - (-1).R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & -1/2 & -1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_2 / (-1/2) \to R_2} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right)$$

$$\underset{\sim}{R_3 - (-1).R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\underset{\sim}{R_1 - (1/2).R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

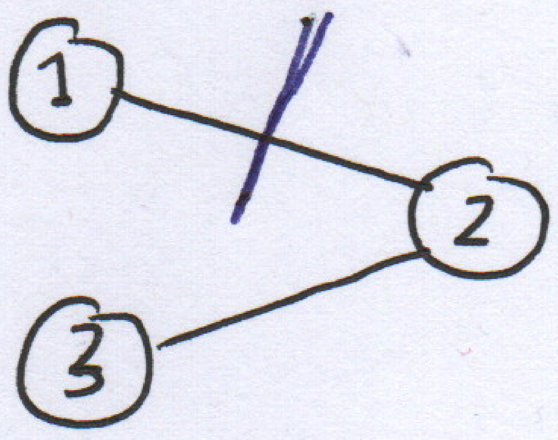
$$x_1 - x_3 = 0$$

$$x_2 + 2 \cdot x_3 = 0$$

$$\therefore X = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix}; \quad x_3 = 1, \quad v_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Ratio cut:

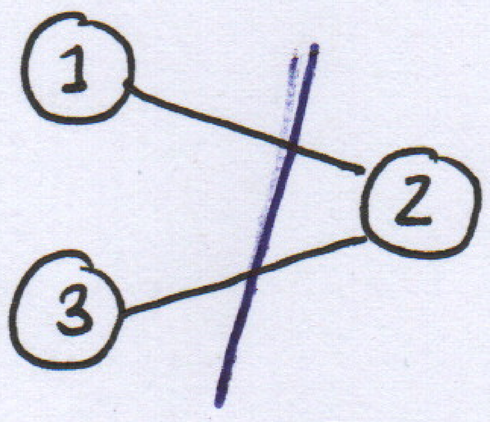
Case 1:



$$\text{Ratiocut} = \sum_{i=1}^{k} \frac{\text{cut}(A_i \cdot \bar{A}_i)}{|A_i|}$$

$\therefore$ Ratiocut $= \frac{1}{1} + \frac{1}{2} = 1.5$

Case 2:

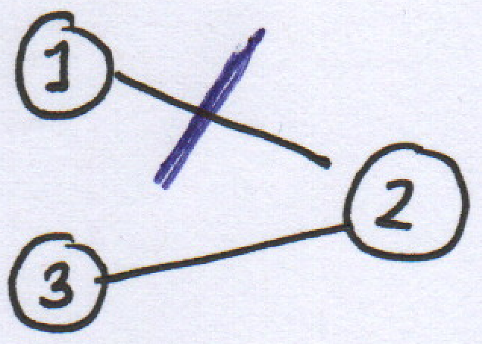


$\therefore$ Ratiocut $= \frac{2}{2} + \frac{2}{1} = 3 > 1.5$

So, Case 1 Ratiocut is minimal.

Ncut:

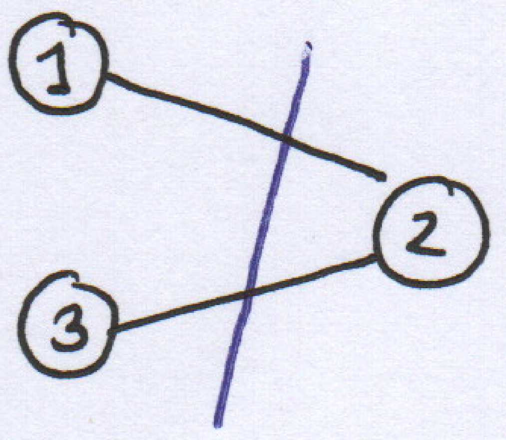
Case 1:



$$\text{Ncut} = \sum_{i=1}^{k} \frac{\text{cut}(A_i, \bar{A}_i)}{\text{vol}(A_i)}$$

$\therefore$ Ncut $= \frac{1}{1} + \frac{1}{3} = 1.33333$

Case 2:



$\therefore$ Ncut $= \frac{2}{2} + \frac{2}{2} = 2 > 1.33333$

So, case 1 Ncut is Minimal.