Let $x_1, \dots, x_n$ be an i.i.d. sample from $U[a,b]$.
Estimate the unknown a and b using Maximum Spacing Estimator.

We have the order statistics of the samples

$$\{x_{(1)}, \dots, x_{(n)}\}$$

and the CDF of $U[a,b]$

$$F(x;a,b) = \begin{cases} 0 & \text{for } x<a \\ \dfrac{x-a}{b-a} & \text{for } x\in[a,b] \\ 1 & \text{for } x>b \end{cases}.$$

Therefore, the spacing $D_i(a,b) = F(x_{(i)};a,b) - F(x_{(i-1)};a,b)$, $i=1,\dots,n+1$
is determined by

$$D_1(a,b) = \frac{x_{(1)}-a}{b-a}$$

$$D_i(a,b) = \frac{x_{(i)}-x_{(i-1)}}{b-a} \qquad i=2,\dots,n.$$

$$D_{n+1}(a,b) = \frac{b-x_{(n)}}{b-a}.$$

Then, the product of the spacing $PS(a,b)$ is given by

$$PS(a,b) = \prod_{i=1}^{n+1} \left(F(x_{(i)};a,b) - F(x_{(i-1)};a,b)\right)$$

$$= \frac{x_{(1)}-a}{b-a} \prod_{i=2}^{n} \frac{x_{(i)}-x_{(i-1)}}{b-a} \, \frac{b-x_{(n)}}{b-a}$$

$$= \frac{1}{(b-a)^{n+1}} (x_{(1)}-a) \prod_{i=2}^{n} (x_{(i)}-x_{(i-1)}) (b-x_{(n)})$$

Note that if $a \geq x_{(1)}$ or $b \leq x_{(n)}$, then $PS(a,b)=0$.
For $a<x_{(1)}$ and $b>x_{(n)}$,

$$\ln PS(a,b) = -(n+1)\ln(b-a) + \ln(x_{(1)}-a) + \ln(b-x_{(n)}) + \text{O.T.}$$

By taking derivatives w.r.t a and b, respectively and setting to zero:

$$\frac{\partial \ln PS(a,b)}{\partial a} = -(n+1)\frac{-1}{(b-a)} - \frac{1}{(x_{(1)}-a)} \overset{!}{=} 0$$

by multiplying $(b-a)(x_{(1)}-a)$ (since $a \neq b$ and $a < x_{(1)}$)

$$\Rightarrow (n+1)(x_{(1)}-a) - (b-a) = 0$$

$$\Rightarrow (-n-1+1)a + (n+1)x_{(1)} - b = 0$$

$$\Rightarrow a = \frac{(n+1)x_{(1)} - b}{n} \quad \cdots ①$$

$$\frac{\partial \ln PS(a,b)}{\partial b} = -(n+1)\frac{1}{(b-a)} + \frac{1}{(b-x_{(n)})} \overset{!}{=} 0$$

(since $a \neq b$ and $b > x_{(n)}$)

$$\Rightarrow -(n+1)(b-x_{(n)}) + (b-a) = 0$$

$$\Rightarrow (-n-1+1)b + (n+1)x_{(n)} - a = 0$$

$$\Rightarrow b = \frac{(n+1)x_{(n)} - a}{n} \quad \cdots ②$$

by plugging ② into ①

$$a = \frac{(n+1)x_{(1)} - \left(\frac{(n+1)x_{(n)} - a}{n}\right)}{n}$$

$$\Rightarrow n^2 a = n(n+1)x_{(1)} - (n+1)x_{(n)} + a$$

$$\Rightarrow (n^2-1)a = (n+1)(n x_{(1)} - x_{(n)})$$

$$\Rightarrow (n-1)a = n x_{(1)} - x_{(n)}$$

$$\Rightarrow \hat{a}_{MS} = \frac{n x_{(1)} - x_{(n)}}{(n-1)} \quad \cdots ③$$

by plugging ③ into ②

$$b = \frac{(n+1)x_{(n)} - \left(\frac{n x_{(1)} - x_{(n)}}{(n-1)}\right)}{n}$$

$$\Rightarrow (n-1)nb = (n-1)(n+1)x_{(n)} - (n x_{(1)} - x_{(n)})$$

$$= (n^2-1+1)x_{(n)} - n x_{(1)}$$

$$\Rightarrow (n-1)nb = n^2 x_{(n)} - n x_{(1)}$$

$$\Rightarrow (n-1)b = n x_{(n)} - x_{(1)}$$

$$\Rightarrow \hat{b}_{MS} = \frac{n x_{(n)} - x_{(1)}}{n-1}$$

Verify $(\hat{a}_{MS}, \hat{b}_{MS})$ is indeed a global maximum using Hessian matrix!