Monte Carlo Simulation in American Option Pricing

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1. Motivation

Financial Engineering Problem

- Need of estimating the payoff of option contracts in order to analyse and determine the most profitable decision path for investors and traders.
- Quantify the risk exposure of financial investments respect to sensitivities or changes on underlying assets

Research Questions

1. How can options be "fairly" evaluated?
2. How to ensure unbiased estimators with low variability?.

2. Background Knowledge

How is the Payoff of an Option determined?

- **Objective:** Estimate the expected value of the american call option V_t at time t = 0.

$$V_0 = \max_{i=0,\dots,d} E\left\{e^{-rt_i}\max(S_i - K, 0)\right\}, \quad \text{where } e^{-rt_i} := \text{discount factor} \tag{1}$$

S_i:= asset value at time t_i, K := strike price, payoff := $\max(S_i - K, 0)$

- **Pricing Problem:** Determine the present value of the future payoff dependent on the future value of the underlying.

Monte Carlo Simulation Application

- Generate sample paths following a Brownian motion.
- Evaluate the payoff function of the option along each path.
- Calculate the expected value (average of the path) to obtain the estimator.

3. Broadie and Glasserman simulation method (Random Tree)

- **Simulation:** generating a tree where at each node b different new branches are created.
- The random tree with b branches per node: $\{S_t^{i_1 i_2\cdots i_t} : \; t = 0, 1, \dots, T; i_j = 1, \dots, b; j = 1, \dots, t\}$
- Broadie and Glasserman introduce two estimators of the option prices.
- The high estimator Θ is defined recursively as follows:

$$\Theta_T^{i_1\cdots i_T} = f_T(S_T^{i_1 i_2\cdots i_T}), \quad \text{and for } t = 0, \dots, T-1: \quad \Theta_t^{i_1\cdots i_t} = \max\left\{h_t(S_t^{i_1\cdots i_t}), \frac{1}{b}\sum_{j=1}^{b} e^{-r}\Theta_{t+1}^{i_1\cdots i_t j}\right\},$$

h_t, f_t the value of the option and payoff at time t, respectively and e^{-r} is the constant discount (special case) factor.

- The low estimator θ is defined recursively as follows:

$$\theta_T^{i_1\cdots i_T} = f_T(S_T^{i_1 i_2\cdots i_T}), \qquad \text{and for } t = 0, \dots, T-1 \tag{2}$$

$$\theta_t^{i_1\cdots i_t} = \frac{1}{b}\sum_{j=1}^{b}\eta_t^{i_1 i_2\cdots i_t j}, \quad t = 1, \dots, T-1, \eta_t^{i_1 i_2\cdots i_t j} = \begin{cases} f_t(S_T^{i_1 i_2\cdots i_t}) & \text{if } f_t(S_T^{i_1 i_2\cdots i_t}) \geq \frac{1}{b}\sum_{i\neq j}^{b} e^{-r}\theta_{t+1}^{i_1\cdots i_t i} \\ e^{-r}\theta_{t+1}^{i_1\cdots i_t j} & \text{otherwise} \end{cases} \tag{3}$$

- Both estimators Θ_0 and θ_0 are biased but asymptotically unbiased. (1- α)% MC - CI:

$$\left[\max\left\{\max(S_0 - K, 0), \bar{\theta}_{0,n} - z_{\frac{\alpha}{2}} s(\bar{\theta}_{0,n})\right\}, \bar{\Theta}_{0,n} + z_{\frac{\alpha}{2}} s(\bar{\Theta}_{0,n})\right] \tag{4}$$

- The final point estimate is given by the following formula: $C = \frac{1}{2}\max\left\{\max(S_0 - K, 0), \theta_0\right\} + \frac{1}{2}\Theta_0$
- Θ_0, θ_0 and C are the estimate in a single simulation and $\bar{\Theta}_{0,n}$, $\bar{\theta}_{0,n}$ and $\bar{C}_n$ are the Monte Carlo estimate after n replications.

4. Longstaff and Schwartz Least Squares Method (LSM)

Aim: Estimate the continuation value by a simple regression via least squares.

- This method combines Monte Carlo Simulation with the Least Squares Method by using backwards induction to approximate the conditional continuation values of the option.
- This method requires the simulation of one single path on a grid of times t_i, with $i = 0, 1, ..., d$
- Let $V_i(x)$ and $f_i(x)$ be the value and the payoff function of the option at time t_i given $S_{t_i} = x$. The continuation value is:

$$C_i(x) = E\left\{V_{i+1}(S_{t+1}) | S_{ti} = x\right\} = \sum_{r=1}^{M}\beta_{ir}\psi_r(x) \tag{5}$$

for some basis functions ψ_r and coefficients $r = 1, .., M$.

- β can be estimated with a simple regression using the values $(S_{ti}, V_{i+1}(S_{t_{i+1}}))$.
- The algorythm proceeds as follows:
  1. Simulate n independent paths on the grid of times.
  2. at $t = T$ set $V_{i+1,j} = f_d(S_{t_j}), j = 1, ..., n$.
  3. for $i = d-1, ..., 1$: (i) Specify the I (index) of paths that are in the money; (ii) Discount the value $\widehat{V}_{i+1,j}, j\epsilon I$ to input in the regression; (iii) Given the estimates $\widehat{V}_{i+i,j}$, run regression to get $\widehat{\beta}_j$; (iv) Estimate the continuation value in eq (5); (v) If $f_i(S_i^j) \geq \widehat{C}_i(S_i^j)$, set $V_{ij}(S_i^j) = f_i(S_i^j)$ else $\widehat{V}_{ij}(S_i^j) = \widehat{V}_{i+1,j}$
  4. Calculate $(\widehat{V}_{11}, ..., \widehat{V}_{1n})/n$ and discount it to get $\widehat{V}_0$
- The estimator convergences in mean square as the number of in the money paths goes to infinity.

5. Black-Scholes Model

- The Black-Scholes model describes the evolution of the stock price through the SDE

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW(t) \iff dS(t) = S(t) + rS(t)dt + \sigma S(t)dW(t) \tag{6}$$

- W is a standard brownian motion. The parameter σ is the standard deviation of asset price and r is the interest rate. In discrete time notation eq.(6) can be equivalently written as:

$$S_{t+\Delta t} = S_t \exp\left(\left[r + \frac{1}{2}\sigma^2\right]\Delta t + \sigma\sqrt{\Delta t}Z\right), \text{ with } Z \sim N(0, 1) \tag{7}$$

- eq.(7) is the path constructing formula for the Monte Carlo simulation of the stock price.

6.Antithetic Variates

- **Antithetic Variates (AV)** attempt to reduce the variance by introducing negative dependence between pairs of replications. In the framework of AV we replace Z in eq. (7) by $-Z$ and get $\tilde{S}_{t+\Delta t}$
- Compute the pair of estimators $(\theta_i, \tilde{\theta}_i)$, $i = 1, \dots, n$, using the path constructing formulas $S_{t+\Delta t}$ and $\tilde{S}_{t+\Delta t}$, respectively. The final MC estimator with antithetic variates is given by $\widehat{\theta}_{AV} = \frac{1}{n}\sum_{i=1}^{n}\frac{\theta_i + \tilde{\theta}_i}{2}$.
- With this approach one can reduce the variance of a simple Monte Carlo estimator.

7. Model Setup

- **Simulation setup:** Asset price (S_0): 36, Expire date (T in year): 1, Number of periods (d): 7 Δt: $\frac{T}{d}$, interest rate (r): 0.05, σ: 0.4, strike price(K):30 and number of samples (n): 10000
- **Random Tree Method:** number of branches (b): 3 and the remaining parameters are taken from Simulation setup.
- **Least Square Method:** The basis function for the regression are the first three Laguerre polynomials. The other parameters are taken from the simulation setup.

8. Simulation results

							CI ($\bar{\theta}_{0,n}$, $\bar{\Theta}_{0,n}$)		CI($\bar{C}_n$)	
simul. method	$\bar{\theta}_{0,n}$	$\bar{\Theta}_{0,n}$	$\bar{C}_n$	$\widehat{\sigma}(\bar{\theta}_{0,n})$	$\widehat{\sigma}(\bar{\Theta}_{0,n})$	$\widehat{\sigma}(\bar{C}_n)$	lower	upper	lower	upper
Simple MC	6.60	9.55	8.35	0.03	0.03	0.02	6.55	9.61	8.31	8.40
MC with AV	6.60	9.54	8.35	0.02	0.01	0.01	6.57	9.56	8.33	8.37

Table: Results of Random Tree method with different simulation methods

			CI ($\widehat{V}_0$)	
simul. method	$\widehat{V}_0$	$\widehat{\sigma}(\widehat{V}_0)$	lower	upper
Simple MC	9.79	0.13	9.53	10.05
MC with AV	9.78	0.06	9.66	9.90

Table: Results of Least Squares method with different simulation methods

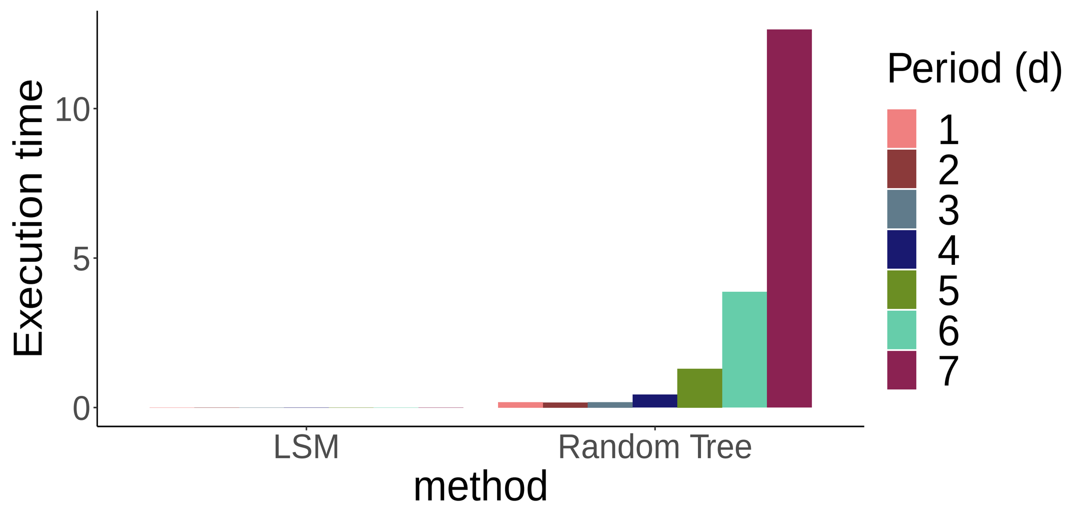


Figure: Comparison of two stock pricing models with respect to computation cost in minutes.

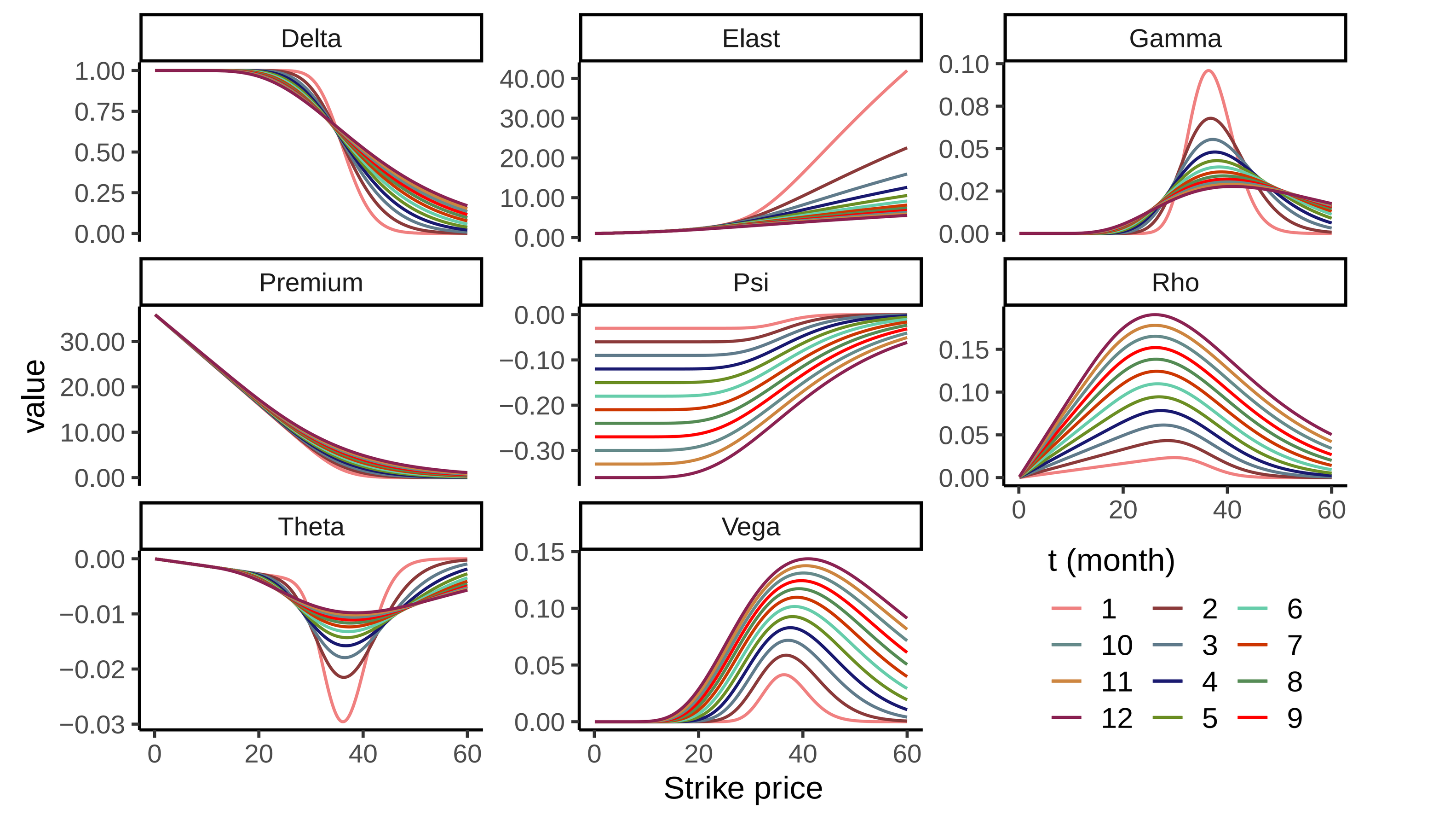
9. Greeks

- **Greeks** represent the sensitivity of the option price (V) with respect to a change in underlying dependent parameters. V is calculated by The Black-Scholes formula.

$$V = Se^{-dt}N(d_1) - Ke^{-rt}N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{(t)}}[ln(\frac{S}{K}) + (r - d + 0.5\sigma^2)t], \quad d_2 = d_1 - \sigma\sqrt{t}$$

- First order Greeks: 1) Delta (Δ) = $\frac{\partial V}{\partial S}$; 2) Elast ($\lambda$) = $\frac{\partial V}{\partial S} \times \frac{S}{V}$; 3) Psi ($\psi$) = $\frac{\partial V}{\partial d}$ (d := dividend); 4) Rho (ρ) = $\frac{\partial V}{\partial r}$; 5) Theta ($\Theta$) = $\frac{\partial V}{\partial \tau}$ (τ: time decay); 6) Vega (ν) = $\frac{\partial V}{\partial \sigma}$; 7) Premium ($\pi$) = $\frac{\partial V}{\partial p}$ (p := premium)
- Second order Greek: Gamma (γ) = $\frac{\partial \Delta}{\partial S_0}$ = $\frac{\partial^2 V}{\partial^2 S_0}$



10. Discussion and Outlook

- Both considered methods provide consistent estimator of the payoff function of the American call option.
- The use of variance reduction methods such antithetic variables are effective by increasing the precision of the estimator.
- The least squares method is simpler to implement and requires lower computational effort than the random tree method. It requires the simulation of a single path instead of generating a ramification of paths.
- The Greeks play a crucial role by decision making. Small changes in volatility, time or in the price of the underlying affect the price of the option. Their values are not observable in the market.
- The development of simulation algorithms which deliver realistic, precise and consistent sensibility estimates are object of study.

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