Lecture # 03

Linear Algebra

Outline

- Vectors
  - Operations
- Matrix
  - Operations
- Transformations
  - Scaling
  - Rotation
  - Translation
- Singular value decomposition

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Vector

- Scalar: $x \in$ R
- Vector: $\boldsymbol{x} \in$ R^N
  - Row Vector v $\in$ R$^{1\times n}$

    $\boldsymbol{x} = [\boldsymbol{x}_1 \quad \boldsymbol{x}_2 \quad \ldots \quad \boldsymbol{x}_n]$
- Column vector v$\in$R$^{n\times 1}$: $$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x_1} \\ \boldsymbol{x_2} \\ \vdots \\ \boldsymbol{x_n} \end{bmatrix} = [\boldsymbol{x_1} \quad \boldsymbol{x_2} \quad \cdots \quad \boldsymbol{x_n}]^{\boldsymbol{T}}$$
- Transpose

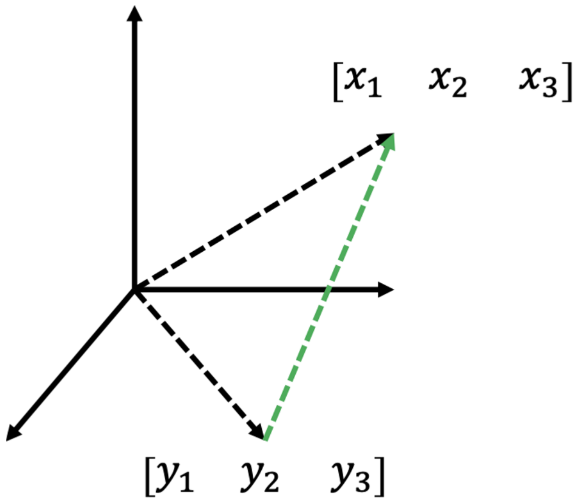
Vectors – use

- Store data in memory
  - Feature vectors
  - Pixel values
  - Any other data for processing

- Any point in coordinate system
  - Can be **n** dimensional

- Difference between two points

$$[x1 - y1 \; x2 - y2 \; x3 - y3]$$



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Vector operations

- Norm – size of the vector
- P-norm

$$\|x\|_p = \left(\sum_i |a_i|^p \right)^{\frac{1}{p}} \qquad p \geq 1$$

- Euclidean norm

$$\|x\|_2 = \left(\sum_i |a_i|^2 \right)^{1/2}$$

- L1-norm

$$\|x\|_1 = \left(\sum_i |a_i| \right)$$

Cont'd

- Inner product (dot product)
  - Scalar number
  - Multiply corresponding entries and odd

$$\boldsymbol{x}^T\,\boldsymbol{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_k^n x_k\ y_k$$

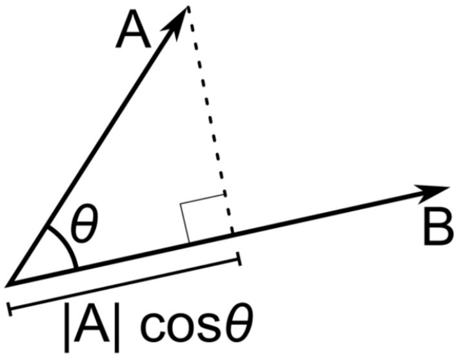
Cont'd

- Inner product (dot product)

$$\boldsymbol{x}_i^T \boldsymbol{x}_i = \sum_k^n \left(x_k^i\right)^2 = \text{squared norm of } \boldsymbol{x}_i$$

- A.B is also *|A||B|cos* (angle between A and B)

- If B is a unit vector, A.B gives projection of A on B



Cont'd

- Outer product

$$\boldsymbol{x}_i \boldsymbol{x}_j^T = \begin{bmatrix} x_1^i x_1^j & x_1^i x_2^j & \cdots & x_1^i x_n^j \\ x_2^i x_1^j & x_2^i x_2^j & \cdots & x_2^i x_2^j \\ \vdots & \vdots & \ddots & \vdots \\ x_n^i x_1^j & x_n^i x_2^j & \cdots & x_n^i x_m^j \end{bmatrix} \Rightarrow$$

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Matrix

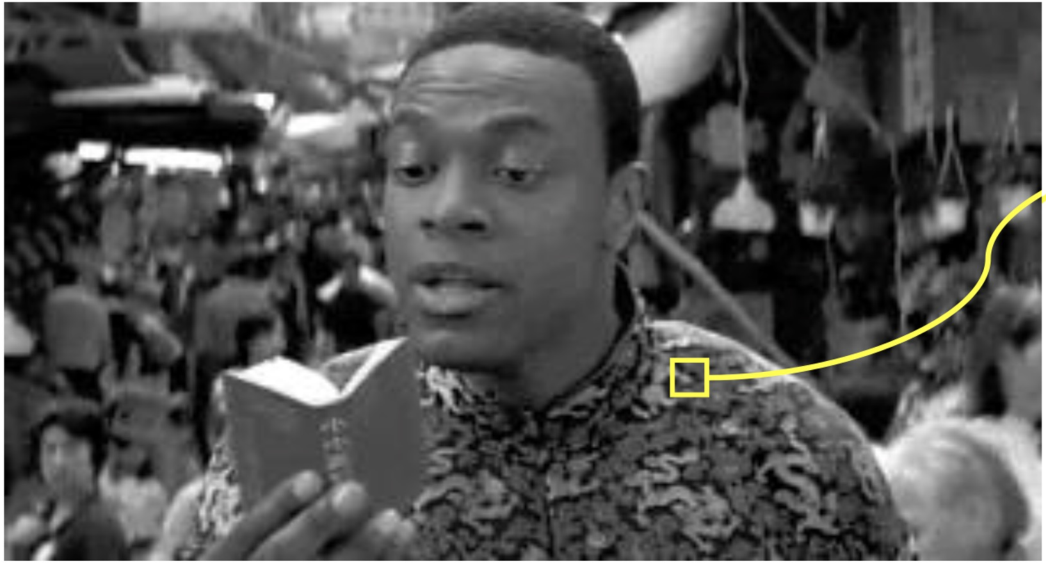
- Array $\boldsymbol{A} \in \mathrm{R}^{m \times n}$ of numbers with shape m by n
  - m rows and n columns

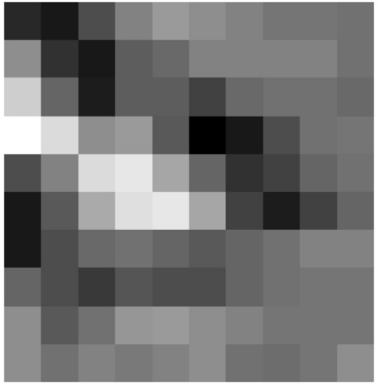
$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A row vector is a matrix with single row
- A column vector is a matric with single column

Matrix — Use

- Image representation - grayscale
  - One number per pixel
  - Stored as **n x m** matrix





0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

Cont’d

- Image representation - RGB
  - 3 numbers per pixel
  - Stored as **n x m x 3** matrix

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

0	3	2	5	4	7	6	9	8
3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
6	5	4	3	2	1	0	3	2
9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

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3	0	1	2	3	4	5	6	7
2	1	0	3	2	5	4	7	6
5	2	3	0	1	2	3	4	5
4	3	2	1	0	3	2	5	4
7	4	5	2	3	0	1	2	3
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9	6	7	4	5	2	3	0	1
8	7	6	5	4	3	2	1	0

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Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

- Both matrices should have same shape, except with a scalar

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 2 = \begin{bmatrix} a+2 & b+2 \\ c+2 & d+2 \end{bmatrix}$$

- Same with subtraction

Cont'd

- Scaling

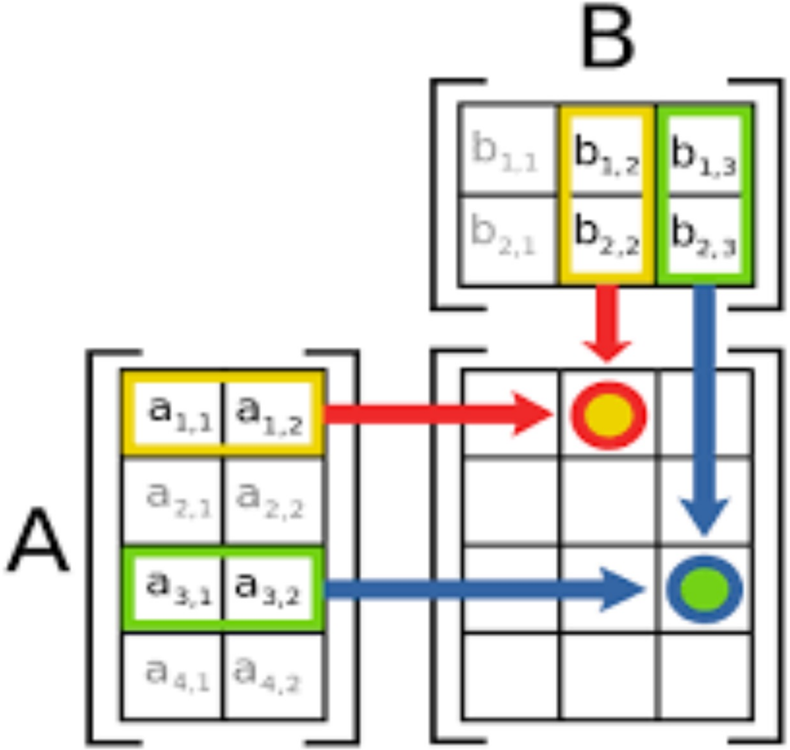
$$s \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s\times a & s\times b \\ s\times c & s\times d \end{bmatrix}$$

- Hadamard product

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \odot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a\times e & b\times f \\ c\times g & d\times h \end{bmatrix}$$

Matrix Operations

- Matrix Multiplication
  - Complexity?
  - **m x n** and **n x p**
  - Results in **m x p** matrix



Cont'd

- Matrix multiplication
- Let $\boldsymbol{a}_i$ denote the i-th column of the matrix $\boldsymbol{A}$, and
- $\boldsymbol{b}_{\mathrm{j}}$ denote the j-th column of the matrix B

$$\boldsymbol{A} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2 \quad \cdots \quad \boldsymbol{a}_n], \text{ and } \boldsymbol{B} = [\boldsymbol{b}_1 \quad \boldsymbol{b}_2 \quad \cdots \quad \boldsymbol{b}_{\boldsymbol{m}}]$$

- The product of the two matrices is defined as

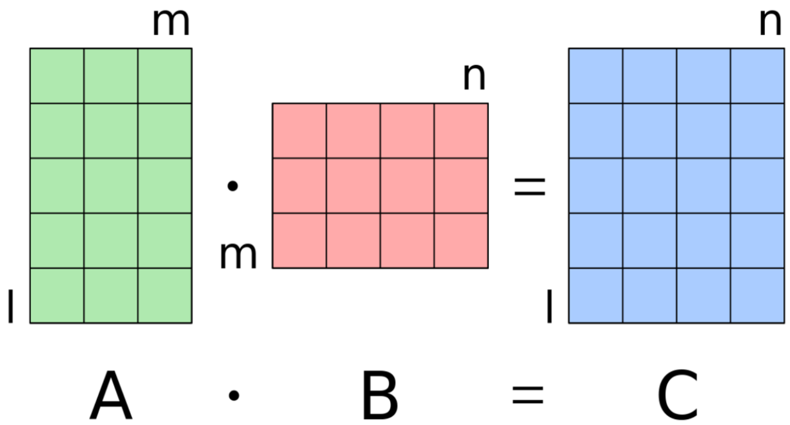
$$\boldsymbol{A}^T\boldsymbol{B} = \begin{bmatrix} \boldsymbol{a}_1^T\boldsymbol{b}_1 & \boldsymbol{a}_1^T\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_1^T\boldsymbol{b}_m \\ \boldsymbol{a}_2^T\boldsymbol{b}_1 & \boldsymbol{a}_2^T\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_2^T\boldsymbol{b}_m \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{a}_n^T\boldsymbol{b}_1 & \boldsymbol{a}_n^T\boldsymbol{b}_2 & \cdots & \boldsymbol{a}_n^T\boldsymbol{b}_m \end{bmatrix}$$

Matrix Operations

- Matrix Multiplication - another interpretation (very intuity)
- The first of column of AB
  - Linear combination of all the columns in **A**

$$[\boldsymbol{a}_1 b_{11} + \boldsymbol{a}_2 b_{12} + \ldots + \boldsymbol{a}_n b_{1n}]$$

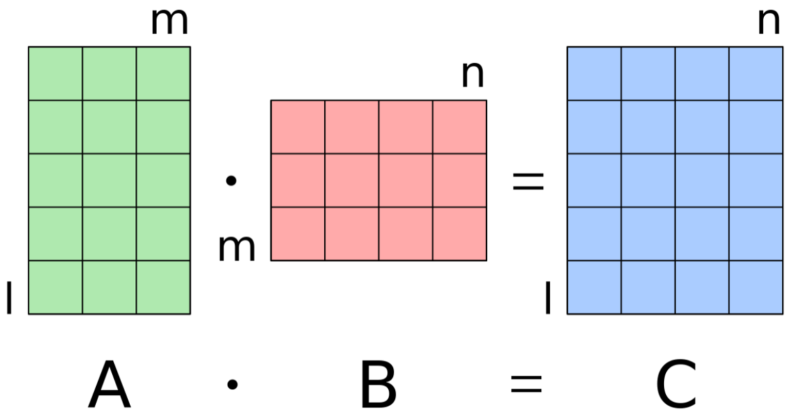
- Similarly we can get other columns…



$$[\boldsymbol{a_1} b_{11} + \boldsymbol{a_2} b_{12} + \ldots + \boldsymbol{a_n} b_{1n}]$$

Cont'd

- How about linear combination of all rows of **B**?
  - • Each row of **C = AB** is a linear combination of rows of **B**



$$[\boldsymbol{b_1} a_{11} + \boldsymbol{b_2} a_{21} + \ \dots + \boldsymbol{b_n} a_{n1}]$$

Matrix Operations

- Transpose

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

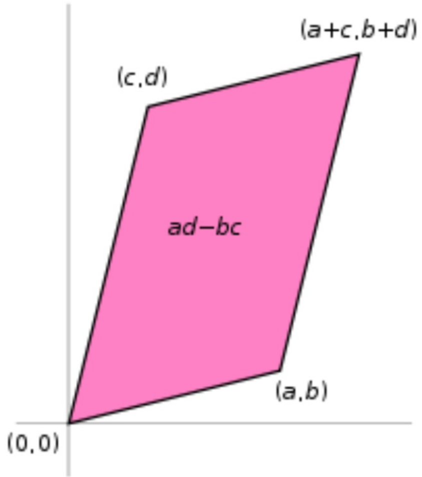
$$\boldsymbol{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Cont'd

- Determinant
  - A scalar

- For A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, det(A) = ad-bc

- Represents area of the parallelogram described by the vectors in the rows of the matrix



Cont'd

- Determinant

$$|\boldsymbol{A}| = \sum_{i=1}^{K} a_{ij} C_{ij}$$

where $\boldsymbol{C_{ij}}$ is the cofactor of $\boldsymbol{a_{ij}}$ defined by

C_{ij} = $(-1)^{i+1}$ $|M_{ij}|$, and

M_{ij} is the minor of matrix $\boldsymbol{A}$ formed by eliminating row i and column j of $\boldsymbol{A}$

- Some Properties
  - $|AB| = |A||B|$
  - $|AB| = |BA|$
  - $|A^T| = |A|$

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Matrix Operations

- Trace
  - Tr(A) = Sum of diagonal elements

$$Tr\begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = a_{00} + a_{11} + a_{22}$$

- Properties
  - tr(AB) = tr(BA)
  - tr(A+B) = tr(A) + tr(B)

Cont'd

- Inverse
  - Given a matrix **A**, its inverse A^{-1} is a matrix such that

$$\mathbf{AA^{-1} = A^{-1}A = I}$$

- Inverse does not always exist
  - Singular vs non-singular

- Properties
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$

Special matrices

- Symmetric matrix

$$A^T = A$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 5 \end{bmatrix}$$

- Skew- symmetric matrix

$$A^T = -A$$

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$$

Cont'd

- Diagonal matrix
  - Used for row scaling

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n \end{bmatrix}$$

- Identity matrix
  - Special diagonal matrix
  - 1 along diagonals

$$I.A = A$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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Transformation - scaling

- Matrices are useful for vector transformations
- Matrix multiplication

$$X' = AX$$

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

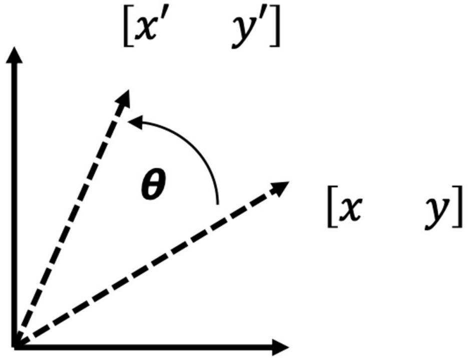
- Linear combination of columns

Transformation

- Rotation
  - Matrix multiplication to rotate a vector

- Rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\boldsymbol{\theta} & -sin\boldsymbol{\theta} \\ sin\boldsymbol{\theta} & cos\boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

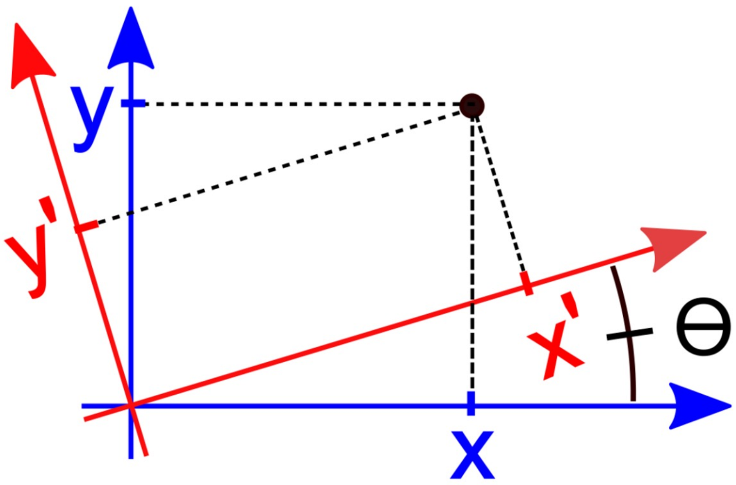


$$x' = \cos\boldsymbol{\theta}x - \sin\boldsymbol{\theta}y$$

$$y' = \sin\boldsymbol{\theta}x + \cos\boldsymbol{\theta}y$$

Transformation - rotation

- Rotating axis first
- Vector v = [x y]
  - x – projection of v on x axis
  - y– projection of v on y axis



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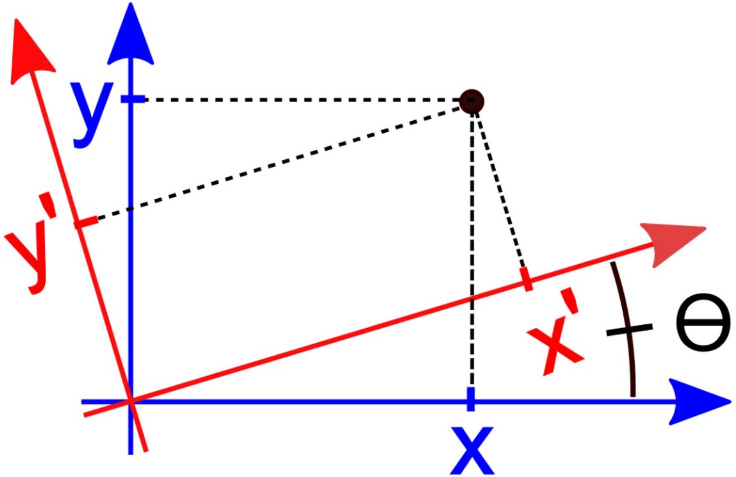
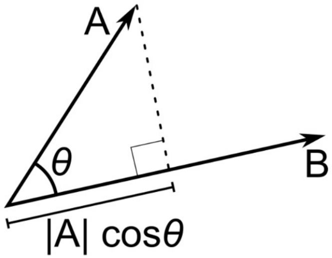
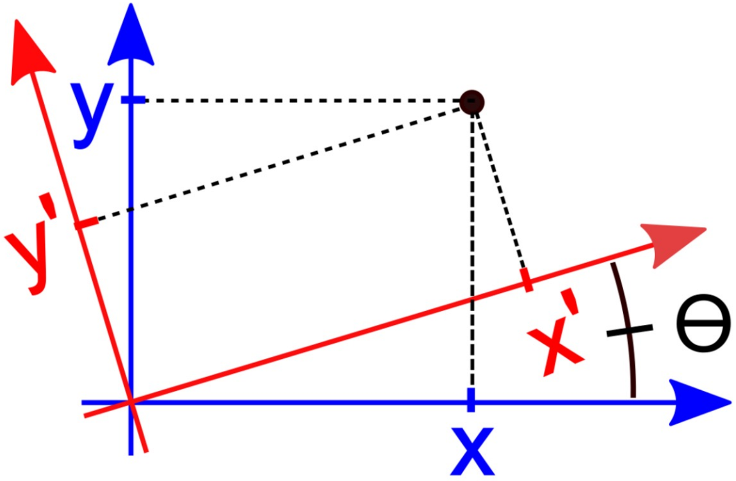


- Remember vector dot product?



Transformation - rotation

- Now we need new x and y axis
  - x′ = v **dot** new x-axis
  - y′ = v **dot** new y-axis



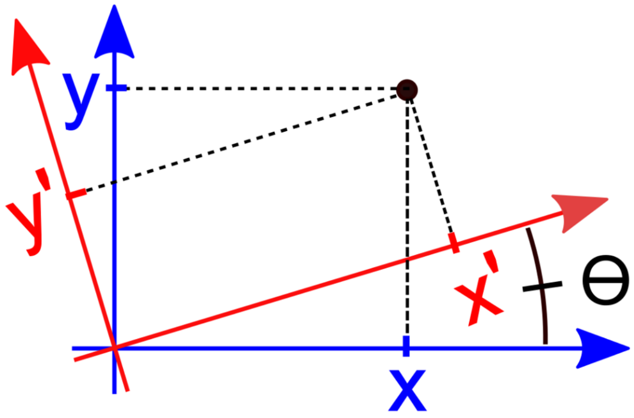
- For rotation of $\boldsymbol{\theta}$
  - New x-axis = [cos$\boldsymbol{\theta}$, sin$\boldsymbol{\theta}$]
  - New y-axis = [-sin$\boldsymbol{\theta}$, cos$\boldsymbol{\theta}$]
- We can form a matrix using new axis

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\boldsymbol{\theta} & sin\boldsymbol{\theta} \\ -sin\boldsymbol{\theta} & cos\boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Cont'd

- When we rotate the axis to left
- We are rotating the vector to right
- We can use rotation matrix to rotate the vector
- We need new x y axis coordinates
  - When we rotate the axis right
- Updated rotation matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\boldsymbol{\theta} & -sin\boldsymbol{\theta} \\ sin\boldsymbol{\theta} & cos\boldsymbol{\theta} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Transformation

- Linear combination

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mathbf{x} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

- Sufficient for
  - Scaling
  - Rotating and
  - Skew transformations
- But no shifting

Cont'd

- Linear combination

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \\ 1 \end{bmatrix}$$

- Still be able to:
  - Scaling
  - Rotating and
  - Skew transformations

Transformation

- Homogeneous System

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- We can also perform shifting now
- This is called homogeneous coordinates

Scaling + rotation + translation

- Careful about the order
  - *V' = (TRS)V*

$$\begin{bmatrix} 1 & 0 & t_y \\ 0 & 1 & t_x \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\boldsymbol{\theta} & -\,sin\boldsymbol{\theta} & 0 \\ sin\boldsymbol{\theta} & cos\boldsymbol{\theta} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{x} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

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Linear independence

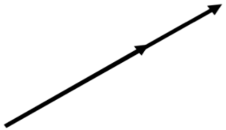
- Linear Independence
  - $\{\boldsymbol{x}_1, \boldsymbol{x}_2 \cdots, \boldsymbol{x}_M\}$ is a set of linearly independent vectors provided

$$a_1\boldsymbol{x}_1 + a_2\boldsymbol{x}_2 + \cdots + a_M\boldsymbol{x}_M = \mathbf{0} \Rightarrow a_1 = 0 = a_2 = a_M$$

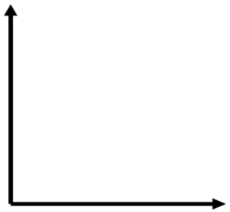
- In other words, none of the vectors can be expressed as a linear combination of the the other vectors
  - Each vector is perpendicular to every other vector
  - For example axis in cartesian coordinate system

Intuition

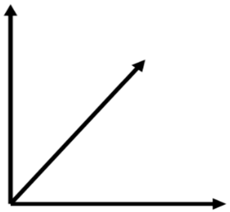
- In terms of features
    - Person recognition - [height, hair color, weight, specs, eye color, etc.]



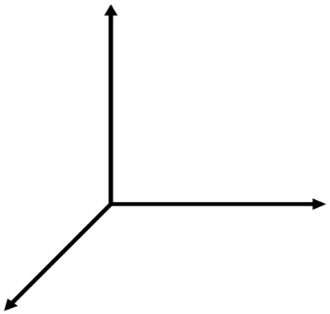
Linearly dependent



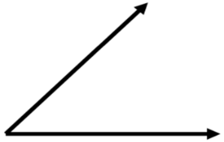
Linearly independent



Linearly dependent



Linearly independent



Linearly independent

Matrix factorization

- Singular value decomposition (SVD)

$$A = U\Sigma V^T$$

- If A is mxn matrix, then
    - U will be m x m,
    - Σ will be m x n, and
    - VT will be n x n

- U and V are unitary matrices
    - Each Column is a unit Vector

- Σ is a diagonal matrix

Singular value decomposition

- Interpretation

$$A = U\Sigma V^T$$

- Columns of U are scaled by values in Σ
- The resultant columns are linearly combined by V
- **A** is formed as a linear combination of columns of U
- If we use all column, we will get original **A**
- We can just use few columns of U and we get an approximation
    - We call these columns principal components

Application