# Control Systems - 7th Semester - Week 2

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# Model

A model is representation or abstraction of reality/system

Who invent model? We, human beings, invent model based on our knowledge

This means the more knowledge a person has, the better he/she can write a model

What is mathematical model?

- A set of equations (linear or differential) that describes the relationship between input and output of a system

# Types of Model and System

In mathematics, we broadly classify systems into 2 types, namely stochastic (random, probabilistic, uncertain) and deterministic (fixed relation between input and output)

To write model for a deterministic system, there are three techniques

- ❑ Black Box
- ❑ Grey Box
- ❑ White Box

# Black Box Model

It is used when only input and output data are available

The internal dynamics are either too complex or totally unknown (sometimes for cyber security purposes, we do not want to label/show the hardware)

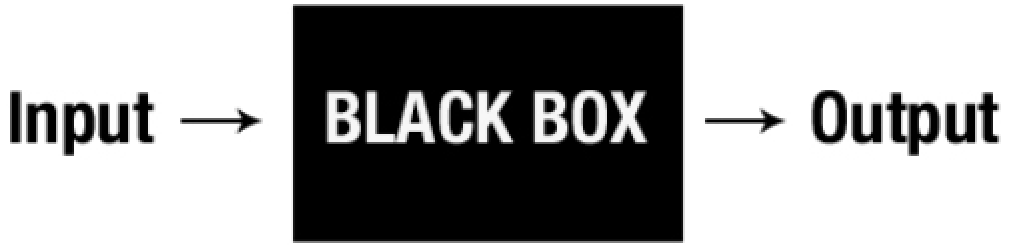


Figure: Black Box Model of a System

It is very hard to analyze or conclude something based on I/O data without having knowledge about the system

# Grey Box Model

It is used when input and output data is known (known means labeled), plus some information (information means knowledge) about internal dynamics of the system are known


Figure: Grey Box Model of a System

In complex systems, we use grey box modeling to identify or estimate the system model

# White Box Model

It is used when the input, output and internal dynamics of the system are known

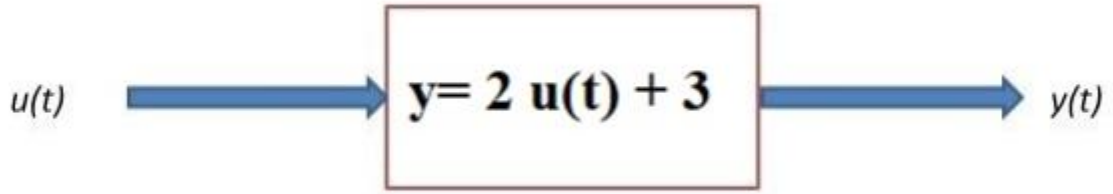


Figure: Grey Box Model of a System

White box models are very easy to predict any future values

Obtaining white box models requires us to know exact mathematical formulas and equations

# Summary of Model Types

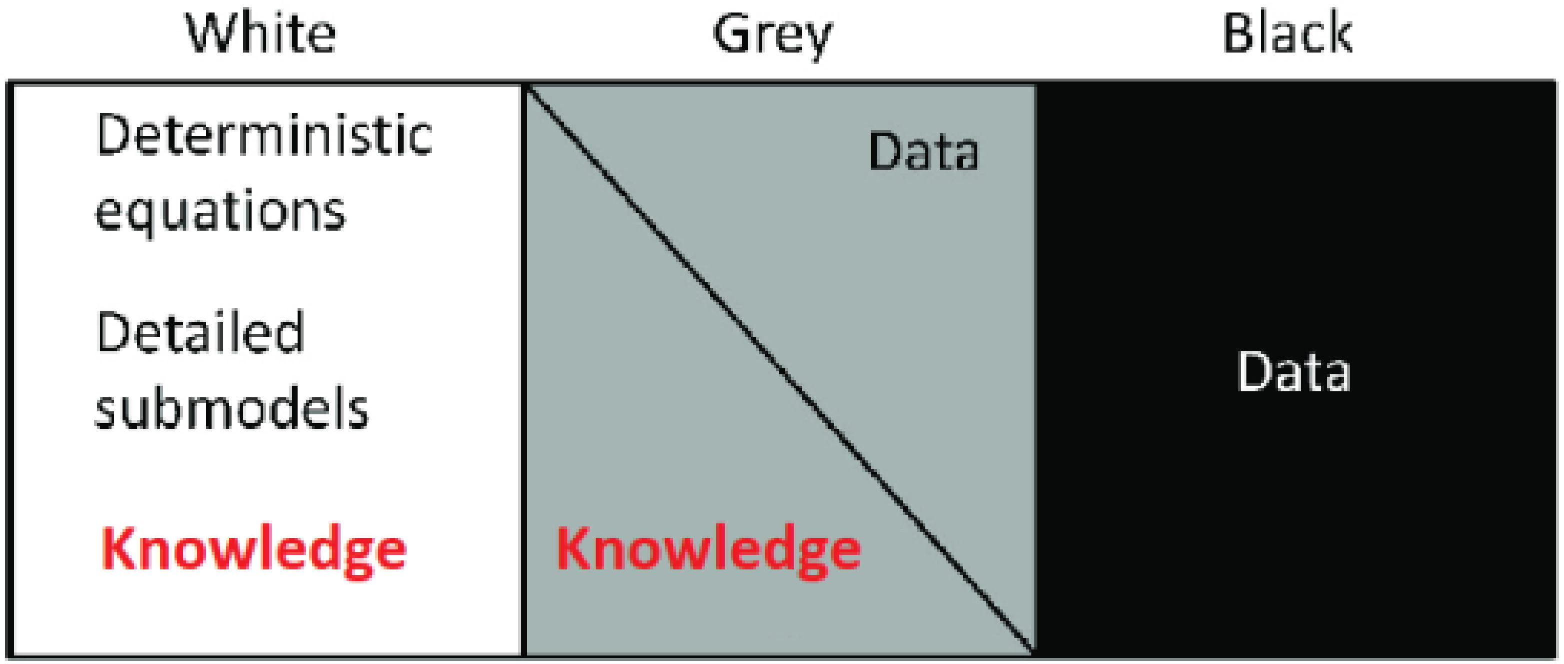


Figure: Techniques for obtaining models of a system

# Equation Writing

In mathematics, we can write static equations as follows:

$$\boldsymbol{y = 2u + 3}$$

If the equations are a function of time (means time-varying), then we equations as follows:

$$\boldsymbol{y(t) = 2u(t) + 3}$$

Remember: If one variable is changing with time (like ***u(t)***), then whole equation changes with time. Mathematically, the following equation is incorrect

$$\boldsymbol{y = 2u(t) + 3}$$

# Equation Example

For example: if ***u(t)*** is given as follows:

| Time | Value |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 5 |
| 4 | 8 |

Table: Example of *u*(*t*)

MATLAB code for plotting the above signal ***u(t)***

```
clear;
clc;
u=[1 3 5 8];
stem(u)
```

# Equation Plot

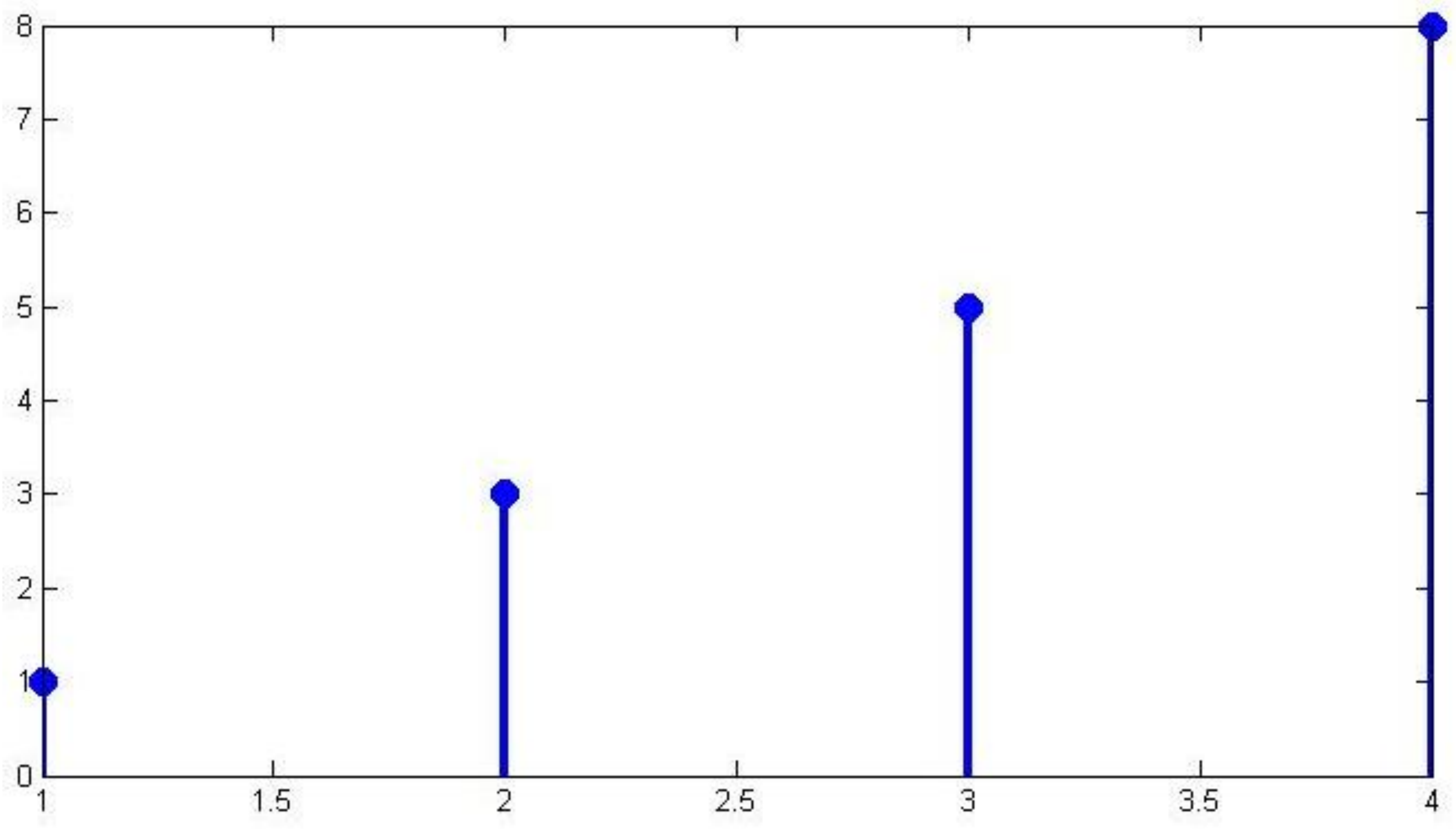


Figure: Plot of *u(t)*

# Equation Plot

Let us put axis function in MATLAB code as follows:

```
clear;

clc;

u=[1 3 5 8];

stem(u)

axis([0 6 0 9])
```

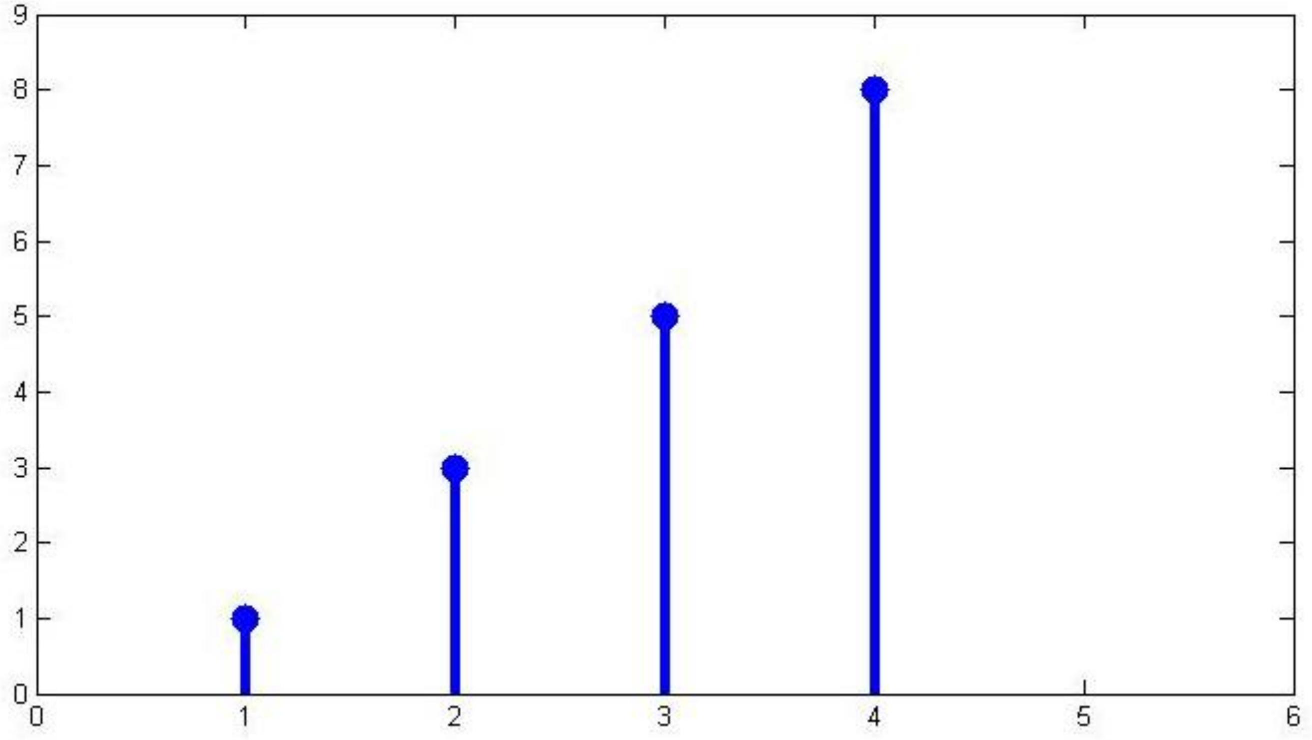


Figure: Plot of *u(t)* with extended axis

Still missing something: the labels for x-axis and y-axis

# Equation Plot

Let us put xlabel and ylabel function in MATLAB code as follows:

```
clear;

clc;

u=[1 3 5 8];

stem(u)

axis([0 6 0 9])

xlabel('Time (sec)')

ylabel('Amplitude')
```

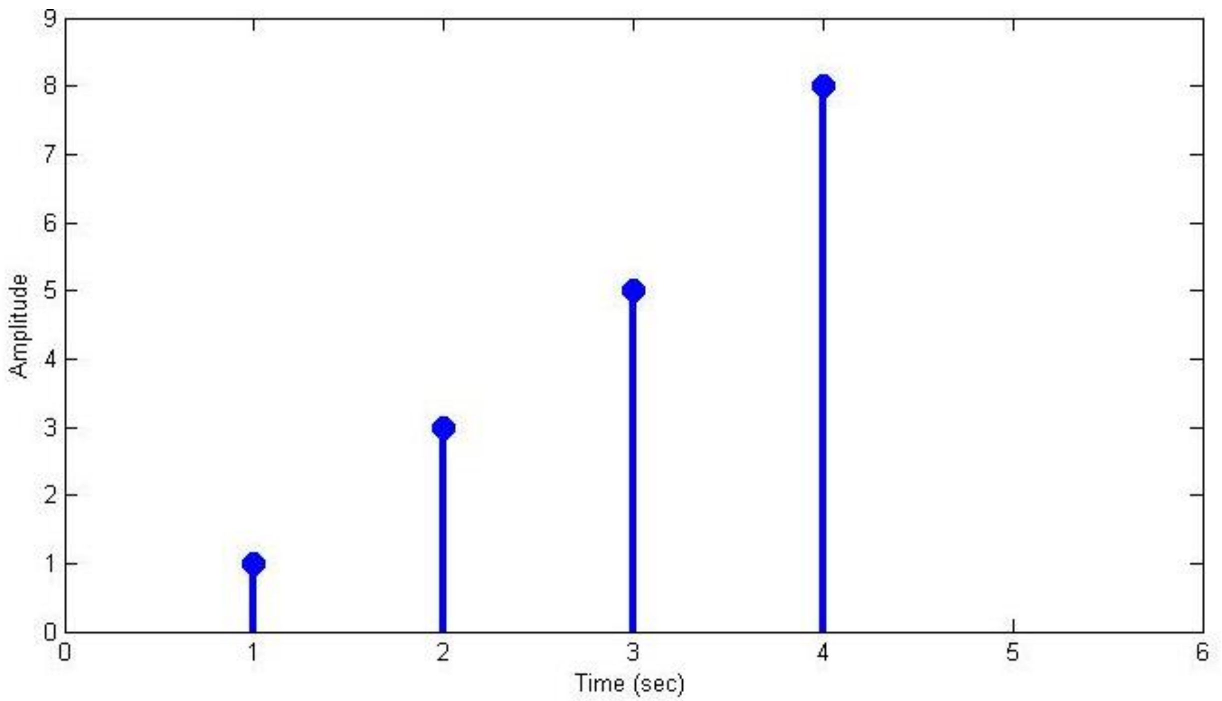


Figure: Plot of *u(t)* with correct labels of axes

# Equation Plot

Now, we have the following equation:

$$y(t) = 2u(t) + 3$$

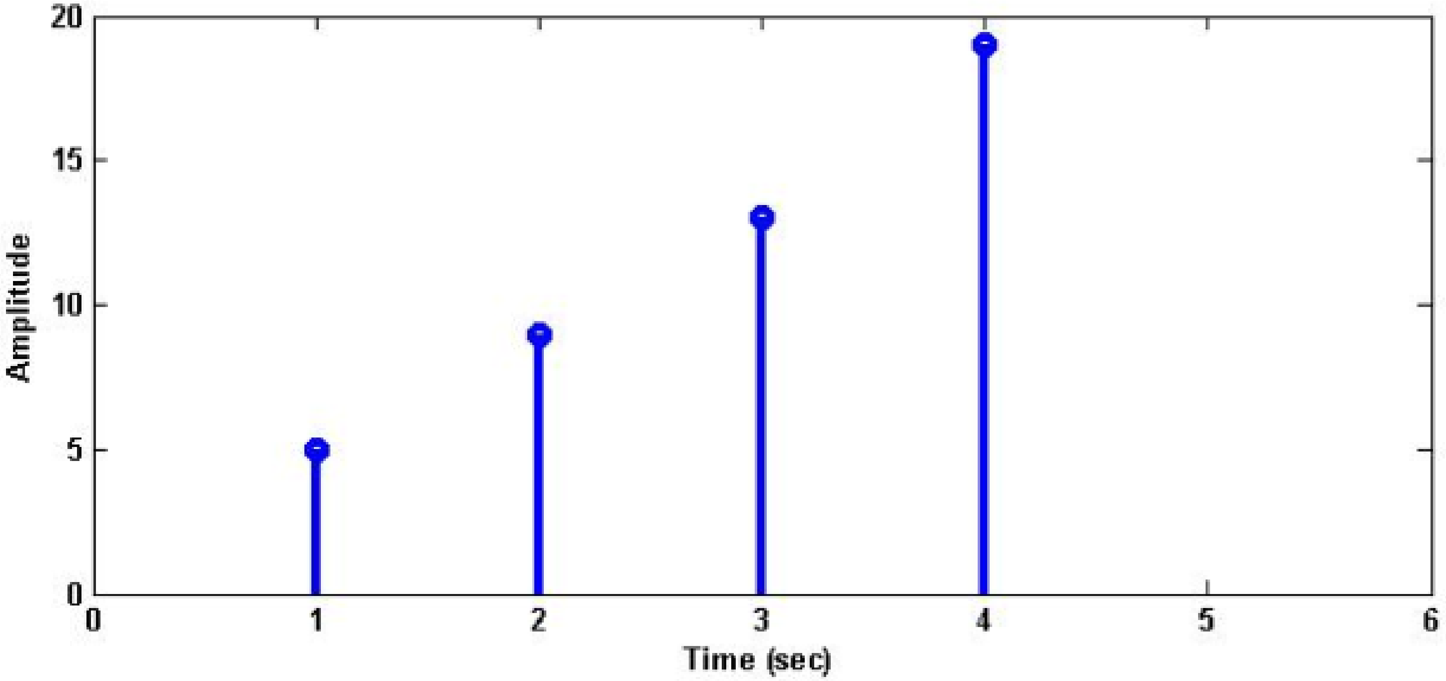


Figure: Plot of *y(t) = 2u(t) + 3*

# Equation Plots

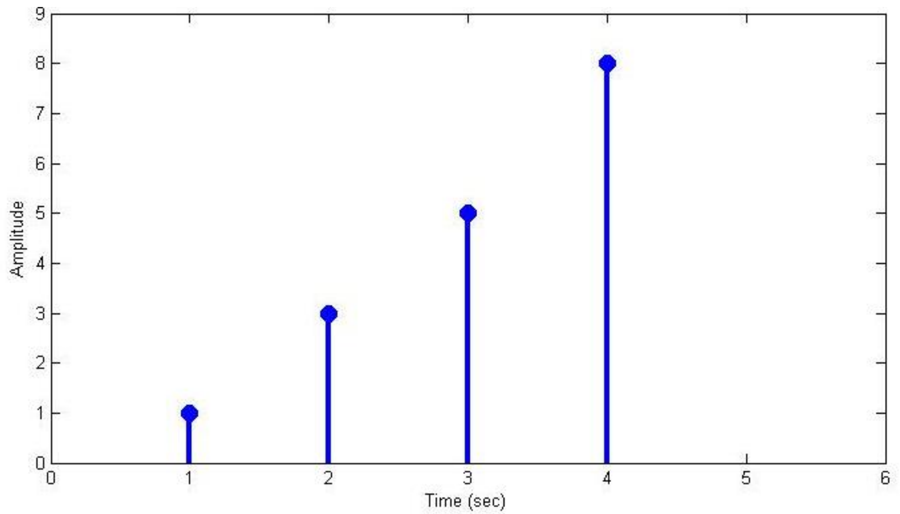


Figure: Plot of *u(t)*

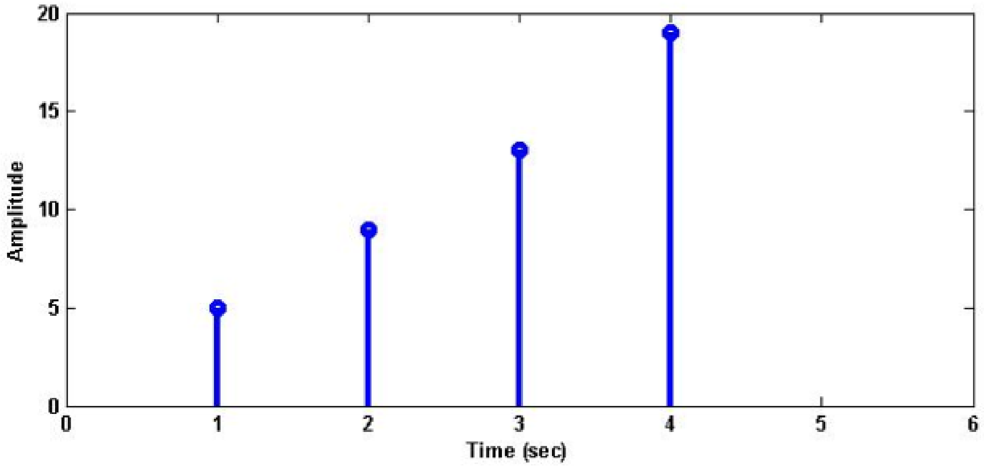


Figure: Plot of $y(t) = 2u(t) + 3$

# Transfer Function

Transfer function: Mathematical model (or relationship) between input and output of a system

What is relationship called in urdu?


Figure: Transfer function

# Transfer Function Symbol

Ohm Law: $\boldsymbol{V = IR}$

Is $\boldsymbol{A} = \boldsymbol{BC}$ Ohm Law?

The answer is yes, but if ***A*** denotes voltage, ***B*** denotes current and ***C*** denotes resistance

***V*** is popular symbol to denote voltage and similarly ***I*** for current and ***R*** for resistance

A transfer function can be denoted by any alphabet from ***A*** to ***Z***, but popular symbols are ***G(s), P (s), H(s)*** and ***T (s)***

# Transfer Function Analysis

**Zeros:** Roots of numerator of a transfer function

**Poles:** Roots of denominator of a transfer function

What information do poles and zeros convey (you have already learnt this in DSP)?

A continuous-time system is stable if all poles are negative

A discrete-time system is stable if all poles are within unit circle

# Stability Definition

Absence of input: If the output goes towards zero (or an equilibrium point),then the system is stable

Presence of bounded input: If the output remains bounded, then the system is stable

Sometimes, we call it BIBO stability. Going back towards our discussion:

If all poles are negative, then the system is stable

# Stability Analysis

Example: Compute the poles, zeros and analyze stability of the following seven transfer functions

$$G_1(s) = \frac{(s-3)}{(s+5)}$$

$$G_2(s) = \frac{(s-3)}{(s-5)}$$

$$G_3(s) = \frac{(s-3)(s+2)}{(s+5)(s-10)}$$

$$G_4(s) = \frac{s(s+2)}{(s+5)(s-10)}$$

$$G_5(s) = \frac{3s}{(s+5)(s-10)}$$

$$G_6(s) = \frac{3s}{2s(s+5)(s-10)}$$

# System Parameters

To recap, in control systems literature, a system has

- input
- output
- variables
- constants

Among the variables present in a system, we choose some variables as state-space variable (based on certain criteria which we will study later on), and call them state-space variables

State-space variables: Those variables which completely describe the behavior of a system

State-space variables are abbreviated as ss variables (or sometimes state variable)

# State-space Model

State-space variables are used to obtain mathematical model of a system

In a system, we can have one or two or many state-space variables

If there is one (1) state-space variable, then we denote it by $x$

In case of more than one (1) state-space variables, then we stack them in a vector and denote it by $\boldsymbol{x}$

# State-space Model

In control systems literature, we use the symbol $\boldsymbol{u(t)}$ to denote input and $\boldsymbol{y(t)}$ to denote output

Before showing you mathematics, I summarize again the main points:

- input denoted by $\boldsymbol{u(t)}$
- output denoted by $\boldsymbol{y(t)}$
- variables
- constants
- state-space variables denoted by $\boldsymbol{x(t)}$

If any parameter is just constant (not a function of time $\boldsymbol{t}$), then we do not write the term $(\boldsymbol{t})$

# State-space Model

The standard template for state-space model is as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax}(\boldsymbol{t}) + \boldsymbol{Bu}(\boldsymbol{t})$$

$$\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{Cx}(\boldsymbol{t}) + \boldsymbol{Du}(\boldsymbol{t})$$

Where $\boldsymbol{x}(\boldsymbol{t})$ denotes the vector having state-space variables, $\dot{\boldsymbol{x}}$ or $\frac{\boldsymbol{dx}}{\boldsymbol{dt}}$ represent the derivative of state-space variables, $\boldsymbol{u}(\boldsymbol{t})$ denotes the input to a system, and $\boldsymbol{y}(\boldsymbol{t})$ denotes the output of a system

The state-space model is sometimes called as ss model also

# State-space Model

A system is composed of variables, constants, inputs and outputs

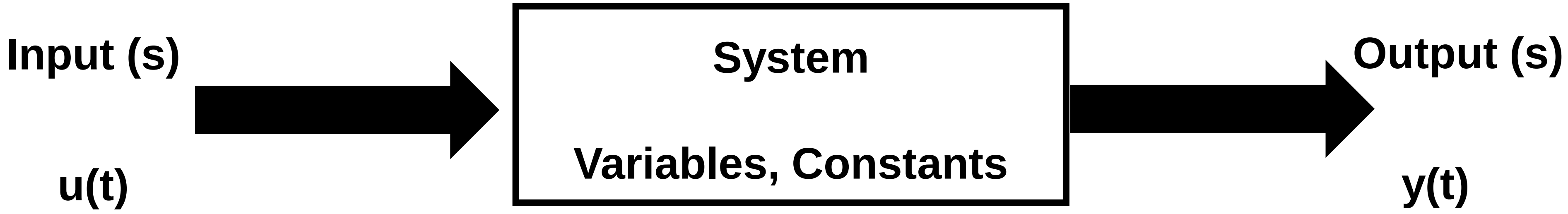


Figure: Graphical sketch of a system

# State-space Model

A system is composed of variables, constants, inputs and outputs

**Input (s)**

**u(t)**

**System**

$$\dot{x} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

**Output (s)**

**y(t)**

Figure: Graphical sketch of a system

# Converting ss to transfer function

The general form or template of ss model is as follows:

$$\dot{\boldsymbol{x}} = \boldsymbol{Ax} + \boldsymbol{Bu}(\boldsymbol{t})$$

$$\boldsymbol{y} = \boldsymbol{Cx} + \boldsymbol{Du}(\boldsymbol{t})$$

Let $\boldsymbol{G}(\boldsymbol{s})$ denote the transfer function after converting to transfer function domain. The formula is:

$$\boldsymbol{G}(\boldsymbol{s}) = \boldsymbol{D} + \boldsymbol{C}[(\boldsymbol{sI} - \boldsymbol{A})^{-\boldsymbol{1}}\boldsymbol{B}]$$

# Example of conversion from ss to tf

Convert the following state-space model to transfer function

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} x + \begin{bmatrix} 5 \\ 6 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} x$$

Let us first obtain $(sI - A)^{-1}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix}$$

# Example of conversion from ss to tf

$$sI - A = \begin{bmatrix} s-1 & -2 \\ -3 & s-4 \end{bmatrix}$$

Now let us find $(sI - A)^{-1}$

$$(sI - A)^{-1} = \frac{\text{adjoint}(sI - A)}{\det(sI - A)}$$

$$\text{adjoint}(sI - A) = \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}$$

$$\begin{aligned} \det(sI - A) &= (s-1)(s-4) - (-2)(-3) \\ &= (s^2 - 5s + 4) - (6) \\ &= s^2 - 5s + 4 - 6 \\ &= s^2 - 5s - 2 \end{aligned}$$

$$(sI - A)^{-1} = \frac{\text{adjoint}(sI - A)}{\det(sI - A)} = \frac{1}{s^2 - 5s - 2} \begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}$$

# Example of conversion from ss to tf

Next, we post-multiply with matrix $\boldsymbol{B}$ as follows:

$$(s\boldsymbol{I}-\boldsymbol{A})^{-1}\times B=\frac{1}{s^2-5s-2}\begin{bmatrix} s-4 & 2 \\ 3 & s-1 \end{bmatrix}\times\begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$=\frac{1}{s^2-5s-2}\begin{bmatrix} \big((s-4)\times 5\big)+\big(2\times 6\big) \\ \big(3\times 5\big)+\big((s-1)\times 6\big) \end{bmatrix}$$

$$=\frac{1}{s^2-5s-2}\begin{bmatrix} 5s-20+12 \\ 15+6s-6 \end{bmatrix}$$

$$=\frac{1}{s^2-5s-2}\begin{bmatrix} 5s-8 \\ 6s+9 \end{bmatrix}$$

# Example of conversion from ss to tf

Now, let us pre-multiply with matrix $\boldsymbol{C}$ as follows:

$$C(sI - A)^{-1}B = \frac{1}{s^2 - 5s - 2}\begin{bmatrix}1 & 2\end{bmatrix} \times \begin{bmatrix}5s - 8 \\ 6s + 9\end{bmatrix}$$

$$= \frac{1}{s^2 - 5s - 2}\left[1 \times (5s - 8) + 2 \times (6s + 9)\right]$$

$$= \frac{1}{s^2 - 5s - 2}\left[5s - 8 + 12s + 18\right]$$

$$= \frac{1}{s^2 - 5s - 2}\left[17s + 10\right]$$

$$= \frac{17s + 10}{s^2 - 5s - 2}$$