Markov Chains

Lecture 23 & 24

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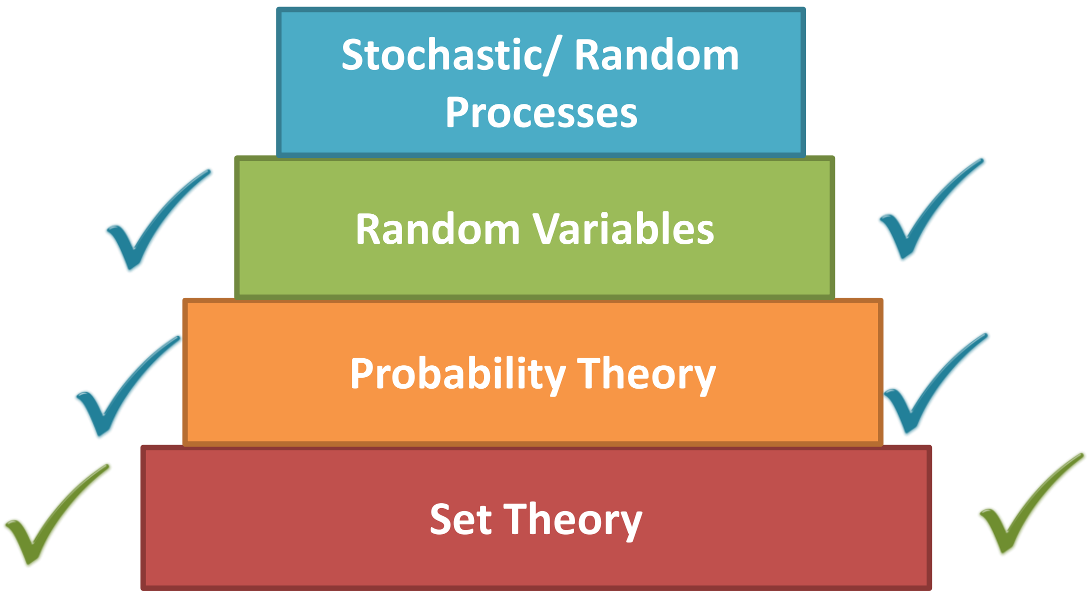

Stochastic/ Random Processes
Random Variables
Probability Theory
Set Theory

Stochastic Processes

Stochastic Process: Collection of random variables. $\{X(t), t \geq 0\}$

or

Stochastic Process: A random variable that changes through time

or

Stochastic process: A process involving the operation of chance.

Stochastic process is a system which evolves in time while undergoing chance fluctuations.

For example:

In radioactive decay every atom is subject to a fixed probability of breaking down in any given time interval.

Stochastic Processes

A stochastic model represents a situation where uncertainty is present. In other words, it’s a model for a process that has some kind of randomness.

All stochastic models have the following in common:

- They reflect all aspects of the problem being studied
- Probabilities are assigned to events within the model
- Those probabilities can be used to make predictions or supply other relevant information about the process.

Stochastic Processes

Steps for Building a Stochastic Model

The basic steps to build a stochastic model are:

1. Create the sample space (Ω) — a list of all possible outcomes
2. Assign probabilities to sample space elements
3. Identify the events of interest
4. Calculate the probabilities for the events of interest.

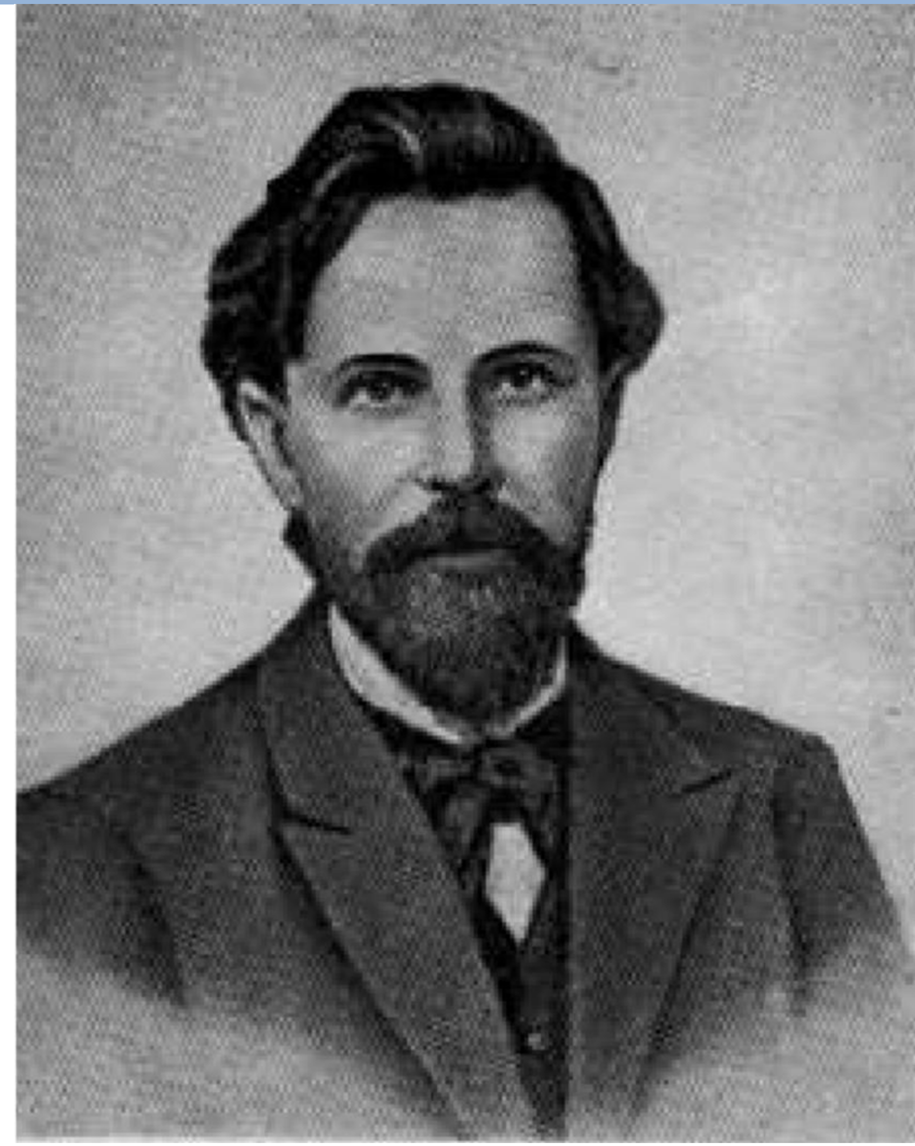
Example

A very simple example of this process in action: You are rolling a die in a casino. If you roll a six or a one, you win $10. The steps would be:

Solution

1. The sample space includes all possibilities for die roll outcomes: $\Omega = \{1,2,3,4,5,6\}$.
2. The probability for any number being rolled is 1/6.
3. The event of interest is “roll a 6 or roll a 1”.
4. The probability for “roll a 6 or 1” is $1/6 + 1/6 = 2/6 = 1/3$.

- Markov Chains
- Discrete Markov Chains



Markov Chains

- A Markov model is a Stochastic method for randomly changing systems where it is assumed that future states do not depend on past states.
- Markov models are often used to model the probabilities of different states and the rates of transitions among them used to model systems.
- Markov models can be expressed in equations or in graphical models. Graphic Markov models typically use circles (each containing states) and directional arrows to indicate possible transitional changes between them. The directional arrows are labeled with the rate or the variable one for the rate.
- Applications of Markov modeling include modeling languages, natural language processing (NLP), image processing, Bioinformatics, speech recognition and modeling computer hardware and software systems.

Markov Chains

- When we study a system that can change over time, we need a way to keep track of those changes.
- A **Markov chain** is a particular model for keeping track of systems that change according to given probabilities.
- The state $\mathbf{X_{n+1}}$ of the system at time $\mathbf{n + 1}$ depends only on the state $\mathbf{X_n}$ at time **n**. It does not depend on the past history of the system.
- A **state** is any particular situation that is possible in the system. For example, if we are studying rainy days, then there are two states:
    - It's raining today.
    - It's not raining today

Markov Chains

- **How Markov Models Work**

- **Starter Sentence**

"One fish two fish red fish blue fish."

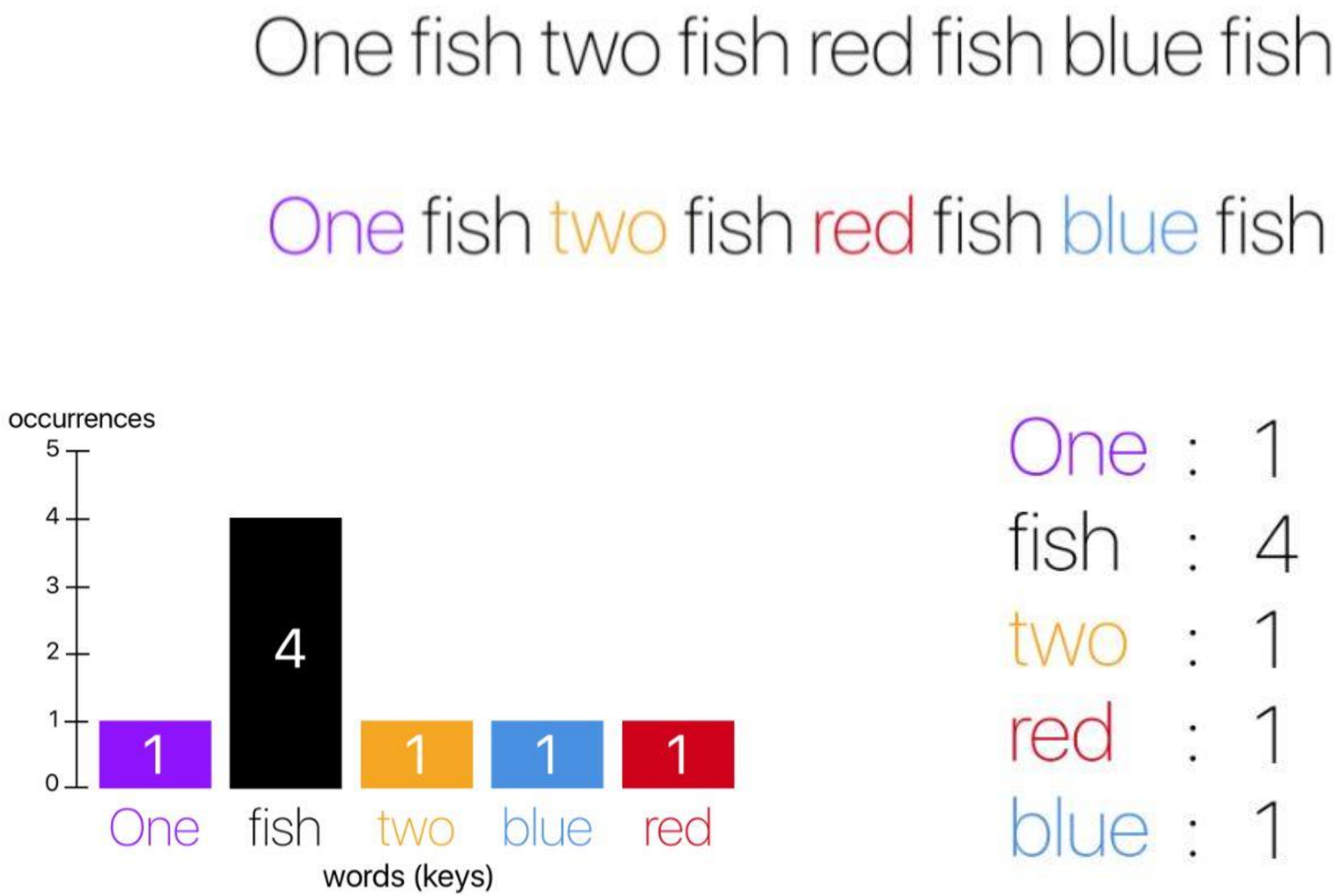
-Dr. Seuss

- **Weighted Distributions**

One fish two fish red fish blue fish

Markov Chains

- **How Markov Models Work**



Markov Chains

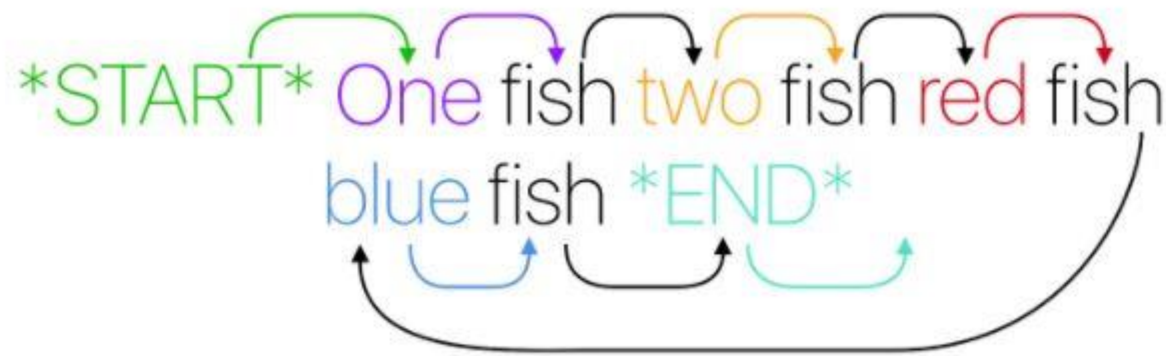
- **How Markov Models Work**

START One fish two fish red fish blue fish *END*

START	:	1
One	:	1
fish	:	4
two	:	1
red	:	1
blue	:	1
END	:	1

Markov Chains

- **How Markov Models Work**



(*Start*, One)
(One, fish)
(fish, two)
(two, fish)
(fish, red)
(red, fish)
(fish, blue)
(blue, fish)
(fish, *END*)
(*END*, none)

Markov Chains

- **How Markov Models Work**

(*Start*, One)

(One, fish)

(fish, two) (fish, red) (fish, blue) (fish, *END*)

(two, fish)

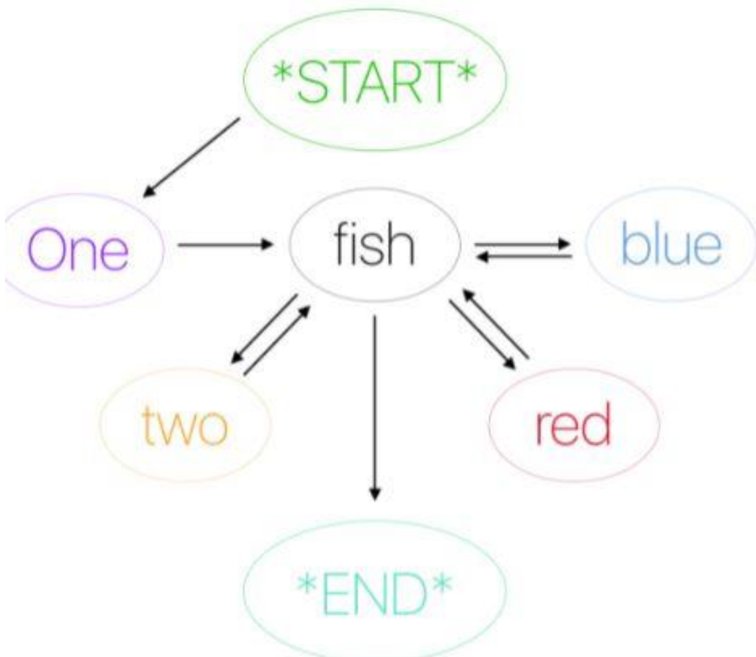
(red, fish)

(blue, fish)

Markov Chains

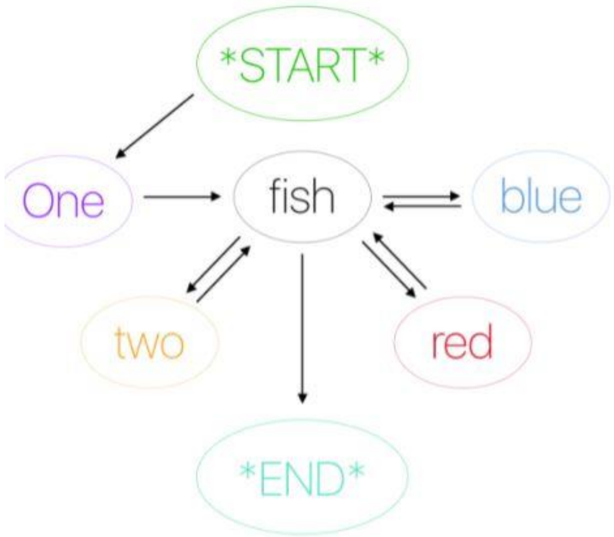
- **How Markov Models Work**

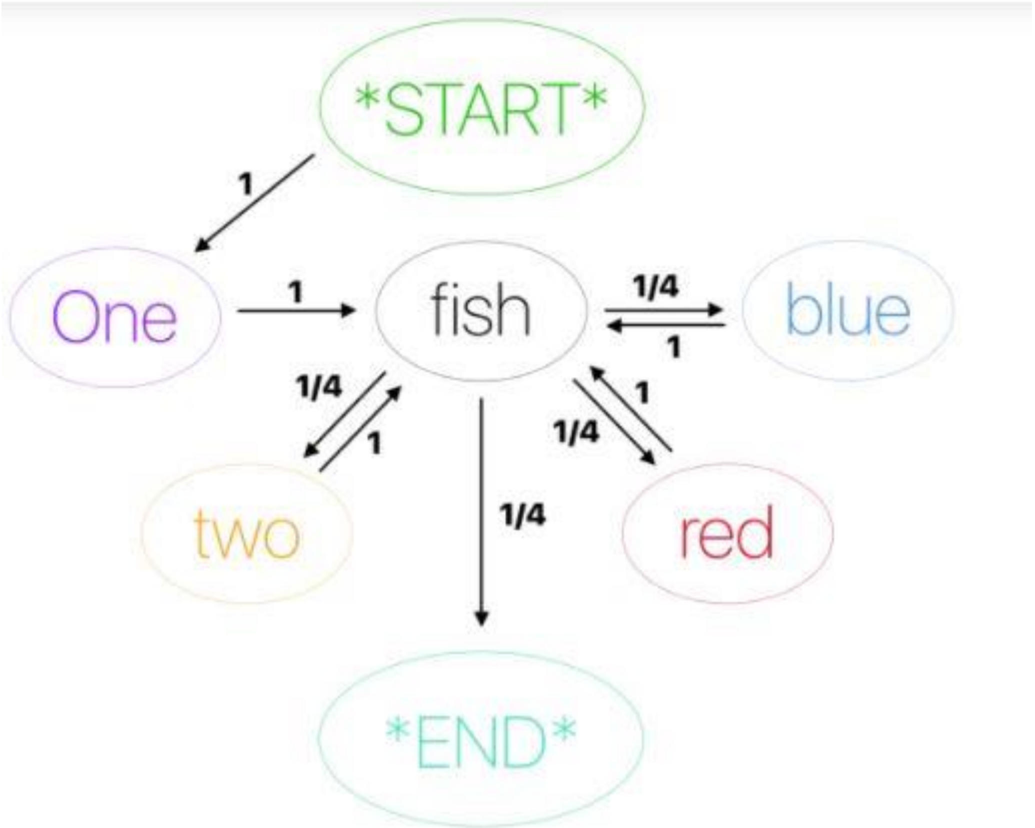
Start	:	[One]
One	:	[fish]
fish	:	[two, red, blue, *END*]
two	:	[fish]
red	:	[fish]
blue	:	[fish]
END	:	[none]



Markov Chains

- **How Markov Models Work**





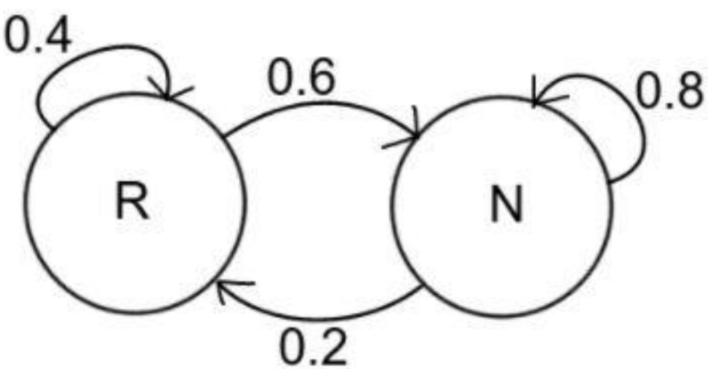
Markov Chains

- The term **Markov chain** refers to any system in which there are a certain number of states and given probabilities that the system changes from any state to another state.
- That's a lot to take in at once, so let's illustrate using our rainy days example. The probabilities of system

- If it rains today (R), then there is a 40% chance it will rain tomorrow and 60% chance of no rain.
- If it doesn't rain today (N), then there is a 20% chance it will rain tomorrow and 80% chance of no rain

Markov Chains

- The left circle represents rain (R), and the right represents no rain (N).
- The arrows indicate the probability to change state.
- For example, the arrow from R to N is labeled 0.6 because there is a 60% chance that if it rains today, then it won't rain tomorrow.



Markov Chains

- **The Transition Matrix**
- Transition diagrams and matrix provide a good techniques for solving some problems about Markov chains, especially for students with poor mathematical background.
- A homogeneous finite Markov chain is entirely defined by its initial state distribution and its transition matrix $S = [p_{ij}]$, where pij = $P(X_1 = i \mid X_0 = j)$ is the transition probability from state j to state i.
- The graphical representation of a Markov chain is a transition diagram, which is equivalent to its transition matrix.

Markov Chains

- **The Transition Matrix**
- If a Markov chain consists of k states, the transition matrix is the k by k matrix (a table of numbers) whose entries record the probability of moving from each state to another state
- The transition from R to R is 0.4, so we put 0.4 in the upper left of the matrix. The transition from R to N is 0.6 (upper right). N to R is 0.2 (lower left) and N to N is 0.8 (lower right)

	R	N
R	0.4	0.6
N	0.2	0.8

$$P = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix}$$

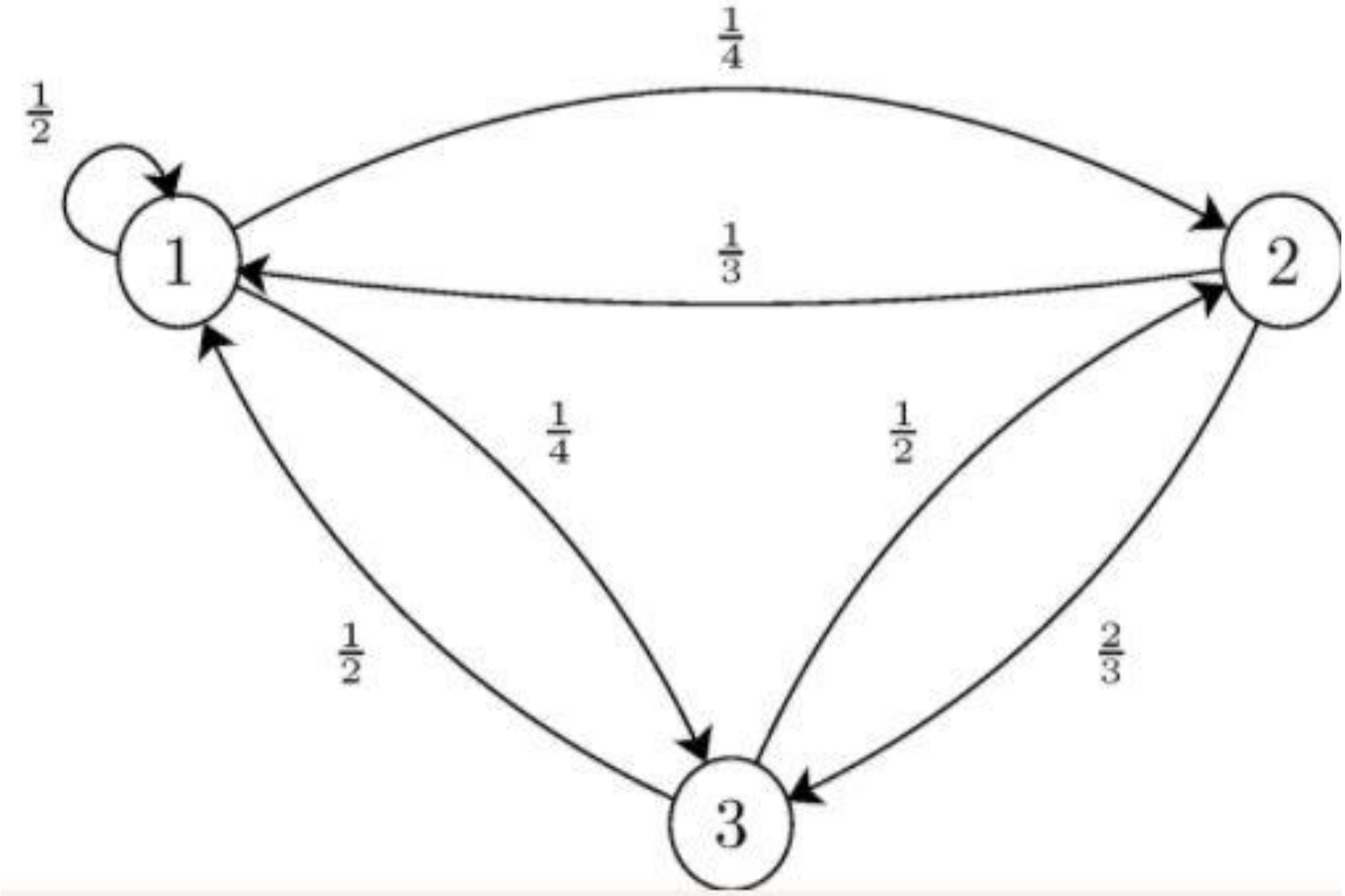
Markov Chain

- **Problem** Consider the Markov chain with three states, $S=\{1,2,3\}$, that has the following transition matrix

$$P=\begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

a. Draw the state transition diagram for this chain.

Markov Chain



Markov Chain

- **Problem**
- A Markov chain has states 1, 2, 3, 4, 5, 6 and the following transition matrix: Draw the state transition diagram.

$$S = \begin{bmatrix} 0.4 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0.3 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

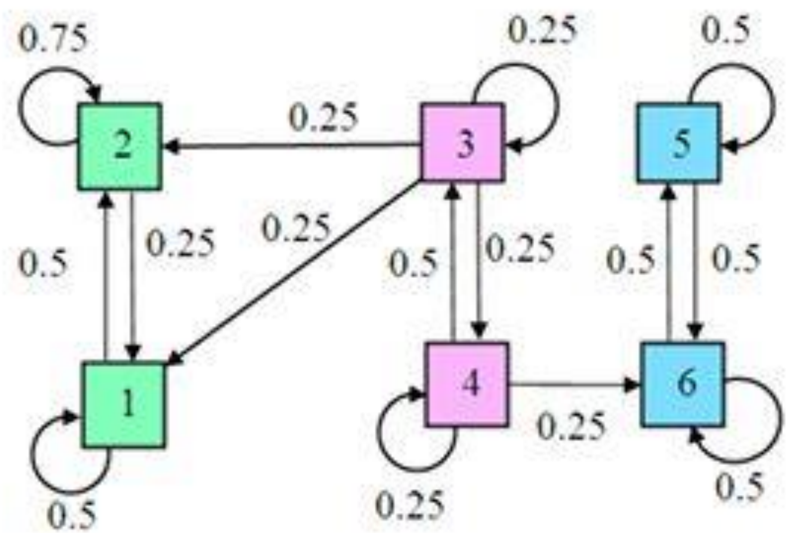
Markov Chain

- **Problem**
- Use a transition matrix and draw the state transition diagram.

$$S = \begin{bmatrix} 0.5 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0.5 & 0.75 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.25 & 0.5 & 0.5 \end{bmatrix}.$$

Markov Chain

- **Solution**
- Transition diagram with transition matrix:



$$S = \begin{bmatrix} 0.5 & 0.25 & 0.25 & 0 & 0 & 0 \\ 0.5 & 0.75 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0.5 & 0 & 0 \\ 0 & 0 & 0.25 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.25 & 0.5 & 0.5 \end{bmatrix}.$$

Markov Chain

- **Problem**
- Use a transition matrix and draw the state transition diagram.

$$P = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.1 & 0.8 & 0.1 \\ 0.4 & 0.4 & 0.2 \end{bmatrix}$$

Markov Chain

- **Problem**
- Use a transition matrix and draw the state transition diagram.

$$P = \begin{bmatrix} 0.5 & 0.25 & 0.25 \\ 0.3 & 0.3 & 0.4 \\ 0.25 & 0.5 & 0.25 \end{bmatrix}$$

Markov Chains

Problem: Assume that people in a particular society can be classified as belonging to the upper class (U), middle class (M), and lower class (L). Membership in any class is inherited in the following probabilistic manner. Given that a person is raised in an upper-class family, he or she will have an upper class family with probability 0.7, a middle-class family with probability 0.2, and a lower-class family with probability 0.1. Similarly, given that a person is raised in a middle-class family, he or she will have an upper-class family with probability 0.1, a middle-class family with probability 0.6, and a lower-class family with probability 0.3. Finally, given that a person is raised in a lower-class family, he or she will have a middle-class family with probability 0.3 and a lower-class family with probability 0.7. Determine

(a) the transition probability matrix

(b) the state-transition diagram for this problem.