



# Control Systems - 7<sup>th</sup> Semester - Week 3

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# Model of Systems

Last week, we studied about transfer function models

Last week, we also studied how to obtain poles, zeros, and analyze stability of transfer function model

Last week, we also studied a new language of modelling which is called as state-space modelling

This week, we will study the conversion techniques from state-space models to transfer function models (and vice versa)



# Example of conversion from ss to tf

Convert the following state-space model to transfer function

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} u(t)$$
$$y = [1 \quad 0 \quad 0]x$$

Let us first obtain  $(sI - A)$

$$(sI - A) = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 & 2 & s+3 \end{bmatrix}$$



# Example of conversion from ss to tf

Now let us find  $(s\mathbf{I} - \mathbf{A})^{-1}$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{\text{adj}(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})} = \frac{\begin{bmatrix} (s^2 + 3s + 2) & s + 3 & 1 \\ -1 & s(s + 3) & s \\ -s & -(2s + 1) & s^2 \end{bmatrix}}{s^3 + 3s^2 + 2s + 1}$$

Next, we post-multiply with matrix  $\mathbf{B}$  as follows:

$$(s\mathbf{I} - \mathbf{A})^{-1} \times \mathbf{B} = \frac{1}{s^3 + 3s^2 + 2s + 1} \begin{bmatrix} (s^2 + 3s + 2) & s + 3 & 1 \\ -1 & s(s + 3) & s \\ -s & -(2s + 1) & s^2 \end{bmatrix} \times \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}$$



# Example of conversion from ss to tf

$$(sI - A)^{-1} \times B = \frac{1}{s^3 + 3s^2 + 2s + 1} \begin{bmatrix} 10s^2 + 30s + 20 \\ -10 \\ -10s \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{1}{s^3 + 3s^2 + 2s + 1} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10s^2 + 30s + 20 \\ -10 \\ -10s \end{bmatrix}$$

$$C(sI - A)^{-1}B = \frac{1}{s^3 + 3s^2 + 2s + 1} [1 \times (10s^2 + 30s + 20) + 0 \times (-10) + 0 \times (-10s)]$$

$$G(s) = C(sI - A)^{-1}B = \frac{10(s^2 + 3s + 2)}{s^3 + 3s^2 + 2s + 1}$$



# Example of conversion from ss to tf

MATLAB code for conversion of ss to tf

```
A = [0 1 0; 0 0 1; -1 -2 -3];
```

```
B = [10 ; 0; 0];
```

```
C = [1 0 0];
```

```
D = [0];
```

```
[num,den] = ss2tf(A,B,C,D);
```

```
g = tf(num,den)
```



# Conversion from tf to ss

Converting from tf to state-space is not **unique** process

There are various techniques to convert from transfer function domain to state-space domain

We call each technique as canonical form. Let us study the first canonical form which is topic 3.5 in book



# Conversion from tf to ss - Canonical Form 1

For a 2<sup>nd</sup> order transfer function:

$$G(s) = \frac{b_1 s^1 + b_0}{s^2 + a_1 s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}$$

$$C = [1 \quad 0]$$





# Conversion from tf to ss - Canonical Form 1

For a 3<sup>rd</sup> order transfer function:

$$G(s) = \frac{b_2s^2 + b_1s^1 + b_0}{s^3 + a_2s^2 + a_1s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$$

$$B = \begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0]$$



# Conversion from tf to ss - Canonical Form 1

For a 4<sup>th</sup> order transfer function:

$$G(s) = \frac{b_3s^3 + b_2s^2 + b_1s^1 + b_0}{s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 \end{bmatrix} \quad B = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad 0]$$



# Conversion from tf to ss - Canonical Form 1

For  $n^{\text{th}}$  order transfer function:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots b_1s^1 + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots a_1s + a_0}$$

We write the following state-space model (using Canonical Form 1):

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad \dots \quad 0]$$



# Conversion from tf to ss - Canonical Form 1

Example 3.4 Page 128: Convert the following transfer function to state-space domain

$$G(s) = \frac{24}{s^3 + 9s^2 + 26s + 24}$$

Solution: In this example,  $a_0 = 24$ ,  $a_1 = 26$ ,  $a_2 = 9$ , and  $b_0 = 24$ , we can obtain the following:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0]$$



# Conversion from tf to ss - Canonical Form 1

Example 3.5 Page 128: Convert the following transfer function to state-space domain

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution: In this example,  $a_0 = 24$ ,  $a_1 = 26$ ,  $a_2 = 9$ , and  $b_0 = 2$ ,  $b_1 = 7$ , and  $b_2 = 1$ , we can obtain the following:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0]$$



# Conversion from tf to ss - Canonical Form 2

This is another canonical form which is as follows:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots b_1s^1 + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots a_1s + a_0}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_{n-2} & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = [b_0 \quad b_1 \quad b_2 \quad \dots \quad b_{n-1}]$$

What is the difference between this canonical form and the previous one?

Matrix B in canonical form 2 seems like transpose of matrix C in the (previous) canonical form 1 and vice versa



# Conversion from tf to ss - Canonical Form 2

Example 3.5 Page 128: Convert the following transfer function to state-space domain using canonical form 2

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [2 \quad 7 \quad 1]$$



# Controller Canonical Form - Canonical Form 3

This is another canonical form called controller canonical form which is as follows:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots b_1s^1 + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots a_1s + a_0}$$

$$A = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \dots & -a_1 & -a_0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

$$C = [b_{n-1} \quad b_{n-2} \quad b_{n-3} \quad \dots \quad b_0]$$





# Conversion from tf to ss - Controller Canonical Form

Example 3.5 Page 128: Convert the following transfer function to state-space domain using controller canonical form

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution:

$$A = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [1 \quad 7 \quad 2]$$



# Observer Canonical Form - Canonical Form 4

Another canonical form is observer canonical form which is as follows:

$$G(s) = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots b_1s^1 + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots a_1s + a_0}$$

$$A = \begin{bmatrix} -a_{n-1} & 1 & 0 & 0 & 0 \\ -a_{n-2} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \\ b_0 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad \dots \quad 0]$$



# Conversion from tf to ss - Observer Canonical Form

Example 3.5 Page 128: Convert the following transfer function to state-space domain using observer canonical form

$$G(s) = \frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

Solution:

$$A = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0]$$



# MATLAB code for conversion from tf to ss

MATLAB code for conversion of tf to ss

```
num=[1 7 2] ;
```

```
den=[1 9 26 24];
```

```
[A, B, C, D]=tf2ss(num,den)
```

```
Aobsv=A'
```

```
Bobsv=B'
```

```
Cobsv=C'
```

```
Dobsv=D'
```



# Common mistake by students using MATLAB code

Students think they are good programmers. They think they may use short variables. So, use the alphabet ***n*** for ***num*** and ***d*** for ***den***

```
n=[1 7 2] ;
```

```
d=[1 9 26 24];
```

```
[a, b, c, d] = tf2ss(n, d)
```

**Problem in above code:** state-space matrix ***d*** and transfer function denominator ***d*** have same alphabets