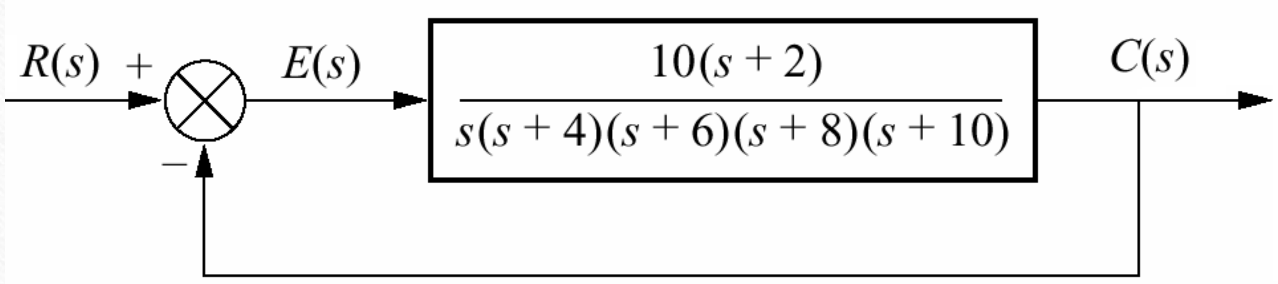
Week 11 – DCSE
7^{th} Semester

In Class Exercise

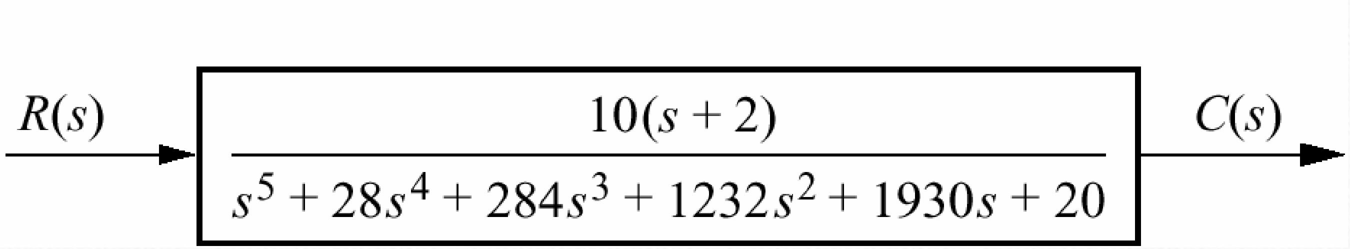
Find the closed loop transfer function or C(s)/R(s) – and is it a stable transfer function



$$T(s) = \frac{G(s)}{1+G(s)}$$

$$T(s) = \frac{10(s+2)}{s(s+4)(s+6)(s+8)(s+10)+10(s+2)}$$

$$T(s) = \frac{10(s+2)}{s^5+28s^4+284s^3+1232s^2+1930s+20}$$



Stability Checking techniques

If you recall, there are 3 main techniques of checking stability

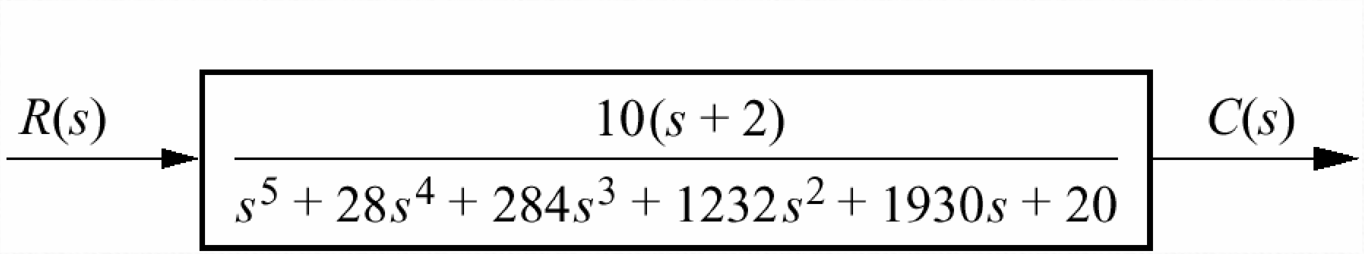
1 – Poles of transfer function (***all*** poles must be ***negative*** for stable tf)

2 – Eigen values of matrix A in State-space Domain

(***all*** eigenvalues must be ***negative*** for stable ss model)

3 – Step Response (should be bounded response for stability)

Who will compute poles of 5^{th} Order Transfer function by hand?



Routh Hurwitz Stability Criteria

- In today's lecture, we will study a technique to check stability of higher order systems

- If you CAN compute poles of high order systems, that's perfect
- For example

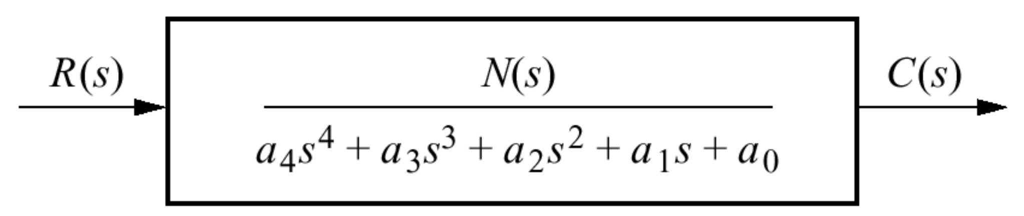
$$\frac{10(s+2)}{s(s+4)(s+6)(s+8)(s+10)}$$

Routh Hurwitz Stability Criteria

- But if you can NOT easily compute poles, then what to do?
- How to check stability of high order systems?
- Routh Hurwitz is a technique to check stability of high order systems
  - (Remember: Its is a technique to check stability – NOT to compute poles)
- Let us study this technique in next slide

Steps of Routh Hurwitz Stability Criteria

Step 1: Compute Closed Loop *tf*



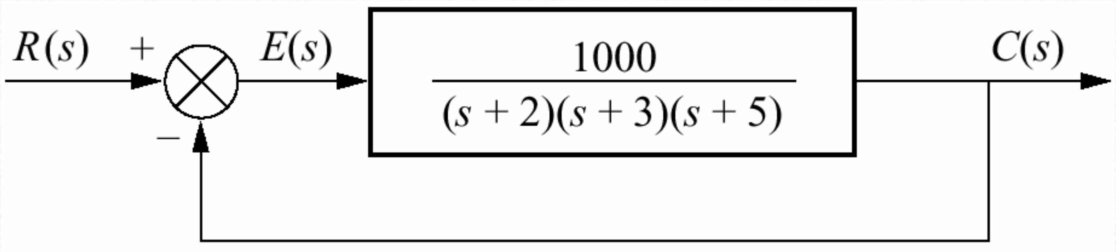
Step 2: Develop Routh-Hurwitz Table

$$P(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$b_1 = \dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3}$	$b_2 = \dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3}$	$b_3 = \dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3}$
s^1	$c_1 = \dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1}$	$c_2 = \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$c_3 = \dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s	$d_1 = \dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1}$	$d_2 = \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$d_3 = \dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

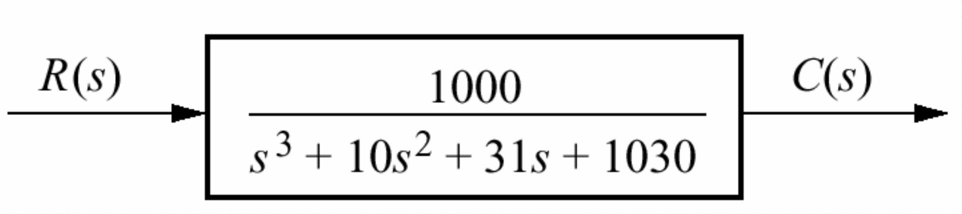
Step 3: Interpretation: The number of RHP poles = The number of SIGN CHANGES of COL 1

Example: Determine whether the close-loop system is stable.



Step 1: Find the closed-loop transfer function

$$T(s) = \frac{G(s)}{1 + G(s)}$$



Step 2: Develop Routh Table

$$P(s) = s^3 + 10s^2 + 31s + 1030$$

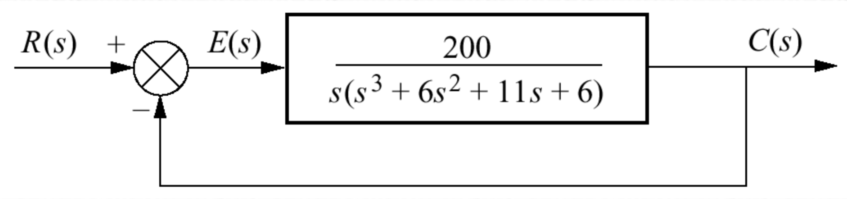
s^3	1	31	0
s^2	10	1030	0
s^1	$b_1 = \frac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = \mathbf{-\ 72}$	$b_2 = \frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = \mathbf{0}$	$b_3 = \frac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = \mathbf{0}$
s^0	$c_1 = \frac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} =$ $\mathbf{103}$	$c_2 = \frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = \mathbf{0}$	$c_3 = \frac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = \mathbf{0}$

Step 3: Interpret Routh Table

TWO sign changes: Thus **2 RHP Poles**

Therefore, system is **UNSTABLE**

Example 2:



$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

s^4	1	11	200
s^3	6 **1**	6 **1**	**0**
s^2	10 **1**	200 **20**	0
s^1	-19	0	0
s^0	20	0	0

2 sign changes: 2 RHP (UNSTABLE)

Poles: 2 LHP and 2 LHP

Example 3: Zero only in the first column

$$T(s)=\frac{10}{s^5+2s^4+3s^3+6s^2+5s+3}$$

s^5	1	3	5
s^4	2	6	3
s^3	$\cancel{0}\quad\varepsilon$	7/2	0
^{2}	$\frac{6\varepsilon-7}{\varepsilon}$	3	0
s^1	$\frac{42\varepsilon-49-6\varepsilon^2}{12\varepsilon-14}$	0	0
s^0	3	0	0

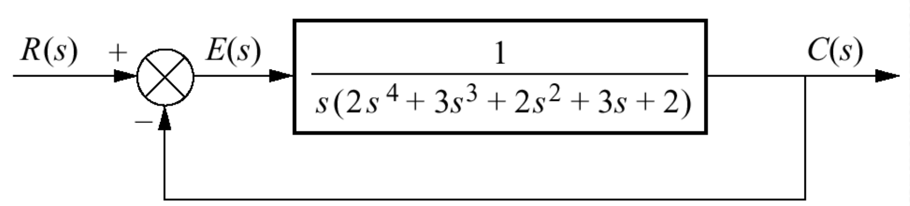
Assume ε is small POSITIVE (follow the previous signage) :

TWO sign changes at first column

Poles: 2 RHP, 3 LHP

System in unstable

Example 4:



$$T(s)=\frac{1}{2s^5+3s^4+2s^3+3s^2+2s+1}$$

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0}\quad \varepsilon$	4/3	0
s^2	$\frac{3\varepsilon-4}{\varepsilon}$	1	0
s^1	$\frac{12\varepsilon-16-3\varepsilon^2}{9\varepsilon-12}$	0	0
s^0	1	0	0

$$T(s)=\frac{1}{2s^5+3s^4+2s^3+3s^2+2s+1}$$

Assume ε is small POSITIVE : TWO sign changes

Poles: 2 RHP, 3 LHP

Example 5: Entire Row is Zero

$$T(s)=\frac{10}{s^5+7s^4+6s^3+42s^2+8s+56}$$

s^5	1	6	8
s^4	$\cancel{7}$ **1**	$\cancel{42}$ **6**	$\cancel{56}$ **8**
s^3	$\cancel{0}$ $\cancel{4}$ **1**	$\cancel{0}$ $\cancel{12}$ **3**	$\cancel{0}$ $\cancel{0}$ **0**
s^2	3	8	0
s^1	1/3	0	0
s^0	8	0	0

Polynomial above the row of zeros

$$P(s)=s^4+6s^2+8$$

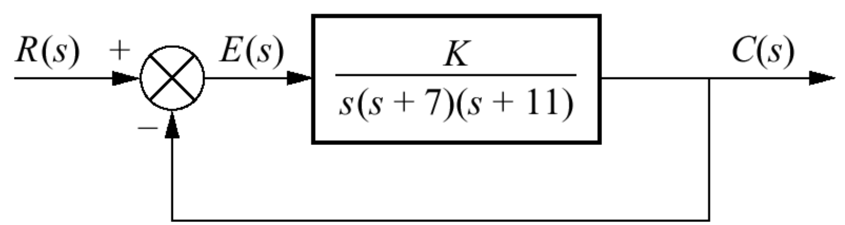
$$\frac{dP(s)}{ds}=4s^3+12s+0$$

Used to replace row of zeros

1on LHP + 4 on jw axis = 5 poles

Example 6: Stability Design via Routh-Hurwitz

Find range of K that will cause the system to be stable / unstable / marginally stable for the given diagram.



Step 1: Find closed-loop transfer function:

$$T(s) = \frac{K}{s^3 + 18s^2 + 77s + K}$$

Step 2: Develop Routh-Hurwitz table

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	0
s^0	K	0

Step 3: Analyze table

STABLE: $0 < K < 1386$

UNSTABLE: $K > 1386$

MARGINALLY STABLE: $K = 1386$

Example 7: Stability in State-Space

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$

Values of the system's poles are equal to the eigenvalues of matrix A

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 3 & 1 \\ 2 & 8 & 1 \\ -10 & -5 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

$$|s\mathbf{I} - \mathbf{A}| = \begin{vmatrix} s-0 & -3 & -1 \\ -2 & s-8 & -1 \\ 10 & 5 & s+2 \end{vmatrix}$$

$$|s\mathbf{I} - \mathbf{A}| = s^3 - 6s^2 - 7s - 52$$

s^3	1	-7
s^2	-3	-26
s^1	-1	0
s^0	-26	0

1 sign change: 1 RHP

Poles: 1 RHP, 2 LHP

$\therefore$ System is UNSTABLE