Midterm 2 Exam - ECE 503 Fall 2017

- Date: Wednesday, November 1, 2017.
- Time: 11:00 am -11:50 am (in class)
- Maximum Credit: 100 points

1. [25 points] The random variable X has the following PDF:

$$f_X(x) = \begin{cases} xe^{-x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Given $X = x$, the random variable Y is distributed as $Y \sim \text{Uniform}(0, x)$.

(a) Find the MMSE estimator of X given $Y = y$.

(b) Find the Linear MMSE estimator of X given $Y = y$.

$$f_{X,Y}(x,y) = \begin{cases} e^{-x} & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases} \quad \left(\because f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x)\right)$$

$$\Rightarrow \underbrace{f_Y(y)}_{\text{marginal of } Y} = \int_{x=y}^{\infty} e^{-x}\,dx = \begin{cases} e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f_{X,Y}}{f_Y} = \begin{cases} e^{-(x-y)} & \text{if } x \geq y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \underbrace{E[X|Y=y]}_{\text{MMSE estimator}} = \int_y^{\infty} x \cdot f_{X|Y}(x|y)\,dx = \int_{x=y}^{\infty} x e^{-(x-y)}\,dx$$

Let $u = x - y$

$$= \int_{u=0}^{\infty} (u+y) e^{-u}\,du$$

$$= \underbrace{\int_{u=0}^{\infty} u e^{-u}\,du}_{1} + y \underbrace{\int_0^{\infty} e^{-u}\,du}_{1}$$

$$= y + 1$$

$\Rightarrow$ $\boxed{\text{Optimal MMSE estimator} = Y + 1}$ $\Rightarrow$ This is LINEAR $\Rightarrow$ same as L-MMSE!

2. [25 points] Let $\mathbf{X} = [X_1, X_2, X_3]^T$ be a Gaussian random vector with the following mean vector and covariance matrix:

$$\mu_X = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad C_X = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}.$$

(a) Find the PDF of the random variable $Z = X_1 + 2X_2 + 3X_3$.

(b) Find the probability that $P(X_1 \geq X_2 + X_3)$.

(Hint: Recall that any linear combination of jointly Gaussian random variables is also a Gaussian random variable).

(a) $Z \rightarrow$ Linear comb. of $(x_1, x_2, x_3) \longrightarrow Z \sim \mathcal{N}(\mu_z, \sigma_z^2)$

$$\mu_z = E[z] = E[x_1] + 2E[x_2] + 3E[x_3]$$
$$= 3 + 2\times 2 + 3\times 1 = 3+4+3 = 10$$

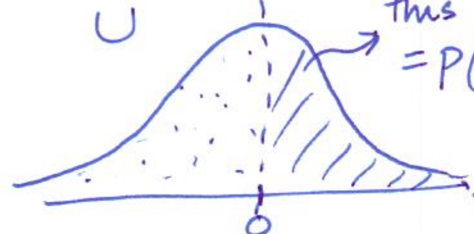
$$\sigma_z^2 = Var(z) = \sigma_{x_1}^2 + 4\sigma_{x_2}^2 + 9\sigma_{x_3}^2 + 4\,Cov(x_1,x_2) + 6\,Cov(x_1,x_3) + 12\,Cov(x_2,x_3)$$
$$= 1+4+9 + 4\times(-1) + 6\times(0) + 12\times(2)$$
$$= 13 - 4 + 0 + 24 = 37 - 4 = 33$$

$$\Rightarrow \boxed{Z \sim \mathcal{N}(10, 33)}$$

(b) $P(\cancel{x}_1 \geq x_2 + x_3) = P(\underbrace{x_1 - x_2 - x_3}_{= U} \geq 0)$

Say $U = x_1 - x_2 - x_3$

$U \sim \mathcal{N}(\mu_u, \sigma_u^2)$



this area $= P(U \geq 0)$.

$$\mu_u = E[U] = E[x_1] - E[x_2] - E[x_3]$$
$$= 3 - 2 - 1 = 0$$

PDF is symmetric around 0

$\Rightarrow$ $U = x_1 - x_2 - x_3 \Rightarrow$ zero mean. & Gaussian.

$$\Rightarrow P(x_1 - x_2 - x_3 \geq 0)$$
$$= P(U \geq 0)$$
$$= 1/2$$

$$\Rightarrow \boxed{P(x_1 \geq x_2 + x_3) = \frac{1}{2}}$$

3. [25 points] During each day, the probability that your computer's operating system crashes at least once is 0.05, independent of every other day. We are interested in the probability of at least 45 crash-free days out of the next 50 days. Using the Central Limit Theorem, find an approximation to this probability (express your answer in terms of $\Phi(.)$ function).

Let S_n = # of crash-free days out of n days.

$$= X_1 + X_2 + \cdots + X_n$$

$$X_n = \begin{cases} 0 & \text{if OS crashes} \\ 1 & \text{if OS does not crash.} \end{cases}$$

$P(X_n = 0) = 0.05$ ⤳ Prob. of crash.

$P(X_n = 1) = \underbrace{0.95}_{p}$ ⤳ Prob. of not crash.

We are interested in the Prob $(S_{50} \geq 45)$

i.e $P(S_{50} \geq 45)$

$E[S_n] = np$

$Var(S_n) = np(1-p)$.

CLT tells us that

$$\frac{S_n - np}{\sqrt{np(1-p)}} \xrightarrow[n \to \infty]{} N(0,1)$$

i.e. for $n=50$, $\dfrac{S_{50} - 50 \times p}{\sqrt{50 \times p(1-p)}} \approx N(0,1)$.

$$\Rightarrow P(S_{50} \geq 45) = P\Big(\overbrace{\frac{S_{50} - 50p}{\sqrt{50p(1-p)}}}^{\approx N(0,1)} \geq \frac{45 - 50p}{\sqrt{50p(1-p)}}\Big)$$

$$\cong 1 - \Phi\left(\frac{45-50p}{\sqrt{50p(1-p)}}\right)$$

$$= 1 - \Phi\left(\frac{45 - 47.5}{1.54}\right)$$

$$= 1 - \Phi(-1.62)$$

$$= \Phi(1.62) = \boxed{0.9474}$$

$$\boxed{P(S_{50} \geq 45) \approx 1 - \Phi(-1.62) = \Phi(1.62)}$$

4. [25 points] Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables. Each one of them is drawn from the following distribution (PDF):

$$f_X(x) = \begin{cases} \theta\left(x - \frac{1}{2}\right) + 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

The parameter $\theta \in [-2, 2]$ is unknown and our goal is to design an estimator $(\hat{\theta}_n)$ to estimate θ from the observations $(X_1, X_2, \ldots, X_n)$. Let the sample mean be $\bar{X} = (X_1 + X_2 + \ldots + X_n)/n$ and consider the following estimator:

$$\hat{\theta}_n = 12\bar{X} - 6 = 12\left(\frac{X_1 + X_2 + \ldots + X_n}{n}\right) - 6.$$

(a) Is $\hat{\theta}_n$ an unbiased estimator of θ ? (i.e., does $E(\hat{\theta}_n) = \theta$)

(b) Find the mean squared error (MSE) of $\hat{\theta}_n$, i.e., find $E((\hat{\theta}_n - \theta)^2)$. Does the MSE converge to 0 as $n \to \infty$?

(a)

$$\underbrace{E[X_i]}_{\text{mean of any (one).}} = \int_{x=0}^{1} x\, f_X(x)\, dx = \int_{x=0}^{1} x\left[\theta(x-\tfrac{1}{2}) + 1\right] dx = \int_{x=0}^{1} \left[\theta x^2 - \frac{\theta}{2}x + x\right] dx$$

$$= \frac{\theta}{3}x^3 - \frac{\theta}{2}\frac{x^2}{2} + \frac{x^2}{2}\Big|_0^1 = \frac{\theta}{3} - \frac{\theta}{4} + \frac{1}{2} = \frac{1}{2} + \frac{\theta}{12} = \boxed{\frac{\theta+6}{12}}.$$

$$\Rightarrow E[\hat{\theta}_n] = E(12\bar{X} - 6) = 12\,E[\bar{X}] - 6$$

$\bar{X}$ is sample mean $\Rightarrow E[\bar{X}] = E[X_i]$

$$= 12\,E[X_i] - 6$$

$$= 12 \times \left(\frac{\theta+6}{12}\right) - 6$$

$$= \theta$$

$\Rightarrow \hat{\theta}_n$ is an unbiased estimator

(b)

$$MSE = E\left[(\hat{\theta}_n - \theta)^2\right]$$

$$= Var(\hat{\theta}_n) = Var\left[12\bar{X} - 6\right] = Var\left[12\bar{X}\right]$$

$$= 144 \times Var(\bar{X}) = \frac{144}{n} \times Var(X_i)$$

$$E(X_i^2) = \frac{\theta}{4} - \frac{\theta}{6} + \frac{1}{3} = \frac{4+\theta}{12}$$

$$= \frac{144}{n}\left\{E[X_i^2] - (E[X_i])^2\right\}$$

$$= \frac{144}{n}\left[\frac{(\theta+4)}{12} - \left(\frac{\theta+6}{12}\right)^2\right]$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$