Application 1 : Slotted Aloha (Multiple Access Protocol).

$$\text{Tx} \longrightarrow \boxed{\text{Channel}} \longrightarrow \text{Rx}$$

Prob (Packet is received) = p

Prop (Packet dropped) = $(1-p)$

If a packet is dropped, then Tx re-transmits immediately.

What is the Expected Time to successfully deliver a packet?

Let T denote the random variable representing the time it takes to deliver.

$$T = \begin{cases} 1 & \text{w.p. } p \\ 2 & \text{w.p. } (1-p)p \\ 3 & \text{w.p. } (1-p)^2 p \\ \vdots & \end{cases}$$

T is a Geometric Random variable

$\Rightarrow$ PMF of $T \Rightarrow P(T=t) = (1-p)^{t-1} p \quad t = 1, 2, 3, \ldots \infty$

$$E[T] = \sum_{t=1}^{\infty} t\, P(T=t) = \sum_{t=1}^{\infty} t(1-p)^{t-1} p = p \left\{ \sum_{t=1}^{\infty} t(1-p)^{t-1} \right\}$$

$$= p \times \frac{1}{p^2} = \frac{1}{p}$$

Recall: $1 + r + r^2 \ldots = \dfrac{1}{1-r}$

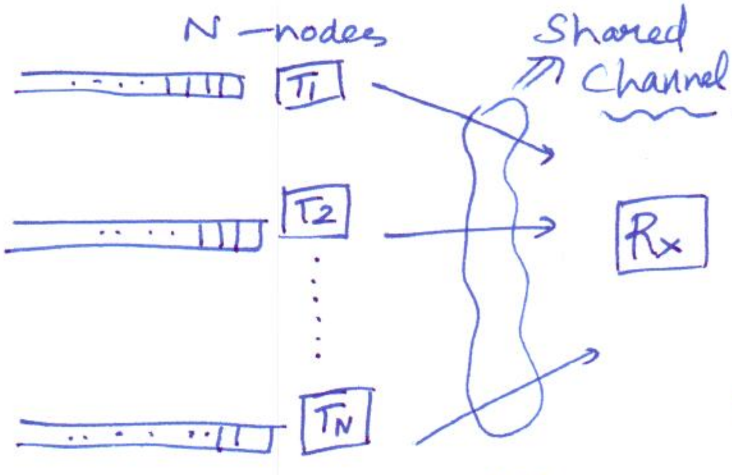
$\dfrac{d}{dr}(\cdot) \Rightarrow 1 + 2r + 3r^2 \ldots = \dfrac{1}{(1-r)^2}$

$$\Rightarrow \sum_{t=1}^{\infty} t r^{t-1} = \frac{1}{(1-r)^2}$$

$$\Rightarrow \sum_{t=1}^{\infty} t(1-p)^{t-1} = \frac{1}{p^2}$$

$\Rightarrow$ Expected time

$$\boxed{E(T) = \frac{1}{p}}$$



Slotted Aloha (Multiple Access Protocol)

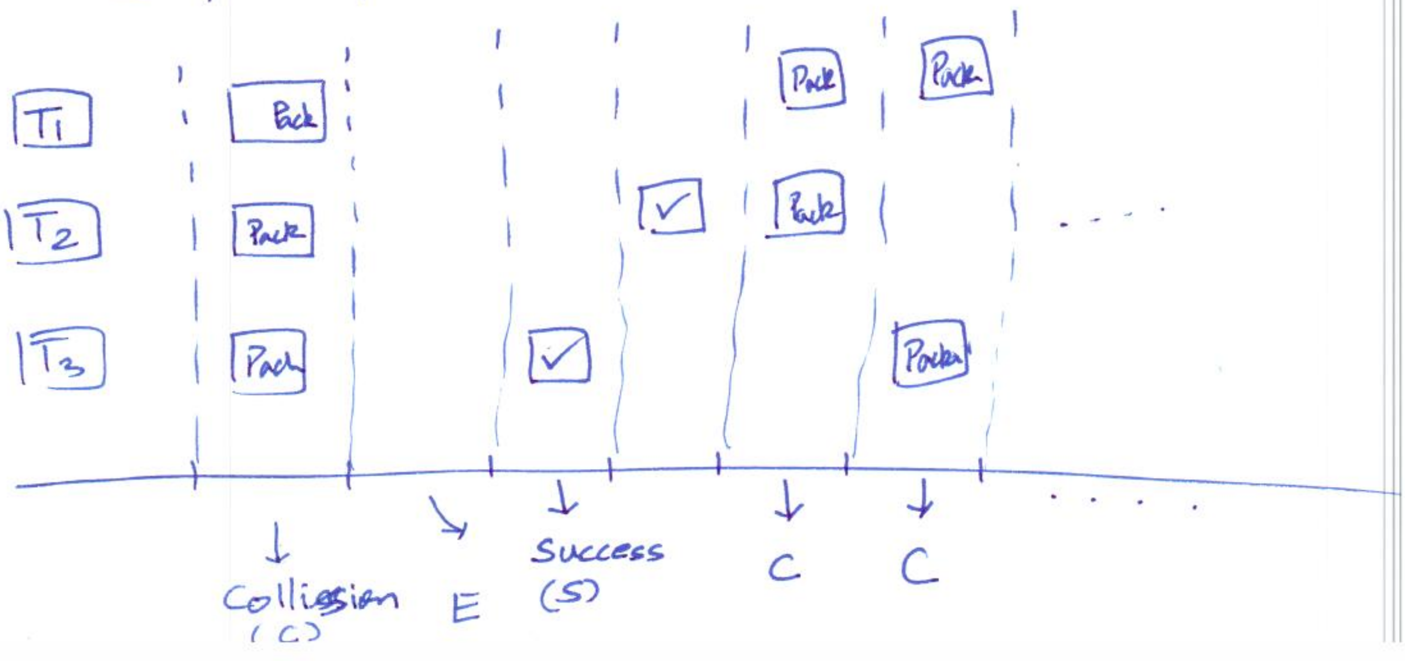
* Time divided into slots
* Each node transmits w/ prob p & does not transmit w/. prob. $(1-p)$
* If more than one nodes transmit, we have a "collision")

Types of slots:
- (S) Successful (only one node transmits)
- (C) Collision (more than one node transmits)
- (E) Empty (No one transmits)

$$P(S) = \binom{N}{1} \cdot p^{1} \cdot (1-p)^{N-1} = Np\,(1-p)^{N-1}$$

$$P(\mathcal{E}) = \binom{N}{N}(1-p)^{N} = (1-p)^{N}$$

$$P(C) = 1 - P(S) - P(\mathcal{E})$$
$$= 1 - Np(1-p)^{N-1} - (1-p)^{N}.$$



Throughput = Expected # of successful transmissions per unit time.

$$= \; ?$$

$(\#S)^{(L)}$ = # of Successful ~~per~~ transmissions in L time slots. $(L \geq 1)$

$$= \mathbb{1}_{[O_1=S]} + \mathbb{1}_{[O_2=S]} \cdots + \mathbb{1}_{[O_L=S]}$$

$$E\left[(\#S)^{(L)}\right] = E\left[\mathbb{1}_{[O_1=S]}\right] + \cdots + E\Big[\underbrace{\mathbb{1}_{[O_L=S]}}_{\text{Indicator Function.}}\Big]$$

$$= L \times E\left[\mathbb{1}_{[O_1=S]}\right]$$

$$\Rightarrow \quad = L \times \left\{ 1 \times P(O_1 = \text{Success}) + \underbrace{0 \times P(O_1 \neq \text{Success})}_{=0} \right\}$$

$$= L \times P(\text{Success})$$

$$= L \times NP(1-P)^{N-1}$$

$$\Rightarrow \quad \text{Expected \# of Successes per unit time} = \frac{E\left[(\#S)^{(L)}\right]}{L} = NP(1-P)^{N-1}$$

$$\text{Expected \# of Successes per unit time} = \text{Throughput}$$

$$\boxed{\mathbb{1}_A = \begin{cases} 1 & \text{if Event } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}}$$

Analysis of Throughput $= Np(1-p)^{N-1}$ when $\underset{\sim}{N}$ is large (# of users)

$$= \left(\frac{NP}{(1-P)}\right) \times (1-P)^N$$

when N becomes large ⊛ $\frac{NP}{1-P} \approx Np$

⊛ $(1-P)^N \approx e^{-NP}$

→ Why ??

$$(1+x)^N = \left(1 + \frac{(Nx)}{N}\right)^N$$

$$= 1 + \binom{N}{1} \times \frac{(Nx)}{N} + \frac{\binom{N}{2}}{N^2} \times (Nx)^2 + \frac{\binom{N}{3}}{N^3}(Nx)^3$$

$$--- + \frac{\binom{N}{N}}{N^N}(Nx)^N$$

$$\frac{\binom{N}{i}}{N^i} = \frac{N!}{i!\,(N-i)!} \times \frac{1}{N^i}$$

$$= \left(\frac{1}{i!}\right) \times \left[\frac{N \times (N-1) \times \qquad \times (N-(i-1))}{\underset{\approx 1}{\underbrace{N}} \times \underset{\approx 1}{\underbrace{N}} \times \cdots \times \underset{\approx 1}{\underbrace{N}}}\right] \approx \frac{1}{i!}$$

(for N large)

$$\Rightarrow (1+x)^N \approx 1 + \frac{(Nx)}{1!} + \frac{(Nx)^2}{2!} + \frac{(Nx)^3}{3!} + \cdots$$

$$= e^{(Nx)}.$$

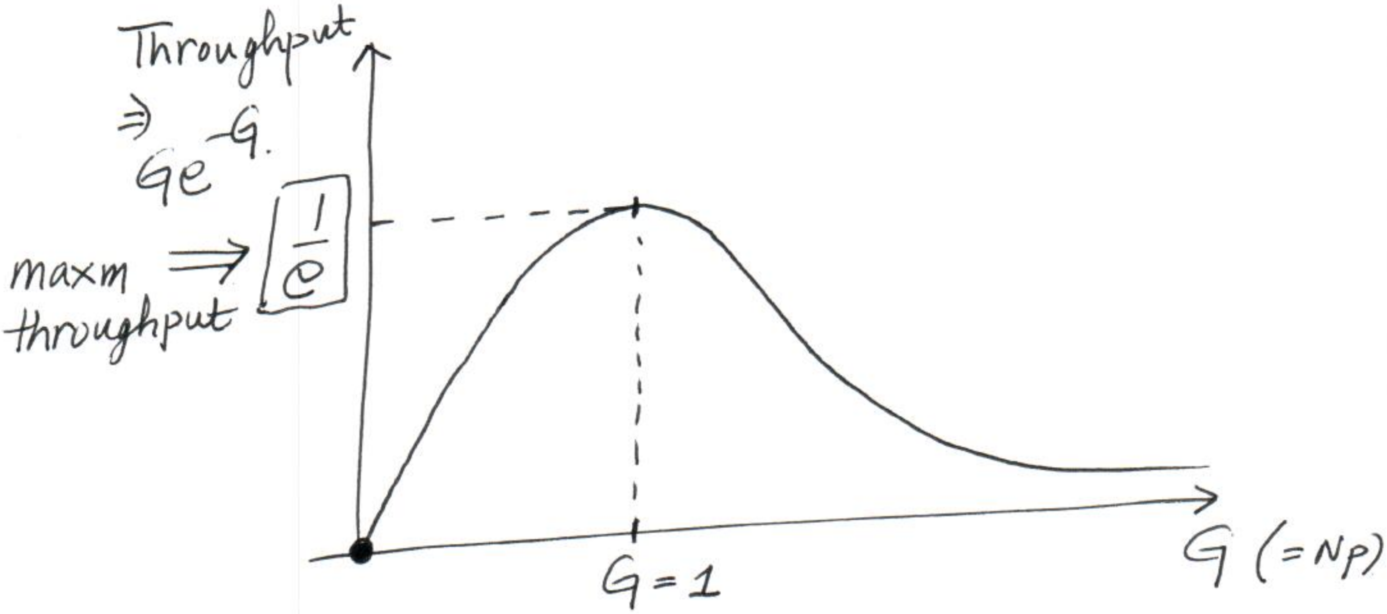
[& Nx is a constant]

$$\Rightarrow \boxed{(1-P)^N \approx e^{-Np}}$$ for N large & p small.

$\Rightarrow$ Throughput $\approx (Np) \times e^{-(Np)}$
(N large)

Let $G = Np$. $\longrightarrow$ G can be viewed as the attempted transmission "rate" of all N nodes.

$\approx G e^{-G}$.



$$\max_G G e^{-G} \Rightarrow \frac{d(Ge^{-G})}{dG} = e^{-G} - Ge^{-G} = 0$$
$$\Rightarrow G = 1$$

$$@\ G = 1 \Rightarrow \text{Max Throughput} = e^{-1} = \frac{1}{e} \approx 0.368.$$

$\Rightarrow$ the shared channel is successfully utilized approximately 37 % of the time.