ECE 503

Lecture 1

What are some of the $\boxed{\text{uses of probability}}$ in everyday life and Engineering applications?

* Wireless Communication (mobility/fading)
* Medical diagnosis and treatment
* Spread of Infectious diseases (CDC)
  — preventive measures
* Information Systems Reliability & Security
  — Dropbox, Azure ....
* Financial Investment Strategies
* Modeling Social Science
* Online Search & Advertising (PageRank)
  — What AD to display to John Doe??

What does Probability Theory help us with??

- language for to discuss/aggregate knowledge about uncertainty
- Statistical decision making / estimation / inference
- modeling tools & dealing with complexity.

AXIOMS of PROBABILITY

* While there are several "philosophies" about probability, we will focus on the widely used axiomatic framework.

* WHY? — Facing a real world example, we use a mathematical model satisfying the axioms. We then use the properties implied by axioms to reason about the real world example.

* The axiomatic approach relies on set theory.

Review of Set Theory

* Set $\triangleq$ collection of elements

  $A = \{H, T\}$ or $A = \{13, 14, 15\}$

* Empty or Null Set $\triangleq \{\phi\} \Rightarrow$ contains NO element

* Total Number of Subsets of a set of size $n = 2^n$

eg: $S = \{1, 2, 3\} \Rightarrow$ # of Subsets $= 8 = 2^3$

$\{0\}, \{1\}, \{2\}, \{3\},$
$\{1, 2\}, \{1, 3\}, \{2, 3\},$
$\{1, 2, 3\}$

* All sets under consideration will be subsets of a set $\boxed{S} \Rightarrow$ we call it $\boxed{\text{Space}}$

Set Operations

$B \subset A$ or $A \supset B \Rightarrow B$ is a sub-set of A
(or) B belongs to A

Transivity If $C \subset B$ and $B \subset A \Rightarrow C \subset A$.

Equality $A = B$ iff $A \subset B$ and $B \subset A$.
(if and only if)

Union or Sum of Two Sets

Notation:- $A + B$ or $A \cup B$

Consists of elements of A, or
of B, or
of both A and B.

Eg: $A = \{1, 3, \alpha, \beta\}$; $B = \{3, \alpha, 7\}$

$A + B = \{1, 3, 7, \alpha, \beta\}$

Union is Commutative: $A \cup B = B \cup A$

Union is Associative: $(A \cup B) \cup C = A \cup (B \cup C)$

Intersection : $\overbrace{AB \quad \text{or} \quad A \cap B}^{\text{Notation}}$

eg: $A = \{1, 3, \alpha, \beta\}$; $B = \{3, \alpha, 7\}$

$$AB = A \cap B = \{3, \alpha\}$$

Intersection is commutative : $AB = BA$

" is associative : $(AB)C = A(BC)$

" is distributive : $A(B \cup C) = AB \cup AC$

Mutually Exclusive Sets / Disjoint Sets

A and B are mutually exclusive if $AB = \{\phi\}$
{or they have no common elements)

$A_1, A_2, \ldots$ are mutually exclusive if

$$A_i A_j = \{\phi\} \text{ for every } i \text{ and } j \neq i$$

Complement of a Set $\Rightarrow$ $\overline{A}$ is complement of A

$\overline{A} \rightarrow$ consists of all elements in the "Space" S, which are not in A.

$$A \cup \overline{A} = S \qquad A\overline{A} = \{\phi\} \qquad \overline{(\overline{A})} = A$$

$$\overline{S} = \{\phi\} \; ; \qquad \overline{\{\phi\}} = S.$$

Note: If $B \subset A$, then $\overline{A} \subset \overline{B}$

<u>De Morgan's Law</u>: $\overline{(A \cup B)} = \bar{A}\,\bar{B}$

$$\overline{AB} = \bar{A} \cup \bar{B}$$

Lecture 1-continued

PROBABILITY SPACE

* Space, S or Ω $\longrightarrow$ certain event

* Elements of S $\Rightarrow$ experimental outcomes

* Subsets of S $\Rightarrow$ events

* $\{\phi\}$ $\Rightarrow$ impossible event

Eg: Roll a die experiment $\rightarrow$ Six outcomes (1, or 2 .. or 6)

$$\Rightarrow \underbrace{S = \Omega}_{\boxed{\text{Space}}} = \{1, 2, 3, 4, 5, 6\}$$

Experimental outcomes

$E_1 = \{2, 4, 6\} \Rightarrow$ Event of observing an even number.

$E_2 = \{1, 3, 5\} \Rightarrow$ " " " odd " .

The AXIOMS To each event A, the number $P(A)$ is defined as the "Probability of event A".

$P(A)$ satisfies:

(I) $P(A) \geqslant 0$ — Probability is non-negative.

(II) $P(S) = 1$ — "Space" is a certain event

(III) If $AB = A \cap B = \{\phi\}$, then

$$P(A \cup B) = P(A) + P(B)$$

Probability of mutually exclusive events is additive.

Using these axioms, let us prove that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We write $A \cup B$, as

$$A \cup B = A \cup (\bar{A} B) \quad \text{and hence using}$$

$$\text{Axiom (III)}, \quad P(A \cup B) = P(A \cup (\bar{A} B))$$

$$= P(A) + P(\bar{A} B) \quad \text{— (i)}$$

We can also write B as

$$B = (AB) \cup (\bar{A} B)$$

$$\Rightarrow \quad P(B) = P((AB) \cup (\bar{A} B))$$
$$= P(AB) + P(\bar{A} B) \quad \text{— (ii)}$$

Eliminate $P(\bar{A} B)$ from (i) and (ii) gives

$$P(A \cup B) = P(A) + P(B) - P(AB)$$

(Ω or S) $\Rightarrow$ denote Sample space

Recall that Ω was the sample space.

In order to fully define a probability space, we define a second component F, which is a set of subsets of Ω.

$\boxed{\text{Event Axioms}}$ The set of ~~axioms~~ events F, is required to satisfy the following axioms:

E.1 Ω is an event (i.e. $\Omega \in F$)

E.2 If A is an event, then $\bar{A}$ is an event (or if $A \in F$, then $\bar{A} \in F$)

E.3 If A and B are events, then. $A \cup B$ is an event.

More generally, if $A_1, A_2, \ldots$ is a list of events, then the union of all of these events $A_1 \cup A_2 \cup \ldots$ is also an event.

* If F satisfies E.1, E.2 and E.3, it is called a BOREL FIELD.

* One choice of F is the set of all subsets of Ω. This choice is OKAY when Ω is finite or countably infinite (i.e elements of Ω can be arranged in an infinite set, indexed by positive integers).

* When Ω is uncountably infinite, it is mathematically impossible to define a $P(\cdot)$ on sets of all subsets of Ω so that we satisfy all the Axiom of Probability.

To avoid such problems, we do not allow all subsets of such an Ω to be events, but the set of events in F.

Example: $S \,/\, \Omega = \{\overbrace{a, b, c, d}^{\text{4 outcomes}}\}$

(or)

Sample Space

Smallest field containing $\{a\}$ and $\{b\}$??

$$\boxed{\begin{array}{c} \{a\},\ \{b, c, d\} \\ \{b\},\ \{a, c, d\} \\ \{a, b, c, d\} \\ \{a, b\},\ \{c, d\} \\ \{\phi\} \end{array}} \Longrightarrow \text{this is a Field.}$$

(Satisfies E.1, E.2 & E.3)

Axiomatic Definition of an Experiment

$$(\Omega, F, P) \quad \text{or} \quad (S, F, P)$$

1. The sample space, Ω / S, or the set of all experimental outcomes.
2. The Borel Field of all events in S.
3. The probabilities of these events. $\{P(\cdot)\}$.

Eg of a <u>COUNTABLE SPACE</u>.

$\boxed{\text{Eg:}}$ Consider a coin-toss exp.

$$S = \{h, t\}$$

$$F = \{\phi\}, \{h\}, \{t\}, \{h, t\}$$

$$P(h) = p; \quad P(t) = q \quad \Rightarrow P(S) = 1$$
$$\Rightarrow p + q = 1$$

Now consider the experiment of tossing a coin 2 times

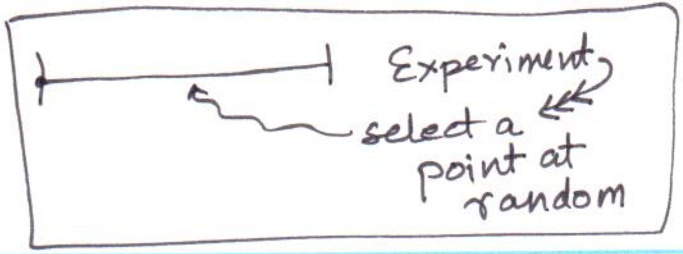
$$S = \{hh, ht, th, tt\}$$

$$F = \{\phi\} \; \{hh\}, \{ht\}, \{th\}, \{tt\}, \{hh, ht\}$$
$$\ldots\ldots, S \Rightarrow 2^4 = 16 \text{ events}$$

What is the Probabity of a Single head in two tosses. $= ?? = P(ht) + P(th)$

$$\{\text{Single head in two tosses}\} = \{ht, th\}$$

$S =$ all points on a real line



* The set S is uncountably infinite

* We cannot determine probabilities in terms of probabilities of the individual outcomes...

* To construct a prob. space, we choose F to consist of events of the form

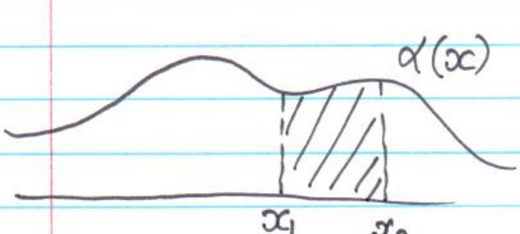
$\{x_1 \le x \le x_2\}$, i.e intervals and their countable unions & intersections.

* To complete the specification of prob. space, we need to define/assign prob. to intervals

For any function $\alpha(x)$ such that

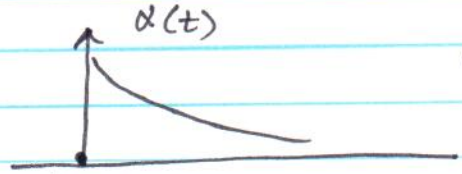
$$\int_{-\infty}^{\infty} \alpha(x)\,dx = 1, \quad \alpha(x) \ge 0$$

$$P\{x_1 \le x \le x_2\} \triangleq \int_{x_1}^{x_2} \alpha(x)\,dx$$



Probability of the event $x_1 \le x \le x_2$ is the area of the "Shaded" region.

$\boxed{\text{Eg 1}}$. Radioactive Emission at a random time t



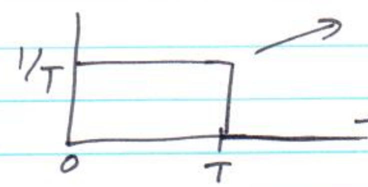
$$\alpha(t) = \begin{cases} c e^{-ct} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Prob that particle is emitted in time interval $(0, 2)$ $\overset{?}{=} \displaystyle\int_0^2 c e^{-ct} \, dt$

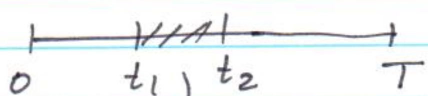
$$= 1 - e^{-2c}$$

$\boxed{\text{Eg 2.}}$ Telephone Call

Uniformly at Random $\rightarrow$ $\alpha(t)$



check $\displaystyle\int_{-\infty}^{\infty} \alpha(t) \, dT = 1.$



Prob that call arrives in (t_1, t_2) $= \displaystyle\int_{t_1}^{t_2} \frac{dt}{T}$

$$= \frac{(t_2 - t_1)}{T}.$$