Today

1) Markov & Chebychev's Inequalities

2) Moment Generating Function (MGF)

Recap from last lec.

$m_n = E[X^n]$ ($n^{th}$ moment)

$\mu_n = E[(X-\mu)^n]$ ($n^{th}$ central moment)

$\mu = E[X] \rightarrow$ mean/first-order

$\sigma^2 = E[(X-\mu)^2] \rightarrow$ second-order (Centered) moment.

$f_X(x)$ is not available.

but we may estimate $\mu$ & $\sigma^2$.

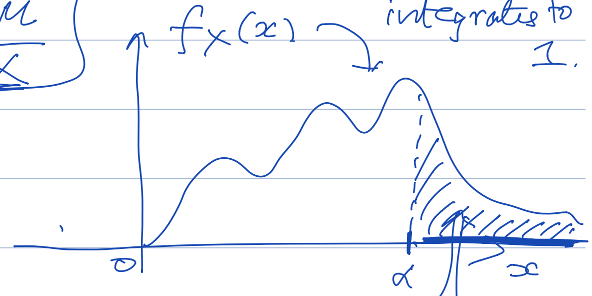
Markov Inequality

$X$ is non-negative r.v.

we only know the mean $\mu$ of the r.v.

$$\boxed{P(X \geq \alpha) \leq \frac{\mu}{\alpha}}$$

$f_X(x)$ integrates to 1.




$P(X \geq \alpha)$

"tail probability"

$\leq \frac{\mu}{\alpha}$

Proof: $\mu = E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$ $\quad (X \geq 0)$

$= \int_{0}^{\infty} x f_X(x)\, dx$

$= \underbrace{\int_{0}^{\alpha} x f_X(x)\, dx}_{\geq 0} + \int_{\alpha}^{\infty} x f_X(x)\, dx.$

$$\mu \geq \int_{x=\alpha}^{\infty} x\, f_X(x)\,dx.$$

$$\geq \int_{\alpha}^{\infty} \alpha\, f_X(x)\,dx$$

$$= \alpha \boxed{\int_{\alpha}^{\infty} f_X(x)\,dx.} = \alpha\, P(X \geq \alpha).$$

$$\boxed{P(X \geq \alpha) = \int_{\alpha}^{\infty} f_X(x)\,dx}$$

$$\mu \geq \alpha\, P(X \geq \alpha)$$

$$\Rightarrow \boxed{P(X \geq \alpha) \leq \frac{\mu}{\alpha}}$$

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**Chebychev's Inequality** For any r.v. $X$ and for any $\varepsilon > 0$

$$\underbrace{P(|X-\mu| \geq \varepsilon)}_{LHS} \leq \frac{\sigma^2}{\varepsilon^2}$$

$\sigma^2 = 1.$


$\varepsilon = 10^{-3}.$

$$\underbrace{P(\quad)}\leq \frac{\sigma^2}{\varepsilon^2} = \frac{1}{10^{-6}} = 10^6.$$

area shaded "tail" $= P(|X-\mu| \geq \varepsilon)$

$$\leq \frac{\sigma^2}{\varepsilon^2}.$$

Proof:

$$\sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$= \int_{-\infty}^{\mu-\varepsilon} (x-\mu)^2 f_X(x)\,dx + \underbrace{\int_{\mu-\varepsilon}^{\mu+\varepsilon} (x-\mu)^2 f_X(x)\,dx}_{} + \int_{\mu+\varepsilon}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$\geq \int_{-\infty}^{(\mu-\varepsilon)} \underbrace{(x-\mu)^2}_{} f_X(x)\,dx + \int_{\mu+\varepsilon}^{\infty} \underbrace{(x-\mu)^2}_{\geq \varepsilon^2} f_X(x)\,dx.$$

$(x-\mu)^2 \geq \varepsilon^2$

$x \leq \mu-\varepsilon.$

$\varepsilon \leq (\mu - x)$

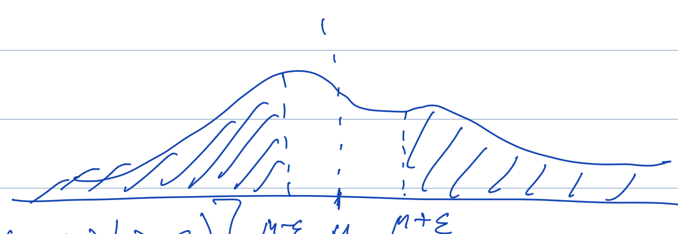
$\varepsilon^2 \leq (\mu-x)^2 = (x-\mu)^2$

$$\boxed{\begin{aligned}&P(|X-\mu| \geq \varepsilon)\\ &= P(X \leq \mu-\varepsilon) + \\ &\quad P(X \geq \mu+\varepsilon).\end{aligned}}$$

$$\geq \int_{-\infty}^{\mu-\varepsilon} \varepsilon^2 f_X(x)\,dx + \int_{\mu+\varepsilon}^{\infty} \varepsilon^2 f_X(x)\,dx$$

$$= \varepsilon^2 \left[\underbrace{\int_{-\infty}^{\mu-\varepsilon} f_X(x)\,dx}_{} + \quad\right]$$

$$= \varepsilon^2 \cdot \left[\underbrace{P(X \leq \mu-\varepsilon) + P(X \geq \mu+\varepsilon)}_{}\right].$$




$$\sigma^2 \geq \varepsilon^2 \left[P(|(X-\mu)| \geq \varepsilon)\right]$$

Proves Chebychev.

Moment Generating Function (MGF)
Characteristic Function

$X$ Complex valued $\Phi_X(\omega) = E[e^{j\omega X}]$ $j=\sqrt{-1}$
(Characteristic fn)

$S(t) = E[e^{tX}]$ $t$ is real valued.
(MGF)
real valued function.

$\Phi_X(\omega) = E[e^{j\omega X}]$

$X \sim Ber(p)$

$\boxed{X} = \begin{cases} 0 & w.p\ (1-p) \\ 1 & w.p\ p \end{cases}$

$$\boxed{E[g(X)] = \sum_i g(X=x_i) \cdot P(X=x_i)}$$

"Law of the unconcious statistician." "(LOTUS)"

$\Phi_X(\omega) = e^{j\omega \times 0} \times P(X=0) + e^{j\omega \times 1} \times P(X=1)$

$= \sum_i e^{j\omega x_i} P(X=x_i)$ (if $X$ was discrete)

$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx$ (if $X$ was continuous)

is $\Phi_X(\omega)$ periodic? $\Phi_X(\omega+2\pi) \overset{?}{=} \Phi_X(\omega)$

$$\boxed{e^{j(\omega+2\pi)X} = e^{j\omega X}.}$$

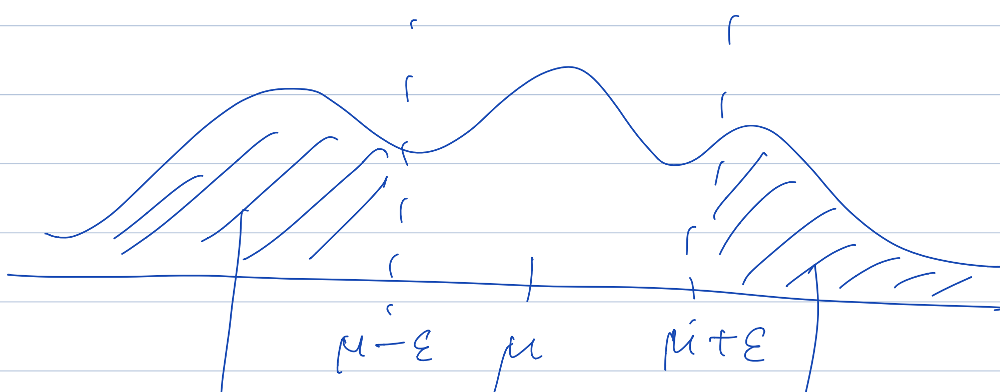
$\Phi_X(\omega)$ is periodic w/ period $2\pi$

↳ need to char. this for $-\pi \leq \omega \leq \pi$.

(1) Using $\Phi_X(\omega)$ we can compute the $n^{th}$ moment of the r.v. $X$.

Markov ineq → works only for $X \geq 0$ (non-neg r.v.).

Chebychev inequality → works for any r.v. $X$.




shaded area $\leq \dfrac{\sigma^2}{\varepsilon^2}$