Midterm 2 Exam - ECE 503 Fall 2020

- Due Date and Time: Wednesday, Nov. 11, 2020, by Noon.
- Submit your answers on D2L.
- Maximum Credit: 100 points

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1. (a) [5 points] (True/False) Covariance matrix is always Positive-semi-definite (PSD). → True

   (b) [5 points] (True/False) Mean-square convergence always implies convergence almost surely. → False

   (c) [10 points] Random variables (X, Y) have the following joint PDF (defined for all x, y):

$$f_{X,Y}(x, y) = ce^{-(2x^2-4xy+4y^2)}$$

- What is c?
- Find $E[X], E[Y]$ and the correlation coefficient ρ.
- Are X and Y independent?

(X, Y) are jointly Gaussian.

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-r^2)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}\right)\right\}$$

We are given the PDF, with exponential term

$$\exp\left\{-\left(2x^2 + 4y^2 - 4xy\right)\right\}$$

$$\Rightarrow \quad 2 = \frac{1}{2(1-\rho^2)\sigma_1^2} \qquad 4 = \frac{1}{2(1-\rho^2)\sigma_2^2} \; ; \quad 4 = \frac{2\rho}{2(1-\rho^2)\sigma_1\sigma_2}$$

can solve for all parameters of PDF

$$\underbrace{\mu_1}_{E[X]} = 0 \; ; \quad \underbrace{\mu_2}_{E[Y]} = 0 \qquad \sigma_1 = \frac{1}{\sqrt{2}} \; ; \quad \sigma_2 = \frac{1}{2}$$

$$\boxed{\rho = \frac{1}{\sqrt{2}}} \quad \Rightarrow \quad c = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$\boxed{c = \frac{2}{\pi}}$$

Since $\rho \neq 0$

$\Rightarrow$ X & Y are <u>NOT</u> independent.

2. [20 points] Let X, Y be a pair of random variables with the following joint PDF:

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x), & 0 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the MMSE estimate of X given Y.

(b) Find the Linear (L-MMSE) estimate of X given Y.

(c) Compare the resulting errors of the estimators.

(a) $\hat{X}_{MMSE} = E[X|Y]$

First find $f_Y(y) = \int_{x=0}^{y} 6(y-x)\,dx = 3y^2$

$\Rightarrow f_{X|Y}(x|y) = \dfrac{f_{XY}}{f_Y} = \begin{cases} \dfrac{6(y-x)}{3y^2} & \text{if } 0 \le x \le y \\ 0 & \text{o.wise} \end{cases}$

conditional PDF of X given Y

$\Rightarrow E[X|Y=y] = \int_0^y x\, f_{X|Y}(x|y)\,dx$

$= \int_0^y \dfrac{2x(y-x)}{y^2}\,dx = y/3$

$\Rightarrow E[X|Y] = \dfrac{Y}{3} \Rightarrow \boxed{\hat{X}_{MMSE} = Y/3}$

(b) Since MMSE estimator is linear, so

$\hat{X}_{L\text{-}MMSE} = \hat{X}_{MMSE} = Y/3.$

(c) Resulting Error:

$= E\left[(X - \hat{X}_{MMSE})^2\right] = E\left[\left(X - \frac{Y}{3}\right)^2\right]$

$= \underbrace{(Var(X))} \times (1-\rho^2)$

$= \dfrac{3}{80} \times \left(1 - \left(\frac{1}{3}\right)^2\right) = \dfrac{3}{80} \times \dfrac{8}{9} = \boxed{\dfrac{1}{30}}$

(Same error for both MMSE & Linear-MMSE estimators)

3. [20 points] Suppose N sensors are placed in a geographical area to detect seismic activity. Each sensor i, for $i = 1, 2, \ldots, N$ has a battery with a lifetime X_i distributed as an exponential random variable with mean 100 days. You can assume that the sensor lifetimes $X_1, X_2, \ldots, X_N$ behave in an i.i.d. manner and all sensors are turned on simultaneously. Let X denote the time when the batteries of all the N sensors die.

(a) Find the CDF and PDF of X. (Hint: try to express X in terms of $X_1, X_2, \ldots, X_N$).

(b) Compute the probability that all sensors will die within one year (365 days).

$N \to$ sensors, iid lifetimes $X_i \sim \exp(\lambda)$ $\quad \lambda = \frac{1}{\mu}$ for exp

$\Rightarrow X_i \sim \exp(1/100)$ $\quad \Rightarrow \lambda = \frac{1}{100}$

$X = \max(X_1, X_2, \ldots, X_N)$

CDF of $X \Rightarrow P(X \le x) = P(\max(X_1 \ldots X_N) \le x)$

$= P(X_1 \le x, \ldots, X_N \le x)$ indep

$= P(X_1 \le x) \ldots P(X_N \le x)$

$= (1 - e^{-\lambda x})(1 - e^{-\lambda x}) \ldots (1 - e^{-\lambda x})$

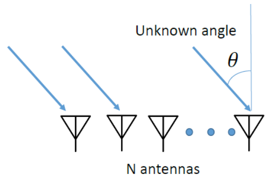
$P(X \le x) = (1 - e^{-\lambda x})^N.$

(b) Prob(all sensors die within one year)

$= P(X \le 365)$

$= \left(1 - e^{-\frac{1}{100} \times 365}\right)^N = (0.974)^N.$

4. [20 points] Consider an antenna array composed of N antennas, which receives a signal arriving from an unknown angle θ along with noise (as shown in the Figure below).



The N-dimensional signal received at the antenna array is given as follows:

$$\mathbf{Y} = \mathbf{a}(\theta) + \mathbf{Z} \tag{1}$$

where $\mathbf{a}(\theta)$ is also known as a steering vector in the direction θ, and the noise $\mathbf{Z}$ is a Gaussian random vector with zero mean and identity covariance matrix. The steering vector is assumed to be unit norm, i.e., it satisfies $\mathbf{a}^T(\theta)\mathbf{a}(\theta) = 1$ for all θ. In this problem, we are interested in devising the maximum likelihood estimator for estimating the unknown angle of arrival θ.

(a) Compute the mean and covariance of $\mathbf{Y}$ as a function of the angle of arrival θ.

(b) Write down the PDF $f_{\mathbf{Y}}(\mathbf{y};\theta)$ of the received vector $\mathbf{Y}$ as a function of angle of arrival θ.

(c) The maximum likelihood (ML) estimate of θ is defined as follows:

$$\hat{\theta}_{ML}(\mathbf{y}) = \arg\max_{\phi} f_{\mathbf{Y}}(\mathbf{y};\phi) \tag{2}$$

i.e., the Maximum-likelihood estimate picks the angle which maximizes the PDF of the received signal. Using the result (i.e., PDF) from part (b), compute the ML estimate of the unknown angle of arrival θ.

$$\vec{Y} = \vec{a}(\theta) + \vec{Z} \sim \mathcal{N}(\vec{0}, \underbrace{I}_{\to \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix}})$$

$$E[\vec{Y}] = \underbrace{E[\vec{a}(\theta)]}_{\vec{a}(\theta)} + \underbrace{E[\vec{Z}]}_{\parallel \; 0} = \vec{a}(\theta)$$

$$C_{\vec{y}} = E[(\vec{y}-\vec{a}(\theta))(\vec{y}-\vec{a}(\theta))^T] = E[\vec{Z}\vec{Z}^T] = C_{\vec{Z}} = I$$

$$\vec{y} \sim \mathcal{N}(\vec{a}(\theta), I)$$

PDF of $\vec{y}$ $\Rightarrow$ $f_{\vec{y}}(\vec{y};\theta) = \dfrac{1}{(2\pi)^{N/2}} \exp\left\{-\frac{1}{2}(\vec{y}-\vec{a}(\theta))^T \cdot I \times (\vec{y}-\vec{a}(\theta))\right\}$

$$f_Y(\vec{y},\theta) = \frac{1}{(2\pi)^{N/2}} \cdot \exp\left\{-\frac{1}{2}\left(\underbrace{\vec{y}^T\vec{y}}_{\text{indep of }\theta} - 2\vec{y}^T\vec{a}(\theta) + \underbrace{\vec{a}^T(\theta)\vec{a}(\theta)}_{=1}\right)\right\}$$

$$\hat{\theta}_{ML} = \arg\max_{\phi} f(\vec{y},\phi) = \arg\max_{\phi} \log(f(\vec{y},\phi)) = \underbrace{\arg\max_{\phi} \vec{y}^T\vec{a}(\phi)}$$

pick the angle ϕ which has maximum correlation with the received vectors

5. [20 points] The latest Tesla factory is supposed to start production of model-S cars. On each day, the number of cars produced is a random variable, with mean 5 and variance 9. Production efficiency is assumed to be independent and identically distributed across days.

   (a) Find an approximation to the probability that the total number of model-S cars produced in 100 days is less than 440.

   (b) Let N be the first day since launching the factory on which the total number of cars produced exceeds 1000. Find an approximation to the probability that $N \geq 220$.

(Hint: use the Central Limit Theorem for the approximations)

Let $P_D = X_1 + X_2 \cdots + X_D$ be the total cars produced in D days.

$X_1, \ldots X_D$ are iid with mean 5 & variance 9.

(a) $$P(P_{100} \leq 440) = P\Big(\underbrace{\frac{P_{100} - 100\times 5}{\sqrt{100\times 9}}}_{\mathcal{N}(0,1) \text{ by CLT}} \leq \frac{440 - 100\times 5}{\sqrt{100\times 9}} \Big)$$

$$\approx \Phi\left(\frac{440 - 500}{\sqrt{100\times 9}} \right) = \boxed{\Phi(-2) \approx 0.023}$$

(b) Event $(N \geq 220)$ = (Produce no more than 1000 cars in 220 days.)

N: # of days to produce 1000 cars.

$$= \text{Prob}(P_{220} \leq 1000)$$

$$= \text{Prob}\left(\frac{P_{220} - 5\times 220}{\sqrt{220\times 9}} \leq \frac{1000 - 1100}{\sqrt{1980}} \right)$$

$$= \Phi\left(\frac{-100}{\sqrt{1980}} \right)$$

$$= \Phi(-2.247) \boxed{\approx 0.0123}$$