Solutions

Final Exam - ECE 503 Fall 2016

- Date: Wednesday, December 14, 2016.
- Time: 10:30 am -12:30 pm
- Maximum Credit: 100 points

1. [10 points] In the Arizona State Lottery, a player picks 6 different numbers from a sequence of 1 through 15. At a lottery drawing, 6 balls are drawn at random from a box containing 15 balls numbered 1 through 15.

   (a) What is the probability that the player has 0 matches and wins nothing ?

   (b) What is the probability that the player has exactly 6 matches and wins the jackpot ?

Let $n = 15$ be total # of balls

Suppose that the player picks some m balls $(m=6)$

Total # of ways to pick m balls out of $n = \binom{n}{m}$

Total " " " k matching balls $= \binom{m}{k}\binom{n-m}{k}$

$\Rightarrow$ Probability of k-matching balls $= \dfrac{\binom{m}{k}\binom{n-m}{k}}{\binom{n}{m}}$

(a) Prob. of 0 matching $= \dfrac{\binom{6}{0}\binom{15-6}{6-0}}{\binom{15}{6}} = \dfrac{\binom{9}{6}}{\binom{15}{6}}$

(b) Prob of 6 matching $= \dfrac{\binom{6}{6}\binom{15-6}{6-6}}{\binom{15}{6}} = \dfrac{1}{\binom{15}{6}}$

2. [20 points] Suppose X and Y are independent random variables, and each one of them is uniformly distributed in $[0,1]$. Let $Z = \min(X,Y)$, and $W = \max(X,Y)$.

(a) Find the joint density (PDF) of (Z,W).

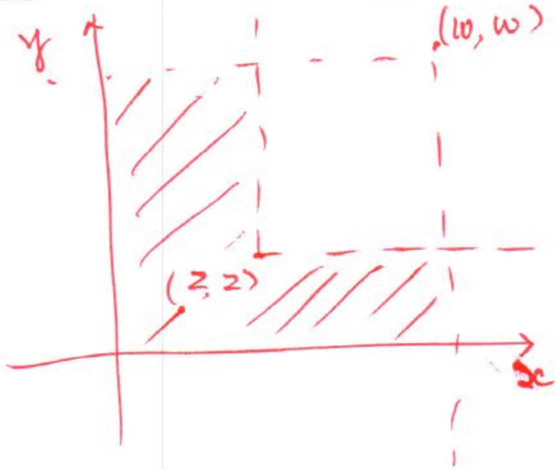
(b) Are Z and W independent random variables ?

$Z = \min(X,Y) \qquad W = \max(X,Y)$

$F_{Z,W}(z,w) = 0$ if $z<0$, $w<0$

$F_{Z,W}(z,w) = P(Z \le z, W \le w) = P(\min(X,Y) \le z, \max(X,Y) \le w)$

Case A $\quad w \ge z$



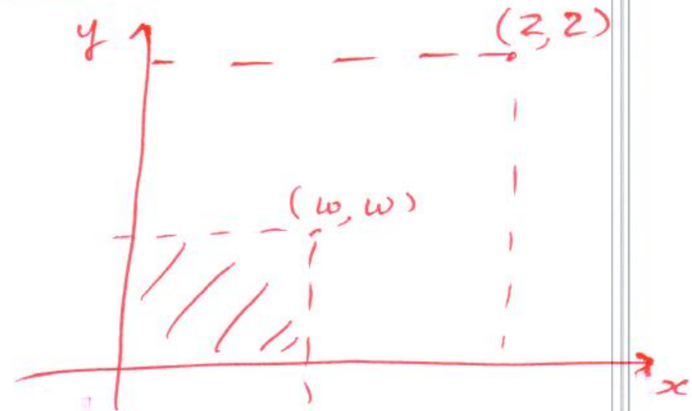
$F_{Z,W}(z,w) =$ shaded area

$= F_{X,Y}(z,w) + F_{X,Y}(w,z) - F_{X,Y}(z,z)$

$= wz + wz - z^2$

$= 2wz - z^2$

Case B $\quad w < z$



$F_{Z,W}(z,w) = F_{X,Y}(w,w) = w^2$

$\Rightarrow F_{Z,W}(z,w) = \begin{cases} 2wz - z^2 & 0<z<w<1 \\ w^2 & 0<w<z<1 \end{cases}$

$\Rightarrow f_{Z,W}(z,w) = \begin{cases} 2 & 0<z<w<1 \\ 0 & \text{otherwise.} \end{cases}$

(b) $f_Z(z) = 2(1-z),\ 0<z<1$

$f_W(w) = 2w,\ 0<w<1$

clearly $f_{Z,W} \neq f_Z \cdot f_W \Rightarrow$ Z and W are NOT independant.

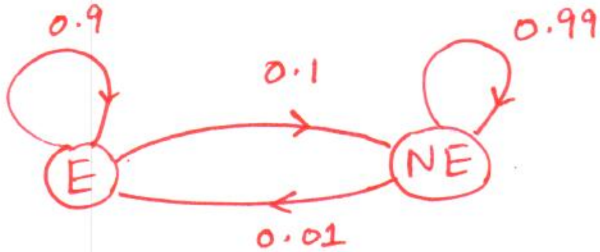
3. [20 points] A wireless communication channel suffers from a clustered error pattern. Whenever a packet has an error, the next packet will have an error with probability 0.9. Whenever a packet is error-free, the next packet is error free with probability 0.99. We want to use a Markov chain to analyze the above process, assuming two states, E (error), and NE (no error).

   (a) Draw a state-transition diagram showing the transition probabilities, and write the one-step transition matrix P.

   (b) Find the stationary probability distribution (and comment on its uniqueness).

   (c) Suppose that the initial probability distribution was $p(0) = [p_E(0) \;\; p_{NE}(0)] = [0 \;\; 1]$. Find the probability distribution at time ~~$n = 100$.~~ $n = 5$.

(a)



$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.01 & 0.99 \end{bmatrix}$$

$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \quad \text{with} \quad \begin{aligned} p &= 0.1 \\ q &= 0.01 \end{aligned}$$

(b) $\vec{\pi} = \vec{\pi} P$

$$\vec{\pi} = \left[\frac{q}{q+p} \quad \frac{p}{q+p} \right] = \left[\frac{0.01}{0.11} \quad \frac{0.1}{0.11} \right] = \left[\frac{1}{11} \quad \frac{10}{11} \right]$$

The M.C. is irreducible and aperiodic $\Rightarrow$ $\vec{\pi}$ is unique & does not depend on $\vec{p}(0)$.

$$\lambda_2 = 1-(p+q) = 1 - 0.11 = 0.89$$

(c)

$$\vec{p}(n) = \left[\frac{1}{11} \quad \frac{10}{11} \right] + \frac{(0.89)^n}{0.11} \times \left[-0.01 \quad 0.01 \right]$$

$\vec{p}(0) = [0 \quad 1]$

$$\vec{p}(5) = \left[\frac{1}{11} - \frac{(0.89)^5}{11} \quad \frac{10}{11} + \frac{(0.89)^5}{11} \right]$$

$$= \left[\frac{1}{11} - 0.05 \quad \frac{10}{11} + 0.05 \right] = \left[0.04 \quad 0.96 \right]$$

4. [15 points]

(a) Let $X(t)$ be a wide sense stationary random process, and we define $Y(t) = X(t+5)$. Are $X(t)$ and $Y(t)$ jointly wide sense stationary ?

(b) Let X_n have the following PDF,

$$f_n(x) = g_n(x)(1 - 1/n^3) + h_n(x)/n^3$$

where $g_n(x) \sim N(0, 1/n^2)$, and $h_n \sim N(n, 1)$. Does X_n converge to 0 in the mean square sense ?

(c) A W.S.S. process $X(t)$, of mean $\mu_X = 10$ is the input to a LTI filter, with impulse response

$$h(t) = \begin{cases} e^{t/0.2} & 0 \le t \le 0.2 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the output process $Y(t)$?

(a) $Y(t) = X(t+5) \quad E[Y(t)] = E[X(t+5)] = \text{const.}$

$E[Y(t)\,Y(t+\tau)] = E[X(t+5)\,X(t+\tau+5)] = R_X(\tau)$

$\Rightarrow Y(t)$ is W.S.S.

$E[X(t_1)\,Y(t_2)] = E[X(t_1)\,X(t_2+5)] = R_X((t_2-t_1)+5) \rightarrow$ only dep. on $(t_2 - t_1)$

$\Rightarrow X(t), Y(t)$ are jointly W.S.S.

(b). $E[|X_n - 0|^2] = E[|X_n|^2] = \int_{-\infty}^{\infty} x_n^2 f_n(x)\,dx.$

$= \int_{-\infty}^{\infty} x_n^2 \left[g_n(x)\left(1 - \frac{1}{n^3}\right) + \frac{h_n(x)}{n^3} \right] dx = \underbrace{\frac{1}{n^2}\left(1-\frac{1}{n^3}\right)}_{\rightarrow 0} + \underbrace{\frac{1+n^2}{n^3}}_{\rightarrow 0}$ as $n \rightarrow \infty$

$\Rightarrow X_n \xrightarrow{M.S.} 0$

(c). $Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(\tau)\,X(t-\tau)\,d\tau.$

$E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\,E[X(t-\tau)]\,d\tau = \int_{-\infty}^{\infty} h(\tau) \times 10 \,.\, d\tau.$

$= 10 \times \int_{0}^{0.2} e^{\tau/0.2}\,d\tau = 2 \int_{0}^{1} e^{+\tau}\,d\tau = \boxed{2(e-1)}.$

5. [20 points] Weather analysis in Florida revealed that hurricanes occur according to a Poisson process of intensity $\lambda = 2$ per week.

(a) What is the average time between two hurricanes ?

(b) What is the probability that no hurricanes occur during a given 2 week period ?

(c) If the peak of hurricane season lasts 12 weeks, what is the expected number of hurricanes ?

(d) Find the probability that during at least one of the 12 weeks of peak season, there are at least five hurricanes.

(a) $E[X_i] = \frac{1}{\lambda} = \boxed{0.5 \text{ weeks}}$

(b) $P(N_2 = 0) = e^{-\lambda \times 2} = e^{-4} = \boxed{0.0183}$

(c) $E[N_{12}] = 12\lambda = \boxed{24 \text{ hurricanes}}$

(d) $P\left(\bigcup_{i=1}^{12} \{N_i - N_{i-1} \geq 5\}\right) = 1 - P\left(\bigcap_{i=1}^{12} (N_i - N_{i-1} \leq 4)\right)$

$= 1 - \left[e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2} + \frac{\lambda^3}{6} + \frac{\lambda^4}{24}\right)\right]^{12}$

$= 1 - \left[e^{-2}\left(1 + 2 + 2 + 4/3 + 2/3\right)\right]^{12}$

$= 1 - \left[7e^{-2}\right]^{12} = 1 - 0.523 = \boxed{0.477}$

6. [15 points] For a zero-mean WSS random process $X(t)$, with auto-correlation function $R_X(\tau) = 1/(1+\tau^2)$, we perform the following sequence of operations. The continuous time process is first sampled at the rate of $f_s = \cancel{10^4}\ 1$ sample~~s~~ per second. This yields a discrete time random process X_n. We then pass X_n through a discrete-time filter, whose output is denoted by $Y_n = X_n - X_{n-1}$.

(a) Find the mean and auto-correlation function of X_n.

(b) Is the discrete time process X_n also WSS?

(c) Find the mean and auto-correlation function of Y_n.

(d) Are X_n and Y_n jointly WSS ?

(a) $X_n = X(nT_s)$ T_s = sampling period = 1 sec.

$$E[X_n] = E[X(nT_s)] = 0$$

$$R_X(k) = E[X_n X_{n+k}] = E[X(nT_s)\, X((n+k)T_s)]$$
$$= R_X(kT_s) = \frac{1}{1+k^2 T_s^2} = \frac{1}{1+k^2}$$

(b) $\Rightarrow$ X_n is W.S.S.

(c) $Y_n = X_n - X_{n-1}$

$$E[Y_n] = 0$$

$$E[Y_n Y_{n+k}] = E[(X_n - X_{n-1})(X_{n+k} - X_{n-1+k})]$$
$$= R_X(k) - R_X(k-1) - R_X(k+1) + R_X(k)$$
$$= \frac{2}{1+k^2} - \frac{1}{1+(k-1)^2} - \frac{1}{1+(k+1)^2}$$

$$E[\cancel{X}_n X_{n+k}] = E[(X_n - X_{n-1}) X_{n+k}] = R_X(k) - R_X(k+1)$$
$$= \frac{1}{1+k^2} - \frac{1}{1+(k+1)^2}$$

X_n, Y_n are also W.S.S.