Today

1) Wrap up Characteristic Function (or MGF)

2) Two Random Variables
- → Joint CDF & Joint PDF
- → Marginal CDF & Marginal PDF
- → Independent Random Variables.

Characteristic Function $\boxed{\Phi_X(\omega) = E[e^{j\omega X}]}$

ω is real
$\Phi_X(\omega)$ is complex valued

if X is discrete

$$\Phi_X(\omega) = \sum_{i \in \mathcal{X}} P(X=x_i)\, e^{j\omega x_i}$$

if X is continuous

$$\Phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

how to compute the characteristic function.

Moment Generating Function $M_X(s) = E[e^{sX}]$ (s is real) & $M_X(s)$ is real valued

$\Phi_X(s) / M_X(t)$

Computing the $\boxed{n^{th} \text{ moment of } X}$ from either $\Phi_X(\omega)$ or $M_X(s)$.

$$\rightarrow \frac{d}{d\omega}\Phi_X(\omega) = \frac{d}{d\omega}\Big[\underbrace{\sum_k P(X=k)\, e^{j\omega k}}_{\substack{\shortparallel \\ E[e^{j\omega X}]}}\Big]$$

$$= \sum_k P(X=k) \cdot \frac{d(e^{j\omega k})}{d\omega}$$

$$= \sum_k P(X=k) \cdot (jk) \cdot e^{j\omega k} \leftarrow$$

$$\boxed{\frac{1}{j}\frac{d}{d\omega}\Phi_X(\omega) = \sum_k P(X=k) \times k \times e^{j\omega k}} \leftarrow$$

$$E[X] = \sum_k P(X=k) \times k$$

Evaluating @ $\omega = 0$

$$\frac{1}{j} \frac{d\Phi(\omega)}{d\omega}\Bigg|_{\omega=0} = \sum_k k \times P(X=k) = E[X].$$

$$\frac{d^2 \Phi_X(\omega)}{d\omega^2} = \sum_k P(X=k)(jk)^2 e^{j\omega k}.$$

$$\boxed{\frac{1}{j^2} \frac{d^2 \Phi_X(\omega)}{d\omega^2} = \sum_k P(X=k)\, k^2 e^{j\omega k}.}$$

$$E[X^2] = \frac{1}{j^2} \frac{d^2 \Phi_X(\omega)}{d\omega^2}\Bigg|_{\omega=0}.$$

$\vdots$

$$\boxed{\underbrace{E[X^n]}_{n^{th} \text{ moment of } X} = \frac{1}{j^n} \frac{d^n \Phi_X(\omega)}{d\omega^n}\Bigg|_{\omega=0}.}$$

Eg Expected value of a Binomial r.v. $\rightarrow$

$$\Phi_X(\omega) = \sum_{k=0}^{n} P(X=k) \cdot e^{j\omega k} \qquad P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$k = 0, 1, 2, \ldots, n$

$$= \sum \binom{n}{k} p^k (1-p)^{n-k} \cdot e^{j\omega k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (p e^{j\omega})^k \cdot (1-p)^{n-k}.$$

$$= \left(p e^{j\omega} + (1-p)\right)^n$$

$$\boxed{\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n.}$$

$$E[X] = \frac{1}{j} \frac{d\,\Phi_X(\omega)}{d\omega}\Big|_{\omega=0}.$$

$$= \frac{1}{\cancel{j}}\, n\left(pe^{j\omega} + (1-p)\right)^{n-1} \times p \times \cancel{j} \cdot e^{j\omega}\Big|_{\omega=0}.$$

$$= np \times (p + 1 - p)^{n-1}$$

$$E[X] = \boxed{np}.$$

$$E[X^2] = \ldots\ldots$$

try this yourself.

$$\boxed{Var(X) \overset{?}{=} E[X^2] - (E[X])^2} = np(1-p)$$

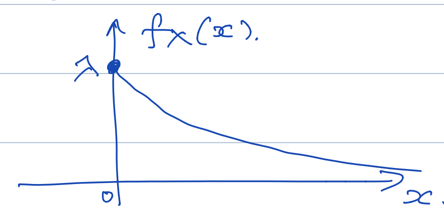
Eg 2 n^{th} moment of an $\underbrace{Exp(\lambda)}$ $\to$ Exponential r.v. w/ parameter λ.

$$\Phi_X(\omega) = E[e^{j\omega X}]$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \int_0^\infty e^{j\omega x} \times \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{(-\lambda + j\omega)x}\, dx$$



$$\boxed{\Phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}.}$$

(See my notes.....)

Two Random Variables

Temp & Humidity.

T → 60°F, 90°F

H → 50%, 100%.

$P(T = 60°F) = 0.3$ $\quad$ $P(H = 50\%) = 0.8$

$P(T = 90°F) = 0.7.$ $\quad$ $P(H = 100\%) = 0.2.$

For discrete r.v.'s. $\boxed{\text{Joint PMF}}$

$$\boxed{\begin{aligned} P(T = 60, H = 50) &\overset{?}{=} 0.1 \\ P(T = 60, H = 100) &\overset{?}{=} 0.2 \\ P(T = 90, H = 50) &\overset{?}{=} 0.6 \\ P(T = 90, H = 100) &\overset{?}{=} 0.1 \end{aligned}}$$

(T, H)

$S = \{(60, 50), (60, 100), (90, 50), (90, 100)\}$

4 mutually exclusive events.

T	H
60	50
60	100
90	50
90	100.

$P(S) = 1.$

joint PMF

T \ H	50	100
60	0.1	0.2
90	0.6	0.1.

$P(T = 90) = 0.7$

$P(H = 50) =$

$P(T = 60) \overset{?}{=} P(T=60, H=50) + P(T=60, H=100)$

$= 0.1 + 0.2 = 0.3.$

$$P(H=50) = \underbrace{P(H=50,\ T=60)} + \underbrace{P(H=50,\ T=90)}$$
$$= 0.1 + 0.6 = 0.7.$$

$$P(H=100) = 0.3.$$

→ "Marginal" behavior of H.

X, Y Joint PMF

(suppose X, Y are discrete valued)

$$\boxed{P_{X,Y}(x,y) = P(X=x,\ Y=y) \quad \text{for } x \in \mathcal{X},\ y \in \mathcal{Y}.}$$

$\mathcal{X} = \{1, 2, 3\}$.

$\mathcal{Y} = \{13, 19, 44, 88\}$.

X \ Y	13	19	44	88
1	0	0	0	0
2	0	0	●	0
3				

$P(X=2,\ Y=44)$.

$\boxed{\text{Sum of all entries} = 1}$.

$$\boxed{\text{First property}: \quad \sum_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} P(X=x,\ Y=y) = 1.}$$

$\boxed{P_X(x)}$ ⟶ Marginal PMF of X

$\boxed{P_Y(y)}$ ⟶ " Marginal PMF of Y.

Computing marginal PMF of X.

$$P_X(x) = \boxed{P(X=x) = \sum_{y \in \mathcal{Y}} P(X=x,\ Y=y)}$$

↑ Total prob. theorem.

$$P_Y(y) = P(Y=y) = \sum_{x \in \mathcal{X}} P(X=x,\ Y=y).$$

Continuous Valued X, Y

Joint PDF of $(X, Y) \Rightarrow \underbrace{f_{X,Y}(x, y)}$. joint density

$$\boxed{\int_{y=-\infty}^{\infty}\int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1.}$$

Marginal PDF of X

$$\underline{f_X(x)} = \int_{y=-\infty}^{\infty} \underbrace{f_{X,Y}(x,y)}\,dy$$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Joint CDF of two random variables

$$\boxed{F_{X,Y}(x,y) = P(X \le x,\ Y \le y)}$$

comma = "and" or $\cap$

$F_X(x) = P(X \le x)$.

$F_Y(y) = P(Y \le y)$.

$$F_{X,Y}(\infty, \infty) \overset{?}{=} 1.$$

$$F_{X,Y}(x, \infty) \overset{?}{=} P(\underbrace{X \le x}, \underbrace{Y \le \infty}) = P(X \le x) = F_X(x).$$

$$F_{X,Y}(\infty, y) = F_Y(y).$$

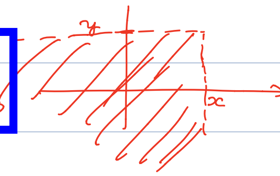
$$F_{X,Y}(-\infty, y) \overset{?}{=} 0 \Longleftarrow$$

$$F_{X,Y}(x, -\infty) = 0.$$

Computing Joint PDF from Joint CDF.

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}, \quad F_{X,Y}(x,y).$$

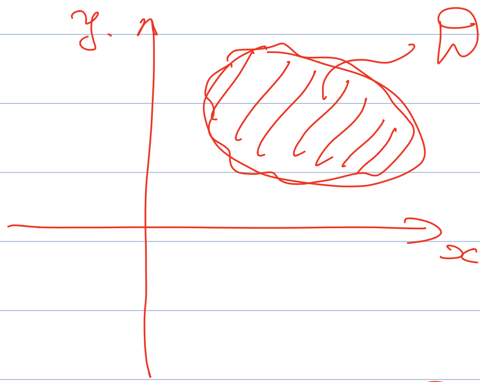
Joint CDF from joint PDF.

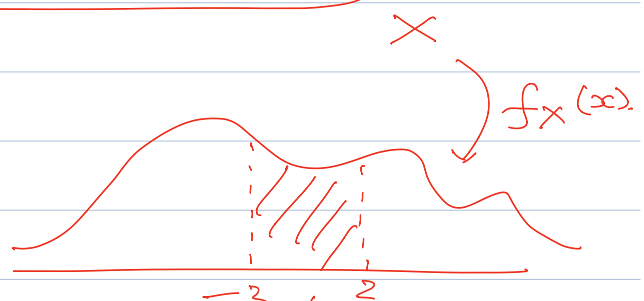


$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(\alpha,\beta)\,d\alpha\,d\beta.$$

Operational meaning of joint PDF





$P(-3 \le X \le 2)$

$$P((X,Y) \in W)$$

$$= \iint_{(x,y)\in W} f_{X,Y}(x,y)\,dx\,dy$$

if $W = \{-\infty \le X \le \infty, -\infty \le Y \le \infty\} \longrightarrow 1$

Sample space.

Suppose we are given $f_{X,Y}(x,y)$.

Find: $P(-10 \le X \le 1)$

Joint PDF $\longrightarrow$ marginal PDF of X?

PDF of X. $\longrightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$$= \int_{-10}^{1} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy.$$