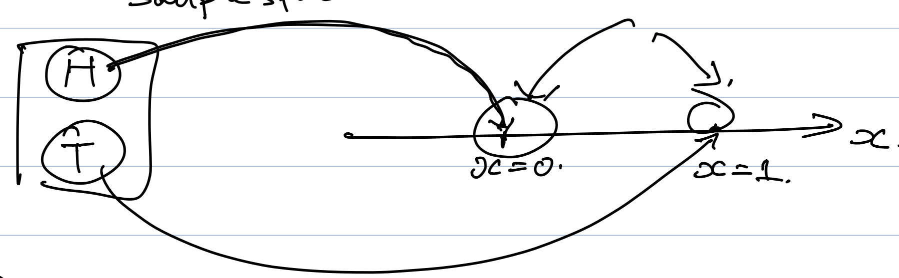
1) Random Variable (R.V.)
2) Probability Mass Function (PMF)
3) Eg's of Discrete valued R.V.'s.

RV: mapping from outcomes of an exp ⟶ real line.

Exp ⟶ $\underbrace{S}_{\text{sample space}} = \{ H, T \}$

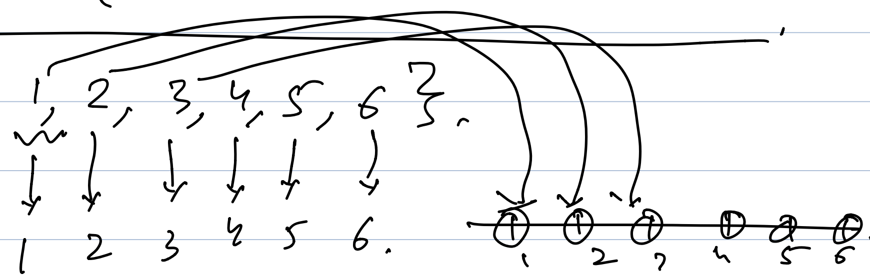
S



Notation

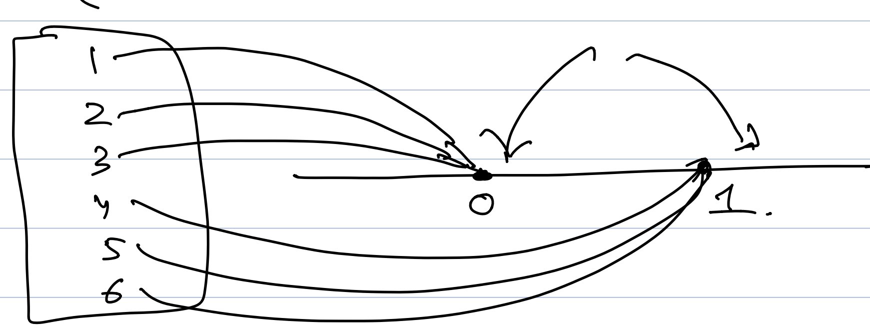
$$X(H) = 0$$
$$X(T) = 1.$$

Eg2 $S = \{ 1, 2, 3, 4, 5, 6 \}$.



Many-to-(one) mapping is allowed.
(onto)

Eg 3 S ⇒



$$\underline{X}(s) = \begin{cases} 0 & \text{if } s \in \{1,2,3\} \\ \underline{1} & \text{if } s \in \{4,5,6\}. \end{cases}$$

<u>Probability Mass Function</u>

$X \longrightarrow$ takes finite / countably finite values.
from a set $\mathcal{X}$ $\hookrightarrow$ alphabet of the r.v.

$$P(X = x) \triangleq P_X(x) \longleftarrow \text{Probability Mass function}$$

will be defined for all $x \in \mathcal{X}$.

<u>Eg 1</u>: $S = \{H, T\}$. $H \to 0$, $T \to 1$. $\mathcal{X} = \{0, 1\}$

$P(H) = 0.3$ (p)
$P(T) = 0.7.$ (1-p). PMF of X

$$P(X=0) \stackrel{?}{=} P(H) = 0.3 \quad (p)$$
$$P(X=1) = P(T) = 0.7. \quad (1-p).$$

<u>Eg 2</u>: $S = \{1, 2, 3, 4, 5, 6\}$. equally likely

$$\boxed{\underline{X} = \begin{cases} 0.7 & \text{if } s = 1, 2 \\ 93 & \text{if } s = 3, 4, 5 \\ 108 & \text{if } s = 6. \end{cases}}$$

Q: $\mathcal{X} = \{0.7, 93, 108\}$

Q: PMF of X ?

$P(X=0.7) \overset{?}{=} 1/6 + 1/6 = 2/6$

$P(X=93) \overset{?}{=} 3 \times 1/6.$

$P(X=108) = 1/6$

Basic Properties of PMF $P_X(x)$.

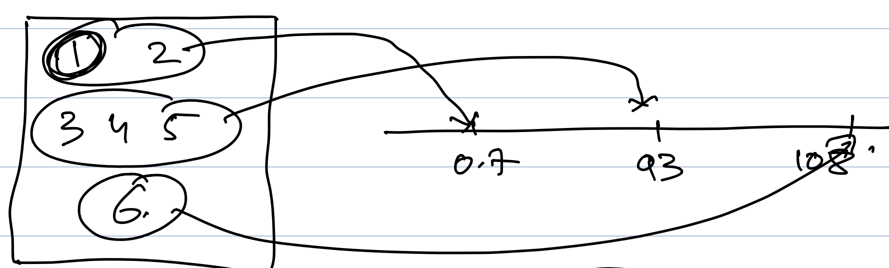
① $0 \le P_X(x) \le 1$ for all $x \in \mathcal{X}$.

$0 \le P_X(x) = P(\underbrace{\{\ldots\}}_{\subseteq S})$

$\le \underbrace{P(S)}_{=1}$.

② $\sum_{x \in \mathcal{X}} P_X(x) = 1.$??

$\boxed{\mathcal{X} \to \text{finite, countably finite}}$



$1 = P(S) = P(\{1,2\} \cup \{3,4,5\} \cup \{6\})$

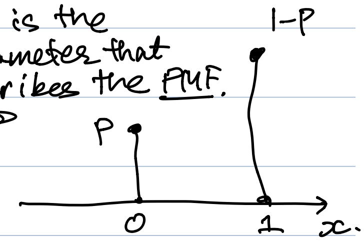
$= P(\{1,2\}) + P(\{3,4,5\}) + P(\{6\})$

$= P_X(0.7) + P_X(93) + P_X(108).$

$\mathcal{X}$

Some common R.V.'s

Bernoulli R.V. $\longleftrightarrow$ $\boxed{\mathrm{Ber}(p)}$ → p is the parameter that describes the PMF.



$\mathcal{X} = \{0, 1\}.$

PMF: $$P_X(x) = \begin{cases} p & \text{if } x = 0 \\ 1-p & \text{if } x = 1. \end{cases}$$

Q: Valid PMF ??

Binomial R.V. $\longleftrightarrow$ $\boxed{\mathrm{Bin}(n, p)}$

$\mathcal{X} = \{0, 1, 2, 3, \ldots, n\}.$

PMF: $$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad \text{for } k = 0, 1, 2, \ldots, n.$$

Q: is this a valid PMF?

1) $0 \leq P_X(k) \leq 1 \quad \forall k.$ ✓

2) $\sum_{k=0}^{n} P_X(k) = 1.$

$$1. \overset{?}{=} \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = \left(p + (1-p)\right)^n = 1$$

↑ Binom. thm.

Geometric Random Variable $\longleftrightarrow$ $\mathrm{Geom}(p)$

$\mathcal{X} = \{1, 2, 3, \ldots\ldots\}$

PMF → $$P_X(k) = (1-p)^{k-1} p \quad \text{for } k = 1, 2, 3, \ldots\ldots$$

imagine success prob $= p$

$P(1^{st} \text{ success @ } k^{th} \text{ trial}) = (1-p)^{k-1} \cdot p$

Q: is this a valid PMF?

$$1 \stackrel{?}{=} \sum_{k=1}^{\infty} (1-p)^{k-1}\, p = p\left[\,1 + \overbrace{(1-p)}^{r} + \overbrace{(1-p)^2}^{r^2} + \cdots\cdots \overbrace{}^{r^3\cdots}\right]$$

$$= p\left[\frac{1}{1-(1-p)}\right] \qquad (p<1) \qquad r=(1-p)$$

$$= p \times \frac{1}{p} = 1.$$

$$S_n = 1 + r + r^2 + r^3 + \cdots + r^{n-1}$$

$(r<1)$

$$(-)\quad r S_n = r + r^2 + r^3 + \cdots + r^{n-1} + r^n$$

$$S_n(1-r) = 1 - r^n$$

$$\Rightarrow \boxed{S_n = \frac{1-r^n}{1-r}} \qquad (0 < r < 1)$$

$$1 + r + r^2 + \cdots\cdots \stackrel{?}{=} \lim_{n\to\infty} S_n$$

$$= \frac{1 - \lim_{n\to\infty} r^n}{(1-r)} \qquad \frac{1}{1-r}$$

$$\boxed{\lim_{n\to\infty} r^n = (0.5)^n} = 0.$$

$\boxed{\text{Poisson Random Variable}} \longleftrightarrow \text{Poisson}(\lambda)$

$\uparrow$ Parameter of Poisson

$$\mathcal{X} = \{0, 1, 2, \ldots, \infty\}.$$

PMF : $P_X(k) = \dfrac{e^{-\lambda} \lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$

Q: is this a valid PMF?

$$0 \le P_X(k) \le 1$$

$$1 \stackrel{?}{=} \sum_{k=0}^{\infty} \left(e^{-\lambda} \frac{\lambda^k}{k!} \right) = e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right).$$

$$1 = e^{-\lambda} \left(\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \right)$$

$$\boxed{e^{\lambda} = \sum_{k=0}^{\infty} \lambda^k / k!}$$?? is this true?

Proof:

Taylor Series Exp. $f(x)$ ∞-diff.

$$f(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \cdots$$

Taylor for e^x at $\underline{a=0}$

$$\left.\begin{aligned} (e^x)' &= e^x \\ (e^x)'' &= e^x \end{aligned}\right\} \rightarrow \underline{\underline{\frac{d^n(e^x)}{dx^n} = (e^x)}}$$

$$e^x = e^0 + \frac{e^0 \times (x-0)}{1!} + \frac{e^0 \cdot (x-0)^2}{2!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\boxed{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}} \longrightarrow \text{Put } \underline{\underline{x=\lambda}}$$

$$\left\{\begin{aligned} n &= 6 \\ r &= 3 \end{aligned}\right\} \longrightarrow$$

$$\left(\cancel{x} \atop \cancel{x}\right) \stackrel{?}{=}$$

r numbers $\qquad n = 50.$

$r = 4$ $\qquad \underbrace{\{7, 13, 18, 24\}}.$

A_1

$\boxed{24}\ \boxed{18}\ \boxed{13}\ \boxed{7}$

$\boxed{7}\ \boxed{24}\ \boxed{18}\ \boxed{13}$

$\vdots$

$P\left(\begin{array}{l} r \text{ numbers are} \\ \text{drawn in an increasing order} \end{array}\right) = ? \longrightarrow$