$\boxed{\text{Today}}$

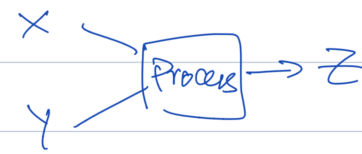
1) Functions of two random Variables

2) Jointly Gaussian R.V.'s.

Suppose (X, Y) is a pair of random variables with a joint PDF $f_{X,Y}(x,y)$; joint CDF $F_{X,Y}(x,y)$.

Let $Z = g(X,Y)$ be a function of X and Y.

How to compute the CDF and PDF of Z ?

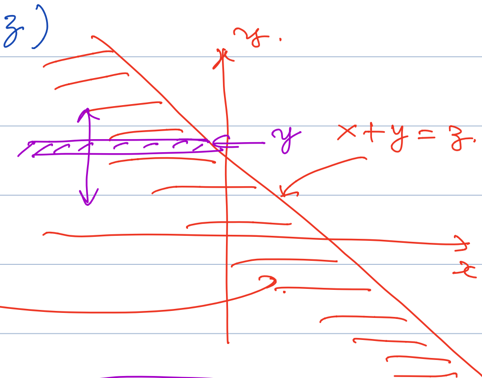


$$F_Z(z) = P(Z \le z)$$
$$= P(g(X,Y) \le z)$$
$$= \iint_{(x,y):\, g(x,y) \le z} f_{X,Y}(x,y)\, dx\, dy.$$

$$f_Z(z) = \frac{d F_Z(z)}{dz}.$$

Ex 1: $Z = X + Y$

$$F_Z(z) = P(X + Y \le z)$$
$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{(z-y)} f_{X,Y}(x,y)\, dx\, dy$$



$$F_Z(z) = \int_{y=-\infty}^{\infty} \left[\int_{x=-\infty}^{(z-y)} f_{X,Y}(x,y)\, dx \right] dy.$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{y=-\infty}^{\infty} \left[\frac{d}{dz} \int_{x=-\infty}^{z-y} f_{X,Y}(x,y)\, dx \right] dy.$$

can apply Leibniz rule...

$$\frac{d}{dz}\left[\int_{a(z)}^{b(z)} f(x,z)\,dx\right]$$

(Leibniz rule for differentiation under integral)

$$= \left(\frac{d\,b(z)}{dz}\right) \times f(b(z), z) - \left(\frac{d\,a(z)}{dz}\right) \times f(a(z), z) + \int_{a(z)}^{b(z)} \left(\frac{\partial f(x,z)}{dz}\right) dx$$

$$\frac{d}{dz}\int_{x=-\infty}^{(z-y)} f_{X,Y}(x,y)\,dx$$

$$= 1 \times f_{X,Y}(z-y, y) - (0) \times f(\dots) + \int_{x=-\infty}^{z-y} \overbrace{\frac{d(f_{X,Y}(x,y))}{dz}}^{=0} dx$$

$$= f_{X,Y}(z-y, y)$$

Plugging it back:

$$f_Z(z) = \int_{y=-\infty}^{\infty} f_{X,Y}(z-y, y)\,dy.$$

$(Z = X+Y)$

if X & Y were independent

$$f_{X,Y}(z-y, y) = f_X(z-y) \times f_Y(y).$$

then,

$$f_Z(z) = \int_{y=-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy.$$

$\Downarrow$

convolution of the marginal densities of X, Y.

$$= \int_{\tau=-\infty}^{\infty} f_X(z-\tau) f_Y(\tau)\, d\tau.$$

Eg 2 $\quad Z = \max(X, Y)$

Compute PDF, CDF of Z.

$$F_Z(z) = P(Z \le z) = P(\max(X,Y) \le z)$$

$$= P(X \le z, Y \le z)$$

$$= F_{X,Y}(z, z)$$

$$\boxed{\begin{aligned} &F_{X,Y}(x,y) \\ &= P(X \le x, Y \le y) \end{aligned}}$$

(if X, Y were independent)
then $\quad = F_X(z) F_Y(z)$

$$f_Z(z) = \frac{d F_Z(z)}{dz} = \frac{d}{dz}\left[F_X(z)\, F_Y(z) \right]$$

$$= f_X(z) \times F_Y(z) + F_X(z) \times f_Y(z).$$

③ Work out $Z = \min(X, Y)$.

Jointly Gaussian / Jointly Normal R.V.'s

(X, Y) are jointly Gaussian if their joint PDF is: $(\mu_1, \mu_2, \sigma_1, \sigma_2, r, \ |r| < 1)$

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{ \frac{-1}{2(1-r^2)} \times \left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - 2r\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} \right) \right\}$$

defined for all $-\infty < x, y < \infty$.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \Rightarrow X \sim N(\mu_1, \sigma_1^2)$$

$$f_Y(y) = \text{- - - - -} \Rightarrow Y \sim N(\mu_2, \sigma_2^2)$$

$|r| < 1 \Rightarrow r \rightarrow$ correlation coefficient.

if $r = 0$ then $f_{X,Y}(x, y) = f_X(x) f_Y(y)$.

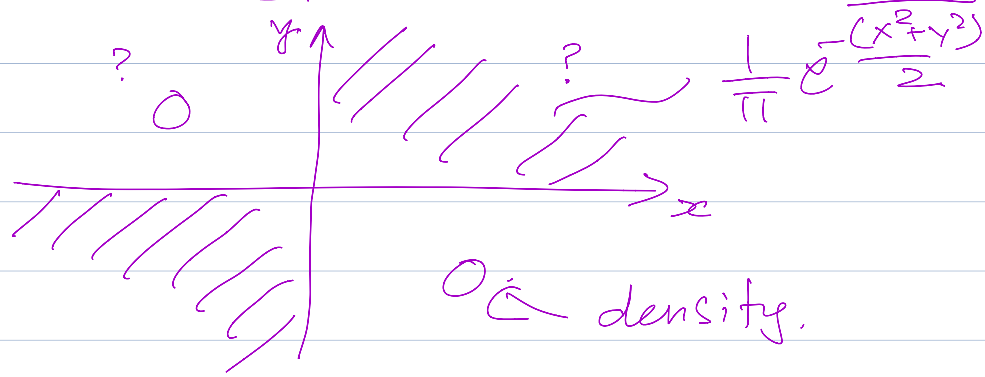
① X & Y are jointly Gauss. if any linear combination $aX + bY$ is also Gaussian.

② If 2 r.v.'s are J.G. then each one is also Gaussian.
The reverse statement is NOT true in general.

$$\left\{ \begin{array}{l} X \sim \mathcal{N}(\ldots) \\ Y \sim \mathcal{N}(\ldots) \\ \text{but } (X,Y) \text{ may NOT necessarily be jointly Gaussian.} \end{array} \right.$$

$$\boxed{f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} e^{-\frac{(x^2+y^2)}{2}} & xy \geq 0 \\ 0 & xy < 0. \end{cases}}$$

Claim: (X,Y) are NOT J.G.



$f_X(x) \longrightarrow$ will turn out to be a Gaussian PDF.

$f_Y(y) \longrightarrow$ " " "

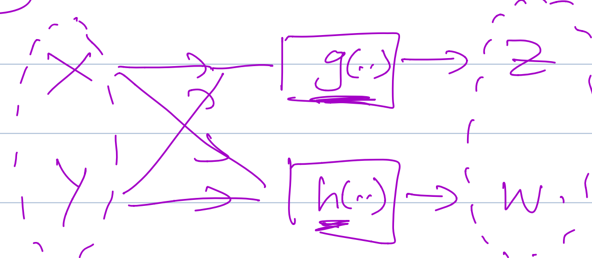
Two functions of Two R.V.'s (X, Y)

$$\left.\begin{aligned} Z &= g(X, Y) \\ W &= h(X, Y) \end{aligned}\right\}$$

($g(\cdot,\cdot)$ & $h(\cdot,\cdot)$ are continuous & differentiable)

Goal: Compute the joint PDF of (Z, W)

Freeze (z, w).



$Z = X + Y$
$W = 3X + 4Y$
$f_{Z,W}(z, w)$

Step 1:

$$\boxed{\begin{aligned} g(x, y) &= z \\ h(x, y) &= w \end{aligned}}$$

simultan sol^n's be.

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Step 2:

$$J(x_i, y_i) = \begin{vmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} \end{vmatrix}_{\substack{x = x_i \\ y = y_i}} \longleftarrow \quad |\cdot| \Rightarrow \text{determinant of the } 2\times 2 \text{ matrix}$$

for $(x_1, y_1), \ldots, (x_n, y_n)$.

Step 3:

$$f_{Z,W}(z,w) = \sum_{\substack{i \\ \text{(all "roots"} \\ \text{from Step 1)}}} \frac{1}{|J(x_i,y_i)|} \cdot f_{X,Y}(x_i,y_i)$$

Eg 1:

$$Z = \underbrace{3X + 9Y} = g(X,Y)$$

$$W = 8X + Y = h(X,Y).$$

Compute $f_{Z,W}(z,w)$.

$$\left.\begin{aligned} z &= 3x + 9y \\ 9(w &= 8x + y) \end{aligned}\right\} \longrightarrow \text{Solve for } x, y \text{ in terms of } z.$$

$$9w - z = 72x - 3x$$

$$x_1 = \left(\frac{9w - z}{69}\right)$$

$$\boxed{\begin{aligned} x_1 &= \frac{(9w - z)}{69} \\ y_1 &= \ldots\ldots \end{aligned}}$$

$$J(x,y) = \begin{vmatrix} 3 & 9 \\ 8 & 1 \end{vmatrix}$$

$$|J| = |3 - 72| = 69.$$

$$f_{Z,W}(z,w) = \frac{1}{69} \times f_{X,Y}\left(\frac{9w-z}{12}, \ldots x \ldots\right)$$

Ex 2 (X, Y) zero mean, independent Gaussian r.v.'s w/ same variance σ^2.

$$r = \sqrt{X^2+Y^2}$$

$$\theta = \tan^{-1}(Y/X), \quad |\theta| < \pi.$$

Compute: Joint PDF of (r, θ).

Step 1: Find joint roots...

$$\begin{cases} r = \sqrt{x^2+y^2} = g(x,y) \\ \theta = \tan^{-1}(y/x) = h(x,y). \end{cases}$$

$$(x_1, y_1) = (r\cos\theta, r\sin\theta)$$

Step 2: Jacobian $= \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix}$

$$= \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}.$$

Step 3 :

$$f_{r,\theta}(r,\theta) = \frac{1}{|J|} \times f_{X,Y}(r\cos\theta, r\sin\theta)$$

$$= r \times f_{X,Y}(r\cos\theta, r\sin\theta)$$

since (X, Y) were indep.

$$= r \times \underbrace{f_X(r\cos\theta)} \times \underbrace{f_Y(r\sin\theta)}$$

$$= \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}$$