Solutions to HW 4
ECE 503: Fall 2016

1. [20 points] The moment generating function (MGF) of a random variable X is defined as $\phi_X(s) = E[e^{sX}]$.

   (a) Prove that the nth moment of X, $E[X^n]$ can be obtained from the MGF as following:

$$E[X^n] = \frac{d^n \phi_X(s)}{ds^n}\Big|_{s=0}$$

Solution:

$$\frac{d^n \phi_X(s)}{ds^n} = \frac{d^n \mathbb{E}\{e^{sX}\}}{ds^n} = \frac{d}{ds^n}[\int e^{sx} f_X(x)dx] = \int \frac{d}{ds^n}[e^{sx} f_X(x)]dx = \int x^n e^{sx} f_X(x)dx.$$

Then, at $s = 0$, we have

$$\frac{d^n \phi_X}{ds^n}|_{s=0} = \int x^n f_X(x)dx = \mathbb{E}X^n.$$

(b) Find the MGF for the following random variables:

- $\mathcal{N}(\mu, \sigma^2)$
- Poisson(λ)
- Uniform(a, b) (i.e., a uniform random variable in the interval $[a, b]$)

Solutions:

The mgf of a Normal RV $X \sim N(\mu, \sigma^2)$ is $m(t) = \exp(\mu t + t^2\sigma^2/2)$

Proof.

$$m(t) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{tr} e^{-(r-\mu)^2/(2\sigma^2)} dr = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{t(u+\mu)} e^{-u^2/(2\sigma^2)} du = \frac{1}{\sqrt{2\pi\sigma^2}} e^{t\mu} \int_{\mathbb{R}} e^{tu} e^{-u^2/(2\sigma^2)} du$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{t\mu} \int_{\mathbb{R}} e^{-(u^2 - tu2\sigma^2)/(2\sigma^2)} du$$

where we used the change of variables $u = r - \mu$. To solve the above integral we complete the square by multiplying and dividing by $e^{t^2\sigma^2/2}$

$$m(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{t\mu} e^{t^2\sigma^2/2} \int_{\mathbb{R}} e^{-(u^2 - tu2\sigma^2 + t^2\sigma^4)/(2\sigma^2)} du = \frac{1}{\sqrt{2\pi\sigma^2}} e^{t\mu} e^{t^2\sigma^2/2} \int_{\mathbb{R}} e^{-(u-t\sigma^2)^2)/(2\sigma^2)} du$$
$$= e^{t\mu + t^2\sigma^2/2}.$$

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$X \sim Pois(\lambda)$

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} = e^{\lambda(e^t-1)}$$

$X \sim U(a,b)$

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a} dx = \frac{e^{bt}-e^{at}}{(b-a)t}$$

(c) Let $X_1, X_2, \ldots, X_n$ be independent random variables. We define $W = X_1 + X_2 + \ldots + X_n$ as a new random variable. Show that the MGF of W is the product of the MGF(s) of the individual random variables $X_1, \ldots, X_n$.

Proof. The mgf $m(t)$ of the sum $\sum_{i=1}^{n} X_i$ is by definition

$$m(t) = \mathsf{E}\left(\exp\left(t\sum_i X_i\right)\right) = \mathsf{E}\left(\prod_i \exp(tX_i)\right) = \prod_i \mathsf{E}(\exp(tX_i)) = \prod_i m_i(t).$$

□

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2. [20 points] Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} c, & x+y \leq 1, x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of constant c.

(b) What is $P(X < Y)$?

(c) What is $P(X + Y \leq 1/2)$?

Solution:

a)

$$\int\int f(x,y)\ dx\ dy = 1$$
$$\int_{x=0}^{1}\int_{y=0}^{1-x} c\ dx\ dy = 1$$
$$\int_{x=0}^{1} c(1-x)\ dx = 1$$
$$c = 2$$

b)

$$p(X < Y) = \int_{x=0}^{\frac{1}{2}}\int_{y=x}^{1-x} 2\ dy\ dx$$
$$= \int_{x=0}^{\frac{1}{2}} 2(1-2x)\ dx = 2x - 2x^2 \mid_0^{\frac{1}{2}} = \frac{1}{2}$$

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c)

$$
\begin{aligned}
p(X+Y \le \frac{1}{2}) &= \int_{x=0}^{\frac{1}{2}} \int_{y=0}^{\frac{1}{2}-x} 2 \; dy \; dx \\
&= \int_{x=0}^{\frac{1}{2}} (1-2x) \; dx = \frac{1}{4}
\end{aligned}
$$

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1. [4 points] Let X and Y be independent exponential random variables with the same mean $\mu_X = \mu_Y = 1$. Find the PDF of the following random variables:

   (a) $X + Y$

   (b) XY

   (c) X/Y

   (d) $\min(X, Y)/\max(X, Y)$

(a) [we did this in class]

$$Z = X + Y$$

Note that both X and Y positive random variables hence (use Eq. (6-45))

$$f_Z(z) = \int_0^z f_{XY}(z-y, y)dy = \int_0^z e^{-(z-y+y)}dy$$
$$= z\,e^{-z}U(z).$$

(b)

$$Z = XY.$$

$$F_Z(z) = P\{Z \le z\} = P\{XY \le z\}$$
$$= \int_0^\infty \int_0^{z/y} f_{XY}(x, y)dx\,dy$$

or (see Eq. (6-148)) (apply Leibniz rule)

$$f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}(\frac{z}{y}, y)dy = \int_0^\infty \frac{1}{y} e^{-((z/y)+y)}dy$$

Solutions to HW 4
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1. [4 points] Let X and Y be independent exponential random variables with the same mean $\mu_X = \mu_Y = 1$. Find the PDF of the following random variables:

   (a) $X + Y$

   (b) XY

   (c) X/Y

   (d) $\min(X, Y)/\max(X, Y)$

(c)

$$Z = X/Y$$

$$\begin{aligned} F_Z(z) &= P\{Z \leq z\} = P\{\tfrac{X}{Y} \leq z\} \\ &= \int_0^\infty \int_0^{yz} f_{XY}(x, y)dx\,dy \end{aligned}$$

(apply Leibniz rule)

$$\begin{aligned} f_Z(z) &= \int_0^\infty y f_{XY}(yz, y)dy = \int_0^\infty ye^{y(z+1)}dy = \int_0^\infty ye^{(1+z)y} \\ &= \left[y\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty + \left(\frac{1}{1+z}\right)\int_0^\infty e^{(1+z)y}dy \\ &= \left(\frac{1}{1+z}\right)\left[\frac{e^{-(1+z)y}}{-(1+z)}\right]_0^\infty = \frac{1}{(1+z)^2}U(z) \end{aligned}$$

Solutions to HW 4
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1. [4 points] Let X and Y be independent exponential random variables with the same mean $\mu_X = \mu_Y = 1$. Find the PDF of the following random variables:

   (a) $X + Y$

   (b) XY

   (c) X/Y

   (d) $\min(X,Y)/\max(X,Y)$

(d) $Z = \dfrac{\min(X,Y)}{\max(X,Y)}, \quad 0 < z < 1$

$$\begin{aligned}
F_Z(z) &= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \cap ((X \le Y) \cup (X > Y))\right\} \\
&= P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \cap (X \le Y)\right\} + P\left\{\left(\frac{\min(X,Y)}{\max(X,Y)} \le z\right) \cap (X > Y)\right\} \\
&= P\{\tfrac{X}{Y} \le z, X \le Y\} + P\{\tfrac{Y}{X} \le z, X > Y\} \\
&= P\{X \le Yz, X \le Y\} + P\{Y \le Xz, X > Y\} \\
&= \int_0^\infty \int_0^{yz} f_{XY}(x,y)\,dx\,dy + \int_0^\infty \int_0^{xz} f_{XY}(x,y)\,dy\,dx
\end{aligned}$$

(apply Leibniz rule)

$$\begin{aligned}
f_Z(z) &= \int_0^\infty y\, f_{XY}(yz,y)dy + \int_0^\infty x f_{XY}(x,xz)dx \\
&= \int_0^\infty y\, f_{XY}(yz,y)dy + \int_0^\infty y\, f_{XY}(y,yz)dy \\
&= \int_0^\infty y\left(e^{-(yz+y)} + e^{-(y+yz)}\right)dy \\
&= 2\int_0^\infty ye^{-y(1+z)}dz = \begin{cases} \dfrac{2}{(1+z)^2}, & 0 \le z \le 1 \\ 0, & \text{otherwise} \end{cases}
\end{aligned}$$

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2. [2 point] X and Y are independent Rayleigh random variables with a common parameter σ^2. Find the density of X/Y.

$$f_{XY}(x,y) = \frac{xy}{\sigma^4}\, e^{-(x^2+y^2)/2\sigma^2}, \quad x, y \geq 0$$

(since X and Y are independent)

$$Z = \frac{X}{Y}$$

$$F_Z(z) = P(Z \leq z) = P(X/Y \leq z) = \int_0^\infty \int_0^{zy} f_{XY}(x,y)\,dx\,dy.$$

This gives the density function of z to be

$$\begin{aligned} f_Z(z) &= \int_0^\infty y\, f_{XY}(zy, y)\,dy = \int_0^\infty \frac{zy^3}{\sigma^4}\, e^{-(z^2y^2+y^2)/2\sigma^2}\,dy \\ &= \frac{z}{\sigma^4}\int_0^\infty y^3\, e^{-y^2(z^2+1)/2\sigma^2}\,dy \quad \text{Let, } t = y^2(z^2+1)/2\sigma^2 \\ &= \frac{2z}{(z^2+1)^2}\int_0^\infty t\, e^{-t}\,dt = \frac{2z}{(z^2+1)^2}, \quad 0 \leq z \leq \infty. \end{aligned}$$

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3. [3 points] Let X and Y be independent and identically distributed normal random variables with zero mean and variance σ^2. Define

$$U = \frac{X^2 - Y^2}{\sqrt{X^2+Y^2}}, \quad V = \frac{2XY}{\sqrt{X^2+Y^2}}$$

(a) Find the joint PDF $f_{U,V}(u,v)$ of the random variables (U, V)

(b) Show that U and V are independent normal random variables

(c) Show that $\frac{(X-Y)^2-2Y^2}{\sqrt{X^2+Y^2}}$ is also a normal random variable

---

Let

$$R = \sqrt{X^2+Y^2}, \qquad \theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

(recall that we found the joint density of (R, \theta) in class)

$$U = \frac{X^2 - Y^2}{\sqrt{X^2+Y^2}} = R\cos 2\theta$$

$$V = \frac{2XY}{\sqrt{X^2+Y^2}} = R\sin 2\theta$$

$\Rightarrow$

$$r = \sqrt{u^2+v^2}, \quad \theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{v}{u}\right), \quad 2\theta_2 = \pi + 2\theta_1.$$

There are two sets of solutions (r, θ_1) and (r, θ_2).

This gives

$$J = \begin{vmatrix} \cos 2\theta & -2r\sin 2\theta \\ \sin 2\theta & 2r\cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2+v^2}$$

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3. [3 points] Let X and Y be independent and identically distributed normal random variables with zero mean and variance σ^2. Define

$$U = \frac{X^2 - Y^2}{\sqrt{X^2+Y^2}}, \quad V = \frac{2XY}{\sqrt{X^2+Y^2}}$$

   (a) Find the joint PDF $f_{U,V}(u,v)$ of the random variables (U,V)
   (b) Show that U and V are independent normal random variables
   (c) Show that $\frac{(X-Y)^2-2Y^2}{\sqrt{X^2+Y^2}}$ is also a normal random variable

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There are two sets of solutions (r,θ_1) and (r,θ_2). $\quad r = \sqrt{u^2+v^2}, \quad \theta_1 = \frac{1}{2}\tan^{-1}\left(\frac{v}{u}\right), \quad 2\theta_2 = \pi + 2\theta_1.$

$$\begin{aligned}
f_{UV}(u,v) &= \frac{1}{|J|}\left\{f_{r,\theta}(r,\theta_1) + f_{r,\theta}(r,\theta_2)\right\} = \frac{2}{|J|}\, f_{r,\theta}(r,\theta_1) \\
&= \frac{2}{2\sqrt{u^2+v^2}}\, \frac{\sqrt{u^2+v^2}}{2\pi\sigma^2}\, e^{-(u^2+v^2)/2\sigma^2} \\
&= \frac{1}{2\pi\sigma^2}\, e^{-(u^2+v^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-u^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-v^2/2\sigma^2} \\
&= f_U(u) f_V(v)
\end{aligned}$$

Thus U and V are independent normal random variables. Hence it follows that $U = \frac{X^2-Y^2}{\sqrt{X^2+Y^2}}$ and $V/2 = \frac{XY}{\sqrt{X^2+Y^2}}$ are independent random variables.

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3. [3 points] Let X and Y be independent and identically distributed normal random variables with zero mean and variance σ^2. Define

$$U = \frac{X^2 - Y^2}{\sqrt{X^2+Y^2}}, \quad V = \frac{2XY}{\sqrt{X^2+Y^2}}$$

(a) Find the joint PDF $f_{U,V}(u,v)$ of the random variables (U,V)

(b) Show that U and V are independent normal random variables

(c) Show that $\frac{(X-Y)^2-2Y^2}{\sqrt{X^2+Y^2}}$ is also a normal random variable

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(c)

$$\begin{aligned} Z &= \frac{(X-Y)^2 - 2Y^2}{\sqrt{X^2+Y^2}} = \frac{(X^2-Y^2) - 2XY}{\sqrt{X^2+Y^2}} \\ &= \frac{X^2-Y^2}{\sqrt{X^2+Y^2}} - \frac{2XY}{\sqrt{X^2+Y^2}} \\ &= U - V \sim N(0, 2\sigma^2). \end{aligned}$$

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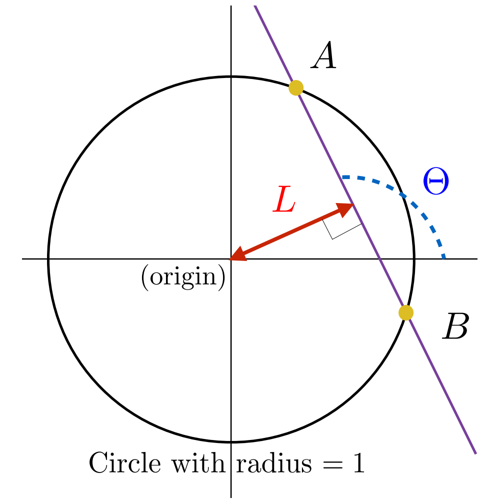
4. [3 points] Two points A and B are picked independently at random on the circumference of a circle C. The circle is of unit radius and is centered at $(0,0)$. Let L denote the length of the perpendicular from the origin to the line AB (i.e., the line joining A and B). Let Θ denote the angle that the line AB makes with the horizontal axis.



Show that the joint density of (L, Θ) is given as:

$$f_{L,\Theta}(\ell,\theta) = \frac{1}{\pi^2\sqrt{1-\ell^2}}, \quad 0 \le \ell \le 1, \quad 0 \le \theta \le 2\pi$$

Points A, B can be described by their angular coordinates, say Θ_1 and Θ_2

Θ_1 and Θ_2 are independent and each is $\sim \text{Uniform}(0, 2\pi)$ Joint Density of (Θ_1, Θ_2) is then $f_{\Theta_1,\Theta_2}(\theta_1,\theta_2) = \frac{1}{4\pi^2}$

Check that you can write: $\Theta = (\pi + \Theta_1 + \Theta_2)/2$ and $L = \cos\left(\frac{\Theta_1-\Theta_2}{2}\right)$

In other words, we have a change of variables from $(\Theta_1, \Theta_2) \to (L, \Theta)$

Jacobian: $$\begin{vmatrix} 1/2 & -1/2\sin((\theta_1-\theta_2)/2) \\ 1/2 & +1/2\sin((\theta_1-\theta_2)/2) \end{vmatrix} = 1/2\sin((\theta_1-\theta_2)/2) = 1/2\sqrt{1-\cos^2((\theta_1-\theta_2)/2)} = \frac{\sqrt{1-\ell^2}}{2}$$

Given a (ℓ, θ), there are two pairs of solutions for (θ_1, θ_2)

Solution 1 $\to (\theta_1,\theta_2)^{(1)} = (\theta - \pi/2 - \cos^{-1}\ell, \quad \theta - \pi/2 + \cos^{-1}\ell)$

Solution 2 $\to (\theta_1,\theta_2)^{(2)} = (\theta - \pi/2 + \cos^{-1}\ell, \quad \theta - \pi/2 - \cos^{-1}\ell)$

$\Rightarrow$ Joint Density of (L, Θ)

$= f_{(L,\Theta)}(\ell,\theta) = \frac{2}{\sqrt{1-\ell^2}} \times 2 \times \frac{1}{4\pi^2} = \frac{1}{\pi^2\sqrt{1-\ell^2}}$

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5. [4 points] Let X and Y have the joint density $f_{X,Y}(x,y) = cx(y-x)e^{-y}, \quad 0 \leq x \leq y < \infty$.

(a) Find c

(b) Find the conditional PDF of X given Y

(c) Find the conditional PDF of Y given X

(d) Show that $E[X|Y] = \frac{Y}{2}$ and $E[Y|X] = X + 2$

By integration, for $x, y > 0$,

$$f_Y(y) = \int_0^y f(x,y)\,dx = \tfrac{1}{6}cy^3e^{-y}, \quad f_X(x) = \int_x^\infty f(x,y)\,dy = cxe^{-x}, \quad \text{whence } c = 1.$$

It is simple to check the values of $f_{X|Y}(x \mid y) = f(x,y)/f_Y(y)$ and $f_{Y|X}(y \mid x)$,

$$f_{X|Y}(x \mid y) = 6x(y-x)y^{-3}, \qquad 0 \leq x \leq y,$$
$$f_{Y|X}(y \mid x) = (y-x)e^{x-y}, \qquad 0 \leq x \leq y < \infty.$$

and then deduce by integration that

$$E(X|Y=y) = y/2 \text{ and } E(Y|X=x) = x+2$$

Solutions to HW 4
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6. [4 points] Random variables X and Y have the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1, \quad 0 \le y \le x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{Y \le 1/4\}$ denote an event.

(a) What is the conditional PDF $f_{X,Y|A}(x,y)$?

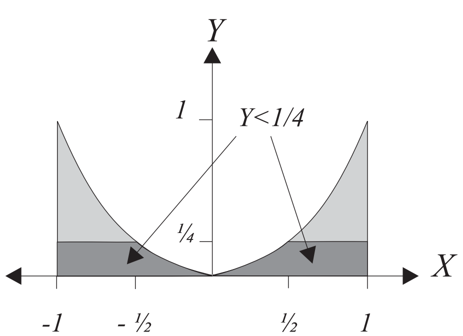
(b) What is $f_{Y|A}(y)$?

(c) What is $E[Y|A]$?

(d) What is $f_{X|A}(x)$?

(e) What is $E(X|A)$?

(a) The event $A = \{Y \le 1/4\}$ has probability



$$P[A] = 2\int_0^{1/2}\int_0^{x^2} \frac{5x^2}{2}\,dy\,dx + 2\int_{1/2}^{1}\int_0^{1/4} \frac{5x^2}{2}\,dy\,dx \tag{1}$$

$$= \int_0^{1/2} 5x^4\,dx + \int_{1/2}^{1} \frac{5x^2}{4}\,dx \tag{2}$$

$$= x^5\Big|_0^{1/2} + 5x^3/12\Big|_{1/2}^{1} \tag{3}$$

$$= 19/48 \tag{4}$$

This implies

$$f_{X,Y|A}(x,y) = \begin{cases} f_{X,Y}(x,y)/P[A] & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$= \begin{cases} 120x^2/19 & -1 \le x \le 1, 0 \le y \le x^2, y \le 1/4 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

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6. [4 points] Random variables X and Y have the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1, \quad 0 \le y \le x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{Y \le 1/4\}$ denote an event.

(a) What is the conditional PDF $f_{X,Y|A}(x,y)$?

(b) What is $f_{Y|A}(y)$?

(c) What is $E[Y|A]$?

(d) What is $f_{X|A}(x)$?

(e) What is $E(X|A)$?

(b)

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\, dx = 2\int_{\sqrt{y}}^{1} \frac{120x^2}{19}\, dx = \begin{cases} \frac{80}{19}(1 - y^{3/2}) & 0 \le y \le 1/4 \\ 0 & \text{otherwise} \end{cases}$$

(c) The conditional expectation of Y given A is

$$E[Y|A] = \int_0^{1/4} y \frac{80}{19}(1 - y^{3/2})\, dy = \frac{80}{19}\left(\frac{y^2}{2} - \frac{2y^{7/2}}{7}\right)\Bigg|_0^{1/4} = \frac{65}{532}$$

Solutions to HW 4
ECE 503: Fall 2015

6. [4 points] Random variables X and Y have the following joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1, \quad 0 \leq y \leq x^2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $A = \{Y \leq 1/4\}$ denote an event.

(a) What is the conditional PDF $f_{X,Y|A}(x,y)$?

(b) What is $f_{Y|A}(y)$?

(c) What is $E[Y|A]$?

(d) What is $f_{X|A}(x)$?

(e) What is $E(X|A)$?

(d) To find $f_{X|A}(x)$, we can write $f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dy$. However, when we substitute $f_{X,Y|A}(x,y)$, the limits will depend on the value of x. When $|x| \leq 1/2$,

$$f_{X|A}(x) = \int_0^{x^2} \frac{120x^2}{19}\,dy = \frac{120x^4}{19} \tag{9}$$

When $-1 \leq x \leq -1/2$ or $1/2 \leq x \leq 1$,

$$f_{X|A}(x) = \int_0^{1/4} \frac{120x^2}{19}\,dy = \frac{30x^2}{19} \tag{10}$$

The complete expression for the conditional PDF of X given A is

$$f_{X|A}(x) = \begin{cases} 30x^2/19 & -1 \leq x \leq -1/2 \\ 120x^4/19 & -1/2 \leq x \leq 1/2 \\ 30x^2/19 & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

(e) The conditional mean of X given A is

$$E[X|A] = \int_{-1}^{-1/2} \frac{30x^3}{19}\,dx + \int_{-1/2}^{1/2} \frac{120x^5}{19}\,dx + \int_{1/2}^{1} \frac{30x^3}{19}\,dx = 0$$