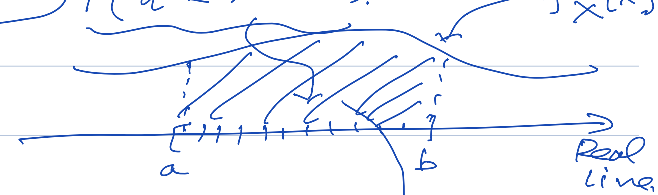
Today:

1) Continuous valued R.V.
2) PDF (Probability Density Function)
3) Examples of Continuous R.V.'s
   - Gaussian R.V.
   - Exponential R.V.
   - Rayleigh R.V.
   - Laplace R.V.
4) Relationship between PDF & CDF.

$\boxed{\text{Mixed Valued R.V.'s}}$

Continuous Valued R.V.'s

$P(a \le X \le b)$



$f_X(x) \Rightarrow$ PDF

density

$$\underline{P(a \le X \le b)} = \int_a^b f_X(x)\,dx$$

area under $f_X(x)$ in $[a, b]$ gives the corresp. probability

$\boxed{\text{What properties should a PDF have?}}$

(1) $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

$$\| $$

$$P(\underbrace{-\infty < X < \infty}_{S}) = 1.$$

(2) $$f_X(x) \ge 0.$$

$\rightarrow$ for $f_X(x)$ to be a valid PDF

$X \rightsquigarrow$ given the PDF $f_X(x)$.

How to compute the CDF?

$$\text{CDF} \Rightarrow F_X(x) = P(X \le x)$$

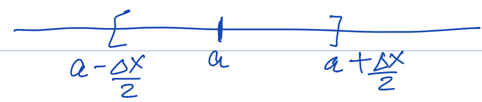
$$= \int_{-\infty}^{x} f_X(t)\,dt \Rightarrow \text{integral of PDF}$$

given the PDF $\longrightarrow$ compute the CDF by integrating $f_X(x)$

give you the CDF $\longrightarrow$ compute the PDF ?

$$\boxed{f_X(x) = \frac{d\,F_X(x)}{dx}}$$

Δx (Δx small).



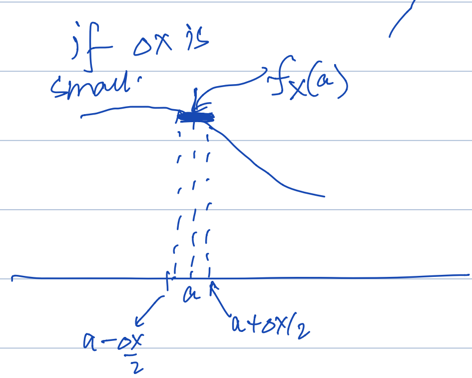
$\Longrightarrow F_X\left(a+\frac{\Delta x}{2}\right) - F_X\left(a-\frac{\Delta x}{2}\right)$

$F_X(x)$

$$= \int_{-\infty}^{(a+\Delta x/2)} f_X(t)\,dt - \int_{-\infty}^{(a-\Delta x/2)} f_X(t)\,dt.$$

$$\left(\int_{-\infty}^{a-\Delta x/2} + \int_{a-\Delta x/2}^{a+\Delta x/2}\right) - \left(\;''\;\right)$$

if Δx is small.



$$= \int_{a-\Delta x/2}^{a+\Delta x/2} f_X(t)\,dt. \approx f_X(a) \int_{a-\Delta x/2}^{a+\Delta x/2} dt$$

$$= f_X(a)\cdot\Delta x.$$

$$F_X(a+\Delta x/2) - F_X(a-\Delta x/2) \approx f_X(a)\,\Delta x.$$

$$f_X(a) = \lim_{\Delta x\to 0} \left(\frac{F_X(a+\Delta x/2) - F_X(a-\Delta x/2)}{\Delta x}\right)$$

$$= \frac{d}{dx} F_X(x).$$

Gaussian Random Variable / Normal Distribution   Discrete/continuous

PMF → CDF
PDF

PDF $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

defined for $-\infty < x < \infty$.

Shorthand: 1) $\mathcal{N}(\mu, \sigma^2)$ $\mathcal{N}(3, 8)$
Normal

$f_X(x) = \frac{1}{\sqrt{2\pi \times 8}} e^{-\frac{(x-3)^2}{2\times 8}}$

2) Standard Normal Distribution.
$\mathcal{N}(0, 1)$.

$\mathcal{N}(0,1)$

$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x^2)}{2}}$

$\left\{ \begin{array}{l} \mu = \text{mean} \\ \sigma = \text{standard deviation} \\ \sigma^2 = \text{Variance.} \end{array} \right\}$

1) $f_X(x) \geq 0 \quad \forall\ x$. ✓

⟹ 2) $\int_{-\infty}^{\infty} f_X(t)\,dt = 1$.

$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$.

Goal: Show $I = 1$

① Change of Variable: $z = \left(\frac{t-\mu}{\sigma}\right)$

$I = \int_{z=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$.

② $I^2 = I \times I = \left(\int \frac{1}{\sqrt{2\pi}} e^{-z_1^2/2} dz_1\right)\left(\int \frac{1}{\sqrt{2\pi}} e^{-z_2^2/2} dz_2\right)$

$\left.\begin{array}{l} z_1 = r\cos\theta \\ z_2 = r\sin\theta. \end{array}\right\}$ Polar coordinates. } $\boxed{I = 1}$

$$F_X(x) \overset{?}{=} \left\{ \underbrace{\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt.}_{} \right\}$$ No closed form expression

$$\Downarrow$$

$X \sim \mathcal{N}(35, 50)$

$F_X(43.4)$

Numerical computn for $\boxed{\mathcal{N}(0,1)}$.

$$\boxed{\Phi(x) = \text{CDF of } \mathcal{N}(0,1).}$$

Goal: given $X \sim \mathcal{N}(\mu, \sigma^2)$, can we compute $F_X(x)$ in terms of $\Phi(x)$?

$$F_X(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt.$$

(Change of variable)

$$\boxed{y = \frac{t-\mu}{\sigma}}$$

$$dy = \frac{dt}{\sigma}$$

$$-\infty < t \le a.$$

$$= \int_{-\infty}^{(a-\mu)/\sigma} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-y^2/2} \cdot \sigma \, dy.$$

$$= \int_{-\infty}^{(a-\mu)/\sigma} \left(\frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy. \right) = \int_{-\infty}^{(a-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

PDF of $\mathcal{N}(0,1)$

= CDF of $\mathcal{N}(0,1)$ evaluated at $(a-\mu)/\sigma$.

$$\boxed{F_X(a) = \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$$\Phi(a') \quad a' \Leftarrow \frac{(a-\mu)}{\sigma}$$

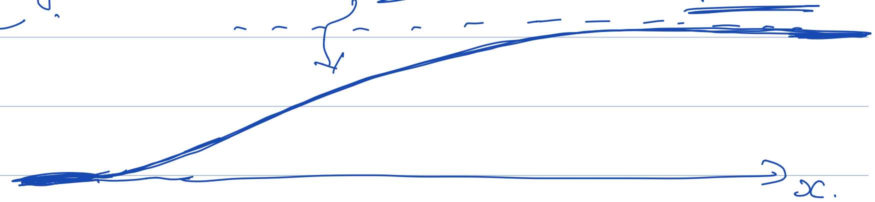
$$\Phi(a') = F_X(a).$$

$Q(x)$ = Complementary CDF of $\mathcal{N}(0,1)$

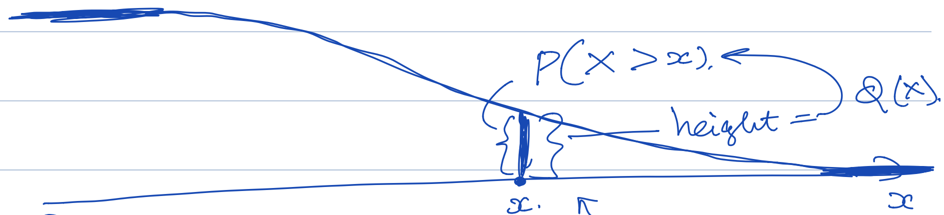
$= 1 - \Phi(x) = P(X > x)$

tail probability

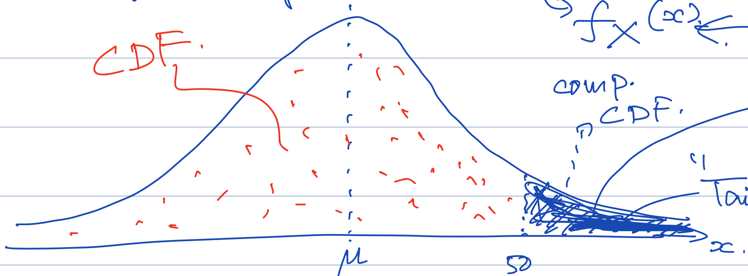
$\Phi(x) \rightarrow$ CDF of $\mathcal{N}(0,1)$



$$P(X>x) = Q(x) = 1 - \Phi(x)$$



PDF of Gaussian $\rightarrow f_X(x)$



$P(X > 50)$

"Tail" probability @ 50.

$P(X \leq 50)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \rightarrow 0. \quad x \rightarrow \infty \text{ or } x \rightarrow -\infty.$$

arg $= 0$ if $\boxed{x = \mu}$.

symmetric around μ.

$$f_X(x+\mu) = f_X(x-\mu)$$

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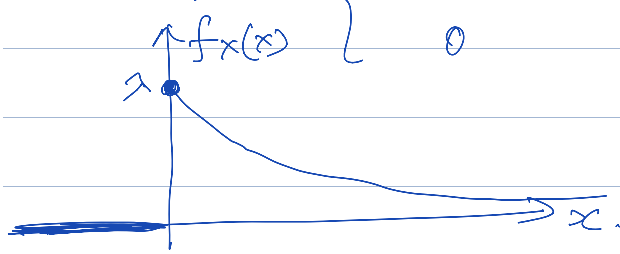
* Exponential R.V. (PDF)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$



modeled as exponential

(diff in time) of arrival