Final Exam - ECE 503 Fall 2015

- Date: Monday, December 16, 2015.
- Time: 10:30 am -12:30 pm (2 hours)
- Maximum Credit: **40 points**

1. **[6 points]** True or False

   (a) (1 point) Poisson process is a Markov process

   (b) (1 point) Mean square convergence always implies almost sure convergence

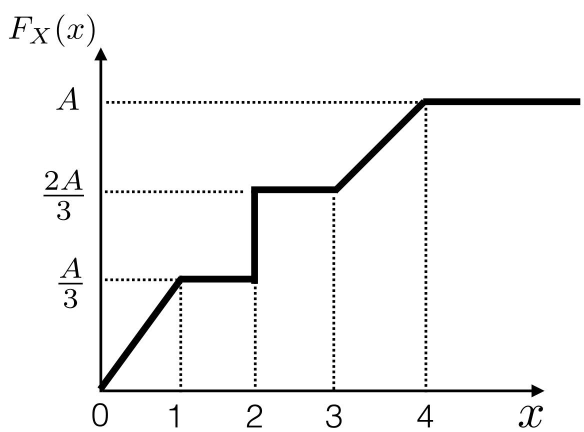
   (c) (1 point) If a random process is stationary of order 3, then it is always 1st order stationary

   (d) (1 point) The PSD of a WSS random process is always symmetric.

   (e) (1 point) If any two random variables X and Y are orthogonal, then they are also uncorrelated.

   (f) (1 point) If a WSS random process with $R_X(\tau) = \eta\delta(\tau)$ is passed through an LTI filter with impulse response $h(t)$, then the output power is $\eta \int_{-\infty}^{\infty} |h(t)|^2 dt$

2. **[6 points]** A random variable X has the following CDF, $F_X(x)$ as shown in the figure below:



   (a) What is the value of A ? Why ?
   (b) Sketch the probability density function (PDF) of this random variable.
   (c) Find $E[X]$ and the variance of X.
   (d) What is $P(X < 2)$?

3. **[3 points]** Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables such that

$$X_n \sim \text{Poisson}(n\lambda), \text{ for } n = 1, 2, 3, \ldots$$

where $\lambda > 0$ is a constant. Define a new sequence of random variables Y_n as

$$Y_n = \frac{X_n}{n} \text{ for } n = 1, 2, 3, \ldots$$

Show that Y_n converges in the mean square sense to λ.

4. **[5 points]** A WSS discrete time and zero mean random process X_n has the following autocorrelation function

$$R_X(m) = E[X_n X_{n+m}] = \begin{cases} 1 - \frac{|m|}{3}, & |m| \le 3 \\ 0 & \text{otherwise} \end{cases}$$

We would like to form a linear minimum mean-squared error (LMMSE) estimator of each sample X_n, from the previous two samples, X_{n-1} and X_{n-2}. Find this estimator $\hat{X}_n$.

5. **[5 points]** Let $X(t)$ be a WSS process with mean $\mu_X = 0$ and

$$R_X(\tau) = \delta(\tau)$$

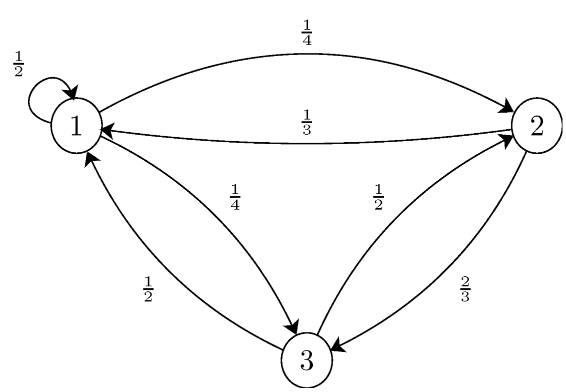
where $\delta(\tau)$ is the Dirac delta function. We define the random process $Y(t)$ as

$$Y(t) = \int_{t-2}^{t} X(u)du$$

(a) Find $\mu_Y(t)$

(b) Find $R_{XY}(t_1, t_2)$

6. **[5 points]** Consider the state-transition diagram for a discrete-time Markov chain in Figure 2



(a) Is the Markov chain irreducible ?

(b) Is the Markov chain aperiodic ?

(c) Find the stationary distribution of the Markov chain.

(d) Is the stationary distribution a limiting distribution of the Markov chain and is it unique ?

7. **[4 points]** Let $N(t)$ be a Poisson process with rate λ, and X_1 be the first arrival time. Show that given $N(t) = 1$, then X_1 is uniformly distributed in $(0, t]$. That is, show that

$$P(X_1 \le x | N(t) = 1) = \frac{x}{t}, \text{ for } 0 \le x \le t$$

8. **[6 points]** A computer wants to download a movie consisting of n distinct packets, $P_1, \ldots, P_n$ hosted on a server. The server works in a peculiar way. Every time slot, the server sends the computer a packet chosen independently and uniformly at random from the set of n packets. We are interested in analyzing the time it will take the computer to download the entire movie. We denote T_n as the random variable denoting the time it takes for the computer to receive all n packets.

   (a) Show that T_n can be written as

   $$T_n = X_1 + X_2 + \ldots + X_n$$

   where X_i's are independent geometric random variables.

   (b) Find $E[T_n]$, i.e., the mean of T_n.

   (c) Find $\mathrm{Var}(T_n)$, i.e., the variance of T_n.

   (d) Using all the above, show that

   $$P\left(\left|T_n - n\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right| \geq cn\right) \leq \frac{\pi^2}{6c^2}$$

   (You may find the following fact useful for part (d) : $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$)