$\boxed{\text{Today}}$

(1) WSS Random Processes ←
(wide sense stationary)

2) Jointly WSS Random Processes

3) Gaussian Random Process

4) Poisson Random Process.

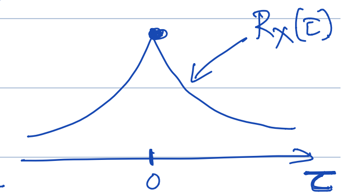
Recap: A random process $X(t)$ is WSS if

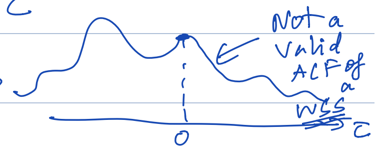
$$\begin{cases} 1)\ \mu_X(t) = E[X(t)] = \mu_X & (\text{mean is time invariant}) \\ 2)\ R_X(t_1,t_2) = E[X(t_1)X(t_2)] = R_X(t_1-t_2) = R_X(\tau) \end{cases}$$

(ACF) (Autocorrelation function only depends on time difference) $\tau = t_1 - t_2$

For a WSS Random process, $R_X(\tau)$ satisfies:

$$\begin{cases} (1) & R_X(0) \geq 0 \\ (2) & R_X(\tau) = R_X(-\tau) \quad (\text{Symmetric around } \tau = 0) \\ (3) & R_X(0) \geq |R_X(\tau)| \text{ for every } \tau \end{cases}$$





Proofs (1) $R_X(0) = E[X(t) \times X(t)] = E[X^2(t)] \geq 0$

(2) $E[X(t)X(t-\tau)] = E[X(t-\tau)X(t)]$

(3) (Cauchy-Schwartz) $E[X(t_1)X(t_2)]^2 \leq E[X^2(t_1)]E[X^2(t_2)]$

(Proved in previous lecture) $R_X(\tau)^2 \leq R_X(0) \times R_X(0)$

$(\tau = t_1 - t_2)$

(Wide Sense Stationary)

Example 1: $X(t) = \cos(t + U)$, where $U \sim \text{unif}[0, 2\pi]$

Show that $X(t)$ is a WSS random process.

1) $\mu_X(t)$ is a constant?

2) $R_X(t_1,t_2)$ is a fn of $(t_1 - t_2)$?

$$\mu_X(t) = E[X(t)] = E[\cos(t+U)] = \int_0^{2\pi} \cos(t+u)\cdot\frac{1}{2\pi}\,du = 0 \checkmark$$

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = E[\cos(t_1+U)\cdot\cos(t_2+U)] = E\Big[\underbrace{\tfrac{1}{2}\cos(t_1+t_2+2U)}_{0} + \tfrac{1}{2}\cos(t_1-t_2)\Big]$$

$$R_X(t_1,t_2) = \frac{1}{2}\cos(t_1 - t_2) \Rightarrow X(t) \text{ is WSS}$$

Power of $X(t) \stackrel{?}{=} 1/2$

Average Power of a WSS Random Process

For a deterministic signal $x(t)$

Energy

$$\int_{-\infty}^{\infty} x(t)^2 \, dt$$

Power

$$\lim_{T\to\infty} \frac{\int_{-T}^{T} x^2(t)\, dt}{2T}$$

For a random signal $X(t)$

$$\boxed{P_X \text{ average/Expected Power}} = \lim_{T\to\infty} \frac{E\left[\int_{-T}^{T} X^2(t)\,dt\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{\int_{-T}^{T} E[X^2(t)]\,dt}{2T} = R_X(0)$$

$R_X(0)$

if $X(t)$ is a WSS Random process

$$E[X^2(t)] = E[X(t)X(t)] = R_X(0) \quad \forall \text{ time } t.$$

$$\boxed{P_X = R_X(0)}$$

Jointly WSS Processes

$X(t)$ & $Y(t)$ are jointly WSS if

✓ (a) both $X(t)$ & $Y(t)$ are WSS, **AND**

✓ (b) $R_{XY}(t, t-\tau) = R_{XY}(\tau)$ --> ONLY depends on time difference τ.

$$\| \\ E[X(t)\, Y(t-\tau)] \longrightarrow \boxed{\text{Cross-correlation function}}$$

Example 2 Let X_n be a discrete-time (WSS) random process, with an autocorrel. function $R_X[k]$ and with zero mean.

Let $Y_n = (-1)^n X_n$ be another random process.

Are (X_n, Y_n) jointly WSS ?

is X_n a WSS $\rightarrow$ Yes

is Y_n a WSS $\rightarrow$? Yes

$$E[Y_n] = E[(-1)^n X_n] = (-1)^n \underbrace{E[X_n]}_{=0.}$$

$$= 0.$$

$\Rightarrow R_Y(n, n+k) = E[Y_n \cdot Y_{n+k}]$ $\rightsquigarrow$ does this depend ONLY on k. ??

(ACF of Y_n)

$$= E\left[(-1)^n X_n (-1)^{n+k} \cdot X_{n+k}\right]$$

$$= (-1)^{(2n+k)} \cdot E[X_n X_{n+k}].$$

$$= (-1)^k \cdot \underbrace{R_X[k]}_{\text{(ACF of } X_n)} \Longrightarrow Y_n \text{ is } \underline{\text{WSS}}$$

Cross-corr.

$$R_{XY}(n, n+k) = E[X_n Y_{n+k}] = E(X_n (-1)^{n+k} \cdot X_{n+k}]$$

$$= (-1)^{n+k} \cdot E[X_n X_{n+k}].$$

$$= (-1)^{n+k} R_X[k] \rightsquigarrow \text{depends on both } (n, k).$$

$\Rightarrow$ X_n & Y_n are $\boxed{\text{NOT}}$ Jointly WSS.

Gaussian Random Process (GRP)

A random process $X(t)$ is a GRP if for all times $(t_1, t_2, t_3, \dots, t_n)$, the random variables $(X(t_1), X(t_2), \dots, X(t_n))$ are jointly Gaussian for all integers n.

$(X(3), X(5), X(9))$

$\sim \mathcal{N}([\ ], \Sigma)$

If a GRP is WSS then it is also SSS

Strict-Sense Stationary.

SSS vs WSS

$X(t_1)\,X(t_2)$

$X(t_1+\Delta)\,X(t_2+\Delta)$

have the same distrib. for every Δ

$E[X(t_1)]$ is const. (wrt time)

$\parallel$

$E[X(t_1+\Delta)]$

$E[X(t_1)X(t_2)] = R_X(t_1-t_2)$

$\parallel$

$E[X(t_1+\Delta)X(t_2+\Delta)]$

$= R_X(t_1-t_2)$

Example 3: Let $X(t)$ be a zero-mean WSS GRP with $R_X(\tau) = e^{-\tau^2}$

(1) Find $P(X(1) < 1)$

(2) Find $P(X(1) + X(2) < 1)$.

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(1) $X(1) \to$ Gaussian r.v.

$E[X(1)] = 0$.

$$Var(X(1)) = E[X^2(1)] - 0^2$$
$$= E[X^2(1)]$$
$$= R_X(0)$$
$$= 1.$$

$$P(\underbrace{X(1)}_{\mathcal{N}(0,1)} < 1) = \Phi(1) \approx 0.84.$$

(2). $P(X(1)+X(2) < 1) = P(Y < 1)$

$Y = X(1) + X(2)$.

$\hookrightarrow \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$

$X(t)$ is a GRP

$\Downarrow$

$(X(1), X(2))$ are jointly Gaussian.

$X(1) + X(2)$ is also a Gaussian

$\mu_Y = E[Y] = 0$.

$\sigma_Y^2 = E[(X(1)+X(2))^2]$.

$$= \underbrace{E[X^2(1)]} + \underbrace{E[X^2(2)]} + 2\,\underbrace{E[X(1)X(2)]}$$

$$= R_X(0) + R_X(0) + 2 \times R_X(1)$$

$$= 1 + 1 + 2 \times \frac{1}{e}$$

$\left(R_X(\tau) = e^{-\tau^2}\right)$

$$\sigma_Y^2 = 2 + \frac{2}{e} = 2\left(1 + \frac{1}{e}\right)$$

$$P(Y < 1) = \Phi\left(\frac{1 - 0}{\sqrt{2\left(1 + \frac{1}{e}\right)}}\right)$$

$$\approx 0.73.$$

Example 4. $X(t)$ is GRP with

$$\boxed{\mu_X(t) = t}$$

$$R_X(t_1, t_2) = 1 + 2t_1 t_2.$$

(1) is $X(t)$ WSS? $\longrightarrow$ NO

(2) Find $P(2X(1) + X(2) < 3)$.

$Y = 2X(1) + X(2)$.

$\hookrightarrow \mathcal{N}(\mu_Y, \sigma_Y^2)$

$$E[Y] = 2E[X(1)] + E[X(2)] = 2 + 2 = 4$$

$$\sigma_Y^2 = \underline{25} = E[(Y-4)^2] \dashrightarrow$$ use $R_X(t_1, t_2)$ to find $E[X(t_1)X(t_2)]$

$$P(\underset{\mathcal{N}(4,25)}{Y} < 3) = \Phi\left(\frac{3-4}{5}\right)$$

$$= \Phi(-1/5) \approx 0.42.$$
