HW-1 Solution

1. we want to show

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

Solution: Let $B = A_2 \cup A_3$,

we know that

$$P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup B) = P(A_1) + P(B) - P(A_1 \cap B) \quad \text{--- (1)}$$

By distributive law,

$$A_1 \cap B = A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

$$\Rightarrow P(A_1 \cap B) = P\big((A_1 \cap A_2) \cup (A_1 \cap A_3)\big)$$

$$= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P\big((A_1 \cap A_2) \cap (A_1 \cap A_3)\big)$$

$$= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3) \quad \text{--- (2)}$$

Substituting (2) in (1),

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + \overbrace{P(A_2 \cup A_3)}^{\Rightarrow P(A_2) + P(A_3) - P(A_2 \cap A_3)} - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

—x—x—

$\boxed{2.}$ We are given $P(A) = 0.7$ and $P(B) = 0.6$ and we are interested in showing that

$P(A \cap B) \geq 0.3$

Recall: $$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.7 + 0.6 - P(A \cap B) \\ &= 1.3 - P(A \cap B) \end{aligned}$$

$$\Rightarrow \quad P(A \cap B) = 1.3 - P(A \cup B)$$

we also know that $P(A \cup B) \leq 1$

$$\begin{aligned} \Rightarrow \quad P(A \cap B) &= 1.3 - P(A \cup B) \\ &\geq 1.3 - 1 \\ &= 0.3 \end{aligned}$$

$$\Rightarrow \quad P(A \cap B) \geq 0.3$$

———— α ———— α ————

$\boxed{3.}$ We are given that A, B and C are independent events. To show that A and $B \cup C$ are independent :

$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$

$$= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$$

A, B, C are independent events

$$= P(A)P(B) + P(A)P(C) - P(A \cap B \cap C)$$

$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$$

$$= P(A)\left[P(B) + P(C) - P(B)P(C)\right]$$

$$= P(A)\left[P(B) + P(C) - P(B \cap C)\right]$$

$$= P(A)\,P(B \cup C)$$

$\Rightarrow$ A and $(B \cup C)$ are independent.

———x———x———

4. A fair coin is tossed repeatedly till a head appears

(a) Sample Space

$$S = \{ e_1, e_2, e_3, \ldots\ldots \}$$

$e_k \Rightarrow$ denotes the event that the first head appears on the k^{th} toss.

or

$$S = \{ H, TH, TTH, TTTH, \ldots\ldots \}$$

(b) Probability that the first head appears on the k^{th} toss ?

$\Rightarrow$ this is the probability of the event e_k

$$P(e_k) = P(\underbrace{TTT\cdots T}_{\substack{(k-1)\\ \text{tails}}}H)$$

$$= \left(\frac{1}{2}\right)^{k-1} \times \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k$$

(c) Probability that first head appears on a odd-numbered toss

$$= P(\{e_1, e_3, e_5 \ldots\})$$

$$= P(e_1) + P(e_3) + \ldots\ldots$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = 2/3$$

Probability that the first head appears on even-numbered toss

$$= P(\{e_2, e_4, \dots\})$$

$$= P(e_2) + P(e_4) + P(e_6) + \dots$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k - 1$$

$$= \left(\frac{1}{1-\frac{1}{4}}\right) - 1$$

$$= \frac{4}{3} - 1 = 1/3$$

$\Rightarrow$ These two are different, $2/3$ vs $1/3$.

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$\boxed{5.}$

$$S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$$

$P(A) = P(1^{st} \text{ die is odd}) = \frac{18}{36} = \frac{1}{2}$

$P(B) = P(2^{nd} \text{ die is odd}) = \frac{18}{36} = \frac{1}{2}$

$P(C) = P(\text{Sum is odd}) = \frac{18}{36.} = \frac{1}{2}$

$P(A \cap B) = P(1^{st} \text{ and } 2^{nd} \text{ die are both odd}) = \frac{9}{36} = \frac{1}{4} = P(A)\,P(B)$

$P(A \cap C) = P(1^{st} \text{ is odd and Sum is odd}) = 1/4 = P(A)\,P(C)$

$P(B \cap C) = P(2^{nd} \text{ is odd and Sum is odd}) = 1/4 = P(B)\,P(C)$

A, B are indep

A, C are independent

B, C are independent

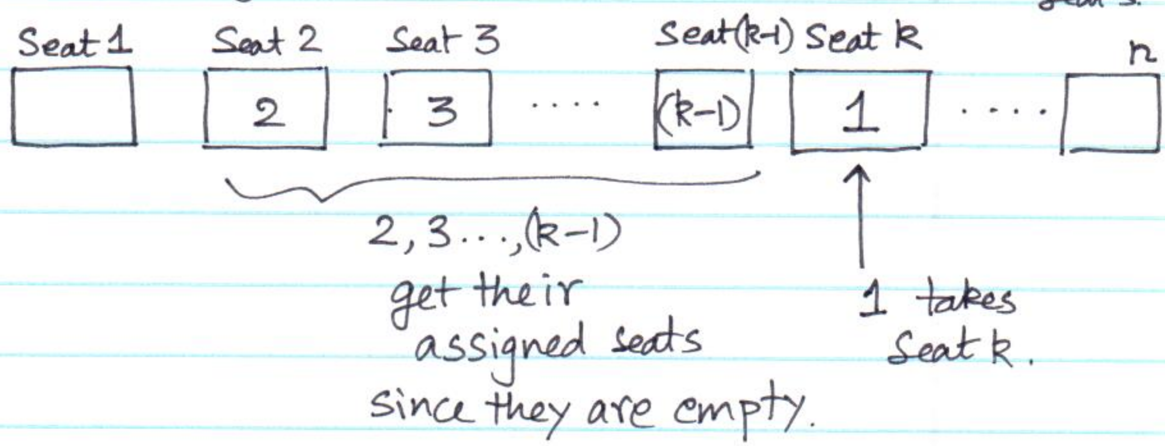
$P(A \cap B \cap C) = \mathbf{0} \neq P(A)P(B)P(C) = \frac{1}{8}$

Why? → Sum of two odd numbers is Even.... $\Rightarrow A \cap B \cap C = \phi$ Null Set.

$\Rightarrow$ $\boxed{\text{A, B, C are NOT independent.}}$

$\boxed{6.}$ Let us number the passenges by $1, 2, \ldots, n$ and assume that passenger i is alloted the seat i. (without any loss of generality).

Let $E_k \rightarrow$ denote the event that the 1^{st} passenger sits on seat k. If Event E_k occurs,
Note that the passengers $2, 3, \ldots, (k-1)$ will find their assigned seats.



* Let A denote the event that the last passenger finds his seat free.

* We are interested in $P(A)$.

$$P(A) = P(A|E_2)\,P(E_2) + P(A|E_3)\,P(E_3) + \cdots + P(A|E_n)\,P(E_n)$$

If we denote $\alpha_k = P(A|E_k)$, then

$$P(A) = \sum_{k=2}^{n} \alpha_k P(E_k)$$

Also, the 1^{st} passenger selects the wrong seat at random.

So, $P(E_k) = \dfrac{1}{n-1}$ for all k.

$$\Rightarrow \quad P(A) = \left(\frac{1}{n-1}\right) \times \left(\sum_{k=2}^{n} \alpha_k\right)$$

$$P(A) = \frac{(\alpha_2 + \alpha_3 + \ldots + \alpha_n)}{(n-1)}$$

where $\alpha_k = P(A \mid E_k)$

$= P(A \mid 1^{st}$ Passenger selects the k^{th} seat$)$.

Conditioned on the event E_K, what are the options for passenger number k?

Seat#	1	2	3		$(k-1)$	k	$(k+1)$		n
		$\boxed{2}$	$\boxed{3}$	$\ldots$	$\boxed{k-1}$	$\boxed{1}$		$\ldots$	

Option 1 $\longrightarrow$ if it selects seat # 1, then all the remaining passengers will get their assigned seats.

Option 2 $\longrightarrow$ if it selects seat # $(k+1)$, then we face the same problem starting from passenger $(k+1)$ onwards.

Option 3 $\longrightarrow$ if it selects seat # $(k+2)$ then $(k+1)^{th}$ gets its seat & we face a same problem from passenger $(k+2)$ onwards

⋮

Last option $\longrightarrow$ if it selects seat # n, then $(k+1), (k+2), \ldots, (n-1)$ get their seat and the n^{th} passenger <u>does NOT</u> get his seat.

How many such options ?? $\Rightarrow (n-k+1)$

$$\Rightarrow \cancel{\alpha_k = P(\text{Option } 1 \mid E_k) P(A \mid \text{option } 1, E_k) + P(\text{option } 2)\, P}$$

$$\alpha_k = P(\text{Option } 1) \cdot P(A \mid \text{Option } 1, E_k) + P(\text{Option } 2) \cdot P(A \mid \text{Option } 2, E_k) + \text{------} + P(\text{Last option}) \cdot P(A \mid \text{Last option}, E_k)$$

$$= \frac{1}{n-k+1} \times 1 + \frac{1}{(n-k+1)} \times \alpha_{k+1} + \cdots + \frac{1}{(n-k+1)} \times \alpha_n$$

$$\Rightarrow \boxed{\alpha_k = \frac{1 + \alpha_{k+1} + \alpha_{k+2} \cdots + \alpha_n}{(n-k+1)}}$$

Also, $\quad \alpha_n = P(A \mid E_n) = 0$

Using these, one can show that

$$\alpha_k = \frac{1}{2} \text{ for all } 2 \le k < n$$

and hence

$$P(A) = \frac{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1} + \alpha_n}{(n-1)}$$

$$= \frac{1/2 \times (n-2)}{(n-1)} = \frac{(n-2)}{2(n-1)}.$$

$\boxed{7.}$ Let W denote the event of Winning, i.e winning a total of N dollars.

Suppose that we start with m dollars, and we denote $P_m(W)$ as the probability of winning if we start with m dollars.

$P_0(W) = 0$ [Why? You cannot play the game since you have No money :( ]

$P_N(W) = 1$ [Why? You already started with N dollars, which was the goal :)]

Let's say we start with m dollars, where $0 < m < N$

Consider the outcome of 1st toss

Original wealth		New Wealth	
m	Head $\nearrow$	$m+1$ $\rightarrow$	continue to Play
	Tail $\searrow$	$m-1$. $\rightarrow$	continue to play.

$$\Rightarrow P_m(W) = P(\text{Head})\, P_{m+1}(W) + P(\text{Tail})\, P_{m-1}(W)$$

$$\Rightarrow \quad P_m(W) = \frac{1}{2} P_{m+1}(W) + \frac{1}{2} P_{m-1}(W).$$

$$\Rightarrow \quad \boxed{P_m(W) = \frac{1}{2}\left(P_{m+1}(W) + P_{m-1}(W)\right)}$$

$P_m(W)$ satisfies the above recursion...

Now, from the above,

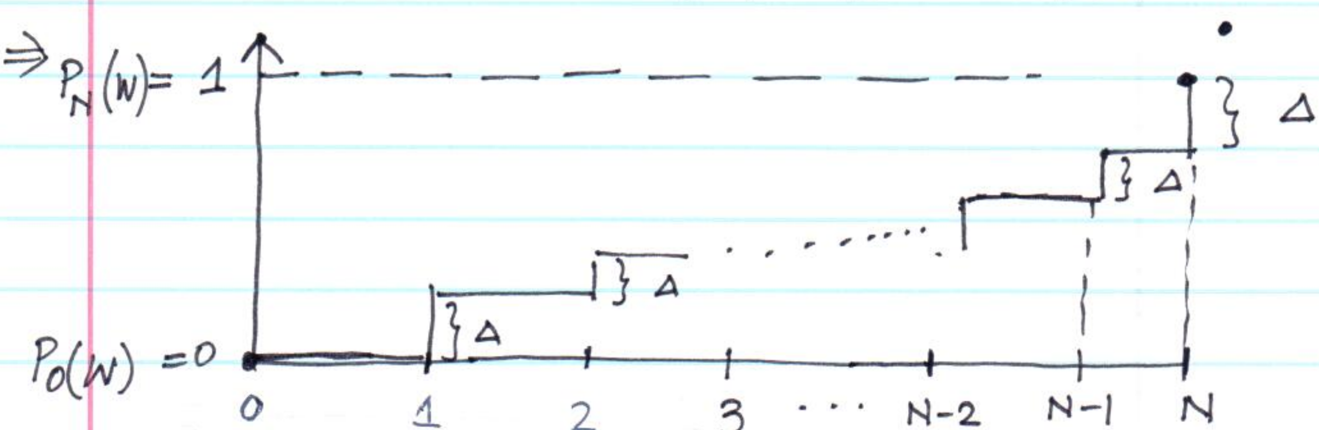
$$2\, P_m(W) = P_{m+1}(W) + P_{m-1}(W)$$

$$\Rightarrow \quad \left[P_{m+1}(W) - P_m(W)\right] = \left[P_m(W) - P_{m-1}(W)\right]$$

$\Rightarrow \quad P_{m+1}(W) - P_m(W)$ is independent of m!!

$\Rightarrow$ it is a constant.

$\Rightarrow$



$$\Rightarrow \quad \Delta = P_{m+1}(W) - P_m(W) = \frac{1}{N}$$

$$\Rightarrow \quad P_m(W) = m\Delta = m/N.$$

(b) $$P_m(W) = m\Delta = \frac{m}{N}$$

Consequence of increasing N ??

as $N \longrightarrow \infty \Rightarrow P_m(W) = \frac{m}{N} \longrightarrow 0$

i.e Probability of winning goes to zero as $N \to \infty$ for a fixed budget m.

8.

(a) Taking independent random guesses

Let A denote the event that Alice guesses correctly.

Let B denote the event that Bob guesses correctly.

Let W denote the event of winning, i.e., both Alice and Bob guess correctly.

Then, the winning probability of taking independent random guesses is:

$$P(W) = P(A \cap B) = P(A) \cdot P(B)$$

As they win if both guess correctly

As they decide to take independent guess.

$$= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

As each of them ~~is~~ donned with either a black hat or a white hat equally likely and they randomly guess.

$\Rightarrow$ The winning probability of taking indep. random guesses is $\frac{1}{4}$.

8.

(b) There are two strategies better than taking independent random guesses:

(I) Each of them decides to guess the color of its own hat the same as the other's.

(II) Each of them decides to guess the color of its own hat the opposite of the other's.

Here, we write the consequences of using Strategy (I) and Strategy (II)

"W" = white, "B" = Black

Alice	Bob	Strategy (I)	Strategy (II)
W	W	win	lose
W	B	lose	win
B	W	lose	win
B	B	win	lose

- winning probability of Strategy (I) $= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

- winning probability of Strategy (II) $= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

Therefore, the devised strategies are better than taking indep. random guesses.