Today

Applications
1) Coupon Collector Problem
2) Slotted Aloha (Multi-Access Protocol)

Recap:

$X \sim \text{Geometric}(p)$ $\quad X \to \{1, 2, \ldots\}$

$$P(X=i) = \underbrace{(1-p)^{i-1}}_{(i-1)\text{ unsucc.}} \times \underbrace{p}_{i^{th}\text{ successful.}}$$

$$E[X] = \sum_{i=1}^{\infty} i\, P(X=i) = \frac{1}{p}.$$

Coupon Collector Problem

n-distinct coupons $\longrightarrow$ $\boxed{c_1, c_2, \ldots, c_n}$ $\longleftarrow$ Box of n distinct coupons.

* Draw a coupon uniformly at random.
* Look @ coupon, record the coupon # & put it back.
* Keep repeating this experiment.

T_n = # of draws till we collect/observe all n coupons.

Goal: $\boxed{\text{Find } E[T_n],}$ i.e. the expected # of draws till we observe all n distinct coupons.

$\boxed{n=4}$

$\boxed{c_1 \quad c_2 \quad c_3 \quad c_4}$

$1^{st} \to c_2$

$2^{nd} \to c_2$

$3^{rd} \to c_1$

$4^{th} \to c_2$

$5^{th} \to c_3$

$6^{th} \to c_4$

$\boxed{\text{Stop}}$

List

2
1
3
4.

$C_1 \mid C_2 \ldots \ldots \mid C_n$

list

13

1^{st} draw $\longrightarrow C_{13}$

2^{nd} Draw $\longrightarrow$

$P(C_{13} \text{ shows up}) \overset{?}{=} \frac{1}{n}$

$P(C_{13} \text{ does not show up}) = 1 - \frac{1}{n} = \frac{(n-1)}{n}$

C_*, $\quad$ C_{13}, C_*, $\quad$ C_{13}, C_{13}, C_* $\ldots\ldots$

$\frac{(n-1)}{n}$, $\quad$ $\left(\frac{1}{n}\right) \times \left(\frac{n-1}{n}\right)$, $\quad$ $\left(\frac{1}{n}\right)^2 \times \left(\frac{n-1}{n}\right)$,

$(C_1 / C_2 \ldots (C_n)$ but not C_{13}

$$T_n = Y_1 + Y_2 + Y_3 + \ldots + Y_n.$$

Y_1: # of draws to see a first new coupon

Y_2: # of dr to see a second new cou..

Y_3: # of dr to see a 3rd new cou

...

$$\boxed{T_n = 1 + Y_2 + Y_3 + \ldots + Y_n.}$$

$Y_2 \sim \text{Geometric}\left(\frac{n-1}{n}\right)$ $\quad$ $\boxed{\text{Geometric}(p)}$

$\boxed{\frac{(n-1)}{n}}$

$Y_3 \sim \text{Geometric}\left(\frac{n-2}{n}\right)$

C_{13}

C_{21}

C_7

n

$\left(\frac{n-2}{n}\right)$ $\quad$ $\left(\frac{2}{n}\right) \longrightarrow Pr(\text{getting } C_{13} \text{ or } C_{21})$.

$Y_k \sim \text{Geometric}\left(\frac{(n-(k-1))}{n}\right)$ $\quad$ Y

$$\boxed{T_n = 1 + Y_2 + Y_3 + \cdots + Y_n.}$$

$$E[T_n] = \underbrace{E[1]} + E[Y_2] + \cdots + E[Y_n].$$

$$= 1 + \left(\frac{n}{n-1}\right) + \frac{n}{n-2} + \cdots + \frac{n}{n-(n-1)}.$$

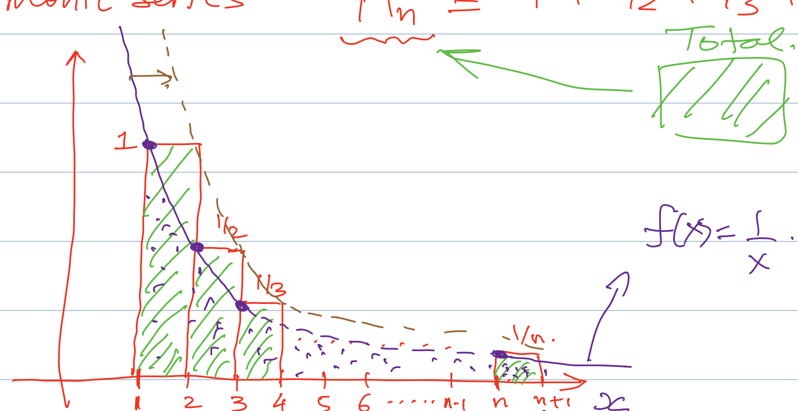
$$\boxed{\begin{array}{l} Y_2 \sim Geom\left(\underset{P}{\frac{n-1}{n}}\right) \\ E[Y_2] = \frac{1}{p} = \frac{n}{n-1} \end{array}}$$

$$E[T_n] = \frac{n}{n} + \frac{n}{n-1} + \frac{n}{n-2} + \cdots + \frac{n}{1}.$$

$$= n\underbrace{\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{n-1} + \frac{1}{n}\right]}_{\text{Harmonic Series } (H_n) \approx \ln(n)}$$

$$\approx n \times \log(n).$$

Harmonic series $\quad H_n = 1 + 1/2 + 1/3 + \cdots + 1/n.$



$1/x$

$$1 + \ln(n) > H_n > \int_1^{(n+1)} \frac{dx}{x} = \ln(x)\Big|_1^{(n+1)} = \ln(n+1).$$

$$\ln(n+1) < \underline{H_n} < \ln(n) + 1$$

$$H_n \approx \ln(n).$$

big-O.

upper $\rightarrow \underline{O(\ln(n))}$

lower $\rightarrow \Omega(\ln(n))$

$\rightarrow \Theta(\ln(n)).$

$$\underline{E[T_n] \cong n\ln(n).}$$

Collect 1^{st} distinct $\rightarrow$ Collect 2^{nd} distn comp $\rightarrow$ Collect 3^{rd} d. $\rightarrow$ - - - nth distinc comp

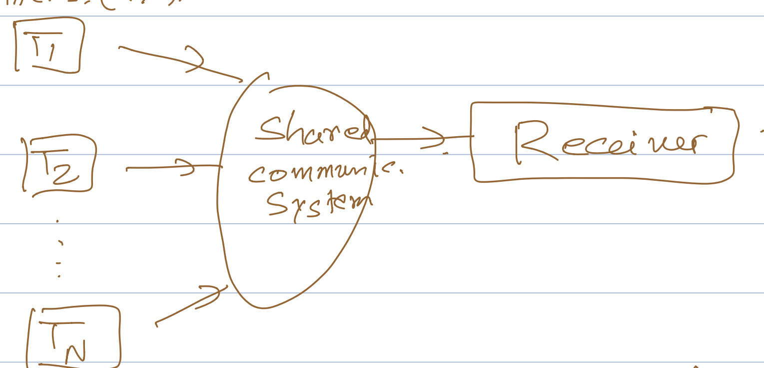
$$T_n = Y_1 + Y_2 + Y_3 + \cdots\cdots + Y_n.$$

$(Y_1 = 1)$

Slotted Aloha $\rightarrow$ (Multi-Access Protocols)

$N \rightarrow$ transmitters. (Tx).

$\boxed{\text{Each Tx sends a packet with prob } \boxed{p}}$.



if more than one $\underline{Tx}$ sends simultaneously $\longrightarrow$ "$\underline{\text{Collision}}$"

if only one Tx " $\longrightarrow$ "Success"

if no-one Tx. $\longrightarrow$ "Empty"

T_1	Pack	–	Pack	–	–	
T_2	Pack	–	–	Pack	–	. . . . . .
T_3	–	–	–	–	–	
	C	E	S	S	E	$\rightarrow$ time slots.

$$P(\underbrace{S}_{\boxed{\text{one Tx sends, } (N-1) \text{ are silent}}}) = \binom{N}{1} \times p \times (1-p)^{N-1}$$

$$P(\mathcal{E}) = \binom{N}{N} \times p^0 \times (1-p)^{N} = (1-p)^{N}$$

Empty slot $\rightarrow$ (No one sends)

$$P(C) = 1 - (\underbrace{P(S) + P(\mathcal{E})}_{P(C^c)})$$

$$= 1 - \binom{N}{1} p(1-p)^{N-1} - (1-p)^{N}.$$

$$\boxed{\text{Throughput of the Scheme} = \text{Expected \# of Successful Transmissions per unit slot.}}$$

$(\#S)^{(L)}$ (Consider running this protocol over L slots)

$$= \frac{E\left[(\#S)^{(L)}\right]}{L}.$$

$\text{Outcome}_i = O_i$

$$\underbrace{(\#S)^{L}}_{(R.V.)} = \underbrace{\mathbb{1}(O_1 = S) + \mathbb{1}(O_2 = S) + \dots + \mathbb{1}(O_L = S)}$$

$$\text{Indicator function} \rightarrow \mathbb{1}(O_i = S) = \begin{cases} 1 & \text{if } O_i = S \\ 0 & \text{otherwise} \end{cases}$$

$$E\left[(\#S)^{L}\right] = \underbrace{E\left[\mathbb{1}(O_1 = S)\right]} + \underbrace{E\left[\mathbb{1}(O_2 = S)\right]} + \dots + \underbrace{E\left[\mathbb{1}(O_L = S)\right]}$$

$$= L \times E\left[\mathbb{1}(O_1 = S)\right]$$

$$\frac{E[(\#S)^2]}{L} = \underbrace{E[\mathbb{1}(O_1 = S)]}$$

$\mathbb{1}(O_1 = S) \to U \to 1$, $U \to 0$.

$$= 1 \times P(U=1) + \underbrace{0 \times P(U=0)}$$

$$= 1 \times P(U=1)$$

$$= 1 \times P(O_1 = S) = 1 \times P(\text{success})$$

Throughput of slotted Aloha. $= \underbrace{NP(1-P)^{N-1}}$

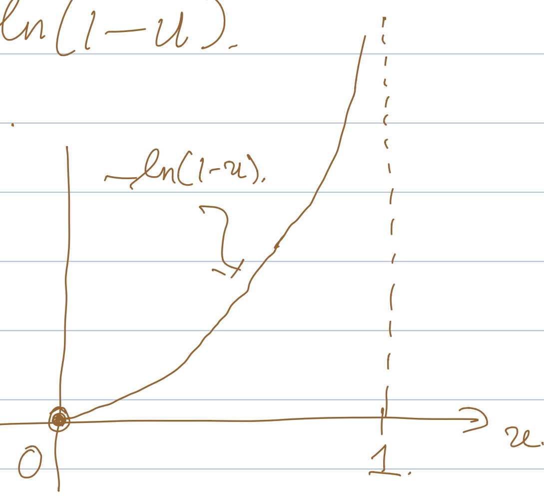
$\Downarrow$ how to design P to maximize througḥ.

Prob 3: $U \sim \underline{\text{unif}\,[0,1]}$

$\underline{X} = -\ln(1-u)$.

$\uparrow$

CDF, PDF, μ.



$F_X(x) = 0$. when $x < 0$.

when $x \geq 0$. $\to$ use defn.

CDF $\to$ PDF $\to$ μ.

$$\underline{E[g(x)]} = \int_{x=-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

$$E[X] = E[-\ln(1-U)].$$

$$= \int_{u=0}^{1} -\ln(\underbrace{1-u}) \, du.$$

$1-u = z.$

$-du = dz.$

$$= \int_{1}^{0} \ln(z) \, dz = \Big[(z\ln(z) - z.) \Big]_{1}^{0}$$

$$= 1$$

$$\boxed{\lim_{z \to 0} z \log(z) \overset{?}{=} 0.}$$

$0 \times \log(0).$