Today

1) Covariance Matrix, Cross-Covariance
2) Gaussian Random Vectors
3) Linear Transformations of Gaussian R.V.'s
4) Properties of Covariance Matrices.

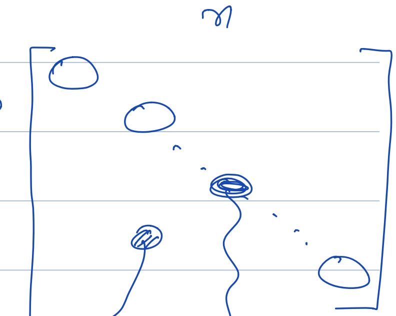
Recap X is a random vector $(n \times 1)$ $\qquad \mu_X \rightarrow$ mean $(n \times 1)$

$$R_X = E[XX^T]$$

correlation matrix $(n \times n)$

$$C_X = E[(X-\mu_X)(X-\mu_X)^T] \rightarrow$$

covariance matrix $(n \times n)$



$$C_X(i,j) = \text{cov}(X_i, X_j).$$

j^{th} diagonal entry ? $\text{Var}(X_j)$

$$= E[(X-\mu_X)(X^T-\mu_X^T)]$$

$$= E[XX^T - X\mu_X^T - \mu_X X^T + \mu_X\mu_X^T].$$

$$= E[XX^T] - \mu_X\mu_X^T$$

$$\boxed{C_X = R_X - \mu_X\mu_X^T}$$

$\underbrace{\vec{X}}_{n\times 1}$, $\underbrace{\vec{Y}}_{m\times 1}$ $\qquad$ Cross-Covariance Matrix

$$\underbrace{C_{XY}}_{n\times m} = E[\underbrace{(X-\mu_X)}_{n\times 1}\underbrace{(Y-\mu_Y)^T}_{1\times m}]$$

$$C_{XY}(i,j) = \text{cov}(X_i, Y_j)$$

$\underline{\underline{Ex}}$ $\underline{\underline{2\text{-dim}}}$ r. vector $\underline{f_X}(\vec{x}) = \begin{cases} 2 & \underline{\underline{\text{if } 0 \le x_1 \le x_2 \le 1}} \\ 0 & \text{otherwise.} \end{cases}$

$\underline{\underline{\text{Find}}}$ $\vec{\mu}_X$, C_X, R_X $\quad \bar{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$$\bar{\mu}_X = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \mu_1 = E[X_1] = \iint x_1 f_{\bar{X}}(\vec{x})\,dx_1\,dx_2$$

$$= \boxed{\int_{x_2=0}^{1} \int_{x_1=0}^{x_2} 2x_1\,dx_1\,dx_2} = 1/3.$$

$$\mu_2 = E[X_2] \cdots = 2/3.$$

$$R_X = E[X X^T] \qquad \vec{\mu}_X = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}.$$

$$\underset{2\times 2}{R_X} = \begin{bmatrix} E[X_1^2] & E[X_1 X_2] \\ E[X_1 X_2] & E[X_2^2] \end{bmatrix} = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

$$C_X \overset{?}{=} R_X - \mu_X \mu_X^T = \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix}.$$

Gaussian Random Vectors (GRV)

$\underset{(n\times 1)}{\vec{X}}$ is a GRV $\longrightarrow \mathcal{N}(\vec{\mu}_X, C_X)$

$\mathcal{N}$: normal. $\quad \vec{\mu}_X$: mean vector $\quad C_X$: covariance matrix.

PDF of GRV.

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2} \cdot \left(\det(C_X)\right)^{1/2}} \cdot \exp\left\{ -\frac{1}{2} (\bar{x} - \mu_X)^T C_X^{-1} (\bar{x} - \mu_X) \right\}$$

where $\det(C_X) > 0$.

determinant of the Cov. matrix

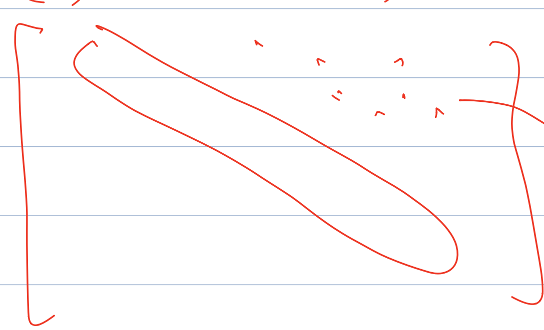
if $n=1$, we have a Gaussian r. variable.

$$\frac{1}{\sqrt{2\pi}\,\sigma} \times \exp\left\{-\frac{1}{2}(x-\mu)\frac{1}{\sigma^2}(x-\mu)\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

A GRV $\vec{X}$ has independent components if and only if C_X is a diagonal matrix.

$C_X(i,j) = \mathrm{Cov}(X_i, X_j)$ $\quad \vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$



off-diag. entries. $= 0$

$\Rightarrow$ all the components are uncorrelated.

A holds iff B holds.

if A is true $\Longrightarrow$ B is true.

if B is true $\Longrightarrow$ A is true.
(only if).

if C_X is diagonal $\Rightarrow$ $\vec{X}$ has independent components.

$$\underset{\sim}{C_X} = \begin{bmatrix} \sigma_1^2 & & & 0 \\ & \sigma_2^2 & & \\ & & \ddots & \\ 0 & & & \sigma_n^2 \end{bmatrix}$$

$$f_{\vec{X}}(\vec{x}) \overset{?}{=} \prod_{i=1}^{n} f_{X_i}(x_i)$$

$$\underline{\det(C_X)} \overset{?}{=} \prod_{i=1}^{n} \sigma_i^2$$

$f_{\vec{X}}(\vec{x})$,

$$\boxed{\det(C_X) > 0}$$

$$\underline{C_X^{-1}} = \begin{bmatrix} 1/\sigma_1^2 & & & 0 \\ & 1/\sigma_2^2 & & \\ & & \ddots & \\ 0 & & & 1/\sigma_n^2 \end{bmatrix}$$

$$\underbrace{C_X C_X^{-1}} = I$$

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2} \prod_{i=1}^{n} \sigma_i} \times \exp\left\{ -\frac{1}{2} \underline{(\vec{x} - \mu_X)} \begin{bmatrix} 1/\sigma_1^2 & & 0 \\ & 1/\sigma_2^2 & \\ 0 & \ddots & 1/\sigma_n^2 \end{bmatrix} (\vec{x} - \mu_X)^T \right\},$$

$$-\frac{(x_i - \mu_i)^2}{2\sigma_i^2}.$$

$$= \underbrace{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(x_i - \mu_i)^2}{2\sigma_i^2}}}_{\text{Product of marginals}}$$

Transformation of a Random Vector

$$\vec{X} \longrightarrow \vec{Y} = \begin{bmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \\ \vdots \\ g_K(\vec{x}) \end{bmatrix}$$

we know PDF of $\vec{X}$

Find PDF of $\vec{Y}$

$$J(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_K}{\partial x_1} & \cdots & \frac{\partial g_K}{\partial x_n} \end{bmatrix}$$

$$f_{\vec{Y}}(\vec{y}) = \sum_{\vec{x}_i \text{ are roots}} \frac{f_{\vec{X}}(\vec{x}_i)}{|J(\vec{x}_i)|}$$

$f_{\vec{Y}}$: PDF of R.V. $\vec{Y}$

$|J(\vec{x}_i)|$: determinant of Jacobian

$$\vec{X} \longrightarrow \vec{Y} = A\vec{X} + \vec{b}$$

(A is an invertible matrix)

$$f_{\vec{Y}}(\vec{y}) = \frac{1}{|\det(A)|} \cdot f_{\vec{X}}\left(A^{-1}(\vec{y} - b)\right)$$

$$\vec{y} = A\vec{x} + \vec{b} \Rightarrow \vec{x}_{root} = A^{-1}(\vec{y} - b)$$

Linear Transformation of a GRV

(Gaussian R.V.)

$*$ $\boxed{\text{Linear Transformations preserve Gaussianity}}$

$$\boxed{\vec{X} \sim \mathcal{N}(\vec{\mu}_x, C_x)}$$

$$\vec{Y} = A\vec{X} + \vec{b} \qquad \left(f_{\vec{Y}}(\vec{y}) \text{ is Gaussian}\right)$$

$$\vec{Y} \sim \mathcal{N}(\underbrace{\mu_Y}_{(A\mu_x + \vec{b})}, C_Y)$$

$$\mu_Y = E[\vec{Y}] = E[A\vec{X} + \vec{b}]$$

$$= E[A\vec{X}] + E[\vec{b}]$$

$$= E[A\vec{X}] + \vec{b}$$

$$= A \times E[\vec{X}] + \vec{b}$$

$$\mu_Y = A \times \vec{\mu}_x + \vec{b}$$

$$C_Y = E\left[(Y - \mu_Y)(Y - \mu_Y)^T\right]$$

$$= E\left[(AX + \cancel{b} - (A\mu_x + \cancel{b}))(\ldots\ldots)^T\right]$$

$$= E\left[\left(A(X - \mu_X) \right) \underbrace{\left(A(X-\mu_X) \right)^T} \right]$$

$$= E\left[\underbrace{A} (X - \mu_X)(X-\mu_X)^T A^T \right]$$

$$\boxed{(AB)^T = B^T A^T}$$

$$= A\, \underbrace{E\left[(X-\mu_X)(X-\mu_X)^T \right]} A^T$$

$$= A\, C_X\, A^T$$

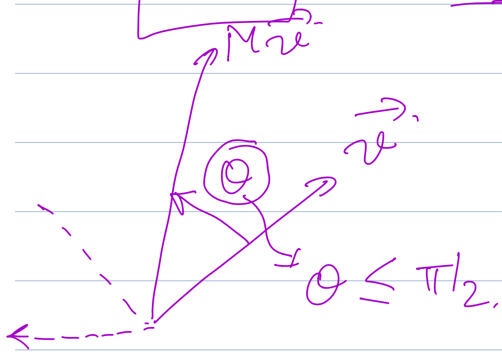
$$C_Y = A\, C_X\, A^T$$

$$\vec{Y} \sim \mathcal{N}\left(A\vec{\mu}_X + b,\ A C_X A^T \right)$$

$$(\vec{Y} = A\vec{X} + b)$$

Property of C_X (Covariance Matrix).

$\boxed{C_X}$ is a Positive Semi-definite matrix.
(PSD).



$$\underbrace{M}_{n \times n}\ \underbrace{\vec{v}}_{n \times 1} \longrightarrow \text{another vector of } n \times 1$$

$$(v^T)(M\vec{v}) \geq 0.$$

for all vectors $\vec{v}$

then we say that M is a PSD matrix.

Claim: C_X is a PSD matrix

Proof: $\forall \vec{v}$ (for every), $\underbrace{v^T C_X \vec{v}}$ $\geq 0.$

LHS

$$= \vec{v}^T E\left[(X-\mu_X)(X-\mu_X)^T\right]\vec{v}$$

$$= E\Big[\underbrace{\vec{v}^T(X-\mu_X)}_{\text{scalar} \Rightarrow W}\ \underbrace{(X-\mu_X)^T\vec{v}}\Big]$$

$$= E\left[W \times W^T\right]$$

$$= E\left[W^2\right]$$ since W is a scalar $\geq 0.$
