Solution HW 8 ECE 503

1 $X_1, X_2, \ldots, X_{120} \rightarrow$ call durations in minutes

$X_i \sim \text{Exponential}(\lambda)$ $\quad E[X_i] = \frac{1}{\lambda} = 150 \text{ sec} = 2.5 \text{ min}$

$Var[X_i] = \frac{1}{\lambda^2} = 6.25 \text{ min}^2.$

$\underbrace{Y}_{\text{Total \# of minutes used}} = X_1 + X_2 + \cdots + X_{120}$

$E[Y] = 120 \times 2.5 = 300$

$Var[Y] = 120 \times 6.25 = 750.$

$\text{Subscriber bill} = \underbrace{30}_{\text{fixed cost}} + 0.4 \times \underbrace{(y-300)^+}$ $\quad (x)^+ = \begin{cases} 0 & \text{if } x \le 0 \\ x & \text{if } x > 0 \end{cases}$

For bill to be $\$36 = 30 + 0.4 \times (y - 300)$

$\Rightarrow y = 315$ minutes.

(a) $P(\text{Bill} \geq \$36) = P(Y \geq 315)$

$= P\left(\frac{Y-300}{\sigma_Y} \geq \frac{315-300}{\sigma_Y}\right)$

$= Q\left(\frac{15}{\sqrt{750}}\right) \approx \boxed{0.2919}$

(b) In this part, if duration is X_i then subscriber is billed for $M_i = \lceil X_i \rceil$ (ceiling function) minutes.

$X_i \sim Exp(\lambda) \Rightarrow \underbrace{M_i}_{\text{discrete valued r.v.}} = \lceil X_i \rceil \sim \text{Geometric}(p)$, with $p = 1 - e^{-\lambda}$ $= 0.3297$

$E[M_i] = \frac{1}{p} = 3.033$; $Var[M_i] = \frac{1-p}{p^2} = 6.167$

Total minutes $= B = M_1 + M_2 + \cdots + M_{120}$ & $E[B] = 120 \times 3.033 = 364$

$Var[B] = 740.08$

$$\Rightarrow P(B > 315) = P\left(\frac{B-364}{\sqrt{740.08}} > \frac{315-364}{\sqrt{740.08}}\right)$$

$$= Q(-1.8) = \Phi(1.8) \boxed{= 0.964}.$$

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2 (a) Exp. # of tests needed to get <u>one</u> acceptable circuit $= \frac{1}{0.8} = 1.25$

$\Rightarrow$ Exp # of tests to get 500 acceptable circuits $= 500 \times 1.25 = \boxed{600}$

(b) Let K = denote the # of acceptable circuits in 600 tests.

$$= \underbrace{X_1 + X_2}_{} \cdots\cdots + X_{600}$$

$$\longrightarrow X_i = \begin{cases} 1 & \text{if } i^{th} \text{ circuit acceptable} \\ 0 & \text{otherwise.} \end{cases}$$

$$E[K] = 600 \times \underbrace{0.8}_{p} = 600\,p = 480$$

$$Var[K] = 600 \times p(1-p) = 96.$$

$$P(K \geq 500) = P\left(\frac{K-480}{\sqrt{96}} \geq \frac{500-480}{\sqrt{96}}\right) \approx Q\left(\frac{20}{\sqrt{96}}\right)$$

$$\boxed{= 0.0206}$$

(c) From MATLAB $\rightsquigarrow$ $P(K \geq 500) = 0.0215$.

(d) To find smallest batch size n for finding 500 acep. circuits with prob 0.9 or greater.

$K = X_1 + \cdots\cdots + X_n$ $\quad E[K] = np$; $Var[K] = np(1-p)$

$$P(K \geq 500) \geq 0.9.$$

$$P(K \geq 500) = P\left(\frac{K - np}{\sqrt{np(1-p)}} \geq \underbrace{\frac{500 - np}{\sqrt{np(1-p)}}}_{= z}\right) \approx 1 - \Phi(z) = 0.9$$

For $z = -1.29$, $\quad 1 - \Phi(z) = \Phi(-z) \geq 0.9$

$\Rightarrow$ we want.

$$\frac{500 - nP}{\sqrt{np(1-p)}} = -1.29$$

Putting $p = 0.8$ & solving for n, we get

$n = 641.3 \Rightarrow \boxed{n = 642 \text{ tests}}$

$\boxed{3}$ $\quad W_1, \dots, W_n$ iid $\sim$ mean μ, Variance σ^2

$$\bar{\mu}_{(n)} = \frac{W_1 + \dots + W_n}{n}$$

(sample mean)

To find $(1-\alpha)\%$ confidence interval, we want to find c,

such that $\Rightarrow P(\bar{\mu}_{(n)} - c \leq \mu \leq \bar{\mu}_{(n)} + c) = 1 - \alpha$

$\Rightarrow P(-c \leq \bar{\mu}_{(n)} - \mu \leq c) = (1-\alpha)$

$$\Rightarrow P\left(-\frac{c\sqrt{n}}{\sigma} \leq \underbrace{\left(\frac{\bar{\mu}_{(n)} - \mu}{(\sigma/\sqrt{n})}\right)}_{\longrightarrow\ \mathcal{N}(0,1) \text{ by CLT.}} \leq +\frac{c\sqrt{n}}{\sigma}\right) = 1 - \alpha.$$

$\Rightarrow$ First find c s.t. $P\left(-\frac{c\sqrt{n}}{\sigma} \leq \mathcal{N}(0,1) \leq \frac{c\sqrt{n}}{\sigma}\right) = 1 - \alpha.$

& then confidence interval $\Rightarrow$ $[\bar{\mu}_{(n)} - c, \bar{\mu}_{(n)} + c]$

For 95% Confid interval $\frac{c\sqrt{n}}{\sigma} = 1.96$

$$\Rightarrow c = \frac{1.96 \times \sigma}{\sqrt{n}} = \frac{1.96 \times 2}{10}$$

$$= 0.392$$

$$\Rightarrow [\bar{\mu}_{(n)} - c, \bar{\mu}_{(n)} + c] \Rightarrow \boxed{14.846 \pm 0.392}$$

4 Same as prev. problem, except replace σ^2 by sample variance.

$$[\bar{\mu}_{(n)} - c, \bar{\mu}_{(n)} + c] = 10.083 \pm 0.1477.$$

$$c = \frac{1.96 \times (\hat{\sigma})}{\sqrt{n}} = \frac{1.96 \times \sqrt{0.568}}{10} = 0.1477$$

5 For 93% confidence interval

$$c = \frac{(1.812) \times \hat{\sigma}}{\sqrt{n}} = \frac{1.812 \times \sqrt{0.957}}{10} = 0.1772$$

$$[\bar{\mu}_{(n)} - c, \bar{\mu}_{(n)} + c] = 4.442 \pm 0.1772$$

6 Likelihood = Joint PDF (of $n = N_F + N_S$ trials)

$= \theta^{N_S} \times (1-\theta)^{N_F}$ (since trials are iid)

$$\hat{\theta}_{ML} = \arg\max \log\left(\theta^{N_S} (1-\theta)^{N_F}\right)$$

$$\frac{d}{d\theta}\left[N_S \log(\theta) + N_F \log(1-\theta) \right]$$

$$= \frac{N_S}{\theta} - \frac{N_F}{1-\theta} = 0$$

$$N_S(1-\hat{\theta}) = N_F\,\hat{\theta}$$

$$\boxed{\hat{\theta}_{ML} = \frac{N_S}{N_S + N_F}} = \frac{N_S}{n}$$

$$\Rightarrow \quad \hat{\theta}_{ML} = \frac{X_1 + X_2 \cdots + X_n}{n} \rightsquigarrow X_i = \begin{cases} 1 & \text{if success} \\ 0 & \text{if failure.} \end{cases}$$

$$X_i \sim \mathrm{Ber}(\theta).$$

$$E[\hat{\theta}_{ML}] = \theta$$

$$\mathrm{Var}[\hat{\theta}_{ML}] = \frac{1}{n^2} \times n \times \theta(1-\theta) = \frac{\theta(1-\theta)}{n}.$$

From Chebychev's

$$P(|\hat{\theta}_{ML} - \theta| \geq c) \leq \frac{\mathrm{Var}(\hat{\theta}_{ML})}{c^2}.$$

$$= \frac{\theta(1-\theta)}{n c^2}.$$

$$\theta(1-\theta) \leq \frac{1}{4} \text{ for any } \theta \in [0,1].$$

$$\leq \frac{1}{4nc^2}.$$

$$\Rightarrow \quad P(|\hat{\theta}_{ML} - \theta| < 0.1) = 1 - P(|\hat{\theta}_{ML} - \theta| \geq 0.1)$$

$$\geq 1 - \frac{1}{4n \times (0.1)^2} \geq 0.99.$$

$$\Rightarrow \quad \frac{100}{4n} \leq 0.01 \Rightarrow \boxed{n \geq 2500 \text{ trials}}.$$

7 (a) $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$; $W \sim \mathcal{N}(\mu_2, \sigma_2^2)$

$Y = X + W$ $\quad (X, W)$ independent.

$\Rightarrow (X, Y)$ are jointly Gaussian.

MMSE estimate of $X \mid Y = y$ (true for jointly Gaussians)

$=$ Linear-MMSE estimate of $X \mid Y = y$.

$$\boxed{\hat{x}_{MMSE} = \mu_x + \rho_x \frac{\sigma_x}{\sigma_y}(y - \mu_y).}$$

[You can compute ρ_x, μ_y, σ_y etc...]

(b) ML estimate of $X \mid Y = y$.

Since $Y = X + W$,

$$Y \mid X = x \sim \mathcal{N}(x + \mu_2, \sigma_2^2)$$

$$\Rightarrow \underbrace{f(y \mid x)}_{\text{Likelihood function}} = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{(y-(x+\mu_2))^2}{2\sigma_2^2}}$$

$$\Rightarrow \boxed{\hat{x}_{ML} = y - \mu_2}$$

[Since maximizing $f(y|x)$ is equivalent to minimizing $(y - x - \mu_2)^2$]

(c) MAP estimate of $X \mid Y = y$

$$\hat{x}_{MAP} = \arg\max_x f(x \mid y) = \arg\max_x f_X(x) \times f_{Y|X}(y \mid x).$$

$$= \arg\max_x \left[e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \cdot e^{-\frac{(y-x-\mu_2)^2}{2\sigma_2^2}} \right]$$

$$= \arg\min_x \left[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-x-\mu_2)^2}{\sigma_2^2} \right]$$

$$\frac{d}{dx}\left[\quad\right] = \frac{(x-\mu_1)}{\sigma_1^2} - \frac{(y-x-\mu_2)}{\sigma_2^2} = 0$$

$$\Rightarrow \quad \sigma_2^2(x-\mu_1) = \sigma_1^2(y-x-\mu_2)$$

$$\Rightarrow \quad \boxed{\hat{x}_{MAP} = \frac{\sigma_2^2\mu_1 + \sigma_1^2(y-\mu_2)}{\sigma_1^2+\sigma_2^2}}$$