HW-1 Solution

1. We want to show

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

Solution: Let $B = A_2 \cup A_3$,

we know that

$$P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup B) = P(A_1) + P(B) - P(A_1 \cap B) \quad (1)$$

By distributive law,

$$A_1 \cap B = A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$$

$$\Rightarrow \quad P(A_1 \cap B) = P\big((A_1 \cap A_2) \cup (A_1 \cap A_3)\big)$$

$$= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P\big((A_1 \cap A_2) \cap (A_1 \cap A_3)\big)$$

$$= P(A_1 \cap A_2) + P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3) \quad (2)$$

Substituting (2) in (1),

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + \overbrace{P(A_2 \cup A_3)}^{\Rightarrow\, P(A_2) + P(A_3) - P(A_2 \cap A_3)} - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

$$= P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

———x—x———

2. We are given $P(A) = 0.7$ and $P(B) = 0.6$ and we are interested in showing that

$$P(A \cap B) \geq 0.3$$

Recall:
$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.7 + 0.6 - P(A \cap B) \\ &= 1.3 - P(A \cap B) \end{aligned}$$

$$\Rightarrow \quad P(A \cap B) = 1.3 - P(A \cup B)$$

we also know that $P(A \cup B) \leq 1$

$$\begin{aligned} \Rightarrow \quad P(A \cap B) &= 1.3 - P(A \cup B) \\ &\geq 1.3 - 1 \\ &= 0.3 \end{aligned}$$

$$\Rightarrow \quad P(A \cap B) \geq 0.3$$

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3. We are given that A, B and C are independent events. To show that A and $B \cup C$ are independent :

$$P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$$

$$= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$$

A, B, C are independent events

$$= P(A)P(B) + P(A)P(C) - P(A \cap B \cap C)$$

$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$$

$$= P(A)\left[P(B) + P(C) - P(B)P(C)\right]$$

$$= P(A)\left[P(B) + P(C) - P(B \cap C)\right]$$

$$= P(A)\,P(B \cup C)$$

$\Rightarrow$ A and $(B \cup C)$ are independent.

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4. A fair coin is tossed repeatedly till a head appears

(a) Sample Space

$$S = \{ e_1, e_2, e_3, \ldots\ldots \}$$

$e_k \Rightarrow$ denotes the event that the <u>first</u> head appears on the k^{th} toss.

or

$$S = \{ H, TH, TTH, TTTH, \ldots\ldots \}$$

(b) Probability that the first head appears on the k^{th} toss ?

$\Rightarrow$ this is the probability of the event e_k

$$P(e_k) = P(\underbrace{TTT\cdots T}_{\substack{(k-1)\\ \text{tails}}}H)$$

$$= \left(\frac{1}{2}\right)^{k-1} \times \left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^k$$

(c) Probability that first head appears on a odd-numbered toss

$$= P(\{e_1, e_3, e_5 \ldots\})$$

$$= P(e_1) + P(e_3) + \ldots\ldots$$

$$= \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k = 2/3$$

Probability that the first head appears on even-numbered toss

$$= P(\{e_2, e_4, \ldots\})$$

$$= P(e_2) + P(e_4) + P(e_6) + \cdots$$

$$= \sum_{k=1}^{\infty} \frac{1}{2^{2k}} = \sum_{k=1}^{\infty} \left(\frac{1}{4}\right)^k$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k - 1$$

$$= \left(\frac{1}{1-\frac{1}{4}}\right) - 1$$

$$= \frac{4}{3} - 1 = 1/3$$

$\Rightarrow$ These two are different, $2/3$ vs $1/3$.

$\boxed{5.}$ $S = \left\{ \begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array} \right\}$

$P(A) = P(1^{st} \text{ die is odd}) = \frac{18}{36} = \frac{1}{2}$

$P(B) = P(2^{nd} \text{ die is odd}) = \frac{18}{36} = \frac{1}{2}$

$P(C) = P(\text{Sum is odd}) = \frac{18}{36}. = \frac{1}{2}$

$P(A \cap B) = P(1^{st} \text{ and } 2^{nd} \text{ die are both odd}) = \frac{9}{36} = \frac{1}{4} = P(A)P(B)$

$P(A \cap C) = P(1^{st} \text{ is odd and Sum is odd}) = 1/4 = P(A)P(C)$

$P(B \cap C) = P(2^{nd} \text{ is odd and Sum is odd}) = 1/4 = P(B)P(C)$

A, B are indep

A, C are independent

B, C are independent

$P(A \cap B \cap C) = 0 \neq P(A)P(B)P(C) = \frac{1}{8}$

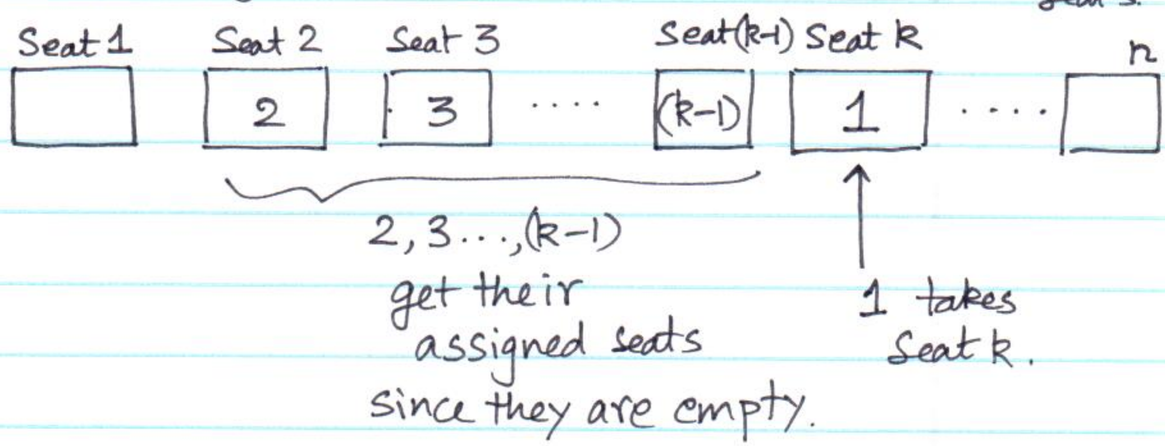
Why? → Sum of two odd numbers is Even.... $\Rightarrow A \cap B \cap C = \phi$ Null Set.

$\Rightarrow$ $\boxed{\text{A, B, C are NOT independent.}}$

$\boxed{6.}$ Let us number the passenges by $1, 2, \ldots, n$ and assume that passenger i is alloted the seat i. (without any loss of generality).

Let $E_k \rightarrow$ denote the event that the 1^{st} passenger sits on seat k. If Event E_k occurs,

Note that the passengers $2, 3, \ldots, (k-1)$ will find their assigned seats.



* Let A denote the event that the last passenger finds his seat free.
* We are interested in $P(A)$.

$$P(A) = P(A|E_2)P(E_2) + P(A|E_3)P(E_3) + \cdots + P(A|E_n)P(E_n)$$

If we denote $\alpha_k = P(A|E_k)$, then

$$P(A) = \sum_{k=2}^{n} \alpha_k P(E_k)$$

Also, the 1^{st} passenger selects the wrong seat at random.

So, $P(E_k) = \dfrac{1}{n-1}$ for all k.

$$\Rightarrow \quad P(A) = \left(\frac{1}{n-1}\right) \times \left(\sum_{k=2}^{n} \alpha_k\right)$$

$$P(A) = \frac{(\alpha_2 + \alpha_3 + \ldots + \alpha_n)}{(n-1)}$$

where $\alpha_k = P(A \mid E_k)$

$= P(A \mid 1^{st}$ Passenger selects the k^{th} seat$)$.

Conditiond on the event E_k, what are the options for passenger number k ?

Seat#	1	2	3		(k-1)	k	(k+1)		n
		2	3	...	k-1	1		----	

Option 1 $\longrightarrow$ if it selects seat # 1, then all the remaining passengers will get their assigned seats.

Option 2 $\longrightarrow$ if it selects seat # $(k+1)$, then we face the same problem starting from passenger $(k+1)$ onwards.

Option 3 $\longrightarrow$ if it selects seat # $(k+2)$ then $(k+1)^{th}$ gets its seat & we face a same problem from passenger $(k+2)$ onwards

⋮

Last option $\longrightarrow$ if it selects seat # n, then $(k+1), (k+2) \ldots, (n-1)$ get their seat and the n^{th} passenger <u>does NOT</u> get his seat.

How many such options ?? $\Rightarrow (n-k+1)$

$\Rightarrow$ ~~$\alpha_k = P(\text{Option } 1 | E_k) P(A | \text{option } 1, E_k) + P(\text{option } 2)\ P$~~

$$\alpha_k = P(\text{Option } 1) \cdot P(A | \text{Option } 1, E_k) + P(\text{option } 2) \cdot P(A | \text{option } 2, E_k) + \text{-----} + P(\text{Last option}) \cdot P(A | \text{Last option}, E_k)$$

$$= \frac{1}{n-k+1} \times 1 + \frac{1}{(n-k+1)} \times \alpha_{k+1} + \cdots + \frac{1}{(n-k+1)} \times \alpha_n$$

$$\Rightarrow \boxed{\alpha_k = \frac{1 + \alpha_{k+1} + \alpha_{k+2} \cdots + \alpha_n}{(n-k+1)}}$$

Also, $\alpha_n = P(A \mid E_n) = 0$

Using these, one can show that

$$\alpha_k = \frac{1}{2} \text{ for all } 2 \le k < n$$

and hence

$$P(A) = \frac{\alpha_2 + \alpha_3 + \cdots + \alpha_{n-1} + \alpha_n}{(n-1)}$$

$$= \frac{1/2 \times (n-2)}{(n-1)} = \frac{(n-2)}{2(n-1)}.$$

$\boxed{7.}$ Let W denote the event of Winning, i.e winning a total of N dollars.

Suppose that we start with m dollars, and we denote $P_m(W)$ as the probability of winning if we start with m dollars.

$P_0(W) = 0$ [Why? You cannot play the game since you have No money :( ]

$P_N(W) = 1$ [Why? You already started with N dollars, which was the goal :)]

Let's say we start with m dollars, where $0 < m < N$

Consider the outcome of 1^{st} toss

Original wealth		New Wealth	
m	Head $\nearrow$	$m+1$ $\rightarrow$	continue to Play
	Tail $\searrow$	$m-1$. $\rightarrow$	continue to play.

$$\Rightarrow \quad P_m(W) = P(\text{Head})\, P_{m+1}(W) + P(\text{Tail})\, P_{m-1}(W)$$

$$\Rightarrow \quad P_m(W) = \frac{1}{2} P_{m+1}(W) + \frac{1}{2} P_{m-1}(W).$$

$$\Rightarrow \quad \boxed{P_m(W) = \frac{1}{2}\left(P_{m+1}(W) + P_{m-1}(W)\right)}$$

$P_m(W)$ satisfies the above recursion...

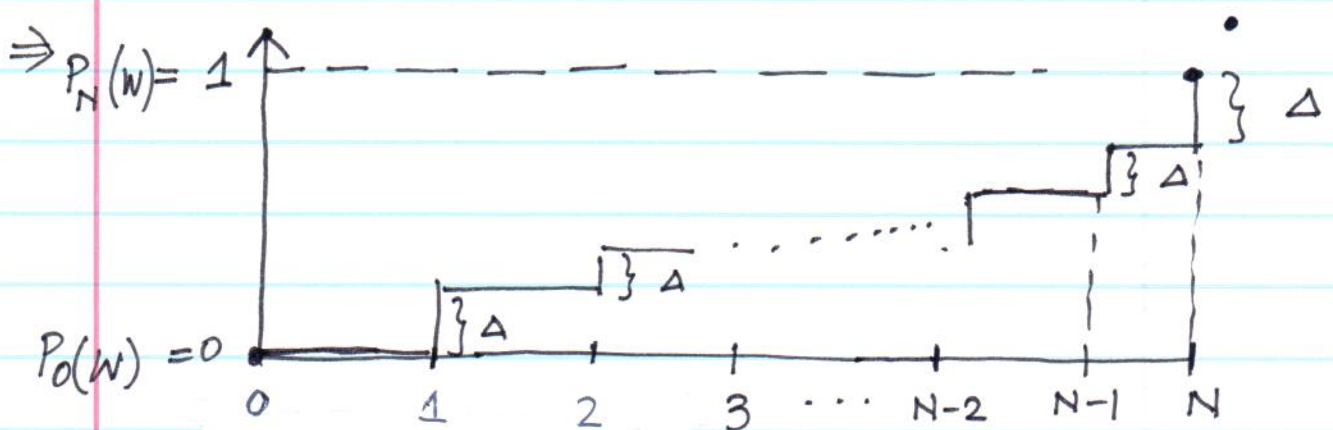
Now, from the above,

$$2 P_m(W) = P_{m+1}(W) + P_{m-1}(W)$$

$$\Rightarrow \left[P_{m+1}(W) - P_m(W)\right] = \left[P_m(W) - P_{m-1}(W)\right]$$

$\Rightarrow$ $P_{m+1}(W) - P_m(W)$ is independent of m !!

$\Rightarrow$ it is a constant.



$$\Rightarrow \quad \Delta = P_{m+1}(W) - P_m(W) = \frac{1}{N}$$

$$\Rightarrow \quad P_m(W) = m\Delta = m/N.$$

(b) $$P_m(W) = m\Delta = \frac{m}{N}$$

Consequence of increasing N ??

as $N \longrightarrow \infty \Rightarrow P_m(W) = \frac{m}{N} \longrightarrow 0$

i.e Probability of winning goes to zero as $N \to \infty$ for a fixed budget m.