Solutions

Midterm 1 Exam - ECE 503 Fall 2020

- Due Date and Time: Monday, Oct. 5, 2020, by Noon.
- Submit your answers on D2L.
- Maximum Credit: 100 points

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1. [25 points]

   (a) (5 points) Mutually exclusive events are always independent. (True or False ?)

   (b) (10 points) Six cards are drawn at random (with replacement) from a deck of 52 cards. What is the probability that there are at least two Aces?

   (c) (10 points) Let X be a discrete random variable with the following PMF:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{(n-k)}, \quad k = 0, 1, \ldots, n$$

   Compute the mean of X.

(a) $\boxed{\text{False}}$

(b) $P(\text{Atleast 2 Aces}) = 1 - P(\text{No Ace}) - P(\text{Exactly One Ace})$

$$= 1 - \left(\frac{48}{52}\right)^6 - \binom{6}{1} \times \left(\frac{4}{52}\right)^1 \times \left(\frac{48}{52}\right)^5$$

$$= 1 - \left(\frac{12}{13}\right)^6 - \left(\frac{6}{13}\right)\left(\frac{12}{13}\right)^5 = \boxed{0.072}$$

(c) $X \sim \text{Binomial}(n, p)$

$$\boxed{E[X] = np}$$

$$E[X] = \sum_{k=0}^{n} k \times P(X=k) = \sum_{k=0}^{n} k \times \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{k \times n!}{k!\,(n-k)!} \times p^k \times (1-p)^{n-k}$$

$$= np \times \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} \times p^{k-1} \times (1-p)^{n-k}$$

$$= np \times \underbrace{\sum_{k=1}^{n} \binom{n-1}{k-1} \times p^{k-1} (1-p)^{n-k}}_{= (p + (1-p))^{n-1} = 1}$$

$$= \boxed{np}$$

2. [25 points] Let X be a random variable with the following CDF:

$$F_X(x) = \begin{cases} 0, & x < -3 \\ \frac{x}{12} + \frac{1}{2}, & -3 \le x < 0 \\ \frac{x}{12} + \frac{3}{4}, & 0 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$

(a) Find $P(X = -2), P(X = 0)$ and $P(0 < X \le 2)$

(b) Find $P(X \le 2 | X > -1)$.

(c) If $Y = X^2$, find the CDF of the random variable Y.

(a) $P(X=-2) = F_X(-2) - F_X((-2)^-) = \boxed{0}$

$P(X=0) = F_X(0) - F_X(0^-) = 3/4 - 1/2 = \boxed{1/4}$

$P(0 < X \le 2) = F_X(2) - F_X(0) = \left(\frac{2}{12} + \frac{3}{4}\right) - \left(\frac{3}{4}\right) = \boxed{\frac{1}{6}}$

(b) $P(X \le 2 | X > -1) = \dfrac{P(X > -1, \; X \le 2)}{P(X > -1)}$

$= \dfrac{P(-1 < X \le 2)}{P(-1 < X)} = \dfrac{F_X(2) - F_X((-1))}{1 - F_X(-1)}$

$= \dfrac{\left(\frac{2}{12} + \frac{3}{4}\right) - \left(-\frac{1}{12} + \frac{1}{2}\right)}{1 - \left(-\frac{1}{12} + \frac{1}{2}\right)}$

$= \dfrac{6/12}{7/12} = \boxed{\frac{6}{7}}$

(c) When $x \in [-3, 3]$ $\quad y = x^2 \in [0, 9]$

Y is non-negative $(= x^2)$ $\quad F_Y(y) = 0$ if $y < 0$

$F_Y(y) = 1$ if $y \ge 9$

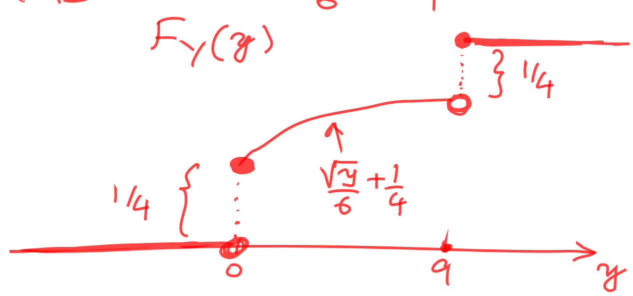
$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le +\sqrt{y})$

$= F_X(\sqrt{y}) - F_X((-\sqrt{y})^-)$

When $y = 0$ $\quad F_Y(0) = F_X(0) - F_X(0^-) = 3/4 - 1/2 = 1/4$

For $0 < y < 9$ $\quad F_Y(y) = \left(\frac{\sqrt{y}}{12} + \frac{3}{4}\right) - \left(\frac{-\sqrt{y}}{12} + \frac{1}{2}\right) = \frac{\sqrt{y}}{6} + \frac{1}{4}$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{\sqrt{y}}{6} + \frac{1}{4} & \text{if } 0 \le y < 9 \\ 1 & \text{if } y \ge 9 \end{cases}$$



3. [25 points] To sign up for a new COVID-19 contact tracing app, users are asked to pick a password of length 8, with the following guidelines. The password must have

- exactly 4 upper-case letters (can be chosen with replacement) from $\{A, B, \cdots, Z\}$
- exactly 2 lower-case letters (can be chosen with replacement) from $\{a, b, \cdots, z\}$
- exactly 2 special characters (chosen without replacement) from the following list $\{\#, \$, \%, \&, !, @\}$

(a) How many distinct passwords are possible?

(b) Suppose there are N users that sign up for the app. Each user independently picks a valid password at random. What is the probability that none of the users share the same password?

$$\overbrace{\square\ \square\ \cdots\cdots\ \square}^{\text{8 slots to fill}}$$

(a)

Total # of distinct Passwords $= K = \left(\binom{8}{4} \times 26^4\right) \times \left(\binom{4}{2} \times (26)^2\right) \times \left(\binom{2}{2} \times 6 \times 5\right)$

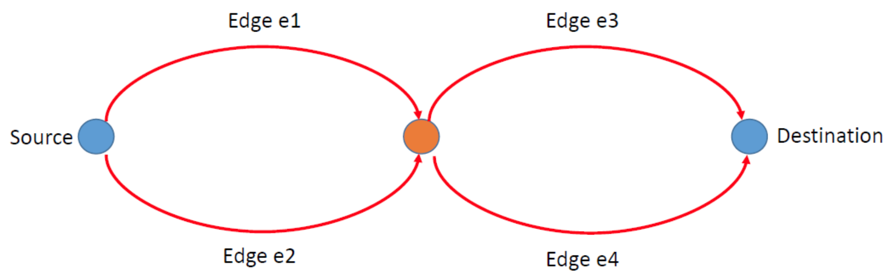
(b) $N = $ # of users.

if $N > K \Rightarrow$ there will be at least one password common to two users $\Rightarrow P\left(\begin{matrix}\text{No users}\\ \text{share same p/w}\end{matrix}\right) = 0$

if $N \le K$

$\Rightarrow P\left(\begin{matrix}\text{No users}\\ \text{share the same}\\ \text{p/w}\end{matrix}\right) = \dfrac{K \times (K-1) \times \cdots\cdots \times (K-(N-1))}{K^N}$

4. [25 points] Consider a source (S) and a destination (D) connected through the network shown in the figure below. A path from S to D is defined as a sequence of edges that connect S to D. For instance, the path $P_{1,3} = e_1 \to e_3$ is a valid path that can allow data transfer from S to D. Each edge in the network is functional independently with probability p (and does not work with probability $1-p$). In order to send data from S to D, one needs a working path, i.e., a path with all functional edges. For instance, the path $P_{1,3}$ is a working path only if both the edges e_1 and e_3 are functional.



(a) Enumerate all the valid paths for this network.

(b) What is the probability that there is at least one working path from S to D?

(c) What is the expected number of working paths?

(a) All valid paths P_{13} $e_1 \to e_3$; P_{23} $e_2 \to e_3$
P_{14} $e_1 \to e_4$; P_{24} $e_2 \to e_4$

(b)

Prob.	e_1	e_2	e_3	e_4	(# of Working Paths)	
$(1-p)^4$	0	0	0	0	0	
	0	0	0	1	0	
	0	0	1	0	0	
$(1-p)^2 p^2$	0	0	1	1	0	
	0	1	0	0	0	
	0	1	0	1	1	(P_{24})
	0	1	1	0	1	(P_{23})
	0	1	1	1	2	(P_{23}, P_{24})
	1	0	0	0	0	
	1	0	0	1	1	
	1	0	1	0	1	
	1	0	1	1	2	(P_{13}, P_{14})
	1	1	0	0	0	
	1	1	0	1	2	
	1	1	1	0	2	
	1	1	1	1	4	

P(# of working paths = 1)
$= 4p^2(1-p)^2$

P(# of working paths = 2)
$= 4p^3(1-p)$

P(# of working paths = 4)
$= p^4$

P(At least 1 working path) $= 1 - P$(No working path)
$= 1 - P$(# of working Paths = 0)
$= 1 - \left[(1-p)^2 + 2p(1-p)^3 + p^2(1-p)^2\right]$

(c) E[# of working Paths] $= 4 \times p^4 + 2 \times [4p^3(1-p)] + 1 \times [4p^2(1-p)^2] = \boxed{4p^2}$