Convergence of Random Variables

Recall, a random variable X is a function $X : \Omega \to \mathbb{R}$, which maps any point $\omega \in \Omega$ in the sample space to a real number $X(\omega)$.

Recall that a sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to converge to a limit $x \in \mathbb{R}$,

or $\quad x_n \longrightarrow x$

if for any $\epsilon > 0$ there exists N such that

$$|x_n - x| < \epsilon \quad \text{for all } n \geq N.$$

$\boxed{\text{MODES of CONVERGENCE}}$

1. Convergence almost surely (a.s.) $X_n \xrightarrow{a.s.} X$

$$P\left(\{\omega : X_n(\omega) \to X(\omega)\}\right) = 1$$

this is a set of events described in terms of the outcomes, for which X_n (outcome) $\longrightarrow X$ (outcome) ie $\omega \in \Omega$

Another method to check a.s. convergence:

For all $\delta > 0$,

$$\lim_{n \to \infty} P\{|X_m - X| \leq \delta \text{ for all } m \geq n\} = 1$$

2. Convergence in Mean-Squared Sense

$X_n \xrightarrow{m.s.} X$

$\{X_n\}$ converges to X in m.s. sense if

$$E(|X_n - X|^2) \to 0 \quad \text{as } n \to \infty$$

3. Convergence in Probability

$X_n \xrightarrow{P} X$

$\{X_n\}$ converges to X in probability if for any $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0 \quad \text{as } n \to \infty$$

4. Convergence in Distribution $X_n \xrightarrow{D} X$

Let F_n and F denote the CDFs of X_n and X.
$\{X_n\}$ converges to X in distribution if

$$F_n(x) \to F(x) \quad \text{as } n \to \infty$$

for any x such that F is continuous at x.

Relationship between Modes of Convergence

$$\begin{tikzcd}
\boxed{\begin{array}{c} X_n \xrightarrow{a.s.} X \\ \text{Almost Sure} \\ \text{Convergence} \end{array}} \arrow[dr, Rightarrow] & & \\
& \boxed{\begin{array}{c} X_n \xrightarrow{P} X \\ \text{Convergence} \\ \text{in Probability} \end{array}} \arrow[r, Rightarrow] & \boxed{\begin{array}{c} X_n \xrightarrow{D} X \\ \text{Convergence} \\ \text{in} \\ \text{Distribution} \end{array}} \\
\boxed{\begin{array}{c} \text{Mean Square} \\ \text{Convergence} \\ X_n \xrightarrow{m.s.} X \end{array}} \arrow[ur, Rightarrow] & &
\end{tikzcd}$$

<u>Remark</u>: In general, $X_n \xrightarrow{a.s.} X$ <u>does not</u> imply that $X_n \xrightarrow{m.s.} X$, and also $X_n \xrightarrow{m.s.} X$ does not imply $X_n \xrightarrow{a.s.} X$.

<u>Example 1</u>: Let $\Omega = [0, 1]$

$$P([a, b]) = b - a \text{ for } 0 \le a \le b \le 1$$

Prob. of an interval.

Define the sequence of r.v.'s

$$X_n(\omega) = \omega + \omega^n$$

$$(0 \le \omega \le 1)$$

and $X(\omega) = \omega$, ω = length of the interval.

For any $\omega \in [0, 1)$, $X_n(\omega) = \omega + \omega^n \to \omega$ as $n \to \infty$

However for $\omega = 1$, $X_n(\omega) = 2\omega$ for all ω, whereas $X(\omega) = \omega$.

$$\Rightarrow P(\{\omega : X_n \to X\}) = P([0, 1)) = 1$$

$$\Rightarrow X_n \to X \text{ almost surely.}$$

Example 2:

$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \le 1/n \\ 0 & \text{otherwise} \end{cases}$$

For any $\omega > 0$ $X_n(\omega) = 0$ if $\omega > \frac{1}{n}$,
i.e there always exist a $n_0(\omega) = \frac{1}{\omega}$ s.t that for all $n > n_0(\omega)$, $X_n(\omega) = 0$.

$\Rightarrow$ $X_n \to 0$ for $\omega > 0$

$\Rightarrow P(\{\omega : X_n \to 0\}) = P((0,1]) = 1$

$\Rightarrow$ $X_n \to 0$ almost surely.

Does $X_n \to 0$ in m.s. ?

$P(X_n = 1) = 1/n$
$P(X_n = 0) = 1 - 1/n$

$$E[|X_n - 0|^2] = E[|X_n|^2] = E[X_n^2] = (1)^2 \times \frac{1}{n} + (0)^2 \times \left(1 - \frac{1}{n}\right)$$

$$= \frac{1}{n}$$

$\Rightarrow E[|X_n - 0|^2] = \frac{1}{n} \longrightarrow 0$ as $n \to \infty$

$\Rightarrow X_n \xrightarrow{m.s.} 0$ ✓.

$X_n \xrightarrow{a.s.} 0$ but $X_n \not\xrightarrow{m.s.} 0$

Example 3:

$$X_n(\omega) = \begin{cases} \sqrt{n} & \text{if } \omega \le \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

You can check that $X_n \to 0$ almost surely (as in Example 2).

$P(X_n = \sqrt{n}) = 1/n$
$P(X_n = 0) = 1 - 1/n$

$$E(|X_n - 0|^2) = E(|X_n|^2) = E(X_n^2)$$
$$= (\sqrt{n})^2 \times \left(\frac{1}{n}\right) + (0)^2\left(1 - \frac{1}{n}\right)$$
$$= \frac{n}{n} + 0$$
$$= 1$$

$\Rightarrow E(|X_n - 0|^2) = 1$ for all n
$\not\to 0$

$\Rightarrow \quad X_n \not\xrightarrow{m.s.} 0$

i.e. X_n can converge almost surely but may not converge in m.s. sense.

Example 4 $\quad U \sim \text{unif}[0,1]$ (uniform r.v.)

$$X_n = n\, I_{[0,\frac{1}{n}]}(U), \quad n = 1, 2, \ldots\ldots$$

or

$$X_n = \begin{cases} n & \text{if } 0 \le U \le \frac{1}{n} \\ 0 & \text{otherwise.} \end{cases}$$

$$\boxed{X_n \xrightarrow{\text{in Probability}} 0 \quad \text{but} \quad X_n \overset{m.s.}{\not\longrightarrow} 0}$$

To show that X_n converges to 0 in probability, we first note that for any $\epsilon > 0$,

$$\{|X_n| \geqslant \epsilon\} = \begin{cases} \{U \le 1/n\}, & n \geqslant \epsilon \\ \phi & n < \epsilon \end{cases}$$

null event

$$\Rightarrow \quad P(|X_n - 0| \geqslant \epsilon) \le P(U \le 1/n)$$

$$= \frac{1}{n} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \quad X_n \xrightarrow{\text{in Prob.}} 0$$

However, $E[|X_n - 0|^2] = E[|X_n|^2] = n^2 \times P(U \in [0, 1/n])$

$$= n^2 \times \frac{1}{n} = n \longrightarrow \infty$$

as $n \to \infty$

$$\Rightarrow \quad X_n \overset{m.s.}{\not\longrightarrow} 0.$$

$X_n \xrightarrow{\text{in Distrib}} X$ but $X_n \overset{\text{in Prob.}}{\not\longrightarrow} X$

Example 5 Let $U \sim \text{uniform}[0,1]$

Let $X_n = U$ and $X = 1 - U$

You can easily check that X is also uniform$[0,1]$

Hence $F_{X_n}(x) = F_X(x)$ for all n and all x

$\Rightarrow \quad X_n \longrightarrow X$ in distribution.

However, for any $0 < \epsilon < 1$, $|X_n - X| < \epsilon$ iff

$$-\epsilon < X_n - X < \epsilon$$

$$\Rightarrow \quad -\epsilon < U - (1-U) < \epsilon$$

$$\Rightarrow \quad -\epsilon < 2U - 1 < \epsilon$$

$$\Rightarrow \quad \frac{1-\epsilon}{2} < U < \frac{1+\epsilon}{2}$$

Thus,

$$P(|X_n - X| < \epsilon) = P\left(\frac{1-\epsilon}{2} < U < \frac{1+\epsilon}{2}\right)$$

$$= \frac{1+\epsilon}{2} - \frac{(1-\epsilon)}{2} = \epsilon$$

$$\Rightarrow P(|X_n - X| \geq \epsilon) = 1 - \epsilon \not\longrightarrow 0 \text{ as } n \to \infty$$

$$\Rightarrow \quad X_n \overset{\text{in Prob}}{\not\longrightarrow} X.$$

Convergence in M.S. $\Rightarrow$ Convergence in Probability

$$X_n \xrightarrow{m.s.} X \quad \Rightarrow \quad X_n \xrightarrow{P} X$$

Proof: Assume $X_n \xrightarrow{m.s.} X$, i.e $E(|X_n - X|^2) \to 0$ as $n \to \infty$.

For any $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) = P(|X_n - X|^2 > \epsilon^2)$$

$$\begin{pmatrix}\text{Markov's}\\ \text{Inequality}\end{pmatrix} \leq \frac{E(|X_n - X|^2)}{\epsilon^2}$$

$\longrightarrow 0$ as $n \to \infty$.

Recall:
Markov's Inequality
For a non-negative random variable X, and any $t > 0$

$$P(X \geq t) \leq \frac{E(X)}{t}$$

Eg 1. Suppose the Sample space $\Rightarrow$ $S \Rightarrow$ $\boxed{\omega_1 \quad \omega_2 \quad \omega_3}$

$P(\omega_1) = 1/6$
$P(\omega_2) = 1/6$
$P(\omega_3) = 2/3$

we construct a sequence of r.v.'s as follows:

$$X_n(\omega_1) = 1 + \left(\tfrac{1}{2}\right)^n$$
$$X_n(\omega_2) = 2 + \left(\tfrac{1}{3}\right)^n$$
$$X_n(\omega_3) = 3 + \left(\tfrac{1}{4}\right)^n$$

i.e., we have a sequence of r.v.'s as:

$$X_1 = \begin{cases} 1 + 1/2 & \text{w.p. } 1/6 \\ 2 + 1/3 & \text{w.p. } 1/6 \\ 3 + 1/4 & \text{w.p. } 2/3 \end{cases} \qquad X_2 = \begin{cases} 1 + (1/2)^2 & \text{w.p. } 1/6 \\ 2 + (1/3)^2 & \text{w.p. } 1/6 \\ 3 + (1/4)^2 & \text{w.p. } 2/3 \end{cases} \quad \cdots\cdots$$

we also have a r.v. X:

$$X(\omega_1) = 1 \qquad\qquad \text{ie} \quad X = \begin{cases} 1 & \text{w.p. } 1/6 \\ 2 & \text{,, } 1/6 \\ 3 & \text{,, } 2/3 \end{cases}$$
$$X(\omega_2) = 2$$
$$X(\omega_3) = 3$$

Claim: $X_n \xrightarrow{\text{almost surely}} X$

To prove this claim we must show $\rightarrow$ $P\left(\{\omega : X_n(\omega) \to X(\omega)\}\right) = 1$

$\{\omega : X_n(\omega) \to X(\omega)\}$ = Set of all ω's for which the sequence $X_n(\omega) \to X(\omega)$

$$X_n(\omega_1) = 1 + \left(\tfrac{1}{2}\right)^n \xrightarrow[n \to \infty]{} 1 = X(\omega_1)$$
$$X_n(\omega_2) = 2 + (1/3)^n \xrightarrow[n \to \infty]{} 2 = X(\omega_2)$$
$$X_n(\omega_3) = 3 + (1/4)^n \xrightarrow[n \to \infty]{} 3 = X(\omega_3)$$

$$\Rightarrow P\left(\{\omega : X_n(\omega) \to X(\omega)\}\right) = P\left(\{\omega_1, \omega_2, \omega_3\}\right) = P(\omega_1) + P(\omega_2) + P(\omega_3)$$
$$= \underline{1}$$

$$\Rightarrow \boxed{X_n \xrightarrow{A.S.} X}$$

Q: Does $X_n \xrightarrow[\text{square}]{\text{mean}} X$?

We must find $E\left[\,|X_n - X|^2\right] \xrightarrow{n\to\infty} 0$?

$$E\left[\,|X_n - X|^2\right] = P(w_1)\cdot|X_n(w_1) - X(w_1)|^2 + P(w_2)\,|X_n(w_2) - X(w_2)|^2 + P(w_3)\cdot|X_n(w_3) - X(w_3)|^2$$

$$= \underbrace{\frac{1}{6}\times\left(\frac{1}{2}\right)^{2n}}_{\to 0} + \underbrace{\frac{1}{6}\times\left(\frac{1}{3}\right)^{2n}}_{\to 0} + \underbrace{\frac{2}{3}\times\left(\frac{1}{4}\right)^{n}}_{\to \infty} \longrightarrow 0 \text{ as } n\to\infty$$

$$\Rightarrow \boxed{X_n \xrightarrow[\text{square}]{\text{mean}} X}$$

Eg 2 Let $X_2, X_3, X_4, \dots$ be a sequence of r.v.'s withe the CDF:

$$F_{X_n}(x) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nx} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$P(X_n \le x)$

show that $\boxed{X_n \xrightarrow{\text{Distribution}} \underbrace{\text{Exponential}(1)}}$

$$F_X(x) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

CDF of Exponential

Solution:

For $x \le 0$

$$F_{X_n}(x) = F_X(x) = 0 \quad \forall\, n = 2, 3, 4, \dots$$

For $x \geq 0$,

$$\lim_{n\to\infty} F_{X_n}(x) = \lim_{n\to\infty} 1 - \left(1 - \frac{1}{n}\right)^{nx} = 1 - \lim_{n\to\infty}\left(1 - \frac{1}{n}\right)^{nx}$$

$$= 1 - e^{-x}$$

$$= F_X(x)$$

$$\Rightarrow \boxed{X_n \xrightarrow{\text{Dist.}} X}$$

$\boxed{\text{Eg 3:}}$ Let $X_n \sim \text{Exponential}(\underset{\llcorner \to \text{parameter}}{n})$

Show that $X_n \xrightarrow{\text{in Probability}} \underset{\sim}{0}$ $\hookrightarrow X = 0$, i.e a constant r.v. which takes value $= 0$.

For $X_n \xrightarrow{\text{Prob.}} 0$; we must show:

$$\lim_{n\to\infty} P(|X_n - 0| \geq \epsilon) = 0$$

$$\lim_{n\to\infty} P(|X_n - 0| \geq \epsilon) = \lim_{n\to\infty} P(|X_n| \geq \epsilon)$$

$$= \lim_{n\to\infty} P(X_n \geq \epsilon)$$

since X_n is exponenti. ($X_n \geq 0$).

$$= \lim_{n\to\infty} \left(1 - P(X_n < \epsilon)\right)$$

$$= \lim_{n\to\infty} \left(1 - (1 - e^{-n\epsilon})\right)$$

$$= \lim_{n\to\infty} e^{-n\epsilon} = 0 \quad \text{for all } \epsilon > 0.$$

$$\Rightarrow X_n \xrightarrow{\text{in Prob.}} 0$$

Q: Does $\boxed{X_n \xrightarrow[\text{Square}]{\text{Mean}} 0 \ ?}$

$$\Rightarrow E[|X_n - 0|^2] = E[|X_n|^2] = E[X_n^2] = Var(X_n) + (E[X_n])^2$$

$$= \left(\frac{1}{\lambda^2}\right) + \left(\frac{1}{\lambda}\right)^2$$

$$= \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}$$

$$\longrightarrow 0 \text{ as } n \to \infty.$$

$\Rightarrow$ Yes! $X_n \xrightarrow{\text{m.s.}} 0$