

## Homework 6 - ECE 503 Fall 2017

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- Assigned on: Tuesday, October 24, 2017.
  - Due Date: **Monday, October 30, 2017 by 11:00 am Tucson Time.**
  - Maximum Credit: **120 points**
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1. **[20 points]** Let  $U \sim \text{uniform}[0, 1]$  denote a uniformly distributed random variable. We define a sequence of random variables as

$$X_n = \begin{cases} n^\alpha, & 0 \leq U \leq \frac{1}{n}, \\ 0, & \text{otherwise} \end{cases}$$

For what values of  $\alpha$  does  $X_n$

- (a) converge to 0 in the mean square sense ?
  - (b) converge to 0 in probability ?
2. **[10 points]** Let  $X_n$  be a Laplacian random variable with zero mean and variance  $2/n^2$  (i.e., scale parameter  $1/n$ ). Show that  $X_n$  converges to 0 almost surely.
3. **[10 points]** Let  $X_n$  be a Rayleigh random variable with parameter  $1/n$ . Show that  $X_n$  converges to 0 almost surely.
4. **[20 points]** Let  $f(x)$  be a probability density function. Let  $X_n$  have the density

$$f_n(x) = nf(nx)$$

Determine whether or not  $X_n$  converges in distribution to zero.

5. **[20 points]**  $X_1, X_2, \dots, X_n$  are i.i.d. uniform random variables, all with expected value  $\mu_X = 7$ , and variance  $\text{Var}(X) = 3$ .
- (a) What is the PDF of  $X_1$  ?
  - (b) What is  $\text{Var}(M_{16}(X))$ , i.e., the variance of the sample mean based on 16 trials ?
  - (c) What is  $P(X_1 > 9)$ , i.e., the probability that one outcome exceeds 9 ?
  - (d) Do you expect  $P(M_{16}(X) > 9)$  to be bigger or smaller than  $P(X_1 > 9)$  ? To check your intuition, use the Central Limit Theorem (CLT) to estimate  $P(M_{16}(X) > 9)$ .
6. **[20 points]** For an arbitrary random variable  $X$ , use the Chebyshev inequality to show that the probability that  $X$  is more than  $k$  standard deviations from its expected value  $E(X)$  satisfies

$$P(|X - E[X]| \geq k\sigma) \leq \frac{1}{k^2}$$

For a Gaussian random variable  $Y$ , use the  $\Phi(\cdot)$  function to calculate the probability that  $Y$  is more than  $k$  standard deviations from its expected value  $E[Y]$ . Compare and comment upon your results to the upper bound obtained via Chebyshev inequality for  $k = 1, \dots, 5$ .

7. [20 points] In this problem, we develop a weak law of large numbers for a *correlated sequence*  $X_1, X_2, \dots$  of identically distributed random variables. Each random variable  $X_i$  has expected value  $E[X_i] = \mu$ , and the random sequence has the covariance function given as:

$$\text{Cov}[X_i, X_j] = \sigma^2 a^{|i-j|}$$

where  $a$  is a constant such that  $|a| < 1$ . For this correlated random sequence, we define the sample mean as:

$$M(X_1, X_2, \dots, X_n) = \frac{X_1 + X_2 + \dots + X_n}{n}$$

Show that  $M(X_1, X_2, \dots, X_n)$  converges to  $\mu$  in probability.