HW8 Solution ECE 503 Fall 2017

Problem 1

(a) Since $\sin\tau$ is odd, it is NOT a valid correlation function.

(b) Since the Fourier transform of $\cos\tau$ is $[\delta(f-1)+\delta(f+1)]/2$, which is real, even, and nonnegative, $\cos\tau$ IS a valid correlation function.

(c) Since the Fourier transform of $e^{-\tau^2/2}$ is $\sqrt{2\pi}\,e^{-(2\pi f)^2/2}$, which is real, even, and nonnegative, $e^{-\tau^2/2}$ IS a valid correlation function.

(d) Since the Fourier transform of $e^{-|\tau|}$ is $2/[1+(2\pi f)^2]$, which is real, even, and nonnegative, $e^{-|\tau|}$ IS a valid correlation function.

(e) Since the value of $\tau^2 e^{-|\tau|}$ at $\tau = 0$ is less than the value for other values of τ, $\tau^2 e^{-|\tau|}$ is NOT a valid correlation function.

(f) Since the Fourier transform of $I_{[-T,T]}(\tau)$ is $(2T)\sin(2\pi T f)/(2\pi T f)$ is not nonnegative, $I_{[-T,T]}(\tau)$ is NOT a valid correlation function.

Problem 2

First consider the mean function,

$$\mathsf{E}[q(t+T)] = \frac{1}{T_0}\int_0^{T_0} q(t+\theta)\,d\theta = \frac{1}{T_0}\int_t^{t+T_0} q(\tau)\,d\tau = \frac{1}{T_0}\int_0^{T_0} q(\tau)\,d\tau,$$

where we have used the fact that since q has period T_0, the integral of q over any interval of length T_0 yields the same result. The second thing to consider is the correlation function. Write

$$\begin{aligned}
\mathsf{E}[q(t_1+T)q(t_2+T)] &= \frac{1}{T_0}\int_0^{T_0} q(t_1+\theta)q(t_2+\theta)\,d\theta \\
&= \frac{1}{T_0}\int_{t_2}^{t_2+T_0} q(t_1+\tau-t_2)q(\tau)\,d\tau \\
&= \frac{1}{T_0}\int_0^{T_0} q([t_1-t_2]+\tau)q(\tau)\,d\tau,
\end{aligned}$$

where we have used the fact that as a function of τ, the product $q([t_1-t_2]+\tau)q(\tau)$ has period T_0. Since the mean function does not depend on t, and since the correlation function depends on t_1 and t_2 only through their difference, X_t is WSS.

Problem 3

(a) $S_X(f) = \sqrt{2\pi}\, e^{-(2\pi f)^2/2}$.

(b) $S_X(f) = \pi e^{-2\pi|f|}$.

Problem 4

Since $R_X(\tau) = 1/(1+\tau^2)$, we have from the transform table that $S_X(f) = \pi e^{-2\pi|f|}$. Similarly, since $h(t) = 3\sin(\pi t)/(\pi t)$, we have from the transform table that $H(f) = 3I_{[-1/2,1/2]}(f)$. We can now write

$$S_Y(f) \;=\; |H(f)|^2 S_X(f) \;=\; 9I_{[-1/2,1/2]}(f)\cdot \pi e^{-2\pi|f|} \;=\; 9\pi e^{-2\pi|f|} I_{[-1/2,1/2]}(f).$$

Problem 5

Let $S_0(f)$ denote the Fourier transform of $R_0(\tau)$, and let $S(f)$ denote the Fourier transform of $R(\tau)$.

(a) The derivation in the text showing that the transform of a correlation function is real and even uses only the fact that correlation functions are real and even. Hence, $S_0(f)$ is real and even. Furthermore, since R is the convolution of R_0 with itself, $S(f) = S_0(f)^2$, which is real, even, and nonnegative. Hence, $R(\tau)$ is a correlation function.

(b) If $R_0(\tau) = I_{[-T,T]}(\tau)$, then

$$S_0(f) \;=\; 2T\frac{\sin(2\pi Tf)}{2\pi Tf} \quad \text{and} \quad S(f) \;=\; 2T\cdot 2T\left[\frac{\sin(2\pi Tf)}{2\pi Tf}\right]^2.$$

Hence, $R(\tau) = 2T\cdot\big(1-|\tau|/(2T)\big)I_{[-2T,2T]}(\tau)$.

Problem 6

First note that since $R_X(\tau) = e^{-\tau^2/2}$, $S_X(f) = \sqrt{2\pi}e^{-(2\pi f)^2/2}$.

(a) $S_{XY}(f) \;=\; H(f)^* S_X(f) \;=\; \left[e^{-(2\pi f)^2/2}\right]^* \sqrt{2\pi}e^{-(2\pi f)^2/2} \;=\; \sqrt{2\pi}\,e^{-(2\pi f)^2}$.

(b) Writing

$$S_{XY}(f) \;=\; \frac{1}{\sqrt{2}}\sqrt{2\pi}\sqrt{2}e^{-(\sqrt{2})^2(2\pi f)^2/2},$$

we have from the transform table that

$$R_{XY}(\tau) \;=\; \frac{1}{\sqrt{2}}e^{-(\tau/\sqrt{2})^2/2} \;=\; \frac{1}{\sqrt{2}}e^{-\tau^2/4}.$$

(c) Write

$$\mathsf{E}[X_{t_1}Y_{t_2}] \;=\; R_{XY}(t_1 - t_2) \;=\; \frac{1}{\sqrt{2}}e^{-(t_1-t_2)^2/4}.$$

(d) $S_Y(f) \;=\; |H(f)|^2 S_X(f) \;=\; e^{-(2\pi f)^2} \cdot \sqrt{2\pi}e^{-(2\pi f)^2/2} \;=\; \sqrt{2\pi}e^{-3(2\pi f)^2/2}$.

(e) Writing

$$S_Y(f) \;=\; \frac{1}{\sqrt{3}}\sqrt{2\pi}\sqrt{3}e^{-(\sqrt{3})^2(2\pi f)^2/2},$$

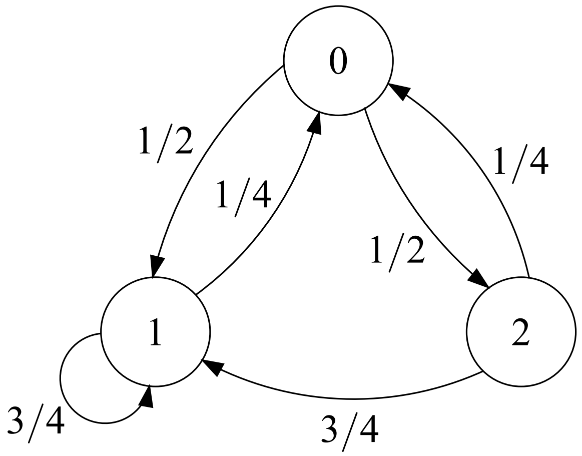
we have from the transform table that

$$R_Y(\tau) \;=\; \frac{1}{\sqrt{3}}e^{-(\tau/\sqrt{3})^2/2} \;=\; \frac{1}{\sqrt{3}}e^{-\tau^2/6}.$$

Problem 7

(a)

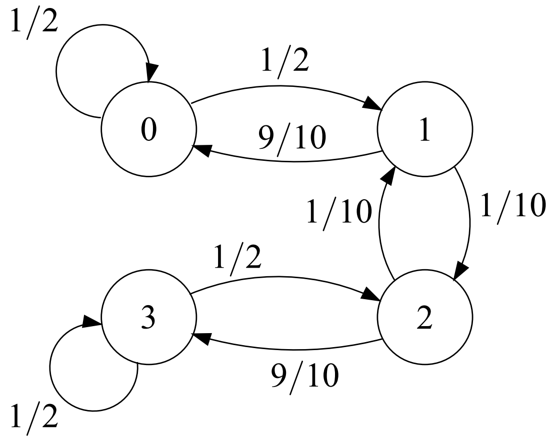
The state transition diagram is



The stationary distribution is $\pi_0 = 1/5$, $\pi_1 = 7/10$, $\pi_2 = 1/10$.

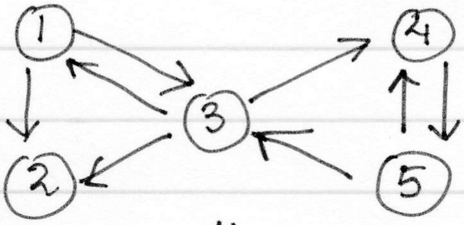
(b)

The state transition diagram is

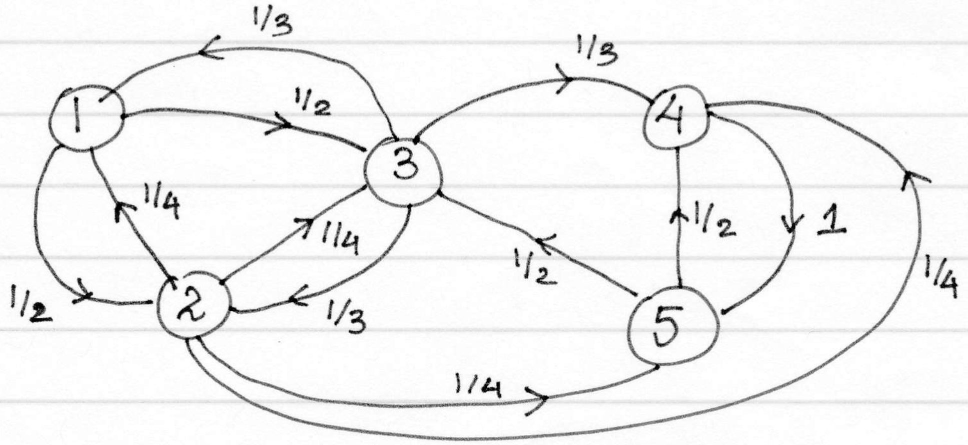


The stationary distribution is $\pi_0 = 9/28$, $\pi_1 = 5/28$, $\pi_2 = 5/28$, $\pi_3 = 9/28$.

8 "Web-page" problem.



$\Downarrow$ State transition diagram.



$$\Rightarrow \quad P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/4 & 0 & 1/4 & 1/4 & 1/4 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

(b) The MC is irreducible;

$$C = \{1, 2, 3, 4, 5\}$$

$\uparrow$ single communicating class.

The MC is aperiodic

Period of State 1 :

$1 \longrightarrow 3 \longrightarrow 1$ 2-steps

$1 \rightarrow 3 \rightarrow 2 \rightarrow 1$ 3-Steps

$$GCD(2, 3) = 1 \Rightarrow \text{Period} = 1$$

$\Downarrow$

Aperiodic

$\Rightarrow$ M.C has a unique limiting state probability distribution, which is independent of initial probability.

$$\pi = \pi P$$

Solving $\pi = \pi P$

$$\pi_1 = \frac{\pi_2}{4} + \frac{\pi_3}{3} \quad \text{—— (1)}$$

$$\pi_2 = \frac{\pi_1}{2} + \frac{\pi_3}{3} \quad \text{—— (2)}$$

$$\pi_3 = \frac{\pi_1}{2} + \frac{\pi_2}{4} + \frac{\pi_5}{2} \quad \text{—— (3)}$$

$$\pi_4 = \frac{\pi_2}{4} + \frac{\pi_3}{3} + \frac{\pi_5}{2} \quad \text{—— (4)}$$

$$\pi_5 = \frac{\pi_2}{4} + \pi_4 \quad \text{—— (5)}$$

From (1) and (2), $\pi_1 - \pi_2 = \frac{\pi_2}{4} - \frac{\pi_1}{2}$

$$\Rightarrow \frac{3}{2}\pi_1 = \frac{5}{4}\pi_2 \Rightarrow \pi_2 = \frac{12}{10}\pi_1$$

$$\Rightarrow \boxed{\pi_2 = \frac{6}{5}\pi_1}$$

From (1),

$$\pi_3 = 3\pi_1 - \frac{3}{4}\pi_2 = 3\left(1 - \frac{1}{4}\times\frac{6}{5}\right)\pi_1$$

$$= 3\left(1 - \frac{3}{10}\right)\pi_1 = \frac{21}{10}\pi_1 \Rightarrow \boxed{\pi_3 = \frac{21}{10}\pi_1}$$

From (3),

$$\pi_5 = 2\pi_3 - \pi_1 - \frac{\pi_2}{2}$$

$$= \frac{(2\times 21 - 10 - 6)}{10}\pi_1$$

$$\pi_5 = \frac{(42-16)}{10}\pi_1 = \frac{26}{10}\pi_1$$

$$\Rightarrow \boxed{\pi_5 = \frac{26}{10}\pi_1}$$

and $\Pi_4 = \Pi_5 - \frac{\Pi_2}{4} = \frac{26}{10}\Pi_1 - \frac{1}{\cancel{4}_2} \times \frac{\cancel{6}^3}{5}\Pi_1$

$= \left(\frac{26}{10} - \frac{3}{10}\right)\Pi_1$

$= \frac{23}{10}\Pi_1$

$\Rightarrow \quad \Pi_2 = \frac{12}{10}\Pi_1$

$\Pi_3 = \frac{21}{10}\Pi_1$

$\Pi_4 = \frac{23}{10}\Pi_1$

$\Pi_5 = \frac{26}{10}\Pi_1$

and also $\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 = 1$

$\Rightarrow \Pi_1\left(\frac{10}{10} + \frac{12}{10} + \frac{21}{10} + \frac{23}{10} + \frac{26}{10}\right) = 1$

$\Rightarrow \Pi_1 = \frac{10}{92}$

$\Pi_1 = 10/92$

$\Pi_2 = \frac{12}{10} \times \frac{10}{92} = \frac{12}{92}$

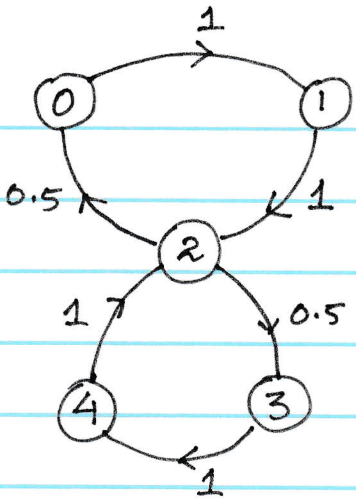
$\Pi_3 = \frac{21}{92}$

$\Pi_4 = \frac{23}{92}$

$\Pi_5 = \frac{26}{92}$

$\Rightarrow \underbrace{\Pi_5}_{\text{Most important}} > \Pi_4 > \Pi_3 > \Pi_2 > \underbrace{\Pi_1}_{\text{Least Important}}$

9 (i)



$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

This is irreducible MC. (paths from $i \to j$ and $j \to i$) & a single communicating class.

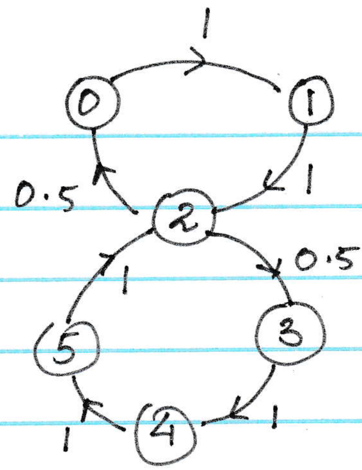
Period of state 0:

$$\left.\begin{array}{ll} 0 \to 1 \to 2 \to 0 & \text{3-steps} \\ 0 \to 1 \to 2 \to 3 \to 4 \to 2 \to 0 & \text{6-steps} \end{array}\right\} \text{GCD} = 3$$

$\Rightarrow$ ~~Aperiodic~~ $\boxed{\text{PERIODIC}}$

$$\pi = \pi P \Rightarrow \underset{\text{(unique)}}{\pi} = \begin{bmatrix} 1/6 & 1/6 & 1/3 & 1/6 & 1/6 \end{bmatrix}$$

<u>Note</u>: This M.C converges to above π as $n \to \infty$.

(ii) 

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

MC is irreducible.

<u>Period of 0:</u> $0 \to 1 \to 2 \to 0$ 3-steps

$0 \to 1 \to 2 \to 3 \to 4 \to 5 \to 2 \to 0$ 7-Steps

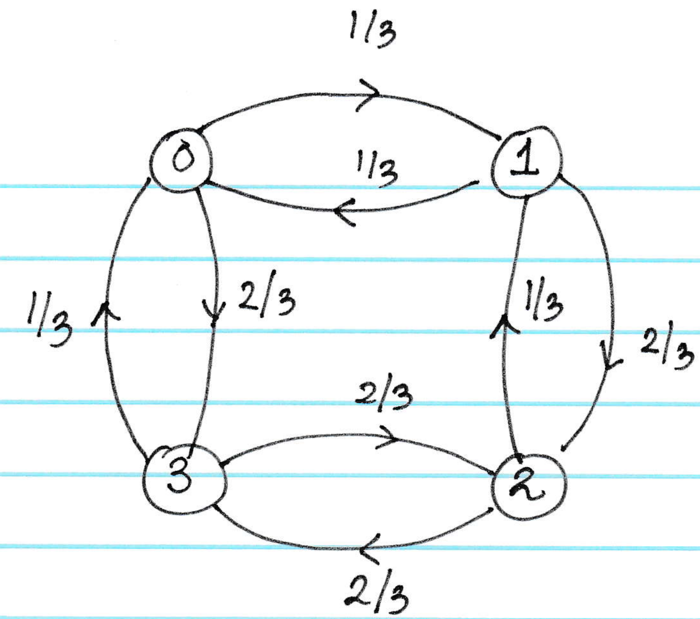
$GCD(3,7) = 1$

$\Rightarrow$ Aperiodic

Solving $\pi = \pi P \Rightarrow \pi = \left[\frac{1}{7} \;\; \frac{1}{7} \;\; \frac{2}{7} \;\; \frac{1}{7} \;\; \frac{1}{7} \;\; \frac{1}{7}\right]$

$\lim_{n \to \infty} p(n) = \pi$ & does not depend on $p(0)$.

(iii)



$$P = \begin{bmatrix} 0 & 1/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/3 & 0 & 2/3 & 0 \end{bmatrix}$$

MC is $\boxed{\text{irreducible.}}$

Period of State 0:

$$\left.\begin{array}{ll} 0 \to 1 \to 0 & \text{2-steps} \\ 0 \to 1 \to 2 \to 1 \to 0 & \text{4-steps} \\ \vdots & \end{array}\right\} GCD = 2$$

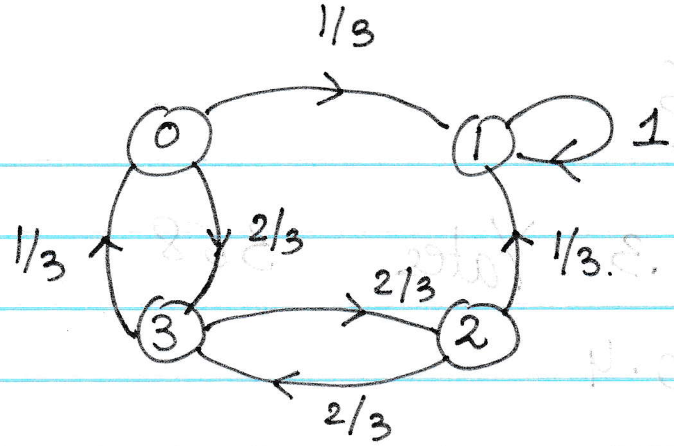
(all other paths are multiples of 2)

$\boxed{\text{Periodic}}$

$$\pi = \pi P \Rightarrow \pi = \begin{bmatrix} 1/6 & 1/6 & 1/3 & 1/3 \end{bmatrix}$$

(unique)

(iv).



$$P = \begin{bmatrix} 0 & 1/3 & 0 & 2/3 \\ 0 & 1 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 \\ 1/3 & 0 & 2/3 & 0 \end{bmatrix}$$

<u>MC is reducible</u>

$$\underbrace{C_1 = \{1\} \;;\quad C_2 = \{0, 2, 3\}}_{\text{Two communicating classes.}}$$

Solving $\Pi = \Pi P$ has a unique solution

$$\Pi = [\underbrace{0}_{\pi_0}\;\; \underbrace{1}_{\pi_1}\;\; \underbrace{0}_{\pi_2}\;\; \underbrace{0}_{\pi_3}].$$