Today

1) Examples of Conditional Expectation (Notes for Lec. 17)

2) Conditional Expectation & MMSE Estimation

HW5 Problem #3

$Z = XY$ find PDF of Z.

CDF of Z: $F_Z(z) = P(Z \le z)$

$= P(XY \le z)$ product.

$X, Y \sim$ exponential. $= \iint f_{X,Y}(x,y)\,dx\,dy$.

X, Y are indep.

(over $(x,y): xy \le z$)

$$F_Z(z) = \int_{y=0}^{\infty} \int_{x=0}^{z/y} f_X(x) f_Y(y)\,dx\,dy.$$

PDF of Z: $\frac{d}{dz} F_Z(z)$ ---> apply Leibniz rule...

Eg X, Y $f_{X,Y}(x,y) = \begin{cases} 6xy & 0 \le x \le 1, \; 0 \le y \le \sqrt{x} \\ 0 & \text{otherwise.} \end{cases}$

(i) Are X & Y independent r.v.'s?

(ii) Find $E[X|Y]$

(iii) Find $Var[X|Y]$.

(i) $f_X(x) = \int_{y=0}^{\sqrt{x}} 6xy\,dy = 3x^2$.

$y \le \sqrt{x} \le 1$

$f_Y(y) = \int_{x=y^2}^{1} 6xy\,dx = 3y(1-y^4)$. $y^2 \le x \le 1$.

$$f_{X,Y}(x,y) \stackrel{?}{=} f_X(x) f_Y(y).$$

$$6xy \neq (3x^2)(3y(1-y^4)).$$

<u>NOT</u> <u>independent</u>

<u>Are they uncorrelated?</u> $\qquad \rho_{XY} \stackrel{?}{=} 0$

$$\underbrace{E[XY]}_{\iint xy f_{XY}(x,y)} \stackrel{?}{=} \underbrace{E[X]}_{(\ \ )} \times \underbrace{E[Y]}_{(\ \ )}.$$

(ii) $E[X|Y] \rightsquigarrow E[X|Y=y]$

$$= \int_{y^2}^{1} x\, \underbrace{f_{X|Y}(x|y)}\, dx$$

$$\frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{2x}{1-y^4}.$$

$$E[X|Y=y] = \int_{y^2}^{1} \frac{x \times 2x}{(1-y^4)}\, dx$$

$$= \frac{2}{3}\frac{(1-y^6)}{(1-y^4)}$$

$y=3$

$y=4$

$$E[X|Y] \stackrel{?}{=} \frac{2}{3}\frac{(1-Y^6)}{(1-Y^4)} = \boxed{g(Y)}.$$

RV

This is a R.V.

& this is a function of Y.

$Y=y: \ g(Y)=g(y).$

$$\mathrm{Var}(X|Y=y) = \underbrace{E[X^2|Y=y]} - (E[X|Y=y])^2$$

$$= \int x^2 f_{X|Y}(x|y)\,dx. - (E[X|Y=y])^2$$

$$\mathrm{Var}(X \mid Y=y) = \frac{1-y^8}{2(1-y^4)} - \left(\frac{2}{3}\,\frac{(1-y^6)}{(1-y^4)}\right)^2.$$

$$\underline{\mathrm{Var}(X \mid Y)} = \frac{1-Y^8}{2(1-Y^4)} - \left(\frac{2}{3}\,\frac{(1-Y^6)}{(1-Y^4)}\right)^2.$$

also a random variable & fun. of $\underline{Y}$.

Iterated Expectation Theorem

$$E\big[\,E[X \mid Y]\,\big] = E[X].$$

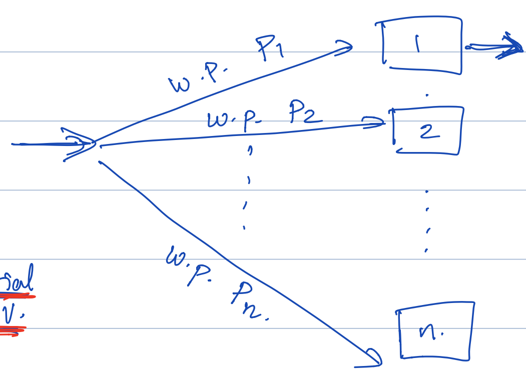
R.v. & fun. of Y.

$$E[g(Y)] = E[X].$$

(Y)

Eg.

user/customer



Time for service by i^{th} teller is exponential r.v. with parameter λ_i.

(1) What is the PDF of T, the time it takes to serve the user?

(2) $E[T]$, $\mathrm{Var}[T]$

$X \to$ teller that is picked by the user

$$X = \begin{cases} 1 & \text{w.p. } P_1 \\ 2 & \text{w.p. } P_2 \\ 3 & \vdots \\ \vdots & \\ n & \text{w.p. } P_n \end{cases}$$

$(P_1 + P_2 \dots + P_n = 1)$ $\to$ must be a valid pmf.

CDF, PDF of T

Total Prob. Theorem

$$\underbrace{P(T \le t)}_{F_T(t)} = P(X=1) \times P(T \le t \mid X=1) + P(X=2) \times P(T \le t \mid X=2) + \dots + P(X=n) \times P(T \le t \mid X=n).$$

???

$$F_T(t) = P_1 \times \underbrace{P(T \le t \mid X=1)}_{\text{1st teller is picked.}} + \dots + P_n \times P(T \le t \mid X=n)$$

How to write this? integral?

$$F_T(t) = P_1 \times \boxed{\text{CDF of } \exp(\lambda_1)} + \dots + (P_n) \times (\text{CDF of } \exp(\lambda_n))$$

$$f_T(t) = \frac{d}{dt} F_T(t) = P_1 \times f_{T|X}(t \mid x=1) + \dots + P_n \times f_{T|X}(t \mid x=n)$$

$$P(\underbrace{T}_{\text{CDF of}} \le t \mid \underbrace{X=1}_{\text{1st teller is sele.}})$$

User $\to$ Teller 1 $\to$ Time $\sim \exp(\lambda_1)$

$$f_T(t) = P_1 \times (\lambda_1 e^{-\lambda_1 t}) + P_2 (\lambda_2 e^{-\lambda_2 t}) + \dots + P_n (\lambda_n e^{-\lambda_n t}).$$

(ii) $$E[T] = \int_{t=0}^{\infty} t \, f_T(t) \, dt.$$

$$= E\left[\underbrace{E[T \mid X]}_{\text{function of } \boxed{X}} \right]$$

$$= \underline{E[\, g(X) \,]}.$$

$$X = \begin{cases} 1 & P_1 \\ 2 & P_2 \\ \vdots & \vdots \\ n & P_n. \end{cases}$$

$$= \sum_{i=1}^{n} P(X=i) \cdot g(X=i) \quad \text{(LOTUS)}$$

$$= \sum P_i \, g(X=i)$$

$$= \sum P_i \left(\underline{E[T \mid X=i]} \right).$$

$$T \mid X=i \sim \text{exponential}(\lambda_i)$$

$$E[T \mid X=i] = \frac{1}{\lambda_i}$$

$$\underline{E[T]} = \sum_{i=1}^{n} P_i \times \frac{1}{\lambda_i} = \sum_{i=1}^{n} \frac{P_i}{\lambda_i}$$

$$E[T^2] = E\left[E[T^2 \mid X] \right].$$

$$= \sum_{i=1}^{n} P_i \, \underbrace{E[T^2 \mid X=i]}_{\text{can readily compute.}}.$$

$T \mid X=i \sim \exp(\lambda_i).$

$$\int_{t=0}^{\infty} t^2 \times (\lambda_i e^{-\lambda_i t})\, dt = \frac{2}{\lambda_i^2}.$$

$$= \sum_{i=1}^{n} P_i \times \frac{2}{\lambda_i^2}$$

Minimum Mean Squared Error (MMSE) Estimation

Y random variable.

User

* do not observe anything.
* Know the PDF & CDF of Y.
* Estimate Y

Suppose user estimates Y by a constant value c.

Error $= (Y - c)$ (can be +ve or −ve)

Squared Error $= (Y-c)^2 \longrightarrow$ function of Y.

Mean Squared Error (MSE) $= E[(Y-c)^2].$

MMSE Estimator: find a const c.

s.t. MSE is minimized.

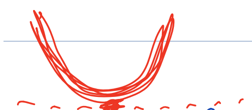
$$c^* \overset{?}{=} E[Y]$$

↑ optimal c.

$$MSE = E[(Y-c)^2] = \int_{y=-\infty}^{\infty} (y-c)^2 f_Y(y)\,dy$$

$$E[g(Y)] = \int g(y) f_Y(y)\,dy \quad \text{(LOTUS)}$$

Find c to minimize $MSE = \int_{y=-\infty}^{\infty} (y-c)^2 f_Y(y)\,dy$



convex in c.

$$\frac{d(MSE)}{dc} = 0 = \int \left(\frac{d(y-c)^2}{dc}\right) \times f_Y(y)\,dy$$

$$0 = \int -2(y-c) f_Y(y)\,dy$$

$$\Rightarrow \int (y-c) f_Y(y)\,dy = 0$$

$$\Rightarrow \underbrace{\int y f_Y(y)\,dy}_{E[Y]} - c \underbrace{\int_{y=-\infty}^{\infty} f_Y(y)\,dy}_{1} = 0$$

PDF of Y

optimal

$$\Rightarrow E[Y] - c = 0 \Rightarrow \boxed{c^* = E[Y]}$$

Plug back c^*. $\longrightarrow$ MMSE (when $c = c^*$) $= E[(Y - c^*)^2]$

$$= E\left[\underbrace{\left(Y - E[Y]\right)^2}\right].$$

$$= Var(Y).$$