Lecture 8

* Gaussian (Normal) R.V.

$$\boxed{\text{PDF} \qquad f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

CDF of Gaussian:

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_X(x)\,dx$$

CDF is the integral of PDF

CDF of Gaussian:

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

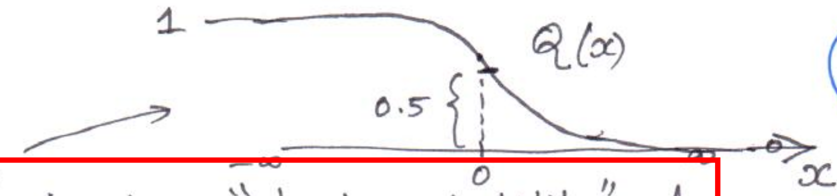
No closed form expression.

Recall: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{x} f_X(x)\,dx + \int_{x}^{\infty} f_X(x)\,dx$

$$1 = \int_{-\infty}^{x} f_X(x)\,dx + \int_{x}^{\infty} f_X(x)\,dx$$

We often numerically use the Q-function, which is defined as

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2}\,du$$



Q-function is the "tail probability" of Standard Normal $\mathcal{N}(0,1)$.

For a $\mathcal{N}(\mu, \sigma^2)$, we can evaluate its CDF in terms of the $Q(\cdot)$ function

$$F_X(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx.$$

Change of variable: $y = \frac{(x-\mu)}{\sigma} \Rightarrow dx = \sigma\, dy$

$$F_X(a) = \int_{-\infty}^{(a-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= \int_{-\infty}^{(a-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy + \int_{(a-\mu)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$- \int_{(a-\mu)/\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

$$= 1 - \int_{\frac{(a-\mu)}{\sigma}}^{\infty} e^{-y^2/2} dy$$

$$\boxed{F_X(a) = 1 - Q\left(\frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right)}$$

$\Phi(x) = \text{CDF of } \underbrace{\mathcal{N}(0,1)}_{\text{Standard Normal}}$.

* Exponential R.V. (Parameter $= \lambda$)

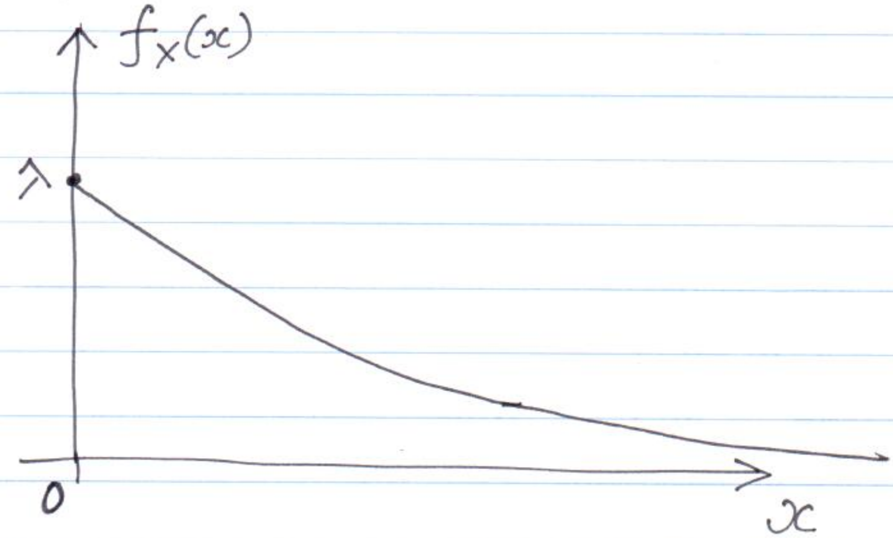
$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

(PDF)

$$\text{CDF} = F_X(a) = \int_0^a \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^a$$

$$\boxed{F_X(a) = 1 - e^{-\lambda a}}$$

If occurances of events over non-overlapping intervals are independent, (eg. arrival times of telephone calls, arrival times of a customer at McD's ...), then the waiting time distribution of these events can be modeled by Exponential.



Memoryless Property of Exponential

$$P(X > t+s \mid X > t) \overset{?}{=} P(X > s)$$

this property is true for Exponential

Probability of waiting for $(t+s)$ given that we have already waited for t ?

$$P(X > t+s \mid X > t) = \frac{P(X > t+s \cap X > t)}{P(X > t)}$$

$$= \frac{P(X > t+s)}{P(X > t)}$$

$$= \frac{1 - P(X \le (t+s))}{1 - P(X \le t)}$$

$$= \frac{1 - (1 - e^{-\lambda(t+s)})}{1 - (1 - e^{-\lambda t})}$$

$$= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s)$$

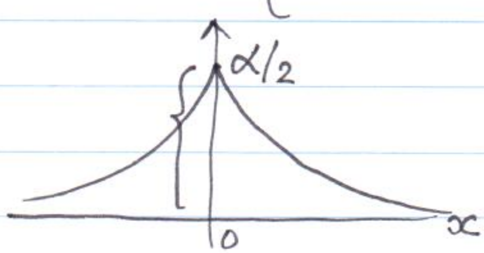
* <u>Rayleigh R.V.</u> (Parameter = σ)

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-x^2/2\sigma^2}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Rayleigh is useful in communication systems, to model the <u>amplitude</u> of a randomly received signal.

* <u>Laplace R.V.</u> (Parameter = α)

$$f_X(x) = \begin{cases} \dfrac{\alpha}{2} e^{-\alpha|x|} & |x| < \infty \end{cases}$$



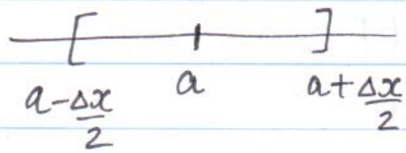
Laplace distrib. is useful in modeling speech signals. Recent uses in Privacy preserving data analysis (Google → <u>Differential Privacy!</u>)

Recovering PDF from CDF

* Recall for a discrete r.v.

$$P_X(a) = F_X(x^+) - F_X(x^-)$$

* For a continuous valued r.v., consider a small interval $\left[a - \frac{\Delta x}{2},\ a + \frac{\Delta x}{2}\right]$



$$F_X\left(a + \frac{\Delta x}{2}\right) - F_X\left(a - \frac{\Delta x}{2}\right) = \int_{-\infty}^{a+\Delta x/2} f_X(t)\,dt - \int_{-\infty}^{a-\Delta x/2} f_X(t)\,dt$$

$$= \int_{a-\Delta x/2}^{a+\Delta x/2} f_X(t)\,dt$$

For Δx small, $f_X(t) \approx \overbrace{\text{Constant}}^{f_X(a)}$ in the range

$$\approx f_X(a) \cdot \int_{a-\Delta x/2}^{a+\Delta x/2} dt$$

$$= f_X(a)\,\Delta x$$

$$\Rightarrow \quad f_X(a) \approx \frac{F_X(a+\Delta x/2) - F_X(a-\Delta x/2)}{\Delta x}$$

$$\rightarrow \quad \left.\frac{d F_X(x)}{dx}\right|_{x=a} \quad \text{as } \Delta x \to 0$$

Hence, we can obtain PDF from CDF via differentiation, i.e.

$$\boxed{f_X(x) = \frac{d F_X(x)}{dx}}$$

Mixed Random Variables

* which can have non-zero probability at some points but not others
* Eg

$$X = \begin{cases} \mathcal{N}(0,1) & \text{if Heads} \\ 0 & \text{if Tails} \end{cases}$$

To find the CDF of such mixed r.v.'s, we can use the Law of Total Probability

$$F_X(x) = P(X \le x)$$

$$= P[\text{heads}] \cdot P[X \le x \mid \text{heads}] + P[X \le x \mid \text{tail}] \cdot P[\text{tail}]$$

<u>Case 1</u>: if $x \geq 0$

$$F_X(x) = \frac{1}{2} \times \underbrace{P[X \le x \mid \text{heads}]}_{\Downarrow \\ \text{CDF of } N(0,1)} + \frac{1}{2} \times \underbrace{P[X \le x \mid \text{tail}]}_{=1}$$

Since $X=0$ if outcome is Tail...

$$= \frac{1}{2} \times \Phi(x) + \frac{1}{2} \times 1$$

$$= (1 + \Phi(x))/2$$

<u>Case 2</u>: if $x < 0$

$$F_X(x) = \frac{1}{2} \times \Phi(x) + \frac{1}{2} \times \underbrace{P(X \le x \mid \text{tail})}_{=0}$$

Since $X=0$ if outcome is tail.

$$= \Phi(x)/2$$

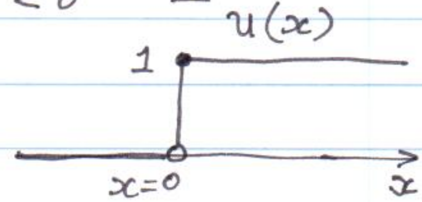
$$\Rightarrow \quad F_X(x) = \begin{cases} (1+\Phi(x))/2 & \text{if } x \geq 0 \\ \Phi(x)/2 & \text{if } x < 0 \end{cases}$$

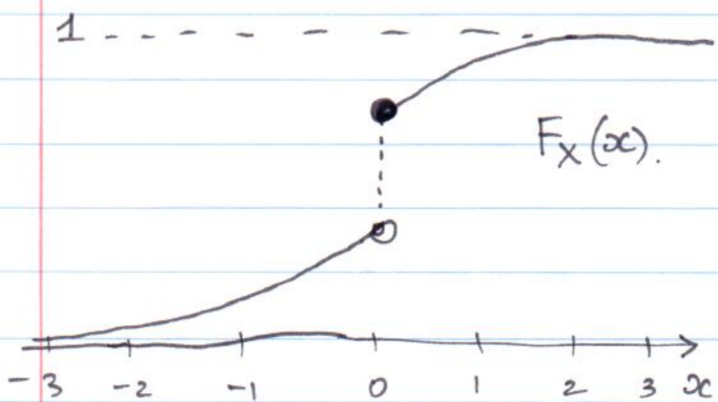
we can write this as

$$F_X(x) = \frac{\Phi(x)}{2} + \frac{u(x)}{2} \quad -\infty < x < -\infty$$

where $u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$

UNIT-STEP Function.





For mixed r.v.'s, CDF is in general continuous except for a countable number of jumps. (It is always right-continuous).

PDF from CDF ??

$$f_X(x) = \frac{d}{dx}\left[\frac{\phi(x)}{2} + \frac{u(x)}{2}\right]$$

$u(x)$ is discontinuous at $x = 0$ & thus formally its derivative does not exist there.

~~anw~~

We can, however, define a derivative for conceptualization & probability calculations

$$\underbrace{\delta(x)}_{\text{Dirac delta function}} \triangleq \frac{d\,u(x)}{dx}, \quad \text{where } u(x) \text{ is unit step function.}$$

* narrow pulse with large amplitude centered at $x=0$
* $\delta(x)=0$ for all $x \neq 0$, however $\int_{-\epsilon}^{\epsilon} \delta(t)\,dt = 1$ for ϵ a small positive number.

$$\Rightarrow \quad f_X(x) = \frac{d}{dx}\left[\frac{1}{2}\Phi(x) + \frac{1}{2}u(x)\right]$$

$$\boxed{f_X(x) = \frac{1}{2}\cdot\frac{1}{\sqrt{2\pi}}e^{-x^2/2} + \frac{1}{2}\delta(x)}$$

(Eg For discrete r.v.'s, we can also write the PMF in terms of Dirac Delta Functions)