Final Exam - ECE 503 Fall 2020

- Due Date & Time: Monday, December 14, 2020 by Noon.
- Upload your scanned exam on D2L (Final Exam Folder).
- Maximum Credit: 100 points

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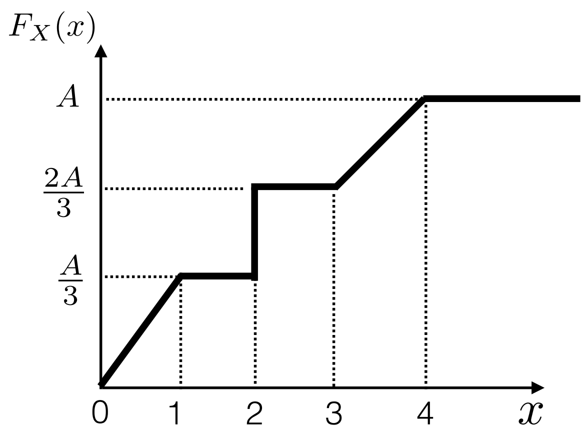
1. [10 points] A random variable X has the following CDF, $F_X(x)$ as shown in the figure below:



   (a) Find the value A.
   (b) Sketch the probability density function (PDF) of this random variable.
   (c) Find the mean and variance of X.
   (d) What is $P(X < 2)$?

2. [10 points] Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables such that

$$X_n \sim \text{Poisson}(n\lambda), \text{ for } n = 1, 2, 3, \ldots$$

where $\lambda > 0$ is a constant. Define a new sequence of random variables Y_n as

$$Y_n = \frac{X_n}{n} \text{ for } n = 1, 2, 3, \ldots$$

Show that Y_n converges in the mean square sense to λ.

3. [20 points] In order to obtain FDA authorization for a new COVID-19 vaccine, one needs to be 97% sure that side effects do not occur more than 10% of the time.

   (a) In order to estimate the probability p of side effects, the vaccine is tested on 100 volunteers. Side effects are experienced by 6 volunteers and the sample variance is observed as 0.239. Find the 97% confidence interval estimate for p. Based on your analysis of the results from above 100 trials, are you convinced with 97% confidence that $p \leq 0.1$?

   (b) Another study is performed, this time with 1000 volunteers. Side effects occur in 71 volunteers. Find the 97% confidence interval for the probability p of side effects if the sample variance is 0.257. Are you now convinced with 97% confidence that $p \leq 0.1$?

4. [20 points] Diners arrive at a popular restaurant according to a Poisson process (denoted as $N(t)$) with rate λ.

   (a) What is the expected time for first n customers to arrive?

   (b) Find the variance of the time it takes for first n customers to arrive.

   (c) Due to social distancing measures, every customer is independently seated with a probability p, or turned away with probability $(1-p)$. Let $M(t)$ be the resulting random process, denoting the total number of customers that are seated at time t. Prove that $M(t)$ is also a Poisson process. Find the rate of $M(t)$.

5. [20 points] A mobile sensor sends a radio signal to a receiver situated at a distance R from it. The distance R is a random variable with the following PDF:

$$f_R(r) = \begin{cases} 2r/10^6, & 0 \le r \le 1000, \\ 0, & \text{otherwise.} \end{cases}$$

The resulting signal power (measured in dB) seen at the receiver as a function of the distance R is modeled as follows:

$$X = Y - 40 - 40 \log_{10}(R),$$

where Y captures a fading phenomenon, modeled as a Gaussian random variable $\mathcal{N}(0, 8)$ which is independent of the distance R. The goal of receiver is to use the received signal power X to estimate the distance R from the sensor.

(a) Write down the joint PDF of (X, R), i.e., $f_{X,R}(x, r)$.

(b) Find the MAP estimate of R given the observation $X = x$.

(c) Find the ML estimate of R given the observation $X = x$.

6. [20 points] A WSS random process $X(t)$ with the following PSD

$$S_X(f) = \begin{cases} 10^{-4} & |f| \leq 100, \\ 0 & \text{otherwise.} \end{cases}$$

is given as an input to a LTI filter with the following transfer function:

$$H(f) = \frac{1}{100\pi + j2\pi f}$$

The output of the filter is the random process $Y(t)$.

(a) Find the power of the input signal $X(t)$.

(b) Find the PSD of the output signal $Y(t)$.

(c) What is the power of the output signal?