① $X_n = \begin{cases} n^\alpha & \text{if } 0 \leq U \leq \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < U \leq 1 \end{cases}$

(a) $P(X_n = n^\alpha) = P(0 \leq U \leq \frac{1}{n}) = \frac{1}{n}$

$P(X_n = 0) = 1 - \frac{1}{n}$

$E[|X_n - 0|^2] = E[X_n^2] = (n^{2\alpha}) \times \frac{1}{n} + (0)(\frac{1-1}{n})$

$\Rightarrow E(X_n^2) = n^{(2\alpha - 1)}$

$\Rightarrow$ if $2\alpha - 1 < 0$

$\Rightarrow \quad \alpha < \frac{1}{2} \quad \Rightarrow$ then $E(X_n^2) \to 0$
as $n \to \infty$

$\Rightarrow$ for $\alpha < \frac{1}{2}$, $X_n \to 0$ in m.s. sense.

(b) $P(|X_n - 0| \geq \epsilon)$

~~for $\epsilon > 1$;~~ $P(X_n \geq \epsilon) \leq P(X_n = n^\alpha)$

For any $0 < \epsilon$

$= \frac{1}{n} \to 0$ as $n \to \infty$

$\Rightarrow X_n \to 0$ in Probability for any α

$$X_n \sim \text{Laplace}(\mu = 0, \sigma^2 = \tfrac{2}{n^2}).$$

② To show a.s. convergence, it suffices to show that

$$\underline{\overline{\text{Note}}} \quad \left[\begin{array}{c}\text{SEE} \\ \text{LECTURE 27,} \\ \text{THEOREM 1}\end{array}\right] \quad \sum_{n=1}^{\infty} P(|X_n| \geqslant \epsilon) < \infty$$

$$P(|X_n| \geqslant \epsilon) = P(X_n \geqslant \epsilon) + P(X_n \leq -\epsilon)$$

$$= 2 \int_{\epsilon}^{\infty} f_{X_n}(x)\, dx$$

$$= 2 \int_{\epsilon}^{\infty} \frac{n}{2} e^{-nx}\, dx$$

$$= -e^{-nx}\Big|_{\epsilon}^{\infty} = e^{-n\epsilon}$$

Then,

$$\sum_{n=1}^{\infty} P(|X_n| \geqslant \epsilon) = \sum_{n=1}^{\infty} \left(e^{-n\epsilon}\right) = \sum_{n=1}^{\infty} \left(e^{-\epsilon}\right)^n$$

$$= \frac{e^{-\epsilon}}{1 - e^{-\epsilon}} < \infty$$

$$\Rightarrow X_n \xrightarrow{a.s.} 0$$

③ For Rayleigh $\left(\frac{1}{n}\right)$, $\begin{bmatrix} \text{SEE} \\ \text{LECTURE 27,} \\ \text{Theorem 1} \end{bmatrix}$

$$P(|X_n| \geq \epsilon) = P(X_n \geq \epsilon)$$

$$= \overline{e}^{\, n^2 \epsilon^2 / 2}$$

$$\Rightarrow \sum_{n=1}^{\infty} P(|X_n| \geq \epsilon) = \sum_{n=1}^{\infty} e^{-\epsilon^2 n^2 / 2}$$

$$\leq \sum_{n=1}^{\infty} \left(e^{-\epsilon^2 / 2} \right)^n$$

$$= \frac{\overline{e}^{-\epsilon^2 / 2}}{1 - \overline{e}^{-\epsilon^2 / 2}} \quad < \infty$$

$$\Rightarrow \quad X_n \longrightarrow \cancel{0} \quad \text{almost surely}$$

(4) X_n has the density $f_n(x) = n f(nx)$
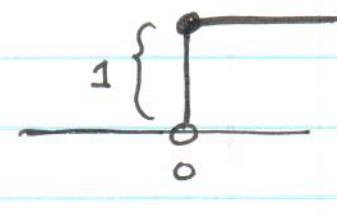
$$F_n(x) = P(X_n \leq x) = \int_{-\infty}^{x} f_n(t)\, dt$$

$\uparrow$

CDF of X_n

$$= \int_{-\infty}^{x} n f(nt)\, dt$$

$$= \int_{-\infty}^{nx} f(y)\, dy$$

$$= F(nx).$$

$\Rightarrow$ $F_n(x) = F(nx)$

for $x > 0$, as $n \to \infty$ $F_n(x) \longrightarrow F(\infty) = 1$

for $x < 0$, as $n \to \infty$ $F_n(x) \longrightarrow F(-\infty) = 0$

(for $x = 0$, $F_n(0) = F(0)$)

$\Rightarrow$ $F_n(x) \longrightarrow$ CDF of 0 $\Rightarrow$

for all $x \neq 0$

$\Rightarrow$ $X_n \longrightarrow 0$ in distribution

as $n \to \infty$.

(5) $X_1, X_2 \ldots, X_n$ are iid, uniform with $\mu = 7$, $\sigma^2 = 3$.

(a) Since X_1 is uniform $\Rightarrow$ uniform PDF over $[a, b]$

$$\left. \begin{array}{l} \mu = \dfrac{a+b}{2} = 7 \\[4mm] \sigma^2 = \dfrac{(b-a)^2}{12} = 3 \end{array} \right\} \Rightarrow \text{Solving for } a, b$$

$$a = 4,$$
$$b = 10$$

$$\Rightarrow \quad f_X(x) = \begin{cases} \dfrac{1}{6} & 4 \leq x \leq 10 \\[3mm] 0 & \text{otherwise.} \end{cases}$$

(b) $M_{16} = \dfrac{X_1 + X_2 \cdots + X_{16}}{16}$, X_i's are iid

$$\text{Var}(M_{16}) = \frac{\text{Var}(X)}{16} = \frac{3}{16}.$$

(c) $P(X_1 > 9) = \displaystyle\int_9^\infty f_{X_1}(x)\,dx = \int_9^{10} \frac{1}{6}\,dx = \frac{1}{6}$

(d)

$$P(M_{16} > 9) = 1 - P(M_{16} \leq 9)$$

$$= 1 - P\left(\boxed{\frac{M_{16} - 7}{\sqrt{3/16}}} \leq \frac{9 - 7}{\sqrt{3/16}} \right)$$

↓ in distribution

$\mathcal{N}(0,1).$ by CLT

$$\cong 1 - \Phi\left(\frac{8}{\sqrt{3}}\right)$$

$$= 1 - \Phi(2.66) = \quad 1.93 \times 10^{-6}$$

$$\Rightarrow P(M_{16} > 9) \ll P(X_1 > 9)$$

⑥ we know from Chebyshev's inequality that

$$P(|X - E(x)| \geq c) \leq \frac{\sigma_x^2}{c^2}$$

choosing $c = k\sigma_x \Rightarrow P(|X - E(x)| \geq k\sigma) \leq \frac{1}{k^2}$

on the other hand, the actual probability that the Gaussian r.v. Y is more than $k\sigma_Y$ from its expected value is

$$P(|Y - E(Y)| \geq k\sigma_y)$$

$$= P(Y - E(Y) \leq -k\sigma_y)$$

$$+ P(Y - E(Y) \geq k\sigma_y)$$

$$= 2 P\left(\boxed{\frac{Y - E(Y)}{\sigma_y}} \geq k \right)$$

$$\mathcal{N}(0,1)$$

$$= 2 Q(k).$$

	k = 1	k = 2	k = 3	k = 4	k = 5
Upper bound (by chebyshev)	1	0.250	0.111	0.0625	0.04
$2Q(k)$	0.317	0.046	0.0027	6.33×10^{-5}	5.73×10^{-7}

what we observe is that Chebyshev based bound gets increasingly weak as k increases. For eg:, at k = 4, the bound exceeds the true probability by a factor of 1000. for k = 5, the bound exceeds the true probability by a factor of nearly 100,000 !!!

⑦

$\text{Var}(x_1 + x_2 + \dots + x_n)$

$$= E\left((x_1 + x_2 \dots + x_n - n\mu)^2\right)$$

$$= E\left(\left((x_1-\mu) + (x_2-\mu) + \dots + (x_n-\mu)\right)^2\right)$$

$$= \sum_{i=1}^{n} \text{Var}(x_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \text{Cov}(x_i, x_j)$$

$$= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a^{j-i}$$

$$= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} \left(a + a^2 + \dots + a^{n-i}\right)$$

$$= n\sigma^2 + \frac{2a\sigma^2}{1-a} \sum_{i=1}^{n-1} \left(1 - a^{n-i}\right)$$

$$\vdots$$

$$= \frac{n(1+a)\sigma^2}{(1-a)} - \frac{2a\sigma^2}{(1-a)} - 2\sigma^2 \left(\frac{a}{1-a}\right)^2 \left(1 - a^{n-1}\right)$$

$$\leq \frac{n\sigma^2(1+a)}{(1-a)}$$

$\Rightarrow \quad \text{Var}(x_1 + \dots + x_n) \leq \dfrac{n\sigma^2(1+a)}{(1-a)}$

Now, $\qquad E(M_n) = \mu; \quad Var(M_n) = \dfrac{Var(X_1 + .. + X_n)}{n^2}$

$$P(|M_n - \mu| \geq \epsilon) \qquad\qquad \leq \dfrac{\sigma^2(1+a)}{n(1-a)}$$

$$\leq \dfrac{Var(M_n)}{\epsilon^2} \leq \dfrac{\sigma^2(1+a)}{n(1-a)\epsilon^2} \longrightarrow 0$$

$$\text{as}$$

$$n \longrightarrow \infty$$

$$\Rightarrow \qquad M_n \overset{\text{in Prob.}}{\longrightarrow} \mu.$$