Homework 6 - ECE 503 Fall 2020

- Assigned on: Monday, October 26, 2020.
- Due Date: **Monday, November 2, 2020 by 11:59 pm Tucson Time.**
- Maximum Credit: 100 points

1. [15 points] The PDF of the 3-dimensional random vector $X = (X_1, X_2, X_3)$ is

$$f_X(x) = \begin{cases} e^{-x_3} & 0 \leq x_1 \leq x_2 \leq x_3, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal PDFs of X_1, X_2 and X_3

   (b) Are the components of X independent ?

2. [20 points] Let X be a 3-dimensional Gaussian random vector with expected value $\mu_X = [4 \quad 8 \quad 6]^T$, and covariance

$$C_X = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

   Calculate

   (a) the correlation matrix R_X

   (b) the PDF of the first two components of X, i.e., $f_{X_1,X_2}(x_1, x_2)$

   (c) the probability that $X_1 > 8$

3. [15 points] Random variables X_1 and X_2 both have zero expected value and variances $\text{Var}(X_1) = 4$, $\text{Var}(X_2) = 9$. Their covariance is $\text{Cov}(X_1, X_2) = 3$.

   (a) Find the covariance matrix of $X = (X_1, X_2)^T$.

   (b) X_1 and X_2 are transformed to new variables Y_1 and Y_2 according to

$$Y_1 = X_1 - 2X_2$$
$$Y_2 = 3X_1 + 4X_2$$

   Find the covariance matrix of $Y = (Y_1, Y_2)^T$.

4. [20 points] The voltage V of a position sensor is a random variable with PDF:

$$f_V(v) = \begin{cases} 1/12 & -6 \leq v \leq 6, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   A receiver obtains $R = V + X$, where the random variable X is a Gaussian $(\mu, \sigma) = (0, \sqrt{3})$ noise voltage that is independent of V. The receiver uses R to estimate the original voltage V. Find

   (a) the expected received voltage $E(R)$

   (b) the variance $\text{Var}(R)$ of the received voltage

   (c) the covariance $\text{Cov}(V, R)$ of the transmitted and received voltages

(d) the LMMSE (linear MMSE) estimator of V from R

(e) the resulting error of the LMMSE estimator

5. [20 points] Given the set $\{U_1, U_2, \ldots, U_n\}$ of i.i.d. uniform $(0, T)$ random variables, we define

$$X_k \triangleq \text{small}_k(U_1, U_2, \ldots, U_n)$$

as the kth "smallest" element of the set. For example, X_1 is the smallest element, X_2 is the second smallest element, and so on, up to X_n, which is the maximum element of $\{U_1, U_2, \ldots, U_n\}$.

(a) Find the joint PDF of $(X_1, X_2, \ldots, X_n)$.

(b) Find the marginal PDF of X_2.

6. [10 points] Let N be a positive, integer valued random variable, and let $X_1, X_2, \ldots$ be i.i.d. random variables. Further, assume that N is independent of $X_1, X_2, \ldots, X_n$ for every n. Consider the random sum,

$$S_N = \sum_{i=1}^{N} X_i$$

Note that the number of terms in the sum above is a random variable.

(a) Find the expected value of S_N.

(b) The number of jobs N submitted to the CPU is a geometric random variable with parameter p. The execution time of each job is an exponential random variable with mean λ. What is the expected total execution time ? (Hint: use the above result)