MARKOV CHAINS

Discrete-Time Markov Chains (DTMC)

A discrete-time Markov chain $\{X_n, n = 0, 1, 2, \cdots\}$ is a discrete-time, discrete-valued random process such that given $X_0, X_1, \ldots, X_n$, the next random variable X_{n+1} depends only on X_n through the probability

$$P(X_{n+1} = i_{n+1} \mid X_n = i_n, \ldots, X_0 = i_0)$$
$$= P(X_{n+1} = i_{n+1} \mid X_n = i_n)$$

In other words, given the sequence of values $i_0, i_1, \ldots, i_n$, the conditional probability of what value X_{n+1} takes depends only on the value of i_n.

Example: The Random Walk is a Markov process

$$X_n = X_0 + W_1 + W_2 + \cdots + W_{n-1} + W_n$$

$$P(X_6 = 11 \mid X_5 = 10, X_4 = 9, \ldots, X_0 = 0)$$
$$= P(X_6 = 11 \mid X_5 = 10)$$
$$= p$$

Using the Markov Property, we can show that

$$\boxed{\begin{aligned} & P(X_{n+m} = j_m, \ldots, X_{n+1} = j_1 \mid X_n = i_n, \ldots, X_0 = i_0) \\ & \quad = P(X_{n+m} = j_m \mid X_{n+m-1} = j_{m-1}) \\ & \qquad \times P(X_{n+m-1} = j_{m-1} \mid X_{n+m-2} = j_{m-2}) \\ & \qquad \times \ \vdots \\ & \qquad \quad \times P(X_{n+2} = j_2 \mid X_{n+1} = j_1) \\ & \qquad \qquad \times P(X_{n+1} = j_1 \mid X_n = i_n) \end{aligned}}$$

To prove this claim,

$$P(\underbrace{X_{n+m} = j_m}_{A}, \underbrace{X_{n+m-1} = j_{m-1}, \ldots, X_{n+1} = j_1}_{B} \mid \underbrace{X_n = i_n, \ldots, X_0 = i_0}_{C})$$

$$= P(A, B \mid C)$$

$$= P(A \mid B, C) \times P(B \mid C)$$

$$= \underbrace{P(X_{n+m} = j_m \mid X_{n+m-1} = j_{m-1}, \cdots}_{\parallel}) \times P(B \mid C)$$

$$P(X_{n+m} = j_m \mid X_{n+m-1} = j_{m-1}) \times \underbrace{P(B \mid C)}_{\Downarrow}$$

apply the same trick recursively

$\vdots$

STATE SPACE and TRANSITION PROBABILITIES

The set of possible values that the random variables X_n can take is called the state space of the chain. We focus on the case in which the state space is either the set of integers or a subset of integers.

The conditional probabilities $P(X_{n+1} = j \mid X_n = j)$ are called transition probabilities. When the transition probabilities do not depend on time n, then the M.C is said to be homogeneous (or have stationary transition probabilities). For a homogeneous M.C., we use the notation

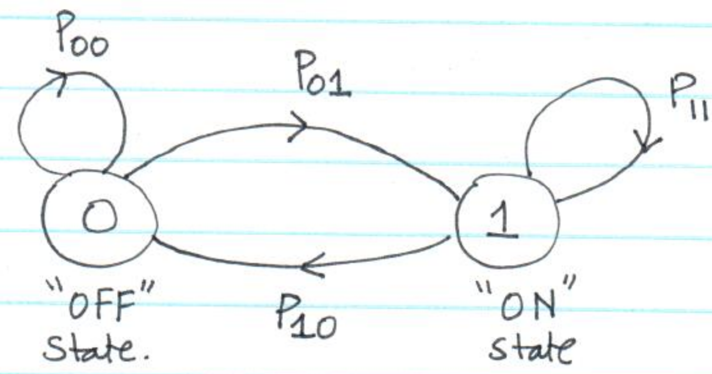
$$P_{ij} = P(X_{n+1} = j \mid X_n = i)$$

for the transition probabilities.

* P_{ij} are also called the one-step transition probabilities since they are the prob. of going from state i to state j in one time step.

* One of the most common ways to specify the transition probabilities is with a state transition diagram.

Example: Two-state Markov chain.



$$
\begin{array}{cccccc|c}
X_0 & X_1 & X_2 & X_3 & \cdots & X_{n-1} & X_n \\
0 & 1 & 1 & 0 & \cdots & 1 \rightarrow & 0 \\
 & & & & & \text{or} \searrow & 1.
\end{array}
$$

State Space $\Rightarrow \{0, 1\} \Longrightarrow$ set of values taken by $\{X_n\}$.

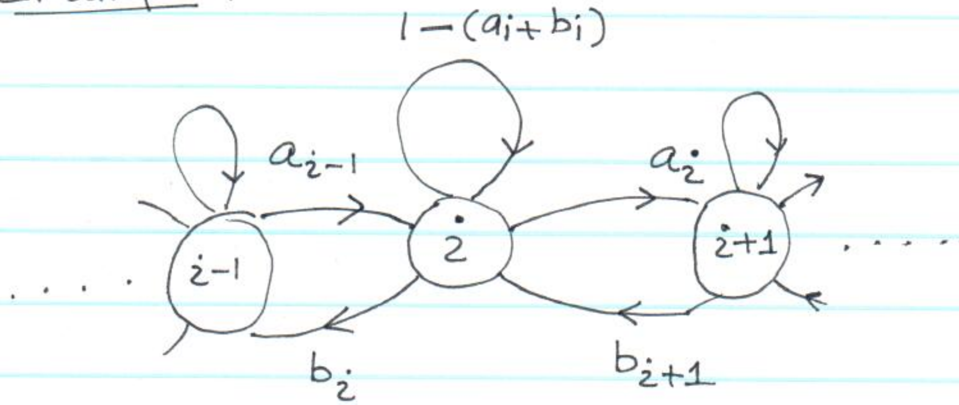
FACT

> The sum of all the probabilities leaving a state must be 1

$$\sum_j P_{ij} = 1$$

WHY?
$$\sum_j P_{ij} = \sum_j P(X_{n+1} = j \mid X_n = i) = 1$$

Example: A general Random Walk.



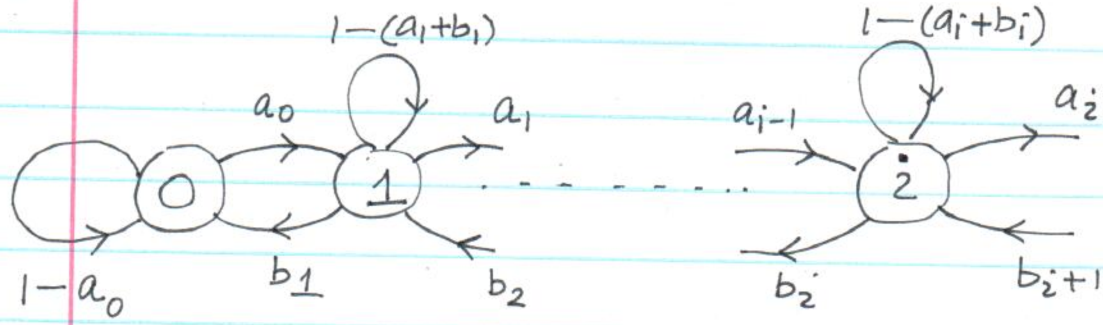
$$P_{ij} = \begin{cases} b_i & \text{if } j = i-1 \\ 1-(a_i+b_i) & \text{if } j = i \\ a_i & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\underset{\text{Transition Matrix}}{P} = \begin{array}{c} \begin{bmatrix} \cdots & & & & & \\ & \ddots & & & & \\ \cdots & 0 & b_i & 1-(a_i+b_i) & a_i & 0 \cdots \\ & & & & \ddots & \\ & & & & & \cdots \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \leftarrow i^{th} \text{ row} \\ \\ \\ \end{array}$$

infinite, tri-diagonal matrix.

(also known as a Birth-death process; eg population counts.)

Example: A Random Walk with a barrier at origin



for $i = 0$ $\qquad P_{0j} = \begin{cases} 1-a_0 & j=0 \\ a_0 & j=1 \\ 0 & \text{otherwise.} \end{cases}$

$$P = \begin{bmatrix} 1-a_0 & a_0 & 0 & 0 & 0 & \cdots \\ b_1 & 1-(a_1+b_1) & a_1 & 0 & 0 & \cdots \\ 0 & b_2 & 1-(a_2+b_2) & a_2 & 0 & \cdots \\ 0 & 0 & b_3 & 1-(a_3+b_3) & a_3 & 0 \cdots \\ & & & & \ddots & \end{bmatrix}$$

semi-infinite, tri-diagonal matrix.

If $a_0 = 1 \rightarrow$ barrier is "reflecting"

If $a_0 = 0 \rightarrow$ barrier is "absorbing", i.e

once a chain hits an absorbing state, it remains in that state from that time onward..