Today

1) Covariance between two random Variables
2) Correlation Coefficient < operational meaning. / properties.
3) Uncorrelated-ness vs Independence
4) Iterated Expectation Theorem

Covariance between X & Y $\longrightarrow$ $f_{X,Y}(x,y)$

$$C_{XY} = E[(X-\mu_X)(Y-\mu_Y)]$$

(C)
$$= E[XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y]$$

AB is true

AB is -ve,

$$= E[XY] - E[\mu_Y \cdot X] - E[\mu_X \cdot Y] + E[\mu_X \mu_Y]$$

$$= E[XY] - \mu_Y \cdot \cancel{\mu_X} - \mu_X \mu_Y + \cancel{\mu_X}\mu_Y$$

$$\boxed{C_{XY} = E[XY] - \mu_X \cdot \mu_Y}$$

Y = const.

X = random var.

Correlation Coefficient ρ_{XY} (or ρ).

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X) \times Var(Y)}} = \frac{Cov(X,Y)}{\sigma_X \cdot \sigma_Y}.$$

① $-1 \le \rho_{XY} \le 1.$

$\Updownarrow$

② $-\sigma_X\sigma_Y \le Cov(X,Y) \le \sigma_X\sigma_Y$

} Basic Property of ρ_{XY}.

Interpretation of ρ_{XY}

If $\rho_{XY} > 0 \Longrightarrow$ if X is high then Y will tend to be high.

If $\rho_{XY} < 0 \implies$ — "opposite" —.

If $\rho_{XY} = 0 \implies$ X, Y are uncorrelated.

X → randomly chosen digits for Telephone #. Y → SSN.

$\rho_{XY} \stackrel{?}{=} 0$

X → height , Y → weight.

$0 < \rho_{XY} \leq 1$

X → distance of a cell phone tower from a user. Y → Power of the received signal.

$-1 \leq \rho_{XY} < 0$ (ρ is -ve)

When is $\rho_{XY} = 1$ or $\rho_{XY} = -1$?

$\implies$ When X and Y have an exact linear relationship.

$\boxed{X = aY + b}$.

$\rho_{XY} = +1$ or -1.

X → temp in Kelvin. Y → temp in Celcius.

$\rho_{XY} = +1$.

$\boxed{\rho_{XY} = \rho_{YX}}$.

$\boxed{\rho_{XY} = 0.3}$

$$\left\{ \begin{array}{l} \boxed{X = Y^3 + 3Y^2 + \cos(Y),} \\ \rho_{XY} = 0.3. \Longrightarrow \end{array} \right.$$

Uncorrelated R.V.'s

X, Y are uncorrelated if.

$Cov(X,Y) = 0.$

(or) $\rho_{XY} = 0$

(or) $E[XY] = E[X]E[Y].$

$\boxed{\text{Independent}}$ **R.V.'s**.

X, Y are independent if

$$\underbrace{f_{X,Y}(x,y)}_{\text{Joint PDF}} = \underbrace{f_X(x) \cdot f_Y(y)}_{\text{Product of marginals.}}$$

$\boxed{\text{If } X \text{ \& } Y \text{ are independent} \Longrightarrow X \text{ \& } Y \text{ are uncorrelated.}}$ (or $\rho_{XY} = 0$)

How to prove this ?

$$E[XY] = \iint x \cdot y \cdot f_{X,Y}(x,y)\,dx\,dy.$$

$$\boxed{E[g(X,Y)] = \iint g(x,y) \cdot f_{X,Y}(x,y)\,dx\,dy.}$$

Since X, Y are indep.

$$= \iint x \cdot y \cdot f_X(x) f_Y(y)\,dx\,dy$$

$$= \underbrace{\left(\int x f_X(x) dx\right)}\underbrace{\left(\int y f_Y(y) dy\right)},$$

$$= E[X]\;E[Y].$$

If X and Y are uncorrelated. $\not\Longrightarrow$ X and Y are independent.

(not true in general)

$$\rho_{XY} = 0 \not\Longrightarrow f_{XY} = f_X f_Y.$$

Example: $X \sim \text{unif}[0,1]$, $Y \sim \text{unif}[0,1]$ and X, Y are independent.

$$\begin{cases} Z = X + Y \\ W = X - Y. \end{cases}$$

Prove: (Z, W) are uncorrelated, however they are NOT independent.

$$\rho_{Z,W} = 0. \qquad \underline{f_{ZW}(z,w)} \overset{?}{\neq} \underline{f_Z(z) f_W(w)}$$

(exercise)

$$E[ZW] \overset{?}{=} E[Z]\, E[W] \rightarrow \text{RHS.}$$

$$E[(X+Y)(X-Y)]$$

$$= E[X^2 - Y^2]$$

$$= E[X^2] - E[Y^2]$$

$$= 0.$$

$$= E[Z] \cdot \underbrace{E[X-Y]}_{=0.}$$

$$E[X-Y] = E[X] - E[Y] = 0.$$

$$E[Z] \times E[W]$$

$$= (E[X+Y]) \times E[X-Y].$$

$$= [\quad] \times \{ E[X] - E[Y]. \}$$

$X \sim \text{unif}[0,1]$ $\quad$ $Y \sim \text{unif}[0,1]$. $\rightarrow$ same mean

$Y = \text{const.} \Rightarrow Var(Y) = 0.$

$X \rightsquigarrow$ r.v.

$$Cov(X, Y) \stackrel{?}{=} E[(X - \mu_X) \times (Y - \mu_Y)]$$

$$= E[(X - \mu_X) \times (C - C)]$$

$$= 0.$$

$\rho_{XY} = 0$

$Cor(X, Y) = 0.$

$\boxed{\text{Iterated Expectation Theorem}}$ (X, Y)

$$E[E[X \mid Y]] = E[X]$$

r.v.

$E[X] \to$ mean of X. $E[X] = \int x f_X(x)\,dx.$

$E[X \mid Y = y] \to$ Conditional mean of X given $Y = y$.

$$= \int_{x=-\infty}^{\infty} x \times f_{X|Y}(x|y)\,dx$$

$\boxed{\text{function of } y}$

$X \mid Y = y. \Longrightarrow f_{X|Y}(x|y).$

Conditional PDF of X given $Y = y$.

as y changes, $E[X \mid Y = y]$ changes.

$\boxed{E[X|Y]} \Rightarrow$ is a random variable. & it is a function of $\boxed{Y}$.

* What values does it take.?

Suppose $Y \in \{0, 1, 2\}$.

$Y=0 \longrightarrow E[X|Y=0]$.

$Y=1 \longrightarrow E[X|Y=1]$

$Y=2 \longrightarrow E[X|Y=2]$.

$$E\Big[\underbrace{E[X|Y]}_{\text{is a R.V.} \Rightarrow g(Y)}\Big] = E[X].$$

$$LHS = E[g(Y)].$$

$$= \int_{y=-\infty}^{\infty} g(y) \cdot f_Y(y)\, dy.$$

OK ?

$$= \int_{y=-\infty}^{\infty} \Big[E[X|Y=y] \Big] f_Y(y)\, dy.$$

$$= \int_{y=-\infty}^{\infty} f_Y(y) \left[\int_{x=-\infty}^{\infty} x \times f_{X|Y}(x|y)\, dx. \right] dy.$$

$$= \int_{x=-\infty}^{\infty} x \times \left[\int_{y=-\infty}^{\infty} \underbrace{f_Y(y)\, f_{X|Y}(x|y)}_{f_{X,Y}(x,y)}\, dy \right] dx.$$

$$= \int_{x=-\infty}^{\infty} x\, f_X(x)\, dx. = E[X].$$