$\boxed{\text{Today}}$

1) Random Vectors

2) IID Random Variables

3) Covariance Matrix, Correlation Matrix.

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n random variables $(X_1, X_2, \ldots, X_n)$

Joint CDF $F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)$

Joint PMF (if X_i's are discrete)

$$P_{X_1, \ldots X_n}(x_1, \ldots x_n) = P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)$$

Joint PDF (for continuous).

$$\underbrace{f_{X_1, X_2 \ldots X_n}(x_1, \ldots x_n)} = \frac{\partial^n F_{X_1 \ldots X_n}(x_1, \ldots, x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

Age, Salary, Zip-code, Race, Gender ....

$$\underbrace{(X_1, \quad X_2, \quad X_3, \quad X_4, \quad X_5)}$$

Joint CDF/PDF/PMF of these objects/features.

Joint PDF: (Properties)

1) $f_{X_1 \ldots X_n}(x_1, \ldots x_n) \ge 0$.

2) $\displaystyle\int_{x_1=-\infty}^{\infty} \int \cdots \int_{x_n=-\infty}^{\infty} f_{X_1 \ldots X_n}(x_1 \ldots x_n)\, dx_1 \ldots dx_n = 1$.

3) Joint CDF from Joint PDF

$$F_{X_1 \ldots X_n}(x_1 \ldots x_n) = \int_{u_1=-\infty}^{x_1} \cdots \int_{u_n=-\infty}^{x_n} f_{X_1 \ldots X_n}(u_1, \ldots u_n)\, du_1 \ldots du_n.$$

$$= P(X_1 \le x_1, X_2 \le x_2 \ldots \quad)$$

$$P(A) = \underbrace{\iint \cdots \int}_{(x_1,\dots x_n)\in A} f_{X_1,\dots,X_n}(x_1,\dots,x_n)\,dx_1\dots dx_n.$$

A is expressed as a function of $(X_1, X_2, \dots, X_n)$

$$\boxed{A = \left\{ (x_1, x_2, x_3, x_4) : \begin{array}{l} x_1^2 + 3x_2 \geq 0 \\ x_2^3 + 9x_2 + x_4 \leq -1 \end{array} \right\}},$$

$$P(A) \stackrel{?}{=} \sum_{(x_1\dots x_n)\in A} \cdots \sum P_{X_1\dots X_n}(x_1\dots x_n)$$

Random Vector

$$\vec{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

($\mathbf{X}$)

where each element X_i is a random variable.

$$\vec{X} = \vec{x} \iff \begin{array}{l} X_1 = x_1 \\ X_2 = x_2 \\ \vdots \\ X_n = x_n \end{array}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

CDF of a Random Vector $\quad F_{\underline{X}}(\bar{x}) = \underbrace{F_{X_1,\dots X_n}(x_1,\dots x_n)}_{\text{joint CDF of } n \text{ r.v.'s}}$

PMF " " $\quad P_{\underline{X}}(\bar{x}) = P_{X_1,\dots X_n}(x_1\dots x_n)$

PDF " " " $\quad f_{\underline{X}}(\bar{x}) = f_{X_1,\dots X_n}(x_1\dots x_n)$

Eg: A $\bar{X}$ has the following PDF

random vector 1×3 (r.v.)

$$f_{\bar{X}}(\bar{x}) = \begin{cases} 6\, e^{-a^T \bar{x}}, & \vec{x} \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Compute the CDF of $\bar{X}$

$$f_{\bar{X}}(\bar{x}) = 6\, e^{-[1\ 2\ 3]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}$$

$$= \begin{cases} \underline{6\, e^{-(x_1 + 2x_2 + 3x_3)}}, & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\underbrace{F_{\bar{X}}(\bar{x})}_{\text{CDF of random vector } \bar{X}} = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} 6\, e^{-(u_1 + 2u_2 + 3u_3)}\, du_1\, du_2\, du_3$$

$$= \begin{cases} (1 - e^{-x_1})(1 - e^{-x_2})(1 - e^{-x_3}) & x_i \geq 0,\ i = 1, 2, 3 \\ 0 & \underline{\text{otherwise}} \end{cases}$$

Eg 2 $\vec{X}$ with the PDF $f_{\underline{X}}(\bar{x}) = \begin{cases} 1 & \text{if } x_i \in [0,1] \ \forall i \\ 0 & \text{otherwise.} \end{cases}$

($1 \times n$ vector).

Compute.

$$P\left(\max_i X_i \le 1/2\right) \qquad P(A)$$

$$= P(X_1 \le 1/2, X_2 \le 1/2, \ldots, X_n \le 1/2)$$

$$= \int_{x_1=0}^{1/2} \cdots \int_{x_n=0}^{1/2} (1)\, dx_1\, dx_2 \cdots dx_n = \left(\frac{1}{2}\right)^n.$$

Eg 3 (Y_1, Y_2, Y_3, Y_4)

$$\underbrace{f_{Y_1 \ldots Y_4}(y_1 \ldots y_4)}_{\text{joint PDF of all 4}} = \begin{cases} 4 & \text{if } 0 \le y_1 \le y_2 \le 1 \\ & \quad 0 \le y_3 \le y_4 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$\underbrace{f_{Y_1}(y_1)}_{\text{marginal PDF of } Y_1} \overset{?}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1 \ldots Y_4}(y_1, y_2, y_3, y_4)\, dy_2\, dy_3\, dy_4$$

$$f_{Y_1, Y_3}(y_1, y_3) \overset{?}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1, \ldots Y_4}(y_1 \ldots y_4)\, dy_2\, dy_4$$

given a subset X_S of r.v.'s from $(X_1 \ldots X_n)$

$S = \{1, 3, 5\}$.

we can compute the joint PDF of

$X_S \Rightarrow \{X_{i_1}, \ldots X_{i_{|S|}}\} \rightarrow (X_1, X_3, X_5)$

Independent R.V.'s

n r.v.'s $(X_1, X_2, \ldots, X_n)$ are independent

if $f_{X_1 \ldots X_n}(x_1 \ldots x_n) = f_{X_1}(x_1) \times f_{X_2}(x_2) \ldots f_{X_n}(x_n)$.

Joint PMF or PDF = Product of Marginal PMF/PDF.

IID R.V.'s.

$\boxed{\text{Independent and Identically Distributed.}}$

$X_1, X_2 \ldots X_n \longrightarrow$ joint PDF = Product of marginals.

$$f_{X_1 \ldots X_n}(x_1 \ldots x_n) = f_X(x_1) \times f_X(x_2) \times f_X(x_3) \cdots \times f_X(x_n)$$

all marginal PDF(s) are identical.

$$P_{X_1 \ldots X_n} = P_X(x_1)\, P_X(x_2) \cdots P_X(x_n).$$

$\boxed{n \quad Ber(p) \quad \text{iid r.v.'s.}}$

What is the joint PMF?

x_i: $0 \rightarrow (1-p)$, $1 \rightarrow p$

$P(X_1 = x_1, \ldots, X_n = x_n)$

$= P(X_1 = x_1) \times P(X_2 = x_2) \cdots P(X_n = x_n).$

$=$

$n = 6$

$$P(\vec{X} = (100010)) \overset{?}{=} p^2 \times (1-p)^4$$

Mean of a Random Vector

$$E[\bar{X}] = \mu_{\bar{X}} = \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix} \rightsquigarrow 1 \times n \text{ vector.}$$

Correlation Matrix of a R. Vector.

$\bar{X} \rightarrow$ r.v. of $n \times 1$.

$$\boxed{\underbrace{R_{\bar{X}}}_{n \times n.} = E[\underbrace{\bar{X}}_{n \times 1} \underbrace{\bar{X}^T}_{1 \times n}]}$$

$$E\left[\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \right].$$

$$= E\begin{bmatrix} X_1^2 & X_1X_2 & X_1X_3 \\ X_2X_1 & X_2^2 & X_2X_3 \\ X_3X_1 & X_3X_2 & X_3^2 \end{bmatrix}$$

j^{th}

$$R_{\bar{X}} = \begin{bmatrix} E[X_1^2] & E[X_1X_2] & E[X_1X_3] \\ & E[X_2^2] & \\ & & E[X_3^2] \end{bmatrix}$$

i^{th}

$(i,j)^{th}$ entry $\longrightarrow$ $\underbrace{E[X_iX_j]}_{\text{Correlation. } X_i \,\&\, X_j.} = R_{\bar{X}}(i,j).$

Covariance Matrix of a random vector $\underline{X}$.

$$C_X = E\left[(X - \mu_X)(X - \mu_X)^T \right]$$

if $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow$

$$C_X = \begin{bmatrix} Var(x_1) & Cov(x_1, x_2) & Cov(x_1, x_3) \\ Cov(x_1, x_2) & Var(x_2) & Cov(x_2, x_3) \\ Cov(x_1, x_3) & Cov(x_2, x_3) & Var(x_3) \end{bmatrix}$$

$C_X^T = C_X$ --- (Symmetric matrix)