Lecture 18

Sequences of Random Variables

Joint CDF of n random variables

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n) = P[X_1 \le x_1, \ldots, X_n \le x_n]$$

Joint PMF (for discrete)

$$P_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = P[X_1 = x_1, \ldots, X_n = x_n]$$

Joint PDF (for continuous)

$$f_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = \frac{\partial^n F_{X_1, \ldots, X_n}(x_1, \ldots, x_n)}{\partial x_1 \ldots \partial x_n}$$

Properties

(Joint PMF)

1. $P_{X_1, \ldots X_n}(x_1, \ldots x_n) \ge 0$
2. $\sum_{x_1} \sum_{x_2} \cdots \sum_{x_n} P_{X_1, \ldots X_n}(x_1, \ldots, x_n) = 1$

(Joint PDF)

1. $f_{X_1, \ldots X_n}(x_1, \ldots, x_n) \ge 0$
2. $F_{X_1, \ldots, X_n}(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f_{X_1, \ldots, X_n}(u_1, \ldots, u_n) du_1 \ldots du_n$
3. $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1 \ldots X_n}(x_1, \ldots, x_n) dx_1 \ldots dx_n = 1.$

Probability of an Event A expressed in terms of $X_1, X_2, \ldots, X_n$

$$P(A) = \sum_{(x_1, \ldots x_n) \in A} P_{X_1 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

or

$$P(A) = \int_A \cdots \int f_{X_1, \ldots X_n}(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

RANDOM VECTOR

A random vector is a vector

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix},$$ where each X_i is a random variable.

(A random vector is a random variable for $n=1$)

A sample value of a random vector will be denoted by

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

ie $\{X = x\} \Leftrightarrow \left\{ \begin{matrix} X_1 = x_1 \\ \vdots \\ X_n = x_n \end{matrix} \right\}$.

CDF of a random vector X is $F_X(x) = F_{X_1 \ldots X_n}(x_1, \ldots, x_n)$

PMF of a discrete random vector X

$$P_X(x) = P_{X_1, \ldots, X_n}(x_1, \ldots, x_n)$$

PDF of a continuous random vector X

$$f_X(x) = f_{X_1 \ldots X_n}(x_1, \ldots, x_n)$$

Eg1: The r.v.'s $X_1, \ldots, X_n$ have the joint PDF

$$f_{X_1 \ldots X_n}(x_1, \ldots, x_n) = \begin{cases} 1 & , 0 \leq x_i \leq 1, \; i = 1, 2, \ldots, n \\ 0 & , \text{otherwise} \end{cases}$$

Let A denote the event $\max_i X_i \leq 1/2$. Find $P(A)$

$$P(A) = P\left(\max_i X_i \leq \frac{1}{2} \right)$$

$$= P(X_1 \leq 1/2, X_2 \leq 1/2, \ldots, X_n \leq 1/2)$$

$$= \int_0^{1/2} \cdots \int_0^{1/2} 1 \cdot dx_1 \, dx_2 \ldots dx_n$$

$$= \underbrace{\frac{1}{2} \times \frac{1}{2} \times \cdots \frac{1}{2}}_{n\text{-times}} = \frac{1}{2^n}$$

Eg 2 The random vector X has the PDF

$$f_X(x) = \begin{cases} 6e^{-a^T x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Where $a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. What is the CDF of $\mathbf{X}$?

Since $a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ has 3 components, X is a 3-dim random vector.

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$f_X(x) = \begin{cases} 6e^{-x_1 - 2x_2 - 3x_3}, & x_i \geq 0, \ i = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \text{CDF} = F_X(x) = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} 6e^{-u_1 - 2u_2 - 3u_3} \, du_1 \, du_2 \, du_3$$

$$F_X(x) = \begin{cases} (1 - e^{-x_1})(1 - e^{-2x_2})(1 - e^{-3x_3}), & x_i \geq 0, \ i = 1, 2, 3 \\ 0, & \text{otherwise.} \end{cases}$$

Eg 3 The random variables Y_1, Y_2, Y_3, Y_4 have the joint PDF

$$f_{Y_1,\dots Y_4}(y_1,\dots,y_4) = \begin{cases} 4 & , 0 \le y_1 \le y_2 \le 1, \\ & \; 0 \le y_3 \le y_4 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the marginal PDFs $f_{Y_1,Y_4}(y_1, y_4)$, $f_{Y_2,Y_3}(y_2, y_3)$ and $f_{Y_3}(y_3)$

$$f_{Y_1,Y_4}(y_1, y_4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{Y_1,\dots Y_4}(y_1, y_2, y_3, y_4)\, dy_2\, dy_3$$

$$= \int_{y_2=y_1}^{1} \int_{y_3=0}^{y_4} f_{Y_1,Y_2,Y_3,Y_4}(y_1, y_2, y_3, y_4)\, dy_2\, dy_4$$

$$= \int_{y_1}^{1}\int_{0}^{y_4} 4\, dy_3\, dy_2 = 4(1-y_1)\, y_4$$

$$\Rightarrow f_{Y_1,Y_4}(y_1, y_4) = \begin{cases} 4(1-y_1)\,y_4 & , \; 0 \le y_1 \le 1, \\ & \;\; 0 \le y_4 \le 1 \\ 0 & , \; \text{otherwise}. \end{cases}$$

INDEPENDENT Random Variables

R.V.'s $X_1, \ldots, X_n$ are independent if for all $(x_1, \ldots, x_n)$

(continuous) $$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdot \ldots \cdot f_{X_n}(x_n)$$

(discrete) $$P_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = P_{X_1}(x_1) \cdot P_{X_2}(x_2) \cdot \ldots \cdot P_{X_n}(x_n)$$

INDEPENDENT and IDENTICALLY DISTRIBUTED (I.I.D.) random variables.

R.V.'s $X_1, \ldots, X_n$ are I.I.D. if

$$f_{X_1,\ldots X_n}(x_1,\ldots,x_n) = f_X(x_1) \cdot f_X(x_2) \cdots\cdots f_X(x_n)$$

$$P_{X_1,\ldots X_n}(x_1,\ldots,x_n) = P_X(x_1) \cdot P_X(x_2) \cdot \ldots \cdot P_X(x_n)$$

Remark: Independence of n random variables is typically a property of an experiment consisting of n independent sub-experiments. In this case, sub-experiment i produces the random variable X_i.

If all sub-experiments follow the same procedure, all of the X_i's have the same PDF or PMF. In such a case, we say that X_i's are identically distributed.

For a random vector X,

$$E[g(X)] = \sum_{x_1} \cdots \sum_{x_n} g(x) \overbrace{P_X(x)}^{P_{X_1,\ldots X_n}(x_1,\ldots,x_n)}$$

$$E[g(X)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x) f_X(x)\, dx_1\, dx_2 \ldots dx_n$$

EXPECTED VALUE (or mean) of a Random Vector

$$E[X] = \mu_X = \begin{bmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{bmatrix}$$

CORRELATION of a Random Vector

(also known as autocorrelation)

$$\underset{\boxed{n \times n \text{ matrix}}}{R_X} = E[X X^T], \quad \text{with} \quad R_X(i,j) = E[X_i X_j]$$

Eg If $X = [X_1 \; X_2 \; X_3]^T$

$$X X^T = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} [X_1 \; X_2 \; X_3] = \begin{bmatrix} X_1^2 & X_1X_2 & X_1X_3 \\ X_2X_1 & X_2^2 & X_2X_3 \\ X_3X_1 & X_3X_2 & X_3^2 \end{bmatrix}$$

$$\Rightarrow R_X = \begin{bmatrix} E(X_1^2) & E(X_1X_2) & E(X_1X_3) \\ E(X_2X_1) & E(X_2^2) & E(X_2X_3) \\ E(X_3X_1) & E(X_3X_2) & E(X_3^2) \end{bmatrix}$$

$\boxed{\text{COVARIANCE of a RANDOM VECTOR } X}$

(also known as autocovariance

$$C_X = E\left[(X - \mu_X)(X - \mu_X)^T \right]$$

* $C_X(i,j) = Cov(x_i, x_j)$
* C_X is a $n \times n$ matrix

If $X = [x_1 \; x_2 \; x_3]^T$

$$C_X = \begin{bmatrix} Var(x_1) & Cov(x_1, x_2) & Cov(x_1, x_3) \\ Cov(x_2, x_1) & Var(x_2) & Cov(x_2, x_3) \\ Cov(x_3, x_1) & Cov(x_3, x_2) & Var(x_3) \end{bmatrix}$$

* $\boxed{C_X = R_X - \mu_X \mu_X^T}$

Proof: $C_X = E\left[(X - \mu_X)(X - \mu_X)^T \right]$

$$= E\left[XX^T - X\mu_X^T - \mu_X X^T + \mu_X \mu_X^T \right]$$

$$= E\left[XX^T \right] - 2\mu_X \mu_X^T + \mu_X \mu_X^T$$

$$= E\left[XX^T \right] - \mu_X \mu_X^T$$

$$= R_X - \mu_X \mu_X^T$$

this is a generalization

[Recall, for a pair of r.v.'s
$Cov(X, Y) = E(XY) - E(X)E(Y)$]

Eg A 2-dimensional r.v. has the PDF

$$f_X(x) = \begin{cases} 2 & 0 \le x_1 \le x_2 \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

* Find $E[X]$
* Correlation matrix R_X
* Covariance matrix C_X

$$E(X_i) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i\, f_X(x)\, dx_1\, dx_2 = \int_0^1\int_0^{x_2} 2x_i\, dx_1\, dx_2, \quad i = 1, 2$$

$$E(X_1) = \frac{1}{3} \quad \text{and} \quad E(X_2) = 2/3$$

$$\mu_X = E(X) = \begin{bmatrix} 1/3 & 2/3 \end{bmatrix}^T$$

$$E(X_1^2) = \int_0^1\int_0^{x_2} 2x_1^2\, dx_1\, dx_2 = 1/6$$

$$E(X_2^2) = \int_0^1\int_0^{x_2} 2x_2^2\, dx_1\, dx_2 = 1/2$$

$$E(X_1X_2) = \int_0^1\int_0^{x_2} 2x_1x_2\, dx_1\, dx_2 = 1/4$$

$$\Rightarrow \quad R_X = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix}$$

$$C_X = R_X - \mu_X \mu_X^T = \begin{bmatrix} 1/6 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} - \begin{bmatrix} 1/9 & 2/9 \\ 2/9 & 4/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1/18 & 1/36 \\ 1/36 & 1/18 \end{bmatrix}.$$

CROSS-CORRELATION

Random vector X with n components
" " Y with m "

$$\underbrace{R_{XY}}_{n \times m \text{ matrix}} = E[XY^T] \qquad \begin{pmatrix} X: & n \times 1 \\ Y: & m \times 1 \end{pmatrix}$$

$$R_{XY}(i,j) = E(X_i Y_j)$$

CROSS-COVARIANCE

$$\underbrace{C_{XY}}_{n \times m} = E[(X - \mu_X)(Y - \mu_Y)^T]$$

$$C_{XY}(i,j) = Cov(X_i Y_j)$$

Remark: when $X = Y$, $R_{XY} = \overset{\substack{R_X \\ \shortparallel}}{R_{XX}}$ (auto-correlation)

$C_{XY} = \underset{\substack{\shortparallel \\ C_X}}{C_{XX}}$ (auto-covariance)

Gaussian Random Vectors

$\underset{(n\times 1)}{\vec{X}}$ is a Gaussian (μ_X, C_X) random vector with expected value $\underbrace{\mu_X}_{n\times 1}$ and covariance $\underbrace{C_X}_{n\times n}$ if and only if

$$f_{\vec{X}}(\vec{x}) = \frac{1}{(2\pi)^{n/2}\left(\det(C_X)\right)^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x}-\mu_X)^T C_X^{-1}(\vec{x}-\mu_X)\right\}$$

where $\det(C_X)$, the determinant of C_X, satisfies $\det(C_X) > 0$.

Theorem: A Gaussian random vector $\vec{X}$ has independent components if and only if C_X is a diagonal matrix.

Proof:

"If Part" (i.e. if $\vec{X}$ has independent components, then C_X is diagonal).

If $\vec{X}$ has independent components $\Rightarrow$ X_i and X_j are independent

$\Rightarrow \operatorname{cov}(X_i, X_j) = 0$

$\Rightarrow$ off-diagonal elements of C_X are zero

$\Rightarrow C_X$ is diagonal.

"Only if" If C_x is diagonal then components of $\vec{x}$ are independent.

For a diagonal C_x,

$$C_x = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} \Rightarrow C_x^{-1} = \begin{bmatrix} 1/\sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & 1/\sigma_n^2 \end{bmatrix}$$

$$\Rightarrow \det(C_x) = \prod_{i=1}^{n} \sigma_i^2$$

and $$(\vec{x} - \mu_x)^T C_x^{-1} (\vec{x} - \mu_x) = \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{\sigma_i^2}$$

hence,

$$f_{\vec{x}}(\vec{x}) = \frac{1}{(2\pi)^{n/2} \sqrt{\prod_{i=1}^{n} \sigma_i^2}} \cdot \exp\left(- \sum_{i=1}^{n} \frac{(x_i - \mu_i)^2}{2\sigma_i^2} \right)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \sigma_i^2}} \cdot \exp\left(- \frac{(x_i - \mu_i)^2}{2\sigma_i^2} \right).$$

$\Rightarrow$ Joint = Product of Marginals

$\Rightarrow$ components of $\vec{x}$ are independent

— x —

Transformation. $\vec{X} \longrightarrow \vec{Y} = \begin{bmatrix} g_1(\vec{x}) \\ \vdots \\ g_k(\vec{x}) \end{bmatrix}$.

$$f_{\vec{Y}}(\vec{y}) = \sum_{\substack{i:\text{ all solutions} \\ (\text{or "roots"})}} \frac{f_{\vec{X}}(\vec{x}_i)}{|J(\vec{x}_i)|}$$

$$J(x_1, \ldots, x_n) = \begin{vmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{vmatrix}.$$

If $\vec{X}$ is a continuous random vector, and A is an invertible matrix, then $\vec{Y} = A\vec{X} + b$ has the PDF:

$$f_{\vec{Y}}(\vec{y}) = \frac{1}{|\det(A)|} \cdot f_{\vec{X}}\left(A^{-1}(\vec{y} - b)\right)$$