Today $\longrightarrow$ Poisson Random Process

Plan for remaining lectures

1) Poisson Process (Today) } $\leftarrow$ HW9, [Poiss + WSS + St...]

2) Filtering of Random Processes } Homework #10

3) Markov Chains

Final Exam $\to$ Take Home, 24 hrs (Cumulative)

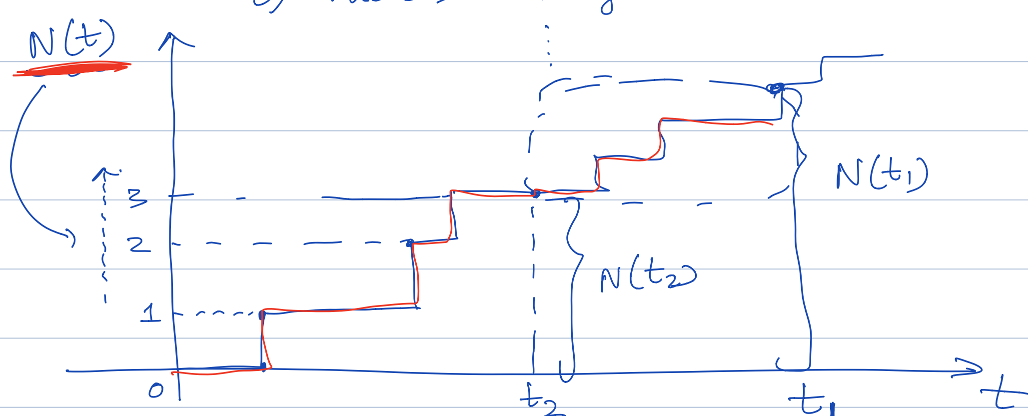
## Counting Process

$N(t)$ which starts at $t=0$.

$N(t) \longrightarrow$ # of occurences of an event till time $t$.

Applications:

1) Hits on a lik/website
2) Photons arriving @ a detector
3) Packets arriving at a router




$\boxed{N(t) = \# \text{ of counts in the interval } (0,t]}$ (Continuous) $t$ is

$N(5) = \#$ of counts in $(0,5]$.

$N(5) - N(3) = \#$ of counts in $(3,5] \longleftarrow$

$N(t) - N(s) =$ # of counts in the interval $(s, t]$
$(t \geq s)$

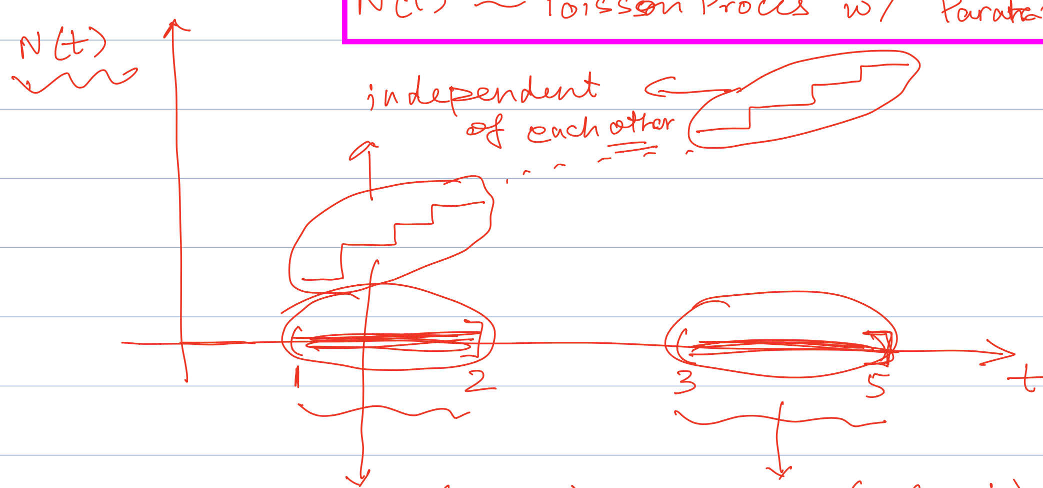
**Poisson Process** (PP) $\quad \lambda =$ rate or intensity of the PP $\quad (3,5] \rightarrow \lambda(5-3) = 2\lambda$

$N(t)$ is a PP if

(a) Number of arrivals (counts) in $(t_0, t_1]$, i.e. $N(t_1) - N(t_0)$ is a Poisson random variable with expected value of $\lambda(t_1 - t_0)$.

(b) For any two non-overlapping intervals $(t_0, t_1]$, $(t_0', t_1']$, the # of arrivals in these intervals, $(N(t_1) - N(t_0))$, $(N(t_1') - N(t_0'))$ are independent of each other.

$N(t) \sim$ Poisson Process w/ Parameter $\lambda$




$\text{Poisson}(\lambda(2-1)) \Rightarrow \text{Poisson}(\lambda)$ $\qquad$ $\text{Poisson}(\lambda(5-3)) \Rightarrow \text{Poiss.}(2\lambda)$.

**Recap** $X \sim \text{Poiss}(\alpha)$

PMF $\Rightarrow \qquad P(X = k) = \dfrac{\alpha^k \cdot e^{-\alpha}}{k!}$

$$\boxed{\begin{aligned} E[X] &= \alpha \\ Var[X] &= \alpha. \end{aligned}}$$

$k = 0, 1, 2, 3, \ldots$

PMF of $\underbrace{N(13) - N(9)}$.

$N(t)$ is a PP with intensity $\lambda$

$$N(13) - N(9) \sim \text{Poiss}(\underbrace{4\lambda})$$

$$P((N(13) - N(9)) = k) = \frac{e^{-(4\lambda)} \cdot (4\lambda)^k}{k!}$$

$k = 0, 1, 2, \ldots$

$$\underbrace{P((N(13) - N(9)) = 5)}$$

$\underbrace{\text{PMF of } N(2)}$ ?

$$N(2) - N(0) \sim \text{Poiss}(2\lambda).$$

Joint PMF of $(\underbrace{N(2), N(3)})$.

$$\underbrace{P(N(2) = \underline{k_1},\ N(3) = \underline{k_2})}.$$

$$= \underbrace{P(N(2)=k_1)}_{\text{PMF of } N(2)} \cdot \underbrace{P(N(3)=k_2 \mid N(2)=k_1)}$$

PMF of $N(2)$.

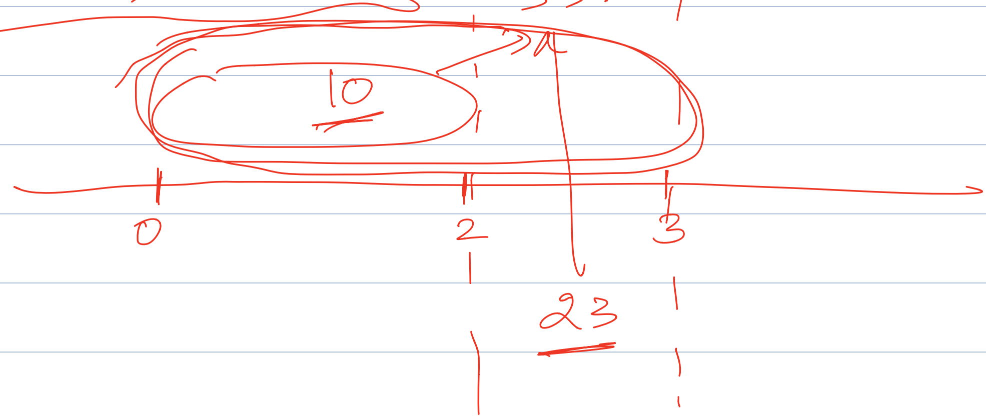
$\sim \text{Poisson}(2\lambda)$

$$\| \quad P((N(3)-N(2)) = k_2-k_1 \mid N(2)=k_1).$$

$k_1 = 10$

$k_2 = 33.$

$$P(N(2)=10,\ N(3)=33)$$




$$P(\underbrace{N(2)=10},\ \underbrace{N(3)-N(2)}=23).$$

$$= P(\underbrace{N(2)=10}_{\text{Poiss}(2\lambda)}) \times P(\underbrace{N(3)-N(2)}_{\text{Poiss}(\lambda)}=23).$$

Joint PMF of

$$\left(N(t_1), N(t_2), N(t_3), \ldots, N(t_k)\right)$$

$$t_1 < t_2 < t_3 \ldots < t_k.$$

$$\underbrace{P\left(N(t_1)=n_1,\ N(t_2)=n_2, \ldots, N(t_k)=n_k\right)}$$

$$= \left(\frac{\alpha_1^{n_1} \cdot e^{-\alpha_1}}{n_1!}\right)\left(\frac{\alpha_2^{(n_2-n_1)} \cdot e^{-\alpha_2}}{(n_2-n_1)!}\right) \cdots \left(\frac{\alpha_k^{(n_k-n_{k-1})} \cdot e^{-\alpha_k}}{(n_k-n_{k-1})!}\right),$$

$$[0,t_1]\ (t_1,t_2],\ (t_2,t_3] \ldots \ldots (t_{k-1},t_k]$$

$$n_1 \qquad (n_2-n_1) \qquad (n_3-n_2) \qquad (n_k-n_{k-1})$$

$$\alpha_1 = \lambda t_1 \qquad \alpha_2 = \lambda(t_2-t_1) \qquad \alpha_3 = \lambda(t_3-t_2) \ldots$$

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Is PP a WSS process? <u>NO</u>

$$E[N(t)] = \lambda t$$ (mean is a function of time)

$$N(t) \sim \text{Poiss}(\lambda t)$$

$$\text{Var}[N(t)] = \lambda t \quad ✓$$

$\boxed{\text{ACF}}$ of a PP.

$$E[N(t)N(s)]$$

$0 \le s < t$

$$= E[(N(t) - N(s) + N(s))\, N(s)]$$

$$= \underbrace{E[(N(t)-N(s))\, N(s)]} + E[N^2(s)]$$

indep. r.v.'s
non-overlapp.

$N(s) \sim \text{Poisson}(\lambda s)$

$$= E[(N(t)-N(s))] \cdot E[N(s)]$$

$E[N(s)] = \lambda s$

$Var[N(s)] = \lambda s.$

$$= \lambda(t-s) \times \lambda s$$

$$= \lambda^2(ts - s^2).$$

$E[N^2(s)] = \lambda s + \lambda^2 s^2$

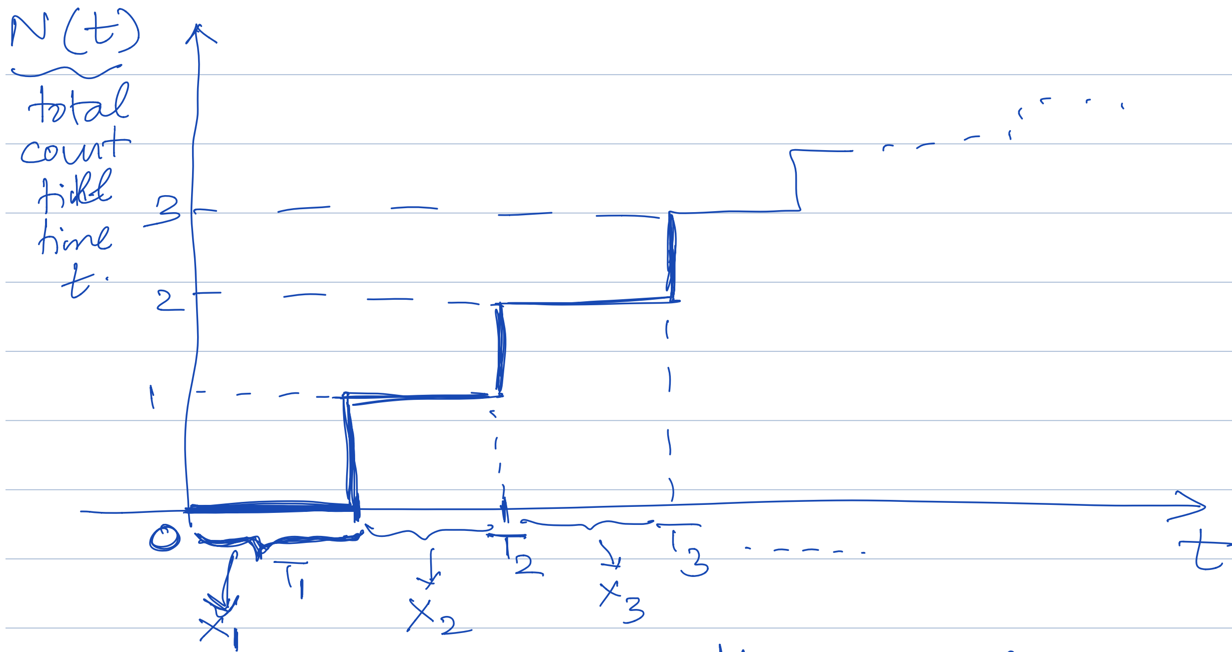
$$Var(X) = E[X^2] - (E[X])^2.$$

$$E[X^2] = Var(X) + (E(X))^2$$

$$= \lambda s + (\lambda s)^2.$$

$$E[N(t)N(s)] = \lambda^2(ts - s^2) + \lambda s + \lambda^2 s^2$$

$$= \lambda s + \lambda^2 ts.$$




$T_i$ = time of the $i^{th}$ arrival.

$$\left.\begin{aligned} X_1 &= T_1 - 0 \\ X_2 &= T_2 - T_1 \\ X_3 &= T_3 - T_2 \\ &\vdots \end{aligned}\right\} \text{inter-arrival times.}$$

For a Poisson Process w/ intensity $\lambda$,

Claim: $X_1, X_2, X_3, \ldots$ are all iid

(PDF) and each is Exponential R.V. with parameter $\lambda$.

$$f_{X_i}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

if we want to find PDF of $X_1$

$$\underbrace{X_1}_{\text{time of } 1^{st} \text{ arrival}} = T_1$$

$N(t)$

$$P(T_1 > t) = P(N(t) < 1)$$

$$= P(N(t) = 0)$$

$$1 - \underbrace{P(T_1 \le t)}_{\text{CDF of } T_1} = \frac{e^{-\lambda t} \cdot (\lambda t)^0}{0!}$$

CDF of $T_1$ = CDF of $X_1$ → PDF of $X_1$

$$= e^{-\lambda t}$$

CDF of $T_1 = 1 - e^{-\lambda t}$

CDF of $X_1 = 1 - e^{-\lambda t}$.

$$\boxed{\text{PDF of } X_1 = \lambda e^{-\lambda t}}$$

$\{T_1 > t\} \Rightarrow$ $1^{st}$ arrival occurs after time $t$

$T_1$ = time of $1^{st}$ arrival.

$\{N(t) < 1\} = \{N(t) = 0\}$.

$\{N(t) < 1\} \Rightarrow$ # of arrivals till time $t < 1$.

$N(t)$: # of arrivals until time $t$.