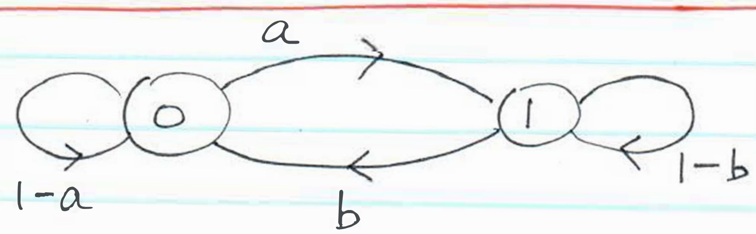
Transition Matrix

The transition probabilities P_{ij} can be arranged in a matrix P, called the transition matrix. The $(i,j)^{th}$ entry of P is P_{ij}.



$$P = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \begin{array}{l} \to \text{sum of row} = 1 \\ \to \text{sum of row} = 1 \end{array}$$

$$\left(\sum_j P_{ij} = 1 \right)$$

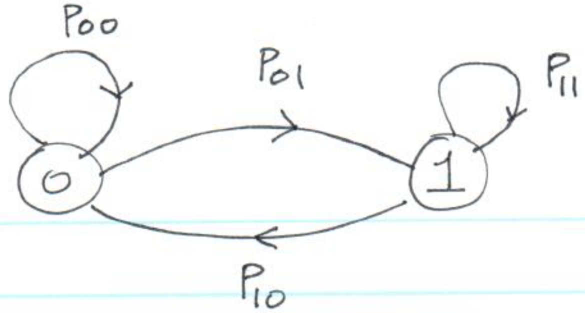
n-step Transition Probabilities

$P_{ij} = P_{ij}^{(1)} \to$ 1-step transition probability

$P_{ij}^{(n)} \to$ n-step transition probability.

$$P_{ij}^{(n)} = P\left[X_{n+m} = j \mid X_m = i \right]$$

$\underbrace{P(n)}_{\text{n-step transition matrix}} \Rightarrow (i,j)^{th}$ element is $P_{ij}^{(n)}$.



$P_{01}^{(2)}$ = Probability of going from state 0 to State 1 in <u>exactly 2</u> steps.

There are two possibilities $\Rightarrow \begin{cases} 0 \longrightarrow 1 \longrightarrow 1 \} \rightarrow P_{01} \times P_{11} \\ 0 \longrightarrow 0 \longrightarrow 1 \} \rightarrow P_{00} \times P_{01} \end{cases} \Rightarrow$ $P_{01}^{(2)}$ is the sum of these two probabilities.

$$\Rightarrow \quad P_{01}^{(2)} = P_{01}P_{11} + P_{00}P_{01}$$

$$P_{00}^{(2)} = \, ?$$

Two "possible paths" $\begin{cases} 0 \longrightarrow 1 \longrightarrow 0 \\ 0 \longrightarrow 0 \longrightarrow 0 \end{cases}$ $\begin{matrix} \rightarrow P_{01} \times P_{10} \\ \rightarrow P_{00} \times P_{00} \end{matrix}$

$$\Rightarrow \quad P_{00}^{(2)} = P_{01}P_{10} + P_{00}^2$$

$$P_{10}^{(2)} = P_{10}P_{00} + P_{11}P_{10}$$

$$\begin{pmatrix} 1 \rightarrow 0 \rightarrow 0 \\ 1 \rightarrow 1 \rightarrow 0 \end{pmatrix}$$

$$P_{11}^{(2)} = P_{11}^2 + P_{10}P_{01} \qquad \begin{matrix} 1 \rightarrow 1 \rightarrow 1 \\ 1 \rightarrow 0 \rightarrow 1 \end{matrix}$$

$$\Rightarrow \quad P(2) = \begin{bmatrix} P_{00}^{(2)} & P_{01}^{(2)} \\ P_{10}^{(2)} & P_{11}^{(2)} \end{bmatrix} = \begin{bmatrix} P_{01}P_{10} + P_{00}^2 & P_{01}P_{11} + P_{00}P_{01} \\ P_{10}P_{00} + P_{11}P_{10} & P_{11}^2 + P_{10}P_{01} \end{bmatrix}$$

$$\underbrace{P(2)}_{\text{2-Step Trans. Matrix}} = \begin{bmatrix} P_{01}P_{10}+P_{00}^2 & P_{00}P_{01}+P_{01}P_{11} \\ P_{00}P_{10}+P_{11}P_{10} & P_{10}P_{01}+P_{11}^2 \end{bmatrix} = \underbrace{\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}}_{\substack{\text{1-Step Trans. Matrix} \\ P(1)}} \times \underbrace{\begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}}_{\substack{\text{1-Step Trans. Matrix} \\ P(1)}}$$

$$\Rightarrow \quad \boxed{P(2) = P(1) \times P(1) = P(1)^2}$$

We note that $P_{ij}(n)$ must account for the probability of <u>every n-step</u> path from state i to state j. It is easier to define than to calculate these probabilities. The Chapman-Kolmogorov equations give a recursive procedure to calculate the n-step transition probabilities.

CHAPMAN-KOLMOGOROV Equations

For a finite Markov Chain with K states, the n-step transition probabilities satisfy:

$$P_{ij}(n+m) = \sum_{k=0}^{K} P_{ik}(n)\, P_{kj}(m)$$

<u>OR</u> Equivalently

$$\underbrace{P(n+m)}_{\text{(n+m)-step Transition Matrix}} = \underbrace{P(n)}_{\text{n-step Trx. Matrix}} \times \underbrace{P(m)}_{\text{m-step Transition Matrix.}}$$

Proof:

$$P_{ij}(n+m) = \sum_{k=0}^{K} P[X_{n+m}=j, X_n=k \mid X_0=i]$$

$$= \sum_{k=0}^{K} \underbrace{P[X_n=k \mid X_0=i]}_{\Downarrow} \times \underbrace{P[X_{n+m}=j \mid X_n=k, X_0=i]}_{\text{due to Markov property}}$$

$$= \sum_{k=0}^{K} P_{ik}(n) \times P[X_{n+m}=j \mid X_n=k]$$

$$= \sum_{k=0}^{K} P_{ik}(m) \times P_{kj}(m).$$

or in a matrix form

$$P(n+m) = P(n)\,P(m)$$

$$P(n) = P(n-1)\,P(1)$$
$$= P(n-2)\,P(1)^2$$
$$= P(n-3)\,P(1)^3 \cdots = P^n$$

$$\Rightarrow \quad \boxed{P(n) = P^n}$$

State Probabilities at Time n

So far, we talked about conditional probabilities P_{ij} and $P_{ij}^{(n)}$. We can use the law of total prob. to write:

$$\underbrace{P(X_n = j)}_{\substack{\downarrow \\ \text{we denote it by } p_j^{(n)}}} = \sum_i P(X_n = j \mid X_0 = i)\, P(X_0 = i)$$

$p(n) = [P_0(n) \; P_1(n) \dots P_K(n)]$ if the state space takes values $\{0, 1, 2, \dots, K\}$

$p(n)$ is called the State Probability vector at time n. $\left[\sum_{j=0}^{K} P_j(n) = 1, \right]$

$$0 \le P_j(n) \le 1.$$

$$P_j(n) = P(X_n = j) = \sum_i P(X_n = j \mid X_0 = i]\, P(X_0 = i]$$

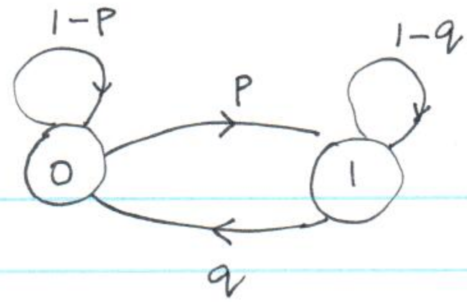
$$= \sum_i P_{ij}(n)\, P_i(0) = p(0) \times P^{(n)}$$

or

$$= \sum_i P[X_n = j \mid X_{n-1} = i]\, P(X_{n-1} = i)$$

$$= \sum_i P_{ij}\, P_i(n-1) = p(n-1) \cdot P$$

Example:



$$P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \quad \left(\begin{array}{c}\text{transition} \\ \text{matrix}\end{array}\right)$$

---

we can write the eigen-decomposition of P as:

$$P = U \Lambda U^{-1} \qquad \Lambda = \text{diag}\left(\text{eigen-values of } P\right)$$

$$\Rightarrow \quad P^2 = P \times P$$
$$= (U \Lambda U^{-1})(U \Lambda U^{-1}) = U \Lambda^2 U^{-1}$$

$$\Rightarrow \quad P^3 = P^2 \times P = (U \Lambda^2 U^{-1})(U \Lambda U^{-1})$$
$$= U \Lambda^3 U^{-1}$$

$$\Rightarrow \quad \boxed{P^n = U \Lambda^n U^{-1}}$$

---

for the 2 state M.C.,

$$\det(P - \lambda I) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 1-p-\lambda & p \\ q & 1-q-\lambda \end{bmatrix}\right) = 0$$

$\Rightarrow$ Two solutions: $\lambda_1 = 1$

$$\lambda_2 = 1 - (p+q).$$

$$U = \begin{pmatrix} 1 & -p \\ 1 & q \end{pmatrix} ; \qquad U^{-1} = \frac{1}{(p+q)} \begin{pmatrix} q & p \\ -1 & 1 \end{pmatrix}$$

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1-(p+q) \end{pmatrix}$$

$$\Lambda^n = \begin{pmatrix} 1 & 0 \\ 0 & (1-(p+q))^n \end{pmatrix}$$

$$\Rightarrow \quad P^n = U \Lambda^n U^{-1}$$

$$= \begin{pmatrix} 1 & -p \\ 1 & +q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1-(p+q))^n \end{pmatrix} \frac{\begin{pmatrix} q & p \\ -1 & 1 \end{pmatrix}}{(p+q)}$$

$$\Rightarrow \cancel{P^n = \frac{1}{(p+q)} \begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{(1-(p+q))^n}{(p+q)} \begin{pmatrix} p & -p \\ -q & q \end{pmatrix}}$$

we can write P^n as:

$$P^n = \frac{1}{(p+q)} \begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{(1-(p+q))^n}{(p+q)} \begin{pmatrix} p & -p \\ -q & q \end{pmatrix}$$

Eg: What is $P_{00}(34)$?

$$P_{00}(34) = \frac{q}{(p+q)} + \frac{(1-(p+q))^{34} p}{(p+q)}$$

Q: What is the state probability vector at time n?

$$p(n) = [p_0(n) \quad p_1(n)] = p(0) \times P^n$$

$$= [p_0(0) \quad p_1(0)] \times [P^n]$$

$$= [p_0 \quad p_1] \left\{ \frac{1}{(p+q)} \begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{\lambda_2^n}{(p+q)} \begin{pmatrix} p & -p \\ -q & q \end{pmatrix} \right\}$$

$$p(n) = \frac{1}{(p+q)} [q \quad p] + \frac{\lambda_2^n}{(p+q)} [p_0 p - p_1 q \quad -p_0 p + p_1 q]$$

State-prob vector at time $\underline{\underline{n}}$.

Limiting State Probabilities for a Finite M.C.

An important task in analyzing MCs is to examine the state probability vector $p(n)$, as n becomes large

Limiting State Probabilities

For a finite M.C., with initial state probability vector $p(0)$, the limiting state probabilities, when they exist, are defined to be the vector

$$\pi = \lim_{n\to\infty} p(n)$$

Eg. From the previous example, $p = 1/140$, $q = 1/100$

$$p(n) = \left[\, 7/12 \quad 5/12 \,\right] + \left(\frac{344}{350}\right)^n \left[\, \frac{5P_0 - 7P_1}{12} \quad \frac{7P_1 - 5P_0}{12} \,\right]$$

as $n \to \infty$

$$\underbrace{p(0) = [P_0 \ P_1]}_{\text{initial state probabilities}}$$

$$\pi = \lim_{n\to\infty} p(n) = \left[\, \frac{7}{12} \quad \frac{5}{12} \,\right]$$

For this example, the limiting state probabilities are the same regardless of $p(0)$.

In general, π may or may not exist, and if it exists, it may or may not depend on the initial state probability vector $p(0)$.

V. Imp

$\boxed{\text{Theorem}}$ If a finite M.C. with transition matrix P and initial state probability P(0) has a limiting state probability vector $\pi = \lim_{n\to\infty} p(n)$, then

$$\boxed{\pi = \pi P}$$

(row-vector) ← $[\pi_1 \dots \pi_N]$

Proof:

$$p(n+1) = p(n) P$$

$$\Rightarrow \underbrace{\lim_{n\to\infty} p(n+1)}_{\Downarrow \; \pi} = \underbrace{\left(\lim_{n\to\infty} p(n)\right)}_{\Downarrow \; \pi} P$$

$$\Rightarrow \boxed{\pi = \pi P}$$

V. Imp

Stationary Probability Vector

A state probability vector π is stationary if $\pi = \pi P$.

For a finite Markov Chain, there can be three possibilities

Case (A) $\lim_{n\to\infty} p(n)$ exists, and is independent of the initial state probability $p(0)$.

Case (B) $\lim_{n\to\infty} p(n)$ exists, and depends on $p(0)$

Case (C) $\lim_{n\to\infty} p(n)$ does NOT exist.

Case (A) → the Markov chain is "well-behaved" and it has a unique π.

Cases (B) and (C) are considered "ill-behaved".

Case (B) occurs when the MC has multiple stationary probability vectors.

Case (C) occurs when there is no stationary probability vector.

Example: 2 state M.C. $\qquad \lambda_2 = 1-(p+q)$

$$P^n = \frac{\begin{bmatrix} q & p \\ q & p \end{bmatrix}}{(p+q)} + \frac{\lambda_2^n}{(p+q)} \begin{bmatrix} p & -p \\ -q & q \end{bmatrix}$$

n-step transition matrix.

$\boxed{\text{Case(A): If } 0 < p+q < 2} \Rightarrow |\lambda_2| < 1$

$$\Rightarrow \lim_{n\to\infty} P^n = \frac{1}{(p+q)} \begin{bmatrix} q & p \\ q & p \end{bmatrix}$$

$\Rightarrow$ for any $p(0) = [p_0 \;\; p_1]$

$$\Rightarrow \lim_{n\to\infty} p(n) = \frac{[p_0 \;\; p_1]}{(p+q)} \begin{bmatrix} q & p \\ q & p \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}}_{\Downarrow}$$

π is unique.

$\boxed{\text{Case (B)} \quad p = q = 0}$ $\Rightarrow$ $\lambda_2 = 1-(p+q) = 1$

$$\Rightarrow P^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ for all } n.$$

$$\Rightarrow p(n) = p(0) P^n = p(0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = p(0)$$

$$\Rightarrow p(n) = p(0) \text{ for all } n$$

$\Rightarrow$ initial conditions completely dictate the limiting state probabilities.

$\boxed{\underline{\text{Case}}\text{ (C)} \quad p+q = 2}$ $\Rightarrow$ $\lambda_2 = 1-(p+q) = -1$

$$\Rightarrow P^n = \frac{1}{2} \begin{bmatrix} 1+(-1)^n & 1-(-1)^n \\ 1-(-1)^n & 1+(-1)^n \end{bmatrix}$$

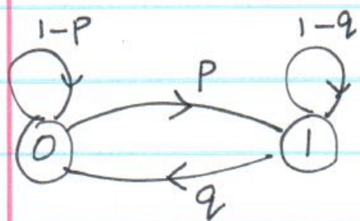
$$\Rightarrow P^{2n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } P^{2n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow p(2n) = [p_0 \;\; p_1], \text{ and } p(2n+1) = [p_1 \;\; p_0]$$

i.e, the chain has a periodic behavior and does not permit the existence of limiting state probabilities.

Case (A)

$(0 < p+q < 2)$

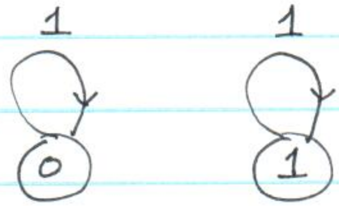


"well-behaved"

$$\pi = \left[\frac{q}{p+q} \quad \frac{p}{p+q}\right]$$

Case (B)

$(p+q=0)$

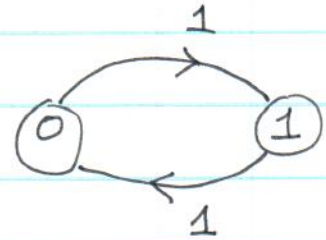


$p(n) = p(0)$

$(p+q=0)$

Case (C)

$(p+q=2)$



π does not exist!

$(p+q=2)$

Eg :
$$P = \begin{bmatrix} 0 & 1/4 & 3/4 \\ 0 & 1/2 & 1/2 \\ 2/5 & 2/5 & 1/5 \end{bmatrix}$$

Stationary distribution : $\Rightarrow [\pi_0 \quad \pi_1 \quad \pi_2] = \left[\frac{1}{6} \quad \frac{5}{12} \quad \frac{5}{12}\right]$

$$\pi = \pi P$$

$$\Rightarrow \quad \cancel{\pi_0} \; \cancel{\pi_1} \; \cancel{\pi_2} = [\pi_0 \quad \pi_1 \quad \pi_2] = \left[\frac{2}{5}\pi_2 \quad \frac{\pi_0}{4} + \frac{\pi_1}{2} + \frac{2}{5}\pi_2 \quad \frac{3}{5}\pi_0 + \frac{\pi_1}{2} + \frac{\pi_2}{5}\right]$$

Solving these $\boxed{\pi_0 = 2/5\, \pi_2}$

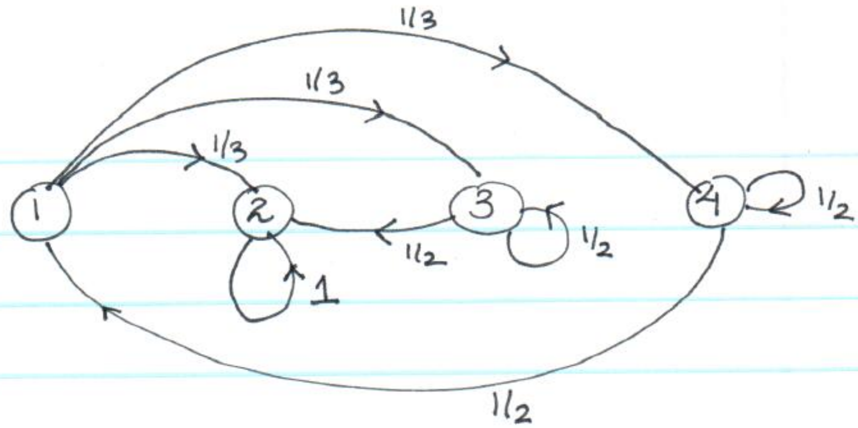
$$\pi_1 = \frac{\pi_0}{4} + \frac{\pi_1}{2} + \frac{2}{5}\pi_2 = \frac{\pi_0}{4} + \frac{\pi_1}{2} + \pi_0$$

$$\Rightarrow \frac{\pi_1}{2} = \frac{5}{4}\pi_0 = \frac{5}{4} \times \frac{2}{5}\pi_2 \Rightarrow \boxed{\pi_1 = \pi_2}$$

also, $\pi_0 + \pi_1 + \pi_2 = 1 \Rightarrow \frac{2}{5}\pi_2 + \pi_2 + \pi_2 = 1$

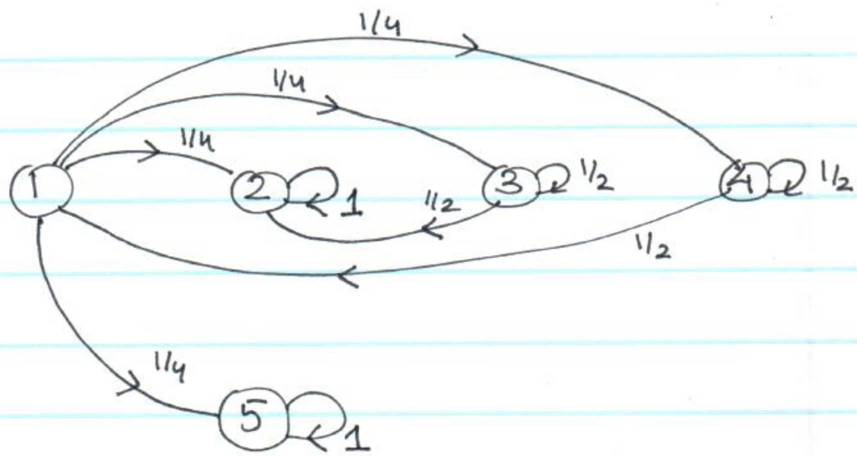
$$\Rightarrow \pi_2 = 5/12$$

Example:



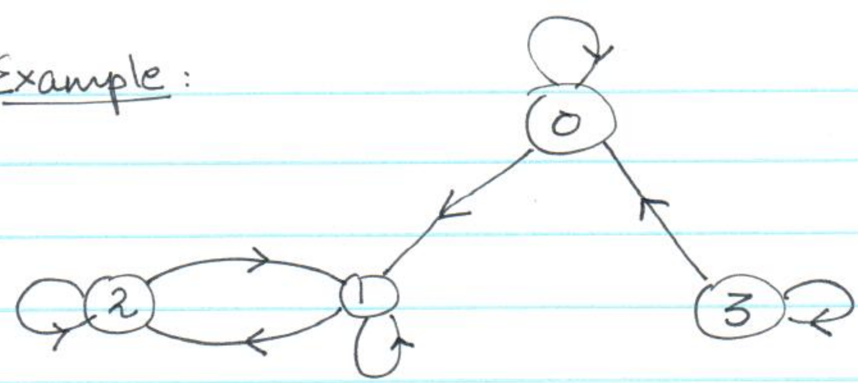
In the above ~~MC~~ example, the Markov chain will converge to state 2 no matter where it started from.

Example.



In this example, the stationary distribution will depend on the starting state. If one starts at 5, it will remain there, if it starts at 2, it remains there.

Example:



Does this MC converge?

* If we start in state 3, after some $n > 0$, eventually we will go to state 0, and then go to states 1 and 2 and stay there

* If we start in state 0, after some $n > 0$, we ~~we~~ eventually go to states 1 & 2 and stay there.

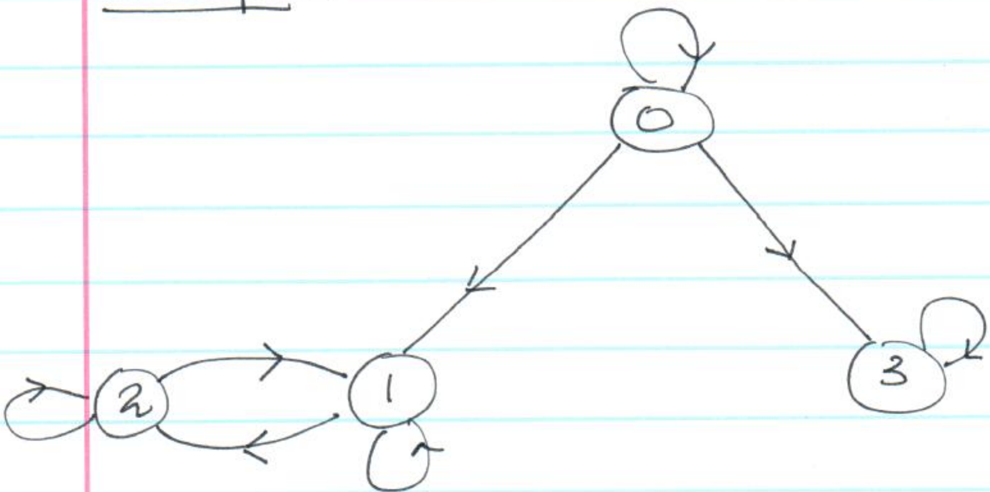
* If initial state is either 1 or 2, the MC stays in these states and never visits 0, 3.

$\Rightarrow$ No matter where the MC starts, as $n \to \infty$, the chain will converge to a unique stationary distribution

$$\cancel{\pi = [\underbrace{*}_{\pi_1} \quad \underbrace{*}_{\pi_2} \quad \underbrace{0}_{\pi_0} \quad \underbrace{0}_{\pi_3}]}$$

$$\pi = [\underbrace{0}_{\pi_0} \quad \underbrace{*}_{\pi_1} \quad \underbrace{*}_{\pi_2} \quad \underbrace{0}_{\pi_3}]$$

Example:

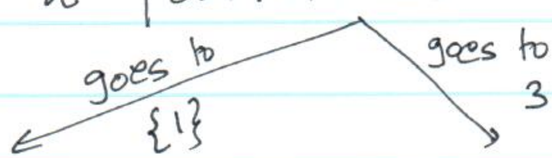


* If initial state is 3, MC stays there forever $\Rightarrow p(n) = [0\ 0\ 0\ 1]$

* If initial state is 1 or 2, MC stays there &

$$p(n) = [0\ *\ *\ 0]$$

* If initial state is 0, after some $n > 0$, there are 2 possibilities



$[0\ *\ *\ 0]$ $\qquad$ $[0\ 0\ 0\ 1]$.

i.e, this M.C. does not converge.

To formally determine the convergence properties of M.C.'s, we will see that MC(s) with certain structural properties will converge to a unique stationary probability vector, which is independent of the initial distribution $p(0)$.

State Classification

Accesibility

State j is accessible from state i, i.e

$$i \longrightarrow j, \quad \text{if} \quad P_{ij}(n) > 0 \text{ for some } n > 0.$$

$i \to j$ if in the MC graph, there is a path from i to j

Communicating States

States i and j communicate, if $i \to j$ and $j \to i$

Also written as $\quad i \longleftrightarrow j$

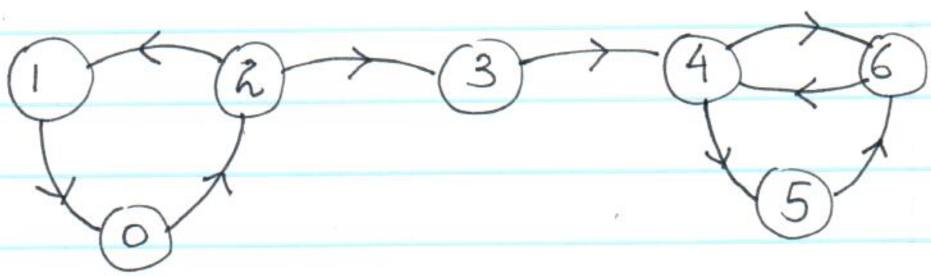
$i \leftrightarrow j$ if $\Rightarrow$ there is a path from i to j
$\Rightarrow$ there " " " " j to i.

Remark: Easy to check that if states j and k communicate with i $\Rightarrow$ $j \leftrightarrow k$

Communicating Class

A communicating class is a non-empty set of states C such that if $i \in C$, and $j \in C$ if and only if $i \leftrightarrow j$

Eg:



This chain has three communicating classes.

* We note that states 0, 1, 2 form a communicating class
  $C_1 = \{0, 1, 2\}$

* $C_2 = \{4, 5, 6\}$.

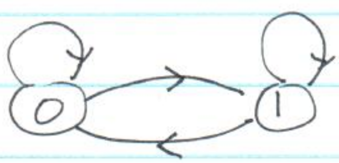
* $C_3 = \{3\}$.

Irreducible Markov Chain

A Markov Chain is called irreducible if it has only one communicating class.

(or: for every pair of states (i, j), $i \leftrightarrow j$)

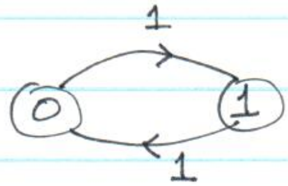
Eg



Irreducible.

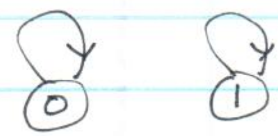
$C = \{0, 1\}$

one-communicating class



Irreducible

$C = \{0, 1\}$



NOT Irreducible.

$C_0 = \{0\}$, $C_1 = \{1\}$ Two comm. classes.

If a M.C. is irreducible, we have the guarantee that it will not get stuck in one state.

To guarantee convergence to a unique stationary distribution, we also need to ~~define~~ look at another structural property, namely periodicity.

Period of a State

A State i has a period d if

$$d_i = \underbrace{GCD}_{\substack{\text{greatest}\\ \text{common}\\ \text{divisor.}}} \{ n : P_{ii}(n) > 0 \}$$

If $d_i = 1$, then the state is APERIODIC.

FACT: All states in the same class have the Same period.

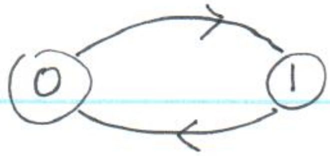
FACT: In an irreducible Markov chain, (which is formed of a single class), all the states have the same period. Hence, it is sufficient to check the periodicity of one state.

Main Theorem

An irreducible MC, where all the states are aperiodic has a UNIQUE Stationary distribution π, i.e $\lim_{n \to \infty} p(n) = \pi$, $\pi = \pi P$.

and π is independent of the initial condition $p(0)$.

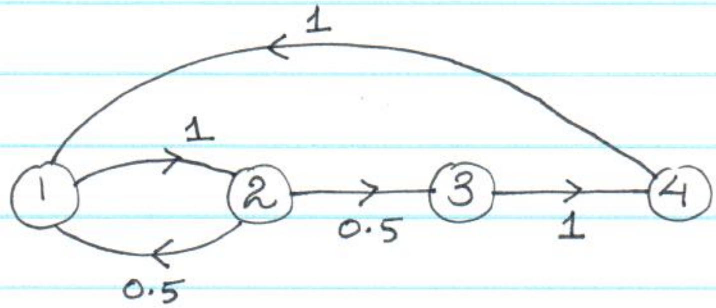
Eg:



$\cancel{P_{00}(2)}$ Irreducible ✓.

$0 \to 1 \to 0 \Rightarrow$ period $= 2 \Rightarrow$ Periodic
(of state 0)

Eg



for any (i,j), $i \leftrightarrow j \Rightarrow$ this chain is irreducible.

Period of state 1:

Start at 1 — Return at 1

$1 \longrightarrow 2 \longrightarrow 1 \Rightarrow 2$ steps

$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 1 \Rightarrow 4$ steps

$GCD(2,4) = 2$

$\Rightarrow$ Period of State 1 $= 2 \Rightarrow$ Periodic.

Period of State 4:

$4 \to 1 \to 2 \to 3 \to 4 \Rightarrow 4$ steps

$4 \to 1 \to 2 \to 1 \to 2 \to 3 \to 4 \Rightarrow 6$ steps

$GCD(4,6) = 2.$

Period of State 3:

$3 \to 4 \to 1 \to 2 \to 3$ 4 steps

$3 \to 4 \to 1 \to 2 \to 1 \to 2 \to 3$ ⇒ 6 steps

$GCD(4, 6) = 2$

Period of State 2:

$2 \to 1 \to 2$ 2 steps

$2 \to 3 \to 4 \to 1 \to 2$ 4 steps

$GCD(2, 4) = 2$

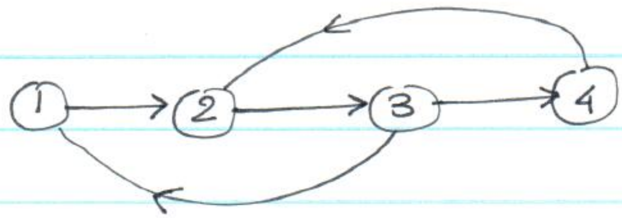
⇒ Period of MC = 2

~~⇒ this MC is N~~

MC is irreducible & Periodic

& does NOT have a unique stationary distribution.

Eg:



check:
MC is irreducible.

Period of state 1 :

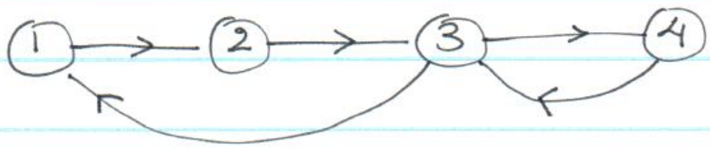
$1 \to 2 \to 3 \to 1$ (3 steps)

$1 \to 2 \to 3 \to 4 \to 2 \to 3 \to 1$ (6 steps)

$GCD(3, 6) = 3$

$\Rightarrow$ MC is irreducible and Periodic.

Eg:



MC is irreducible

Period of State 1 :

$1 \to 2 \to 3 \to 1$ (3 steps)

$1 \to 2 \to 3 \to 4 \to 3 \to 1$ (5 steps)

$GCD(3, 5) = 1$

$\Rightarrow$ MC is irreducible and APERIODIC

$\Rightarrow$ $\boxed{\text{has a unique } \pi}$