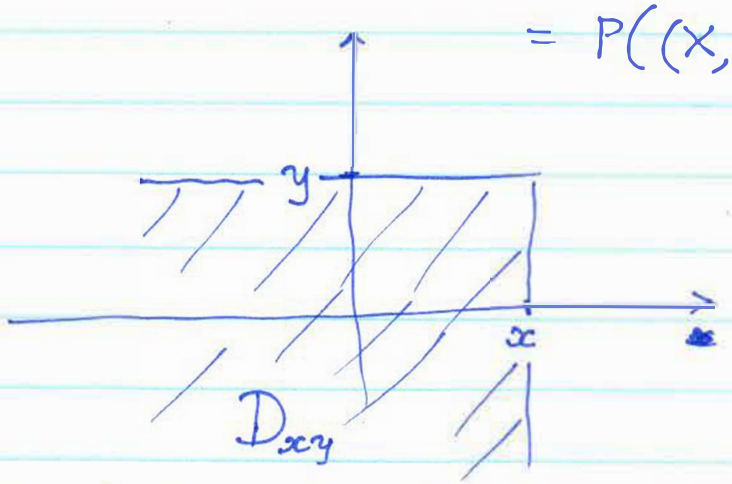
Lecture 14

TWO RANDOM VARIABLES

For two random variables X and Y, the joint distribution $F_{XY}(x,y)$ is the probability:

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$

$$= P((X,Y) \in D_{xy})$$



If we denote the region D corresponding to $X \le x$, $Y \le y$.

Properties of Joint Distribution

P1 :

$F_{XY}(-\infty, y) = 0 \longrightarrow P(X = -\infty) = 0$

$\Rightarrow P(X = -\infty, Y \le y) \le P(X = -\infty) = 0$

$F_{XY}(x, -\infty) = 0$

$F_{XY}(\infty, \infty) = 1$

$\longrightarrow \{X \le +\infty, Y \le +\infty\} = S$

$\Rightarrow P(\quad \downarrow \quad) = P(S) = 1$

$F(\infty, \infty)$.

P2 $P(x_1 < X \le x_2,\ Y \le y)$

$$= F(x_2, y) - F(x_1, y)$$

Event:

$$\{X \le x_2, Y \le y\} = \underbrace{\{X \le x_1, Y \le y\}} \cup \underbrace{\{x_1 < X \le x_2, Y \le y\}}$$

Mutually Exclusive Events.

$$\underbrace{P(X \le x_2, Y \le y)}_{= F(x_2, y)} = \overbrace{P(X \le x_1, Y \le y)}^{F(x_1, y) =} + P(x_1 < X \le x_2, Y \le y)$$

$$\Rightarrow \boxed{P(x_1 < X \le x_2, Y \le y) = F(x_2, y) - F(x_1, y)}$$

Similarly, easy to show that

$$P(X \le x, y_1 < Y \le y_2) = F(x, y_2) - F(x, y_1)$$

P3 $P(x_1 < X \le x_2 \,,\, y_1 < Y \le y_2)$

$$= F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1)$$

Proof $\rightarrow$ as before; breaking it into mutually exclusive events....

JOINT DENSITY

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \, \partial y}$$

$$F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(\alpha, \beta)\, d\alpha\, d\beta$$

$$P((X, Y) \in R) = \iint_R f_{XY}(x, y)\, dx\, dy.$$

MARGINAL STATISTICS

The statistics (PDF/CDF) of each r.v. are called its marginals.

$F_X(x)$ is the marginal distribution of X

$f_X(x)$ is the " density of X.

$F_Y(y)$ is the " distribution of Y

$f_Y(y)$ " " " density of Y

$\boxed{\text{Marginals can be obtained from the JOINT}}$

$$F_X(x) = F_{XY}(x, \infty) \qquad F_Y(y) = F_{XY}(\infty, y)$$

$$\underbrace{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy}_{\Downarrow} \qquad \underbrace{f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}_{\Downarrow}$$

Marginals of X | Marginals of Y.

INDEPENDENT RANDOM VARIABLES

Two random variables X and Y are independent if the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for <u>all</u> A, B.

i.e. $P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$

Applying this to $\{X \leq x\}$ and $\{Y \leq y\}$ if X and Y are independent, then

$$P(X \leq x, Y \leq y) = P(X \leq x) \cdot P(Y \leq y)$$

or

$$\boxed{F_{XY}(x, y) = F_X(x) \cdot F_Y(y)}$$

$$\text{Joint} = \begin{pmatrix} \text{Marginal} \\ \text{of } X \end{pmatrix} \cdot \begin{pmatrix} \text{Marginal} \\ \text{of} \\ Y \end{pmatrix}$$

Also,

$$\boxed{f_{XY}(x, y) = f_X(x) \cdot f_Y(y)}$$

For independent r.v.'s X and Y,

Joint (CDF/PDF) = Product of Marginals (CDF/PDF)

FACT: If the random variables X and Y are independent, then the random variables $Z = g(X)$ and $W = h(Y)$ are also independent.

Proof: Let A_z be the points on x-axis such that $g(x) \le z$

Let B_ω be the points on Y-axis such that $h(y) \le \omega$

$$\Rightarrow \{Z \le z\} = \{X \le A_z\}$$

and

$$\{W \le \omega\} = \{Y \le B_\omega\}$$

$$\Rightarrow F_{ZW}(z, w) = P(Z \le z, W \le w)$$

$$= P(X \le A_z, Y \le B_w)$$

(since X & Y are independent)

$$= P(X \le A_z) \cdot P(Y \le B_\omega)$$

$$= P(Z \le z) \cdot P(W \le \omega)$$

$$= F_Z(z) \cdot F_W(w)$$

$\Rightarrow$ Z and W are also independent

JOINT PMF (for discrete r.v.'s X and Y)

$$P_{X,Y}(x,y) = P(X=x, Y=y)$$

Eg: $P_{Q,G}(q,g)$ is given as in the following table:

$P_{Q,G}(q,g)$	$g=0$	$g=1$	$g=2$	$g=3$
$q=0$	0.06	0.18	0.24	0.12
$q=1$	0.04	0.12	0.16	0.08

$$P(Q=0) = P(0,0) + P(0,1) + P(0,2) + P(0,3)$$
$$= 0.06 + 0.18 + 0.24 + 0.12$$
$$= 0.6$$

$$P(Q=1) = 0.4$$

$$P(Q=G) = P(0,0) + P(1,1) = 0.06 + 0.12$$
$$= 0.18$$

$$P(G>1) = (0.18 + 0.24 + 0.12) + (0.12 + 0.16 + 0.08).$$

Marginal PMF(s)

$$\boxed{\begin{aligned} P_X(x) &= \sum_y P_{X,Y}(x,y) \\ P_Y(y) &= \sum_x P_{X,Y}(x,y) \end{aligned}}$$

Independence

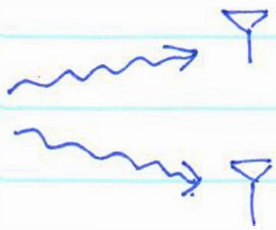
X and Y are independent r.v.'s if

$$P_{XY}(x,y) = P_X(x)\,P_Y(y).$$

Functions of Two Random Variables

In many situations, we observe 2 r.v.'s & use their values to compute a new r.v.

Eg: cellular base stations with 2 antennas



amplitude of signals at antenna 1 $\Rightarrow X$

antenna 2 $\Rightarrow Y$.

* We can choose the signal with a larger amplitude & receiver uses

$$W = \max(X, Y) \leftarrow \{\text{Selection diversity combining}\}.$$

* Receiver can add the signals

$$W = X + Y \leftarrow \{\text{equal gain combining}\}$$

* Receiver can unequally combine the signals

$$W = aX + bY \quad \{\text{Maximal ratio combining}\}.$$

All of such processes appear in practical radio receivers.

For discrete r.v.'s X and Y, the random variable $W = g(X,Y)$ has the pmf

$$P_W(w) = \sum_{(x,y):\, g(x,y)=w} P_{X,Y}(x,y).$$

When X and Y are continuous r.v's, and $g(x,y)$ is a continuous function, $W = g(X,Y)$ is a continuous r.v.. To find its PDF, i.e $f_W(w)$, it is usually helpful to first find the CDF, ie $F_W(w)$ & then take its derivative.

$$F_W(w) = P(W \le w) = \iint_{g(x,y)\le w} f_{X,Y}(x,y)\,dx\,dy$$

and $f_W(w) = \dfrac{d\,F_W(w)}{dw}$.

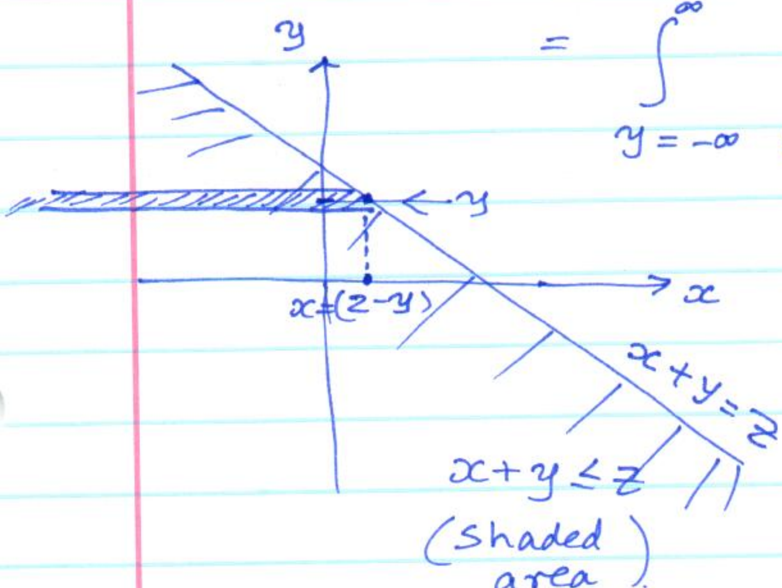
Examples → Function of 2 r.v.'s

Eg 1 $Z = X + Y$. PDF of Z $(f_Z(z))$

$$F_Z(z) = P(Z \le z)$$

$$= P(X + Y \le z)$$

$$= \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{(z-y)} f_{X,Y}(x,y)\,dx\,dy.$$



$$F_Z(z) = \int_{y=-\infty}^{\infty} \left[\int_{x=-\infty}^{(z-y)} f_{X,Y}(x,y)\,dx \right] dy$$

$$\Rightarrow \quad f_Z(z) = \frac{d}{dz} F_Z(z) = \int_{y=-\infty}^{\infty} \left[\frac{d}{dz} \int_{x=-\infty}^{(z-y)} f_{X,Y}(x,y)\,dx \right] dy.$$

Leibniz Rule for Differentiation under Integral Sign

$$\frac{d}{dz}\left\{\int_{a(z)}^{b(z)} f(x,y)\,dx\right\}$$

$$= \left(\frac{d\,b(z)}{dz}\right)\cdot f(b(z),y) - \left(\frac{d\,a(z)}{dz}\right)\cdot f(a(z),y)$$

$$+ \int_{a(z)}^{b(z)} \frac{\partial}{\partial z} f(x,y)\cdot dx$$

Returning to

$$\frac{d}{dz}\int_{x=-\infty}^{z-y} f_{X,Y}(x,y)\,dx.$$

$$= 1\cdot f_{X,Y}(z-y,y) - 0 + \int_{-\infty}^{z-y} (0)\cdot dx$$

$$= f_{X,Y}(z-y,y).$$

$$\Rightarrow \quad f_Z(z) = \int_{y=-\infty}^{\infty} f_{X,Y}(z-y,\, y)\, dy$$

for $Z = X + Y$.

If X and Y are independent r.v.'s, then

$$f_{X,Y}(z-y,\, y) = f_X(z-y)\, f_Y(y)$$

$$\Rightarrow \quad f_Z(z) = \int_{y=-\infty}^{\infty} f_X(z-y) f_X(y)\, dy$$

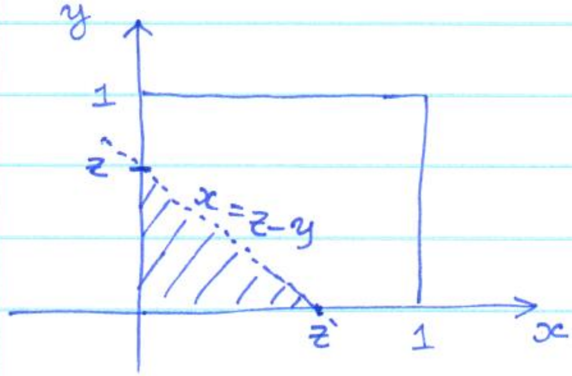
$$= \int_{x=-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$\Rightarrow$ Convolution of the functions $f_X(z)$ and $f_Y(z)$. (expressed in two ways).

$\Rightarrow$ If X and Y are independent, then density of $Z = X+Y$ is the convolution of their densities.

Ex X and Y are independent uniform r.v.'s in the common interval $[0,1)$

$$Z = X + Y, \text{ find } f_Z(z)$$



$$\underbrace{f_{X,Y}(x,y) = f_X(x) f_Y(y)}$$

Since X and Y are independent...

For $0 \le z < 1$

$$F_Z(z) = P(Z \le z)$$

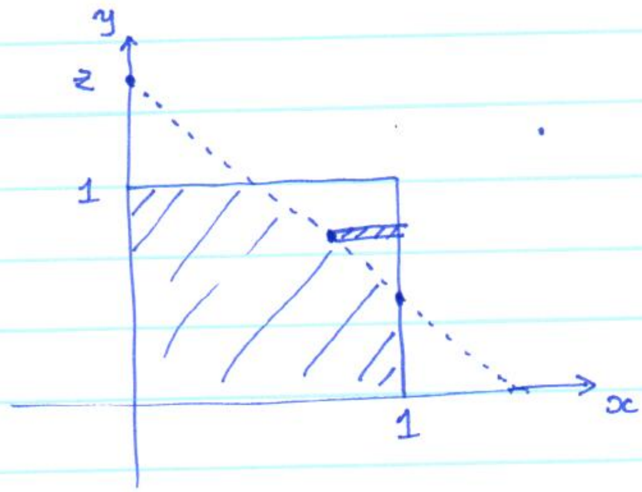
$$= P(X + Y \le z)$$

$$= \int_{y=0}^{z} \int_{x=0}^{z-y} \underbrace{f_{X,Y}(x,y)}_{\substack{\shortparallel \\ 1}} \, dx \, dy$$

$$= \int_{y=0}^{z} (z-y)\, dy = zy - \frac{y^2}{2} \Big|_0^z$$

$$= z^2 - \frac{z^2}{2} - (0-0) = \frac{z^2}{2} \quad 0 \le z < 1$$

For $1 \leq z < 2$



$$\underbrace{P(Z \leq z)}_{\text{shaded}} = 1 - \underbrace{P(Z > z)}_{\text{unshaded}}$$

$$= 1 - \int_{y=z-1}^{1} \int_{x=z-y}^{1} dx\, dy$$

$$= 1 - \int_{z-1}^{1} \left(x \Big|_{z-y}^{1} \right) dy = 1 - \int_{z-1}^{1} (1 - z + y)\, dy.$$

$$\cancel{= 1 - zy - \frac{y^2}{2} \Big|_{z-1}^{1}}$$

$$= 1 - \frac{(2-z)^2}{2} \qquad 1 \leq z < 2$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \begin{cases} z & 0 \leq z < 1 \\ 2 - z & 1 \leq z < 2 \\ 0 & \text{otherwise.} \end{cases}$$