Combinatorics, Counting Methods & Bernoulli Trials

* Counting

For a finite sample space S, with equally likely outcomes,

$$P(\underset{\text{Event}}{A}) = \frac{|A| \rightarrow \text{\# of elements in } A}{|S| \rightarrow \text{\# of elements in } S}$$

* Multiplication Principle

Eg: You wish to purchase Iphone 7 & have the following choices:

L or S screen | 64 GB or 128 GB or 256 GB storage | B or W or R color.

How many choices do you have?

Ans: $2 \times 3 \times 3 = 18$ choices.

$$\left\{ \begin{array}{l} L-64-B \\ L-64-W \\ L-64-R \\ \vdots \\ S-256-R \end{array} \right\}$$

$\boxed{\begin{array}{l}\text{If we perform } r \text{ experiments, such that } k^{th} \text{ experiment} \\ \text{has } n_k \text{ possible outcomes, } k = 1, 2, \ldots, r, \\ \text{Total \# of possible outcomes} = n_1 \times n_2 \times \cdots \times n_r.\end{array}}$

Eg: You are supposed to choose a password as follows:

→ two lower case letters (a to z) followed by
→ one capital letter (A to Z) followed by
→ four digits (0, 1, .., 9).

For instance xyT1319 is a valid password.

(a) Find the total number of passwords N.

$$N = 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^3 \times 10^4.$$

(b) A hacker writes a program to randomly & independently generate 10^8 passwords. (same p.w. can be generated twice). If one of the above passwords match your p.w., your account will be hacked. What is the probability that the hacker is successful?

Let G_i be the event that hacker's i^{th} guess matches your p.w, $i = 1, 2, \ldots, 10^8$

$$P(G_i) = \frac{1}{N}$$

P_{hack} = Prob. that at least one of hacker's p.w. matches

$$= P(G_1 \cup G_2 \cup G_3 \cdots \cup G_{10^8}) = P\left(\bigcup_i G_i\right)$$

$$P\left(\left(\bigcup_i G_i\right)^c\right) = P\left(\bigcap_i G_i^c\right)$$

$$= P\left(G_1^c \cap G_2^c \cap \dots \cap G_{10^8}^c\right)$$

$$= P(G_1^c)\cdot P(G_2^c) \cdots P(G_{10^8}^c)$$

Since G_i's are independent events

$$= \left(1-\frac{1}{N}\right)\cdot\left(1-\frac{1}{N}\right)\cdots\cdot\left(1-\frac{1}{N}\right)$$

$$= \left(1-\frac{1}{N}\right)^{10^8}$$

$$P_{hack} = P\left(\bigcup_i G_i\right)$$

$$= 1 - P\left(\left(\bigcup_i G_i\right)^c\right)$$

$$= 1 - \left(1-\frac{1}{N}\right)^{10^8} = 1 - \left(1-\frac{1}{26^3\times 10^4}\right)^{10^8} \approx 0.4339.$$

General Terminology for Counting Problems

* Sampling — choosing an element from a set.
* With or Without repacement
  - if we draw multiple samples from a set
  - With replacement ⇒ we put each object back after each draw.
  - Without replacement ⇒ we do not put the drawn object back.
* Ordered or Unordered
  - if order matters (i.e $(1,3,4) \neq (3,1,4)$) ⇒ Ordered
  - if ordering does Not matter (i.e $(1,3,4) = (3,1,4)$)

There are 4 possibilities

- Ordered Sampling with replacement
- Ordered Sampling without replacement
- Unordered " with "
- Unordered " without "

(I) Ordered Sampling with Replacement (OS, wR)

- Set → $A = \{1, 2, \ldots, n\} \rightarrow n$ elements.
- Draw k samples (OS, wR) from set A with n elements.

$$\boxed{\# \text{ of Possible ways} = n \times n \times \cdots \times n = n^k}$$

Eg: $A = \{1, 2, 3\}$; $k = 2$

$\{1,1\}, \{1,2\}, \{1,3\}, \{2,1\}, \{2,2\}, (2,3), (3,1), (3,2), (3,3)$

3 ways 3 ways ⇒ Total 9 ways.

(II) Ordered Sampling without Replacement (OS, w/oR)

$$\boxed{\begin{aligned}\# \text{ of Possible ways} &= n \times (n-1) \times \cdots \times (n-(k-1)) \\ &= \frac{n!}{(n-k)!} = P_k^n\end{aligned}}$$

k-boxes

n ways, $(n-1)$ ways, ..., $(n-k+1)$ ways

Eg: $A = \{1, 2, 3\}$; $k=2$

$(1,2)$
$(1,3)$ $(3,1)$
$(2,1)$ $(3,2)$
$(2,3)$,

Permutation of n-elements

of n-permutations of n elements

$$P^n_n = n \times (n-1) \times \dots \times (n-(n-1)) \qquad [0! = 1]$$
$$= n!$$

of k-permutations of n elements

$$P^n_k = \frac{n!}{(n-k)!} \qquad 0 \leq k \leq n$$

Eg: In a group of k people, what is the probability that at least two of them have the same birthday? Suppose that there are $n = 365$ days in a year and all days are equally likely to be the birthday of a specific person.

Ans: A = Event that at least two people have same birthday.

Case 1: if $k > n$ then $P(A) = 1$ (Eg if you have 400 people)

Case 2: if $k \leq n$ then this becomes interesting!

A^c = Event that no two people have the same birthday.

$$P(A) = 1 - P(A^c)$$
$$= 1 - \frac{|A^c|}{|S|} = \boxed{1 - \frac{n!}{(n-k)!\, n^k}}$$

$|S| \stackrel{?}{=}$ Total # of possible sequences of birthdays of k people?

$$= n \times n \times \dots \times n = n^k$$

$$|A^c| = n \times (n-1) \times \dots \times (n-k+1)$$
$$= \frac{n!}{(n-k)!}$$

Eg: $k = 23$ people

$$P(A) = 1 - \frac{365!}{342! \times (365)^{23}} \approx 0.5073$$

$k = 57 \dashrightarrow P(A) \approx 0.99$

Eg: You are in a party with $(k-1)$ people. What is the probability that at least one person in the party has the same birthday as yours?

Ans: Your birthdate is fixed $\dashrightarrow$ say Aug. 28

A = Event that atleast one person has same birthday as Aug. 28.

     = Event that at least one person has Aug. 28 as their birthday.

A^c = Event that No one has Aug. 28 as their birthday.

$$P(A) + P(A^c) = 1 \Rightarrow P(A) = 1 - P(A^c)$$
$$= 1 - \frac{|A^c|}{|S|}$$

$$|A^c| = (n-1) \times (n-1) \times \cdots \times (n-1) = (n-1)^{k-1}$$
$$|S| = n \times \cdots\cdots \times (n) = n^{k-1}$$

$$\Rightarrow P(A) = 1 - \frac{(n-1)^{k-1}}{n^{k-1}} = 1 - \left(\frac{n-1}{n}\right)^{k-1}$$

$k = 23 \Rightarrow P(A) \approx 0.0586$

(III) Unordered Sampling without Replacement. (Combinations).

- $A = \{1, 2, \ldots, n\}$
- k-elements drawn. (US w/o R)

$$\boxed{\# \text{ of ways} = \frac{\left(\frac{n!}{(n-k)!}\right)}{k!} = \frac{n!}{(n-k)!\,k!} = \binom{n}{k} \text{ or } C^n_k}$$

Eg: $A = \{1, 2, 3\}$; $k = 2$

$\underbrace{\{1,2\}, \{1,3\}, \{2,3\}}_{3 \text{ combinations}}$

$\binom{n}{k} \Rightarrow$ Binomial coefficient

$0 \le k \le n$

Binomial Theorem: For any integer $n \ge 0$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Other important identities: (Try to prove these yourself...)

1. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

2. For $0 \le k < n$, $$\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$$ (Pascal's Rule)

3. $$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$ (Vandermonde's Identity)

Eg: We choose 3 cards from a deck of 52 cards. What is the prob. that they contain atleast one Ace?

Ans: $P(A) = 1 - P(A^c) = 1 - \dfrac{|A^c|}{|S|}$

$|S| = \binom{52}{3}$

A^c = Event that there is <u>No ace</u> in the three cards.

# of non-Ace cards $= 52-4 = 48$

$|A^c| = \binom{48}{3}$. $\Rightarrow P(A) = 1 - \dfrac{\binom{48}{3}}{\binom{52}{3}}$.

Eg: How many distinct sequences can we make using 3 letter "A"s & 5 letter "B"s?
(Eg: AAABBBBB, AABABBBB etc...)

Ans: $\binom{8}{3}$ $\underbrace{\square\ \square\ \square\ \cdots\ \square}_{8 \text{ positions to fill.}}$

$\left.\begin{array}{l}\text{3 - positions for A's} \\ \text{5 - " " B's}\end{array}\right\}$ once we pick 3 postns for A's B's will automatically be <u>filled</u>.

$\Rightarrow$ Total # of ways $= \binom{8}{3} = \binom{8}{5}$.

Bernoulli Trials & Binomial Distribution

Bernoulli Trial → random experiment with two outcomes
"success" or "failure"
or Eg: "head" or "taie"

Prob of "success" $= p$
Prob of "failure" $= q = 1-p$

Binomial Experiment (or Repeated Bernoulli Trials) → if we perform n independent Bernoulli trials and count the total number of successes.

Eg: For a coin, $P(H) = p$. $P(T) = 1-p$. We toss the coin 5 times.

(a) What is Prob. of outcome THHHH ?

$$P(THHHH) = P(T) \times P(H) \times P(H) \times P(H) \times P(H) \quad \left[\text{since trials are independent}\right]$$
$$= (1-p) \cdot p^4$$

(b) What is Prob. of HTHHH ?

$$P(HTHHH) = (1-p)p^4$$

(c) What is Prob. of HHTHH ? $\Rightarrow (1-p)p^4$

(d) What is the Prob. that we observe exactly four heads and one tails?

$$B = \{THHHH, HTHHH, HHTHH, HHHTH, HHHHT\}$$

Event

$$P(B) = 5 \times (1-p)p^4$$

(e) What is the Prob. that we observe exactly three heads & two tails?

$$C = \{HHHTT, HHTHT, \ldots\ldots, TTHHH\}$$

$$P(C) = |C| \times (1-p)^2 p^3$$

How do we find $|C|$?

$|C|$ = # of distinct sequences of length 5 we can create using two tails & three heads.

$$= \binom{5}{3}$$

$$\Rightarrow P(C) = \binom{5}{3}(1-p)^2 p^3$$

(f) If we toss the coin n times, what is the prob of observing exactly k heads & $(n-k)$ tails?

$$P(k \text{ heads } \& \ (n-k) \text{ tails}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial Formula

For n independent Bernoulli trials, where each trial has success prob. p, the probability of k successes is

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

From Binomial to Multinomial.

Eg: For a flood relief effort, 8 agencies need to provide support.
- 4 agencies provide food.
- 3 agencies provide shelter
- 1 agency provides security.

How many ways the agencies can be divided into such groups?

Ans: We can choose 4 out of 8 agencies for food

$$\Rightarrow \# \text{ of ways} = \binom{8}{4}$$

We can then choose 3 out of remaining 4 for shelter

$$\Rightarrow \# \text{ of ways} = \binom{4}{3}$$

We can then choose 1 " " " 1 " security

$$\Rightarrow \# \text{ of ways} = \binom{1}{1}$$

$$\Rightarrow \text{Total } \# \text{ of ways} = \binom{8}{4} \times \binom{4}{3} \times \binom{1}{1}$$ [by the multiplication principle]

$$= \frac{8!}{4! \times \cancel{4!}} \times \frac{\cancel{4!}}{3! \times \cancel{1!}} \times \frac{\cancel{1!}}{1! \cdot 0!}$$

$$= \frac{8!}{4!\, 3!\, 1!}$$

If we have n people. & we want to divide them into r groups, so that

$$n = \underbrace{n_1}_{\substack{\#\text{ of}\\ \text{people in}\\ \text{group } 1}} + \underbrace{n_2}_{\substack{\#\text{ of}\\ \text{people in}\\ \text{group } 2}} + \cdots + \underbrace{n_r}_{\substack{\#\text{ of}\\ \text{people in}\\ \text{group } r.}}$$

$$\#\text{ of ways to do this} = \underbrace{\binom{n}{n_1, n_2, \ldots, n_r}}_{\text{Multi-nomial coefficient.}} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$$

Multinomial Theorem

$$(x_1 + x_2 \ldots + x_r)^n = \sum_{n_1 + \cdots + n_r = n} \binom{n}{n_1, n_2, \ldots, n_r} \cdot x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}.$$

Multi-nomial Formula

Suppose that an experiment has r possible outcomes,

$$\left(\begin{array}{c}\text{sample}\\ \text{space}\end{array}\right) \rightarrow S = \{s_1, s_2, \ldots, s_r\} \qquad P(s_i) = P_i, \quad i = 1, 2, \ldots, r.$$

For $n = n_1 + n_2 + \cdots + n_r$ independent trials of this experiment, the probability that each s_i appears n_i times is given by $\Rightarrow$

$$\binom{n}{n_1, n_2, \ldots, n_r} \cdot P_1^{n_1} \cdot P_2^{n_2} \cdots \cdots P_r^{n_r}.$$

(IV) Unordered Sampling $\boxed{\text{with}}$ Replacement.

Sample k times from a set $A = \{a_1, a_2, \ldots, a_n\}$

— order does not matter

— with replacement (i.e; repetition is allowed)

Eg: $A = \{1, 2, 3\}$; $k = 2$

$$\left.\begin{array}{l} (1,1) \\ (1,2) \\ (1,3) \\ (2,2) \\ (2,3) \\ (3,3) \end{array}\right\} \quad 6 \text{ ways}$$

General Answer (# of possible ways) $\Rightarrow \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$

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one way to represent the outcomes.

(x_1, x_2, x_3) — # of 1's, # of 2's, # of 3's.

$(1, 1) \rightarrow (x_1, x_2, x_3) = (2, 0, 0)$

$(1, 2) \rightarrow (x_1, x_2, x_3) = (1, 1, 0)$

$(1, 3) \rightarrow (x_1, x_2, x_3) = (1, 0, 1)$

$(2, 2) \rightarrow (x_1, x_2, x_3) = (0, 2, 0)$ * x_i's are integers

$(2, 3) \rightarrow (x_1, x_2, x_3) = (0, 1, 1)$ * $x_1 + x_2 + x_3 = 2$

$(3, 3) \rightarrow (x_1, x_2, x_3) = (0, 0, 2)$.

$2 = k$, the number of draws.

# of integer solutions to the equation $x_1 + x_2 + x_3 = 2$

$= 6 =$ # of possible outcomes

Fact: The number of distinct solutions to the equation

$$x_1 + x_2 + \dots + x_n = k, \quad \text{where } x_i \in \{0, 1, 2, 3, \dots\}$$

is equal to

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

Proof: Let's define the following mapping.

$$\text{integers} \left\{ \begin{array}{l} 1 \longrightarrow | \\ 2 \longrightarrow || \\ 3 \longrightarrow ||| \\ \vdots \end{array} \right\} \text{vertical lines.}$$

We can replace x_i's by vertical lines.

Eg: $x_1 + x_2 + x_3 + x_4 = 3 + 0 + 2 + 1 = 6$

$\Rightarrow$ $|||\ +\ +\ ||\ +\ |$

$\Rightarrow$ Each Equation's solution can be represented by.

- k "vertical lines" ($|$) and
- $(n-1)$ "plus signs" ($+$).

$\Rightarrow$ How many sequences can we make by using k vertical lines ($|$) and $(n-1)$ plus ($+$) signs?

$$\text{Answer} = \binom{n+k-1}{k}$$