

Combinatorics, Counting Methods & Bernoulli Trials

* Counting

For a finite sample space S , with equally likely outcomes,

$$P(A) = \frac{|A|}{|S|}$$

\downarrow Event \rightarrow # of elements in A \rightarrow # of elements in S

* Multiplication Principle

Eg:

You wish to purchase iPhone 7 & have the following choices:

| | | |
|------------------|--------------------------------------|-----------------------|
| L or S Screen | 64 GB or 128 GB or 256 GB Storage | B or W or R color. |
|------------------|--------------------------------------|-----------------------|

How many choices do you have?

Ans: $2 \times 3 \times 3 = 18$ choices.

- $\left\{ \begin{array}{l} L-64-B \\ L-64-W \\ L-64-R \\ \vdots \\ S-256-R \end{array} \right\}$



If we perform r experiments, such that k^{th} experiment (2) has n_k possible outcomes, $k=1, 2, \dots, r$,
Total # of possible outcomes $= n_1 \times n_2 \times \dots \times n_r$.

Eg: You are supposed to choose a password as follows:
→ two lower case letters (a to z) followed by
→ one capital letter (A to Z) followed by
→ four digits (0, 1, ..., 9).

For instance xyT1319 is a valid password.

(a) Find the total number of passwords N .

$$N = 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 26^2 \times 10^4$$

(b) A hacker writes a program to randomly & independently generate 10^8 passwords. (same p.w. can be generated twice).
If one of the above passwords match your p.w., your account will be hacked. What is the probability that the hacker is successful?

Let G_i be the event that hacker's i^{th} guess matches your p.w., $i=1, 2, \dots, 10^8$

$$P(G_i) = \frac{1}{N}$$

$$P_{\text{hack}} = \text{Prob. that at least one of hacker's p.w. matches} \\ = P(G_1 \cup G_2 \cup G_3 \dots \cup G_{10^8}) = P\left(\bigcup_i G_i\right)$$

$$P\left(\left(\bigcup_i G_i\right)^c\right) = P\left(\bigcap_i G_i^c\right)$$

$$= P(G_1^c \cap G_2^c \cap \dots \cap G_{10^8}^c)$$

$$= P(G_1^c) \cdot P(G_2^c) \cdot \dots \cdot P(G_{10^8}^c)$$

$$= \left(1 - \frac{1}{N}\right) \cdot \left(1 - \frac{1}{N}\right) \cdot \dots \cdot \left(1 - \frac{1}{N}\right)$$

$$= \left(1 - \frac{1}{N}\right)^{10^8}$$

Since G_i^c 's
are
independent
events

$$P_{\text{hack}} = P\left(\bigcup_i G_i\right)$$

$$= 1 - P\left(\left(\bigcup_i G_i\right)^c\right)$$

$$= 1 - \left(1 - \frac{1}{N}\right)^{10^8} = 1 - \left(1 - \frac{1}{26^3 \times 10^4}\right)^{10^8} \approx 0.4339.$$

General Terminology for Counting Problems

* Sampling — choosing an element from a set.

* With or Without replacement

— if we draw multiple samples from a set

— With replacement \Rightarrow we put each object back after each draw.

— Without replacement \Rightarrow we do not put the drawn object back.

* Ordered or Unordered

— if order matters (i.e. $(1,3,4) \neq (3,1,4) \Rightarrow$ ordered)

— if ordering does not matter (i.e. $(1,3,4) = (3,1,4)$)

There are 4 possibilities

(4)

- Ordered Sampling with replacement
- Ordered Sampling without replacement
- Unordered " with "
- Unordered " without "

(I) Ordered Sampling with Replacement (OS, wR)

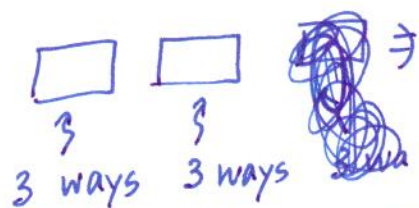
- Set $\rightarrow A = \{1, 2, \dots, n\}$ n elements.

- Draw k samples (OS, wR) from set A with n elements.

$$\# \text{ of Possible ways} = n \times n \times \dots \times n = n^k$$

Eg: $A = \{1, 2, 3\}$; $k = 2$

$\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 1\}, \{2, 2\}, \{2, 3\}, \{3, 1\}, \{3, 2\}, \{3, 3\}$



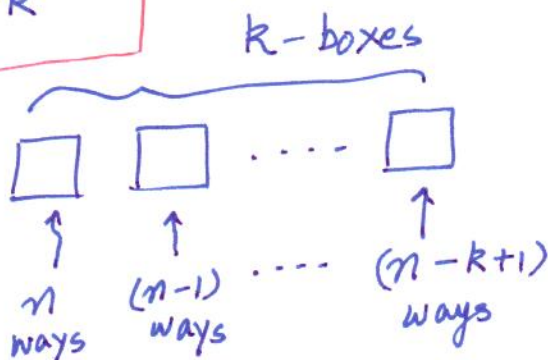
Total 9 ways.

(II) Ordered Sampling without Replacement (OS, w/oR)

$$\begin{aligned} \# \text{ of Possible ways} &= n \times (n-1) \times \dots \times (n-(k-1)) \\ &= \frac{n!}{(n-k)!} = P_k^n \end{aligned}$$

Eg: $A = \{1, 2, 3\}$; $k = 2$

$(1, 2)$
 $(1, 3) \quad (3, 1)$
 $(2, 1) \quad (3, 2)$
 $(2, 3)$



Permutation of n -elements

of n -permutations of n elements

$$P_n^n = n \times (n-1) \times \dots \times (n-(n-1)) \quad [0! = 1]$$

$$= n!$$

of k -permutations of n elements

$$P_k^n = \frac{n!}{(n-k)!} \quad 0 \leq k \leq n$$

Eg: In a group of k people, what is the probability that at least two of them have the same birthday? Suppose that there are $n = 365$ days in a year and all days are equally likely to be the birthday of a specific person.

Ans: A = Event that at least two people have same birthday.

Case 1: if $k > n$ then $P(A) = 1$ (Eg if you have 400 people)

Case 2: if $k \leq n$ then this becomes interesting!

$$P(A) = 1 - P(A^c)$$

$$= 1 - \frac{|A^c|}{|S|} = \boxed{1 - \frac{n!}{(n-k)! n^k}}$$

A^c = Event that no two people have the same birthday.

$|S|$? Total # of possible sequences of birthdays of k people?

$$= n \times n \times \dots \times n = n^k$$

$$|A^c| = n \times (n-1) \times \dots \times (n-k+1)$$

$$= \frac{n!}{(n-k)!}$$

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Eg: $k = 23$ people

$$P(A) = 1 - \frac{365!}{342! \times (365)^{23}} \approx 0.5073$$

$$k = 57 \Rightarrow P(A) \approx 0.99$$

Eg: You are in a party with $(k-1)$ people.
What is the probability that at least one person in the party has the same birthday as yours?

Ans: Your birthdate is fixed \rightarrow say Aug. 28
 A = Event that at least one person has same birthday as Aug. 28.

A^c = Event that No one has Aug. 28 as their birthday.

$$P(A) + P(A^c) = 1 \Rightarrow P(A) = 1 - P(A^c)$$

$$= 1 - \frac{|A^c|}{|S|}$$

$$|A^c| = (n-1) \times (n-1) \times \dots \times (n-1) = (n-1)^{k-1}$$

$$|S| = n \times \dots \times (n) = n^{k-1}$$

$$\Rightarrow P(A) = 1 - \frac{(n-1)^{k-1}}{n^{k-1}} = 1 - \left(\frac{n-1}{n}\right)^{k-1}$$

$$k = 23 \Rightarrow P(A) \approx 0.0586$$

(III) Unordered Sampling without Replacement.
(Combinations).

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- $A = \{1, 2, \dots, n\}$

- k -elements drawn. (US w/o R)

$$\# \text{ of ways} = \frac{\frac{n!}{(n-k)!}}{k!} = \frac{n!}{(n-k)! k!} = \binom{n}{k} \text{ or } C_k^n$$

Eg: $A = \{1, 2, 3\}; k = 2$

$\{1, 2\}, \{1, 3\}, \{2, 3\}$
3 combinations

$\binom{n}{k} \Rightarrow$ Binomial coefficient

$$0 \leq k \leq n$$

Binomial Theorem: For any integer $n \geq 0$,

$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

Other important identities: (Try to prove these yourself...)

1. $\sum_{k=0}^n \binom{n}{k} = 2^n$

2. For $0 \leq k < n$, $\binom{n+1}{k+1} = \binom{n}{k+1} + \binom{n}{k}$
(Pascal's Rule)

3. $\binom{m+n}{k} = \sum_{i=0}^k \binom{m}{i} \binom{n}{k-i}$ (Vandermonde's Identity)

Eg: We choose 3 cards from a deck of 52 cards. What is the prob. that they contain at least one Ace? (8)

Ans: $P(A) = 1 - P(A^c) = 1 - \frac{|A^c|}{|S|}$

$$|S| = \binom{52}{3}$$

A^c = Event that there is No ace in the three cards.

of non-Ace cards = $52 - 4 = 48$

$$|A^c| = \binom{48}{3} \Rightarrow P(A) = 1 - \frac{\binom{48}{3}}{\binom{52}{3}}$$

Eg: How many distinct sequences can we make using 3 letter "A"s & 5 letter "B"s? (Eg: AAABBBBB, AABABBBB etc...)

Ans: $\binom{8}{3}$

3 - positions for A's } 8 positions to fill.
5 - " " B's } once we pick 3 posns for A's, B's will automatically be filled.

$$\Rightarrow \text{Total \# of ways} = \binom{8}{3} = \binom{8}{5}$$

Bernoulli Trials & Binomial Distribution

(9)

Bernoulli Trial \rightarrow random experiment with two outcomes
"success" or "failure"

or "head" or "tail"
Eg:

Prob of "success" = P

Prob of "failure" = $q = 1 - P$

Binomial Experiment \rightarrow if we perform n independent Bernoulli trials and count the total number of successes.
(or Repeated Bernoulli Trials)

Eg: For a coin, $P(H) = P$, $P(T) = 1 - P$. We toss the coin 5 times.

(a) What is Prob. of outcome THHHH?

$$P(THHHH) = P(T) \times P(H) \times P(H) \times P(H) \times P(H) \quad \left[\begin{array}{l} \text{since trials} \\ \text{are} \\ \text{independent} \end{array} \right]$$
$$= (1 - P) \cdot P^4$$

(b) What is Prob. of HTHHH?

$$P(HTHHH) = (1 - P) P^4$$

(c) What is Prob. of HHTHH? $\Rightarrow (1 - P) P^4$

(d) What is the prob. that we observe exactly four heads and one tails?

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$$B = \{T H H H H, H T H H H, H H T H H, H H H T H, H H H H T\}$$

\uparrow
event

$$P(B) = 5 \times (1-p)^4 p$$

(e) What is the Prob. that we observe exactly three heads & two tails?

$$C = \{H H H T T, H H T H T, \dots, T T H H H\}$$

$$P(C) = |C| \times (1-p)^2 p^3$$

How do we find $|C|$?

$|C|$ = # of distinct sequences of length 5 we can create using two tails & three heads.

$$= \binom{5}{3}$$

$$\Rightarrow P(C) = \binom{5}{3} (1-p)^2 p^3$$

(f) If we toss the coin n times, what is the prob of observing exactly k heads & $(n-k)$ tails?

$$P(k \text{ heads \& } (n-k) \text{ tails}) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial Formula

For n independent Bernoulli trials, where each trial has success prob. p , the probability of K successes is

$$P(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

From Binomial to Multinomial.

Eg: For a flood relief effort, 8 agencies need to provide support.

- 4 agencies provide food.
- 3 agencies provide shelter
- 1 agency provides security.

How many ways the agencies can be divided into such groups?

Ans: We can choose 4 out of 8 agencies for food

$$\Rightarrow \# \text{ of ways} = \binom{8}{4}$$

We can then choose 3 out of remaining 4 for shelter

$$\Rightarrow \# \text{ of ways} = \binom{4}{3}$$

We can then choose 1 " " " 1 " security

$$\Rightarrow \# \text{ of ways} = \binom{1}{1}$$

$$\Rightarrow \text{Total } \# \text{ of ways} = \binom{8}{4} \times \binom{4}{3} \times \binom{1}{1}$$

$$= \frac{8!}{4! \times 4!} \times \frac{4!}{3! \times 1!} \times \frac{1!}{1! \times 0!}$$

$$= \frac{8!}{4! 3! 1!}$$

[by the multiplication principle]

If we have n people. & we want to divide them into r groups, so that

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$$n = \underbrace{n_1}_{\substack{\# \text{ of} \\ \text{people in} \\ \text{group 1}}} + \underbrace{n_2}_{\substack{\# \text{ of} \\ \text{people in} \\ \text{group 2}}} + \dots + \underbrace{n_r}_{\substack{\# \text{ of} \\ \text{people in} \\ \text{group } r.}}$$

$$\# \text{ of ways to do this} = \underbrace{\binom{n}{n_1, n_2, \dots, n_r}}_{\text{Multi-nomial coefficient.}} = \frac{n!}{n_1! n_2! \dots n_r!}$$

Multinomial Theorem

$$(x_1 + x_2 + \dots + x_r)^n = \sum_{n_1 + \dots + n_r = n} \binom{n}{n_1, n_2, \dots, n_r} \cdot x_1^{n_1} x_2^{n_2} \dots x_r^{n_r}.$$

Multi-nomial Formula

Suppose that an experiment has r possible outcomes,

(sample space) $\rightarrow S = \{s_1, s_2, \dots, s_r\}$ $P(s_i) = p_i, i=1, 2, \dots, r.$

For $n = n_1 + n_2 + \dots + n_r$ independent trials of this experiment, the probability that each s_i appears n_i times is given by $\Rightarrow \binom{n}{n_1, n_2, \dots, n_r} \cdot p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_r^{n_r}.$

(IV) Unordered Sampling with Replacement.

(13)

Sample k times from a set $A = \{a_1, a_2, \dots, a_n\}$

— order does not matter

— with replacement (i.e; repetition is allowed).

Eg: $A = \{1, 2, 3\}$; $k = 2$

$\left. \begin{array}{l} (1, 1) \\ (1, 2) \\ (1, 3) \\ (2, 2) \\ (2, 3) \\ (3, 3) \end{array} \right\} \quad 6 \text{ ways}$

General Answer
(# of possible ways). $\Rightarrow \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$

one way to represent the outcomes.

$$(1, 1) \rightarrow (x_1, x_2, x_3) = (2, 0, 0)$$

$$(1, 2) \rightarrow (x_1, x_2, x_3) = (1, 1, 0)$$

$$(1, 3) \rightarrow (x_1, x_2, x_3) = (1, 0, 1)$$

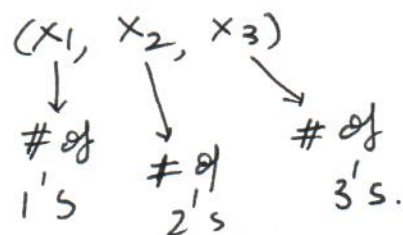
$$(2, 2) \rightarrow (x_1, x_2, x_3) = (0, 2, 0) \quad * \quad x_i \text{'s are integers}$$

$$(2, 3) \rightarrow (x_1, x_2, x_3) = (0, 1, 1) \quad * \quad x_1 + x_2 + x_3 = 2$$

$$(3, 3) \rightarrow (x_1, x_2, x_3) = (0, 0, 2).$$

of integer solutions to the equation $x_1 + x_2 + x_3 = 2$

$$= 6 = \text{\# of possible outcomes}$$



k , the number of draws.

Fact: The number of distinct solutions to the equation

$$x_1 + x_2 + \dots + x_n = k, \text{ where}$$

is equal to $x_i \in \{0, 1, 2, 3, \dots\}$

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

Proof: Let's define the following mapping.

$$\text{integers} \left\{ \begin{array}{l} 1 \rightarrow | \\ 2 \rightarrow || \\ 3 \rightarrow ||| \\ \vdots \end{array} \right\} \text{vertical lines.}$$

We can replace x_i 's by vertical lines.

Eg: $x_1 + x_2 + x_3 + x_4 = 3 + 0 + 2 + 1 = 6$

$$\Rightarrow ||| + + || + |$$

\Rightarrow Each Equation's solution can be represented by.

- ⊙ k "vertical lines"(|) and
- ⊙ $(n-1)$ "plus signs"(+).

\Rightarrow How many sequences \rightarrow can we make by using k vertical lines(|) and $(n-1)$ plus(+) signs?

$$\text{Answer} = \binom{n+k-1}{k}$$