Homework 10 - ECE 503 Fall 2020

- Assigned on: Wednesday, Dec 2, 2020.
- Due Date: **Saturday, December 12, 2020 by 11:59 pm Tucson Time.**
- Maximum Credit: 200 points

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1. [25 points] For each of the following functions, determine and explain whether or not it is a valid auto-correlation function of a WSS random process:
   (a) $\sin(\tau)$
   (b) $\cos(\tau)$
   (c) $e^{-\tau^2/2}$
   (d) $e^{-|\tau|}$
   (e) $\tau^2 e^{-|\tau|}$
   (f) $I_{[-T,T]}(\tau)$

2. [10 points] Let $q(t)$ be a periodic function, with period T_0, and let $T \sim \text{uniform}[0, T_0]$. We define a random process $X(t)$ as $X(t) = q(t + T)$. Prove that $X(t)$ is WSS.

3. [20 points] Find the Power Spectral Density (PSD), i.e., $S_X(f)$ for a WSS random process $X(t)$ with the following auto-correlation functions:
   (a) $R_X(\tau) = e^{-\tau^2/2}$
   (b) $R_X(\tau) = 1/(1 + \tau^2)$

4. [15 points] A WSS process $X(t)$ with auto-correlation function $R_X(\tau) = 1/(1 + \tau^2)$ is passed through an LTI system with impulse response $h(t) = 3\sin(\pi t)/(\pi t)$. Let $Y(t)$ denote the system output. Find the PSD of the output random process.

5. [20 points] Let $R_0(\tau)$ be a real-valued, even function, but not necessarily a auto-correlation function. Let $R(\tau)$ denote the convolution of R_0 with itself, i.e.,

$$R(\tau) = \int_{-\infty}^{\infty} R_0(\theta) R_0(\tau - \theta) d\theta$$

   (a) Show that $R(\tau)$ is a valid auto-correlation function.
   (b) Now, suppose that $R_0(\tau) = I_{[-T,T]}(\tau)$. What is $R(\tau)$, and what is its Fourier transform ?

6. [30 points] A WSS random process $X(t)$ with auto-correlation function $R_X(\tau) = e^{-\tau^2/2}$ is passed through an LTI system with transfer function $H(f) = e^{-(2\pi f)^2/2}$, and denote the system output by $Y(t)$. Find:
   (a) $S_{XY}(f)$
   (b) the cross-correlation $R_{XY}(\tau)$
   (c) $E(X_{t_1} Y_{t_2})$
   (d) $S_Y(f)$
   (e) the output auto-correlation function $R_Y(\tau)$

7. [20 points] Draw the state-transition diagrams and find the stationary distribution of the Markov chains with the following transition matrices:

(a) $P = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 3/4 & 0 \\ 1/4 & 3/4 & 0 \end{bmatrix}$

(b) $P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 9/10 & 0 & 1/10 & 0 \\ 0 & 1/10 & 0 & 9/10 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$

8. [30 points] A web surfer browses pages in a five-page Web universe as shown in Figure 1. The surfer selects a page to view by selecting with equal probability from the pages pointed to by the current page. If a page has no outgoing link (for example, page 2), then the surfer selects any of the other pages in the universe with equal probability.

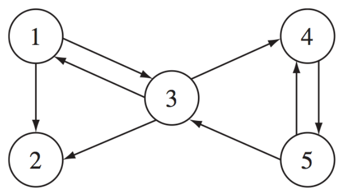

Figure 1: Five page web.

(a) Find the transition matrix P of the corresponding Markov chain.

(b) Find the probability that the surfer views page i after a sufficiently long time surfing the Web. Does this probability depends on the initial page that the surfer has started from ?

(c) How would you rank these pages in terms of importance (from most important, i.e., most visited, to least important) ?

9. [30 points]

(a) For the Markov chains shown in Figure 2, find the transition probability matrices.

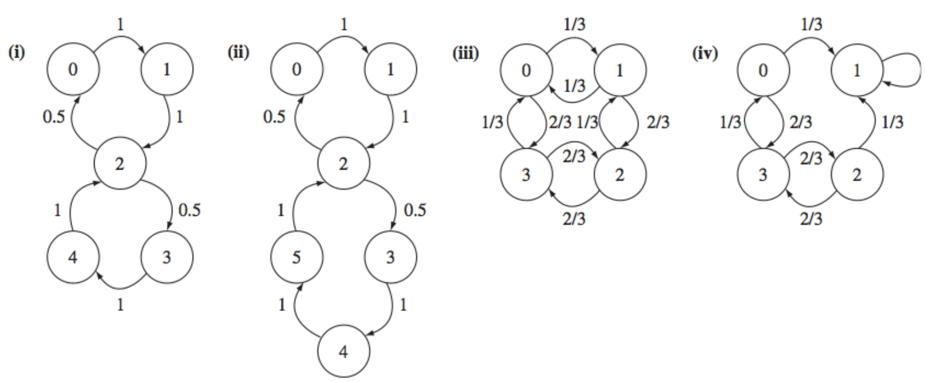

Figure 2: Markov chains.

(b) Specify the classes of Markov chains, and classify them as recurrent or transient; periodic or aperiodic

(c) Find the stationary PMF where applicable and determine whether it is unique.