Solutions

Midterm 2 Exam - ECE 503 Fall 2016

- Date: Wednesday, November 3, 2016.
- Time: 11:00 am -11:50 am (in class)
- Maximum Credit: 100 points

1. [~~25~~ points] Let the random variables (X, Y) have the following joint PDF:

(25). points

$$f_{X,Y}(x,y) = \begin{cases} 6(y-x) & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal PDF of X.

(b) Find the conditional distribution $f_{Y|X}(y|x)$.

(c) What is the optimal MMSE estimator of Y given X ?

(d) Is the above estimator linear ?

(a) $f_X(x) = \int_{y=x}^{1} 6(y-x)\,dy = 6\left[\frac{y^2}{2} - xy\right]\Big|_{y=x}^{1} = 6\left[\frac{1}{2} - x - \left(\frac{x^2}{2} - x^2\right)\right]$

$f_X(x) = 6\left[\frac{1}{2} - x + \frac{x^2}{2}\right] = 3(1-x)^2, \quad 0 \le x \le 1$

(b) $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{6(y-x)}{3(1-x)^2} = \begin{cases} \frac{2(y-x)}{(1-x)^2} & 0 \le x \le y \le 1 \\ 0 & \text{o.w.} \end{cases}$

(c) $\hat{Y}(x) = E[Y|X=x] = \int_{y=x}^{1} y f_{Y|X}(y|x)\,dy = \frac{2}{(1-x)^2} \cdot \int_{y=x}^{1} y(y-x)\,dy$

$= \frac{2}{(1-x)^2}\left[\frac{y^3}{3} - \frac{xy^2}{2}\right]_{y=x}^{1} = \frac{2}{(1-x)^2}\left[\frac{1}{3} - \frac{x}{2} - \frac{x^3}{3} + \frac{x^3}{2}\right] = \frac{1\left[x^3 - 3x + 2\right]}{3(1-x)^2}$

$= \frac{(x+2)(1-x)^2}{3(1-x)^2} = \frac{x+2}{3}$

$\Rightarrow E[Y|X] = \frac{X+2}{3}$

(d) Yes, this is a linear Estimator.

(25 points)

2. (~~20 points~~) A 3-dimensional random vector $X = [X_1 \ X_2 \ X_3]$ has zero mean, i.e., $E[X] = [0 \ 0 \ 0]$, and a auto-correlation matrix as follows:

$$R_X = \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{bmatrix}$$

We observe a 2-dimensional random vector $Y = [Y_1 \ Y_2]$, where

$$Y_1 = X_1 + X_2$$
$$Y_2 = X_2 + X_3$$

Find the Linear Minimum Mean Squared Error (LMMSE) Estimator for X_1 from Y.

(Hint: Use the orthogonality principle, i.e., the error must be orthogonal to the observations.)

Let the estimator be $\hat{X}_1 = \alpha_1 Y_1 + \alpha_2 Y_2$

Error $= \hat{X}_1 - X_1$

From orthogonality principle: $E[(\hat{X}_1 - X_1)Y_1] = 0$;

$E[(\hat{X}_1 - X_1)Y_2] = 0$

$E[(\hat{X}_1 - X_1)Y_1] = E[(\alpha_1 Y_1 + \alpha_2 Y_2 - X_1)Y_1] = \alpha_1 E[Y_1^2] + \alpha_2 E[Y_2 Y_1] - E[X_1 Y_1] = 0$

$E[(\hat{X}_1 - X_1)Y_2] = E[(\alpha_1 Y_1 + \alpha_2 Y_2 - X_1)Y_2] = \alpha_1 E[Y_1 Y_2] + \alpha_2 E[Y_2^2] - E[X_1 Y_2] = 0$

$E[X_1 Y_1] = E[X_1(X_1 + X_2)] = E[X_1^2] + E[X_1 X_2] = 1 + 1/2 = 3/2$

$E[X_1 Y_2] = E[X_1(X_2 + X_3)] = E[X_1 X_2] + E[X_1 X_3] = 1/2 + 1/2 = 1$.

$E[Y_1^2] = E[(X_1 + X_2)^2] = 1 + 1 + 2 \times \frac{1}{2} = 3$

$E[Y_2^2] = E[(X_2 + X_3)^2] = 3$

$E[Y_1 Y_2] = E[(X_1 + X_2)(X_2 + X_3)] = \frac{1}{2} + \frac{1}{2} + 1 + \frac{1}{2} = 5/2$

$\Rightarrow 3\alpha_1 + \frac{5}{2}\alpha_2 = \frac{3}{2} \Rightarrow 6\alpha_1 + 5\alpha_2 = 3 \quad \times 5$

and.

$\frac{5}{2}\alpha_1 + 3\alpha_2 = 1 \Rightarrow 5\alpha_1 + 6\alpha_2 = 2 \quad \times 6$

$36\alpha_2 - 25\alpha_2 = 12 - 15 = -3$

$11\alpha_2 = -3 \Rightarrow \boxed{\alpha_2 = -\frac{3}{11}}$

$\boxed{\alpha_1 = \frac{8}{11}}$

optimal.
LMMSE Estimator for X_1
$= \frac{8}{11} Y_1 - \frac{3}{11} Y_2$

(30 points)

3. [30 points] Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. (independent and identically distributed) random variables. Each one of these random variables is uniformly distributed over $[-1, 1]$.

- Show that $Y_n = \frac{X_n}{n}$ converges to 0 in probability.
- Show that $Y_n = X_1 \cdot X_2 \cdots X_n$ converges to 0 in the mean square sense.
- Show that $Y_n = \max\{X_1, X_2, \ldots, X_n\}$ converges to 1 in distribution.

(a) $P(|Y_n - 0| \geq \epsilon) = 1 - P(|Y_n| < \epsilon)$

$= 1 - P\left(\left|\frac{X_n}{n}\right| < \epsilon\right) = 1 - P\left(X_n \in [-n\epsilon, n\epsilon]\right)$

for a fixed ϵ, and <u>any</u> $n > \frac{1}{\epsilon}$ $\quad P\left[X_n \in [-n\epsilon, n\epsilon]\right] = 1$

$\Rightarrow P(|Y_n - 0| \geq \epsilon) = 1 - 1 = 0$ for all $n > 1/\epsilon$.

$\Rightarrow Y_n \longrightarrow 0$ in probability.

(b) $E\left[|Y_n - 0|^2\right] = E\left[|Y_n|^2\right] = E\left[|X_1 X_2 \ldots X_n|^2\right]$

$= E[X_1^2] \cdot E[X_2^2] \cdots E[X_n^2]$ $\qquad X_i \sim \text{unif}[-1, 1]$

$= \frac{1}{3} \cdot \frac{1}{3} \cdots \frac{1}{3} = \left(\frac{1}{3}\right)^n \rightarrow 0$ as $n \to \infty$ $\qquad E[X_i^2] = \int_{-1}^{1} x_i^2 dx = 1/3$.

$\Rightarrow Y_n = X_1 \ldots X_n \longrightarrow 0$ in mean-square sense.

(c) CDF of Y_n

$F_n(y) = P(Y_n \leq y) = P(\max(X_1, \ldots, X_n) \leq y) = P(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y)$

$= P(X_1 \leq y) \cdot P(X_2 \leq y) \cdot \ldots \cdot P(X_n \leq y)$

$= \left(\frac{1+y}{2}\right)\left(\frac{1+y}{2}\right) \cdots \left(\frac{1+y}{2}\right) = \begin{cases} \left(\frac{1+y}{2}\right)^n & \text{for } y < 1 \\ 1 & \text{for } y \geq 1 \end{cases}$

$\left.\begin{array}{l} \left(\frac{1+y}{2}\right)^n \longrightarrow 0 \text{ for any } y < 1 \\ \quad \longrightarrow = 1 \text{ for any } y \geq 1 \end{array}\right\} \Rightarrow$ CDF converges to CDF of $\underline{\underline{1}}$

Problem 4 (20 points)

Problem ~~5~~ 4. ~~[15 points]~~ Let p be the fraction of Arizona voters who will support a particular candidate in the 2016 presidential election. We survey n randomly selected voters, and record M_n, the fraction of the voters who support this candidate. We can view M_n as our estimate of p and would like to investigate its properties. In particular, we can interpret the responses of voters as i.i.d. Bernoulli random variables, with probability of voting for the candidate as p. How many voters should we include in the survey, so that our estimate M_n is within 0.02 confidence interval of p, and with high confidence (probability at least 95%) ?

$$X \sim Ber(p)$$

$$Var(X) = p(1-p) \leq \frac{1}{4}$$

for any value of p.

$$P(|M_n - p| \geq c) \leq \frac{Var(X)}{nc^2}$$

$$P(|M_n - p| < c) > 1 - \frac{Var(x)}{nc^2}$$

$$> 1 - \frac{1}{4nc^2}$$

we are given $2c = 0.02$

$$\Rightarrow c = 0.01.$$

and confidence prob $= 0.95 = 1 - \frac{1}{4nc^2}$

$$\Rightarrow 0.95 = 1 - \frac{1}{4n \times (0.01)^2} = 1 - \frac{10^4}{4n}.$$

$$\Rightarrow \frac{10^4}{4n} = 1 - 0.95 = 0.05 = 5 \times 10^{-2}$$

$$n = 10^6 \times \frac{1}{20} = 10^4 \times \frac{\cancel{100}^{5}}{\cancel{20}} = 50{,}000$$

$\Rightarrow$ we must ~~sample~~ survey 50,000 voters.