Homework 5 - ECE 503 Fall 2020

- Assigned on: Tuesday, October 13, 2020.
- Due Date: **Wednesday, October 21, 2020 by 11:59 pm Tucson Time.**
- Maximum Credit: 150 points

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1. [20 points] The moment generating function (MGF) of a random variable X is defined as $\phi_X(s) = E[e^{sX}]$.

   (a) Prove that the nth moment of X, $E[X^n]$ can be obtained from the MGF as following:

   $$E[X^n] = \frac{d^n \phi_X(s)}{ds^n}\Big|_{s=0}$$

   (b) Find the MGF for the following random variables:

   - $\mathcal{N}(\mu, \sigma^2)$
   - Poisson(λ)
   - Uniform(a, b) (i.e., a uniform random variable in the interval $[a, b]$)

   (c) Let X_1, X_2, ..., X_n be independent random variables. We define $W = X_1 + X_2 + \ldots + X_n$ as a new random variable. Show that the MGF of W is the product of the MGF(s) of the individual random variables $X_1, \ldots, X_n$.

2. [20 points] Random variables X and Y have the joint PDF

   $$f_{X,Y}(x,y) = \begin{cases} c, & x + y \leq 1, x \geq 0, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the value of constant c.

   (b) What is $P(X < Y)$?

   (c) What is $P(X + Y \leq 1/2)$?

3. [20 points] Let X and Y be independent exponential random variables with the same mean $\mu_X = \mu_Y = 1$. Find the PDF of the following random variables:

   (a) $X + Y$

   (b) XY

   (c) X/Y

   (d) $\min(X, Y)/\max(X, Y)$

4. [10 points] X and Y are independent Rayleigh random variables with a common parameter σ^2. Find the PDF of X/Y.

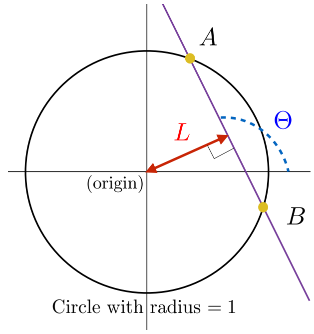
5. [20 points] Let X and Y be independent and identically distributed normal random variables with zero mean and variance σ^2. Define

   $$U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}}, \quad V = \frac{2XY}{\sqrt{X^2 + Y^2}}$$

   (a) Find the joint PDF $f_{U,V}(u, v)$ of the random variables (U, V)

   (b) Show that U and V are independent normal random variables

   (c) Show that $\frac{(X-Y)^2 - 2Y^2}{\sqrt{X^2+Y^2}}$ is also a normal random variable



6. [20 points] Two points A and B are picked independently at random on the circumference of a circle C. The circle is of unit radius and is centered at $(0, 0)$. Let L denote the length of the perpendicular from the origin to the line AB (i.e., the line joining A and B). Let Θ denote the angle that the line AB makes with the horizontal axis.

   Show that the joint density of (L, Θ) is given as:

   $$f_{L,\Theta}(\ell, \theta) = \frac{1}{\pi^2\sqrt{1-\ell^2}}, \quad 0 \le \ell \le 1, \quad 0 \le \theta \le 2\pi$$

   [Hint: describe the points A and B in terms of their angular coordinates and use their joint density]

7. [20 points] Let X and Y have the joint density $f_{X,Y}(x, y) = cx(y - x)e^{-y}, \quad 0 \le x \le y < \infty$.
   (a) Find c
   (b) Find the conditional PDF of X given Y
   (c) Find the conditional PDF of Y given X
   (d) Show that $E[X|Y] = \frac{Y}{2}$ and $E[Y|X] = X + 2$

8. [20 points] Random variables X and Y have the following joint PDF

   $$f_{X,Y}(x, y) = \begin{cases} 5x^2/2 & -1 \le x \le 1, \quad 0 \le y \le x^2, \\ 0 & \text{otherwise.} \end{cases}$$

   Let $A = \{Y \le 1/4\}$ denote an event.
   (a) What is the conditional PDF $f_{X,Y|A}(x, y)$?
   (b) What is $f_{Y|A}(y)$?
   (c) What is $E[Y|A]$?
   (d) What is $f_{X|A}(x)$?
   (e) What is $E(X|A)$?