$\boxed{\text{Today}}$

(1) Confidence Interval Estimation.

(2) Maximum Likelihood (ML) Estimation

(3) Maximum a Posteriori (MAP) Estimation

Recap from last Lecture

* Want to estimate a parameter θ (Eg. mean, variance)

* $\hat{\Theta}_n \longrightarrow$ estimate for θ using n samples (Eg. Sample mean, Sample Variance)

* Desirable properties:
  - (1) Unbiased ✓ $E[\hat{\Theta}_n] = \theta$
  - (2) Consistency ✓ $\hat{\Theta}_n \xrightarrow[n\to\infty]{P} \theta$
  - (3) Accuracy $\hat{\Theta}_n \xrightarrow[n\to\infty]{M.S.} \theta$

(Point Estimates)

Confidence Interval Estimates

θ

$$\underbrace{(\hat{\Theta}_L, \hat{\Theta}_H)}_{\text{interval estimate.}}$$

$(-\infty, \infty)$

(1) want length (size of interval to be small.

$\theta \rightarrow$ unkn $(\theta = 3)$

(2). High Confidence that θ lies in the interval estimate.

We say that $[\hat{\Theta}_L, \hat{\Theta}_H]$ is a $(1-\alpha)100\%$ confidence interval for a unknown parameter θ if

$$P(\hat{\Theta}_L \le \theta \le \hat{\Theta}_H) \ge (1-\alpha)$$

($\alpha \rightarrow$ small)

$[-3, 4]$ is a 95% confidence int estimate of θ

$$P(\underbrace{-3 \le \theta \le 4}) \ge 0.95$$

Length of interval $\Rightarrow$ $\underline{\hat{\theta}_H - \hat{\theta}_L}$ $\longrightarrow$ desired accuracy of prediction.

$(1-\alpha)$ $\Longrightarrow$ confidence of our prediction.

Suppose $\longrightarrow$ goal: estimate the unknown mean μ.

$(X_1, X_2, \ldots, X_n)$

$$\longrightarrow \bar{X} = \frac{X_1 + X_2 \cdots + X_n}{n}.$$

Point estimate.

$[\bar{X} - c, \bar{X} + c] \longrightarrow$ interval Estimate.

$$P(\underbrace{\bar{X} - c}_{\hat{\mu}_{Low}} \le \mu \le \underbrace{\bar{X} + c}_{\hat{\mu}_{High}}) \ge \underline{(1-\alpha)}$$

Digression. $Z \sim N(0,1)$. $\longleftarrow$ Standard Gau.

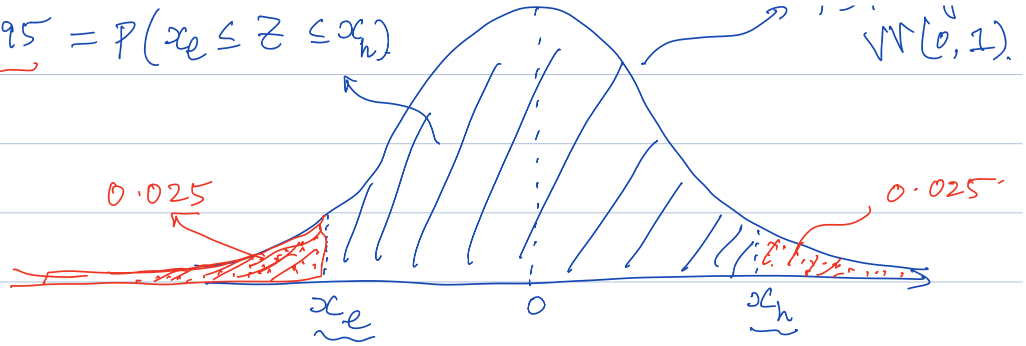
$$P(x_l \le Z \le x_h) = 0.95 = (1-\alpha)$$

Find x_l and x_h so that is satisfied.

(Area)

PDF of

$0.95 = P(x_l \le Z \le x_h)$

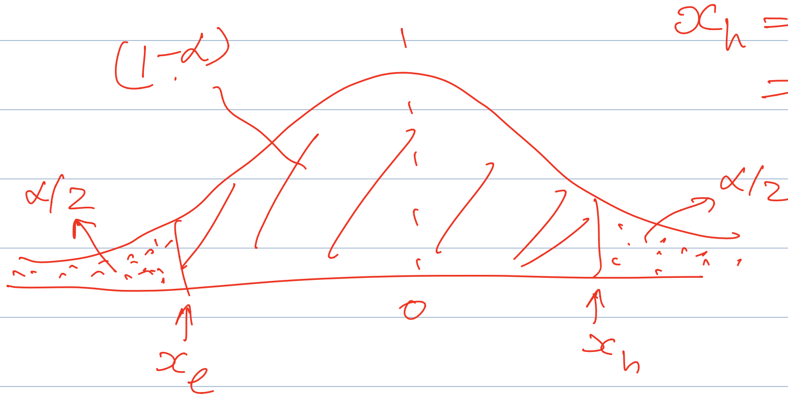


$$\underbrace{\Phi(x_l)}_{\Downarrow \; P(Z \le x_l)} = 0.025 \Rightarrow x_l = \Phi^{-1}(0.025)$$

$$\boxed{x_l = -1.96}$$

$$\Phi(x_h) \overset{?}{=} 1 - 0.025 = 0.975$$

$$x_h = \Phi^{-1}(0.975) = +1.96$$



$$\Phi(x_l) = \alpha/2 \Rightarrow x_l = \Phi^{-1}(\alpha/2)$$

$$\& \quad x_h = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

Eg $X_1, X_2, \ldots, X_n$ iid from a $N(\theta, 1)$.
Find a 95% confidence interval estimate for θ.

Point Estimate of θ : $\underline{\bar{X}} = \frac{X_1 + \dots + X_n}{n}$.

$$\bar{X} \sim \mathcal{N}\left(\theta, \frac{1}{n}\right)$$

$$\Rightarrow \boxed{\left(\frac{\bar{X} - \theta}{1/\sqrt{n}}\right) = \underbrace{\sqrt{n}(\bar{X} - \theta)}_{\text{Standard Gaussian}} \sim \mathcal{N}(0,1)}$$

(Digression)

$$P\Big(\underbrace{\underbrace{-1.96}\le \overset{\text{Standard Gau.}}{\sqrt{n}(\bar{X}-\theta)} \le \underbrace{1.96}}\Big) = 0.95$$

$$P\left(-\frac{1.96}{\sqrt{n}} \le \bar{X} - \theta \le +\frac{1.96}{\sqrt{n}}\right) = 0.95$$

$$P\left(\bar{X} - \frac{1.96}{\sqrt{n}} \le \theta \le \bar{X} + \frac{1.96}{\sqrt{n}}\right) = 0.95.$$

95% confid int estimate is

$$\left[\bar{X} - \underbrace{\frac{1.96}{\sqrt{n}}}, \ \bar{X} + \underbrace{\frac{1.96}{\sqrt{n}}}\right].$$

Maximum Likelihood Estimation

Maximum a Posteriori (MAP) Estimation.

$X_1\ X_2\ X_2 \ldots X_n \rightarrow$ samples.

(iid) $\rightarrow$ some distribution with an unknown parameter θ

Likelihood function:

$$= f_{\underbrace{X_1 \ldots X_n}_{\text{data}}}(x_1, \ldots x_n; \theta)$$

$$\underbrace{\log\left(f_{X_1, \ldots X_n}(x_1, \ldots x_n; \theta)\right)}_{}$$

$\downarrow$

Log-likelihood function.

ML estimate for θ =

$$\hat{\Theta}_{ML} = \arg\max_{\phi} f_{X_1 \ldots X_n}(x_1, \ldots x_n; \phi)$$

Example: $\underbrace{X_1, X_2 \ldots X_n}$ $\boxed{\text{iid}}$ $\sim \mathrm{Exp}(\theta)$.

Goal: Find the ML estimate of θ.

Likelihood func. $=$ Joint PDF of $(X_1 \dots X_n)$

$$f_{X_1 \dots X_n}(x_1, \dots, x_n) = \prod_{i=1}^{n} \left(\theta e^{-\theta x_i} \right)$$

$$= \theta^n \prod_{i=1}^{n} e^{-\theta x_i}$$

$$= \theta^n e^{-\theta [x_1 + x_2 \dots + x_n]}$$

$$\ln(\text{Likel.}) = n \ln(\theta) - \theta \sum_{i=1}^{n} x_i$$

$\rightarrow$ func of θ.

$$\hat{\theta}_{ML} \overset{?}{=}$$

$$\boxed{\frac{d \ln(\text{likelihood})}{d\theta} = 0}$$

$$\frac{n}{\theta} - \sum_{i=1}^{n} x_i = 0.$$

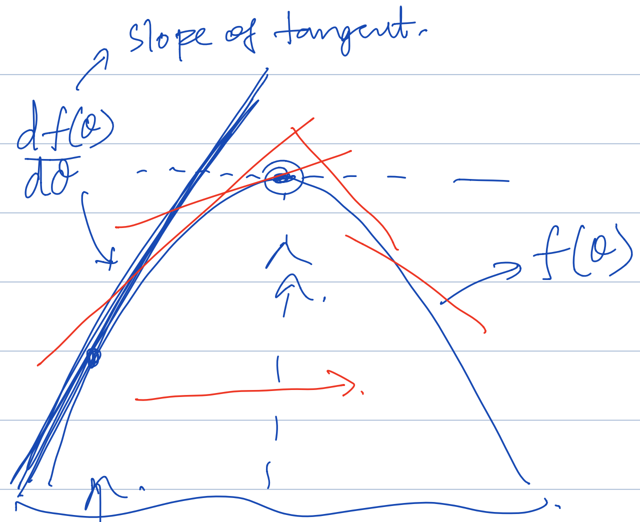
$$\frac{d^2 \ln(\text{li.})}{d\theta} = -\frac{n}{\theta^2} < 0 \Rightarrow$$

$$\boxed{\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i}}$$

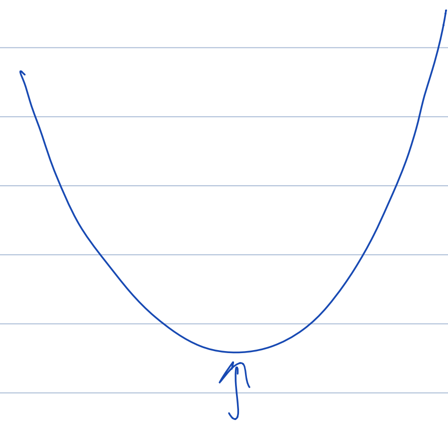
$n = 5$.

$$(\underset{X_1}{3}, \underset{X_2}{1}, \underset{X_3}{8}, \underset{X_4}{9}, \underset{X_5}{15})$$

$$\hat{\theta}_{ML} = \frac{5}{(3+1+8+9+15)}$$



Concave ?



Convex.

$$\frac{d^2 f(\theta)}{d\theta^2} < 0.$$