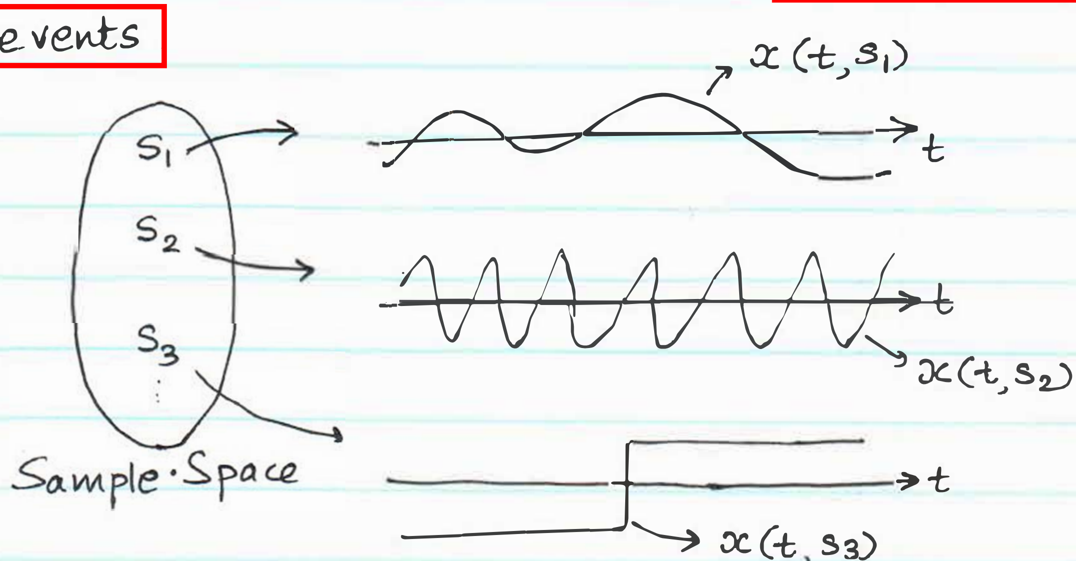
Lecture 23

Random Processes / Stochastic Processes

The word "process" in this context means a function of time. Thus, when we study stochastic processes, we study random functions of time. Almost all practical applications of probability involve multiple observations over time. When we talked about random variables, we were concerned about how frequently an event occurs. When we study stochastic processes, we also pay attention to the time sequence of the events



Sample Functions

Def 1

STOCHASTIC PROCESS

A Stochastic Process $X(t)$ consists of an experiment with a Probability measure $P[\cdot]$ defined on a Sample Space S, and a function that assigns a time function $x(t, s)$ to each outcome s in the Sample Space.

Def 2 Sample Function

A sample function $x(t, s)$ is the time function associated with outcome s of an experiment.

Def 3 Ensemble

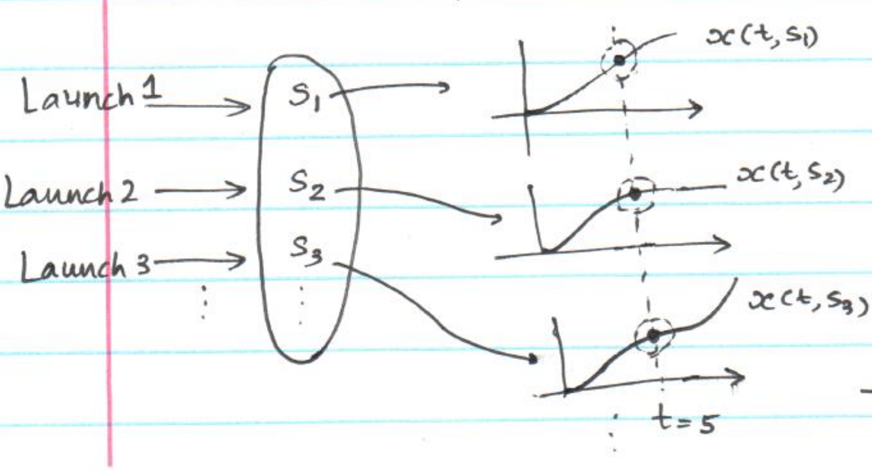
The ensemble of a stochastic process is the set of all possible time functions that ~~th~~ can result from an experiment.

With $t = t_0$ fixed, $\boxed{X(t_0) \text{ is a random variable,}}$ and we have the averages (eg. mean, variance etc). These are known as the ensemble averages.

Other type of average applies to a specific sample function $x(t, s_0)$ → time averages.

EXAMPLE 1

Eg Starting at launch time $t = 0$, let $X(t)$ denote the temp on surface of a space shuttle. For each launch s, we record a temperature sequence $x(t, s)$.



$E(X(5)) = 217°K$

↓

ensemble average at $t = 5$.

$$\frac{1}{5}\int_0^5 x(t, s_1)\,dt = \text{Time average for } s_1$$

$$\frac{\int_{t=0}^{5} x(t,s)\,dt}{5} \Rightarrow \text{time average}$$

EXAMPLE 2: Starting Jan 1, we measure noontime Temp. at Tucson airport every day for 1 year from 1920-1980.

$$\begin{array}{cccc}
C^{1920}(1) & \boxed{C^{1920}(2)} & \cdots & C^{1920}(365) \\
C^{1921}(1) & C^{1921}(2) & \cdots & C^{1921}(365) \\
\vdots & & & \\
C^{1980}(1) & C^{1980}(2) & \cdots & C^{1980}(365)
\end{array}$$

Avg noontime Temp. in 1921

$$= \underbrace{\frac{C^{1921}(1) + \cdots + C^{1921}(365)}{365}}$$

$\Downarrow$

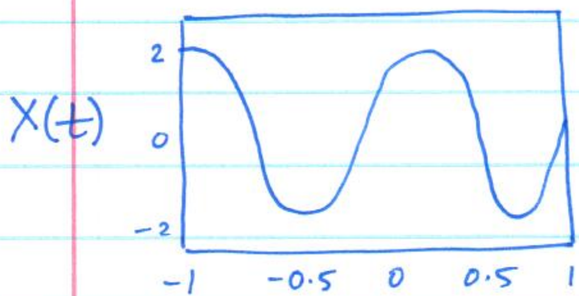
"Time" Average

Average noontime temperature on Jan 2

$$= \frac{C^{1920}(2) + \cdots + C^{1980}(2)}{61} \Rightarrow \text{"Ensemble" Average}$$

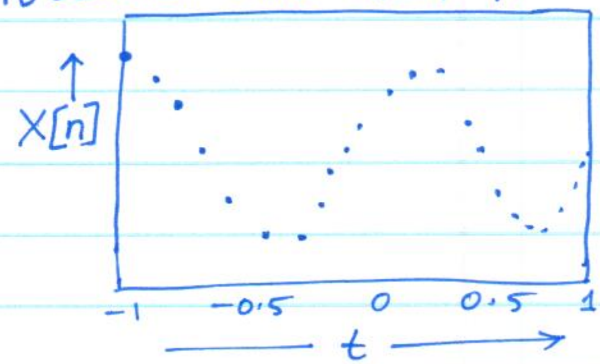
Types of Random Processes

Continuous Time, Continuous Valued

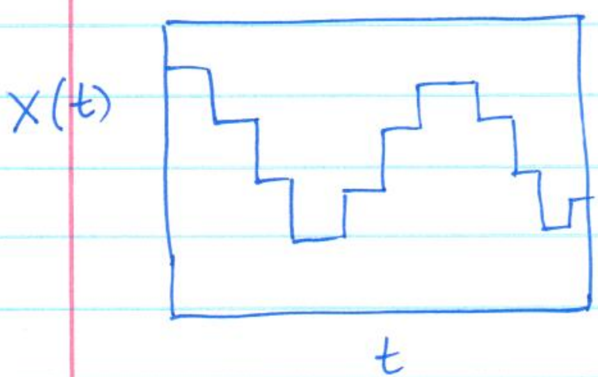


Discrete Time, discrete Continuous Valued.

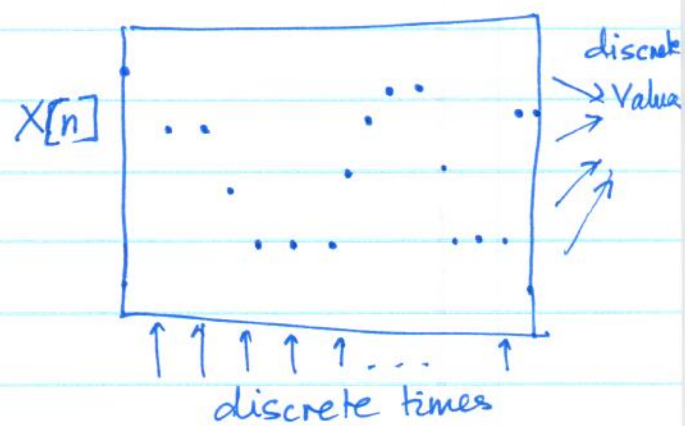
"n" obtained → via $t_n = nT$ for some. T



Continuos Time Discrete Valued



Discrete Time Discrete Valued



$X(t)$ is discrete-valued if set of all possible values $X(t)$ can take is a countable set. otherwise, $X(t)$ is continuous-valued

$X(t)$ is a discrete-time process if it is defined only for a set of time instants $t_n = nT$, where T is a constant and n is an integer.

Statistics of Random Processes.

For a single r.v. $X \longrightarrow f_X(x)$ determined its properties.

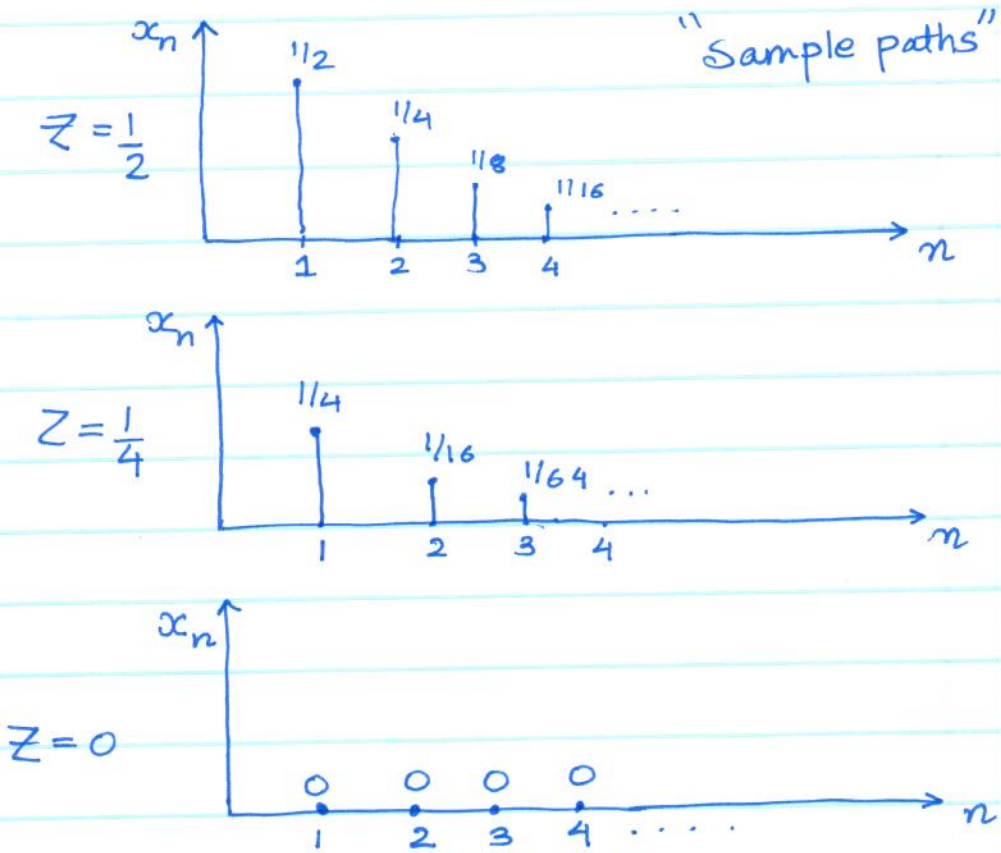
For 2 r.v.s $X, Y \rightarrow f_{X,Y}(x,y)$ "

For a random process, $X(t)$, if we sample it at k time instants $t_1, t_2, \ldots, t_k$, we obtain k random variables $X(t_1), X(t_2) \ldots, X(t_k)$
For the determination of the statistical properties of a random process, $\boxed{\text{we need to describe the}}$ $\boxed{\text{joint PDF of } (X(t_1), \ldots, X(t_k))}$ for any value of k, and any set of time instants $t_1, t_2, \ldots, t_k$.

or $f_{X(t_1), X(t_2), \ldots X(t_k)}(x_1, x_2, \ldots, x_k)$

However, specifying this is challenging & in several applications, $\boxed{\text{we work with second-order}}$ $\boxed{\text{and first-order properties of the random process.}}$

Eg 1 Let $Z \sim \text{unif}[0,1]$, and define the discrete time process $X_n = Z^n$ for $n \geq 1$



First-order PDF (or PMF) of the process:
For each n, $X_n = Z^n$ is a random variable and the sequence of PDFs (or PMFs) are called the FIRST-ORDER PDF (or PMF) of the process.

To find
First-order PDF:

$$P(X_n \leq x) = P(Z^n \leq x) = P(Z \leq x^{1/n}) = x^{1/n}$$

$$\Rightarrow f_{X_n}(x) = \begin{cases} \frac{1}{n} x^{\frac{1}{n}-1} & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Eg 2. Let $\overbrace{X(t) = R\,|\cos(2\pi f t)|}^{\nearrow\text{(continuous-time, continuous-valued)}}$ be a rectified cosine signal with a random amplitude R, with

$$f_R(r) = \begin{cases} \frac{1}{10} e^{-r/10} & r \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the ~~FIRST-ORDER~~ PDF $f_{X(t)}(x)$?

$X(t) \geq 0$ for all $t \Rightarrow P(X(t) \leq x) = 0$ for $x < 0$

If $x \geq 0$ and $\cos(2\pi f t) \neq 0$,

$$\begin{aligned} P(X(t) \leq x) &= P(R\,|\cos(2\pi f t)| \leq x) \\ &= P\left(R \leq \frac{x}{|\cos(2\pi f t)|}\right) \\ &= \int_0^{\frac{x}{|\cos(2\pi f t)|}} f_R(r)\,dr \\ &= 1 - e^{-\frac{x}{10|\cos(2\pi f t)|}} \end{aligned}$$

$\Rightarrow$ when $\cos(2\pi f t) \neq 0$, the CDF of $X(t)$ is

$$F_{X(t)}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\frac{x}{10|\cos(2\pi f t)|}} & x \geq 0. \end{cases}$$

When $\cos(2\pi f t) = 0$, i.e. $t = k\pi + \pi/2$,

then $X(t) = 0$ for any value of R.

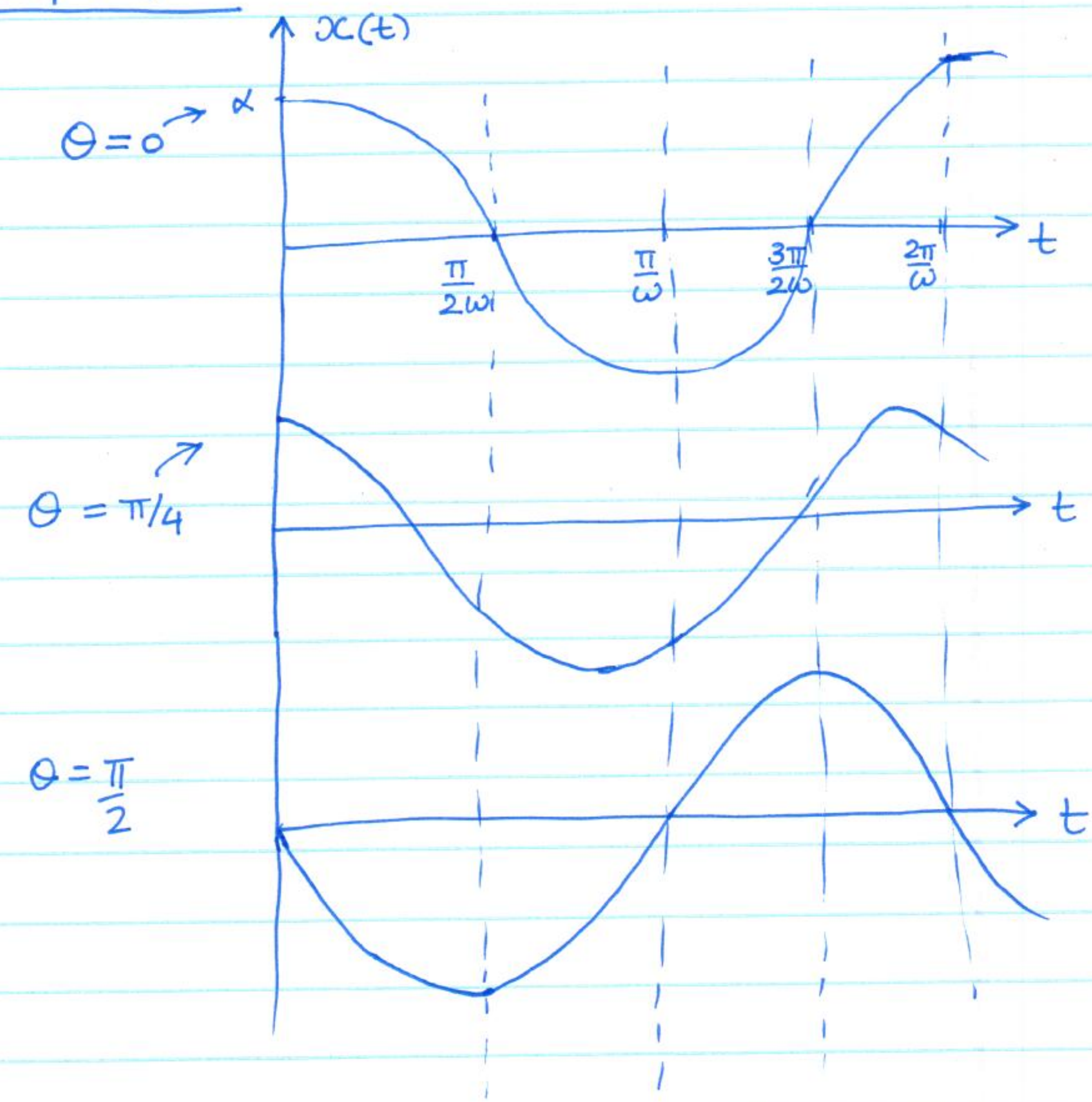
$\Rightarrow \quad f_{X(t)}(x) = \delta(x)$ at these values of t.

Eg 3 Sinusoidal signal with Random Phase

$$X(t) = \alpha \cos(\omega t + \Theta), \quad t \geq 0$$

where $\Theta \sim$ unif $[0, 2\pi]$, and α, ω are constants.

Sample functions :



FIRST ORDER PDF $\Rightarrow$ PDF of

$X(t) = \alpha \cos(\omega t + \theta)$

$$f_{X(t)}(x) = \begin{cases} \dfrac{1}{\alpha\pi\sqrt{1-(x/\alpha)^2}} & -\infty < x < +\alpha \\ 0 & \text{otherwise} \end{cases}$$

Discrete-time, Discrete-Valued.

Bernoulli Random Process

A Bernoulli (p) process X_n is an iid random sequence, in which each X_n is a Bernoulli(p) random variable.

Joint PMF.

For a single sample X_i, $X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

$$\underbrace{P_{X_i}(x_i)}_{\text{Prob}(X_i = x_i)} = \begin{cases} p^{x_i}(1-p)^{1-x_i} & \text{if } x_i \in \{0,1\} \\ 0 & \text{otherwise.} \end{cases}$$

$$P_{X_1, X_2, .., X_n}(x_1, x_2, .., x_n) = P_{X_1}(x_1) P_{X_2}(x_2) \cdots P_{X_n}(x_n) \quad (\text{iid})$$

$$= \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$= p^{(x_1+x_2 .. + x_n)} \cdot (1-p)^{n-(x_1+..+x_n)}$$

$$= p^k \cdot (1-p)^{n-k}, \text{ where}$$

$$x_1 + x_2 .. + x_n = k.$$

$$= \begin{cases} p^{\Sigma x_i}(1-p)^{n-\Sigma x_i}, & x_i \in \{0,1\} \\ & i = 1, .., n \\ 0 & \text{otherwise} \end{cases}$$