HW-7 Solutions Fall 2017

Solution to Problem # 1

The autocorrelation function $R_X(\tau) = \delta(\tau)$ is mathematically valid in the sense that it meets the conditions required in Theorem 10.12. That is,

$$R_X(\tau) = \delta(\tau) \geq 0 \tag{1}$$

$$R_X(\tau) = \delta(\tau) = \delta(-\tau) = R_X(-\tau) \tag{2}$$

$$R_X(\tau) \leq R_X(0) = \delta(0) \tag{3}$$

However, for a process $X(t)$ with the autocorrelation $R_X(\tau) = \delta(\tau)$, Definition 10.16 says that the average power of the process is

$$E\left[X^2(t)\right] = R_X(0) = \delta(0) = \infty \tag{4}$$

Processes with infinite average power cannot exist in practice.

Solution to Problem # 2

(a) $Y(t)$ has autocorrelation function

$$\begin{aligned} R_Y(t,\tau) &= E\left[Y(t)Y(t+\tau)\right] && (1)\\ &= E\left[X(t-t_0)X(t+\tau-t_0)\right] && (2)\\ &= R_X(\tau). && (3) \end{aligned}$$

(b) The cross correlation of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t,\tau) &= E\left[X(t)Y(t+\tau)\right] && (4)\\ &= E\left[X(t)X(t+\tau-t_0)\right] && (5)\\ &= R_X(\tau-t_0). && (6) \end{aligned}$$

(c) We have already verified that $R_Y(t,\tau)$ depends only on the time difference τ. Since $E[Y(t)] = E[X(t-t_0)] = \mu_X$, we have verified that $Y(t)$ is wide sense stationary.

(d) Since $X(t)$ and $Y(t)$ are wide sense stationary and since we have shown that $R_{XY}(t,\tau)$ depends only on τ, we know that $X(t)$ and $Y(t)$ are jointly wide sense stationary.

Solution to Problem # 3

$$W(t) = X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t)$$

X, Y are uncorrelated, $E(X) = E(Y) = 0$

$$Var(Y) = Var(X) = \sigma^2$$

Since X, Y are uncorrelated, $\Rightarrow E(X^2) = E(Y^2) = \sigma^2$

$$E(XY) = E(X)E(Y) = 0$$

$$R_W(t, t+\tau) = E[W(t)W(t+\tau)]$$

$$= E\Big[\big(X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t)\big)\big(X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau))\big)\Big]$$

$$\Downarrow$$

$$= E[X^2]\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) + E[Y^2]\sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau))$$

$\left\{ \begin{array}{c} \text{using} \\ E(XY)=0 \end{array} \right\}$

Solution to Problem # 3

$$= \sigma^2 \left\{ \cos(\omega t)\cos(\omega(t+\tau)) + \sin(\omega t)\sin(\omega(t+\tau)) \right\}$$

$$(\omega = 2\pi f_0)$$

$$2\cos A \cos B = \cos(A-B) + \cos(A+B)$$

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$= \sigma^2 \cos(\omega\tau) = \sigma^2 \cos(2\pi f_0 \tau).$$

$$\Rightarrow \begin{cases} R_W(t, t+\tau) = \sigma^2 \cos(2\pi f_0 \tau) \\ E[W(t)] = 0 \end{cases}$$

$\rightarrow$ depends only on difference in times.

$$\Rightarrow \boxed{W(t) \text{ is W.S.S.}}$$

Solution to Problem # 4

(a) $\underbrace{X(t)}_{\text{WSS}}$ has average power $= 1$

$\Rightarrow E[X^2(t)] = \text{Avg Power} = \boxed{1}$

(b) $\Theta \sim \text{unif}[0, 2\pi]$, $\quad f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow E[\cos(2\pi f_c t + \theta)] = \int_{-\infty}^{\infty} \cos(2\pi f_c t + \theta) f_\theta(\theta)\, d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \cos(2\pi f_c t + \theta)\, d\theta$$

$$= \frac{1}{2\pi} \sin(2\pi f_c t + \theta) \Big|_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\sin(2\pi + 2\pi f_c t) - \sin(2\pi f_c t)\right]$$

$$= \boxed{0}$$

(c) Since $X(t)$ and Θ are independent

$$E[Y(t)] = E[X(t)] \cdot \underbrace{E[\cos(2\pi f_c t + \theta)]}_{0}$$

$$= E[X(t)] \times 0$$

$$= \boxed{0} \quad \text{(no matter what the mean of } X(t) \text{ is..)}$$

(d) $E[Y^2(t)] = E[X^2(t) \cos^2(2\pi f_c t + \theta)]$

$$= E(X^2(t)] \cdot E[\cos^2(2\pi f_c t + \theta)] = \boxed{1/2}$$

$$= 1 \times 1/2$$

Solution to Problem # 5

(3) We are given $\quad Y(t) = \int_0^t N(u)\,du$

$$Y(t+\tau) = \int_0^{t+\tau} N(u)\,du = \int_0^{t+\tau} N(v)\,dv.$$

$$R_Y(t, t+\tau) = E[Y(t)\,Y(t+\tau)] = E\left[\int_0^t N(u)\,du \int_0^{t+\tau} N(v)\,dv\right]$$

$$= E\left[\int_{u=0}^{t}\int_{v=0}^{t+\tau} N(u)\,N(v)\,dv\,du\right]$$

$$= \int_0^t\int_0^{t+\tau} \underbrace{E[N(u)\,N(v)]}_{\alpha\,\delta(u-v)}\,dv\,du$$

$$R_Y(t, t+\tau) = \int_{u=0}^{t}\int_{v=0}^{t+\tau} \alpha\,\delta(u-v)\,dv\,du$$

Solution to Problem # 5

At this point, it matters whether $\tau \geq 0$ or if $\tau < 0$. When $\tau \geq 0$, then v ranges from 0 to $t+\tau$ and at some point in the integral over v, we will have $v = u$.

$\Rightarrow$ for $\tau \geq 0$, $\quad R_Y(t, t+\tau) = \int_{u=0}^{t} \alpha \, du = \alpha t$

For $\tau < 0$, we can reverse the order of integration

$$R_Y(t, t+\tau) = \int_{v=0}^{t+\tau} \int_{u=0}^{t} \alpha \, \delta(u-v) \, du \, dv.$$

$\boxed{[\tau < 0]}$

$$= \int_{v=0}^{t+\tau} \alpha \, dv = \alpha(t+\tau)$$

$$\Rightarrow \quad R_Y(t, t+\tau) = \begin{cases} t & \text{if } \tau \geq 0 \\ t+\tau & \text{if } \tau < 0 \end{cases}$$

$\Rightarrow R_Y(t, t+\tau) = \min(t, t+\tau)$.

Not WSS.

$\Rightarrow$ Not Stationary.

Solution to Problem # 6

X_n is a WSS random process,

i.e $\quad E[X_n] \rightarrow$ does NOT depend on n

$E[X_n X_m] \rightarrow$ depends on $(n-m)$

$$\mu_Y = E[Y_n] = E[X_n - X_{n-1}] = E[X_n] - E[X_{n-1}] = 0$$

$$R_Y(n,m) = E[Y_n Y_m] = E[(X_n - X_{n-1})(X_m - X_{m-1})]$$

$$= E[X_n X_m] + E[X_{n-1} X_{m-1}] - E[X_{n-1} X_m] - E[X_{m-1} X_n]$$

define $\tau = (n-m)$

$$= R_X(\tau) + R_X(\tau) - R_X(\tau-1) - R_X(\tau+1)$$

$$\Rightarrow \quad R_Y(n-m) = R_Y(\tau) = 2R_X(\tau) - R_X(\tau-1) - R_X(\tau+1)$$

$$R_Y(n-m) = R_Y(n,m)$$

$\Rightarrow$ Y_n is also W.S.S.

Solution to Problem # 7

) Avg. time between hit songs = 7 months

$\Rightarrow \quad \lambda = \frac{1}{7}$ per-month.

(a) Since 1 Year = 12 month,

$$P(N(12) > 2) = 1 - P(N(12) \leq 2)$$

$$= 1 - \{P(N(12)=0) + P(N(12)=1) + P(N(12)=2$$

$$= 1 - e^{-12\lambda}\left\{1 + 12\lambda + \frac{(12\lambda)^2}{2!}\right\}.$$

$$= 1 - e^{-12/7}\left\{1 + 12/7 + \frac{(12/7)^2}{2}\right\}$$

$$= 0.247$$

(b). Let T_n = time of n^{th} hit Song

$$T_n = X_1 + X_2 \ldots + X_n$$

$$E[T_n] = E[X_1] + \ldots + E[X_n]$$

$$= n \times E[X_1] = n \times \frac{1}{\lambda} = 7n.$$

$$E[T_{10}] = 7 \times 10 = 70 \text{ months}.$$

Solution to Problem # 8

Inter-arrival time $\rightarrow Exp(\lambda) \Rightarrow E(X_i) = \frac{1}{\lambda} = 2$ months

$\Rightarrow \lambda = \frac{1}{2}$ per-month

(a) Prob(No Launch in 4-months)

$$= P(N(4) = 0) = e^{-4\lambda} \cdot \frac{(4\lambda)^0}{0!}$$

$$= e^{-4\lambda}$$

$$= e^{-4/2} = e^{-2}$$

$$= 0.135$$

(b) P(during at least one month out of 4 consecutive moths, there are 2 launches)

$$= P\left(((N_1 - N_0) \geq 2) \cup (N_2 - N_1 \geq 2) \cup (N_3 - N_2 \geq 2) \cup (N_4 - N_3 \geq 2) \right)$$

$$= P\left(\bigcup_{i=1}^{4} (N_i - N_{i-1} \geq 2) \right)$$

$$= 1 - P\left(\bigcap_{i=1}^{4} (N_i - N_{i-1} < 2) \right)$$

$$= 1 - P\left(\bigcap_{i=1}^{4} (N_i - N_{i-1}) \leq 1 \right)$$

$$= 1 - \prod_{i=1}^{4} P(N_i - N_{i-1} \leq 1)$$

$$= 1 - \prod_{i=1}^{4} \left\{ P(N_i - N_{i-1} = 0) + P(N_i - N_{i-1} = 1) \right\}$$

$$= 1 - \prod_{i=1}^{4} \left\{ e^{-\lambda} \cdot \frac{(\lambda)^0}{0!} + e^{-\lambda} \cdot \frac{(\lambda)^1}{1!} \right\}$$

$$= 1 - \prod_{i=1}^{4} \left\{ e^{-\lambda} (1 + \lambda) \right\} = 1 - \left(e^{-1/2} \times 3/2 \right)^4$$

$$= 1 - \frac{81}{16} e^{-2}$$

$$= 0.315$$

Solution to Problem # 9

We note that the process of packet departures that are received successfully can be obtained by **splitting** the original Poisson process.

In particular, the *successfully* received packets, follow a Poisson process with rate p x lambda.

a) For this part, this is the time until the first successfully received packet in the split Poisson Process, which is an exponentially distributed random variable with parameter p x lambda.

b) This is the probability of no packets received in the split process during one hour. Hence, this probability is exp(-p x lambda).

c) This is the expected number of packets received in the split Poisson process during an hour, which is equal to p x lambda.