**Today:**
1) Cumulative Distribution Function (CDF) $\leftarrow$
2) Properties of CDF $\leftarrow$
3) Continuous valued R.V.'s. $\leftarrow$

$$\boxed{\begin{array}{l} \text{PMF} \rightarrow \text{Prob. Mass function.} \qquad \mathcal{X} \rightarrow \text{alphabet of } X \\ P_X(x) = \Pr(X = x) \end{array}}$$

## CDF of a random variable

$$F_X(x) = P(X \le x)$$

defined for all $-\infty < x < \infty$

$$\underbrace{\{X \le x\}}_{\text{Event.}} = \{ s \in \overset{\text{sample space}}{\Omega} : X(s) \le x \}.$$

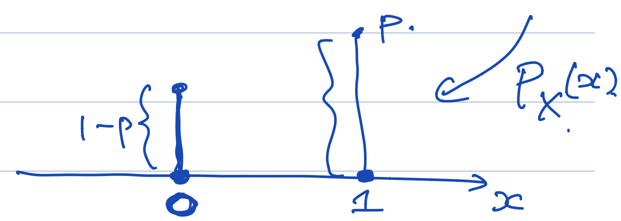
$$P(X \le x) = P(\{ s \in \Omega : X(s) \le x \}).$$

**Eg 1:** $X \sim Ber(p) \longrightarrow$ Bernoulli r.v. with parameter p.

$$P(X=0) = 1-p$$
$$P(X=1) = p$$

$$P(X=0.5) \overset{?}{=} 0$$
$$P(X=-0.9) \overset{?}{=} 0$$
$$P(X=1.9) \overset{?}{=} 0$$

✓




$\boxed{F_X(x)}$ for Ber(p) r.v.

$F_X(x)$ for Ber(p)

$F_X(-3) \overset{?}{=} P(X \le -3) = 0$

$F_X(-1) \overset{?}{=} P(X \le -1) = 0$

$F_X(-0.00001) \overset{?}{=} 0$

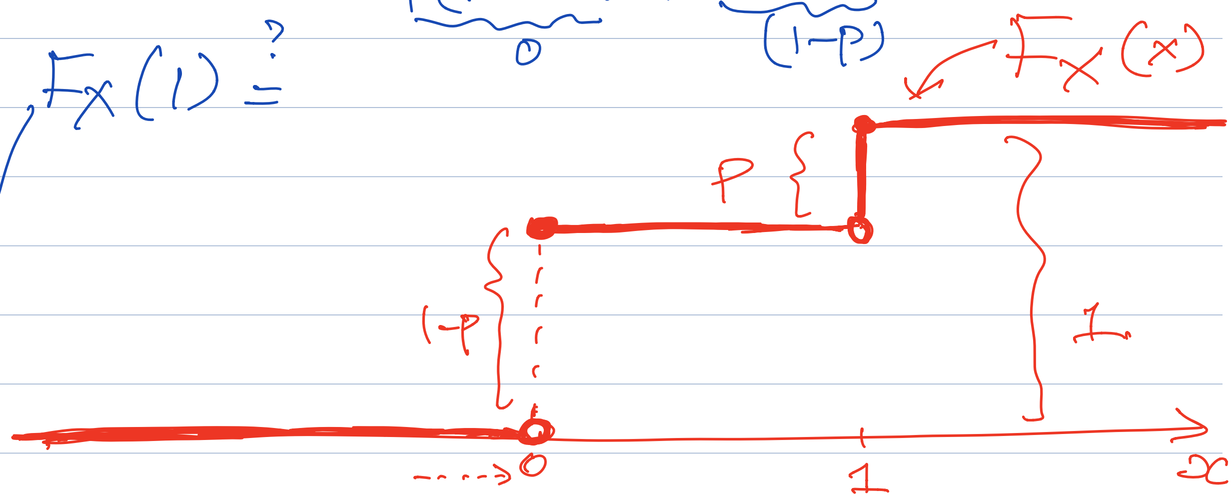
$F_X(0) \overset{?}{=} P(X \le 0) = 1-p \Longleftarrow$

$= \underbrace{P(X<0)}_{0} + \underbrace{P(X=0)}_{1-p}$

$F_X(0.999) \overset{?}{=} P(X \le 0.999)$

$= \underbrace{P(X<0)}_{0} + \underbrace{P(X=0)}_{(1-p)} + \overbrace{P(0<X\le 0.999)}^{0}$

$F_X(1) \overset{?}{=}$




$F_X(3900) \overset{?}{=} 1$

$F_X(1) \overset{?}{=} P(X \le 1)$

$= \underbrace{P(X=0)}_{} + P(X=1)$

$= 1-p + p = 1.$

$$F_X(3.8) \stackrel{?}{=} P(X \le 3.8) = 1$$

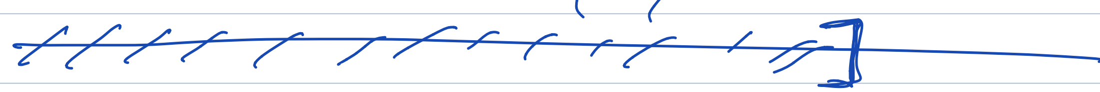
$$= P(\cancel{X<0}) + \overbrace{P(X=0)}^{1-p} + P(\cancel{X\in(0,1)}) + \overbrace{P(X=1)}^{p} + P(\cancel{X\in(1,3.8]})$$

$$\stackrel{?}{=} 1$$

$$P(X \le 3.8) = P(\{X \le 3.8\})$$

$$= P(\{X<0\} \cup \{X=0\} \cup \{X \in (0,1)\} \cdots)$$

$$= P(\quad) + P(\quad) + \cdots$$



3.8

Eg2 $\quad \Omega = \{\underbrace{1,2}, 3, 4, 5, 6\}$ equally likely.

$$X = \begin{cases} 0 & \text{if } s = \underbrace{1 \text{ or } 2} \\ 1.3 & \text{if } s = \underbrace{3 \text{ or } 4} \\ 2.9 & \text{if } s = \underbrace{5 \text{ or } 6}. \end{cases}$$

$P_X(x) \stackrel{?}{=}$ $\qquad \mathcal{X} = \{0, 1.3, 2.9\}$

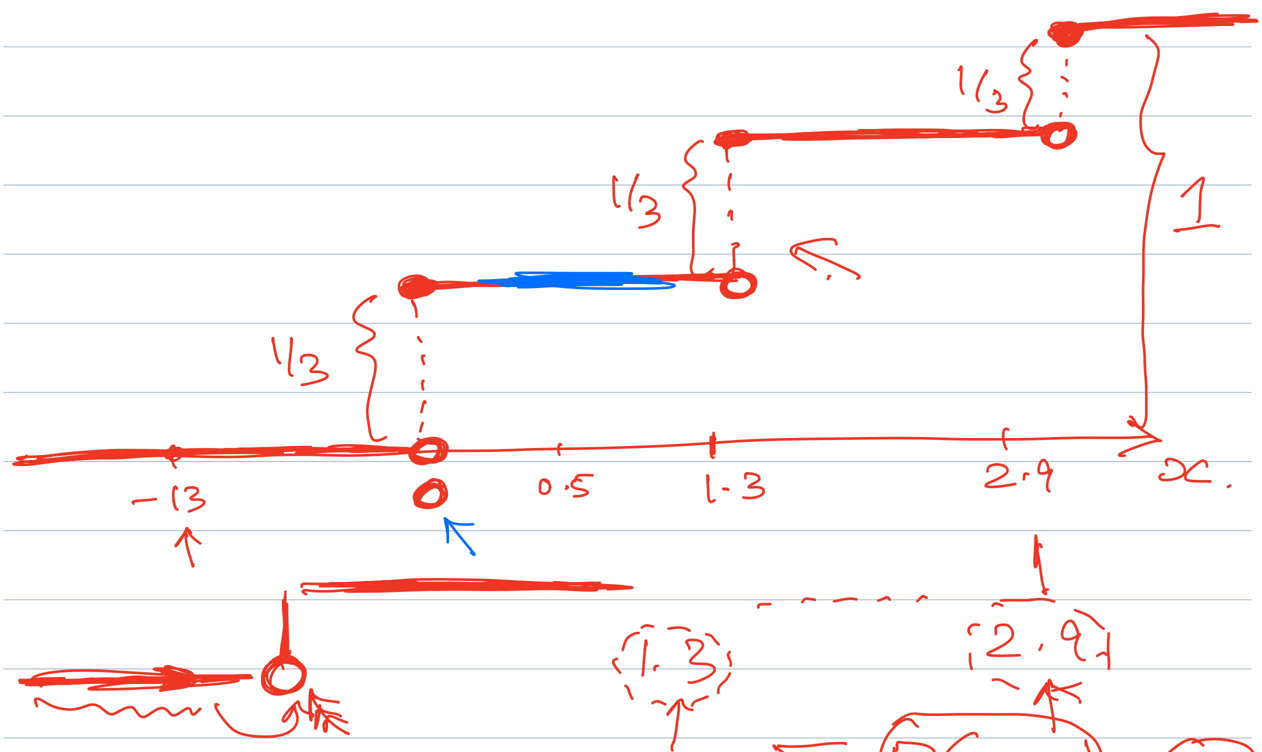
PMF

$$P(X=0) \stackrel{?}{=} P(\{1,2\}) = \frac{2}{6} = 1/3$$

$$P(X=1.3) = P(\{3,4\}) = 1/3$$

$$P(X=2.9) = P(\{5,6\}) = 1/3.$$

CDF ? $\Longrightarrow$ $P(X \le x) = F_X(x)$




$$\sum_{x \in \mathcal{X}} P(X=x) = 1.$$

$P(X \le x)$.

$$P(X=0) + P(X=1.3) + P(X=2.9) = 1$$

# Properties of CDF

$g(x)$ has a limit at some point $\underline{b}$

$$g(x^+) = \boxed{\lim_{\varepsilon \to 0} g(\cancel{x}+\varepsilon)} \leftarrow \text{Right lim}$$

$$g(x^-) = \lim_{\varepsilon \to 0} g(x-\varepsilon). \rightarrow \text{left lim.}$$

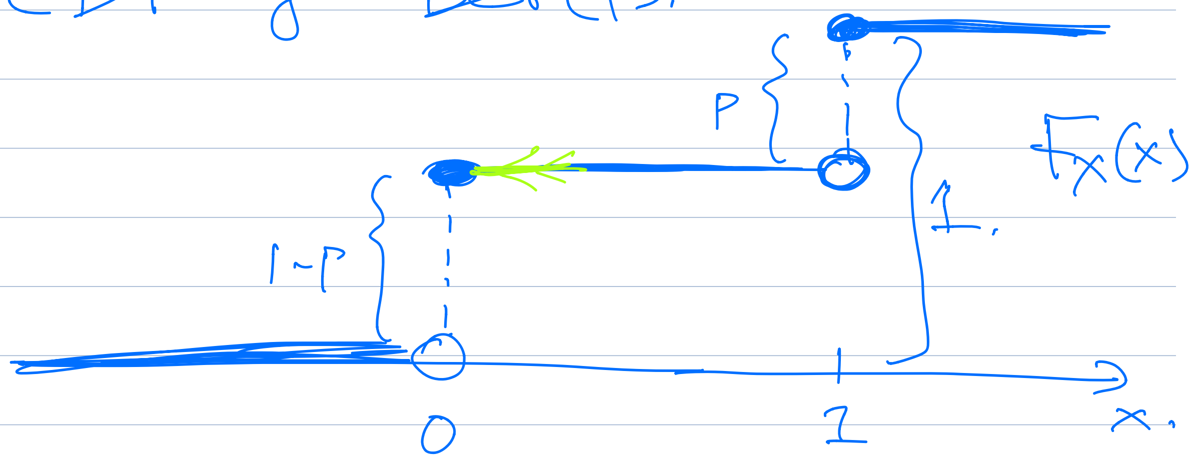
1) non-decreasing

2) always betw. $[0, 1]$.

$g(x) \Rightarrow$ value of fn at $\underline{x}$.

limit exists @ b. if $\underline{g(x)} = \underline{g(x^-)} = \underline{g(x^+)}$.

CDF of Ber(P).



$F_X(0^+) \overset{?}{=} 1-p$

$F_X(0^-) \overset{?}{=} 0$

$F_X(0) = 1-p.$

LIMIT does not exist

Right limit exists if $F_X(0^+) = F_X(0)$ ✓

Left limit does not exist.

1) CDF is NOT a continuous function.

2) is CDF right continuous ?

is this true $F_X(x^+) = F_X(x)$ for all $x$ ??

YES

# Properties of CDF

$$F_X(x^+) = \lim_{\varepsilon \to 0} F_X(x+\varepsilon)$$

$$F_X(x^-) = \lim_{\varepsilon \to 0} F_X(x-\varepsilon)$$

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① $F_X(\infty) = 1$

$P(X \le \infty)$

$P(S) = 1.$

$F_X(-\infty) = 0$

② $F_X(x)$ is a non-decreasing func of $x$.

if $x_1 < x_2$ then.

$$F_X(x_1) \le F_X(x_2).$$

$$P(X \le x_1) \le P(X \le x_2).$$

③ $$\underbrace{P(X > x) + P(X \le x)}_{P(\underbrace{\{X > x\} \cup \{X \le x\}}_{S})} = 1.$$

$$P(X > x) + P(X \le x) = 1.$$

$$P(X > x) + F_X(x) = 1$$

$$P(X > x) = 1 - F_X(x).$$

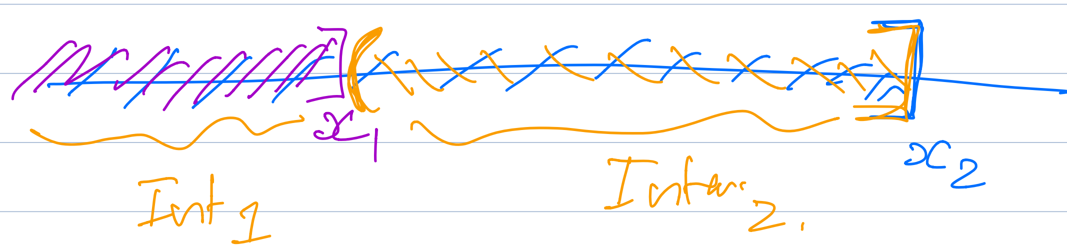
④ . $F_X(x)$ is a right continuous function.

$$F_X(x^+) = F_X(x)$$

⑤ $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

$$\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$$

Int.




$$P(X \le x_2) = P(X \le x_1) + P(x_1 < X \le x_2).$$

$$P(x_1 < X \le x_2) = \underbrace{P(X \le x_2)} - P(X \le x_1)$$

$$= F_X(x_2) - F_X(x_1).$$

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⑥ $$P(x_1 \le X \le x_2) \overset{?}{=} F_X(x_2) - \underbrace{F_X(x_1^-)}$$

(try this)

left limit of CDF @ $x_1$.

⑦ $$\underbrace{F_X(x)} = F_X(x^-) + P(X=x).$$

$$P(X \le x) = P(\{X \le x\})$$

$$= P(\underbrace{\{X < x\}} \cup \underbrace{\{X = x\}})$$

mut. excl.

$$= P(X < x) + P(X = x).$$

$$= F_X(x^-) + P(X=x).$$

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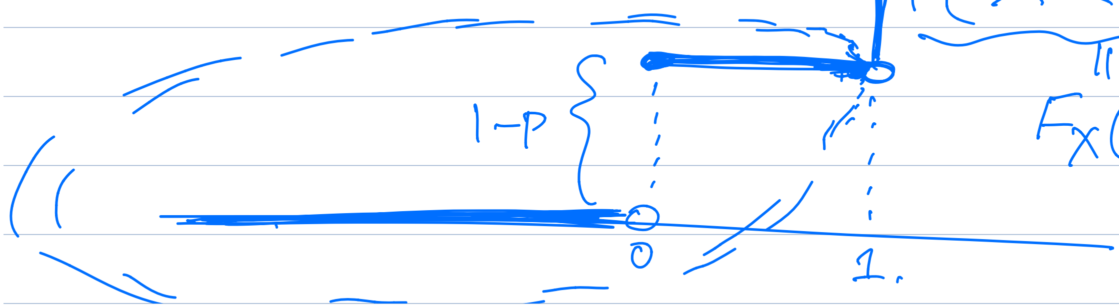
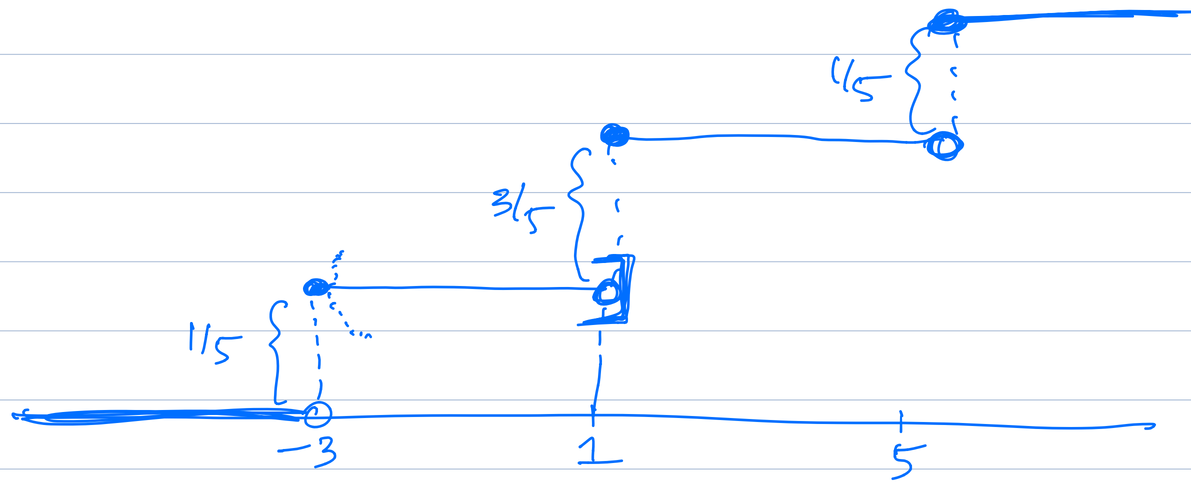


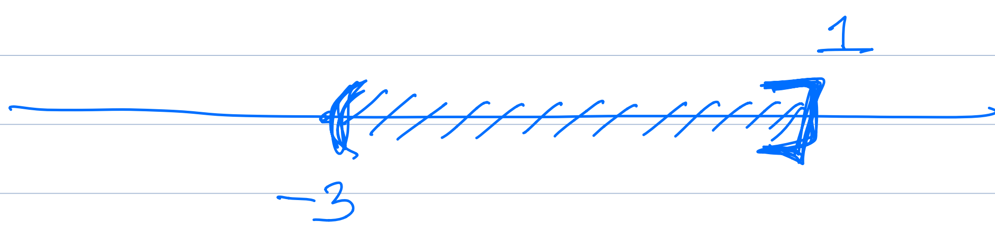

$$\underbrace{P(X < 1)}.$$

$$\| $$

$$F_X(1^-).$$




$$P(-3 < X \le 1)$$




$$X \in (-\infty, 1] = X \in (-\infty, -3] \cup X \in (-3, 1]$$

$$P(X \le 1) = P(X \le -3) + P(-3 < X \le 1)$$

$$F_X(1) - F_X(-3) = P(-3 < X \le 1).$$