Lecture 10

FUNCTIONS of ONE RANDOM VARIABLE

Suppose X is a r.v. and $g(x)$ is a function of the real variable x.

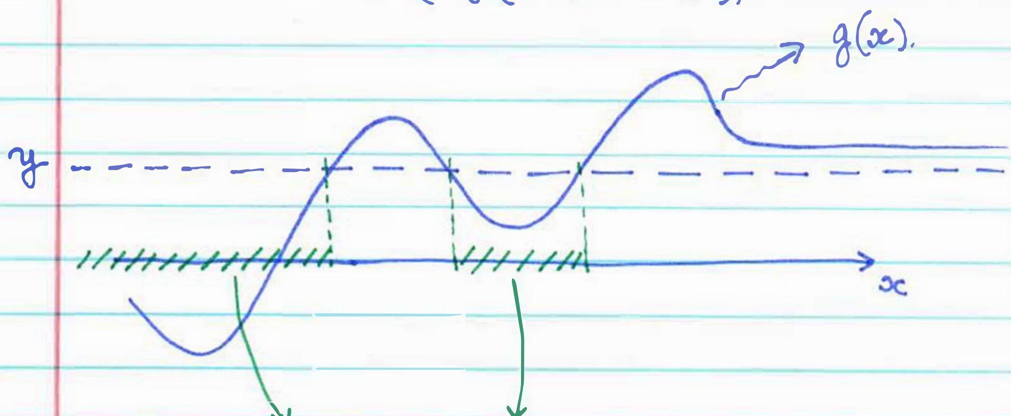
$$Y = g(X) \text{ is also a random variable.}$$

To formally define Y, recall X was a mapping from S (sample space) to real line, i.e $X(s)$ for $s \in S$. Hence, for an outcome $s \in S$, the value $Y(s) = g(X(s))$ is assigned to the random variable Y.

The distribution $F_Y(y)$ of the r.v. Y is then the probability of the event $\{Y \leq y\}$

$$\{Y = y\} = \{s : g(X(s)) \leq y\}$$

$$F_Y(y) = P(Y \leq y)$$
$$= P(g(X) \leq y)$$



These are the values of x for which $g(x) \leq y$.

For a fixed y, define this set as: $R_y = \{x : g(x) \leq y\}$.

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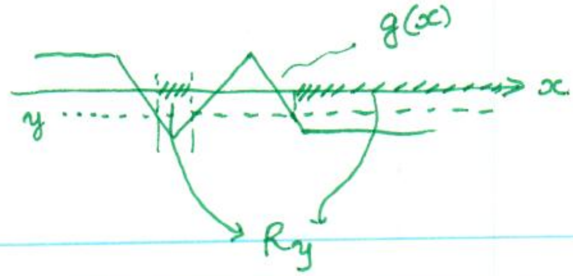
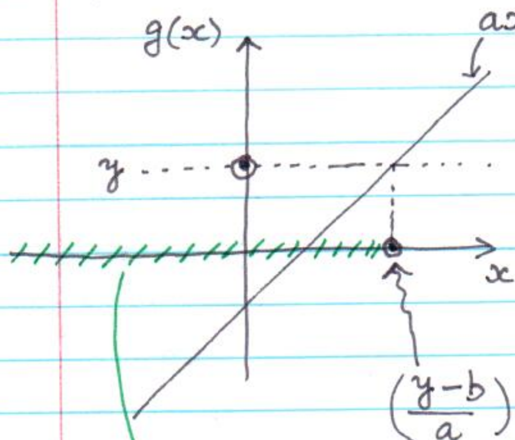


$$\Rightarrow F_Y(y) = P(Y \le y)$$
$$= P(g(X) \le y)$$
$$= P\Big(X \in \underbrace{\{x : g(x) \le y\}}_{R_y}\Big)$$
$$= P(X \in R_y)$$

$\boxed{\text{Eg 1} \quad Y = aX + b}$

To find $F_Y(y)$, we must find the values of x for which $ax + b \le y$. We consider ~~two~~ THREE cases:



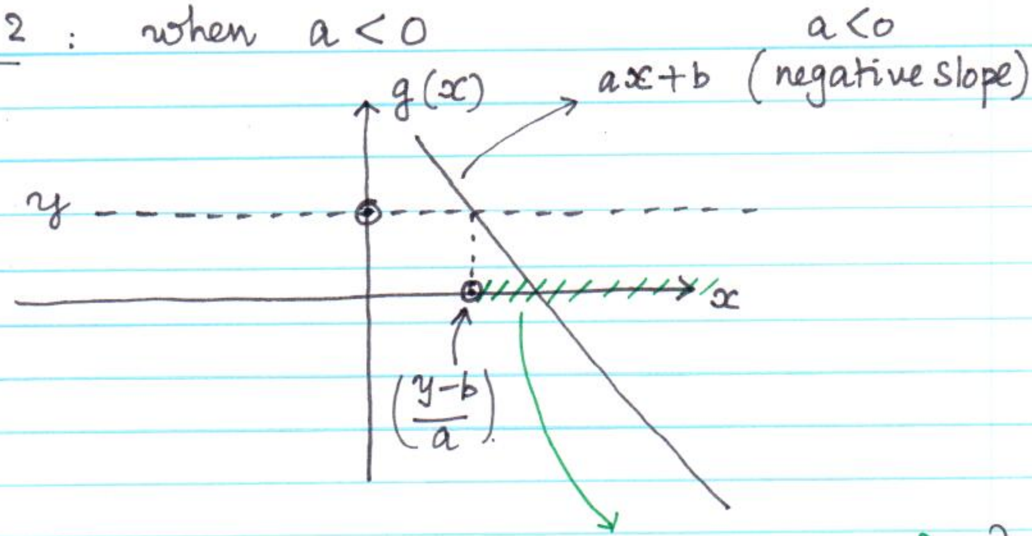
$a > 0$ (positive slope) $\Leftarrow$ CASE 1 when $a > 0$

$$R_y = \left\{x : \ x \le \left(\frac{y-b}{a}\right)\right\}$$

$$\Rightarrow F_Y(y) = P(X \in R_y) = P\left(X \le \left(\frac{y-b}{a}\right)\right)$$

(for $a > 0$) $\quad = F_X\left(\frac{y-b}{a}\right)$ [by definition of CDF of X]

CASE 2 : when $a < 0$



$R_y = \left\{ \begin{array}{l} \text{values of } x \text{ for} \\ \text{which } g(x) \le y \end{array} \right\}$

$\Rightarrow \quad R_y = \left\{ x : \; x \geq \left(\frac{y-b}{a}\right) \right\}$

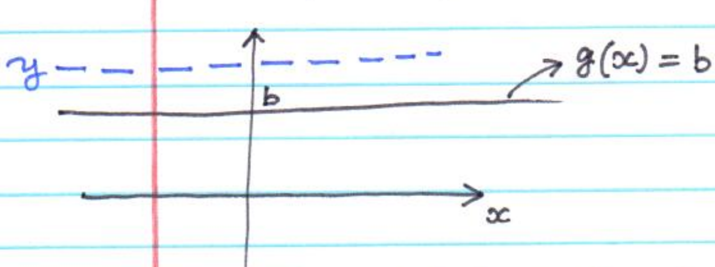
$\Rightarrow F_Y(y) = P(X \in R_y) = P\left(X \geq \left(\frac{y-b}{a}\right)\right)$
(for $a<0$)

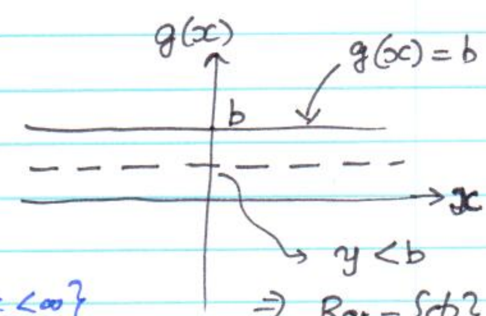
$= 1 - P\left(X < \left(\frac{y-b}{a}\right)\right)$

$= 1 - F_X\left(\frac{y-b}{a}\right).$

CASE 3 : when $a = 0$

$\Rightarrow \; Y = b$, i.e Y is a constant r.v.





case 1 : $y \geq b \Rightarrow R_y = \{x : -\infty < x < \infty\}$

$\Rightarrow F_Y(y) = P(-\infty < X < +\infty)$

$= 1$

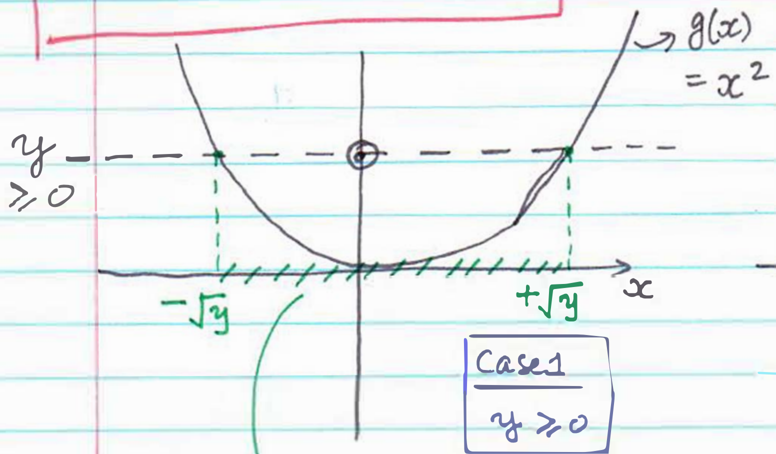
$\Rightarrow R_y = \{\phi\}$

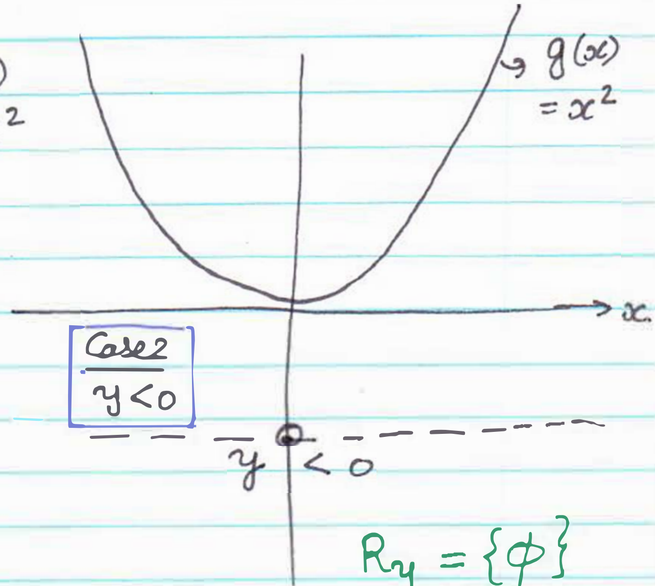
Case 2 : $y < b$

$\Rightarrow F_Y(y) = P(\{\phi\})$

$= 0.$

Eg 2 $\quad Y = X^2$





$R_y = \{x : -\sqrt{y} \leq x \leq +\sqrt{y}\}$

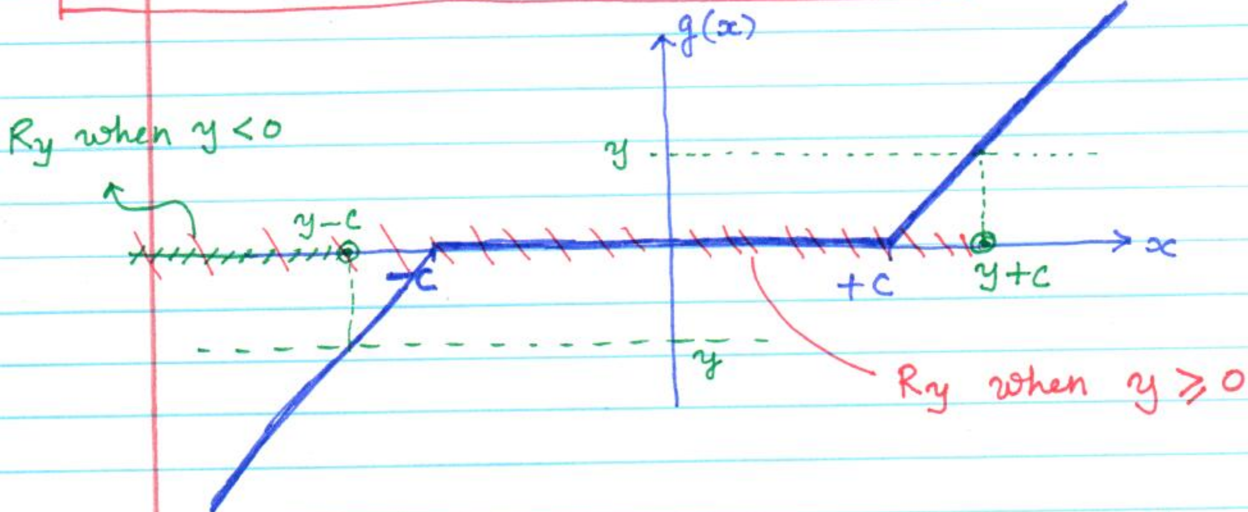
$R_y = \{\phi\}$
Null-set.

For $y \geq 0$: $\quad F_Y(y) = P(X \in R_y)$
$$= P(-\sqrt{y} \quad X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

For $y < 0$: $\quad F_Y(y) = P(X \in R_y)$
$$= P(X \in \{\Phi\}) = 0$$

$$\Rightarrow \quad F_Y(y) = \begin{cases} F_X(\sqrt{y}) - F_X(-\sqrt{y}) & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

$$\underline{\underline{\text{Eg 3}}}: \quad g(x) = \begin{cases} x - c & x > c \\ 0 & -c \le x \le c \\ x + c & x < -c \end{cases}$$



<u>Case 1</u> : $\quad y < 0 \Rightarrow R_y = \{x : \; x \le y - c\}$

$$\Rightarrow F_Y(y) = P(X \in R_y) = P(X \le y - c)$$

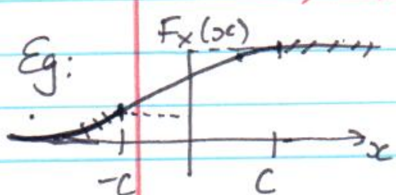
$$= F_X(y-c) \quad \text{for} \quad y < 0$$

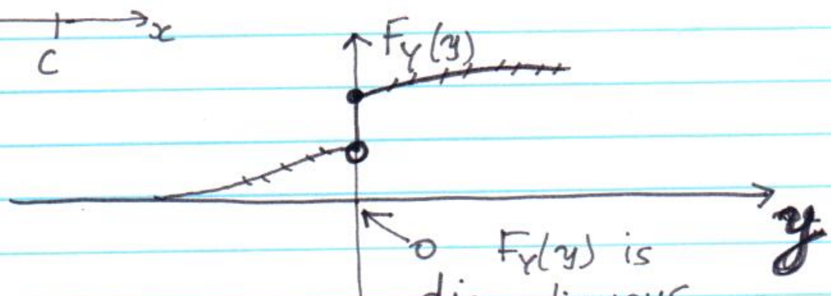
<u>Case 2</u> : $\quad y \geqslant 0 \Rightarrow R_y = \{x : \; x \le y + c\}$

$$\Rightarrow F_Y(y) = P(X \in R_y) = P(X \le y + c)$$

$$= F_X(y+c) \quad \text{for} \quad y \geqslant 0$$

Eg:





o $F_Y(y)$ is discontinuous at $y=0$. [Still, it is Right-continuous].

Quantization

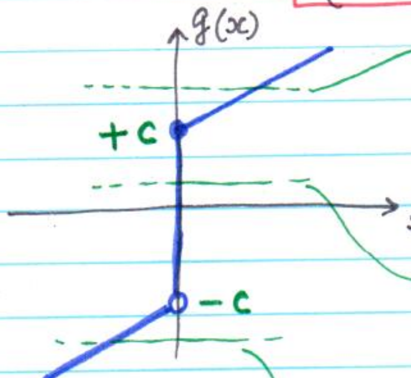
Eg 4 $g(x) = ns \quad , \quad \text{if } (n-1)s < x \le ns$

The random variable Y takes the value $y_n = ns$ when $(n-1)s < x \le ns$

PMF of Y

$$P_Y(y_n) = P_Y(ns) = P((n-1)s < X \le ns)$$
$$= F_X(ns) - F_X((n-1)s)$$

Eg 5 $g(x) = \begin{cases} x+c & x \ge 0 \\ x-c & x < 0 \end{cases}$



Case 1: $y \ge c$

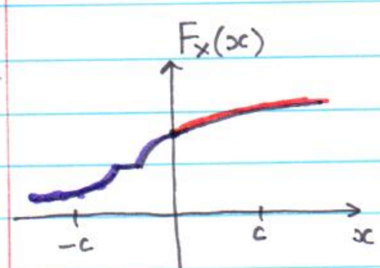
$F_Y(y) = F_X(y-c)$

Case 2: $-c \le y \le c$

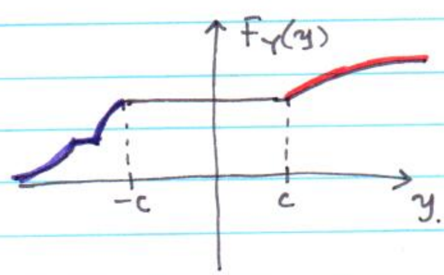
$F_Y(y) = F_X(0)$

Case 3: $y < -c$

$F_Y(y) = F_X(y+c)$



$\Rightarrow$



$\boxed{\underline{\text{Eg6}}:}$ Suppose X is a discrete r.v. taking values x_k, with probability P_k.

Then, the r.v. $Y = g(X)$ is also a discrete r.v. taking values $y_k = g(x_k)$.

* If $y_k = g(x)$ for only one $x = x_k$, then

$$P(Y = y_k) = P(X = x_k) = P_k$$

* If, however $y_k = g(x)$ for $x = x_k$ and $x = x_\ell$, then

$$P(Y = y_k) = P(X = x_k) + P(X = x_\ell) = P_k + P_\ell.$$

Determination of $\overbrace{F_Y(y)}^{CDF}$ or $\overbrace{f_Y(y)}^{PDF}$, when $Y = g(X)$.

Fundamental Theorem: To find $f_Y(y)$, for a specific value of y, we solve the equation $y = g(x)$.

Denote the real roots by $x_1, x_2, \ldots$, i.e

$$y = g(x_1) = g(x_2) = \cdots = g(x_n) = \cdots$$

Then

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} + \cdots + \frac{f_X(x_n)}{|g'(x_n)|} + \cdots$$

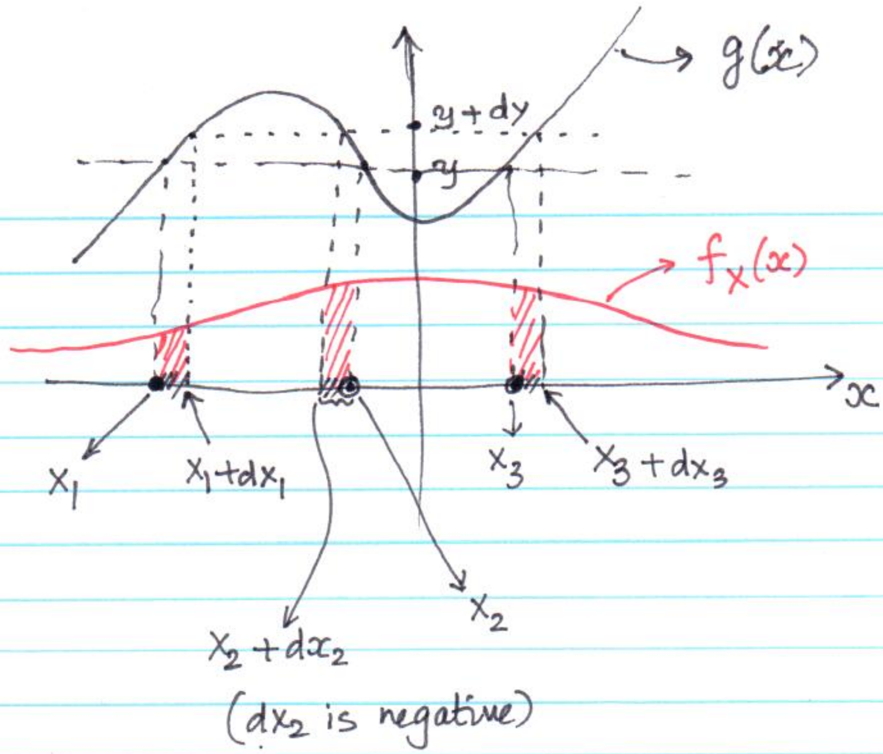
where $g'(x)$ is the derivative of $g(x)$.

Proof: Recall,

$$f_Y(y)\,dy = P(y < Y \le y + dy)$$

$$= P(X \in \{x : y < g(x) \le y + dy\})$$

i.e. we need to find the set of values of x such that $y < g(x) \le y + dy$ and the probability that X is in this set.



$$P(y < Y \le y+dy) = P(x_1 < X < x_1 + dx_1)$$
$$+ \; P(x_2 + dx_2 < X < x_2)$$
$$+ \; P(x_3 < X < x_3 + dx_3)$$

$$= f_X(x_1)\,dx_1 + f_X(x_2)\cdot|dx_2| + f_X(x_3)\cdot dx_3$$

$g'(x_1)\,dx_1 = dy_1$

$g'(x_2)\,dx_2 = dy_2$

$g'(x_3)\,d(x_3) = dy_3$

$$\Rightarrow \quad \underbrace{P(y < Y < y+dy)}_{\overset{\parallel}{f_Y(y)\,dy}} = \frac{f_X(x_1)}{g'(x_1)}\,dy + \frac{f_X(x_2)}{|g'(x_2)|}\,dy + \frac{f_X(x_3)}{g'(x_3)}\,dy$$

—— α ——

Ex $\quad Y = aX + b \qquad g'(x) = a$

$y = ax + b$ has a $\boxed{\text{single solution } x_0 = \left(\frac{y-b}{a}\right)}$

$$\Rightarrow \quad f_Y(y) = \frac{f_X\left(\frac{y-b}{a}\right)}{|g'(x_0)|} = \frac{1}{|a|} \cdot f_X\left(\frac{y-b}{a}\right)$$

Ex $\quad Y = aX^2, \quad a > 0$

$g'(x) = 2ax$

$y = ax^2 \to$ if $y \le 0 \Rightarrow$ this equation has NO REAL ROOTS

$\Rightarrow f_Y(y) = 0$ for $y < 0$

$\to$ if $y > 0 \Rightarrow$ Roots $\Rightarrow$

$$x = \pm\sqrt{\frac{y}{a}}$$

$$x_1 = \sqrt{\frac{y}{a}}; \quad x_2 = -\sqrt{\frac{y}{a}}.$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{ay}}\left[f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right] & \text{if } y > 0 \\ 0 & \text{if } y \le 0 \end{cases}$$

Eg $Y = e^X$ $\qquad g'(x) = e^x$

If $y < 0 \Rightarrow y = e^x$ has no Real root $\Rightarrow f_Y(y) = 0$

If $y > 0 \Rightarrow x = \ln(y)$

$$\Rightarrow f_Y(y) = \begin{cases} \dfrac{f_X(\ln(y))}{y} & y > 0 \\ 0 & y < 0 \end{cases}$$

Eg $Y = a\,\mathrm{Sin}(X + \theta)$, $a > 0$

If $|y| > a$, then $y = a\,\mathrm{Sin}(x+\theta)$ has No solutions

$$\Rightarrow f_Y(y) = 0$$

If $|y| < a \Rightarrow x_n = \sin^{-1}\left(\frac{y}{a}\right) - \theta$

infinite Solutions

$n = \ldots, -2, -1, 0, 1, 2, \ldots$

$g(x) = a\sin(x+\theta)$

$g'(x) = a\cos(x+\theta) = a\cos\left(\sin^{-1}(y/a)\right) = \sqrt{a^2 - y^2}$

$$\Rightarrow f_Y(y) = \begin{cases} \dfrac{1}{\sqrt{a^2-y^2}} \displaystyle\sum_{n=-\infty}^{\infty} f_X(x_n), & |y| < a \\ 0, & |y| > a. \end{cases}$$