Cumulative Distribution Function (CDF)
or
Distribution Function

An alternative means of summarizing the probabilities of a discrete r.v. is the CDF

CDF or Distribution Function: $F_X(x) = P(X \le x)$, defined for $-\infty < x < \infty$

The notation $X \le x$ represents a subset of Ω which consists of all outcomes s, for which $X(s) \le x$.

or $\{X \le x\} = \{s \in \Omega : X(s) \le x\}$.

and then, $F_X(x) = P(X \le x)$

Example Coin Toss

$$X = \begin{cases} 0, & s = T \\ 1, & s = H \end{cases}$$

$P(H) = p$
$P(T) = 1-p$

If $x \ge 1$, then depending on the outcome

either $s = T$, $\Rightarrow X = 0 < 1 \le x$

or $s = H$, $\Rightarrow X = 1 = x$

$\Rightarrow$ for both outcomes (T or H), $X \le x$ when $x \ge 1$.

$\Rightarrow$ For $x \geq 1$

$$F_X(x) = P(X \leq x)$$
$$= P(\{h, t\}) = p + 1-p = 1.$$

What about when $x < 1$?

If $0 \leq x < 1$

$X(H) = 1 > x$. $\longrightarrow$ this outcome does Not contribute to $X \leq x$

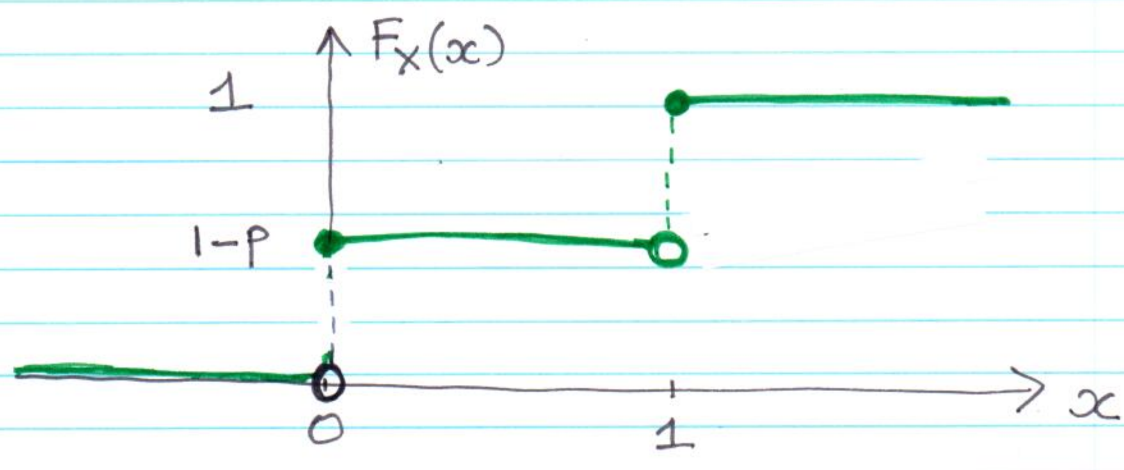
$X(T) = 0 \leq x$. $\longrightarrow$ contributes to $X \leq x$.

$$\Rightarrow F_X(x) = P(X \leq x) = P(\{t\}) = 1-p \quad \text{for } 0 \leq x < 1$$

If $x < 0$

$\left.\begin{array}{l} X(H) = 1 > x \\ X(T) = 0 > x \end{array}\right\}$ Neither outcomes contribute to $X \leq x$

$\Rightarrow F_X(x) = P(X \leq x) = P(\phi) = 0$ for $x < 0$



Another Example

Suppose $\Omega = \{1, 2, 3, 4, 5, 6\}$
and each outcome is equally likely.
We define the following random variable:

$$X = \begin{cases} 0 & \text{if } s = 1, 2 \\ 1.3 & \text{if } s = 3, 4 \\ 2.9 & \text{if } s = 5, 6 \end{cases}$$

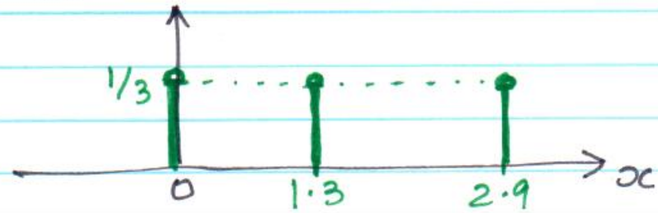
(a) PMF of X

$P_X(x)$??

$$P_X(0) = P(X=0)$$
$$= P(\{1,2\}) = \frac{2}{6} = \frac{1}{3}$$
$$P_X(1.3) = P(X=1.3)$$
$$= P(\{3,4\}) = 1/3$$
$$P_X(2.9) = P(\{5,6\}) = 1/3.$$

$$P_X(0) + P_X(1.3) + P_X(2.9) = 1$$

$P_X(x) \rightsquigarrow$ PMF



(b) CDF of X ??

$F_X(x) = P(X \le x)$

If $x \geq 2.9$ $\Rightarrow X \le x = \{1, 2, 3, 4, 5, 6\}$
(little x)

$\Rightarrow P(X \le x) = P(\{1, 2, \dots, 6\})$

$\Rightarrow F_X(x) = 1$ for $x \geq 2.9$

If $1.3 \le x < 2.9$

$\Rightarrow X \le x = \{1, 2, 3, 4\}$

$\Rightarrow F_X(x) = P(\{3, 4, 1, 2\}) = 2/3$ for $1.3 \le x < 2.9$

If $0 \le x < 1.3$

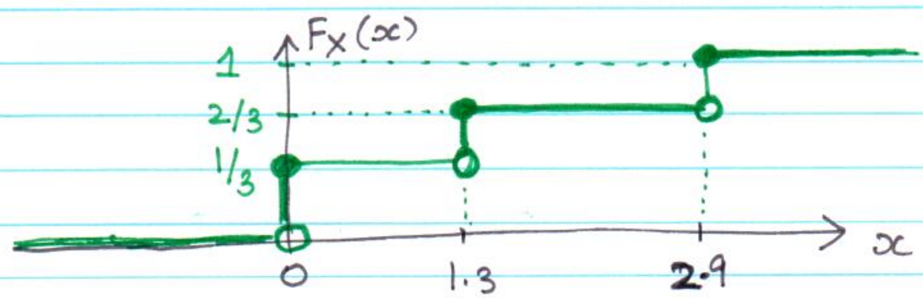
$\Rightarrow X \le x = \{1, 2\}$

$\Rightarrow F_X(x) = P(\{1, 2\}) = 1/3$ for $0 \le x < 1/3$

If $x < 0$

$\Rightarrow X \le x \Rightarrow \phi$

$\Rightarrow F_X(x) = P(\phi) = 0$ for $x < 0$



Properties of CDF

$$F_X(x^+) = \lim_{\epsilon \to 0} F_X(x+\epsilon)$$

$$F_X(x^-) = \lim_{\epsilon \to 0} F_X(x-\epsilon)$$

1. $\underbrace{F_X(+\infty)}_{\substack{\parallel \\ P(S)=1}} = 1 \qquad F_X(-\infty) = 0$

2. $F_X(x)$ is a non-decreasing function of x.

   If $x_1 < x_2 \Rightarrow F_X(x_1) \leq F_X(x_2)$

   $[P(X \leq x_1) \leq P(X \leq x_2)]$

3. $P(X > x) = 1 - \underbrace{F_X(x)}_{P(X \leq x)}$

4. $F_X(x)$ is continuous from the right

   $F_X(x^+) = F_X(x)$

   [$F_X(x)$ may or may not be left continuous]

5. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$

6. $P(x_1 \leq X \leq x_2) = F_X(x_2) - F_X(x_1^-)$

7. $P(X = x) = F_X(x) - F_X(x^-)$

Classification of R.V.'s based on CDF

Continuous R.V. $\Rightarrow$ $F_X(x)$ is continuous for all values of x.

$\Rightarrow$ $P(X=x) = F_X(x) - F_X(x^-) = 0$

Discrete R.V. $\Rightarrow$ $F_X(x)$ is constant except for a finite number of jump discontinuities (piecewise constant; step type shape).

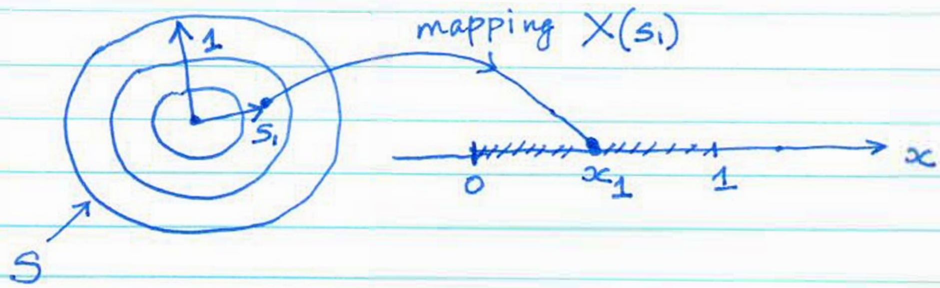
Lecture 8

Continuous Random Variables

In several applications, outcomes of experiments do not produce a discrete set of values but rather a continuum of values, for eg. noise in a receiver circuit. The # of outcomes can be infinite and uncountable.

A continuas R.V. is a mapping from S to a numerical sample space (or subset of the real line).

Eg Throwing a dart



Outcome of Experiment = location of dart s_1

$$X(s_1) = x_1 = \text{distance of dart from bullseye.}$$

We cannot assign a non-zero probability to each value of X and expect the sum to be 1.
We assign probabilities to intervals.

$$P(a \leq X \leq b)$$

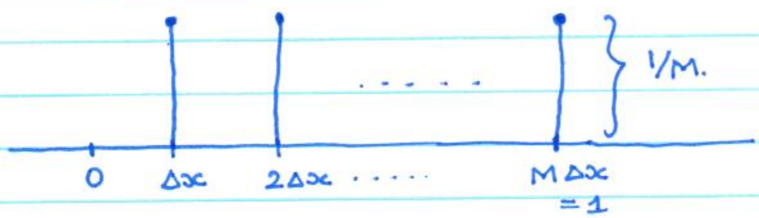
If each value of X is equally likely so that intervals of same length are equally likely, we could assign

$$(*) \quad P(a \leq X \leq b) = b - a, \quad 0 \leq a \leq b \leq 1$$

But what would we do if the probability of all equal length intervals were not the same? For eg, a champion dart thrower would be more likely to approach $x = 0$ than $x = 1$. We need a more general approach.

Consider approximation of $(*)$ by a uniform PMF

$$\left(\Delta x = \frac{1}{M}\right).$$



$$P_X(x_i) = \frac{1}{M}, \quad x_i = i\,\Delta x, \quad \text{for } i = 1, 2, 3, \ldots, M$$

Probability of an interval

$$P(a \leq X \leq b) = \sum_{\substack{i: \\ \{a \leq x_i \leq b\}}} (1/M)$$

From ⊛ $P(0.38 \le X \le 0.52) = 0.52 - 0.38 = 0.14$

Eg if $M = 10$, $\Delta x = 0.1$

$$P(0.38 \le X \le 0.52) = \frac{2}{M} = 0.2$$

if we increase M

$$M = 20, \quad \Delta x = 0.05$$

$$P(0.38 \le X \le 0.52) = \frac{3}{M} = 0.15$$

as $M \to \infty$ or $\Delta x \to 0$, the approximation to ⊛ becomes exact.

If we define a function

$$f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise}, \end{cases}$$

we can write

$$P(a \le X \le b) = \sum_{\{i : a \le x_i \le b\}} 1 \cdot \Delta x$$

$$= \sum_{\{i : a \le x_i \le b\}} f_X(x_i) \Delta x \quad \text{——} \quad (**)$$

Letting $\Delta x \to 0$ to yield no error in approximation, the sum in $(**)$ becomes an integral.

and $f_X(x_i) \rightarrow f_X(x)$

$$\Rightarrow \quad P(a \le X \le b) = \int_a^b f_X(x)\,dx$$

Since we defined $f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.wise} \end{cases}$,

the above integral yields the same result as $P(a \le X \le b) = b - a$.

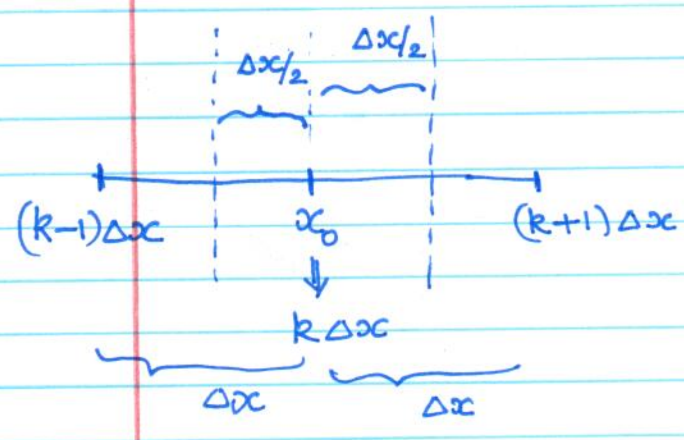
Interpretation of $f_X(x)$

Consider: $$P(a \le X \le b) = \sum_{\{i:\ a \le x_i \le b\}} f_X(x_i)\,\Delta x$$

For some $x_0 = k\,\Delta x$, with k an integer

$$P\left(x_0 - \Delta x/2 \le X \le x_0 + \Delta x/2\right) = f_X(x_0)\cdot \Delta x$$

$$\Downarrow$$

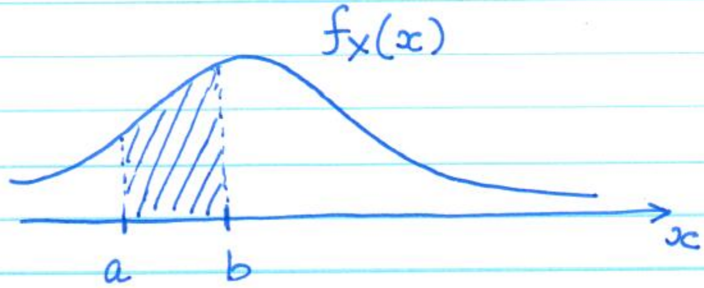


$$f_X(x_0) = \left\{ \frac{P\left(x_0 - \frac{\Delta x}{2} \le X \le x_0 + \frac{\Delta x}{2}\right)}{\Delta x} \right.$$

Probability of X being in the interval $\left[x_0 - \frac{\Delta x}{2}, x_0 + \frac{\Delta x}{2}\right]$ divided by the interval length Δx.

"Probability per unit length"

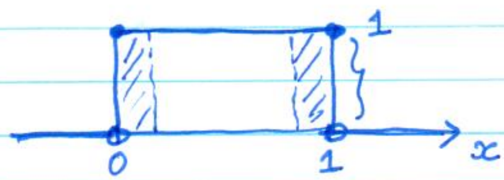
Hence $f_X(x)$ is termed as probability density function (PDF)



$$\boxed{P(a \leq X \leq b) = \int_a^b f_X(x)\,dx = \text{Shaded Area}.}$$

Returning to the "Expert Dart thrower".

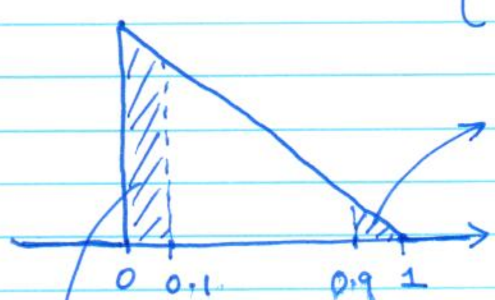
Option 1: $f_X(x) \Rightarrow$ (uniform)



$P(0 \leq X \leq 0.1) = 0.1$

$P(0.9 \leq X \leq 1) = 0.1$

Option 2: $f_X(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise.} \end{cases}$ (non-uniform).



$$P(0.9 \leq X \leq 1) = \int_{0.9}^{1} 2(1-x)\,dx = 0.01.$$

$$P(0 \leq X \leq 0.1) = \int_0^{0.1} 2(1-x)\,dx = 2\left(x - \frac{x^2}{2}\right)\Big|_0^{0.1} = 0.19$$

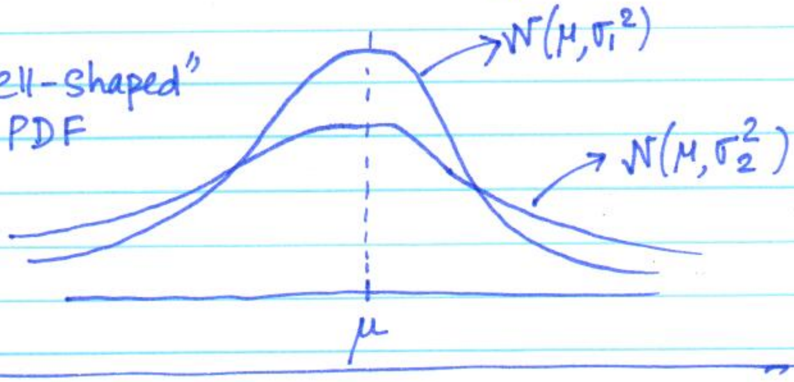
Commonly Encountered R.v.'s

* Gaussian (Normal) R.v.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Shorthand: $\boxed{X \sim \mathcal{N}(\mu, \sigma^2)}$

"Bell-shaped" PDF



$\sigma_1^2 < \sigma_2^2$

Proof that $f_X(x)$ is a valid PDF $\left(\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \right)$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Step 1: change of variable: $z = \left(\frac{x-\mu}{\sigma}\right)$

$$dz = \frac{dx}{\sigma}$$

$$\Rightarrow I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz.$$

Consider

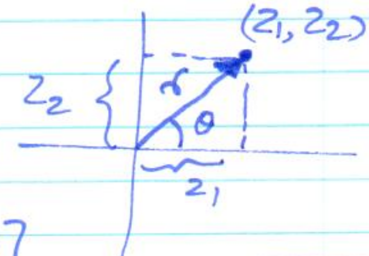
<u>Step 2</u>: $$I^2 = I \times I$$

$$= \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z_1^2/2} \, dz_1\right) \cdot \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z_2^2/2} \, dz_2\right)$$

$$I^2 = \frac{1}{2\pi} \int_{z_1=-\infty}^{\infty} \int_{z_2=-\infty}^{\infty} e^{-\frac{(z_1^2+z_2^2)}{2}} \, dz_1 \, dz_2.$$

<u>Now What ???</u> $\rightarrow$ Polar Coordinates

$$z_1 = r \cos\theta$$
$$z_2 = r \sin\theta$$



$$\Rightarrow z_1^2 + z_2^2 = r^2$$
$$\Rightarrow dz_1 \, dz_2 = r \, dr \, d\theta \quad [\text{why ??}]$$

via <u>Jacobi Transform</u>

$$dz_1 \, dz_2 = \det(\text{Jacobi}) \cdot dr \, d\theta. \quad \frac{\partial(z_1, z_2)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial z_1}{\partial r} & \frac{\partial z_1}{\partial \theta} \\ \frac{\partial z_2}{\partial r} & \frac{\partial z_2}{\partial \theta} \end{vmatrix}$$

Wiki / Google yourself....... ☺

$$\Rightarrow I^2 = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2/2} \, r \, dr \, d\theta$$

$$= \frac{1}{2\pi} \left(\int_{\theta=0}^{2\pi} d\theta\right) \cdot \left(\int_{r=0}^{\infty} e^{-r^2/2} \, r \, dr\right) = \int_{r=0}^{\infty} e^{-r^2/2} \, r \, dr$$

$$\Rightarrow \quad I^2 = \int_{r=0}^{\infty} e^{-r^2/2}\, r\,dr$$

Another change of variable

$$\lambda = r^2/2$$

$$d\lambda = r\,dr$$

$$I^2 = \int_{\lambda=0}^{\infty} e^{-\lambda}\,d\lambda = -e^{-\lambda}\Big|_{\lambda=0}^{\infty}$$

$$= (1-0) = 1$$

$$\Rightarrow \quad I^2 = 1$$

$$\Rightarrow \quad I = 1$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

———— α ————.