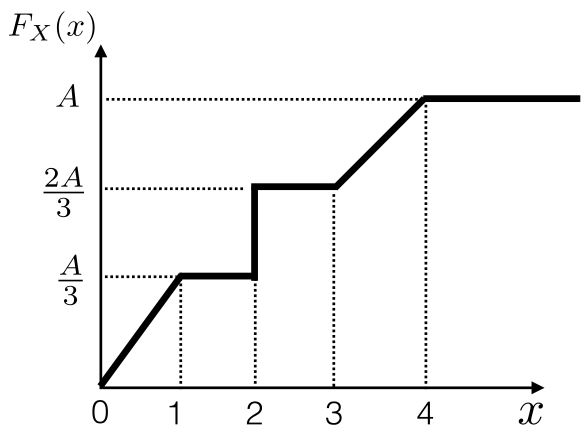
Final Exam - ECE 503 Fall 2020

- Due Date & Time: Monday, December 14, 2020 by Noon.
- Upload your scanned exam on D2L (Final Exam Folder).
- Maximum Credit: 100 points

1. [10 points] A random variable X has the following CDF, $F_X(x)$ as shown in the figure below:



(a) Find the value A.

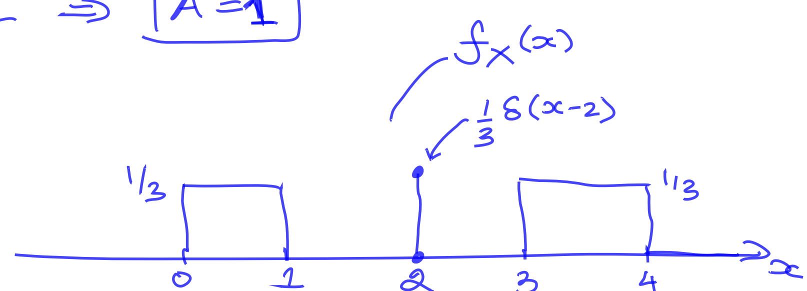
(b) Sketch the probability density function (PDF) of this random variable.

(c) Find the mean and variance of X.

(d) What is $P(X < 2)$?

(a) Since $\lim_{x \to \infty} F_X(x) = 1 \Rightarrow \boxed{A = 1}$

(b) $\underbrace{f_X(x)}_{\text{PDF}} = \frac{d}{dx} F_X(x)$



(c) $E[X] = \frac{1}{3}\int_0^1 x\,dx + 2 \times \frac{1}{3} + \frac{1}{3} \times \int_3^4 x\,dx = \boxed{2}$

$E[X^2] = 50/9 \Rightarrow Var[X] = E(X) - (E(X))^2 = \frac{50}{9} - 4$

$= \boxed{14/9}$

(d) $P(X < 2) = 1/3$

2. [10 points] Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables such that

$$X_n \sim \text{Poisson}(n\lambda), \text{ for } n = 1, 2, 3, \ldots$$

where $\lambda > 0$ is a constant. Define a new sequence of random variables Y_n as

$$Y_n = \frac{X_n}{n} \text{ for } n = 1, 2, 3, \ldots$$

Show that Y_n converges in the mean square sense to λ.

For a Poisson r.v. $E[X_n] = n\lambda$

$Var[X_n] = n\lambda$

To prove

$Y_n \xrightarrow{m.s.} \lambda \Rightarrow$ we need to show $E[|Y_n - \lambda|^2] \to 0$ as $n \to \infty$.

$$E[|Y_n - \lambda|^2] = E\left[\left|\frac{X_n}{n} - \lambda\right|^2\right] = \frac{1}{n^2} E[|X_n - n\lambda|^2]$$

$$= \frac{Var[X_n]}{n^2} = \frac{n\lambda}{n^2} = \frac{\lambda}{n} \to 0 \text{ as } n \to \infty.$$

3. [20 points] In order to obtain FDA authorization for a new COVID-19 vaccine, one needs to be 97% sure that side effects do not occur more than 10% of the time.

   (a) In order to estimate the probability p of side effects, the vaccine is tested on 100 volunteers. Side effects are experienced by 6 volunteers and the sample standard deviation is observed as 0.239. Find the 97% confidence interval estimate for p. Based on your analysis of the results from above 100 trials, are you convinced with 97% confidence that $p \leq 0.1$?

   (b) Another study is performed, this time with 1000 volunteers. Side effects occur in 71 volunteers. Find the 97% confidence interval for the probability p of side effects if the sample standard deviation is 0.257. Are you now convinced with 97% confidence that $p \leq 0.1$?

(a) Sample mean (for # of side-effects) $= \bar{\mu}_{(n)} = \frac{6}{100}$

$\hat{\sigma}_n = 0.239$ (Standard deviation)

97% confidence interval $= [\bar{\mu}_{(n)} - c, \bar{\mu}_{(n)} + c]$

where $c = \frac{(2.170) \times \hat{\sigma}_n}{\sqrt{100}} = 0.0519.$

$\Rightarrow$ 97 % CI for p $\Rightarrow [0.06 - 0.0519, 0.06 + 0.0519]$

$\Rightarrow [0.0081, 0.1119]$

$\Rightarrow$ We are NOT 97 % sure that $p < 0.1$ (Since the interval exceeds 0.1).

(b) For second study,

$\hat{\mu}_{(n)} = \frac{71}{1000}$

$\hat{\sigma}_{(n)} = 0.257$

$c = \frac{(2.170) \times \hat{\sigma}_n}{\sqrt{1000}} = 0.018$

$\Rightarrow$ 97% CI for p $\Rightarrow [0.071 - 0.018, 0.071 + 0.018]$

$\Rightarrow [0.053, 0.089]$

Since 0.1 is outside & larger than this interval, $\Rightarrow$ we are 97 % sure that $p < 0.1$.

4. [20 points] Diners arrive at a popular restaurant according to a Poisson process (denoted as $N(t)$) with rate λ.

(a) What is the expected time for first n customers to arrive?

(b) Find the variance of the time it takes for first n customers to arrive.

(c) Due to social distancing measures, every customer is independently seated with a probability p, or turned away with probability $(1-p)$. Let $M(t)$ be the resulting random process, denoting the total number of customers that are seated at time t. Prove that $M(t)$ is also a Poisson process. Find the rate of $M(t)$.

For a poisson Process, we know that inter arrival times (X_i's) are iid, exponential (λ) random variables.

$\Rightarrow$ (a) Exp. time for n arrivals $= n \times E[X_i] = n \times \frac{1}{\lambda} = \boxed{\frac{n}{\lambda}}$

(b) Variance of Exp. time for n arrivals $= n \times Var[X_i] = n \times \frac{1}{\lambda^2} = \boxed{\frac{n}{\lambda^2}}$

(c) $N(t) =$ # of arrivals by time $t \sim$ Poisson(λt)

$M(t) =$ # of seated customers by time t.

$$P(M(t) = i) = \sum_{j=i}^{\infty} P(M(t) = i, N(t) = j)$$ (From Total Probability Theorem).

$$= \sum_{j=i}^{\infty} P(N(t)=j) \times P(M(t) = i \mid N(t) = j)$$

$$= \sum_{j=i}^{\infty} \left(\frac{e^{-\lambda t} \times (\lambda t)^j}{j!}\right) \times \underbrace{\left[\binom{j}{i} \times p^{i} (1-p)^{j-i}\right]}_{\text{Probability that } i \text{ customers seated given that } j \text{ arrived}}$$

$$= \left(e^{-\lambda t}\right) \times \sum_{(j-i)=0}^{\infty} \left[\frac{(\lambda t)^j}{j!} \times \frac{j!}{i!\,(j-i)!} \times p^{i} \times (1-p)^{j-i}\right]$$

$$= e^{-\lambda t} \times \frac{(\lambda p t)^i}{i!} \times \underbrace{\sum_{(j-i)=0}^{\infty} \frac{(\lambda p t)^{j-i}}{(j-i)!} \times (1-p)^{j-i}}$$

$$= e^{-\lambda t} \times \frac{(\lambda p t)^i}{i!} \times e^{+\lambda t(1-p)}.$$

$$P(M(t) = i) = \frac{e^{-(\lambda p) t} \times (\lambda p t)^{i}}{i!} \Rightarrow$$ $\boxed{M(t) \text{ is a Poisson process with rate } = \lambda p}$

5. [20 points] A mobile sensor sends a radio signal to a receiver situated at a distance R from it. The distance R is a random variable with the following PDF:

$$f_R(r) = \begin{cases} 2r/10^6, & 0 \le r \le 1000, \\ 0, & \text{otherwise.} \end{cases}$$

The resulting signal power (measured in dB) seen at the receiver as a function of the distance R is modeled as follows:

$$X = Y - 40 - 40\log_{10}(R),$$

where Y captures a fading phenomenon, modeled as a Gaussian random variable $\mathcal{N}(0, 8)$ which is independent of the distance R. The goal of receiver is to use the received signal power X to estimate the distance R from the sensor.

(a) Write down the joint PDF of (X, R), i.e., $f_{X,R}(x, r)$.

(b) Find the MAP estimate of R given the observation $X = x$.

(c) Find the ML estimate of R given the observation $X = x$.

(a) $f_{X,R}(x,r) = f_R(r) \times f_{X|R}(x|r)$ for $0 \le r \le 1000$

$= \frac{2r}{10^6} \times \frac{1}{\sqrt{2\pi \times 8}} \times e^{-\frac{(x+40+40\log_{10}(r))^2}{2\times 8}}$

$(X|R=r) \sim \underbrace{Y}_{\mathcal{N}(0,8)} - 40 - 40\log_{10}(r).$

$\Rightarrow X|R=r \sim \mathcal{N}(-40-40\log_{10}(r), 8).$

$\Rightarrow f_{X,R}(x,r) = \begin{cases} (\text{const}) \times r \times e^{-\frac{(x+40+40\log_{10}(r))^2}{16}} & \text{if } 0 \le r \le 1000 \\ 0 & \text{otherwise.} \end{cases}$

(c) $\hat{r}_{ML}(x) = \underset{r \ge 0}{\text{argmax}}\ f_{X|R}(x|r).$ $\Rightarrow$ find r that maximizes $e^{-\frac{(x+40+40\log_{10}(r))^2}{16}}$

(b) $\hat{r}_{MAP}(x) = \underset{r \ge 0}{\text{argmax}}\ f_{R|X}(r|x)$

$= \underset{r \ge 0}{\text{argmax}}\ f(r) \times f_{X|R}(x|r)$

$= \underset{r \ge 0}{\text{argmax}} \left(r \times e^{-\frac{(x+40+40\log_{10}(r))^2}{16}} \right)$

$\Rightarrow$ find r that minimizes $(x+40+40\log_{10}(r))^2$

$\Rightarrow x + 40 + 40\log_{10}(r) = 0.$

$\Rightarrow \log_{10}(r) = -\frac{x}{40} - 1$

$\boxed{\hat{r}_{ML} = 10^{-\left(\frac{x}{40}+1\right)}}$

6. [20 points] A WSS random process $X(t)$ with the following PSD

$$S_X(f) = \begin{cases} 10^{-4} & |f| \le 100, \\ 0 & \text{otherwise.} \end{cases}$$

is given as an input to a LTI filter with the following transfer function:

$$H(f) = \frac{1}{100\pi + j2\pi f}$$

The output of the filter is the random process $Y(t)$.

(a) Find the power of the input signal $X(t)$.

(b) Find the PSD of the output signal $Y(t)$.

(c) What is the power of the output signal?

Solution for Problem (5) continued: $\hat{\gamma}_{MAP}(x) = \underset{\gamma \ge 0}{\arg\max} \; \log(\gamma) - \frac{(x + 40 + 40\log_{10}(\gamma))^2}{16}$

$$\frac{d(\cdots)}{d\gamma} = \frac{1}{\gamma} - \frac{2}{16}(x + 40 + 40\log_{10}(\gamma)) \times \frac{d(\log_{10}(\gamma))}{d\gamma}$$

$$= \frac{1}{\gamma} - \frac{1}{8}(x + 40 + 40\log_{10}(\gamma)) \times \frac{1}{\ln(10)} \times \frac{1}{\gamma} = 0.$$

$$\Rightarrow \frac{1}{\gamma}\left\{1 - \frac{1}{8\ln(10)}(x + 40 + 40\log_{10}(\gamma))\right\} = 0$$

$$\Rightarrow 8\ln(10) = x + 40 + 40\log_{10}(\gamma) \Rightarrow \boxed{\hat{\gamma}_{MAP}(x) = 10^{\left\{\frac{(8\ln(10) - 40 - x)}{40}\right\}}}$$

Problem 6 Solution:

(a) Power of $X(t) = \int_{-\infty}^{\infty} S_X(f)\,df = \int_{-100}^{100} 10^{-4} \times df = 0.02$ Watts.

(b) PSD of $Y(t) = S_Y(f) = |H(f)|^2 \times S_X(f)$

$$= \begin{cases} \frac{10^{-4}}{(10^4\pi^2 + (2\pi f)^2)} & \text{if } |f| \le 100 \\ 0 & \text{otherwise.} \end{cases}$$

(c) Power of $Y(t) = \int_{-\infty}^{\infty} S_Y(f)\,df = 1.12 \times 10^{-7}$ Watts