# Lecture 24



## $\boxed{\text{Mean and Correlation Functions}}$

If $X(t)$ is a random process, then for every value of $t$, $X(t)$ is a random variable, which has a mean $E[X_t]$. We call

$$\mu_X(t) \text{ or } m_X(t) = E[X(t)]$$

as the **mean function** of the random process. The mean function reflects the average behavior of the process with time.

---

Example: In a communication system, the carrier signal at the receiver is modeled as

$$X(t) = \cos(2\pi f t + \Theta), \text{ where } \Theta \sim \text{unif}[-\pi, \pi]$$

$$m_X(t) = E[X(t)] = E[\cos(2\pi f t + \Theta)]$$

$$= \int_{-\pi}^{\pi} \cos(2\pi f t + \theta) f_\Theta(\theta)\, d\theta$$

$$= \int_{-\pi}^{\pi} \cos(2\pi f t + \theta) \frac{1}{2\pi}\, d\theta = 0$$

$$\Rightarrow m_X(t) = 0 \text{ for all } \underline{\underline{t}}.$$

If $X(t_1)$, $X(t_2)$ are two random variables of a process $X(t)$, their correlation is denoted by

$$\underbrace{R_X(t_1, t_2)}_{\boxed{\text{Auto-Correlation function}}} = E[X(t_1)X(t_2)]$$

$\boxed{\text{Auto-Correlation function}}$ of the random process $X(t)$.

The value of $R(t_1, t_2)$ on the "diagonal" $t_1 = t_2 = t$ is the average power of $X(t)$

$$R_X(t, t) = E[X^2(t)]$$

$\boxed{\text{Autocovariance function}}$ $C_X(t_1, t_2)$ of $X(t)$

$$C_X(t_1, t_2) = Cov(X(t_1), X(t_2))$$

$$= E[X(t_1)X(t_2)] - \mu_X(t_1)\mu_X(t_2)$$

$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\mu_X(t_2).$$

$$C_X(t,t) = E[X^2(t)] - \mu_X^2(t) = Var(X(t)).$$

## Properties: of auto-correlation and auto-covariance functions.

1. $R_X(t_1, t_2) = R_X(t_2, t_1)$, i.e $R_X(\cdot,\cdot)$ is a symmetric function of $t_1$ and $t_2$.

Recall that we had the problem in Mid-Term 2, $(E[AB])^2 \leq E(A^2)E(B^2)$

$$\Rightarrow \quad \underbrace{E[X(t_1)X(t_2)]}_{\Downarrow}{}^2 \leq E[X^2(t_1)]E[X^2(t_2)]$$

$$\Rightarrow \quad (R_X(t_1,t_2))^2 \leq E[X^2(t_1)]E[X^2(t_2)]$$

$$\boxed{|R_X(t_1,t_2)| \leq \sqrt{E(X^2(t_1))E(X^2(t_2))}}$$

2. $C_X(t_1, t_2) = C_X(t_2, t_1)$, i.e the auto-covariance function is also symmetric.

Let $X(t)$ and $Y(t)$ be two random processes. Their $\boxed{\text{cross-correlation function}}$ is

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

Their $\boxed{\text{cross-covariance function}}$ is

$$C_{XY}(t_1, t_2) = E[(X(t_1) - \mu_X(t_1))(Y(t_2) - \mu_Y(t_2))]$$

$$= R_{XY}(t_1, t_2) - \mu_X(t_1)\mu_Y(t_2).$$

~~Eg~~:

**Eg:** $X(t) = \cos(2\pi f t + \theta)$,
where $\theta \sim \text{unif}\,[-\pi, \pi]$

Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

$$= E[\cos(2\pi f t_1 + \theta)\cos(2\pi f t_2 + \theta)]$$

$$\left\{ \begin{array}{l} \text{(using)} \\ \cos A \cos B = \dfrac{\cos(A+B) + \cos(A-B)}{2} \end{array} \right\}$$

$$= \frac{1}{2} E\left[\cos\left(2\pi f(t_1+t_2) + \theta\right) + \cos(2\pi f(t_1 - t_2))\right]$$

$$= \underbrace{\frac{1}{2} E\left[\cos\left(2\pi f(t_1+t_2) + \theta\right)\right]}_{= 0} + \frac{1}{2}\cos(2\pi f(t_1 - t_2))$$

auto-correlation

$$\Rightarrow \boxed{R_X(t_1, t_2) = \frac{1}{2}\cos(2\pi f(t_1 - t_2))}$$

**Eg:** $X(t) = \cos(2\pi f t + \theta_1)$
$Y(t) = \cos(2\pi f t + \theta_2)$, $\theta_1, \theta_2$ are independent unif $[-\pi, \pi]$ random variables.

Cross-correlation between $X(t)$ and $Y(t)$

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)] = E[\cos(2\pi f t_1 + \theta_1)\cos(2\pi f t_2 + \theta_2)]$$

$$= E[\cos(2\pi f t_1 + \theta_1)]\, E[\cos(2\pi f t_2 + \theta_2)]$$ since $\theta_1, \theta_2$ are independent

$$= 0 \times 0 = 0$$

# Strict-Sense Stationary and Wide-Sense Stationary Processes

A random process is $\boxed{n^{th} \text{ order strictly stationary}}$ if for any collection of $n$ times, $t_1, t_2, \ldots, t_n$, the joint distribution of $(X(t_1), X(t_2), \ldots, X(t_n))$ is the same as the joint distribution of $X(t_1+\Delta), X(t_2+\Delta), \ldots, X(t_n+\Delta)$ for any $\Delta$.

A random process is $\boxed{\text{strictly stationary}}$ if it is $n^{th}$ order strictly stationary for every positive, finite integer $n$.

$1^{st}$ order S.S. $\quad f_{X(t)}(x) = f_{X(t+\Delta)}(x)$ for all $\Delta$

$2^{nd}$ order S.S. $\quad f_{X(t_1)X(t_2)}(x_1, x_2) = f_{X(t_1+\Delta)X(t_2+\Delta)}(x_1, x_2)$ for all $\Delta$

$\vdots$

---

Eg $X_n$ be a sequence of iid r.v.'s with a common density $f(\cdot)$

$\Rightarrow$ $X_n$ is $\left.\begin{array}{l} 1^{st} \text{ order SS} \\ 2^{nd} \text{ " "} \\ \vdots \\ n^{th} \text{ " " } \forall n \end{array}\right\} \Rightarrow X_n$ is strictly stationary.

# Wide-Sense Stationary (WSS)

A random process is WSS if the following two properties (both) hold:

(i) The mean function $\mu_X(t)$ does not depend on time $t$, i.e.

$$\boxed{E[X(t)] = \mu_X}$$

(ii) The auto-correlation function $R_X(t_1, t_2)$ depends on $t_1$ and $t_2$ only through the time difference $t_1 - t_2$.

$$\cancel{R_X(t_1, t_2) = R_X(t, t+\tau) = R_X(\tau)}, \text{ ~~where~~ } \cancel{t_1 - t_2 = \tau}$$

$$\boxed{R_X(t_1, t_2) = R_X(t_1, t_1 - \tau) = R_X(\tau)} \quad \begin{pmatrix} t_2 = t_1 - \tau \\ t_1 - t_2 = \tau \end{pmatrix}$$

Fact: Every Strictly Stationary process is WSS.
However WSS does NOT imply S.S.

$$R_X(t_1, t_2) = R_X(\tau) \; ; \; \tau = t_1 - t_2$$ 3

$\boxed{\text{For a WSS random process,}}$ the following properties always hold:

P1: $R_X(0) \geq 0$

P2: $R_X(\tau) = R_X(-\tau)$

P3: $R_X(0) \geq |R_X(\tau)|$

---

Proof of $P_1$, $P_2$, and $P_3$:

$\boxed{P_1}$ $R_X(0) = R_X(t,t) = E[X^2(t)] \geq 0$

$\boxed{P_2}$ $R_X(\tau) = R_X(t, t-\tau) = E[X(t)X(t-\tau)]$

$$\parallel$$

$R_X(-\tau) = R_X(t-\tau, t) = E[X(t-\tau)X(t)]$

$\Rightarrow R_X(\tau) = R_X(-\tau)$ ie for a WSS, $R_X(\cdot)$ is symmetric around $\tau = 0$

~~or $R_X(\cdot)$ is an even function (Symmetric)~~

$\boxed{P_3}$ $\left(\underbrace{R_X(t_1, t_2)}\right)^2 \leq E[X^2(t_1)]\,E[X^2(t_2)]$

$(R_X(\tau))^2 \leq R_X(0)\,R_X(0)$

$\Rightarrow |R_X(\tau)| \leq R_X(0).$

Eg: Input to a digital filter is an iid random sequence .... $X_{-2}$ $X_{-1}$ $X_0$ $X_1$ $X_2$... with $E[X_i]=0$, $Var[X_i]=1$ for all $i$.
The output of the filter is a random sequence ... $Y_{-2}$ $Y_{-1}$ $Y_0$ $Y_1$ $Y_2$...

$$Y_n = X_n + X_{n-1} \quad \text{for all integers } n.$$

Find $\underbrace{\mu_Y(n)}_{\text{mean function}}$, and $\underbrace{C_Y(m,n)}_{\text{auto-correlation function}}$ of $Y_n$.

---

$$E[Y_n] = E[X_n + X_{n-1}] = \underbrace{E[X_n]}_{0} + \underbrace{E[X_{n-1}]}_{0}$$

$$\Rightarrow E[Y_n] = 0 \text{ for all } n.$$

$$\Rightarrow \boxed{\mu_Y(n) = E[Y_n] = 0}$$

$$C_Y(m,n) = E[Y(m)\,Y(n)] - \underbrace{\mu_Y(m)}_{0}\underbrace{\mu_Y(n)}_{0}$$

$$= E[Y_m Y_n]$$

$$= E[(X_m + X_{m-1})(X_n + X_{n-1})]$$

$$= E[X_m X_n + X_m X_{n-1} + X_{m-1} X_n + X_{m-1} X_{n-1}]$$

$$= C_X(m,n) + C_X(m,n-1) + C_X(m-1,n) + C_X(m-1,n-1).$$

To find this, we note that

$$C_X(m,n) = E[X_m X_n] = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n. \end{cases}$$

$$C_Y(m,n) = C_X(m,n) + C_X(m,n-1) + C_X(m-1,n) + C_X(m-1,n-1).$$

If $m = n$ $\Rightarrow$ $C_Y(m,n) = C_X(m,m) + C_X(m-1,m-1)$
(or $|m-n| = 0$)
$= 2$

If $m = n-1 \Rightarrow C_Y(m,n) = C_X(m,m) = 1$

If $m-1 = n \Rightarrow C_Y(m,n) = C_X(n,n) = 1$

or $|m-n| = 1$.

For $|m-n| > 1$ $C_Y(m,n) = 0$

$$\Rightarrow \boxed{C_Y(m,n) = \begin{cases} 2 & |m-n| = 0 \\ 1 & |m-n| = 1 \\ 0 & |m-n| > 1. \end{cases}}$$

# Jointly Wide-Sense Stationary Processes

$X(t)$ and $Y(t)$ are jointly WSS if

(a) both $X(t)$ and $Y(t)$ are WSS, and

(b) cross-correlation depends only on time diffrence

$$R_{XY}(t, t-\tau) = R_{XY}(\tau)$$

---

Eg Let $X_n$ be a WSS discrete-time random process, with auto-correlation function $R_X[k]$, and zero-mean

Let $Y_n$ be a random process as:

$$Y_n = (-1)^n X_n$$

(i) Is $Y_n$ WSS ?

$$E[Y_n] = (-1)^n E[X_n] = 0$$ (does not depend on time)

$$R_Y(n, n+k) = E[Y_n Y_{n+k}]$$

$$= E[(-1)^n X_n (-1)^{n+k} X_{n+k}]$$

$$= \underbrace{(-1)^{2n+k}}_{\downarrow} \underbrace{E[X_n X_{n+k}]}_{\downarrow}$$

$$= (-1)^k R_X[k]$$ (auto-correlation depends only on time-difference $k$)

(ii) Cross-correlation

$$R_{XY}(n, n+k) = E[X_n Y_{n+k}]$$

$$= E[X_n (-1)^{n+k} X_{n+k}]$$

$$= (-1)^{n+k} \underbrace{E[X_n X_{n+k}]}$$

$$= (-1)^{n+k} R_X[k]$$

depends on n and k <u>both</u>

$\Rightarrow X_n, Y_n$ are <u>NOT</u> jointly WSS.

even though $X_n$ is WSS
and $Y_n$ is WSS.