Solutions

1. $$X_n = \begin{cases} n^{\alpha} & \text{if } 0 \le U \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < U \le 1 \end{cases}$$

(a) $P(X_n = n^{\alpha}) = P(0 \le U \le \frac{1}{n}) = \frac{1}{n}$

$P(X_n = 0) = 1 - \frac{1}{n}$

$E[|X_n - 0|^2] = E[X_n^2] = (n^{2\alpha}) \times \frac{1}{n} + (0) \times (1 - \frac{1}{n})$

$\Rightarrow E(X_n^2) = n^{(2\alpha - 1)}$

$\Rightarrow$ if $2\alpha - 1 < 0$

$\Rightarrow \alpha < \frac{1}{2} \Rightarrow$ then $E(X_n^2) \to 0$ as $n \to \infty$

$\Rightarrow$ for $\alpha < \frac{1}{2}$, $X_n \to 0$ in m.s. sense.

(b) $P(|X_n - 0| \ge \epsilon)$

~~for $\epsilon > 1$~~

For any $0 < \epsilon$: $P(X_n \ge \epsilon) \le P(X_n = n^{\alpha}) = \frac{1}{n} \to 0$ as $n \to \infty$

$\Rightarrow X_n \to 0$ in Probability for any α

$X_n \sim Laplace(\mu = 0, \sigma^2 = \frac{2}{n^2})$.

(2) To show a.s. convergence, it suffices to show that

Note
[SEE LECTURE 27, THEOREM 1]

$$\sum_{n=1}^{\infty} P(|X_n| \geq \epsilon) < \infty$$

$$P(|X_n| \geq \epsilon) = P(X_n \geq \epsilon) + P(X_n \leq -\epsilon)$$

$$= 2 \int_{\epsilon}^{\infty} f_{X_n}(x)\, dx$$

$$= 2 \int_{\epsilon}^{\infty} \frac{n}{2} e^{-nx} dx$$

$$= -e^{-nx} \Big|_{\epsilon}^{\infty} = e^{-n\epsilon}$$

Then,

$$\sum_{n=1}^{\infty} P(|X_n| \geq \epsilon) = \sum_{n=1}^{\infty} \left(e^{-n\epsilon}\right) = \sum_{n=1}^{\infty} \left(e^{-\epsilon}\right)^n$$

$$= \frac{e^{-\epsilon}}{1 - e^{-\epsilon}} < \infty$$

$$\Rightarrow \quad X_n \xrightarrow{a.s.} 0$$

3 For Rayleigh $\left(\frac{1}{n}\right)$, [SEE LECTURE 27, Theorem 1]

$$P(|X_n| \geq \epsilon) = P(X_n \geq \epsilon)$$
$$= e^{-n^2\epsilon^2/2}$$

$$\Rightarrow \sum_{n=1}^{\infty} P(|X_n| \geq \epsilon) = \sum_{n=1}^{\infty} e^{-\epsilon^2 n^2/2}$$
$$\leq \sum_{n=1}^{\infty} \left(e^{-\epsilon^2/2}\right)^n$$
$$= \frac{e^{-\epsilon^2/2}}{1 - e^{-\epsilon^2/2}} < \infty$$

$\Rightarrow$ $X_n \longrightarrow 0$ almost surely

(4) X_n has the density $f_n(x) = n f(nx)$

$$F_n(x) = P(X_n \le x) = \int_{-\infty}^{x} f_n(t)\,dt$$

CDF of X_n

$$= \int_{-\infty}^{x} n f(nt)\,dt$$

$$= \int_{-\infty}^{nx} f(y)\,dy$$

$$= F(nx).$$

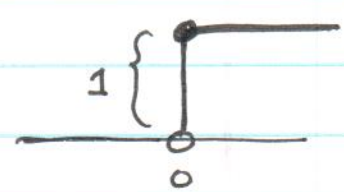
$\Rightarrow \quad F_n(x) = F(nx)$

for $x > 0$, as $n \to \infty$ $\quad F_n(x) \longrightarrow F(\infty) = 1$

for $x < 0$, as $n \to \infty$ $\quad F_n(x) \longrightarrow F(-\infty) = 0$

(for $x = 0$, $F_n(0) = F(0)$)

$\Rightarrow \quad F_n(x) \longrightarrow$ CDF of $0 \Rightarrow$



for all $x \neq 0$

$\Rightarrow \quad X_n \longrightarrow 0$ in distribution as $n \to \infty$.

5 $X_1, X_2, \ldots, X_n$ are iid,
uniform with $\mu = 7$, $\sigma^2 = 3$.

(a) Since X_1 is uniform $\Rightarrow$ uniform PDF over $[a, b]$

$$\left.\begin{aligned} \mu &= \frac{a+b}{2} = 7 \\ \sigma^2 &= \frac{(b-a)^2}{12} = 3 \end{aligned}\right\} \Longrightarrow \text{Solving for } a, b$$

$a = 4$,
$b = 10$

$$\Rightarrow \quad f_X(x) = \begin{cases} \frac{1}{6} & 4 \le x \le 10 \\ 0 & \text{otherwise.} \end{cases}$$

(b) $M_{16} = \dfrac{X_1 + X_2 \cdots + X_{16}}{16}$, X_i's are iid

$$\mathrm{Var}(M_{16}) = \frac{\mathrm{Var}(X)}{16} = \frac{3}{16}.$$

(c) $P(X_1 > 9) = \displaystyle\int_9^\infty f_{X_1}(x)\,dx = \int_9^{10} \frac{1}{6}\,dx = \frac{1}{6}$

(d)

$$P(M_{16} > 9) = 1 - P(M_{16} \le 9)$$

$$= 1 - P\left(\frac{M_{16} - 7}{\sqrt{3/16}} \le \frac{9-7}{\sqrt{3/16}} \right)$$

$$\frac{M_{16} - 7}{\sqrt{3/16}} \xrightarrow[\text{by CLT}]{\text{in distribution}} \mathcal{N}(0,1).$$

$$\cong 1 - \Phi\left(\frac{8}{\sqrt{3}}\right)$$

$$= 1 - \Phi(2.66) = $$ [redacted] 1.93×10^{-6}

$$\Rightarrow P(M_{16} > 9) \ll P(X_1 > 9)$$

---

(6) We know from Chebyshev's inequality that

$$P(|X - E(X)| \ge c) \le \frac{\sigma_X^2}{c^2}$$

choosing $c = k\sigma_X \Rightarrow P(|X - E(X)| \ge k\sigma) \le \frac{1}{k^2}$

On the other hand, the actual probability that the Gaussian r.v. Y is more than $k\sigma_Y$ from its expected value is

$$P(|Y-E(Y)| \geq k\sigma_Y)$$
$$= P(Y-E(Y) \leq -k\sigma_Y)$$
$$+ P(Y-E(Y) \geq k\sigma_Y)$$
$$= 2P\Big(\underbrace{\frac{Y-E(Y)}{\sigma_Y}}_{\mathcal{N}(0,1)} \geq k\Big)$$
$$= 2Q(k).$$

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
upper bound (by Chebyshev)	1	0.250	0.111	0.0625	0.04
$2Q(k)$	0.317	0.046	0.0027	6.33×10^{-5}	5.73×10^{-7}

What we observe is that Chebyshev based bound gets increasingly weak as k increases. For eg:, at $k=4$, the bound exceeds the true probability by a factor of 1000. for $k=5$, the bound exceeds the true probability by a factor of nearly 100,000 !!!

7

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n)$$

$$= E\left((X_1 + X_2 \cdots + X_n - n\mu)^2 \right)$$

$$= E\left(\left((X_1-\mu) + (X_2-\mu) + \ldots + (X_n-\mu) \right)^2 \right)$$

$$= \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \mathrm{cov}(X_i, X_j)$$

$$= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} a^{j-i}$$

$$= n\sigma^2 + 2\sigma^2 \sum_{i=1}^{n-1} (a + a^2 + \ldots + a^{n-i})$$

$$= n\sigma^2 + \frac{2a\sigma^2}{1-a} \sum_{i=1}^{n-1} (1 - a^{n-i})$$

$$\vdots$$

$$= \frac{n(1+a)\sigma^2}{(1-a)} - \frac{2a\sigma^2}{(1-a)} - 2\sigma^2 \left(\frac{a}{1-a}\right)^2 (1 - a^{n-1})$$

$$\leq n\sigma^2 \frac{(1+a)}{(1-a)}$$

$$\Rightarrow \mathrm{Var}(X_1 + \ldots + X_n) \leq n\sigma^2 \frac{(1+a)}{(1-a)}$$

Now, $E(M_n) = \mu$; $Var(M_n) = \dfrac{Var(X_1 + \cdots + X_n)}{n^2}$

$$\leq \frac{\sigma^2(1+a)}{n(1-a)}$$

$$P(|M_n - \mu| \geq \epsilon) \leq \frac{Var(M_n)}{\epsilon^2} \leq \frac{\sigma^2(1+a)}{n(1-a)\epsilon^2} \longrightarrow 0$$

as $n \to \infty$

$$\Rightarrow \quad M_n \xrightarrow{\text{in Prob.}} \mu .$$