Lecture 26

Random Walk

{ discrete-time random process }

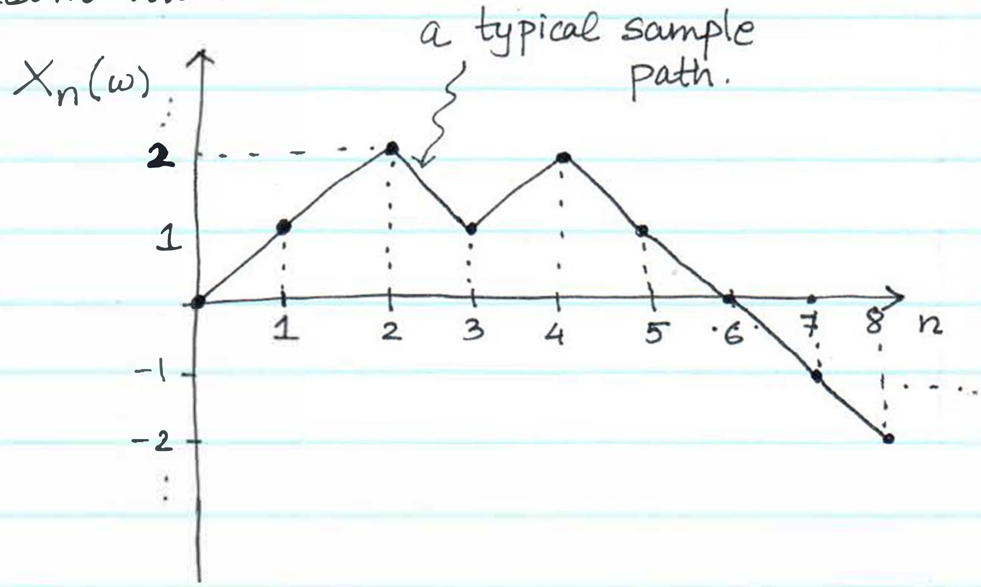
Let $W_1, W_2, \ldots$ be iid random variables.

$$W_i = \begin{cases} +1 & \text{with prob. } p \\ -1 & \text{'' '' } 1-p \end{cases}$$

$$X_0 = 0$$

$$X_n = W_1 + W_2 + \cdots + W_n$$

The above random-process X_n is called a random walk.



$$E[X_n] = n(2p-1)$$

$$Var[X_n] = 4np(1-p)$$

$$\text{Prob}(X_n = j-(n-j)) = \binom{n}{j} p^j (1-p)^{n-j} \quad \text{for } 0 \leq j \leq n$$

"Gambler's Ruin Problem" is a nice example of a random walk.

Assume initial wealth $X_0 = k$, $k \geq 0$.

X_n = # of units of money a gambler has at time n.

Suppose that the goal is to accumulate b units of \$\$. for some $b \geq k$.

R_b = denote the event that the gambler is eventually ruined, i.e. random walk reaches 0 without first reaching b.

we are interested in $P_k(R_b)$, i.e the ganbler's ruin probability.

Let $r_k = P_k(R_b)$ for $0 \leq k \leq b$

r_k → ruin probability with initial wealth k and target wealth b.

<u>Boundary Conditions:</u>

$r_0 = 1$

$r_b = 0$

$$r_k = P(W_1 = 1) P_k[R_b | W_1 = 1] + P(W_1 = -1) P_k[R_b | W_1 = -1]$$

$$\Rightarrow r_k = p\, r_{k+1} + (1-p)\, r_{k-1}$$

for $p \neq 1/2$

$$r_k = \frac{\left(\frac{1-p}{p}\right)^k - \left(\frac{1-p}{p}\right)^b}{1 - \left(\frac{1-p}{p}\right)^b}, \quad 0 \le k \le b$$

for $p = \frac{1}{2}$; $r_k = 1 - \frac{k}{b}$.

for $p = \frac{1}{2}$; Expected wealth $\cancel{\Rightarrow b \times (1-k)}$

$$\Downarrow$$

$$\underbrace{P(\text{ruin})} \times 0 + \underbrace{P(\text{Not-ruin})} \times b$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\left(1 - \frac{k}{b}\right) \times 0 + \frac{k}{b} \times b = k$$

$\downarrow$

Same as initial wealth!!!

What happens for $p > 1/2$??

Mean function for Random Walk.

$$\mu_X(n) = E[X_n]$$

$$= E[W_1 + W_2 \cdots + W_n]$$

$$= \underbrace{E[W_1]}_{=0} + \cdots + \underbrace{E[W_n]}_{=0} = 0$$

$$\Rightarrow \boxed{\mu_X(n) = 0}$$

Auto-correlation function for Random Walk

$$R_X(n_1, n_2) = E[X_{n_1} X_{n_2}]$$

(assume $n_2 \geq n_1$)

$$= E[X_{n_1}(X_{n_2} - X_{n_1} + X_{n_1})]$$

$$= \underbrace{E[X_{n_1}(X_{n_2} - X_{n_1})]} + E[X_{n_1}^2]$$

$$= 0 + E[X_{n_1}^2]$$

$$= n_1$$

In, general $\boxed{R_X(n_1, n_2) = \min(n_1, n_2)}$

Markov Processes

(More on these later....)

A discrete-time process X_n is said to be a Markov process if the future and past are conditionally independent given its present value.

$$P_{X_{n+1}|X^n}(x_{n+1} \mid x_n, X^{n-1})$$

$$= P_{X_{n+1}|X_n}(x_{n+1} \mid x_n).$$

$$\forall\, n.$$

* IID processes are Markov

* Random Walk is a Markov process.

Independent Increment Process

A discrete-time process X_n is said to be independent increment if the increment random variables

$$X_{n_1},\ X_{n_2}-X_{n_1},\ X_{n_3}-X_{n_2},\ \dots,\ X_{n_k}-X_{n_{k-1}}$$

are independent for all sequences of indices such that $n_1 < n_2 < n_3 < \dots < n_k$

* Random Walk is an idependent increment process.

Eg: Applications:

- Photon arrivals at optical detector
- Packet arrivals at router
- Hits on a Website....

POISSON PROCESS

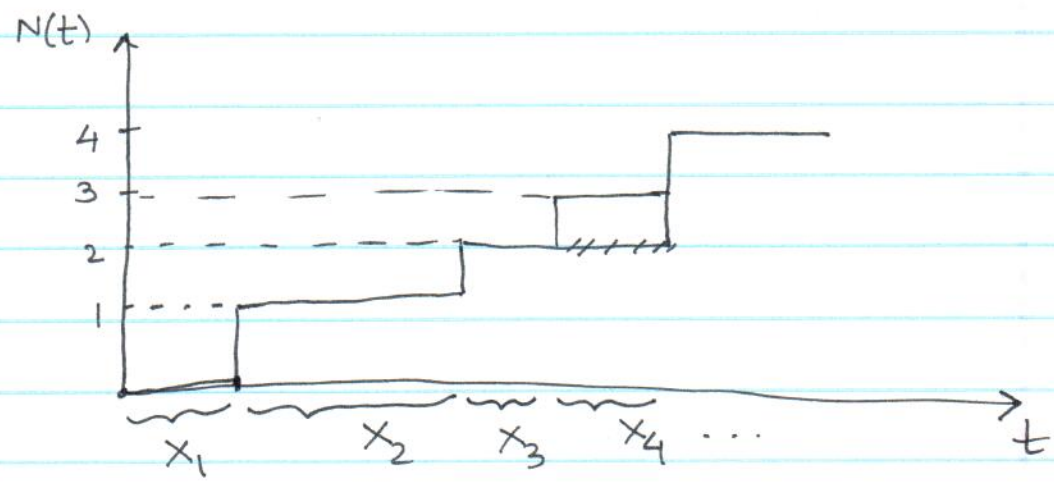
A counting process $N(t)$ starts at time 0 and counts the occurence of events. These events are generally called <u>arrivals</u>. Since we start at time $t=0$, $n(t,s)=0$ for all $t \le 0$, and the number of arrivals up to any $t>0$ is an integer that cannot decrease with time.

Def: Counting Process

A stochastic process ~~is~~ $N(t)$ is a counting process if for every sample function, $n(t,s)=0$ for $t<0$ and $n(t,s)$ is integer-valued and non-decreasing with time.

$N(t)$ = # of customers arriving in the interval $[0,t]$

$N(t_1) - N(t_0)$ = # of customers arriving in $(t_0, t_1]$



Poisson Approximation to Binomial

$$\binom{m}{n}\left(\frac{\alpha}{m}\right)^{n}\left(1-\frac{\alpha}{m}\right)^{m-n}$$

$$= \frac{m!}{n!\,(m-n)!} \times \frac{1}{m^{n}}\;\alpha^{n}\left(1-\frac{\alpha}{m}\right)^{m-n}$$

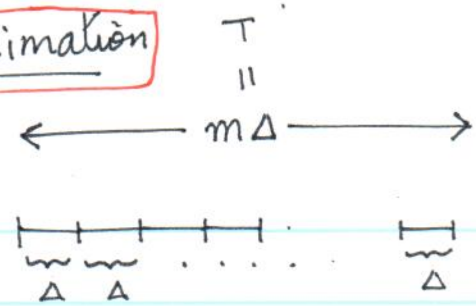
$$= \frac{m!}{(m-n)!} \times \frac{1}{m^{n}} \times \left(1-\frac{\alpha}{m}\right)^{m-n} \times \left(\frac{\alpha^{n}}{n!}\right)$$

$$= \underbrace{\left(\frac{m\times(m-1)\times\cdots\times(m-n+1)}{m^{n}}\right)}_{\lim\limits_{m\to\infty} = 1} \times \underbrace{\left(1-\frac{\alpha}{m}\right)^{m-n}}_{\lim\limits_{m\to\infty} = e^{-\alpha}} \times \left(\frac{\alpha^{n}}{n!}\right)$$

$$\lim_{m\to\infty} \frac{\overbrace{(1-\alpha/m)^{m}}^{\to\, e^{-\alpha}}}{\underbrace{\left(1-\frac{\alpha}{m}\right)^{n}}_{\to\, 1}} = \frac{e^{-\alpha}}{1}.$$

$$\underset{\left(\lim\limits_{m\to\infty}\right)}{=} 1 \times e^{-\alpha} \times \frac{\alpha^{n}}{n!} = e^{-\lambda T}\cdot\frac{(\lambda T)^{n}}{n!}$$

Poisson Approximation Via Binomial



λ = avg. arrival rate.

Prob. of 1 arrival in Δ $= \lambda\Delta$

$= \lambda \frac{T}{m}$

Probability of n arrivals in $m\Delta = T$ units of time.

$$= \binom{m}{n} \cdot \left(\lambda \frac{T}{m}\right)^{n} \left(1 - \frac{\lambda T}{m}\right)^{m-n}$$

$$\Delta = \lambda \frac{T}{m}$$

as $m \to \infty$

$$P_{N(T)}(n) = \begin{cases} \dfrac{(\lambda T)^{n} e^{-\lambda T}}{n!} & n = 0, 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

$$\binom{m}{n} (\lambda\Delta)^{n} (1 - \lambda\Delta)^{m-n}$$

for m large, $\Delta = \frac{T}{m}$, $\lambda\Delta = \frac{\lambda T}{m}$ is $\ll 1$.

Prob of n arrivals in T $= \binom{m}{n} (\lambda\Delta)^{n} (1 - \lambda\Delta)^{m-n}$

$$= \binom{m}{n} \left(\frac{\alpha}{m}\right)^{n} \left(1 - \frac{\alpha}{m}\right)^{m-n}. \qquad (\alpha = \lambda T)$$

Def: Poisson Process

A process $N(t)$ is a Poisson process if

(a) Number of arrivals in any interval $(t_0, t_1]$, i.e $N(t_1) - N(t_0)$ is a Poisson random variable, with expected value $\lambda(t_1 - t_0)$.

Independent increments property!!

(b) For any pair of nonoverlapping intervals $(t_0, t_1]$ and $(t_0', t_1']$, the number of arrivals in each interval, i.e $N(t_1) - N(t_0)$ and $N(t_1') - N(t_0')$ are independent random variables.

$M = N(t_1) - N(t_0)$ is a Poisson r.v.

$$P_M(m) = \begin{cases} \dfrac{(\lambda(t_1 - t_0))^m}{m!} e^{-\lambda(t_1 - t_0)}, & m = 0, 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Joint PMF of $(N(t_1), N(t_2) \cdots, N(t_k))$ for $t_1 < t_2 \cdots < t_k$ is

$$P_{N(t_1)\dots N(t_k)}(n_1, n_2, \dots, n_k) = \left(\frac{\alpha_1^{n_1} e^{-\alpha_1}}{n_1!}\right) \times \left(\frac{\alpha_2^{n_2 - n_1} \cdot e^{-\alpha_2}}{(n_2 - n_1)!}\right) \times \cdots \left(\frac{\alpha_k^{n_k - n_{k-1}} \cdot e^{-\alpha_k}}{(n_k - n_{k-1})!}\right)$$

$\alpha_1 = \lambda t_1$, $\alpha_i = \lambda(t_i - t_{i-1})$, for $i = 2, 3, \dots, k$

Recall, for a poisson r.v. with PMF

$$P(X=k) = \frac{\alpha^k e^{-\alpha}}{k!}, \quad k = 0, 1, 2 \ldots$$

$$E[X] = \alpha \; ; \quad Var(X) = \alpha$$

For the poisson process $N(t)$,

$$E[N(t) - N(s)] = \lambda(t-s).$$

$$Var[N(t) - N(s)] = \lambda(t-s)$$

$\boxed{\lambda \text{ is called the rate or the intensity of the process.}}$

Mean, and Auto-Correlation of Poisson Process

$$E[N(t)] = \lambda t$$

$$Var[N(t)] = \lambda t \Rightarrow E[(N(t))^2] = \lambda t + (\lambda t)^2$$

$$R_N(t,s) = E[N(t)N(s)] = E\left[\left(N(t) - N(s) + N(s)\right)N(s)\right]$$

for $0 \leq s < t$

$$= E[(N(t) - N(s))N(s)] + E[N^2(s)]$$

$$= E[(N(t) - N(s))(N(s) - N(0))] + \text{"}$$

$$= E[(N(t) - N(s))] \cdot E[(N(s) - N(0))] + \text{"}$$

$$= \lambda(t-s) \times \lambda(s-0) + \lambda s + (\lambda s)^2$$

$$= \lambda s + \lambda^2 st$$

$\boxed{\text{Distribution of Inter-arrival Times}}$
and $\boxed{\text{Time to the } n^{th} \text{ Arrival}}$

Let us denote

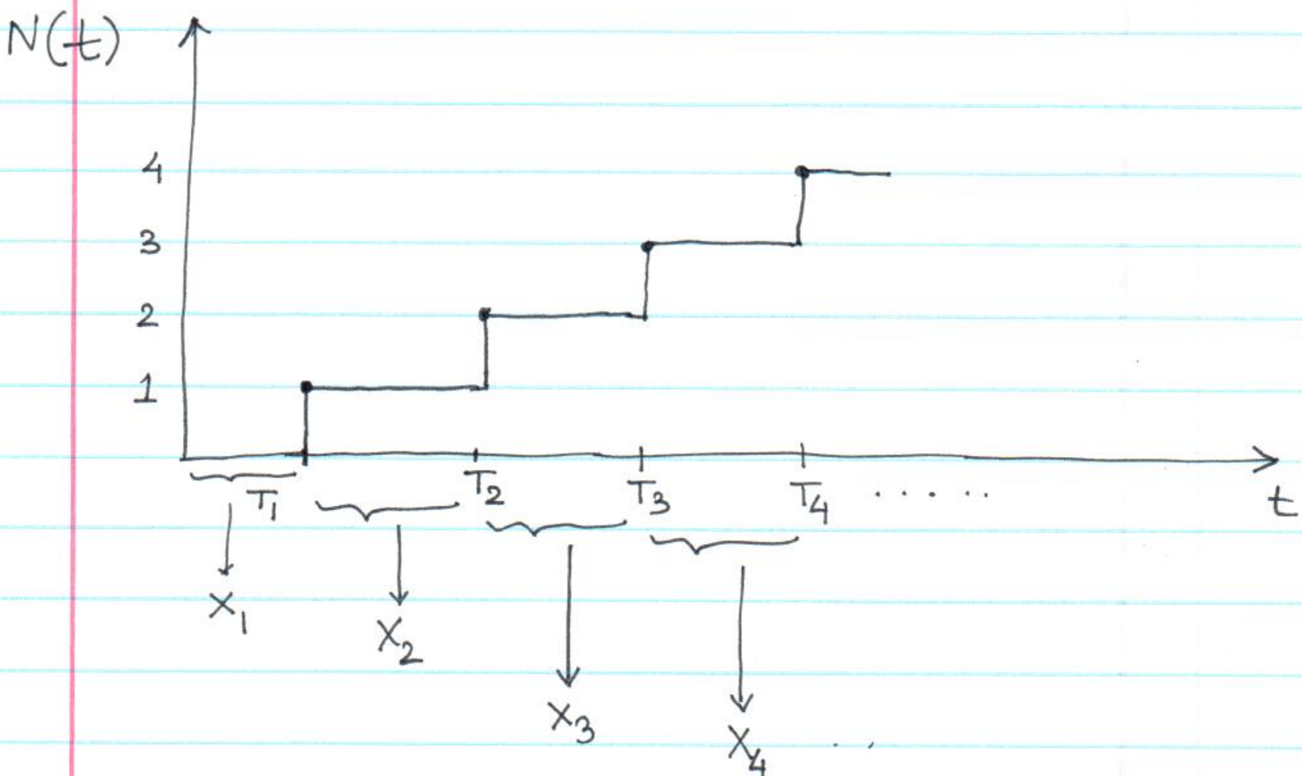
T_1 = time of 1^{st} arrival

T_2 = time of 2^{nd} arrival

$\vdots$

T_i = time of i^{th} arrival.

$$\left.\begin{aligned} X_1 &= (T_1 - 0) \\ X_2 &= (T_2 - T_1) \\ X_3 &= (T_3 - T_2) \\ &\ \vdots \\ X_i &= (T_i - T_{i-1}) \end{aligned}\right\} \rightarrow X_i\text{'s denote the inter-arrival times.}$$



$\boxed{\begin{array}{l}\underline{\text{Claim}}: \quad X_1, X_2, X_3, \ldots \text{ are i.i.d.} \\ \quad \text{and each is Exponential}(\lambda), \\ \text{i.e.} \quad f_{X_i}(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}\end{array}}$

INTER-ARRIVAL Times are iid, exponential(λ) r.v.'s

$$T_n = X_1 + X_2 + \ldots + X_n$$

* $\boxed{T_n \text{ is the time of the } n^{th} \text{ arrival.}}$

* T_n is the sum of n iid $\exp(\lambda)$ r.v.'s.

PDF of T_n

$$f_{T_n}(t) = \begin{cases} \lambda e^{-\lambda t} \dfrac{(\lambda t)^{n-1}}{(n-1)!} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\downarrow$

$\boxed{\text{Erlang}(\lambda, n) \text{ distribution}}$.

We will show that $T_n \sim \text{Erlang}(\lambda, n)$.

Proof:
To do this, we note that the following two events are equivalent:

$$\{T_n > t\} = \{N(t) < n\}$$

Why ??

$\Rightarrow$ If $T_n > t$, then the n^{th} arrival occurs after time t and hence at time t, $N(t) < n$, ie number of arrivals till time t must be $< n$.

Conversely

$\Rightarrow$ If $N(t) < n$, then the n^{th} arrival has not happened yet, and it must occur after time t, ie $T_n > t$.

$$\Rightarrow P(T_n > t) = P(N(t) < n)$$

$$= P(N(t) \leq n-1)$$

$$= \sum_{i=0}^{n-1} P(N(t) = i)$$

$$= \sum_{i=0}^{n-1} \frac{(\lambda t)^i e^{-\lambda t}}{i!}$$

$$\Rightarrow \quad P(T_n > t) = \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

$$\Rightarrow \quad \underbrace{P(T_n \le t)}_{\text{CDF of } T_n} = 1 - P(T_n > t)$$

$$= 1 - \sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

$$\Rightarrow \text{PDF of } T_n \Rightarrow f_{T_n}(t) = \frac{d}{dt} P(T_n \le t)$$

$$f_{T_n}(t) = -\sum_{i=0}^{n-1} \frac{d}{dt}\left(\frac{(\lambda t)^i e^{-\lambda t}}{i!}\right)$$

$$= -\sum_{i=0}^{n-1}\left\{\frac{i\lambda (\lambda t)^{i-1} e^{-\lambda t}}{i!} - \lambda \frac{(\lambda t)^i e^{-\lambda t}}{i!}\right\}$$

$$= \lambda e^{-\lambda t}\left\{\sum_{i=0}^{n-1} \frac{(\lambda t)^i}{i!} - \sum_{i=0}^{n-1} \frac{(\lambda t)^{i-1}}{(i-1)!}\right\}$$

$$= \lambda e^{-\lambda t} \cdot \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \Rightarrow \text{Erlang}(\lambda, n).$$

Example: Suppose that micrometeors strike a space shuttle according to a Poisson process. The expected time between two strikes is 30 minutes. Find the probability that during at least one hour out of five consecutive hours, three or more micrometeors strike the shuttle.

Solution: We are told that the expected interarrival time is 30 minutes = 0.5 hours. Since the interarrival times are $\exp(\lambda)$, their mean is $1/\lambda$

$$\Rightarrow \quad \frac{1}{\lambda} = 0.5 \text{ hours} \quad \Rightarrow \lambda = \frac{1}{0.5} = 2$$

or $\lambda = 2$ strikes/hour.

The number of strikes during the i^{th} hour is $N(i) - N(i-1)$. The probability that during at least 1 hour out of five consecutive hours, three or more micrometeors strike is :

$$P\Big((N(1)-N(0) \geq 3) \cup (N(2)-N(1) \geq 3) \cup \cdots \cup (N(5)-N(4) \geq 3) \Big)$$

$$= P\left(\bigcup_{i=1}^{5} \{ N(i) - N(i-1) \geq 3 \} \right).$$

$$P\left(\bigcup_{i=1}^{5}\{N(i)-N(i-1)\geq 3\}\right)$$

$$= 1 - P\left(\bigcap_{i=1}^{5}\{N(i)-N(i-1)<3\}\right)$$

$$= 1 - \prod_{i=1}^{5} P\left(N(i)-N(i-1)\leq 2\right)$$

due to independent increments property of Poisson process....

Since $N(i)-N(i-1) \sim \text{Poisson}\left(\lambda(i-(i-1))\right)$
$\sim \text{Poisson}(\lambda)$

$$\Rightarrow P\left(N(i)-N(i-1)\leq 2\right)$$

$$= P\left((N(i)-N(i-1))=0\right) + P\left((N(i)-N(i-1))=1\right) + P\left((N(i)-N(i-1))=2\right)$$

$$= e^{-\lambda}\frac{\lambda^0}{0!} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!}$$

$$= e^{-\lambda}\left(1+\lambda+\lambda^2/2\right) = 5e^{-2}$$

$$\Rightarrow P\left(\bigcup_{i=1}^{5}(N(i)-N(i-1)\geq 3)\right) = 1-\left(5e^{-2}\right)^5$$

$$\approx 0.86.$$

———— α ————.