Final Exam - ECE 503 Fall 2017

- Date: Wednesday, December 13, 2017.
- Time: 10:30 am -12:30 pm
- Maximum Credit: 100 points

1. [10 points] A jet aircraft's autopilot has conditional probability 1/3 of failure given that it employs a faulty microprocessor chip. The autopilot has conditional probability of failure of 1/10 given that it employs a non-faulty chip. According to the chip manufacturer, the probability of a customer receiving a faulty chip is 1/4.

   (a) Find the probability of failure of autopilot.

   (b) Given that an autopilot failure has occurred, find the conditional probability that a faulty chip was used.

Let $A_F = \{\text{autopilot fails}\}$; $C_F = \{\text{faulty chip was used}\}$.

We are given: $P(A_F|C_F) = \frac{1}{3}$; $P(C_F) = \frac{1}{4}$; $P(A_F|\overline{C_F}) = \frac{1}{10}$

(a) $$P(A_F) = P(C_F)\cdot P(A_F|C_F) + P(\overline{C_F})\cdot P(A_F|\overline{C_F})$$
$$= \frac{1}{4}\times\frac{1}{3} + \frac{3}{4}\times\frac{1}{10} = \boxed{\frac{19}{120}}$$

(b) $$P(C_F|A_F) = \frac{P(A_F, C_F)}{P(A_F)} = \frac{P(C_F)\cdot P(A_F|C_F)}{P(A_F)}$$
$$= \frac{(1/4)\times(1/3)}{19/120} = \frac{120}{12\times 19} = \boxed{\frac{10}{19}}$$

2. [20 points]

(a) Let X and Y be independent exponential random variables (both distributed as $\exp(\lambda)$). Find $E[\max(X, Y)]$.

(b) Let $f(x)$ denote a probability density function. We define a sequence of random variables X_n which have the following density (PDF):

$$f_n(x) = n f(nx).$$

Find the CDF of X_n and its limiting behavior as $n \to \infty$. Does this sequence of random variables converge in distribution ? If yes, then which random variable does this sequence converge to ?

(a) Let $Z = \max(X, Y)$ CDF of Z: $F_Z(z) = P(Z \leq z) = P(\max(X,Y) \leq z)$

$$F_Z(z) = P(\max(X,Y) \leq z) = P(X \leq z, Y \leq z) = P(X \leq z) \cdot P(Y \leq z) = F_X(z) \cdot F_Y(z)$$

$$\Rightarrow f_Z(z) = f_X(z) F_Y(z) + f_Y(z) F_X(z) = 2 f_X(z) F_X(z) = 2\lambda e^{-\lambda z} \cdot (1 - e^{-\lambda z}) \quad z \geq 0.$$

$$E[Z] = \int_0^\infty z f_Z(z)\, dz = 2 \int_0^\infty \lambda z e^{-\lambda z}(1 - e^{-\lambda z})\, dz = \frac{2}{\lambda} - \frac{1}{2\lambda} = \boxed{\frac{3}{2\lambda}}$$

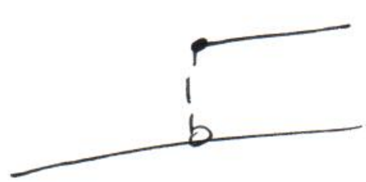
(b). $$F_n(x) = P(X_n \leq x) = \int_{-\infty}^{x} n f(nt)\, dt = \int_{-\infty}^{nx} f(y)\, dy = F(nx).$$

$\Rightarrow F_n(x) = F(nx)$.

for $x > 0$ as $n \to \infty$ $F_n(x) \to F(\infty) = 1$

$x < 0$ as $n \to \infty$ $F_n(x) \to F(-\infty) = 1$

(for $x = 0$, $F_n(0) = F(0)$).



$\Rightarrow X_n \to 0$ in distribution as $\underline{n \to \infty}$.

3. [15 points] A mixed random variable X has the following density:

$$f_X(x) = \frac{1}{3}e^{-x}u(x) + \frac{1}{2}\delta(x) + \frac{1}{6}\delta(x-1)$$

where $u(x)$ is the unit step function, and $\delta(x)$ is the Dirac delta function.

(a) Sketch the density function $f_X(x)$.

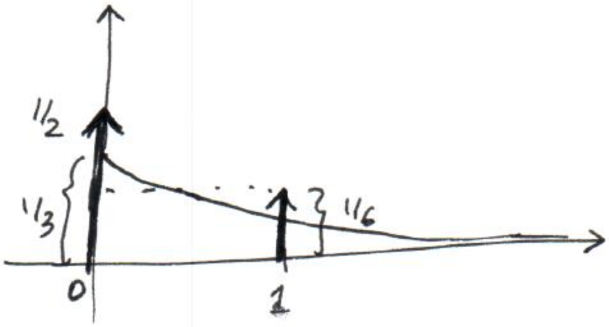
(b) Compute $P(X=0)$ and $P(X=1)$.

(c) Compute $P(0 < X < 1)$ and $P(X > 1)$.

(d) Compute $P(0 \le X \le 1)$ and $P(X \ge 1)$.

(e) Compute $E[X]$.

(a)



(b) $P(X=0) = \boxed{\frac{1}{2}}$; $P(X=1) = \boxed{\frac{1}{6}}$

(c) $P(0 < X < 1) = \int_{0^+}^{1^-} f_X(x)\,dx = \int_{0^+}^{1^-}\left(\frac{1}{3}e^{-x}u(x) + \frac{1}{3}\delta(x) + \frac{\delta(x-1)}{6}\right)dx$

$= \frac{1}{3}\int_0^1 e^{-x}dx = \boxed{\frac{1-1/e}{3}}$

$P(X>1) = \int_{1^+}^{\infty} f(t)\,dt = \frac{1}{3}\int_1^{\infty} e^{-x}dx = \boxed{\frac{1/e}{3}} = \frac{1}{3e}$

(d) $P(0 \le X \le 1) = P(X=0) + P(0<X<1) + P(X=1) = \frac{1}{2} + \frac{1-1/e}{3} + \frac{1}{6}$

$= \boxed{1 - \frac{1}{3e}}$

$P(X \geq 1) = P(X=1) + P(X>1) = \boxed{\frac{1+2/e}{6}}$

(e) $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{x}{3}e^{-x}dx + 0 \times P(X=0) + 1 \times P(X=1)$

$= \frac{1}{3} + 0 + \frac{1}{6} = \boxed{\frac{1}{2}}$

4. [20 points] Consider the random process

$$W(t) = X\cos(2\pi t) + Y\sin(2\pi t)$$

where X and Y are uncorrelated Gaussian random variables. Both X and Y have zero mean, and unit variance.

(a) Find the auto correlation function of the random process $W(t)$.

(b) Is $W(t)$ a wide sense stationary random process ?

(c) What is the average power of this random process ?

(d) Find the probability $P\left(W(1) + W(\frac{3}{4}) \le 0\right)$.

(a) $E[W(t)W(t+\tau)] = E\Big[\big(X\cos(2\pi t) + Y\sin(2\pi t)\big)\big(X\cos(2\pi(t+\tau)) + Y\sin(2\pi(t+\tau))\big)\Big]$

$$= E[X^2]\cdot\cos(2\pi t)\cos(2\pi(t+\tau)) + E[Y^2]\sin(2\pi t)\sin(2\pi(t+\tau))$$

$$+ \underbrace{E[XY]}_{\substack{\| \\ E[X]E[Y] \\ \| \quad \| \\ 0 \quad 0}} \times [\,- - -\,]$$

$$= \frac{E[X^2]}{2}\cdot\left[\cos(2\pi\tau) + \cos(2\pi(2t+\tau))\right]$$

$$+$$

$$\frac{E[Y^2]}{2}\left[\cos(2\pi\tau) - \cos(2\pi(2t+\tau))\right]$$

$$E[X^2] = E[Y^2] = 1$$

$$= \frac{1}{2}\times\cos(2\pi\tau)\times 2 = \boxed{\cos(2\pi\tau)}$$

(b) $\boxed{\text{Yes, } W(t) \text{ is WSS}}$ $\left\{ E[W(t)] = 0,\ R_X(\tau) \right\}$ only on time diff.

(c) Avg. Power $= R_W(0) = \cos(0) = \boxed{1}$

(d) $P\left(W(1) + W(3/4) \le 0\right)$

$$W(1) + W(3/4) = X\underbrace{\cos(2\pi)}_{1} + Y\underbrace{\sin(2\pi)}_{0} + X\underbrace{\cos(\tfrac{3\pi}{2})}_{0} + Y\underbrace{\sin(\tfrac{3\pi}{2})}_{-1}$$

$$= X - Y$$

$$= P(X - Y \le 0) = P(Z \le 0) = P\left(\frac{Z-0}{\sqrt{2}} \le \frac{0-0}{\sqrt{2}}\right) = \boxed{\frac{1}{2}}$$

$Z = X - Y \sim N(0, 2)$

5. [20 points] Suppose X_n is a random sequence satisfying

$$X_n = cX_{n-1} + Z_{n-1}$$

where $Z_1, Z_2, \ldots$ is an i.i.d. sequence with $E[Z_n] = 0$ and $\text{Var}[Z_n] = \sigma^2$, and c is a constant with $|c| < 1$. We also know that $E[X_0] = 0$, and $\text{Var}[X_n] = \sigma^2/(1-c^2)$. We make noisy observations as follows:

$$Y_{n-1} = dX_{n-1} + W_{n-1}$$

where $W_1, W_2, \ldots$ is a i.i.d. sequence of measurement noises, with $E[W_n] = 0$ and $\text{Var}[W_n] = \eta^2$. We also know that W_n's are independent of X_n and Z_n.

(a) Find the optimal linear MMSE estimator $\hat{X}_n$ of X_n using the noisy observation Y_{n-1}.

(b) Find the resulting mean square estimation error $E[(X_n - \hat{X}_n)^2]$.

(a) $$\hat{X}_n = \underbrace{\hat{X}_n(Y_{n-1})}_{\text{optimum linear MMSE-Estimator}} = \rho_{(X_n, Y_{n-1})} \left(\frac{Var(X_n)}{Var(Y_{n-1})}\right)^{1/2} \left(Y_{n-1} - E[Y_{n-1}]\right) + E[X_n]$$

$$\left.\begin{array}{l} E[Y_{n-1}] = 0 \\ E[X_n] = 0 \end{array}\right\} \Rightarrow \hat{X}_n = \left[\rho_{X_n, Y_{n-1}} \left(\frac{Var(X_n)}{Var(Y_{n-1})}\right)^{1/2} \right] \cdot Y_{n-1}.$$

$$Var(Y_{n-1}) = d^2 \cdot Var(X_{n-1}) + Var(W_{n-1})$$
$$= \frac{d^2\sigma^2}{1-c^2} + \eta^2$$

$$\rho_{X_n, Y_{n-1}} = \frac{E[X_n Y_{n-1}]}{\sqrt{Var(X_n)\, Var(Y_{n-1})}} \quad \& \quad E[X_n Y_{n-1}] = cd\, Var[X_{n-1}]$$

$$X_n = cX_{n-1} + Z_{n-1}$$
$$= c(cX_{n-2} + Z_{n-2}) + Z_{n-1}$$
$$= c^2 X_{n-2} + cZ_{n-2} + Z_{n-1}$$
$$\vdots$$
$$X_n = c^n X_0 + \sum_{j=1}^{n} c^{j-1} Z_{n-j}$$

$$Var(X_n) = c^{2n} Var(X_0) + \sum_{j=1}^{n} (c^2)^{j-1} Var[Z_{n-j}]$$
$$= \frac{c^{2n}\cdot\sigma^2}{1-c^2} + \sigma^2 \cdot [1 + c^2 \cdots + (c^2)^{n-1}]$$
$$= \frac{c^{2n}\cdot\sigma^2}{1-c^2} + \sigma^2\frac{(1-c^{2n})}{1-c^2} = \frac{\sigma^2}{1-c^2}$$

$$\Rightarrow \hat{X}_n = \left(\frac{c}{d}\right)\left(\frac{1}{1+\beta^2(1-c^2)}\right) Y_{n-1}$$

where
$$\beta^2 = \frac{\eta^2}{d^2\sigma^2}$$

6. [15 points] Consider the following Markov chain with three states $S = \{A, B, C\}$, which has the following one-step transition matrix:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

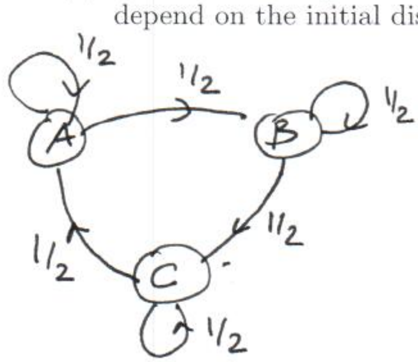
(a) Draw the state transition diagram for this Markov chain.

(b) If you are given that $P(X_1 = A) = 1/2$, $P(X_1 = C) = 1/6$, find the probability $P(X_1 = B, X_2 = C, X_3 = A)$.

(c) Is this MC irreducible and aperiodic ?

(d) Find the limiting distribution of this Markov chain. Is this distribution unique ? Does it depend on the initial distribution ?

(a)



(b) $P(X_1 = B, X_2 = C, X_3 = A)$

$$= \underbrace{P(X_1=B)}_{\Downarrow \; (1-\frac{1}{2}-\frac{1}{6})} \cdot \underbrace{P(X_2=C \mid X_1=B)}_{\Downarrow \; \frac{1}{2}} \cdot \underbrace{P(X_3=A \mid X_2=C)}_{\Downarrow \; 1/2}$$

$$= \frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{12}$$

(c) Yes, MC is irreducible & aperiodic

(d) Unique limiting distrib $\Rightarrow$ $\pi = \pi P$

$\Downarrow$

solving gives

$\pi = [1/3 \quad 1/3 \quad 1/3]$