Contents

1 Week 1: 8/26- 8/30

1. The following identities are true:

$$
\begin{aligned}
A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\
A \cap (B \cap C) &= (A \cap B) \cap C, \\
A \backslash (B \cap C) &= (A \backslash B) \cup (A \backslash C).
\end{aligned}
$$

Hint. Let us show

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{1.1}$$

Indeed, $x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $(x \in B$ or $x \in C) \Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C) \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$.

Answer. The equality $A \cap (B \cap C) = (A \cap B) \cap C$ is obvious. In order to prove

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

we show that the complements of the sets on both sides are equal. By DeMorgan's law and (1.1),

$$
\begin{aligned}
(A \cup (B \cap C))^c &= A^c \cap (B^c \cup C^c) \\
&= (A^c \cap B^c) \cup (A^c \cap C^c),
\end{aligned}
$$

and

$$
\begin{aligned}
((A \cup B) \cap (A \cup C))^c &= (A \cup B)^c \cup (A \cup C)^c \\
&= (A^c \cap B^c) \cup (A^c \cap C^c).
\end{aligned}
$$

Now, by de Morgan's law and (1.1) again,

$$
\begin{aligned}
A \backslash (B \cap C) &= A \cap (B \cap C)^c = A \cap (B^c \cup C^c) \\
&= (A \cap B^c) \cup (A \cap C^c) = (A \backslash B) \cup (A \backslash C).
\end{aligned}
$$

2. Let $\{A_i, i \in I\}$ be a collection of sets. Prove De Morgan's Laws:

$$(\cup_i A_i)^c = \cap_i A_i^c, \ (\cap_i A_i)^c = \cup_i A_i^c.$$

3. a) Let $\mathcal{F}$ be a σ-field, $A, B \in \mathcal{F}$. Show that $A \backslash B$ and $A \triangle B \in \mathcal{F}$.

b) Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be σ-fields of subsets of Ω. Show that $\mathcal{F}_1 \cap \mathcal{F}_2$, the collection of subsets of Ω that belong to both $\mathcal{F}_1$ and $\mathcal{F}_2$, is σ-field.

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4. Let $\mathcal{A}$ be a collection of subsets of Ω, and let $\mathcal{F}_i, i \in I$, be all σ-fields that contain $\mathcal{A}$. Show that $\mathcal{F} = \cap_i \mathcal{F}_i$ is a σ-field.

Comment. Note that $\mathcal{P}(\Omega)$, the σ-field of all subsets of Ω, is among $\mathcal{F}_i$. The collection $\mathcal{F} = \cap_i \mathcal{F}_i$ is called the smallest σ-field containing $\mathcal{A}$.

5. Describe the sample spaces for the following experiments:

(a) Two balls were drawn without replacement from an urn which originally contained two red and two black balls.

Answer. Mark balls in the urn: $B_1, B_2, R_1.R_2$. If drawn one by one, and going with ordered pairs, then Ω consists of $4 \cdot 3 = 12$ ordered pairs:

$$\{R_1 R_2, R_2 R_1, R_1 B_1, R_1 B_2, R_2 B_1, R_2 B_2, B_1 B_2, B_2 B_1, B_1 R_1, B_1 R_2, B_2 R_1, B_2 R_2\}.$$

If drawn loosely as group of two, then Ω consists of $\binom{4}{2} = 6$ unordered pairs:

$$\{\{R_1, R_2\}, \{R_1, B_1\}, \{R_2, B_1\}, \{R_1, B_2\}, \{R_2, B_2\}, \{B_1, B_2\}\}.$$

(b) A coin is tossed three times.

Answer. $\Omega = \{HHH, HHT, HTT, HTH, TTT, TTH, THT, THH\}$ consists of $2^3 = 8$ different outcomes.

6. A fair die is thrown twice. What is the probability that:

(a) a six turns up exactly once?

Answer. The event A consists of all $(6, i)$ and $(j, 6)$ with $1 \leq i, j \leq 5$. Hence $\mathbf{P}(A) = 10/36 = 5/18$.

(b) both numbers are odd?

Answer. The event A consists of all (i, j) with $i, j \in \{1, 3, 5\}$; $\#A = 3^2 = 9$ and $\mathbf{P}(A) = 9/36 = 1/4$.

(c) the sum of the scores is 4?

Answer. $A = \{(1, 3), (2, 2), (3, 1)\}$, and $\mathbf{P}(A) = 3/36 = 1/12$.

(d) the sum of the scores is divisible by 3?

Answer. $A = \{(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6.6)\}$, and

$$\mathbf{P}(A) = 12/36 = 1/3.$$

7. A fair coin is thrown repeatedly. What is the probability that on the nth throw:

(a) a head appears for the first time?

Answer. The sample space Ω consists of all "words" of length n made by H and T: $\#\Omega = 2^n$. The event $A =$" H appears 1st on the nth throw" is a single word with $n - 1$ T before H at the end:

$$\mathbf{P}(A) = 1/2^n = 2^{-n}.$$

(b) the number of heads and tails to date are equal?

Answer. If n is odd the probability is zero. If n is even, then there are $\binom{n}{n/2}$ different "words" with $n/2$ H:

$$\mathbf{P}(A) = \binom{n}{n/2} 2^{-n}.$$

(c) exactly two heads have appeared altogether to date?

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Answer. There are $\binom{n}{2}$ different words with exactly two H :

$$\mathbf{P}(A) = \binom{n}{2} 2^{-n}.$$

(d) at least two heads have appeared to date?

Answer. Let B="at most one H" consists of one word with all T in it and n words with a single H in them: $\mathbf{P}(B) = (n+1) 2^{-n}$ and $\mathbf{P}(A) = 1 - (n+1) 2^{-n}$.

8. Show that the probability that exactly one of the events A and B occurs is

$$\mathbf{P}(A) + \mathbf{P}(B) - 2\mathbf{P}(A \cap B).$$

(note: it is the event $A \triangle B = (A \backslash B) \cup (B \backslash A)$)

Answer. Since $A \triangle B$ is the union of two disjoint sets,

$$A \triangle B = (A \backslash B) \cup (B \backslash A) = (A \backslash (A \cap B)) \cup (B \backslash (A \cap B)),$$

and $A \cap B \subseteq A$, $A \cap B \subseteq B$, we have

$$
\begin{aligned}
\mathbf{P}(A \triangle B) &= \mathbf{P}(A \backslash (A \cap B)) + \mathbf{P}(B \backslash (A \cap B)) \\
&= \mathbf{P}(A) - \mathbf{P}(A \cap B) + \mathbf{P}(B) - \mathbf{P}(A \cap B).
\end{aligned}
$$

9. a) A fair coin is tossed n times. What is the probability of H in the last toss?

Answer. The sample space Ω consists of all "words" of length n made by H and T: $\#\Omega = 2^n$. The event A="H in the last toss" consists of all words of length n formed by H and T with H at the end; $\#A = 2^{n-1}$ and

$$\mathbf{P}(A) = \frac{2^{n-1}}{2^n} = \frac{1}{2}.$$

Similarly, probability of H in any toss is 1/2.

b) An urn contains 9 whites and one red ball. All ten balls are randomly drawn out without replacement one by one. What is the probability that:

(i) the red ball is taken out first?

Answer. The event A="red ball is taken out first" consists of all orderings of 10 balls with red ball first: $\#A = (n-1)!$. Hence

$$\mathbf{P}(A) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

(ii) the red ball is taken out last?

Answer. The event A="red ball is taken out last" consists of all orderings of 10 balls with red ball last: $\#A = (n-1)!$. Hence

$$\mathbf{P}(A) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

(ii) the red ball is taken in the kth draw ($1 \leq k \leq 10$)?

Answer. The event A="red ball is taken out last" consists of all orderings of 10 balls with red ball in the kth position: $\#A = (n-1)!$. Hence

$$\mathbf{P}(A) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

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10. Six cups and saucers come in pairs: there are two cups and saucers which are red, two white, and two with stars on. If the cups are placed randomly onto the saucers (one each), find the probability that no cup is upon a saucer of the same pattern.

Hint. Put the saucers in a row. For instance, $RRWWSS$. The sample space Ω is the space of all cup $(R_1, R_2, W_1, W_2, S_1, S_2)$ arrangements along the row of saucers $RRWWSS$.

Answer. Put the saucers in a row. For instance, $RRWWSS$. The sample space Ω is the space of all cup $(R_1, R_2, W_1, W_2, S_1, S_2)$ arrangements along the row of saucers $RRWWSS$: $\#\Omega = 6!$ Let $A=$"no cup is upon a saucer of the same pattern". Then $A = A_1 \cup A_2$, where A_1 are all arrangements where one pattern cups go to the same different pattern of saucers, and A_2 are all arrangements where one pattern cups go to the different patterns of saucers.

Note A_1 consists of $W_1 W_2 S_1 S_2 R_1 R_2$ and $S_1 S_2 R_1 R_2 W_1 W_2$ and their arrangements with switching R_1 and R_2, W_1 and W_2, S_1 and S_2 : $\#A_1 = 2 \cdot 2! \cdot 2!2! = 16$.

Counting A_2. For R_1 we have 4 choices (4 positions in $WWSS$ part), for R_2 we have remaining 2 choices: total number of R arrangements is $4 \cdot 2 = 8$. After both R are placed we have 4 ways to arrange both W. After R and W are placed there are only 2 ways to arrange remaining S. Hence $\#A_2 = 8 \cdot 4 \cdot 2 = 64$, and $\#A = \#A_1 + \#A_2 = 16 + 64 = 80$. Thus $\mathbf{P}(A) = 80/6! = 1/9$.

11. You choose k of the first n positive integers, and a lottery chooses a random subset L of the same size (k numbers in it). What is the probability that:

(a) L includes no consecutive integers? *Hint about counting nonconsecutive integers.* Let $n = 7, k = 3$. (i) Put $7 - 3 = 4$ white balls in a row with spaces between them, in the beginning and at the end (there are 5 spaces). (ii) Choose 3 spaces and put 3 black balls there. Number all balls from the left to the right. For instance, if the 2nd, fourth and fifth space were chosen, then L consists of the numbers $2, 5, 7$. Realize that number of ways to have non consecutive integers in L equals to the number of ways to choose 3 spaces among 5 available.

Answer. As a sample space Ω take all possible groups of k numbers from $\{1, 2, \ldots, n\}$: $\#\Omega = \binom{n}{k}$. Let $A=$"L includes no consecutive integers". In order to count outcomes in A, imagine we put $n-k$ white balls in a row with spaces between them, also one space before the first ball and one space after the last ball: there are $n - k + 1$ spaces. Choose k spaces and place black balls there: there are $\binom{n-k+1}{k}$ different ways to do it. Write the numbers $1, 2, \ldots, n$ from the left to the right on the balls in the row. The numbers written on the black balls are numbers in L (no consecutive integers). Thus $\#A = \binom{n-k+1}{k}$, and

$$\mathbf{P}(A) = \frac{\binom{n-k+1}{k}}{\binom{n}{k}}.$$

Comment. We were choosing a group of k numbers. The counting of the outcomes in A goes with a row of the balls because we are choosing from the set of numbers $\{1, 2, 3, \ldots, n\}$ that has an order (one number is bigger than the other and A cannot contain consecutive numbers).

(b) L includes exactly one pair of consecutive integers? *Hint.* Like (a) but think about one pair of consecutive integers as one entity.

Answer. The sample space Ω is the same as in (a): $\#\Omega = \binom{n}{k}$. Let $A =$"L includes exactly one pair of consecutive integers". In order to count the outcomes in A, imagine we put $n - k$ white balls in a row with spaces between them, also one space before the first ball and one space after the last ball: there are $n - k + 1$ spaces. Take $k - 2$ black balls and a narrow box with two black balls. First put the box into one of the spaces: there are $n - k + 1$ ways to do it. Then choose $k - 2$ spaces from the remaining $n - k$ spaces and put the black balls there: there are $\binom{n-k}{k-2}$ different ways. Write the

numbers from 1 to n on the balls including those in the box. The k numbers on the black balls are the numbers in L (there is exactly one pair of consecutive numbers). Thus

$$\#A = (n - k + 1)\binom{n-k}{k-2} = (k-1)\binom{n-k+1}{k-1}$$

and

$$\mathbf{P}(A) = \frac{(k-1)\binom{n-k+1}{k-1}}{\binom{n}{k}}.$$

Question. If you play lottery, (a) or (b) strategy would be better?

(c) the numbers in L are drawn in increasing order? Hint. Any ordering of k numbers is equally likely.

Answer. Since any ordering of k numbers is equally likely, the probability is $1/k!$.

(d) your choice of numbers is the same as L?

Answer. The probability is, obviously,

$$\frac{1}{\binom{n}{k}}.$$

(e) there are exactly l of your numbers matching members of L?

Answer. In your collection of k numbers there are $\binom{k}{l}$ different groups of size l that could match with l numbers in L; you particular group of l numbers must be in L but the remaining $k-l$ numbers in L do not match; you remove from $\{1, 2, \ldots, n\}$ the whole your group of k numbers and choose $k-l$ numbers from the remaining $n-k$ numbers in $\binom{n-k}{k-l}$ different ways. The probability is

$$\frac{\binom{k}{l}\binom{n-k}{k-l}}{\binom{n}{k}}.$$

12. 10% of the surface of a sphere is colored blue, the rest is red. Show that it is possible to inscribe a cube in S with all its vertices red. Hint: select a random inscribed cube and think what is the probability that a kth vertex is blue. Then estimate from above the probability that at least one vertex is blue.

Answer. The inscribed cube has 8 vertices. Let B_k be the event that kth vertex is blue, $1 \le k \le 8$. Then

$$\mathbf{P}(\text{at least one vertex is blue }) = \mathbf{P}\left(\cup_{k=1}^{8} B_k\right) \le \sum_{k=1}^{8}\mathbf{P}(B_k) \le 8 \cdot 0.1 = 0.8.$$

Hence $\mathbf{P}$ (all vertices red) > 0.2.

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2 Week 2: 9/4- 9/6

1. A rare disease affects one person in 10^3. A test for the disease shows positive with probability 0.99 when applied to an ill person, and with probability 0.01 when applied to a healthy person. What is the probability that you have the disease given that the test shows negative?

2. English and American spellings are *rigour* and *rigor*, respectively. At a certain hotel, 40% of guests are from England and the rest are from America. A guest at the hotel writes the word (as either *rigour* or *rigor*), and a randomly selected letter from that word turns out to be a vowel. Compute the probability that the guest is from England.

3. In a certain community, 36% of all the families have a dog and 30% have a cat. Of those families with a dog, 22% also have a cat. Compute the probability that a randomly selected family

(a) has both a dog and a cat;

(b) has a dog given that it has a cat.

Hint. Interpret 22% as conditional probability.

4. (Galton's paradox) You flip three fair coins. At least two are alike, and it is an even chance that the third is a head or a tail. Therefore $\mathbf{P}$(all alike)=$\frac{1}{2}$. Do you agree? (all alike means all heads or all tails).

5. The event A is said to be repelled by the event B if $\mathbf{P}(A|B) < \mathbf{P}(A)$, and to be attracted by B if $\mathbf{P}(A|B) > \mathbf{P}(A)$. Show that if B attracts A, then A attracts B, and B^c repels A. If A attracts B, and B attracts C, does A attract C?

6. Calculate the probability that a hand of 13 cards dealt from a normal shuffled pack of 52 contains exactly two kings and one ace. What is the probability that it contains exactly one ace given that it contains exactly two kings?

7. A woman has n keys, of which two will open her door. (a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try? (b) What if she does not discard previously tried keys? What is the probability of no right key in k tries ($k \geq 1$)? If $n = 9$, how many tries are needed to be 90% sure that the door is opened?

8. Three prisoners are informed by their jailer that one of them has been chosen at random to be executed and the other two are to be freed. Prisoner A asks the jailer to tell him privately which of his fellow prisoners will be set free, claiming that there would be no harm in divulging this information because he already knows that at least one of the two will go free. The jailer refuses to answer the question, pointing out that if A knew which of his fellow prisoners were to be set free, then his own probability of being executed would rise from 1/3 to 1/2 because he would then be one of two prisoners. What do you think of the jailer's reasoning?