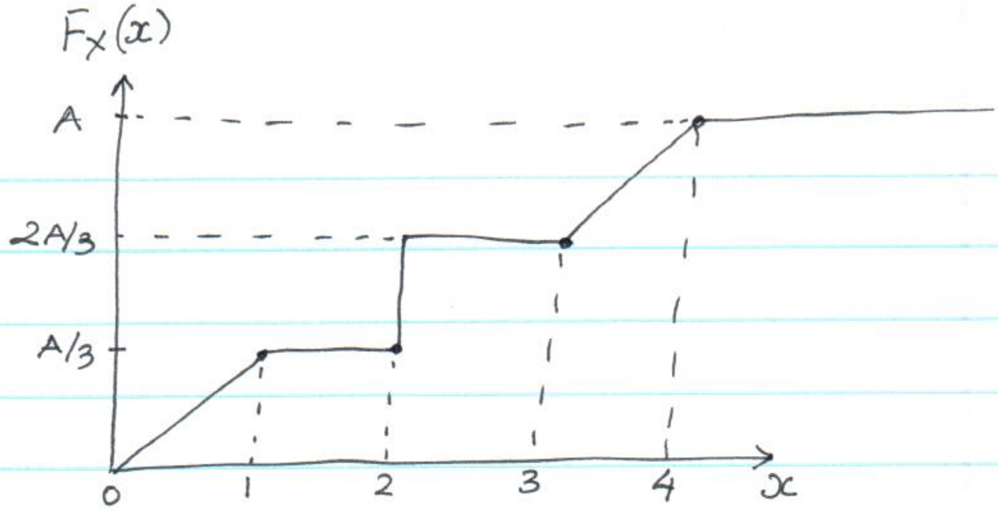
Solutions to Final Exam

ECE 503
Fall 2015

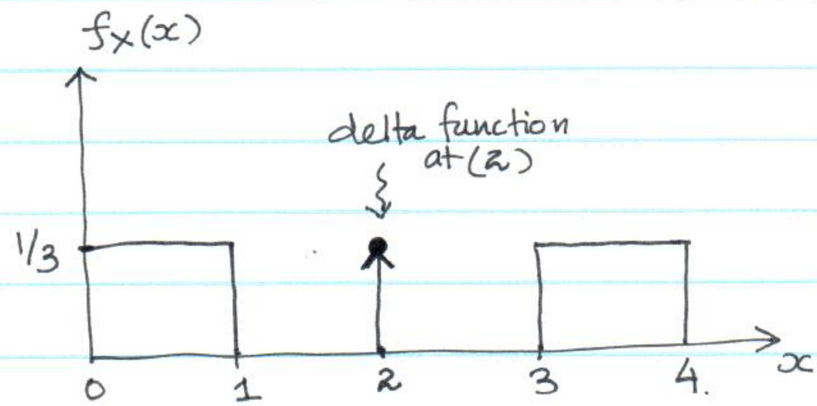
1. (a) True
   (b) False
   (c) True
   (d) True
   (e) False
   (f) True.

(2)



(a) Since $\lim_{x \to \infty} F_X(x) = 1 \Rightarrow A = 1$

(b) $f_X(x) = \frac{d}{dx} F_X(x) \Rightarrow$ PDF



(c) $E[X] = \frac{1}{3}\int_0^1 x\,dx + 2 \times \left(\frac{1}{3}\right) + \frac{1}{3}\int_3^4 x\,dx$

$= \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} + \frac{1}{6}(16-9)$

$= \frac{1}{6} + \frac{4}{6} + \frac{7}{6} = \frac{12}{6} = \boxed{2}$

$E[X^2] = \frac{1}{3}\int_0^1 x^2\,dx + \frac{4}{3} + \frac{1}{3}\int_3^4 x^2\,dx$

$= \frac{1}{9} + \frac{12}{9} + \frac{1}{9}(64-27) = \frac{1+12+37}{9} = \frac{50}{9}$

$$\Rightarrow \mathrm{Var}(X) = E[X^2] - (E(X))^2$$

$$= \frac{50}{9} - 4 = \frac{50-36}{9} = \frac{14}{9}$$

(d) $P(X < 2) = \frac{1}{3}$

(3) $X_1, X_2 \ldots\ldots \quad X_n \sim \text{Poisson}(n\lambda)$

Recall, for a poisson r.v.

$$Y_n = \frac{X_n}{n} \qquad E[X_n] = n\lambda$$

$$Var[X_n] = n\lambda$$

To show $\underbrace{\frac{X_n}{n}}_{Y_n} \xrightarrow{m.s.} \lambda$, i.e. $E\left[|Y_n - \lambda|^2\right] \xrightarrow[n\to\infty]{} 0$

$\Rightarrow$ We need to show $E\left[\left|\frac{X_n}{n} - \lambda\right|^2\right] \to 0$ as $n \to \infty$

$$E\left[\left|\frac{X_n}{n} - \lambda\right|^2\right] = \frac{1}{n^2} E\left[|X_n - n\lambda|^2\right] = \frac{Var(X_n)}{n^2}$$

$$= \frac{1}{n^2} \underbrace{\sum_{k=0}^{\infty} (k - n\lambda)^2 \frac{(n\lambda)^k e^{-n\lambda}}{k!}}_{\Rightarrow Var(X_n)}$$

$$= \frac{1}{n^2} Var(X_n) = \frac{n\lambda}{n^2} = \frac{\lambda}{n} \to 0$$

as $n \to \infty$

$$\Rightarrow \boxed{Y_n = \frac{X_n}{n} \xrightarrow{m.s.} \lambda}$$

(4) (a) $\hat{X}_n = \alpha_1 X_{n-1} + \alpha_2 X_{n-2} + b$

$$\underbrace{E[\hat{X}_n]}_{0} = \alpha_1 \underbrace{E[X_{n-1}]}_{0} + \alpha_2 \underbrace{E[X_{n-2}]}_{0} + b$$

$$\Rightarrow \boxed{b = 0}$$

$$\Rightarrow \hat{X}_n = \alpha_1 X_{n-1} + \alpha_2 X_{n-2}$$

Orthogonality principle:

$$E[(X_n - \hat{X}_n) X_{n-1}] = 0$$

$$\Rightarrow E[X_n X_{n-1}] = \alpha_1 E[X_{n-1}^2] + \alpha_2 E[X_{n-1} X_{n-2}]$$

$$\Rightarrow \left(1 - \frac{1}{3}\right) = \alpha_1 \times 1 + \alpha_2 \times \left(1 - \frac{1}{3}\right)$$

$$\Rightarrow \frac{2}{3} = \alpha_1 + \frac{2}{3}\alpha_2$$

$$\Rightarrow \boxed{2 = 3\alpha_1 + 2\alpha_2}$$

$$E[(X_n - \hat{X}_n) X_{n-2}] = 0.$$

$$\Rightarrow E[X_n X_{n-2}] = \alpha_1 E[X_{n-1} X_{n-2}] + \alpha_2 E[X_{n-2}^2]$$

$$\Rightarrow \left(1 - \frac{2}{3}\right) = \alpha_1 \left(1 - \frac{1}{3}\right) + \alpha_2$$

$$\Rightarrow \frac{1}{3} = \frac{2\alpha_1}{3} + \alpha_2$$

$$\Rightarrow \boxed{1 = 2\alpha_1 + 3\alpha_2}$$

$$\Rightarrow (3\alpha_1 + 2\alpha_2 = 2) \times 2 \Rightarrow 6\alpha_1 + 4\alpha_2 = 4$$

$$(2\alpha_1 + 3\alpha_2 = 1) \times 3 \Rightarrow 6\alpha_1 + 9\alpha_2 = 3$$

$$5\alpha_2 = -1$$

$$\boxed{\alpha_2 = -1/5}$$

$$2\alpha_1 = 1 - 3\alpha_2 = 1 + \frac{3}{5} = \frac{8}{5}$$

$$\Rightarrow \alpha_1 = \frac{4}{5}.$$

$$\Rightarrow \hat{x}_n = \frac{4}{5} x_{n-1} - \frac{x_{n-2}}{5}.$$

MSE Error

$$= E\left[(x_n - \hat{x}_n)^2\right]$$

$$= E\left[(x_n - \hat{x}_n)(x_n - \hat{x}_n)\right]$$

$$= E\left[(x_n - \hat{x}_n)x_n\right] - \underbrace{E\left[(x_n - \hat{x}_n)\hat{x}_n\right]}_{\equiv 0}.$$

$$= E[x_n^2] - E\left[x_n(\alpha_1 x_{n-1} + \alpha_2 x_{n-2})\right]$$

$$= E[x_n^2] - \alpha_1 E[x_n x_{n-1}] - \alpha_2 E[x_n x_{n-2}]$$

$$= 1 - \alpha_1\left(1 - \frac{1}{3}\right) - \alpha_2\left(1 - \frac{2}{3}\right)$$

$$= 1 - \frac{2\alpha_1}{3} - \frac{\alpha_2}{3}$$

$$= 1 - \frac{2}{3} \times \frac{4}{5} + \frac{1}{3} \times \frac{1}{5}$$

MSE

$\Downarrow$

$$= \frac{15 - 8 + 1}{15} = \frac{16 - 8}{15} = \boxed{\frac{8}{15}}$$

Answer

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(a)
$$\mu_Y(t) = E[Y(t)]$$
$$= E\left[\int_{t-2}^{t} X(u)\,du\right]$$
$$= \int_{t-2}^{t} \underbrace{E[X(u)]}_{=0}\,du$$
$$= 0$$

(b)
$$R_{XY}(t_1, t_2) = E[X(t_1)\,Y(t_2)]$$
$$= E\left[X(t_1)\int_{u=t_2-2}^{t_2} X(u)\,du\right]$$
$$= \int_{u=t_2-2}^{t_2} \underbrace{E[X(t_1)\,X(u)]}_{\Downarrow\ \delta(t_1-u)}\,du = \int_{u=t_2-2}^{t_2} \delta(t_1-u)\,du$$
$$= \begin{cases} 1 & \text{if } t_2-2 < t_1 < t_2 \\ 0 & \text{otherwise} \end{cases}$$

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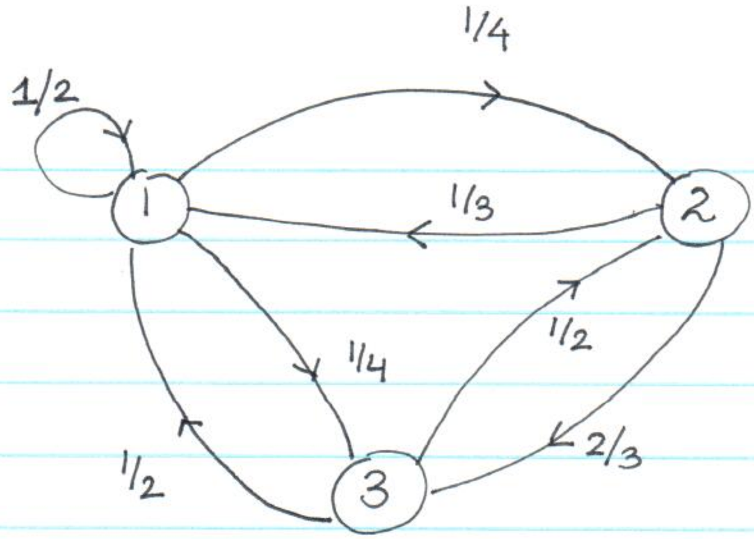

(a) Yes, the chain is irreducible, since there is a path from every state to every other state.

(b) Yes, the chain is aperiodic (self-loop at state 1).

(c) To find the stationary distribution, $\pi = \pi P$

$$\Rightarrow \quad \pi_1 = \frac{\pi_1}{2} + \frac{\pi_2}{3} + \frac{\pi_3}{2} \quad — ①$$

$$\pi_2 = \frac{\pi_1}{4} + \frac{\pi_3}{2} \quad — ②.$$

$$\pi_3 = \frac{\pi_1}{4} + \frac{2\pi_2}{3} \quad \text{and} \quad \pi_1 + \pi_2 + \pi_3 = 1$$

$$① - ② \Rightarrow \quad \pi_1 - \pi_2 = \frac{\pi_1}{2} + \frac{\pi_2}{3} - \frac{\pi_1}{4}$$

$$\frac{\pi_1}{2} = \pi_2 \left\{ 1 + \frac{1}{3} - \frac{1}{4} \right\} = \pi_2 \left(\frac{12+4-3}{12} \right)$$

$$\frac{\pi_1}{\cancel{2}} = \frac{\pi_2 \times 13}{\cancel{12}\ 6} \Rightarrow \boxed{\pi_2 = \frac{6}{13}\pi_1}$$

$$\Rightarrow \quad \pi_1\left(\frac{1}{2}+\frac{1}{4}\right) = \pi_2\left(1+\frac{1}{3}\right)$$

$$\Rightarrow \quad \frac{3}{4}\pi_1 = \frac{4}{3}\pi_2 \quad \Rightarrow \quad \boxed{\pi_2 = \frac{9}{16}\pi_1}$$

$$\pi_3 = \frac{\pi_1}{4} + \frac{2}{3}\pi_2 = \frac{\pi_1}{4} + \frac{\cancel{2}}{\cancel{3}} \times \frac{\cancel{9}^{\,3}}{\cancel{16}_{\,8}}\pi_1$$

$$\pi_3 = \left(\frac{1}{4} + \frac{3}{8}\right)\pi_1 = \left(\frac{5}{8}\right)\pi_1$$

also,

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\Rightarrow \pi_1\left(1 + \frac{9}{16} + \frac{10}{16}\right) = 1$$

$$\pi_1\left(\frac{16+19}{16}\right) = 1 \quad \Rightarrow \quad \boxed{\pi_1 = \frac{16}{35}} \approx 0.457$$

$$\boxed{\pi_2 = \frac{9}{35}} \approx 0.257$$

$$\boxed{\pi_3 = \frac{10}{35}} \approx 0.286.$$

(d) Yes, π is unique & is the limiting distribution since the M.C. is irreducible and aperiodic.

(7)

For $0 \le x \le t$, we write

$$P(X_1 \le x \mid N(t)=1) = \frac{P(X_1 \le x,\ N(t)=1)}{P(N(t)=1)}$$

we know $P(N(t)=1) = \lambda t e^{-\lambda t}$

Also

$P(X_1 \le x, N(t)=1)$

$= P($one arrival in $(0, x]$ and no arrivals in $(x, t])$

$$= \left(\lambda x e^{-\lambda x}\right) \times \left(e^{-\lambda(t-x)}\right)$$

$$= \lambda x e^{-\lambda t}$$

$$\Rightarrow P(X_1 \le x \mid N(t)=1) = \frac{\lambda x e^{-\lambda t}}{\lambda t e^{-\lambda t}}$$

$$= \frac{x}{t}$$ ✓

"Coupon Collector"

(8) As in Practice Problem, we have already shown $T_n = X_1 + X_2 + \dots + X_n$,

where $X_i \sim \text{Geometric}(P_i)$,

$$E[X_i] = 1/P_i \qquad P_i = \frac{n-i+1}{n}.$$

$$Var[X_i] = \frac{1-P_i}{P_i^2}.$$

$$Var(T_n) = Var(X_1) + \cdots + Var(X_n)$$

$$= \frac{1-P_1}{P_1^2} + \frac{1-P_2}{P_2^2} + \cdots + \frac{1-P_n}{P_n^2}$$

$$< \left(\frac{n^2}{n^2} + \frac{n^2}{(n-1)^2} \cdots + \frac{n^2}{1^2}\right)$$

$$= n^2\left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2}\right)$$

$$\leq n^2\Big(\underbrace{1/1^2 + 1/2^2 \cdots \cdots \infty}_{= \pi^2/6}\Big)$$

$$= n^2\pi^2/6.$$

Using Chebyshev's $\quad E[T_n] =$

$$P\left(|T_n - \underbrace{n H_n}_{E(T)}| \geq \epsilon \right) \leq \frac{Var(T_n)}{\epsilon^2}$$

$$P\left(|T_n - n x_n| \geq \underset{\downarrow \, cn}{\epsilon} \right) \leq \frac{Var(T_n)}{\epsilon^2}$$

$$\frac{\cancel{n^2} \pi^2}{c^2 \cancel{n^2} 6}$$

$$= \frac{\pi^2}{6 c^2}$$