Jointly Normal R.V.'s

2 r.v.'s X and Y are jointly Normal (or jointly Gaussian) if their joint density is

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \cdot \exp\left\{ \frac{-1}{2(1-r^2)} \left(\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2} - 2r\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} \right) \right\}$$

$|r| < 1$. ($r \Rightarrow$ Correlation coefficient)

Marginal densities:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \; ; \; f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}.$$

Important Facts:

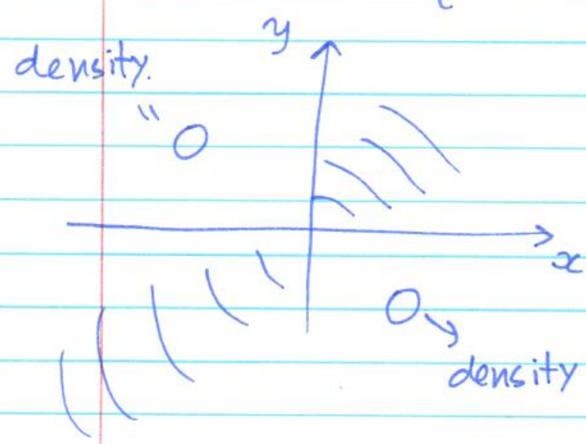
1. If two random variables are jointly normal, they are also marginally normal. The reverse statement is not true in general.

2. Joint normality can also be defined as follows:

   X and Y are jointly normal if the sum $aX + bY$ is normal for every a and b.

Counter-Example:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi} e^{-\frac{(x^2+y^2)}{2}} & xy \geq 0 \\ 0 & xy < 0 \end{cases}$$

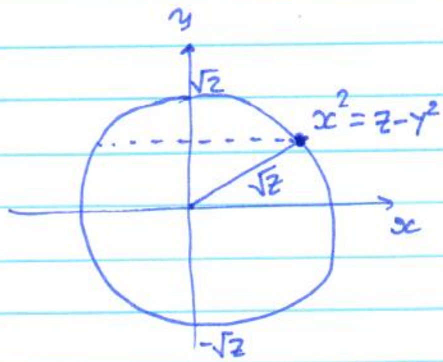


Check if $Z = X + Y$ is a Normal r.v.

Answer → No.   [Try this Yourself]

Eg $\quad Z = X^2 + Y^2$

$$F_Z(z) = P(X^2 + Y^2 \le z) = \iint_{x^2+y^2 \le z} f_{X,Y}(x,y)\, dx\, dy$$



$$= \int_{y=-\sqrt{z}}^{\sqrt{z}} \int_{x=-\sqrt{z-y^2}}^{+\sqrt{z-y^2}} f_{X,Y}(x,y)\, dx\, dy$$

$$F_Z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \left(\int_{x=-\sqrt{z-y^2}}^{+\sqrt{z-y^2}} f_{X,Y}(x,y)\, dx \right) dy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z) \quad \text{(Leibniz Rule)}$$

$$= \left(\frac{d(\sqrt{z})}{dz}(0) - \frac{d(-\sqrt{z})}{dz}(0) \right) + \int_{-\sqrt{z}}^{\sqrt{z}} \frac{d}{dz} \left(\int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_{XY}(x,y)\, dx \right) dy$$

again Leibniz Rule....

$$= \int_{-\sqrt{z}}^{\sqrt{z}} \left[\frac{1}{2\sqrt{z-y^2}} f_{X,Y}(\sqrt{z-y^2}, y) - \left(\frac{-1}{2\sqrt{z-y^2}} \right) \cdot f_{X,Y}(-\sqrt{z-y^2}, y) + 0 \right] dy$$

$$= \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \left(f_{X,Y}(\sqrt{z-y^2}, y) + f_{X,Y}(-\sqrt{z-y^2}, y) \right) dy.$$

Eg. continued....

Suppose X and Y are independent normal r.v.'s with 0 mean & common variance σ^2. Find $f_Z(z)$ when $Z = X^2 + Y^2$.

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$= \frac{1}{2\pi\sigma^2} \cdot e^{-\frac{x^2}{2\sigma^2}} \cdot e^{-y^2/2\sigma^2}$$

From prev. page,...

$$f_{X,Y}(\sqrt{z-y^2}, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(z-y^2)}{2\sigma^2}} \cdot e^{\frac{-y^2}{2\sigma^2}}$$

$$= \frac{e^{-z/2\sigma^2}}{2\pi\sigma^2}$$

$$f_{X,Y}(-\sqrt{z-y^2}, y) = e^{-z/2\sigma^2}/2\pi\sigma^2$$

$$\Rightarrow \quad f_Z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \cdot \frac{2}{2\pi\sigma^2} e^{-z/2\sigma^2}\, dy$$

$$= \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \left(\int_{-\sqrt{z}}^{\sqrt{z}} \frac{dy}{2\sqrt{z-y^2}} \right)$$

even function

$$= \left(\frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \right) \cdot 2 \int_0^{\sqrt{z}} \frac{dy}{2\sqrt{z-y^2}} = \underbrace{\frac{e^{-z/2\sigma^2}}{(2\sigma^2)}}_{\text{Exponential r.v}}, \quad z \geq 0$$

Worth Remembering....

X, Y are independent normal r.v.'s with 0 mean & variance σ^2 each

$$\boxed{Z = X^2 + Y^2 \sim \text{exponential}\left(\frac{1}{2\sigma^2}\right)}$$

$$Z = \sqrt{X^2+Y^2} \sim \underbrace{\text{Rayleigh r.v.}}$$

$$f_Z(z) = \begin{cases} \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} & z \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$W = X + iY$

X, Y are real, independent normal r.v.'s with 0 mean, equal variance σ^2.

Amplitude $\Rightarrow |W| = \sqrt{X^2+Y^2} \rightarrow$ Rayleigh

Phase $\Rightarrow \theta = \tan^{-1}\left(\frac{Y}{X}\right) \rightarrow \text{Uniform}(-\pi, \pi)$

If X & Y do not have zero mean, say μ_X and μ_Y (still independent and normal),

$\sqrt{X^2+Y^2} \sim$ Rician random variable.

Eg $$Z = \max(X, Y) = \begin{cases} X & \text{if } X > Y \\ Y & \text{if } X \le Y \end{cases}$$

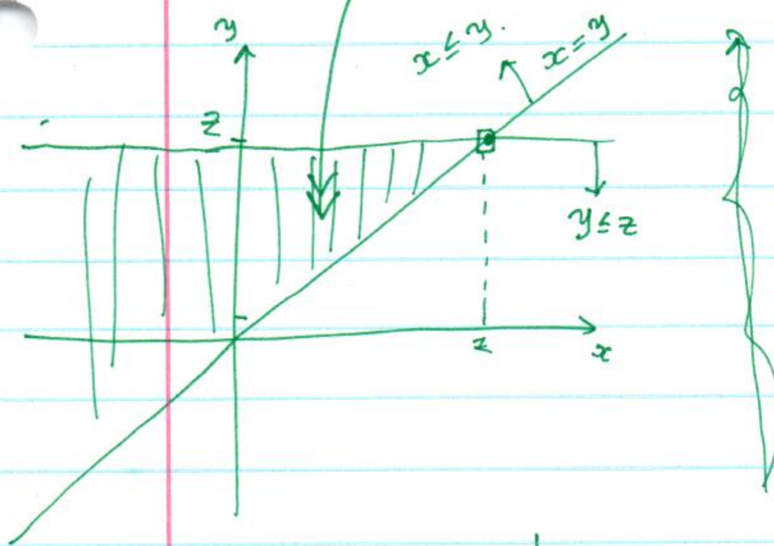
$$F_Z(z) = P(\max(X,Y) \le z)$$

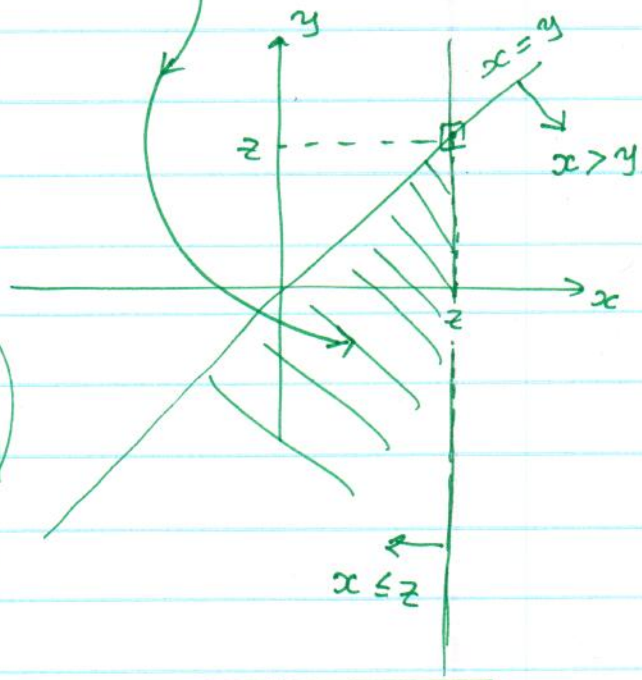
mutually exclusive

$$= P\Big((\max(X,Y) \le z) \cap \underbrace{\{ (X \le Y) \cup (X > Y) \}}_{\text{Sample space.}} \Big)$$

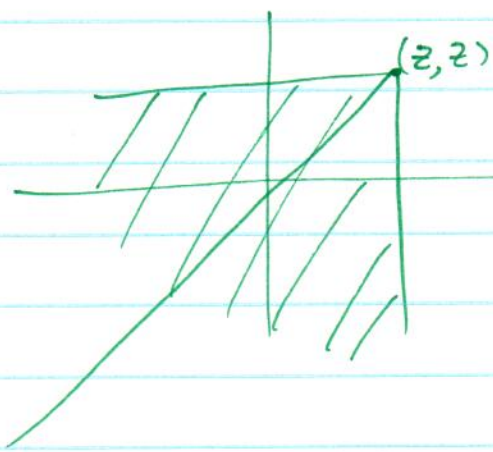
$$= P(X \le Y,\ \max(X,Y) \le z) + P(X > Y,\ \max(X,Y) \le z)$$

$$= P(X \le Y,\ Y \le z) + P(X > Y,\ X \le z)$$





$\Rightarrow$



$$\Rightarrow \boxed{F_Z(z) = F_{X,Y}(z,z)}$$

If X and Y are independent...

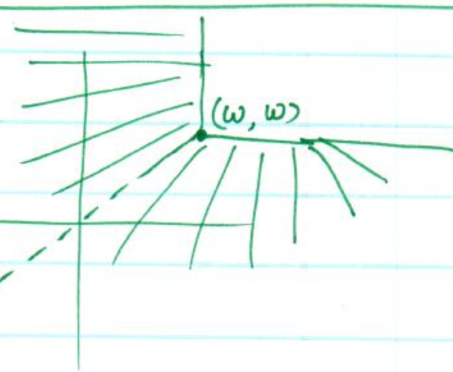
$$F_Z(z) = F_{XY}(z,z) = F_X(z)F_Y(z)$$

$$\Rightarrow \frac{d}{dz}F_Z(z) = f_X(z)F_Y(z) + F_X(z)f_Y(z)$$

$\Rightarrow$ For X, Y independent, $Z = \max(X,Y)$

$$\rightarrow f_Z(z) = f_X(z)F_Y(z) + F_X(z)f_Y(z).$$

$W = \min(X,Y) \rightarrow$



$$F_W(w) = 1 - P(W > w)$$

$$= 1 - P(X > w, Y > w)$$

$$F_W(w) = F_X(w) + F_Y(w) - F_{X,Y}(w,w).$$

Two functions of 2 r.v.'s

Let $Z = g(X, Y)$

$W = h(X, Y)$

If the functions $g(x, y)$ and $h(x, y)$ are continuous and differentiable, then we can directly obtain the joint density of (Z, W) from the joint density of (X, Y) as follows: →

Step 1: Consider the equations

$$g(x, y) = z \;;\quad h(x, y) = w$$

For a given pair (z, w), Let $(x_1, y_1), (x_2, y_2), \ldots\ldots, (x_n, y_n)$ denote the Simultaneous solutions of the above, i.e

$$g(x_i, y_i) = z, \quad h(x_i, y_i) = w \text{ for } i = 1, 2, \ldots, n.$$

Step 2:

Find the Jacobian

$$J(x_i, y_i) = \left| \begin{matrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{matrix} \right|_{\substack{x = x_i, \\ y = y_i}}$$

Step 3:

$$f_{Z,W}(z,w) = \sum_i \frac{1}{|J(x_i, y_i)|} \cdot f_{X,Y}(x_i, y_i)$$

Eg 1:

$$Z = aX + bY$$
$$W = cX + dY$$

If $ad-bc \neq 0$, then the above system has only one solution:

$$\left.\begin{aligned} x &= Az + Bw \\ y &= Cz + Dw \end{aligned}\right\} \rightarrow (x_1, y_1)$$

$$J(x,y) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = (ad - bc)$$

$$\Rightarrow f_{Z,W}(z,w) = \frac{1}{|ad-bc|} f_{X,Y}(Az + Bw, Cz + Dw)$$

Eg 2: X, Y are zero mean, independent Gaussian r.v.'s with the same variance σ^2. Find joint density of (r, θ), where

$$r = \sqrt{X^2+Y^2}$$

$$\theta = \tan^{-1}(Y/X), \quad |\theta| < \pi$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{(x^2+y^2)}{2\sigma^2}}$$

$$\left.\begin{aligned} r &= g(x,y) = \sqrt{x^2+y^2} \\ \theta &= h(x,y) = \tan^{-1}(y/x) \end{aligned}\right\} \rightarrow \text{only 1 solution } (x_1, y_1) = (r\cos\theta, r\sin\theta)$$

$$J(x,y) = \begin{vmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ \dfrac{-y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$$

$\Rightarrow$

$$f_{r,\theta}(r,\theta) = \frac{1}{|J(x_1,y_1)|} \cdot f_{X,Y}(x_1, y_1)$$

$$= r \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(r^2\cos^2\theta + r^2\sin^2\theta)}$$

$$f_{r,\theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-r^2/2\sigma^2}, \quad 0 < r < \infty, \quad |\theta| < \pi$$

$$f_{r,\theta}(r,\theta) = \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} \qquad \begin{array}{l} 0 < r < \infty \\ |\theta| < \pi \end{array}$$

Marginal of r

$$\underbrace{f_r(r)}_{\downarrow \atop \text{Rayleigh r.v.}} = \int_{\theta=-\pi}^{\pi} f_{r,\theta}(r,\theta)\, d\theta = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad 0 < r < \infty$$

Marginal of θ.

$$\underbrace{f_\theta(\theta)}_{\downarrow \atop \text{uniform r.v.}} = \int_{r=0}^{\infty} f_{r,\theta}(r,\theta)\, dr = \frac{1}{2\pi}, \quad |\theta| < \pi$$

$$\Rightarrow \quad f_{r,\theta}(r,\theta) = f_r(r) \cdot f_\theta(\theta)$$

$\Rightarrow$
r & θ are independent r.v.'s
(magnitude) (phase).

$\boxed{\text{IMPORTANT}}$

Eg 3 Let X and Y be independent exponential r.v.'s with (same) parameter λ.

$$U = X + Y$$
$$V = X - Y$$

$$f_{X,Y}(x,y) = \left(\frac{1}{\lambda} e^{-\frac{x}{\lambda}}\right)\left(\frac{1}{\lambda} e^{-y/\lambda}\right) \qquad \begin{matrix} x > 0 \\ y > 0 \end{matrix}$$

$$= \frac{1}{\lambda^2} e^{-(x+y)/\lambda}$$

$$\left.\begin{matrix} u = x + y \\ v = x - y \end{matrix}\right\} \rightarrow \text{one solution} \left(\begin{matrix} x_1 = \left(\frac{u+v}{2}\right) \\ y_1 = \left(\frac{u-v}{2}\right) \end{matrix}\right)$$

$$J(x,y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2$$

$\Rightarrow$

$$f_{U,V}(u,v) = \frac{1}{|-2|} f_{X,Y}(x_1, y_1)$$

$$= \frac{1}{2} \frac{1}{\lambda^2} e^{-\frac{\left(\frac{u+v}{2} + \frac{u-v}{2}\right)}{\lambda}}$$

$$= \frac{1}{2\lambda^2} e^{-u/\lambda}, \quad \underbrace{0 < |v| < u}_{} < \infty.$$

Why?? $\Rightarrow$ $x > 0$, $y > 0$

$$\left.\begin{matrix} u = x + y \\ v = x - y \end{matrix}\right\} \Rightarrow |v| < u.$$

Marginal of U

$$f_U(u) = \int_{v=-u}^{u} f_{U,V}(u,v)\,dv = \frac{1}{2\lambda^2}\int_{-u}^{u} e^{-u/\lambda}\,dv$$

$$= \frac{1}{2\lambda^2} \cdot e^{-u/\lambda} \cdot 2u = \frac{u}{\lambda^2} e^{-u/\lambda},$$

$$0 < u < \infty$$

Marginal of V

$$f_V(v) = \int_{u=|v|}^{\infty} f_{U,V}(u,v)\,du = \int_{|v|}^{\infty} \frac{1}{2\lambda^2} e^{-u/\lambda}\,du$$

$$= \frac{e^{-|v|/\lambda}}{2\lambda}, \qquad -\infty < v < \infty$$

$$f_{U,V}(u,v) \neq f_U(u)\, f_V(v)$$

$\Rightarrow$ U and V are $\boxed{\text{NOT}}$ independent r.v.'s

EXPECTED VALUE of A FUNCTION of 2 R.V.'s $g(X,Y)$

$$E\left(g(X,Y)\right) = \int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

LINEARITY of EXPECTATION

$$E\left(a_1 g_1(X,Y) + a_2 g_2(X,Y) + \dots + a_n g_n(X,Y)\right)$$
$$= \sum_{i=1}^{n} a_i E\left(g_i(X,Y)\right)$$

Recall, $E(X+Y) = E(X) + E(Y)$

However, in general $E(XY) \neq E(X)E(Y)$ ✓

COVARIANCE of 2 RANDOM VARIABLES

C or C_{XY} or σ_{XY}

$$\boxed{C_{XY} = E\left[(X-\mu_X)(Y-\mu_Y)\right]}$$

$$= E\left[XY - \mu_Y X - \mu_X Y + \mu_X \mu_Y\right]$$

$$= E(XY) - \mu_Y \mu_X - \mu_X \mu_Y + \mu_X \mu_Y$$

$$= E(XY) - \mu_X \mu_Y$$

$$\boxed{C_{XY} = E(XY) - E(X)E(Y)}$$

$$Var(X+Y) = E\left[\left(X+Y-(\mu_X+\mu_Y)\right)^2\right]$$

$$= E\left[\left((X-\mu_X)+(Y-\mu_Y)\right)^2\right]$$

$$= \underbrace{E\left[(X-\mu_X)^2\right]}_{Var(X)} + \underbrace{E\left[(Y-\mu_Y)^2\right]}_{Var(Y)} + 2\underbrace{E\left[(X-\mu_X)(Y-\mu_Y)\right]}_{Cov(X,Y) \text{ or } C_{XY}}$$

$$\Rightarrow \boxed{V(X+Y) = V(X) + V(Y) + 2C_{XY}}$$