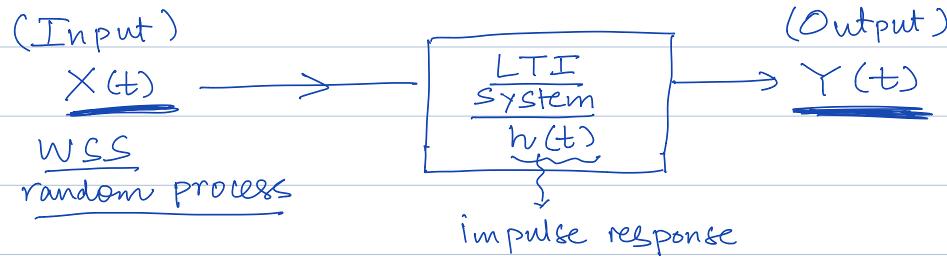
Today

(1) WSS random process over LTI systems.

(2) White Noise, White Gaussian Noise.

(3) Examples.



$$Y(t) = h(t) * X(t) \quad \text{(Convolution)}$$

$$= \int_{-\infty}^{\infty} h(\alpha)\, X(t-\alpha)\, d\alpha$$

$$\boxed{\text{Claim: } X(t) \text{ and } Y(t) \text{ are Jointly WSS.}}$$

(1) $Y(t)$ is WSS

(2) $R_{XY}(t_1, t_2)$ depends only on $(t_1 - t_2)$.

$$\mu_Y(t) = E[Y(t)] = E\left[\int_{-\infty}^{\infty} h(\alpha)\, X(t-\alpha)\, d\alpha\right].$$

$$= \int_{-\infty}^{\infty} h(\alpha)\, \underbrace{E[X(t-\alpha)]}_{=\mu_X}\, d\alpha.$$

(Since $X(t)$ is WSS).

$$\Longrightarrow \boxed{\mu_Y = \mu_X \int_{-\infty}^{\infty} h(\alpha)\, d\alpha.}$$

$\Rightarrow$ $\mu_Y(t)$ is indep of time

$$R_{XY}(t_1, t_2) = E[X(t_1)\, Y(t_2)].$$

$$= E\left[X(t_1) \int_{-\infty}^{\infty} h(\alpha)\, X(t_2 - \alpha)\, d\alpha\right].$$

$$= E\left[\int_{-\infty}^{\infty} \underbrace{h(\alpha)}_{\text{const.}} \underbrace{X(t_1)\,X(t_2-\alpha)}\,d\alpha\right].$$

$$= \int_{-\infty}^{\infty} h(\alpha)\cdot \underbrace{E\left[X(t_1)\,X(t_2-\alpha)\right]}_{R_X(t_1-(t_2-\alpha))}\,d\alpha \quad \left(\text{since } X(t) \text{ is } \underline{\text{WSS}}\right)$$

$$\underline{R_{XY}(t_1,t_2)} = \int_{-\infty}^{\infty} h(\alpha)\,R_X(t_1-t_2+\alpha)\,d\alpha$$

$\tau = \underline{t_1 - t_2}.$

$$R_{XY}(\tau) = \underbrace{\int_{-\infty}^{\infty} h(\alpha)\,R_X(\tau+\alpha)\,d\alpha}$$

$$= h(\tau) * R_X(-\tau)$$

$$\boxed{R_{XY}(\tau) = h(-\tau) * R_X(\tau).}$$

$$\boxed{R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau).}$$

(DIY).

Freq. Domain Analysis of the output Signal

(recap)

$X(t) \longrightarrow \mu_X,\quad \underline{R_X(\tau)}.$

input (WSS)

$$\underset{\text{Fourier transform.}}{\mathcal{F}}\left(R_X(\tau)\right) = \underset{\text{PSD of } X(t).}{\underline{S_X(f)}}$$

Power of $X(t) = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\,df.$

$h(t)$ impulse resp. of LTI system. $\longrightarrow$ $\mathcal{F}(h(t)) = H(f)$

$$H(f) = \int_{-\infty}^{\infty} h(t) e^{-j2\pi ft} dt.$$

$H(f) \Rightarrow$ Transfer function of this system.

(mean of output $Y(t)$)

$$\mu_Y = \mu_X \times \int_{-\infty}^{\infty} h(\alpha) d\alpha.$$

$$\boxed{\mu_Y = \mu_X \times H(0)}$$

$h(t)$ is typically assumed to be a real valued signal

$$\mathcal{F}\{h(-t)\} = H(-f) = H^*(f).$$

(please verify using definition of F.T.)

$$R_{XY}(\tau) = h(-\tau) * R_X(\tau)$$

Take F.T. of both sides

$$\boxed{S_{XY}(f) = H^*(f) \times S_X(f)}$$

(cross PSD)

$$R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau).$$

Take F.T. of both sides.

$$S_Y(f) = H(f) \times H^*(f) \times S_X(f)$$

$$\boxed{S_Y(f) = |H(f)|^2 \times S_X(f)}$$

Eg. $X(t)$ WSS, zero-mean, $R_X(\tau) = e^{-|\tau|}$

$$X(t) \longrightarrow \boxed{\begin{array}{c}\text{LTI} \\ H(f)\end{array}} \longrightarrow Y(t).$$

$$H(f) = \begin{cases} \sqrt{1+4\pi^2 f^2} & \text{if } -2 \le f \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Find

(i) $\mu_Y = 0 \quad (= \overset{0.}{\mu_X} \cdot H(0))$.

(2) $R_Y(\tau)$.

(3) $E[Y^2(t)]$.

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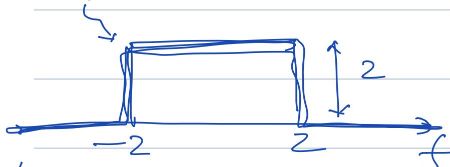
$$S_Y(f) = S_X(f) \times |H(f)|^2.$$

$$= \mathcal{F}\left(e^{-|\tau|}\right) \times (1+4\pi^2 f^2)$$

$$= \frac{2}{\cancel{(1+(2\pi f)^2)}} \times \cancel{(1+4\pi^2 f^2)}$$

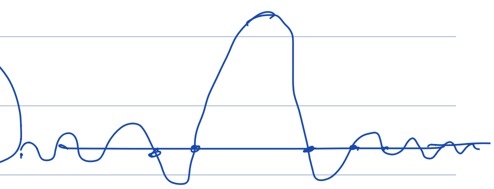
$S_Y(f)$



(rectangular)

$$S_Y(f) = \begin{cases} 2 & \text{if } -2 \le f \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$R_Y(\tau) = \mathcal{F}^{-1}(S_Y(f))$$



$$= 8 \operatorname{sinc}(4\tau).$$

$E[Y(\tilde{t})\,Y(\tilde{t})] \to \cancel{R_Y}(t-t)$ $\qquad \left(\operatorname{sinc}(f) = \dfrac{\sin(\pi f)}{\pi f}\right)$

(c) $E[Y^2(t)] \overset{?}{=} R_Y(0) = 8 \operatorname{sinc}(0) = 8$

Fact: if $X(t)$ is a Stationary Gaussian random process

$$X(t) \longrightarrow \boxed{\begin{matrix} LTI \\ h(t) \end{matrix}} \longrightarrow Y(t) = \int h(\alpha)\, X(t-\alpha)\, d\alpha$$

then $Y(t)$ is also a stationary Gaussian process.

($X(t)$, $Y(t)$ are jointly Gaussian)

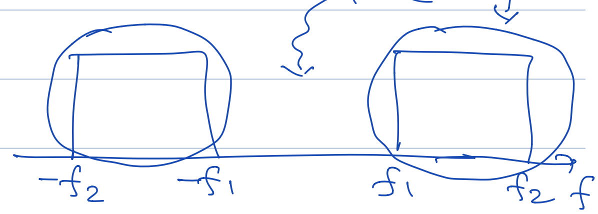
Eg: $X(t)$ is a zero-mean Gaussian proc with $R_X(\tau) = 8 \operatorname{Sinc}(4\tau)$.

$$X(t) \longrightarrow \boxed{\begin{matrix} LTI \\ H(f) \end{matrix}} \longrightarrow Y(t)$$

$$H(f) = \begin{cases} 1/2 & -1 < f < 1 \\ 0 & \text{otherwise.} \end{cases}$$

LPF
Low-pass filter

$H(f)$.



BPF
(Band-pass filter)

Find

$$P\big(Y(2) < 1 \mid Y(1) = 1\big)$$

$Y(t) \longrightarrow$ stationary Gaussian random process

$\mu_Y = 0$ ✓

$$R_Y(\tau) = \mathcal{F}^{-1}\left(S_Y(f) \right)$$

$$= \mathcal{F}^{-1}\left(|H(f)|^2 \times S_X(f) \right)$$

$$S_X(f) = \mathcal{F}\left(R_X(\tau)\right) = \mathcal{F}\left(8\,\mathrm{sinc}\,(4\tau) \right).$$

$$= \begin{cases} 2 & \text{if } |f| < 2 \\ 0 & \text{otheswise.} \end{cases}$$

$$S_Y(f) = \begin{cases} 1/2 & \text{if } |f| < 1 \\ 0 & \text{otheswise.} \end{cases}$$

$$\mathcal{F}^{-1}\left(S_Y(f)\right) = R_Y(\tau) = \mathrm{sinc}(2\tau).$$

$\boxed{\mu_Y = 0}$; $R_Y(\tau) = \mathrm{sinc}(2\tau)$.

$$\mathrm{sinc}(f) = \frac{\sin(\pi f)}{\pi f}.$$

$$\boxed{R_Y(\tau) = \frac{\sin(2\pi\tau)}{2\pi\tau}.}$$

if $Y(1), Y(2)$ were independent.

$$P\left(Y(2) < 1 \mid Y(1) = 1 \right) \neq P\left(Y(2) < 1 \right)$$

$(Y(2), Y(1)) \longrightarrow$ jointly Gaussian.

$\underbrace{(A, B)} \longrightarrow \underline{\underline{\text{check indep}}} \longrightarrow f_{AB} \overset{?}{=} f_A f_B.$

$\text{if } (A, B) \text{ are } \underline{\underline{J \cdot G}} \longrightarrow \rho_{AB} \overset{?}{=} 0$

$\text{uncorrelated} \Longrightarrow \text{independence.}$

$\underbrace{\rho_{Y(2)Y(1)}} \overset{?}{=} 0 \iff \underbrace{\mathrm{Cov}(Y(1), Y(2))} \overset{?}{=} 0.$

$$\mathrm{Cov}_{Y(1),Y(2)} = E\big[(Y(1) - 0)(Y(2) - 0)\big]$$

$$= E[Y(1)Y(2)]. = R_Y(-1).$$

$\underline{Y(t) \text{ is a WSS}}$

$$= \frac{\sin(-2\pi)}{(-2\pi)}$$

$$= 0$$

$\Rightarrow Y(1), Y(2) \text{ are independent.}$

$$\Rightarrow P(Y(2) < 1 \mid Y(1) = 1)$$

$$= \underbrace{P(Y(2) < 1)}. \quad \mathcal{N}(0, 1)$$

$$= \Phi(1) \simeq \underline{\underline{0.84}}$$

$$\mathrm{Var}(Y(2)) = \underbrace{E[Y^2(2)]}_{E[Y(2)Y(2)]} = R_Y(0) = \underline{1}.$$