ECE503

HOMEWORK 3 Solution

Problem 1. (1 point)

$S = \{ s_i : s_i = -3, -2, -1, 0, 1, 2, 3 \}$

$\uparrow$ Sample Space

$\underbrace{-3, -2, -1, 0, 1, 2, 3}_{\text{all equally likely, i.e } P(s_i) = 1/7}$

$$X(s) = \begin{cases} 0 & \text{if } s_i = 0 \\ 1 & \text{if } s_i = \pm 1 \\ 4 & \text{if } s_i = \pm 2 \\ 9 & \text{if } s_i = \pm 3 \end{cases}$$

PMF of X

$P(X=0) = 1/7 \quad P(X=1) = 2/7 \quad P(X=4) = 2/7$

and $P(X=9) = 2/7$

CDF of X : $F_X(x) = P(X \leq x)$

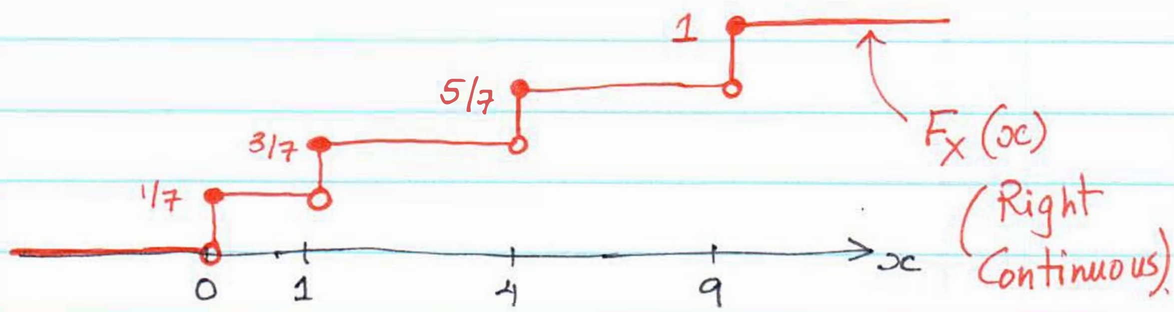
Case 1: $x \geq 9$ $\quad F_X(x) = P(X \in \{0,1,4,9\}) = 1$

Case 2: $4 \leq x < 9$ $\quad F_X(x) = P(X=0) + P(X=1) + P(X=4) = 5/7$

Case 3: $1 \leq x < 4$ $\quad F_X(x) = P(X=0) + P(X=1) = 3/7$

Case 4: $0 \leq x < 1$ $\quad F_X(x) = P(X=0) = 1/7$

Final Case : $x < 0$ $\quad F_X(x) = 0$



Problem 2

Arrival Rate of calls = 0.2 / second

Probability of k calls in a T second interval

$$= e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad \text{where}$$

$$\lambda = \begin{pmatrix}\text{Arrival} \\ \text{rate}\end{pmatrix} \times T$$

$\Rightarrow$ for T = 60 seconds,

$$\lambda = 0.2 \times 60 = 12$$

$$\Rightarrow \quad P(k \text{ calls in } T \text{ sec}) = e^{-12} \cdot 12^k / k!$$

$$\Rightarrow \quad P\begin{pmatrix}\text{No more than} \\ \text{2 calls in T} \\ \text{seconds}\end{pmatrix} = P\begin{pmatrix}\text{0 call} \\ \text{in T sec}\end{pmatrix} + P\begin{pmatrix}\text{1 call in} \\ \text{T sec}\end{pmatrix} + P\begin{pmatrix}\text{2 calls in} \\ \text{T sec}\end{pmatrix}$$

$$= e^{-12} \left\{ \frac{12^0}{0!} + \frac{12^1}{1!} + \frac{12^2}{2!} \right\}$$

$$= \underline{5.22 \times 10^{-4}}$$

Problem 3 $\quad X$ is geometric, i.e.

$$P(X=k) = (1-p)^{k-1} p \quad \text{for } k = 1, 2, 3, \ldots$$

To prove that X (geometric) is memoryless, we must show that

$$P(X > m+n \mid X > m) = P(X > n).$$

LHS $= P(X > m+n \mid X > m)$

$$= \frac{P(X > m+n,\ X > m)}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)} \quad \text{—— } (*)$$

To find above, we must obtain $P(X > k)$

$$P(X > k) = \sum_{i=k+1}^{\infty} P(X=i) = p \sum_{i=k+1}^{\infty} (1-p)^{i-1}$$

$$= p(1-p)^k \cdot \sum_{i=0}^{\infty} (1-p)^i$$

$$= p(1-p)^k \cdot \frac{1}{1-(1-p)} = (1-p)^k. \quad \text{—— } (\star)$$

Using $(\star)$ in $(*)$,

$$\text{LHS} = \frac{P(X > m+n)}{P(X > m)} = \frac{(1-p)^{m+n}}{(1-p)^m} = (1-p)^n = P(X > n)$$

—— α ——.

$\boxed{\text{Problem 4.}}$ We denote Y as the random variable equal to the call duration.

Since Y denotes call duration,

$$F_Y(y) = 0 \quad \text{if} \quad y < 0$$

[Why? Call duration cannot be −ve]

We also know that no call is for more than 3 minutes, i.e.

$$F_Y(y) = 1 \quad \text{if} \quad y > 3$$

So, we focus on the case when $0 \le y \le 3$.
First, let A denote the event that the call is answered.

⇒ By total Probability Theorem

$$F_Y(y) = P[Y \le y]$$
$$= P(A)\,P(Y \le y \mid A) + P(A^c)\,P(Y \le y \mid A^c)$$

$P(A) = P(\text{call is answered}) = 2/3$
$P(A^c) = P(\text{call is NOT answered}) = 1/3$

Given that the event A^c has occured (i.e call <u>not</u> answered), $Y = 0$

$$\Rightarrow P(Y \le y \mid A^c) = 1 \quad \text{for } 0 \le y \le 3$$

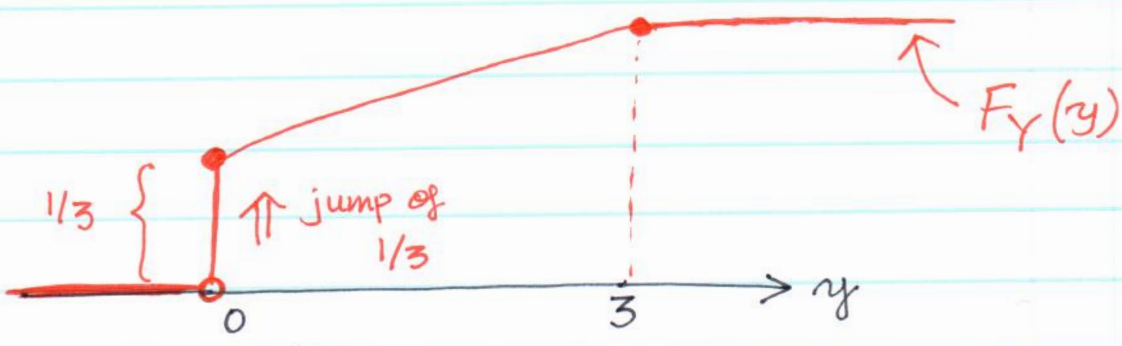
Given that event A occurs (i.e call is answered), call duration $\sim$ unif $[0,3]$

$$\Rightarrow P(Y \le y | A) = \int_0^y f_{Y|A}(y|A)\,dy = \int_0^y \frac{1}{3}\,dy$$

$$= y/3$$

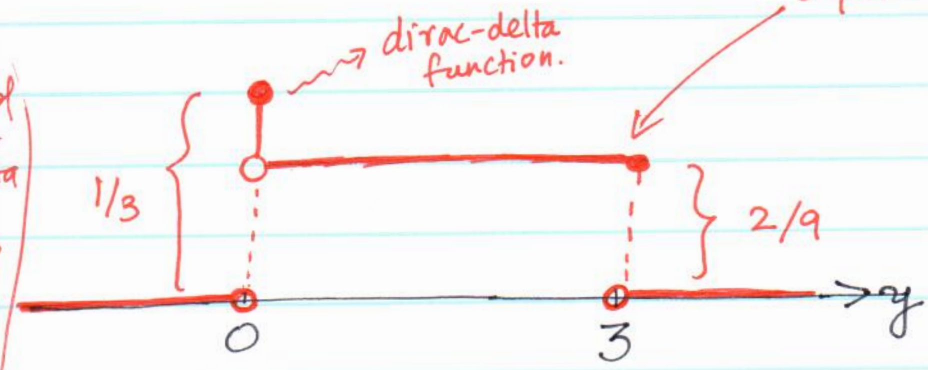
$$\Rightarrow F_Y(y) = \frac{2}{3} \times \frac{y}{3} + \frac{1}{3} \times 1 = \frac{1}{3} + \frac{2y}{9}$$

$$\Rightarrow F_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 + 2y/9 & 0 \le y \le 3 \\ 1 & y > 3 \end{cases}$$



$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ 1/3 & y = 0 \\ 2/9 & 0 < y \le 3 \\ 0 & y > 3 \end{cases}$$



height of dirac-delta = Jump in CDF

Problem 5

We first note that $-10 \leq W \leq 10$

$\Rightarrow$ Case 1: If $w \geq 10 \Rightarrow P(W \leq w) = 1$

and

$\Rightarrow$ Case 2: If $w < -10 \Rightarrow P(W \leq w) = 0$.

Case 3: $-10 \leq w < 10$

For $-10 \leq v \leq 10$, $W = V$

$$\Rightarrow \underbrace{P(W \leq w)}_{\substack{\parallel \\ F_W(w)}} = P(V \leq w) = F_V(w).$$

$\Rightarrow \sigma^2 = 25 \Rightarrow \sigma = 5$

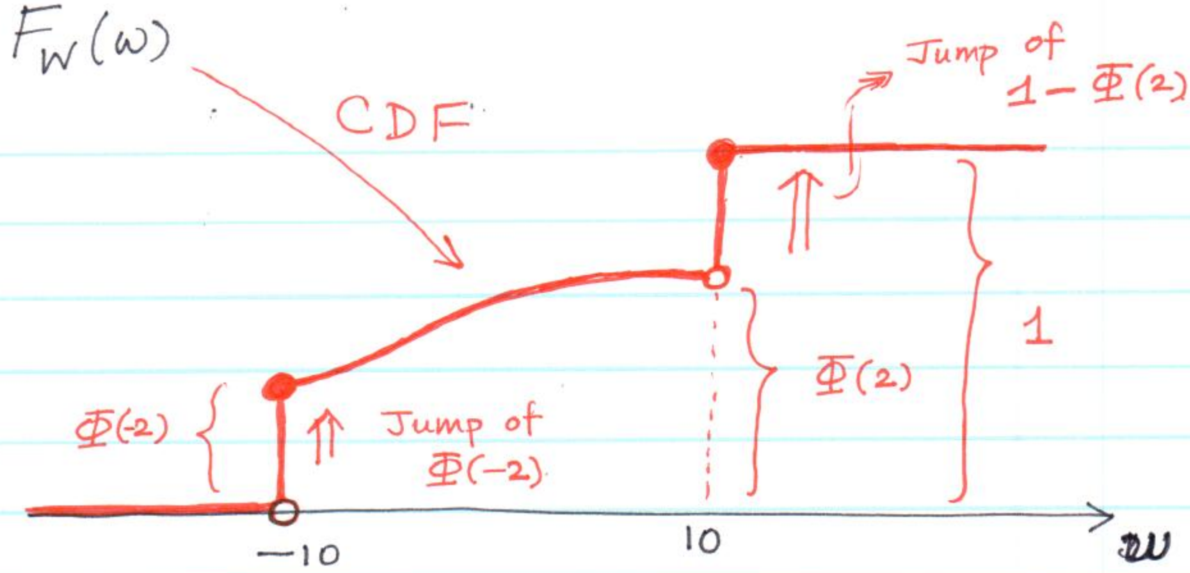
Since $V \sim N(0, 25)$

$$\Rightarrow F_V(v) = \Phi\left(\frac{v-\mu}{\sigma}\right) = \Phi\left(\frac{v-0}{5}\right) = \Phi\left(\frac{v}{5}\right)$$

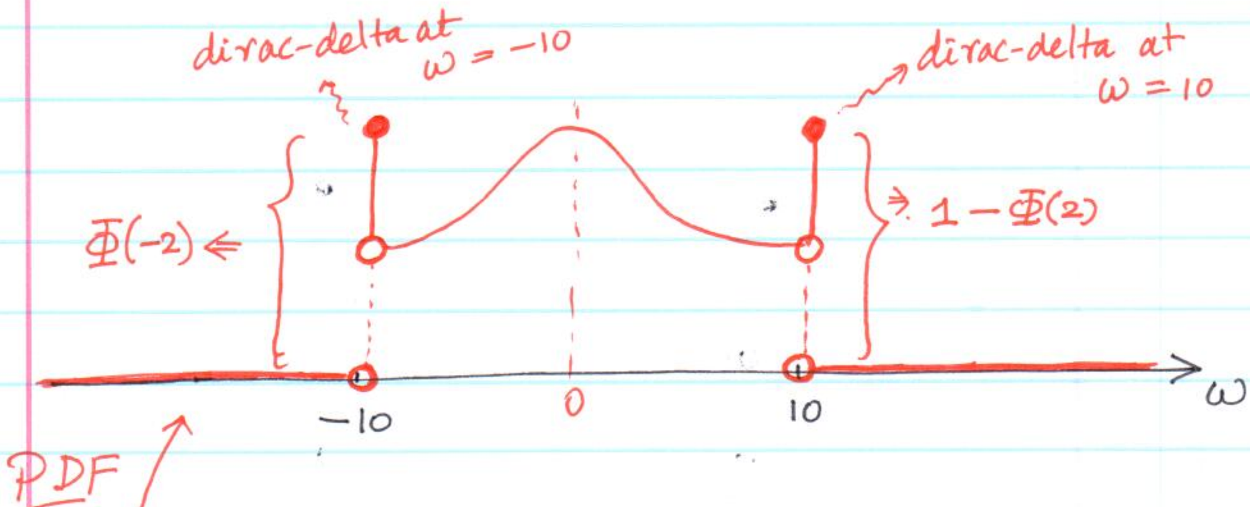
CDF of Standard normal.

Hence,

$$F_W(w) = \begin{cases} 0 & w < -10 \\ \Phi\left(\frac{w}{5}\right) & -10 \leq w < 10 \\ 1 & w \geq 10 \end{cases}$$



$$f_W(\omega) = \frac{d}{d\omega} F_W(\omega)$$



$$f_W(\omega) = \begin{cases} \Phi(-2)\cdot\delta(\omega+10) & \omega = -10 \\ \dfrac{e^{-\omega^2/50}}{5\sqrt{2\pi}} & -10 < \omega < 10 \\ (1-\Phi(2))\,\delta(\omega-10) & \omega = +10 \\ 0 & \text{otherwise.} \end{cases}$$

Problem 6
3 points

$$X \sim \text{unif}(0.4, 0.6)$$

(a) $P(\text{Click} \mid X = p) = p$

$$\Rightarrow P(\text{Click}) = \int_{0.4}^{0.6} P(\text{Click} \mid X=p)\, \underbrace{f_X(p)}_{= \frac{1}{(0.6-0.4)} = 5}\, dp$$

$$= 5 \int_{0.4}^{0.6} p\, dp = \boxed{0.5}$$

(b) Let A = { Jane clicked on the first 60 of 100 ad-links }
(Event)

$$f_X(p \mid A) = \begin{cases} \dfrac{p^{60}(1-p)^{40}}{\int_{0.4}^{0.6} p^{60}(1-p)^{40}\, dp} & 0.4 \le p \le 0.6 \\ 0 & \text{otherwise} \end{cases}$$

we did this in class....

$$\Rightarrow P(\text{Click} \mid A) = \int_{0.4}^{0.6} \overbrace{P(\text{Click} \mid A, X=p)}^{p} \cdot f_X(p \mid A)\, dp$$

$$= \frac{\int_{0.4}^{0.6} p \cdot p^{60}(1-p)^{40}\, dp}{\int_{0.4}^{0.6} p^{60}(1-p)^{40}\, dp} = 0.56$$

Even if you do not get this number, its OKAY

Problem 7

$U \sim \text{unif}(0,1)$

(a) $P(Y=0) = P(U \le 1/4) = 1/4$

$P(Y=1/2) = P(1/4 \le U \le 3/4) = 1/2$

$P(Y=1) = P(U > 3/4) = 1/4.$

(b) You will observe that as n increases, (or should) the quality of estimate of PMF improves.

Problem 8
4 points

$$X = \underbrace{\,}_{\text{\# of arriving photons detected}} \begin{cases} \text{Poisson}(\lambda_1) & \text{if bit sent} = 1 \\ \text{Poisson}(\lambda_0) & \text{if bit sent} = 0 \end{cases}$$

$P(\text{bit}_{\text{sent}} = 1) = p$

$P(\text{bit}_{\text{sent}} = 0) = 1 - p$

(a) $P(\text{bit } 1 \text{ is sent} \mid X = k)$

$$= \frac{P(\text{bit } 1 \text{ is sent}, \ X = k)}{P(X = k)}$$

$$= \frac{P(\text{bit } 1 \text{ is sent}) \cdot \overbrace{P(X = k \mid \text{bit } 1 \text{ sent})}^{\text{Poisson}(\lambda_1)}}{P(X = k)}$$

$$= \frac{p \times \left(\frac{e^{-\lambda_1} \cdot \lambda_1^k}{k!} \right)}{P(X = k)}$$

$$= \frac{p \times (e^{-\lambda_1} \lambda_1^k / k!)}{P(\text{bit } 1 \text{ sent}) \cdot P(X = k \mid \text{bit } 1_{\text{sent}}) + P(\text{bit } 0_{\text{sent}}) \cdot P(X = k \mid \text{bit } 0_{\text{sent}})}$$

$$= \frac{p\,(e^{-\lambda_1} \lambda_1^k / k!)}{p\,e^{-\lambda_1} \lambda_1^k / k! + (1-p) e^{-\lambda_0} \lambda_0^k / k!}$$

$$= (p\,e^{-\lambda_1} \lambda_1^k) / (p\,e^{-\lambda_1} \lambda_1^k + (1-p) e^{-\lambda_0} \lambda_0^k)$$

(b) Detect bit 1 if $P(\underset{\text{sent}}{\text{bit}=1} \mid X=k) > P(\underset{\text{sent}}{\text{bit}=0} \mid X=k)$.

$$LHS = P(\text{bit}=1 \text{ sent} \mid X=k)$$
$$= \frac{p e^{-\lambda_1} \lambda_1^k}{p e^{-\lambda_1} \lambda_1^k + (1-p) e^{-\lambda_0} \lambda_0^k}$$

$$RHS = P(\text{bit}=0 \text{ sent} \mid X=k)$$
$$= \frac{(1-p) e^{-\lambda_0} \lambda_0^k}{p e^{-\lambda_1} \lambda_1^k + (1-p) e^{-\lambda_0} \lambda_0^k}$$

(by a similar calculation...)

$\Rightarrow$ $LHS > RHS$

$$\Rightarrow \quad p e^{-\lambda_1} \lambda_1^k > (1-p) e^{-\lambda_0} \lambda_0^k$$

$$\Rightarrow \quad \left(\frac{\lambda_1}{\lambda_0}\right)^k > \left(\frac{1-p}{p}\right) e^{(\lambda_1 - \lambda_0)}$$

$$\Rightarrow \quad k \ln\left(\frac{\lambda_1}{\lambda_0}\right) > \ln\left(\frac{1-p}{p}\right) + (\lambda_1 - \lambda_0)$$

$$\Rightarrow \quad k > \frac{\ln\left(\frac{1-p}{p}\right) + (\lambda_1 - \lambda_0)}{\ln\left(\frac{\lambda_1}{\lambda_0}\right)}.$$

$\lceil x \rceil$ = "Ceiling of x)" or Smallest integer not less than x.

Let $k^{\text{threshold}} = \left\lceil \dfrac{\ln\left(\frac{1-p}{p}\right) + \lambda_1 - \lambda_0}{\ln\left(\frac{\lambda_1}{\lambda_0}\right)} \right\rceil$

$\Rightarrow$ Decision Rule.

Decode bit 1 if $\underset{\sim}{\overset{X}{}}\; k > k^{\text{threshold}}$

Decode bit 0 if $k \leq k^{\text{threshold}}$

(c) <u>Prob of Error</u>.

$$P(\text{Error}) = P(\text{bit 1 sent}) \cdot P(\text{Error} \mid \text{bit 1 sent}) + P(\text{bit 0 sent}) \cdot P(\text{Error} \mid \text{bit 0 sent})$$

$$= p \times P(\text{Error} \mid \text{bit 1 sent}) + (1-p)\, P(\text{Error} \mid \text{bit 0 sent})$$

$$= p \cdot P(X \leq k^{\text{threshold}} \mid \text{bit 1 sent}) + (1-p)\, P(X > k^{\text{threshold}} \mid \text{bit 0 sent})$$

$$= p \times \sum_{i=0}^{k^{\text{threshold}}} e^{-\lambda_1} \cdot \frac{\lambda_1^i}{i!} + (1-p) \sum_{i=k^{\text{threshold}}+1}^{\infty} e^{-\lambda_0} \cdot \frac{\lambda_0^i}{i!}$$

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