Homework 3 - ECE 503 Fall 2017

- Assigned on: Tuesday, September 12, 2017.
- Due Date: **Tuesday, September 19, 2017 by 11:00 am Tucson Time.**
- Maximum Credit: 100 points

1. [10 points] Consider a random experiment with the sample space $S = \{s_i : s_i = -3, -2, -1, 0, 1, 2, 3\}$. The outcomes are equally likely. If a random variable is defined as $X(s_i) = s_i^2$, find its PMF and CDF.

2. [10 points] The arrival rate of calls at a mobile station is 0.2 per second. The probability of k calls in a T second interval is given by a Poisson PMF with the parameter $\lambda = (\text{arrival rate}) \times T$. What is the probability that there will be no more than 2 calls placed in a 1-minute interval ?

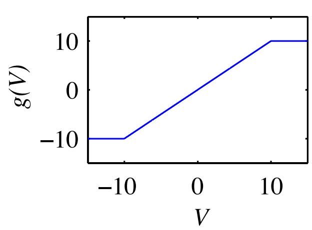
3. [10 points] The probability distribution of a random variable X is said to be "memoryless" if it satisfies the following property for any positive numbers m, n:

$$P(X > m + n | X > m) = P(X > n)$$

Show that the geometric distribution is memoryless.

4. [10 points] Suppose, we can observe someone making a phone call and record the duration of the call. In a simple model of the experiment, 1/3 of the calls never begin either because no one answers or the line is busy. The duration of these calls is 0 minutes. Otherwise, with probability 2/3, a call duration is uniformly distributed between 0 and 3 minutes. Let Y denote the random variable denoting the call duration. Find the CDF, $F_Y(y)$ and the PDF, $f_Y(y)$.

5. [10 points] The output voltage of a microphone is a Gaussian random variable $\mathcal{N}(0, 25)$, i.e., $\mu = 0$ Volt, and $\sigma = 5$ Volts. The microphone signal is the input to a limiter circuit with a cut-off value of ± 10 Volts. The random variable W is the output of the limiter as shown in the figure below:



$$W = g(V) = \begin{cases} -10 & V < -10, \\ V & -10 \le V \le 10, \\ 10 & V > 10. \end{cases}$$

Find the CDF and the PDF of W.

6. [20 points] The probability of Jane Doe clicking an ad-link in her e-mail is a random variable X, which is uniformly distributed in the interval $(0.4, 0.6)$.

   (a) What is the probability that she will click on a ad-link ?

   (b) Over the past year, Jane has clicked on the first 60 out of 100 advertisement links. Given this information, what is the probability that she will click on the next presented link ? Is it the same or different from part (a) ?

7. [10 points] A communication engineer is supposed to generate a discrete random variable X with the following PMF:

$$P_X(0) = 1/4, \quad P_X(1/2) = 1/2, \quad P_X(1) = 1/4. \tag{1}$$

However, due to budget constraints, he only has access to a *primitive* device which can only generate a uniform random variable (denoted by $U \sim \text{unif}(0,1)$). Under such constraints, the engineer instead generates a random variable Y by utilizing the *primitive* device as follows:

$$Y = \begin{cases} 0 & \text{if } U \leq 1/4 \\ 1/2 & \text{if } 1/4 \leq U \leq 3/4 \\ 1 & \text{if } U > 3/4 \end{cases} \tag{2}$$

(a) Show that the random variable Y has the same PMF as the PMF of the target random variable X which the engineer was supposed to generate.

(b) (Matlab) Use the unifrnd$(0,1)$ function in Matlab to generate n random numbers uniformly distributed between 0 and 1. For each randomly generate number $U = u$, calculate the value taken by $Y = y$ using the relationship in equation (2). For each value of n, compute the estimated PMF for Y as follows:

$$\hat{P}_Y^{(n)}(y) = \begin{cases} \frac{\#\text{ of times } Y=0}{n}, & \text{for } y = 0 \\ \frac{\#\text{ of times } Y=1/2}{n}, & \text{for } y = 1/2 \\ \frac{\#\text{ of times } Y=1}{n}, & \text{for } y = 1. \end{cases} \tag{3}$$

In a single figure, plot the following quantities: PMF of Y and estimated PMF $\hat{P}_Y^{(n)}(y)$ for $n = 10, 50, 100$ and 1000. Comment upon your observations.

(You do not need to submit your Matlab code)

8. [20 points] Receiver of an optical communication system uses a photodetector that counts the number of photons arriving during one time unit. Suppose that if a signal is present (i.e., if bit 1 is transmitted), the number of arriving photons X can be modeled as a Poisson random variable with rate λ_1. In the absence of the signal (bit 0 is transmitted), it can be modeled as a Poisson random variable with rate λ_0 (with $\lambda_0 < \lambda_1$). Let p be the probability that the bit 1 was transmitted.

(a) What is the probability that a bit 1 was transmitted given that the photodetector detects k photons ?

(b) In order to decode the transmitted bit, the receiver implements the following rule (also known as Maximum Likelihood (ML) detection): if $P(\text{bit } = 1|X = k) > P(\text{bit } = 0|X = k)$ then decide that bit 1 was sent, otherwise bit 0. Find the corresponding (threshold) decision rule based on the value of X.

(c) What is the probability of error of this receiver ?