Solutions:

Midterm 1 Exam - ECE 503 Fall 2016

- Date: Monday, September 26, 2016.
- Time: 11:00 am -11:50 am (in class)
- Maximum Credit: 100 points

1. [25 points] A continuous valued random variable, X has the following PDF:

$$f_X(x) = \begin{cases} k_1 x + k_2 x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What condition must (k_1, k_2) satisfy so that $f_X(x)$ is a valid PDF ?

(b) Suppose that you are given that $P(X \leq 1/2) = 1/2$. Determine k_1 and k_2.

(c) Determine the CDF of X.

(a) $$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_0^1 (k_1 x + k_2 x^2)\,dx = \left[k_1 \frac{x^2}{2} + k_2 \frac{x^3}{3}\right]_0^1 = \frac{k_1}{2} + \frac{k_2}{3}$$

$\Rightarrow (k_1, k_2)$ must satisfy $\boxed{\frac{k_1}{2} + \frac{k_2}{3} = 1}$

(b) $$\frac{1}{2} = P\left(X \leq \frac{1}{2}\right) = \int_0^{1/2} f_X(x)\,dx = \left[k_1 \frac{x^2}{2} + k_2 \frac{x^3}{3}\right]_0^{1/2} = \frac{k_1}{8} + \frac{k_2}{24}$$

$$\Rightarrow \quad \frac{k_1}{4} + \frac{k_2}{12} = 1. \quad ; \quad \underbrace{\frac{k_1}{2} + \frac{k_2}{3} = 1}_{\Rightarrow \ \frac{k_1}{4} + \frac{k_2}{6} = \frac{1}{2}}$$

$$\Rightarrow \quad \frac{k_2}{6} - \frac{k_2}{12} = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$+\frac{k_2}{12} = -\frac{1}{2} \Rightarrow k_2 = -\frac{12}{2} = \boxed{-6} \Rightarrow \frac{k_1}{2} - \frac{6}{3} = 1$$

$$\frac{k_1}{2} = 1 + 2 \Rightarrow k_1 = 6$$

$$\boxed{k_1 = 6 \ ; \ k_2 = -6}$$

(c) CDF of $X \Rightarrow F_X(x) = P(X \leq x) = \int_0^x 6(\alpha - \alpha^2)\,d\alpha$ (for $0 < x \leq 1$)

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right] = 3x^2 - 2x^3$$

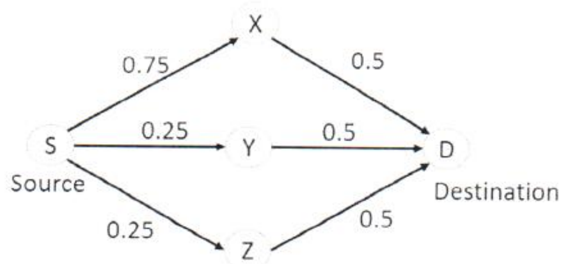
$$\boxed{F_X(x) = \begin{cases} 0 & x < 0 \\ 3x^2 - 2x^3 & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}}$$

2. [25points] A computer network connects a source (S) and a destination (D) through intermediate nodes X, Y, and Z as shown in the Figure below. For every pair of directly connected nodes, say i and j, the probability that the link from node i to node j is working is given by p_{ij}. These probabilities are shown in the figure. We assume that the link failures are independent of each other.

(a) What is the probability that all the paths from S to D fail ?

(b) What is the probability that there is exactly one working path connecting S to D ?

(c) What is the probability that there is at least one working path from S to D ?



3 total paths.

P_1: $S \to X \to D$ $\quad P_1 = \text{Prob}(\text{Path SXD is working}) = 0.75 \times 0.5 = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$

P_2: $S \to Y \to D$ $\quad P_2 = \text{Prob}(\text{SYD is working}) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$

P_3: $S \to Z \to D$ $\quad P_3 = \text{Prob}(\text{SZD " "}) = \frac{1}{8}$

(a) Probability that all paths fail $= (1-P_1)(1-P_2)(1-P_3)$

$$= \frac{5}{8} \times \frac{7}{8} \times \frac{7}{8} = \boxed{\frac{245}{512} = 0.4785}$$

(b) Probability that there is exactly one working path

$$= P_1(1-P_2)(1-P_3) + P_2(1-P_1)(1-P_3) + P_3(1-P_2)(1-P_1)$$

$$= \frac{3}{8} \times \frac{7}{8} \times \frac{7}{8} + \frac{1}{8} \times \frac{5}{8} \times \frac{7}{8} + \frac{1}{8} \times \frac{7}{8} \times \frac{5}{8} = \frac{147+35+35}{512}$$

$$\boxed{= \frac{217}{512} = 0.4238}$$

(c) Prob. that there is at least one working path

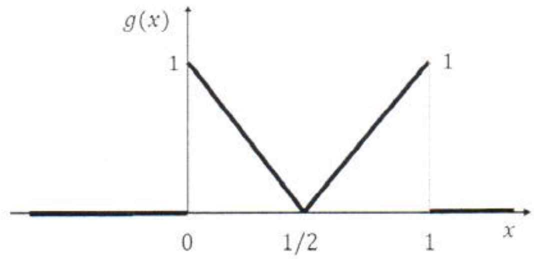
$= 1 - [\text{Prob there is No working Path}]$

$= 1 - [\text{Prob that all paths fail}]$

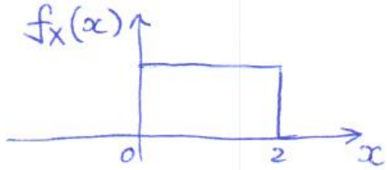
$$= 1 - \frac{245}{512} = \boxed{\frac{267}{512} = 0.5214}$$

25 CDF and PDF

3. [~~20~~ points] Let X be a uniform random variable in $[0, 2]$. Compute the ~~PDF~~ of the random variable $Y = g(X)$, where the function $g(.)$ is shown in the figure below.



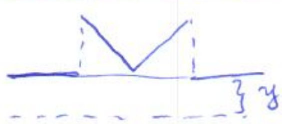
$X \sim \text{unif}\ [0,2]$



$Y = g(X)$.

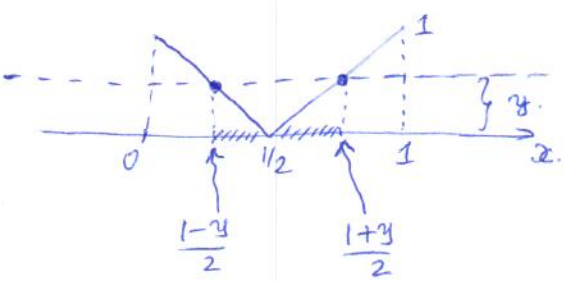
$F_Y(y) = P(Y \le y)$
$\quad = P(g(X) \le y)$.

Case 1 : if $y < 0$



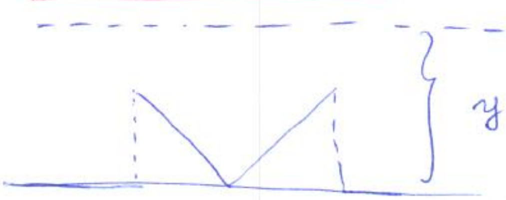
$\Rightarrow F_Y(y) = 0$.

Case 2 : if $0 \le y < 1$.

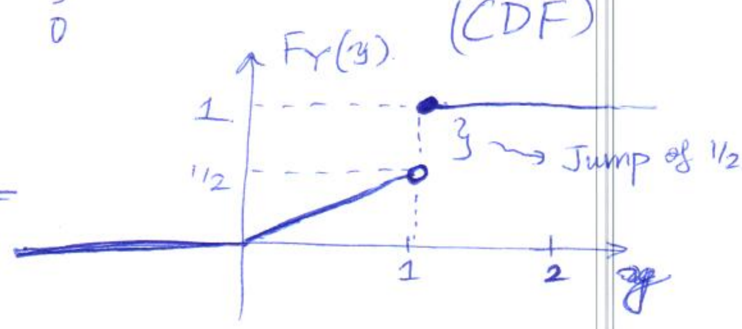


$$\Rightarrow F_Y(y) = P\left(\frac{1-y}{2} \le X \le \frac{1+y}{2}\right)$$
$$= \int_{\frac{1-y}{2}}^{\frac{1+y}{2}} f_X(x)\,dx = \int_{\frac{1-y}{2}}^{(1+y)/2} \frac{1}{2}\,dx$$
$$= \frac{1}{2}\left[\frac{1+y}{2} - \left(\frac{1-y}{2}\right)\right] = y/2.$$

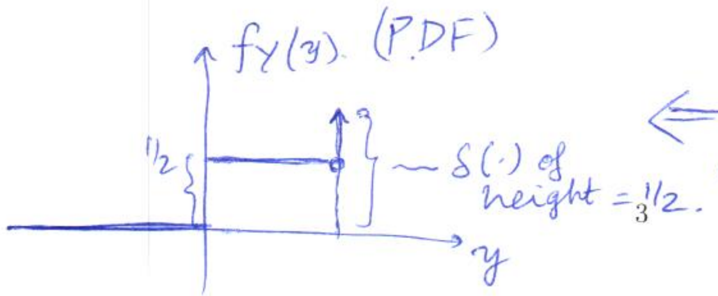
Case 3 : $1 \le y \le 2$.



$$F_Y(y) = \int_0^2 f_X(x)\,dx = 1.$$



$\Leftarrow$



4. [25 points] The random variable X models the duration of the call made by a typical cell phone user. Assume that X is distributed as an exponential random variable, with parameter $\lambda = 1$, i.e., the PDF of the call duration is

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Verizon and AT&T have different mechanisms of charging a user based on the call duration. Verizon uses the following charging plan (i.e., if the call duration is X, then $Y_{\text{Verizon}}(X)$ denotes the amount of money charged as a function of X):

$$Y_{\text{Verizon}}(X) = \begin{cases} 3X, & 0 \leq X \leq 1 \\ 5, & X > 1 \end{cases}$$

On the other hand, AT&T uses the following charging plan:

$$Y_{\text{AT\&T}}(X) = \begin{cases} 4X, & 0 \leq X \leq 1 \\ 4, & X > 1 \end{cases}$$

(a) Find the expected amount you will pay if you pick the Verizon plan, i.e., $E[Y_{\text{Verizon}}(X)]$.

(b) Find $E[Y_{\text{AT\&T}}(X)]$. Which one would you prefer ?

(a) $E[Y_{\text{Verizon}}(X)] = \int_{-\infty}^{\infty} Y_{\text{Verizon}}\, f_X(x)\,dx = \int_0^1 3x e^{-x}\,dx + \underbrace{\int_1^{\infty} 5 e^{-x}\,dx}_{=\frac{5}{e}}$

$= 3\left[\int_0^1 x e^{-x}\,dx\right] + 5/e$

Recall $\int u\,dv = uv - \int v\,du$

$\left.\begin{aligned} u &= x \\ dv &= -e^{-x}dx \\ v &= -e^{-x} \end{aligned}\right\} \Rightarrow \int x e^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} = -(1+x)e^{-x}$

$\int_0^1 x e^{-x}\,dx = -(1+x)e^{-x}\Big|_0^1 = -2e^{-1} - (-1) = +1 - \frac{2}{e}$

$E[Y_{\text{Verizon}}(X)] = 3\left(1 - \frac{2}{e}\right) + \frac{5}{e} = 3 - \frac{6}{e} + \frac{5}{e} = \boxed{3 - \frac{1}{e}}$

(b) $E[Y_{\text{AT\&T}}(X)] = 4\int_0^1 x e^{-x}\,dx + 4\int_1^{\infty} e^{-x}\,dx$

$= 4\left(1 - \frac{2}{e}\right) + \frac{4}{e} = \boxed{4 - \frac{4}{e}}$

Which one is better ??

⇓

Pick AT&T
(Lower expected Cost)

$3 - \frac{1}{e} \overset{?}{\gtrless} 4 - \frac{4}{e}$

$\frac{3}{e} \overset{?}{\gtrless} 1 \Rightarrow 3 \overset{?}{\gtrless} e = 2.718281$

$\Rightarrow$ cost of Verizon $\geq$ cost of AT&T