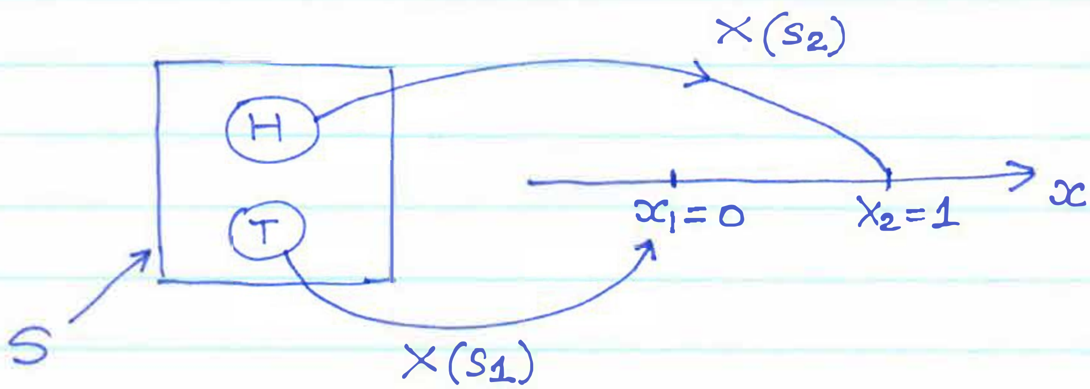
RANDOM VARIABLE

A random variable (r.v.) is a mapping of the outcomes of a random experiment to the set of real numbers. With such an association, we are able to use real number description to quantify items of interest.

We will first focus on Discrete r.v.'s, which take finite or countably infinite number of values. Then, we will discuss continuous r.v.'s.

$$\boxed{X : S \rightarrow R}$$ → Real Numbers

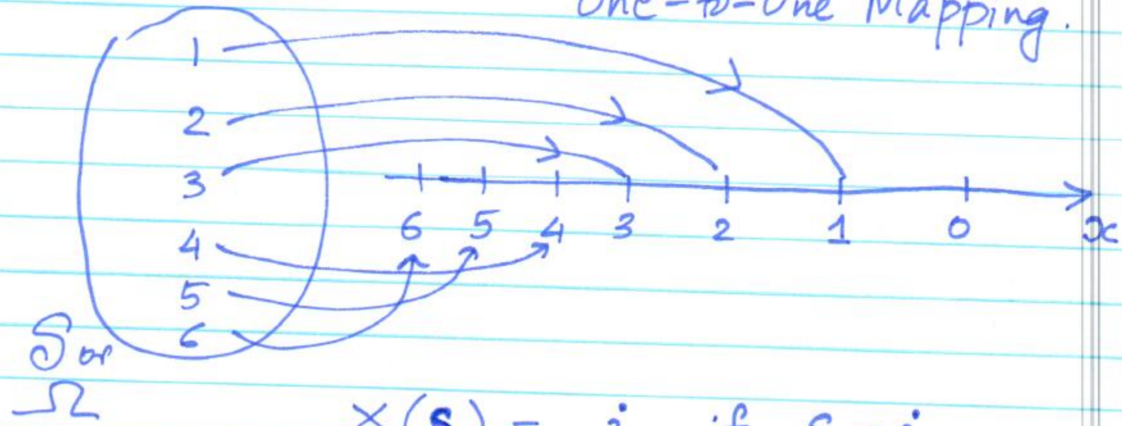
Coin Toss $\quad S = \{H, T\}$



$$X(s_i) = \begin{cases} 0 & s_1 = \text{tail} \\ 1 & s_2 = \text{head.} \end{cases}$$

Die Experiment

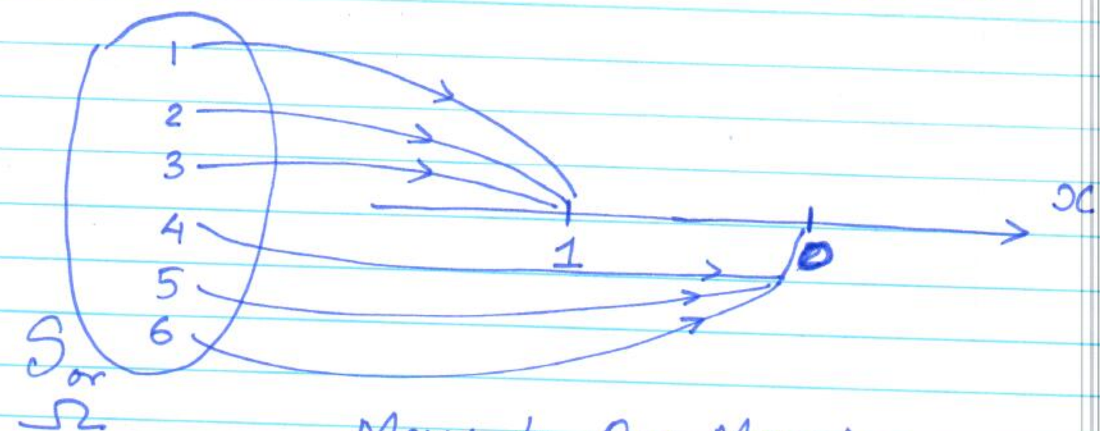
One-to-One Mapping.



$$X(s) = i \quad \text{if} \quad s = i$$

outcome.

s denotes the outcome



Many-to-One Mapping.

Eg → we are interested in the event that the outcome is ≤ 3.

$$X(s) = \begin{cases} 0 & \text{if } s > 3 \\ 1 & \text{if } s \leq 3. \end{cases}$$

Note that the <u>Mapping</u> which defines the r.v. is NOT RANDOM; the outcome of the Experiment is <u>RANDOM</u>.

<u>Eg</u>. $S = \{HH, HT, TH, TT\}$.

For any outcome $s \in S$,

$X(s) =$ Number of heads in s.

$\Rightarrow X(\cdot)$ takes values 0, 1 or 2

$$X(s) = \begin{cases} 0 & \text{if } s = TT \\ 1 & \text{if } s = HT \text{ or } TH \\ 2 & \text{if } s = HH. \end{cases}$$

We often suppress the dependence $X(\cdot)$ and denote the Random Variable by X

$x \rightarrow$ denotes a value or instance of the random variable X

Eg: $X, Y, Z \rightarrow$ r.v.'s

$x, y, z \rightarrow$ instance or values taken by the r.v.'s.

Probability Mass Function (PMF)

(discrete)

PMF is the probability that the random variable X takes on the value x, for each possible x.

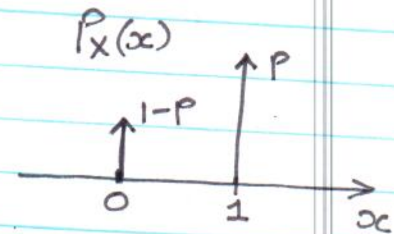
$$\boxed{PMF \Rightarrow P_X(x) = P(X = x)}$$

Eg: Coin Toss Example. $S = \{H, T\}$

$$X = \begin{cases} 0, & s = T \\ 1, & s = H \end{cases}$$

$P(H) = p$
$P(T) = 1-p$

$$P_X(0) = P(X=0) = 1-p$$
$$P_X(1) = P(X=1) = p$$

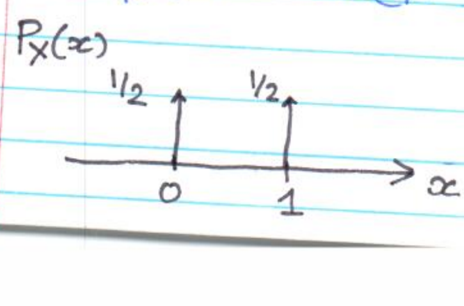


Eg: Die Toss. $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$X = \begin{cases} 0 & \text{if } s = 1, 3, 5 \\ 1 & \text{if } s = 2, 4, 6 \end{cases}$$

$$P_X(0) = P(X=0) = \sum_{s=1,3,5} P(s) = \frac{3}{6} = \frac{1}{2}$$

$$P_X(1) = P(X=1) = \sum_{s=2,4,6} P(s) = \frac{3}{6} = \frac{1}{2}.$$



Properties of PMF (Probability Mass Function).

Property 1:
Range of Values

$0 \le P_X(x) \le 1$ for all x

Property 2:
Sum of Values

If the random variable X takes finite number of values, say M,

$$\sum_{i=1}^{M} P_X(x_i) = 1$$

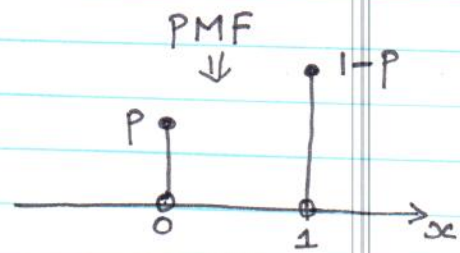
If the random variable X takes countably infinite values,

$$\sum_{i=1}^{\infty} P_X(x_i) = 1$$

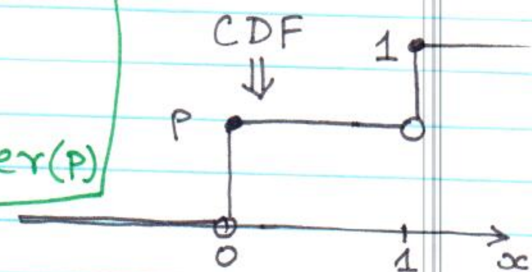
Commonly Encountered Discrete R.V.'s and their PMF(s)

Bernoulli

PMF $\Rightarrow$ $P_X(x) = \begin{cases} p & \text{if } x = 0 \\ 1-p & \text{if } x = 1. \end{cases}$



Shorthand Notation
$X \sim Ber(p)$
"X is distributed according to Ber(p)



Binomial

r.v. takes $\cancel{n}$ $(n+1)$ values

PMF $\Rightarrow$ $P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k = \underbrace{0, 1, 2, \dots, n}_{(n+1) \text{ values}}$

Shorthand Notation: $X \sim Bin(n, p)$

Geometric Random Variable

PMF $\Rightarrow$ $P_X(k) = (1-p)^{k-1} p$ for $k = 1, 2, 3, \dots\dots \infty$

Shorthand Notation: $X \sim Geom(p)$

Poisson Random Variable

$e = 2.71828\dots \rightarrow \exp(1)$

PMF $\Rightarrow$ $P_X(k) = e^{-\lambda} \cdot \dfrac{\lambda^k}{k!}$, for $k = 0, 1, 2, \dots \infty$

Shorthand: $X \sim Poi(\lambda)$ or $Pois(\lambda)$ or $Poisson(\lambda)$

Verification of properties of PMF

* Consider the Binomial r.v.

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots, n$$

$$\sum_{k=0}^{n} P_X(k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \left(p + (1-p)\right)^n$$

$$= 1^n = 1.$$

* Consider Geometric r.v.

$$P_X(k) = (1-p)^{k-1} p, \quad k = 1, 2, 3, \ldots, \infty$$

$$\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

$$= p \left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots \right]$$

Recall, the sum of first n terms of a geometric series:

$$1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = \frac{1-\alpha^n}{1-\alpha} \quad \text{[Why ??]}$$

for $\alpha \neq 1$

$\Rightarrow$

$$1 + \alpha + \alpha^2 + \ldots\ldots = \sum_{k=0}^{\infty} \alpha^k = \frac{1}{1-\alpha}$$

for $|\alpha| < 1$

$\Rightarrow$ Returning to Geometric r.v.

$$\sum_{k=1}^{\infty} P_X(k) = p\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \ldots\ldots\right]$$

$$= p \times \frac{1}{(1-(1-p))}$$

$$= \frac{p}{p} = 1.$$

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* Consider the Poisson r.v.

$$P_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, 3, \ldots\ldots, \infty$$

$$\Rightarrow \sum_{k=0}^{\infty} P_X(k) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda}\Big(\underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}\Big)$$

$$= e^{-\lambda} \times e^{\lambda} \quad \left[\text{Why ???}\right]$$

$$= 1$$

Taylor series Expansion