Midterm1

Sunday, October 4, 2020 11:56 AM

Safwan Elmadani

Midterm1-ECE503...

Midterm 1 Exam - ECE 503 Fall 2020

- Due Date and Time: Monday, Oct. 5, 2020, by Noon.
- Submit your answers on D2L.
- Maximum Credit: 100 points

1. [25 points]

   (a) (5 points) Mutually exclusive events are always independent. (True or False ?)

   (b) (10 points) Six cards are drawn at random (with replacement) from a deck of 52 cards. What is the probability that there are at least two Aces?

   (c) (10 points) Let X be a discrete random variable with the following PMF:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{(n-k)}, \quad k = 0, 1, \ldots, n$$

   Compute the mean of X.

(a) False.

(b)

E: at least two Aces

$$P(E) = 1 - P(\text{no Aces}) - P(1 \text{ Ace})$$

$$= 1 - \left(\frac{48}{52}\right)^6 - \left(\frac{4}{52}\right) \cdot \left(\frac{48}{52}\right)^5$$

$$\approx 0.330$$

(c) $$E[X] = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \cdot \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} k \frac{n!}{k(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$, chang variables

let $x = k-1 \Rightarrow k = x+1$

$$= \sum_{x=0}^{n-1} \frac{n!}{x!\,(n-x-1)!} p^{x+1} (1-p)^{n-x-1}$$

let $m = n-1 \Rightarrow n = m+1$

$$= \sum_{x=0}^{m} \frac{(m+1)\,m!}{x!\,(m-x)!} p^{x+1} (1-p)^{m-x}$$

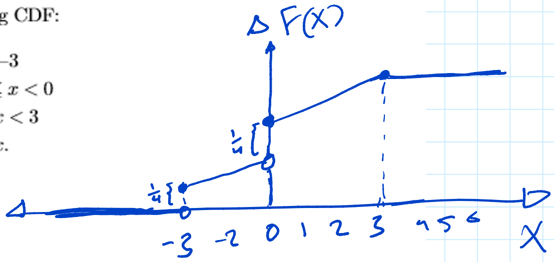
$$= (m+1)\,p \underbrace{\sum_{x=0}^{m} \frac{m!}{x!\,(m-x)!} p^{x} (1-p)^{m-x}}_{=\underbrace{(p+(1-p))^n}_{=1}}$$

using binomial theorm

$$\mu_x = E[X] = np$$

2. [25 points] Let X be a random variable with the following CDF:

$$F_X(x) = \begin{cases} 0, & x < -3 \\ \frac{x}{12} + \frac{1}{2}, & -3 \le x < 0 \\ \frac{x}{12} + \frac{3}{4}, & 0 \le x < 3 \\ 1, & 3 \le x. \end{cases}$$



(a) Find $P(X = -2), P(X = 0)$ and $P(0 < X \le 2)$

(b) Find $P(X \le 2 | X > -1)$.

(c) If $Y = X^2$, find the CDF of the random variable Y.

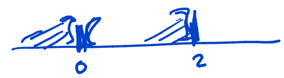
(a) $P(X=-2) = F_X(-2) - F_X(-2^-)$

$= 0$

$P(X=0) = F_X(0) - F_X(0^-)$

$= \frac{3}{4} - \frac{1}{2}$

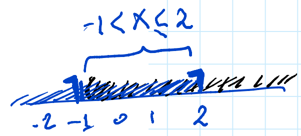
$= \frac{1}{4}$



$P(0 < X \le 2) = F_X(2) - F_X(0)$

$= \left(\frac{2}{12} + \frac{3}{4}\right) - \left(\frac{3}{4}\right) = \frac{1}{6}$

(b) $P(X \le 2 | X > -1) = \frac{P(X \le 2, X > -1)}{P(X > -1)}$



$= \frac{P(-1 < X \le 2)}{1 - P(X \le -1)}$

$= \frac{F_X(2) - F(-1)}{1 - F_X(-1)}$

$= \frac{\left(\frac{2}{12} + \frac{3}{4}\right) - \left(\frac{-1}{12} + \frac{1}{2}\right)}{1 - \left(\frac{-1}{12} + \frac{1}{2}\right)}$

$= \frac{6}{7} = 0.857$

(c) $Y = X^2$ $\qquad 0 \le y \le 9$

(c) $Y = X^2$, $0 \leq y \leq 9$

$F_Y(y) = 0$, $y < 0$

For $y \geq 0$, $F_Y(y) = P(Y \leq y)$

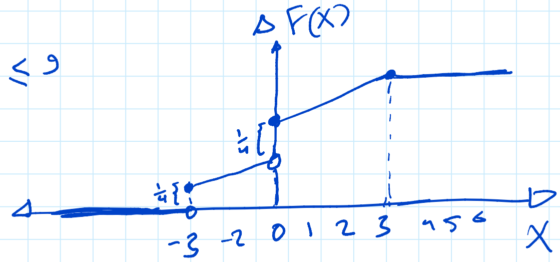
$$= P(X^2 \leq y)$$

$$= P(-\sqrt{y} \leq X \leq \sqrt{y})$$

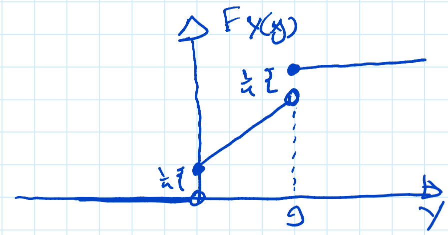
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

$$F_X(\sqrt{y}) = P(X \leq \sqrt{y}) = \frac{\sqrt{y}}{12} + \frac{3}{4}$$

$$F_y(-\sqrt{y}) = P(X \leq -\sqrt{y}) = \frac{-\sqrt{y}}{12} + \frac{1}{2} \quad , \quad 0 \leq y \leq 9$$



$$F_Y(y) = \begin{cases} 0 & , \quad y < 0 \\ \left[\frac{\sqrt{y}}{12} + \frac{3}{4}\right] - \left[\frac{-\sqrt{y}}{12} + \frac{1}{2}\right] & , \quad 0 \leq y < 9 \\ 1 & , \quad y \geq 9 \end{cases}$$



3. [25 points] To sign up for a new COVID-19 contact tracing app, users are asked to pick a password of length 8, with the following guidelines. The password must have

- exactly 4 upper-case letters (can be chosen with replacement) from $\{A, B, \cdots, Z\}$
- exactly 2 lower-case letters (can be chosen with replacement) from $\{a, b, \cdots, z\}$
- exactly 2 special characters (chosen without replacement) from the following list $\{\#, \$, \%, \&, !, @\}$

(a) How many distinct passwords are possible?

(b) Suppose there are N users that sign up for the app. Each user independently picks a valid password at random. What is the probability that none of the users share the same password?

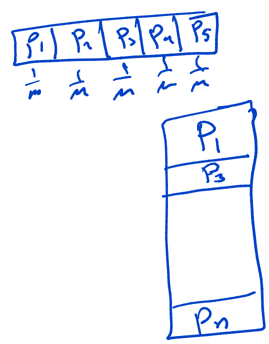
(a) # of distinct passwords $= [26^4 \cdot 26^2 \cdot 6 \cdot 5] = M$

(b) $\boxed{N}$ user signed up

$\boxed{N}$ users means there are N password at the time.

A: none of users share the same password

U_i: event that user i chooses unique password



$P(U_1) = 1$

$P(U_2) = \frac{M-1}{M}$

$P(U_3) = \frac{M-2}{M}$

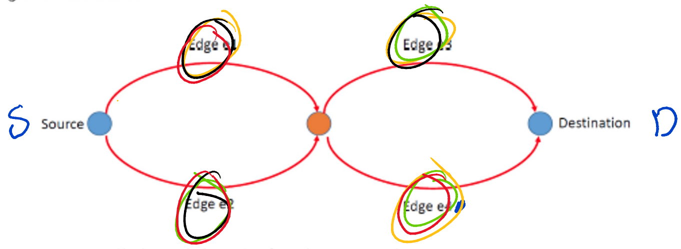
$P(U_4) = \frac{M-3}{M}$

$\vdots$

$P(U_N) = \frac{M-(N-1)}{M}$

$$P(A) = P(U_1 \cap U_2 \cap U_3 \ldots \cap U_N)$$
$$= P(U_1) \cdot P(U_2) \ldots P(U_N)$$
$$= \frac{M}{M} \cdot \frac{M-1}{M} \cdot \frac{M-2}{M} \cdots \frac{M-(N-1)}{M}$$
$$= \frac{M!}{M^N (M-N)!}$$

4. [25 points] Consider a source (S) and a destination (D) connected through the network shown in the figure below. A path from S to D is defined as a sequence of edges that connect S to D. For instance, the path $P_{1,3} = e_1 \to e_3$ is a valid path that can allow data transfer from S to D. Each edge in the network is functional independently with probability p (and does not work with probability $1-p$). In order to send data from S to D, one needs a working path, i.e., a path with all functional edges. For instance, the path $P_{1,3}$ is a working path only if both the edges e_1 and e_3 are functional.



(a) Enumerate all the valid paths for this network.

(b) What is the probability that there is at least one working path from S to D?

(c) What is the expected number of working paths?

$\begin{cases} \text{work}, & p \\ \text{not work}, & (1-p) \end{cases}$

(a) $P_{1,3},\ P_{1,4},\ P_{2,4},\ P_{2,3}$

(b) E: at least one working path

let X be r.v. representing # of working path

$$X \to \begin{cases} 1, & 4p^2(1-p)^2 \\ 2, & 4p^3(1-p) \\ 4, & p^4 \end{cases}$$

$X \neq 3$, because 2 edges gives us 1 path (not e_1e_2 and e_3e_4)
3 " " 2 paths
4 " " 4 paths

$$P(E) = P(X \geq 1) = P(X=1) + P(X=2) + P(X=4)$$
$$= 4p^2(1-p)^2 + 4p^3(1-p) + p^4$$
$$= p^2(p^2 - 4p + 4)$$
$$= p^2(p-2)^2$$

$$= p^2(p^2 - 4p + 4)$$

$$= p^2(p-2)^2$$

$$E[X] = \sum_i x \, P(X = x)$$

$$= 1\,(4p^2(1-p)^2) + 2\cdot(4p^3(1-p)) + 4p^4$$

$$E[X] = 4p^2$$