$\boxed{\text{Today}}$

1) Mean and Autocorrelation functions of a random process (R.P.)

2) Stationary Random Process
   Wide-Sense Stationary (WSS) Random Process
   Cyclo Stationary Random Process

$X(t) \longrightarrow$ Random Process

$\boxed{\text{First order}} \longrightarrow$ PDF of $X(t)$ $\forall$ time instants t.

$\boxed{\text{Second Order}} \longrightarrow$ Joint PDF of $(X(t_1), X(t_2))$ $\forall$ pair of times (t_1, t_2).

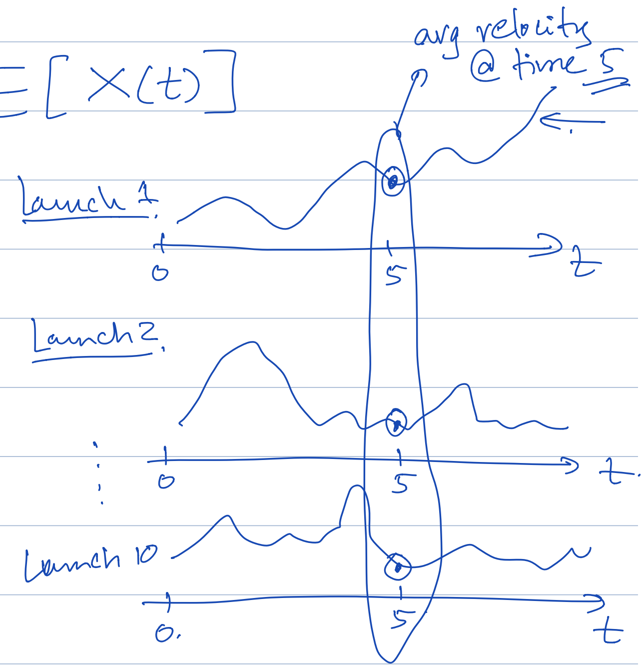
$(X(t_1) \dots X(t_k))$

Full description: joint distrib $\forall$ $t_1, t_2 \dots t_k$ & $\forall$ integer k.

Mean Function

$$\mu_X(t) = E[X(t)]$$

$\mu_X(5)$



Autocorrelation Function

$$R_X(t_1, t_2) = E[X(t_1)\,X(t_2)]$$

X, Y

Auto Covariance Function. $\quad \operatorname{cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$

$\downarrow$

$$\underbrace{C_X(t_1, t_2)} = \mathrm{Cov}\left(X(t_1)\, X(t_2) \right)$$

$$= E\left[\left(X(t_1) - \mu_X(t_1)\right)\left(X(t_2) - \mu_X(t_2)\right) \right]$$

$$\vdots$$

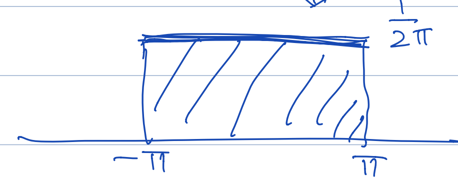
$$C_X(t_1, t_2) = R_X(t_1, t_2) - \mu_X(t_1)\,\mu_X(t_2)$$

Eg: Comm. system,

$$\boxed{X(t) = \cos(2\pi f t + \Theta)}\,, \quad \boxed{\Theta \sim \mathrm{unif}[-\pi, \pi]}$$

f: fixed freq.

$f_\Theta(\theta)$



$$\begin{cases} \mu_X(t) \overset{?}{=} \\ R_X(t_1, t_2) \overset{?}{=} \\ C_X(t_1, t_2) \overset{?}{=} \end{cases}$$

$$\mu_X(t) = E[X(t)] = E\left[\cos(2\pi f t + \Theta)\right]$$

$$= \int_{-\pi}^{\pi} \cos(2\pi f t + \theta) \cdot \boxed{f_\Theta(\theta)}\, d\theta$$

$$= \int_{-\pi}^{\pi} \cos(2\pi f t + \theta) \cdot \frac{1}{2\pi}\, d\theta = \boxed{0}$$

$$R_X(t_1, t_2) = E\left[X(t_1)\, X(t_2)\right]$$

$$= E\left[\cos(\underbrace{2\pi f t_1 + \Theta}_{A})\, \cos(\underbrace{2\pi f t_2 + \Theta}_{B})\right]$$

$\cos(A)\cos(B)$

$$= \frac{1}{2} E\left[\cos\left(2\pi f(t_1 + t_2) + 2\Theta\right) + \cos\left(2\pi f(t_1 - t_2)\right)\right]$$

$$= 0 + \frac{1}{2} \underbrace{E\left[\cos\left(2\pi f(t_1 - t_2)\right)\right]}$$

$$R_X(t_1, t_2) = \frac{1}{2} \cos(2\pi f(t_1 - t_2))$$

$$\underbrace{C_X(t_1, t_2)} = R_X(t_1, t_2) - \underbrace{\mu_X(t_1)}_{0}\underbrace{\mu_X(t_2)}_{0}.$$

$$= R_X(t_1, t_2).$$

If we have two random processes

$X(t)$, $Y(t)$.

<u>Cross-Correlation Function</u>

$$R_{X,Y}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$

<u>Cross-Covariance Function</u>

$$C_{X,Y}(t_1, t_2) = Cov(X(t_1)\, Y(t_2)).$$ simplify

$$\underbrace{\qquad\qquad} = R_{XY}(t_1, t_2) - \mu_X(t_1)\, \mu_Y(t_2)$$

$\underline{X(t)}$ → Stock prices over time.

$\underline{Y(t)}$ → Car sales over time.

Eg

→ $X(t) = \cos(2\pi f t + \theta_1)$

→ $Y(t) = \cos(2\pi f t + \theta_2)$

θ_1, θ_2 are $\boxed{\text{indep}}$, r.v.'s.
uniform $[-\pi, \pi]$.

$$R_{X,Y}(t_1, t_2) = E[X(t_1)\, Y(t_2)]$$

$$= E[\cos(2\pi f t_1 + \theta_1)\cos(2\pi f t_2 + \theta_2)]$$

indep of θ_1, θ_2.

$$= \underbrace{E[\cos(2\pi f t_1 + \theta_1)]}\, E[\cos(2\pi f t_2 + \theta_2)].$$

$$= 0 \times 0 = \boxed{0}$$

$$R_X(t_1, t_2) = R_X(t_2, t_1) \longleftarrow \text{(Symmetric)}$$

$$C_X(t_1, t_2) = C_X(t_2, t_1)$$

$$E[X(t_1)X(t_2)] = E[X(t_2)X(t_1)] = R_X(t_2, t_1)$$

Imp. Property

$$\boxed{|R_X(t_1, t_2)| \le \sqrt{E[X^2(t_1)]\,E[X^2(t_2)]}}$$

Proof: For any two R.V.'s. A, B

$$|(E[AB])| \le \sqrt{E[A^2]\,E[B^2]}$$

(Cauchy-Schwartz Ineq.)

$$\rho_{AB.} = \frac{Cov(A,B)}{\sigma_A \cdot \sigma_{B.}}$$

$$-1 \le \rho_{AB} \le 1 \implies \boxed{\rho_{AB}^2 \le 1.}$$

$$\frac{Cov(A,B)^2}{\sigma_A^2 \sigma_B^2} \le 1.$$

$$(Cov(A,B)^2) \le \sigma_A^2 \sigma_B^2.$$

$$(E[(A-\mu_A)(B-\mu_B)])^2 \le (E[A^2]-\mu_A^2)(E[B^2]-\mu_B^2)$$

$\vdots$

DIY (Try it yourself).

$$\left\{ \begin{aligned} |R_X(t_1,t_2)| &= \left|E[X(t_1)X(t_2)]\right| \\ &\le \sqrt{E[X^2(t_1)]\,E[X^2(t_2)]} \end{aligned} \right.$$

from Cauchy-Schwartz.

Classification of Random Processes

$X(t)$

Stationary R.P.

$\boxed{1^{st}\text{ order}}$ stationary if $X(t)$ & $X(t+\Delta)$ have the same PDF. for EVERY Δ

$\boxed{2^{nd}\text{ order stationary}}$ if $(X(t_1), X(t_2))$ & $(X(t_1+\Delta), X(t_2+\Delta))$ have the same joint distribution for EVERY Δ.

⋮

Strict Sense Stationary (SSS) R.P. $\Rightarrow$ if it is k^{th} order stationary for all integers k.

Wide Sense Stationary (Weak Sense ~~stationary~~ stationary. (WSS R.P.))

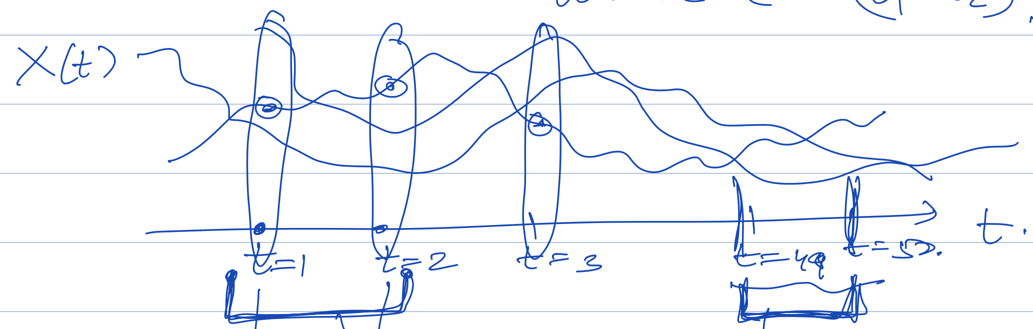
A R.P. $X(t)$ is WSS. if.

(1) $\mu_X(t) = \text{constant} = \mu_X$ is independent of time.

(2) $R_X(t_1, t_2) = R_X(t_1 - t_2)$. (ACF) is only depends on the time difference.

$= R_X(\tau)$

where $\tau = (t_1 - t_2)$.



length = 1

$R_X(1,2)$

$\Downarrow$

$\underbrace{E[X(1)\,X(2)]}$.

$R_X(49, 50)$

$= \underbrace{E[X(49)\,X(50)]}$.

If a R.P. is S.S.S. is it WSS? YES

$E[(X(t_1)X(t_2))] \leftarrow$

$\downarrow$

$E[(X(t_1+\Delta)\,X(t_2+\Delta))]$,

Same joint distrib.

If a R.P. is WSS, is it always S.S.S.? $\boxed{\text{NO}}$.

ACF of a WSS Random Process
(Auto correlation Function)

→ P1 → $\underline{R_X(0)} \geq 0$

→ P2 → $R_X(\tau) = R_X(-\tau)$

P3 → $R_X(0) \geq |R_X(\tau)| \;\; \forall \; \tau$

$$R_X(t_1, t_2) = R_X(\underbrace{t_1 - t_2}_{\text{time difference.}})$$

if X is a WSS

$$R_X(0) \overset{?}{=} R_X(t, t) = R_X(t-t)$$

$$= R_X(0)$$

$$= E[X(t) \cdot X(t)] = E[X^2(t)] \geq 0.$$

$$R_X(t_1, t_2) = E[X(t_1) X(t_2)] = R_X(t_1 - t_2)$$

$$= E[X(t_2) X(t_1)] = R_X(t_2 - t_1)$$

if $\tau = t_1 - t_2$ then $R_X(\tau) = R_X(-\tau)$

$X(t) \rightsquigarrow$ stock price of tesla.

$Y(t) \rightsquigarrow$ umbrella sales in Tucson

$$C_{X,Y}(t_1, t_2) = 0.$$

$$\mathrm{cor}(X(t_1)\, Y(t_2)) = 0 \quad \forall (t_1, t_2)$$

$X(t_1)$ & $Y(t_2)$ are uncorrelated R.V.'s $\forall$ (t_1, t_2)