Solved Problems on Joint Distribution & Conditional Expectation.

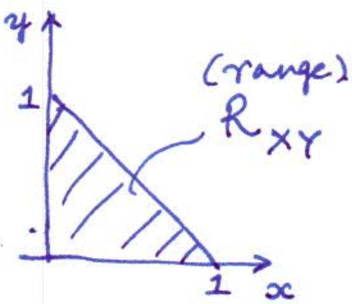
P1: Let (X, Y) have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx + 1 & x \geq 0,\ y \geq 0,\ x + y < 1 \\ 0 & \text{otherwise} \end{cases}$$

1. show the range of (X, Y).
2. Find the constant c.
3. Find the marginals $f_X(x)$ & $f_Y(y)$.
4. Find $P(Y < 2X^2)$.

Solution:

1.



$\left(\iint \text{Joint PDF} = 1 \right)$

2. $$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx\, dy = \int_0^1 \int_0^{1-x} (cx+1)\, dy\, dx$$

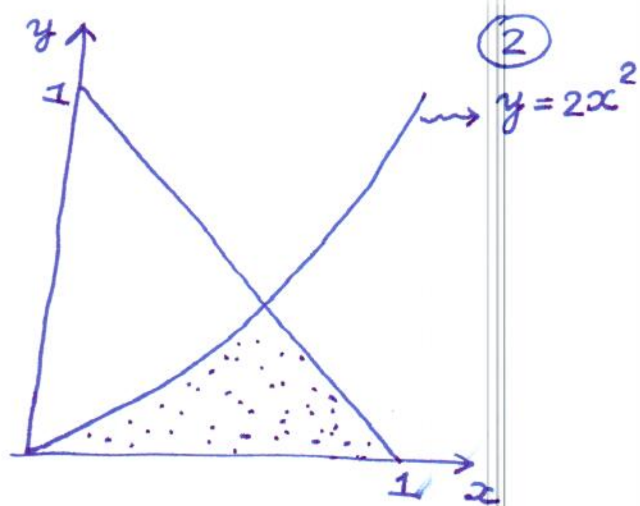
$$= \frac{c}{6} + \frac{1}{2}$$

$$\Rightarrow \boxed{c = 3}$$

3. $$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^{1-x} (3x+1)\, dy = (3x+1)(1-x)$$ for $x \in [0,1]$

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_{x=0}^{1-y} (3x+1)\, dx = \frac{(1-y)(5-3y)}{2}$$ for $y \in [0,1]$

4. To find $P(Y < 2X^2)$, we integrate $f_{X,Y}(x,y)$ over the "dotted" region.



$$P(Y < 2X^2) = \int_{-\infty}^{\infty} \int_{-\infty}^{2x^2} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{\min(2x^2,\, 1-x)} (3x+1)\, dy\, dx$$

$$= \int_{x=0}^{1} (3x+1) \cdot \min(2x^2, 1-x)\, dx$$

$$= \int_{x=0}^{1/2} 2x^2(3x+1)\, dx + \int_{x=1/2}^{1} (3x+1)(1-x)\, dx = \frac{53}{96}.$$

P2. Let X be a continuous r.v. with PDF

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

We know that given $X = x$, the r.v. Y is uniformly distributed in $[-x, x]$.

1. Find the joint PDF $f_{X,Y}(x,y)$.
2. Find the marginal $f_Y(y)$.
3. Find $P(|Y| < X^3)$.

Solution:

1. We are given: $$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2x} & -x \le y \le x \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x)$$

$$= \begin{cases} 1 & 0 \le x \le 1, \\ & -x \le y \le x \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & |y| \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

2. The r.v. Y takes values in $[-1, 1]$.

$$f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{x=|y|}^{1} 1 \cdot dx = 1 - |y|$$

$\Rightarrow$ Marginal of $Y \Rightarrow$ $$f_Y(y) = \begin{cases} 1 - |y| & |y| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

3. To find $P(|Y| < X^3)$, we can use the total prob. theorem: (4)

$$P(A) = \int_{x=-\infty}^{\infty} P(A \mid X = x) \cdot f_X(x)\, dx$$

Here: $A = |Y| < \cancel{x}^3$

$$\Rightarrow P(|Y| < X^3) = \int_{x=0}^{1} P(|Y| < X^3 \mid X = x) \cdot (2x)\, dx$$

$$= \int_{x=0}^{1} P(|Y| < x^3 \mid X = x) \cdot (2x)\, dx$$

Recall: $Y \mid X = x \sim \text{uniform } [-x, x]$

$$\Rightarrow P(|Y| < x^3 \mid X = x) = P(-x^3 < Y < x^3 \mid X = x)$$

$$= \frac{2x^3}{2x} = x^2$$

$$\Rightarrow P(|Y| < X^3) = \int_{x=0}^{1} (x^2) \cdot (2x)\, dx = 2 \int_0^1 x^3\, dx$$

$$= 2 \times \left. \frac{x^4}{4} \right|_0^1 = \frac{2}{4}$$

$$= \frac{1}{2}$$

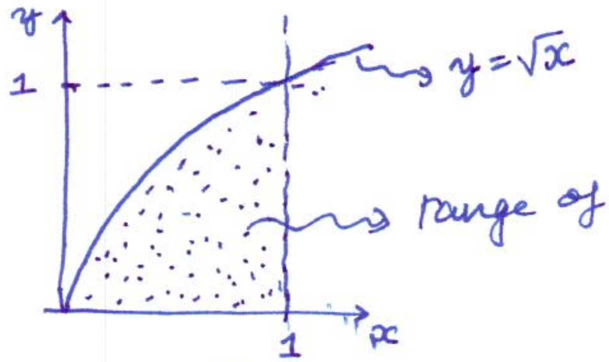
$$\Rightarrow \boxed{P(|Y| < X^3) = \frac{1}{2}}$$

P3: Let X & Y be r.v.'s with the joint PDF: (5)

$$f_{X,Y}(x,y) = \begin{cases} 6xy & 0 \le x \le 1 \\ & 0 \le y \le \sqrt{x} \\ 0 & \text{otherwise.} \end{cases}$$

1. Show the range of (X, Y).
2. Find $f_X(x)$ & $f_Y(y)$.
3. Are X & Y independent?
4. Find the conditional PDF of X given $Y=y$, i.e $f_{X|Y}(x|y)$
5. Find $E[X|Y=y]$, for $0 \le y \le 1$.
6. Find $Var[X|Y=y]$, for $0 \le y \le 1$.

1.



2. $$f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{y=0}^{\sqrt{x}} 6xy\,dy = 3x^2$$

$$\boxed{f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

$$f_Y(y) = \int_{x=y^2}^{1} 6xy\,dx = 3y(1-y^4)$$

$$\boxed{f_Y(y) = \begin{cases} 3y(1-y^4) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}}$$

3. $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$

$\Rightarrow X, Y$ are <u>NOT</u> independent.

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4. $$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \dfrac{2x}{1-y^4} & y^2 \le x \le 1 \\ & \text{otherwise.} \\ 0 & \end{cases}$$

5. $$E[X|Y=y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx = \int_{y^2}^{1} x \cdot \frac{2x}{1-y^4}\,dx$$

$$= \frac{2}{3}\frac{(1-y^6)}{(1-y^4)}$$

6. To find $Var[X|Y=y] = E[X^2|Y=y] - (E[X|Y=y])^2$

$$E[X^2|Y=y] = \int_{y^2}^{1} x^2 \cdot \frac{2x}{1-y^4} \cdot dx = \frac{1-y^8}{2(1-y^4)}$$

$$\Rightarrow Var(X|Y=y) = \frac{1-y^8}{2(1-y^4)} - \left(\frac{2}{3}\frac{(1-y^6)}{(1-y^4)}\right)^2$$

$$E[X|Y] = \frac{2}{3}\frac{(1-Y^6)}{(1-Y^4)} \rightsquigarrow$$ this is a Random Variable

this is a function of $\underline{\underline{Y}}$.

P4: A customer entering a bank picks one of n tellers with probability P_i, $i = 1, 2, \cdots, n$. The time taken by teller i to serve the customer is exponential r.v. with parameter λ_i.

1. Find the PDF of T, the time taken to service a customer.
2. Find $E[T]$
3. Find $Var[T]$

Solution: Let X be the "random teller" which Services the customes.

1.

$$X = \begin{cases} 1 & \text{w.p. } P_1 \\ 2 & \text{w.p. } P_2 \\ \vdots & \vdots \\ n & \text{w.p. } P_n \end{cases}$$

$\Rightarrow$

$$\underbrace{f_T(t)}_{\substack{\text{PDF} \\ \text{of } T}} = P_1 f_{T|X}(t|1) + P_2 \cdot f_{T|X}(t|2) + \dots + P_n f_{T|X}(t|n)$$

$$= P_1(\lambda_1 e^{-\lambda_1 t}) + P_2 (\lambda_2 e^{-\lambda_2 t}) + \dots + P_n \lambda_n e^{-\lambda_n t}$$

$$= \sum_{i=1}^{n} P_i \lambda_i e^{-\lambda_i t}$$

2.

$$\underbrace{E[T] = E_{(X)}[E[T|X]]}_{\text{Iterated Expectation Theorem}} = \sum_{i=1}^{n} P_i \cdot \underbrace{E[T|X=i]}$$

$T|X=i \sim exp(\lambda_i)$

$\Rightarrow E[T|X=i] = \frac{1}{\lambda_i}$

$$= \sum_{i=1}^{n} P_i \times \frac{1}{\lambda_i} = \sum_{i=1}^{n} \left(\frac{P_i}{\lambda_i}\right)$$

3. $Var(T) = E[T^2] - (E[T])^2$

$$= E[T^2] - \left(\sum_{i=1}^{n}\left(\frac{p_i}{\lambda_i}\right)\right)^2$$

To find $E[T^2] = E_{(X)}\left[E[T^2|X]\right]$ (again by Iterated Expectation Theorem).

$$= \sum_{i=1}^{n} p_i \underbrace{E[T^2|X=i]}_{= \frac{2}{\lambda_i^2}}$$

$$= \sum_{i=1}^{n} \frac{2p_i}{\lambda_i^2}$$

$$\Rightarrow \quad Var(T) = \sum_{i=1}^{n} \frac{2p_i}{\lambda_i^2} - \left(\sum_{i=1}^{n}\left(\frac{p_i}{\lambda_i}\right)\right)^2$$