HW 2- Solution

1.

We denote by B_1 and B_2 the balls in boxes 1 and 2, respectively. R and W are the sets of red and white balls. We have (assumption)

$$
\begin{aligned}
&p(B_1) = p(B_2) = 0.5 \\
&p(R|B_1) = 0.96 \quad p(R|B_2) = 0.2 \\
&p(W|B_1) = 0.04 \quad p(W|B_2) = 0.8
\end{aligned}
$$

Hence (Bayes' theorem)

$$
\begin{aligned}
(a) p(B_1|R) &= \frac{p(R|B_1)p(B_1)}{p(R|B_1)p(B_1) + p(R|B_2)p(B_2)}, \\
(b) p(B_2|W) &= \frac{p(W|B_2)p(B_2)}{p(W|B_1)p(B_1) + p(W|B_2)p(B_2)}.
\end{aligned}
$$

2.

(a)

$$p_1 = 1 - \left(\frac{5}{6}\right)^6 = 0.665$$

(b)

$$1 - \left(\frac{5}{6}\right)^{12} - \binom{12}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{11} = 0.619$$

(c) $$1 - \left(\frac{5}{6}\right)^{18} - \binom{18}{1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{17} - \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16} = 0.597$$

3.

(a)

$$
\begin{aligned}
& P(\text{A } \textit{occurs atleast twice in n trials}) \\
= \; & 1 - P(\text{A } \textit{never occurs in n trials}) - P(\text{A } \textit{occurs once in n trials}) \\
= \; & 1 - (1-p)^n - np(1-p)^{n-1}
\end{aligned}
$$

(b)

$$
\begin{aligned}
& P(\text{A } \textit{occurs atleast thrice in n trials}) \\
= \; & 1 - P(\text{A never occurs in } n \text{ trials}) - P(\text{A occurs once in } n \text{ trials}) \\
& -P(\text{A occurs twice in } n \text{ trials}) \\
= \; & 1 - (1-p)^n - np(1-p)^{n-1} - \frac{n(n-1)}{2} p^2 (1-p)^{n-2}
\end{aligned}
$$

4.

(a) $\binom{n-r+1}{r} \Big/ \binom{n}{r}$.

(b) $(r-1)\binom{n-r+1}{r-1} \Big/ \binom{n}{r}$.

(c) $1 \Big/ \binom{n}{r}$.

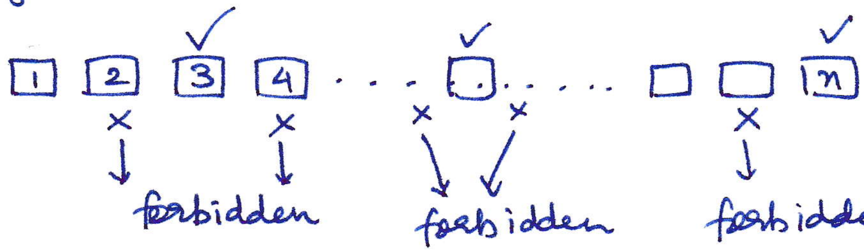
(d) $\dfrac{1}{r!}$.

5.

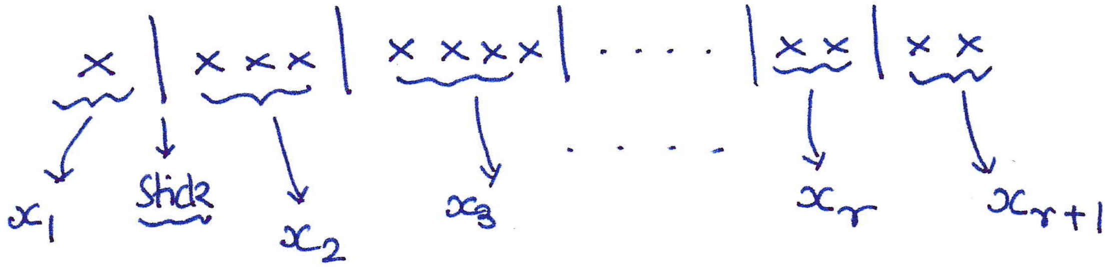
$$
\begin{aligned}
p(\text{at least one white ball}) &= 1 - p(\text{no white ball}) \\
&= 1 - \frac{\binom{n}{k}}{\binom{n+m}{k}}
\end{aligned}
$$

Detailed Solution to Problem #4

(a) # of ways there are <u>no</u> consecutive integers ??



$\Downarrow$ same as placing r sticks so that sticks have at least one space apart.



$x_1 \geq 0$
$x_2 \geq 1$
$x_3 \geq 1$
$\vdots$
$x_r \geq 1$
$x_{r+1} \geq 0$

and; $x_1 + x_2 \cdots + x_r + x_{r+1} + r = n.$

$\Rightarrow x_1 + x_2 + \cdots + x_r + x_{r+1} = (n-r)$

Let us make a change of variables.

$$y_1 = x_1 \qquad\qquad y_1 \geq 0$$
$$y_2 = x_2 - 1 \quad \text{so that} \quad y_2 \geq 0$$
$$y_3 = x_3 - 1 \qquad\qquad \vdots$$
$$\vdots$$
$$y_r = x_r - 1 \qquad\qquad y_r \geq 0$$
$$y_{r+1} = x_{r+1} \qquad\qquad y_{r+1} \geq 0$$

$\Rightarrow y_1 + (y_2+1) + (y_3+1) \cdots + (y_r+1) + y_{r+1} = n - r$

$\Rightarrow y_1 + y_2 \cdots + y_{r+1} = (n-r) - (r-1)$

$$\Rightarrow \boxed{\begin{array}{l} y_1 + y_2 \cdots + y_{r+1} = n - 2r + 1 \\ \text{and } y_1 \geq 0,\ y_2 \geq 0, \ldots, y_{r+1} \geq 0 \end{array}}$$

(2)

How many solutions for

$$x_1 + x_2 \cdots + x_k = n$$

if $x_1 \geq 0,\ x_2 \geq 0 \ \ldots\ x_k \geq 0$

[Recall Lec. 3-5]

$$\text{\# of sol}^{\text{n}}\text{s} \overset{?}{=} \binom{n+k-1}{k-1}$$

$\Rightarrow$ # of sol^{n}s for $y_1 + y_2 \cdots + y_{r+1} = n - 2r + 1$

s.t. $y_1 \geq 0, \ldots, y_{r+1} \geq 0$

$$= \binom{(n-2r+1) + (r+1-1)}{(r+1) - 1}$$

$$= \binom{n-r+1}{r}$$

Total # of ways to select r out of $n = \binom{n}{r}$

$$\Rightarrow \text{Prob}(\text{there are no consecutive integers}) = \frac{\binom{n-r+1}{r}}{\binom{n}{r}}$$

(b) $\text{Prob}\left(\begin{array}{c}\text{there is } \underline{\text{exactly}} \text{ one pair of} \\ \text{consecutive integers}\end{array}\right) = ?$

\# of ways there is exactly one pair of consec. integers

must be together. (view as a "single" bar)

$$\underbrace{\times}_{\substack{x_1 \\ \geq 0}} \,\|\, \underbrace{\times\times\times}_{\substack{x_2 \\ \geq 1}} \,|\, \underbrace{\times\times}_{x_3} \,|\, \underbrace{\times\times}_{x_4} \,|\, \cdots\cdots \underbrace{\times\times}_{\substack{x_{r-1} \\ \geq 1}} \,|\, \underbrace{\times\times}_{\substack{x_r \\ \geq 0}}$$

$\Rightarrow$ \# of ways $= ?$

$$x_1 + x_2 + \cdots + x_{r-1} + x_r + r = n$$

$$x_1 + x_2 \cdots + x_r = (n-r)$$

$$\Rightarrow y_1 + (y_2+1) + \cdots\cdots + (y_{r-1}+1) + y_r = n-r$$

$$\Rightarrow y_1 + y_2 \cdots + y_{r-1} + y_r = (n-r) - (r-2)$$
$$= n - 2r + 2$$

$$\text{\# of sol}^n\text{(s)} = \binom{(n-2r+2) + (r-1)}{r-1}$$

$$= \binom{n-r+1}{r-1}.$$

<u>However</u>, the "$\|$" bar can be arranged in $(r-1)$ ways

i.e first two #'s can be consec.
$2^{nd}/3^{rd}$ " "
$3^{rd}/4^{th}$ " " "
⋮
$(r-1)/r^{th}$ #'s " "

$\Rightarrow$ Total # of ways $= (r-1) \times \binom{n-r+1}{r-1}.$

4

$\Rightarrow$ Prob(there is exactly one pair of consec. intg.) $= \dfrac{(r-1)\binom{n-r+1}{r-1}}{\binom{n}{r}}$.

(c). P(Your choice of #'s is same as that of the Lottery)

$$= \frac{1}{\binom{n}{r}}.$$

(d) P(Numbers in L are drawn in an increasing order) $= \dfrac{1}{r!}$

Eg: suppose $n = 15$; $r = 3$

& lottery draws ~~10 17 19~~ 9 6 11

$\Rightarrow$ # of ways to draw $\{9, 6, 11\}$

$$\Rightarrow \left.\begin{matrix} \boxed{6 \quad 9 \quad 11} \\ 6 \quad 11 \quad 9 \\ 9 \quad 6 \quad 11 \\ 9 \quad 11 \quad 6 \\ 11 \quad 6 \quad 9 \\ 11 \quad 9 \quad 6 \end{matrix}\right\} r! \text{ ways}$$

Only one way to draw in an increasing order

$\Rightarrow$ Prob $= \dfrac{1}{r!}$

[Recall lecture 3-5]

Sol^n to Problem (6):

$$\text{Total \# of (all possible groups)} = \binom{16}{4,4,4,4} = \frac{16!}{4!\,4!\,4!\,4!}$$

Let's now focus on the event that each group has a grad student. We can do this in two stages:

(a) Take the four Grad students & distribute them to the four groups

$$\boxed{\text{Group 1}} \quad \boxed{\text{Group 2}} \quad \boxed{\text{Group 3}} \quad \boxed{\text{Group 4}}$$

$$\underset{\text{ways to choose}}{4} \times 3 \times 2 \times 1$$

$\Rightarrow$ $4!$ ways/choices for this stage.

(b) Now we take the remaining 12 students & distribute them to Groups 1, 2, 3, 4 (3 students per group)

$$\text{\# of ways to do this} = \binom{12}{3,3,3,3} = \frac{12!}{3!\,3!\,3!\,3!}$$

$$\Rightarrow \text{Total \# of ways each grp. has a grad} = (4!) \times \frac{12!}{3!\,3!\,3!\,3!}$$

$$\Rightarrow \text{Desired Probability} = \frac{\left(4! \times \frac{12!}{3!\,3!\,3!\,3!}\right)}{\left(\frac{16!}{4!\,4!\,4!\,4!}\right)} = \frac{12 \times 8 \times 4}{15 \times 14 \times 13} = \frac{384}{2730}$$

$$\boxed{\approx 0.14}$$

Sol^n to Prob (7)

(a) # of DNA Sequences $= 4^N$

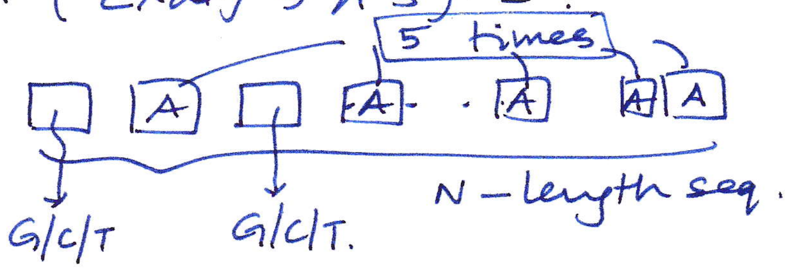
(b) $\Pr(\text{A appears at least once}) = 1 - P(\text{A does not appear at all})$

$$P(\text{A does not appear at all}) = \frac{\text{\# of DNA seq. comprised of } \underline{\text{only}} \{G, C, T\}}{\text{Total \# of DNA seq.}}$$

$$= \frac{3^N}{4^N}$$

$$\Rightarrow P(\text{A appears at least once}) = 1 - \left(\tfrac{3}{4}\right)^N.$$

(c) $P(\text{Exactly 5 A's}) = ?$



for a single nucleotide $\begin{cases} P(A) = 1/4 \\ P(G \text{ or } C \text{ or } T) = 3/4. \end{cases}$

We can view this as a "Bernoulli" trial

$N \rightarrow$ # of trials.

$A \rightarrow$ Success $\rightsquigarrow P(\text{success}) = 1/4$

$G/C/T \rightarrow$ Failure. $\rightsquigarrow P(\text{failure}) = 3/4.$

$$P(\text{Exactly 5 A's}) = P(\text{5 success \& } (N-5) \text{ failures})$$

$$= \binom{N}{5} (1/4)^5 (3/4)^{N-5}.$$

(d) if $P(A) = 1/2$; $P(G) = 1/4$
$P(C) = 1/8$
$P(T) = 1/8$

Repeat a, b, c

(a) $\longrightarrow$ same as before

(b) $P\left(\begin{matrix}\text{A appears at}\\ \text{least once}\end{matrix}\right) = 1 - P\left(\begin{matrix}\text{A does}\\ \text{not appear at all}\end{matrix}\right)$

$$P\left(\begin{matrix}\text{A does not}\\ \text{appear at all}\end{matrix}\right) = \binom{N}{0}(P(A))^0 (P(G \text{ or } C \text{ or } T))^{N-0}$$

$$= 1 \times \left(\tfrac{1}{2}\right)^0 \times (1/4 + 1/8 + 1/8)^{N-0}$$

$$= 1 \times 1 \times \left(\tfrac{1}{2}\right)^N = \left(\tfrac{1}{2}\right)^N.$$

(c) $P\left(\begin{matrix}\text{A appears exactly}\\ \text{5 times}\end{matrix}\right)$

$$= \binom{N}{5}(P(A))^5 (P(G|C|T))^{N-5}$$

$$= \binom{N}{5}\left(\tfrac{1}{2}\right)^5 \cdot \left(\tfrac{1}{2}\right)^{N-5} = \binom{N}{5}\left(\tfrac{1}{2}\right)^N.$$