Some important facts about variance

$$\boxed{\text{Var}(\text{constant}) = 0}$$

$$\text{Var}(X + \text{constant}) = \text{Var}(X)$$

$$\text{Var}((\text{constant}) \times X) = (\text{constant})^2 \times \text{Var}(X)$$

Quiz

not equal in general

$$\text{Var}(X_1 + X_2) \neq \text{Var}(X_1) + \text{Var}(X_2)$$

Counterexample

$$\begin{aligned}\text{Var}(X + X) &= \text{Var}(2X) \\ &= 2^2 \times \text{Var}(X) \\ &= 4\,\text{Var}(X) \neq \text{Var}(X) + \text{Var}(X)\end{aligned}$$

MOMENTS of a R.V.

$$m_n = E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)\,dx \qquad (n^{th} \text{ moment})$$

Central moments

$$\mu_n = E((X-\mu)^n) = \int_{-\infty}^{\infty} (x-\mu)^n f_X(x)\,dx$$

Absolute moments

$$E(|X|^n) \qquad E(|X-\mu|^n)$$

$$\mu_0 = E((X-\mu)^0) = 1$$

$$\mu_1 = E((X-\mu)) = E(X) - \mu = \mu - \mu = 0$$

$$\mu_2 = E((X-\mu)^2) = \sigma^2 \quad (\text{variance})$$

$$\mu_n = E((X-\mu)^n) = E\left\{ \sum_{k=0}^{n} \binom{n}{k} X^k (-\mu)^{n-k} \right\}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \underbrace{E(X^k)}_{= m_k} \cdot (-\mu)^{n-k}$$

Moments of Normal R.V.

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}} \quad \text{(zero mean)}.$$

$$E(X^n) = \begin{cases} 0, & n = 2k+1 \quad \text{(if } n \text{ is odd)} \\ 1\times 3\times \cdots \times (n-1)\times \sigma^n, & n = 2k \quad \text{(if } n \text{ is even)}. \end{cases}$$

Proof: For n = odd

$$E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)\,dx.$$

$\Rightarrow$ (integration of odd function from $-\infty$ to ∞) $\quad x^n f_X(x) = -(-x)^n f_X(-x)$ for n odd

$$= 0$$

For n = even

Recall, $$\int_{-\infty}^{\infty} e^{-\alpha x^2}\,dx = \sqrt{\frac{\pi}{\alpha}}$$

Take derivative w.r.t. α k times

$$\int_{-\infty}^{\infty} x^{2k} e^{-\alpha x^2}\,dx = \frac{1\times 3\times\cdots\times(2k-1)}{2^k}\cdot\sqrt{\frac{\pi}{\alpha^{2k+1}}}$$

Select $\alpha = 1/2\sigma^2$ and we get the moments.

Chebychev Inequality.

For any $\epsilon > 0$, The probability that x deviate from its m

$$\boxed{P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2}}$$

Proof: $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)\,dx$

$$= \int_{-\infty}^{\mu-\epsilon} (x-\mu)^2 f_X(x)\,dx + \overbrace{\int_{\mu-\epsilon}^{\mu+\epsilon} (x-\mu)^2 f_X(x)\,dx}^{\geq 0} + \int_{\mu+\epsilon}^{\infty} (x-\mu)^2 f_X(x)\,dx$$

$$\geq \int_{-\infty}^{(\mu-\epsilon)} \underbrace{(x-\mu)^2}_{\geq \epsilon^2 \text{ in the range of integral}} f_X(x)\,dx + \int_{\mu+\epsilon}^{\infty} \underbrace{(x-\mu)^2}_{\geq \epsilon^2 \text{ in the range of integral.}} f_X(x)\,dx.$$

$$\geq \epsilon^2 \times \left[\int_{-\infty}^{(\mu-\epsilon)} f_X(x)\,dx + \int_{\mu+\epsilon}^{\infty} f_X(x)\,dx \right]$$

$$= \epsilon^2 \left[P(X \leq \mu-\epsilon) + P(X \geq \mu+\epsilon) \right]$$

$$= \epsilon^2 \left[P(|X-\mu| \geq \epsilon) \right]$$

done.....

Markov Inequality

If $f_X(x) = 0$ for $x < 0$ (i.e, the r.v. X is ≥ 0).

then for any $\alpha > 0$,

$$\boxed{P(X \geq \alpha) \leq \frac{\mu}{\alpha}}$$

<u>Proof</u>: $\mu = E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$

$= \int_{0}^{\infty} x f_X(x)\,dx$ [since $f_X(x) = 0$ for $x < 0$]

$\cancel{\geq \int_{\alpha}^{\infty} \alpha f_X(}$

$\geq \int_{\alpha}^{\infty} x f_X(x)\,dx.$

$\geq \int_{\alpha}^{\infty} \alpha f_X(x)\,dx$

$= \alpha \cdot \left[\int_{\alpha}^{\infty} f_X(x)\,dx \right]$

$= \alpha \cdot P[X \geq \alpha]$

$\Rightarrow \boxed{P[X \geq \alpha] \leq \mu/\alpha}$

Characteristic Functions

Characteristic function of a r.v. X

$$\phi_X(\omega) = E\left[e^{j\omega X}\right], \quad j = \sqrt{-1}$$

Note that the function $g(x) = e^{j\omega x}$ is complex, however we can write it as

$$g(x) = e^{j\omega x} = \cos(\omega x) + j\sin(\omega x)$$

$$\Rightarrow \phi_X(\omega) = E\left(e^{j\omega X}\right) = E\left(\cos(\omega X)\right) + j\,E\left(\sin(\omega X)\right)$$

* for discrete valued r.v. X,

$$E\left(\cos(\omega X)\right) = \sum_i \cos(\omega x_i)\, P(X = x_i)$$

$$E\left(\sin(\omega X)\right) = \sum_i \sin(\omega x_i)\, P(X = x_i)$$

$$\Rightarrow \boxed{\phi_X(\omega) = \sum_i e^{j\omega x_i} P(X = x_i)}$$

~~if we take x~~

$$\boxed{\phi_X(\omega) = \sum_{k=-\infty}^{\infty} P_X(X=k) \cdot e^{j\omega k}}$$

(by relabeling X) and setting $P(X=k)=0$ for those k not included in range of X.

In this form, $\phi_X(\cdot)$ can be viewed as the (discrete) Fourier Transform of the sequence $P(X=k)$, $k \in (-\infty, \infty)$. The definition of $\phi_X(\cdot)$ has a slight difference from $\underbrace{\text{F.T.}}$ since it uses $e^{-j\omega k}$ (as opposed to $e^{j\omega k}$).

(Fourier Transform)

Similar to F.T., the $\phi_X(\cdot)$ is periodic with a period of 2π, i.e

$$\phi_X(\omega + 2\pi) = \phi_X(\omega)$$

$\Rightarrow$ We only need to understand the characteristic function over the interval $-\pi \le \omega \le \pi$, (or the fundamental period).

For a continuous valued r.v.;

$$\Phi_X(\omega) = E\left(e^{j\omega X}\right)$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\,dx.$$

Application of $\phi_X(\omega)$

Using Characteristic Function to find Moments of a r.v.

For discrete r.v. X,

$$\frac{d\,\phi_X(\omega)}{d\omega} = \frac{d}{d\omega} \sum_{k=-\infty}^{\infty} P(X=k)\cdot e^{j\omega k}$$

$$= \sum_{k=-\infty}^{\infty} P(X=k)\cdot \frac{d\,e^{j\omega k}}{d\omega}$$

$$= \sum_{k=-\infty}^{\infty} P(X=k)\cdot jk\cdot e^{j\omega k}$$

$$\Rightarrow \boxed{\left.\frac{1}{j}\frac{d\,\phi_X(\omega)}{d\omega}\right|_{\omega=0} = \sum_{k=-\infty}^{\infty} k \times P(X=k) = E(X).}$$

By repeatedly differentiation,

$$\frac{d^2 \phi_X(\omega)}{d\omega^2} = \sum_{k=-\infty}^{\infty} P(X=k)(jk)^2 e^{j\omega k}$$

$$\Rightarrow \quad \frac{1}{j^2} \frac{d^2 \phi_X(\omega)}{d\omega^2}\bigg|_{\omega=0} = \sum_{k=-\infty}^{\infty} k^2 P(X=k) = E(X^2)$$

& more generally,

$$\boxed{E(X^n) = \frac{1}{j^n} \frac{d^n \phi_X(\omega)}{d\omega^n}\bigg|_{\omega=0}}$$

Examples showing applications of $\underset{\sim}{\phi_X(\omega)}$

Eg 1. $\boxed{\text{First two moments of a Geometric r.v.}}$

$$P(X=k) = (1-p)^{k-1}p \quad , \quad k = 1, 2, 3, \ldots$$

$$\boxed{\phi_X(\omega) = \sum_{k=-\infty}^{\infty} P(X=k) \cdot e^{j\omega k}}$$

$$= \sum_{k=1}^{\infty} (1-p)^{k-1} p \, e^{j\omega k}$$

$$= p e^{j\omega} \sum_{k=1}^{\infty} \Big(\underbrace{(1-p)e^{j\omega}}_{Z = (1-p)e^{j\omega}}\Big)^{k-1}$$

~~Since $Z = |(1-p)e^{j\omega}|$~~

$$= p e^{j\omega} \sum_{k=1}^{\infty} Z^{k-1}$$

$$= p e^{j\omega} \sum_{k=0}^{\infty} Z^{k}$$

$$= p e^{j\omega} \frac{1}{1-Z} \quad \Big(\text{valid since } \underbrace{|Z|}_{(1-p)} < 1\Big)$$

$$= \frac{p e^{j\omega}}{1-(1-p)e^{j\omega}}$$

$$\boxed{\phi_X(\omega) = \frac{p}{e^{-j\omega} - (1-p)}}$$

$$\Rightarrow E(X) = \frac{1}{j}\frac{d\phi_X(\omega)}{d\omega}\bigg|_{\omega=0}$$

$$= \frac{1}{j}\frac{(-1)\times(-j)e^{-j\omega}\cdot p}{\left(e^{-j\omega}-(1-p)\right)^2}\bigg|_{\omega=0}$$

$$= \frac{e^{-j\omega}\cdot p}{\left(e^{-j\omega}-(1-p)\right)^2}\bigg|_{\omega=0}$$

$$= \frac{p}{\left(1-(1-p)\right)^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$E(X^2) = \frac{1}{j^2}\frac{d^2\phi_X(\omega)}{d\omega^2}\bigg|_{\omega=0}$$

$$= \frac{1}{j}\times\frac{d}{d\omega}\left(\frac{p\,e^{-j\omega}}{(e^{-j\omega}-(1-p))^2}\right)\bigg|_{\omega=0}$$

$$= \frac{p}{j}\left\{\frac{-j\,e^{-j\omega}}{D^2} - \frac{2}{D^3}\times(-j)e^{-j\omega}\right\}\bigg|_{\omega=0}$$

denote
$D = e^{-j\omega} - (1-p)$

$$= p\left\{\frac{2e^{-j\omega}}{D^3} - \frac{e^{-j\omega}}{D^2}\right\}\bigg|_{\omega=0}$$

$$E(X^2) = p\left\{\frac{2}{p^3} - \frac{1}{p^2}\right\} = \frac{2}{p^2} - \frac{1}{p} \Rightarrow Var(X) = \frac{2}{p^2} - \frac{1}{p} - \frac{1}{p^2} = \frac{(1-p)}{p^2}$$

$$P(X=k)=\binom{M}{k}p^k(1-p)^{M-k}$$
$k=0,1,2,\ldots,M$

Eg 2 Expected value of a BINOMIAL R.V.

$$\phi_X(\omega)=\sum_{k=-\infty}^{\infty}P(X=k)\,e^{j\omega k}$$

$$=\sum_{k=0}^{M}\binom{M}{k}p^k(1-p)^{M-k}\cdot e^{j\omega k}$$

$$=\sum_{k=0}^{M}\binom{M}{k}\cdot(\underbrace{pe^{j\omega}}_{a})^k\,(\underbrace{1-p}_{b})^{M-k}.$$

$$=(a+b)^M \qquad \left\{\text{from Binomial Theorem}\right\}.$$

$$=\left(pe^{j\omega}+(1-p)\right)^M$$

$$\Rightarrow\quad E(X)=\frac{1}{j}\left.\frac{d\phi_X(\omega)}{d\omega}\right|_{\omega=0}$$

$$=\frac{1}{j}\left.M\left(pe^{j\omega}+1-p\right)^{M-1}\times pje^{j\omega}\right|_{\omega=0}$$

$$=M(p+1-p)^{M-1}\times p$$

$$=Mp.$$

DIY $\Rightarrow$ Exercise $\Rightarrow$ Show that $Var(X)=Mp\times(1-p)$.

Eg 3

n^{th} Moment of the Exponential R.V.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_X(\omega) = \int_0^\infty e^{j\omega x} f_X(x)\,dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \cdot e^{j\omega x}\,dx$$

$$= \lambda \int_0^\infty e^{(-\lambda + j\omega)x}\,dx$$

$$= \lambda \int_0^\infty e^{-(\lambda - j\omega)x}\,dx$$

$$\boxed{\phi_X(\omega) = \frac{\lambda}{\lambda - j\omega}}$$

$$\frac{d\phi_X(\omega)}{d\omega} = \lambda(-1)(\lambda - j\omega)^{-2}(-j)$$

$$\frac{d^2\phi_X(\omega)}{d\omega^2} = \lambda(-1)(-2)(\lambda - j\omega)^{-3}(-j)^2$$

$$\vdots$$

$$\frac{d^n\phi_X(\omega)}{d\omega^n} = \lambda(-1)(-2)\cdots(-n)(\lambda - j\omega)^{-n-1}(-j)^n$$

$$= \lambda j^n n!\,(\lambda - j\omega)^{-n-1}$$

Hence,

$$E(X^n) = \frac{1}{j^n} \left.\frac{d^n\phi_X(\omega)}{d\omega^n}\right|_{\omega=0}$$

$$= \frac{1}{j^n} \cdot \lambda j^n n!\,\left.(\lambda - j\omega)^{-n-1}\right|_{\omega=0}$$

$$\Rightarrow \boxed{E(X^n) = \frac{n!}{\lambda^n}}$$