Today

1) Two random variables

2) Joint distribution, Conditional distribution & marginal distribution

3) Independence of two R.V.'s.

4) Functions of two R.V.'s.

Recap

$\underbrace{(X, Y)} \sim f_{X,Y}(x,y)$ — Joint PDF

X and Y are jointly distributed as $f_{X,Y}(x,y)$

$\boxed{\text{Marginal PDF of } X \Rightarrow}$ $f_X(x) = \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\,dy$

$\boxed{\text{Marginal PDF of } Y \Rightarrow}$ $f_Y(y) = \int_{x=-\infty}^{\infty} f_{X,Y}(x,y)\,dx$.

analogue for discrete X, Y as well

Conditional Distribution

Suppose X, Y are discrete valued:

$\underbrace{P(X = x, Y = y)}_{\boxed{\text{joint PMF}}} \longrightarrow$ described for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$.

$\{P(X = x \mid \underbrace{Y = y}) \longrightarrow$ $\boxed{\text{Conditional PMF of } X \text{ given } Y.}$

given an event. event corresp to $\{Y = y\}$.

$P(Y = y \mid X = x) \longrightarrow$ $\boxed{\text{Conditional PMF of } Y \text{ given } X.}$

$$\underbrace{P(X = x \mid Y = y)}_{\text{Condition of } X \text{ given } Y.} = \frac{P(X = x, Y = y)}{P(Y = y)} = \frac{\boxed{\text{Joint PMF}}}{\boxed{\text{Marginal of } Y.}}$$

$$P(X=x, Y=y) = P(Y=y) \times P(X=x \mid Y=y).$$

$$\boxed{\text{Joint PMF} = (\text{Marg of } Y) \times (\text{cond. of } X \text{ given } Y)}$$

$$= \underbrace{P(X=x) \times P(Y=y \mid X=x)}_{(\text{Marginal of } X) \times (\text{Cond. of } Y \text{ given } X)}$$

$$P(Y=y \mid X=x) = \frac{P(Y=y, X=x)}{P(X=x)}.$$

$$\boxed{\begin{array}{llll} \text{joint.} & & \text{marg} & \text{cond } Y \mid X \\ (X, Y) & \longrightarrow & (X) \times & (Y \mid X) \longleftarrow \\ & & \text{marg} & \\ & \longrightarrow & (Y) \times & (X \mid Y). \longleftarrow \end{array}}$$

For continuous r.v.'s.

$$f_{X,Y}(x,y) = \underbrace{f_X(x)}_{\text{marginal PDF of } X} \times \underbrace{f_{Y|X}(y|x)}_{\text{Conditional PDF of } Y \text{ given } \cancel{x}}.$$

$$= f_Y(y) \times f_{X|Y}(x|y).$$

Suppose we are given $f_{X,Y}(x,y)$.

Goal: compute $f_{X|Y}(x|y)$?

$$\underbrace{f_{X,Y}(x,y)}_{} \longrightarrow \text{compute } \underline{f_Y(y)}. \longrightarrow \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Compute: $P(-3 \leq X \leq 4 \mid Y=y)$

Compute : $P(-3 \le X \le 4) = \int_{-3}^{4} f_X(x)\,dx.$

Compute : $$\boxed{\begin{aligned} &P(-3 \le X \le 4 \mid Y=y) \\ &= \int_{x=-3}^{4} f_{X|Y}(x|y)\,dx \end{aligned}}$$

Compute : $E[X \mid Y=y]$

$X \to$ Temp

$Y \to$ Humidity.

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

$E[$Temp. $\mid$ Humidity $= 30\%]$

$$E[X \mid Y=y] = \int_{-\infty}^{\infty} x \times f_{X|Y}(x|y)\,dx$$

$$E[g(X) \mid Y=y] = \int_{-\infty}^{\infty} g(x)\, f_{X|Y}(x|y)\,dx$$

$$Var[X \mid Y=y] \stackrel{?}{=} E[X^2 \mid Y=y] - \left(E[X \mid Y=y]\right)^2$$

$$Var[X] = E[X^2] - (E[X])^2.$$

Eg: $$f_{X,Y}(x,y) = \begin{cases} \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{xy}{6}, & 0 \le x \le 1,\ 0 \le y \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1}$$

Conditional PDF of X given $Y=y$.

$$\underline{f_Y(y)} = \int_0^1 \left(\frac{x^2}{4} + \frac{y^2}{4} + \frac{xy}{6}\right) dx$$

Marginal of Y.

$$= \frac{3y^2+y+1}{12} \checkmark$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{3x^2+3y^2+2xy}{3y^2+y+1} \checkmark$$

$$\boxed{P(X < 1/2 \mid Y=y) = \int_0^{1/2} f_{X|Y}(x|y)\, dx}$$

$$= \frac{3/2\, y^2 + y/4 + 1/8}{3y^2 + y + 1}$$

$$\boxed{0 \le y \le 2}$$

Independent Random Variables

When are two events independent?
$\mathcal{E}_1, \mathcal{E}_2$

$$P(\mathcal{E}_1 \cap \mathcal{E}_2) = P(\mathcal{E}_1) \times P(\mathcal{E}_2)$$

X, Y are indep r.v.'s if $\{X \in A\}$ & $\{Y \in B\}$ are independent events for **every** A, B

$\{X \le x\}$ $\{Y \le y\}$. must be indep.

$$\underbrace{P(X \le x, Y \le y)}_{\text{Joint CDF of } X, Y} = \underbrace{P(X \le x)}_{\text{Marginal CDF of } X} \times \underbrace{P(Y \le y)}_{\text{Marg. CDF of } Y.}$$

$\dfrac{\partial^2}{\partial x \partial y}$.

$$\boxed{\underbrace{f_{X,Y}(x,y)}_{\text{Joint PDF}} = \underbrace{f_X(x)}_{\text{marginal PDF of } X} \times \underbrace{f_Y(y)}_{\text{marg PDF of } Y}}$$

$$f_{X,Y}(x,y) = f_X(x) \times f_{Y|X}(y|x)$$

if X, Y are independent. then.

$$f_{Y|X}(y|x) = f_Y(y).$$

conditional = marginal.

& $f_{X|Y}(x|y) = f_X(x).$

Q: suppose we are given $f_{X,Y}(x,y)$. Check if X, Y are independent or not?

$f_{X,Y} \to f_X$

$f_{X,Y} \to f_Y.$

Check if $f_{X,Y}(x,y) \overset{?}{=} f_X(x) \times f_Y(y)$

or if $f_{X|Y} \overset{?}{=} f_X$

or if. $\boxed{f_{Y|X} \overset{?}{=} f_Y}$

If X, Y are indep. r.v.'s then.

$Z = g(X)$ & $W = h(Y)$. are

also independent r.v.'s. (see the notes)

Eg: Suppose X & Y are both independent & uniform $[0,1]$ r.v.'s.

Find $P(X^3+Y>1)$.

$$f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y).$$

$$= \begin{cases} 1 & 0\le x\le 1,\ 0\le y\le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$P(\underbrace{X^3+Y>1}_{A})$$

$$P(A) = \int_{-\infty}^{\infty} P(A \mid X=x)\times f_X(x)\,dx.$$

$$= \int_0^1 P(x^3+Y>1 \mid X=x)\,dx$$

$$P(A \mid X = x)$$

$$= P(X^3 + Y > 1 \mid \underbrace{X = x}) \quad ??$$

$$= P(x^3 + Y > 1 \mid \underbrace{X = x})$$

$$= \boxed{P(Y > 1 - x^3 \mid \underline{X = x})}$$

$$\boxed{P(Y > a \mid X = b) = \int_{y=a}^{\infty} f_{Y|X}(y \mid b)\, dy.}$$

$$= \underbrace{P(Y > 1 - x^3)}_{Y \sim \text{unif}[0,1]} \quad \text{(Since } X \text{ \& } Y \text{ are independent)}$$

$$= x^3. \qquad \int_{1-x^3}^{1} 1\, dy = x^3.$$

$$P(A) = \int_{x=0}^{1} x^3\, dx = \frac{1}{4}.$$