Today: 1) Continuous R.V(s) {Examples: Gaussian, Exponential, Rayleigh, Laplace}

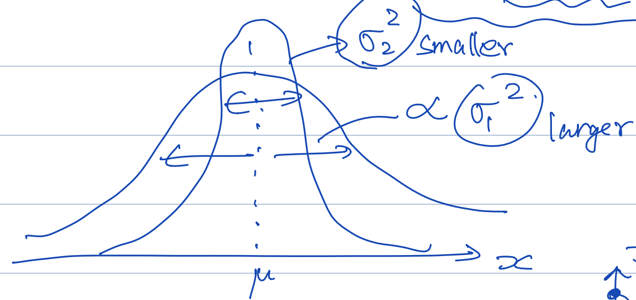
2) Mixed R.V.(s)

3) Conditional PDF & Conditional CDF

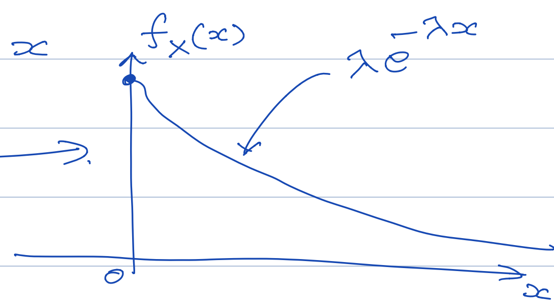
4) Continuous versions of → Total Probability Theorem, → Bayes' Theorem.

Gaussian r.v. $\mathcal{N}(\mu, \sigma^2) \longrightarrow$ using $\Phi(x) \to$ CDF of $\mathcal{N}(0,1)$.

$Q(x) = 1 - \Phi(x)$

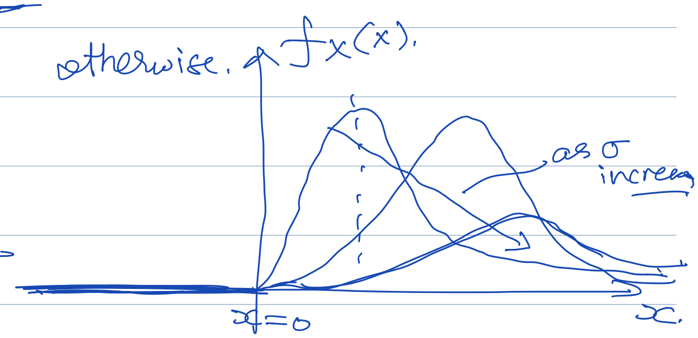


Exponential r.v. $\longrightarrow$



Rayleigh R.V.

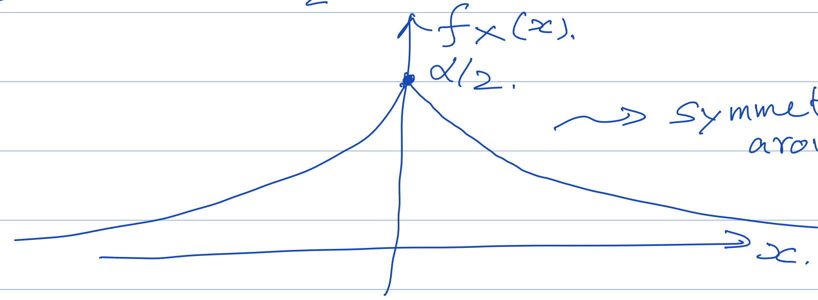
$$\underbrace{f_X(x)}_{\text{PDF}} = \begin{cases} \dfrac{x}{\sigma^2} e^{-x^2/2\sigma^2}, & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$



modeling the amplitude of a wireless signal.

Laplace R.V.

$$f_X(x) = \frac{\alpha}{2} e^{-\alpha|x|} \qquad |x| < \infty.$$



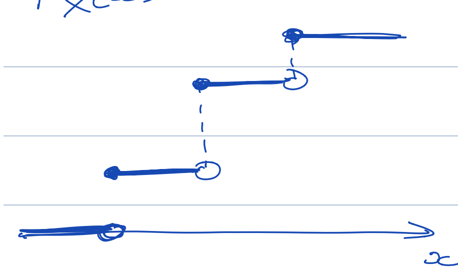
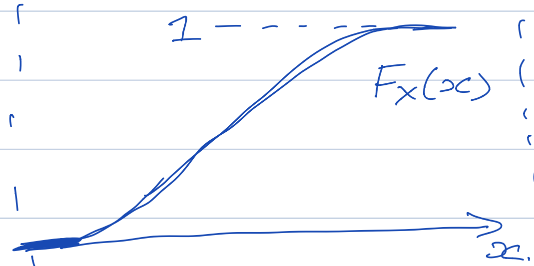
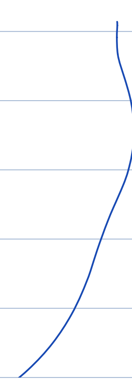
$\rightsquigarrow$ symmetric around origin

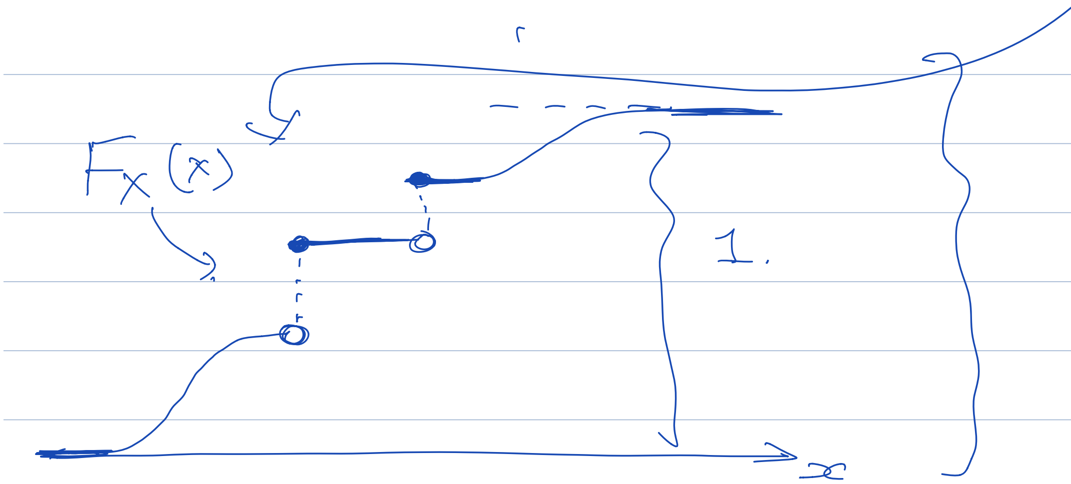
- modeling speech signals.
- in privacy preserving data release.

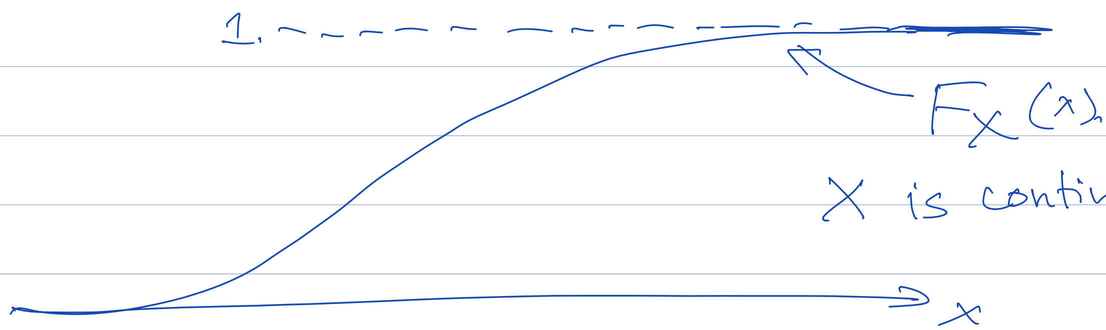
User $\rightarrow$ X $\longrightarrow$ reveal X but in a way "privacy" is preserved

reveal $\Rightarrow$ X + Noise. $\rightarrow$ laplacian noise

$\boxed{\text{Differential Privacy}}$ $\Longleftarrow$

Discrete R.V.	Continuous R.V.'s	Mixed R.V.
$F_X(x)$ 		
if $F_X(x)$ is Flat except with a finite # of discontinuities.	if $F_X(x)$ is continuous.	





X is contin.

$$F_X(x) = P(X \le x)$$

$$P(X=3) \overset{?}{=} 0$$

$$\boxed{P(X=3) \overset{?}{=} F_X(3) - F_X(3^-)}$$

since $F_X(\cdot)$ is a continuous func.

$$\boxed{\text{For a continuous r.v. } P(X=a) = 0.}$$

$\forall\, a$

$$X = \begin{cases} \mathcal{N}(0,1) & \text{if Heads} \\ 0 & \text{if Tails} \end{cases}$$

Goal: find CDF & PDF of X.

$$F_X(x) = P(\underbrace{X \le x}_{\mathcal{E}})$$

$$= P(\text{Heads}) \cdot P(X \le x \mid \text{Heads})$$

$$+ P(\text{Tails}) \cdot P(X \le x \mid \text{Tails})$$

Total Prob. Thm

$$= \frac{1}{2} P(X \le x \mid \text{Heads}) + \frac{1}{2} P(X \le x \mid \text{Tails})$$

$$= \frac{1}{2} \underbrace{\Phi(x)}_{\text{CDF of } \mathcal{N}(0,1)} + \frac{1}{2} P(X \le x \mid \text{Tails})$$

when tails $\longrightarrow$ $\boxed{X = 0}$

$x \to -5$ $\qquad P(\underbrace{X}_{0} \le -5 \mid \text{Tails}) \stackrel{?}{=} 0$

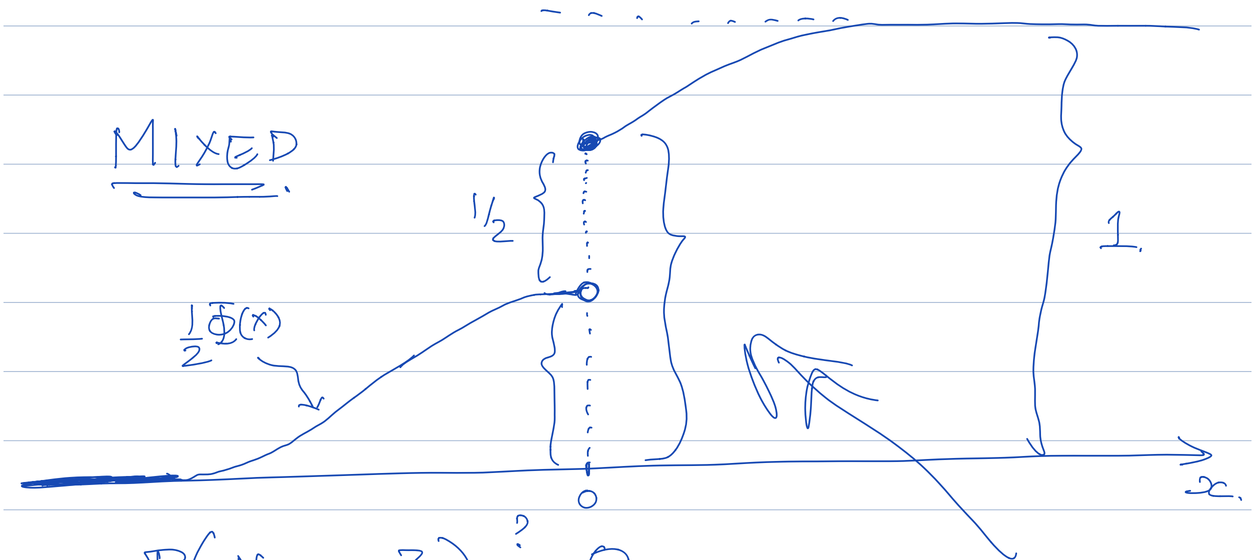
for any $x < 0$ $\qquad P(X \le x \mid \text{Tails}) = 0$

$x \to +3$ $\qquad P(\underbrace{X}_{0} \le +3 \mid \text{Tails}) \stackrel{?}{=} 1$

for any $x \ge 0$ $\qquad P(X \le x \mid \text{Tails}) = 1$.

$$\boxed{F_X(x) = \begin{cases} \frac{1}{2}\Phi(x) & x < 0 \\ \frac{1}{2} + \frac{1}{2}\Phi(x), & x \geq 0 \end{cases}}$$

MIXED



$P(X=-3) \stackrel{?}{=} 0$

$P(X=-1) \stackrel{?}{=} 0$

$\boxed{P(X=0)} \stackrel{?}{=} F_X(0) - F_X(0^-) = 1/2.$

$P(X=+1.5) \stackrel{?}{=} 0$

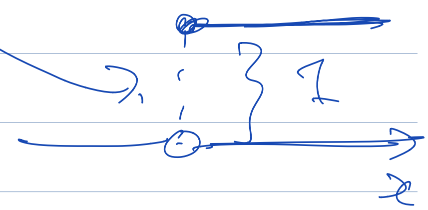
$\vdots$

Find PDF of this r.v.

$$f_X(x) = \frac{d}{dx} F_X(x).$$

$$F_X(x) = \frac{\Phi(x)}{2} + \frac{u(x)}{2}$$

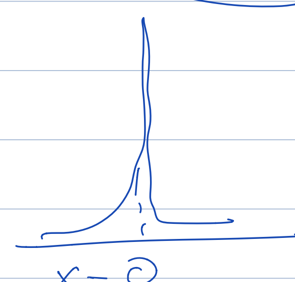
$$u(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$



$$\delta(x) \triangleq \frac{d}{dx} u(x)$$

Dirac delta function:

$\delta(x) = 0$ almost ev.



$$\int_{-\varepsilon}^{\varepsilon} \delta(t)\,dt = 1$$

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{1}{2} \frac{d}{dx} \Phi(x) + \frac{1}{2}\delta(x)$$

$$f_X(x) = \frac{1}{2} \times \frac{1}{\sqrt{2\pi}} e^{-x^2/2} + \frac{1}{2} \underbrace{\delta(x)}_{\text{delta function}}$$

Conditional Distributions

$$\Rightarrow P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A, B)}{P(B)}.$$

Conditional CDF of X given an event M

$$F_X(x \mid M) = P(X \leq x \mid M)$$

$$= \frac{P(X \leq x, M)}{P(M)}$$

Conditional PDF $\longrightarrow$ derivative of conditional CDF.

$$f_X(x \mid M) = \frac{d}{dx} F_X(x \mid M)$$

$M \Rightarrow$ Heads.

$$F_X(x \mid \text{Heads}) \stackrel{?}{=} \Phi(x).$$

$\boxed{F_X(x) \rightarrow \text{Mixed r.v.}}$

Eg 1: Suppose we are given a r.v. X & we want to find conditional CDF of X given that $\underbrace{X \le a}_{M}$.

$$F_X(x|M) = F_X(x \mid X \le a)$$

$$= P(X \le x \mid X \le a)$$

$$= \frac{P(X \le x,\ X \le a)}{P(X \le a)}$$

if $x < a$

$\{X \le x,\ X \le a\}$

$= \{X \le x\}$

$$F_X(x|X \le a) = \frac{P(X \le x)}{P(X \le a)}$$

$$= \frac{F_X(x)}{F_X(a)}.$$

if $x \ge a$

$\{X \le x,\ X \le a\}$

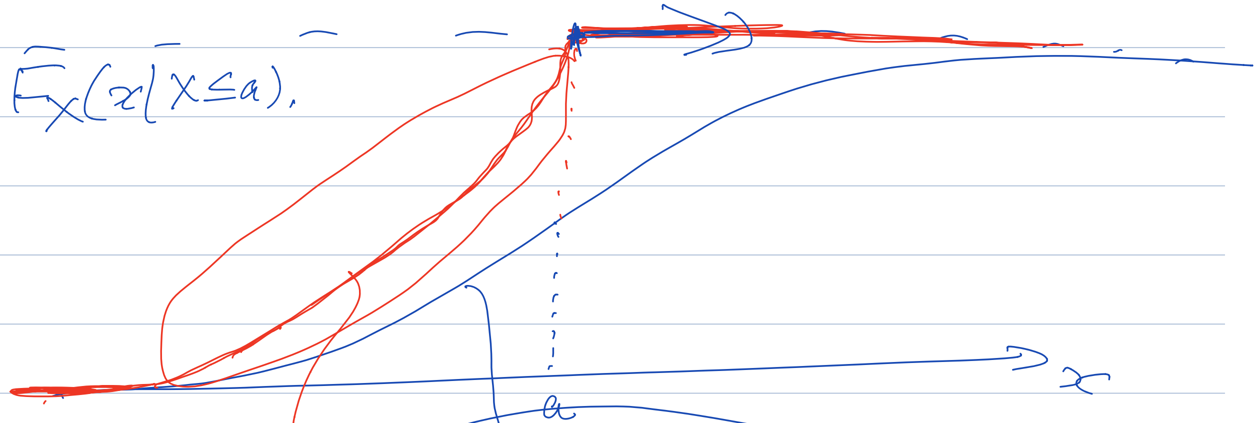
$= \{X \le a\}$.

$$F_X(x|X \le a) = \frac{P(X \le a)}{P(X \le a)}$$

$$= 1.$$

$$F_X(x / X \le a) = \begin{cases} \dfrac{F_X(x)}{F_X(a)} & \text{if } x < a \\ 1 & \text{if } x \ge a. \end{cases}$$

$F_X(x \mid X \le a)$.



$\dfrac{F_X(x)}{F_X(a)}$ $\quad F_X(x)$

$F_X(a) \rightarrow P(X \le a) \le 1$.

$\dfrac{F_X(x)}{F_X(a)} \le 1$.

for $x \le a$.