Homework 7 - ECE 503 Fall 2020

- Assigned on: Tuesday, November 3, 2020.
- Due Date: **Saturday, November 7, 2020 by 11:59 pm Tucson Time.**
- Maximum Credit: 120 points

1. [20 points] Let $U \sim \text{uniform}[0,1]$ denote a uniformly distributed random variable. We define a sequence of random variables as

$$X_n = \begin{cases} n^\alpha, & 0 \le U \le \frac{1}{n}, \\ 0, & \text{otherwise} \end{cases}$$

For what values of α does X_n

(a) converge to 0 in the mean square sense ?

(b) converge to 0 in probability ?

2. [10 points] Let X_n be a Laplacian random variable with zero mean and variance $2/n^2$ (i.e., scale parameter $1/n$). Show that X_n converges to 0 almost surely.

3. [10 points] Let X_n be a Rayleigh random variable with parameter $1/n$. Show that X_n converges to 0 almost surely.

4. [20 points] Let $f(x)$ be a probability density function. Let X_n have the density

$$f_n(x) = nf(nx)$$

Determine whether or not X_n converges in distribution to zero.

5. [20 points] $X_1, X_2, \ldots, X_n$ are i.i.d. uniform random variables, all with expected value $\mu_X = 7$, and variance $\text{Var}(X) = 3$.

(a) What is the PDF of X_1 ?

(b) What is $\text{Var}(M_{16}(X))$, i.e., the variance of the sample mean based on 16 trials ?

(c) What is $P(X_1 > 9)$, i.e., the probability that one outcome exceeds 9 ?

(d) Do you expect $P(M_{16}(X) > 9)$ to be bigger or smaller than $P(X_1 > 9)$? To check your intuition, use the Central Limit Theorem (CLT) to estimate $P(M_{16}(X) > 9)$.

6. [20 points] For an arbitrary random variable X, use the Chebyshev inequality to show that the probability that X is more than k standard deviations from its expected value $E(X)$ satisfies

$$P(|X - E[X]| \ge k\sigma) \le \frac{1}{k^2}$$

For a Gaussian random variable Y, use the $\Phi(\cdot)$ function to calculate the probability that Y is more than k standard deviations from its expected value $E[Y]$. Compare and comment upon your results to the upper bound obtained via Chebyshev inequality for $k = 1, \ldots, 5$.

7. [20 points] In this problem, we develop a weak law of large numbers for a *correlated sequence* $X_1, X_2, \ldots$ of identically distributed random variables. Each random variable X_i has expected value $E[X_i] = \mu$, and the random sequence has the covariance function given as:

$$\text{Cov}[X_i, X_j] = \sigma^2 a^{|i-j|}$$

where a is a constant such that $|a| < 1$. For this correlated random sequence, we define the sample mean as:

$$M(X_1, X_2, \ldots, X_n) = \frac{X_1 + X_2 + \ldots + X_n}{n}$$

Show that $M(X_1, X_2 \ldots, X_n)$ converges to μ in probability.