Lecture 11

MEAN & VARIANCE

Mean of a R.v. X or Expected Value of X.

$$\boxed{E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx}$$

(~~doned~~ denoted as μ or μ_X)

Recall, for a discrete valued r.v., the density can be written as

$$f_X(x) = \sum_i p_i\, \delta(x - x_i) \quad \text{where} \quad \underbrace{P(X = x_i) = p_i}_{\text{comes from PMF}}$$

also,

$$\int_{-\infty}^{\infty} x\, \delta(x - x_i)\,dx = x_i$$

$\Rightarrow$

$$\boxed{E(X) = \sum_i p_i x_i, \quad \text{where } p_i = P(X = x_i) \text{ (PMF)}}$$

for a discrete R.v.

Eg. $X \sim \text{Unif}(a,b) \longrightarrow f_X(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{o.w} \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_a^b x\left(\frac{1}{b-a}\right)dx$$

$$= \frac{1}{(b-a)} \left.\frac{x^2}{2}\right|_a^b = \frac{(b^2 - a^2)}{2(b-a)} = \frac{(b+a)}{2}.$$

Eg X is discrete r.v. taking values 1, 2, 3, 4, 5, 6, each with probability 1/6

$$E(X) = \sum_{i=1}^{6} P_i x_i = \sum_{i=1}^{6} \left(\frac{1}{6}\right) x_i = \frac{1}{6}(1+2+\ldots+6)$$
$$= 3.5$$

CONDITIONAL MEAN

Conditional Mean of a r.v. X assuming an event M is given by

$$\boxed{E(X|M) = \int_{-\infty}^{\infty} x \underbrace{f_X(x|M)}_{\text{conditional density.}} dx}$$

For discrete valued r.v.'s,

$$\boxed{E(X|M) = \sum_i x_i \underbrace{P(X = x_i|M)}_{\text{Conditional PMF}}}$$

Frequency Interpretation of Mean.

Suppose a random experiment is associated with a r.v. X

Exp. is conducted N times and the outcomes are ~~$x_1, x_2, x_3, \ldots, x_n$~~

$x^{(1)}, x^{(2)}, \ldots, x^{(N)}$

Suppose X (r.v.) takes M values $x_1, x_2, .., x_M$

$$\underbrace{\overline{X}}_{\text{average of outcomes}} = \frac{x^{(1)} + x^{(2)} + \dots + x^{(N)}}{N}$$

$$= \frac{n_1 x_1 + n_2 x_2 \dots + n_M x_M}{N}$$

$$= \sum_{i=1}^{M} x_i \left(\frac{n_i}{N}\right), \quad n_i = \text{\# of times } x_i \text{ is the outcome}$$

for N large. $\rightarrow \frac{n_i}{N} \rightarrow P(X = x_i)$

$$E(X) = \sum_{i=1}^{M} x_i P(X = x_i)$$

$\boxed{\text{Mean of } g(X)}$

$$Y = g(X) \Rightarrow E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

this is approach 1, i.e find CDF/PDF of Y & then find its Mean

<u>Another approach</u>

$$\boxed{\text{Theorem: } E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx}$$

i.e, to find the mean of $g(X)$, one does not necessarily need to obtain the PDF of $g(X)$.

Similarly, for discrete valued r.v.

$$\boxed{E[g(X)] = \sum_i g(x_i)\, P(X = x_i)}$$

IMPORTANT PROPERTY of MEAN

LINEARITY: For ANY n r.v.'s $X_1, X_2, \ldots, X_n$

$$E(X_1 + X_2 \cdots + X_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

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Notations $\Rightarrow Var(X),\ \sigma_X^2$

VARIANCE of a Random Variable

For a R.v. X with mean $\mu = E(X)$, the variance is defined as

$$\boxed{Var(X) = \sigma_X^2 \triangleq E\left[(X-\mu)^2\right]} \geq 0$$

Interpretation: For a R.v. X with a mean μ,

* $(X-\mu) \rightarrow$ represents the deviation from the mean
* This deviation could be positive or negative
* Expected value of $\underbrace{(\text{deviation})^2}_{\text{or } (X-\mu)^2}$ = Variance

or Expected Squared Deviation $\Leftrightarrow$ Variance.

* $$\boxed{\underset{\substack{\uparrow \\ \text{Standard deviation}}}{\sigma_X} = \sqrt{Var(X)} = \sqrt{E((X-\mu)^2)}}$$

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represents the Root of the mean-squared deviation of X around its mean μ.

$$
\begin{aligned}
Var(X) &= E\left[(X-\mu)^2\right] \\
&= E\left[\,X^2 - 2\mu X + \mu^2\right] \\
&= E[X^2] - 2\,E[\mu X] + E[\mu^2] \\
&= E[X^2] - 2\mu\, E[X] + \mu^2 \\
&= E[X^2] - 2\mu \times \mu + \mu^2 \\
&= E[X^2] - 2\mu^2 + \mu^2
\end{aligned}
$$

$$
\boxed{\begin{aligned}
Var(X) &= E[X^2] - \mu^2 \\
&= E[X^2] - \left(E(X)\right)^2
\end{aligned}}
$$

$\Rightarrow$ Since

$Var(X) \geq 0$

$\Rightarrow \quad E(X^2) - \left(E(X)\right)^2 \geq 0$

$\Rightarrow \quad \underline{\boxed{E(X^2) \geq \left(E(X)\right)^2}}$ this holds for <u>any</u> random variable X.

(Uniform)

Eg $X \sim \text{unif}(-c, c) \Rightarrow \mu = E(x) = \int_{-c}^{c} 1 \cdot dx = \underset{=0}{c - c}$

i.e $\mu = 0$

$$\Rightarrow \sigma^2 = E(X^2) - \mu^2$$

$$= E(X^2) - 0 = E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_{-c}^{c} \frac{x^2 dx}{2c} = \frac{1}{2c}\left(\frac{x^3}{3}\right)\Big|_{-c}^{c} = \frac{2c^3}{2 \times 3c}$$

$$\boxed{\sigma^2 = \frac{c^2}{3}}$$

(Bernoulli)

Eg: $X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } q = (1-p) \end{cases}$

$$\mu = E(x) = 1 \times p + 0 \times q = p$$

$$E(X^2) = 1^2 \times p + 0^2 \times q = p$$

$$\Rightarrow \sigma^2 = E(X^2) - (E(x))^2$$

$$= p - p^2 = p(1-p) = pq$$

Eg $\boxed{\text{(Poisson)}}$

$$P(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, 3, \ldots\ldots$$

$$E(X) = \lambda$$

$$E(X^2) = \lambda^2 + \lambda$$

$$\sigma^2 = Var(X) = \lambda$$

To prove the above:

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \quad (\text{Taylor expansion of } e^{\lambda})$$

Differentiate w.r.t. λ

$$e^{\lambda} = \sum_{k=0}^{\infty} \frac{k\lambda^{k-1}}{k!} = \frac{1}{\lambda} \cdot \sum_{k=0}^{\infty} \left(\frac{k \cdot \lambda^k}{k!} \right)$$

$$\Rightarrow \quad \lambda = \sum_{k=0}^{\infty} k \left(\frac{e^{-\lambda} \cdot \lambda^k}{k!} \right)$$

$$= \sum_{k=0}^{\infty} k \cdot P(X=k) = E(X)$$

Again differentiate w.r.t λ

$$\cancel{e^{\lambda} = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^{k-1}}{k!}}$$

$$e^{\lambda} = \sum_{k=0}^{\infty} k \frac{\lambda^{k-1}}{k!} = \frac{1}{\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k}}{k!}$$

$$\frac{d}{d\lambda}\left(\quad \right) = \frac{d}{d\lambda}\left(\quad \right)$$

(again differentiate w.r.t. λ)

$$\Rightarrow \quad e^{\lambda} = \frac{-1}{\lambda^2} \cdot \sum_{k=1}^{\infty} k \frac{\lambda^{k}}{k!} + \frac{1}{\lambda} \sum_{k=1}^{\infty} k^2 \frac{\lambda^{k-1}}{k!}$$

$$e^{\lambda} = \frac{-1}{\lambda^2} \cdot \sum_{k=1}^{\infty} k \frac{\lambda^{k}}{k!} + \frac{1}{\lambda^2} \sum_{k=1}^{\infty} k^2 \frac{\lambda^{k}}{k!}$$

$$\Rightarrow \quad \frac{\displaystyle\sum_{k=1}^{\infty} \frac{k^2 \lambda^k e^{-\lambda}}{k!}}{\lambda^2} = 1 + \frac{1}{\lambda^2} \cdot \sum_{k=1}^{\infty} \frac{k \lambda^k e^{-\lambda}}{k!}$$

$$\Rightarrow \quad \underbrace{\sum_{k=1}^{\infty} k^2 \left(\frac{\lambda^k e^{-\lambda}}{k!}\right)}_{\Downarrow \; E(X^2)} = \lambda^2 + \underbrace{\sum_{k=1}^{\infty} k \left(\frac{\lambda^k e^{-\lambda}}{k!}\right)}_{\Rightarrow E(X) = \lambda}$$

$$\Rightarrow \quad E(X^2) = \lambda^2 + \lambda$$

$$\Rightarrow \quad Var(X) = E(X^2) - \left(E(X)\right)^2$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

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