Today:

1) Conditional PDF & Conditional CDF
2) Continuous version of → Total Probability Theorem, → Bayes' Theorem
3) Examples

(1) Conditional CDF of a r.v. $\boxed{X}$ given an event M

$$\underbrace{F_X(x \mid M)}_{\text{Conditional CDF}} = P(X \le x \mid M) = \frac{P(X \le x, M)}{P(M)}$$

$$\underbrace{f_X(x \mid M)}_{\substack{\text{conditional PDF}\\ \text{of } X \text{ given event } M}} = \frac{d}{dx} F_X(x \mid M)$$

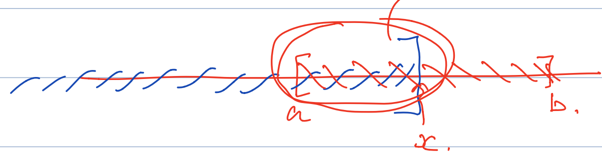
Example: $F_X(x \mid \underbrace{a \le X \le b}_{\text{event } M}) \longrightarrow$ express in terms of $F_X(x)$.

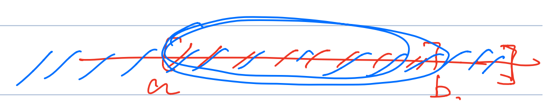
$$= \frac{P(X \le x,\ a \le X \le b)}{P(a \le X \le b)}$$

$$= \frac{P(X \le x,\ a \le X \le b)}{F_X(b) - F_X(a)} \Rightarrow (*)$$

$x < a$	$a \le x < b$	$x \ge b$
$(*) = 0$.	$(*) = F_X(x) - F_X(a)$	$(*) = F_X(b) - F_X(a)$.

$$P(X \le x,\ a \le X \le b)$$
$$= P(a \le X \le x) = F_X(x) - F_X(a).$$

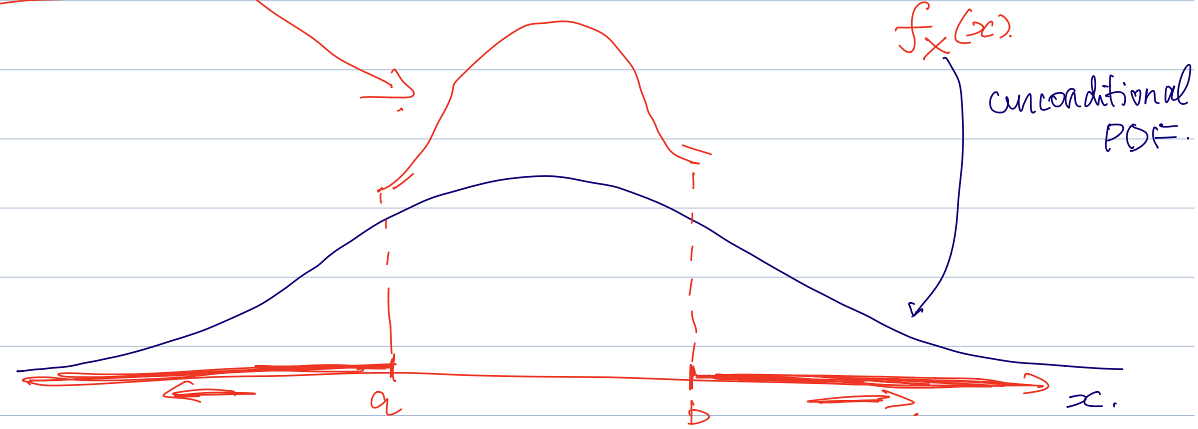




$$\underbrace{F_X(x \mid a \le X \le b)}_{\text{Cond. CDF}} = \begin{cases} 0 & \text{if } x < a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & \text{if } a \le x < b \\ 1. & \text{if } x \ge b \end{cases}$$

$$\frac{d(\ldots)}{dx}$$

$$f_X(x \mid a \le X \le b) \stackrel{?}{=} \begin{cases} \dfrac{f_X(x)}{F_X(b) - F_X(a)}. & \text{if } a \le x < b \\ 0 & \text{otherwise} \end{cases}$$



$$\frac{d}{dx}\left[\frac{F_X(x)}{F_X(b)-F_X(a)} - \frac{F_X(a)}{F_X(b)-F_X(a)}\right]$$

$$= \frac{d/dx\, F_X(x)}{F_X(b) - F_X(a)} - \underbrace{\frac{d}{dx}(\text{constant})}_{=0}$$

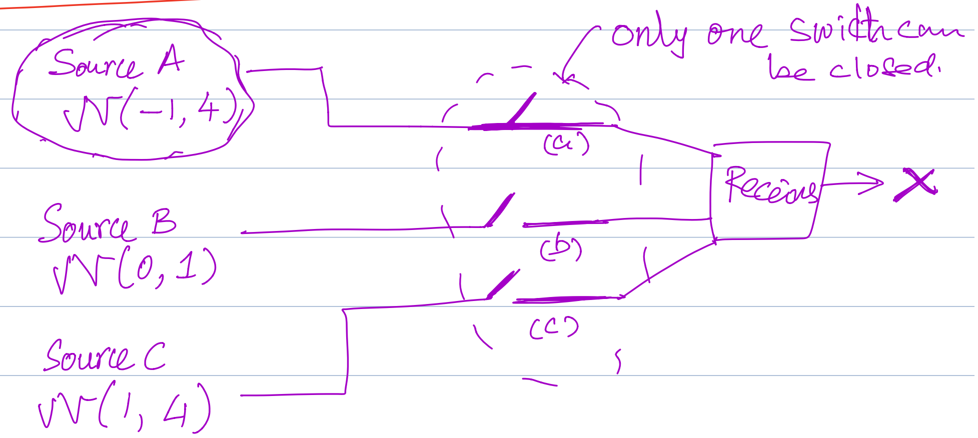
$$= \frac{f_X(x)}{F_X(b) - F_X(a)}.$$

$S \longrightarrow$ Consider a partition of S.S. $(A_1, A_2 \ldots A_n) \longrightarrow$ partition.

$$\underbrace{P(\underbrace{X \le x}_{E})}_{F_X(x)} = \boxed{P(X \le x \mid A_1)} \cdot P(A_1) + P(X \le x \mid A_2) \cdot P(A_2) + \ldots\ldots + P(X \le x \mid A_n) \cdot P(A_n).$$

$$\boxed{F_X(x) = F_X(x \mid A_1) \cdot P(A_1) + F_X(x \mid A_2) \cdot P(A_2) + \ldots\ldots + F_X(x \mid A_n) \cdot P(A_n)}$$

Example 2:



$P(\text{Swicth (a) is closed}) = 2P(\text{switch (b) is closed})$

$P(\text{sw (b) is " }) = 2P(\text{ " (c) is closed}).$

① PDF & CDF of X.

$F_X(x) = P(X \le x).$

$$\left\{\begin{array}{l} A \to \text{Switch (a) is closed} \\ B \to \text{" (b) " "} \\ C \to \text{" (c) " "} \\ P(A) = 2P(B) = 4P(C). \\ P(A) + P(B) + P(C) = 1 \end{array}\right\} \Longrightarrow \boxed{\begin{array}{l} P(A) = 4/7 \\ P(B) = 2/7 \\ P(C) = 1/7. \end{array}}$$

$$F_X(x) = F_X(x|A)\cdot \underbrace{P(A)} + F_X(x|B)\cdot P(B) + F_X(x|C)\cdot P(C).$$

$$= \frac{4}{7}\underbrace{F_X(x|A)} + \frac{2}{7}F_X(x|B) + \frac{1}{7}F_X(x|C)$$

$$F_X(x|A) \overset{?}{=} \Phi\left(\frac{x+1}{2}\right) \qquad \mathcal{N}(-1,4)$$

$$F_X(x|B) \overset{?}{=} \Phi(x).$$

$(\mathcal{N}(0,1))$

$$\boxed{\begin{array}{c}\text{From } \underline{\text{last lecture.}} \\ \mathcal{N}(\mu,\sigma^2). \\ F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)\end{array}}$$

$$F_X(x|C) \overset{?}{=} \Phi\left(\frac{x-1}{2}\right).$$

$(\mathcal{N}(1,4))$

$$F_X(x) = \frac{4}{7}\Phi\left(\frac{x+1}{2}\right) + \frac{2}{7}\Phi(x) + \frac{1}{7}\Phi\left(\frac{x-1}{2}\right)$$

(2). Compute.

$$P(X \le -1) = \underbrace{F_X(-1)}$$

$$= \frac{4}{7}\Phi(0) + \frac{2}{7}\Phi(-1) + \frac{1}{7}\Phi(-1)$$

$$= \underline{0.354}.$$

(3) Suppose we are given that $X > -1$, which source was this signal most likely?

$$\boxed{P(X > -1) = 1 - P(X \le 1) = 1 - (0.354) =}$$

$$\left.\begin{array}{l} P(A \mid X > -1) \\ P(B \mid X > -1) \\ P(C \mid X > -1) \end{array}\right\} \rightarrow$$

which one is the maximum.

$$P(A \mid X > -1) = \frac{P(A, X > -1)}{P(X > -1)}$$

$$= \frac{P(A) \cdot P(X > -1 \mid A)}{P(X > -1)}$$

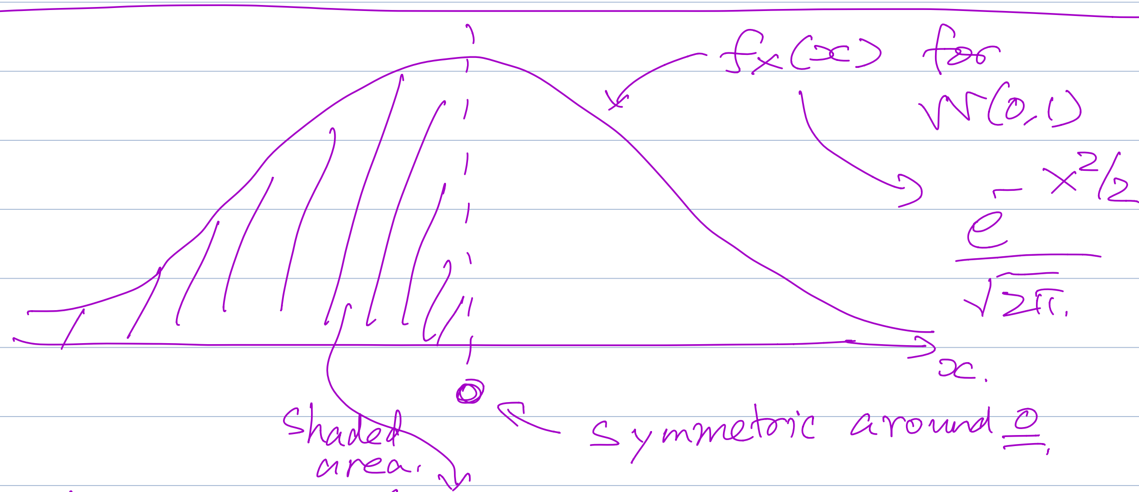
$P(A) \stackrel{?}{=} 4/7$

$P(X > -1) \stackrel{?}{=} 1 - P(X \le -1) = 1 - 0.356$

$$\boxed{P(X > -1 \mid A) \stackrel{?}{=} 1 - P(X \le -1 \mid A)}$$

$$P(E \mid A) + P(E^c \mid A) = 1.$$

$$= 1 - \Phi\left(\frac{(-1) - (-1)}{2}\right) = 1 - \Phi(0).$$
$$= 1/2.$$



$$\Phi(0) = P(X \le 0) = 1/2.$$

$$\Phi(3.4).$$

$$P(A \mid X \le x) = \frac{P(A, X \le x)}{P(X \le x)}.$$

$$= \frac{P(A) \cdot P(X \le x \mid A)}{P(X \le x)}.$$

$$\boxed{P(A \mid X \le x) = P(A) \cdot \frac{F_X(x \mid A)}{F_X(x)}.}$$

$$P(A \mid \underbrace{X=x}) = \lim_{\Delta x \to 0} P(A \mid \underbrace{x < X \le x+\Delta x})$$

$\left(\begin{array}{l} X \text{ is cont. r.v.} \\ \underline{P(X=x)} \overset{?}{=} 0. \end{array} \right)$

$$= \frac{P(A,\ x < X \le x+\Delta x)}{P(\ x < X \le x+\Delta x)}$$

$$= P(A) \cdot \lim_{\Delta x \to 0} \left[\frac{\boxed{\Delta x} \boxed{F_X(x+\Delta x \mid A) - F_X(x \mid A)}}{\boxed{\Delta x} \boxed{F_X(x+\Delta x) - F_X(x)}} \right]$$

$$= P(A) \cdot \frac{f_X(x \mid A)}{f_X(x)}$$

$$\boxed{\underline{P(A \mid X=x)} = \underline{P(A)} \cdot \frac{f_X(x \mid A)}{f_X(x)}}$$

$$\rightarrow P(A) f_X(x \mid A) = P(A \mid X=x) f_X(x).$$

$$\int_{x=-\infty}^{\infty} P(A) f_X(x \mid A)\, dx = \int_{-\infty}^{\infty} \underline{P(A \mid X=x)}\, \underline{f_X(x)}\, dx$$

$$P(A) \underbrace{\int_{-\infty}^{\infty} f_X(x \mid A)\, dx}_{1} = \int_{-\infty}^{\infty} P(A \mid X=x) f_X(x)\, dx.$$

$$\boxed{P(A) = \int_{-\infty}^{\infty} P(A \mid X=x) \cdot f_X(x)\,dx.}$$

Continuous version of the total Prob. theorem.

Expl: if $(E_1, E_2, \dots)$ is a partition.

$$P(A) = P(A \mid E_1)\,P(E_1) + P(A \mid E_2)\,P(E_2) + \dots$$

$$\boxed{f_X(x \mid A) = f_X(x) \cdot \frac{P(A \mid X=x)}{P(A)}}$$

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$X \sim$ uniformly distrib in $[a, b]$.

$f_X(x)$.