

Today

Statistical Inference.

$$f_X(x; \theta)$$

prob. model.

parameters
of the
model

Estimated from observations

Classical methods.
(frequentist)

Treat θ as
unknown vector/scalar.
(constant)

Point Estimation
Interval Estimation
MLE
(Maxm-likelihood)

Bayesian methods

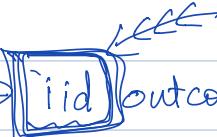
* θ is random.
* make assumption on the
distribution of θ

MMSE - estimator.
L-MMSE - "
MAP - estimator.

① Point Estimation

$$\underline{x_1}, \underline{x_2}, \dots, \underline{x_n}$$

Goal: Estimate the mean.



$$E[X_i] = \mu$$

(We do not know μ)

$$\hat{\mu}_{(n)} = \frac{x_1 + x_2 + \dots + x_n}{n}$$

$\hat{\mu}_{(n)}$

Sample mean

this is an estimate of
the unknown mean μ .

is a Random Variable.

(Desired) Properties of a Point Estimate

① Unbiased Estimator

$\hat{\gamma}$ is an estimate of γ

$$E[\hat{\gamma}] = \gamma$$

$$(\text{Bias} = 0), \quad \text{Bias} = (E[\hat{\gamma}] - \gamma)$$

is $\underbrace{\hat{\mu}_n}_{\text{AVG}_n}$ an unbiased estimate of μ ?

$$\text{is } E[\hat{\mu}_n] = ? \quad \mu$$

$$E\left[\frac{\underbrace{x_1 + \dots + x_n}_n\right] = \frac{n\mu}{n} = \mu. \quad \checkmark$$

Yes

$$\hat{\mu}^{(my)} = x_1 \quad \text{Yes this is also unbiased.}$$

$$E[\hat{\mu}^{(my)}] = E[x_1] = \mu \quad \checkmark$$

② Consistency of an Estimator

$\gamma \rightsquigarrow \hat{R}_1, \hat{R}_2, \dots, \hat{R}_n$ (Sequence of estimator)

The seq. is consistent if for any $\varepsilon > 0$

$$\lim_{n \rightarrow \infty} P(|\hat{R}_n - \gamma| \geq \varepsilon) = 0$$

$$\hat{R}_n \xrightarrow[n \rightarrow \infty]{\text{(in Prob)}} \gamma$$

(3) MSE of estimate \hat{R}

$$\text{error} = E[(\hat{R} - \tau)^2].$$

if a sequence of estimators $\hat{R}_1, \hat{R}_2, \dots, \hat{R}_n$.

$$e_n = E[(\hat{R}_n - \tau)^2] = \underline{\text{Var}(\hat{R}_n)}.$$

if \hat{R}_n is also an unbiased estimate of τ .

$$E[\hat{R}_n] = \tau.$$

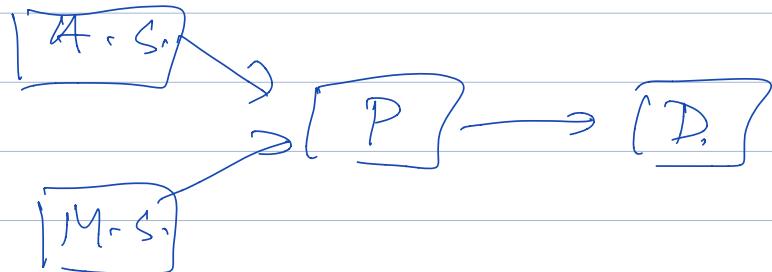
if

$$\lim_{n \rightarrow \infty} e_n = 0.$$

then the sequence of estimators

\hat{R}_n is consistent.

$$\hat{R}_n \xrightarrow[n \rightarrow \infty]{\text{in M.S.}} \tau$$



Sample mean. $\bar{X}_{(n)} = \frac{x_1 + x_2 + \dots + x_n}{n}$

① Unbiased ✓.

② Consistent?

$$E[(\bar{X}_{(n)} - \mu)^2] = \text{Var}(\bar{X}_{(n)})$$

$= \frac{\sigma^2}{n} \xrightarrow[n \rightarrow \infty]{\text{as}} 0$

$$\Rightarrow \bar{X}_{(n)} \xrightarrow[\text{as } n \rightarrow \infty.]{\text{in M. S.}} \mu$$

$$\Rightarrow \bar{X}_{(n)} \xrightarrow[\text{as } n \rightarrow \infty.]{\text{in Prob.}} \mu$$

$\Rightarrow \bar{X}_{(n)}$ is a consistent estimate of μ .

$$\hat{\mu}_{(n)}^{(\text{my})} = \bar{x}_1$$

$$\rightarrow \text{MSE of this estimator} = E[(\bar{x}_1 - \mu)^2]$$

$$= \underbrace{\sigma^2}_{\text{const.}} \cancel{\xrightarrow[n \rightarrow \infty.]{\text{as}}} 0$$

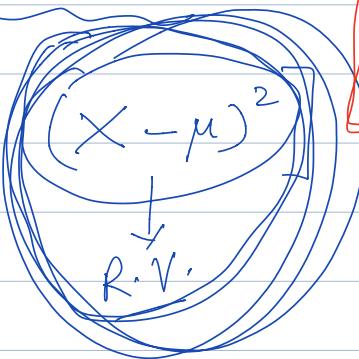
Estimating Variance

X_1, X_2, \dots, X_n iid.

↓
we know μ .

Goal: estimate σ^2 .

$$\sigma^2 = E[(X - \mu)^2]$$



(1) Known mean,
unknown
variance.

(2) Unknown mean,
Unknown
Variance.

$$\hat{\sigma}_{(n)}^2 = \frac{(X_1 - \mu)^2 + (X_2 - \mu)^2 + \dots + (X_n - \mu)^2}{n}$$

$$\hat{\sigma}_{(n)}^2 = \frac{\sum_{i=1}^n (X_i - \mu)^2}{n}$$

① Unbiased (?) $E[\hat{\sigma}_{(n)}^2] = \sigma^2$

$$E\left[\frac{\sum_{i=1}^n (X_i - \mu)^2}{n}\right] = \sigma^2$$

$$\frac{\sum_{i=1}^n E[(X_i - \mu)^2]}{n} = \frac{n\sigma^2}{n} = \sigma^2$$

(2) consistent (?)

$$P(|\hat{\sigma}_{(n)}^{(2)} - \sigma^2| \geq \varepsilon) \xrightarrow[\text{as } n \rightarrow \infty]{}$$

Chebychev's inequality

Homework \rightarrow DIY

Unknown mean, Unknown Variance.

Estimate of mean $\hat{\bar{X}} = \frac{\sum_{i=1}^n x_i}{n}$

(Sample mean.)

x_i 's
are
iid

$$\hat{\sigma}_{(n)}^{(2)} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

① Is the above estimator unbiased?

$$E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right] = \sigma^2.$$

$$\text{LHS} = E\left[\frac{1}{n} \sum_{i=1}^n (x_i^2 + \bar{x}^2 - 2x_i\bar{x})\right].$$

$$= \frac{1}{n} \sum_{i=1}^n \left[E[x_i^2] + E[\bar{x}^2] - 2E[x_i\bar{x}] \right]$$

$$= \frac{1}{n} \sum_{i=1}^n \left[(\sigma^2 + \mu^2) + \frac{\sigma^2 + \mu^2}{n} - 2E[x_i\bar{x}] \right]$$

$$\text{Var}(A) = E[A^2] - (E[A])^2.$$

$$\text{Var}(\bar{x}) = E[\bar{x}^2] - (E[\bar{x}])^2.$$

$$E[\bar{x}^2] = \underbrace{\text{Var}(\bar{x})}_{\sigma^2/n} + \underbrace{(E[\bar{x}])^2}_{\mu^2}$$

$$= \sigma^2/n + \mu^2$$

$$\left(\bar{x} = \frac{x_1 + \dots + x_n}{n} \right)$$

$$E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right] = \left(\frac{n-1}{n}\right) \sigma^2$$

$$E[\hat{\sigma}_{(n)}^{(2)}]$$

Biased Estimate of σ^2

As $n \rightarrow \infty$.

$$\text{Bias} = E[\hat{\sigma}_{(n)}^{(2)}] - \sigma^2$$

$$= \frac{(n-1)\sigma^2}{n} - \sigma^2$$

$$= -\frac{\sigma^2}{n} \xrightarrow[n \rightarrow \infty]{\text{as}} 0$$

Asymptotically unbiased estimate

$$E\left[\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2\right] = \left(\frac{n-1}{n}\right) \sigma^2.$$

divide by $(n-1)$

$$E\left[\frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2\right] = \sigma^2.$$

Unbiased

$$\hat{\sigma}_{(n)}^2 = \frac{1}{(n-1)} \sum_{i=1}^n (x_i - \bar{x})^2$$

Sample Variance

② Check if this is a
consistent estimator

