Lecture 9

Conditional Distributions

Recall that Prob. of A given another event B is
an event

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A, B)}{P(B)}$$

$\underbrace{P(A|B)}_{\text{Conditional probability}}$

CONDITIONAL DISTRIBUTION (CDF) of a random variable X given an event M

$$F_X(x|M) = P(X \le x | M) = \frac{P(X \le x, M)}{P(M)}$$

Conditional Density

$$f_X(x|M) = \frac{d F_X(x|M)}{dx}$$

Eg 1: We are interested in finding the conditional distribution of a r.v. X given that $X \le a$

$$F_X(x \mid X \le a) = P(X \le x \mid X \le a)$$

$$= \frac{P(X \le x,\ X \le a)}{P(X \le a)} \quad \text{—— } (*)$$

To further simplify $(*)$, consider two cases

$x < a$ | $x \ge a$

$\Rightarrow \{X \le x,\ X \le a\} = \{X \le x\}$

$\Rightarrow \{X \le x,\ X \le a \le x\} = \{X \le a\}$

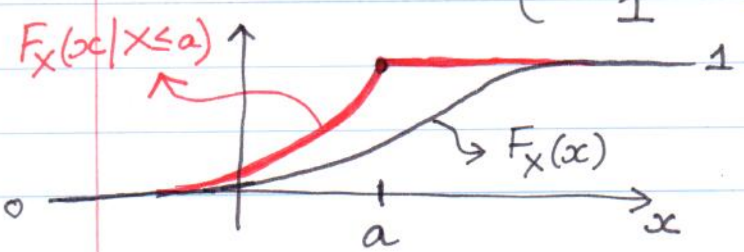
Hence,

$$F_X(x \mid X \le a) = \frac{P(X \le x)}{P(X \le a)} = \frac{F_X(x)}{F_X(a)}$$

$\Rightarrow P(X \le x, X \le a) = P(X \le a)$

$\Rightarrow F_X(x \mid X \le a) = \frac{P(X \le a)}{P(X \le a)} = 1$

$$F_X(x \mid X \le a) = \begin{cases} \dfrac{F_X(x)}{F_X(a)} & \text{if } x < a \\ 1 & \text{if } x \ge a \end{cases}$$



$\boxed{\text{Eg. 2}}$ $F_X(x \mid a \le X \le b) \stackrel{?}{=}$

$$F_X(x \mid a \le X \le b) = P(X \le x \mid a \le X \le b)$$

$$= \frac{P(X \le x,\ a \le X \le b)}{P(a \le X \le b)}$$

What is $P(a \le X \le b) \stackrel{?}{=} F_X(b) - F_X(a)$.

What about $P(X \le x,\ a \le X \le b) \stackrel{?}{=} (*)$

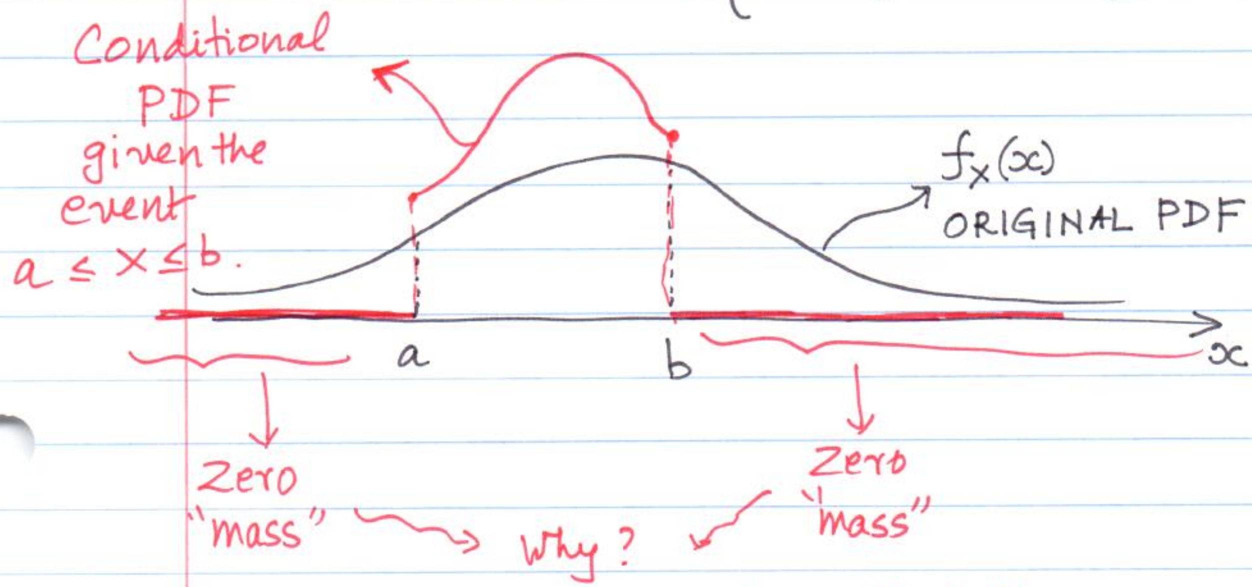
Three cases.

- $b \le x$ $\Rightarrow (*) = P(a \le X \le b)$
- $a \le x < b$ $(*) = P(a \le X \le x)$ $= F_X(x) - F_X(a)$
- $x < a$ $(*) = 0$

$$\Rightarrow \quad F_X(x \mid a \le X \le b) = \begin{cases} 0 & x < a \\ \dfrac{F_X(x) - F_X(a)}{F_X(b) - F_X(a)} & a \le x < b \\ 1. & b \le x \end{cases}$$

What about the conditional PDF?

$$f_X(x \mid a \le X \le b) = \begin{cases} \dfrac{f_X(x)}{F_X(b) - F_X(a)} & a \le x < b \\ 0 & \text{otherwise} \end{cases}$$



Since we are given that the event $a \le X \le b$ has occured.

Note: Recall, in the past, we showed the "memoryless" property of exponential distribution

$\boxed{\text{Eg 3}}$: Memoryless Property of a Geometric R.V.

Geometric R.V. $P(X=k) = (1-p)^{k-1} \cdot p$ for $k = 1, 2, 3, \ldots$

$$\boxed{\text{Memoryless Property}: \quad P(X > m+n \mid X > m) = P(X > n)}$$

Check yourself....

Total Probability & Bayes' Theorem

$$P(X \le x) = P(X \le x \mid A_1) P(A_1) + \ldots \ldots + P(X \le x \mid A_n) P(A_n)$$

for $A_1, A_2 \ldots, A_n$ being a partition of sample space.

or

$$F_X(x) = F_X(x \mid A_1) \cdot P(A_1) + F_X(x \mid A_2) P(A_2) + \ldots \ldots + F_X(x \mid A_n) \cdot P(A_n)$$

and, also

$$f_X(x) = f_X(x \mid A_1) P(A_1) + \ldots + f_X(x \mid A_n) P(A_n).$$

$$P(\underbrace{A}_{\text{event.}} \mid X \le x) = \frac{P(A,\ X \le x)}{P(X \le x)}$$

$$= \frac{P(A) \cdot P(X \le x \mid A)}{P(X \le x)}$$

$$\boxed{P(\underbrace{A}_{\text{Event.}} \mid X \le x) = \frac{F_X(x \mid A)}{F_X(x)} \cdot P(A)}$$

What about

$$P(A \mid X = x) \overset{?}{=}$$

$\left\{ \begin{array}{l} \text{if } X \text{ is} \\ \text{continuous} \\ \text{valued, then} \\ P(X = x) = 0 \end{array} \right\}$

We define it as follows:

$$P(A \mid X = x) = \lim_{\Delta x \to 0} P(A \mid x < X \le x + \Delta x)$$

Let's find $P(A \mid x \le X \le x + \Delta x)$

$$= \frac{P(A,\ x < X \le x + \Delta x)}{F_X(x + \Delta x) - F_X(x)}$$

$$P(A \mid x < X \le x + \Delta x) = \underbrace{P(A) \cdot \left\{ \frac{F_X(x + \Delta x \mid A) - F_X(x \mid A)}{F_X(x + \Delta x) - F_X(x)} \right\}}_{\text{take } \lim_{\Delta x \to 0}} \Rightarrow \frac{f_X(x \mid A)}{f_X(x)} \cdot P(A)$$

or

$$\boxed{P(A \mid X = x) = \frac{f_X(x \mid A)}{f_X(x)} \cdot P(A)}$$

$$\Rightarrow \cancel{f_X(x \mid A) = P(A)}.$$

$$P(A) \cdot f_X(x \mid A) = P(A \mid X = x)\, f_X(x).$$

$$\Rightarrow P(A) \cdot \underbrace{\int_{-\infty}^{\infty} f_X(x \mid A)\, dx}_{=1.} = \int_{-\infty}^{\infty} P(A \mid X = x)\, f_X(x)\, dx.$$

$$\Rightarrow \boxed{P(A) = \int_{-\infty}^{\infty} P(A \mid X = x)\, f_X(x)\, dx.}$$

CONTINUOUS VERSION of TOTAL PROBABILITY THEOREM.

from here,

CONTINUOUS VERSION OF BAYES' THEOREM

$$f_X(x \mid A) = \left(\frac{P(A \mid X = x)}{P(A)} \right) f_X(x)$$

$$\boxed{f_X(x \mid A) = \left(\frac{P(A \mid X = x)}{\int_{-\infty}^{\infty} P(A \mid X = x)\, f_X(x)\, dx} \right) \cdot f_X(x)}$$

Applications of Conditional distribution

Eg1 Prob. of heads in a coin-tossing Experiment is a r.v. P with density $f_P(p)$. What is Prob. of Heads ?

$$\text{Prob}\{H \mid P = p\} = p$$

$$\Rightarrow \text{Prob}\{H\} = \int_{p=0}^{1} \text{Prob}\{H \mid P = p\}\, f_P(p)\, dp$$

$$= \int_0^1 p\, f_P(p)\, dp.$$

Eg2 Let $P(H) = p$ represent the prob. of obtaining a head in a toss.

→ p can take any value in $(0,1)$

→ we can treat P as a random variable.

→ without any information about the coin, we assume the a priori pdf of P to be uniform.

i.e $f_P(p) \sim \text{uniform}(0,1)$.

→ we conduct an experiment → toss this coin n times & observe k heads

Q → How should we update $f_P(p)$??

Let A = "k heads in n tosses" (ie the Event that occured) ⑨

for a specific value of $P = p$,

$$P(A \mid P=p) = p^k (1-p)^{n-k}$$

Google: Euler's Integral $\boxed{\text{WHY??}}$ or (Beta Function)

$$\Rightarrow \quad P(A) = \int_0^1 P(A \mid P=p) \cdot f_P(p)\, dp$$

$$= \int_0^1 p^k (1-p)^{n-k}\, dp = \frac{(n-k)!\, k!}{(n+1)!}$$

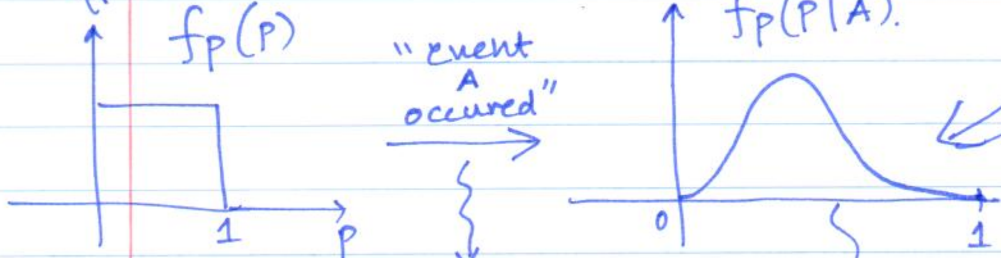
We are interested in a-posteriori pdf...

i.e

$$f_P(p \mid A) = \frac{P(A \mid P=p) \cdot f_P(p)}{P(A)}$$

$$= \frac{(n+1)!}{(n-k)!\, k!}\, p^k (1-p)^{n-k}, \quad 0 < p \le 1$$

a-priori



or "after observing" some data

a posteriori

We can use the a posteriori PDF to predict future outcomes.

Eg → B = "head occuring in $(n+1)^{th}$ toss given that k heads occured in first n tosses"

$$P(B) = \int_0^1 \underbrace{P(B \mid P=p)}_{\Downarrow \; p} \cdot \underbrace{f_P(p \mid A)}_{\Downarrow \; \frac{(n+1)!}{(n-k)!\,k!} p^k (1-p)^{n-k}} \, dp$$

$$P(B) = \frac{k+1}{n+2}$$

Eg $n = 10, \quad k = 6$

$P(B) = 7/12 = 0.58$ ⇝ Prediction using a-posteriori PDF

vs

0.5 ⇝ Prediction via (or non-informative) a-priori PDF