HW5 Solution ECE 503

1. The PDF of the 3-dimensional random vector $X = (X_1, X_2, X_3)$ is

$$f_X(x) = \begin{cases} e^{-x_3} & 0 \le x_1 \le x_2 \le x_3, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the marginal PDFs of X_1, X_2 and X_3

   (b) Are the components of X independent ?

$$f_{X_1}(x_1) = \int_{x_1}^{\infty} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_2 = \int_{x_1}^{\infty} e^{-x_2}\, dx_2 = e^{-x_1} \tag{1}$$

Similarly, for $x_2 \ge 0$, X_2 has marginal PDF

$$f_{X_2}(x_2) = \int_{0}^{x_2} \left(\int_{x_2}^{\infty} e^{-x_3}\, dx_3 \right) dx_1 = \int_{0}^{x_2} e^{-x_2}\, dx_1 = x_2 e^{-x_2} \tag{2}$$

Lastly,

$$f_{X_3}(x_3) = \int_{0}^{x_3} \left(\int_{x_1}^{x_3} e^{-x_3}\, dx_2 \right) dx_1 = \int_{0}^{x_3} (x_3 - x_1) e^{-x_3}\, dx_1 \tag{3}$$

$$= -\frac{1}{2}(x_3 - x_1)^2 e^{-x_3} \Big|_{x_1=0}^{x_1=x_3} = \frac{1}{2} x_3^2 e^{-x_3} \tag{4}$$

The complete expressions for the three marginal PDFs are

b) Clearly, they are not independent since the joint is not the same as product of marginals

$$f_{X_1}(x_1) = \begin{cases} e^{-x_1} & x_1 \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$f_{X_2}(x_2) = \begin{cases} x_2 e^{-x_2} & x_2 \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

$$f_{X_3}(x_3) = \begin{cases} (1/2) x_3^2 e^{-x_3} & x_3 \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

In fact, each X_i is an Erlang $(n, \lambda) = (i, 1)$ random variable.

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2. [3 points] Let X be a 3-dimensional Gaussian random vector with expected value $\mu_X = [4 \quad 8 \quad 6]^T$, and covariance

$$C_X = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \tag{1}$$

Calculate

(a) the correlation matrix R_X

(b) the PDF of the first two components of X, i.e., $f_{X_1,X_2}(x_1, x_2)$

(c) the probability that $X_1 > 8$

$$\begin{aligned}
\mathbf{R}_X &= \mathbf{C}_X + \boldsymbol{\mu}_X \boldsymbol{\mu}'_X \\
&= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 8 \\ 6 \end{bmatrix} \begin{bmatrix} 4 & 8 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} + \begin{bmatrix} 16 & 32 & 24 \\ 32 & 64 & 48 \\ 24 & 48 & 36 \end{bmatrix} = \begin{bmatrix} 20 & 30 & 25 \\ 30 & 68 & 46 \\ 25 & 46 & 40 \end{bmatrix}
\end{aligned}$$

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(b) Let $\mathbf{Y} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$. Since $\mathbf{Y}$ is a subset of the components of $\mathbf{X}$, it is a Gaussian random vector with expected velue vector

$$\boldsymbol{\mu}_Y = \begin{bmatrix} E[X_1] & E[X_2] \end{bmatrix}' = \begin{bmatrix} 4 & 8 \end{bmatrix}'. \tag{4}$$

and covariance matrix

$$\mathbf{C}_Y = \begin{bmatrix} \text{Var}[X_1] & \text{Cov}[X_1, X_2] \\ \mathbf{C}_{X_1} X_2 & \text{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \tag{5}$$

We note that $\det(\mathbf{C}_Y) = 12$ and that

$$\mathbf{C}_Y^{-1} = \frac{1}{12} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}. \tag{6}$$

This implies that

$$(\mathbf{y} - \boldsymbol{\mu}_Y)' \mathbf{C}_Y^{-1} (\mathbf{y} - \boldsymbol{\mu}_Y) = \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} y_1 - 4 \\ y_2 - 8 \end{bmatrix} \tag{7}$$

$$= \begin{bmatrix} y_1 - 4 & y_2 - 8 \end{bmatrix} \begin{bmatrix} y_1/3 + y_2/6 - 8/3 \\ y_1/6 + y_2/3 - 10/3 \end{bmatrix} \tag{8}$$

$$= \frac{y_1^2}{3} + \frac{y_1 y_2}{3} - \frac{16 y_1}{3} - \frac{20 y_2}{3} + \frac{y_2^2}{3} + \frac{112}{3} \tag{9}$$

The PDF of $\mathbf{Y}$ is

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi\sqrt{12}} e^{-(\mathbf{y} - \boldsymbol{\mu}_Y)' \mathbf{C}_Y^{-1} (\mathbf{y} - \boldsymbol{\mu}_Y)/2} \tag{10}$$

$$= \frac{1}{\sqrt{48\pi^2}} e^{-(y_1^2 + y_1 y_2 - 16 y_1 - 20 y_2 + y_2^2 + 112)/6} \tag{11}$$

Since $\mathbf{Y} = \begin{bmatrix} X_1, X_2 \end{bmatrix}'$, the PDF of X_1 and X_2 is simply

$$f_{X_1, X_2}(x_1, x_2) = f_{Y_1, Y_2}(x_1, x_2) = \frac{1}{\sqrt{48\pi^2}} e^{-(x_1^2 + x_1 x_2 - 16 x_1 - 20 x_2 + x_2^2 + 112)/6} \tag{12}$$

(c) We can observe directly from $\boldsymbol{\mu}_X$ and $\mathbf{C}_X$ that X_1 is a Gaussian $(4, 2)$ random variable. Thus,

$$P[X_1 > 8] = P\left[\frac{X_1 - 4}{2} > \frac{8 - 4}{2}\right] = Q(2) = 0.0228 \tag{13}$$

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3. [3 points] Random variables X_1 and X_2 both have zero expected value and variances $\mathrm{Var}(X_1) = 4$, $\mathrm{Var}(X_2) = 9$. Their covariance is $\mathrm{Cov}(X_1, X_2) = 3$.

   (a) Find the covariance matrix of $X = (X_1, X_2)^T$.

   (b) X_1 and X_2 are transformed to new variables Y_1 and Y_2 according to

$$\begin{aligned} Y_1 &= X_1 - 2X_2 \\ Y_2 &= 3X_1 + 4X_2 \end{aligned}$$

   Find the covariance matrix of $Y = (Y_1, Y_2)^T$.

(a) The coavariance matrix of $\mathbf{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}'$ is

$$\mathbf{C_X} = \begin{bmatrix} \mathrm{Var}[X_1] & \mathrm{Cov}[X_1, X_2] \\ \mathrm{Cov}[X_1, X_2] & \mathrm{Var}[X_2] \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix}.$$

(b) From the problem statement,

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \mathbf{X} = \mathbf{AX}.$$

$$\mathbf{C_Y} = \mathbf{AC_XA}' = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 28 & -66 \\ -66 & 252 \end{bmatrix}.$$

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4. [4 points] The voltage V of a position sensor is a random variable with PDF:

$$f_V(v) = \begin{cases} 1/12 & -6 \le v \le 6, \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

A receiver obtains $R = V + X$, where the random variable X is a Gaussian $(\mu, \sigma) = (0, \sqrt{3})$ noise voltage that is independent of V. The receiver uses R to estimate the original voltage V. Find

(a) the expected received voltage $E(R)$

(b) the variance $\text{Var}(R)$ of the received voltage

(c) the covariance $\text{Cov}(V, R)$ of the transmitted and received voltages

(d) the LMMSE estimator of V from R

(e) the resulting error of the LMMSE estimator

The problem statement tells us that

$$f_V(v) = \begin{cases} 1/12 & -6 \le v \le 6, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Furthermore, we are also told that $R = V + X$ where X is a Gaussian $(0, \sqrt{3})$ random variable.

(a) The expected value of R is the expected value V plus the expected value of X. We already know that X has zero expected value, and that V is uniformly distributed between -6 and 6 volts and therefore also has zero expected value. So

$$E[R] = E[V + X] = E[V] + E[X] = 0. \tag{2}$$

(b) Because X and V are independent random variables, the variance of R is the sum of the variance of V and the variance of X.

$$\text{Var}[R] = \text{Var}[V] + \text{Var}[X] = 12 + 3 = 15. \tag{3}$$

(c) Since $E[R] = E[V] = 0$,

$$\text{Cov}[V, R] = E[VR] = E[V(V + X)] = E\left[V^2\right] = \text{Var}[V]. \tag{4}$$

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(d) The correlation coefficient of V and R is

$$\rho_{V,R} = \frac{\mathrm{Cov}\left[V,R\right]}{\sqrt{\mathrm{Var}[V]\,\mathrm{Var}[R]}} = \frac{\mathrm{Var}[V]}{\sqrt{\mathrm{Var}[V]\,\mathrm{Var}[R]}} = \frac{\sigma_V}{\sigma_R}. \tag{5}$$

The LMSE estimate of V given R is

$$\hat{V}(R) = \rho_{V,R}\frac{\sigma_V}{\sigma_R}\left(R - E\left[R\right]\right) + E\left[V\right] = \frac{\sigma_V^2}{\sigma_R^2}R = \frac{12}{15}R. \tag{6}$$

Therefore $a^* = 12/15 = 4/5$ and $b^* = 0$.

(e) The minimum mean square error in the estimate is

$$e^* = \mathrm{Var}[V](1 - \rho_{V,R}^2) = 12(1 - 12/15) = 12/5$$

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5. [4 points] Given the set $\{U_1, U_2, \ldots, U_n\}$ of i.i.d. uniform $(0, T)$ random variables, we define

$$X_k \triangleq \text{small}_k(U_1, U_2, \ldots, U_n)$$

as the kth "smallest" element of the set. For example, X_1 is the smallest element, X_2 is the second smallest element, and so on, up to X_n, which is the maximum element of $\{U_1, U_2, \ldots, U_n\}$.

(a) Find the joint PDF of $(X_1, X_2, \ldots, X_n)$.

(b) Find the marginal of X_2

We can observe that $(X_1, \ldots, X_n)$ are functions of $(U_1, \ldots, U_n)$

Note that the mapping from $(U_1, \ldots, U_n) \to (X_1, \ldots, X_n)$ is one-to-one

However, mapping from $(X_1, \ldots, X_n) \to (U_1, \ldots, U_n)$ is one-to-many

For a given $(X_1, \ldots, X_n) = (x_1, x_2, \ldots, x_n)$, $(U_1, \ldots, U_n)$ can take $n!$ values. $x_1 \leq x_2 \leq \ldots \leq x_n$

The absolute value of Jacobian of the transformation is 1 for any n.

Hence, $f_{X_1,\ldots,X_n}(x_1, x_2, \ldots, x_n) = n! \times \left(\frac{1}{T} \times \ldots \times \frac{1}{T}\right) = \frac{n!}{T^n}$ $\quad 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq T$

$$f_{X_1,\ldots,X_n}(x_1, \ldots, x_n)$$
$$= \begin{cases} n!/T^n & 0 \leq x_1 < \cdots < x_n \leq T, \\ 0 & \text{otherwise.} \end{cases}$$

Part (b) of Problem 5:

k^{th} smallest r.v.

The distribution F_k of $X_{(k)}$ is given by

$$F_k(x) = \sum_{j=k}^{n} \binom{n}{j} (F(x))^j (1-F(x))^{n-j},$$

where $F(x)$ is the common CDF of $U_1, U_2 .., U_n$'s.

For any x, let

$$N_x = I(U_1 \le x) + I(U_2 \le x) \cdots + I(U_n \le x)$$

i.e, N_x is the number of r.v.'s that are less than or equal to x, where

indicator function.

$$I(U_i \le x) = \begin{cases} 1 & \text{if } U_i \le x \\ 0 & \text{otherwise.} \end{cases}$$

N_x has a binomial distribution with

$$P(N_x = j) = \binom{n}{j} (F(x))^j (1-F(x))^{n-j}$$

Now, note that

$$X_{(k)} \le x \iff N_x \ge k$$

$$\Rightarrow P(X_{(k)} \le x) = P(N_x \ge k)$$

$$= \sum_{j=k}^{n} P(N_x = j)$$

distribution of the k^{th} smallest.

$$F_k = P(X_{(k)} \le x) = \sum_{j=k}^{n} \binom{n}{j} (F(x))^j (1 - F(x))^{n-j}$$

Density of the k^{th} smallest.

$$f_k(x) = \frac{n!}{(k-1)!\,(n-k)!} (F(x))^{k-1} (1 - F(x))^{n-k} f(x), \quad x \in R.$$

To prove this claim,

$$f_k(x) = \frac{d}{dx} P(X_{(k)} \le x).$$

$$\frac{d}{dx}\left(F(x)^j (1 - F(x))^{n-j}\right)$$

$$= j F(x)^{j-1} f(x) (1 - F(x))^{n-j}$$

$$- (n-j) F(x)^j f(x) (1 - F(x))^{n-j-1}$$

~~Part (b)~~

$\Rightarrow$

$$f_k(x) = \frac{d}{dx} P(X_{(k)} \le x)$$

$$= \sum_{j=k}^{n} \binom{n}{j} \left[j\, F(x)^{j-1} (1-F(x))^{n-j} - (n-j)\, F(x)^{j} (1-F(x))^{n-j-1} \right] \times f(x)$$

$$= \left\{ \sum_{j=k}^{n} \binom{n}{j} j\, F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{n} \binom{n}{j} (n-j)\, F(x)^{j} (1-F(x))^{n-j-1} \right] f(x)$$

Now, we use the identities

$$j \binom{n}{j} = n \binom{n-1}{j-1} \quad \text{and} \quad (n-j)\binom{n}{j} = n \binom{n-1}{j}$$

$$= n \left\{ \sum_{j=k}^{n} \binom{n-1}{j-1} F(x)^{j-1} (1-F(x))^{n-j} - \sum_{j=k}^{n} \binom{n-1}{j} F(x)^{j} (1-F(x))^{n-j-1} \right] f(x)$$

all terms cancel except $j=k$ term in the First Σ.

$$= n \binom{n-1}{k-1} F(x)^{k-1} (1-F(x))^{n-k} f(x)$$

$$= \frac{n!}{(k-1)!\,(n-k)!} (F(x))^{k-1} (1-F(x))^{n-k} f(x).$$

Solution to Problem 6

(a) $S_N = X_1 + X_2 \dots + X_N$

$$= \sum_{i=1}^{N} X_i$$

$$E(S_N) = E\big(\underbrace{E(S_N \mid N)}\big).$$

Via (Iterated Expectation Theorem)

this is a r.v., which takes value $E(S_N / N=n)$, when $N=n$

$$E(S_N \mid N=n) = E\left(\sum_{i=1}^{n} X_i \mid N=n\right)$$

$$= E(X_1 + X_2 \dots + X_n \mid N=n)$$

$$= E(X_1 \mid N=n) + \dots + E(X_n \mid N=n).$$

$$= E(X_1) + \dots + E(X_n)$$ [since X_i and N are independent

$$= n\, E[X_1]$$

$$\Rightarrow \quad E(S_N) = E[N\, E(X_1)]$$

$$= E[N] \times E[X_1].$$

(b) Part b is straight forward, $N \sim Geom(p)$. $X_1 \sim exp(mean = \lambda)$

$E(N) = 1/p \quad E(X_1) = \lambda.$