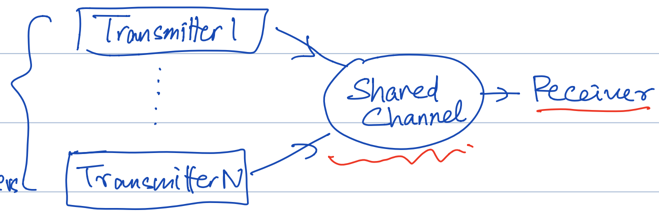
Today

1) Wrap up Analysis of Slotted Aloha protocol
2) Moments of a R.V.
3) Markov Inequality & Chebychev's Inequality
4) Moment Generating Function (MGF)

From last lecture

(Slotted Aloha) MAC-protocol.

N = # of transmitters



p = Prob(Transmitter sends in a slot).

$(1-p)$ = Prob (Transmitter is silent).

We saw that each slot could be
- S (success) ⟺ only one Tx sends
- C (collision) ⟺ more than one Tx sends
- E (empty) ⟺ No one sends.

$E[\text{Throughput}]$ = Exp # of successful transmissions per unit time.

$$= Np \times (1-p)^{N-1}$$

$N \to$ # of Transmitters

$p \to$ prob of transmission

Goal: To maximize throughput.

$N = 100$ users. , what is the optimal p?

$$Np(1-p)^{N-1} = \left(\frac{Np}{1-p}\right) \times (1-p)^N$$

analyze the throughput

→ $\boxed{N \text{ is large } \& \ p \text{ is small}}$

& Np = constant = G.

$$* \ \frac{(Np)}{(1-p)} \approx Np \quad \text{for } (p \text{ small})$$

$*$ $\boxed{(1-p)^N \approx e^{-Np}}$ $\longleftarrow$ $x \rightsquigarrow (-p)$

$Nx \rightsquigarrow -Np.$

why?

$$(1+x)^N = \left(1+\frac{Nx}{N}\right)^N \longleftarrow \text{(Binomial Theorem)}$$

$$= 1 + \binom{N}{1}\times\frac{(Nx)}{N} + \binom{N}{2}\frac{(Nx)^2}{N^2} + \dots\dots$$

i^{th} term in the above expansion.

$Nx \to \text{const} \Rightarrow G$

$(Nx)^2 \to G^2$

$\vdots$

$$\frac{\binom{N}{i}(Nx)^i}{N^i} = \boxed{\frac{\binom{N}{i}}{N^i}}\, G^i$$

Claim: if N is large, then $\dfrac{\binom{N}{i}}{N^i} \approx \dfrac{1}{i!}$

$$\frac{\binom{N}{i}}{N^i} = \left(\frac{N!}{i!\,(N-i)!}\right) \times \frac{1}{N^i}$$

$$= \frac{1}{i!}\left[\frac{N \times (N-1) \times (N-2) \times \dots \times (N-(i-1))}{N \times N \times N \times \dots \times N}\right]$$

$$= \frac{1}{i!}\left[1 \times \underbrace{\left(1-\frac{1}{N}\right)}_{\approx 1} \times \underbrace{\left(1-\frac{2}{N}\right)}_{\approx 1} \times \dots \times \underbrace{\left(1-\frac{(i-1)}{N}\right)}_{\approx 1.}\right]$$

$$\approx \frac{1}{i!}$$

$$(1+x)^N \approx \underbrace{1 + \frac{(Nx)}{1!} + \frac{(Nx)^2}{2!} + \frac{(Nx)^3}{3!} + \dots}$$

$$= e^{Nx}.$$

$$(1-p)^N \approx e^{N(-p)} = e^{-Np}$$

N is large
p is small
$Np = G$ (const)

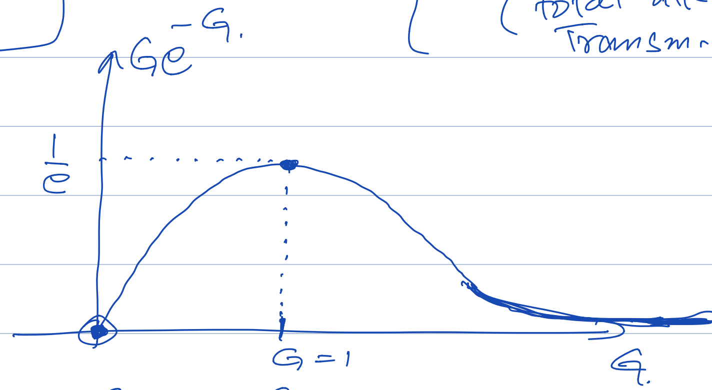
$$\text{Exp Throughput} = \left(\frac{Np}{1-p}\right) \times (1-p)^N$$

$$\approx (Np) \times e^{-(Np)}$$

$$= G \times e^{-G}$$

$$\boxed{\max_{G} \; G \times e^{-G}}$$

$\{ G = Np$ (total attempted Transmn. prob.) $\}$



$$\lim_{x\to\infty} x e^{-x} = 0$$

$$\frac{d(Ge^{-G})}{dG} \Rightarrow e^{-G} - Ge^{-G} = 0.$$

$$\Rightarrow \boxed{G = 1} \Rightarrow Np = 1.$$

$$\Rightarrow \boxed{p = 1/N}$$

Maxm throughput (@ $G=1$ or $p = 1/N$) $= \frac{1}{e} \approx 0.368$.

$\Rightarrow$ $\boxed{37\%}$ of time we will have successful transmission.

$\boxed{p = 0.5}$

Moments of a R.V.

$$m_n \overset{\Delta}{=} E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$$

definition.

$f_X(x)$ → PDF of X.

(n^{th} moment of a R.V.)

$\boxed{\mu}$ = mean = $E[X]$

$\boxed{\text{(Central moment)}}$

$$\mu_n \overset{\Delta}{=} E[(X-\mu)^n] = \int_{-\infty}^{\infty} (x-\mu)^n f_X(x)\,dx$$

$$\mu_0 = 1$$

$$\mu_1 = E[(X-\mu)] = E[X] - E[\mu]$$
$$= \mu - \mu = 0.$$

$$\mu_2 = \underbrace{E[(X-\mu)^2]}. = \sigma^2 \text{ (Variance).}$$

$$\mu_3 = E[(X-\mu)^3]$$

Binomial Thm.

$$\mu_n = E[(X-\mu)^n]$$
$$= E\left[\sum_{k=0}^{n} \binom{n}{k} X^k (-\mu)^{n-k}.\right]$$
$$= \sum_{k=0}^{n} \binom{n}{k}. \underbrace{E[X^k]}.(-\mu)^{n-k}$$

$$\mu_n. = \sum_{k=0}^{n} \binom{n}{k} \times m_k \times (-\mu)^{n-k}.$$

if we can compute non-centered moments $m_0, m_1, \ldots, m_n \longrightarrow$ we can compute μ_n (centered moment).

Gaussian R.V. $\mathcal{N}(0, \sigma^2)$.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}$$

μ_n, m_n.

$E[(X-\mu)^n]$ $E[X^n]$

if $\mu = 0$ then $m_n = \mu_n$.

$$E[X^n] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 \times 3 \times \dots \times (n-1)\sigma^2 & \text{if } n \text{ is even.} \end{cases}$$

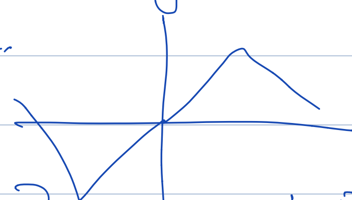
$E[X] = 0$. $E[X^3] = 0$ $E[X^5] = 0$

$E[X^{2k+1}] = 0$.

$$E[X^3] = \int_{-\infty}^{\infty} x^3 f_X(x)\, dx. = 0.$$

odd function. $g(x) = -g(-x)$.

$x^3 \rightsquigarrow$ odd. $f_X(x) \rightsquigarrow$ even. $\} \to$ odd fun.



$X^{2k+1} \rightsquigarrow$ odd, $f_X(x) \sim$ even $\}$ multiply $\to$ odd func. $\} \Rightarrow E[X^{2k+1}] = 0$.

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n is even.

$$\boxed{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}}.$$

$$\mathcal{N}(0, \sigma^2) \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\sigma^2}} dx = 1.$$

$$\alpha = \frac{1}{2\sigma^2}.$$

$$\sigma^2 = \frac{1}{2\alpha}.$$

$$\sigma = \frac{1}{\sqrt{2\alpha}}.$$

$$\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{2\pi} \times \sigma.$$

$$= \sqrt{2\pi} \times \frac{1}{\sqrt{2\alpha}} = \sqrt{\frac{\pi}{\alpha}}.$$

$$\boxed{\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}.}$$ ← Given this.

$$E[x^0] = 1.$$

$$E[x^2] = c \cdot \int x^2 e^{-\alpha x^2} dx$$

$$E[x^4] = c \cdot \int x^4 e^{-\alpha x^2} dx$$

$$\vdots$$

$$LHS = \int_{x=-\infty}^{\infty} e^{-\alpha x^2} dx.$$

$$\frac{d(\dots)}{d\alpha} = \int_{x=-\infty}^{\infty} \left(\frac{d\, e^{-\alpha x^2}}{d\alpha}\right) dx.$$

$$= \int_{-\infty}^{\infty} -x^2 \cdot e^{-\alpha x^2} dx.$$

$$\frac{d(\dots)}{d\alpha^2} = \int (-x^2) \times \frac{d(e^{-\alpha x^2})}{d\alpha} dx.$$

$$= \int (-x^2) \times (-x^2) e^{-\alpha x^2} dx.$$

$$= \int x^4 e^{-\alpha x^2} dx.$$

if $n = 2k$.
then we can take deriv wrt α k times

$$Y = \text{Data} + \underbrace{\text{Noise}}_{\text{(Gaussian)}}$$

Goal: remove eff. of (Noise)

$$\underbrace{Y \times Y \times Y} = (D + N)^3$$

$$E[\qquad] = D^3 + \underbrace{E(N^3)}_{= 0} + \dots$$

all odd moments go to zero..

$\boxed{\text{24 hour window}}$

$$T = \text{Poisson}(\boxed{96}),$$

$0, 1, 2, \dots, \boxed{k}$.

$\boxed{T = k} \Longrightarrow k$ ads were shown

$$\text{Profit} \mid T = \underline{k} \overset{?}{=} P_1 + P_2 + \cdots + P_K$$

$$\text{Profit}(T) = \underbrace{P_1 + P_2 + \cdots + P_{T}}$$

$$E\big[\underbrace{\text{Profit}(T)}_{\text{func of } T}\big]$$

$$E\big[E[g(T)]\big] = \sum_{k=0}^{\infty} P(T=k)\, E\, \underline{g(T=k)}$$