Today

1) Poisson Process (wrap up)

(focus on WSS) 2) LTI filtering of random processes
3) Power Spectral Density (PSD)

Recap (Poisson process) $\boxed{N(t)} \longrightarrow \lambda \longrightarrow$ (rate or intensity)

$N(t)$ is a poisson Process

(a) $(t_0, t_1] \rightsquigarrow N(t_1) - N(t_0) \longrightarrow$ # of counts / arrivals in $(t_0, t_1]$

$\sim \text{Poisson}(\lambda(t_1 - t_0))$

(b) For non-overlapping intervals, the counts are independent of each other.

Property: interarrival times for a Poisson process are iid $\sim$ Exponential(λ).

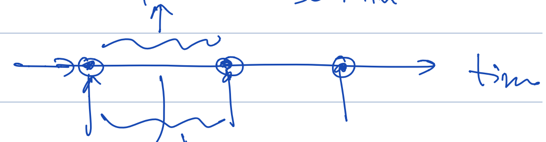
Eg: * Space Shuttle $\longrightarrow$ particles strike this shuttle according to a Poisson process.

* Expected time between any 2 strikes = 30 minutes.

* Probability that during at least one out of 5 consecutive hours, 3 or more particles strike the shuttle.

Q: what is λ ?

Exp time = 30 min

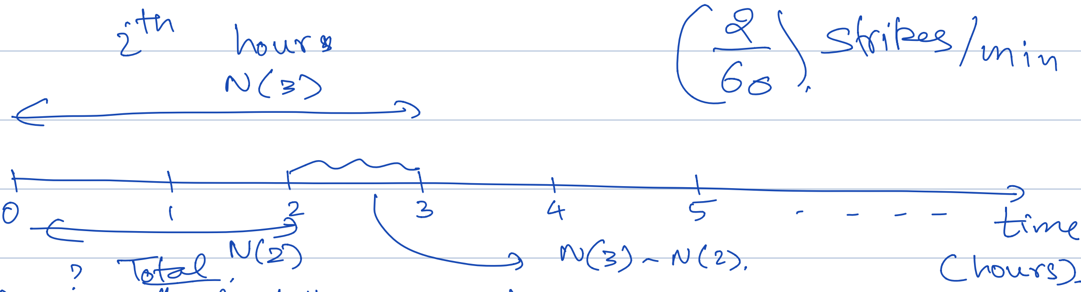


interarriv. $\sim \text{Exp}(\lambda)$

$E[X] = 30$ min.

$$\text{Mean} = \frac{1}{\lambda} = 0.5 \text{ hours.}$$

$$\boxed{\lambda = \frac{1}{0.5} = 2 \text{ strikes/hr.}}$$



$\left(\frac{2}{60}\right)$ strikes/min

$N(3) \overset{?}{=}$ # of strikes in 3 hours.

# of strikes in 3^{rd} hour $\overset{?}{=}$ $N(3) - N(2)$. $\mathcal{E}$.

P(during at least 1 hour out of 5 consec. hours, 3 or more strikes happen).

3 or more strikes in 1^{st} hour $\iff$ $N(1) - N(0) \geq 3$.

3 " 2^{nd} " $\iff$ $N(2) - N(1) \geq 3$

3 " 3^{rd} h $\iff$ $N(3) - N(2) \geq 3$

⋮

$$\mathcal{E} \iff \bigcup_{i=1}^{5} \{N(i) - N(i-1) \geq 3\}.$$

$$P\Big(\bigcup_{i=1}^{5} \underbrace{\{N(i)-N(i-1) \geq 3\}}_{\mathcal{E}_i}\Big) = 1 - P\Big(\bigcap_{i=1}^{5} \overbrace{\{N(i) - N(i-1) < 3\}}^{\mathcal{E}_i^c}\Big)$$

$$= 1 - P\Big(\bigcap_{i=1}^{5} (N(i) - N(i-1) < 3)\Big)$$

$$= 1 - \prod_{i=1}^{5} P\big(N(i) - N(i-1) \leq 2\big)$$

$$\boxed{X \sim Poiss(2) \qquad P(X = k) = \frac{e^{-2} \cdot 2^{k}}{k!}}$$

$N(i) - N(i-1) \sim$ Poiss $(\lambda \times 1)$

$P(X \leq 2)$. = Poiss (2).

$\frac{2}{60} \times 60$

$$= 1 - \prod_{i=1}^{5} \left\{ \underbrace{P(X=0)} + \underbrace{P(X=1)} + \underbrace{P(X=2)} \right\}^{60}$$

$$= 1 - \prod_{i=1}^{5} \left\{ \underbrace{e^{-2} \cdot 1 + \frac{e^{-2} \times 2^1}{1!} + \frac{e^{-2} \times 2^2}{2!}} \right\}$$

$$= 1 - \left(\frac{5}{e^2}\right)^5 \approx \underline{\underline{0.86}}$$

"Processing" of WSS Random Processes → Signal Processing
→ Digital Comm.

Frequency Domain Analysis of Random Signals

WSS R.P. $X(t) \longrightarrow \underbrace{R_X(\tau)}$ → Autocorrelation function

τ → time difference.

$X(t)$, $X(t+\tau)$ → $R_X(\tau)$

$R_X(\tau) = R_X(-\tau)$. (Even)

$R_X(0) \geq |R_X(\tau)| \; \forall \tau$.

$R_X(0) \geq 0$.

$R_X(\tau) = E[X(t)X(t+\tau)]$.

$\underbrace{P_X}_{\text{Power of } X(t)} = R_X(0)$

PSD (Power Spectral Density)

$$S_X(f) = \mathcal{F}\left(R_X(\tau) \right)$$

(PSD) Fourier Transform

$$= \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$j = \sqrt{-1}$.

$$R_X(\tau) = \mathcal{F}^{-1}\left(S_X(f) \right)$$

$$= \int_{-\infty}^{\infty} S_X(f) \cdot e^{+j2\pi f\tau}\, df$$

$$R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df.$$

Power of Signal.

Power Spectral Density

$X(t) \longrightarrow$ How much power does this signal have.

[0, 10 kHz],

$$\int_0^{10000} S_X(f)\,df$$

Cross-Spectral Density.

For two jointly WSS $X(t)$, $Y(t)$, $\rightarrow R_{XY}(\tau)$.

Cross-correlation func.

$$\underbrace{S_{XY}(f)}_{\text{cross spectral density.}} = \mathcal{F}\left(R_{XY}(\tau)\right).$$

$$X(t) \longrightarrow \boxed{\begin{array}{c} LTI \\ h(t) \end{array}} \longrightarrow Y(t).$$

R. Process (WSS)

$h(t) \Rightarrow$ impulse response of the system.

(Continuous time)

$$Y(t) = h(t) * X(t)$$

$$= \int_{-\infty}^{\infty} h(\alpha)\, X(t-\alpha)\, d\alpha$$

(Discrete time)

$$X(n) \longrightarrow \boxed{h(n)} \longrightarrow Y(n)$$

$$Y(n) = h(n) * X(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k)\, X(n-k)$$

If $X(t)$ is WSS.

what can we say about $Y(t)$?

Q: is $Y(t)$ also WSS ??

Q: Are $X(t)$ & $Y(t)$ jointly WSS ??

Q: if $X(t)$ has $S_X(f)$, what is PSD of output $S_Y(f)$??