Hw1

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part I: maximum posterior vs. probability of chance:

$P(\omega_{max} | x) \geq \frac{1}{C}$, C # of classes

(a) Show / explain $P(\omega_{max}|x) \geq \frac{1}{C}$?

(b) Derive expression for $P(err)$?

(c) show that $P(err) \leq \frac{C-1}{C}$?

(a) $P(\omega_{max}|x) \geq P(\omega_i|x)$

using probability axioms

$$\Rightarrow \sum_{i=1}^{C} P(\omega_{max}|x) \geq \sum_{i=1}^{C} P(\omega_i|x) = 1$$

$$\Rightarrow C\, P(\omega_{max}|x) \geq 1$$

$$P(\omega_{max}|x) \geq \frac{1}{C} \quad \#$$

(b) We know that $P(err|x) = 1 - P(\omega_{max}|x)$

$$P(err) = \int_{-\infty}^{\infty} P(err|x)\, P(x)\, dx$$

$$= \int_{-\infty}^{\infty} [1 - P(\omega_{max}|x)]\, P(x)\, dx$$

$$= \int_{-\infty}^{\infty} P(x)\, dx - \int_{-\infty}^{\infty} P(\omega_{max}|x) \cdot P(x)\, dx$$

$$P(err) = 1 - \int_{-\infty}^{\infty} P(\omega_{max}|x)\, P(x)\, dx$$

(c) $$P(err) = 1 - \int_{-\infty}^{\infty} P(\omega_{max}|x)\, P(x)\, dx$$

$$P(err) \leq 1 - \frac{1}{C} \int_{-\infty}^{\infty} P(x)\, dx$$

$$p(err) \leqslant 1 - \frac{1}{c}(1)$$

$$p(err) \leqslant \frac{c-1}{c} \quad \#$$

part II :

choose w_1 if $p(w_1)\,p(x|w_1) > p(w_2)\,p(x|w_2)$

$p(x_i = 1 \mid w_1) = p \Rightarrow p(x_i = 0 \mid w_1) = 1-p$

$p(x_i = 1 \mid w_2) = 1-p \Rightarrow p(x_i = 0 \mid w_2) = p$

since the elements of the vector x are independent

$$p(x|w_1) = \prod_{i=1}^{d} p(x_i|w_1)$$

$$p(x|w_2) = \prod_{i=1}^{d} p(x_i|w_2)$$

since the elements of the Vector X are binary, we can sum them

let $s = \sum_{i=1}^{d} x_i$

Thus, there are s ones and (d-s) zeros

Now we can simplify the previous equations more

$$p(x|w_1) = \prod_{i=1}^{d} p(x_i|w_1) = p^s \cdot (1-p)^{d-s}$$

$$p(x|w_2) = \prod_{i=1}^{d} p(x_i|w_2) = (1-p)^s \; p^{d-s}$$

choose w_1 if $p(w_1)\,p(x|w_1) > p(w_2)\,p(x|w_2)$

since $p(w_1) = p(w_2) = \frac{1}{2}$

choose w_1 if $p(x|w_1) > p(x|w_2)$

$$p^s(1-p)^{d-s} > (1-p)^s \; p^{d-s}$$

$$\left(\frac{1-p}{p}\right)^{d-s} > \left(\frac{1-p}{p}\right)^{s}$$

$$\left(\frac{1-p}{p}\right)^{d} \left(\frac{1-p}{p}\right)^{-s} > \left(\frac{1-p}{p}\right)^{s}$$

$$\left(\frac{1-p}{p}\right)^{d}\left(\frac{1-p}{p}\right)^{-s} > \left(\frac{1-p}{p}\right)^{s}$$

$$\left(\frac{1-p}{p}\right)^{d} > \left(\frac{1-p}{p}\right)^{2s}$$

since $p > \frac{1}{2}$, then $2s > d$

$$2\sum_{i=1}^{d} x_i > d$$

$$\sum_{i=1}^{d} x_i > \frac{d}{2}$$

part III:

$$P(\text{boy}) = P(\text{girl}) = \frac{1}{2}$$

Let BB be the event of boy and boy

BG " " " " boy and Girle

GB ⋮ ⋮

GG

We want to know $P(BB|B) = \dfrac{P(B|BB)\,P(BB)}{P(B)}$?

$P(BB) = \frac{1}{4}$, $P(BG) = \frac{1}{4}$

$P(GB) = \frac{1}{4}$, $P(GG) = \frac{1}{4}$

		B							G						
		M	T	W	Th	F	S	Su	M	T	W	Th	F	S	Su
B	M	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	T	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	W	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	Th	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	F	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	S	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	Su	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
G	M	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	T	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	W	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	Th	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	F	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	S	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	Su	–	–	✓	–	–	–	–	–	–	–	–	–	–	–

from the information above, we can find the following probabilites:

$$P(B|BB) = \frac{13}{49}$$

$$P(B|BG) = \frac{1}{7}$$

$$P(B|GB) = \frac{1}{7}$$

$$P(B|GG) = 0$$

using the total probability thm. we can find $P(B)$.

$$P(B) = P(B|BB) \cdot P(BB) + P(B|BG)\,P(BG) + P(B|GB)\,P(GB) + P(B|GG)\,P(GG)$$

$$= \frac{13}{49} \cdot \frac{1}{4} + \frac{1}{7} \cdot \frac{1}{4} + \frac{1}{7} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4}$$

$$= \frac{27}{49} \cdot \frac{1}{4}$$

$$P(BB|B) = \frac{P(B|BB) \cdot P(BB)}{P(B)} = \frac{\frac{13}{49} \cdot \frac{1}{4}}{\frac{27}{49} \cdot \frac{1}{4}} = \frac{13}{27}$$

part IV:

$$p(x|w_i) = \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma_i|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1} (x-\mu_i) \right]$$

$$\log(p(x|w_i)\,P(w_i)) = \log(p(x|w_i)) + \log(p(w_i))$$

$$\Rightarrow \log(p(x|w_i)) = \log\left(\frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma_i|^{\frac{1}{2}}}\right) + \left(-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1} (x-\mu_i)\right)$$

$$= -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log(|\sigma|) - \frac{1}{2\sigma^2}(x-\mu_i)^T(x-\mu_i)$$

$$\log(p(x|w_i)\,p(w_i)) = -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log(\sigma) - \frac{1}{2\sigma^2}(x^T x - 2\mu_i^T x + \mu_i^T \mu_i) + \log(p(w_i))$$

The term $x^T x$ does not depend on variable i.

from the above equation

$$w_i = \frac{1}{\sigma^2} \mu_i^T x$$

$$w_{0i} = \frac{-d}{2} \log(2\pi) - \frac{1}{2} \log(\sigma) - \frac{1}{2\sigma^2} \mu_i^T \mu_i + \log(p(\omega_i))$$

$$\Rightarrow \quad w_i = \frac{1}{\sigma^2} \mu_i^T x$$

$$w_{0i} = \frac{-1}{2\sigma^2} \mu_i^T \mu_i + \log(p(\omega_i))$$