part I: maximum posterior vs. probability of chance:

$P(\omega_{max} | x) \geq \frac{1}{C}$, C # of classes

ⓐ Show/explain $P(\omega_{max}|x) \geq \frac{1}{C}$?

ⓑ Derive expression for $P(err)$?

ⓒ show that $P(err) \leq \frac{C-1}{C}$?

ⓐ $P(W_{max}|x) \geq P(\omega_i|x)$

using probabilty axioms

$\Rightarrow \sum_{i=1}^{C} P(W_{max}|x) \geq \sum_{i=1}^{C} P(\omega_i|x) = 1$

$\Rightarrow C\, P(W_{max}|x) \geq 1$

$P(W_{max}|x) \geq \frac{1}{C}$ #

ⓑ We know that $P(err|x) = 1 - P(W_{max}|x)$

$$P(err) = \int_{-\infty}^{\infty} P(err|x)\, P(x)\, dx$$

$$= \int_{-\infty}^{\infty} [1 - P(W_{max}|x)]\, P(x)\, dx$$

$$= \int_{-\infty}^{\infty} P(x)\, dx - \int_{-\infty}^{\infty} P(W_{max}|x) \cdot P(x)\, dx$$

$$P(err) = 1 - \int_{-\infty}^{\infty} P(W_{max}|x)\, P(x)\, dx$$

ⓒ $P(err) = 1 - \int_{-\infty}^{\infty} P(W_{max}|x)\, P(x)\, dx$

$P(err) \leq 1 - \frac{1}{C} \int_{-\infty}^{\infty} P(x)\, dx$

$$p(err) \leqslant 1 - \frac{1}{c}(1)$$

$$p(err) \leqslant \frac{c-1}{c} \quad \#$$

part II :

choose w_1 if $p(w_1)\,p(x|w_1) > p(w_2)\,p(x|w_2)$

$p(x_i=1 \mid w_1) = p \Rightarrow p(x_i=0|w_1) = 1-p$

$p(x_i=1 \mid w_2) = 1-p \Rightarrow p(x_i=0|w_2) = p$

since the elements of the vector x are independent

$$p(x|w_1) = \prod_{i=1}^{d} p(x_i|w_1)$$

$$p(x|w_2) = \prod_{i=1}^{d} p(x_i|w_2)$$

since the elements of the Vector X are binary, we can sum them

let $s = \sum_{i=1}^{d} x_i$

Thus, there are s ones and $(d-s)$ zeros

Now we can simplify the previous equations more

$$p(x|w_1) = \prod_{i=1}^{d} p(x_i|w_1) = p^s \cdot (1-p)^{d-s}$$

$$p(x|w_2) = \prod_{i=1}^{d} p(x_i|w_2) = (1-p)^s \; p^{d-s}$$

choose w_1 if $p(w_1)\,p(x|w_1) > p(w_2)\,p(x|w_2)$

since $p(w_1) = p(w_2) = \frac{1}{2}$

choose w_1 if $p(x|w_1) > p(x|w_2)$

$$p^s (1-p)^{d-s} > (1-p)^s \; p^{d-s}$$

$$\left(\frac{1-p}{p}\right)^{d-s} > \left(\frac{1-p}{p}\right)^{s}$$

$$\left(\frac{1-p}{p}\right)^{d} \left(\frac{1-p}{p}\right)^{-s} > \left(\frac{1-p}{p}\right)^{s}$$

$$\left(\frac{1-p}{p}\right)^{d}\left(\frac{1-p}{p}\right)^{-s} > \left(\frac{1-p}{p}\right)^{s}$$

$$\left(\frac{1-p}{p}\right)^{d} > \left(\frac{1-p}{p}\right)^{2s}$$

since $p > \frac{1}{2}$, then $2s > d$

$$2\sum_{i=1}^{d} x_i > d$$

$$\sum_{i=1}^{d} x_i > \frac{d}{2}$$

part III:

$$P(\text{boy}) = P(\text{girl}) = \frac{1}{2}$$

Let BB be the event of boy and boy

BG " " " " boy and Girle

GB ⋮ ⋮

GG

We want to know $P(BB|B) = \dfrac{P(B|BB)\,P(BB)}{P(B)}$?

$P(BB) = \frac{1}{4}$, $P(BG) = \frac{1}{4}$

$P(GB) = \frac{1}{4}$, $P(GG) = \frac{1}{4}$

		B M	B T	B W	B Th	B F	B S	B Su	G M	G T	G W	G Th	G F	G S	G Su
B	M	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	T	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	W	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
	Th	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	F	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	S	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	Su	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
G	M	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	T	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	W	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	Th	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	F	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	S	–	–	✓	–	–	–	–	–	–	–	–	–	–	–
	Su	–	–	✓	–	–	–	–	–	–	–	–	–	–	–

from the information above, we can find the following probabilites:

$$P(B|BB) = \frac{13}{49}$$

$$P(B|BG) = \frac{1}{7}$$

$$P(B|GB) = \frac{1}{7}$$

$$P(B|GG) = 0$$

using the total probability thm. we can find $P(B)$.

$$P(B) = P(B|BB) . P(BB) + P(B|BG)\,P(BG) + P(B|GB)\,P(GB) + P(B|GG)\,P(GG)$$

$$= \frac{13}{49} \cdot \frac{1}{4} + \frac{1}{7} \cdot \frac{1}{4} + \frac{1}{7} \cdot \frac{1}{4} + 0 \cdot \frac{1}{4}$$

$$= \frac{27}{49} \cdot \frac{1}{4}$$

$$P(BB|B) = \frac{P(B|BB) . P(BB)}{P(B)} = \frac{\frac{13}{49} \cdot \frac{1}{4}}{\frac{27}{49} \cdot \frac{1}{4}} = \frac{13}{27}$$

part IV:

$$p(x|w_i) = \frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma_i|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1} (x-\mu_i) \right]$$

$$\log(p(x|w_i)\,P(w_i)) = \log(P(x|w_i)) + \log(P(w_i))$$

$$\Rightarrow \log(p(x|w_i)) = \log\left(\frac{1}{(2\pi)^{\frac{d}{2}} |\Sigma_i|^{\frac{1}{2}}}\right) + \left(-\frac{1}{2}(x-\mu_i)^T \Sigma_i^{-1} (x-\mu_i)\right)$$

$$= -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log(|\sigma|) - \frac{1}{2\sigma^2}(x-\mu_i)^T(x-\mu_i)$$

$$\log(p(x|w_i)\,p(w_i)) = -\frac{d}{2}\log(2\pi) - \frac{1}{2}\log(\sigma) - \frac{1}{2\sigma^2}(x^T x - 2\mu_i^T x + \mu_i^T \mu_i) + \log(p(w_i))$$

The term $x^T x$ does not depend on variable i.

from the above equation

$$w_i = \frac{1}{\sigma^2} \mu_i^T x$$

$$w_{0i} = \frac{-d}{2} \log(2\pi) - \frac{1}{2} \log(\sigma) - \frac{1}{2\sigma^2} \mu_i^T \mu_i + \log(p(\omega_i))$$

$$\Rightarrow \quad w_i = \frac{1}{\sigma^2} \mu_i^T x$$

$$w_{0i} = \frac{-1}{2\sigma^2} \mu_i^T \mu_i + \log(p(\omega_i))$$

Safwan Elmadani ECE 523

Linear and Quadratic Classifiers:

In [1]:
```
import numpy as np
import plotly.express as px
import pandas as pd
import plotly.graph_objects as go
import seaborn as sns
import matplotlib.pyplot as plt
import math
import scipy as sp
%matplotlib notebook
```

- Write a general function to generate random samples from $\mathcal{N}(\mu, \Sigma)$ in d-dimensions (i.e., $\mu \in \mathbb{R}^d$ and $\Sigma \in \mathbb{R}^{d\times d}$).

In [2]:
```
def data_gen_multivariate_gaussian(mean, cov, sample_size):
    # mean: mean of the d-dimensional distribution
    # cov: covariance matrix and must be PSD matrix. 2 dimensions array (d,d)
    # sample_size: the number of samples
    # this function returns a random vector of d-dimensions
    x = np.random.multivariate_normal(mean,cov, sample_size)
    return x
```

Testing the function by generating random samples from a normal distribution in 2-dimentsions

In [3]:
```
x = data_gen_multivariate_gaussian([-1,-1], [[1, -0.25],[-0.25, 1]], 500)
y = data_gen_multivariate_gaussian([1,1], [[1, -0.25],[-0.25, 1]], 500)
x.shape
```

Out[3]: (500, 2)

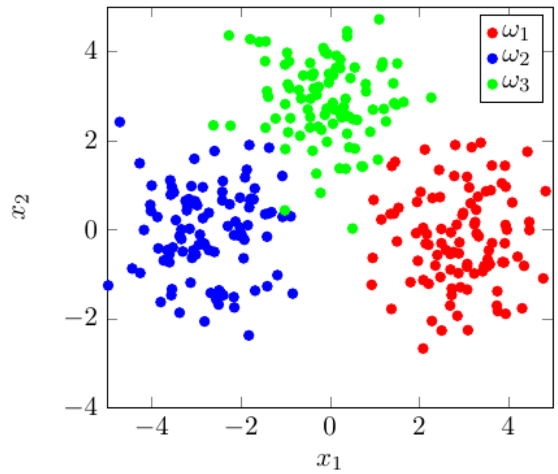
- Write a procedure of the discriminant of the following form

$$g_i(\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\mu_i)^{\mathsf{T}}\Sigma_i^{-1}(\mathbf{x}-\mu_i) - \frac{d}{2}\log(2\pi) - \frac{1}{2}\log(|\Sigma_i|) + \log(P(\omega_i)) \tag{1}$$

In [4]:
```
def discriminant(x, mu_i, cov_i, prior_prob):
    x_minus_mu = x - mu_i
    inv_sigma = np.linalg.inv(cov_i)
    temp= np.dot(x_minus_mu.T, inv_sigma)
    mahalanobis_distance= np.dot(temp, x_minus_mu)
    d = len(x)
    g_i = - (1/2) * mahalanobis_distance - (d/2) * math.log(2 * math.pi) \
            - (1/2) * math.log(np.linalg.det(cov_i)) \
            + prior_prob

    return g_i
```

- Generate a 2D dataset with three classes and use the quadratic classifier above to learn the parameters and make predictions. As an example, you should generate training data shown below to estimate the the parameters of the classifier in (1) and you should test the classifier on another randomly generated dataset. It is also sufficient to show the dataset used to train your classifier and the decision boundary it produces.



Generating 2D training dataset with 3 classes

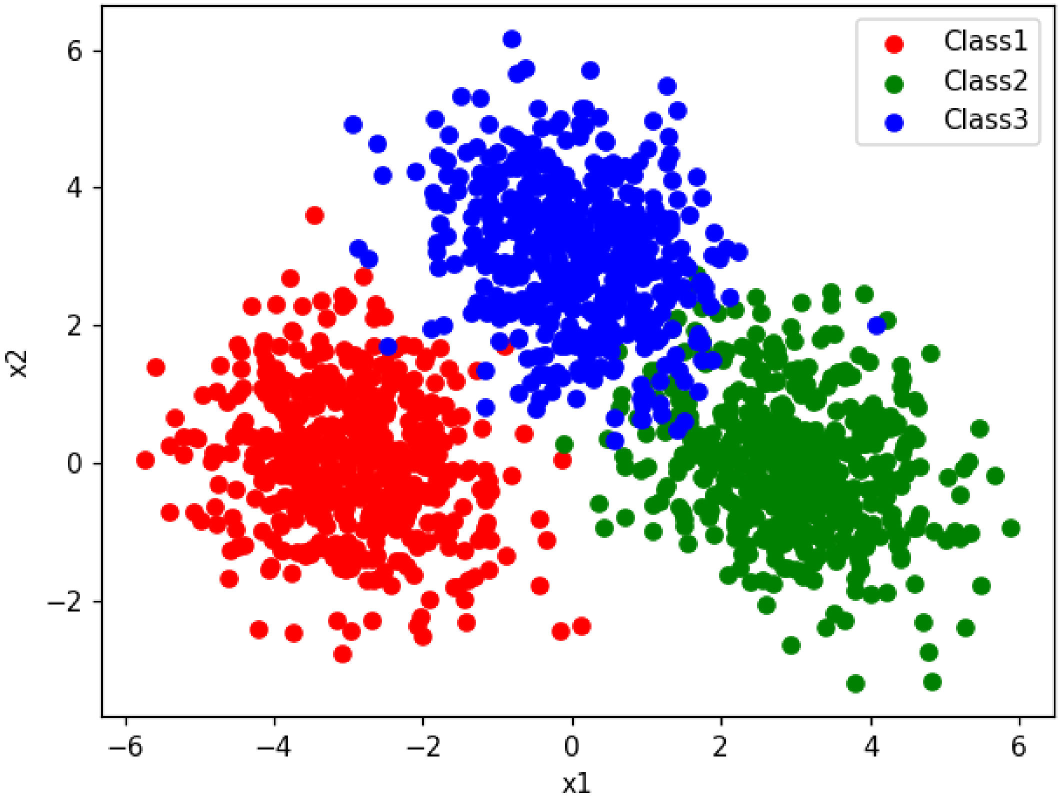
In [5]:

```
mu_1=[-3,0]
mu_2 = [3,0]
mu_3 = [0,3]
cov_1 = [[1, -0.25],[-0.25, 1]]
cov_2 = [[1, -0.25],[-0.25, 1]]
cov_3 = [[1, -0.25],[-0.25, 1]]
num_samples= 500

w1 = data_gen_multivariate_gaussian(mu_1, cov_1, num_samples)
w2 = data_gen_multivariate_gaussian(mu_2, cov_2, num_samples)
w3 = data_gen_multivariate_gaussian(mu_3, cov_3, num_samples)
#prior probability is the same for each class
prior_p=500/1500
#concatenate whole samples in one array
train_dataset=np.concatenate([w1,w2,w3])
```

In [6]:

```
figure1 = plt.figure()
plt.scatter(w1[:,0], w1[:,1], color='r', label='Class1')
plt.scatter(w2[:,0], w2[:,1], color='g', label='Class2')
plt.scatter(w3[:,0], w3[:,1], color='b', label='Class3')
# plt.rcParams['figure.figsize'] = [20, 20]
plt.xlabel('x1')
plt.ylabel('x2')
plt.legend()
# plt.grid()
plt.show()
```



Ploting the generated dataset

In [ ]:

Using the training data to estimate the parameters of the classifier

In [7]:
```
resutls =[]
belong_to = ''
for x in train_dataset:
    g_1 = discriminant(x,mu_1, cov_1, prior_p)
    g_2 = discriminant(x,mu_2, cov_2, prior_p)
    g_3 = discriminant(x,mu_3, cov_3, prior_p)
    #finding the index of the maximum value
    class_index= [g_1, g_2, g_3].index(max([g_1, g_2, g_3]))
    if class_index == 0:
        belong_to = 'class1'
    if class_index == 1:
        belong_to = 'class2'
    if class_index == 2:
        belong_to = 'class3'
    resutls.append([g_1, g_2, g_3, belong_to])
```

Classifing the traning data afer using the discriminat function

In [8]:
```
#elements in class 1
train_dataset = train_dataset.tolist()
w1_new = []
w2_new = []
```

```
w3_new = []

for i in resutls:
    if i[-1] == 'class1':
        w1_new.append(train_dataset[resutls.index(i)])
    if i[-1] == 'class2':
        w2_new.append(train_dataset[resutls.index(i)])
    if i[-1] == 'class3':
        w3_new.append(train_dataset[resutls.index(i)])
```

comparing classifier parameters

Original parameters:

mu_1=[-3,0]

mu_2 = [3,0]

mu_3 = [0,3]

cov_1 = [[1, -0.25],[-0.25, 1]]

cov_2 = [[1, -0.25],[-0.25, 1]]

cov_3 = [[1, -0.25],[-0.25, 1]]

In [9]:

```
w1_new_=np.array(w1_new)
mu_1new=np.mean(w1_new_, axis=0)
print( 'estimated mean 1:', mu_1new)
print('estimated cov. 1: \n',np.cov(w1_new_.T))
```

```
estimated mean 1: [-2.96926469  0.04866827]
estimated cov. 1:
 [[ 0.95113247 -0.19430101]
 [-0.19430101  1.01851877]]
```

In [10]:

```
w2_new_=np.array(w2_new)
mu_2new=np.mean(w2_new_, axis=0)
print( 'estimated mean 2:', mu_2new)
print('estimated cov. 2: \n',np.cov(w2_new_.T))
```

```
estimated mean 2: [ 2.9725136 -0.0280828]
estimated cov. 2:
 [[ 0.99857554 -0.18850639]
 [-0.18850639  0.93203163]]
```

In [11]:

```
w3_new_=np.array(w3_new)
mu_3new=np.mean(w3_new_, axis=0)
print( 'estimated mean 3:', mu_3new)
print('estimated cov. 3: \n',np.cov(w3_new_.T))
```

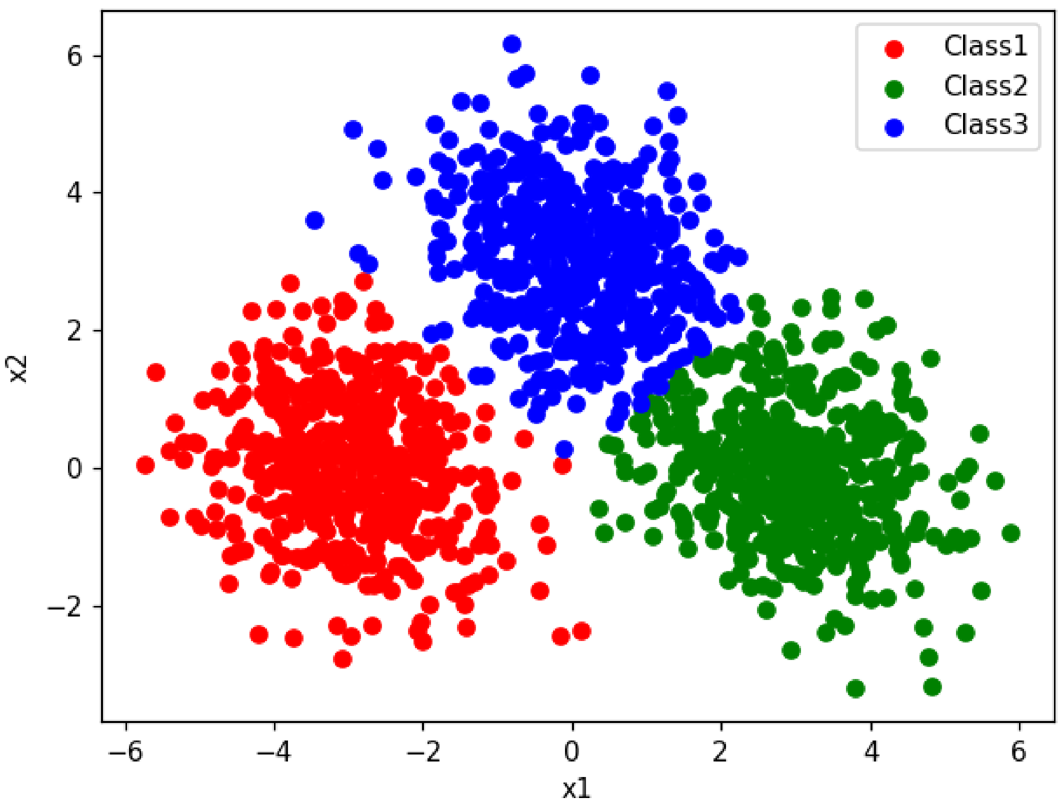
```
estimated mean 3: [0.0589787  3.03773288]
estimated cov. 3:
 [[ 0.85716762 -0.21489694]
 [-0.21489694  1.00718631]]
```

Ploting the classified dataset to show the decision boundary

In [12]:

```
figure2 = plt.figure()
plt.scatter(w1_new_[:,0], w1_new_[:,1], color='r', label='Class1')
plt.scatter(w2_new_[:,0], w2_new_[:,1], color='g', label='Class2')
plt.scatter(w3_new_[:,0], w3_new_[:,1], color='b', label='Class3')
# plt.rcParams['figure.figsize'] = [20, 20]
```

```
plt.xlabel('x1')
plt.ylabel('x2')
plt.legend()
# plt.grid()
plt.show()
```



In [ ]:

- Write a procedure for computing the Mahalanobis distance between a point $\mathbf{x}$ and some mean vector μ, given a covariance matrix Σ.

$$d_{\text{Mahal}}(\mathbf{x}, \mu, \Sigma) = (\mathbf{x} - \mu)^{\mathsf{T}} \Sigma^{-1} (\mathbf{x} - \mu)$$

In [13]:

```
def mahalanobi_dist(x, mu, cov):
    x_minus_mu = x - mu
    inv_sigma = np.linalg.inv(cov)
    temp= np.dot(x_minus_mu.T, inv_sigma)
    mahalanobis_distance= np.dot(temp, x_minus_mu)
    return mahalanobis_distance
```

Compating mahalanobis_dist function to the scipy implementation

https://docs.scipy.org/doc/scipy/reference/generated/scipy.spatial.distance.mahalanobis.html

In [14]:

```
some_x = np.array([2,0,0])
```

```
some_mu = np.array([0,1,0])
some_cov =np.array([[ 1.5, -0.5, -0.5],
                    [-0.5,  1.5, -0.5],
                    [-0.5, -0.5,  1.5]])
print ('my function: ', mahalanobi_dist(some_x, some_mu, some_cov) **0.5)
print ('scipy function: ', sp.spatial.distance.mahalanobis(some_x, some_mu, \
                                        np.linalg.inv(some_cov)))
```

```
my function:  1.7320508075688772
scipy function:  1.7320508075688772
```

- Implement the naïve Bayes classifier from scratch and then compare your results to that of Python's built-in implementation. Use different means, covariance matrices, prior probabilities (indicated by relative data size for each class) to demonstrate that your implementations are correct.

In [32]:

```
def cal_pdf(x_i, mu_w, sigma_w):
    """

    This function is used to calculate the pdf of normal distribution

    """
    e_power = - (1/(2 * sigma_w**2)) * (x_i - mu_w)**2
    pdf = (1/ (math.sqrt(2 * math.pi * sigma_w**2 ))) * math.exp(e_power)
    return pdf
```

In [33]:

```
#The cov matrix is not diagonal but we assume it is.
def Gaussian_naive_bayes(x, mu, cov, prior_p):
    # the summation is with respect the number of features in a vector.
    d=len(x)
    sigmas = np.diagonal(cov)
    liklihood = []
    for i in range(len(x)):
        g_pdf = cal_pdf(x[i], mu[i], sigmas[i])
        liklihood.append(np.log(g_pdf))
#     print (liklihood)
    score = np.log(prior_p) + np.sum(np.array(liklihood))
    return score
```

I will be using the same dataset to test Naive Bayes classifier

In [34]:

```
prior_p=500/1500
train_dataset=np.concatenate([w1,w2,w3])
w1.shape
```

Out[34]: (500, 2)

In [35]:

```
w1_l = w1.tolist()
w2_l = w2.tolist()
w3_l = w3.tolist()
datalist = []
for i in range(len(w1_l)):
    w1_l[i].append(1)
    w2_l[i].append(2)
    w3_l[i].append(3)

datalist.extend(w1_l)
```

```
datalist.extend(w2_l)
datalist.extend(w3_l)
```

In [ ]:

In [ ]:

In [36]:

```
bayes_results = []
for j in train_dataset:
    belong_to = ''
    s1 = Gaussian_naive_bayes(j, mu_1, cov_1, prior_p)
    s2 = Gaussian_naive_bayes(j, mu_2, cov_2, prior_p)
    s3 = Gaussian_naive_bayes(j, mu_3, cov_3, prior_p)
    #finding the index of the maximum value
    class_index= [s1, s2, s3].index(max([s1, s2, s3]))
    if class_index == 0:
        belong_to = 'class1'
    if class_index == 1:
        belong_to = 'class2'
    if class_index == 2:
        belong_to = 'class3'
    bayes_results.append([s1, s2, s3, belong_to])
```

In [37]:

```
w1b = []
w2b = []
w3b = []
train_dataset = train_dataset.tolist()
for i in bayes_results:

    if i[-1] == 'class1':
        w1b.append(train_dataset[bayes_results.index(i)])
    if i[-1] == 'class2':
        w2b.append(train_dataset[bayes_results.index(i)])
    if i[-1] == 'class3':
        w3b.append(train_dataset[bayes_results.index(i)])
```

comparing classifier parameters

Original parameters:

mu_1=[-3,0]

mu_2 = [3,0]

mu_3 = [0,3]

cov_1 = [[1, -0.25],[-0.25, 1]]

cov_2 = [[1, -0.25],[-0.25, 1]]

cov_3 = [[1, -0.25],[-0.25, 1]]

In [38]:

```
w1b=np.array(w1b)
print( 'estimated mean 1:', np.mean(w1b, axis=0))
print('estimated cov. 1: \n',np.cov(w1b.T))
```

```
estimated mean 1: [-2.97547076  0.05349335]
estimated cov. 1:
 [[ 0.93378841 -0.17971947]
 [-0.17971947  1.00892735]]
```

In [77]:
```
w2b=np.array(w2b)
print( 'estimated mean 2:', np.mean(w2b, axis=0))
print('estimated cov. 2: \n',np.cov(w2b.T))
```

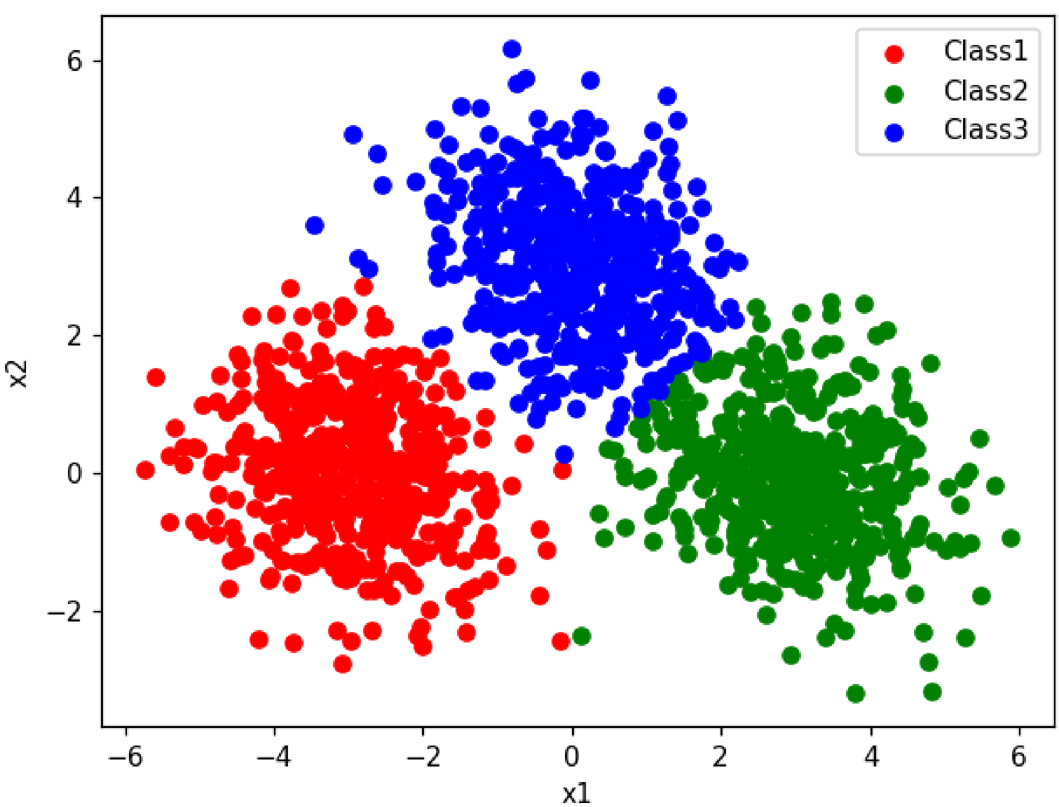
```
estimated mean 2: [ 2.96677687 -0.03276315]
estimated cov. 2:
 [[ 1.01291859 -0.17478193]
 [-0.17478193  0.94103966]]
```

In [78]:
```
w3b=np.array(w3b)
print( 'estimated mean 3:', np.mean(w3b, axis=0))
print('estimated cov. 3: \n',np.cov(w3b.T))
```

```
estimated mean 3: [0.0589787  3.03773288]
estimated cov. 3:
 [[ 0.85716762 -0.21489694]
 [-0.21489694  1.00718631]]
```

Ploting the classified dataset to show the decision boundary

In [41]:
```
figure3 = plt.figure()
plt.scatter(w1b[:,0], w1b[:,1], color='r', label='Class1')
plt.scatter(w2b[:,0], w2b[:,1], color='g', label='Class2')
plt.scatter(w3b[:,0], w3b[:,1], color='b', label='Class3')
# plt.rcParams['figure.figsize'] = [20, 20]
plt.xlabel('x1')
plt.ylabel('x2')
plt.legend()
# plt.grid()
plt.show()
```



In [ ]:

Problem II: Sampling from a Distribution Let the set $\mathcal{N} \in [1, \dots, n, \dots, N]$ be a set of integers and $\mathbf{p}$ be a probability distribution $\mathbf{p} = [p_1, \dots, p_n, \dots, p_N]$ such that p_k is the probability of observing $k \in \mathcal{N}$. Note that since $\mathbf{p}$ is a distribution then $1^\mathsf{T}\mathbf{p} = 1$ and $0 \leq p_k \leq 1\ \forall n$. Write a function $\texttt{sample}(M, \mathbf{p})$ that returns M indices sampled from the distribution $\mathbf{p}$. Provide evidence that your function is working as desired. Note that all sampling is assumed to be i.i.d. You must include a couple of paragraphs and documented code that discusses how you were able to accomplish this task.

The approach that was used to solve this problem was to first find the cdf of the probability distribution, then draw a sample from uniform distribution. Then using the cdf we find the range in which the sample lies. We repeat this process until we gather M samples.

In [42]:

```
def sample(M , p):
    """
    This function return a list of M indices sampled from the distribution p

    M: sample size
    p: distribution to be used in sampling
    """
    temp = 0.0
    cdf_p = [] #list to hold the cdf ranges
    index_list = [] #list of indices to be returned
    for i in p:
        cdf_p.append(temp + i)
        temp = cdf_p[-1]

    for j in range(M):
        #for every iteration
        #draw a sample from a uniform distribution
        sample_uniform = np.random.uniform(0.0,1.0,1)
        temp = 0.0
        #Now we need to find in what range the sample lies with respect to the c
        for k in range(len(cdf_p)):
            if (temp <= sample_uniform) and (sample_uniform <= cdf_p[k]):
                index_list.append(k)
            temp = cdf_p[k]

    return index_list
```

In [75]:

```
#tesitng the function
M = 1000
N = 4
r = np.random.uniform(0.0,1.0,N)
p = r/r.sum()
print ("Original distribution p: ", p)
```

```
Original distribution p:  [0.02559924 0.13567054 0.46809595 0.37063427]
```

In [76]:

```
est_dist = []
list_indices = sample(M, p)
for i in range(len(p)):
    est_dist.append(list_indices.count(i))
estimated_p = np.array(est_dist)/M
print('Estimated distribution p\': ', estimated_p)
```

```
Estimated distribution p':  [0.021 0.138 0.469 0.372]
```

In [ ]:

```

```

In [ ]:

```

```