Received: 16 March 2020 | Accepted: 9 December 2020

DOI: 10.1002/mana.202000126

MATHEMATISCHE NACHRICHTEN

ORIGINAL PAPER

Four-dimensional gradient almost Ricci solitons with harmonic Weyl curvature

Jongsu Kim

Dept. of MathematicsSogang University, Seoul, Korea

Correspondence
Jongsu Kim, Dept. of Mathematics, Sogang University, Seoul, Korea.
Email: jskim@sogang.ac.kr

Funding information
National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT), Grant/Award Number: 2020R1A2B5B01001862

Abstract

In this article we make a classification of four-dimensional gradient almost Ricci solitons with harmonic Weyl curvature. We prove first that any four-dimensional (not necessarily complete) gradient almost Ricci soliton (M, g, f, λ) with harmonic Weyl curvature has less than four distinct Ricci-eigenvalues at each point. If it has three distinct Ricci-eigenvalues at each point, then (M, g) is locally a warped product with 2-dimensional base in explicit form, and if g is complete in addition, the underlying smooth manifold is $\mathbb{R}^2 \times M_k^2$ or $\left(\mathbb{R}^2 - \{(0,0)\}\right) \times M_k^2$. Here M_k^2 is a smooth surface admitting a complete Riemannian metric with constant curvature k. If (M, g) has less than three distinct Ricci-eigenvalues at each point, it is either locally conformally flat or locally isometric to the Riemannian product $\mathbb{R}^2 \times N_\lambda^2$, $\lambda \neq 0$, where $\mathbb{R}^2$ has the Euclidean metric and N_λ^2 is a 2-dimensional Riemannian manifold with constant curvature λ. We also make a complete description of four-dimensional gradient almost Ricci solitons with harmonic curvature.

KEYWORDS
gradient Ricci soliton, harmonic curvature, harmonic Weyl curvature

MSC (2020)
53C21, 53C25

1 | INTRODUCTION

Ricci solitons are natural generalizations of Einstein metrics and also serve as singularity models in the Ricci flow theory introduced by Hamilton [16]. These solitons have been intensely studied in recent years. In [21] Pigola et al. have defined a modified concept; a Riemannian manifold (M, g) is called *an almost Ricci soliton* (or a *Ricci almost soliton* in some literature) if there exist a vector field X and a function λ on M satisfying

$$\frac{1}{2}\mathcal{L}_X g + Rc = \lambda g \tag{1.1}$$

where $\mathcal{L}$ and Rc denote, respectively, the Lie derivative and the Ricci tensor of g. When X is the gradient of a function f, the manifold (M, g) is called a gradient almost Ricci soliton and the preceding equation becomes

$$\nabla df + Rc = \lambda g. \tag{1.2}$$

We may denote the above gradient almost Ricci soliton by (M, g, f, λ). If λ is a constant in (1.1) or (1.2), respectively, then (M, g) is a Ricci soliton or a gradient Ricci soliton. A gradient almost Ricci soliton is said to be trivial if f is a constant.

In [21] the authors presented some introductory properties of gradient almost Ricci solitons, and in particular, classified all those which are Einstein metrics. In [4] Hitchin–Thorpe inequality was shown for almost Ricci solitons under some condition. Compact gradient almost Ricci solitons were studied in [3], so that if they have either constant scalar curvature or an associated conformal vector field, then they are isometric to a Euclidean sphere; see also [2]. It was proved in [9] that a locally conformally flat gradient almost Ricci soliton of dimension $n \geq 3$, near any regular point of f, is a warped product with an $(n-1)$-dimensional fiber of constant curvature. Locally homogeneous almost Ricci solitons were characterized in [7], and half conformally flat gradient ones in [5]. In [10] a particular type of almost Ricci solitons, called gradient Einstein solitons, were investigated. In [19] some duality among gradient almost Ricci solitons was explored. Any Kähler gradient almost Ricci soliton of complex dimension ≥ 2 was proved to be a gradient Ricci soliton in [13, 19].

We also note that some physical interpretation of semi-Riemannian almost Ricci solitons was given as imperfect fluid spacetimes in [14] and that almost Ricci solitons were generalized to h-almost Ricci soltons in [15].

In this paper we classify four-dimensional gradient almost Ricci solitons which have harmonic Weyl curvature. This work generalizes the previous studies in [5, 9, 17, 21] and provides explicit new examples.

The followings are main theorems of this paper. We present them according to the number of distinct Ricci-eigenvalues.

Theorem 1.1. *Any four-dimensional (not necessarily complete) gradient almost Ricci soliton with harmonic Weyl curvature has less than four distinct Ricci-eigenvalues at each point.*

Theorem 1.2. *Let (M, g, f, λ) be any four-dimensional (not necessarily complete) gradient almost Ricci soliton with harmonic Weyl curvature and exactly three distinct Ricci-eigenvalues at each point. Then the following holds.*

In a neighborhood V_{p_0} of each point p_0 in an open dense subset, there exist coordinates (τ, t, x_3, x_4) in which, for constants $c_1 \neq 0$, k and $\tilde{c}$, we may write g as

$$g = \frac{1}{\left(c_1 e^{-\tau} - k\right)^2 Q(\tau)} d\tau^2 + e^{2\tau} Q(\tau) dt^2 + e^{2\tau} \tilde{g}, \tag{1.3}$$

where

$$Q(\tau) = -\frac{2}{c_1^2}\left(c_1 e^{-\tau} - k + k \ln |c_1 e^{-\tau} - k|\right) + \tilde{c} \geq 0,$$

and $\tilde{g}$ is (a pull-back of) a Riemannian metric of constant curvature k on a 2-dimensional domain with x_3, x_4 coordinates. The soliton functions are as follows; $f = -\ln |c_1 e^{-\tau} - k| + C$, for a constant C, and $\lambda = \left(c_1 e^{-\tau} - k\right)\left(3k - c_1 e^{-\tau}\right) Q - \left(2c_1 e^{-\tau} - 3k\right) e^{-2\tau}$.

Furthermore, in the four cases (i)–(iv) below, g in (1.3) yields a complete Riemannian metric and (g, f, λ) form a gradient almost Ricci soliton on the smooth manifold $\mathbb{R}^2 \times M_k^2$ or $\left(\mathbb{R}^2 - \{(0,0)\}\right) \times M_k^2$. Here M_k^m is an m-dimensional manifold admitting a complete Riemannian metric with constant curvature k.

(i) $k < 0$, $c_1 > 0$,
(ii) $k < 0$, $c_1 < 0$ *and* $c_1 - k e^{\tau} > 0$,
(iii) $k > 0$, $c_1 > 0$,
(iv) $k = 0$, $c_1 > 0$.

Theorem 1.3. *Let (M, g, f, λ) be any four-dimensional (not necessarily complete) gradient almost Ricci soliton with harmonic Weyl tensor and exactly two distinct Ricci-eigenvalues at each point. Then in a neighborhood V_{p_0} of each point p_0 in an open dense subset, $\left(V_{p_0}, g\right)$ is isometric to one of the following:*

(i) *A domain in the Riemannian product $\mathbb{R}^2 \times N_\lambda^2$, $\lambda \neq 0$, where $\mathbb{R}^2$ has the Euclidean metric and N_λ^m is an m-dimensional Riemannian manifold with constant curvature λ. Moreover, $f = \frac{\lambda}{2}|x|^2$ modulo a constant on the Euclidean factor $\mathbb{R}^2$.*

(ii) *A domain in a warped product $I \times_h M_k^3$ with the metric $ds^2 + h(s)^2\tilde{g}$, where I is an open interval in $\mathbb{R}$ and $\tilde{g}$ has constant curvature k. In particular, g is locally conformally flat.*

The case of exactly one eigenvalue is explained in [21]. We just write a simple local version in Lemma 6.4 below.
We give a classification of four-dimensional complete gradient almost Ricci solitons with harmonic curvature.

Theorem 1.4. *Let (M, g, f, λ) be any nontrivial complete connected four-dimensional gradient almost Ricci soliton with harmonic curvature. Then (M, g) has less than three distinct Ricci-eigenvalues at each point and is isometric to one of the followings:*

(i) *A finite quotient of the Riemannian product $\mathbb{R}^2 \times N_\lambda^2$, $\lambda \neq 0$, as in Theorem 1.3 (i).*
(ii) *A finite quotient of a warped product $I \times_h M_k^3$ with the metric $ds^2 + h(s)^2\tilde{g}$, where I is an open interval in $\mathbb{R}$ and $\tilde{g}$ has constant curvature k. The function h satisfies $(h')^2 + \frac{R}{12}h^2 + \frac{\tilde{a}}{h^2} = k$ for a constant $\tilde{a}$ and the scalar curvature R, under the parameter conditions of Kobayashi's cases (I), (II), (IV), (V) and (VI) in [18, p. 670]. Moreover, f and λ are given by (7.11) and (7.6).*

To prove the above theorems, we first derive geometric properties from (1.2) and the harmonic Weyl condition. In particular, we show that ∇f is a Ricci-eigen vector field and that all Ricci eigenvalues as well as most geometric quantities depend only on one variable $s = \int \frac{df}{|df|}$. Then we apply these to characterize the metric and analyze the eigenvalues, and eventually convert (1.2) and the harmonic Weyl condition into a system of ordinary differential equations.

As such, we follow the argument framework of [17] on gradient Ricci solitons. In the current work we had to overcome new difficulties arising from the non-constancy of the function λ. One main difference is that in the three-eigenvalues case of Theorem 1.2 we obtain families of explicit examples.

This paper is organized as follows. In Section 2 we explain some properties of gradient almost Ricci solitons with harmonic Weyl curvature. In Section 3 we consider a local Ricci-eigen orthonormal frame field $\left\{E_1 := \frac{\nabla f}{|\nabla f|}, \dots, E_4\right\}$ and the corresponding eigenvalues λ_i. And we prove that no four-dimensional gradient almost Ricci solitons with harmonic Weyl curvature can have pairwise distinct $\lambda_2, \lambda_3, \lambda_4$. This proves Theorem 1.1. In Section 4 we develop properties in the case of $\lambda_2 \neq \lambda_3 = \lambda_4$. In Section 5, we continue analyzing in detail the case of $\lambda_2 \neq \lambda_3 = \lambda_4$ and prove Theorem 1.2. In Section 6, we treat the case of one or two eigenvalues. In particular, we prove Theorem 1.3. In Section 7, we make a classification of four-dimensional gradient almost Ricci solitons with harmonic curvature.

2 | PRELIMINARIES

Recall that a Riemannian metric is said to have harmonic Weyl curvature or harmonic curvature if the divergence of the Weyl curvature tensor or the Riemannian curvature tensor, respectively, is zero. In this section we present some properties of gradient almost Ricci soliton with harmonic Weyl curvature. These can be obtained by slight modification of arguments in [8], [12], [17] and formulas of [3].

Lemma 2.1. *For an n-dimensional gradient almost Ricci soliton $\left(M^n, g, f, \lambda\right)$ with harmonic Weyl curvature, it holds that*

$$R(X, Y, Z, \nabla f) = d\left(\frac{R}{2(n-1)} - \lambda\right)(X)g(Y, Z) - d\left(\frac{R}{2(n-1)} - \lambda\right)(Y)g(X, Z)$$
$$= \frac{1}{n-1}R(\nabla f, X)g(Y, Z) - \frac{1}{n-1}R(\nabla f, Y)g(X, Z).$$

Proof. The tensor $\mathcal{A} := Rc - \frac{R}{2(n-1)}g$ of a Riemannian metric with harmonic Weyl curvature is a Codazzi tensor i.e., $d^\nabla \mathcal{A} = 0$, written in local coordinates as $\nabla_k \mathcal{A}_{ij} = \nabla_i \mathcal{A}_{kj}$. By Ricci identity, $\nabla_i \nabla_j \nabla_k f - \nabla_j \nabla_i \nabla_k f = -R_{ijkl}\nabla^l f$.

(1.2) gives

$$
\begin{aligned}
-R_{ijkl}\nabla^l f &= \nabla_i\{-R_{jk}+\lambda g_{jk}\} - \nabla_j\{-R_{ik}+\lambda g_{ik}\} \\
&= \nabla_i\lambda g_{jk} - \nabla_j\lambda g_{ik} - \frac{1}{2(n-1)}\left(\nabla_i R g_{jk} - \nabla_j R g_{ik}\right).
\end{aligned}
$$

We have got

$$
R(X,Y,Z,\nabla f) = d\left(\frac{R}{2(n-1)} - \lambda\right)(X)g(Y,Z) - d\left(\frac{R}{2(n-1)} - \lambda\right)(Y)g(X,Z).
$$

We recall $\frac{1}{n-1}R(\nabla f,\cdot) = d\left(\frac{R}{2(n-1)} - \lambda\right)$ from Proposition 1 in [3]. This yields the lemma. □

Lemma 2.2. *Let (M^n, g, f, λ) be a gradient almost Ricci soliton with harmonic Weyl curvature and non-constant f. Let c be a regular value of f and $\Sigma_c = \{x \in M \mid f(x) = c\}$ be the level surface of f. Then the followings hold;*

(i) Where $\nabla f \neq 0$, $E_1 := \frac{\nabla f}{|\nabla f|}$ is an eigenvector field of Rc.
(ii) $|\nabla f|$ is constant on a connected component of Σ_c.
(iii) There is a function s locally defined with $s(x) = \int \frac{df}{|\nabla f|}$, so that $ds = \frac{df}{|\nabla f|}$ and $E_1 = \nabla s$.
(iv) $E_1E_1 f = -R(E_1,E_1) + \lambda$, in particular $-R(E_1,E_1) + \lambda$ is constant on a connected component of Σ_c.
(v) Near a point in Σ_c, the metric g can be written as

$$
g = ds^2 + \sum_{i,j>1} g_{ij}(s, x_2, \dots, x_n)dx_i \otimes dx_j,
$$

where $x_2, \dots x_n$ is a local coordinates system on Σ_c.
(vi) $\nabla_{E_1}E_1 = 0$.

Proof. In Lemma 2.1, put $Y = Z = \nabla f$ and $X \perp \nabla f$ to get

$$
0 = R(X,\nabla f,\nabla f,\nabla f) = \frac{1}{n-1}R(\nabla f, X)g(\nabla f,\nabla f).
$$

So, $R(\nabla f, X) = 0$. Hence $E_1 = \frac{\nabla f}{|\nabla f|}$ is an eigenvector of Rc, proving (i). Also, $\frac{1}{2}\nabla_X|\nabla f|^2 = \langle\nabla_X\nabla f,\nabla f\rangle = -R(\nabla f, X) = 0$ for $X \perp \nabla f$. We proved (ii).

Next

$$
d\left(\frac{df}{|\nabla f|}\right) = -\frac{1}{2|\nabla f|^{\frac{3}{2}}}d|\nabla f|^2 \wedge df = 0
$$

as $\nabla_X(|\nabla f|^2) = 0$. So, (iii) is proved. We have $\nabla_{E_1}E_1(f) = \sum_i\langle\nabla_{E_1}E_1, E_i\rangle E_i(f) = \langle\nabla_{E_1}E_1, E_1\rangle E_1(f) = 0$ for an orthonormal frame field $\{E_i\}$ extending E_1. We have from (1.2) $\nabla df(E_1,E_1) = -R(E_1,E_1) + \lambda$, which equals $E_1E_1f - (\nabla_{E_1}E_1)f = E_1E_1f$. As f is a function of the local variable s only, $E_1 f = \nabla s(f) = \frac{df}{ds}$ and $E_1E_1 f = E_1\left(\frac{df}{ds}\right) = \frac{d^2 f}{ds^2}$ depend on s only. So, $-R(E_1,E_1) + \lambda$ is constant on a connected component of Σ_c.

As ∇f and the level surfaces of f are perpendicular, one gets (v). One uses (v) to compute the Christoffel symbols and gets (vi). □

Following Derdziński [12], for the Codazzi tensor $\mathcal{A} := Rc - \frac{R}{2(n-1)}g$ and a point x in M, let $E_{\mathcal{A}}(x)$ be the number of distinct eigenvalues of $\mathcal{A}_x$, and set $M_{\mathcal{A}} = \{x \in M \mid E_{\mathcal{A}}$ is constant in a neighborhoodof $x\}$, so that $M_{\mathcal{A}}$ is an open dense subset of M. Then we recall the following; see the Section 2 of [17].

Lemma 2.3. *For a Riemannian metric g of dimension $n \geq 4$ with harmonic Weyl curvature, consider orthonormal vector fields E_i, $i = 1, \ldots n$ such that $Rc(E_i, \cdot) = \lambda_i g(E_i, \cdot)$. Then the followings hold in each connected component of $M_{\mathcal{A}}$;*

(i) $(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k \rangle + \nabla_{E_i}(\mathcal{A}(E_j, E_k)) = (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k \rangle + \nabla_{E_j}(\mathcal{A}(E_k, E_i))$, *for any $i, j, k = 1, \ldots n$.*
(ii) If $k \neq i$ and $k \neq j$, $(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k \rangle = (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k \rangle$.
(iii) Given distinct eigenfunctions λ, μ of $\mathcal{A}$ and local vector fields v, u such that $\mathcal{A}v = \lambda v$, $\mathcal{A}u = \mu u$ with $|u| = 1$, it holds that $v(\mu) = (\mu - \lambda)\langle \nabla_u u, v \rangle$.
(iv) For each eigenfunction λ of $\mathcal{A}$, the λ-eigenspace distribution is integrable and its leaves are totally umbilic submanifolds of M.

3 | FOUR-DIMENSIONAL GRADIENT ALMOST RICCI SOLITON WITH DISTINCT $\lambda_2, \lambda_3, \lambda_4$

Consider a local Ricci-eigen orthonormal frame field E_i, $i = 1, \ldots, n$, with $E_1 = \nabla s$ as in Lemma 2.2. This shall be called an *adapted frame field* of (M, g, f, λ). We prove:

Lemma 3.1. *Let (M, g, f, λ) be a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature and non-constant f. Then the scalar curvature R, $R(E_1, E_1)$ and λ are constant on a connected component of $\Sigma_c = f^{-1}(c)$ for a regular value c of f.*

Proof. From Lemma 2.3 (i), apply $j = k = 1$, $i \geq 2$ to get

$$\nabla_{E_i}\left(\lambda_1 - \frac{R}{6}\right) = (\lambda_i - \lambda_1)\langle \nabla_{E_1} E_i, E_1 \rangle = -(\lambda_i - \lambda_1)\langle E_i, \nabla_{E_1} E_1 \rangle = 0$$

from Lemma 2.2 (vi). So, $\lambda_1 - \frac{R}{6}$ is constant on a connected component of Σ_c; equivalently it depends only on the local variable s. From Lemma 2.2 (iv), $\lambda_1 - \lambda$ depends only on s and so does $\frac{R}{6} - \lambda$.

Put $X = E_1$, $Y = Z = E_i$ into Lemma 2.1, and the sectional curvature corresponding to E_1 and E_i is

$$R_{1ii1} := R(E_1, E_i, E_i, E_1) = \frac{E_1\left(\frac{R}{6} - \lambda\right)}{|\nabla f|}, \tag{3.1}$$

which depends on s only by Lemma 2.2. Then $\lambda_1 = \sum_{i=2}^4 R_{1ii1}$ depends only on s. And R and λ depend only on s. □

For an adapted frame field we set $\zeta_i := -\langle \nabla_{E_i} E_i, E_1 \rangle = \langle E_i, \nabla_{E_i} E_1 \rangle$, for $i > 1$. As

$$\nabla_{E_i} E_1 = \nabla_{E_i}\left(\frac{\nabla f}{|\nabla f|}\right) = \frac{\nabla_{E_i} \nabla f}{|\nabla f|} = \frac{-R(E_i, \cdot) + \lambda g(E_i, \cdot)}{|\nabla f|},$$

so we write

$$\nabla_{E_i} E_1 = \zeta_i E_i \quad \text{where} \quad \zeta_i = \frac{-\lambda_i + \lambda}{|\nabla f|}. \tag{3.2}$$

Lemma 3.2. *Let (M, g, f, λ) be a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. The Ricci eigen-functions λ_i associated to an adapted frame field E_i are constant on a connected component of a regular level hypersurface Σ_c of f, and so depend on the local variable s only. And ζ_i, $i = 2, 3, 4$, in (3.2) also depend on s only. In particular, we have $E_i(\lambda_j) = E_i(\zeta_k) = 0$ for $i, k > 1$ and any j.*

Proof. We write $R_{ij} := R(E_i, E_j)$. First, R and $\lambda_1 = R_{11}$ depend on s only by Lemma 3.1. Writing the Hessian $\nabla_j \nabla_i R := \nabla_{E_j} \nabla_{E_i} R$, by (3.2) we compute the following

$$
\begin{aligned}
(\nabla_k \nabla_j (R-6\lambda)) R_{jk} &= (\nabla_1 \nabla_1 (R-6\lambda))\lambda_1 + \sum_{i>1} (\nabla_i \nabla_i (R-6\lambda))\lambda_i \\
&= (R-6\lambda)'' \lambda_1 + \sum_{i>1} \{E_i E_i (R-6\lambda) - (\nabla_{E_i} E_i)(R-6\lambda)\}\lambda_i \\
&= (R-6\lambda)'' \lambda_1 - \sum_{i>1} \frac{(R-6\lambda)'}{|\nabla f|} (\lambda_i^2 - \lambda \cdot \lambda_i). \qquad (3.3)
\end{aligned}
$$

In the above F' denotes the derivative of a function $F = F(s)$. Similar computation gives

$$
\sum_{j=1}^{4} \nabla_j \nabla_j (R-6\lambda) = (R-6\lambda)'' - \frac{(R-6\lambda)'}{|\nabla f|}(R - \lambda_1 - 3\lambda),
$$

which depends only on s. We drop summation symbols using the Einstein summation convention below. From $2R(\nabla f, \cdot) = \nabla(R-6\lambda)$,

$$
\sum_{j=1}^{4} \frac{1}{2} \nabla_j \nabla_j (R-6\lambda) = \nabla_j (f_i R_{ij}) = f_{ij} R_{ij} + f_i \nabla_j R_{ij} = -R_{ij} R_{ij} + \lambda R + \frac{1}{2} f' R'.
$$

So, $R_{ij} R_{ij}$ depends only on s.

We shall use the Codazzi equation $\nabla_k R_{ij} = \nabla_i R_{kj} - \frac{R_i}{6} g_{kj} + \frac{R_k}{6} g_{ij}$.

$$
\begin{aligned}
\nabla_k (f_i R_{ij} R_{jk}) &= f_{ik} R_{ij} R_{jk} + f_i (\nabla_k R_{ij}) R_{jk} + f_i R_{ij} \nabla_k R_{jk} \\
&= -(R_{ik} - \lambda g_{ik}) R_{ij} R_{jk} + f_i \left(\nabla_i R_{kj} - \frac{R_i}{6} g_{kj} + \frac{R_k}{6} g_{ij} \right) R_{jk} + \frac{1}{2} f_i R_{ij} R_j \\
&= -R_{ik} R_{ij} R_{jk} + L(s), \qquad (3.4)
\end{aligned}
$$

where $L(s)$ consists of terms depending on s only; note that $f_i(\nabla_i R_{kj}) R_{jk} = \frac{1}{2} f_i \nabla_i (R_{kj} R_{jk}) = \frac{1}{2} f' (R_{kj} R_{jk})'$. From $2R(\nabla f, \cdot) = \nabla(R-6\lambda)$ and (3.3), we get

$$
\begin{aligned}
2\nabla_k (f_i R_{ij} R_{jk}) &= \nabla_k \{\nabla_j (R-6\lambda) R_{jk}\} \\
&= (R-6\lambda)'' R_{11} - \sum_{i>1} \frac{(R-6\lambda)'}{|\nabla f|} (\lambda_i^2 - \lambda \lambda_i) + \frac{1}{2} (R-6\lambda)' R',
\end{aligned}
$$

which depends only on s. So, we compare this with (3.4) to see that $\sum_{i=1}^4 (\lambda_i)^3$ depends only on s. As R, λ_1 and $\sum_{i=1}^4 (\lambda_i)^2$ depend only on s, each λ_i depends only on s. By (3.2), ζ_i, $i = 2, 3, 4$ depend on s only. □

We set $\Gamma_{ij}^k := \langle \nabla_{E_i} E_j, E_k \rangle$. Lemma 2.3 (ii) and (3.2) gives $\Gamma_{ij}^k = \Gamma_{ji}^k \frac{\zeta_k - \zeta_i}{\zeta_k - \zeta_j}$, for distinct $i, j, k > 1$.

Lemma 3.3. *Let (M, g, f, λ) be a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. Suppose that for an adapted frame field $\{E_j\}$ in an open subset W of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, the eigenfunctions $\lambda_2, \lambda_3, \lambda_4$ are pairwise different at any point. Then in W we have the following.*

For $i, j > 1$, $i \neq j$,

$$\nabla_{E_1} E_1 = 0, \quad \nabla_{E_i} E_1 = \zeta_i E_i, \quad \nabla_{E_i} E_i = -\zeta_i E_1, \quad \nabla_{E_1} E_i = 0, \quad \nabla_{E_i} E_j = \sum_{k \neq 1,i,j} \Gamma_{ij}^k E_k,$$

$$\begin{aligned}
R_{1ii1} &= -\zeta_i' - \zeta_i^2, \quad R_{1ij1} = 0,\\
R_{11} &= -\zeta_2' - \zeta_2^2 - \zeta_3' - \zeta_3^2 - \zeta_4' - \zeta_4^2,\\
R_{22} &= -\zeta_2' - \zeta_2^2 - \zeta_2\zeta_3 - \zeta_2\zeta_4 - 2\Gamma_{34}^2\Gamma_{43}^2,\\
R_{33} &= -\zeta_3' - \zeta_3^2 - \zeta_3\zeta_2 - \zeta_3\zeta_4 + 2\frac{(\zeta_2 - \zeta_4)}{\zeta_3 - \zeta_4}\Gamma_{34}^2\Gamma_{43}^2,\\
R_{44} &= -\zeta_4' - \zeta_4^2 - \zeta_4\zeta_2 - \zeta_4\zeta_3 + 2\frac{(\zeta_2 - \zeta_3)}{\zeta_4 - \zeta_3}\Gamma_{34}^2\Gamma_{43}^2.
\end{aligned}$$

Proof. $\nabla_{E_1} E_1 = 0$ from Lemma 2.2 (vi) and $\nabla_{E_i} E_1 = \zeta_i E_i$ from (3.2). Let $i, j > 1$ be distinct. From Lemma 2.3 (iii) and Lemma 3.2, $\langle \nabla_{E_i} E_i, E_j \rangle = 0$. And $\langle \nabla_{E_i} E_i, E_1 \rangle = -\langle E_i, \nabla_{E_i} E_1 \rangle = -\zeta_i$. So, we get $\nabla_{E_i} E_i = -\zeta_i E_1$. Now, $\langle \nabla_{E_i} E_j, E_i \rangle = -\langle \nabla_{E_i} E_i, E_j \rangle = 0$, $\langle \nabla_{E_i} E_j, E_j \rangle = 0$. And $\langle \nabla_{E_i} E_j, E_1 \rangle = -\langle \nabla_{E_i} E_1, E_j \rangle = 0$. So, $\nabla_{E_i} E_j = \sum_{k \neq 1,i,j} \Gamma_{ij}^k E_k$. Clearly $\Gamma_{ij}^k = -\Gamma_{ik}^j$. From Lemma 2.3 (ii), $(\lambda_i - \lambda_j)\langle \nabla_{E_1} E_i, E_j \rangle = (\lambda_1 - \lambda_j)\langle \nabla_{E_i} E_1, E_j \rangle$. As $\langle \nabla_{E_i} E_1, E_j \rangle = 0$, $\langle \nabla_{E_1} E_i, E_j \rangle = 0$. This gives $\nabla_{E_1} E_i = 0$. Then from Lemma 3.2 we can compute other formulas. □

Proposition 3.4. *Let* (M, g, f, λ) *be a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. Consider an adapted frame field* $\{E_j\}$ *on an open set* U *in* $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$*. Then it does not hold that the eigenfunctions* $\lambda_2, \lambda_3, \lambda_4$ *are pairwise different at any point of* U.

Proof. We set $a := \zeta_2$, $b := \zeta_3$ and $c := \zeta_4$. Suppose that a, b, c are pairwise different, i.e. $a \neq b, b \neq c$ and $c \neq a$. We have the three equations $\zeta_i f' = -R_{ii} + \lambda$, $i = 2, 3, 4$, obtained by feeding (E_i, E_i) to (1.2). From these three equations and Lemma 3.3, we can eliminate λ and get

$$4\Gamma_{34}^2\Gamma_{43}^2 = \frac{1}{P}(a-b)(a-c)(b-c)^2, \tag{3.5}$$

$$f' = \frac{1}{2P}\left(a^2b + a^2c + ab^2 + ac^2 + b^2c + c^2b - 6abc\right), \tag{3.6}$$

where $P := a^2 + b^2 + c^2 - ab - bc - ac$. Lemma 3.3 and (3.5) tell that Γ_{ij}^k, $i, j, k > 1$, all depend on s only. Using $b' - a' = a^2 - b^2$, $c' - a' = a^2 - c^2$ and (3.6), we compute

$$(\ln P)' = \frac{\left(a^2 + b^2 + c^2 - ab - bc - ac\right)'}{a^2 + b^2 + c^2 - ab - bc - ac} = -2(a + b + c) + 2f'.$$

From Lemma 3.3, we may write

$$[E_2, E_3] = \alpha E_4, \quad [E_3, E_4] = \beta E_2, \quad [E_4, E_2] = \gamma E_3. \tag{3.7}$$

We compute $\alpha = \Gamma_{23}^4 - \Gamma_{32}^4 = \Gamma_{34}^2 \frac{\zeta_3 - \zeta_2}{\zeta_3 - \zeta_4}$. And from (3.5) $\alpha^2 = \frac{(a-b)^4}{4P}$. Take log of this and differentiate to get

$$(\ln|\alpha|)' = c - a - b - f'. \tag{3.8}$$

From Jacobi identity $[[E_1, E_2], E_3] + [[E_2, E_3], E_1] + [[E_3, E_1], E_2] = 0$,

$$E_1(\alpha) = \alpha(c - a - b). \tag{3.9}$$

From (3.8) and (3.9) we get $f' = 0$. So, g is an Einstein metric and $\lambda_2 = \lambda_3 = \lambda_4$. □

Proof of Theorem 1.1.. Let (M, g) be a four-dimensional (not necessarily complete) gradient almost Ricci soliton with harmonic Weyl curvature.

If there are four pairwise different Ricci eigenvalues at p, then it will be so in a neighborhood U_p of p by continuity. By definition $U_p \subset M_{\mathcal{A}}$. By Proposition 3.4, there is no point q in U_p where $\nabla f(q) \neq 0$. So, $\nabla f = 0$ at every point of U_p. This means g is Einstein, a contradiction. This proves the theorem. □

4 | FOUR-DIMENSIONAL GRADIENT ALMOST RICCI SOLITON WITH $\lambda_2 \neq \lambda_3 = \lambda_4$

In this section we study the case when exactly two of $\lambda_2, \lambda_3, \lambda_4$ are equal for an adapted frame field $\{E_i\}$. We may well assume that $\lambda_2 \neq \lambda_3 = \lambda_4$.

Lemma 4.1. *Let (M, g, f, λ) be a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. Suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field $\{E_i\}$ on an open subset W of M. Then on W we have:*

(i) For $i \in \{3, 4\}$ and for $j \in \{1, 2, 3, 4\}$, $\left\langle \nabla_{E_i} E_j, E_2 \right\rangle = 0$. And $\left\langle \nabla_{E_2} E_2, E_3 \right\rangle = \left\langle \nabla_{E_2} E_2, E_4 \right\rangle = 0$.

(ii) $\nabla_{E_1} E_2 = 0$, $\left[E_1, E_2\right] = -\zeta_2 E_2$ and $\left[E_3, E_4\right] = \beta_3 E_3 + \beta_4 E_4$, for some functions β_3, β_4. In particular, the distribution spanned by E_1 and E_2 is integrable. So is that spanned by E_3 and E_4.

Proof. From Lemma 2.3 (ii) and (3.2), $(\lambda_2 - \lambda_i)\left\langle \nabla_{E_1} E_2, E_i \right\rangle = (\lambda_1 - \lambda_i)\left\langle \zeta_2 E_2, E_i \right\rangle = 0$, for $i = 3, 4$. This implies $\nabla_{E_1} E_2 = 0$, and so $\left[E_1, E_2\right] = -\zeta_2 E_2$. From Lemma 2.3 (ii), $(\lambda_2 - \lambda_4)\left\langle \nabla_{E_3} E_2, E_4 \right\rangle = (\lambda_3 - \lambda_4)\left\langle \nabla_{E_2} E_3, E_4 \right\rangle = 0$. So, $\left\langle \nabla_{E_3} E_2, E_4 \right\rangle = -\left\langle E_2, \nabla_{E_3} E_4 \right\rangle = 0$. This and (3.2) yields $\nabla_{E_3} E_4 = \beta_3 E_3$, for some function β_3. Similarly, $\nabla_{E_4} E_3 = -\beta_4 E_4$ for some function β_4. Then $[E_3, E_4] = \beta_3 E_3 + \beta_4 E_4$. From Lemma 2.3 (iii) and Lemma 3.2, $\left\langle \nabla_{E_i} E_i, E_2 \right\rangle = 0$ for $i = 3, 4$ and $\left\langle \nabla_{E_2} E_2, E_3 \right\rangle = \left\langle \nabla_{E_2} E_2, E_4 \right\rangle = 0$. □

The computations of Lemma 4.1 are sufficient to obtain the following lemma.

Lemma 4.2. *Let (M, g, f, λ) be a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. Suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field $\left\{E_i\right\}$ on an open subset W of M.*

Then for each point p_0 in W, there exists a neighborhood V of p_0 in W with coordinates (s, t, x_3, x_4) such that $\nabla s = \frac{\nabla f}{|\nabla f|} = E_1$, $\frac{1}{p(s)} \frac{\partial}{\partial t} = E_2$ and g can be written on V as

$$g = ds^2 + p(s)^2 dt^2 + h(s)^2 \tilde{g}, \tag{4.1}$$

where $p := p(s)$ and $h := h(s)$ are smooth functions and $\tilde{g}$ is (a pull-back of) a Riemannian metric of constant curvature, say k, on a 2-dimensional domain with x_3, x_4 coordinates.

Proof. Let D^1 and D^2 be the 2-dimensional distribution spanned by E_1, E_2 and E_3, E_4, respectively. These are integrable by Lemma 4.1. Based on the computations of Lemma 4.1, the proof then follows verbatim those of Lemma 4.3 and Lemma 5.1 in [17]. □

From (3.1), we have $R_{1221} = R_{1331} = R_{1441}$. Setting $a := \zeta_2$ and $b := \zeta_3 = \zeta_4$, we can compute $a = \frac{p'}{p}$, $b = \frac{h'}{h}$, $R_{1221} = -a' - a^2 = -\frac{p''}{p}$ and $R_{1331} = R_{1441} = -b' - b^2 = -\frac{h''}{h}$. So, it holds that

$$a' + a^2 = b' + b^2. \tag{4.2}$$

Now the Ricci curvature components $R_{ij} = R(E_i, E_j)$ and the scalar curvature of g can be readily computed.

$$\begin{aligned}
R_{11} &= -a' - a^2 - 2b' - 2b^2 = -\frac{p''}{p} - 2\frac{h''}{h},\\
R_{22} &= -a' - a^2 - 2ab = -\frac{h''}{h} - 2\frac{p'}{p}\frac{h'}{h},\\
R_{33} = R_{44} &= -b' - 2b^2 - ab + \frac{k}{h^2} = -\frac{h''}{h} - \frac{p'}{p}\frac{h'}{h} - \frac{(h')^2}{h^2} + \frac{k}{h^2},\\
R &= -2a' - 2a^2 - 4b' - 6b^2 - 4ab + 2\frac{k}{h^2}.
\end{aligned} \tag{4.3}$$

We can write non-trivial components of the soliton equation $\nabla df\big(E_i, E_i\big) = -(Rc - \lambda g)\big(E_i, E_i\big)$, $i = 1, 2, 3$ as follows;

$$f'' = a' + a^2 + 2b' + 2b^2 + \lambda, \tag{4.4}$$

$$f'a = a' + a^2 + 2ab + \lambda, \tag{4.5}$$

$$f'b = b' + 2b^2 + ab - \frac{k}{h^2} + \lambda. \tag{4.6}$$

We shall understand three simple cases in Lemma 4.3~4.5. These are needed in later sections.

Lemma 4.3. *Let (M, g, f, λ) be a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. Suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field $\{E_j\}$ on an open subset W of M. Assume that $b = 0$.*

Then (W, g) is locally isometric to a domain in $\mathbb{R}^2 \times (N, \tilde{g})$ with metric $ds^2 + s^2dt^2 + \tilde{g}$, where $\tilde{g}$ is a Riemannian metric of constant curvature $\lambda \neq 0$ on a two-dimensional manifold N. And $f_{|V} = \frac{\lambda}{2}s^2 + C_1$, for a nonzero constant C_1.

Proof. As $b = 0$ on W, $a' + a^2 = 0$ by (4.2). As $a = \frac{p'}{p}$, we have $\frac{p''}{p} = 0$. So, $p = c_p\big(s - c_1\big)$, for constants $c_p \neq 0$ and c_1. As h is constant, we set $h = h_0 > 0$. From (4.6), we have $\lambda = \frac{k}{h_0^2}$, a constant. From (4.5), $f' = \lambda\big(s - c_1\big)$. Then we get $f(s) = \frac{1}{2}\lambda\big(s - c_1\big)^2 + C_1$ for a constant C_1. If $\lambda = 0$, then f is constant, a contradiction. So, $\lambda \neq 0$. And by absorbing a constant to the variable t, we can write the metric $g = ds^2 + \big(s - c_1\big)^2 dt^2 + h_0^2\tilde{g}$, where $h_0^2\tilde{g}$ is a Riemannian metric of constant curvature $\frac{k}{h_0^2} = \lambda$. The metric g is isometric to $ds^2 + s^2dt^2 + h_0^2\tilde{g}$. □

Lemma 4.4. *For the gradient almost Ricci soliton (M, g, f, λ) satisfying the hypothesis of Lemma 4.2 on W, the set*

$$\{p \in W \mid a(p) = 0\}$$

has empty interior.

Proof. If $a = 0$ on an open subset, then $b' + b^2 = \frac{h''}{h} = 0$ there. So, $h = c_h(s - c)$, for constants $c_h \neq 0$ and c. From (4.5), $\lambda = 0$. From (4.4), $f'' = 0$ and f' is constant. From (4.6) we get $f' = \frac{1}{s-c}\left(1 - \frac{k}{c_h^2}\right)$. Then, $c_h^2 = k > 0$ and $f' = 0$, a contradiction. □

Lemma 4.5. *For the gradient almost Ricci soliton (M, g, f, λ) satisfying the hypothesis of Lemma 4.2 on W, the set*

$$\{p \in W \mid (a + b)(p) = 0\}$$

has empty interior.

Proof. Suppose $a + b = 0$ on an open subset. Then $a' - b' = b^2 - a^2 = 0$. So, $a - b = C$, a constant. Then $a = \frac{p'}{p} = \frac{C}{2}$, $b = \frac{h'}{h} = -\frac{C}{2}$. As $a \neq b$, $C \neq 0$. Then $h = c_h e^{-\frac{C}{2}s}$ for a constant $c_h > 0$. Put it into (4.5) and (4.6), we get $\frac{k}{h^2} = 2\lambda$. (4.4) and (4.5) gives $\lambda' = \frac{C}{2}\lambda + \frac{3}{8}C^3$. Solving this ODE, we get $\lambda = -\frac{3}{4}C^2 + C_2 e^{\frac{C}{2}s}$ for a constant C_2. As $\frac{k}{h^2} = 2\lambda$, we get $\frac{k}{c_h^2 e^{-Cs}} = 2\left(-\frac{3}{4}C^2 + C_2 e^{\frac{C}{2}s}\right)$. So, $C = 0$, a contradiction. □

Now for the soliton metric g of (4.1), from (4.2), (4.5) and (4.6),

$$(a-b)f' = b(a-b) + \frac{k}{h^2}. \tag{4.7}$$

Differentiating, $(a-b)'f' + (a-b)f'' = b'(a-b) + b(a-b)' - 2\frac{kh'}{h^3}$.

Meanwhile, from (4.4), (4.7) and $a' - b' = -a^2 + b^2$,

$$(a-b)'f' + (a-b)f'' = (a+b)\left\{-b(a-b) - \frac{k}{h^2}\right\} + (a-b)(-\lambda_1 + \lambda).$$

Equating the above two expressions for $\{(a-b)f'\}'$ and recalling $b = \frac{h'}{h}$, we get

$$b'(a-b) = (a-b)\left\{-\frac{k}{h^2}\right\} + (a-b)\big(-\lambda_1 + \lambda\big),$$

which gives

$$\lambda = -2\big(b' + b^2\big) - b^2 + \frac{k}{h^2}. \tag{4.8}$$

From (4.5), (4.6), we have $b\big(a' + a^2 + 2ba + \lambda\big) = a\Big(b' + b^2 + ba + b^2 - \frac{k}{h^2} + \lambda\Big)$. By (4.2) we reduce this to

$$(a-b)\big(a' + a^2 + ab + \lambda\big) = a\frac{k}{h^2}. \tag{4.9}$$

Putting (4.8) into (4.9), we get

$$(a-b)\big(a' + a^2 - ab + b^2\big) = -b\frac{k}{h^2}. \tag{4.10}$$

If we consider the harmonic Weyl curvature condition, i.e. $\nabla_k R_{ij} - \nabla_j R_{ik} = -\frac{R_j}{6}g_{ki} + \frac{R_k}{6}g_{ij}$, then in the case $(k,i,j) = (1,2,2)$ we get

$$0 = \nabla_1 R_{22} - \nabla_2 R_{21} - \frac{R'}{6} = \big(R_{22}\big)' + aR_{22} - aR_{11} - \frac{R'}{6}. \tag{4.11}$$

But (4.11) does not provide a new equation, as it reduces to (4.10) by (4.3). Actually we can check that no new equation is obtained from any (k,i,j).

5 | DETAILED ANALYSIS OF THE CASE $\lambda_2 \neq \lambda_3 = \lambda_4$

In this section we analyze (4.10) in detail and prove Theorem 1.2. Put $a = \frac{p'}{p}$ and $b = \frac{h'}{h}$ into (4.10) and get

$$\left(\frac{p'}{p} - \frac{h'}{h}\right)\left(\frac{p''}{p} - \frac{p'}{p}\frac{h'}{h} + \frac{(h')^2}{h^2}\right) = -\frac{kh'}{h^3}. \tag{5.1}$$

Set $p = uh$. Then $\frac{h''}{h} = \frac{p''}{p} = \frac{u''}{u} + \frac{2u'h'}{uh} + \frac{h''}{h}$, so that $0 = \frac{u''}{u} + \frac{2u'h'}{uh}$. From this, for a constant c,

$$u' = \frac{c}{h^2}. \tag{5.2}$$

If $c = 0$, then $p = u_0 h$ for a constant u_0, which leads to a contradiction to the hypothesis $\lambda_2 \neq \lambda_3$. So, we must have $c \neq 0$.

Again using $p = uh$, (5.1) becomes $\left(\frac{u'}{u}\right)\left\{\frac{h''}{h} - \frac{u'}{u}\frac{h'}{h}\right\} = -\frac{kh'}{h^3}$. Multiplying this by $\frac{h}{h'}\frac{u}{u'}$ and using (5.2) gives $\frac{h''}{h'} = \frac{u'}{u} - k\frac{u}{c}$. From this and (5.2), we compute

$$\frac{h}{h'}\frac{d}{ds}\left(c\frac{h'}{p}\right) = c\frac{h}{h'}\left(\frac{h''}{p} - \frac{h'p'}{p^2}\right) = c\left\{\left(\frac{u'}{u} - \frac{ku}{c}\right)\frac{h}{p} - \frac{h}{p}\left(\frac{h'}{h} + \frac{u'}{u}\right)\right\} = -k - \frac{ch'}{p}.$$

We define

$$\tau := \ln h, \quad \text{so that } d\tau = \frac{h'}{h}ds. \tag{5.3}$$

Then the previous equality becomes $\frac{d}{d\tau}\left(\frac{ch'}{p}\right) = -k - \frac{ch'}{p}$. We view it as a first order ordinary differential equation for the solution $c\frac{h'}{p}$ with the variable τ. Solving for $c\frac{h'}{p}$ gives $\frac{ch'}{p} = c_1 e^{-\tau} - k$ for a constant c_1, i.e.

$$ch' = \left(c_1 e^{-\tau} - k\right)p. \tag{5.4}$$

The case when $c_1 = 0$ or $k = 0$ will be treated later in Lemmas 5.7~5.9. Until then, we assume that $c_1 \neq 0$ and $k \neq 0$. We shall keep using the prime symbol $'$ to denote the derivative $\frac{d}{ds}$.

We get

$$\frac{dp}{d\tau} = \frac{h}{h'}\frac{dp}{ds} = \frac{h}{h'}\left(\frac{h'}{h} + \frac{c}{uh^2}\right)p = p + \frac{c}{h'} = p + \frac{c^2}{\left(c_1 e^{-\tau} - k\right)p}.$$

Then $\frac{d}{d\tau}\left\{\left(\frac{p}{c}\right)^2 e^{-2\tau}\right\} = \frac{2e^{-2\tau}}{c_1 e^{-\tau} - k}$. So, $p^2 = c^2 e^{2\tau}\int \frac{2e^{-2\tau}}{c_1 e^{-\tau} - k}\,d\tau$. We set

$$Q(\tau) := \int \frac{2e^{-2\tau}}{c_1 e^{-\tau} - k}\,d\tau,$$

i.e. for a constant $\tilde{c}$,

$$Q(\tau) = -\frac{2}{c_1^2}\left(c_1 e^{-\tau} - k + k\ln\left|c_1 e^{-\tau} - k\right|\right) + \tilde{c} \geq 0.$$

Basic properties of Q are as follows;

The function Q is smooth on $\mathbb{R}$ except at $\tau = \ln\left(\frac{c_1}{k}\right)$ if $\frac{c_1}{k} > 0$.

- $\lim_{\tau\to\infty} Q = -\frac{2}{c_1^2}(-k + k\ln|k|) + \tilde{c}$. This number will be denoted by $\hat{c}$.
- $\lim_{\tau\to-\infty} Q = -\infty$ for $c_1 > 0$ and $\lim_{\tau\to-\infty} Q = \infty$ for $c_1 < 0$.
- And $\frac{dQ}{d\tau} = \frac{2e^{-2\tau}}{c_1 e^{-\tau} - k}$.

We have $p = \pm c e^{\tau}\sqrt{Q}$ and, from (5.4), $\frac{dh}{ds} = \frac{p}{c}\left(c_1 e^{-\tau} - k\right) = \pm\left(c_1 - k e^{\tau}\right)\sqrt{Q}$. We may only consider

$$p = c e^{\tau}\sqrt{Q}, \quad \text{with } c > 0,$$

$$\frac{dh}{ds} = h' = \left(c_1 - k e^{\tau}\right)\sqrt{Q}, \tag{5.5}$$

since the other case would lead to the same result; indeed, the metric form $g = ds^2 + p(s)^2dt^2 + h(s)^2\tilde{g}$ in (4.1) and the soliton equations (4.4)∼(4.6) are invariant under the sign change from (p, h) to $(-p, -h)$. From (5.3) and (5.5) we have

$$g = \frac{1}{\left(c_1e^{-\tau} - k\right)^2 Q(\tau)}d\tau^2 + c^2e^{2\tau}Q(\tau)dt^2 + e^{2\tau}\tilde{g}. \tag{5.6}$$

We begin analyzing this g under conditions on the parameters k, c_1 and $\tilde{c}$ (equivalently $\hat{c}$). Recall that we denote by M^2_k a smooth surface admitting a complete Riemannian metric with constant curvature k.

Lemma 5.1. *Suppose $k < 0$ and $c_1 > 0$. Then the metric g is defined for $\tau_0 < \tau < \infty$ for some number τ_0. Moreover, g can be extended smoothly at $\tau = \tau_0$ and becomes a complete Riemannian metric on the smooth manifold $\mathbb{R}^2 \times M^2_k$. Here $\mathbb{R}^2$ arises as the quotient space $\left\{(\tau, t) \mid \tau \geq \tau_0,\ \ 0 \leq t \leq 2\pi\right\}/_\sim$, where the equivalence relation $\sim$ identifies $(\tau, 0)$ with $(\tau, 2\pi)$ and the subset $\left\{(\tau_0, t) \mid 0 \leq t \leq 2\pi\right\}$ with a point.*

Proof. From $k < 0$ and $c_1 > 0$, Q is a smooth function on $\mathbb{R}$, $\frac{dQ}{d\tau} = \frac{2e^{-2\tau}}{c_1e^{-\tau}-k} > 0$ and $\lim_{\tau\to-\infty} Q = -\infty$. If $\lim_{\tau\to\infty} Q = \hat{c} \leq 0$, then $Q(\tau) < 0$, so this case is void.

If $\hat{c} > 0$, then there is τ_0 such that $Q\left(\tau_0\right) = 0$, and $Q > 0$ on $\tau_0 < \tau < \infty$. As $c_1e^{-\tau} - k > 0$, g in (5.6) is defined at least for $\tau_0 < t < \infty$.

From Lemma 2.2 (vi), any unit-speed curve tangent to the $\frac{d}{ds}$ direction, equivalently the $\frac{d}{d\tau}$ direction, is a geodesic. We observe from (5.6) that the Riemannian manifold (M, g) will be a complete metric space if such geodesics can be extended infinitely.

We compute the length of curves tangent to the $\frac{d}{d\tau}$ direction. From $dQ = \frac{2e^{-2\tau}}{c_1e^{-\tau}-k}d\tau$, we have

$$\int_a^b \frac{1}{\left(c_1e^{-\tau} - k\right)\sqrt{Q(\tau)}}\, d\tau = \int_{Q(a)}^{Q(b)} \frac{e^{2\tau(Q)}}{2\sqrt{Q}}\, dQ,$$

where $\tau(Q)$ is the inverse function of $Q(\tau)$. For some $\tau_1 > \tau_0$, the length of the curve from τ_1 to ∞ equals $\int_{\tau_1}^{\infty} \frac{1}{(c_1e^{-\tau}-k)\sqrt{Q(\tau)}}\, d\tau = \infty$, since $\lim_{\tau\to\infty} Q = \hat{c} > 0$. The length from τ_1 to τ_0 equals

$$\int_{\tau_0}^{\tau_1} \frac{1}{\left(c_1e^{-\tau} - k\right)\sqrt{Q(\tau)}}\, d\tau = \int_0^{Q(\tau_1)} \frac{e^{2\tau(Q)}}{2\sqrt{Q}}\, dQ < \infty,$$

since $\lim_{Q\to 0} \tau(Q) = \tau_0$.

It is easy to see from (5.5) that $\lim_{\tau\to\tau_0^+}(h', p) = (0, 0)$. So, we will check whether g can be smooth at $\tau = \tau_0$. We set

$$s(\tau) = \int_{\tau_0}^{\tau} \frac{1}{\left(c_1e^{-\tau} - k\right)\sqrt{Q(\tau)}}\, d\tau$$

and check below the smoothness of g at $s = 0$, which corresponds to $\tau = \tau_0$.

Using

$$\frac{d\{\cdot\}}{ds} = \frac{d\tau}{ds}\frac{d\{\cdot\}}{d\tau} = \left(c_1e^{-\tau} - k\right)\sqrt{Q}\frac{d\{\cdot\}}{d\tau},$$

we can get

$$\frac{d}{ds}\left\{e^{a\tau}Q^{\frac{m}{2}}\right\} = a\left(c_1e^{-\tau} - k\right)e^{a\tau}Q^{\frac{m+1}{2}} + me^{(a-2)\tau}Q^{\frac{m-1}{2}}, \tag{5.7}$$

for a constant a and an integer $m \geq 0$. We note that the right hand side (RHS) is a linear combination of terms of the form $e^{\alpha\tau}Q^{\frac{l}{2}}$, for a constant α and an integer $l \geq 0$; when $m = 0$, the second term in the RHS is 0. As $h = e^{\tau}$ and $p = ce^{\tau}\sqrt{Q}$ are both of this form, by induction we have $\frac{d^j h}{ds^j} = \sum_{i=1}^{i_j} \alpha_i e^{a_i\tau}Q^{\frac{m_i}{2}}$ with constants α_i, a_i and an integer $m_i \geq 0$, for any integer $j \geq 0$, and likewise for $\frac{d^j p}{ds^j}$. Note from (5.7) that if j is odd, then m_i is all odd in $\frac{d^j h}{ds^j} = \sum_{i=1}^{i_j} \alpha_i e^{a_i\tau}Q^{\frac{m_i}{2}}$, and if j is even, n_i is all odd in $\frac{d^j p}{ds^j} = \sum_{i=1}^{i_j} \beta_i e^{b_i\tau}Q^{\frac{n_i}{2}}$. For instance,

$$
\begin{aligned}
\frac{dh}{ds} &= \left(c_1 - ke^{\tau}\right)\sqrt{Q}, \\
\frac{d^2h}{ds^2} &= \left(c_1 - ke^{\tau}\right)\left(-kQ + e^{-2\tau}\right), \qquad (5.8)\\
\frac{d^3h}{ds^3} &= \left(c_1 - ke^{\tau}\right)\sqrt{Q}\left\{k^2Q - ke^{-2\tau} - 2c_1e^{-3\tau}\right\}.
\end{aligned}
$$

$$
p = ce^{\tau}\sqrt{Q}, \quad \frac{dp}{ds} = c\left(c_1 - ke^{\tau}\right)Q + ce^{-\tau}. \qquad (5.9)
$$

Therefore, for the function $h(s) = e^{\tau(s)}$ we have $h(0) = e^{\tau_0} > 0$ and $\frac{d^j h}{ds^j}(0) = 0$ for odd j. And for $p(s)$, $\frac{d^j p}{ds^j}(0) = 0$ for even j, and $\frac{dp}{ds}(0) = ce^{-\tau_0}$, which equals 1 as we let $c = e^{\tau_0}$.

So, g satisfies the hypothesis of Petersen's lemma in the Subsection 1.4.1 of [20]. Note that his lemma is stated for $k > 0$, but it should hold for any sign of k. Now g is smooth at $s = 0$, i.e. at $\tau = \tau_0$, and g is smooth on $\mathbb{R}^2 \times N_k^2$. Those curves tangent to $\frac{d}{d\tau}$ can be extended infinitely. So, g is complete. □

From (4.3), (5.8) and (5.9), the scalar curvature of g is as follows.

$$
R = 6\left(c_1e^{-\tau} - k\right)\left(2k - c_1e^{-\tau}\right)Q - 10c_1e^{-3\tau} + 12ke^{-2\tau}. \qquad (5.10)
$$

Lemma 5.2. *Suppose $k < 0$ and $c_1 < 0$. Assume further $\frac{dh}{ds} = \left(c_1 - ke^{\tau}\right)\sqrt{Q} < 0$. Then the metric g cannot be extended to become a complete metric.*

Proof. By hypothesis, $\tau < \ln\left(\frac{c_1}{k}\right)$. Q is a smooth function on $-\infty < \tau < \ln\left(\frac{c_1}{k}\right)$. We have

$$
\lim_{t\to-\infty} Q = \infty, \quad \lim_{t\to\left(\ln\frac{c_1}{k}\right)^-} Q = -\infty \quad \text{and} \quad \frac{dQ}{d\tau} < 0.
$$

So, there is τ_0 with $\tau_0 < \ln\left(\frac{c_1}{k}\right)$ such that $Q(\tau_0) = 0$. So, g as in (5.6) is well defined on $-\infty < \tau < \tau_0$.

For $\tau_1 < \tau_0$, the length of a curve tangent to $\frac{d}{d\tau}$ from τ_1 to $-\infty$ equals $-\int_{-\infty}^{\tau_1} \frac{1}{(c_1e^{-\tau}-k)\sqrt{Q}}\, d\tau < \infty$. From (5.10), we have

$$
R = 6\left(c_1e^{-\tau} - k\right)\left(2k - c_1e^{-\tau}\right)\left\{-\frac{2}{c_1^2}\left(c_1e^{-\tau} - k + k\ln|c_1e^{-\tau} - k|\right) + \tilde{c}\right\} - 10c_1e^{-3\tau} + 12ke^{-2\tau},
$$

from which we get $\lim_{\tau\to-\infty} R = -\infty$. So, the metric cannot be extended to become a complete metric. □

Lemma 5.3. *Suppose $k < 0$ and $c_1 < 0$. Assume $\frac{dh}{ds} > 0$. Then g is defined for $\tau > \tau_0$ where τ_0 is some number with $\tau_0 > \ln\left(\frac{c_1}{k}\right)$. Moreover, g can be extended smoothly at $\tau = \tau_0$ and becomes a complete Riemannian metric on $\mathbb{R}^2 \times M_k^2$, where $\mathbb{R}^2$ arises similarly as in Lemma 5.1.*

Proof. By hypothesis, $\tau > \ln\left(\frac{c_1}{k}\right)$. Q is smooth on $\ln\left(\frac{c_1}{k}\right) < \tau < \infty$. We have

$$\lim_{\tau \to \left(\ln \frac{c_1}{k}\right)^+} Q = -\infty, \quad \lim_{\tau \to \infty} Q = \hat{c}, \quad \text{and} \quad \frac{dQ}{d\tau} > 0.$$

So, if $\hat{c} \leq 0$, then $Q(\tau) < 0$ and this case is void.

If $\hat{c} > 0$, there is a unique τ_0 with $\tau_0 > \ln\left(\frac{c_1}{k}\right)$ such that $Q(\tau_0) = 0$. So, g as in (5.6) is well defined on $\tau_0 < \tau < \infty$. For $\tau_1 > \tau_0$, the length of a curve tangent to $\frac{d}{d\tau}$ from τ_1 to ∞ is $\int_{\tau_1}^{\infty} \frac{1}{(c_1 e^{-\tau} - k)\sqrt{Q}}\, d\tau = \infty$. The distance from τ_1 to τ_0 is $\int_{\tau_0}^{\tau_1} \frac{1}{(c_1 e^{-\tau} - k)\sqrt{Q(\tau)}}\, d\tau < \infty$.

One can check that g is smooth at $\tau = \tau_0$, similarly as in the proof of Lemma 5.1. □

Lemma 5.4. *Suppose $k > 0$, $c_1 > 0$ and $\frac{dh}{ds} > 0$. Then the metric g as in (5.6) is defined on $\tau_0 < \tau < \ln\left(\frac{c_1}{k}\right)$ for some number τ_0. Moreover, g can be extended smoothly at $\tau = \tau_0$ and becomes a complete Riemannian metric on $\mathbb{R}^2 \times M_k^2$, where $\mathbb{R}^2$ arises similarly as in Lemma 5.1.*

Proof. From $\frac{dh}{ds} > 0$, we have $\tau < \ln\left(\frac{c_1}{k}\right)$. The function Q is smooth on $-\infty < \tau < \ln\left(\frac{c_1}{k}\right)$, $\lim_{\tau \to -\infty} Q = -\infty$, $\lim_{\tau \to \left(\ln \frac{c_1}{k}\right)^-} Q = \infty$, and $\frac{dQ}{d\tau} > 0$. There is a unique τ_0 with $\tau_0 < \ln\left(\frac{c_1}{k}\right)$ such that $Q(\tau_0) = 0$. The rest of proof is similar to that of Lemma 5.1. We omit it. □

Recall the constant $\hat{c} = -\frac{2}{c_1^2}(-k + k \ln |k|) + \tilde{c}$.

Lemma 5.5. *Suppose $k > 0$, $c_1 > 0$ and $\frac{dh}{ds} < 0$. Then the following hold.*

(i) If $\hat{c} \geq 0$, Then g is a complete metric defined on the smooth manifold $\left(\mathbb{R}^2 - \{(0,0)\}\right) \times M_k^2$.
(ii) If $\hat{c} < 0$, then g is defined on $\ln\left(\frac{c_1}{k}\right) < \tau < \tau_0$ for some number τ_0. Moreover, g can be extended smoothly at $\tau = \tau_0$ and becomes a complete Riemannian metric on $\mathbb{R}^2 \times M_k^2$, where $\mathbb{R}^2$ arises similarly as in Lemma 5.1.

Proof. As $\frac{dh}{ds} < 0$, we have $\tau > \ln\left(\frac{c_1}{k}\right)$. Q is smooth on $\tau > \ln\left(\frac{c_1}{k}\right)$, $\lim_{\tau \to \ln\left(\frac{c_1}{k}\right)^-} Q = \infty$, $\lim_{\tau \to \infty} Q = \hat{c}$, and $\frac{dQ}{d\tau} < 0$. There are two cases.

(i) If $\hat{c} \geq 0$, then $Q(\tau) > 0$ for $\tau > \ln\left(\frac{c_1}{k}\right)$. For $\tau_1 > \ln\left(\frac{c_1}{k}\right)$, the length of a curve tangent to $\frac{d}{d\tau}$ from τ_1 to $\ln\left(\frac{c_1}{k}\right)$ equals $\int_{\tau_1}^{\ln\left(\frac{c_1}{k}\right)} \frac{1}{(c_1 e^{-\tau} - k)\sqrt{Q}}\, d\tau = \int_{Q(\tau_1)}^{\infty} \frac{e^{2\tau(Q)}}{2\sqrt{Q}}\, dQ = \infty$. The length from τ_1 to ∞ is $\int_{\infty}^{\tau_1} \frac{1}{(c_1 e^{-\tau} - k)\sqrt{Q(\tau)}}\, d\tau = \infty$. So, g is a complete metric defined for $\tau > \ln\left(\frac{c_1}{k}\right)$.
(ii) If $\hat{c} < 0$, then there is a unique τ_0 with $\tau_0 > \ln\left(\frac{c_1}{k}\right)$ such that $Q(\tau_0) = 0$. $Q(\tau) > 0$ on $\ln\left(\frac{c_1}{k}\right) < \tau < \tau_0$.

For $\tau_1 > \tau_0$, the length of a curve tangent to $\frac{d}{d\tau}$ from τ_1 to $\ln\left(\frac{c_1}{k}\right)$ is $\int_{\tau_1}^{\ln\left(\frac{c_1}{k}\right)} \frac{1}{(c_1 e^{-\tau} - k)\sqrt{Q}}\, d\tau = \int_{Q(\tau_1)}^{\infty} \frac{e^{2\tau(Q)}}{2\sqrt{Q}}\, dQ = \infty$. The length from τ_1 to τ_0 is $\int_{\tau_0}^{\tau_1} \frac{1}{(c_1 e^{-\tau} - k)\sqrt{Q(\tau)}}\, d\tau = \int_{Q(\tau_0)}^{Q(\tau_1)} \frac{e^{2\tau(Q)}}{2\sqrt{Q}}\, dQ < \infty$. One can check that g is smooth at $\tau = \tau_0$, similarly as in the proof of Lemma 5.1.

Lemma 5.6. *Suppose $k > 0$ and $c_1 < 0$. Then g cannot be a complete metric.*

Proof. The proof is similar to that of Lemma 5.2, so we omit it. □

Next, we treat the case when $k = 0$ and $c_1 \neq 0$. Then the function $Q(\tau) = \int \frac{2e^{-2\tau}}{c_1 e^{-\tau} - k} d\tau$ is smooth on $\mathbb{R}$;

$$Q(\tau) = -\frac{2}{c_1} e^{-\tau} + \tilde{c}.$$

Lemma 5.7. *Suppose $k = 0$ and $c_1 > 0$. Then g can be extended to a complete metric on $\mathbb{R}^2 \times M_0^2$ when $\tilde{c} > 0$.*

Proof. We have $\lim_{\tau\to\infty} Q = \tilde{c}$, $\lim_{\tau\to-\infty} Q = -\infty$ and $\frac{dQ}{d\tau} = \frac{2e^{-\tau}}{c_1} > 0$. If $\tilde{c} \leq 0$, then $Q(\tau) < 0$. So, this case is void.

If $\tilde{c} > 0$, then $Q(\tau_0) = 0$ where $\tau_0 = -\ln\left(\frac{\tilde{c}c_1}{2}\right)$. Then $Q(\tau) > 0$ when $\tau_0 < \tau < \infty$. For any τ_1 with $\tau_0 < \tau_1$, the length of a curve tangent to $\frac{d}{d\tau}$ from τ_1 to ∞ is $\int_{\tau_1}^{\infty} \frac{1}{(c_1 e^{-\tau})\sqrt{Q}} d\tau = \infty$. The length from τ_1 to τ_0 is $\int_{\tau_0}^{\tau_1} \frac{1}{(c_1 e^{-\tau})\sqrt{Q}} d\tau < \infty$. One can show that g is smooth at $\tau = \tau_0$, as in the proof of Lemma 5.1. So, g can be extended to a complete metric on $\mathbb{R}^2 \times M_0^2$. □

Lemma 5.8. *Suppose $k = 0$ and $c_1 < 0$. Then g cannot be extended to become a complete metric.*

Proof. The proof is similar to that of Lemma 5.2, so we omit it. □

Lemma 5.9. *The case that $c_1 = 0$ does not occur.*

Proof. If $c_1 = k = 0$, we have $h' = 0$ from (5.4). By Lemma 4.3, we can write $g = ds^2 + s^2 dt^2 + \tilde{g}$. Then it contradicts to the hypothesis to $\lambda_2 \neq \lambda_3$.

Now suppose $c_1 = 0$ and $k \neq 0$. Then $p = -\frac{c}{k} h'$ from (5.4). Now, $\frac{h''}{h} = \frac{p''}{p}$ becomes $\frac{h''}{h} = \frac{h'''}{h'}$ to get $\frac{h''}{h} = C$ for a constant C. So, $\frac{p''}{p} = C$.

If $C = 0$, $h = a_1 s + a_2$ for constants $a_1 \neq 0$ and a_2, and $p = -\frac{a_1 c}{k}$. Then $\frac{p'}{p} = 0$. By Lemma 4.4, this cannot occur.

If $C < 0$, then $h = C_1 \sin \sqrt{-C}(s + s_0)$ for constants $C_1 \neq 0$ and s_0. $p = -\frac{cC_1\sqrt{-C}}{k} \cos \sqrt{-C}(s + s_0)$. Putting these into (5.1), we get $k = -CC_1^2$. Then we can get $R_{22} = R_{33}$ from (4.3), a contradiction.

If $C > 0$, we get a contradiction by the same argument as the $C < 0$ case. □

Proof of Theorem 1.2. As (M, g) has exactly three distinct Ricci-eigenvalues at each point, $M = M_A$. If $\nabla f = 0$ on an open subset U, then g is Einstein on U, a contradiction. So, the set $D := \{p \in M \mid \nabla f \neq 0\}$ is dense in M. In a neighborhood of any point in D, we may consider an adapted frame field E_i. By Proposition 3.4, $\lambda_2, \lambda_3, \lambda_4$ cannot be pairwise different. As (M, g) has exactly three distinct Ricci-eigenvalues, we may only have to consider the case that $\lambda_3 = \lambda_4$ but $\lambda_1, \lambda_2, \lambda_3$ are pairwise distinct. Then by Lemma 4.2, the formula (5.6) and Lemma 5.9 we obtain the first clause.

From (4.8) and (5.8), we compute $\lambda = \left(c_1 e^{-\tau} - k\right)\left(3k - c_1 e^{-\tau}\right)Q - \left(2c_1 e^{-\tau} - 3k\right)e^{-2\tau}$. From (4.6), (5.8) and (5.9) we compute $f' = c_1 e^{-\tau}\sqrt{Q}$. Using (5.3) and (5.5) we have

$$f = \int f' ds = \int c_1 e^{-\tau}\sqrt{Q}\, \frac{ds}{d\tau}\, d\tau = \int \frac{c_1}{(c_1 - ke^{\tau})} d\tau = -\ln|c_1 e^{-\tau} - k| + C$$

for a constant C.

According to Lemmas 5.1∼5.9, g in (1.3) yields a complete Riemannian metric in the four cases (i)–(iv). From the description of the extended manifold $\mathbb{R}^2 \times M_k^2$ in Lemmas 5.1∼5.7, f and λ are smooth except possibly at $s = 0$, corresponding to $\tau = \tau_0$.

As shown in the proof of Lemma 5.1, $\frac{d^k p}{ds^k}(0) = 0$ for even k and $\frac{d^k h}{ds^k}(0) = 0$ for odd k. So, h and p^2 are smooth at $s = 0$ and so on the extended manifold. Note that $h > 0$ on the extended manifold in Lemmas 5.1∼5.7, so both $\frac{1}{h}$ and $\tau = \ln h$ are smooth. By (5.5), $Q = \frac{p^2}{c^2 h^2}$ which is smooth. These imply that λ and f are smoothly defined on the extended manifold. Therefore, (g, f, λ) form a gradient almost Ricci soliton on the extended manifold. □

6 | ONE OR TWO EIGENVALUES

Suppose that there are exactly two distinct Ricci-eigenvalues on an open subset of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$. As $E_1 = \frac{\nabla f}{|\nabla f|}$ is distinct from others and we may assume $\lambda_3 = \lambda_4$, we have to consider only three cases (i) $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$, (ii) $\lambda_1 = \lambda_3 = \lambda_4 \neq \lambda_2$ and (iii) $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$.

Lemma 6.1. *Suppose that (M, g, f, λ) is a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature and that $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field in an open subset U of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$. Then $b = 0$ on U.*

Proof. From $\lambda_1 = \lambda_2$ and (4.3), $b' + b^2 = ab$. From (4.8), $\lambda = -2ab - b^2 + \frac{k}{h^2}$. Then (4.9) becomes $(a-b)\left(-b^2 + \frac{k}{h^2}\right) = a\frac{k}{h^2}$ so that $b\left(\frac{k}{h^2} + ab - b^2\right) = 0$.

If $\frac{k}{h^2} + ab - b^2 = 0$ on an open set, (4.8) gives $3ab + \lambda = 0$. Now (4.5) gives $f'a = 0$. As $f' \neq 0$, $a = 0$, a contradiction to Lemma 4.4. Thus b must be zero on W. □

Lemma 6.2. *Suppose that (M, g, f, λ) is a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. Then it cannot hold that $\lambda_1 = \lambda_3 = \lambda_4 \neq \lambda_2$ on any open subset of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$.*

Proof. Suppose that $\lambda_1 = \lambda_3 = \lambda_4 \neq \lambda_2$ on an open subset. From $\lambda_1 = \lambda_3$ and (4.3), we get

$$2b' + 2b^2 = ab + b^2 - \frac{k}{h^2}. \tag{6.1}$$

From (4.8) we get $\lambda = -ab - 2b^2 + 2\frac{k}{h^2}$. Then (4.9) gives $(a - 3b)\left(ab - b^2 + \frac{k}{h^2}\right) = 0$.

If $a = 3b$ on an open subset of V, from $a' + a^2 = b' + b^2$ we get $b' + 4b^2 = 0$. So, either $b = 0$ or $b = \frac{1}{4s+c}$ for a constant c. When $b = 0$, by Lemma 4.3 we get $\lambda_1 = \lambda_2 \neq \lambda_3 = \lambda_4$, a contradiction. When $b = \frac{1}{4s+c}$, we have $h = c_3\left|s + \frac{c}{4}\right|^{\frac{1}{4}}$ for a positive constant c_3. But one can check that this cannot satisfy (6.1).

If $ab - b^2 + \frac{k}{h^2} = 0$ on an open subset of V, then (6.1) gives $b' + b^2 = ab$. Due to (4.8), $\lambda = -3ab$. Now (4.5) gives $f'a = 0$. As $f' \neq 0$, $a = 0$, a contradiction to Lemma 4.4. So, the lemma follows. □

Next we treat the case of $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$. The following lemma has been proved for gradient Ricci soliton case [17, Proposition 7.1], but one can check that the proof still works when λ is not constant, i.e. in gradient almost Ricci soliton case. We present it, omitting its proof.

Lemma 6.3. *Suppose that (M, g, f, λ) is a four-dimensional gradient almost Ricci soliton with harmonic Weyl curvature and that $\lambda_1 \neq \lambda_2 = \lambda_3 = \lambda_4$ for an adapted frame field in an open subset U of $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$.*

Then for each point p_0 in U, there exists a neighborhood V of p_0 in U where g is a warped product

$$g = ds^2 + h(s)^2\tilde{g}, \tag{6.2}$$

for a function h, where the Riemannian metric $\tilde{g}$ has constant curvature, say k. In particular, g is locally conformally flat.

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3.. As in the proof of Theorem 1.2, $M = M_{\mathcal{A}}$ and the set $D := \{p \in M \mid \nabla f \neq 0\}$ is dense in M. In a neighborhood of any point in D, we may consider an adapted frame field E_i.

The proof then follows from the first paragraph of this section and Lemma 6.1 ∼6.3 and Lemma 4.3. □

When all λ_i's, $i = 1, \dots, 4$, are equal, i.e. in the Einstein case, the gradient almost Ricci solitons are thoroughly understood, see [21, Theorem 2.3]. As the latter work uses the completeness assumption of the metric, we would like to just highlight a short local argument. Indeed, as the Einstein metric g satisfies $\nabla df = \left(-\frac{R}{4} + \lambda\right)g$, from the Section 1 of [11], there is a local variable s so that f depends only on s and g becomes locally of the form $g = ds^2 + (f'(s))^2\tilde{g}$ where $\tilde{g}$ has constant sectional curvature. We can then easily get $\frac{f'''}{f'} = -\frac{R}{12}$ and g becomes constant curved. We write it as a statement.

Lemma 6.4. *Suppose that (M, g, f, λ) is a four-dimensional gradient almost Ricci soliton with non constant f and that g is Einstein in an open subset U of $\{\nabla f \neq 0\}$. Then for each point p_0 in U, there exists a neighborhood V of p_0 in U where g is a warped product defined by*

$$g = ds^2 + (f'(s))^2\tilde{g}, \tag{6.3}$$

where $f = f(s)$ and $\tilde{g}$ has constant curvature. Moreover, g itself has constant curvature. It holds that $\frac{f'''}{f'} = -\frac{R}{12}$ and $\lambda = f'' + \frac{R}{4}$.

7 | CLASSIFICATION OF FOUR-DIMENSIONAL GRADIENT ALMOST RICCI SOLITONS WITH HARMONIC CURVATURE

If we assume that (M, g) with harmonic Weyl curvature has constant scalar curvature, then by definition (M, g) satisfies the *harmonic curvature* condition $d^\nabla Rc = 0$. A Riemannian manifold with harmonic curvature is real analytic in harmonic coordinates [6]. Equation (1.2) then implies that f is real analytic in harmonic coordinates. So, the subset $\{\nabla f \neq 0\}$ is dense in M if M is connected and f is not constant.

The case of harmonic curvature can be classified as a by-product of the study of harmonic Weyl case.

Lemma 7.1. *Let (M, g, f, λ) be a (not necessarily complete) nontrivial four-dimensional gradient almost Ricci soliton with harmonic curvature. Then in a neighborhood V_{p_0} of each point p_0 in the open dense subset $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$, $\left(V_{p_0}, g\right)$ is isometric to one of the followings:*

(i) A domain in the Riemannian product $\mathbb{R}^2 \times N^2_\lambda$ where $\mathbb{R}^2$ has the Euclidean metric and $\lambda \neq 0$. Moreover, $f = \frac{\lambda}{2}|x|^2$ modulo a constant on the Euclidean factor.
(ii) A domain in $I \times W^3$ with the warped product metric $g = ds^2 + h(s)^2\tilde{g}$ where I is an open interval in $\mathbb{R}$ and $\tilde{g}$ on a three dimensional manifold W^3 has constant curvature. In particular, g is locally conformally flat.

Proof. By Theorem 1.1, (M, g) cannot have four distinct Ricci-eigenvalues anywhere. If a point $p_0 \in M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$ has two distinct Ricci-eigenvalues, by Theorem 1.3 a neighborhood of p_0 is isometric to either type (i) or (ii). When all λ_i, $i = 1, \ldots, 4$ are equal at p_0, a neighborhood of p_0 is isometric to type (ii) by Lemma 6.4.

Thus to prove the lemma, we shall show that each point p_0 in $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$ cannot have three distinct Ricci-eigenvalues. Suppose that p_0 has three distinct Ricci-eigenvalues. By Proposition 3.4, we may assume that in a neighborhood U of p_0, there is an adapted frame field $\{E_i\}$ such that λ_1, λ_2 and λ_3 are pairwise distinct but $\lambda_3 = \lambda_4$.

Then we can exploit the argument of Section 4. So, subtracting (4.10) from (4.9),

$$(a - b)\left(2ab - b^2 + \lambda\right) = (a + b)\frac{k}{h^2}. \tag{7.1}$$

Then from (7.1), (4.9) and Lemma 4.5, we have

$$\frac{k}{h^2} = \frac{(a - b)\left(2ab - b^2 + \lambda\right)}{a + b}. \tag{7.2}$$

$$a' = -\frac{a^3 + 2ab^2 + b\lambda}{a + b}, \tag{7.3}$$

We obtain $f' = \frac{3ab+\lambda}{a+b}$ from (4.7). Since $R' = 0$, we get from (3.1)

$$\lambda' = \frac{b\big(a^2 - 2ab - \lambda\big)(3ab+\lambda)}{(a+b)^2}. \tag{7.4}$$

Now we apply $\log_e$ to (7.2) and differentiate it, using $(\ln h)' = b$, $a' + a^2 = b' + b^2$ and (7.3). This long computation yields

$$b\big(3a^2 - \lambda\big)(3ab+\lambda) = 0. \tag{7.5}$$

First, b cannot be zero on an open subset of U, by Lemma 4.3 and our pairwise-distinct eigenvalues assumption among λ_i, $i = 1, 2, 3$.

Suppose $3a^2 - \lambda = 0$ on some open subset of U. Then $6aa' - \lambda' = 0$. But if we recompute $6aa' - \lambda'$ from (7.3) and (7.4), putting $\lambda = 3a^2$, then we get $6aa' - \lambda' = -6a^2(a+b)$. Due to Lemma 4.4 and 4.5, this cannot be zero on an open set, a contradiction.

Now suppose $3ab + \lambda = 0$ on some open subset of U. Then we get $a' + a^2 = ab$ from (7.3). Then (4.5) gives $f'a = 0$, which is a contradiction since a cannot be zero by Lemma 4.4.

So, (7.5) cannot hold and p_0 cannot have exactly three distinct Ricci-eigenvalues. This proves the lemma.

Proof of Theorem 1.4. By Lemma 7.1, since $M_{\mathcal{A}} \cap \{\nabla f \neq 0\}$ is dense in M, (M, g) has less than three distinct Ricci-eigenvalues at each point. We need to look into the case (ii) in Lemma 7.1; consider the metric $g = ds^2 + h(s)^2\tilde{g}$ on a domain of $\mathbb{R} \times W^3$ where $\tilde{g}$ has constant curvature k. The functions h, f and λ satisfy the following equations out of $\nabla\nabla f + Rc = \lambda g$:

$$\lambda = f'' - 3\frac{h''}{h}, \tag{7.6}$$

$$\lambda = \frac{h'}{h}f' + \frac{2k}{h^2} - \frac{h''}{h} - 2\frac{(h')^2}{h^2}. \tag{7.7}$$

Eliminating λ and integrating, we get, for some constants s_0 and C,

$$f'(s) = 2h(s)\int_{s_0}^{s}\left(\frac{h''(u)}{h(u)^2} - \frac{(h'(u))^2}{h(u)^3} + \frac{k}{h(u)^3}\right)du + Ch(s). \tag{7.8}$$

The constant scalar curvature is $R = -6\left\{\frac{h''}{h} + \left(\frac{h'}{h}\right)^2 - \frac{k}{h^2}\right\}$. Multiplying by h^3h' on both sides of this formula and integrating, we get

$$(h')^2 + \frac{R}{12}h^2 + \frac{\tilde{a}}{h^2} = k \tag{7.9}$$

for a constant $\tilde{a}$. Differentiating it, we obtain

$$h'' + \frac{R}{12}h = \frac{\tilde{a}}{h^3}. \tag{7.10}$$

Using these two equations, from (7.8) we get

$$f'(s) = 4h(s)\int_{s_0}^{s}\frac{\tilde{a}}{h^5}\,ds + Ch(s). \tag{7.11}$$

Note that (7.9) and (7.10) coincide with Equations (2.2) and (2.1) in [18], respectively. Kobayashi sorted conditions on $\tilde{a}$, R, k and initial values of (7.9) and (7.10) into six cases (I)∼(VI) and described the solutions of each case in [18, p. 670]. Below, we briefly recall the solutions in each of the six cases.

In Kobayashi's case (I), h is a constant function and g is isometric to a domain in the Riemannian product $\mathbb{R} \times N^3_{\frac{\lambda}{2}}$; if $\lambda \neq 0$, then $f = \frac{\lambda}{2}|x|^2$ modulo a constant on the Euclidean factor, and if $\lambda = 0$, then $\nabla df = 0$.

In the case (II), g is isometric to a domain in the Gaussian soliton from (7.6) and (7.7). From (III), we obtain incomplete gradient almost Ricci solitons.

In the case (IV), we get complete warped product manifolds $\mathbb{R} \times_h M^3_{\frac{\lambda}{2}}$ with $h > 0$ on $\mathbb{R}$. Here $R \leq 0$. And f is given by (7.11) and λ by (7.6).

In the case (V) we have warped product metrics over $\mathbb{R}$ with fiber $\mathbb{S}^3$, with periodic warping function $h > 0$ on $\mathbb{R}$. So the metrics may be defined on compact quotients, but the function $f'(s) = 4h(s)\int_{s_0}^{s} \frac{\tilde{a}}{h^5}\, ds + Ch(s)$ is not periodic on $\mathbb{R}$ as $\tilde{a} > 0$ in this case. Therefore we get a complete gradient almost Ricci soliton on $\mathbb{R} \times \mathbb{S}^3$.

The case (VI) produces constant curved metrics with either positive or negative R, so their functions f and λ are described by [21].

We have obtained complete spaces out of Kobayashi's cases of (I), (II), (IV), (V) and (VI).

After h been given, f can be obtained by (7.11). Finally, λ can be defined by (7.6). This finishes the proof of Theorem 1.4. □

Remark 7.2. It is interesting to classify gradient almost Ricci solitons with harmonic curvature in any dimension. Of course, classification of those with harmonic Weyl curvature is more challenging.

Remark 7.3. A gradient almost Ricci soliton with harmonic Weyl curvature is not necessarily real analytic. The conformally flat spaces in Theorem 1.3 (ii) is apparently not so. One may construct a connected gradient almost Ricci soliton with harmonic Weyl curvature on which the number of distinct Ricci-egienvalues change from a non-empty open subset to another. But, since such a space should be simple in topology, one may find a geometric or topological characterization of compact four-dimensional gradient almost Ricci solitons with harmonic Weyl curvature. □

ACKNOWLEDGEMENT

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1A2B5B01001862).

ORCID

Jongsu Kim https://orcid.org/0000-0002-6672-1500

REFERENCES

[1] A. L. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb. (3), Band 10, Springer-Verlag, 1987.
[2] A. Barros, J. N. Gomes, and E. Ribeiro Jr, *A note on rigidity of almost Ricci solitons*, Arch. Math. (Basel) **100** (2013), 481–490.
[3] A. Barros and E. Ribeiro Jr, *Some characterizations for compact almost Ricci solitons*, Proc. Amer. Math. Soc. **140** (2012), no. 3, 1033–1040.
[4] A. Brasil, E. Costa, and E. Ribeiro Jr., *Hitchin–Thorpe inequality and Kaehler metrics for compact almost Ricci soliton*, Ann. Mat. Pura Appl. (4) **193** (2014), 1851–1860.
[5] M. Brozos-Vázquez, E. García-Río, and X. Valle-Regueiro, *Half conformally flat gradient Ricci almost solitons*, Proc. R. Soc. A **472** (2016), Issue 2189. Available at https://doi.org/10.1098/rspa.2016.0043
[6] D. DeTurck and H. Goldschmidt, *Regularity theorems in Riemannian geometry. II. Harmonic curvature and the Weyl tensor*, Forum Math. **1** (1989), 377–394.
[7] E. Calviño-Louzao, M. Fernández-López, E. García-Río, and R. Vázquez-Lorenzo, *Homogeneous Ricci almost solitons*, Israel J. Math. **220** (2017), no. 2, 531–546.
[8] H. D. Cao and Q. Chen, *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. **162** (2013), 1003–1204.
[9] G. Catino, *Generalized quasi-Einstein manifolds with harmonic Weyl tensor*, Math. Z. **271** (2012), 751–756.
[10] G. Catino and L. Mazzieri, *Gradient Einstein solitons*, Nonlinear Anal. **132** (2016), 66–94.
[11] J. Cheeger and T. H. Colding, *Lower bounds on Ricci curvature and the almost rigidity of warped products*, Ann. Math. (2) **144** (1996), no. 1, 189–237.
[12] A. Derdziński, *Classification of certain compact Riemannian manifolds with harmonic curvature and non-parallel Ricci tensor*, Math. Z. **172** (1980), 273–280.
[13] A. Derdziński and G. Maschler, *Local classfification of conformally-Einstein Kähler metrics in higher dimensions*, Proc. Lond. Math. Soc. **87** (2003), 779–819.

[14] K. L. Duggal, *A new class of almost Ricci solitons and their physical interpretation*, International Scholarly Research Notices. 2016, Article ID 4903520, 6 pp.
[15] J. N. Gomes, Q. Wang, and C. Xia, *On the h-almost Ricci soliton*, J. Geom. Phys. **114** (2017), 216–222.
[16] R. S. Hamilton, *Three manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), 255–306.
[17] J. Kim, *On a classification of 4-d gradient Ricci solitons with harmonic Weyl curvature*, J. Geom. Anal. **27** (2017), no. 2, 986–1012.
[18] O. Kobayashi, *A differential equation arising from scalar curvature fucntion*, J. Math. Soc. Japan **34** (1982), no. 4, 665–675.
[19] G. Maschler, *Almost soliton duality*, Adv. Geom. **15** (2015), no. 2, 159–166.
[20] P. Petersen, *Riemannian geometry*, Grad. Texts in Math., vol. 171, Springer-Verlag, New York, 1998.
[21] S. Pigola, M. Rigoli, M. Rimoldi, and A. G. Setti, *Ricci almost solitons*, Ann. Sc. Norm. Super Pisa Cl. Sci. (5) **10** (2011), no. 4, 757–799.

How to cite this article: Kim J. Four-dimensional gradient almost Ricci solitons with harmonic Weyl curvature. *Mathematische Nachrichten*. 2021;294:774–793. https://doi.org/10.1002/mana.202000126