ALMOST RICCI SOLITONS WITH HARMONIC WEYL CURVATURE

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Abstract.

1. Introduction and Main Results

A Riemannian manifold $(\mathcal{M}^n, g)$ is called an almost Ricci soliton if there exist a vector field $X \in \mathfrak{X}(\mathcal{M})$ and a smooth function $\lambda \in \mathscr{C}^\infty(\mathcal{M})$ such that

$$\operatorname{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g$$

If there also exists $f \in \mathscr{C}^\infty(\mathcal{M})$ such that $X = \nabla f$, then $(\mathcal{M}^n, g, f, \lambda)$ is said to be a gradient almost Ricci soliton. In [13], Kim proved a classification of four-dimensional gradient almost Ricci solitons with harmonic Weyl curvature. The key element to his arguments was showing that the Ricci tensor had at most three distinct Ricci eigenvalues.

Theorem (T.1.1). *Any gradient almost Ricci soliton with harmonic Weyl curvature is a multiply warped product metric.*

comentario das fibras como variedades integrais dos autoespaços, generalização do resultado do Kim da quantidade de autovalores

Theorem (T.1.2). *Let $(\mathcal{M}^n, g, f, \lambda)$ be a gradient almost Ricci soliton, not necessarily complete, with harmonic Weyl curvature, nonconstant f and $n \geq 4$. Then its Ricci tensor has at most three distinct eigenvalues at each point.*

outros comentários

Theorem (T.1.3) (Local warped product structure). *Let $(\mathcal{M}^n, g, f, \lambda)$ be an almost Ricci soliton with $n \geq 4$ and f non constant. Assume the soliton has harmonic Weyl curvature and that its Ricci tensor has exactly two distinct eigenvalues. Then for any point $p \in \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ there are a neighborhood $U \subset \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ of p and a warped product $I \times_h \mathcal{N}$ of an interval and an Einstein manifold $\mathcal{N}^{n-1}$ so that U is isometric to a domain of $I \times_h \mathcal{N}$. Furthermore, f and λ are constant on $\mathcal{N}$, through the identification between $I \times \mathcal{N}$ and U.*

possibilidade de construir exemplos dada qualquer h

Theorem (T.1.4) (Local multiply warped product structure). *Let $(\mathcal{M}^n, g, f, \lambda)$ be an almost Ricci soliton with $n \geq 4$ and f non constant. Assume the soliton has harmonic weyl curvature and that its Ricci tensor has exactly three distinct eigenvalues. Then for any point $p \in \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ there are a neighborhood $U \subset \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ of p and a multiply warped product $I \times_{h_1} \mathcal{N}_1 \times_{h_2} \mathcal{N}_2$ of an interval I and two Einstein manifolds $\mathcal{N}_1$ and $\mathcal{N}_2$, so that (U, g) is isometric to a domain of $I \times_{h_1} \mathcal{N}_1 \times_{h_2} \mathcal{N}_2$. Furthermore, f and λ are constant on $\mathcal{N}_1 \times \mathcal{N}_2$, through the identification between $I \times \mathcal{N}_1 \times \mathcal{N}_2$ and U.*

- teorema com edo das ρ-Einstein como aplicação das EDOs correspondentes do T.1.3, contraste com o T.1.3 - não é possível construir exemplos dadas quaisquer h_1 e h_2
- comentário sobre a generalização das estratégias de Li e Kim referente ao T.1.2

2. Preliminaries

Along this work we will adopt the following convention for the curvature

$$\operatorname{Rm}(X, Y, Z, W) = \left\langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, W \right\rangle, \tag{2.1}$$

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for vector fields $X, Y, Z, W \in \mathfrak{X}(M)$. When $\mathcal{M}$ is an Einstein manifold, we will denote its Einstein constant (i.e. the constant value $\operatorname{Ric}_{\mathcal{M}}$ takes on the unit tangent bundle of $\mathcal{M}$) by $\lambda_{\mathcal{M}}$. Similarly, its constant scalar curvature (equal to $\dim(\mathcal{M})\lambda_{\mathcal{M}}$) will be denoted by $\tau_{\mathcal{M}}$.

Lemma (L.2.1) (Barros-Ribeiro). *If $(\mathcal{M}^n, g, f, \lambda)$ is a gradient almost Ricci soliton with harmonic Weyl curvature, then*

$$\begin{aligned}\operatorname{Rm}(\nabla f, X, Y, Z) &= Y\left(\frac{R}{2(n-1)} - \lambda\right) g(X,Z) - Z\left(\frac{R}{2(n-1)} - \lambda\right) g(X,Y)\\ &= \frac{1}{n-1}(\operatorname{Ric}(\nabla f, Y) g(X,Z) - \operatorname{Ric}(\nabla f, Y) g(X,Y))\end{aligned}$$

Lemma (L.2.2) (Cao-Chen). *Let $(\mathcal{M}, g, f, \lambda)$ be a gradient almost Ricci soliton with harmonic Weyl curvature and non-constant f. Let c be a regular value of f and $\Sigma_c = f^{-1}(c)$ be the level surface of f. Then,*

1. *Where $\nabla f \neq 0$, $E_1 = \dfrac{\nabla f}{|\nabla f|}$ is an eigenvector of* Ric.
2. *$|\nabla f|$ is constant on a connected component of Σ_c.*
3. *There is a function s locally defined with $s(x) = \displaystyle\int \frac{\mathrm{d}f}{|\nabla f|}$, so that $\mathrm{d}s = \dfrac{\mathrm{d}f}{|\nabla f|}$ and $E_1 = \nabla s$.*
4. *$E_1E_1 f = -\operatorname{Ric}(E_1, E_1) + \lambda$. In particular, $-\operatorname{Ric}(E_1, E_1) + \lambda$ is constant on a connected component of Σ_c.*
5. *Near a point in Σ_c, the metric g can be written as*
$$g = \mathrm{d}s^2 + \sum_{i,j\geq 2} g_{ij}(s, x_2, \dots, x_n)\, \mathrm{d}x_i \otimes \mathrm{d}x_j.$$
6. *$\nabla_{E_1} E_1 = 0$.*

It is a well known fact that a Riemannian manifold (M^n, g), $n \geq 4$, has harmonic Weyl tensor if and only if its Schouten tensor $\mathcal{A} = \operatorname{Ric} - \dfrac{R}{2(n-1)} g$ is Codazzi. In coordinates this is equivalent to

$$\nabla_i \mathcal{A}_{jk} = \nabla_j \mathcal{A}_{ik}. \tag{2.2}$$

Let $\mathcal{A}$ be a codazzi tensor and denote by $E_{\mathcal{A}}(x)$ the number of distinct eigenvalues of $\mathcal{A}$ at x. In [10], Derdzinski considered the following open dense set

$$\mathcal{M}_{\mathcal{A}} = \{x \in \mathcal{M} \mid E_{\mathcal{A}}(x) \text{ is constant in a neighborhood of } x\}. \tag{2.3}$$

It turns out that in $\mathcal{M}_{\mathcal{A}}$ the eigenvalues of $\mathcal{A}$ are well defined and define smooth functions $\lambda_1, \dots, \lambda_n : \mathcal{M}_{\mathcal{A}} \to \mathbb{R}$. Furthermore, he proved that in such a set the following is true

Lemma (L.2.3) (Derdziński). *Let $(\mathcal{M}^n, g)$, $n \geq 4$, be a Riemannian metric with harmonic Weyl curvature. Let $\{E_i\}_{i=1}^n$ be a local orthonormal frame such that $\operatorname{Ric}(E_i, \cdot) = \lambda_i g(E_i, \cdot)$. Then,*

1. *For any $i, j, k \geq 1$,*
$$\begin{gathered}(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k\rangle + \nabla_{E_i}(\mathcal{A}(E_j, E_k)) =\\ (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k\rangle + \nabla_{E_j}(\mathcal{A}(E_k, E_i)).\end{gathered}$$
2. *If $k \neq i$ and $k \neq j$, then $(\lambda_j - \lambda_k)\langle \nabla_{E_i} E_j, E_k\rangle = (\lambda_i - \lambda_k)\langle \nabla_{E_j} E_i, E_k\rangle$.*
3. *Given distinct eigenfunctions λ and μ of $\mathcal{A}$ and local vector fields U and V such that $AV = \lambda V$ and $AU = \mu U$ with $|U| = 1$, it holds that $V(\mu) = (\mu - \lambda)\langle \nabla_U U, V\rangle$.*
4. *Each distribution D_{λ_i}, defined by $D_{\lambda_i}(p) = \{v \in T_p\mathcal{M} \mid \operatorname{Ric}(v, \cdot) = \lambda_i g(v, \cdot)\}$, is integrable and its leaves are totally umbilic submanifolds of M.*

3. Local structure

Suppose that the potential function of the almost soliton $\mathcal{M}$ is nonconstant and consider the set $\mathcal{R}$ of all regular points of f

$$\mathcal{R} = \{x \in \mathcal{M} \mid \nabla f(x) \neq 0\}.$$

We start with the following result.

Lemma (L.3.1). *Let $(\mathcal{M}^n, g, f, \lambda)$ be a gradient almost Ricci soliton with harmonic Weyl curvature. If f is not constant and W is a connected component of $\mathcal{M}_{\mathcal{A}}$, then either $W \cap \mathcal{R}$ is dense in W, or it is empty.*

Proof. Let W be a connected component of $M_{\mathcal{A}}$ so that $W' = W \cap \mathcal{R} \neq \emptyset$. We will show that W' is dense in W. Suppose by contradiction that this is not true and consider an open set $U \subset W \backslash W'$. Once ∇f vanishes in U, the almost Ricci soliton equation becomes $\mathrm{Ric} = \lambda g$ in U. As the quantity of eigenvalues of Ric is constant in W, it follows that $\mathcal{M}$ is Einstein in this set, i.e., there is $\mu \in \mathbb{R}$ so that $\mathrm{Ric} = \mu g$ in W. In particular, $\nabla\nabla f = (\lambda - \mu) g$ in W. Using the Bochner formula we obtain

$$X(\lambda) = \mathrm{div}(\nabla\nabla f)(X) = \mathrm{Ric}(\nabla f, X) + X(\Delta f) = X(\mu f + n\lambda), \ \forall X \in \mathfrak{X}(W),$$

which implies the existence of $c_0 \in \mathbb{R}$ so that $\lambda = -\dfrac{\mu}{n-1} f + \mu + c_0$. Consequently,

$$\nabla\nabla f = \left(-\frac{\mu}{n-1} f + c_0 \right) g, \text{ in } W.$$

Now observe that if $\mu \neq 0$, then $f - \dfrac{(n-1)c_0}{\mu}$ is an eigenfunction of $-\Delta$. If, on the other hand, $\mu = 0$, then λ is constant, $\mathrm{Ric} = 0$ and $\nabla\nabla f = \lambda g$, which means that $(W, g|_W, f, \lambda)$ is a Ricci soliton. In both cases we conclude that f is analytic in W. As ∇f vanishes in U, f is constant in W, contradicting our assumption.

This means that $W \backslash W'$ has empty interior or, equivalently, that the set W' is dense in W. □

For each point p of the open set $\mathcal{R} \cap M_{\mathcal{A}}$, we will consider the orthonormal frame $\{E_i\}_{i=1}^n$ given in Lemma 2.3 and recall that $E_1 = \dfrac{\nabla f}{|\nabla f|}$. For this frame, we have $R_{ij} = \lambda_i \delta_{ij}$. Furthermore,

Lemma (L.3.2). *Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a gradient almost Ricci soliton with harmonic Weyl curvature and nonconstant f. Then,*

$$\nabla_{E_a} E_1 = \xi_a E_a \quad \textit{and} \quad \xi_a = -\langle \nabla_{E_a} E_a, E_1 \rangle, \tag{3.1}$$

where

$$\xi_a = \frac{\lambda - \lambda_a}{|\nabla f|}. \tag{3.2}$$

Proof. To prove the first identity of (3.1), notice that for any $a \geq 2$ we have

$$\nabla_{E_a} E_1 = \frac{\nabla_{E_a} \nabla f}{|\nabla f|} = \frac{\lambda E_a - \mathrm{Ric}(E_a, \cdot)}{|\nabla f|} = \frac{(\lambda - \lambda_a) E_a}{|\nabla f|} = \xi_a E_a. \tag{3.3}$$

Now we combine it with the equality $\langle \nabla_{E_a} E_1, E_a \rangle = -\langle \nabla_{E_a} E_a, E_1 \rangle$ to get the second identity of (3.1). □

The main goal of this section is to obtain a local representation of M as a multiply warped product. The first step, contained in the next two lemmas, is to prove that most geometric functions involved in our problem are constant on a connected component of a regular level Σ_c of f. This was first proved in [13] for four dimensional Ricci solitons, and later generalized to four dimensional almost Ricci solitons in [14]. The proof of [14] works in any dimension, and we include it here for sake of completeness.

Lemma (L.3.3). *Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a gradient almost Ricci soliton with harmonic Weyl curvature and nonconstant f. Then R, $\mathrm{Ric}(E_1, E_1) = \lambda_1$ and λ are constant on a connected component of Σ_c, for a regular value c of f.*

Proof. We already know that $\lambda - \lambda_1$ depends only on s, once from item (4) of Lemma 2.2, $\lambda - \lambda_1 = f''$. On the other hand, from item (1) of Lemma 2.3 we obtain for $j = k = 1$ and $i \geq 2$

$$\nabla_{E_i} \left(\lambda_1 - \frac{R}{2(n-1)} \right) = (\lambda_i - \lambda_1) \langle \nabla_{E_1} E_i, E_1 \rangle = -(\lambda_i - \lambda_1) \langle \nabla_{E_i} E_1, E_1 \rangle = 0,$$

showing that $\lambda_1 - \dfrac{R}{2(n-1)}$ depends only on s as well.

The two facts above combined imply that λ_1 depend only on s. To see this, take $Y = E_1$ and $X = Z = E_i$, $i \geq 2$, in Lemma 2.1 and notice that

$$\begin{aligned}\lambda_1 = \mathrm{Ric}_{11} = \sum_{i=2}^{n} R_{1i1i} &= \frac{n-1}{f'} E_1 \left(\frac{R}{2(n-1)} - \lambda \right) \\ &= \frac{n-1}{f'} E_1 \left(\frac{R}{2(n-1)} - \lambda_1 \right) + \frac{n-1}{f'} E_1 \left(\lambda_1 - \lambda \right).\end{aligned}$$

As both terms in the last line depend only on s, so does λ_1. Writing,

$$\begin{aligned}\lambda &= (\lambda - \lambda_1) + \lambda_1 \\ R &= (R - 2(n-1)\lambda_1) + 2(n-1)\lambda_1\end{aligned}$$

we see that R and λ depend only on s, and the lemma is proved. □

In what follows we will need the following

Lemma (L.3.4). *Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a gradient almost Ricci soliton with harmonic Weyl curvature and nonconstant f. Then,*

(1) $\nabla_1 R_{11} = \lambda_1'$
(2) $\nabla_1 R_{aa} = \dfrac{1}{f'}(\lambda_a^2 - (\lambda_1 + \lambda)\lambda_a + \lambda\lambda_1) + \dfrac{R'}{2(n-1)}$

Proof. To prove (1) notice that

$$(\nabla_{E_1} \mathrm{Ric})_{11} = E_1(\mathrm{Ric}_{11}) - 2\,\mathrm{Ric}(\nabla_{E_1} E_1, E_1) = \lambda_1'$$

Now, using the fact that the Schouten tensor is Codazzi, we have

$$\begin{aligned}(\nabla_{E_1} \mathrm{Ric})_{aa} &= (\nabla_{E_a} \mathrm{Ric})_{1a} + \frac{E_1(R)}{2(n-1)}\delta_{aa} - \frac{E_a(R)}{2(n-1)}\delta_{1a} \\ &= E_a(\mathrm{Ric}_{1a}) - \mathrm{Ric}(\nabla_{E_a} E_1, E_a) - \mathrm{Ric}(E_1, \nabla_{E_a} E_a) + \frac{E_1(R)}{2(n-1)} \\ &= -\lambda_a g(\nabla_{E_a} E_1, E_a) - \lambda_1 g(E_1, \nabla_{E_a} E_a) + \frac{R'}{2(n-1)} \\ &= (\lambda_1 - \lambda_a)\xi_a + \frac{R'}{2(n-1)} \\ &= \frac{1}{f'}(\lambda_1 - \lambda_a)(\lambda - \lambda_a) + \frac{R'}{2(n-1)},\end{aligned}$$

which proves (2). □

In the next lemma we prove that all the eigenvalues of the Ricci tensor depend only on s. This is a generalization to nonconstant λ of [11, Lemma 3.3], which is already an extension of [13, Lemma 2.7] to higher dimensions of its counterpart for four dimensional Ricci solitons. However, the proof in [11] follows a different path, which was inspired by [19, Lemma 3]. It is this latter approach that we follow here.

Lemma (L.3.5). *Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a gradient almost Ricci soliton with harmonic Weyl curvature and nonconstant f. The functions $\lambda_2, \ldots, \lambda_n$ are constant on each connected component of Σ_c. As a consequence, the functions $\xi_2, \ldots, \xi_n$ are also constant on each connected component of Σ_c.*

Proof. Consider $\mathrm{Ric}^k_{ij} = \sum_{\ell_1 \cdots \ell_k = 1}^{n} R_{i\ell_1} \cdots R_{\ell_{k-1}\ell_k} R_{\ell_k j}$. We will show that

$$\mathrm{tr}(\mathrm{Ric}^k) = \sum_{i=1}^{n} (\mathrm{Ric}_{ii})^k = \sum_{i=1}^{n} \lambda_i^k$$

depends only on s for all $k \in \{1, \ldots, n\}$, which implies that $\lambda_2, \ldots, \lambda_n$ depend only on s, as the lemma claims.

We have already seen that $\mathrm{tr}(\mathrm{Ric}) = R$ depends only on s. Now observe that

$$\mathrm{tr}(\mathrm{Ric}^2) = \|\mathrm{Ric}\|^2. \tag{3.4}$$

Below we show that $\|\mathrm{Ric}\|^2$ depends only on s. Since $\nabla\nabla R(E_1, E_1) = R''$, using (3.1) we get

$$\begin{aligned}\Delta R &= \nabla\nabla R(E_1, E_1) + \sum_{a=2}^{n} \nabla\nabla R(E_a, E_a)\\ &= R'' + \sum_{a=2}^{n}(E_a E_a R - (\nabla_{E_a} E_a)\, R)\\ &= R'' - R' \sum_{a=2}^{n} \xi_a\\ &= R'' - R' \left(\frac{(n-1)\lambda - R + \lambda_1}{f'}\right),\end{aligned}$$

where in the last line we have used Lemma 3.2. Analogously,

$$\Delta\lambda = \lambda'' - \lambda' \left(\frac{(n-1)\lambda - R + \lambda_1}{f'}\right).$$

Using Lemma 3.3 and the equalities above we conclude that ΔR and $\Delta\lambda$ depend only on s. Now, it follows from equation (7.1) of [17, pag 789] that

$$\frac{1}{2}\Delta R - \frac{1}{2} f' R' = \lambda R - \|\mathrm{Ric}\|^2 + (n-1)\Delta\lambda.$$

As a consequence, $\|\mathrm{Ric}\|^2$ depends only on s, since all the other terms of the equation above have this property.

In order to proceed, we will show that if $\mathrm{tr}(\mathrm{Ric}^r)$ depends only on s for all $r \in \{1, \dots, k-1\}$, then $\mathrm{tr}(\mathrm{Ric}^k)$ depends only on s also. In fact, using Lemma 3.4 and $k \geq 2$ we get

$$\begin{aligned}&\frac{(\mathrm{tr}(\mathrm{Ric}^{k-1}))'}{k-1} = \sum_{i=1}^{n} (\mathrm{Ric}_{ii})^{k-2} \left(\nabla_1 \mathrm{Ric}\right)_{ii} =\\ &(\mathrm{Ric}_{11})^{k-2} \left(\nabla_1 \mathrm{Ric}\right)_{11} + \sum_{a=2}^{n} (\mathrm{Ric}_{aa})^{k-2} \left(\nabla_1 \mathrm{Ric}\right)_{aa} =\\ &\lambda_1^{k-2}\lambda_1' + \frac{1}{f'}\sum_{a=2}^{n} \lambda_a^{k-2}(\lambda_a^2 - (\lambda_1 + \lambda)\lambda_a + \lambda\lambda_1) + \frac{R'}{2(n-1)}\sum_{a=2}^{n}\lambda_a^{k-2} =\\ &\frac{1}{f'}\,\mathrm{tr}(\mathrm{Ric}^k) - \frac{\lambda_1 + \lambda}{f'}\,\mathrm{tr}(\mathrm{Ric}^{k-1}) + \left(\frac{R'}{2(n-1)} + \frac{\lambda\lambda_1}{f'}\right) \mathrm{tr}(\mathrm{Ric}^{k-2})\\ &\qquad\qquad + \left(\lambda_1' - \frac{R'}{2(n-1)}\right) \lambda_1^{k-2}.\end{aligned} \tag{3.5}$$

This shows that $\mathrm{tr}(\mathrm{Ric}^k)$ depends only on s, since the equalities above express it in terms of $\mathrm{tr}(\mathrm{Ric}^{k-1})$, $\mathrm{tr}(\mathrm{Ric}^{k-2})$ and quantities that depend only on s.

Now, using (3.5) together with $\mathrm{tr}(\mathrm{Ric})$ and $\mathrm{tr}(\mathrm{Ric}^2)$ being dependent only on s, the proof of the corollary follows using finite induction. □

Following [11], for each $a \geq 2$ we consider the following notation

$$[a] = \{i \in \{2, \dots, n\} \mid \lambda_i = \lambda_a\}. \tag{3.6}$$

From now on we use the convention that $2 \leq a, b, c, \dots, \alpha, \beta, \gamma, \dots \leq n$ satisfy $b, c \in [a]$, $\beta, \gamma \in [\alpha]$ and $[a] \neq [\alpha]$.

Lemma (L.3.6). *Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a gradient almost Ricci soliton with harmonic Weyl curvature and nonconstant f. Let $(x_b)_{b\in[a]}$ and $(x_\beta)_{\beta\in[\alpha]}$ be local coordinate systems of the integral manifolds of the distributions D_a and D_α, respectively . Setting $\partial_1 = E_1$, we have*

(1) $\nabla_{\partial_a}\partial_1 = \xi_a \partial_a$ *and* $\nabla_{\partial_a}\partial_b = -\xi_a g_{ab}\partial_1 + \sum_{c\in[a],c\neq a}\Gamma^c_{ab}\partial_c$.

(2) $\nabla_{\partial_\alpha}\partial_a = 0$.

(3) $R_{1a1b} = -(\xi_a' + \xi_a^2)g_{ab}$.

(4) $R_{a\alpha b\beta} = -\xi_a\xi_\alpha g_{ab}g_{\alpha\beta}$

In particular,

$$\partial_1 g_{ab} = 2\xi_a g_{ab} \qquad \text{and} \qquad \partial_\alpha g_{ab} = 0. \tag{3.7}$$

Proof. Proceeding as in (3.3), for any vector field X and any $a \geq 2$ we have

$$\langle \nabla_{\partial_a}\partial_1, X\rangle = \frac{1}{|\nabla f|}\nabla\nabla f(\partial_a, X) = \frac{\lambda - \lambda_a}{|\nabla f|}\langle \partial_a, X\rangle = \langle \xi_a\partial_a, X\rangle,$$

which proves the first equality of (1). The second one follows from $g_{1a} = 0$ and from

$$\langle \nabla_{\partial_a}\partial_b, \partial_1\rangle = -\langle \nabla_{\partial_a}\partial_1, \partial_b\rangle = -\xi_a g_{ab}.$$

Now we proceed to prove (2). If $a, \alpha, z \in \{2,\dots,n\}$ are so that $[a] \neq [\alpha]$, then

$$\begin{aligned} R_{a\alpha 1z} &= \langle \nabla_{\partial_\alpha}\nabla_{\partial_a}\partial_1 - \nabla_{\partial_a}\nabla_{\partial_\alpha}\partial_1, \partial_z\rangle \\ &= \xi_a\langle \nabla_{\partial_\alpha}\partial_a, \partial_z\rangle - \xi_\alpha\langle \nabla_{\partial_a}\partial_\alpha, \partial_z\rangle \\ &= (\xi_a - \xi_\alpha)\langle \nabla_{\partial_\alpha}\partial_a, \partial_z\rangle. \end{aligned}$$

On the other hand, by using (2.1) we obtain

$$\begin{aligned} (\xi_a - \xi_\alpha)\langle \nabla_{\partial_\alpha}\partial_a, \partial_z\rangle &= R_{1za\alpha} \\ &= \partial_a\left(\frac{R}{2(n-1)} - \lambda\right)g_{z\alpha} - \partial_\alpha\left(\frac{R}{2(n-1)} - \lambda\right)g_{za} \\ &= 0, \end{aligned}$$

where in the last line we have used that $\partial_i\lambda = \partial_i R = 0$ for all $i \geq 2$. As $[a] \neq [\alpha]$ and $z \in \{1,\dots,n\}$, we conclude that $\nabla_{\partial_\alpha}\partial_a = 0$.

Using (1) and (2) we get

$$\begin{aligned} R_{1a1b} &= \langle \nabla_{\partial_a}\nabla_{\partial_1}\partial_1 - \nabla_{\partial_1}\nabla_{\partial_a}\partial_1, \partial_b\rangle \\ &= -\langle \xi_a'\partial_a + \xi_a\nabla_{\partial_1}\partial_a, \partial_b\rangle \\ &= -(\xi_a' + \xi_a^2)g_{ab}, \end{aligned}$$

and

$$\begin{aligned} R_{a\alpha b\beta} &= \langle \nabla_{\partial_\alpha}\nabla_{\partial_a}\partial_b - \nabla_{\partial_a}\nabla_{\partial_\alpha}\partial_b, \partial_\beta\rangle \\ &= \left\langle \nabla_{\partial_\alpha}\left(-\xi_a g_{ab}\partial_1 + \sum_{c\in[a],c\neq a}\Gamma^c_{ab}\partial_c\right), \partial_\beta\right\rangle \\ &= -\xi_a g_{ab}\langle \nabla_{\partial_\alpha}\partial_1, \partial_\beta\rangle + \sum_{c\in[a],c\neq a}\langle \nabla_{\partial_\alpha}(\Gamma^c_{ab}\partial_c), \partial_\beta\rangle \\ &= -\xi_a\xi_\alpha g_{ab}g_{\alpha\beta} + \sum_{c\in[a],c\neq a}\left(\Gamma^c_{ab}\langle \nabla_{\partial_\alpha}\partial_c, \partial_\beta\rangle + \partial_\alpha\Gamma^c_{ab}\langle \partial_c, \partial_\beta\rangle\right) \\ &= -\xi_a\xi_\alpha g_{ab}g_{\alpha\beta}, \end{aligned}$$

proving (3) and (4), respectively.

In order to finish the proof of the lemma, recall that $\xi_a = \xi_b$ and consider the following derivatives of g_{ab}

$$\begin{aligned} \partial_1 g_{ab} &= g(\nabla_{\partial_1}\partial_a, \partial_b) + g(\partial_a, \nabla_{\partial_1}\partial_b) = 2\xi_a g_{ab} \\ \partial_\alpha g_{ab} &= g(\nabla_{\partial_\alpha}\partial_a, \partial_b) + g(\partial_a, \nabla_{\partial_\alpha}\partial_b) = 0. \end{aligned}$$

$\square$

In the next lemma we show that the Ricci tensor of $\mathcal{M}^n$ has at most 3 distinct eigenvalues for all $n \geq 4$. This result was already known for almost Ricci solitons in dimension 4 [14, Lemma 3.4] and for Ricci solitons in any dimensions [19, Theorem 4.2]. Both of them are inspired by a result of Kim for four dimensional Ricci solitons [13, Proposition 3.4]. The proofs go through a long computation in order to express f' in a specific way that implies its vanishment when more than three distinct eigenvalues exist. Here we take a different route and adapt the proof given by Li in [11]. Almost all of the computations regarding local structure in [11] don't use the constancy of λ and hence go through in exactly the same way in our context, the one exception being equation (1) in lemma 3.7, which was easily adapted by using lemma 2.1.

Mutatis mutandis, a straightforward consequence of Lemma (3.6) shows that the distribution $V_i = \mathrm{Span}\{E_\ell \mid \ell \in [i]\}$ is closed under the Lie bracket, hence integrable. We'll denote by N_i its associated integrable r_i-dimensional submanifold. We'll also denote $\xi_a := X$ and $\xi_\alpha := Y$.

Theorem (T.3.1). *Any gradient almost Ricci soliton with harmonic Weyl curvature is a multiply warped product metric of eigenspaces with the Ricci tensor.*

Proof. Let $U \subset \mathcal{M}_A \cap \{\nabla f \neq 0\}$ be an open subset. We'll take local coordinates $(x_1 = s, x_2, x_3, \ldots, x_n)$ in U as in lemma 2.2. Let us fix $s_0 \in I$ and $a, b \in [i]$. Mutatis mutandis, equation (3.7) from lemma (3.6) implies that

$$\partial_1(g_{ab}) = 2\xi_a g_{ab}$$

Therefore, if we define the function $\tilde{h}_i : I \to \mathbb{R}$ by

$$s \in I \mapsto \tilde{h}_i(s) := \exp\left(\int_{s_0}^{s} \xi_a(y)\ \mathrm{d}y\right)$$

Then $\tilde{h}_i$ satisfies

$$\partial_1\left(\tilde{h}_i^{-2} g_{ab}\right) = 0$$

so that $\tilde{h}_i^{-2} g_{ab}$ is constant in s, i.e,

$$(\tilde{h}_i(s))^{-2} g_{ab}(s, x_2, \ldots, x_n) = (\tilde{h}_i(s_0))^{-2} g_{ab}(s_0, x_2, \ldots, x_n)$$

or, equivalently,

$$g_{ab}(s, x_2, \ldots, x_n) = h(s)^2 g_{ab}(s_0, x_2, \ldots, x_n) \tag{3.8}$$

where $h(s) := \dfrac{h_i(s)}{h_i(s_0)}$. Without any loss of generality, we can assume $h_i(s_0) = 1$. Clearly, (3.8) defines a Riemannian metric g_i on N_i. In an entirely analogous manner, we obtain Riemannian metrics g_α on N_α for any $\alpha \notin [i]$. This proves that in U, g can be written as the following warped product (of possibly $n-1$ fibers):

$$g = \mathrm{d}s^2 + h_i^2 g_i + \sum_{\alpha \notin [i]} h_\alpha^2 g_\alpha$$

□

Remark 3.1. As a straightforward consequence of the proof of theorem **(T.3.1)**, we get

$$\frac{\lambda - \lambda_i}{f'} = \xi_i = \frac{h_i'}{h_i} \tag{3.9}$$

This equation will be useful later on.

Remark 3.2. The next two lemmas were initially proven by Li, F in [11] in the context of gradient Ricci solitons. A careful examination shows that with very minor adaptations, both lemmas also hold for gradient almost Ricci solitons, since most of the computations do not use the constancy of λ.

Lemma (L.3.7) (Li, F.). *Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a gradient almost Ricci soliton with harmonic Weyl curvature and nonconstant f. Then the following integrability conditions hold*

(1)

$$\xi_i' + \xi_i^2 = \frac{2(n-1)\lambda' - R'}{2(n-1)f'} = \xi_\alpha' + \xi_\alpha^2$$

(2)
$$\lambda_i = -f'\xi_i + \lambda = -\xi_i' - \xi_i \sum_{j=2}^{n} \xi_j + (r_i - 1)\frac{k_i}{h_i^2}$$

(3)
$$\lambda_1 = -f'' + \lambda = -(n-1)\left(\xi_i' + \xi_i^2\right)$$

(4)
$$\lambda_i' - (\lambda_1 - \lambda_i)\,\xi_i = \frac{R'}{2(n-1)} = \lambda_\alpha' - (\lambda_1 - \lambda_\alpha)\,\xi_\alpha$$

Proof. (1) follows immediately from lemma 2.1. The first equality in (2) (and also in (3)) is a straightforward consequence of the gradient almost Ricci soliton equation, while the second one is a consequence of the Gauss equations applied to the submanifold N_i, as seen in the proof of Theorem 3.6 in [11] (which does not use constancy of λ and therefore holds in our context as well). The second equality in (3) follows from equation (3.8) of Lemma 3.4 in [11] combined with (1). Equation (4) is a straighforward consequence of the three previous ones. □

Lemma (L.3.8) (Li, F.). *Let $(\mathcal{M}^n, g, f), (n \geq 4)$, be an n-dimensional gradient almost Ricci soliton with harmonic Weyl curvature. Suppose that in some neighborhood U of $p \in M_A \cap \{\nabla f \neq 0\}$, λ_a and λ_α are mutually different Ricci eigenvalues with multiplicities r_1 and r_2. Then the following identities hold:*

(1)
$$(r_1 - 1)\frac{k_1}{h_1^2} - (r_2 - 1)\frac{k_2}{h_2^2} = (X - Y)\left[\sum_{i=2}^{n} \xi_i - (X + Y) - f'\right],$$

(2)
$$\begin{aligned}&(r_1 - 1)\frac{k_1}{h_1^2}X - (r_2 - 1)\frac{k_2}{h_2^2}Y\\ =&(X - Y)\left\{\left(X' + X^2\right) + \lambda + (X + Y)\left[\sum_{i=2}^{n} \xi_i - (X + Y) - f'\right] + XY\right\},\end{aligned}$$

(3)
$$-(r_1 - 1)\frac{k_1}{h_1^2}Y + (r_2 - 1)\frac{k_2}{h_2^2}X = (X - Y)\left[\left(X' + X^2\right) + \lambda + XY\right]$$

(4)
$$\begin{aligned}&(r_1 - 1)\frac{k_1}{h_1^2}X - (r_2 - 1)\frac{k_2}{h_2^2}Y\\ =&(X - Y)\left[\left(X' + X^2\right) + \sum_{i=2}^{n} \xi_i^2 - \left(X^2 + Y^2\right) - XY\right]\end{aligned}$$

(5)
$$\sum_{i=2}^{n} \xi_i^2 - \lambda = (X + Y)\left(\sum_{i=2}^{n} \xi_i - f'\right).$$

Proof. All of these equations can be obtained as consequences of the ones in lemma (3.7), as shown in [11]. □

The next result shows that the bound on the amount of eigenvalues of the Ricci tensor also holds for gradient almost Ricci solitons. The proof given in [11], originally only for gradient Ricci solitons, also works perfectly for gradient almost Ricci solitons. For the sake of completeness, we'll include it here as well.

Lemma (L.3.9). *Let $(\mathcal{M}^n, g, f, \lambda)$, $n \geq 4$, be a gradient almost Ricci soliton with harmonic Weyl curvature and nonconstant f. There cannot be more than two distinct λ_i with $i \in \{2, \dots, n\}$.*

Proof. Suppose by contradiction that this is not true. Then there are at least three distinct eigenvalues among $\lambda_2, \lambda_3, \cdots, \lambda_n$, which we'll denote by $\lambda_a, \lambda_\alpha$ and λ_p, with multiplicities r_1, r_2 and r_3. For convenience, we also denote $\xi_a := X, \xi_\alpha := Y$ and $\xi_p := Z$. It follows from (5) of the previous lemma that

$$\sum_{i=2}^{n} \xi_i^2 - \lambda = (X+Y)\left(\sum_{i=2}^{n} \xi_i - f'\right) = (X+Z)\left(\sum_{i=2}^{n} \xi_i - f'\right) = (Y+Z)\left(\sum_{i=2}^{n} \xi_i - f'\right). \tag{3.10}$$

Since $Y \neq Z$, it follows that

$$\sum_{i=2}^{n} \xi_i = f' \tag{3.11}$$

Going back to (3.10), we obtain

$$\sum_{i=2}^{n} \xi_i^2 = \lambda$$

On the other hand, by using lemma 3.7 and differentiating (3.11), we obtain

$$-\sum_{i=2}^{n} \xi_i^2 = \lambda$$

Therefore $\xi_i = 0$ for all $i \in \{2, \ldots, n\}$, which is a contradiction since we assumed X, Y, Z were distinct. □

The lemma above implies Theorem **(T.1.2)**.

Proof of Theorem (T.1.2). Let $(\mathcal{M}^n, g, f, \lambda)$ be an almost Ricci soliton with $n \geq 4$ and let $p \in M_{\mathcal{A}}$. It follows from Lemma 3.1 that there is a connected open set $U \subset M_{\mathcal{A}}$ containing p so that the quantity of eigenvalues of Ric in U is constant and $U \cap \mathcal{R}$ is either empty or dense in U. If it is empty, then the number of eigenvalues of Ric in U is constant equals to one. If it is dense, Lemma 3.9 asserts that Ric has exactly two or three eigenvalues in $U \cap \mathcal{R}$, and consequently, the same is true in U. □

Remark 3.3. We'll recognize the amount of fibers in a multiply warped product in the same way as [20]. Namely, warping functions cannot be constant, two different warping functions cannot be multiples of one another and any two fibers with the same warping function are joined in a single fiber.

Now we obtain a local description of M, which is a consequence of Lemma 3.6 and Lemma 3.9. First, recall the following formulas for the Ricci and scalar curvatures of a multiply warped product:

Lemma (L.3.10). *Let $\mathcal{M} = B \times_{h_1} \mathcal{N}_1^{d_1} \times \cdots \times_{h_k} \mathcal{N}_k^{d_k}$ be a Riemannian multiply warped product with metric $g = g_B \oplus h_1^2 g_{\mathcal{N}_1} \oplus \cdots \oplus h_k^2 g_{\mathcal{N}_k}$. Then the scalar curvature of $\mathcal{M}$ is given by*

$$\begin{aligned} R = R_B &- 2\sum_{1\leq i\leq k} d_i \frac{\Delta_B h_i}{h_i} + \sum_{1\leq i\leq k} \frac{R_{\mathcal{N}_i}}{h_i^2} - \sum_{1\leq i\leq k} d_i(d_i-1)\frac{\|\mathrm{grad}_B h_i\|_B^2}{h_i^2} \\ &- \sum_{1\leq i\leq k} \sum_{\substack{1\leq \ell\leq k \\ \ell\neq i}} d_i d_\ell \frac{g_B(\mathrm{grad}_B h_i, \mathrm{grad}_B h_\ell)}{h_i h_\ell} \end{aligned} \tag{3.12}$$

Also, for any lifted vector fields $X, Y, Z \in \mathcal{L}(B)$, $V \in \mathcal{L}(\mathcal{N}_i)$ and $W \in \mathcal{L}(\mathcal{N}_j)$, we have

(1)
$$\nabla_X V = \nabla_V X = \frac{X(h_i)}{h_i} V$$

(2)
$$\mathrm{Ric}(X,Y) = \mathrm{Ric}_B(X,Y) - \sum_{1\leq \ell\leq k} \frac{d_\ell}{h_\ell} \mathrm{Hess}_B(h_\ell)(X,Y)$$

(3)
$$\mathrm{Ric}(X,V) = 0$$

(4)

$$\operatorname{Ric}(V,W)=0 \text{ if } i\neq j$$

(5)

$$\operatorname{Ric}(V,W)=\operatorname{Ric}_{\mathcal{N}_i}(V,W)-\left(\frac{\Delta_B h_i}{h_i}+(d_i-1)\frac{\|\operatorname{grad}_B h_i\|^2}{h_i^2}+\sum_{\substack{1\leq \ell\leq k\\ \ell\neq i}} d_\ell\cdot\frac{g_B(\operatorname{grad}_B h_i,\operatorname{grad}_B h_\ell)}{h_i h_\ell}\right)g(V,W), \text{ if } i=j$$

Another interesting fact about warped products is the following

Theorem (T.3.2). *Suppose that the multiply warped product $\mathcal{M}=B\times_{h_1}\mathcal{N}_1^{d_1}\times\cdots\times_{h_k}\mathcal{N}_k^{d_k}$ with metric $g=g_B\oplus h_1^2 g_{\mathcal{N}_1}\oplus\cdots\oplus h_k^2 g_{\mathcal{N}_k}$ has harmonic Weyl curvature. Then each fiber $(\mathcal{N}_i,g_{\mathcal{N}_i})$ is an Einstein manifold.*

Proof. Since $\mathcal{M}$ has harmonic Weyl curvature, we have

$$(\nabla_X \operatorname{Ric})(Y,Z)-(\nabla_Y \operatorname{Ric})(X,Z)=\frac{1}{2n-2}\left\{(\nabla_X(Rg))(Y,Z)-(\nabla_Y(Rg))(X,Z)\right\} \tag{3.13}$$

for all vector fields $X,Y,Z\in\Gamma(T\mathcal{M})$. In particular, if we take an unitary vector field $\widetilde{V}\in\mathcal{L}(\mathcal{N}_i)$ and a convenient (in a sense we'll clarify soon) $X\in\Gamma(T\mathcal{M})$, we get

$$\begin{aligned}(\nabla_X \operatorname{Ric})(V,V)-(\nabla_V \operatorname{Ric})(X,V)&=\varphi_1 X(h_i)\operatorname{Ric}^{\mathcal{N}_i}(\widetilde{V},\widetilde{V})\\ &=\varphi_2\end{aligned}$$

where $\varphi_1\neq 0,\varphi_2$ are functions depending only on $\mathcal{B}$ determined by lemma **(L.3.10)**. Clearly, this implies that

$$\operatorname{Ric}^{\mathcal{N}_i}(\widetilde{V},\widetilde{V})=\frac{\varphi_3}{X(h_i)}$$

where φ_3 is a function which depends on $\mathcal{B}$ only. Since h is not constant we can always choose X in a way that makes the right-hand side of the above equation well defined. This completes the proof. ☐

Combining theorems **(T.3.1)**, lemma **(L.3.9)** and **(T.3.2)**, we get

Theorem (T.3.3). *Suppose a multiply warped product $\mathcal{M}=B\times_{h_1}\mathcal{N}_1^{d_1}\times\cdots\times_{h_k}\mathcal{N}_k^{d_k}$ with metric $g=g_B\oplus h_1^2 g_{\mathcal{N}_1}\oplus\cdots\oplus h_k^2 g_{\mathcal{N}_k}$ has harmonic Weyl curvature and also admits a structure of almost Ricci soliton with non constant f. Then $k\leq 2$.*

Proof. This is a straighforward consequence of the equation

$$\lambda_\alpha-\lambda_\beta=f'\left(\frac{h'_\alpha}{h_\alpha}-\frac{h'_\beta}{h_\beta}\right)$$

☐

When there is only a single fiber to be considered, the reciprocal of **(T.3.2)** is the following:

Theorem (T.3.4). *Let $\mathcal{M}=\mathcal{I}\times_h\mathcal{N}^{n-1}$ be a warped product over an Einstein manifold $\mathcal{N}$. Then $\mathcal{M}$ has harmonic Weyl curvature.*

Proof. Let $\{\widetilde{E}_i\}_{1\leq i\leq n}$ be a $g_{\mathcal{N}}$-orthonormal frame and consider the g-orthonormal frame $\left\{E_i\doteq\frac{\widetilde{E_i}}{h}\right\}_{1\leq i\leq n}$.

Letting C denote the Cotton tensor, defined by

$$C(X,Y,Z)=(\nabla_X \operatorname{Ric})(Y,Z)-(\nabla_Y \operatorname{Ric})(X,Z)-\left\{\frac{1}{2n-2}\left\{(\nabla_X(Rg))(Y,Z)-(\nabla_Y(Rg))(X,Z)\right\}\right\}$$

we must check that C vanishes everywhere. Due to the antissimetry $C(X,Y,Z)=-C(Y,X,Z)$, it is enough to verify that

(i) $C(\partial_1,E_a,\partial_1)=0$

(ii) $C(\partial_1, E_a, E_b) = 0$

(i) holds trivially, so we'll only check **(ii)**. Since **(ii)** is obviously true if $a \neq b$, we'll assume $a = b$ in the following computations. We have

$$(\nabla_{\partial_1} \operatorname{Ric})(E_a, E_a) = \partial_1 (\operatorname{Ric}(E_a, E_a)) - 2 \operatorname{Ric}(\nabla_{\partial_1} E_a, E_a) \tag{3.14}$$

Now, since

$$\operatorname{Ric}(E_a, E_a) = \frac{R}{n-1} + \frac{h''}{h}$$

and

$$\begin{aligned}
\nabla_{\partial_1} E_a = \nabla_{\partial_1}\left(h^{-1}\widetilde{E}_a\right) &= -h^{-2}h'\widetilde{E}_a + h^{-1}\nabla_{\partial_1}\widetilde{E}_a \\
&= -h^{-2}h'\widetilde{E}_a + h^{-1}h'h^{-1}\widetilde{E}_a \\
&= 0
\end{aligned}$$

then **(3.14)** becomes

$$(\nabla_{\partial_1} \operatorname{Ric})(E_a, E_a) = \partial_1 (\operatorname{Ric}(E_a, E_a)) = \frac{R'}{n-1} + h^{'''}h^{-1} - h'h''h^{-2} \tag{3.15}$$

Similarly,

$$\begin{aligned}
-(\nabla_{E_a} \operatorname{Ric})(\partial_1, E_a) &= \operatorname{Ric}(\nabla_{E_a}\partial_1, E_a) + \operatorname{Ric}(\partial_1, \nabla_{E_a} E_a) \\
&= \operatorname{Ric}\left(h^{-1}\nabla_{\widetilde{E}_a}\partial_1, h^{-1}\widetilde{E}_a\right) + \operatorname{Ric}\left(\partial_1, h^{-2}\nabla_{\widetilde{E}_a}\widetilde{E}_a\right) \\
&= \operatorname{Ric}\left(h^{-1}h'h^{-1}\widetilde{E}_a, h^{-1}\widetilde{E}_a\right) + h^{-2}\operatorname{Ric}(\partial_1, -hh'\partial_1) \\
&= h^{-1}h'\left(\frac{R}{n-1} + \frac{h''}{h}\right) + (n-1)h'h^{-2}h'' \\
&= h^{-1}h'\frac{R}{n-1} + h^{-2}h'h'' + (n-1)h'h^{-2}h'' \\
&= h^{-1}h'\frac{R}{n-1} + nh^{-2}h'h''
\end{aligned} \tag{3.16}$$

Together, **(3.15)** and **(3.16)** give us

$$(\nabla_{\partial_1} \operatorname{Ric})(E_a, E_a) - (\nabla_{E_a} \operatorname{Ric})(\partial_1, E_a) = \frac{R'}{n-1} + h^{'''}h^{-1} + (n-1)h^{-2}h'h'' + h^{-1}h'\frac{R}{n-1} \tag{3.17}$$

On the other hand, we have

$$\begin{aligned}
&\frac{1}{2(n-1)}\left\{(\nabla_{\partial_1}(Rg))(E_a, E_a) - (\nabla_{E_a}(Rg))(\partial_1, E_a)\right\} = \\
&\frac{R'}{2(n-1)} + \frac{1}{2(n-1)}\left\{Rg(\nabla_{E_a}\partial_1, E_a) + Rg(\partial_1, \nabla_{E_a}E_a)\right\} \\
&= \frac{R'}{2(n-1)} + \frac{1}{2(n-1)}\left\{Rg\left(h'h^{-2}\widetilde{E}_a, h^{-1}\widetilde{E}_a\right) + Rg\left(\partial_1, -h^{-1}h'\partial_1\right)\right\} \\
&= \frac{R'}{2(n-1)} + \frac{1}{2(n-1)}\left\{Rh'h^{-1} - Rh'h^{-1}\right\} \\
&= \frac{R'}{2(n-1)}
\end{aligned} \tag{3.18}$$

Notice that **(3.13)** holds if, and only if

$$\frac{R'}{2(n-1)} + h^{'''}h^{-1} + (n-1)h^{-2}h'h'' + h^{-1}h'\frac{R}{n-1} = 0 \tag{3.19}$$

Now, recall that **(3.12)** gives us

$$\begin{aligned}
\frac{R'}{2(n-1)} &= \frac{1}{2(n-1)}\left(-2(n-1)h''h^{-1} + R^{\mathcal{N}}h^{-2} - (n-1)(n-2)\left(h'\right)^2 h^{-2}\right)' \\
&= \left(-h''h^{-1} + \frac{R^{\mathcal{N}}}{2(n-1)}h^{-2} - \frac{n-2}{2}\left(h'\right)^2 h^{-2}\right)' \\
&= -h^{'''}h^{-1} + h^{-2}h'h'' - \frac{R^{\mathcal{N}}}{n-1}h^{-3}h' - (n-2)\left(h'h''h^{-2} - \left(h'\right)^3 h^{-3}\right) \\
&= -h^{'''}h^{-1} - (n-3)\, h^{-2}h'h'' - \frac{R^{\mathcal{N}}}{n-1}h^{-3}h' + (n-2)\left(h'\right)^3 h^{-3}
\end{aligned}$$

A simple calculation now shows that **(3.19)** is indeed satisfied, since

$$\begin{aligned}
0 = &-h^{'''}h^{-1} - (n-3)\, h^{-2}h'h'' - \frac{R^{\mathcal{N}}}{n-1}h^{-3}h' + (n-2)\left(h'\right)^3 h^{-3} + h^{'''}h^{-1} + (n-1)h^{-2}h'h'' \\
&+ h^{-1}h'\left(-2h''h^{-1} + \frac{R^{\mathcal{N}}}{n-1}h^{-2} - (n-2)\left(h'\right)^2 h^{-2}\right)
\end{aligned}$$

□

We have proven that for any smooth function $h : I \to \mathbb{R}$, the warped product $\mathcal{M}^n = I \times_h \mathcal{N}$ has harmonic Weyl curvature. This allows us to endow any such warped product with an almost Ricci soliton structure.

Theorem (T.3.5). *Let $\left(I \times_h \mathcal{N}^{n-1}, g\right)$ be a warped product, where $\mathcal{N}$ is an Einstein manifold with Einstein constant $\lambda_{\mathcal{N}}$. Let $f, \lambda : I \to \mathbb{R}$ be functions satisfying the system of ODE's*

$$\begin{cases} \lambda = f'' - \dfrac{(n-1)}{h}h'', \\[2ex] \lambda = f'\dfrac{h'}{h} + \dfrac{\lambda_{\mathcal{N}}}{h^2} - \dfrac{h''}{h} - \dfrac{(n-2)}{h^2}(h')^2. \end{cases} \tag{3.20}$$

Then $(I \times_h \mathcal{N}, g, f, \lambda)$ is an almost Ricci soliton with harmonic Weyl curvature and its Ricci tensor has exactly two eingenvalues.

Proof. This is an immediate consequence of lemma **(L.3.10)** combined with theorem **(T.3.4)**.

□

On the other hand, we shall now prove that, in a sense, the almost soliton condition determines the warping function h.

Theorem (T.3.6) (Local warped product structure). *Let $(\mathcal{M}^n, g, f, \lambda)$ be an almost Ricci soliton with $n \geq 4$ and f non constant. Assume the soliton has harmonic Weyl curvature and that its Ricci tensor has exactly two distinct eigenvalues. Then for any point $p \in \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ there are a neighborhood $U \subset \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ of p and a warped product $I \times_h \mathcal{N}$ of an interval and an Einstein manifold $\mathcal{N}^{n-1}$ so that U is isometric to a domain of $I \times_h \mathcal{N}$. Furthermore, f and λ are constant on $\mathcal{N}$, through the identification between $I \times \mathcal{N}$ and U. Finally, it is necessary that h, f and λ satisfy the following ODE system*

$$\begin{cases} \lambda = f'' - \dfrac{(n-1)}{h}h'' \\[2ex] \lambda = f'\dfrac{h'}{h} + \dfrac{\lambda_{\mathcal{N}}}{h^2} - \dfrac{h''}{h} - \dfrac{(n-2)}{h^2}(h')^2 \end{cases} \tag{3.21}$$

Proof. The warped product decomposition follows from **(T.3.1)**, whence it follows $\mathcal{N}$ is Einstein. The first equation is then an immediate consequence of theorem **(T.3.4)** and lemma **(L.2.1)**. Taking a local coordinate frame $(\partial_i)_{i\geq 2} \in \mathfrak{X}(\mathcal{N})$, the formulae **(2)**, **(1)** and **(5)** from Lemma **(L.3.10)**, respectively, yield

$$\begin{cases} \mathrm{Ric}_{11} = -\dfrac{(n-1)}{h}h'' = \lambda g_{11} - (\nabla\nabla f)_{11} = \lambda - f'' \\ \mathrm{Ric}_{22} = \lambda_{\mathcal{N}} g^{\mathcal{N}}_{22} - \left(\dfrac{h''}{h} + (n-2)\dfrac{(h')^2}{h^2}\right) g_{22} = \left(\dfrac{\lambda_{\mathcal{N}}}{h^2} - \dfrac{h''}{h} - (n-2)\dfrac{(h')^2}{h^2}\right) g_{22} \end{cases}$$

as desired.

□

The following example, which we shall aproppriately name as Bryant's almost Ricci soliton (as it is but a very small generalization of R. Bryant's MO post in [reference]), shows that in fact any Einstein manifold gives rise to a warped product with a gradient almost Ricci soliton structure.

Example 3.1. *(Bryant's almost Ricci soliton)*
Let $(\mathcal{N}^n, h)$ be a Riemannian manifold with Einstein constant $(n-1)k$. Consider $I \subset (0, \infty)$ an interval where

$$\varphi(u) \doteq k - au^2 + bu^{1-n} > 0,\ \forall u \in I,$$

where $a, b \in \mathbb{R}$ are constants and $u > 0$ is the coordinate on $\mathbb{R}^+$. Then

$$g = \frac{\mathrm{d}u^2}{(\varphi(u))^2} + u^2\, h$$

is a Riemannian metric on $\mathcal{M}^{n+1} \doteq I \times \mathcal{N}$. Since $\mathcal{N}$ is Einstein, it follows from **(T.3.4)** that $\mathcal{M}$ has harmonic Weyl curvature. After a few straightforward computations, one also sees that

$$\mathrm{Ric}(g) = \left(n\,a - \tfrac{1}{2}\,b\,(n-1)\,u^{-n-1}\right) g + \frac{(n^2-1)b\,\mathrm{d}u^2}{2\left(b\,u^2 + k\,u^{n+1} - a\,u^{n+3}\right)},$$

whence it follows that $\mathcal{M}$ has constant scalar curvature $n(n+1)a$ as well as that g is Einstein if, and only if, $b = 0$. We can also make f explicit. Indeed, the Ricci-almost soliton equation in our case becomes

$$f'' = \lambda - na - \frac{1}{2}\,b\,n\,(n-1)\,u^{-n-1}$$

Therefore, if $q \in \mathscr{C}^\infty(I)$ satisfies $q'' = \lambda$, we have

$$f = q - \frac{1}{2}\,n\,au^2 - \frac{1}{2}\,b\,u^{1-n}$$

When there are three distinct Ricci-eigenvalues (and hence two fibers in the local warped product decomposition of $\mathcal{M}$), theorem **(T.3.4)** no longer holds, as shown in Killing tensors and warped products. Thus arises the need to add the harmonic Weyl curvature hypothesis.

Theorem (T.3.7). *Let $I \times_{h_1} \mathcal{N}_1^{r_1} \times_{h_2} \mathcal{N}_2^{r_2}$ be a multiply warped product, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are Einstein manifolds, with Einstein constants $\lambda_{\mathcal{N}_1}$ and $\lambda_{\mathcal{N}_2}$, respectively. Let $f, \lambda : I \to \mathbb{R}$ be functions satisfying the system of ODE's*

$$\begin{cases} \lambda = f'' - r_1\dfrac{(h_1)''}{h_1} - r_2\dfrac{(h_2)''}{h_2} \\ \lambda = \dfrac{\lambda_{\mathcal{N}_1}}{(h_1)^2} + \dfrac{h_1'}{h_1}f' - \dfrac{(h_1)''}{h_1} - (r_1-1)\dfrac{(h_1')^2}{(h_1)^2} - r_2\dfrac{h_1'h_2'}{h_1h_2} \\ \lambda = \dfrac{\lambda_{\mathcal{N}_2}}{(h_2)^2} + \dfrac{h_2'}{h_2}f' - \dfrac{(h_2)''}{h_2} - (r_2-1)\dfrac{(h_2)'^2}{(h_2)^2} - r_1\dfrac{h_1'h_2'}{h_1h_2} \\ \lambda' = \dfrac{f'}{n-1}\left(r_1\dfrac{h_1''}{h_1} + r_2\dfrac{h_2''}{h_2}\right) \end{cases} \tag{3.22}$$

Then $(I \times_{h_1} \mathcal{N}_1^{r_1} \times_{h_2} \mathcal{N}_2^{r_2}, g, f, \lambda)$ is an almost Ricci soliton with harmonic Weyl curvature and its Ricci tensor has exactly three eingenvalues.

Proof. This an immediate consequence of lemmas **(L.2.1)** and **(L.3.10)**. □

Theorem (T.3.8) (Local multiply warped product structure). *Let $(\mathcal{M}^n, g, f, \lambda)$ be an almost Ricci soliton with $n \geq 4$ and f non constant. Assume the soliton has harmonic weyl curvature and that its Ricci tensor has exactly three distinct eigenvalues. Then for any point $p \in \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ there are a neighborhood $U \subset \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ of p and a multiply warped product $I \times_{h_1} \mathcal{N}_1 \times_{h_2} \mathcal{N}_2$ of an interval I and two Einstein manifolds $\mathcal{N}_1$ and $\mathcal{N}_2$, so that (U, g) is isometric to a domain of $I \times_{h_1} \mathcal{N}_1 \times_{h_2} \mathcal{N}_2$. Furthermore, f and λ are constant on $\mathcal{N}_1 \times \mathcal{N}_2$, through the identification between $I \times \mathcal{N}_1 \times \mathcal{N}_2$ and U. Finally, it is necessary that h_1, h_2, f and λ satisfy the ODE system* **(3.22)**.

Proof. Entirely analogous to the proof of Theorem **(T.3.5)**. □

4. ρ-Einstein gradient Ricci solitons

As a straighforward application of the previous theorems, we can also determine (via ODE's) the warping functions of ρ-Einstein gradient Ricci solitons. Recall that $(\mathcal{M}, g, f)$ is said to be a ρ-Einstein gradient Ricci soliton when there exist constants $\rho, \mu \in \mathbb{R}$ such that

$$\operatorname{Ric} + \nabla^2 f = (\rho R + \mu)\, g$$

Theorem (T.4.1). *Let $\mathcal{M}^n = \left(I \times_h \mathcal{N}^{n-1}, g\right)$ be a warped product, where $\mathcal{N}$ is an Einstein manifold with Einstein constant $\lambda_{\mathcal{N}}$. Let $h : I \to \mathbb{R}$ be a fuction satisfying the ODE (...). Then $(I \times_h \mathcal{N}, g, f)$ is a ρ-Einstein gradient Ricci soliton and its Ricci tensor has exactly two eingenvalues.*

Reciprocally, the next theorem shows that the only possible warping functions are the affine ones.

Theorem (T.4.2) (Local warped product structure). *Let $(\mathcal{M}^n, g, f)$ be a ρ-Einstein gradient Ricci soliton. Assume the soliton has harmonic Weyl curvature and that its Ricci tensor has exactly two distinct eigenvalues. Then for any point $p \in \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ there are a neighborhood $U \subset \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ of p and a warped product $I \times_h \mathcal{N}$ of an interval and an Einstein manifold $\mathcal{N}^{n-1}$ so that U is isometric to a domain of $I \times_h \mathcal{N}$. Furthermore, h satisfies the ODE* referenciar EDO gigante *and f is constant on $\mathcal{N}$, through the identification between $I \times \mathcal{N}$ and U.*

5. Classification of gradient almost Ricci solitons with harmonic curvature

The case of harmonic curvature can be studied as a special case of harmonic Weyl curvature. Indeed, recall that in dimension ≥ 4,

$$\underbrace{\text{Rm is harmonic}}_{\iff \nabla^\ell R_{ijk\ell}=0} \iff \underbrace{W \text{ is harmonic}}_{\iff \nabla^\ell W_{ijk\ell}=0} \text{ and } R \text{ is constant}$$

In fact, this is a straightforward consequence of the decomposition

$$\operatorname{Rm} = \frac{R}{n(n-1)}(g \owedge g) + \frac{2}{n-2}(g \owedge E) + W$$

and the second Bianchi identity.

With this characterization in mind, we can elaborate a bit further on the local structure given by **??**.

Theorem (T.5.1) (Local multiply warped product structure of harmonic gradient almost solitons). *Let $(\mathcal{M}^{n\geq 4}, g, f, \lambda)$ be a gradient almost Ricci soliton with harmonic Riemannian curvature and non-constant f. For any point $p \in \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ there is a neighborhood $U \subset \mathcal{R} \cap \mathcal{M}_{\mathcal{A}}$ of p in which g is a multiply warped product with at most two Einstein fibers. More precisely, U is isometric to a domain of one of the following manifolds*

(1) *$I \times_h \mathcal{N}^{n-1}$, where $\mathcal{N}$ is an Einstein manifold. In this case, $h : I \to \mathbb{R}$ must solve the ODE*

$$h'' = \frac{(n-1)\lambda_2 - \tau_{\mathcal{M}}}{n-1} h$$

(2) *$I \times_{h_1} \mathcal{N}_1^{r_1} \times_{h_2} \mathcal{N}_2^{r_2}$, where $\mathcal{N}_1$ and $\mathcal{N}_2$ are Einstein manifolds and $h_1, h_2 : I \to \mathbb{R}$ are smooth functions which satisfy both* **(3.22)** *and*

$$-r_1 \frac{(h_1)''}{h_1} - 2r_2 \frac{(h_2)''}{h_2} + \frac{\tau_{\mathcal{N}_2}}{(h_2)^2} - r_2(r_2-1)\frac{(h_2')^2}{(h_2)^2} = -r_2 \frac{(h_2)''}{h_2} - 2r_1 \frac{(h_1)''}{h_1} + \frac{\tau_{\mathcal{N}_1}}{(h_1)^2} - r_1(r_1-1)\frac{(h_1')^2}{(h_1)^2}$$

Furthermore, I is the integral manifold of $\partial_1 = \dfrac{\nabla f}{|\nabla f|}$ and $\lambda, \lambda_1, \ldots, \lambda_n, R$ and $|\nabla f|$ restricted to U depend only on $x_1 = s \in I$.

Proof. Suppose Ric has exactly two eigenvalues and Rm is harmonic. Clearly, we are then in case **(1)**. Multiplying the last equation of **(3.21)** by $(n-1)$ and using **(3.12)**, we obtain

$$\begin{cases} \tau_{\mathcal{N}}\left(1 - \dfrac{\lambda_2}{\lambda_{\mathcal{N}}} h^2\right) - (n-1)hh'' - (n-1)(n-2)(h')^2 = 0 \\ \tau_{\mathcal{N}} - 2(n-1)hh'' - (n-1)(n-2)(h')^2 = \tau_{\mathcal{M}} \cdot h^2 \end{cases}$$

which, as desired, imply

$$h'' = \frac{(\tau_{\mathcal{N}} \frac{\lambda_2}{\lambda_{\mathcal{N}}} - \tau_{\mathcal{M}})}{n-1} h,$$

where

$$\lambda_2 = \lambda - f' \frac{h'}{h}.$$

On the other hand, if Ric has three eigenvalues, in an entirely analogous manner, lemma **(L.3.10)** yields

$$\begin{cases} 0 = -\tau_{\mathcal{M}}(h_1)^2 - 2r_1 h_1 (h_1)'' - 2r_2 \dfrac{(h_2)''}{h_2}(h_1)^2 + \tau_{\mathcal{N}_1} + \tau_{\mathcal{N}_2}\dfrac{(h_1)^2}{(h_2)^2} \\ -r_1(r_1-1)(h_1')^2 - r_2(r_2-1)\dfrac{(h_2')^2}{(h_2)^2}(h_1)^2 - r_1 r_2 h_1 h_1' \dfrac{h_2'}{h_2} \\ 0 = -\tau_{\mathcal{N}_1}\left(1 - \dfrac{\lambda_2}{\lambda_{\mathcal{N}_1}}(h_1)^2\right) + r_1 h_1 h_1'' + r_1(r_1-1)(h_1')^2 + r_1 r_2 h_1 h_1' \dfrac{h_2'}{h_2} \end{cases}$$

which implies

$$\tau_{\mathcal{M}} = -r_1 \frac{(h_1)''}{h_1} - 2r_2 \frac{(h_2)''}{h_2} + \frac{\tau_{\mathcal{N}_2}}{(h_2)^2} - r_2(r_2-1)\frac{(h_2')^2}{(h_2)^2}$$

Similarly, from

$$\begin{cases} 0 = -\tau_{\mathcal{M}}(h_2)^2 - 2r_1 h_1 (h_1)'' - 2r_2 \dfrac{(h_2)''}{h_2}(h_1)^2 + \tau_{\mathcal{N}_1} + \tau_{\mathcal{N}_2}\dfrac{(h_1)^2}{(h_2)^2} \\ -r_1(r_1-1)(h_1')^2 - r_2(r_2-1)\dfrac{(h_2')^2}{(h_2)^2}(h_1)^2 - r_1 r_2 h_1 h_1' \dfrac{h_2'}{h_2} \\ 0 = -\tau_{\mathcal{N}_2}\left(1 - \dfrac{\lambda_3}{\lambda_{\mathcal{N}_2}}(h_2)^2\right) + r_2 h_2 h_2'' + r_2(r_2-1)(h_2')^2 + r_1 r_2 h_2 h_2' \dfrac{h_1'}{h_1} \end{cases}$$

we get

$$\tau_{\mathcal{M}} = -r_2 \frac{(h_2)''}{h_2} - 2r_1 \frac{(h_1)''}{h_1} + \frac{\tau_{\mathcal{N}_1}}{(h_1)^2} - r_1(r_1-1)\frac{(h_1')^2}{(h_1)^2}$$

□

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