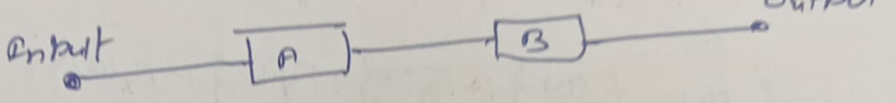
Minimum of Two Independent Random Variables

~~Focus~~ in
Two independent continuous r.v. X and Y.

~~V=~~ Random variable

$$V = \min(X, Y)$$

We want to characterise ~~r~~.v. V. using characteristic of X and Y



- The random variable V can be used to represent the reliability ~~function~~ of systems with series connections as shown in figure.
- The first component to fail causes the system to fail, i.e., system has a single point ~~fun~~ failure.
- In above example the time-to-failure are ~~repres~~ represented by the random variables X and ~~y th~~ Y, then V represents the time until the system fails, which is the minimum of the lifetimes of the two components
- The CDF of V can obtained as follows:

$$F_V(v) = P[V \le v] = P[\min(X,Y) \le v]$$

$$F_V(v) = P[(X \le v, X \le Y) \cup (Y \le v, X > Y)]$$

Since $P[A \cup B] = P[A] + P[B] - P[A \cap B]$, we have

$$F_V(v) = P[X \le v] + P[Y \le v] - P[X \le v, Y \le v]$$

$$\cancel{F_V = F_X(v) + F_Y(v) - F_{XY}(v,v)}$$

$$\boxed{F_V(v) = F_X(v) + F_Y(v) - F_{XY}(v,v)}$$

Also, since X and Y are independent, we obtain the CDF and PDF of U as follows;

$$F_U(u) = F_X(u) + F_Y(u) - F_{XY}(u,u) = F_X(u) + F_Y(u) - F_X(u)F_Y(u)$$

$$f_U(u) = \frac{d}{du} F_U(u) = f_X(u) + f_Y(u) - [f_X(u) \cdot F_Y(u) + F_X(u) \cdot f_Y(u)]$$

$$= f_X(u) + f_Y(u) - f_X(u) \cdot F_Y(u) - F_X(u) \cdot f_Y(u)$$

$$f_U(u) = f_X(u)\{1 - F_Y(u)\} + f_Y(u)\{1 - F_X(u)\}$$

$$\therefore \boxed{f_U(u) = f_X(u)\{1 - F_Y(u)\} + F_Y(u)\{1 - F_X(u)\}}$$

Ex:- Assume that $V = \min(X, Y)$, where X and Y are independent random variables with the respective PDF

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = \mu e^{-\mu y}; \quad y \geq 0$$

where $\lambda > 0$ and $\mu > 0$. What is the PDF of V?

Answer:- We first obtain the CDFs of X and Y, which are as follows:

$$F_X(x) = P[X \leq x] = \int_0^x \lambda e^{-\lambda u} du = 1 - e^{-\lambda x}$$

$$F_Y(y) = P[Y \leq y] = \int_0^y \mu e^{-\mu w} dw = 1 - e^{-\mu y}$$

Thus, the PDF ~~of V~~ of V is given by:

$$f_V(u) = f_X(u)\{1 - F_Y(u)\} + f_Y(u)\{1 - F_X(u)\}$$

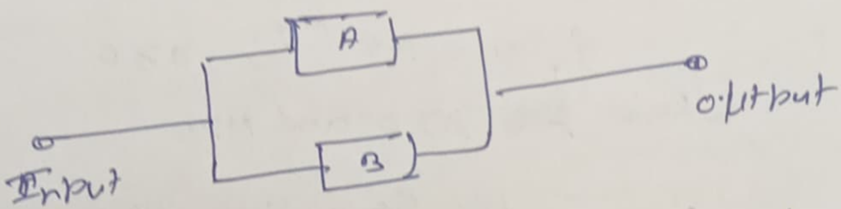
$$f_V(u) = \lambda e^{-\lambda u} \cdot e^{-\mu u} + \mu e^{-\mu u} e^{-\lambda u}$$

$$\boxed{f_V(u) = (\lambda + \mu) e^{-(\lambda + \mu)u}}; \quad u \geq 0$$

— Since λ and μ are the failure rates of the components, the result indicate that the composite system behaves like a single unit whose failure rate is the sum of the two failure rates. More importantly, V is an exponentially distributed random variable whose expected value is $E[V] = \frac{1}{\lambda + \mu}$.

Maximum Of Two Independent Random Variables

Let X and Y are two continuous r.v. We are interested in CDF and PDF of r.v. W that is the maximum of the two random variables; ~~W = X+Y~~ $W = \max(X, Y)$.
The random variable W can be used to represent the reliability of systems with parallel connections.



As long as one or both components are operational, the system is operational.

- The system is fail when both component fail simultaneously.
- The ~~reliabil~~ reliability of the system depends on the reliability of the last component to fail.

CDF and PDF

$$W = \max(X, Y)$$

$$F_W(w) = P[W \le w] = P[\max(X, Y) \le w]$$

$$F_W(w) = P[(X \le w) \cap (Y \le w)] = F_{XY}(w)$$

Since X and Y are independent

$$\boxed{F_W(w) = F_X(w) \cdot F_Y(w)}$$ CDF

PDF

$$f_W(w) = \frac{d}{dw} F_W(w) = \frac{d}{dw}[F_X(w) F_Y(w)]$$

$$= \frac{d}{dw} F_X(w) \cdot F_Y(w) + F_X(w) \frac{d}{dw} F_Y(w)$$

$$\boxed{f_W(w) = f_X(w) \cdot F_Y(w) + F_X(w) \cdot f_X(w)}$$

Ex:- Two components A and B have lifetimes X and Y, respectively, that are independent r.v. The components are connected in parallel to create a system whose lifetime is W. Find the PDF of W if the system needs at least one of the components to be operational and the PDF of X and Y are given respectively by:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$

$$f_Y(y) = \mu e^{-\mu y}; \quad y \geq 0$$

where ~~λ~~ $\lambda > 0$ and $\mu > 0$.

Solution:- We have that $W = \max(X, Y)$. We first obtain the CDFs of X and Y, which are as follows:

$$F_X(x) = P[X \leq x] = \int_0^x \lambda e^{-\lambda u}\, du = 1 - e^{-\lambda x}$$

$$F_Y(y) = P[Y \leq y] = \int_0^y \mu e^{-\mu t}\, dt = 1 - e^{-\mu y}$$

Thus PDF of W is given by

$$f_W(w) = f_X(w) F_Y(w) + F_X(w) f_Y(w)$$

$$= \lambda e^{-\lambda w}(1 - e^{-\mu w}) + (1 - e^{-\lambda w})\mu e^{-\mu w}$$

$$f_W(w) = \lambda e^{-\lambda w} + \mu e^{-\mu w} - (\lambda + \mu) e^{-(\lambda+\mu) w}$$

$$\therefore \boxed{f_W(w) = \lambda e^{-\lambda w} + \mu e^{-\mu w} - (\lambda + \mu) e^{-(\lambda+\mu) w}}$$

Now $E[W]$? (Expected value of W)

$$\therefore \boxed{E[W] = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda + \mu}}$$

Explanation of Result:- The mean time until first failure is $\frac{1}{\lambda + \mu}$, according to the previous example.

~~Interpretation~~

After the first component failed, the mean time until the second failure occur is $\frac{1}{\lambda}$, if the component B was the first to ~~fail~~ fail, and $\frac{1}{\mu}$, if the component A was first fail..

- The probability that a component 'A' fails before component B is $\frac{\lambda}{\lambda+\mu}$, and the probability that component B fails before component A is $\frac{\mu}{\mu+\lambda}$. Thus, mean life of the system is

$$E[W] = \frac{1}{\lambda+\mu} + \frac{1}{\lambda}\left(\frac{\mu}{\lambda+\mu}\right) + \frac{1}{\mu}\left(\frac{\lambda}{\lambda+\mu}\right)$$

$$E[W] = \frac{\lambda\mu+\lambda^2+\mu^2}{\lambda\mu(\lambda+\mu)} = \frac{\lambda^2+\mu^2+2\lambda\mu-\lambda\mu}{\lambda\mu(\lambda+\mu)}$$

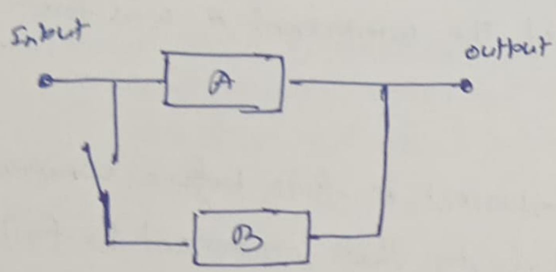
$$E[W] = \frac{(\lambda+\mu)^2-\lambda\mu}{\lambda\mu(\lambda+\mu)}$$

$$= \frac{\lambda+\mu}{\lambda\mu} - \frac{1}{\lambda+\mu}$$

$$\boxed{E[W] = \frac{1}{\lambda} + \frac{1}{\mu} - \frac{1}{\lambda+\mu}}$$

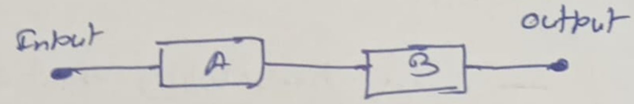
~~Comp~~

Comparision of Inter connection models



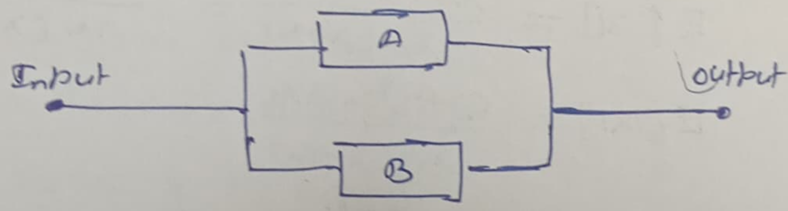
$U = X + Y$

(a)



$V = \min(X, Y)$

(b)



$W = \max(X, Y)$

(c)

Define the following r.v.

$U = X + Y$

$V = \min(X, Y)$

$W = \max(X, Y)$

Assume that the PDF of X and Y are defined respectively as follows:

$f_X(x) = \lambda e^{-\lambda x}; \quad x \geq 0, \lambda > 0$

$f_Y(y) = \mu e^{-\mu y}; \quad y \geq 0, \mu > 0$

Thus PDF of U, V and W in terms of

$f_U(u) = \frac{\lambda\mu}{\lambda - \mu}\{e^{-\mu u} - e^{-\lambda \mu}\}, \quad u \geq 0$

$f_V(v) = (\lambda + \mu) e^{-(\lambda+\mu)v}; \quad v \geq 0$

$f_W(w) = \lambda e^{-\lambda w} + \mu e^{-\mu w} - (\lambda + \mu) e^{-(\lambda+\mu)w}; \quad w \geq 0.$

$E[U] > E[W] > E[V]$

Two functions of Two Random Variables

Let X and Y ~~too~~ be two random variables with a given joint PDF $f_{XY}(x,y)$. Assume that U and W are two functions of X and Y, i.e. $U = g(X,Y)$ and $W = h(X,Y)$. Sometimes it is necessary to obtain the joint PDF of U and W, $f_{UW}(u,w)$, in terms of the PDFs ~~of X~~ of X and Y.

It can be shown that if $(x_1,y_1), (x_2,y_2), \ldots, (x_n,y_n)$ are real solutions to the equation $u=g(x,y)$ and $w=h(x,y)$, then ~~for~~ $f_{UW}(u,w)$ is given by

$$f_{UW}(u,w) = \frac{f_{XY}(x_1,y_1)}{|J(x_1,y_1)|} + \frac{f_{XY}(x_2,y_2)}{|J(x_2,y_2)|} + \cdots + \frac{f_{XY}(x_n,y_n)}{|J(x_n,y_n)|}$$

where $J(x,y)$ is called the Jacobian of the transformation $\{u = g(x,y),\ w = h(x,y)\}$ and is defined by:

$$J(x,y) = \begin{vmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{vmatrix} = \left(\frac{\partial g}{\partial x}\right)\left(\frac{\partial h}{\partial y}\right) - \left(\frac{\partial g}{\partial y}\right)\left(\frac{\partial h}{\partial x}\right)$$

Example:– Let $U = g(X,Y) = X+Y$ and $W = h(X,Y) = X-Y$. Find $f_{UV}(u,w)$.

Solution:– The unique solution to the equations $u=x+y$ and $w=x-y$ is $x = \frac{u+w}{2}$ and $y = \frac{u-w}{2}$. Thus, there is only one set of solutions. Since,

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

we obtain

$$f_{UW}(u,w) = \frac{f_{XY}(x,y)}{|J(x,y)|} = \frac{1}{|-2|} f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right)$$

$$\boxed{f_{UW}(u,w) = \frac{1}{2} f_{XY}\left(\frac{u+w}{2}, \frac{u-w}{2}\right)}$$

Ex:- Find $f_{UW}(u,w)$ if $U = X^2+Y^2$ and $W = X^2$.

Solution:- From the second equation we have that $x = \pm\sqrt{w}$. ~~Substitution~~ Substituting this value of x in the first equation, we obtain $y = \pm\sqrt{u-w}$, which is real only when $u \geq w$. Also

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2x & 0 \end{vmatrix} = -4xy$$

$$f_{UW}(u,w) = \frac{f_{XY}(\sqrt{w}, \sqrt{u-w})}{|J(\sqrt{w}, \sqrt{u-w})|} + \frac{f_{XY}(\sqrt{w}, -\sqrt{u-w})}{|J(\sqrt{w}, -\sqrt{u-w})|}$$

$$+ \frac{f_{XY}(-\sqrt{w}, \sqrt{u-w})}{|J(-\sqrt{w}, \sqrt{u-w})|} + \frac{f_{XY}(-\sqrt{w}, -\sqrt{u-w})}{|J(-\sqrt{w}, -\sqrt{u-w})|}$$

$$f_{UW}(u,w) = \frac{f_{XY}(\sqrt{w}, \sqrt{u-w})}{4|\sqrt{w}\,(\sqrt{u-w})|} + \frac{f_{XY}(\sqrt{w}, -\sqrt{u-w})}{4|\sqrt{w} \times -\sqrt{u-w}|}$$

$$+ \frac{f_{XY}(-\sqrt{w}, \sqrt{u-w})}{4|-\sqrt{w} \times \sqrt{u-w}|} + \frac{f_{XY}(-\sqrt{w}, -\sqrt{u-w})}{4|-\sqrt{w} \times -\sqrt{u-w}|}$$

$$= \frac{f_{XY}(\sqrt{w}, \sqrt{u-w})}{4\sqrt{w(u-w)}} + \frac{f_{XY}(\sqrt{w}, -\sqrt{u-w})}{4\sqrt{w(u-w)}} +$$

$$\frac{f_{XY}(-\sqrt{w}, \sqrt{u-w})}{4\sqrt{w(u-w)}} + \frac{f_{XY}(-\sqrt{w}, -\sqrt{u-w})}{4\sqrt{w(u-w)}}$$

Application of the Transform Method:

→ Assume that $U = g(X,Y)$ and interested to find PDF of U.

→ Using auxiliary method by defining an auxiliary function $W = X$ or $W = Y$ so we can obtain the joint PDF $f_{UW}(u,w)$ of U of U and W. Then we obtain the required marginal PDF $f_U(u)$ as follows:

$$\boxed{f_U(u) = \int_{-\infty}^{+\infty} f_{UW}(u,w)\,dw.}$$

Ex:- Find the PDF of the random variable $U = X+Y$ where the joint PDF of X and Y, $f_{XY}(x,y)$ is given

Soln:- We define the auxiliary r.v. $W = X$.

Then the soln of $U = X+Y$ and $W = X$ is

$x = w$, and $y = u-w$, and Jacobian of The ~~transformat~~ transformation is:

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1$$

Since there is only one solution to the equations, we have that:

$$f_{UW}(u,w) = \frac{f_{XY}(x,y)}{|J(x,y)|} = \frac{1}{|-1|} f_{XY}(w, u-w)$$

$$f_{UW}(u,w) = f_{XY}(w, u-w)$$

$$\boxed{f_U(u) = \int_{-\infty}^{+\infty} f_{UW}(u,w)\,dw = \int_{-\infty}^{+\infty} f_{XY}(w, u-w)\,dw}$$

marginal

This reduces to the convolution integral we obtained earlier when X and Y are independent

~~Ex~~

Example :- Find the PDF of the random variable $U = X - Y$, ~~and~~ where the Joint PDF of X and Y, $f_{XY}(x,y)$ is given

Solution :- We define the auxilary random variable $W = X$. Then the unique solution to the two equations is $x = w$ and $y = w - u$, and the Jacobian of the transform is:

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

Since there is only one solution to equation:

$$f_{UW}(u,w) = \frac{f_{XY}(x,y)}{|J(x,y)|} = f_{XY}(w, w-u)$$

$$f_U(u) = \int_{-\infty}^{+\infty} f_{UW}(u,w)\, dw = \int_{-\infty}^{+\infty} f_{XY}(w, w-u)\, dw$$

Ex :- The joint PDF of two R.V. X and ~~Y is given~~ Y is given by $f_{XY}(x,y)$. If we define the random variable $U = XY$, ~~det~~ determine the PDF of U.

Solution :- We define auxilary random variable $W = X$. Then the unique solution to the two equation is $x = w$ and $y = \frac{u}{x} = \frac{u}{w}$, and the jacobian of the transformation is

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} = -x = -w$$

$$f_{UW}(u,w) = \frac{f_{XY}(x,y)}{|J(x,y)|} = \frac{f_{XY}(w, \frac{u}{w})}{|w|} = \frac{1}{|w|} f_{XY}(w, \frac{u}{w})$$

$$\boxed{f_{UW}(u,w) = \frac{1}{|w|} f_{XY}(w, \frac{u}{w})}$$

Ex :- The joint PDF of two r.v. X and Y is given by $f_{XY}(x,y)$. If we define the random variable $V = X/Y$, determine the PDF of V.

Solution :- We define the auxilary r.v. $W = Y$. Then the unique solution to the two equation is $y = w$ and $x = vy = vw$, and the Jacobian of the transformation is

$$J(x,y) = \begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{y} & -\frac{x}{y^2} \\ 0 & 1 \end{vmatrix}$$

$$J(x,y) = \frac{1}{y} = \frac{1}{w}$$

Since there is only one solution to the equations, we have that

$$f_{VW}(v,w) = \frac{f_{XY}(x,y)}{|J(x,y)|} = |w|\, f_{XY}(vw, w)$$

$$\boxed{f_V(v) = \int_{-\infty}^{+\infty} f_{VW}(v,w)\,dw = \int_{-\infty}^{+\infty} |w|\, f_{XY}(vw,w)\,dw}$$

EG

Laws of Large Numbers :—

There are two fundamental laws that deal with limiting behavior of probabilistic sequences.

"Weak" law of large number

↳ How ~~the~~ a sequence of probabilities converges

"Strong" law of large Number

↳ How a sequence of random variables behaves in the limit

~~Propositn~~

Result ① (Weak law of large Number) :— Let

$X_1, X_2, \ldots, X_n$ be a sequence of mutually independent and identically distributed random variables each of which has a finite mean $E[X_k] = \mu_x < \infty$, $k = 1, 2, 3, \ldots n$.

Let S_n be the linear sum of the n random variables; that is

$$S_n = X_1 + X_2 + X_3 + \cdots + X_n$$

Then for any $\varepsilon > 0$.

$$\lim_{n\to\infty} P\left[\left|\frac{S_n}{n} - \mu_x\right| \geq \varepsilon\right] \longrightarrow 0$$

Alternatively,

$$\lim_{n\to\infty} P\left[\left|\frac{S_n}{n} - \mu_x\right| < \varepsilon\right] \longrightarrow 1$$

Proof :— $S_n = X_1 + X_2 + \cdots + X_n$

$$\bar{S}_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{n\mu_x}{n} = \mu_x$$

$$\operatorname{var}(\bar{S}_n) = \operatorname{var}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

$$= \frac{1}{n^2}\left\{\operatorname{var}(X_1) + \operatorname{var}(X_2) + \cdots + \operatorname{var}(X_n)\right\}$$

$$Var(\bar{S}_n) = \frac{n\sigma_x^2}{n^2}$$

$$\boxed{Var(\bar{S}_n) = \frac{\sigma_x^2}{n}}$$

From chebyshev's inequality

$$P[|\bar{S}_n - \mu_x| \geq \varepsilon] \leq \frac{Var(\bar{S}_n)}{\varepsilon^2} = \frac{\sigma_x^2}{n\varepsilon^2}$$

Thus,

$$\lim_{n \to \infty} P[|\bar{S}_n - \mu_x| \geq \varepsilon] = 0$$

which proves the theorem

~~Re~~ Result (2) :- [Strong law of large Numbers] :— Let $x_1, x_2, \ldots, x_n$ be a sequence of mutually independent and identically distributed r-v. each of which have finite mean

$E[X_k] = \mu_x$, $k = 1, 2, 3, \ldots, n$

Let S_n be linear sum of 'n' random variables, i.e.

$$S_n = X_1 + X_2 + X_3 + \cdots + X_n$$

Then for any $\varepsilon > 0$

$$P[\lim_{n \to \infty} |\bar{S}_n - \mu_x| > \varepsilon] = 0$$

where $\bar{S}_n = \frac{S_n}{n}$. An Alternative statement of the law is:

$$P[\lim_{n \to \infty} |\bar{S}_n - \mu_x| \leq \varepsilon] = 1$$

Note :- (1) Arithmetic average $\bar{S}_n$ of a sequence of independent observations of a random variable X converges with probability 1 to the ~~expect~~ expected value μ_x of X.

Central limit theorem:— Strong law of large number helps to validate the relative-frequency definition of the probability, it says nothing about the limiting distribution of the sum S_n.

· CLT gives answer of this.

Let $X_1, X_2, \ldots, X_n$ be a sequence of mutually independent and identically distributed random variables each of which has a finite mean μ_X and a finite variance σ_X^2. Let

$$S_n = X_1 + X_2 + X_3 + \cdots + X_n$$

CLT states that for large n, the distribution of ~~S_n~~ S_n is approximately normal, regardless of the forms of the distribution of the X_k.

— That is, if we add a large number of ~~iid~~ independent and identically distributed random variable together, the resulting sum will have a normal distribution, regardless of the distribution of the random variables that are added up.

Now $\bar{S}_n = E[S_n] = n\mu_X$

$$\sigma_{S_n}^2 = n\sigma_X^2$$

Converting S_n to standard normal variable $N(0,1)$ we obtain

$$Z_n = \frac{S_n - \bar{S}_n}{\sigma_{S_n}} = \frac{S_n - n\mu_X}{\sqrt{n\sigma_X^2}} = \frac{S_n - n\mu_X}{\sigma_X\sqrt{n}}$$

Then the central limit theorem states that if $F_{Z_n}(z)$ is the CDF of Z_n, then

$$\lim_{n\to\infty} F_{Z_n}(z) = \lim_{n\to\infty} P[Z_n \le z] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{u^2}{2}}du$$
$$= \Phi(z)$$

This means that $\lim_{n\to\infty} Z_n \sim N(0,1)$. This is true for $n \geq 30$.

Ex:- Assume that r.v. S_n is the sum of 48 independent experimental values of the random variable X whose PDF is given by

$$f_X(x) = \begin{cases} \frac{1}{3}; & 1 \le x \le 4 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability that S_n lies in the range $180 \le S_n \le 126$

Solution :- The expected value and variance of X are given by

$$E[X] = \frac{4+1}{2} = \frac{5}{2}$$

$$\sigma_X^2 = \frac{(4-1)^2}{12} = \frac{3}{4}$$

$S_n = X_1 + X_2 + \cdots + X_{48}$ since X_i are i.i.d.

Then mean and variance of S_n is given by

$$E[S_n] = 48\, E[X] = 48 \times \frac{5}{2} = 24 \times 5 = 120$$

$$\sigma_{S_n}^2 = 48\, \sigma_X^2 = 36$$

Assuming that S_n approximate the normal r.v.

CDF of the normalized r.v.

$$P[S_n \le s] = F_{S_n}(s) = \Phi\left(\frac{s - E[S_n]}{\sigma_{S_n}}\right) = \Phi\left(\frac{s-120}{6}\right)$$

$$\therefore P[108 \le S_n \le 126] = F_{S_n}(126) - F_{S_n}(108)$$

$$= \Phi\left(\frac{126-120}{6}\right) - \Phi\left(\frac{108-120}{6}\right)$$

$$= \Phi(1) - \Phi(-2) = \Phi(1) - \{1 - \Phi(2)\}$$

$$= \cancel{\Phi(1)}$$

$$= \Phi(1) + \Phi(2) - 1$$

$$= 0.8413 + 0.9772 - 1$$

$$P[108 \le S_n \le 126] = 0.8185$$

Order Statistics:

- Consider an experiment in which we have n identical bulbs, label 1, 2, 3, ..., n, that we turn on at the same time.

- Find the time until each of the n bulbs fails.
- Assume that the time X until a bulb fails has a PDF $f_X(x)$ and a CDF $F_X(x)$
- Assume we order the lifetimes of these bulbs after the experiment.

Particularly, let the random variables Y_k, $k = 1, 2, 3, \ldots, n$ be defined as follows:

$$Y_1 = \max(X_1, X_2, X_3, \ldots, X_n)$$
$Y_2 =$ second largest of $X_1, X_2, X_3, \ldots X_n$
$Y_3 =$ third largest of $X_1, X_2, X_3, \ldots, X_n$
$\vdots$
$$Y_n = \min(X_1, X_2, \ldots X_n)$$

- The random variables $Y_1, Y_2, Y_3, \ldots, Y_n$ are called the order statistics corresponding to the random variables $X_1, X_2, X_3, \ldots, X_n$. In particular Y_k is called the k^{th} order ~~statistics~~ statistic. It is obvious that $Y_1 \geq Y_2 \geq Y_3 \geq \ldots \geq Y_n$, and in the case where the X_k are continuous random variables, then $Y_1 > Y_2 > Y_3 > \cdots Y_n$ with probability one.
- The cdf of Y_k, $F_{Y_k}(y) = P[Y_k \leq y]$, can be computed as follows:

$$F_{Y_k}(y) = P[Y_k \leq y] = P[\text{at most } (k-1)\ X_i \geq y]$$
$$= P\Big[\{\text{all } X_i \leq y\} \cup \{[(n-1) X_i \leq y] \cap [1\ X_i \geq y]\} \cup \ldots \{[(n-k+1) X_i \leq y] \cap [(k-1) X_i \geq y]\}\Big]$$

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$$F_{Y_k}(y) = P[\text{all } X_i \le y] + P[\{(n-1) X_i \le y\} \cap \{1 \cdot X_i \ge y\}] + \cdots + P[\{(n-k+1) X_i \le y\} \cap \{(k-1) X_i \ge y\}]$$

- If we consider the events as results of n ~~Bernoulli~~ Bernoulli, where in any trial we have that

$$P[\text{success}] = P[X_i \le y] = F_X(y)$$
$$P[\text{Failure}] = P[X_i > y] = 1 - F_X(y)$$

Then we obtain the result:

$$F_{Y_k}(y) = P[n \text{ success}] + P[(n-1) \text{ success}] + \cdots + P[(n-k+1) \text{ success}]$$

$$= [F_Y(y)]^n + \binom{n}{n-1} (F_Y(y))^{n-1} [1 - F_Y(y)] + \cdots + \binom{n}{n-k+1} [F_X(y)]^{n-k+1} [1 - F_X(y)]^{k-1}$$

$$\boxed{F_{Y_k}(y) = \sum_{m=0}^{k-1} \binom{n}{n-m} [F_X(y)]^{n-m} [1 - F_X(y)]^m}$$

PDF:- The PDF can ~~be~~ be obtained by differentiating above $F_{Y_k}(y)$. Another method is:

$$f_{Y_k}(y)\, dy = P[Y_k \approx y] = P[1 \cdot X_i \approx y, (k-1) X_i \ge y, (n-k) X_i \le y]$$

$$f_{Y_k}(y)\, dy = \frac{n!}{1!\,(k-1)!\,(n-k)!} [f_X(y)\, dy]^1 [1 - F_X(y)]^{k-1} (F_X(y))^{n-k}$$

~~when cancelled are~~

when cancel out the dy's we have

$$f_{Y_k}(k) = \frac{n!}{1!\,(k-1)!\,(n-k)!} f_X(y) [1 - F_X(y)]^{k-1} [F_X(y)]^{n-k}$$

Ex :- Assume that the random variables $X_1, X_2, \ldots, X_{10}$ are independent and identically distributed with the common PDF $f_X(x)$ and common CDF $F_X(x)$. Find the PDF and CDF of the following:

(a) The third largest random ~~var~~ variable

(b) The fifth largest random variable

(c) Largest random variable.

(d) The smallest random variable.

Solution :-

(a) For ordered n-random variable the CDF, and PDF of k^{th} random variable is given by

$$\boxed{F_{Y_k}(y) = \sum_{m=0}^{k-1} \binom{n}{n-m} [F_X(y)]^{n-m} [1-F_X(y)]^m}$$

$$\boxed{f_{Y_k}(y) = \frac{n!}{(k-1)!\,(n-k)!} f_X(y) [1-F_X(y)]^{k-1} [F_X(y)]^{n-k}}$$

Third $k=3, \; n=10$

$$F_{Y_3}(y) = \sum_{m=0}^{2} \binom{10}{10-m} [F_X(y)]^{10-m} [1-F_X(y)]^m$$

$$F_{Y_3}(y) = \binom{10}{10} [F_X(y)]^{10} + \binom{10}{9} [F_X(y)]^9 [1-F_X(y)]^1 + \binom{10}{8} [F_X(y)]^8 [1-F_X(y)]^2 + \cancel{\binom{10}{7} [F_X(y)]^7 [1-F_X(y)]^3}$$

$$\boxed{F_{Y_3}(y) = [F_X(y)]^{10} + 10 [F_X(y)]^9 [1-F_X(y)] + 45 [F_X(y)]^8 [1-F_X(y)]^2}$$

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$$f_{Y_3}(y) = \frac{10!}{2!\,7!} f_X(y)\,[1-F_X(y)]^2\,[F_X(y)]^7$$

$$\boxed{f_{Y_3}(y) = 360\, f_X(y)\,[1-F_X(y)]^2\,[F_X(y)]^7}$$

(b). The fifth largest random variable is obtained by substituting $k=5$ and $n=10$, as follows:

$$F_{Y_5}(y) = [F_X(y)]^{10} + \binom{10}{9}[F_X(y)]^9[1-F_X(y)] + \binom{10}{8}[F_X(y)]^8 [1-F_X(y)]^2 + \binom{10}{7}[F_X(y)]^7[1-F_X(y)]^3 + \binom{10}{6}[F_X(y)]^6 [1-F_X(y)]^4$$

$$F_{Y_5}(y) = [F_X(y)]^{10} + 10[F_X(y)]^9[1-F_X(y)] + 45[F_X(y)]^8[1-F_X(y)]^2 + 120[F_X(y)]^7[1-F_X(y)]^3 + 210[F_X(y)]^6[1-F_X(y)]^4$$

$$f_{Y_5}(y) = \frac{10!}{4!\,5!} f_X(y)\,[1-F_X(y)]^4\,[F_X(y)]^5$$

$$\boxed{f_{Y_5}(y) = 1260\, f_X(y)\,[1-F_X(y)]^4\,[F_X(y)]^5}$$

(c) The largest random variable implies $k=1$ in the formula with $n=10$. Thus we obtain

$$F_{Y_1}(y) = [F_X(y)]^{10}$$

$$f_{Y_1}(y) = \frac{10!}{(1-1)!\,(10-1)!} f_X(y)\,[1-F_X(y)]^{1-1}\,[F_X(y)]^{10-1}$$

$$\boxed{f_{Y_1}(y) = 10\, f_X(y)\,[F_X(y)]^9}$$

(d) The smallest random variable $k=10$

$$F_{10}(y) = \sum_{m=0}^{9} \binom{10}{10-m} [F_X(y)]^{10-m} [1-F_X(y)]^{m}$$

$$f_{Y_{10}}(y) = \frac{10!}{(10-1)!\,(10-10)!} f_X(y) [1-F_X(y)]^{10-1} [F_X(y)]^{10-10}$$

$$\boxed{f_{Y_{10}}(y) = 10\, f_X(y) \cdot [1-F_X(y)]^{9}}$$

Example :- Assume that the random variables $X_1, X_2, \ldots, X_{32}$ are independent and identically distributed with the common PDF $f_X(x)$ and the common CDF $F_X(x)$. Find the PDF and CDF of the following.

(a) the 4th largest random variable

(b) the 27th largest random variable

(c) The largest random variable

(d) The smallest random variable

(e) What is the probability that the 4th largest random ~~Va~~ variable has a value between 8 and 9 ?