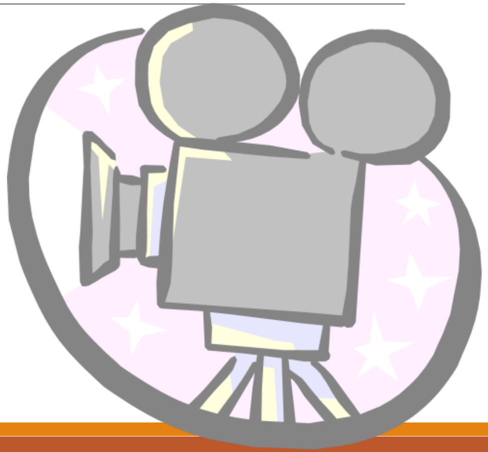
Image Processing

CS-341



Outline

- Image Enhancement in the Frequency Domain
    - Unitary Transformation
        - Fourier Transformation

Transform=Change of Coordinates

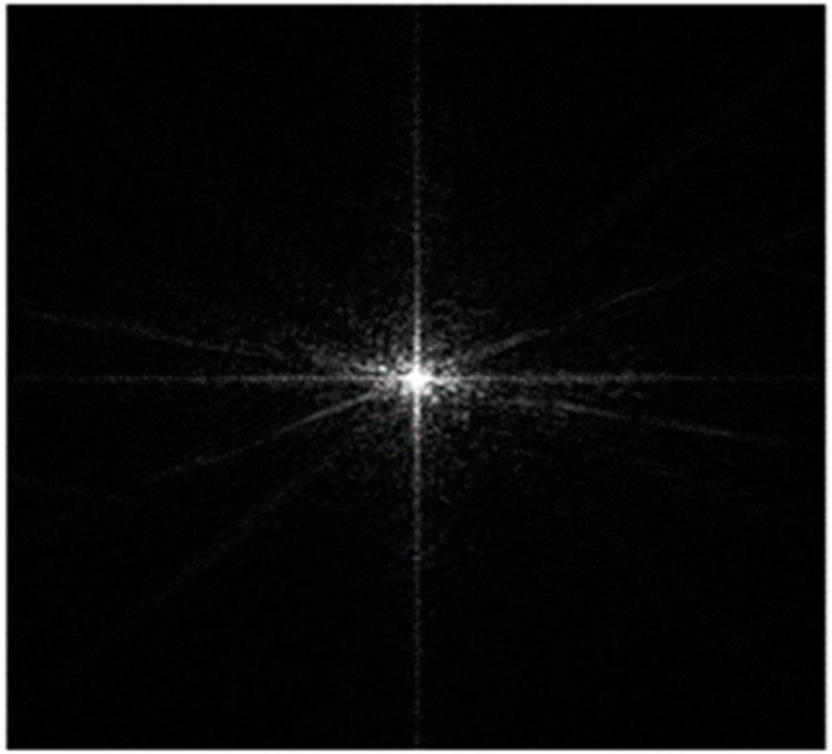
Intuitively speaking, transform plays the role of facilitating the source modeling

- Due to the decorrelating property of transform, it is easier to model transform coefficients (Y) instead of pixel values (X)

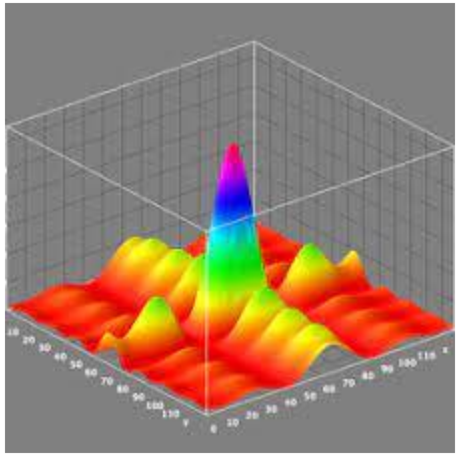
An appropriate choice of transform (transform matrix A) depends on the source statistics P(X)

- We will only consider the class of transforms corresponding to unitary matrices

image power circles



Amplitude



frequency

Hybrid Images

Low Freq Only

High Freq Only

Hybrid (Sum) Image

Unitary Matrix and 1D Unitary Transform

Definition

conjugate transpose

A matrix A is called unitary if $A^{-1}=A^{*T}$

When the transform matrix A is unitary, the defined transform $\vec{y} = A\vec{x}$ is called unitary transform

Example

$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \mathbf{A}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \mathbf{A}^{T}$$

For a real matrix **A**, it is unitary if $\mathbf{A}^{-1}=\mathbf{A}^{T}$

Properties of Unitary Transform

Energy compaction: only few transform coefficients have large magnitude

- Such property is related to the decorrelating role of unitary transform

Energy conservation: unitary transform preserves the 2-norm of input vectors

- Such property essentially comes from the fact that rotating coordinates does not affect Euclidean distance

Energy Compaction Example

Hadamard matrix

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 100 \\ 98 \\ 98 \\ 100 \end{bmatrix}$$

$$\Downarrow$$

$$\vec{y} = \mathbf{A}\vec{x} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 100 \\ 98 \\ 98 \\ 100 \end{bmatrix} = \begin{bmatrix} 198 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

significant

insignificant

Energy Conservation

$$\vec{y} = \mathbf{A}\vec{x}$$

A is unitary

$$\Rightarrow \ \|\vec{y}\|^2 = \|\vec{x}\|^2$$

$$\|\vec{y}\|^2 = \sum_{i=1}^{N} |y_i|^2$$

Numerical Example

$$\mathbf{A} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \vec{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\Downarrow$$

$$\vec{y} = \mathbf{A}\vec{x} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

<u>Check:</u>

$$\|\vec{x}\|^2 = 3^2 + 4^2 = 25, \|\vec{y}\|^2 = \frac{7^2 + 1^2}{2} = 25$$

Summary of 1D Unitary Transform

Unitary matrix: $\mathbf{A}^{-1}=\mathbf{A}^{*T}$

Unitary transform: $\vec{y} = \mathbf{A}\vec{x}$ $\mathbf{A}$ unitary

Properties of 1D unitary transform

- Energy compaction: most of transform coefficients y_i are small
- Energy conservation: quantization can be directly performed to transform coefficients

Definition of 2D Transform

2D forward transform $$\mathbf{Y}_{N\times N} = \mathbf{A}_{N\times N}\mathbf{X}_{N\times N}\mathbf{A}^T_{N\times N}$$

$$\begin{bmatrix} y_{11} & \cdots & \cdots & y_{1N} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ y_{N1} & \cdots & \cdots & y_{NN} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1N} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{N1} & \cdots & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_{11} & \cdots & \cdots & x_{1N} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ x_{N1} & \cdots & \cdots & x_{NN} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & \cdots & a_{N1} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{1N} & \cdots & \cdots & a_{NN} \end{bmatrix}$$

1D column transform

1D row transform

2D Transform (Two Sequential 1D Transforms)

$$\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^T$$

$$\Updownarrow$$

column transform $\quad \mathbf{Y_1} = \mathbf{A}\mathbf{X}$ (left matrix multiplication first)

row transform $\quad \mathbf{Y} = \mathbf{Y}_1\mathbf{A}^T = (\mathbf{A}\mathbf{Y}_1^T)^T$

$$\Updownarrow$$

row transform $\quad \mathbf{Y}_2 = \mathbf{X}\mathbf{A}^T = (\mathbf{A}\mathbf{X}^T)^T$ (right matrix multiplication first)

column transform $\quad \mathbf{Y} = \mathbf{A}\mathbf{Y}_2$

<u>Conclusion:</u>

- 2D separable transform can be decomposed into two sequential
- The ordering of 1D transforms does not matter

2D Unitary Transform

Suppose A is a unitary matrix,

forward transform
$$\mathbf{Y}_{N\times N} = \mathbf{A}_{N\times N}\mathbf{X}_{N\times N}\mathbf{A}^{T}_{N\times N}$$

inverse transform
$$\mathbf{X}_{N\times N} = \mathbf{A}^{*T}_{N\times N}\mathbf{Y}_{N\times N}\mathbf{A}^{*}_{N\times N}$$

Proof

Since **A** is a unitary matrix, we have
$$\mathbf{A}^{-1} = \mathbf{A}^{*T}$$

$$\mathbf{A}^{*T}\mathbf{Y}\mathbf{A}^{*} = \mathbf{A}^{*T}(\mathbf{A}\mathbf{X}\mathbf{A}^{T})\ \mathbf{A}^{*} = \mathbf{I}\cdot\mathbf{X}\cdot\mathbf{I} = \mathbf{X}$$

Energy Compaction Property of 2D Unitary Transform

- Example

$$\mathbf{A} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 100 & 100 & 98 & 99 \\ 100 & 100 & 94 & 94 \\ 98 & 97 & 96 & 100 \\ 100 & 99 & 97 & 94 \end{bmatrix} \xrightarrow{\mathbf{Y} = \mathbf{AXA}^T} \mathbf{Y} = \begin{bmatrix} 391.5 & 0 & 5.5 & 1 \\ 2.5 & -2 & -4.5 & 2 \\ 1 & -0.5 & 2 & -0.5 \\ 2 & 1.5 & 0 & -1.5 \end{bmatrix}$$

A coefficient is called significant if its magnitude is above a pre-selected threshold *th*

insignificant coefficients (*th*=64)

Example: Energy Compaction

- Original Lena image

- 2D DCT

Energy Conservation Property of 2D Unitary Transform

2-norm of a matrix $\mathbf{X}$

$$\|\mathbf{X}\|^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} |x_{ij}|^2$$

$$\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^T \qquad A \text{ unitary} \Rightarrow \|\mathbf{Y}\|^2 = \|\mathbf{X}\|^2$$

<u>Example:</u>

$$\mathbf{A} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^T} \mathbf{Y} = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}$$

$$\|\mathbf{X}\|^2 = 1^2 + 2^2 + 3^2 + 4^2 = 30 = 5^2 + 2^2 + 1^2 + 0^2 = \|\mathbf{Y}\|^2$$

Suggested Readings

- **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

- **Fundamental of Digital image processing by A. K Jain, Pearson Education India, 2015.**

Thank you