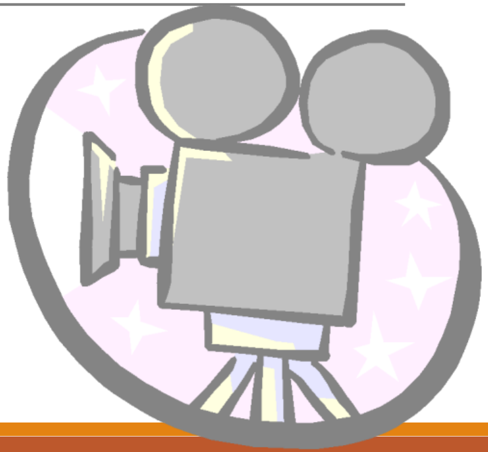
Image Processing

CS-317/CS-341



Outline

- Image Enhancement in the Frequency Domain
    - Relationship between sampling and frequency intervals
    - Properties of Fourier Transformation

Relationship between spatial sampling and frequency intervals

If $f(x)$ consists of M samples of a function $f(t)$ taken ΔT units apart, the duration of the record comprising the set $\{f(x)\}$, $x = 0, 1, 2, \ldots, M-1$, is

$$T = M\Delta T$$

The corresponding spacing, Δu, in the discrete frequency domain follows

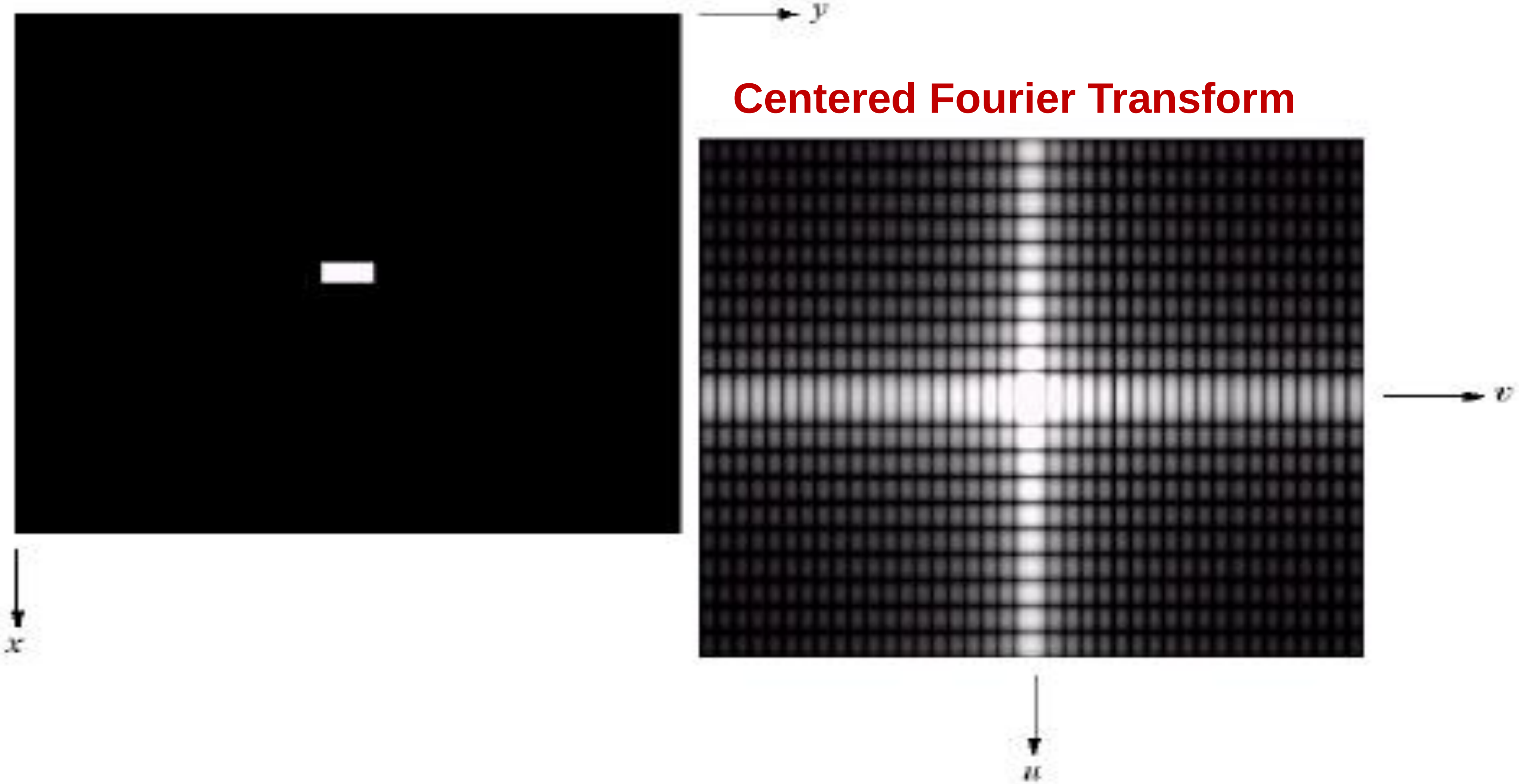
$$\Delta u = \frac{1}{M\Delta T} = \frac{1}{T}$$

The entire frequency range spanned by the M components of the DFT is

$$\Omega = M\Delta u = \frac{1}{\Delta T}$$

2-D Discrete Fourier Transform

Association Between Frequency Domain and Spatial Domain:



a b

FIGURE 4.3 (a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels. (b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

Properties of Fourier Transform

- Relationships between Spatial and Frequency Intervals
- Translation and Rotation
- Periodicity
- Symmetry Properties

Relationships between Spatial and Frequency Intervals

Suppose that a continuous function f(t, z) is sampled to form a digital image, f(x, y) consisting M * N samples Taken in t and z directions respectively.

Let ΔT and ΔZ denote the separation between samples then the separation between the corresponding discrete Frequency domain variables are given by

$$\Delta u = \frac{1}{M\Delta T}$$

$$\Delta v = \frac{1}{N\Delta Z}$$

Translation and Rotation

Multiplying f(x y) by the exponential shifts the origin of the DFT to (u_0, v_0)

$$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$$

$$f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(x_0u/M + y_0v/N)}$$

Rotation

Using polar coordinates

$$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$$

Results in the following transform pair:

$$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$$

Which indicates that rotating f(x, y) by angle θ_0 rotates F(u, v) by the same angle.

Periodicity

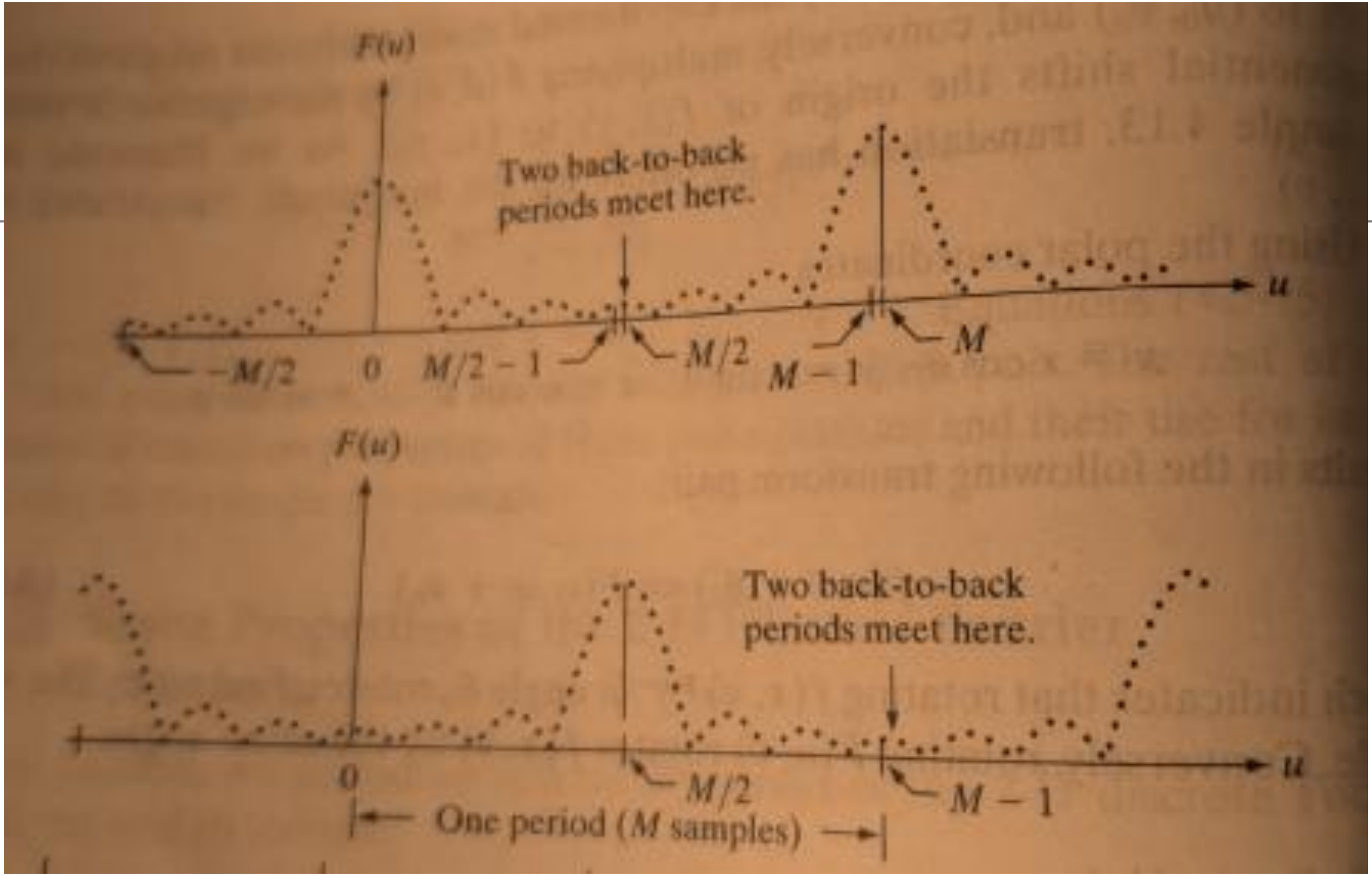
As in the 1D case, the 2-D Fourier Transform and its inverse are infinitely periodic in u and v direction.

$$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$$

and

$$f(x, y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$$

K_1 and k_2 are integers.



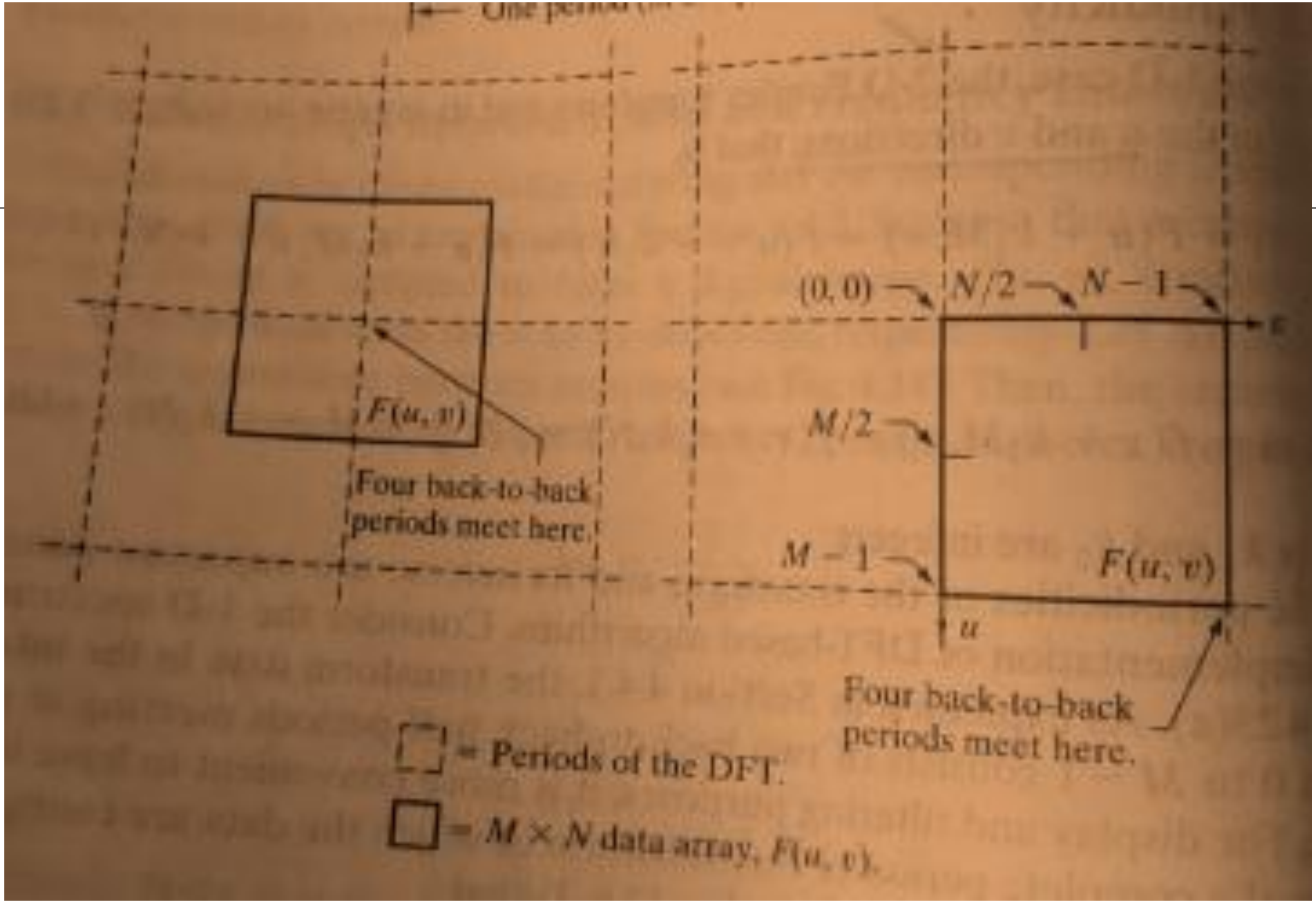
F(u)
Two back-to-back
periods meet here.
u
−M/2
0
M/2 − 1
M/2
M − 1
M
F(u)
Two back-to-back
periods meet here.
u
0
M/2
M − 1
One period (M samples)

$$f(x)e^{j2\pi(u_0x/M)} \Leftrightarrow F(u - u_0)$$

Let u_0=M/2 the exponential becomes $e^{j\pi x}$= $(-1)^x$

$$f(x)(-1)^x \Leftrightarrow F(u - M/2)$$

Multiplying f(x) by (-1)x shifts the data so that F(0) is at the center of the interval [0, M-1]



F(u, v)
Four back-to-back periods meet here.
(0, 0)
N/2
N − 1
M/2
M − 1
F(u, v)
u
Four back-to-back periods meet here.
= Periods of the DFT.
= M × N data array, F(u, v).

2-D Discrete Fourier Transform

- **Shift the origin *F*(*u*,*v*) to (*M/2*, *N/2*) :**

$$FT\left\{ f(x,y)(-1)^{x+y} \right\} = F\left(u - \frac{M}{2}, v - \frac{N}{2} \right)$$

- Multiplying $f(x,y)$ by $(-1)^{x+y}$ shifts the origin of $F(u,v)$, to frequency coordinate $(M/2, N/2)$.
- u, v are integer, so shifted coordinates must be an integer. This requires M and N are even number.

$$F(0,0) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)$$

2-D Discrete Fourier Transform

- If $f(x,y)$ is real and symmetric, its Fourier transform is conjugate symmetric:

$$F(u,v) = F^*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

- Relationship between samples in the spatial domain and frequency domain

$$\Delta u = \frac{1}{M\Delta x} \qquad \Delta v = \frac{1}{N\Delta y}$$

Symmetry Property

Any real or complex function can be expressed as

$$w(x, y) = w_e(x, y) + w_o(x, y) \qquad \text{.....................1}$$

where the even and odd parts are defined as

$$w_e(x, y) \triangleq \frac{w(x, y) + w(-x, -y)}{2} \qquad \text{.....................1(a)}$$

and

$$w_o(x, y) \triangleq \frac{w(x, y) - w(-x, -y)}{2} \qquad \text{.....................1(b)}$$

Sustituting 1(a) and 1(b) in equation 1 gives the identity w(x, y)=w(x, y)

As,

$$w_e(x, y) = w_e(-x, -y)$$

and that

$$w_o(x, y) = -w_o(-x, -y)$$

Even functions are said to be symmetric and odd functions are antisymmetric. It is convenient to think only in terms of nonnegative indices in which case the definition of evenness and oddness become:

$$w_e(x, y) = w_e(M - x, N - y)$$

and

$$w_o(x, y) = -w_o(M - x, N - y)$$

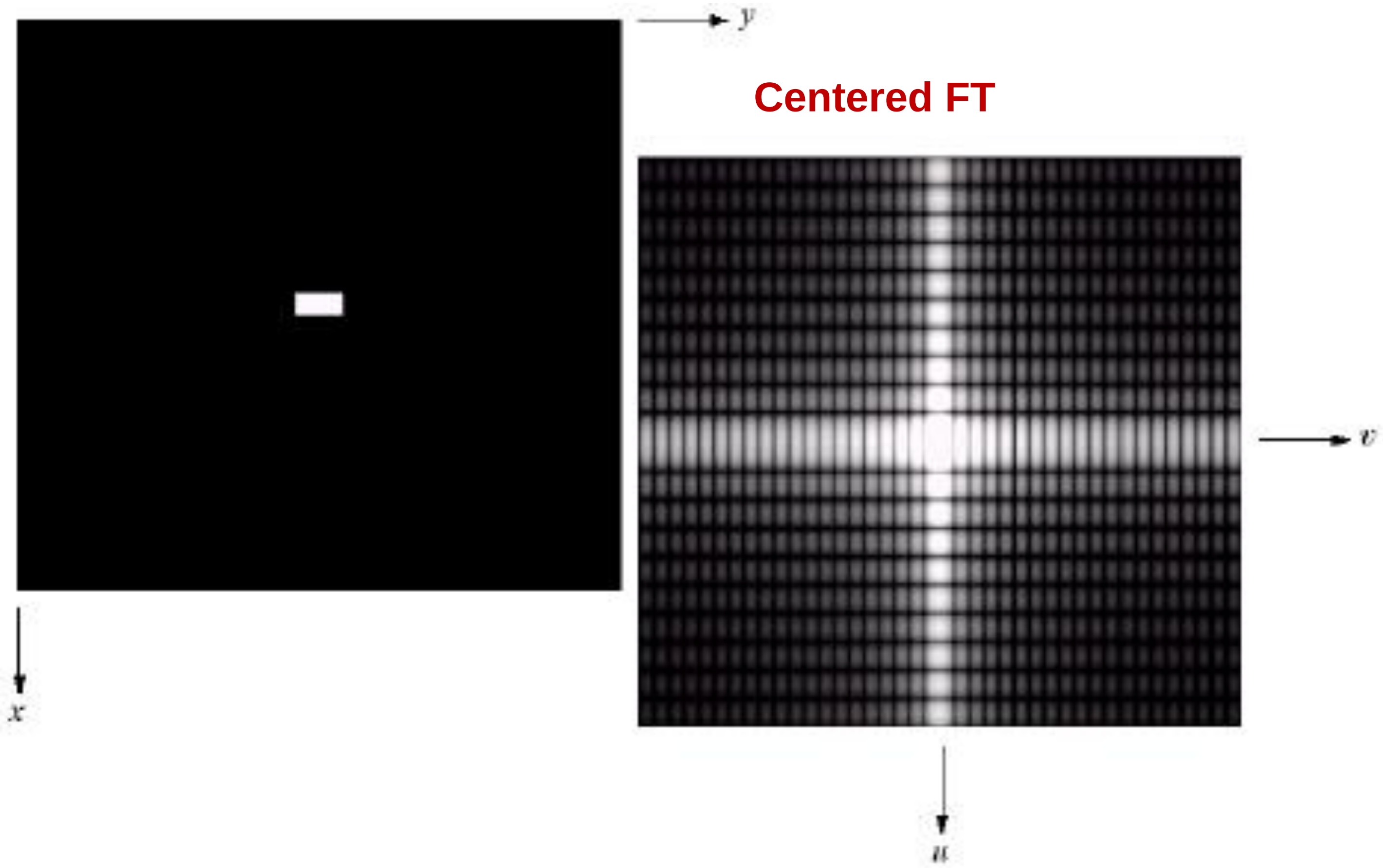
where, as usual, M and N are the number of rows and columns of a 2-D array.

We know from elementary mathematical analysis that the product of two even or two odd functions is even, and that the product of an even and an odd function is odd. In addition, the only way that a discrete function can be odd is if all its samples sum to zero. These properties lead to the important result that

$$\sum_{x=0}^{M-1} \sum_{y=0}^{N-1} w_e(x, y)\, w_o(x, y) = 0$$

2-D Discrete Fourier Transform

Example:



a b

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2-D Discrete Fourier Transform

Frequency component of FT and Spatial Characteristic of an Image:

- Frequency is directly related to rate of change of intensity in the image.
- F(0,0) is the average gray level of image.
- Around origin of the FT, the low frequency corresponds to the slow varying component.
- As move further away from origin, the higher frequency being correspond to faster and faster gray level changes in the image.

Suggested Readings

- **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

- **Fundamental of Digital image processing by A. K Jain, Pearson Education India, 2015.**

Thank you