# Multiple Random Variable And Joint Distribution

Two or more r.v. defined on the same sample space So we have concept of multivariate system. First we consider bivariate random variables.

## Joint CDF of Bivariate Random Variable :—

consider two r.v. X and Y defined on sample space $\Omega$. For example, X can denote the grade of a student and Y can denote the height of the same student. The joint cumulative distribution function (Joint CDF) of X and Y is given by

$$F_{XY}(x, y) = P[\{X \le x\} \cap \{Y \le y\}]$$

$$F_{XY}(x, y) = P[X \le x, Y \le y]$$

The pair $\{X, Y\}$ is referred to as a bivariate random variable. If we define

$F_X(x) = P[X \le x]$ — as the marginal CDF of X.

$F_Y(y) = P[Y \le y]$ — as the marginal CDF of Y.

→ We define r.v. X and Y are ~~and~~ independent if

$$F_{XY}(x, y) = F_X(x) \cdot F_Y(y)$$

for every value of x and y.

Note :— We only consider only bivariate r.v. in which both are discrete r.v. or both contin random variable.

## Properties of Joint CDF :-

As a probability function $F_{XY}(x,y)$ has certain ~~property~~ properties, which include the following:

(a) Since $F_{XY}(x,y)$ is a probability measure, $0 \le F_{XY}(x,y) \le 1$ for $-\infty < x < \infty$ and $-\infty \le y < +\infty$

(b) If $x_1 \le x_2$, and $y_1 \le y_2$ then $F_{XY}(x_1, y_1) \le F_{XY}(x_2, y_2)$

(c) $\lim_{\substack{x \to +\infty \\ y \to +\infty}} F_{XY}(x,y) = F_{XY}(+\infty, \infty) = 1$

(d) $\lim_{x \to -\infty} F_{XY}(x,y) = F_{XY}(-\infty, y) = 0$

(e) $\lim_{y \to -\infty} F_{XY}(x, -\infty) = 0$

(f) $P[x_1 < X \le x_2,\ Y \le y] = F_{XY}(x_2, y) - F_{XY}(x_1, y)$

(g) $P[X \le x,\ y_1 \le Y \le y_2] = F_{XY}(x, y_2) - F_{XY}(x, y_1)$

(h) If $x_1 \le x_2$ and $y_1 \le y_2$, then

$$P[x_1 < X \le x_2,\ y_1 < Y \le y_2] = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1) \ge 0$$

Also, the marginal CDF are obtained as follows:

$$F_X(x) = F_{XY}(x, \infty)$$

$$F_Y(y) = F_{XY}(\infty, y).$$

Ex :- Given two r.v. X and Y with joint CDF $F_{XY}(x,y)$ and marginal CDFs $F_X(x)$ and $F_Y(y)$, respectively, compute the joint probability that X is greater than 'a' and Y is greater than 'b'.

Solution :- We can ~~obtain~~ obtain the desired ~~probab~~ probability as follows: From the De Morgan's ~~law~~ law $\overline{A \cap B} = \bar{A} \cup \bar{B}$. Thus

$$P[X>a,\ Y>b] = P[\{X>a\} \cap \{Y>b\}]$$

$$= 1 - P[\overline{\{X>a\} \cap \{Y>b\}}]$$

$$= 1 - P[\overline{\{X>a\}} \cup \overline{\{Y>b\}}]$$

$$P[X>a,\ Y>b] = 1 - P[\{X \le a\} \cup \{Y \le b\}]$$

$$P[X>a,\ Y>b] = 1 - \{P[X \le a] + P[Y \le b] - P[X \le a, Y \le b]\}$$

$$= 1 - F_X(a) - F_Y(b) + F_{XY}(a,b)$$

$$\boxed{\therefore\ P[X>a,\ Y>b] = 1 - F_X(a) - F_Y(b) + F_{XY}(a,b)}$$

## Discrete Bivariate R.V.:

When both X and Y are discrete r.v., we ~~deter~~ define their joint PMF as follows

$$p_{XY}(x,y) = P[X=x, Y=y]$$

The properties of joint PMF include the following

(a) As a probability measure, the PMF can neither be negative nor exceed unity, which means that $\cancel{0 \le p_{XY}(x,y) \le 1}$ $0 \le p_{XY}(x,y) \le 1$.

(b) $$\sum_{x} \sum_{y} p_{XY}(x,y) = 1$$

(c) $$\sum_{x \le a} \sum_{y \le b} p_{XY}(x,y) = F_{XY}(a,b)$$

The marginal PMFs are obtained as follows:

$$p_X(x) = \sum_{y} p_{XY}(x,y) = P[X=x]$$

$$p_Y(y) = \sum_{x} p_{XY}(x,y) = P[Y=y]$$

If X and Y are independent random variables then

$$\boxed{p_{XY}(x,y) = p_X(x)\, p_Y(y)}$$

Ex: The Joint PMF of two r.v. X and Y is given by:

$$p_{XY}(x,y) = \begin{cases} k(2x+y) ; & x=1,2 ; y=1,2,3. \\ 0 ; & \text{otherwise.} \end{cases}$$

where $k$ is a constant.

(a) What is the value of $k$?

(b) Find the marginal PMF's of X and Y?

(c) Are X and Y independent?

Solution :—

(a) To evaluate $k$, we remember that

$$\sum_x \sum_y p_{XY}(x,y) = 1 = \sum_{x=1}^{2} \sum_{y=1}^{3} k(2x+y)$$

Thus,

$$\sum_{x=1}^{2} \sum_{y=1}^{3} k(2x+y) = k \sum_{x=1}^{2} \{ (2x+1)+(2x+2)+(2x+3) \}$$

$$= k \sum_{x=1}^{2} \{6x+6\} = 30k$$

$$30k = 1 \Rightarrow k = \frac{1}{30}$$

(b) Marginal PMFs are:

$$p_X(x) = \sum_{y=1}^{3} p_{XY}(x,y) = \frac{1}{30} \sum_{y=1}^{3} (2x+y)$$

$$= \frac{1}{30}(6x+6) = \frac{1}{5}(x+1) ; \quad x=1,2$$

$$p_Y(y) = \sum_{x=1}^{2} p_{XY}(x,y) = \frac{1}{15}(y+3), \quad y=1,2,3.$$

(c) Since $p_X(x)\, p_Y(y) = \frac{1}{75}(x+1)(y+2) \neq p_{XY}(x,y)$,
we conclude that X and Y are not independent.

Ex :- The number of emergency calls X to a police station of a certain town has a poisson distribution with mean $\lambda$. The probability that any one of these call is about robbery is p. What is the PMF of Y, the number of calls about robbery?

## — : Example : —

A fair coin is tossed three times. Let X be r.v. that takes the value 0 if the first toss is a tail and the value 1 if the first toss is a head. Also let Y be a r.v. that defines the total number of heads in the three tosses.

(a) Determine the joint PMF of X and Y.

(b) Are X and Y independent?

**Solution**

| Sample space | Value of X | Value of Y |
|---|---|---|
| HHH | 1 | 3 |
| HHT | 1 | 2 |
| HTH | 1 | 2 |
| HTT | 1 | 1 |
| THH | 0 | 2 |
| THT | 0 | 1 |
| TTH | 0 | 1 |
| TTT | 0 | 0 |

(a) Each of these eight samples points is equally likely to be obtained. Thus, since X takes values 0 and 1, and Y takes values 0, 1, 2 and 3, the joint PMF of X and Y can then be constructed as follows: (5)

$$p_{XY}(0,0) = P[X=0, Y=0] = P[TTT] = \frac{1}{8}$$

$$p_{XY}(0,1) = P[X=0, Y=1] = P[\{THT\} \cup \{TTH\}]$$

$$= P[THT] + P[TTH]$$

$$\cancel{p_{XY} = \frac{1}{4}}$$

$$\boxed{p_{XY}(0,1) = \frac{1}{4}}$$

$$p_{XY}(0,2) = P[X=0, Y=2] = P[TTH] = \frac{1}{8}$$

$$p_{XY}(0,3) = P[X=0, Y=3] = 0.$$

$$p_{XY}(1,0) = P[X=1, Y=0] = 0$$

$$p_{XY}(1,1) = P[X=1, Y=1] = P[HTT] = \frac{1}{8}$$

$$p_{XY}(1,2) = P[X=1, Y=2] = P[\{HHT\} \cup \{HTH\}]$$

$$= P[HHT] + P[HTH]$$

$$\boxed{p_{XY}(1,2) = \frac{1}{4}}$$

$$p_{XY}(1,3) = P[X=1, Y=3] = P[HHH] = \frac{1}{8}$$

(b) If X and Y are independent, then for all $x$ and $y$, we have ~~$p_{XY}(x,y)$~~ $p_{XY}(x,y) = p_X(x) \cdot p_Y(y)$.
Thus, to show that X and Y are not independent, all we have to do is to find a pair ~~x and y~~ $x$ and $y$ at which the joint PMF does not satisfy the above condition. Consider the point

5a

$(x, y) = (0, 0)$

$$p_X(0) = \sum_y p_{XY}(0) = p_{XY}(0,0) + p_{XY}(0,1) + p_{XY}(0,2) + p_{XY}(0,3)$$

$$= \frac{1}{2}$$

$$p_Y(0) = \sum_x p_{XY}(0) = p_{XY}(0,0) + p_{XY}(1,0) = \frac{1}{8}$$

Since $p_X(0)\, p_Y(0) = \left(\frac{1}{2}\right) \times \left(\frac{1}{8}\right) = \frac{1}{16} \neq p_{XY}(0,0) = \frac{1}{8}$,

Hence X and Y are not independent

## → Bivariate Continuous Random Variable :-



If both X and Y are continuous random variables, their joint PDF is given by

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x\, \partial y} F_{XY}(x,y)$$

The joint PDF satisfy the ~~foll~~ following condition:

$$\cancel{F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\, dv\, du}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)\, dv\, du$$

The joint PDF also has the following properties:

(a) For all $x$ and $y$, $f_{XY}(x,y) \geq 0$

(b) $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy\, dx = 1$

(c) $f_{XY}(x,y)$ is continuous for all except possibly finitely many values of $x$ ~~for~~ or of $y$.

(d) $P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f_{XY}(x,y)\, dy\, dx$

The marginal PDFs are given by

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy$$

$$\boxed{f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx}$$

If X and Y are independent random variables then

$$\boxed{f_{XY}(x,y) = f_X(x)\, f_Y(y)}$$

Ex :- X and Y are two continuous r.v. whose joint PDF is given by

$$f_{XY}(x,y) = \begin{cases} e^{-(x+y)} & ; \; 0 \le x < +\infty,\; 0 \le y < \infty \\ 0 & ; \; \text{otherwise} \end{cases}$$

Are X and Y are independent ?

Solution :- (Yes)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy = \int_{0}^{+\infty} e^{-(x+y)}\,dy$$

. . . .

$$= e^{-x} \;,\quad x \ge 0$$

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx = \int_{0}^{+\infty} e^{-(x+y)}\,dx$$

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$$= e^{-y} \;;\; y \ge 0.$$

X and Y are independent

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) = e^{-x}.e^{-y} = e^{-(x+y)} = f_{XY}(x,y).$$

Ex :- Determine if r.v. X and Y are independent? when their joint PDF is ~~giv~~ given by

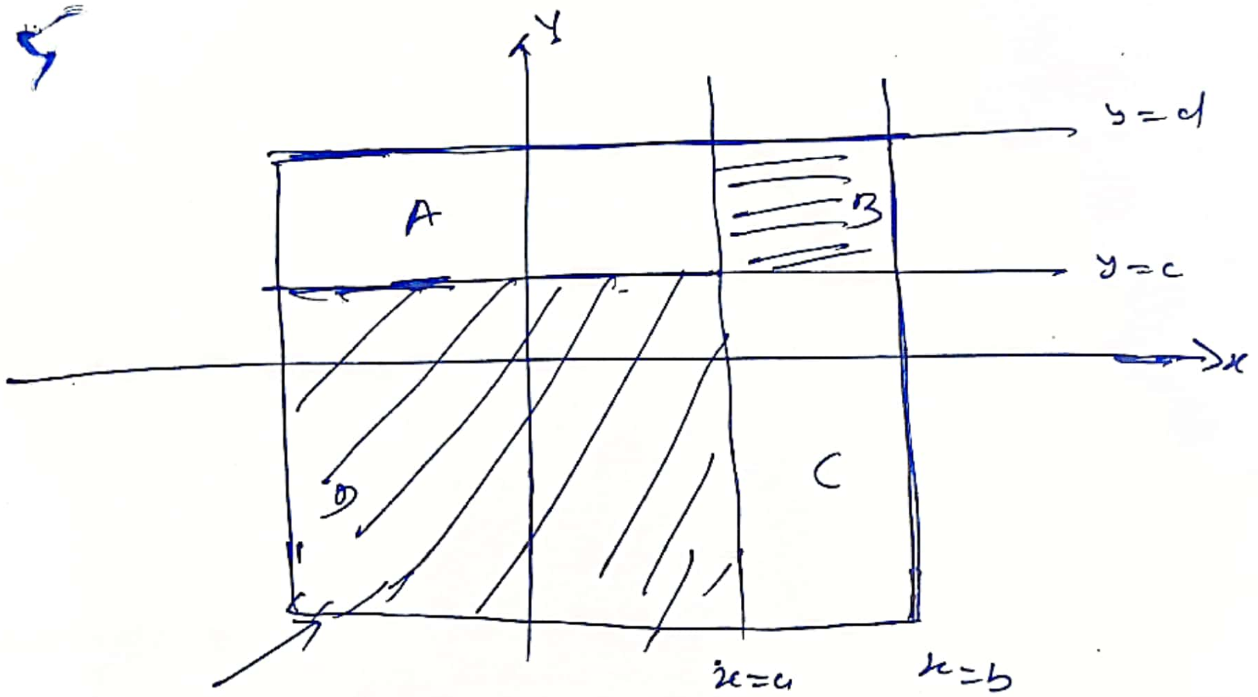
$$f_{XY}(x,y) = \begin{cases} 2e^{-(x+y)} & ; \ 0 \le x \le y, \ 0 \le y < +\infty \\ 0 & ; \ 0, \text{ otherwise} \end{cases}$$

Solution ( Answer No) :-

## Determining Probabilities from A Joint CDF

Let X and Y are two r.v. and we require to determine the probability of a certain event define in terms of X and Y for which the joint CDF is known. For example, assume that to find $P[a < X \leq b, c < Y \leq d]$

Event sketch in X-Y plane




Domain

Consider the following events

$$E_1 = \{ (X \leq b) \cap (Y \leq d) \}$$

$$E_2 = \{ (X \leq b) \cap (Y \leq c) \}$$

$$E_3 = \{ (X \leq a) \cap (Y \leq d) \}$$

$$E_4 = \{ (X \leq a) \cap (Y \leq c) \}$$

$$E_5 = \{ (a < X \leq b) \cap (c < Y \leq d) \}$$

The region of interest is B, which correspond to event $E_5$ that can be obtained as follows:

(8)

$$E_5 = E_1 - E_2 - E_3 + E_4$$

Thus

$$P[a < X \le b, c < Y \le d]$$

$$\boxed{P[a < X \le b, c < Y \le d] = F_{XY}(b,d) - F_{XY}(b,c) - F_{XY}(a,d) + F_{XY}(a,c)}$$

If X and Y are independent R.V.

$$P[a < X \le b, c < Y \le d]$$

$$= F_{XY}(b,d) - F_{XY}(b,c) - F_{XY}(a,d) + F_{XY}(a,c)$$

$$= F_X(b) \cdot F_Y(d) - F_X(b) \cdot F_Y(c) - F_X(a) F_Y(d) + F_X(a) F_Y(c)$$

$$= F_X(b)[F_Y(d) - F_Y(c)] - F_X(a)[F_Y(d) - F_Y(c))$$

$$P[a < X \le b, c < Y \le d]$$

$$= \{F_X(b) - F_X(a)\}\{F_Y(d) - F_Y(c)\}$$

$$= P[a < X \le b] \, P[c < Y \le d]$$

## :- Example 52 :-

The joint CDF of two discrete r.v. X and Y ~~is~~ is given as follows:

$$F_{XY}(x,y) = \begin{cases} \frac{1}{8}, & x=1, y=1 \\ \frac{5}{8}, & x=1, y=2 \\ 1/4; & x=2, y=1 \\ 1; & x=2, y=2 \end{cases}$$

Determine the following:

(a) Joint PMF of X and Y.

(b) Marginal PMF of X

(c) Marginal PMF of Y.

Solution :- The joint PMF is obtained from the relationship

$$F_{XY} = \sum_{m \le x} \sum_{n \le y} P_{xy}(m,n)$$

## Conditional Distributions :

For two events A and B, the conditional probability of an event A given event B is defined by

$$P[A/B] = \frac{P[A \cap B]}{P[B]}$$

which is defined when $P[B] > 0$

↳ Same concept is extended to two random variable X and Y govern by joint CDF, $F_{XY}(x,y)$

### Conditional PMF for Discrete Bivariate Random Variables

Consider the two r.v. X and Y with joint PMF $p_{XY}(x,y)$. The conditional PMF of Y given $X=x$, is given by

$$p_{Y/X}(y/x) = \frac{P[X=x, Y=y]}{P[X=x]}$$

$$\boxed{p_{Y/X}(y/x) = \frac{p_{XY}(x,y)}{p_X(x)}};$$

provided $p_X(x) > 0$

Similarly conditional PMF of X, given $Y=y$ is given by

$$p_{X/Y}(x/y) = \frac{P[X=x, Y=y]}{P[Y=y]} = \frac{p_{XY}(x,y)}{p_Y(y)}$$

- If $X$ and $Y$ are independent r.v., $p_{X|Y}(x|y) = p_X(x)$ and $p_{Y|X}(y|x) = p_Y(y)$

## Conditional CDF

$$F_{X|Y}(x|y) = P[X \le x | Y = y] = \sum_{u \le x} p_{X|Y}(u|y)$$

$$F_{Y|X}(y|x) = P[Y \le y | X = x] = \sum_{v \le y} p_{Y|X}(v|x)$$

Ex The joint PMF of two r.v. $X$ and $Y$ is given by

$$p_{XY}(x,y) = \begin{cases} \frac{1}{30}(2x+y), & x = 1,2;\ y = 1,2,3 \\ 0, & \text{otherwise} \end{cases}$$

(a) What is the conditional PMF of $Y$ given $X$?

(b) What is the conditional PMF of $X$ given $Y$?

Solution :-

$$p_X(x) = \sum_y p_{XY}(x,y) = \frac{1}{30}\sum_{y=1}^{3}(2x+y) = \frac{1}{5}(x+1), \quad x = 1,2$$

$$p_Y(y) = \sum_x p_{XY}(x,y) = \frac{1}{30}\sum_{x=1}^{2}(2x+y) = \frac{1}{15}(y+3); \quad y = 1,2,3$$

Thus, the conditional PMFs are given by:

$$p_{X|Y}(x|y) = \frac{p_{XY}(x,y)}{p_Y(y)} = \frac{2x+y}{2(y+3)}; \quad x = 1,2$$

$$p_{Y|X}(y|x) = \frac{p_{XY}(x,y)}{p_X(x)} = \frac{2x+y}{6(x+1)}; \quad y = 1,2,3$$

## Conditional PDF for Continuous Bivariate Random Variables

Consider two continuous r.v. X and Y with the joint PDF $f_{XY}(x,y)$. The conditional PDF of Y, given $X=x$, is defined by:

$$f_{Y/X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}$$

provided $f_X(x) > 0$. Similarly, the conditional PDF of X, given $Y=y$, is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

provided $f_Y(y) > 0$. If X and Y are independent, then $f_{X|Y}(x|y) = f_X(x)$ and $f_{Y|X}(y|x) = f_Y(y)$

**Example:-** Two r.v. X and Y have the following joint PDF

$$f_{XY}(x,y) = \begin{cases} xe^{-x(y+1)} ; & 0 \le x < \infty, 0 \le y < \infty \\ 0 ; & \text{otherwise} \end{cases}$$

Determine the conditional PDF of X given Y and the conditional PDF of Y given X.

## Conditional Means and Variances :-

If X and Y are discrete r.v. with the joint PMF $p_{XY}(x,y)$, the conditional expected value of Y, given that $X=x$, is defined by

$$\boxed{\mu_{Y/X} = E[Y|X] = \sum_y y\, p_{Y/X}(y/x)}$$

The conditional variance of Y, given $X=x$ is given by

$$\sigma^2_{Y/X} = E\left[(Y-\mu_{Y/X})^2 \middle| X\right]$$

$$= \sum_y (y-\mu_{Y/X})^2\, p_{Y/X}(y/x)$$

$$\boxed{\sigma^2_{Y/X} = E[Y^2/X=x] - \left(E[Y/X=x]\right)^2}$$

- The condn expected value and variance of X given $Y=y$, are given by

$$\mu_{X/Y} = E[X|Y] = \sum_x x\, p_{X/Y}(x/y)$$

$$\sigma^2_{X/Y} = E\left[(X-\mu_{X/Y})^2 \middle| Y\right] = \sum_x (x-\mu_{X/Y})^2\, p_{X/Y}(x/y)$$

$$\boxed{\sigma^2_{X/Y} = E[X^2/Y=y] - \left(E[X/Y=y]\right)^2}$$

Ex :- compute the conditional mean $E[X|Y=y]$ if the joint PDF of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} \dfrac{e^{-x/y}e^{-y}}{y}, & 0 \le x < \infty;\ 0 < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Solution :- $f_Y(y) = \int_0^\infty f_{XY}(x,y)\,dx = \int_0^\infty \dfrac{e^{-x/y}e^{-y}}{y}\,dx$

$$= \frac{e^{-y}}{y}\int_0^\infty e^{-x/y}\,dx = e^{-y}$$

$$f_{X|Y}(y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{e^{-x/y}e^{-y}}{y e^{-y}} = \left(\frac{1}{y}\right)e^{-x/y}$$

Thus, the conditional mean is given by:

$$E[X|Y=y] = \int_0^\infty x f_{X|Y}(x|y)\,dx = \left(\frac{1}{y}\right)\int_0^\infty x e^{-x/y}$$

$$E[X|Y=y] = y$$

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## Rule for Independence :-

Let X and Y are r.v.. The following is a general rule that applies when X and Y are independent

If joint PDF of X and Y is of the form

$$f_{XY}(x,y) = \text{constant} \times \{\text{function of } x\} \times \{\text{function of } y\}$$

in a rectangular region (which can be finite or infinite) $a \le x \le b$, $c \le y \le d$, then X and Y are independent.

— If joint PDF is not separable form as above or the joint sample space is not rectangular, then X and Y are not independent.

## Example :-

Assume that r.v. X and Y have the joint PDF

$$f_{XY}(x,y) = \frac{1}{2} x^3 y; \qquad 0 \le x \le 2; \quad 0 \le y \le 1$$

(a) Determine if X and Y are independent

(b) What are the marginal PDF?

~~Solution~~ .

Solution (13)

(a) Applying the above rule we find that the join PDF is separable and the joint sample space is rectangular.

Therefore X and Y are ~~indepen.~~ independent

$$f_X(x) = x^3 \quad f_Y(y) = y$$

(b) Thus we have

$$f_{XY}(x,y) = f_X(x) \times f_Y(y) = \frac{1}{2} \times x^3 \times y \;; \; 0 \le x \le 2, \; 0 \le y \le 1$$

where $f_X(x) = Ax^3, \quad 0 \le x \le 2$

$f_Y(y) = By \;; \quad 0 \le y \le 1$

$AB = \frac{1}{2}$

we can find the value of A and B as follows

$$\int_0^2 f_X(x)\,dx = 1 \;\Rightarrow\; \int_0^2 Ax^3\,dx = A\left[\frac{x^4}{4}\right]_0^2$$

$\Rightarrow \quad 4A = 1 \;\Rightarrow\; A = \frac{1}{4}$

Thus $AB = \frac{1}{2} \Rightarrow B = \frac{1}{2A} = \frac{1}{2} \times \frac{4}{1} = 2$

$\therefore \; B = 2$

From the we obtain marginal PDF

$f_X(x) = \frac{1}{4}x^3, \quad 0 \le x \le 2$

$f_Y(y) = 2y \;; \quad 0 \le y \le 1$

## Covariance and Correlation coefficient :

consider two r.v. X and Y with expected value $E[X]=\mu_x$ and $E[Y]=\mu_y$, respectively., and variances $\sigma_x^2$ and $\sigma_y^2$, respectively. The covariance of X and Y, which is denoted by $Cov(X,Y)$ or $\sigma_{XY}$ is defined by:

$$\text{cov}(X,Y) = \sigma_{XY} = E[(X-\mu_x)(Y-\mu_y)]$$

$$= E[XY - \mu_y X - \mu_x Y + \mu_x\mu_y]$$

$$= E[XY] - \mu_y E[X] - \mu_x E[Y] + E[\mu_x \mu_y]$$

$$\text{Cov}(X,Y) = E[XY] - \mu_y \mu_x - \mu_x\mu_y + \mu_x\mu_y$$

$$= E[XY] - 2\mu_x\mu_y + \mu_x\mu_y$$

$$\boxed{\text{Cov}(X,Y) = E[XY] - \mu_x\mu_y}$$

Note :- (1) If X and Y are independent, then

$$E[XY] = \mu_x \mu_y$$

$$\therefore \quad \text{Cov}(X,Y) = \mu_x\mu_y - \mu_x\mu_y$$

$$\boxed{\text{Cov}(X,Y) = 0}$$

But converse is not true

i.e. $\text{cov}(X,Y) = 0 \not\Rightarrow$ X and Y are r.v. independent

(2) If the covariance of the two ~~r.v.~~ random variables is zero, we define the two random ~~va~~ variables to be uncorrelated.

## ~~Corr~~ Correlation Coefficient :-

Correlation coefficient of r.v. X and Y, denoted by $\rho(X,Y)$ or $\rho_{XY}$, as follows:

$$\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

$$\boxed{\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}}$$

- The correlation coefficient has the property that

$$|\rho_{XY}| \le 1 \Rightarrow -1 \le \rho_{XY} \le 1$$

Proof :- Since variance is ~~also~~ always non-negative, we have ~~it~~ that if X and Y have variances given by $\sigma_X^2$ and $\sigma_Y^2$ respectively, then

$$0 \le Var\left(\frac{X}{\sigma_X} + \frac{Y}{\sigma_Y}\right) = \frac{1}{\sigma_X^2} Var(X) + \frac{1}{\sigma_Y^2} Var(Y) + \frac{2\, Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$= 2(1 - \rho_{XY})$$

$\therefore \quad 1 - \rho_{XY} \ge 0 \Rightarrow \rho_{XY} \le 1$

Thus $-1 \le \rho_{XY} \le 1$

Note (1) :- The correlation coefficient $\rho_{XY}$ provides a ~~meas~~ measure of how ~~long~~ good a linear prediction of the value of one of the two random variable can be ~~f~~ formed on an observed value of the other.

→ Thus, if represent the relationship between X and Y by the linear equation $Y = a + bX$.

→ a value of $\rho_{XY}$ near −1 or +1 indicates a high degree of linearity between X and Y.

— $\rho_{XY} > 0 \Rightarrow$ high degree of linearity between X and Y.

Infact $\rho_{XY} > 0 \Rightarrow b > 0$

$\rho_{XY} < 0 \Rightarrow b < 0$

$\rho_{XY} > 0 \Rightarrow$ as X increases, Y also tend to increase

$\rho_{XY} < 0 \Rightarrow$ X increases, Y tends to decrease

$\rho_{XY} = 0 \Rightarrow$ There is no linear correlation ~~relationship~~ between X and Y

↳ It does not mean that there is no correlation between X and Y because there may ~~be st~~ still be a high nonlinear correlation between them.

In general $\rho_{XY}$ measure the goodness of fit of the equation that express Y as a function of X ~~to actu~~ to actual value (measured) of Y.

(17) This is indicates how closely the equation that expresses Y as a function of X matches measured (or observed) values of Y.

Example :— The joint PDF of the r.v. X and Y is defined as follows :

$$f_{XY}(x,y) = \begin{cases} 25e^{-5y}; & 0 \le x < 0.2,\ y \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the marginal PDF of X and Y.

(b) What is the covariance of X and Y ?

Solution :—

(a) The marginal PDF are obtained as follows:

$$f_X(x) = \int_0^{+\infty} f_{XY}(x,y)\,dy = \int_0^{+\infty} 25e^{-5y}\,dy = \begin{cases} 5, & 0 \le x < 0.2 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_0^{0.2} f_{XY}(x,y)\,dx = \int_0^{0.2} 25e^{-5y}\,dx = \begin{cases} 5e^{-5y}; & y \ge 0 \\ 0; & \text{otherwise} \end{cases}$$

Thus, X has uniform distribution and Y has an exponential distribution.

(b) The expected values of X and Y are given by:

$$E[X] = \mu_X = \frac{0 + 0.2}{2} = 0.1$$

$$E[Y] = \mu_Y = \frac{1}{5} = 0.2$$

Also,
$$E[XY] = \int_{x=0}^{0.2}\int_{y=0}^{\infty} xy\, f_{XY}(x,y)\,dy\,dx$$

$$= \int_{x=0}^{0.2}\int_{y=0}^{\infty} 25xy\, e^{-5y}\,dy\,dx$$

$$E[XY] = \int_{x=0}^{0.2} x\left\{\int_{y=0}^{+\infty} 25y\, e^{-5y}\,dy\right\}dx = \int_{x=0}^{0.2} x\,dx = \left[\frac{x^2}{2}\right]_0^{0.2}$$

$\boxed{E[XY] = 0.02}$, Thus, the covariance of X and Y is given by

$$\sigma_{XY} = E[XY] - \mu_X\mu_Y = 0.02 - (0.1)(0.2) = 0$$

This means that X and Y are uncorrelated. ~~Note~~ Note that the reason why $\sigma_{XY} = 0$ is because X and Y are independent. This follows from the fact that $f_{XY}(x, y)$ is separable into a function X and a function Y, and ~~...~~ the region of interest is rectangular.

Thus, $\boxed{f_{XY}(x, y) = f_X(x)\, f_Y(y)}$

EXAMPLE :- Hans and Ann planned to meet at their favorite restaurant on a date about 6:30 pm. Both of them will arrive at the restaurant separately by train. They live in different parts of the city and so will be arriving on different train that operate independently of each other's schedule. Han's train will arrive at a stop by the ~~restaur~~ restaurant at a time that is uniformally distributed between 6:00 pm and 7:00 pm. Ann's train will arrive at the same stop at a time that is uniformally distributed between 6:15 pm and 6:45 pm. They agreed that whoever arrives at the restaurant first will wait up to 5 minutes before leaving;

(a) What is probability that they meet?

(b) What is probability that Ann arrives before Hans?

# Multivariate Random Variables :-

Extend the concepts of bivariate system, system of two random variables to systems of more than two random ~~Varia~~ variable.

Let $X_1, X_2, X_3, \ldots, X_n$ be a set of random variables that are defined on the same sample space. Their joint CDF is defined as:

$$F_{X_1,X_2,\ldots X_n}(x_1, x_2, x_3, \ldots, x_n) = P[X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n]$$

If all random variables are discrete r.v., their joint PMF is defined by

$$p_{X_1,X_2,\ldots X_n}(x_1, x_2, x_3, \ldots, x_n) = P[X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n]$$

## Properties of Joint PMF :-

1. $0 \le p_{X_1 X_2 \ldots X_n}(x_1, x_2, x_3, \ldots, x_n) \le 1$

2. $$\sum_{x_1}\sum_{x_2}\sum_{x_3}\cdots\sum_{x_n} p_{X_1,X_2 \ldots X_n}(x_1, x_2, x_3, \ldots, x_n) = 1$$

3. The marginal PMF are obtained by summing the joint PMF over the ~~all~~ appropriate ranges. For example, the marginal PMF of $X_n$ is given by:

$$p_{X_n}(x_n) = \sum_{x_1}\sum_{x_2}\cdots\sum_{x_{n-1}} p_{X_1 X_2 \ldots X_n}(x_1, x_2, \ldots, x_n)$$

4. The conditional PMF are similarly obtained. For example

$$p_{X_n / X_1 X_2 \cdots X_{n-1}}(x_n / x_1 x_2 \cdots x_{n-1})$$
$$= P[X_n = x_n / X_1 = x_1, \cdots, X_{n-1} = x_{n-1}]$$

$$\boxed{p_{X_n / X_1 X_2 \cdots X_{n-1}}(x_n / x_1, x_2 \cdots x_{n-1}) = \frac{p_{X_1 X_2 \cdots X_n}(x_1, x_2, x_3, \cdots, x_n)}{p_{X_1, X_2 \cdots X_{n-1}}(x_1, x_2, x_3 \cdots, x_{n-1})}}$$

provided
$$p_{X_1, X_2, \cdots X_{n-1}}(x_1, x_2, \cdots x_{n-1}) > 0$$

The random variables are defined to be mutually independent if

$$\boxed{p_{X_1 X_2 \cdots X_n}(x_1, x_2 \cdots x_n) = \prod_{k=1}^{n} p_{X_k}(x_k)}$$

If all the random variables are continuous random variables, their joint PDF can be obtained from the joint CDF as follows:

$$f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} F_{X_1, X_2 \cdots X_n}(x_1, x_2, \cdots x_n)$$

The joint PDF has the following properties:-

1. $f_{X_1 X_2 \cdots X_n}(x_1, x_2, \cdots, x_n) \geq 0.$
2. $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty} f_{X_1, X_2 \cdots X_n}(x_1, x_2, \cdots, x_n)\, dx_1\, dx_2 \cdots dx_n = 1$

3. The conditional PDF are similarly obtained, for example.

$$f_{X_n|X_1 X_2 \cdots X_{n-1}}(x_n | x_1, x_2, \ldots, x_{n-1}) = \frac{f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n)}{f_{X_1 X_2 \cdots X_{n-1}}(x_1, x_2, \ldots x_{n-1})}$$

provided $f_{X_1 X_2 \cdots X_{n-1}}(x_1, x_2, \ldots, x_{n-1}) > 0$. If the random variables are mutually independent, then

$$\boxed{f_{X_1 X_2 \cdots X_n}(x_1, x_2, \ldots, x_n) = \prod_{k=1}^{n} f_{X_k}(x_k)}$$

Ex :- A machine have ~~N~~ identical components each of ~~each~~ which has an exponentially distributed lifetime $T$ with PDF $\lambda e^{-\lambda t}$, $t \geq 0$. What is the probability that exactly $n$ of the components have failed by time $v \geq 0$ ?

Solution :- Let $p$ be the probability that a component has failed by time $v$, which is the CDF of $T$ evaluated at $t = v$, that is:

$$p = P[T \leq v] = F_T(v) = 1 - e^{-\lambda v}$$

$$\boxed{p = 1 - e^{-\lambda v}}$$

If we assume that the component ~~fails~~ fail ~~independ~~ independently, then if $q_n$ is the probability that exactly $n$ of them ~~fail~~ have failed by time $v$, we have that

$$q_n = \binom{N}{n} p^n (1-p)^{N-n}$$

$$q_n = \binom{N}{n} (1-e^{-\lambda u})^n (e^{-\lambda u})^{N-n}$$

$$\boxed{q_n = \binom{N}{n} [1-e^{-\lambda u}]^n e^{-\lambda (N-n) u}}$$

# Functions of Random Variables

↳ Events are directly defined using r.v., ~~but many~~ ~~cas~~ But in some cases the events are functions of other ~~event~~ events.

~~Ba~~ e.g. The time until a complex system fails is a function of the time to failure of the individual components that make up the system.

↳ r.v. to represent the time of failure of the complex system is a function of the random variables used to represent the times to failure of the component parts of the ~~syst~~ system.

Function of One R.V.:- Let X be a random variable, and let Y be a new random variable that is a function of X.

$$Y = g(X)$$

Now find PMF or PDF of Y when PMF and PDF of X is known.

e.g. let $g(X) = X + 5$. Then

$$F_Y(y) = P[Y \leq y]$$

$$\boxed{F_Y(y) = P[X + 5 \leq y]}$$

## Linear Functions:-

Consider the function $g(x) = ax+b$, where a and b are constants. The CDF of Y is given by

$$F_Y(y) = P[Y \le y] = \cancel{P[ax+b]}$$
$$= P[ax+b \le y]$$
$$F_Y(y) = P\left[X \le \frac{y-b}{a}\right]$$
$$\boxed{F_Y(y) = F_X\left(\frac{y-b}{a}\right)}$$

where 'a' is positive. The PDF of Y is given by

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X\left(\frac{y-b}{a}\right)}{dy}$$

$$f_Y(y) = \left(\frac{d}{du}F_X(u)\right)\frac{du}{dy}$$

Where $u = \frac{y-b}{a}$ and $\frac{du}{dy} = \frac{1}{a}$. Thus,

$$f_Y(y) = \left(\frac{dF_X(u)}{du}\right)\left(\frac{du}{dy}\right)$$
$$= f_X(u)\cdot\left(\frac{1}{a}\right)$$
$$\therefore \boxed{f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)}$$

When $a<0$ we have that

$$F_Y(y) = P[Y \le y] = \cancel{P[ax+b]}\ P[ax+b \le y]$$
$$= P[ax \le y-b]$$
$$= P\left[X \ge \frac{y-b}{a}\right]$$

$$= 1 - \left\{ P\left[X \le \frac{y-b}{a}\right] - P\left[X = \frac{y-b}{a}\right] \right\}$$

$$\boxed{F_Y(y) = 1 - \left\{ P\left[X \le \frac{y-b}{a}\right] - P\left[X = \frac{y-b}{a}\right] \right\}}$$

The change in the sign on the second line arises from the fact that 'a' ~~is~~ is negative. If X is ~~cont~~ ~~Cont~~ continuous, $P\left[X = \frac{y-b}{a}\right] = 0$.

Thus CDF and PDF for the case of negative a are given by:

$$\boxed{F_Y(y) = 1 - P\left[X \le \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right)}$$

$$\boxed{f_Y(y) = \frac{d}{dy} F_Y(y) = -\left(\frac{1}{a}\right) f_X\left(\frac{y-b}{a}\right)}$$

Therefore, the general PDF of Y is given by

$$\boxed{f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Example:- Find the PDF of Y in terms of the PDF of X if $Y = 2X + 7$

Solution:-

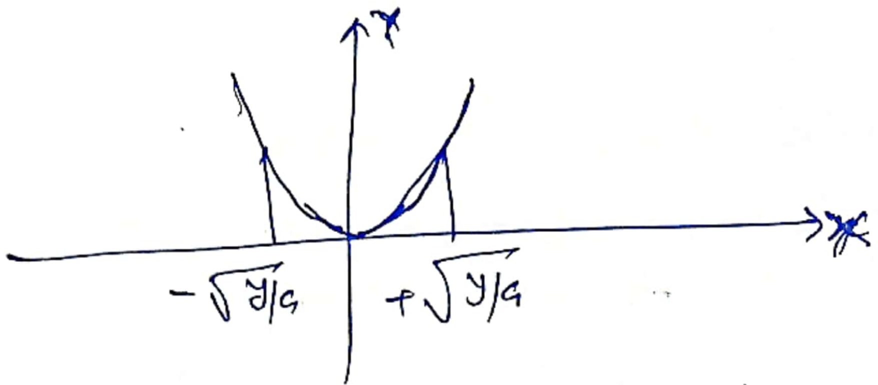
$$F_Y(y) = F_X\left(\frac{y-7}{2}\right)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{2} f_X\left(\frac{y-7}{2}\right)$$

## ~~Pow~~ Power Functions:

Consider the quadratic function $Y = aX^2$, where $a>0$

The plot of $Y$ against $X$ is




For each value of $Y$ there are two corresponding values of $X$; ~~m~~ namely $\sqrt{y/a}$, $-\sqrt{Y/a}$

Thus, the CDF of $Y$ is given by

$$F_Y(y) = P[Y \le y]$$
$$= P[aX^2 \le y]$$
$$= P[X^2 \le \tfrac{y}{a}]$$
$$= P[|X| \le \sqrt{\tfrac{y}{a}}] \quad , \quad y>0$$

$$\therefore F_Y(y) = P\left[-\sqrt{\tfrac{y}{a}} \le X \le \sqrt{\tfrac{y}{a}}\right]$$

$$\boxed{F_Y(y) = F_X\left(\sqrt{\tfrac{y}{a}}\right) - F_X\left(-\sqrt{y/a}\right)}$$

Let $u = \sqrt{y/a} = \left(\frac{y}{a}\right)^{1/2}$

$$\frac{du}{dy} = \frac{1}{2(ay)^{1/2}} = \frac{1}{2\sqrt{ay}}$$

and

$$f_Y(y) = \frac{d}{dy}\left\{F_X\left(\sqrt{\frac{y}{a}}\right) - F_X\left(-\sqrt{\frac{y}{a}}\right)\right\}$$

$$f_Y(y) = \frac{dF_X(u)}{du} \times \frac{du}{dy} + \frac{dF_X(-u)}{du} \cdot \frac{du}{dy}$$

$$f_Y(y) = \frac{1}{2\sqrt{ay}}\left\{\frac{dF_X(u)}{du} + \frac{dF_X(-u)}{du}\right\}$$

$$= \frac{1}{2\sqrt{ay}}\left\{f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)\right\}$$

$$\boxed{f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{a}}\right) + f_X\left(-\sqrt{y/a}\right)}{2\sqrt{ay}}, \quad \frac{y}{a} > 0}$$

If $f_X(x)$ is an even function, then $f_X(-x) = f_X(x)$ and $F_X(-x) = 1 - F_X(x)$. Thus we have

$$f_Y(y) = \frac{f_X\left(\sqrt{y/a}\right) + f_X\left(-\sqrt{\frac{y}{a}}\right)}{2\sqrt{ay}}$$

$$f_Y(y) = \frac{\cancel{2} f_X\left(\sqrt{y/a}\right)}{\cancel{2}\sqrt{ay}}$$

$$\boxed{\therefore\ f_Y(y) = \frac{f_X\left(\sqrt{\frac{y}{a}}\right)}{\sqrt{ay}}, \quad \frac{y}{a} > 0}$$

~~Example~~

## Example

Find the PDF of the random variable $Y = aX^2$, where $X$ is the standard normal random variable and $\Rightarrow a > 0$

Solution :- The PDF of $X$ is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} ; \quad -\infty < x < +\infty$$

which is an ~~event~~ even function. Thus from previous result

$$f_{Y}(y) = \frac{f_X(\sqrt{y/a})}{\sqrt{ay}}$$

$$\boxed{f_Y(y) = \frac{e^{-\frac{1}{2a}y}}{\sqrt{2\pi a y}} \quad \Rightarrow y > 0}$$

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## Expectation of a function of one Random Variable:-

Let X be a r.v. and g(X) be a real-valued function of X. The expected value of g(X) is ~~defin~~ defined by:

$$E[g(X)] = \begin{cases} \sum g(x)\, p_X(x) \;; & X \text{ discrete} \\ \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx \;; & X \text{ continuous.} \end{cases}$$

## Moments of linear function:-

Assume that $g(X) = aX + b$, where X is a contn random variable. Then

$$E[g(X)] = E[aX+b] = \int_{-\infty}^{+\infty} g(x)\, f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} (ax+b)\, f_X(x)\, dx$$

$$E[g(X)] = a\int_{-\infty}^{+\infty} x f_X(x)\, dx + b\int_{-\infty}^{+\infty} f_X(x)\, dx$$

$$\boxed{E[g(X)] = a\,E[X] + b}$$

↳ This means that the expected value of a linear function of a single random variable is the linear function obtained by replacing the random variable by its expectation. When X is discrete r.v., we replace the summation by integration and obtain the same result

The variance of $g(x)$ is given by

$$Var(g(x)) = Var(ax+b)$$

$$Var(g(x)) = E[(ax+b - E[ax+b])^2]$$

$$Var(g(x)) = E[(ax+b - E[ax]-b)^2]$$

$$= E[\{a(x-E[x])\}^2]$$

$$= a^2 E[(x-E[x])^2]$$

$$\boxed{Var(g(x)) = a^2 \sigma_x^2}$$

## Expected Value of a Conditional Expectation:

conditional expectation of a random variable $X$, given that an event $A$ has occurred, is given by:

$$E[X|A] = \begin{cases} \sum_k x_k \, p_{X|A}(x_k|A); & X \text{ discrete} \\ \int x f_{X|A}(x|A)\,dx; & X \text{ continuous.} \end{cases}$$

where the conditional PMF $p_{X|A}(x)$ and the conditional PDF $f_{X|A}(x|A)$ are defined as follows

$$p_{X|A}(x|A) = \frac{p_X(x)}{P[A]}$$

$$f_{X|A}(x|A) = \frac{f_X(x)}{P[A]}$$

where $x \in A$ and $P[A] > 0$ is the probability that event $A$ occurs. When $A$ is a random variable, say, $A = Y$, the conditional expected value, which for the case of ~~contra~~ continuous random variables is given by

$$E[E[X|Y]] = \int_{-\infty}^{+\infty} E[X|Y) f_Y(y)\, dy$$

$$E[E[X|Y]] = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} x\, f_{X|Y}(x|y)\, dx \right\} f_Y(y)\, dy$$

$$E[E[X|Y]] = \int_{-\infty}^{+\infty} x \left\{ \int_{-\infty}^{+\infty} f_{X|Y}(x|y)\, f_Y(y)\, dy \right\} dx$$

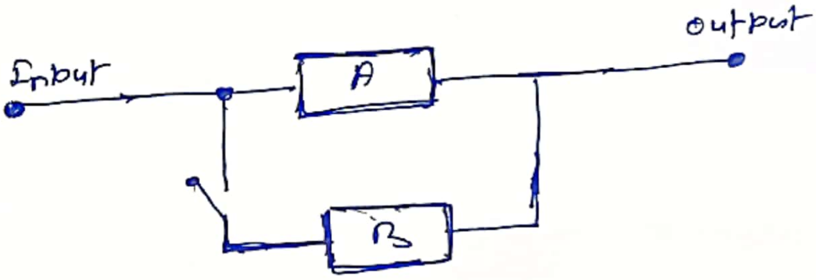
$$= \int_{-\infty}^{+\infty} x \left\{ \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dy \right\} dx$$

$$= \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

$$\boxed{E[E[X|Y]] = E[X]}$$

## Sum of Independent R.V.:

Consider two independent continuous random variable X and Y. We are interested in computing the CDF and PDF of their sum $g(X,Y) = U = X+Y$. The random variable U can be used to model the reliability of system with stand-by connections.




CDF of U can be obtained as follows:

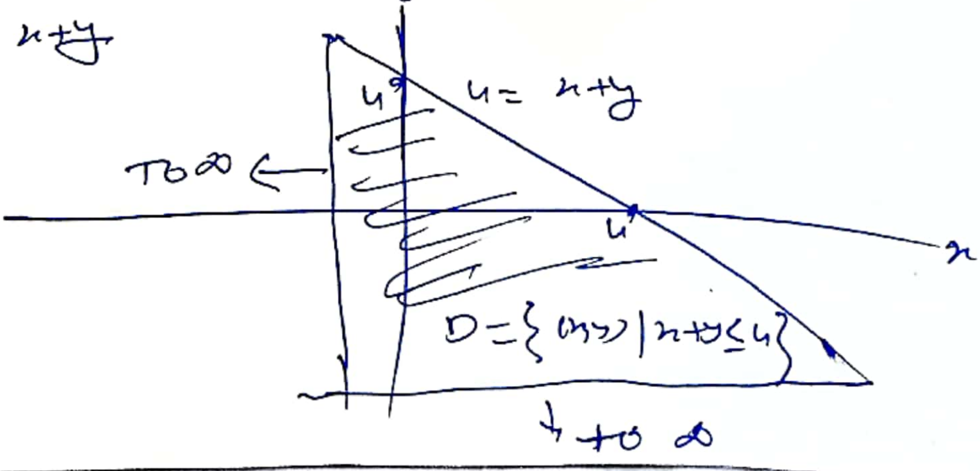
$$\cancel{F_U(u)} =$$

$$F_U(u) = P[U \le u]$$

$$= P[X+Y \le u]$$

$$= \iint_D f_{XY}(x,y)\, dx\, dy$$

where D is the set $D = \{(x,y) \mid x+y \le u\}$ which is the area to the left of the line $u = x+y$.



Thus $F_U(u) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{u-y} f_{XY}(x,y)\,dx\,dy$

$$F_U(u) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{u-y} f_X(x) f_Y(y)\,dx\,dy$$

$$= \int_{y=-\infty}^{+\infty} \left\{ \int_{x=-\infty}^{u-y} f_X(x)\,dx \right\} f_Y(y)\,dy$$

$$\boxed{F_U(u) = \int_{y=-\infty}^{+\infty} F_X(u-y) f_Y(y)\,dy}$$

The PDF of U is obtain by differentiating the CDF, as follows:

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} \int_{y=-\infty}^{+\infty} F_X(u-y) f_Y(y)\,dy$$

$$f_U(u) = \int_{y=-\infty}^{+\infty} \frac{d}{du} F_X(u-y) f_Y(y)\,dy$$

$$\boxed{f_U(u) = \int_{y=-\infty}^{+\infty} f_X(u-y) f_Y(y)\,dy}$$

[ Assume that order of integration can be change ]

The PDF of U is obtained by differentiating the CDF, as follows:

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} \int_{y=-\infty}^{+\infty} F_X(u-y) f_Y(y)\,dy$$

$$= \int_{y=-\infty}^{+\infty} \frac{d}{du} F_X(u-y) f_Y(y)\,dy$$

$$f_U(u) = \underbrace{\int_{y=-\infty}^{+\infty} f_X(u-y)\, f_Y(y)\, dy}$$

Convolution integral.

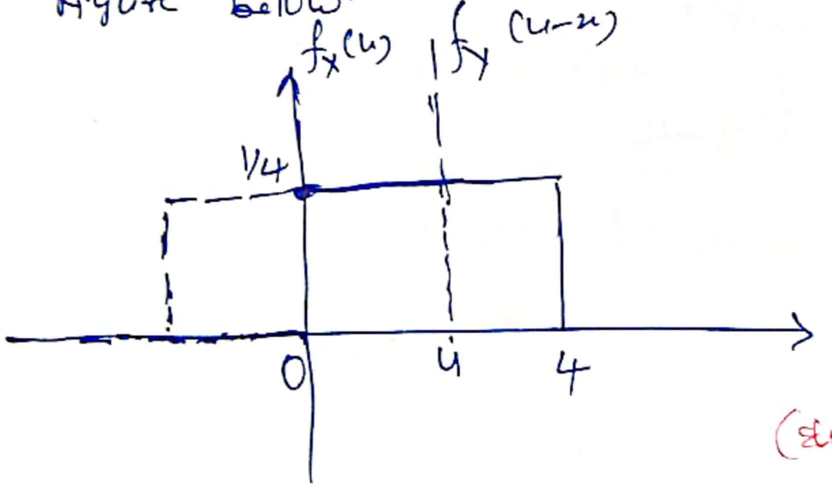
$$\boxed{\therefore f_U(u) = f_X(u) \otimes f_Y(u)}$$

Ex :-

Find the PDF of the sum of X and Y if the two random variables are independent r.v. with the common PDF:

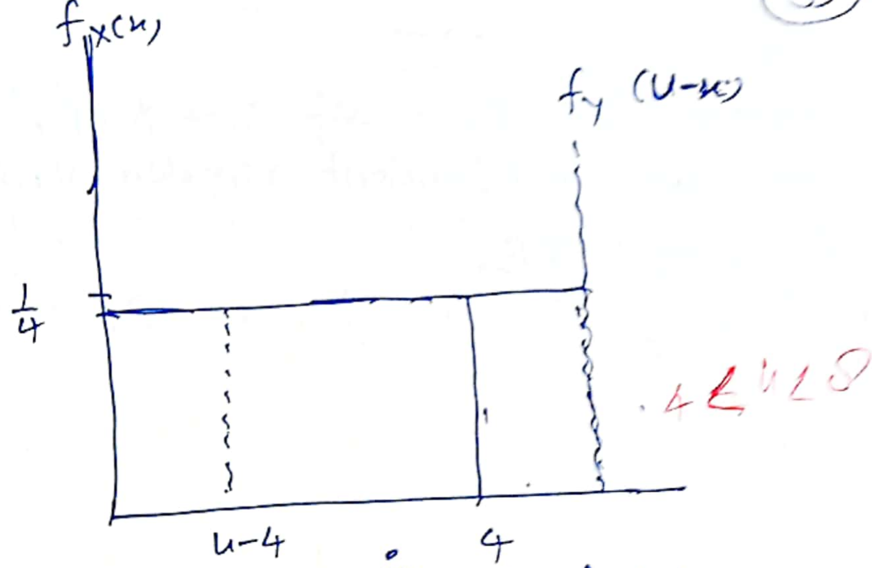
$$f_X(u) = f_Y(u) = \begin{cases} \frac{1}{4}, & 0<u<4 \\ 0, & \text{otherwise} \end{cases}$$

Solution :- The limits of integration of ~~the~~ ~~th~~ the PDF of $U = X+Y$ can be computed with aid figure below.




(1)

(for $0 \le u \le 4$)




Here $f_Y(u-x)$ is shown in dashed line.

$$f_U(u) = \int_{y=-\infty}^{+\infty} f_X(u-y)\, f_Y(y)\, dy$$

$$= \int_{y=0}^{u} \frac{1}{16}\, dy = \frac{u}{16}$$

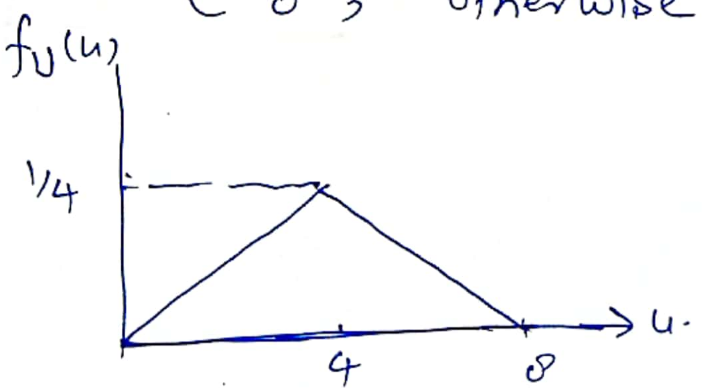
$$\boxed{\therefore f_U(u) = \frac{u}{16}}$$

For $4 \le u < 8$

$$f_U(u) = \frac{1}{16} \int_{u-4}^{4} dy = \frac{8-u}{16}.$$

Thus

$$f_U(u) = \begin{cases} \frac{u}{16}; & 0 \le u \le 4 \\ \frac{8-u}{16}, & 4 \le u < 8 \\ 0, & \text{otherwise} \end{cases}$$



## Example :

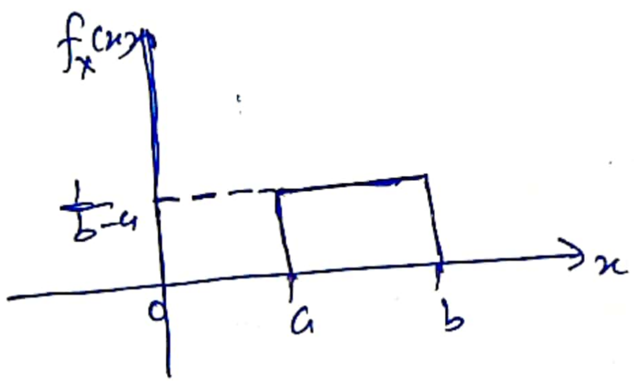
obtain the PDF of $Z = X+Y$, where $X$ and $Y$ are two independent random variables with the following PDFs :

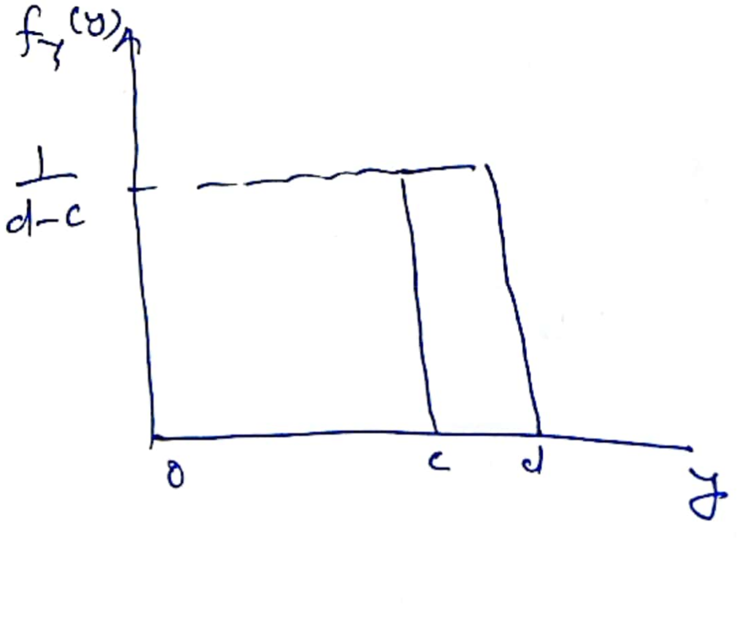
$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

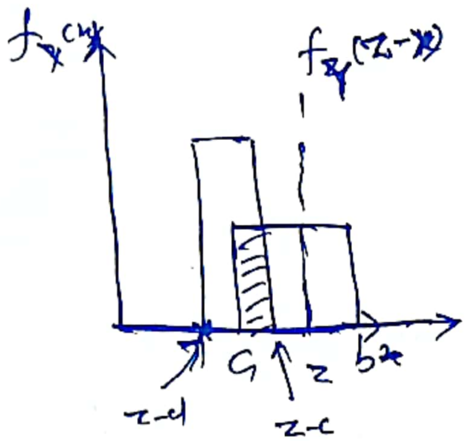
$$f_Y(y) = \begin{cases} \frac{1}{d-c}; & c < y < d, \\ & d-c < b-a \\ 0, & \text{otherwise.} \end{cases}$$

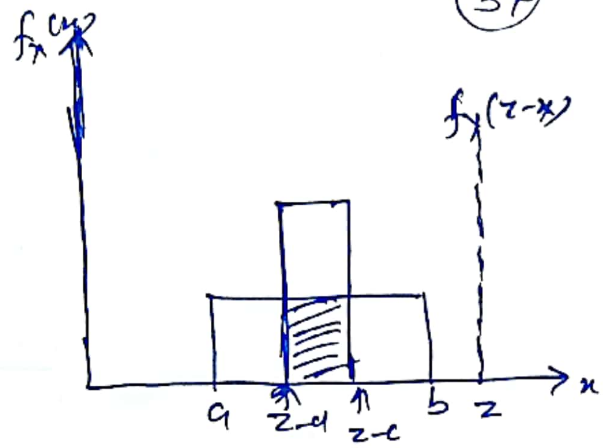
Solution :-

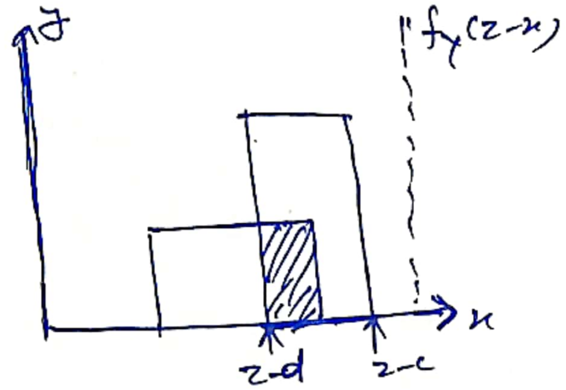
Graph of the ~~the~~ two pdf are as follows:






- Convolution of the PDFs for different values of Z.

When $z < a+c$, $f_Z(z) = 0$ because there is no overlap of the curve $f_X(x)$ and $f_Y(z-x)$

- $a+c \le z \le a+d$

$$f_Z(z) = \frac{1}{(b-a)(d-c)} \int_a^{z-c} dy$$

$$\boxed{f_Z(z) = \frac{z-c-a}{(b-a)(d-c)}}$$

- $a+d \le z \le b+c$

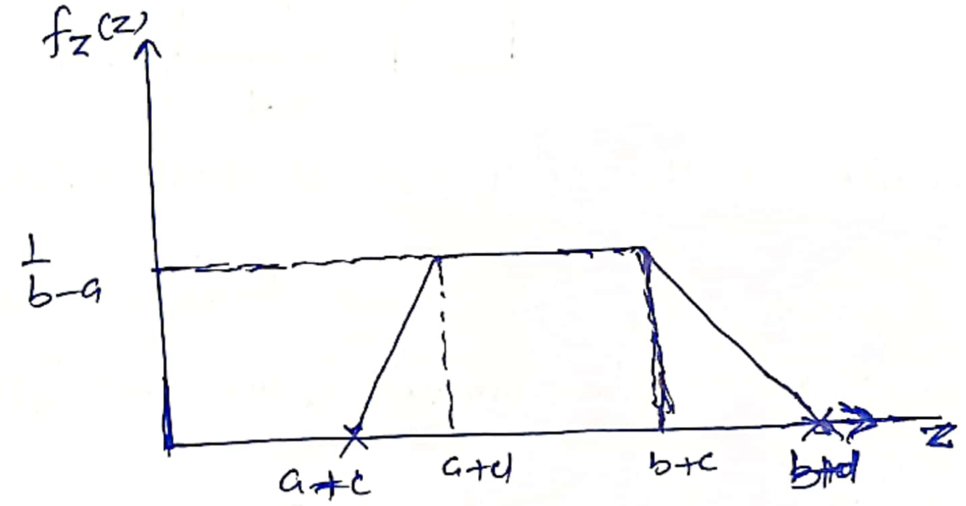
$$f_Z(z) = \frac{1}{(b-a)(d-c)} \int_{z-d}^{z-c} dy = \frac{1}{b-a}$$

- When $b+c \le z \le b+d$

$$f_Z(z) = \frac{1}{(b-a)(d-c)} \int_{z-d}^{b} dy = \frac{b+d-z}{(b-a)(d-c)}$$

Finally, when $z > b+d$, $f_Z(z) = 0$. Thus, the PDF of $Z$ is given by

$$f_Z(z) = \begin{cases} 0 \; ; & z < a+c \\ \dfrac{z-a-c}{(b-a)(d-c)} \; ; & a+c \le z \le a+d \\ \dfrac{1}{b-a} \; ; & a+d \le z \le b+c \\ \dfrac{b+d-z}{(b-a)(d-c)} \; ; & b+c \le z \le b+d \\ 0 \; , & z > b+d. \end{cases}$$




PDF of $Z = X + Y$

**Ex :-** The time $X$ between consecutive snowstorms in winter is a random variable with the PDF

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} \; ; & x \ge 0 \\ 0 \; ; & \text{otherwise} \end{cases}$$

Assume it has not showed up ~~now~~ until now. What is the PDF of the time $W$ until the second ~~sn~~ snowstorm?

Solution :—

X : R.v. denoting the first snow storm from the reference time.

Y : R.V. that denotes the ~~tim~~ time between the first snowstorm and second snowstorm.

— Assume that time between snowstorms are ~~indeben.~~ independent, then X and Y are independent and identically distributed random variable. That is, the PDF of Y is given by :

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Thus $U = X + Y$, and the PDF of U is given by

$$f_U(u) = \int_{x=0}^{+\infty} f_X(x) f_Y(u-x)\, dx$$

→ Lower limit is zero since $f_X(x) = 0$, ~~for~~ $x < 0$

→ Upper limit :— $f_Y(y) = 0$, when $y < 0$, $f_Y(u-x) = 0$ when ~~u x~~ $u - x < 0$ (i.e. when $x > u$). Thus, the range of ~~integration~~ interest in ~~the~~ the integration is $0 \leq x \leq u$., and we obtain:

$$f_U(u) = \int_{x=0}^{u} f_X(x) f_Y(u-x)\, dx$$

$$= \int_{x=0}^{u} \{\lambda e^{-\lambda x}\} \{\lambda e^{-\lambda(u-x)}\}\, dx$$

$$= \lambda^2 e^{-\lambda u} \int_{x=0}^{u} dx$$

$$\boxed{f_U(u) = \lambda^2 u e^{-\lambda u} \; ; \quad u \geq 0}$$

This is the Erlang-2 distribution.