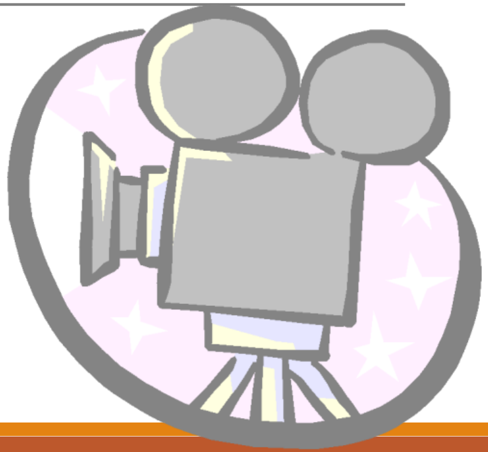
Image Processing

CS-317/CS-341



Outline

- Histogram Processing
    - Histogram Equalization
    - Histogram Matching

Histogram Processing

Histogram of a digital image with gray levels in the range [0,L-1] is a discrete function

$$h(r_k) = n_k$$

Where

- r_k : the k^{th} gray level
- n_k : the number of pixels in the image having gray level r_k
- $h(r_k)$: histogram of a digital image with gray levels r_k

Normalized Histogram

dividing each of its component by the total number of pixels in the image, MN

$$p(r_k) = n_k / MN$$

Where, M, N are the no. of rows and columns of the image.

For k = 0,1,…,L-1

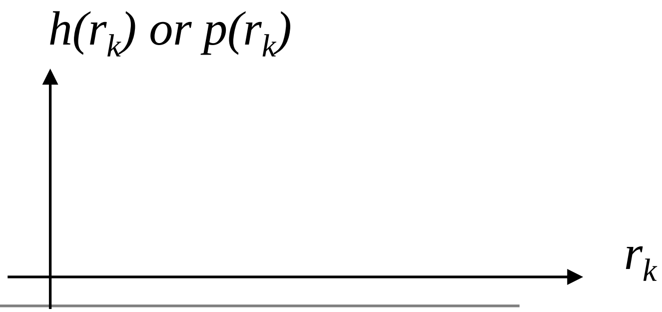
$p(r_k)$ gives an estimate of the probability of occurrence of gray level r_k

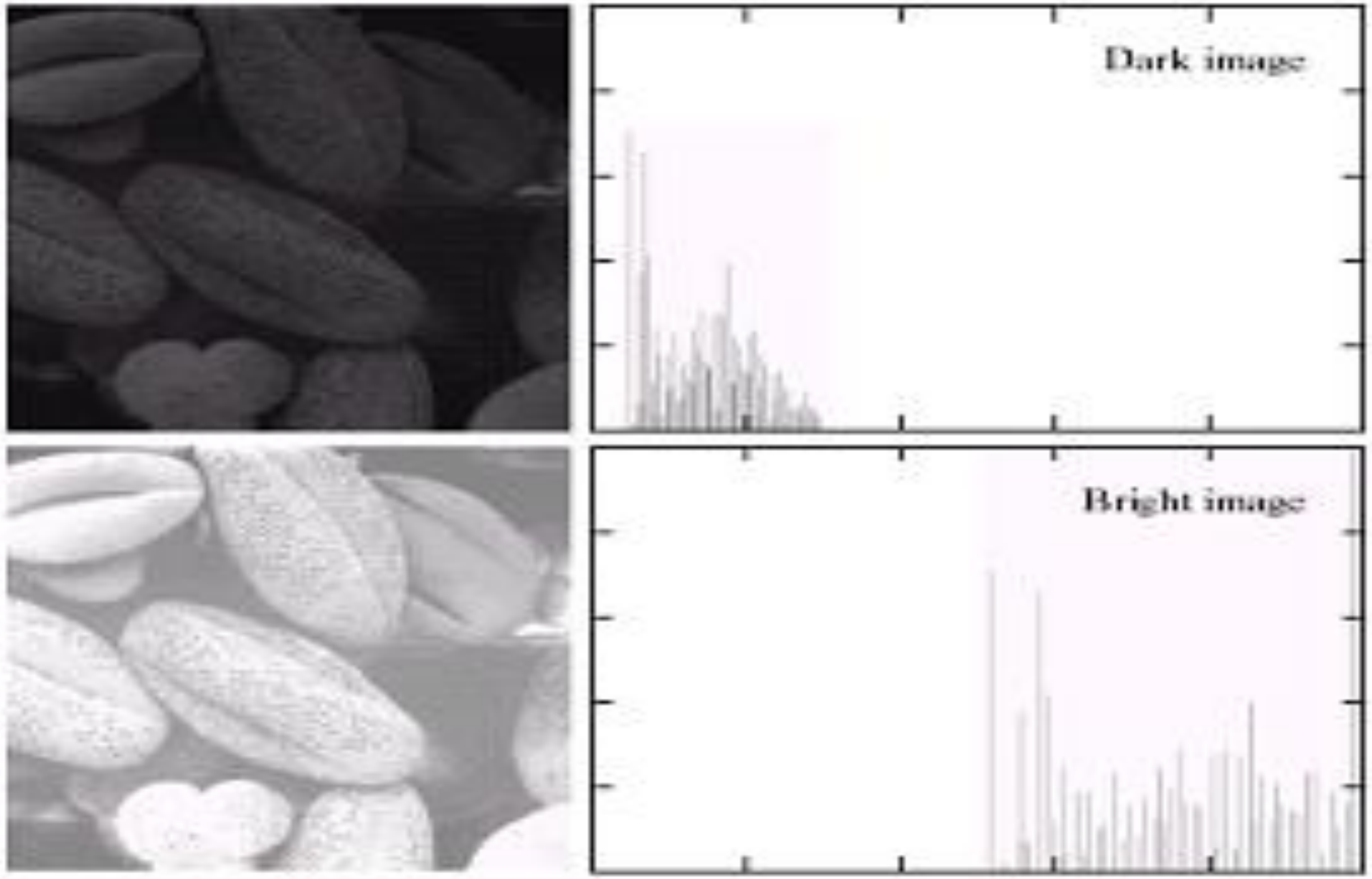
The sum of all components of a normalized histogram is equal to 1

Histogram Processing

- Basic for numerous spatial domain processing techniques
- Used effectively for image enhancement
- Information inherent in histograms also is useful in image compression and segmentation

Example





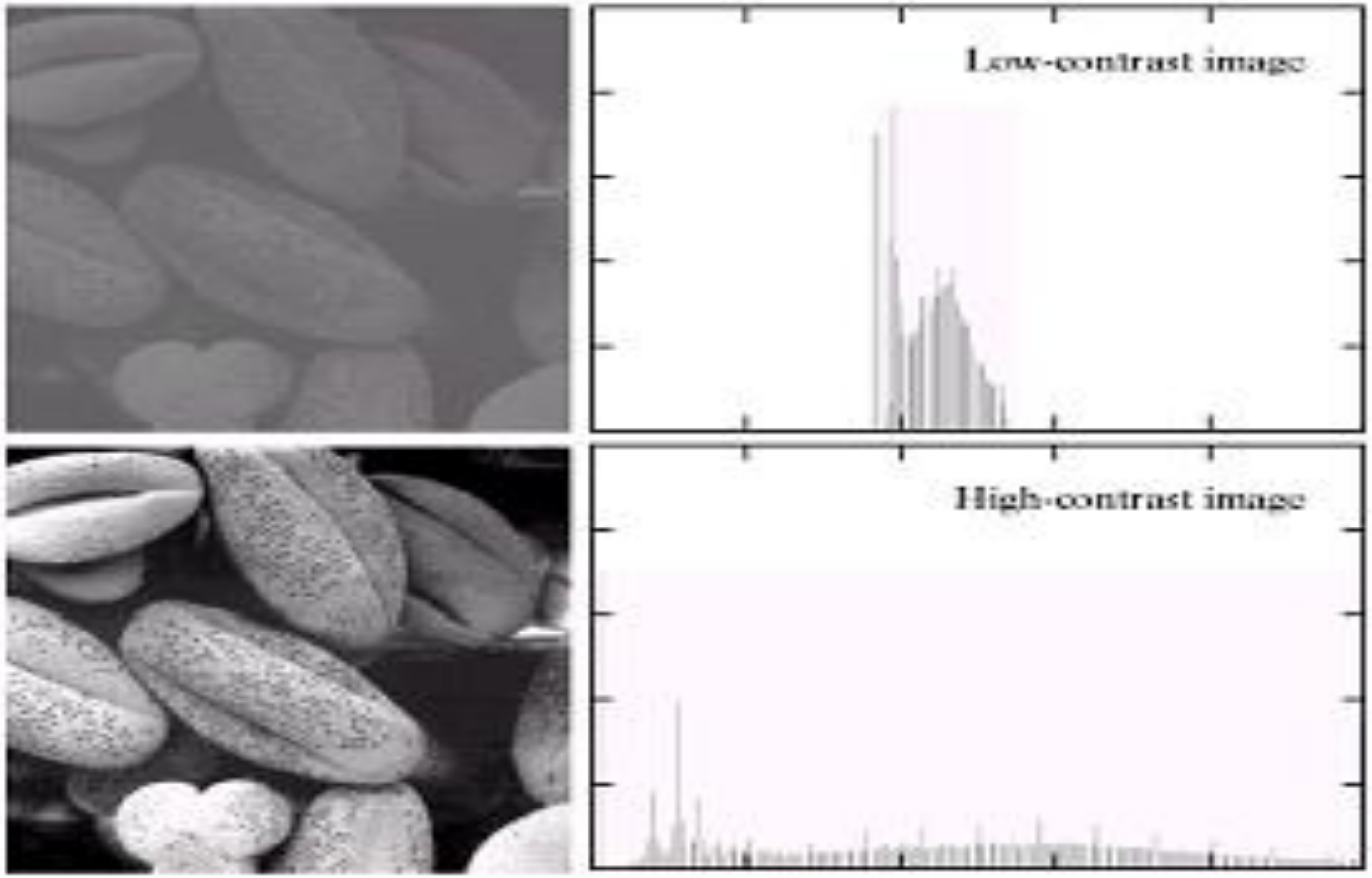
Dark image

Components of histogram are concentrated on the low side of the gray scale.

Bright image

Components of histogram are concentrated on the high side of the gray scale.

Example



Low-contrast image

histogram is narrow and centered toward the middle of the gray scale

High-contrast image

histogram covers broad range of the gray scale and the distribution of pixels is not too far from uniform, with very few vertical lines being much higher than the others

Histogram Equalization

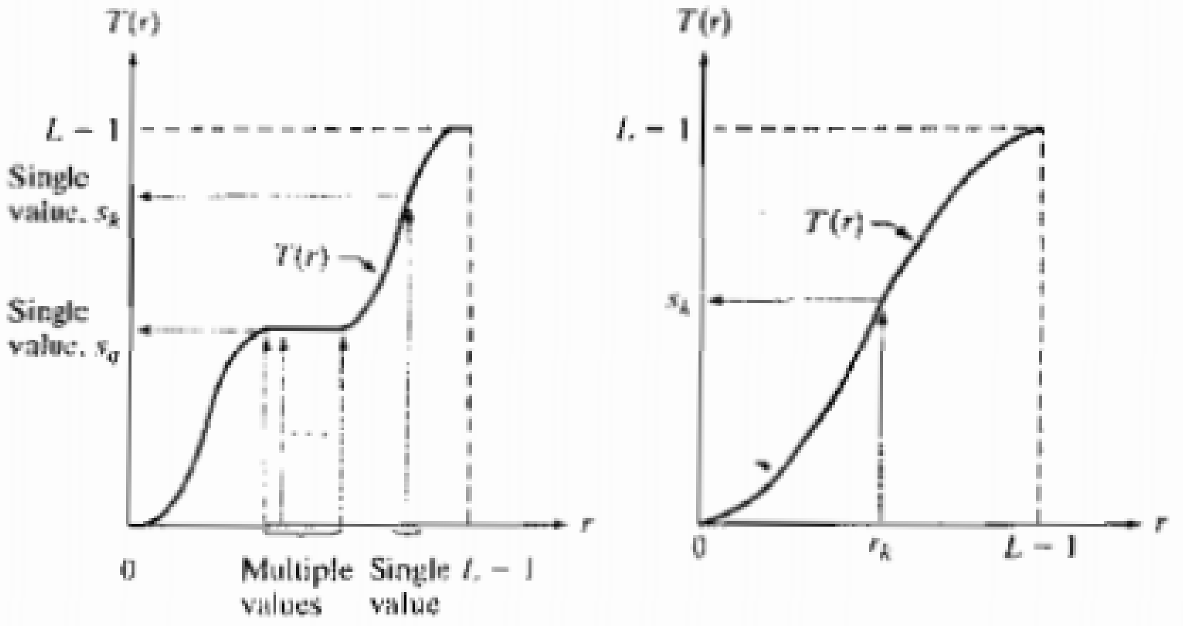
As the low-contrast image's histogram is narrow and centered toward the middle of the gray scale, if we distribute the histogram to a wider range, the quality of the image will be improved.

We can do it by adjusting the probability density function of the original histogram of the image so that the probability spread equally.

Histogram transformation

a b

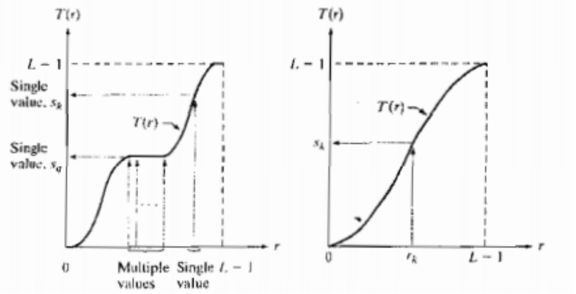
FIGURE 3.17
(a) Monotonically increasing function, showing how multiple values can map to a single value.
(b) Strictly monotonically increasing function. This is a one-to-one mapping, both ways.



Histogram transformation

a b

FIGURE 3.17
(a) Monotonically increasing function, showing how multiple values can map to a single value.
(b) Strictly monotonically increasing function. This is a one-to-one mapping, both ways.



Consider the transformation function of the form

$$s = T(r)$$

Where $0 \leq r \leq L-1$

T(r) satisfies

(a) T(r) is a monotonically increasing function in the interval $0 \leq r \leq L-1$.

(b) $0 \leq T(r) \leq L-1$ for $0 \leq r \leq L-1$.

In some formulations to be discussed later, we use inverse

$r = T^{-1}(s)$ for $0 \leq s \leq L-1$

In which condition (a) is changed to (a')

(a') T(r) is strictly monotonically increasingly in the interval $0 \leq r \leq L- 1$

Histogram Equalization

The intensity levels in an image may be viewed as a random variables in the interval [0-L-1]. A fundamental descriptor of a random variable is its probability density function (PDF). Let $p_r(r)$ and $p_s(s)$ denote the PDFs of r and s respectively.

If $p_r(r)$ and T(r) are known anf T(r) is continuous and differentiable over the range of values of interest then the PDF of the transformed variable s can be obtained using

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right| \qquad \text{……………….(1)}$$

Histogram Equalization

A transformation function of particular importance in image processing has the form

$$s = T(r) = (L-1)\int_0^r p_r(w)\,dw$$

where w is a dummy variable of integration

Note: **T(r)** depends on $\mathbf{p_r(r)}$

Example

To find $p_s(s)$ corresponding to the transformation, we use eq (1) $p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$

$$\frac{ds}{dr} = \frac{dT(r)}{dr}$$

$$\frac{ds}{dr} = (L-1)\frac{d}{dr}\left[\int_0^r p_r(w)dw\right]$$

$$\frac{ds}{dr} = (L-1)\,p_r(r)$$

Substituting this result in eq 1

Example

Substituting this result in eq 1

$$p_s(s) = p_r(r)\left|\frac{dr}{ds}\right|$$

$$p_s(s) = p_r(r)\left|\frac{1}{(L-1)p_r(r)}\right|$$

$$p_s(s) = \frac{1}{(L-1)}, 0 \le s \le L-1$$

Example

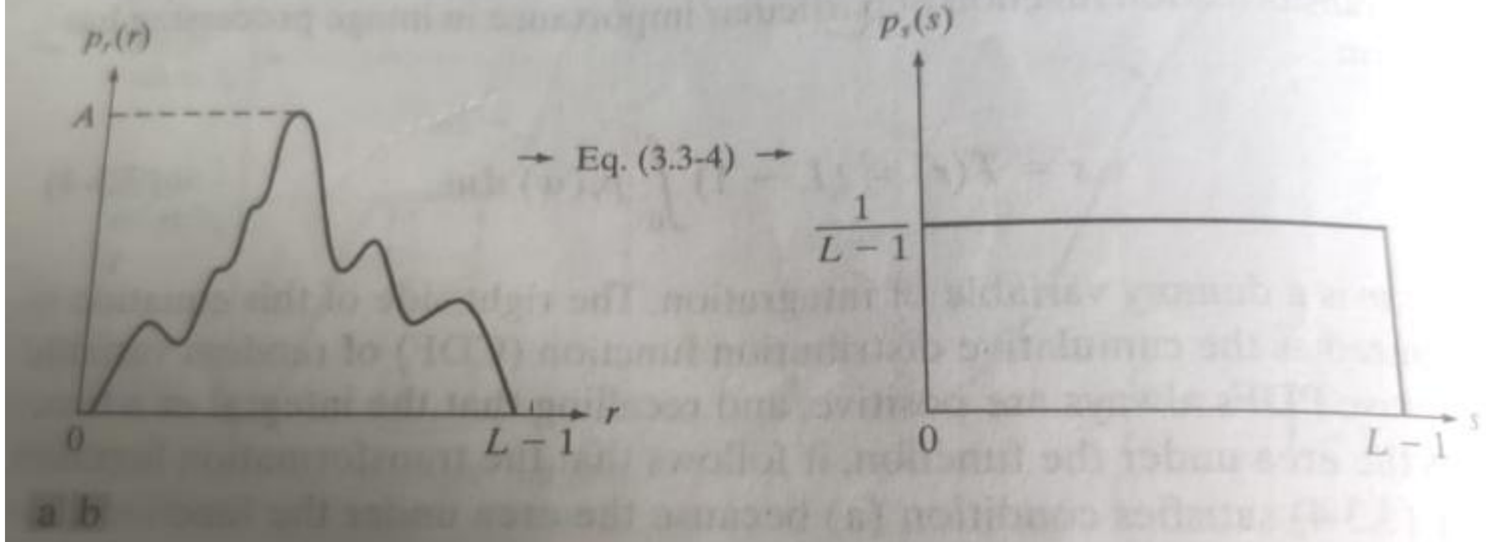


FIGURE 3.18 (a) An arbitrary PDF. (b) Result of applying the transformation in Eq. (3.3-4) to all intensity levels, r. The resulting intensities, s, have a uniform PDF, independently of the form of the PDF of the r's.

* Image is taken from **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

Histogram Equalization

The discrete form of the transformation is

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_k), k = 0,1,2......L-1$$

$$s_k = \frac{(L-1)}{MN}\sum_{j=0}^{k} n_k$$

Example

■ Before continuing, it will be helpful to work through a simple example. Suppose that a 3-bit image ($L = 8$) of size 64×64 pixels ($MN = 4096$) has the intensity distribution shown in Table 3.1, where the intensity levels are integers in the range $[0, L-1] = [0, 7]$.

The histogram of our hypothetical image is sketched in Fig. 3.19(a). Values of the histogram equalization transformation function are obtained using Eq. (3.3-8). For instance,

$$s_0 = T(r_0) = 7\sum_{j=0}^{0} p_r(r_j) = 7p_r(r_0) = 1.33$$

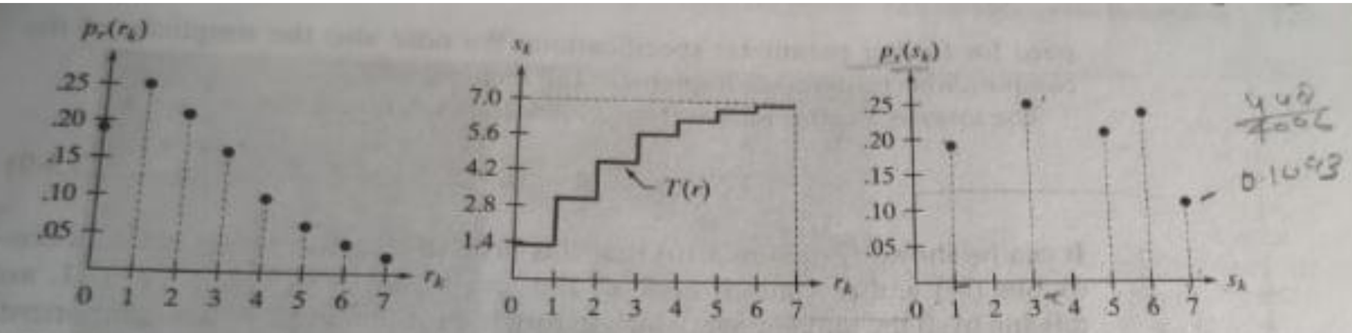
Similarly,

$$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7p_r(r_0) + 7p_r(r_1) = 3.08$$

and $s_2 = 4.55$, $s_3 = 5.67$, $s_4 = 6.23$, $s_5 = 6.65$, $s_6 = 6.86$, $s_7 = 7.00$. This transformation function has the staircase shape shown in Fig. 3.19(b).

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

Example



a b c

FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

At this point, the s values still have fractions because they were generated by summing probability values, so we round them to the nearest integer:

$s_0 = 1.33 \rightarrow 1$ $\quad s_4 = 6.23 \rightarrow 6$

$s_1 = 3.08 \rightarrow 3$ $\quad s_5 = 6.65 \rightarrow 7$

$s_2 = 4.55 \rightarrow 5$ $\quad s_6 = 6.86 \rightarrow 7$

$s_3 = 5.67 \rightarrow 6$ $\quad s_7 = 7.00 \rightarrow 7$

6 – Pixel

r_k	n_k	$p_r(r_k) = n_k/MN$
$r_0 = 0$	790	0.19
$r_1 = 1$	1023	0.25
$r_2 = 2$	850	0.21
$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

245+122+81=448 pixels with a value 7
In the histogram equalized image

Histogram Equalization

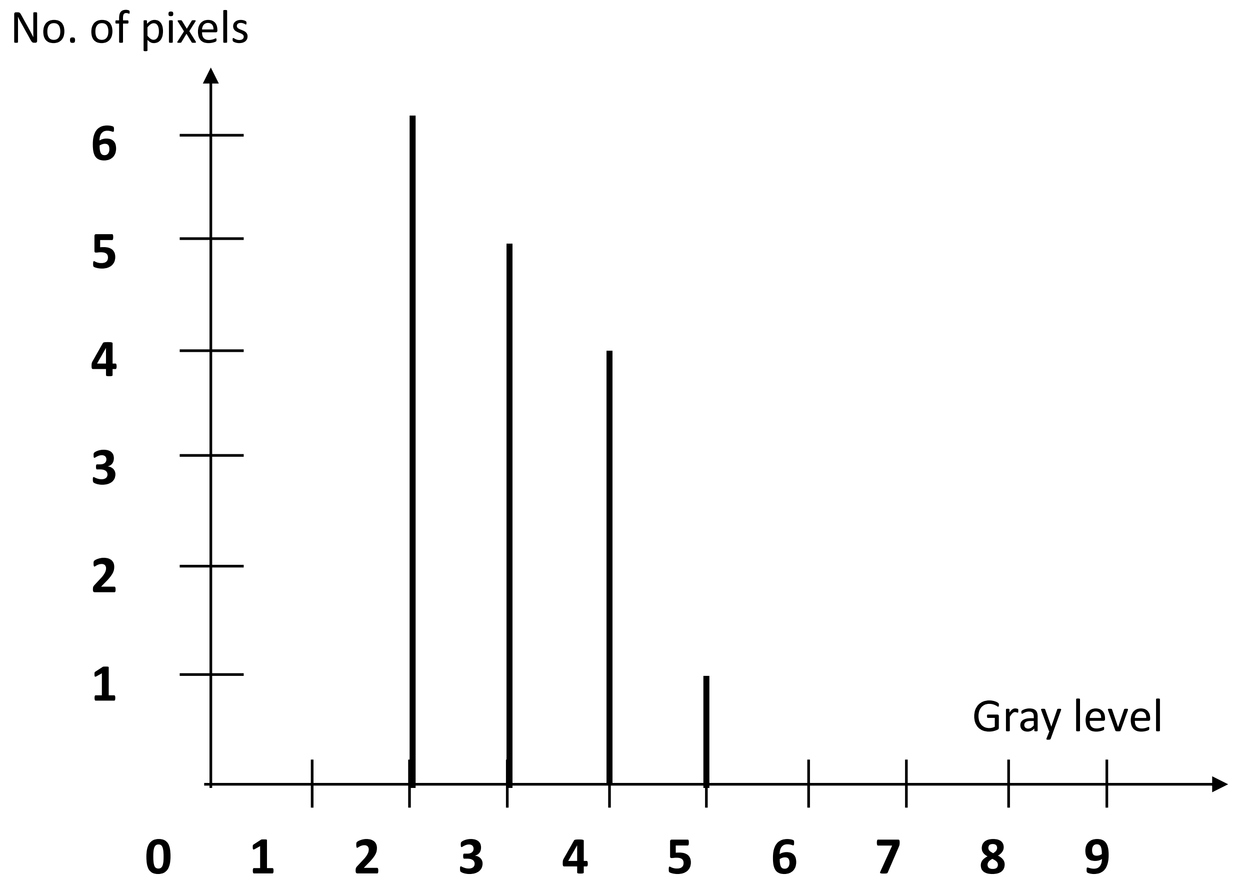
Thus, an output image is obtained by mapping each pixel with level r_k in the input image into a corresponding pixel with level s_k in the output image.

Example

2	3	3	2
4	2	4	3
3	2	3	5
2	4	2	4

4x4 image

Gray scale = [0,9]



histogram

Gray Level(j)	0	1	2	3	4	5	6	7	8	9
No. of pixels	0	0	6	5	4	1	0	0	0	0
$\sum_{j=0}^{k} n_j$	0	0	6	11	15	16	16	16	16	16
$s = \sum_{j=0}^{k} \frac{n_j}{MN}$	0	0	6 / 16	11 / 16	15 / 16	16 / 16	16 / 16	16 / 16	16 / 16	16 / 16
s x 9	0	0	3.3 ≈3	6.1 ≈6	8.4 ≈8	9	9	9	9	9

Example

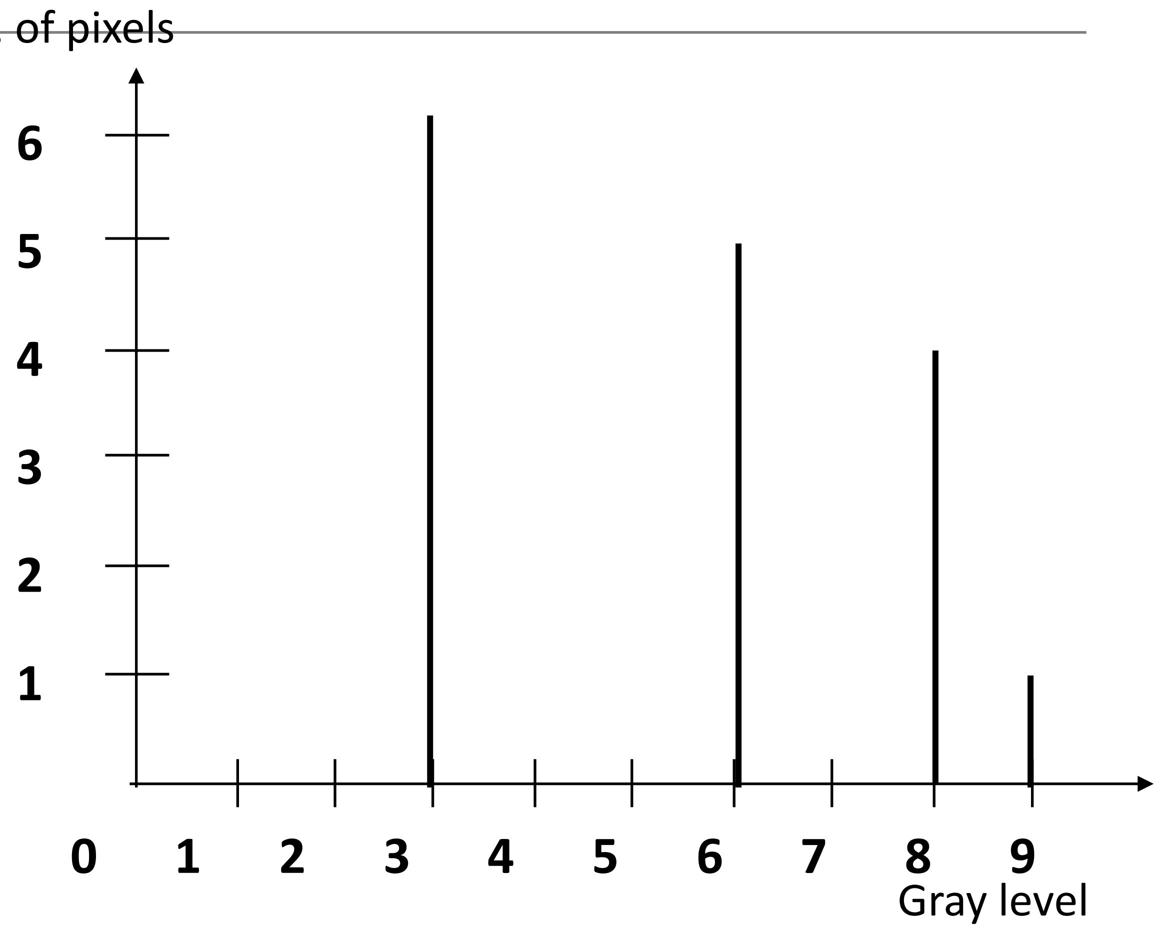
2	3	3	2
4	2	4	3
3	2	3	5
2	4	2	4

Input image

3	6	6	3
8	3	8	6
6	3	6	9
3	8	3	8

Output image

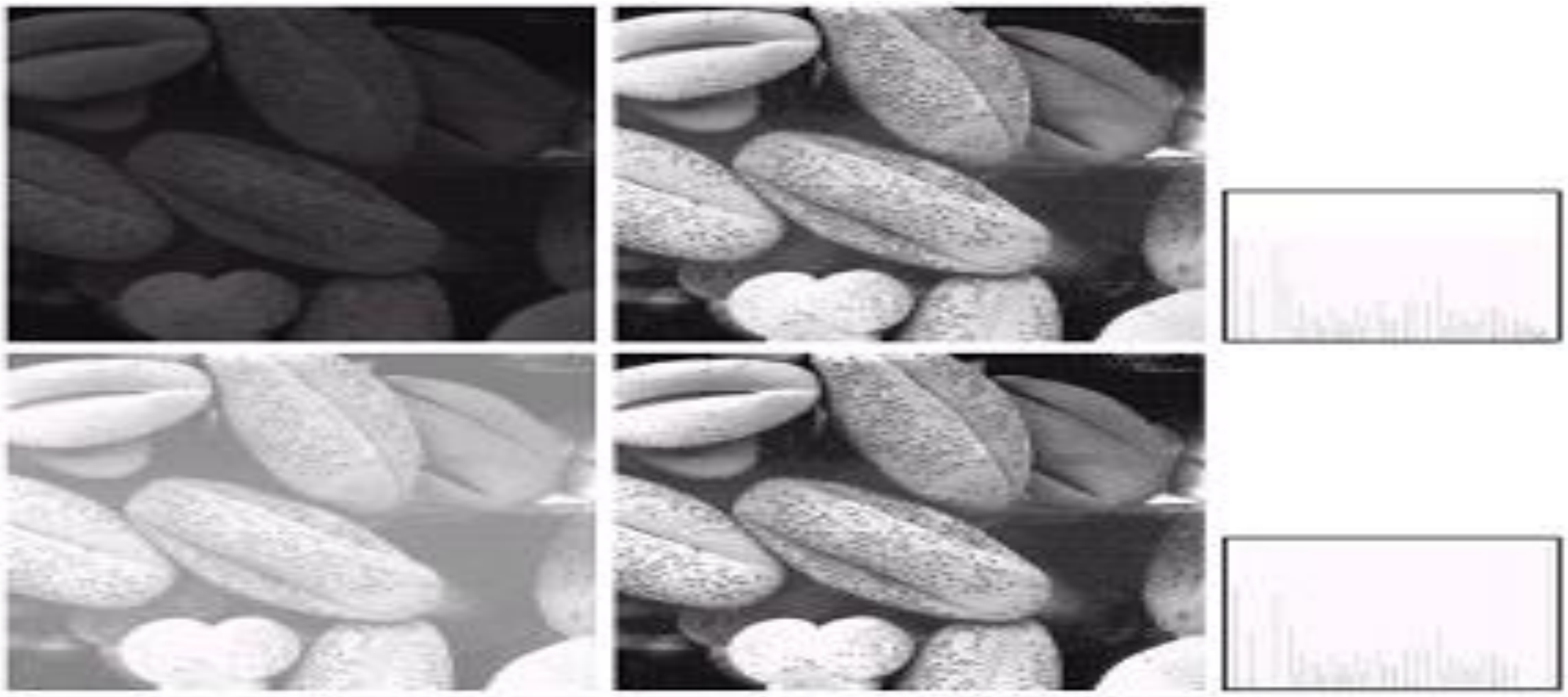
Gray scale = [0,9]



Histogram equalization

Example

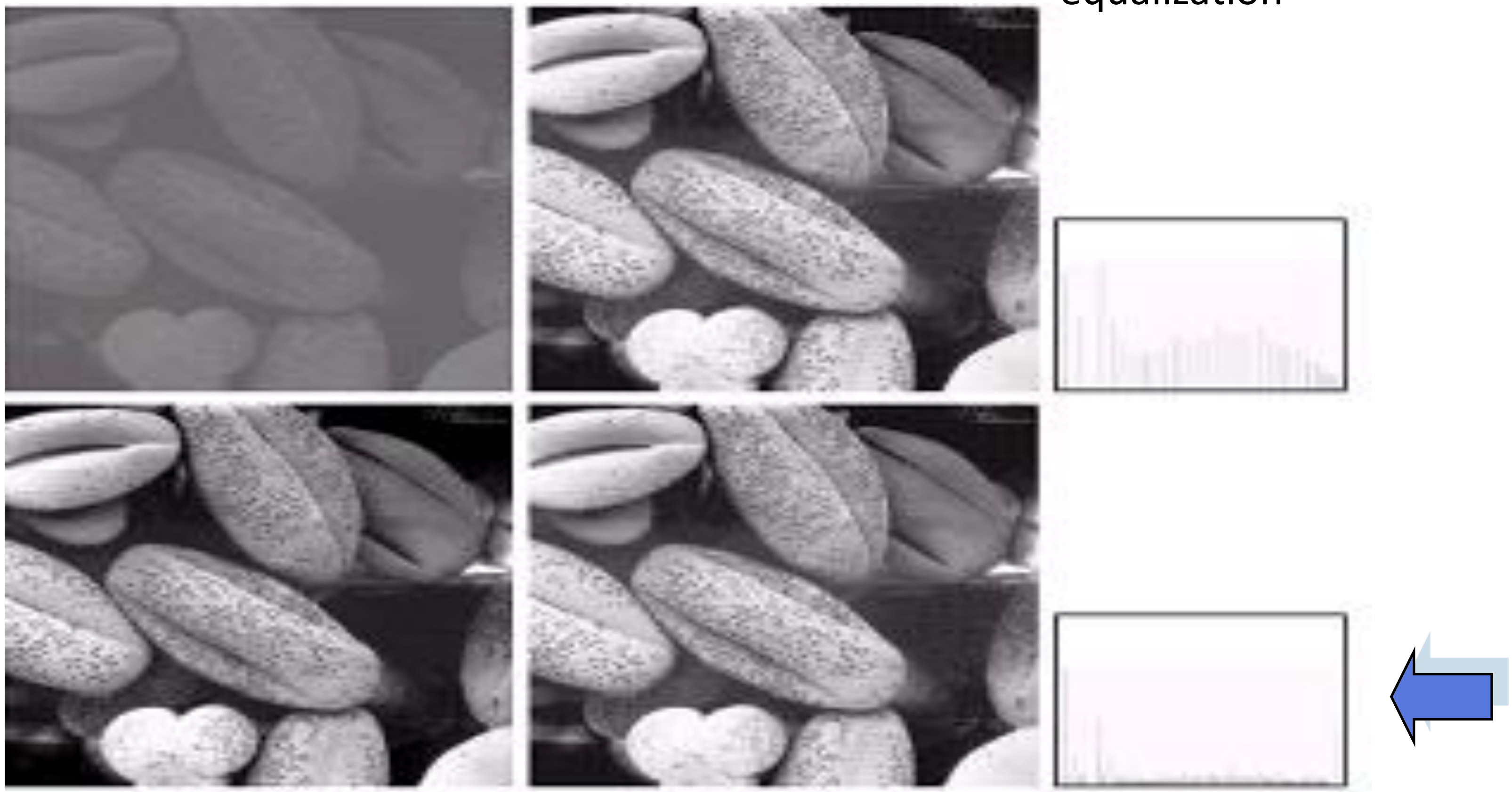
before after Histogram equalization



* Image is taken from **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

Example

before after Histogram equalization



The quality is not improved much because the original image already has a broaden gray-level scale

* Image is taken from **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

Histogram Matching (Specification)

Histogram equalization has a disadvantage which is that it can generate only one type of output image.

With Histogram Specification, we can specify the shape of the histogram that we wish the output image to have.

It doesn’t have to be a uniform histogram

Consider the continuous domain

Let $p_r(r)$ denote continuous probability density function of gray-level of input image, r

Let $p_z(z)$ denote desired (specified) continuous probability density function of gray-level of output image, z

Let s be a random variable with the property

$$s = T(r) = \int_0^r p_r(w)\,dw$$

Histogram equalization

Where w is a dummy variable of integration

Next, we define a random variable z with the property

$$G(z) = \int_0^z p_z(t)\,dt = s$$

Histogram equalization

Where t is a dummy variable of integration

thus

$$\boldsymbol{s = T(r) = G(z)}$$

Therefore, z must satisfy the condition

$$\boldsymbol{z = G^{-1}(s) = G^{-1}[T(r)]}$$

Assume G^{-1} exists and satisfies the condition (a’) and (b)

We can map an input gray level r to output gray level z

Procedure Conclusion

1. Obtain the transformation function T(r) by calculating the histogram equalization of the input image

$$s = T(r) = \int_0^r p_r(w)\,dw$$

2. Obtain the transformation function G(z) by calculating histogram equalization of the desired density function

$$G(z) = \int_0^z p_z(t)\,dt = s$$

Procedure Conclusion

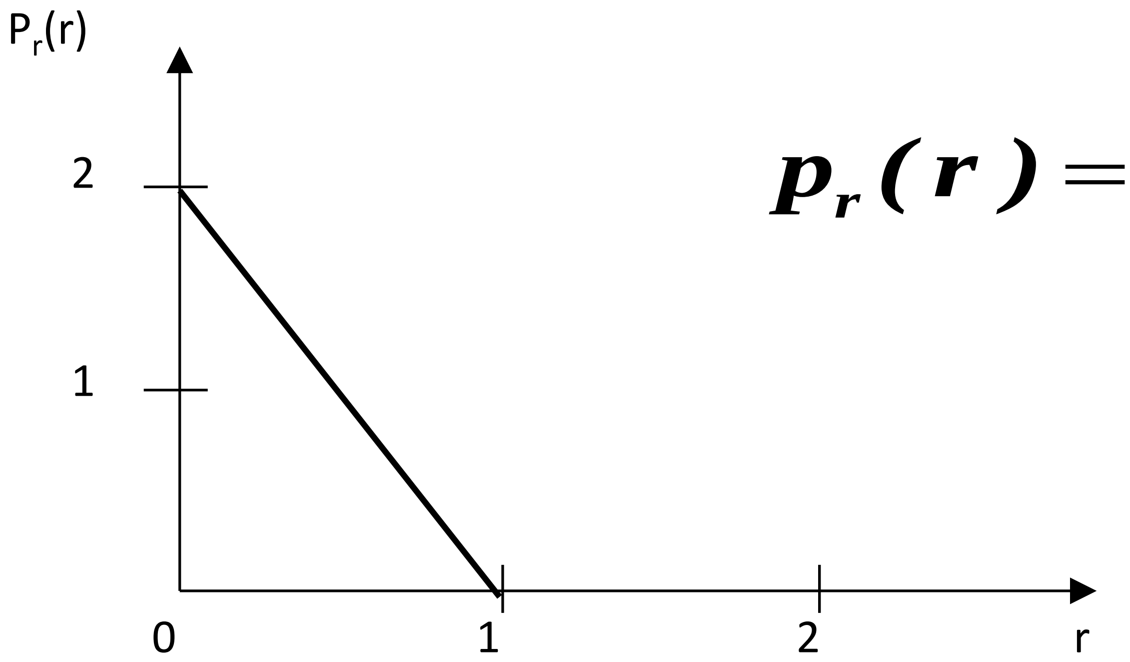
3. Obtain the inversed transformation function G^{-1}

$$z = G^{-1}(s) = G^{-1}[T(r)]$$

4. Obtain the output image by applying the processed gray-level from the inversed transformation function to all the pixels in the input image

Example

Assume an image has a gray level probability density function $p_r(r)$ as shown.

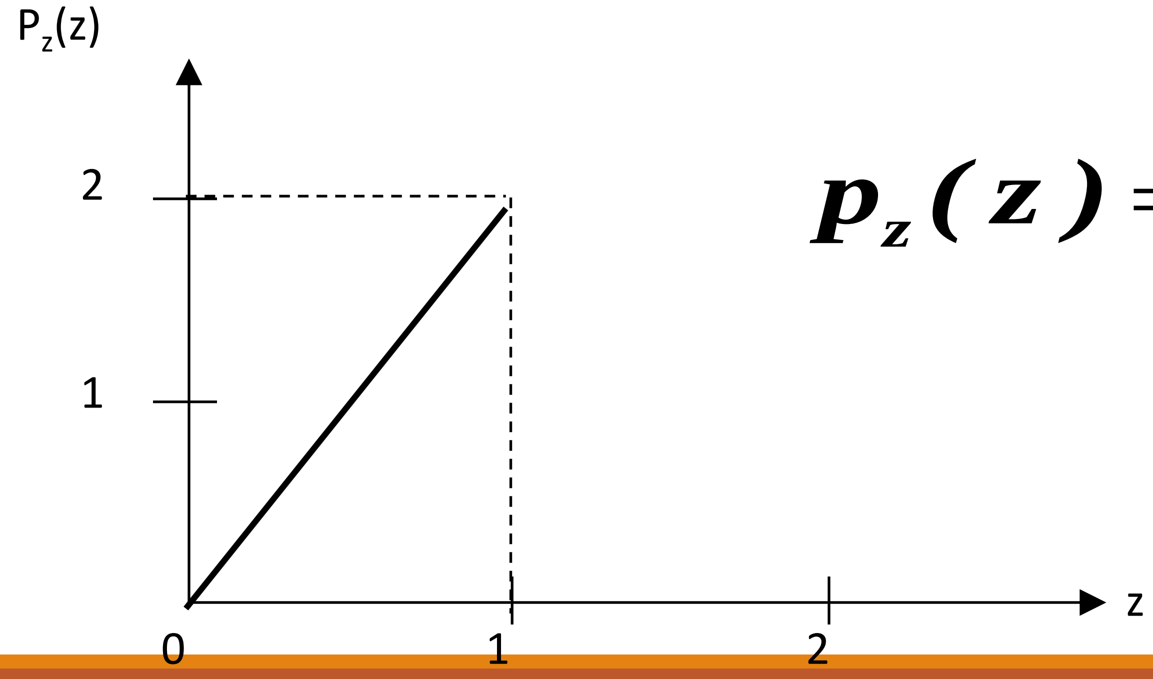


$$p_r(r) = \begin{cases} -2r + 2 & ;0 \le r \le 1 \\ 0 & ;\text{elsewhere} \end{cases}$$

$$\int_0^r p_r(w)\,dw = 1$$

Example

We would like to apply the histogram specification with the desired probability density function $p_z(z)$ as shown.

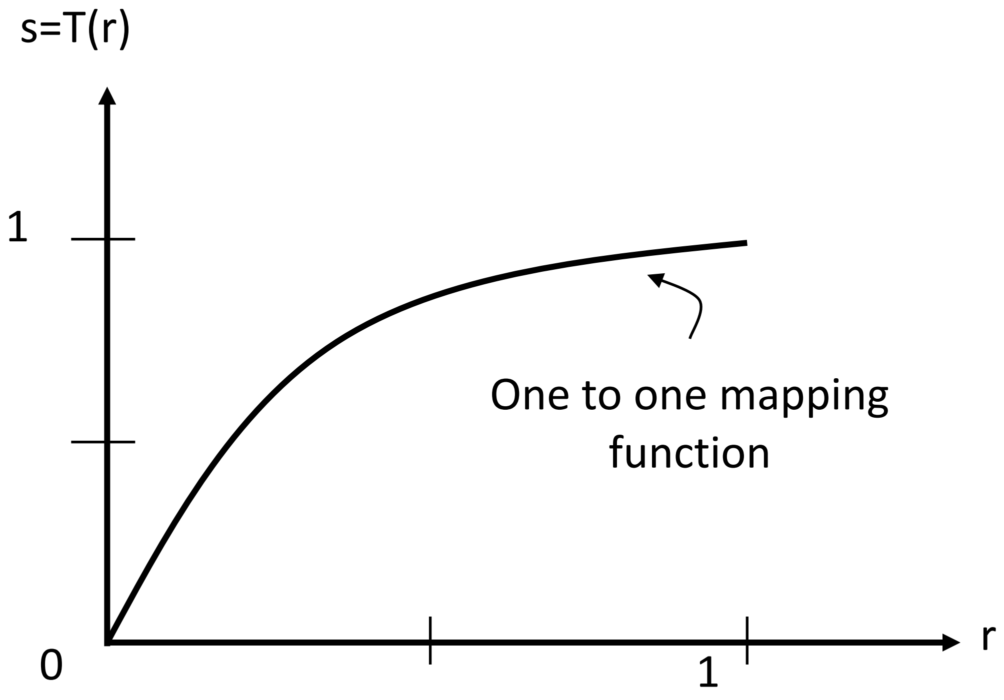


$$p_z(z) = \begin{cases} 2z & ; 0 \le z \le 1 \\ 0 & ; \text{elsewhere} \end{cases}$$

$$\int_0^z p_z(w)\,dw = 1$$

Step 1:

Obtain the transformation function T(r)



$$s = T(r) = \int_0^r p_r(w)dw$$

$$= \int_0^r (-2w + 2)dw$$

$$= -w^2 + 2w\Big|_0^r$$

$$= -r^2 + 2r$$

Step 2:

Obtain the transformation function G(z)

$$G(z) = \int_0^z (2w)dw \qquad = z^2\Big|_0^z \qquad = z^2$$

Step 3:

Obtain the inversed transformation function G^{-1}

$$G(z) = T(r)$$

$$z^2 = -r^2 + 2r$$

$$z = \sqrt{2r - r^2}$$

We can guarantee that $0 \le z \le 1$ when $0 \le r \le 1$

Discrete formulation

$$s_k = T(r_k) = (L-1)\sum_{j=0}^{k} p_r(r_j)$$
$$= (L-1)\sum_{j=0}^{k} \frac{n_j}{MN} \qquad k = 0,1,2,\ldots, L-1$$

$$G(z_k) = (L-1)\sum_{i=0}^{k} p_z(z_i) = s_k \qquad k = 0,1,2,\ldots, L-1$$

$$\boldsymbol{z_k} = \boldsymbol{G}^{-1}\left[\boldsymbol{T}(\boldsymbol{r_k})\right]$$
$$= \boldsymbol{G}^{-1}\left[\boldsymbol{s_k}\right] \qquad \boldsymbol{k} = 0,1,2,\ldots, \boldsymbol{L}-1$$

Example (Histogram specification)

Step-1 Histogram equalization of original image

Before continuing, it will be helpful to work through a simple example. Suppose that a 3-bit image ($L = 8$) of size 64×64 pixels ($MN = 4096$) has the intensity distribution shown in Table 3.1, where the intensity levels are integers in the range $[0, L-1] = [0, 7]$.

The histogram of our hypothetical image is sketched in Fig. 3.19(a). Values of the histogram equalization transformation function are obtained using Eq. (3.3-8). For instance,

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Similarly,

$$s_1 = T(r_1) = 7\sum_{j=0}^{1} p_r(r_j) = 7p_r(r_0) + 7p_r(r_1) = 3.08$$

and $s_2 = 4.55$, $s_3 = 5.67$, $s_4 = 6.23$, $s_5 = 6.65$, $s_6 = 6.86$, $s_7 = 7.00$. This transformation function has the staircase shape shown in Fig. 3.19(b).

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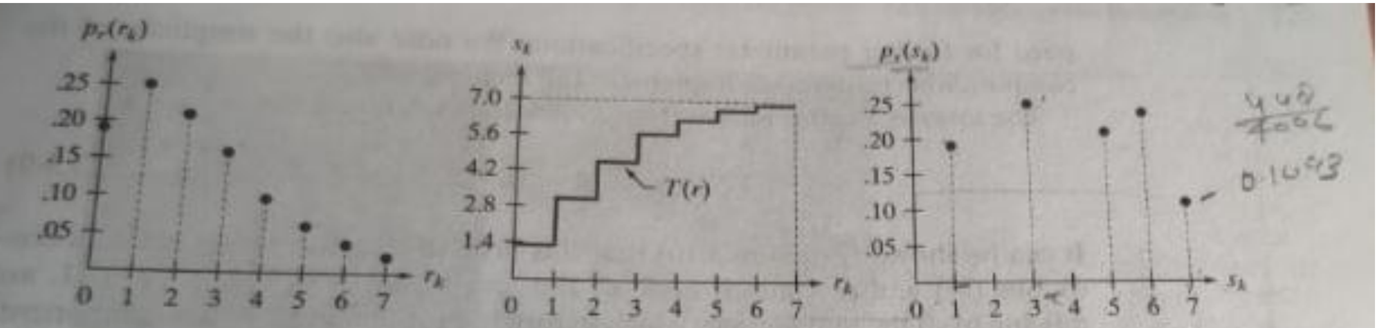
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

* Image is taken from **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

Example



a b c

FIGURE 3.19 Illustration of histogram equalization of a 3-bit (8 intensity levels) image. (a) Original histogram. (b) Transformation function. (c) Equalized histogram.

At this point, the s values still have fractions because they were generated by summing probability values, so we round them to the nearest integer:

$s_0 = 1.33 \rightarrow 1$	$s_4 = 6.23 \rightarrow 6$
$s_1 = 3.08 \rightarrow 3$	$s_5 = 6.65 \rightarrow 7$
$s_2 = 4.55 \rightarrow 5$	$s_6 = 6.86 \rightarrow 7$
$s_3 = 5.67 \rightarrow 6$	$s_7 = 7.00 \rightarrow 7$

r_k	n_k	$p_r(r_k) = n_k/MN$
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$r_3 = 3$	656	0.16
$r_4 = 4$	329	0.08
$r_5 = 5$	245	0.06
$r_6 = 6$	122	0.03
$r_7 = 7$	81	0.02

245+122+81=448 pixels with a value 7
In the histogram equalized image

* Image is taken from **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

Histogram Specification

Step-2 Compute all values of the transformation Function G

In the next step, we compute all the values of the transformation function, G, using Eq. (3.3-14):

$$G(z_0) = 7\sum_{j=0}^{0} p_z(z_j) = 0.00$$

Similarly,

$$G(z_1) = 7\sum_{j=0}^{1} p_z(z_j) = 7\left[p(z_0) + p(z_1)\right] = 0.00$$

and

$$G(z_2) = 0.00 \quad G(z_4) = 2.45 \quad G(z_6) = 5.95$$

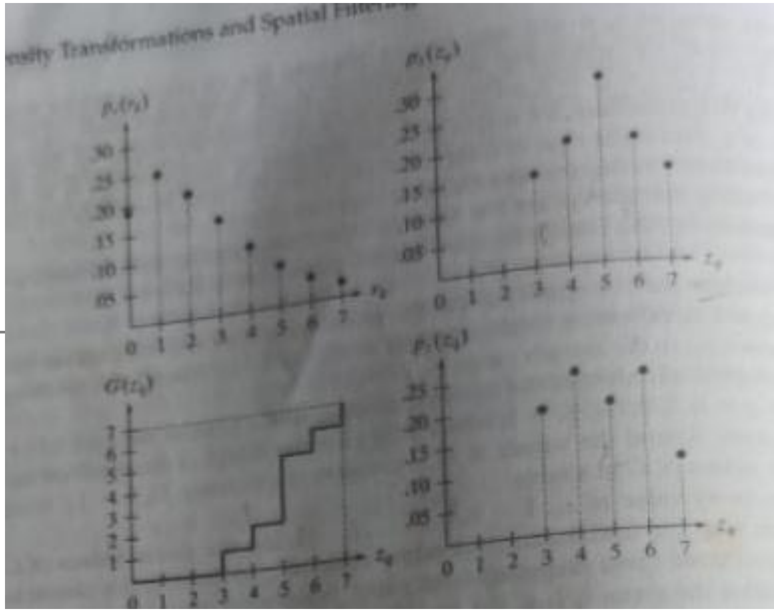
$$G(z_3) = 1.05 \quad G(z_5) = 4.55 \quad G(z_7) = 7.00$$

z_q	Specified $p_z(z_q)$	Actual $p_z(z_k)$
$z_0 = 0$	0.00	0.00
$z_1 = 1$	0.00	0.00
$z_2 = 2$	0.00	0.00
$z_3 = 3$	0.15	0.19
$z_4 = 4$	0.20	0.25
$z_5 = 5$	0.30	0.21
$z_6 = 6$	0.20	0.24
$z_7 = 7$	0.15	0.11

* Image is taken from **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

Step-3 Find the smallest value of z_q so that the value $F(z_q)$ is the closest to s_k.

Do this for each s_k to create the required mapping from s to z.



z_q	$G(z_q)$
$z_0 = 0$	0
$z_1 = 1$	0
$z_2 = 2$	0
$z_3 = 3$	1
$z_4 = 4$	2
$z_5 = 5$	5
$z_6 = 6$	6
$z_7 = 7$	7

$G(z_0) = 0.00 \rightarrow 0$ $\quad$ $G(z_4) = 2.45 \rightarrow 2$

$G(z_1) = 0.00 \rightarrow 0$ $\quad$ $G(z_5) = 4.55 \rightarrow 5$

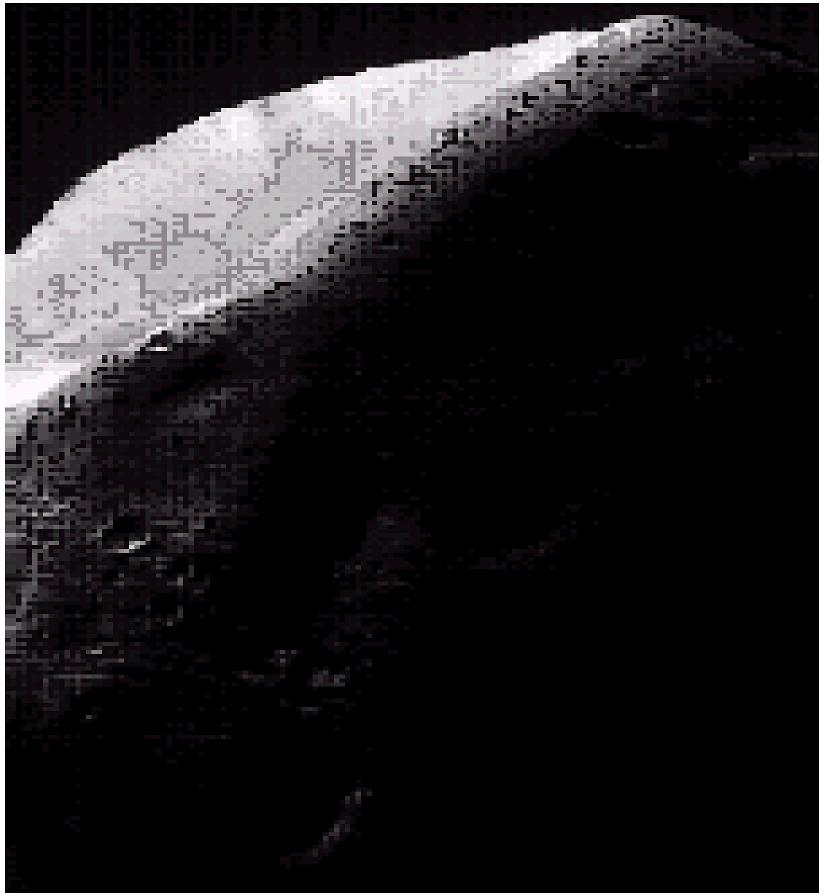
$G(z_2) = 0.00 \rightarrow 0$ $\quad$ $G(z_6) = 5.95 \rightarrow 6$

$G(z_3) = 1.05 \rightarrow 1$ $\quad$ $G(z_7) = 7.00 \rightarrow 7$

Step-4 Finally, use the mapping shown in table to Map every pixel in the histogram equalized image into a Corresponding pixel in the newly created histogram Specified image.

s_k	$\rightarrow$	z_q
1	$\rightarrow$	3
3	$\rightarrow$	4
5	$\rightarrow$	5
6	$\rightarrow$	6
7	$\rightarrow$	7

Example



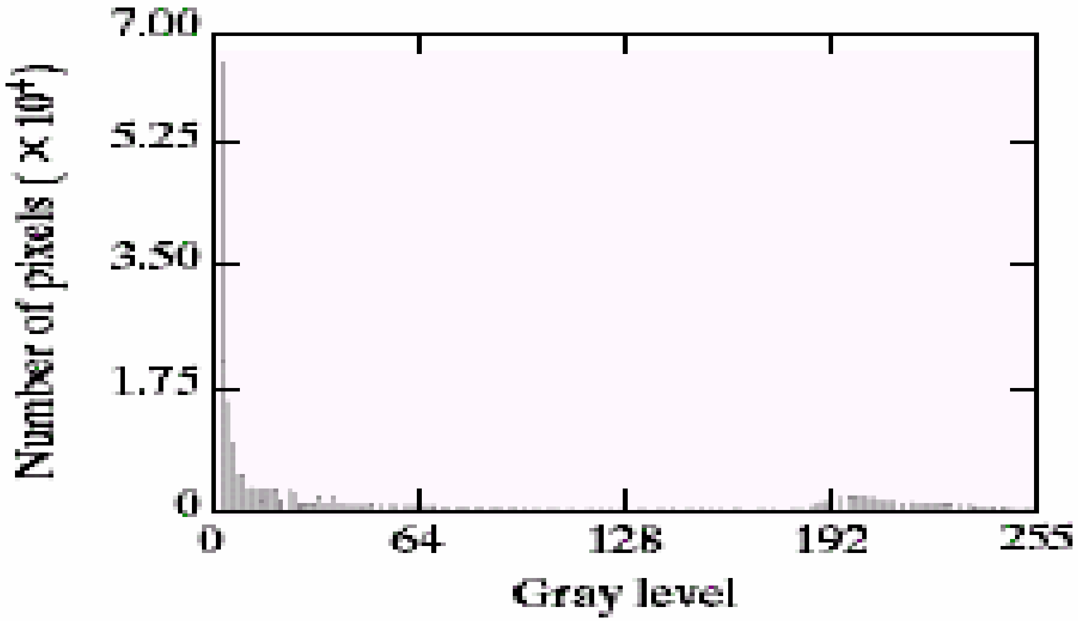
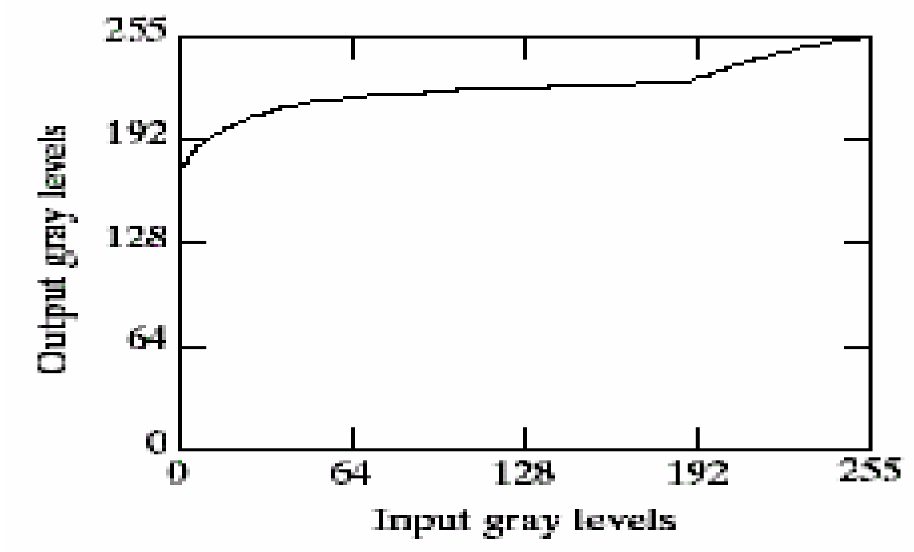


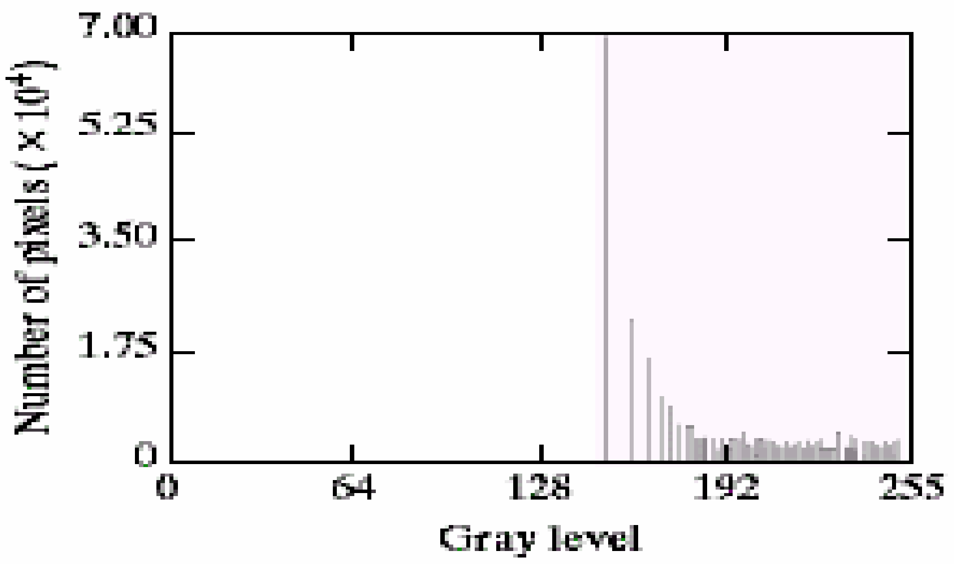
Image is dominated by large, dark areas, resulting in a histogram characterized by a large concentration of pixels in pixels in the dark end of the gray scale

Image of Mars moon

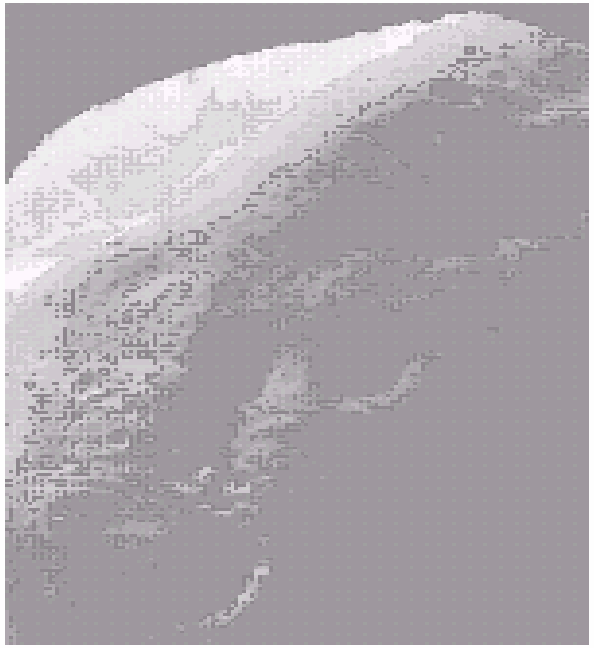
Image Equalization



Transformation function for histogram equalization



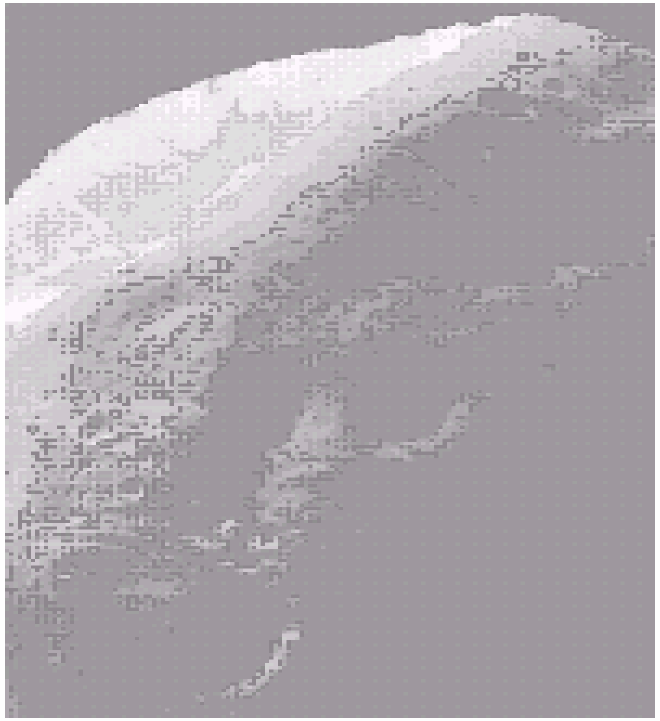
Histogram of the result image



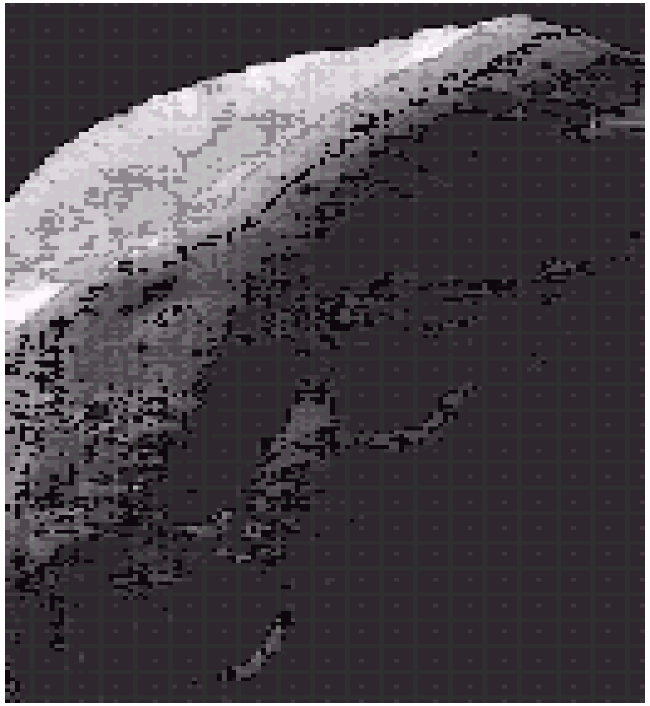
Result image after histogram equalization

The histogram equalization doesn't make the result image look better than the original image. Consider the histogram of the result image, the net effect of this method is to map a very narrow interval of dark pixels into the upper end of the gray scale of the output image. As a consequence, the output image is light and has a washed-out appearance.

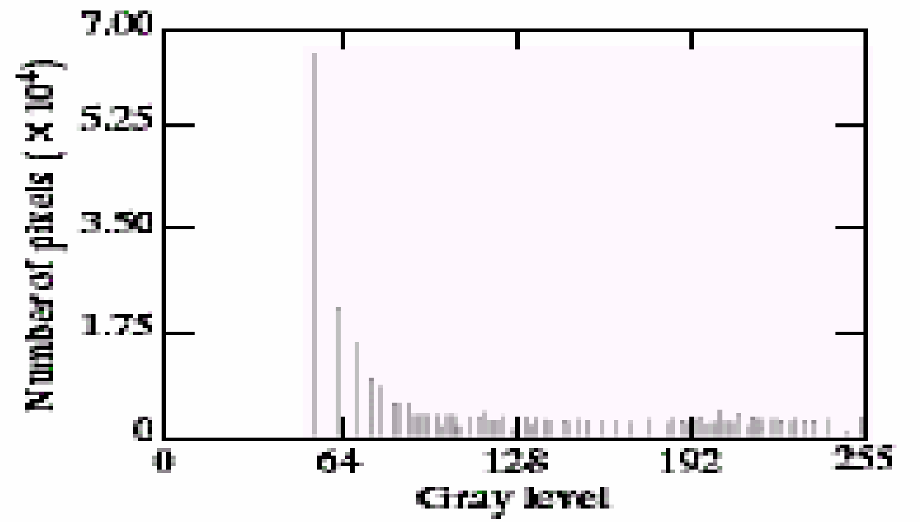
Result image and its histogram



Original image



After applied the histogram equalization



The output image's histogram

Notice that the output histogram's low end has shifted right toward the lighter region of the gray scale as desired.

Note

Histogram specification is a trial-and-error process

There are no rules for specifying histograms, and one must resort to analysis on a case-by-case basis for any given enhancement task.

Note

Histogram processing methods are global processing, in the sense that pixels are modified by a transformation function based on the gray-level content of an entire image.

Sometimes, we may need to enhance details over small areas in an image, which is called a local enhancement.

Suggested Readings

- **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

- **Fundamental of Digital image processing by A. K Jain, Pearson Education India, 2015.**

Thank you