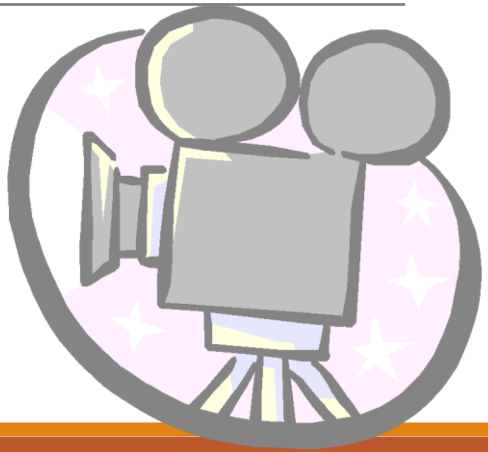
Image Processing

CS-317/CS-341



Outline

- Image Enhancement in the Frequency Domain
    - Fourier Transformation

Background ….

- Jean Baptise Joseph (B. 1768 in town of Auxerre, France)
- Most important contribution was outlined in a memoir in 1807 and published in 1822 in his book, “La Theorie Analitique de la Chaleur (The Analytic Theory of Heat)”

Contribution

Any function that periodically repeats itself and satisfy the some mild mathematical condition, does not matter how much complicated the function is, can be expressed as the sum of sines and / or cosines of different frequencies, each multiplied by a different coefficients.

➢ **Now this sum is called as Fourier Series**

Background….

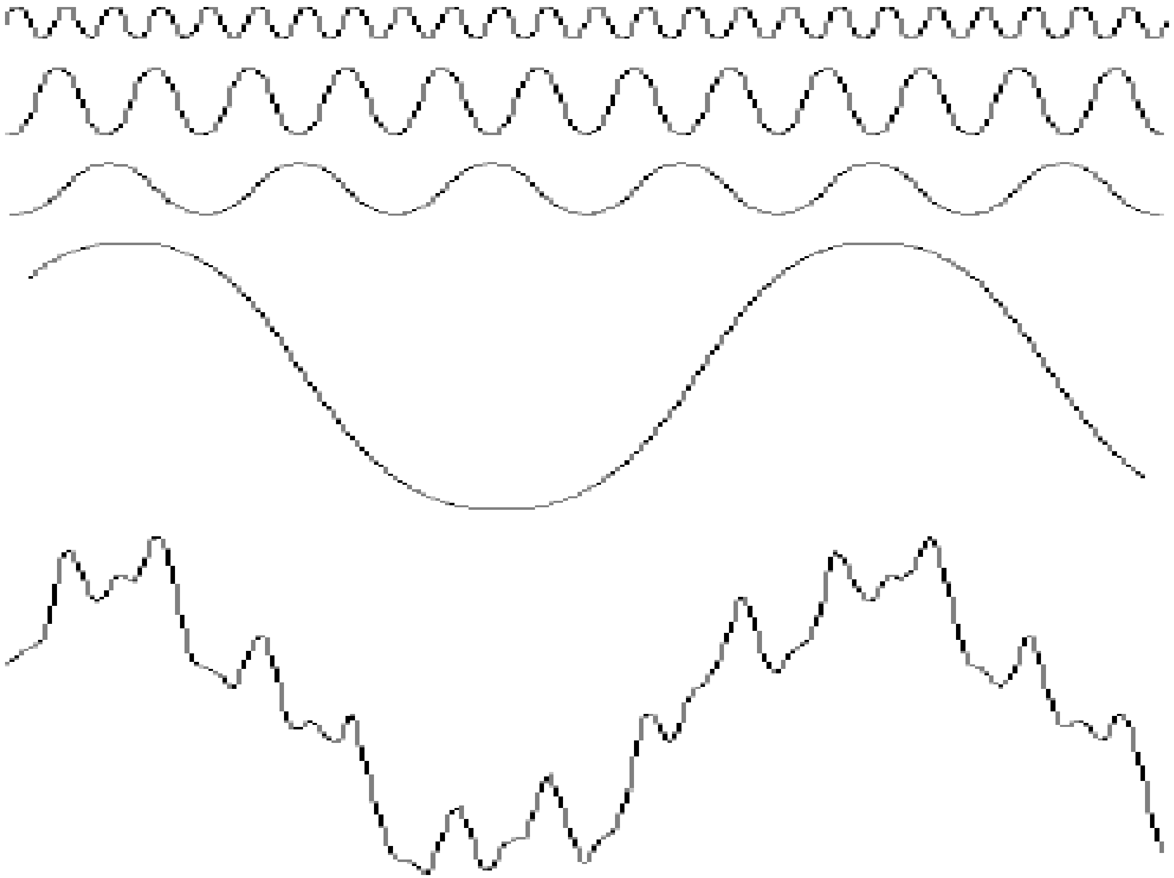
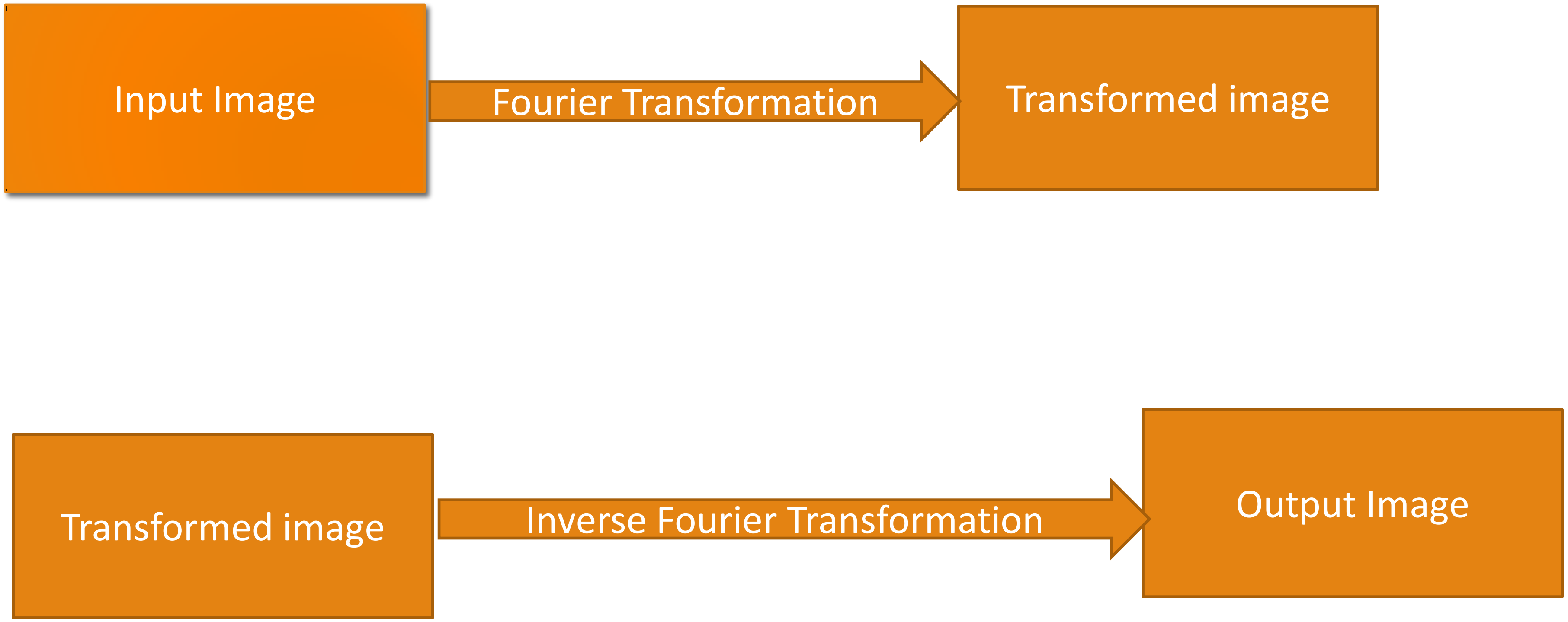


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Background….

- Even functions that are not periodic, but area under the curve is finite, can be expressed as the integral of sines and/or cosines multiplied by a weighting function.
- This representation is called as Fourier Transform. It is used very much in most of the practical applications.
- These representation share most important characteristics that a function can be perfectly reconstructed (recovered) completely via an inverse process, with no loss of information.
- For finite duration / support (e.g. image) Fourier transform is most important tool.

Fourier Transform
&
Frequency Domain


Input Image
Fourier Transformation
Transformed image
Transformed image
Inverse Fourier Transformation
Output Image

1-D Continuous Fourier Transform

Let f(x) is a 1-D function

$$F(u) = \int_{-\infty}^{+\infty} f(x) e^{-j2\pi ux} dx; \quad f(x) = \int_{-\infty}^{+\infty} F(u) e^{j2\pi ux} du$$

$$j = \sqrt{-1}$$

$$e^{j\theta} = \cos\theta + j\sin\theta$$

2-D Continuous Fourier Transform

Let f(x,y) is a 2-D function, the Fourier transform of f(x, y) is given as:

$$F(u,v) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy \ ;$$

The inverse Fourier Transform of F(u,v) is

$$f(x,y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

One Dimensional
Discrete Fourier Transform

1-D Discrete Fourier Transform

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)e^{-j2\pi ux/M} \; ; \quad u = 0,1,2,3,\ldots\ldots,M-1$$

$$f(x) = \sum_{x=0}^{M-1} F(u)e^{j2\pi ux/M} \; ; \quad x = 0,1,2,3,\ldots\ldots,M-1$$

- Some authors include the 1/M term in eq. f(x) instead the way shown in eq. F(u). That does not affect the proof that the two equations form the Fourier series

1-D Discrete Fourier Transform

- **Frequency Domain**

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[\cos 2\pi ux / M - j\sin 2\pi ux / M\right];\ u = 0,1,\cdots,M-1$$

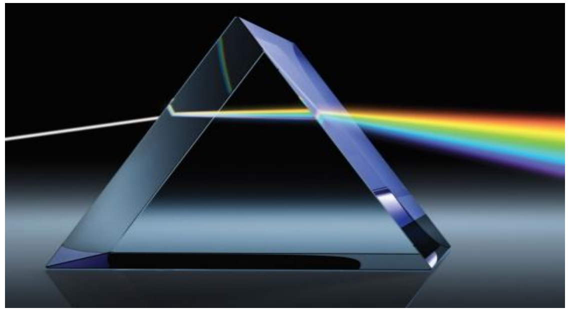
- The domain (value of u) over which the values of $F(u)$ is appropriately called the frequency domain. u determines the frequency of the component of the transform.

- Each of the M terms of $F(u)$ is called frequency component of the transform.

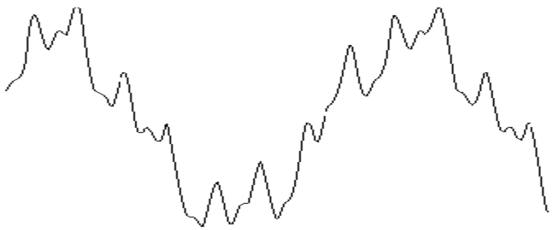
1-D Discrete Fourier Transform

- **Analogy of FT to glass prism:**

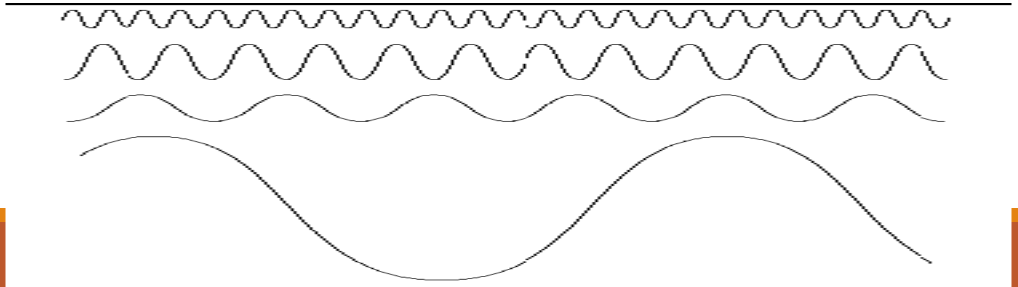
Glass prism → Physical device separates light into various color component depending on its frequency



FT as Mathematical Prism → Separates a function into various component, also based on frequency content



FT



1-D Discrete Fourier Transform

- **Frequency Domain**

Euler's Formula

$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x)\left[\cos 2\pi ux / M - j\sin 2\pi ux / M\right];\; u = 0,1,\cdots,M-1$$

F(u) is Complex

Properties of Fourier Transform

- **Polar Coordinate Representation of *F*(*u*):**

***F*(*u*) is Complex**

$$F(u) = |F(u)| e^{-j\phi(u)} \qquad |F(u)| = \left[R^2(u) + I^2(u) \right]^{1/2}$$

$R(u)$: Real part of $F(u)$, $I(u)$: Imaginary part of $F(u)$, $|F(u)|$: Magnitude spectrum of the Fourier Transform.

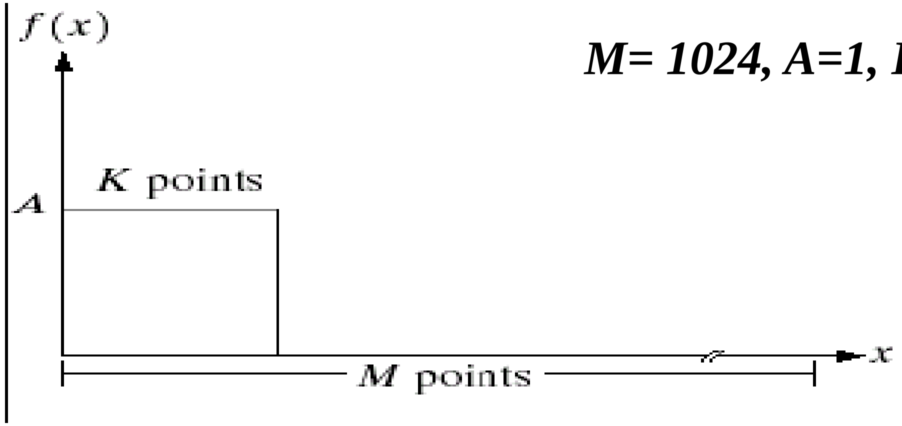
$$\phi(u) = \tan^{-1}\left[I(u) / R(u) \right]$$: phase angle or phase spectrum

$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

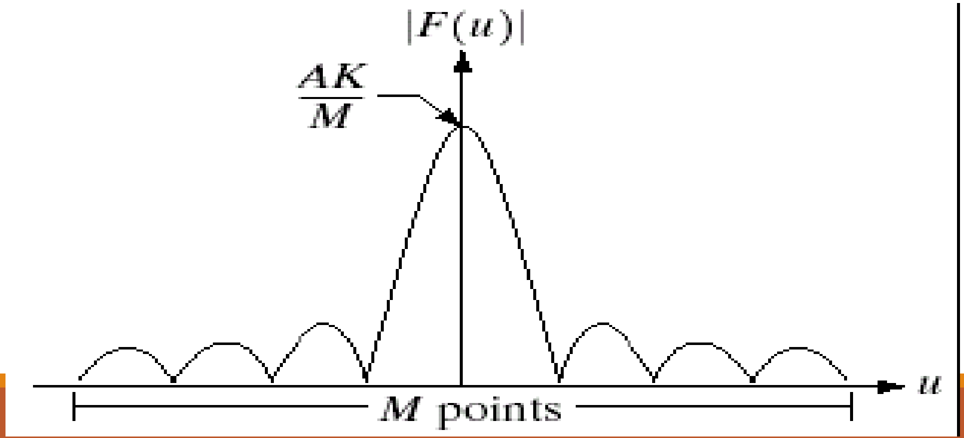
: power spectrum / power spectral density

1-D Discrete Fourier Transform

- **Example:**



M= 1024, A=1, K= 8



$$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M}$$

$$= \frac{1}{M}\sum_{x=0}^{K-1} A e^{-j2\pi ux/M}$$

$$= \frac{A}{M}\sum_{x=0}^{K-1} e^{-j2\pi ux/M}$$

$$F(0) = \frac{A}{M}\sum_{x=0}^{K-1} e^{-j2\pi 0x/M}$$

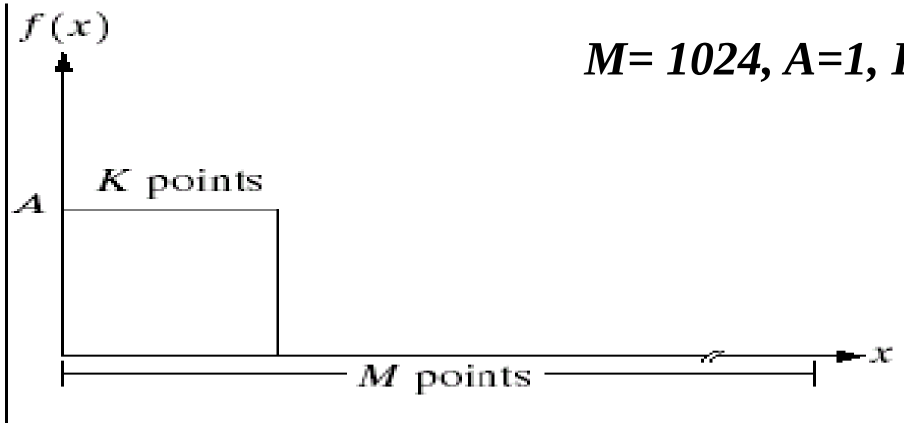
$$= \frac{A}{M}\sum_{x=0}^{K-1} 1$$

$$= \frac{AK}{M}$$

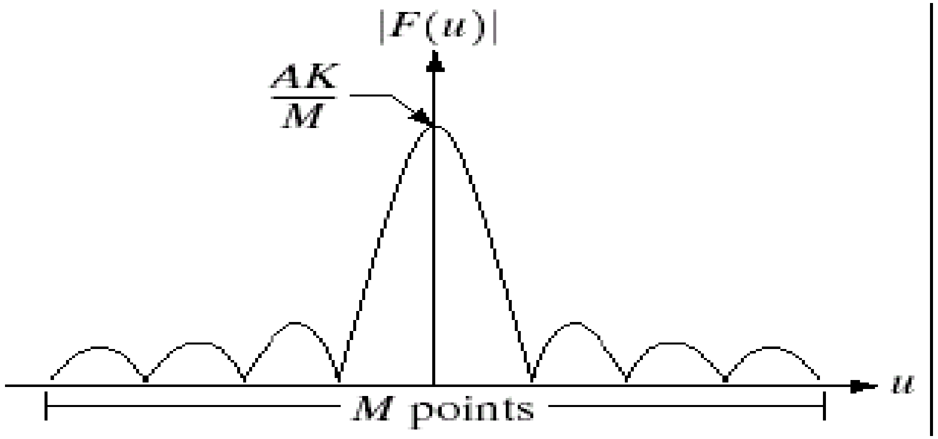
$$
\begin{aligned}
F(0) &= \frac{A}{M}\sum_{x=0}^{K-1} e^{-j2\pi 0x/M} \\
&= \frac{A}{M}\sum_{x=0}^{K-1} 1 \\
&= \frac{AK}{M}
\end{aligned}
$$

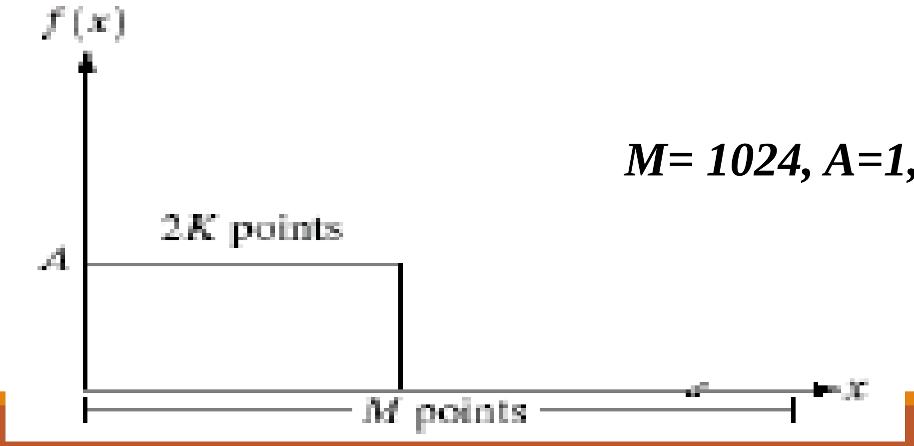
1-D Discrete Fourier Transform

- **Example:**

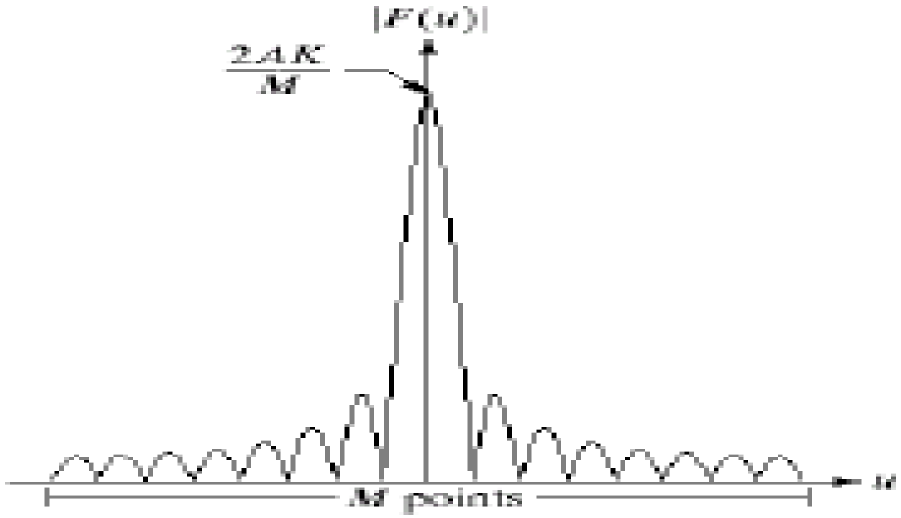


M= 1024, A=1, K= 8



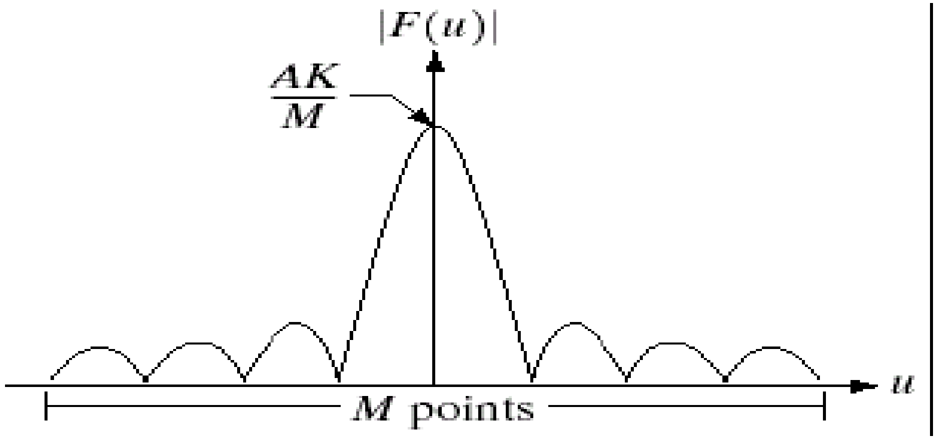


M= 1024, A=1, K= 16

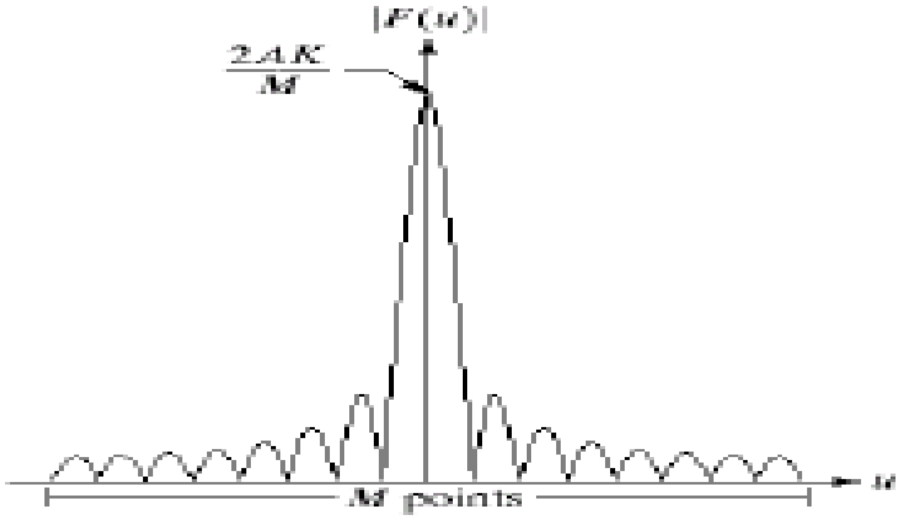


1-D Discrete Fourier Transform

- **Example:**



M= 1024, A=1, K= 8



M= 1024, A=1, K= 16

- Spectrum is centered at *u* =0, this can be accomplish by multiplying $f(x)$ *by* $(-1)^x$ before taking transform.
- Height for K = 16 is twice of K = 8, and area under curve the curve in x - domain doubled.
- Number of zeros in the spectrum in the same interval doubled as the length function doubled.

Two Dimensional Discrete Fourier Transform

Relationship between spatial and frequency intervals

- Δx, Δu are related as $$\Delta u = \frac{1}{M \Delta x}$$

- This relationship is useful when measurement are an issue in the image being processed.

2-D Discrete Fourier Transform

- **2D DFT and its Inverse :**

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

For $u = 0,1,2\ldots, M\text{-}1$, and $v = 0,1,2, \ldots., N\text{-}1$, transform *or* frequency variable.

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$$

For $x = 0,1,2\ldots, M\text{-}1$, and $y = 0,1,2, \ldots., N\text{-}1$, spatial or image variable.

2-D Discrete Fourier Transform

- **Spectrum, Phase angle and Spectrum density:**

$$F(u,v) = |F(u,v)| e^{-j\phi(u,v)} \qquad |F(u,v)| = \left[R^2(u,v) + I^2(u,v) \right]^{1/2}$$

R*(*u*,*v*):** Real part of ***F*(*u*,*v*)**, ***I*(*u*,*v*):** Imaginary part of ***F*(*u*,*v*)**, |F*(*u*,*v*)**|**:** Magnitude spectrum of the Fourier Transform.

$$\phi(u,v) = \tan^{-1}\left[I(u,v)/R(u,v) \right]$$

: phase angle or phase spectrum

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

: power spectrum / power spectral density

Suggested Readings

- **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

- **Fundamental of Digital image processing by A. K Jain, Pearson Education India, 2015.**

Thank you