Note :- Every field of knowledge is collection of ideas described by means of words, and symbols.

- one can not understand these ideas unless one know the exact meaning of the words and symbols that are used.

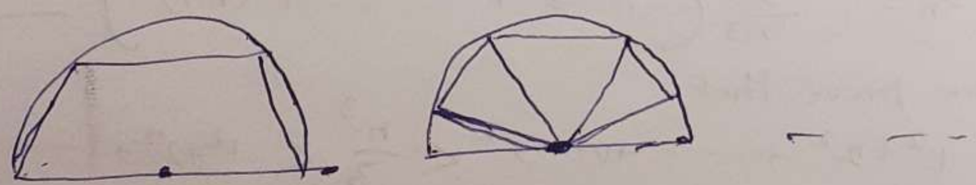
Deductive System :- In this number of "undefined" ~~know~~ concept are chosen in advance and all other concepts in the system are defined in terms of these.

- Certain statements about these undefined concept are taken as axioms or postulates and other statements that can be deduced from the axioms are called theorems.

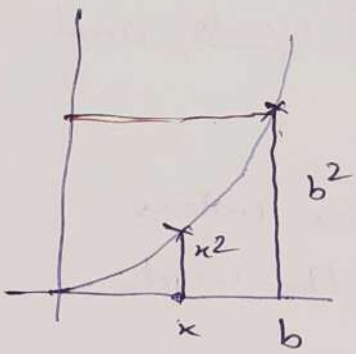
e.g. Euclidean ~~Geometry~~ theory of elementry geometry.

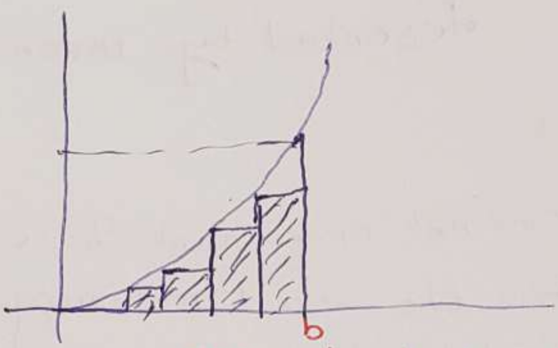
Historical BackGround :- 2000 years ago Greeks attempted to determine the area ~~of~~ by a process called method of exhaustion.

Archimedes (287-212 B.C.) :- To find the area of circle

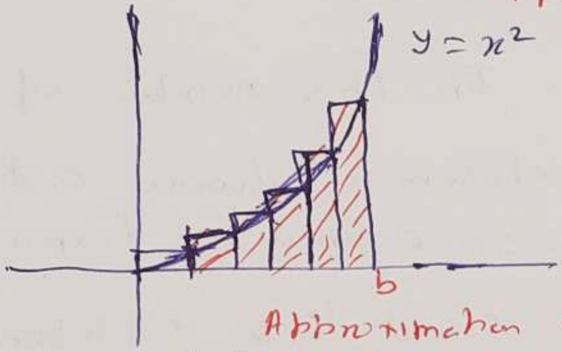


Area of Parabolic Segment :- $y = x^2$





Approximation from below



Approximation from Above.

For n- strip : ~~base length of ea~~ breadth of each strip is $\frac{b}{n}$.

Points of subdivision

$$0,\ \frac{b}{n},\ \frac{2b}{n},\ \cdots\ \frac{kb}{n},\ \cdots\ \frac{(n-1)b}{n},\ \frac{nb}{n} = b$$

~~In general~~

~~a~~ Area with ~~outer App~~ approximation from above

~~S_n~~: area of k^{th} strip

$$\frac{b}{n}\left(\frac{kb}{n}\right)^2 = \frac{b^3}{n^3}k^2$$

$$\therefore S_n = \frac{b^3}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \quad \text{—(1)}$$

Area with lower approximation

$$s_n = \frac{b^3}{n^3}\left(1^2 + 2^2 + \cdots + (n-1)^2\right) \quad \text{—(2)}$$

we can prove that

$$1^2 + 2^2 + \cdots + (n-1)^2 < \frac{n^3}{3} < 1^2 + 2^2 + \cdots + n^2 \quad \text{—(3)}$$

for integer $n \geq 1$

Multiply (5) by $\frac{b^3}{n^3}$ we have

$$s_n \le \frac{b^3}{3} \le S_n \quad \text{——— (4)}$$

* This can be prove that $\frac{b^3}{3}$ is only number which satisfy this ~~ine~~ inequality. for every n.

Note :- (i) This ~~er~~ Archimedes result about the area of parabolic segment ~~as~~ can not be accepted as a theorem until a satisfactory definition of area given first. * It is not clear that Archimedes
(ii) had ~~ever~~ formulated precise definition ~~of area~~ of what he meant by area.

— It seems he assume that every region has a ~~area~~ area associated with it

— on this assumption of for granted

- One region lie inside another, the area of smaller region can not be greater than of the larger region.
- Region is decomposed in two or more part the sum of areas of the individual part is equal to the area of whole.

* It may possible Archimedes may suppose area is as undefined concept and then use the above two properties as Axium about the area.

⇒ Greeks and Archimedes work is not so important in the present day because we need to find the area of an arbitrary figure rather than particular figure

⇓

So we define much more general concept for area and it is known as integral

⇓

This ~~area~~ integral concept is also used to calculate quantities like arc length, volume, work, and others.

*

$$\lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1} \frac{b}{n} f\left(\frac{ib}{n}\right)}_{S_n}$$

$$\lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} \frac{b}{n} f\left(\frac{ib}{n}\right)}_{S_n}$$

*It is read as integral of f from 0 to b

and written as $\int_0^b f(x)\,dx = \int_0^b x^2\,dx = \frac{b^3}{3}$

* $\int$ → integral sign introduced by Leibniz in 1675.

* ~~The~~ The process by which ~~integrati~~ get the above value called integration

$\int_0^X f(x)\,dx$ means integral of $f(x)$, over x from $x=0$ to $x=X$.

* To add many infinitely many "infinitely small quantities"

Field Axiom :– $\{F, +, *\}$ — ~~adentity~~, inverse, Associativity

(i) $(F, +)$ is abelian group

(ii) $(F^*, *)$ is abelian group where $F^* = F - \{0\}$

(iii) Distribution of $*$ over $+$

$x\cdot(y+z) = x\cdot y + x\cdot z \quad \forall x,y,z \in F.$

Order Axioms :– Partial ordering (Partial order set concept)

- Reflexive
- Antisymmetry
- Transitive

Well ordering :– Every non empty set have ~~least~~ smallest member

Concept of area as a set function :– Region on a plane

- When we assign an area to a plane region, we associate a number with a set S in a plane ~~to~~
- In mathematical point of view we have function a ~~from~~ which assign a real no. $a(S)$ to each set S in some given collection of sets.

$a : CS \longrightarrow \mathbb{R}.$

- A function whose domain is ~~set~~ collection of a set and ~~domain~~ ~~set~~ function values are real no is called set function.

- The basic problem

↳ Given a plane set S, what area $a(S)$ shall we assign to S?

- Approach to this problem is to start with a number of properties ~~wee~~ feel area should have and take these as axioms for area?

- Those set function which satisfy these ~~function~~ axioms called area function.

* What will be necessary condn that area function ~~exist~~ exists? Here and onward ~~the~~ discussion I assume that area function exist.

(Reference Chapter 14 and 22 of Edwin E. Moise, Elementary Geometry From an advanced stand point, Addison-Wesley Publishing co. ~~to~~ 1963.).

: Note about Area :

— ~~Measurable sets~~
— Collection of sets in the plane to which an area can be assigned. The sets are called measurable set, and it is denoted by $\mathcal{M}$.

— Rectangle as set of points :—

The term rectangle ~~is~~ refers to any set congruent to a set of the form:

$$\{ (x,y) \mid 0 \le x \le h,\ 0 \le y \le k \}$$

where $h \ge 0$ and $k \ge 0$. The number h and k are called the lengths of edges of the rectangle.

** ~~Point~~ — h and k both zero
* Line — h or k to be zero.) special case of rectangle

* Congruent :— ~~Two~~

↳ concept is same as concept of congruence used in elementary Euclidean geometry.

* Two sets are said to be congruent if their points can be put into one-to-one correspondence in such a way that distances are preserved.

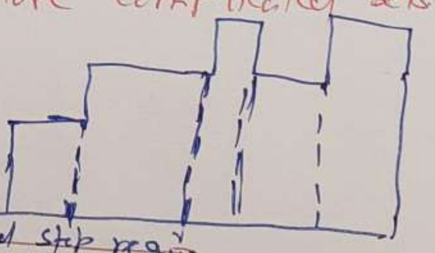
i.e. p and q ~~are~~ in one set correspond to p' and q' in the other, the distance from p to q be equal to distance between p' and q'.

i.e. $d(p,q) = d(p',q')$.

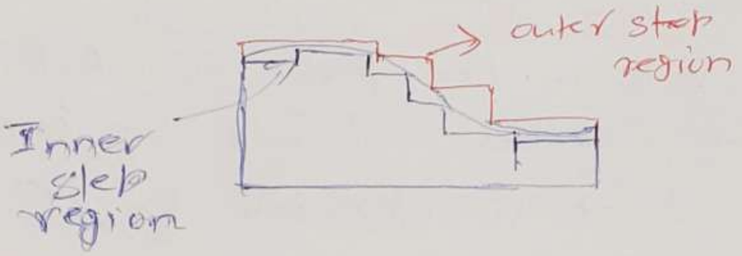
* Rectangle used to buildup more complicated sets

e.g. ~~Step region~~ step region

↳ union of a finite collection of adjacent rectangles with their bases resting on x-axis. ~~called step region~~



↳ Here each ~~re~~ step ~~region~~ region



AXIOMATIC Definition OF AREA :—

Assume that there exists a class $\mathcal{M}$ of ~~et~~ measurable sets in the plane and a set of function a, whose domain is $\mathcal{M}$, with following properties:

1. Nonnegative property:— For each set S in $\mathcal{M}$, we have $a(S) \geq 0$

2. Additive property:— If S and T are in $\mathcal{M}$, then $S \cup T$ and $S \cap T$ are in $\mathcal{M}$ and we have [overlapping region]

$$a(S \cup T) = a(S) + a(T) - a(S \cap T)$$

3. Difference property:— If S and T are in $\mathcal{M}$ with $S \subseteq T$, then $T - S$ is in $\mathcal{M}$, and we have $a(T-S) = a(T) - a(S) \geq 0$
(monotone property) $a(\phi) = 0$ $\quad a(T) \geq a(S)$

4. Invariance under congruence:— Let $S \in \mathcal{M}$ and if T is congruent to S, then T is also in $\mathcal{M}$ and we have $a(S) = a(T)$ [a set having same size and shape have equal area]

5. choice of scale:— Every rectangle in $\mathcal{M}$. If edges of R have lengths h and k then
$$a(R) = hk$$

6. Exhaustion property:— Let Q be a set that can be ~~enclo~~ enclosed between two steps regions S and T,

so that

$$S \subseteq Q \subseteq T \quad \text{—(1.1)}$$

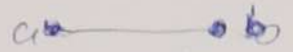
If there is one and only one number 'c' which satisfies the inequalities

$$a(S) \leq c \leq a(T)$$

for all step regions S and T satisfying (1.1) then Q is measurable and $a(Q) = c$.

(This axiom incorporate the Greek method of exhaustion)

Intervals and ordinate Sets:



~~a < b~~

Closed interval

$a < b$

$a \le x \le b$

denoted by $[a, b]$

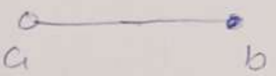
$a < b$

~~$a < x$~~ open

Interval = end point not included

$a < x < b$

denoted by (a, b)



~~$a < b$~~ (a

Half open from left

$a < b$

$a < x \le b$

left end point is not contain in at interval while righ end point is included

Half open from right

$a < b$

$a \le x < b$

~~lef~~ left end point is included while right end point is excluded.

~~Ordidan~~

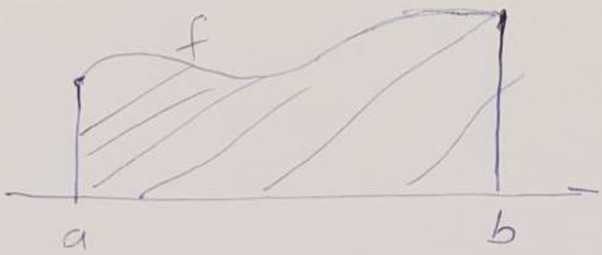
Ordinate Set:

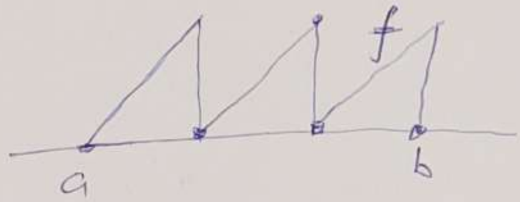
- f is non-negative function with domain closed interva $[a, b]$

- The portion of plane between the graph of f and x-axis etc called ordinate set of f.

i.e. collection of all points (x, y) satisfying the inequalities

$$a \le x \le b, \quad 0 \le y \le f(x)$$





Partition :

closed interval $[a, b]$ decompose into n sub intervals ~~such that~~ by inserting $n-1$ points of subdivision. only restriction is that

$$a < x_1 < x_2 < x_3 < \cdots x_{n-1} < b \quad \text{--- (1)}$$

For convienience denote $x_0 = a$ and $x_n = b$

— A collection of points satisfying (1) is called partition P of $[a, b]$ and we use the symbol

$P = \{x_0, x_1, \ldots, x_n\}$ to designate ~~the~~ this partition. The partition P determines n closed subintervals:

$$[x_0, x_1], [x_1, x_2], \ldots\ldots, [x_{n-1}, x_n]$$

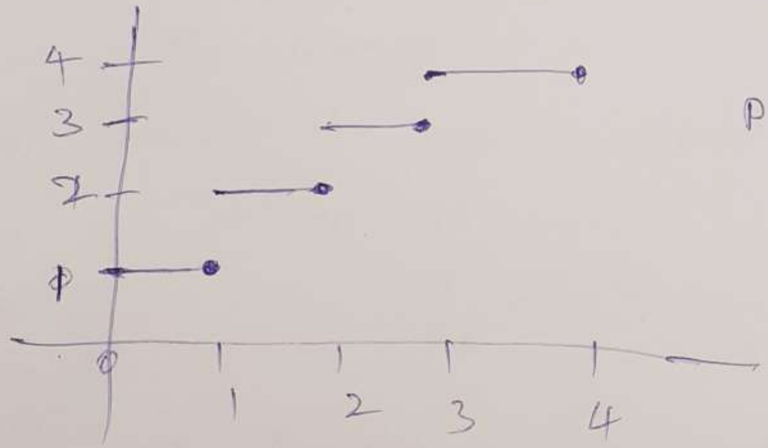
Step function :-

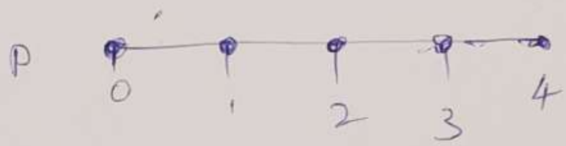
a function s whose domain is closed interval $[a, b]$, is called a step function if there is partition P
$P = \{x_0, x_1, x_2, \ldots x_n\}$ of $[a, b]$ such that s is constant on each open subinterval of P i.e.

$$s(x) = s_k \quad , \quad \text{if } x_{k-1} < x < x_k$$

Note (1) Step function also some time called piecewise constant function.

Note (2) At each of ~~end point~~ endpoints x_{k-1} and x_k the function must have some well defined value but this need not be same as s_k.





Sum and Product of step function :-

Sum of step function is also step function

If s and t are step function on closed interval $[a, b]$. Then their sum and product is also step function on $[a, b)$

$$~~u(x) = s(x) +~~$$

$$u(x) = s(x) + t(x)$$

$$v(x) = s(x) \cdot t(x).$$

Integral for step function :-

Basic ~~institution~~ is idea is integral of non-negative step function is equal to area of its ordinate sets.

- $s : [a,b] \longrightarrow \mathbb{R}$ and Partition of $[a,b]$ is $P = \{x_0, x_1, x_2, \ldots x_n\}$ s.t. s is constant over each open intervals of P.

$$s(x) = s_k \quad \text{if } x_{k-1} < x < x_k, \; k = 1, 2, \ldots n$$

s_k ↳ value at k^{th} sub interval.

Defn (Integral of step function) :-

The integral of step f^n s from a to b is denoted by ~~symo~~ symbol $\int_a^b s(x)\,dx$ and is defined as

$$\int_a^b s(x)\,dx = \sum_{k=1}^{n} s_k (x_k - x_{k-1})$$

i.e. multiply the ~~ka~~ constant value s_k to each k^{th} sub interval ~~length~~ length. and then we add the all these ~~product~~ products.

Properties of Integral of step function :-

① Additive Property :- $\int_a^b [s(x) + t(x)]\,dx = \int_a^b s(x)\,dx + \int_a^b t(x)\,dx$

② Homogeneous Property :- $\int_a^b c\cdot s(x)\,dx = c\int_a^b s(x)\,dx$

where c is a real number

③ Linearity Property :- for every real c_1 and c_2, we have

$$\int_a^b [c_1 s(x) + c_2 t(x)]\,dx = c_1 \int_a^b s(x)\,dx + c_2 \int_a^b t(x)\,dx$$

④ Comparision theorem: if $s(x) < t(x)$ for all x in $[a, b]$, then

$$\int_a^b s(x)\,dx < \int_a^b t(x)\,dx$$

[Monotone property of area]

⑤ Additive w.r.t interval of integration

$$\int_a^c s(x)\,dx + \int_c^b s(x)\,dx = \int_a^b s(x)\,dx \quad \text{if } a<c<b$$

Ordinate set is split into two ordinate set.

⑥ Invariance Under translation

$$\int_a^b s(x)\,dx = \int_{a+c}^{b+c} s(x-c)\,dx \quad \text{for every real } c.$$

Scaling of ordinate.

$$t(x) = s\left(\frac{x}{k}\right), \text{ if } ka \le x \le kb$$

$k>0$

⑦ Expansion or contraction of the interval of Integration:-

$$\int_{ka}^{kb} s\left(\frac{x}{k}\right) dx = k\int_a^b s(x)\,dx \quad \text{for every } k>0$$

Note:- $\int_a^b$ it is assume that lower limit a is less than upper limit b.

$$\cancel{\int_a^b s(x)\,dx = -\int_b^a s(x)\,dx}$$

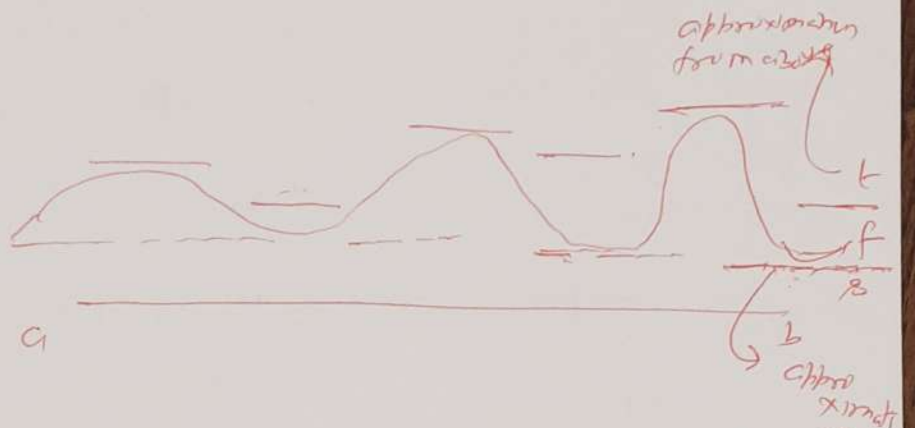
$$\int_b^a s(x)\,dx = -\int_a^b s(x)\,dx \quad \text{if } a<b$$

— Ⓐ

$$\int_a^a f(x)\,dx = 0 \quad \text{——} \quad (B)$$

Put $a = b$ in (A) we get (B)

Integration of More General Functions



- approximating fn f from above and below by step function. (arbitrary t and s step fn)
- In general

$$\int_a^b s(x)\,dx < \int_a^b t(x)\,dx$$

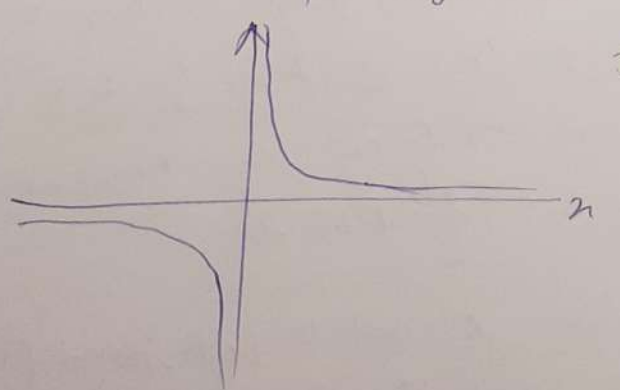
It is not possible to approximate every function from above and from below by step function.

e.g. the function f given by the equation:

$$f(x) = \frac{1}{x} \; ; \text{ if } x \neq 0, \; f(0) = 0$$

This defined for all real x, but on any ~~any~~ interval [a,b] containing the origin we can not ~~surround~~ surround f by step function.

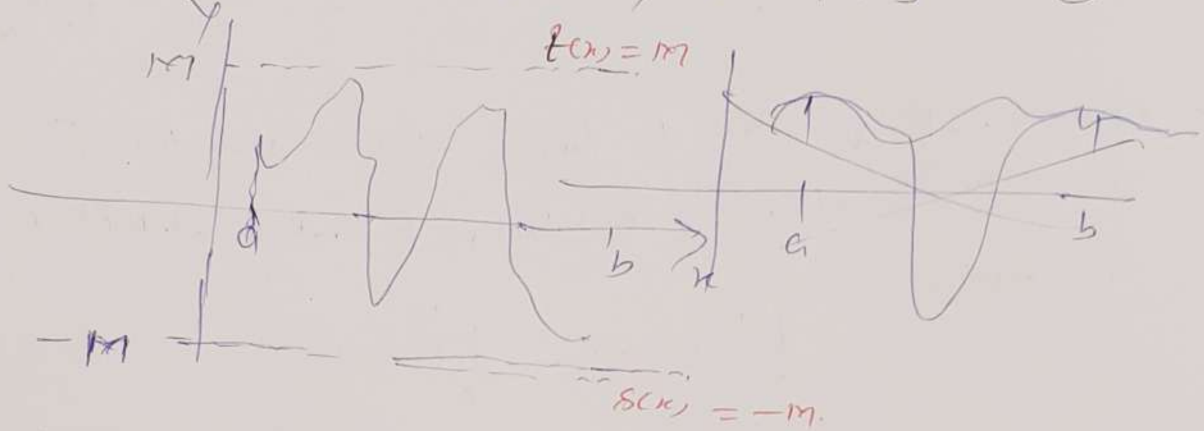
Since f is arbitrary large value near the origin, we say that f is unbounded in every neighbourhood of origin



Restrict first on the function that is Bounded

A function f on $[a,b]$ is said to be bounded if $\exists$ a number $M>0$ s.t.

$$-M \le f(x) \le M \ , \ \forall x \in [a,b] \quad —(1)$$



equivalent (1)

$$|f(x)| \le M \ , \ \forall x \in [a,b]$$

This means the f^n f is bounded by two constant function $s(x) = -M$, $t(x) = M$ $\forall x \in [a,b]$.

~~Defn~~ Definition:— (Integral of a Bounded Function):—

Let f be a function defined on $[a,b]$ and bounded.
Let s and t be an arbitrary step function on $[a,b]$ such that (1)— $s(x) \le f(x) \le t(x)$, $\forall x \in [a,b]$.
If there is ~~only~~ one and only one number I such that

(2)— $\int_a^b s(x)\,dx \le I \le \int_a^b t(x)\,dx$

$\forall$ pair of step function s and t satisfying (1) then number I is called the integral of 'f' from a to b.

and denoted by symbol $\int_a^b f(x)\,dx$ or by $\int_a^b f$.
When such Σ exist then function 'f' ~~is~~ is said to be ~~int~~ integrable on $[a,b]$.

→ The f^n 'f' called integrand, a, b called limit of integration, the interval $[a,b]$ is interval of integration

Upper and lower integral :—

f is bounded on $[a,b]$.
If s and t are step function s.t. that

$$s(x) \le f(x) \le t(x) \quad \forall x \in [a,b]$$

s is below of f → t is above of f.

Now let

$$S = \left\{ \int_a^b s(x)\,dx \;\middle|\; s \le f \right\}$$ = set of all numbers when s runs through all function below f

$$T = \left\{ \int_a^b t(x)\,dx \;\middle|\; f \le t \right\}$$ = set of all numbers when t runs through all function above f.

$$\int_a^b s(x)\,dx \le \sup S \le \inf T \le \int_a^b t(x)\,dx$$

∀ s, t satisfying $s \le f \le t$

⇒ Thus number $\sup S$ and $\inf T$ satisfy (2), ~~Therefor~~

Therefore f is integrable on $[a,b]$ iff $\sup S = \inf T$

$$\int_a^b f(x)\,dx = \sup S = \inf T$$

$\underline{I}(f)$ = ~~lower integral of f~~ $= \sup S = \sup\{\int_a^b s(x)\,dx \mid s \le f\}$

$\bar{I}(f)$ = upper integral of $f = \inf T = \inf\{\int_a^b t(x)\,dx \mid f \le t\}$

Result :- Every function f which bounded on $[a,b]$ has a lower integral $\underline{I}(f)$ and upper integral $\bar{I}(f)$ ~~&~~ satisfy the inequalities

$$\int_a^b s(x)\,dx \le \underline{I}(f) \le \bar{I}(f) \le \int_a^b t(x)\,dx.$$

for all step functions s and t with $s \le f \le t$.

~~The functions 'f' is integrable s and t with $s \le f \le t$.~~
~~The function~~

the function f is integrable on $[a,b]$ if and only if upper and lower integral equal in which case we have

$$\int_a^b f(x) = \underline{I}(f) = \bar{I}(f).$$

Note :- Two fundamental Question Arise at this stage:

1. Which Bounded functions are integrable?

   This is studied as theory of Integration

2. Given that function f is integrable, how do we compute the integral of f?

   Techniques of integration

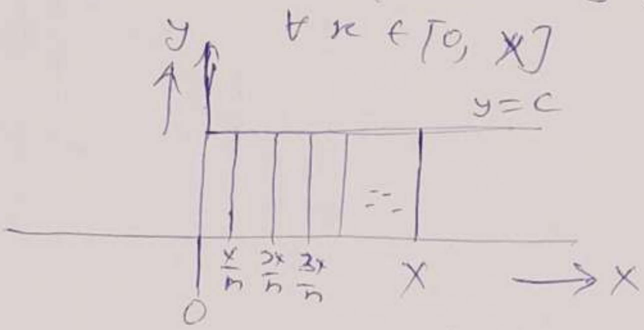
write short notes on the Criterion for Integrability of fn

[Bounded monotonic function on interval $[a,b]$ is integrable]

Area under Various Curves :

Ex1:- Area under constant curve

$y = c$ (where c is constant $\forall x \in [0, X]$



n-division

Divide length 0 to X into n equal parts

↳ Each part length $\frac{X}{n}$. The point on x-axis may be marked as

$$x_0 = 0, \quad x_1 = \frac{X}{n}, \quad x_2 = \frac{2X}{n}, \quad --- \quad x_n = \frac{nX}{n} = X$$

e.g. $X = 1\,cm$ $\quad x_0 = 0, \quad x_1 = \frac{1}{n}\,cm, \quad x_2 = \frac{2}{n}\,cm, \quad -- x_n = \frac{n \cdot 1}{n} = 1\,cm$

e.g. for $X = 5\,cm$. $\quad x_0 = 0, \quad x_1 = \frac{5}{n}, \quad x_2 = \frac{2(5)}{n}, \quad -- x_n = \frac{n(5)}{n}$

The ordinate at each point $\{x_i\}_{i=0}^{m}$ remains the same since $f(x) = c$ whatever be x. That is

$$f(\tfrac{X}{n}) = c, \quad f(\tfrac{2X}{n}) = c, \quad --, f(\tfrac{nX}{n}) = c$$

The areas of infinitesimal rectangles, when n is very large, summed up is then

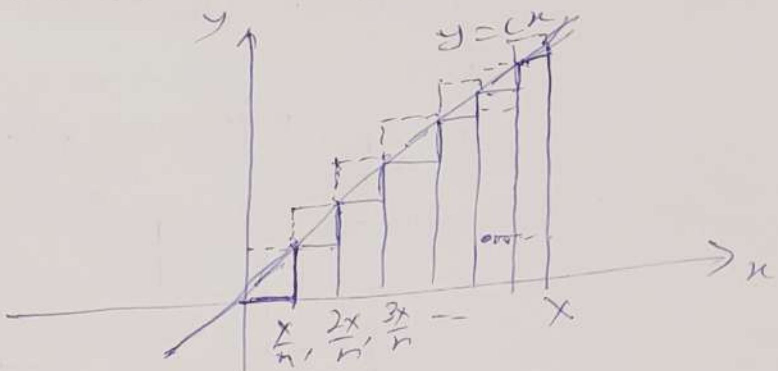
$$A = \sum_{i=0}^{n-1} \frac{X}{n} f\left(\frac{iX}{n}\right) = \sum_{i=0}^{n-1} \frac{X}{n} c = \cancel{n} \cdot \frac{X}{\cancel{n}} \cdot c$$

$$\therefore \boxed{A = cX}$$

Note :- Sum does not depend on ~~the~~ 'i' because ~~$f(\frac{ix}{})$~~ $f(\frac{iX}{n}) = c$ which is free of ~~(i)~~ i.

Ex 2:- Area under $y = cx$ where c is constant between ordinate $x=0$, $x=X$

~~What is Gr~~



Subdivide the ordinate length X into n equal parts

$$x_0 = 0,\ ,\ x_1 = \frac{X}{n},\ x_2 = \frac{2X}{n},\ \ldots,\ x_i = \frac{iX}{n},\ \ldots,\ x_n = \frac{nX}{n} = X$$

Two type of rectangle

below the line $y = cx$

Above the line $y = cx$

Sum of area of rectangle below $y = cx$ = lower sum

$$A_X = \sum_{i=0}^{n-1} \frac{X}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{X}{n} f\left(\frac{iX}{n}\right)$$

$$= \sum_{i=0}^{n-1} \frac{X}{n}\left[c \cdot \frac{iX}{n}\right] = c\left(\frac{X}{n}\right)^2 \sum_{i=0}^{n-1} i$$

$$= c\left(\frac{X}{n}\right)^2 \left[\frac{n(n-1)}{2}\right]$$

$$A_X = c\,\frac{X^2}{n^2}\left(\frac{n(n-1)}{2}\right)$$

Sum of area of rectangle above the $y = cx$ = upper sum

$$A^X = \sum_{i=1}^{n} \frac{X}{n} f(x_i) = \sum_{i=1}^{n} \frac{X}{n}\left[c\,\frac{iX}{n}\right]$$

$$A^* = c\left[\frac{x}{n}\right]^2 \sum_{i=1}^{n} i = c\,\frac{x^2}{n^2}\left[\frac{n(n+1)}{2}\right]$$

Now clear . A must satisfy

$$A_* \leq A \leq A^*$$

and when $n \to \infty$ i.e. when $\frac{x}{n}$ becomes smaller and smaller and finally when $\frac{x}{n}$ becomes differential in x, namely dx, then $A_* = A = A^*$.

⇒ This limiting sum of infinitesimals will be denoted by the sign $\int$, which on elongated or stretched S.

$$\lim_{n\to\infty} A_* = \lim_{n\to\infty} c\,\frac{x^2}{n^2}\left[\frac{n\cdot(n-1)}{2}\right]$$

$$= \lim_{n\to\infty} c\cdot\frac{x^2}{\cancel{n^2}}\left[\frac{\cancel{n^2}}{2}\left(1-\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{cx^2}{2}\left(1-\frac{1}{n}\right)$$

$$= \cancel{\lim_{n\to\infty}} = \frac{cx^2}{2}\lim_{n\to\infty}\left(1-\frac{1}{n}\right)$$

$$A = \frac{cx^2}{2}\cdot 1$$

Similarly $\lim_{n\to\infty} A^* = A = \frac{cx^2}{2}$.

Notation :- ~~Integral~~ Integral $= \int =$ stretched sum; $\int_a^b =$ streched sum from $\int_a^b$.

Defn :- ~~Note~~ Integral defined by the following common limit of the sum of infinitesimals:

$$\int_0^X f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} \frac{X}{n} f\left(\frac{iX}{n}\right) A_3$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{X}{n} f\left(\frac{iX}{n}\right) A^*$$

Notation :-

Note :- (1) For fixed a and b and arbitrary X

$$\int_a^X cx\,dx = \int_0^X cx\,dx - \int_0^a cx\,dx = c\frac{X^2}{2} - c\frac{a^2}{2}$$

and $\int_a^b cx\,dx = c\left[\frac{x^2}{2}\right]_a^b = c\frac{b^2}{2} - c\frac{a^2}{2}$

notation

$\left[c\frac{x^2}{2}\right]_a^b$ means evaluate the f^n at $x=b$ first, ~~and~~ then subtract from it the function evaluated at $x=a$

Notation :- 2 :-

$$F(x)\Big]_a^b = \left[F(x)\right]_a^b = F(b) - F(a).$$

Ex :- Evaluate the area under the curve $y = c\cdot x^2$ and above $y=0$ from

(i) : $x=0$ to $x=X$, where X is arbitrary.,

(ii) : from $x=a$ to $x=b$

Ex :- Evaluate the ~~integral~~ ~~integral~~ area under the the curve $y = cx^p$; $p \geq 0$ above $y > 0$ and from (i) $x = 0$ to $x = X$ where X is arbitrary, (ii) $x = a$ to $x = b$ where a and b are fixed.

Solution :- Interval $[0, X]$ is divided into n equals parts. Then the lower and upper sum are as follows:

$$S_* = \sum_{i=0}^{n-1} \frac{X}{n} f(x_i) = \sum_{i=0}^{n-1} \frac{X}{n} f\left(\frac{iX}{n}\right)$$

$$S_* = \sum_{i=0}^{n-1} \frac{X}{n} \left[c\left(\frac{iX}{n}\right)^p\right]$$

$$= \sum_{i=0}^{n-1} \frac{X}{n} c \frac{i^p X^p}{n^p}$$

$$= \sum_{i=0}^{n-1} c \frac{X^{p+1}}{n^{p+1}} i^p = c \frac{X^{p+1}}{n^{p+1}} \sum_{i=0}^{n-1} i^p$$

$$S_* = \frac{X^{p+1}}{n^{p+1}} c \left[0^p + 1^p + 2^p + 3^p + \cdots + (n-1)^p\right] \quad \text{--- (1)}$$

Similarly upper sum

$$S^* = \frac{X^{p+1}}{n^{p+1}} c \left[1^p + 2^p + \cdots + n^p\right] \quad \text{--- (2)}$$

Since

$$1^p + 2^p + 3^p + \cdots + n^p = \frac{1}{p+1} n(n+1)(n+2)\cdots(n+p) + O(n^p) \quad \text{--- (3)}$$

where $O(n^p)$ means that for the remaining terms the maximum order is n^p. Hence

$$\lim_{n\to\infty} S^* = \frac{cX^{p+1}}{p+1}\left[\lim_{n\to\infty} \frac{n(n+1)(n+2)\cdots(n+p)}{n^{p+1}} + \lim_{n\to\infty} \frac{O(n^p)}{n^{p+1}}\right]$$

$$= \frac{cX^{p+1}}{p+1}\left[\lim_{n\to\infty} \frac{n^{p+1}(1+\frac{1}{n})(1+\frac{2}{n})\cdots(1+\frac{p}{n})}{n^{p+1}} + \lim_{n\to\infty} \frac{O(n^p)}{n^{p+1}}\right]$$

$$= \frac{cX^{p+1}}{p+1}\left[\lim_{n\to\infty} (1)(1+\tfrac{1}{n})(1+\tfrac{2}{n})\cdots(1+\tfrac{p}{n}) + \lim_{n\to\infty} \frac{O(n^p)}{n^{p+1}}\right.$$

$\therefore$

$$\lim_{n\to\infty} S^* = \frac{cX^{p+1}}{p+1} + 0 = \frac{cX^{p+1}}{p+1}$$

using same procedure we can get

$$\lim_{n\to\infty} S_* = \frac{cX^{p+1}}{p+1}$$

Therefore,

$$\int_0^X cx^p\,dx = \frac{cX^{p+1}}{p+1} \quad \text{———} \quad (A)$$

and $$\int_a^b cx^p\,dx = \left.\frac{cx^{p+1}}{p+1}\right]_a^b = \frac{cb^{p+1}}{p+1} - \frac{ca^{p+1}}{p+1}$$

$$\boxed{\int_a^b cx^p\,dx = \frac{c}{p+1}\left[b^{p+1} - a^{p+1}\right]}$$

Note:— It can show that (A) is general result for all p, negative, positive, zero, fraction etc. as long as $p+1 \neq 0$

General Result:—

$$\int_0^X cx^p\,dx = \frac{cX^{p+1}}{p+1}$$

and

$$\int_a^b cx^p\,dx = \frac{c}{p+1}\left[b^{p+1} - a^{p+1}\right]; \quad \cancel{p+1 \neq 0} \quad p+1 \neq 0$$

Definite and indefinite Integral :-

- $\int_a^b f(x)$, where a and b are fixed called definite integral.

- Indefinite :-

$$\int_0^X f(x)\,dx \;, \quad \int f(x)\,dx$$

where X is arbitrary, are called indefinite integral.

* When X is arbitrary we may note that if $\int f(x)\,dx = F(x)$ then

$$\int_0^X f(x)\,dx = F(X) - F(0)$$

where F(0) is only a constant.

- Whatever ~~of~~ be the character of ~~or~~ variable of integration used when integrating, we get same answer i.e.

$$\int_0^X f(t)\,dt = \int_0^X f(u)\,du = \int_0^X f(y)\,dy = F(X) - F(0)$$

Hence reuse of small x again instead of X that will also fine.

In order to avoid confusion from the limit and character of integration being the same we will stick to X ~~and x to~~ denote the limit of integration x: variable of integration

Some General Observation

~~I~~ Integrand :– Quantity which is integrated out is called ~~the~~ the integrand

Integrand written by using any other symbol such as ct or cz or cu the integral would have been same.

$$\int_0^x cx\,dx = \int_0^x ct\,dt = \int_0^x cz\,dz = \int_0^x cu\,du = c\frac{x^2}{2}$$

→ Integrand can be denoted by any symbol.
So without loss of generality use x instead of X and we write

$$\int f(x)\,dx = F(x) + d$$

e.g. $\int cx\,dx = c\frac{x^2}{2} + d$
where 'd' is an arbitrary constant.

— If the limits of integral 0 to x then $d = 0$ and if limit is from a to x then $d = -c\frac{a^2}{2}$

Definite Integral $\int_a^b f(x)$ → upper limit in the integral.

lower limit in the integral

If we do not have definite limits such as a and b then we write $\int f(x)\,dx$ as an indefinite integral.

$$\int g(x)\,dx = G(x) + d \quad \text{—— (1)}$$

where 'd' is an arbitrary constant.

$$\int_a^b g(x)\,dx = G(b) - G(a) \quad \text{—— (2)}$$

for $g(x) = cx$, $G(x) = \frac{cx^2}{2}$

$g(x) = cx^p$, $G(x) = \frac{c\,x^{p+1}}{p+1}$, $p+1 \neq 0$

— $\int_a^b f(x)\,dx$ gives the ~~area~~ ~~under~~ under the curve bounded by $y=0$, $x=a$, $x=b$, $y=f(x)$

— $\int c\,dx = cx + \cancel{d}$ 'c' is constant

— $\int [f(x) + c]\,dx = \int f(x)\,dx + cx + d$; c, d are constant

— $\int K f(x)\,dx = K \int f(x)\,dx$.
where K is constant

— $\int [K_1 f_1(x)\,dx + K_2 f_2(x)\,dx] = K_1 \int f_1(x)\,dx + K_2 \int f_2(x)\,dx$
where f_1 ~~and~~ and f_2 are ~~integral~~ integreble.

— $\int f_1(x) f_2(x)\,dx \neq \left[\int f_1(x)\,dx\right]\left[\int f_2(x)\,dx\right]$

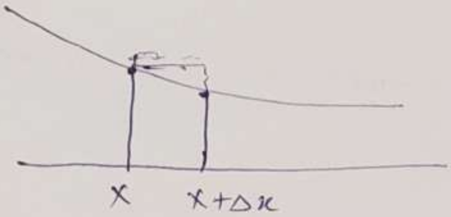
— If f_1 and f_2 are functions of different varicbles and if we are integrating out over individual varicbles and if varicbles are independent (not connected to each other in any way) then we have:

$$\int_x \int_y f_1(x) f_2(y)\,dx \wedge dy = \left[\int_x f_1(x)\,dx\right]\left[\int f_2(y)\,dy\right]$$

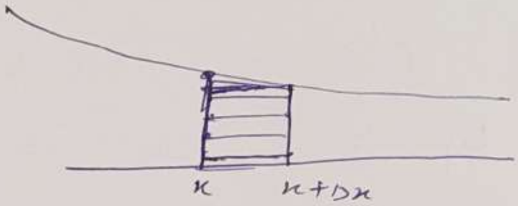
Where $\wedge$: wedge product of differentials.

When integrate out x the function $f_2(y)$ behaves like a constant and vice-versa, where the function f_x means integration ~~over~~ over x.

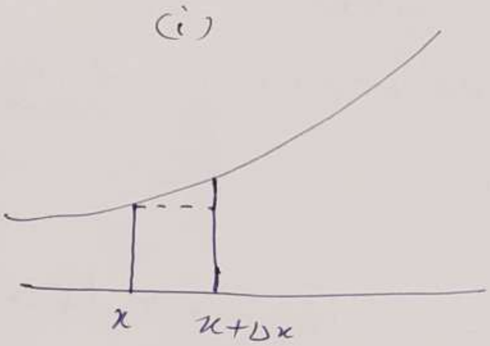
Note :— Consider the following observations on ~~integr~~ integrals:



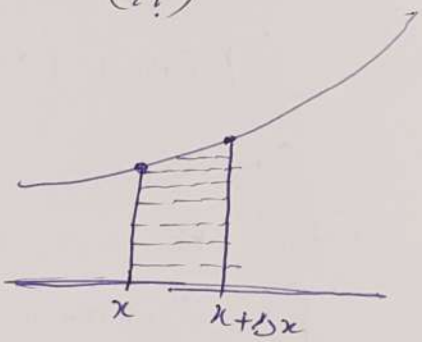
(i)



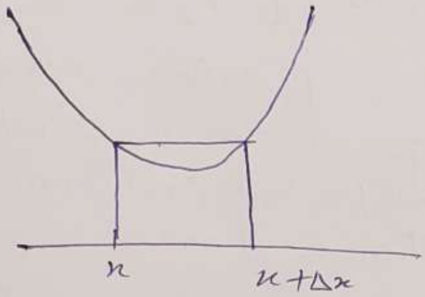
(ii)



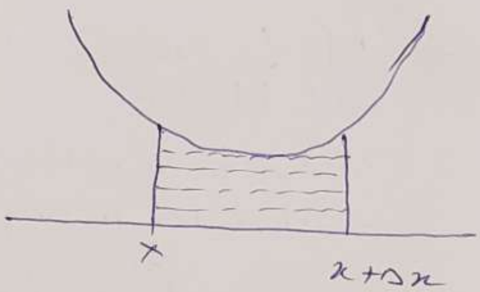
(iii)



(iv)



(V)



(Vi)

What we have seen is that

$\int f(x)\,dx = F(x) + d$, where d is constant, which means that

$$\int_x^{x+\Delta x} f(t)\,dt = \left[F(t) \right]_x^{x+\Delta x} = F(x+\Delta x) - F(x) \quad \text{—(1)}$$

Case (i) and (ii) : $f(x)$ is decreasing function.

$\Rightarrow$ The actual area marked in (ii)

$\Rightarrow$ Approximation is the area of the rectangle $f(x)\Delta x$ in (i) case (iii) and iv) :- Monotonic increasing fn the actual area is in (v) however ~~its approximation~~ approximated by the area of rectangle in (iii)

case (V) and (Vi) :- When the function is decreasing and increasing the approximate area $f(x)\,\Delta x$ is in (v) and exact ~~ea.~~ area in ~~#~~ (Vi).

→ The left side of ① is approximated as

$$f(x)\,\Delta x \longrightarrow f(x)\,dx \quad \text{when } \Delta x \longrightarrow 0$$

Now look at the right side:

$$F(x+\Delta x) - F(x) = \left[\frac{F(x+\Delta x) - F(x)}{\Delta x}\right]\Delta x$$

$$\lim_{\Delta x \to 0}\left[F(x+\Delta x) - F(x)\right] = \lim_{\Delta x \to 0}\left[\frac{F(x+\Delta x) - F(x)}{\Delta x}\right] \cdot \lim_{\Delta x \to 0} \Delta x$$

$$\lim_{\Delta x \to 0}\left[F(x+\Delta x) - F(x)\right] = \left[\frac{d}{dx}F(x)\right]dx$$

Now comparing left and write side

$$f(x)\,dx = \left[\frac{d}{dx}F(x)\right]dx$$

$$\therefore \quad \boxed{\frac{d}{dx}F(x) = f(x)}$$

Hence differential and integration of inverse operations of each other.

⇒ Integral called antiderivative or integration is called anti differentiation.

Result :- $\int_a^x f(t)\,dt = F(x) - F(a)$ then

$$\boxed{\frac{d}{dx}F(x) = f(x)}$$

About Notation

b ← upper limit

↙ Integrand

Integral sign → $\int_a^b f(x)\, dx$

dx ↪ differential element

→ a

lower limit

Example (Modelling using integration) : consider a particle is moving along a straight path and if the distance covered in time t is denoted by $S = S(t)$

Velocity : $v(t)$, Acceleration : $a(t)$,

Jerk : $J(t)$

$S = S(t)$: distance covered in time 't'

$$v(t) = \frac{d}{dt} S(t) = \text{velocity in time } t \equiv \text{Instantaneous rate of change of distance}$$

$$a(t) = \frac{d}{dt} v(t) = \frac{d^2}{dt^2} S(t)$$

= acceleration in time 't'

= instantaneous rate of change of velocity ;

$$J(t) = \frac{d}{dt} a(t)$$

$$= \frac{d^2}{dt^2} v(t)$$

$$= \frac{d^3}{dt^3} S(t)$$

= Jerk in time 't'

= instaneous rate of change of acceleration

$$\int_{c_1}^{t} J(x)\,dx = a(t) - a(c_1) = \text{acceleration}$$

$a(c_1)$ is constant.

$$\int_{c_2}^{t} a(x)\,dx = V(t) - V(c_2) = \text{velocity};$$

$V(c_2)$: constant

$$\int_{c_3}^{t} V(x)\,dx = S(t) - S(c_3) = \text{distance};$$

$S(c_3) = $ a constant

Example 1.9:— A racing car is fast moving and it is timed. Two minutes after clocking, it is found that the car is Jerking at the rate of 2t meters per minute, where t = time is measured in minute and distance in meters. What is the distance travelled during the observation time of 2 minutes to 10 minutes period?

Solution:— Given that the jerk

~~J(t) = 2(t)~~ $J(t) = 2t$

Therefore acceleration from $t=2$ min to an ~~abitrary~~ arbitrary time ~~per~~ t is

$$a(t) = \int_{2}^{t} J(x)\,dx = \int_{2}^{t} [2x]\,dx = [x^2]_2^t = t^2 - 4$$

Now velocity from $t=2$ to an abitrary time t

$$v(t) = \int_{2}^{t} a(x)\,dx = \int_{2}^{t} (x^2-4)\,dx = \left[\frac{x^3}{3} - 4x\right]_2^t$$

$$v(t) = \frac{t^3}{3} - 4t + \frac{16}{3}$$

Distance travelled from $t=2$ to an arbitrary time 't':

$$S(t) = \int_2^t \left[\frac{x^3}{3} - 4x + \frac{16}{3} \right] dx$$

$$= \left[\frac{x^4}{12} - 4 \cdot \frac{x^2}{2} + \frac{16}{3} x \right]_2^t$$

$$S(t) = \frac{t^4}{12} - 2t^2 + \frac{16}{3} t - 4$$

Now $S(10) = \left[\frac{t^4}{12} - 2t^2 - \frac{16}{3} t - 4 \right]_{t=10}$

$$\boxed{S(10) = 682.67}$$

Thus distance covered during $t=2$ to $t=10$ period is approximately 683 meters.

Problem:— A farmer growing tuber crops (such as potato or tropica) has two choices of fertilizers, fertilizer A and fertilizer B. If y denotes the yeild (amount of tuber available) and if growing season is 6 months equal to 2 units of time, 3 months = 1 unit of time, and if the rate of growth under fertilizers A and B are respectively:

$$\frac{dy}{dt} = 2t \quad \text{and} \quad \frac{dy}{dt} = t^2$$

which fertilizer the fermer should chose to obtain better yeilds under fertilizers A and B?

~~Recap:-~~

Recap:- Look the areas under the curve $y = f(x)$

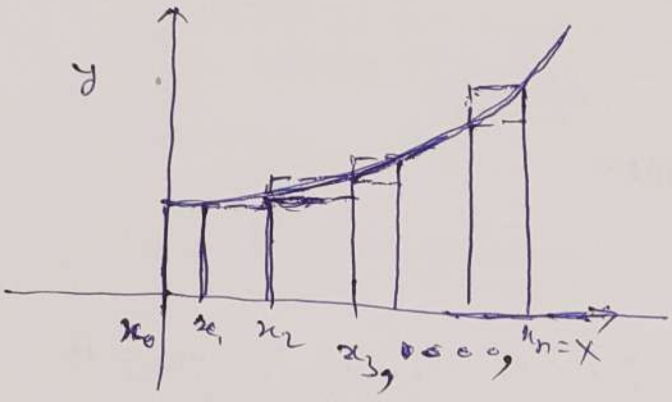


Fig. 2.1 (a)
Indefinite integral

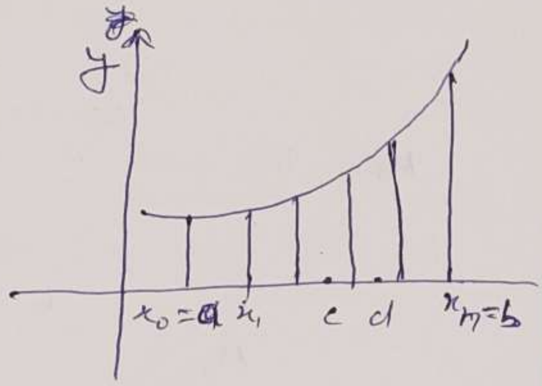


Fig. 2.2 (b)
Definite Integral

Area computed by ~~summing up~~ the infinitely small summation

⟹ Area is to be computed by dividing it into n-equal parts and summing up the area under the curve.

Result (1) If 0 to X or X units is divided into n equals parts then ~~th~~ ~~Each~~ each part is $\frac{X}{n}$.

↳ This small price by Δx

Case 1:- In Figure (1)(a) X then ~~the~~ the points

$x_0 = 0, \quad x_1 = \Delta x, \quad x_2 = 2\Delta x, \dots, \quad x_n = n\Delta x = X$

Case 2:- $x_0 = a, \quad x_1 = a + \Delta x, \quad x_2 = a + 2\Delta x, \dots,$

$x_n = a + n\Delta x = b \quad$ or $\quad \Delta x = \frac{b-a}{n}$

⇒ The areas of lower and upper rectangles and comeup with the upper sum S^* and lower ~~S~~ S_*.

Then the actual area under the curve will be such that

$S_* \leq A \leq S^*$ and as $n \to \infty$ or when $\Delta x \to 0$

$$\lim_{\Delta x \to 0} S_* = \lim_{\Delta x \to 0} S^* = A$$

When the area A is finite

We can also look ~~at~~ at A being approximated by

$$A_n = \sum_{i=1}^{n} f(x_i^*)\,(x_i - x_{i-1}); \quad \lim_{n\to\infty} A_n = A$$

where x_i^* is an arbitrary point in the interval (x_i, x_{i-1}).

In the limit when $n \to \infty$ then A_n ~~will~~ also will go to true area A.

- When $n \to \infty$ then interval (x_i, x_{i-1}) goes to the infinitesimal length or $\Delta x \to dx$. Hence the area A can be written as:

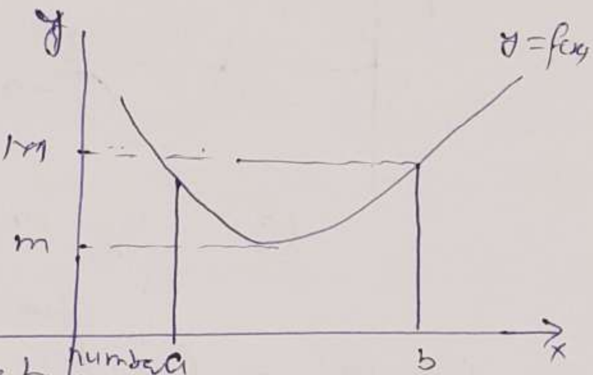
$$A = \int_0^X f(x)\,dx \quad \text{or} \quad \int_a^b f(x)\,dx$$

Indefinite ~~...~~ Integral — Definite Integral

Figure (a) — Figure (b)

Some Basic Results And Its Interpretation :-

Result (1):-



- Integrating function f(x) from a to b. Within this range suppose that-
  (i) f(x) is bounded above by number M
  i.e. $f(x) \le M \quad \forall x \in [a,b]$
  (ii) f(x) is bounded below by number m
  i.e. $f(x) \ge m \quad \forall x \in [a,b]$

- Area of rectangle with height M and width b-a is $(b-a)M$.

- Area of rectangle with height m and width b-a is $(b-a)m$

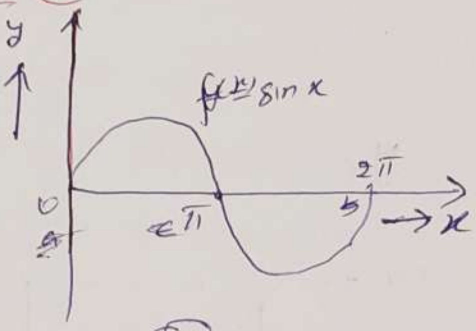
- True Area, A, under the curve $y = f(x)$ ~~and~~ between ordinate $x=a$ and $x=b$ and above x-axis ($y=0$) must satisfy

$$m(b-a) \le A \le M(b-a)$$

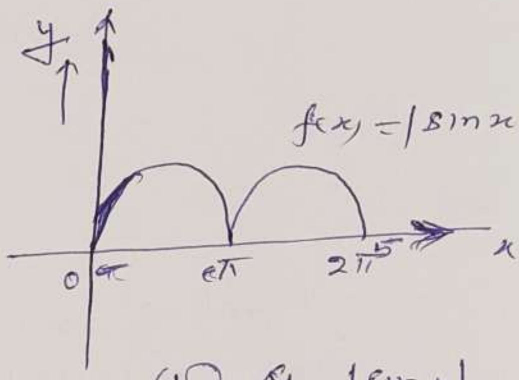
i.e A is in between the area of these upper and lower rectangles

i.e
$$\boxed{m(b-a) \le A = \int_a^b f(x)\,dx \le M(b-a)}$$

Result ②



(a) $f(x) = \sin x$



(b) $f(x) = |\sin x|$

($f(x)<0$)

Case (a) – $f(x)$ can be negative ($f(x)<0$) over the range

c to b, $a < c < b$

– $f(x)$ can be positive ($f(x)>0$) over the range a to c then

— We have following:

$$\int_a^c f(x)\,dx > 0, \quad \text{and} \quad \int_c^b f(x)\,dx \leq 0$$

— The total ~~area~~ Integral $A = \int_a^b f(x)\,dx$ can be

>0

<0

$=0$

depending on the $f(x)$ and the interval $[a, c]$, and $[c, b]$

case (b) :- In case of absolute value of $f(x)$ over the range $a \leq x \leq b$, then the total integral $A = \int_a^b f(x)\,dx$ will be bigger or equal to the true value A. Hence we have following result:

$$\boxed{\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx}$$

Result 3 :–

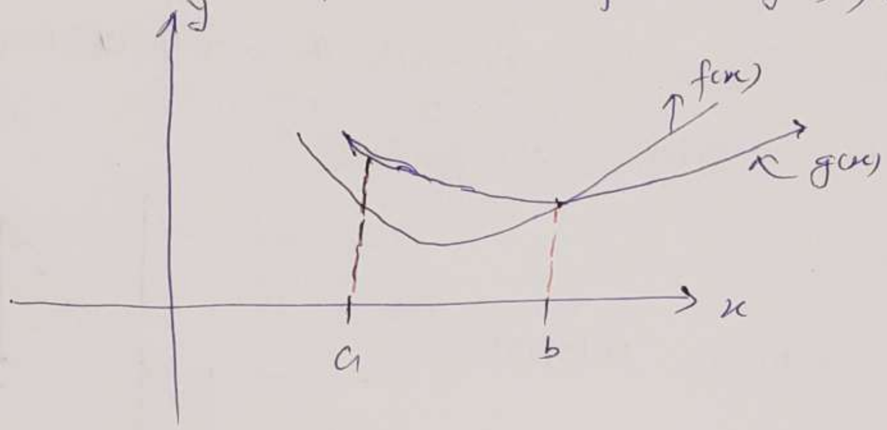
Instead of summing ~~up~~ up infinitesimal rectangles from $\int_a^b$ a to b we start summing up from b to a in opposite direction. Then naturally $\Delta x < 0$ or ~~equivalently $-\Delta x > 0$~~ we may write $-\Delta x$ with $\Delta x > 0$. Hence $f(x)\,dx$ becomes $-f(x)\,dx$ with $\Delta x > 0$. Thus

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

Result 4

Result 4

Let in interval $a \le x \le b$ there is function $g(x)$ which dominates over $f(x)$, i.e., $f(x) \le g(x), \forall x \in [a,b]$



Then

$$\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx, \text{ when}$$

$f(x) \le g(x)$ for $a \le x \le b$

Result 5

Since integral is interpreted as areas, area of a sum is the sum of the areas. ~~and we will restate~~

$$\boxed{\int_a^b [c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)]\, dx = c_1\int_a^b f_1(x)\, dx + c_2\int_a^b f_2(x)\, dx + \cdots + c_k\int_a^b f_k(x)\, dx}$$

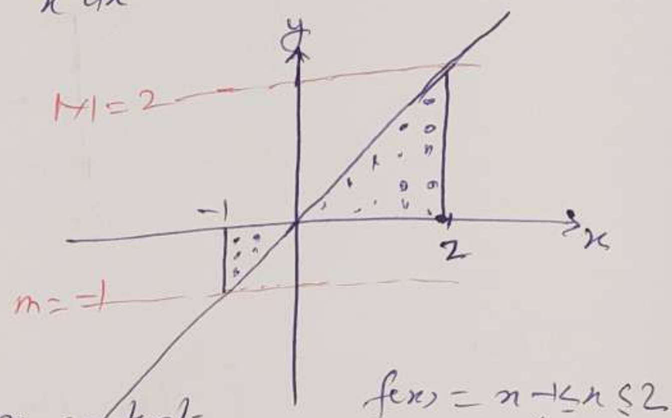
where $c_1, c_2, c_3, \ldots c_k$ are constants and $f_i(x), i = 1, 2, 3 \ldots k$ are some integrable function giving some finite integrals

~~Ex~~

Note :- Results from (1) to (5) are also valid for indefinite integral.

Example :- ~~State the~~ Illustrate the result (1) to (3) for the integral $A = \int_{-1}^{2} x\, dx$

Solution :-



Indefinite integral

$$\int x\, dx = \frac{x^2}{2} + c$$

where 'c' is an arbitrary constant.

— Actual integral under the curve is shown in figure as the shaded region in fig. where one piece of integral is negative.

$$\int_{-1}^{2} f(x)\, dx = \left[\frac{x^2}{2} + c\right]_{-1}^{2} = \frac{2^2}{2} - \frac{(-1)^2}{2} = \frac{3}{2}$$

Now split this integral into two part $\int_{-1}^{0}$ and $\int_{0}^{2}$

$$\int_{-1}^{0} x\, dx = \left[\frac{x^2}{2}\right]_{-1}^{0} = \frac{0^2}{2} - \frac{(-1)^2}{2} = -\frac{1}{2}$$

$$\int_{0}^{2} x\, dx = \left[\frac{x^2}{2}\right] = \frac{(2)^2}{2} - \frac{(0)^2}{2} = 2$$

Hence $\int_{-1}^{0} x\, dx + \int_{0}^{2} x\, dx = \int_{-1}^{2} x\, dx = 2 - \frac{1}{2} = \frac{3}{2}$

— From above figure we can see that $m = -1$ is lower bound and $M = 2$ is upper bound for function $f(x) = x$ in interval $[-1, 2]$

upper limit of integration $b = 2$

lower limit of integration $a = -1$

Thus $b - a = 2 - (-1) = 3$

$\therefore \quad m(b-a) = -1 \times 3 = -3$

$M(b-a) = 2 \times 3 = 6$

The actual integral $A = 3/2$.

$\cancel{-3 \; 6} \quad -3 < A (= 3/2) < 6$

$\therefore \quad \cancel{-3 < \frac{3}{2}(=A) < 6}$

$$\therefore \boxed{m(b-a) < A < M(b-a)}$$

Now consider the integral $\int_{-1}^{2} |f(x)|\, dx$:

$f(x) > 0$ for $0 < x < 2$

$f(x) < 0$ for $-1 < x < 0$

$$\therefore \int_0^2 |f(x)|\, dx = \int_0^2 f(x)\, dx = \int_0^2 x\, dx = \left[\frac{x^2}{2}\right]_0^2$$

$$\int_0^2 |f(x)|\, dx = \frac{4}{2} - \frac{0}{2} = 2.$$

$$\int_{-1}^{0} |f(x)|\, dx = \int_{-1}^{0} -f(x)\, dx = \int_{-1}^{0} (-x)\, dx = \left[-\frac{x^2}{2}\right]_{-1}^{0}$$

$$= -\left[\frac{0^2}{2} - \frac{(-1)^2}{2}\right]$$

$$= -\left[-\frac{1}{2}\right]$$

$$\int_{-1}^{0} |f(x)|\, dx = +\frac{1}{2}$$

$$\therefore \int_{-1}^{2} |f(x)|\, dx = \int_{-1}^{0} |f(x)|\, dx + \int_{0}^{2} |f(x)|\, dx$$

$$= 2 + \frac{1}{2} = \frac{5}{2}$$

$$\boxed{\text{Hence} \quad \int_{-1}^{2} |f(x)|\,dx = \frac{5}{2}}$$

But $\left|\int_{-1}^{2} f(x)\,dx\right| = \left|\frac{3}{2}\right| = \frac{3}{2}$

Since $\frac{3}{2} < \frac{5}{2}$ thus result

$$\boxed{\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx}$$ is verified

Now consider

$$\int_2^{-1} f(x)\,dx = \left[\frac{x^2}{2}\right]_2^{-1} = \left[\frac{(-1)^2}{2} - \frac{(2)^2}{2}\right]$$

$$= \frac{1}{2} - 2$$

$$= -\frac{3}{2}$$

$$\boxed{\int_{-1}^{2} f(x)\,dx = \frac{3}{2} = -\int_2^{-1} f(x)\,dx = -\left[-\frac{3}{2}\right]}$$

Note :- (i) Area under given curve $\Rightarrow$ absolute values of integrals

(ii) If integral gives the negative quantity then the quantity is ~~treated~~ counted as negative quantity, if interested in total integral.

(iii) If it is to be interpreted as area then absolute value is taken because term "negative area" may not convey a proper meaning

$$\boxed{\int_{-1}^{2} |f(x)|\, dx = \int_{-1}^{2} |x|\, dx = \int_{-1}^{0} (-x)\, dx + \int_{0}^{2} x\, dx = \frac{5}{2}}$$

Result 6

odd function: $\cancel{f(x)}\ f(-x) = -f(x)$

Even function: $f(-x) = f(x)$

$$\int_{-a}^{a} f(x)\,dx = 0 \quad \text{for } f(-x) = -f(x) \text{ and } \int_{0}^{a} f(x)\,dx < \infty$$

Result 7

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx, \quad \text{for } \cancel{f(x)}\ f(-x) = f(x)$$

$$\text{and } \int_{0}^{a} f(x)\,dx < \infty$$

Assignment

~~Assign~~ Assignment – I :—

Let $f(x)$ and $g(x)$ be real-valued functions of the real variable x. Let c be a real number. Consider the equation

$$\int_a^b [c f(x) + g(x)]^2 \, dx = 0$$

Evaluate the condn that the quadratic equn above for c has no real root.

Assignment – II :—

With the help of ~~eq~~ Assignment-I prove the Cauchy-Schwartz inequality

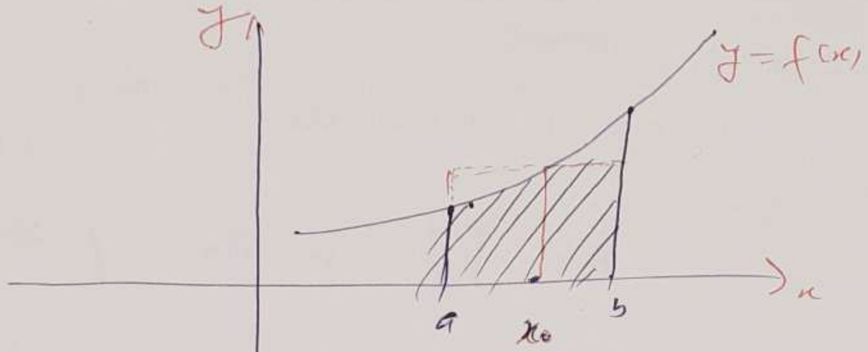
$$\left[\int_a^b f(x) g(x) \, dx \right]^2 \leq \int_a^b f^2(x) \, dx \int_a^b g^2(x) \, dx$$

Assignment – III :—

By using Assignment – II, show that

$$\boxed{\left[\int_a^b f(x) \, dx \right]^2 \leq (b-a) \int_a^b f^2(x) \, dx}$$

The Mean Value Theorem



Let ~~A~~ be area under the curve, between the ordinate $x = a$, $x = b$ and above x-axis.

Since area is positive quantity / number we can always write it as

$$A = (b-a)\theta, \text{ for some constant } \theta$$

e.g. $A = 12$, $b = 6$, $a = 2$ then $b - a = 4$ and $\theta = 3$. If $\theta = f(x_*)$ can be solved for x_* for a given θ, then we can write following:

Result (Mean Value theorem) :—

$$A = (b-a) f(x_*) = \int_a^b f(x)\, dx$$

$(b-a)$ is area of rectangle with width $b-a$ and height $f(x_*)$, here $x_* \in [a, b]$

- At $x = x_*$ area excluded is equal to the area included. and hence $(b-a) f(x_*)$ becomes true area under the curve.
- Such choice of x^* is possible if one can solve for x^* from the equation $\theta = f(x^*)$, for a given θ.

Example — Illustrate the mean value theorem for the integral

$$A = \int_1^3 (2+3x)\, dx$$

Solution —

$$A = \int_1^3 (2+3x)\, dx = \left[2x + \frac{3}{2}x^2 \right]_1^3$$

$$= \left(2\cdot(3) + \frac{3}{2}\cdot(3)^2\right) - \left(2\cdot(1) + \frac{3}{2}(1)^2\right)$$

$$= \left(6 + \frac{3}{2}\cdot 9\right) - \left(2 + \frac{3}{2}\right)$$

$$= \left(6 + \frac{27}{2}\right) - \frac{7}{2}$$

$$= \frac{39}{2} - \frac{7}{2}$$

$$= \frac{32}{2}$$

$$\boxed{A = 16}$$

$A = (b-a)\,\theta \Rightarrow (3-1)\,\theta = 16 \Rightarrow \theta = 8$

Now $\theta = f(x_*) \Rightarrow 2 + 3x_* = 8$

$\Rightarrow x_* = \frac{8-2}{3} = 2$

$\therefore \boxed{x_* = 2}$

$$\boxed{\begin{aligned}(b-a)\,f(x_*) &= (3-1)\,(2+3\times 2)\\ &= 2\times 8\\ &= 16 = A\end{aligned}}$$

Another Result:

Since $\int f(x)\,dx = F(x)+C$, where 'C' is an arbitrary constant. Then

$$\int_x^{x+\Delta x} f(x)\,dx = F(x+\Delta x) - F(x)$$

Now consider

$$\frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{1}{\Delta x}\int_x^{x+\Delta x} f(x)\,dx \quad \text{--- (1)}$$

Now apply the mean value theorem for evaluating

$\int_x^{x+\Delta x} f(x)\,dx$

$$\int_x^{x+\Delta x} f(x)\,dx = (x+\Delta x - x)\cdot f(x^*)$$

$$= \Delta x\, f(x^*), \quad x^* \in [x, x+\Delta x]$$

Now take $x^* = x + k\Delta x$, where k is some constant

$$\therefore \int_x^{x+\Delta x} f(x)\,dx = \Delta x\, f(x+k\Delta x) \quad \text{--- (2)}$$

from (1) and (2) we have

$$\lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\cancel{\Delta x}}\cancel{\Delta x}\cdot f(x+k\Delta x)$$

$$= \lim_{\Delta x \to 0} f(x+k\Delta x)$$

$$\therefore \boxed{\frac{d}{dx}F(x) = f(x)}$$

$k\Delta x \to 0$ as $\Delta x \to 0$

where $\int f(x)\,dx = F(x) + C$ or $\int_a^x f(t)\,dt = F(x) - F(a)$

Problems

(1) Find x^* ~~of~~ for integral

$$\int_2^3 \frac{dx}{x} \quad \text{such that}$$

$$\boxed{(3-2)\frac{1}{x^*} = \int_2^3 \frac{dx}{x}}$$

(2) If oil is consumed at the rate of

$$R(t) = 16\, e^{0.07t}$$

talking 1970 as the sterting year $t=0$, then what is total consumption of oil between the years 1980 and 1995

(3) A factory marginal cost is

$$c'(x) = \frac{1}{300}x^2 - 2x + \cancel{300}\ 120$$

for producing x units per day. Find the cost of producing 200 units per day if the current production level is 100 units per day.

(4) The spread of an epidemic is the rate of

$$P'(t) = 120t - 3t^2$$

peple per day. At $t=0$, 100 peple are affected ($P(0) = 100$) Find the number after 10 days

Methods of Integration:

- Methods of Substitution
- Algebraic ~~manup~~ Manipulation
- Technique of Partial fractions
- Technique of Integration by Parts

Convergence of Integral :—

- Convergence and divergence of integrals is associated with definite ~~int~~ integrals of the form $\int_a^b f(x)\,dx$
- length of interval $[a, b]$ could be infinite also i.e. lower limit a could be $-\infty$ and the upper ~~bound~~ limit b could be $+\infty$

Defn :— Convergence of Integrals :— If a definite integral produces a finite quantity then we say that the ~~in~~ integral is convergent and we write

$$\int_a^b f(x)\,dx < \infty$$

otherwise integral is divergent

Divergence can come in many ways.

- $\int_a^b f(x)\,dx = \pm\infty$
- $\lim_{x \to b} \int_a^x f(x)\,dx$ does not ~~exists~~ exist
- $\lim_{x \to a} \int_x^b f(x)$ does not ~~exist~~ exist

Defn :— ~~Improper~~ ~~and~~ Proper Integral :— In a definite integral of the form $\int_a^b f(x)\,dx$ if a or b or both are infinite, i.e. :

$$\int_{-\infty}^{b} f(x)\,dx,\ b < \infty \ ; \quad \int_a^{\infty} f(x)\,dx,\ a < \infty \ ; \quad \int_{-\infty}^{+\infty} f(x)\,dx$$

Then the integral is called improper integral. All other definite integrals are called proper integrals.

Double Integral :-

Let f be function of two variable

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$z = f(x,y)$$

When $f(x,y) \geq 0$ over a region S on the x-y plane then we can think that $z = f(x,y)$ is surface or hill sitting ~~over~~ over the region S. The double integral over S, that is

$$\iint_S f(x,y)\, dx\, dy$$

— Limits and Continuity —

:— Expression And Functions :—

A. ~~mathematical expression~~

Term :- A term is combination (valid ways) of number/single variable/product of several variable but never use addition and subtraction ~~A term is a product of a scalar/constant with product of one or more variable/ or a term is simply constant~~

Mathematical Expression :- ~~A mathematical expression is the representation in which term are joined by plus, minus &~~

~~When a number~~

When we combine numbers and variable in a valid way, using operations such as addition, subtraction, multiplication, division, exponentiation, and other operations, ~~such as addition,~~ ~~and functions as yet un~~ the resulting combination of mathematical symbols is called mathematical expression.

e.g $2a, x+5$, and y^2

Degree of term :- The degree of a term in an expression or equation is determined by the following convention:

- A constant is assigned degree zero
- A variable is assigned degree ~~one~~ (1) if it appears only once in the term or m if the variable raised to m appears in the term.
- the degree of product ~~term~~ elements/term is the sum of ~~element~~ the degree of the elements.

Note :- Fractional and negative powers are not assigned the degree.

Example :- Here x_1, x_2 and x_3 are Variable.

Term	Degree	Term	Degree
5	0	$2x_1 x_2 x_3$	$0+1+1+1=3$
$5x$	$0+1=1$	$2x_1^4 x_2 x_3^3$	$0+4+1+3=8$
$5x^2y$	$0+2+1=3$	$\frac{1}{2}x_1 x_2^2 x_3$	$0+1+2+1=4$
$5x^{\frac{1}{2}}y$	not assigned	$3x_1 x_2^{\frac{1}{2}} x_3$	not assigned
$5x^2y^4$	$0+2+4=6$	$3x_1^{-1} x_2 x_3^3$	not assigned

Degree of expression containing many term :-

~~Degree of~~ maximum degree of the term present in expression

Expression	Degree
$2+5x-3yx$	2
x^2+y^2-4xy	2
$1+x_1+x_2+3x_3+5x_1^4x_2+4x_2^3$	5

Linear form and Qudratic form :-

If all terms containing the variables have degree one each, then it is called a linear form; and if all terms in an expression are of degree 2 each then it is called a quadratic form.

Note :- "form" is reserved for the homogeneous case or all terms are of the same degree.

- e.g. "Qudratic form" means all terms are of degree two each.
- " Qudratic expression" indicate that the maximum degree of the terms there is two.

For example!

Expression	Degree
$3x+2y$	1, Linear form
$3x+2y-7$	1, Linear expression
0	0, constant
x^2+y^2	2, quadratic form
x^2-3y^2+7xy	2, quadratic form
$x^3+y^3+2x^2y$	3 cubic form.

→ **Function** :—

For given value of a variable (say x) if we get an unique value/only one value (say y), then the relationship between variables (x and y) is called a function and it is denoted as

$$y = f(x)$$

In, otherwords, if a mathematical expression gives a unique value for a particular assignment of value to variable ~~appear~~ appearing ~~int~~ in it then it is called as function.

Explicit function

A variable y is said to be an explicit function of variable x, if for every given value x, there is only one value for y. It is written as $y=f(x)$ and read as y equals f of x or y is function of x

e.g. (i) $u=2v^2+v+4$ (ii) $z=t+5$

(iii) $y^2=4x$ ✗ (iv) $y^2=4x^2$ ✗

(v) $x^2+y^2=1$ ✗

Implicit function

↳ If relationship between any two variables where neither of them can be written as an explicit function of the other then function is called an implicit function of the two variables.

(iii), (iv), (v)

Note :- Explicit function are simply called function removing adjective "explicit".

Ex :- → Height and weight:-

time	Height (h) (meter)	weight (w) (Kg)
birth time	0.5 m	2.5 Kg
6 month	0.8 m	3.5 Kg
1 year	0.9 m	5.0 Kg
11 year	1.5 m.	35.0 Kg
13 year	1.9 meter	50 kg
⋮	≃ 1.9 meter	55 Kg
	≃ 1.9 meter	60 kg

— As we can write weight (w) as function height (h)

we do not know the exact form. $w = f(h)$

* See After age 13 years full height 1.9 m. attains where as the weight w changes.

↳ ~~Therefore~~ Thus there are several values of w for ~~given~~ same h.

⇒ Therefore w can not be explicit function of h after the ~~height h = 1.9 m. when attains the weight w = 50~~ the age of 13 year

* ~~As~~ ~~h = f(t)~~ Plot h against time 't'.

then for every value of t there is only one value for height h.

$\therefore \quad h = f(t)$

Similarly $\quad \underline{w = f(t)}$

Limits

Ex (1) Evaluate $y = 2x - 3$ at (1) $x = 1$, (2) $x = 0$, (3) $x = 1.9$

Solution :- (1) $x = 1$; $y = 2 \times 1 - 3$

$\boxed{y = -1}$

(2) $x = 0$; $y = 2 \times 0 - 3$

$\boxed{y = -3}$

(3) $x = 1.9$

$y = 2 \times 1.9 - 3$
$= 3.8 - 3$

$\boxed{y = 0.8}$

Ex (2) :- If $y = \frac{x^2 - 1}{x - 1}$ evaluate y at (1) $x = 1$ (2) $x = 1.1$, $x = 1.01$, $x = 1.001$, $x = 0.9$, $x = 0.99$, $x = 0.999$

Solution :- (1) ~~y=~~ $x = 1$; $y = \frac{1^2 - 1}{1 - 1}$

$y = \frac{1 - 1}{1 - 1}$

$y = \frac{0}{0}$

Denominator : $x - 1$
at $x = 1$, $x - 1 = 1 - 1 = 0$

$\therefore$ at $x = 1$, $y = \frac{x^2 - 1}{x - 1}$ can not evaluated
(Divide by zero is undefined quantity in mathematics)

(2) $x = 1.1$, $x = 1.01$, $x = 1.001$, $x = 0.9$, $x = 0.99$, ~~x = 0.9.9~~ $x = 0.999$

$x = 1.1$, $y = \frac{(1.1)^2 - 1}{(1.1) - 1} = \frac{1.21 - 1}{(1.1) - 1} = 2.1$

$x = 1.01$; $y = \frac{(1.01)^2 - 1}{1.01 - 1} = \frac{1.0201 - 1}{1.1 - 1} = 2.01$

$x = 1.001$; $y = \frac{(1.001)^2 - 1}{1.001 - 1} = \frac{(1.002001)}{(0.001)} = 2.001$

Clearly, as x closer to ~~1 (x → 1)~~ from the right ($x \to 1^+$), y is comming closer and closer to the value 2.

Going closer to 1 from left :-

At $x = 0.9$, $y = \frac{(0.9)^2 - 1}{0.9 - 1} = \frac{(0.81) - 1}{(-0.1)} = 1.9$

$x = 0.99$, $y = \frac{(0.99)^2 - 1}{0.99 - 1} = 1.99$

$x = 0.999$, $y = \frac{(0.999)^2 - 1}{0.999 - 1} = 1.999$

Thus as ~~x~~ x closer to 1 from left ($x \to 1_-$) then y approaches the value 2.

$\Downarrow$

Thus there is definite value 2 when x approaches 1 either from left or right even though at ~~z~~ $x = 1$, y is undefined.

$\Downarrow$

It can also seen that if $x \neq 1$ we can write

$$y = \frac{x^2 - 1}{x - 1} = \frac{(x+1)(x-1)}{x-1} = x + 1$$

(division $x - 1$ is posible since $x - 1 \neq 0$) at $x = 1$

$\therefore \boxed{y = x + 1}$

over all conclusion

clearly as $x \to 1$ $x + 1 \to 2$

Note :- limit of f(x) when x is going to a number a is not a evaluation of f(x) at $x = a$.

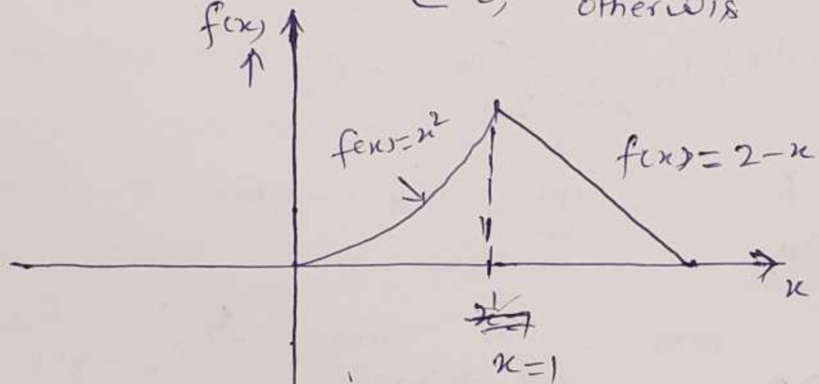
- Some times f(x) at $x = a$, i.e. f(a) may not exisit while both limit may exisit
- Some time f(x) at $x = a$, f(a), may ~~many~~ exist but not equal to the limit.

Definition :- If an explicit function or function $f(x)$ approaches a definite ~~number~~ number L, when x approaches a, from the right or from the left then we say that the limit of $f(x)$ when x approaches a, is L. This written as:

$$\boxed{\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = \lim_{x\to a} f(x) = L}$$

Example:— consider the function

$$f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & \text{otherwis} \end{cases}$$



Evaluate the limit of $f(x)$ when (1) $x \to 1$; (2) $x \to 0$ (3) $x \to 2$ (4) $x \to -10$, (5) $x \to 20$

(1) when $x \to 1_-$ the function near x but $x < 1$ is $f(x) = x^2$. Hence limit is computed from ~~the~~ $f(x) = x^2$. This function has no singularity in $(0, 1]$ since no zero in the denominator

$$\boxed{\lim_{x\to 1_-} f(x) = \lim_{x\to 1_-} x^2 = (1)^2 = 1}$$

- when $x \to 1$ from the right; then $x > 1$ and function $f(x) = 2-x$. This is again has no singularity.

$$\lim_{x\to 1_+} f(x) = \lim_{x\to 1_+} (2-x) = 2-1 = 1.$$

$$\boxed{\lim_{x\to 1_+} f(x) = 1}$$

(2) when $x \to 0$ from left $(x \to 0_-)$ then $f(x) = 0$

~~for~~

for $-\infty < x \le 0$ and hence

$$\lim_{x \to 0_-} f(x) = 0$$

when $x \to 0$ from the right $(x \to 0_+)$ then $f(x) = x^2$

and $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} x^2 = (0)^2 = 0$

$$\therefore \boxed{\lim_{x \to 0-} f(x) = \lim_{x \to 0+} f(x) = \lim_{x \to 0} f(x) = 0}$$

(3) Similarly $\lim_{x \to 2} f(x) = 0$

and

(4) $\lim_{x \to 10_-} f(x) = \lim_{x \to 10+} f(x) = 0$

(5) $\boxed{\lim_{x \to 20-} f(x) = \lim_{x \to 20+} f(x) \Rightarrow \lim_{x \to 20} f(x) = 20}$

Ex:- Oscillating function:- Consider the function

$$f(x) = \begin{cases} 1 + (-1)^x, & x = 0, 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

Evaluate the limit of $f(x)$ when $x \to +\infty$

Solution:- Note that when $x = 0$, $f(x) = 1 + (-1)^0 = 1 + 1 = 2$

$x = 1 \Rightarrow f(x) = 1 + (-1)^1 = 1 - 1 = 0$

$x = 2 \Rightarrow f(x) = 1 + (-1)^2 = 1 + 1 = 2$

When x is even positive integer $f(x) = 2$

When x is an odd positive integer $f(x) = 0$.

$\therefore \lim_{x \to +\infty} f(x)$ does not exist when $x \to -\infty$

$$\boxed{\lim_{x \to -\infty} f(x) = 0}$$

Limit of a Function of Several Variables :—

— Suppose that e.g. Growing child Growing child weight (w) depends on the height (h) and x the total amount of food consumed so far.
(weight may depend on many other factor)

— Let us consider only two variable h and x which could be preassigned. Now consider

$$w = f(h, x)$$

This function 'f' is not known.

— If functional form of 'f' is known and suppose that

$$w = 2 + 5h + \frac{1}{100}x$$

where 'w' is measured in kilogram, height h in meters and x in kilogram. Then when height h approaches 1 meter, w will approach

$$w \to 2 + 5 + \frac{x}{100} = 7 + \frac{x}{100}$$

when total amount of food consumed x approaches 200 kg. Then w approaches:

$$w \to 2 + 5(1) + \frac{200}{100} = 9\,kg$$

This can defined as $w \to 9\,kg$, when $h \to 1\,m$ and $x \to 200\,kg$.

or

$$\boxed{\lim_{h \to 1,\ x \to 200} f(x, h) = 9}$$

Thus when function 'f' is function of many variable say $x_1, x_2, \ldots, x_k$ then $f = f(x_1, x_2, x_3, \ldots, x_k)$ and we may evaluate the limit $f = f(x_1, x_2, \ldots, x_k)$ when individual variable x_k's goes to specific values or some of them go to specific value or all of x_i's go to specific values, provided limit exist.

$$\lim_{x_1 \to a_1} \lim_{x_2 \to a_2} \cdots \lim_{x_k \to a_k} f(x_1, x_2, x_3, \ldots, x_k) = \lim_{x_i \to a_i,\ i = 1, \ldots k} f(x_1, x_2, x_3, \ldots x_k)$$

[here assume the $x_1, x_2, \ldots x_k$ are functionally independent, i.e. none of them are function of each other]

Continuity at a Point :

- The limit of a function f(x) at a point $x = a$ is available if the left limit and right limit, $x \to a_-$ and $x \to a_+$ are equal.

For continuity at $x = a$ we need the following conditions

(i) $\lim_{x \to a_-} f(x)$ exists and it is a finite quantity

(ii) $\lim_{x \to a_+} f(x)$ exists and it is a finite quantity

(iii) $\lim_{x \to a_-} f(x) = \lim_{x \to a_+} f(x) = \left[\lim_{x \to a} f(x)\right]$

(iv) $\lim_{x \to a} f(x) = f(a)$ = function evaluted at $x = a$

Thus condition (iv) is the crucial condition for the continuity of the function at $x = a$

Ex :- Check the continuity of this function over the real line $-\infty < x < +\infty$

$$\boxed{f(x) = \frac{x^2 - x - 2}{x - 2}}$$

Soln :-
$$\lim_{x \to 2-} f(x) = \lim_{x \to 2+} f(x) = 2 \neq f(2)$$

Ex - sharp corner :- consider the function $f(x) = |x-1|$. Check the limit of the function when (1): $x \to 0$ (2): $x \to 2$, (3): $x \to 1$

Soln :-
$$|x-1| = \begin{cases} x-1, & \text{for } x > 1 \\ 1-x, & \text{for } x < 1 \\ 0, & \text{at } x = 1 \end{cases}$$

:- ε, δ Definition of limits :-

Precise mathematical Definition, we illustrate by example:

Ex:- consider the function $f(x) = 4x^2$ and look at the limit when $x \to 1$

Clearly limit is 4.

This means that when x is near 1 either slightly bigger than 1 or slightly smaller than 1, then f(x) will be near 4, may be slightly bigger than 4 or slightly smaller than 4.

⇒ When x is in the neighbourhood of 1 then f(x) will be in the neighbourhood of 4. i.e

$$x \in N(1, \delta) \Rightarrow f(x) \in N(4, \varepsilon)$$

$$N(1, \delta) \equiv |x-1| < \delta$$

e.g.

$$\delta = 0.1 \Rightarrow |x-1| < 0.1 \Rightarrow -0.1 < x-1 < 0.1 \Rightarrow 0.9 < x < 1.1$$

$$\delta = 0.01 \Rightarrow |x-1| < 0.01 \Rightarrow -0.01 < x-1 < 0.01 \Rightarrow 0.99 < x < 1.01$$

$$\delta = 0.001 \Rightarrow |x-1| < 0.001 \Rightarrow -0.001 < x-1 < 0.001 \Rightarrow 0.999 < x < 1.001$$

Now calculate the corresponding neighbourhood of f(x) to the limiting value of 4. Let us start with $\delta = 0.1$ or $0.9 < x < 1.1$

$\underline{\delta = 0.1} \equiv 0.9 < x < 1.1$, when $x = 1.1$ we have $f(x) = 4(1.1)^2$

$$= 4(1.21)$$

$$f(x) = 4.84$$

When $x = 0.9$, we have $f(x) = 4(0.9)^2 = 4(0.81) = 3.24$

i.e.

$$3.24 < f(x) < 4.84 \Rightarrow 3.24 - 4 < f(x) - 4 < 4.84 - 4$$

$$= \underline{0}.76 < f(x) - 4 < 0.84$$

- Thus when $|f(x) - 4|$ is between $\underline{0}.76$ and 0.84 it is definitely between -0.84 and $+0.84$.
- Similar computation can be done for each δ to come up with statement f(x) is ~~not~~ neighborhood of 4 here.

Let us converse question :- Select a number $\varepsilon > 0$ and take that $|f(x)-4| < \varepsilon$, e.g. $\epsilon = 0.2$ then the statement

$$|f(x)-4| < 0.2$$

$$\Rightarrow f(x) \in N(4, 0.2)$$

Can we find a neighborhood of x around 1, or we can find a δ such that $|x-1| < \delta$?

$$|f(x)-4| < 0.2 \Rightarrow |4x^2-4| < 0.2$$

$$\Rightarrow |x^2-1| < \frac{1}{20} = 0.05$$

$$\Rightarrow -0.05 < x^2-1 < 0.05$$

$$\Rightarrow 0.95 < x^2 < 1.05$$

$$\Rightarrow \sqrt{0.95} < x < \sqrt{1.05}$$

$$\Rightarrow 0.97 < x < 1.03$$

$$\Rightarrow 0.97-1 < x-1 < 1.03-1$$

$$\Rightarrow -0.03 < (x-1) < 0.03$$

$$\Rightarrow |x-1| < 0.03$$

Here $\boxed{\delta = 0.03}$ actually $\delta = f(\varepsilon)$.

Thus $\underbrace{|f(x)-4| < \varepsilon}_{(=0.2)} \Rightarrow |x-1| < \delta\ (=0.3)$

Similarly other ε:

Definition:— Suppose that there exist a finite number L and a number a where L is such that for every $\varepsilon>0$, however small it may be, $|f(x)-L|<\varepsilon$. For every such given $\varepsilon>0$ if there exists a δ such that $|x-a|<\delta$ then we say that ~~L~~ L is the limit of f(x) when x goes to a.

i.e.

$$\boxed{\begin{array}{l}\forall\ \varepsilon>0\ ,\ \exists\ \delta>0\ \text{ s.t.}\\ \quad |f(x)-L|<\varepsilon \Rightarrow |x-a|<\delta\end{array}}$$

equivalently :— If two number 'L' and 'a' exist and if we ~~make~~ make the distance between f(x) and L arbitrary small, i.e., $|f(x)-L|<\varepsilon$, $\varepsilon>0$, however small it may be, can we find ~~negi~~ neighbourhood of x around a, i.e., $0<|x-a|<\delta$ for some δ? If this possible then we say that f(x) goes to L when x goes to a or

$$\lim_{x\to a} f(x) = L.$$

Ex:— Let $f(t)=-2t^2+5t+3$. Check the limit of f(t) by using ε, δ notation for t going to zero.

Solution:— $f(0)=3$; we can guess that as $t\to 0$, $f(x)\to 3$.

Now consider $f(x)\in[3-\varepsilon,\ 3+\varepsilon]$

Let $f(x)=3+\varepsilon \Rightarrow -2t^2+5t+3=3+\varepsilon \Rightarrow 2t^2-5t+\varepsilon=0$

$$t=\frac{5\pm\sqrt{25-8\varepsilon}}{4}=\frac{5}{4}\pm\frac{\sqrt{25-8\varepsilon}}{4}$$

$\varepsilon=1$; $\quad t=\frac{5}{4}\pm\frac{\sqrt{17}}{4} \Rightarrow \frac{5}{4}-\frac{\sqrt{17}}{4}<t<\frac{5}{4}+\frac{\sqrt{17}}{4}$ — (1)

$f(x)=3-\varepsilon \Rightarrow t=\frac{5\pm\sqrt{25+8\varepsilon}}{4}=\frac{5}{4}\pm\frac{\sqrt{25+8\varepsilon}}{4}$

take upper and lower value

$$\frac{5}{4}-\frac{\sqrt{25+8\varepsilon}}{4}<t<\frac{5}{4}+\frac{\sqrt{25-8\varepsilon}}{4} \Rightarrow \varepsilon=1\quad \frac{5}{4}-\frac{\sqrt{33}}{4}<t<\frac{5}{4}+\frac{\sqrt{33}}{4}$$

(δ)

Limit and continuity at End Points

↳ When function is defined in a finite interval, e.g. If temperature between 30°c and 40°c in the growing season is going to kill the crops then a farmer will be definitely interested in the interval $30 \le T \le 40$, where T denotes the temperature.

$f(T)$: function of temperature T

Our interest will be on the function $f(T)$ for $30 \le T \le 40$.

↳ At $T = 30$, there is no left limit and similarly at $T = 40$ there is no right limit.

So at boundary point limit and continuity defined ~~dif~~ differently

~~$T \to 30$ w~~ $\lim\limits_{T \to 30} f(T)$ will defined through only right limit

$$\boxed{\lim_{T \to 30} f(T) = \lim_{T \to 30+} f(T)}$$

Similarly

$$\boxed{\lim_{T \to 40} f(T) = \lim_{T \to 40-} f(T)}$$

Left contn:– $f(T)$ is contn at left boundary point $T = 30$ iff

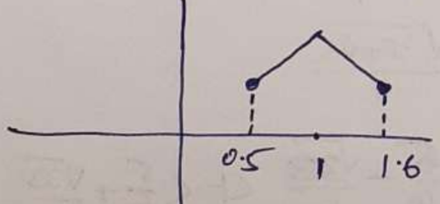
$$\boxed{\lim_{T \to 30} f(T) = \lim_{T \to 30+} f(T) = f(30)}$$

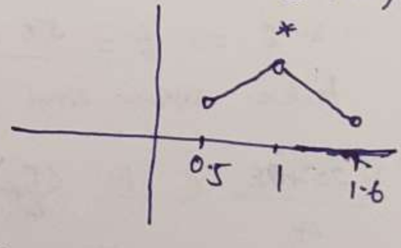
Right contn:– $f(T)$ is contn at right boundary point $T = 40$ iff

$$\boxed{\lim_{T \to 40} f(T) = \lim_{T \to 40-} f(T) = f(40)}$$

Ex:– Consider following two function

$$f_1(x) = \begin{cases} x, & 0.5 \le x < 1 \\ 2-x, & 1 \le x \le 1.6 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} x, & 0.5 < x < 1 \\ 2-x, & 1 < x < 1.6 \\ 1.8, & x = 1 \end{cases}$$





Ex:– Check the continuity of the functions $f_1(x) = [x]$ = largest integr in x and

$$f_2(x) = \begin{cases} 1, & -\infty < x \le 2 \\ -1, & -2 < x < \infty \end{cases}$$

Derivatives And Differentiation

↳ special ~~time~~ type of limits which will measure the rate of change of one variable with respect to another variable or ~~on~~ certain limit which can measure the instantaneous rate of change.

Example:— car moving a constant speed.

Let the car is moving along a straight road at 60 Kilometer (Km) an hour.

$t=1 \Rightarrow S=60;\quad t=2 \Rightarrow S=120;\quad t=\frac{1}{2} \Rightarrow S=30$

$t=0.25 \Rightarrow S=15;\quad t=0.1 \Rightarrow 6;\quad t=0.01 \Rightarrow S=0.6$

$t=0.001 \Rightarrow \cancel{S=0.0001}\ S=0.06;\quad t=0.00001 \Rightarrow S=0.0006.$

and so on.

⇒ Average speed or the change in distance with respect to change in the time or the average rate of change of distance S w.r.t to time or change in S per unit change in t.

⇒ when $t=0$ to $t=1$ then S moves from $S=0$ to $S=60$

⇒ Average rate of change in S with respect to t at $t=0$ is then

$$\text{Average rate of change} = \frac{\text{change in } S}{\text{change in } t} = \frac{60-0}{1-0} = 60$$

When t changes from $t=0$ to $t=0.5$, then the first half an hour

$$\text{Rate of change} = \frac{\text{change in } S}{\text{change in } t} = \frac{30-0}{0.5-0} = \frac{30}{0.5} = 60$$

t changes from $t=0$ to $t=0.1$, S changes from $S=0$ to $S=6$

$$\text{Rate of change} = \frac{6}{0.1} = 60$$

When t changes from $t=0$ to $t=0.001$, S changes from 0 to 0.06

$$\text{Rate of change} = \frac{0.06}{0.001}$$

$$\text{Rate of change} = 60$$

⇒ When the change in t is near zero or the instantaneous rate of change at $t=0$ is the instantaneous rate of change of S with respect to t

$$= \frac{\text{corresponding change in } S}{\text{instantaneous change in } t}$$

$$= 60.$$

In mathematical term above can be stated as follow:

when car moving with constant speed 60 Km/hour then S is written as function of t as:

$$\boxed{S = 60t}$$

$$\Rightarrow \quad \frac{\text{corresponding change in } S}{\text{Infinitesimal change in } t} = 60$$

$$\frac{60(0.1) - 60(0)}{(0.1) - (0)} = \frac{60(0.1)}{(0.1)} = 60$$

Now consider change in $t = 1 + h$, h is very-very small

$$\text{Now} \quad \frac{S(1+h) - S(1)}{(1+h) - 1} = \frac{60(1+h) - 60(1)}{h}$$

$$= 60$$

* When $h \to 0$ we say instantaneous rate of change or the rate of change when $h \to 0$

* The rate of change when h does not go to zero will be simply called an average change.

Notation :- small change in any quantity / variable (say z) is denoted by Δz

$$\Delta s = 60(t+\Delta t) - 60t$$

$$\Delta s = 60\,\Delta t$$

$$\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{60\,\Delta t}{\Delta t}$$

$$= \lim_{\Delta t \to 0} 60$$

$$\boxed{\lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = 60}$$

* This particular limit is called derivative of s. w.r.t t.

and it is written as

$$\boxed{\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}}$$

Left side is read as de es by de te

* Note it is symboly

Differential Coefficients :-

Let $y = f(x)$ be a function of x. Then the derivative of y w.r.t. x or differential coefficient of y w.r.t. x is denoted by $\frac{dy}{dx}$ and is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\frac{f(x+\Delta x) - f(x)}{\Delta x} \right]$$

Whenever limit exist.

Standard Result :-

1. Derivative of constant is zero

2. $y = f(x)$ and $z = ky$ where k is constant

$$\boxed{\frac{dz}{dx} = k \cdot \frac{dy}{dx}}$$

3. If $y = \alpha f_1(x) + \beta f_2(x)$; where f_1 and f_2 are two differentiable function

$$\boxed{\frac{dy}{dx} = \alpha \frac{df_1}{dx} + \beta \frac{df_2}{dx}}$$

4. $f(x) = u(x)\, v(x)$ find $\frac{d}{dx} f(x)$
(Derivative of product)

5. Derivative of ratio : $f = \frac{u}{v} = uv^{-1}$
$\therefore \frac{d}{dx} f(x) = ?$

6. Chain rule :-

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = \cdots$$

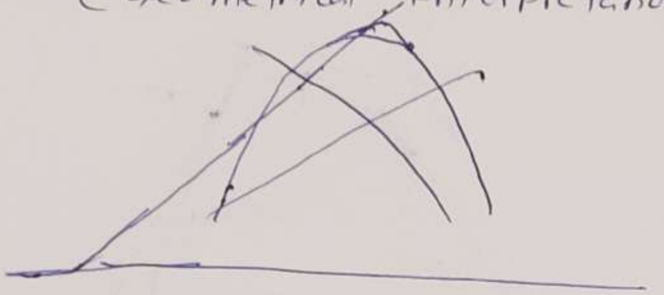
provided $du \neq 0$, $dv \neq 0$, $dx \neq 0$

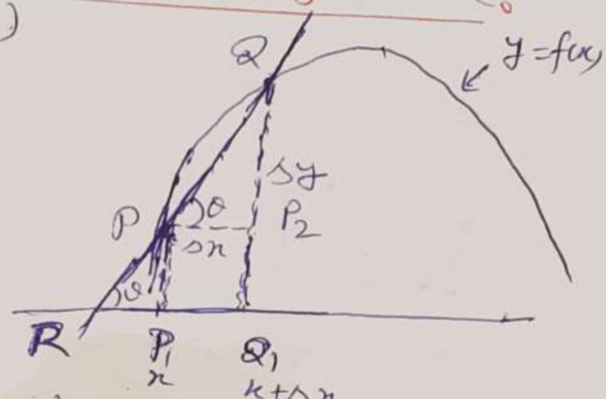
move to page (A)

Derivative As slope

— Derivative As a Slope, and Tangental line :—

(Geometrical Interpretation)





- Corresponding Point P, the point on the x-axis is P_1 is x
- Small ~~element~~ increment Δx in x gives point Q_1 at $x+\Delta x$ on x-axis.
- Corresponding Q_1 the ordinate is $QQ_1 = y+\Delta y$

From above figure.

$$P_2Q = \Delta y, \quad P_1Q_1 = \Delta x$$

- Consider the ~~sect~~ secant passing through ~~PQQ~~ P, Q and cutting the x-axis at R.
- Let the line make the angle θ ~~from~~ with x-axis.

Then $PP_1 = y = f(x)$, $Q_1Q = y+\Delta y = f(x+\Delta x)$

$P_2Q = \Delta y = f(x+\Delta x) - f(x)$, $PP_2 = \Delta x$

Now consider

$$\frac{P_2Q}{PP_2} = \frac{\Delta y}{\Delta x} = \tan\theta = \frac{Q_1Q}{RQ_1}$$

Therefore,

$$\frac{\Delta y}{\Delta x} = \tan\theta = \text{slope of line segment } QP$$

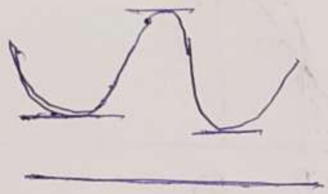
- Now think that Q is moving along the curve $y = f(x)$ towards P ⇒ This is equivalent to projection of Q on x-axis (i.e. Q_1) is moving towards P_1 on the x-axis

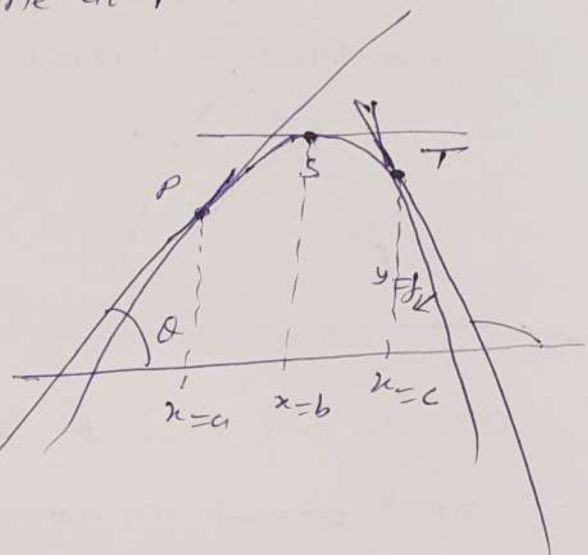
↳ When Q coincides with P the line QP becomes the tangent line to the curve $y=f(x)$ at P on the curve.

↳ $Q \to P$ on the curve is equivalent to $x+\Delta x \to x$ i.e. $\Delta x \to 0$

$\Rightarrow$ Thus we have interpretation $\frac{dy}{dx}$ at x as the slope of the tangent line at P.

Consider





Moving from left to right

- When ~~the~~ curve is increasing (as we move from left to right on the x-axis the value ~~value~~ of ordinate is increasing) θ is ~~an acute~~ an acute angle and $\tan\theta > 0$. Thus

$$\left.\frac{dy}{dx}\right|_{x=a} > 0 \Rightarrow y = f(x) \text{ is increasing at } x = a.$$

- At point S in figure 2, the tangent line is parallel to x-axis.

↳ When tangent line is parallel at a point $x=b$ then the curve has local maxima or local minimum

$$\left.\frac{dy}{dx}\right|_{x=b} = 0 \Rightarrow \cancel{x=b}.$$

- At the point ~~S~~ T on the curve, the tangent line make obtuse angle with the x-axis and hence $\tan\theta < 0$.

Therefore, $\left.\frac{dy}{dx}\right|_{x=c} < 0 \Rightarrow f(x)$ at $x=c$.

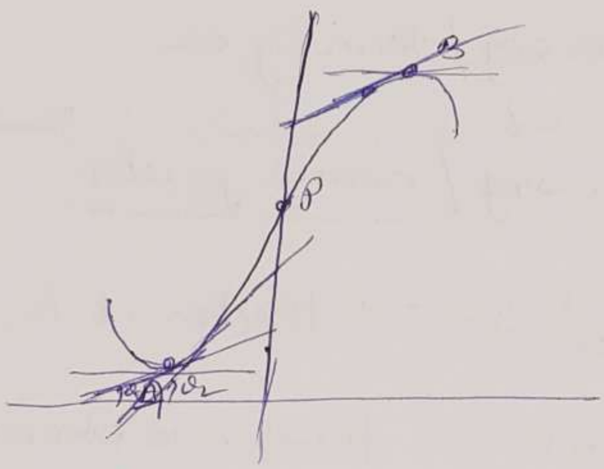
In Summary:

$\frac{dy}{dx} > 0 \Rightarrow$ The function is increasing

$\frac{dy}{dx} < 0 \Rightarrow$ The function is decreasing

$\frac{dy}{dx} = 0 \Rightarrow$ The function is steady.

~~$\frac{dy}{dx}$~~.



Moving from A to B ($A \longrightarrow B$) along the curve :—

Angle θ, that tangent makes from x-axis, starts ~~increasin~~ increasing from ~~0~~ 0, i.e., $\tan\theta$ = slope, is increasing from 0. $\Rightarrow$ At the point P, $\theta = \frac{\pi}{2}$ $\Rightarrow$
$\Rightarrow$ Tangent line cuts the curve and becomes $\perp^r$ to the x-axis.

At Point P : $\tan\theta = \tan\frac{\pi}{2} = \frac{\sin \pi/2}{\cos \pi/2}$
$\Rightarrow$ This not defined since $\cos\frac{\pi}{2} = 0$

Hence when tangent line cuts the curve and becomes perpendicular to the x-axis $\tan\theta$ is undefined and hence

$\boxed{\frac{dy}{dx} \text{ at P does not exist}}$

Increasing / decreasing at Increasing / decreasing rates :—

We see that $\frac{dy}{dx} > 0 \Rightarrow$ function is increasing

$\frac{dy}{dx} < 0 \Rightarrow$ function is decreasing

$\Rightarrow$ Now a function may increase at an increasing rate, In this case

$$\frac{dy}{dx} > 0 \text{ and } \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} > 0.$$

Thus we have following:

$\frac{dy}{dx} > 0$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ increasing at increasing rate

$\frac{dy}{dx} > 0$, $\frac{d^2y}{dx^2} < 0 \Rightarrow$ increasing at decreasing rate

$\frac{dy}{dx} < 0$, $\frac{d^2y}{dx^2} > 0 \Rightarrow$ Decreasing at increasing rate

$\frac{dy}{dx} < 0$, $\frac{d^2y}{dx^2} < 0 \Rightarrow$ Decreasing at decreasing rate.

[Associated with the concept of concavity and convexity of the function]

Point of Inflection :—

A point of inflection is the point where $\frac{d^2y}{dx^2} > 0$ on one side and $\frac{d^2y}{dx^2} < 0$ on the other side.

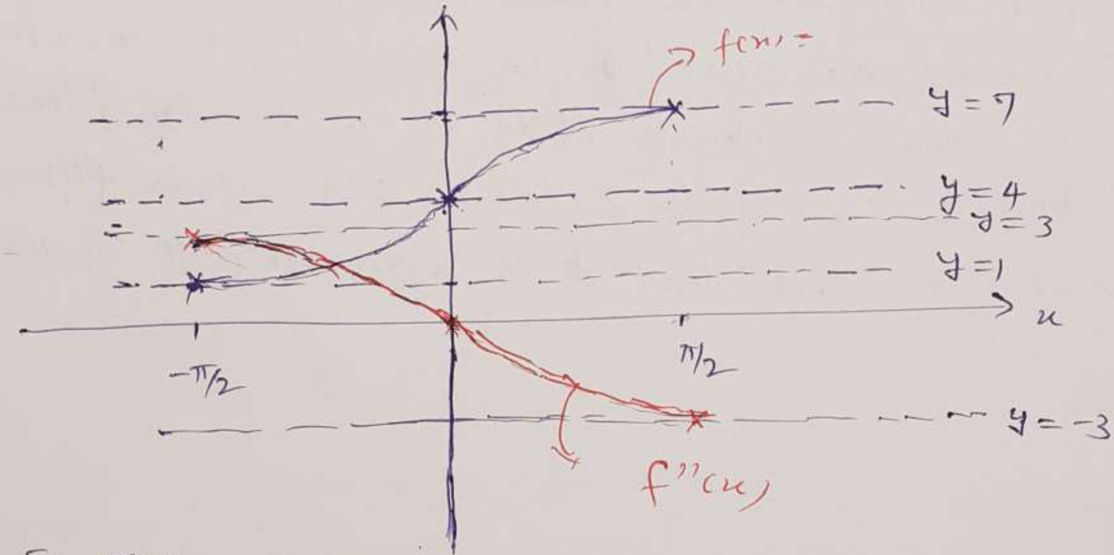
i.e. $\frac{d^2y}{dx^2} = 0$ at point of inflection

Ex:— check for the points of inflection, if any, for

$$f(x) = 4 + 3\sin x, \; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Solution:— $f(x) = 4 + 3\sin x; \; -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$f''(x) = -3\sin x$$



① In range $-\frac{\pi}{2} < x < 0$, $\sin x < 0$ and $f''(x) > 0$

⇒ In range function is increasing at increasing rate.

② $0 < x < \frac{\pi}{2}$, $\sin x > 0$, and $f''(x) < 0$

Therefore in this range $f(x)$ is increasing at decreasing rate.

⇒ $x = 0$ is inflection point

More over $f''(x) = 0$ at $x = 0$.

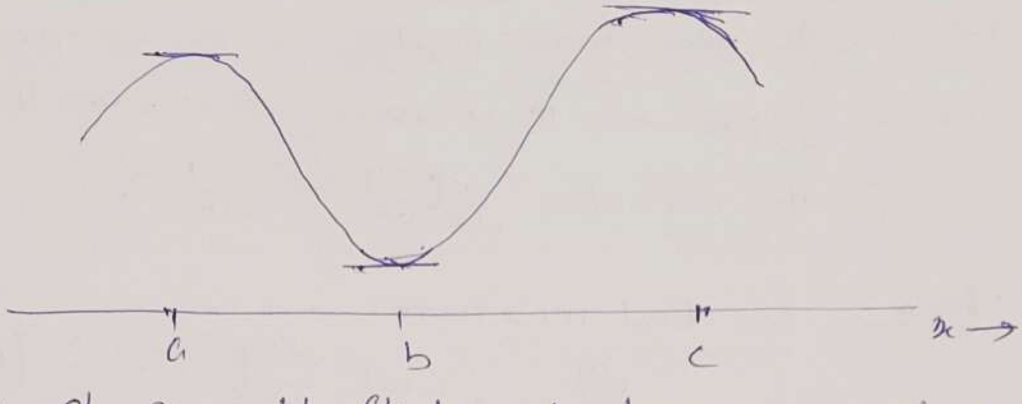
Problem ①: A measure of consumer reaction to change in price of ~~elect~~ electricity of demand. If x is no. of unit sold (demand) and p is the market price then the elasticity of demand E, is measured by:

$$E = \frac{p\,dx}{x\,dp}$$

where x is a function of p. When the price increases demand decreases ~~&~~ and x will be a decreasing function of p. Show that where $E < -1$ then the total revenue xp decreases.

Problem ②: The chance that a variable falls in a small interval ~~dx~~ around x, is denoted by $f(x)\,dx$, where $f(x)$ is called the density of x. If the density of the waiting time t, waiting for a bus at a bus stop, is $f(t) = 3e^{-3t}$, $0 \le t < \infty$ show that it is decreasing at an increasing rate.

Maxima/Minima of a function of one variable :



— clearly at $x=a$ the f^n have local maximum, at $x=b$ the function has local minimum and at $x=c$ the f^n has local maximum.

— Largest maximum at $x=c$.

— Increasing and decreasing behaviour of function $y=f(x)$

from left of $x=a$ moving to a the f^n is increasing i.e. $\frac{dy}{dx}>0$ and at $x=a$ $\frac{dy}{dx}=0$	From $x=a$ to $x=b$ the f^n is decreasing, i.e; $\frac{dy}{dx}<0$ and at $x=b$ $\frac{dy}{dx}=0$	From $x=b$ to $x=c$ function is increasing i.e $\frac{dy}{dx}>0$ and at $x=c$ $\frac{dy}{dx}=0$

- Points $x=a$, $x=b$ and $x=c$ called critical point or turning points in the terminology of maxima and minima.

Case I At $x=a$, $\frac{dy}{dx}=0$ what will be nature of $\frac{dy}{dx}$ around $x=a$?

$\frac{dy}{dx}$ goes from positive to negative,

before $x=a$, the $\frac{dy}{dx}>0$, ~~after~~ after $x=a$ the $\frac{dy}{dx}<0$

Thus $\frac{dy}{dx}$ goes from positive to negative quantity, thus ~~d^2 2nd deri~~ Thus $\frac{dy}{dx}$ is decreasing function and hence its derivative must be negative, ~~d^2y~~ i.e. $\frac{d^2y}{dx^2}<0$

Case at point $x=b$:-

before $x=b$, $\frac{dy}{dx} < 0$, after $x=b$, $\frac{dy}{dx} > 0$

i.e. $\frac{dy}{dx}$ goes from negative to positive quantity,

hence $\frac{dy}{dx}$ ~~at x=~~ is increasing at $x=b$ and thereafter

$\therefore$ its derivative $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} > 0$.

Thus For local maximum

$$\frac{dy}{dx} = 0, \text{ and } \frac{d^2y}{dx^2} < 0 \quad \text{—(A)}$$

For local minimum

$$\frac{dy}{dx} = 0, \text{ and } \frac{d^2y}{dx^2} > 0 \quad \text{—(B)}$$

Note :- Hence for differential function, rule for maxima/ and minima ~~case~~ are given in (A) and (B).

But these rules need not always hold.

Ex :- check for maxima and minima for following function:

(i) $f(x) = 2x^2 - 5x + 7$

(ii) $f(x) = x^3 + 3x^2 + 6x + 8$

(iii) $f(x) = x^3 - 3x^2 + 3x - 5$

(iv) $f(x) = -3x^2 + 4x - 10$

Solution:— (i) $\frac{df}{dx} = 0 \Rightarrow 4x - 5 = 0 \Rightarrow x = \frac{5}{4}$

Hence $x = \frac{5}{4}$ is a critical point or turning point

$$\frac{d^2f}{dx^2} = 4 > 0$$

Therefor $x = \frac{5}{4}$ (critical point) corresponds to the minimum.

(ii)
$$\frac{d}{dx}[x^3+3x^2+6x+8] = 0$$

$\cancel{3x^2+6x^2}$ $3x^2+6x+6 = 0 \Rightarrow x^2+2x+2=0$

or, $x = \frac{-2 \pm \sqrt{4-4(1)(2)}}{2} = \frac{-2 \pm \sqrt{4-8}}{2}$

or, $x = \frac{-2 \pm \sqrt{-4}}{2} \Rightarrow$ ~~or~~ x is complex number, not real. Hence there is no critical point. The function f neigther has a maximum nor a minimum here.

(iii) $\frac{d}{dx}[x^3-3x^2+3x-5] = 3x^2-6x+3$

For maxima and minima

$\because 3x^2-6x+3=0 \Rightarrow x^2-2x+1=0$

$\Rightarrow (x-1)^2=0 \Rightarrow x=1$

$\therefore$ critical point at $x=1$. Now consider $\frac{d^2f}{dx^2}$

$\frac{d^2f}{dx^2} = 6x-6$; Now $\left.\frac{d^2f}{dx^2}\right|_{x=1} = 0$

~~Now at $x=0.99$, $\frac{df}{dx}$~~ $\left.\frac{df}{dx}\right|_{x=0.99} = 3(0.99)^2-6(0.99)+3$

$\left.\frac{df}{dx}\right|_{x=0.99} = \cancel{3\times0.9801-5.94+3=2.9403-5.94+3}$

$=$

And at $x=0.99$ $\left.\frac{d^2f}{dx^2}\right|_{x=0.99} = 6(0.99-1) = 6\times-0.01$
$= -0.06 < 0$

at $x=1.01$ $\left.\frac{d^2f}{dx^2}\right|_{x=1.01} = 6(1.01-1) = 6\times0.01 = 0.06 > 0$

Therefore $x=1$ is inflection point.

(IV) $\frac{df}{dx} = -6x+4 \Rightarrow$ ~~for maxima or maximama~~

for maxima and minima we have $\frac{df}{dx}=0$

$\therefore -6x+4=0 \Rightarrow x = \frac{\cancel{4}2}{\cancel{6}3} \Rightarrow x = \frac{2}{3}$; Thus $x=\frac{2}{3}$ is critical point.

Now $\frac{d^2f}{dx^2} = -6 < 0 \therefore$ Hence $x=\frac{2}{3}$ is local minima.

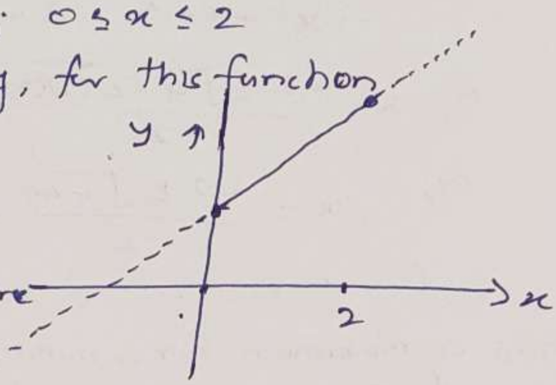
Maxima and Minima at Boundary Point

Ex :- consider the function

$$y = 2x+3; \quad 0 \le x \le 2$$

Find maxima/minima, if any, for this function.

Soln :- $y = 2x+3$



$\frac{dy}{dx} = 2 \neq 0$

Hence method calculus fails here for finding maxima and minima.

⇒ In a problem if $\frac{dy}{dx} \neq 0$, this does not necessarily mean that there is no maximum or minimum for the function.

— If there is no restriction on x, i.e., if $-\infty < x < +\infty$ then from the graph one can see that the line increases to $+\infty$ and decreases to $-\infty$ and there is no finite value of maxima and minima. Hence if $-\infty < x < +\infty$ in this case there is no maximum or minimum.

— But function is defined on interval $0 \le x \le 2$. At $x=0$, $y=3$ and at $x=2$, $y=7$. Hence, there is a minimum and maximum. The minimum is at $x=0$, $y=3$ and at $x=2$, $y=7$. Hence there is a minimum and a maximum. ~~The~~

↳ The maximum is at $x=2$ and the ~~maximum~~ maximum value is 7.

Note 1 :- If x was defined as $0 < x \le 2$ where the point $x=0$ is excluded, then the function has no minimum but there is maximum at $x=2$

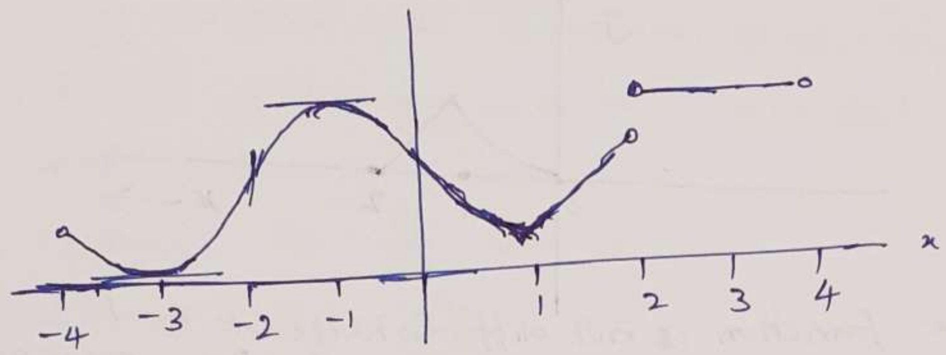
$0 < x \le 2 \Rightarrow$ no minimum, maximum at $x=2$

$0 \le x < 2 \Rightarrow$ no maximum, minimum at $x=0$

$0 < x < 2 \Rightarrow$ no minimum, no maximum.

Note 2 :- This linear function is differentiable but differentiation will not help to determine maximum or minimum. The function is differentiable here at $x=0$, at $x=2$ and over the interval $0 \le x \le 2$

Non smooth function (Maxima and Minima) :-



For above function defined for interval $-4 < x < 4$ find maxima and minima.

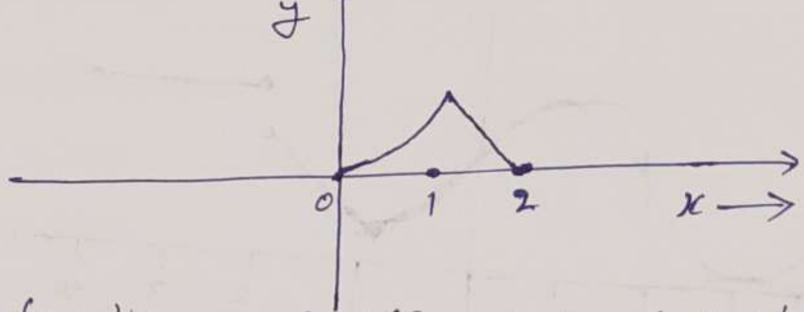
Solun :- — ~~at~~ The f^n in interval $-4 < x < 4$ is smooth and differentiable.

- At $x = -4$ the point is missing, hence the function is not differentiable.
- At $x = +1$, there is sharp corner and hence the function is not differentiable
- At $x = -4$ there ~~is~~ no local maxima
- At $x = -3$, we have local minima
- At $x = -2$, we have point of inflection, because $\frac{d^2y}{dx^2}$ changes sign at $x = -2$
- At $x = -1$, we have local maxima
- At $x = 1$, function is not differentiable ~~⇒ There is no l.~~ But there is a local minimum for the function.
- At $x = 2$ the function is discontinuous and hence not differentiable but there is ~~maximization~~ no maximum or minimum
- At $x = 4$ there is no maximum and minimum; but for every point in the interval $2 < x < 4$ the function attains its maximum possible value.

Ex :- (Sharp corner) :- Consider the function

$$f(x) = \begin{cases} x^2; & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \end{cases}$$

Check for the maxima / minima for this function.



- The function is not differentiable at $x=1$

since $\frac{d}{dx}f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ -1, & 1 \le x \le 2 \end{cases}$

$\therefore$ $\left.\frac{d f(x)}{dx}\right|_{x=1} = \begin{cases} 2 ; & 0 \le x \le 1 \\ -1 ; & 1 \le x \le 2 \end{cases}$

→ But there is local maximum for the function at $x=1$ and the maximum value is $f(1)=1$

- There is local minimum at boundary point $x=0$ and $x=2$.
- The function is also differentiable at the boundary points $x=0$ and $x=2$. If $x=0$ is omitted from the range then there is no local minimum at $x=0$. Similarly, if $x=2$ is omitted from the range of x then there is no local minimum at $x=2$.

Note:— Maxima/minima are not always associated with differentiability of the function

⇓

- A function not differentiable at ~~at~~ a any point may have a maximum or minimum at that point
- A function differentiable everywhere may not have a local maximum or local minimum.

Go to B

(A)

Note:- (e) D denote the differential operator $D = \frac{d}{d\theta}$ or $D = \frac{d}{dx}$, depending upon the independent variable, we may write

$$~~Dy =~~ \quad Dy = \frac{dy}{dx} \quad ~~or~~ ~~\theta~~$$

$$D(Dy) = D^2 y = \frac{d^2y}{dx^2}$$

If we differentiate n-time

$$\boxed{D^n = \left(\frac{d}{d\theta}\right)^n = \frac{d^n}{d\theta^n}}$$

Result:- Derivative of product

$f(x) = u(x)\,v(x)$, show that

$$\frac{d}{dx} f(x) = v(x)\frac{d}{dx}u(x) + u(x)\frac{d}{dx}v(x)$$

i.e. $f(x) = u(x)\,v(x) \Rightarrow f'(x) = v(x)u'(x) + u(x)v'(x)$

where a prime denotes the first derivative.

Result:- ~~thus~~ using ~~above~~ above result, show that

$$\frac{d}{dx}[u(x)\,v(x)\,w(x)] = u'(x)\,v(x)\,w(x) + u(x)\,v'(x)\,w(x) + u(x)\,v(x)\,w'(x)$$

i.e. $$\boxed{(u(x)v(x)\,w(x))' = u'(x)\,v(x)w(x) + u(x)\,v'(x)\,w(x) + u(x)v(x)w'(x)}$$

~~This result can be generalized for a finite no. product of number~~

This result is generalized for product of finite number of differentiable function. Let $u_1(x), u_2(x), u_3(x), \cdots u_k(x)$ be differentiable functions of x and consider the product $u_1 u_2 u_3 \cdots u_k$. Then

$$\boxed{(u_1 u_2 u_3 \cdots u_k)' = u_1' u_2 u_3 \cdots u_k + u_1 u_2' u_3 \cdots u_k + u_1 u_2 u_3' u_4 \cdots u_k + \cdots + u_1 u_2 u_3 \cdots u_k'}$$

Result — n^{th} derivative of a product

$$\frac{d^n}{dx^n}[u(x)\,v(x)] = u^{(n)}v + \binom{n}{1}u^{(n-1)}v^{(1)} + \binom{n}{2}u^{(n-2)}v^{2} + \cdots + \binom{n}{r}u^{(n-r)}v^{(r)} + \cdots + \binom{n}{n}u\,v^{(n)} \quad \text{— (1)}$$

Where $u^{(k)}$ = k^{th} derivative of u w.r.t x.

Let $D_1 = \frac{d\,u(x)}{dx}$, $D_2 = \frac{d\,v(x)}{dx}$

$$(D_1+D_2)^n = D_1^n D_2^0 + \binom{n}{1}D_1^{n-1}D_2^1 + \binom{n}{2}D_1^{n-2}D_2^2 + \cdots + \binom{n}{n}D_1^0 D_2^n$$

Then if $(D_1+D_2)^n$ operate on uv we obtain the result (1)

$$\frac{d^n}{dx^n}u(x)v(x) = (D_1+D_2)^n(uv)$$

$$= u^{(n)}v + \binom{n}{1}u^{n-1}v^{(1)} + \binom{n}{2}u^{n-2}v^{(2)} + \cdots + \binom{n}{n}u\,v^{n}$$

Derivative of Ratios :—

Show that

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{d}{dx}[uv^{-1}]$$

$$= \frac{v u' - u v'}{v^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Differentials :—

Differential coefficient or derivative is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \quad , \text{where } y = f(x)$$

- In case of limits: the limit is ~~not~~ reached not by substituting the point but by checking to see whether the function is approaching a value when the independent variable x approaches the point $x = a$

$$\text{e.g.} \quad f(x) = \frac{x^2 - 4}{x - 2}$$

↳ In this we can not obtain the limit by putting $x = 2$ because at $x = 2$ this function is undefined, but $f(x) \to 4$ approaches definite value 4, when x approaches to ~~to~~ 2 $(x \to 2)$

— Similarly in evaluation of derivative we do not put $\Delta x = 0$, but before Δx attain the value zero ratio $\frac{\Delta y}{\Delta x}$ goes to definite finite quantity when derivative exist.

— At this stage when derivative is attained there is infinitesimal value for Δx. This value is called ~~different~~ differential in x and is denoted by dx. And at this stage value attained by Δy is called differential in y and it is denoted by dy

→ Thus differential coefficient $\frac{dy}{dx}$ can be looked ~~as~~ upon as the ratio of the differential of y over the differential in x.

↳ In earlier discussion $\frac{dy}{dx}$ was standing purely as notation to denote $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, but now quantity can be treated as ratio of two differential where ~~$dx \neq$~~ 0 dx never ~~be~~ be zero $(dx \neq 0)$ But $dy = 0$ / $dy < 0$ / $dy > 0$ depending on the function.

⇒ Now if this approach is used and if u is f^n of x and du is the differential in u and $du \neq 0$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = \cdots$$

provided that $du \neq 0$, $dv \neq 0$, $dx \neq 0$.

This chain rule, this ~~eable~~ enables us deals with function of a function.

Ⓐ Finished

(B)

Differentiability:

~~We~~ Till this point we assume that derivative exist. Now we go little more insight into definition.

→ Δx was a small increment in x, but we did not restrict Δx to be positive all the time.

- If there is decrease, then $\Delta x < 0$
- If there is increase, then $\Delta x > 0$

→ Also defined in terms of limit

(1) → We have to check left limit exist

(2) → We have to ~~check~~ check the right limit exist

(3) → (1) and (2) exist then (1) = (2) and they must be finite.

Thus for existence of derivative: left as well as right limit exists.

We have define

(1) Left derivative at a point

(2) Right derivative at a point.

— At a point $x = a$ consider $f(a+h)$, $h < 0$. When $h \to 0$ x is approaching a from the left. Then for $h < 0$

$$\boxed{\lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right] = \frac{d}{dx} f(x) \Big|_{x = a_-}}$$

→ left derivative $f(x)$ at $x = a$.

Similarly, for $h > 0$

$$\boxed{\lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h} \right] = \frac{d}{dx} f(x) \Big|_{x = a+}}$$

→ right derivative ~~of f(x)~~ at $x = a$

For derivative to exists at $x = a$ we must have left derivative equal to right derivative

Note :- (1) Clearly if function is discontn at $x=a$ then derivative does not exisit.

(2) If $f(x)$ is differential at $x=a$ then it is necessary it is contn at $x=a$. But not all contn f^n at $x=a$ are differentiable at $x=a$.

Example :- consider the function

$$f(x) = \begin{cases} x^2 ; & 0 \le x \le 1 \\ 2-x ; & 1 \le x \le 2 \end{cases}$$

Is $f(x)$ is differentiable at (i) $x=1$; (ii) $x=1.5$; (iii) $x=0$; (iv) $x=2$.

Solun :- (i) For $x \to 1_-$ function is x^2,

$\therefore \frac{d}{dx} f(x)\Big|_{x=1_-} = 2x\Big|_{x=1_-} = 2$ (left derivative)

right deriv. for $x \to 1_+$, function $f(x) = 2-x$

$$\frac{d f(x)}{dx}\Big|_{x \to 1_+} = \cancel{\frac{df}{dx}}\Big| \frac{d}{dx}(2-x)\Big|_{x \to 1_+}$$

$$= (-1)\Big|_{x \to 1_+}$$

$$\therefore \boxed{\frac{d}{dx} f(x)\Big|_{x \to 1_+} = -1}$$

$\therefore$ left derivative $\neq$ Right derivative at $x=1$

function is not differentiable at $x=1$.

But function is contn at $x=1$.

$\Downarrow$

Differentiability is something more than continuity.

(ii) at $x = 1.5$, $f(x) = 2-x$

$$\therefore \frac{d}{dx} f(x)\Big|_{x\to 1.5-} = \frac{d}{dx}(2-x)\Big|_{x\to 1.5-}$$

$$= (-1)\Big|_{x\to 1.5}$$

$$= -1$$

at $x\to 1.5+$ f^n is defined $2-x$

$$\frac{d}{dx} f(x)\Big|_{x\to 1.5+} = -1$$

$\therefore$ Left derivative = Right derivative

hence function is differentiable at $x = 1.5$

(iii) $x = 0$, is the left end point of the function.

Hence limit and differentiability ~~are~~ all are defined ~~from right~~ only from right limit

Thus at $x=0$ only right derivative is consider

as $x \to 0+$ the function $f(x) = x^2$

$$\therefore \frac{d}{dx} f(x)\Big|_{x\to 0+} = \frac{d}{dx}(x^2)\Big|_{x\to 0+} = 2x\Big|_{x\to 0+} = 0$$

(iv) $x=2$ is the ~~left~~ right end point

$\Rightarrow$ only left derivative is consider

$\therefore$ when $x \to 2-$ the $f(x) = 2-x$

$$\therefore \frac{d}{dx} f(x)\Big|_{x\to 2-} = \frac{d}{dx}(2-x)\Big|_{x\to 2-}$$

$$= -1\Big|_{x\to 2-}$$

$$\therefore \frac{d}{dx} f(x)\Big|_{x\to 2-} = -1$$

Differentiability at a terminal point :-

$f(x)$ is defined on finite interval $a \le x \le b$ then a and b are called terminal point or boundary point. Differentiability at $x = a$ is defined ~~as the~~ in terms of ~~left~~ right differentiability and differentiability at $x = b$ is defined as ~~right~~ left differentiability.

Thus $f(x)$ is differentiable at $x = a$ if

$$\frac{d}{dx} f(x) \Big|_{x \to a_+} \text{ exist}$$

and differentiable at $x = b$ if

$$\frac{d}{dx} f(x) \Big|_{x \to b_-} \text{ exists.}$$

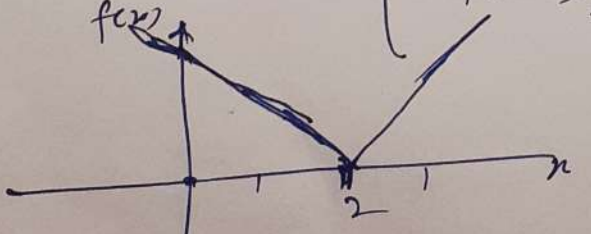
Differentiability Over an Interval :-

A function $f(x)$ is said to be differentiable in an interval $\{ c \le x \le d \}$ if $f(x)$ is differentiable at each point on $\{ c \le x \le d \}$

Ex :- Consider the function $f(x) = 4|x-2|$. check for differentiability and evaluate for first derivative at the point (i) $x = 0$ (ii) $x = 5$ (iii) $x = 2$; over the interval $-\infty < x < \infty$

Soln :- $f(x) = 4|x-2|$ means

$$f(x) = \begin{cases} 4(x-2); & \text{for } x-2 \geq 0 \text{ or for } x \geq 2 \\ -4(x-2); & \text{for } x-2 \leq 0 \text{ or for } x \leq 2 \end{cases}$$



(i) for $x=0$, $f(x) = -4(x-2) = 4(2-x)$

$\therefore$ $\cancel{\frac{df(x)}{dx}} = \frac{d}{dx} f(x) = \frac{d}{dx}[4(2-x)]$

$= -4$

Also $\left.\frac{d}{dx} f(x)\right|_{x=0-} = \left.\frac{d}{dx} f(x)\right|_{x=0+} = -4$

Therefore f^n is differentiable at $x=0$

(ii) for $x=5$, $f(x) = 4(x-2)$

$\frac{d}{dx} f(x) = \frac{d}{dx}[4(x-2)] = 4$

and $\boxed{\left.\frac{d}{dx} f(x)\right|_{x=4-} = \left.\frac{d}{dx} f(x)\right|_{x=4+} = 4}$

(iii) for $x=2$

$\left.\frac{d}{dx} f(x)\right|_{x=2-} = \left.\frac{d}{dx}[-4(x-2)]\right|_{x=2-} = -4$

$\left.\frac{d}{dx} f(x)\right|_{x=2+} = \left.\frac{d}{dx}[4(x-2)]\right|_{x=2+} = 4$

$\therefore$ $\left.\frac{d}{dx} f(x)\right|_{x=2-} \neq \left.\frac{d}{dx} f(x)\right|_{x=2+}$

$\therefore$ f^n is not differentiable at $x=2$
(But f^n is contn).

(iv) f^n is not differentiable in $-\infty < x < +\infty$ since it is not differentible at $x=2$

Notes:- (i) At point corresponding sharp corners of a function $f(x)$ the function is not differentiable, the function at these point may be ~~cont~~ continuous

(ii) At point where tangent line is perpendicular to the x-axis the f^n is not differentiable

(iii) In the def^n of left / right derivative at point $x=a$, the derivative contains a term when the fn is evaluated at $x=\alpha$

$$\lim_{h \to 0} \left[\frac{f(\alpha+h) - f(\alpha)}{h} \right]$$

~~Here~~ $f(\alpha)$ is present or f' is to be evaluated at $x=\alpha$

Hence if ~~f~~ $x=\alpha$ is in not included in definition of $f(x)$ or if $x=\alpha$ is ~~a~~ a discontinuity point the fn is not differentiable.

→ Contn requires the evaluation of $f(x)$ at $x=\alpha$ besides the left and right limits at $x=\alpha$ being equal. Also $f(x)$ must go to $f(\alpha)$ when $x \to \alpha$

Power Series Expansions :-

If a f^n $f(x)$ is differentiable any number of times at $x=0$ and if it can be written as a power series of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + \qquad (1)$$

Then coefficients $a_0, a_1, a_2, \ldots$ can be evaluated by differentiating the right side term by term.

Note :- Here we assume that we are considering the power series of the type where differentiation, term by term of an infinite series is possible.

In (1) type series we assume that $a_0, a_1, a_2, \ldots$ are constant, not containing x. e.g.

$$f_1(x) = 1 + x + x^2 + x^3 + \cdots \quad \text{is series of type (1)}$$

where $a_0 = a_1 = a_2 = a_3 = \cdots = 1$

$$f_2(x) = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Here $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, \ldots, a_n = n+1, \ldots$

$$f_3(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$$

Here $a_0 = 1, a_1 = \frac{1}{1!}, a_2 = \frac{1}{2!}, a_3 = \frac{1}{3!}, \ldots, a_n = \frac{1}{n!}, \ldots$

Now evaluation of $a_0, a_1, a_2, \ldots$ of (1) in terms of f^n value at $x=0$:-

$f(0) = a_0$, $\quad f'(x)\Big|_{x=0} = \frac{d}{dx}\left[a_0 + a_1 x + a_2 x^2 + \cdots\right]\Big|_{x=0}$

~~$f'(0) = a_1$~~, $\quad = \left[0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots\right]\Big|_{x=0}$

$= 0 + a_1 + 0 + 0 + \cdots$

$\therefore a_1 = f'(0)$

$f''(x)\Big|_{x=0} = \frac{d^2}{dx^2}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right]\Big|_{x=0} = \left[2a_2 + (3)(2) a_3 x + \cdots\right]$

$= 2a_2 \quad \therefore 2a_2 = f''(0) \Rightarrow a_2 = \frac{1}{2!} f''(0)$

Proceeding in same way we get

$$a_n = \frac{f^{\overbrace{''''\cdots'}^{n\text{-times}}}(0)}{n!} = \frac{f^{(n)}}{n!}$$

$$\therefore f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

$$\boxed{f(x) = f(0) + \frac{x}{1!}f^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \frac{x^3}{3!}f^{(3)}(0) + \cdots} \quad — (2)$$

Now If we want to expand f(x) at some point $x = a$

we write.

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \cdots \cdots \quad (3)$$

Now differentiating (2) and evaluating at $x = a$ we have the following:

$$\boxed{f(x) = f(a) + \frac{(x-a)}{1!}f^{(1)}(a) + \frac{(x-a)^2}{2!}f^{(2)}(a) + \cdots} \quad (4)$$

Evaluation of f(x) in the neighbourhood of $x = a$ or for $x = a+h$, where $h \to 0$.

Put $x = a+h$ in (4) we have

$$f(a+h) = f(a) + \frac{h}{1!}f^{(1)}(a) + \frac{h^2}{2!}f^{(2)}(a) + \frac{h^3}{3!}f^{(3)}(a) + \cdots \quad — (5)$$

Ex:- Power series Expansions: Expand the following functions in power series of x or expand at $x = 0$

(i) $f(x) = (1-x)^{-1}$; (ii) $(1-x)^{-\alpha}$; (iii) $\sin x$, (iv) $\cos x$;
(v) e^x

Soln :- (i) We use series expansion of $f(x)$ around $x=0$

$$\cancel{f(x) = f(0) + \frac{f'(0)}{1!}}$$

$$f(x) = f(0) + \frac{x}{1!} f^{(1)}(0) + \frac{x^2}{2!} f^{(2)}(0) + \frac{x^3}{3!} f^{(3)}(0) + \cdots$$

Here $f(x) = (1-x)^{-1}$, $f(0) = 1$

$$f^{(1)}(x) = (-1)(1-x)^{-2} \times (-1) \Rightarrow f^{(1)}(0) = 1(1-0)^{-2}$$
$$= (1-x)^{-2} \qquad f^{(1)}(0) = 1$$

$$f^{(2)}(x) = (-2)(1-x)^{-3}(-1) = +2(1-x)^{-3} \Rightarrow f^{(2)}(0) = 1 \cdot 2$$

$$f^{(3)}(x) = (2)(-3)(1-x)^{-4}(-1) \quad \therefore f^{(3)}(0) = (1)(2)(3) = 3!$$
$$= (2)(3)(1-x)^{-4}$$

Similarly $f^{(n)}(x) = n!(1-x)^{-n+1}$; $f^{(n)}(0) = n!$

$$\therefore f(x) = 1 + \frac{x}{1!} + \frac{2!\,x^2}{2!} + \frac{3!\,x^3}{3!} + \cdots + \frac{n!}{n!}x^n + \cdots$$

$$\boxed{f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots}$$

This series is convergence when $|x| < 1$

(ii) $f(x) = (1-x)^{-\alpha}$; $f(0) = (1-0)^{-\alpha} = 1$

$$f^{(1)}(x) = (-\alpha)(1-x)^{-\alpha-1}(-1) = \alpha(1-x)^{-\alpha-1}$$

$$\therefore f^{(1)}(0) = \alpha,$$

$$f^{(2)}(x) = \alpha(-\alpha-1)(1-x)^{-\alpha-2}(-1) = \alpha(\alpha+1)(1-x)^{-\alpha-2}$$

$$\therefore f^{(2)}(0) = \alpha(\alpha+1)$$

$$\therefore f^{(3)}(x) = \alpha(\alpha+1)(\alpha+2)(1-x)^{-\alpha-3} ; \; f^{(3)}(0) = \alpha(\alpha+1)(\alpha+2)$$

$\vdots$

$$f^{(n)}(x) = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)(1-x)^{-\alpha-n+1}$$

$$\therefore f^{(n)}(0) = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$$

$$\therefore (1-x)^{-\alpha} = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha+1)}{2!}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{3!}x^3 + \cdots +$$

$|x| < 1$

The cond^n $|x|<1$ is need for all α except when $-\alpha > 0$

i.e. $\alpha < 0$

when $-\alpha > 0$ then after some term coefficien

$\alpha(\alpha+1) \cdots (\alpha+n-1)$ will become zero.

e.g. $(-2), (-2+1), \underbrace{(-2)(-2+1)(-2+2)}_{0}$

(iii) $f(x) = \sin x$ Here x is in radian.

$f(0) = 0, \quad f^{(1)}(x) = \cos x, \quad f^{(1)}(0) = 1$

$f^{(2)}(x) = -\sin x; \quad f^{(2)}(0) = 0$

~~$f^{(3)}(x) =$~~ $f^{(3)}(x) = -\cos x, \quad f^{(3)}(0) = -1$

$f^{(4)}(x) = -(-\sin x) = \sin x; \quad f^{(4)}(0) = 0$

$\vdots$

$$\boxed{\text{Thus} \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots}$$

(iv) $f(x) = \cos x, \quad f(0) = 1$

$f^{(1)}(x) = -\sin x, \quad f^{(1)}(0) = 0$

$f^{(2)}(x) = -\cos x; \quad f^{(2)}(0) = -1$

$f^{(3)}(x) = +\sin x; \quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x, \quad f^{(4)}(0) = 1$

$f^{(5)}(x) = -\sin x; \quad f^{(5)}(0) = 0$

$f^{(6)}(x) = -\cos x; \quad f^{(6)}(0) = -1$

$\vdots$

$\therefore$ $\cancel{f(x) = \cos x = 1 + \frac{(-1)}{2!}x^2} +$

$$f(x) = f(0) + \frac{f^{(1)}(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(0)}{5!}x^5$$

$$f(x) = 1 + \frac{0}{1!}x + \frac{(-1)}{2!}x^2 + \frac{(0)}{3!}x^3 + \frac{(1)}{4!}x^4 + \frac{(0)}{5!}x^5 + \frac{(-1)}{6!}x^6 + \cdots$$

$\cancel{f(x) = 1+}$

$$\boxed{f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots}$$

(v) $f(x) = e^x$; $\quad f(0) = 1$,

$f^{(n)}(x) = e^x$; $\quad f^{(1)}(0) = 1, \; f^{(2)}(0) = 1, \; f^{(3)}(0) = 1 \ldots$

$$\rightarrow \boxed{f(x) = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

Note:- (1) $$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots}{x}$$

$$= \lim_{x \to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$

$$\cancel{= \lim_{x \to 0}} = 1$$

$$\therefore \boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

(2) We can show that

$$\lim_{x \to 0} \left[\frac{1 - \cos x}{x}\right] = 0;$$

$$\lim_{x \to 0} \left[\frac{1 - \cos}{x^2}\right] = \frac{1}{2!}$$

Partial Derivative :

- Explicit function of many independent variables.
- inflation may depend on many ~~variable~~ market forces/ factors such as price per unit of rice, vegitable, ~~meat~~ meat, fish etc., cooking oil, petrol, diesel, electricity charges and so on.
- Let inflation index is denoted by y as a function of these independent variables, $x_1, x_2, x_3, --, x_k$, Then

$y = f(x_1, x_2, x_3, ---, x_k)$; where f is some specific function such as:

$$y = a_1x_1 + a_2x_2 + a_3x_3 + --- + a_kx_k$$

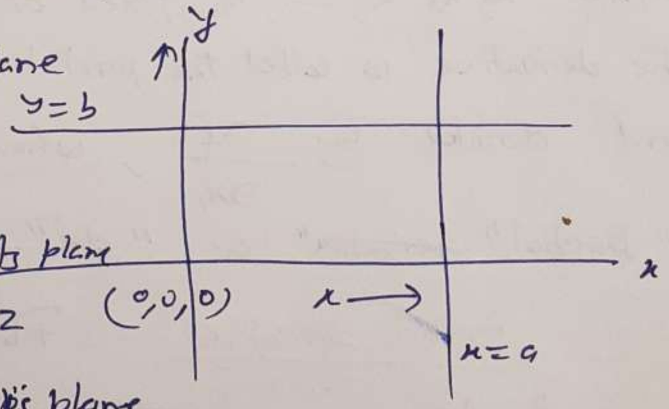
where $a_1, a_2, a_3, \cdots, a_k$ are constants

- Consider the function of two independent Variable, denoted by x and y, and the function by z so that we have

$$z = f(x,y), \text{ i.e., } z \text{ is function of } x \text{ and } y$$

→ This when ploted taking x, y, z coordinate we get surface setting on the (x,y) plane for $z>0$, something like a hill or a mountain. On the mountain there may be several peaks, crevices, dips, depressions and various types of formation.

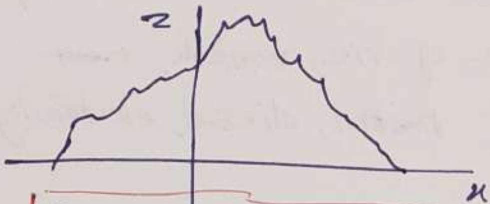
→ $x = a$, represents plane
i.e. $x = a$, for every y and z

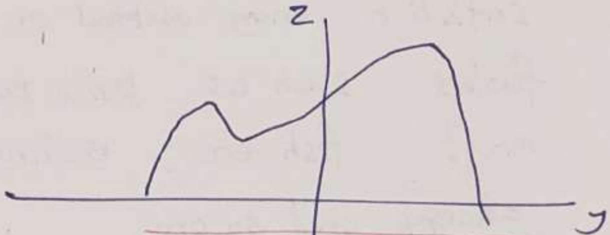


Similarly $y = b$, represents plane for every x and z

- If $z = c$, then this is plane at distance 'c' from (x,y)-plane

– plane $x=a$ cuts the hill, we can trace the shape of mountain at $x=a$. This shape will be curve on ~~(y,z)~~ (y,z)–plane





[$y=b$; cutting the surface]   [$x=a$ cutting the surface]

similarly $y=b$ cuts the mountain ~~we will get shape~~ as a curve on the (x,z)–plane. ~~These shapes may~~

* **Algebraically:–** When $y=b$, we are considering the function $z=f(x,b)$, when 'b' is fixed.

Similarly when $x=a$, we have f^n $z=f(a,y)$ where 'a' is fixed.

$z=f(a,b)$:– We can measure rate of change z w.r.t x, when y is fixed

$z=f(a,y)$ – ______ w.r.t y, when x is fixed.

These derivative is called parial derivatives..

– **In general**, if $f=f(x_1,x_2,x_3,\ldots,x_k)$, function of k independent variable $x_1,x_2,x_3,\text{--}\,x_k$, is differentiated w.r.t. x_1, keeping $x_2,x_3,x_4,\text{--},x_k$ fixed or treated as constants, then the derivative is called the partial derivative of f' w.r.t. x_1 and denoted by $\frac{\partial f}{\partial x_1}$, where ∂ is pronounced as, "partial" ~~derivative~~" or " "del".

Notation :– Partial Derivative :– Partial derivative

$\frac{\partial}{\partial x_i} f(x_1,x_2,x_3,\ldots,x_k)$ = partial derivative of f' with respect to x_i, keeping $x_1, x_2, \text{--}\, x_{i-1}, x_{i+1}, \text{--}\, x_k$ fixed.

~~Ex~~ Ex :- Evaluate the partial derivatives w.r.t. x and w.r.t. y from the following functions:

(i) $z = x^2y - 3xy + y^2 - \frac{x^3}{y} + \frac{y}{x}$

(ii) $z = x + 2y + 7$

Solution :— (i) $\frac{\partial z}{\partial y} = x^2 - 3x + 2y + \frac{x^3}{y^2} + \frac{1}{x}$; Here x is a constant

$\frac{\partial z}{\partial x} = 2xy - 3y + 0 - \frac{3x^2}{y} - \frac{y}{x^2}$; Here y is a constant

(ii) $\frac{\partial z}{\partial x} = 1$, $\frac{\partial z}{\partial y} = 2$

(diff. z w.r.t. x keeping y constant) (diff z. w.r.t. y keeping x constant)

Ex :- Evaluate the all higher order ~~deriv~~ derivatives upto order 3 for the following function

$f(x_1, x_2) = 2x_1^2x_2 - 3x_1x_2 + 5x_1^4 - x_2^3 + 8$

Solution :— $\frac{\partial f}{\partial x_1} = 4x_1x_2 - 3x_2 + 20x_1^3 + 0 + 0$

$\boxed{\frac{\partial f}{\partial x_1} = 4x_1x_2 - 3x_2 + 20x_1^3}$

$\boxed{\frac{\partial f}{\partial x_2} = \cancel{4x_1}\ 2x_1^2 - 3x_1 - 3x_2^2}$

$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial^2 f}{\partial x_1^2} \Rightarrow \frac{\partial^2 f}{\partial x_1^2} = \frac{\partial f}{\partial x_1}(4x_1x_2 - 3x_2 + 20x_1^3)$

$= 4x_2 - 0 + 60x_1^2$

$\therefore \boxed{\frac{\partial^2 f}{\partial x_1^2} = 4x_2 + 60x_1^2}$

$\cancel{\partial^2 f}\ \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}\right) = \frac{\partial^2 f}{\partial x_2 \partial x_1}$

$\cancel{\frac{\partial^2 f}{\partial x_2 \partial x_1}}\ \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}(4x_1x_2 - 3x_2 + 20x_1^3)$

$= 4x_1 - 3$

$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_2}\right)$

$= \frac{\partial}{\partial x_2}(2x_1^2 - 3x_1 - 3x_2^2)$

$\frac{\partial^2 f}{\partial x_2^2} = -6x_2$

$$\frac{\partial}{\partial x_1}\left[\frac{\partial f}{\partial x_2}\right] = \frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left[2x_1^2 - 3x_1 - 3x_2^2\right]$$

$$= 4x_1 - 3$$

$$\boxed{\frac{\partial^2 f}{\partial x_1 \partial x_2} = 4x_1 - 3}$$

Result :-

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}$$

~~prove that~~

$$\cancel{\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1}}$$

provided f has continuous second order partial derivatives at the given point

Thus we can differentiating by taking the variables in any order while considering higher order derivatives

Thus $\frac{\partial^3 f}{\partial x_1^3} = \frac{\partial}{\partial x_1}\left(\frac{\partial^2 f}{\partial x_1^2}\right) = \frac{\partial}{\partial x_1}\left[4x_2 + 60x_1^2\right] = 120x_1$

$$\frac{\partial^4 f}{\partial x_1^4} = 120 ; \quad \frac{\partial^5 f}{\partial x_1^5} = 0$$

$$\frac{\partial^3 f}{\partial x_2 \partial x_1^2} = \frac{\partial}{\partial x_2}\left[\frac{\partial^2 f}{\partial x_1^2}\right] = \frac{\partial}{\partial x_2}\left[4x_2 + 60x_1^2\right] = 4$$

$$\frac{\partial^4 f}{\partial x_2^2 \partial x_1^2} = 0$$

$$\frac{\partial^3 f}{\partial x_1^2 \partial x_2} = \cancel{\frac{\partial}{\partial x_1}\left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]} = \frac{\partial}{\partial x_2}\left[\frac{\partial^2 f}{\partial x^2}\right] = \frac{\partial}{\partial x_2}\left[4x_2 + 60x_1^2\right]$$

$$\therefore \frac{\partial^3 f}{\partial x_1^2 \partial x_2} = 4 ; \quad \frac{\partial^4 f}{\partial x_1^3 \partial x_2} = 0$$

Differential for Function of Many Variables :

let a f^n of two variables, which is differentiable.

$$u = u(x_1, x_2)$$

Small change $\Delta x_1, \Delta x_2$ in x_1 and x_2 produce change in u.

$$\therefore \quad u + \Delta u = u(x_1 + \Delta x_1, x_2 + \Delta x_2)$$

$$\Rightarrow \quad \Delta u = u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1, x_2)$$

Now adding and subtracting $u(x_1, x_2 + \Delta x_2)$ we have the following :

$$\Delta u = [u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1, x_2 + \Delta x_2)] + [u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)]$$

$$\Delta u = \left[\frac{u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1, x_2 + \Delta x_2)}{\Delta x_1}\right] \Delta x_1 + \left[\frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2}\right] \Delta x_2$$

• Now taking limit $\Delta x_1 \to 0, \Delta x_2 \to 0$, we have $\Delta x_1 \to dx_1$ and $\Delta x_2 \to dx_2$

$$\frac{u(x_1 + \Delta x_1, x_2 + \Delta x_2) - u(x_1, x_2 + \Delta x_2)}{\Delta x_1} \to \frac{\partial u}{\partial x_1}$$

$$\frac{u(x_1, x_2 + \Delta x_2) - u(x_1, x_2)}{\Delta x_2} \to \frac{\partial u}{\partial x_2}$$

Hence we have following result

(1) $u = u(x_1, x_2) \Rightarrow du = \dfrac{\partial u}{\partial x_1} dx_1 + \dfrac{\partial u}{\partial x_2} dx_2$

Extending the result we have

(2) $u = u(x_1, x_2, x_3, \dots, x_n) \Rightarrow du = \dfrac{\partial u}{\partial x_1} dx_1 + \dfrac{\partial u}{\partial x_2} dx_2 + \dots + \dfrac{\partial u}{\partial x_n} dx_n$ $\longleftarrow$ (A)

for $k=1$

$$du = \frac{\partial u}{\partial x_1} dx_1 = \frac{du}{dx_1} dx_1$$

we can obtain approximation by using (A) for $k=1,2,3,-$

$$\Delta u \simeq u' \Delta x, \quad u' = \frac{du}{dx}$$

$$\Delta u \simeq \frac{\partial u}{\partial x_1} \Delta x_1 + \frac{\partial u}{\partial x_2} \Delta x_2 + \cdots + \frac{\partial u}{\partial x_k} \Delta x_k$$

$\simeq$ "approximately equal to-"

Ex:- A farmer decided to dig a circular well of radius ~~2m.~~ 2 meters and depth 10 meters. The contract was given at the rate of Rs. 100 per cubic meter of earth removed. The ~~contracto~~ contractor dug the well at 1.99 meter insted of 2 meter and charged according to contract for 2 meter radius well. How much is the farmer's loss approximately.

Solu^n:- The total volume of the earth removed for well of radius 2m and depth 10m is:

$$\pi r^2 h = \pi (2)^2 (10)$$

and cost for this volume:

$$C = 100 \times \pi r^2 h = 100 \pi r^2 h.$$

A slight change in ~~or~~ r produces an approximate change in C by the relation:

$$\Delta C = \frac{\partial}{\partial r}(100 \pi r^2 h) \Delta r$$

$$\Delta C = 200 \pi r h \Delta r$$

Thus for $r = 2m$, $h = 10m$, and $\Delta r = -0.01$ is given

$$\Delta C = (200\pi)(2)(10)(-0.01)$$

$$\Delta C = -40\pi = -125$$

Thus farmer ~~lost~~ loss ~~Rs -125~~ Rs. 125.

Ex :- Personal helth chart of Ajith says that his weight can be maintained at the same level of he takes starch $x_1 = 50g/day$, protein $x_2 = 40g/day$, fat $x_3 = 10g/day$, number of minutes of exercise $x_4 = 30$ minutes/day. w denotes the gain in the weight, and the health eqn is given by:

$$\omega = x_1^2 + 2.5x_2 + 10x_3 - 3x_4^2$$

So that at $x_1 = 50$, $x_2 = 40$, $x_3 = 10$, $x_4 = 30$, $\omega = 0$.

Ajeet decided to take ~~the~~ an extra 1kg. starch per day and reduce the exercise by 5 minutes per day. What is the ~~approximate~~ approximate effect on the gain of weight ω?

Solun :- only changes in the variables ~~x_5~~ x_1 and x_4 are ~~three~~ there whereas x_2 and x_3 remains the same. Hence,

$$\Delta w \simeq \frac{\partial w}{\partial x_1}\Delta x_1 + \frac{\partial w}{\partial x_2}\Delta x_2 + \frac{\partial w}{\partial x_3}\Delta x_3 + \frac{\partial w}{\partial x_4}\Delta x_4$$

$$\therefore \cancel{\frac{\partial w}{\partial x_2} = 0}, \quad \Delta x_2 = 0, \quad \Delta x_3 = 0$$

$$\Delta w \simeq \frac{\partial w}{\partial x_1}\Delta x_1 + \frac{\partial w}{\partial x_4}\Delta x_4$$

$$\frac{\partial w}{\partial x_1} = 2x_1, \quad \frac{\partial w}{\partial x_4} = -6x_4$$

$$\Delta x_1 = 1, \quad \Delta x_4 = -5$$

$\therefore$ change in ω at $x_1 = 50$, $x_4 = 30$

$$\Delta w = 2x_1\Delta x_1 - 6x_4\Delta x_4$$

$$= 2 \times 50 \times 1 - 6 \times 30 \times -5$$

$$= 100 + 900$$

$$\Delta w = 1000 \text{ grams}$$

$\therefore$ There will be increase in wt weight of 1000 grams.

Series Expansion :-

Two variable case :- Parallel to one variable power series expansion, we can write down the a series form for 2-variable case as follows:

$$f(x_1, x_2) = a_{00} + \underbrace{(a_{10}x_1 + a_{01}x_2)}_{\text{Linear form}} + \underbrace{(a_{20}x_1^2 + a_{11}x_1x_2 + a_{02}x_2^2)}_{\text{Quadratic form}}$$

$$+ \underbrace{(\quad\quad)}_{\text{cubic form}} + \cdots \quad \text{——— (A)}$$

Where $a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02} \cdots$ are constant to be determined.

- Two script notation

a_{ij} is the coefficience of $x_1^i x_2^j$

$\therefore$ ~~$x^0y^0 = 1$~~ The coefficient of $x^0y^0 = 1$ or the constant term a_{00}.

- coefficient of $x_1 x_2$ is a_{11},
- coefficient of $x_1^2 \equiv x_1^2 x_2^0$ is a_{20} and so on —
- How to evalute the coefficient a_{ij}?
- If $f(x_1, x_2)$ is differentiable, and partially, any number of times, then all the coefficient can be evaluated.

→ and Term by term differentiation of the series is possible.

- a_{00} $\quad f(0,0) = a_{00}$
- Now partially differentiating w.r.t x_1 and evaluating at $0,0$.

$$a_{10} = \frac{\partial}{\partial x_1} f(x_1, x_2)\bigg|_{(0,0)}$$

$$a_{01} = \frac{\partial}{\partial x_2} f(x_1, x_2)\bigg|_{(0,0)}$$

Similarly

$$a_{20} = \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}\Bigg|_{\substack{x=0\\y=0}} \qquad a_{02} = \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2}\Bigg|_{\substack{x_1=0\\x_2=0}}$$

$$a_{11} = \frac{\partial f(x_1,x_2)}{\partial x_2 \partial x_1}\Bigg|_{\substack{x_1=0\\x_2=0}}$$

$$a_{20} = \frac{1}{2!}\frac{\partial^2}{\partial x_1^2} f(x_1,x_2)\Bigg|_{\substack{x_1=0\\x_2=0}} ; \quad \boxed{a_{20} = \frac{1}{2!} D_1^2 f(0,0)}$$

$$a_{02} = \frac{1}{2!}\frac{\partial^2}{\partial x_2^2} f(x_1,x_2)\Bigg|_{\substack{x_1=0\\x_2=0}} ; \quad \boxed{a_{02} = \frac{1}{2!} D_2^2 f(0,0)}$$

$$\boxed{a_{11} = D_2 D_1 f(0,0)}$$

where D_1 is partial derivative of $f(x_1,x_2)$ w.r.t. x_1

D_2 is partial derivative of $f(x_1,x_2)$ w.r.t. x_2

Thus if we write all the second degree terms in the expansion, we have

$$\frac{x_1^2}{2!} D_1^2 f(0,0) + x_1 x_2 D_1 D_2 f(0,0) + \frac{x_2^2}{2!} D_2^2 f(0,0)$$

$$= \frac{1}{2!}\left[x_1^2 D_1^2 + 2x_1x_2 D_1D_2 + x_2^2 D_2^2\right] f(0,0)$$

$$= \frac{1}{2!}\left[x_1 D_1 + x_2 D_2\right]^2 f(0,0)$$

All the terms of degree 3 :-

$$\left[\frac{1}{3!}x_1^3 D_1^3 + \frac{x_1^2 x_2}{2!\,1!} D_1^2 D_2 + \frac{x_1 x_2^2}{1!\,2!} D_1 D_2^2 + \frac{1}{3!} x_2^3 D_2^3\right] f(0,0)$$

$$= \frac{1}{3!}\left[x_1^3 D_1^3 + 3x_1^2 x_2 D_1^2 D_2 + 3x_1 x_2^2 D_1 D_2^2 + x_2^3 D_2^3\right] f(0,0)$$

$$= \frac{1}{3!}\left[x_1^3 D_1^3 + 3x_1^2 x_2 D_1^2 D_2 + 3x_1 x_2^2 D_1 D_2^2 + x_2^3 D_2^3\right] f(0,0)$$

$$= \frac{1}{3!}\left(x_1 D_1 + x_2 D_2\right)^3 f(0,0)$$

Therefore,

$$f(x_1, x_2) = e^{(x_1 D_1 + x_2 D_2)} f(0,0)$$

$$f(x_1, x_2) = \left[1 + \frac{x_1 D_1 + x_2 D_2}{1!} + \frac{(x_1 D_1 + x_2 D_2)^2}{2!} + \cdots \right] f(0,0)$$

We can make one interesting observation here:
consider all second degree term

$$\frac{1}{2!} \left[x_1^2 \frac{\partial^2 f(0,0)}{\partial x_1^2} + 2 x_1 x_2 \frac{\partial f(0,0)}{\partial x_1 \partial x_2} + x_2^2 \frac{\partial^2 f(0,0)}{\partial x_2^2} \right]$$

$$\cancel{= \frac{1}{2!} \left[x_1 \quad x_2 \right]}$$

$$= \frac{1}{2!} [x_1, x_2] \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}_{x_1=0, x_2=0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{2!} [x_1, x_2] \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix} \left[\frac{\partial}{\partial x_1} \quad \frac{\partial}{\partial x_2} \right] f(x_1, x_2) \Big|_{x_1=0, x_2=0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \frac{1}{2!} X' \left\{ \left(\frac{\partial}{\partial X} \frac{\partial}{\partial X'} \right) f(x_1, x_2) \Big|_{x_1=0, x_2=0} \right\} X \quad \text{— (B)}$$

$$\frac{\partial}{\partial X} = \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{bmatrix}, \quad \frac{\partial}{\partial X'} = \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right]$$

The operator $\left(\frac{\partial}{\partial X} \frac{\partial}{\partial X'} \right)$ is a symmetric ~~matrix~~ matrix.

This matrix operates on $f(x_1, x_2)$ and the evaluate at $x_1=0, x_2=0$

Note :- → form in (B) is general form even if we have k independent variables $x_1, x_2, x_3, \ldots, x_k$ – for $k=2$ we have all the second degree term ~~explicit~~ explicitly in (B)

— In general case, all the second degree terms together are the following:

$$\frac{1}{2!} X' \left[\frac{\partial^2 f_0}{\partial X \partial X'} \right] X \; ; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} ; \quad X' = [x_1, x_2, \ldots, x_k] \quad \text{— (C)}$$

$$\frac{\partial^2 f_0}{\partial X \partial X'} = \left\{ \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_k} \end{bmatrix} \left[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k} \right] \right\} f(x_1, x_2, \ldots, x_k) \Bigg|_{\substack{x_1=0, x_2=0, \ldots \\ x_k=0}} \quad \text{— (D)}$$

This means the matrix of all second order partial derivatives operates on $f(x_1, x_2, \ldots, x_k)$ and then the quantity is evaluated at $x_1=0$, $x_2=0$, $\ldots$ $x_k=0$.

- The ~~quantit~~ quadratic form (C) can be written as $\frac{1}{2!} X' A X$, ~~where~~ $A' = A$, where A is given by (D)

Expansion Around An arbitrary point $(x_1, x_2, \ldots, x_k) = (c_1, c_2, \ldots, c_k)$

Replace x_i in (A) by $(x_i - c_i)$ and all the derivatives are evaluated at $x_1 = c_1$, $x_2 = c_2$, $\ldots$, $x_k = c_k$. The case in (A) is when $c_1=0$, $c_2=0$ and $k=2$. Thus, the expansion around the point $(c_1, c_2, \ldots, c_k)$ is following:

$$f(x_1, x_2, x_3, \ldots, x_k) = e^{(x_1-c_1)D_1 + (x_2-c_2)D_2 + \cdots + (x_k-c_k)D_k} f(x_1, x_2, \ldots, x_k) \Big|_{x_1=c_1, x_2=c_2, \ldots x_k=c_k}$$

$$f(x_1, x_2, \dots, x_k) = f(c_1, c_2, \dots, c_k) + \frac{1}{2!}\left[(x_1-c_1)\frac{\partial f_c}{\partial x_1} + \dots + (x_k-c_k)\frac{\partial f_c}{\partial x_k}\right]$$

$$+\frac{1}{2!}\left[(x_1-c_1)^2\frac{\partial^2 f_c}{\partial x_1^2} + 2(x_1-c_1)(x_2-c_2)\frac{\partial^2 f_c}{\partial x_1 \partial x_2} + (x_2-c_2)^2\frac{\partial^2 f_c}{\partial x_2^2} + \dots + (x_k-c_k)^2\frac{\partial^2 f_c}{\partial x_k^2}\right] + \dots \quad (F)$$

where $X = C \Rightarrow x_1 = c_1,\ x_2 = c_2, \dots, x_k = c_k$

f_c means after taking the derivative put $x_1 = c_1,\ x_2 = c_2, \dots, x_k = c_k$

→ The exponent in (E) can also be written as:

$$(x_1-c_1)D_1 + (x_2-c_2)D_2 + \dots + (x_k-c_k)D_k = (X-C)'D$$

where $X - C = \begin{bmatrix} x_1-c_1 \\ x_2-c_2 \\ \vdots \\ x_k-c_k \end{bmatrix}$; $D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_k \end{bmatrix}$

⇛ If we consider x_i in the neighborhood of c_i, that is, $x_1 = c_1 + h_1, \dots, x_k = c_k + h_k$, then the expansion in (A) for a general k can be written as:

$$f(c_1+h_1, c_2+h_2, \dots, c_k+h_k)$$

$$= f(c_1, c_2, \dots, c_k) + \frac{1}{1!}\left[h_1\frac{\partial f}{\partial x_1} + h_2\frac{\partial f}{\partial x_2} + h_3\frac{\partial f}{\partial x_3} + \dots + h_k\frac{\partial f}{\partial x_k}\right]\Bigg|_{X=C}$$

$$+\frac{1}{2!}\left[h_1^2\frac{\partial^2 f}{\partial x_1^2} + \dots + h_k^2\frac{\partial^2 f}{\partial x_k^2} + 2h_1h_2\frac{\partial^2 f}{\partial x_1 \partial x_2} + \dots + 2h_{k-1}h_k\frac{\partial^2 f}{\partial x_{k-1}\partial x_k}\right]\Bigg|_{X=C} + \dots \quad (F)$$

$$f(C+h) = f(C) + \frac{1}{1!}h'Df_c + \frac{1}{2!}h'(DD'f_c)h + \dots \quad (G)$$

where $f(C+h) = f(c_1+h_1, \dots, c_k+h_k)$

$f(C) = f(c_1, c_2, c_3, \dots, c_k)$; $h' = (h_1, h_2, \dots, h_k)$.

$$D' = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}\right); \quad D'f_c = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}\right) f(x_1, x_2, \cdots, x_k)\Big|_{x=c}$$

– Hence the general ~~expression~~ expansion can be ~~let~~ written ~~as~~ up in a nice form as in (G) where $f(x)$ is expanded around the point $x = c$ with the increment vector denoted by h.

Ex :– Power series : Expand $f(x,y) = (1-x-y)^{-1}$, for for $0 < x+y < 1$, at $(x,y) = (0,0)$

Solution :– $f(0,0) = 1$; $\frac{\partial f}{\partial x}\Big|_{(0,0)} = (-1)(1-x-y)^{-2}(-1)\Big|_{(0,0)} = 1$;

$$\frac{\partial f}{\partial y}\Big|_{(0,0)} = 1; \quad \frac{\partial^2 f}{\partial x^2} = (-2)(1-x-y)^{-3}(-1)\Big|_{(0,0)} = 2!$$

$$\frac{\partial^2 f}{\partial y \partial x}\Big|_{(0,0)} = (-2)(1-x-y)^{-3}(-1)\Big|_{(0,0)} = 2;$$

$$\frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)} = (-2)(1-x-y)^{-3}(-1)\Big|_{(0,0)} = 2, \cdots$$

The coefficient of $x^i y^j$ in the general expression is:

$$a_{ij} = \frac{1}{i!\,j!} \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y)\Big|_{(0,0)}$$

Here in this case,

$$a_{10} = 1, \; a_{01} = 1, \; a_{20} = \frac{1}{2!} \frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} = 1$$

$$a_{11} = \frac{1}{1!\,1!} \frac{\partial^2 f}{\partial x \partial y}\Big|_{(0,0)} = 2; \quad a_{02} = 1.$$

Hence for ~~$(1-x-y)$~~ $0 < x+y < 1$,

$$(1-x-y)^{-1} = 1 + (x+y) + (x^2 + 2xy + y^2) + \cdots$$

$$= 1 + (x+y) + (x+y)^2 + \cdots$$

Examples :– Expand the function in ~~example (1)~~ previous example at $x = 0.5$, $y = 0.25$.

Example :— Expand $(1-x-y)^{-1}$ around the point $\underset{c}{(0.5, 0.25)}$ denoting the small increments by h_1 and h_2.

Soln :— This means we want the expression for $[1-(0.5+h_1) - (0.25+h_2)]^{-1}$ for small values of h_1 and h_2.

This can be obtained by substituting $x = 0.5 + h_1$ and $y = 0.25 + h_2$ on both sides in (3.79). That is:

$$[1-(0.5+h_1)-(0.25+h_2)]^{-1}$$

$$= (0.25)^{-1} + [h_1 + h_2](0.25)^{-2} + [h_1^2 + 2h_1h_2 + h_2^2](0.25)^{-3}$$

$$+ \dots + -$$

~~Log~~ Nor

Lognormal Distribution:— Contn probability distribution of r.v. whose logarithms is normally distributed

i.e. if X is lognormally distributed then

$Y = \log(X)$ is normally distributed

i.e. $Y \sim N(\mu, \sigma)$ for some μ and σ

i.e. If Y is normal distribution then $X = e^Y$ is lognormal distribution.

* lognormally distributed variable takes only positive value

Probability Density function and Cumulative Probability Distribution Function:—

Let x is lognormally distributed with parameter μ_x, σ_x^2 i.e. $X \sim$ Lognormal (μ_x, σ_x^2)

Then $\log(X) \sim N(\mu, \sigma^2)$

Let ϕ is pdf of standard normal i.e $N(0,1)$

$$\cancel{f_X(x)} = \phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$$

and Φ is CDF of standard normal distribution

$$\cancel{f_X(x)} = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}}\, du$$

Relation between (PDF, CDF)

$$\frac{d}{dx} \Phi_x(x) = \cancel{f_x(x)}\ \phi_x(x)$$

PDF. $$f_X(x) = \frac{d}{dx} P(X \le x) = \frac{d}{dx} P(\log X \le \log x)$$

$$= \frac{d}{dx} \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

$\frac{\ln x - \mu}{\sigma} = t$

$\cancel{\frac{1}{x} dx = dt}$

$\frac{1}{x} \cdot \frac{1}{\sigma} = \frac{dt}{dx}$

$$= \phi\left(\frac{\ln x - \mu}{\sigma}\right) \frac{1}{x\sigma}$$

$$= \frac{1}{x\sigma} \phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

$$= \frac{1}{x\sigma} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{(\ln x - \mu)^2}{2\sigma^2}\right]$$

$$f_X(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}$$

~~Cumulat~~ cumulative Distribution function

$$F_X(x) = \Phi\left(\frac{\ln x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}\frac{1}{x}\exp\left\{-\frac{1}{2}\frac{(\ln x-\mu)^2}{\sigma^2}\right\}dx$$

Mole ~~—~~ Follow same procedure as normal distribution

Joint distribution function

$$L(\mu,\sigma) = \prod_{i=1}^{n} f_X(x_i)$$

$$L(\mu,\sigma) = \prod_{i=1}^{n}\frac{1}{\sigma x_i\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(\ln x_i-\mu)^2\right\}$$

$$L(\mu,\sigma) = \prod_{i=1}^{n}\frac{1}{x_i}\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(\ln x_i-\mu)^2\right\}$$

log likelihood function

$$\ell(\mu,\sigma/x_1,x_2,\dots x_n) = \sum_{i=1}^{n}\log\left[\frac{1}{x_i}\cdot\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(\ln x_i-\mu)^2\right\}\right]$$

$$\ell(\mu,\sigma/x_1,x_2,x_3,\dots x_n) = \sum_{i=1}^{n}\ln\frac{1}{x_i} + \sum_{i=1}^{n}\ln\left[\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(\ln x_i-\mu)^2\right\}\right]$$

$$= -\sum_i \ln x_i + \ln\left\{\prod_{i=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(\ln x_i-\mu)^2\right\}\right\}$$

$$= \underbrace{-\sum_i \ln x_i}_{\mathrm{I}} + \underbrace{\ell_N(\mu,\sigma/x_1,x_2,\dots x_n)}_{\mathrm{II}}$$

where $\ell_N(\mu,\sigma/x_1,x_2,\dots x_n) \rightarrow$ log likelihood function of the

- I term is constant with regard μ and σ
- Both ~~ℓ and ℓ_N~~ function ℓ and ℓ_N, reach their maximum with the same μ and σ. Hence ML estimators are identical to those for a normal distribution for the observations $\ln x_1, \ln x_2, \ln x_3, \dots, \ln x_n$

$$\therefore \hat{\mu} = \frac{1}{n}\sum_k \ln x_k, \quad \hat{\sigma}^2 = \frac{1}{n}\sum_k(\ln x_k-\hat{\mu})^2$$

For finite n :-

$\hat{\mu}$ and $\hat{\sigma}$ are baised

Estimator

- $\hat{\mu}$ is Bais for $\hat{\mu}$ is negligible
- less baised estimator for σ is obtained by replacing n by $n-1$ in $\hat{\sigma}$.

- When the individual value $x_1, x_2, \ldots x_n$ are not available, but the sample's mean $\bar{x}$ and standard deviation s is, then the corresponding parameters are determined by the following formulas obtained by solving the equation $E[X]$ and $Var(X)$ for μ and σ.