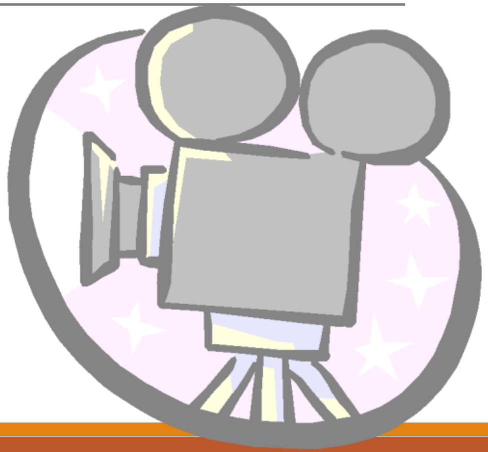
Image Processing

CS-317/CS-341



Outline

- Image reconstruction from projection
  - The reconstruction problem
  - Principal of Computed Tomography
  - The Radon Transformation
  - Fourier Slice Theorem
  - Reconstruction using parallel beam Filter back projection

CT Scan

Image Reconstruction problem

Consider a single object on a uniform background (suppose that this is a cross section of 3D region of a human body).
Background represents soft, uniform tissue and the object is also uniform but with higher absorption characteristics.

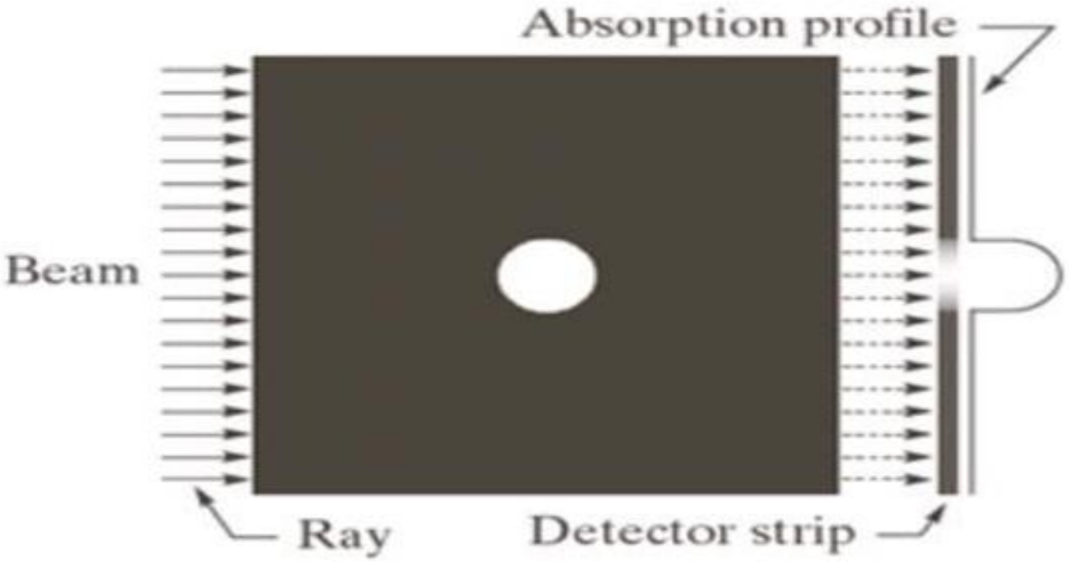


Image Reconstruction problem

A beam of X-rays is emitted and part of it is absorbed by the object.
The energy of absorption is detected by a set of detectors.
The collected information is the absorption signal.

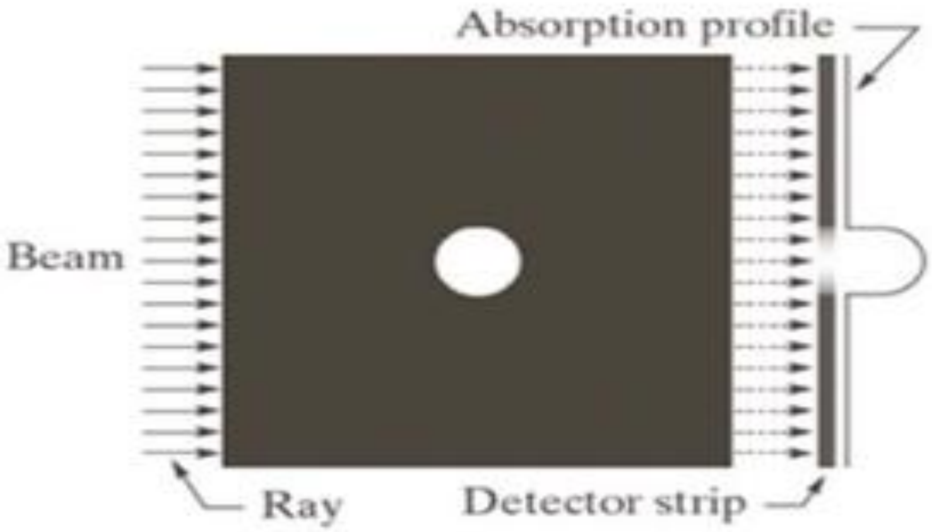


Image Reconstruction problem (Cont…)

We have no means of determining the number of objects from a single projection.

We now rotate the position of the source-detector pair and obtain another 1D signal.

We repeat the procedure and add the signals from the previous back-projections.

We can now tell that the object of interest is located at the central square.

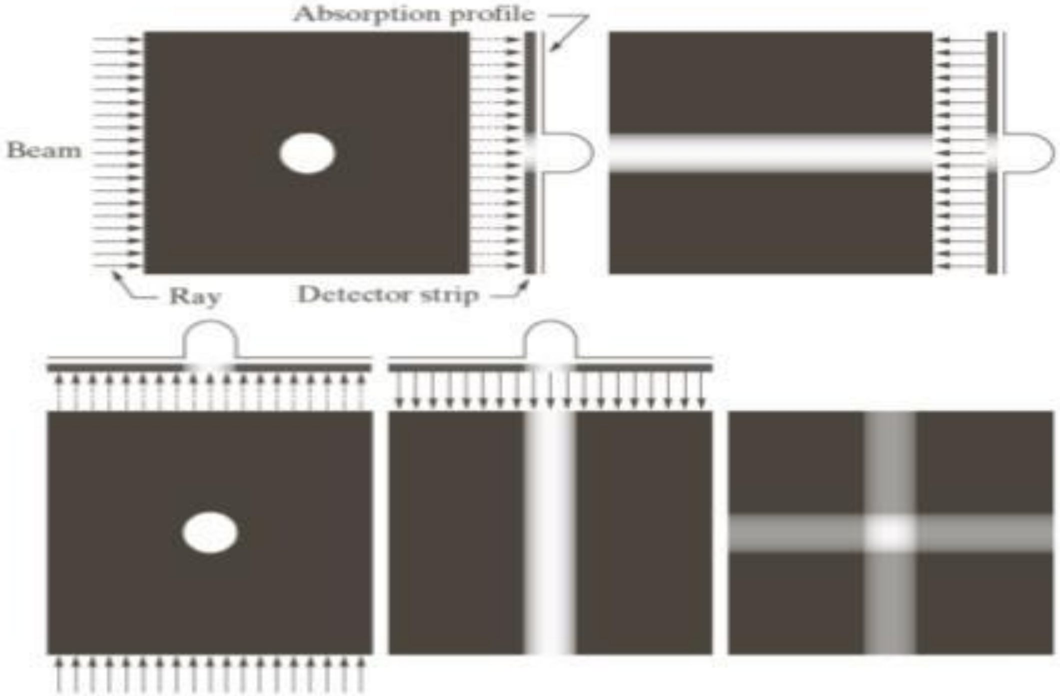


Image Reconstruction problem (Cont…)

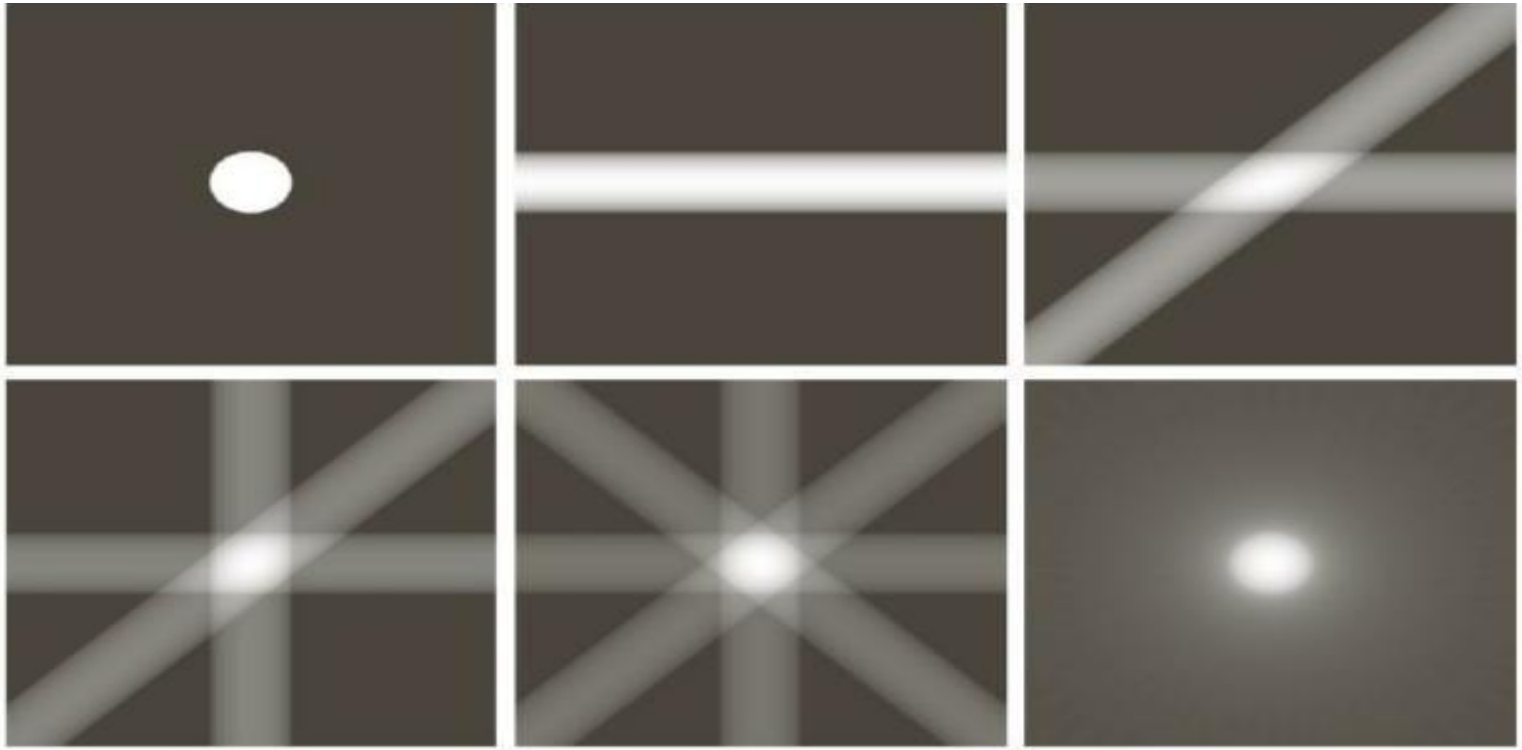


FIGURE 5.33 (a) Same as Fig. 5.32(a). (b)–(e) Reconstruction using 1, 2, 3, and 4 backprojections 45° apart. (f) Reconstruction with 32 backprojections 5.625° apart (note the blurring).

By taking more projections:

- the form of the object becomes clearer as brighter regions will dominate the result
- back-projections with few interactions with the object will fade into the background.

Image Reconstruction problem (Cont…)

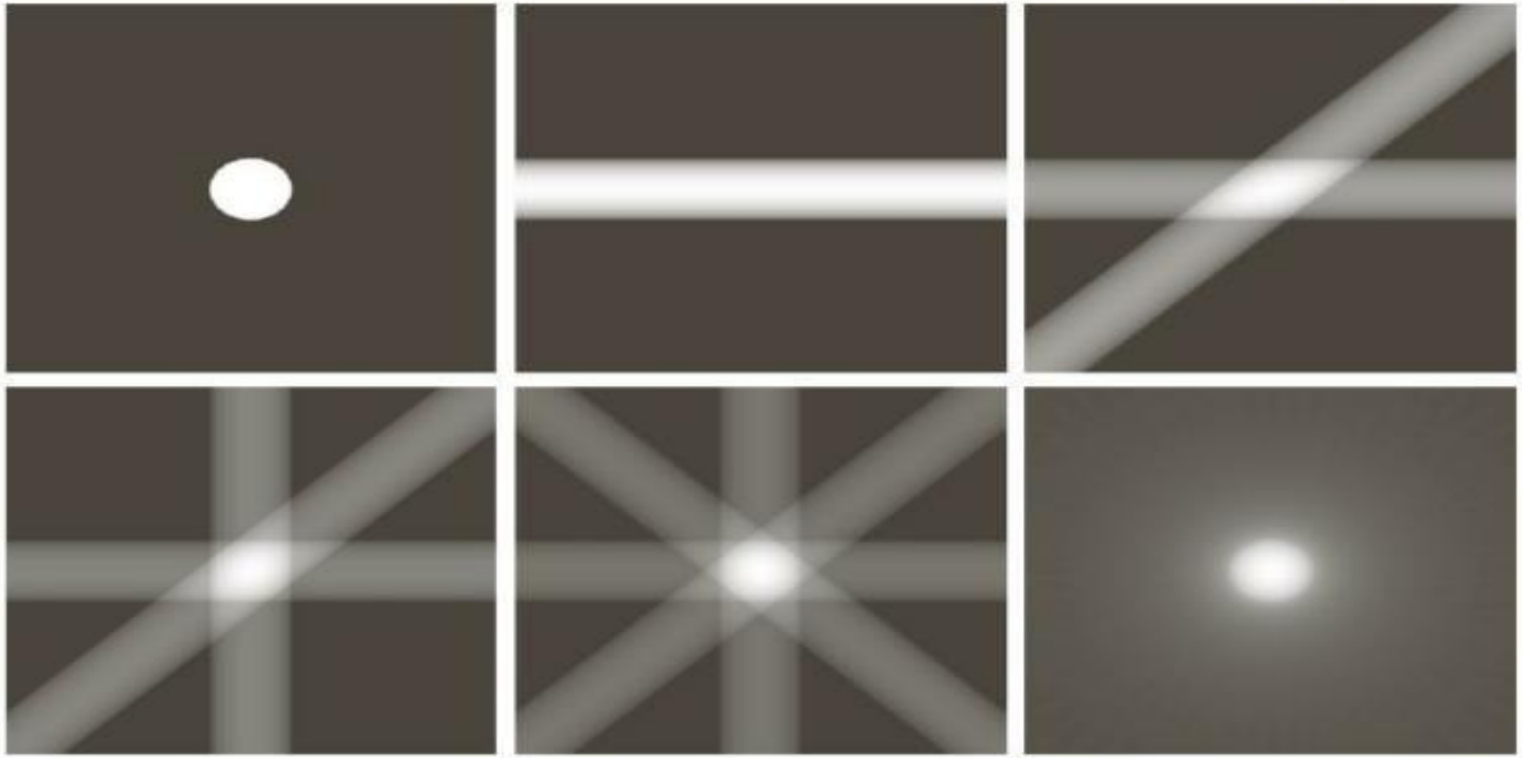
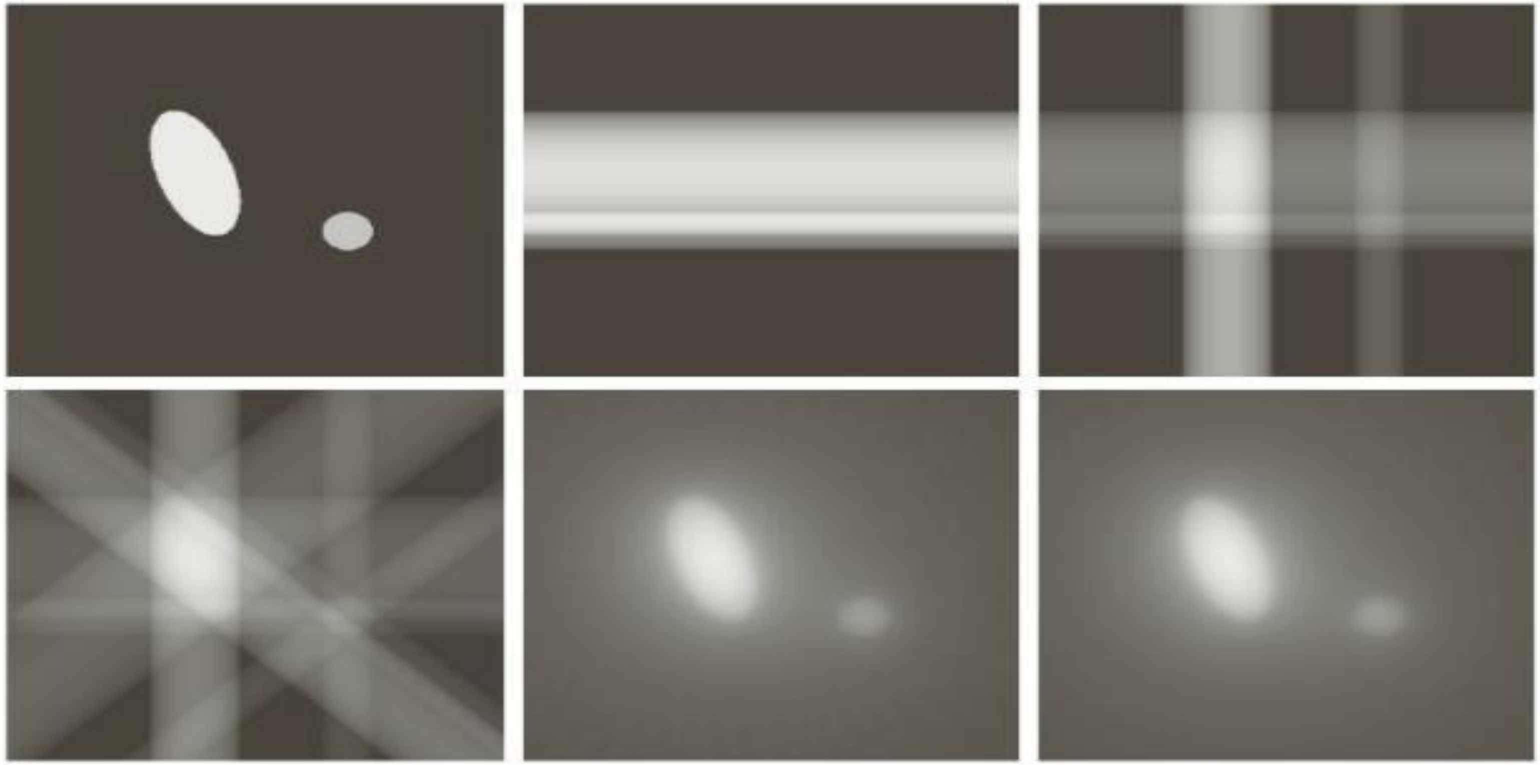


FIGURE 5.33 (a) Same as Fig. 5.32(a). (b)–(e) Reconstruction using 1, 2, 3, and 4 backprojections 45° apart. (f) Reconstruction with 32 backprojections 5.625° apart (note the blurring).

- The image is blurred. Important problem!
- We only consider projections from 0 to 180 degrees as projections differing 180 degrees are mirror images of each other.

Image Reconstruction problem (Cont…)



a b c
d e f

FIGURE 5.34 (a) A region with two objects. (b)–(d) Reconstruction using 1, 2, and 4 backprojections 45° apart. (e) Reconstruction with 32 backprojections 5.625° apart. (f) Reconstruction with 64 backprojections 2.8125° apart.

Principles of Computed Tomography

The goal of CT is to obtain a 3D representation of the internal structure of an object by X-raying it from many different directions.

Imagine the traditional chest X-ray obtained by different directions. The image is the 2D equivalent of a line projections.

Back-projecting the image would result in a 3D volume of the chest cavity.

Principles of Computed Tomography

CT gets the same information by generating slices through the body.

A 3D representation is then obtained by stacking the slices.

More economical due to fewer detectors.

Computational burden and dosage is reduced.

Theory developed in 1917 by J. Radon.

Application developed in 1964 by A. M. Cormack and G. N. Hounsfield independently. They shared the Nobel prize in Medicine in 1979.

Principles of Computed Tomography

a b
c d

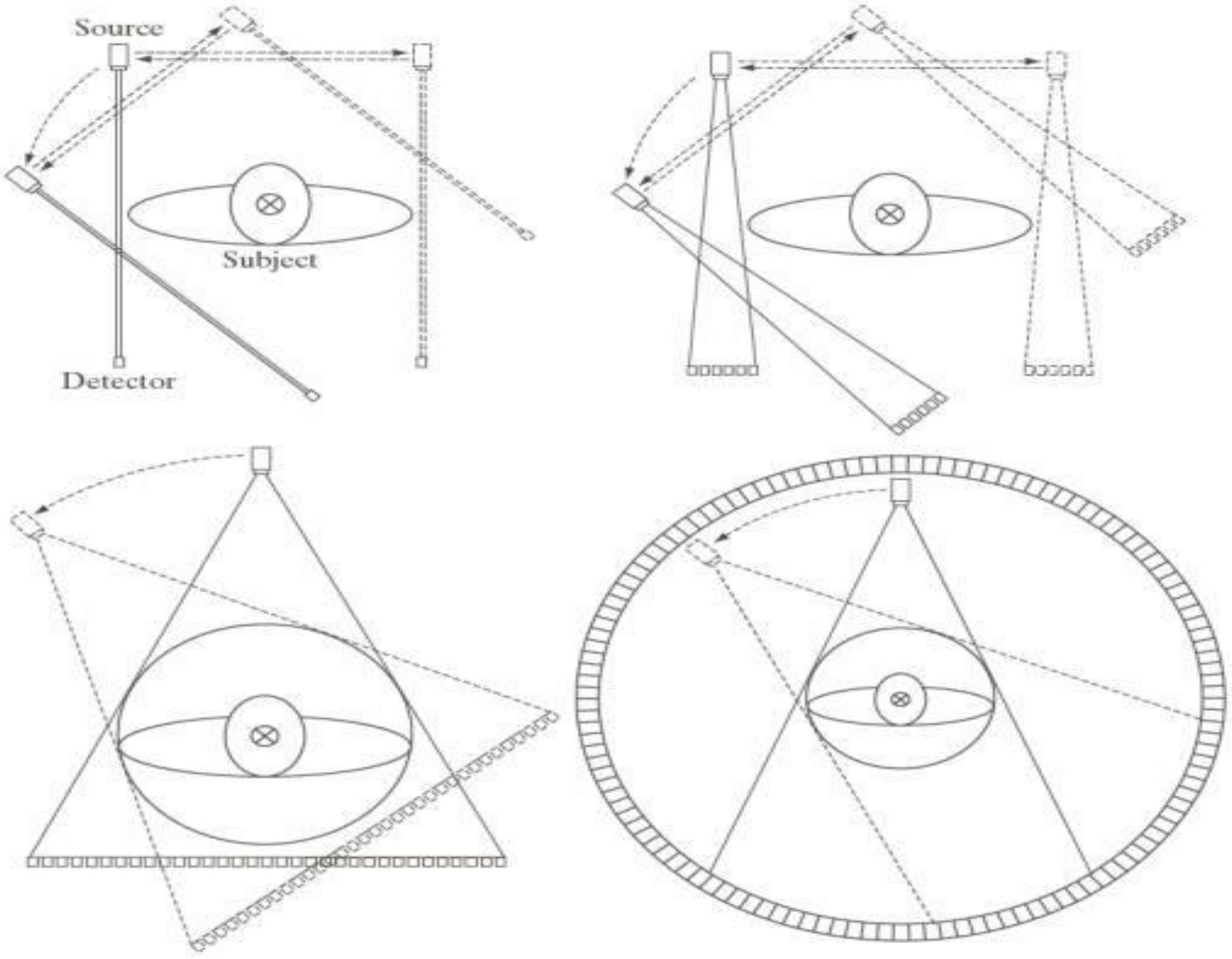


FIGURE 5.35 Four generations of CT scanners. The dotted arrow lines indicate incremental linear motion. The dotted arrow arcs indicate incremental rotation. The cross-mark on the subject's head indicates linear motion perpendicular to the plane of the paper. The double arrows in (a) and (b) indicate that the source/detector unit is translated and then brought back into its original position.

The Radon Transform

A straight line in Cartesian coordinates may be described by its *slope-intercept* form:

$$y = ax + b$$

or by its *normal representation*:

$$x\cos\theta + y\sin\theta = \rho$$

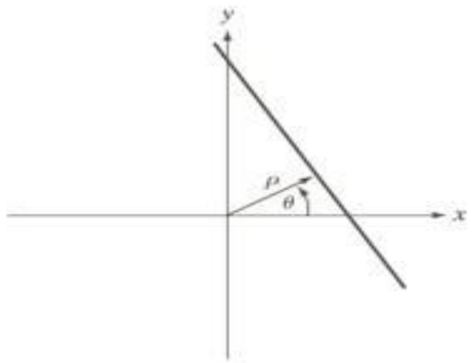
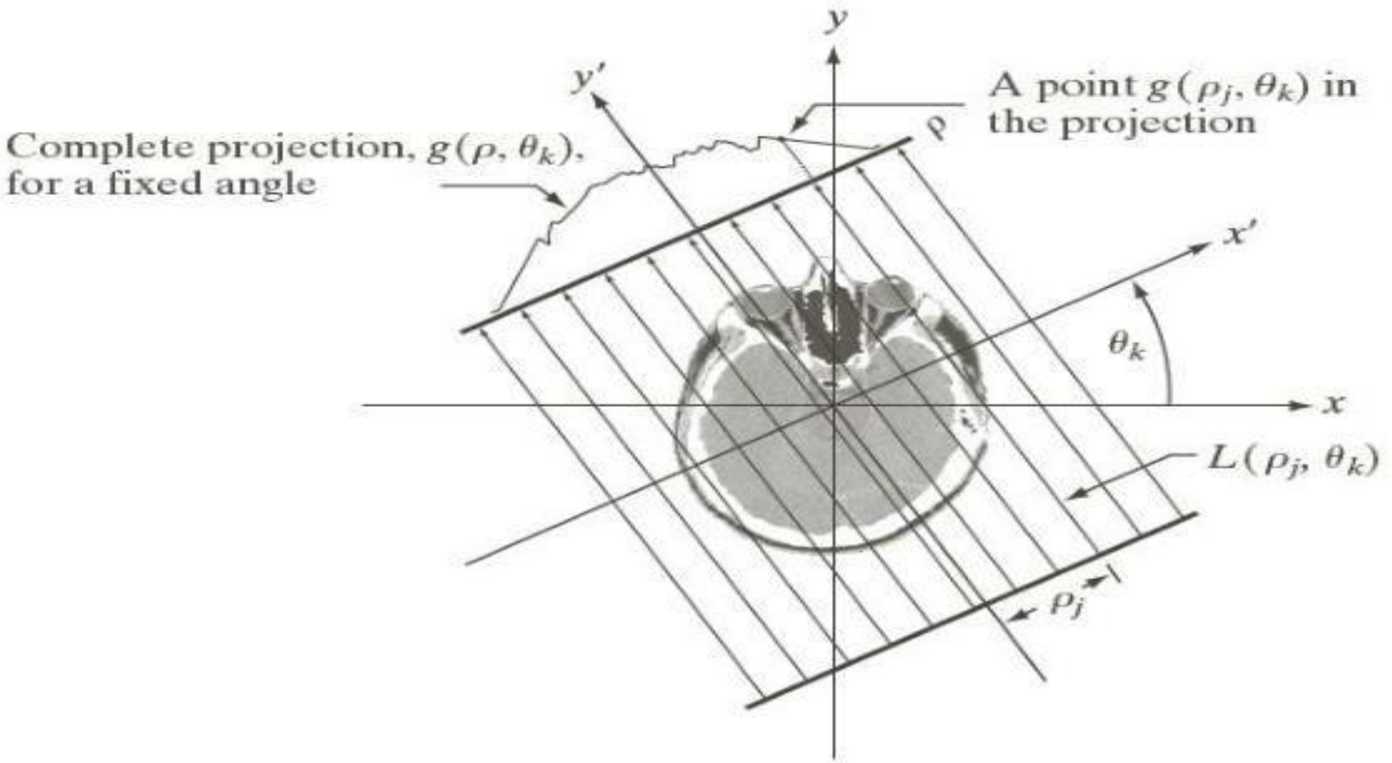


FIGURE 5.36 Normal representation of a straight line.

The Radon Transform (Cont…)

The projection of a parallel-ray beam may be modelled by a set of such lines.

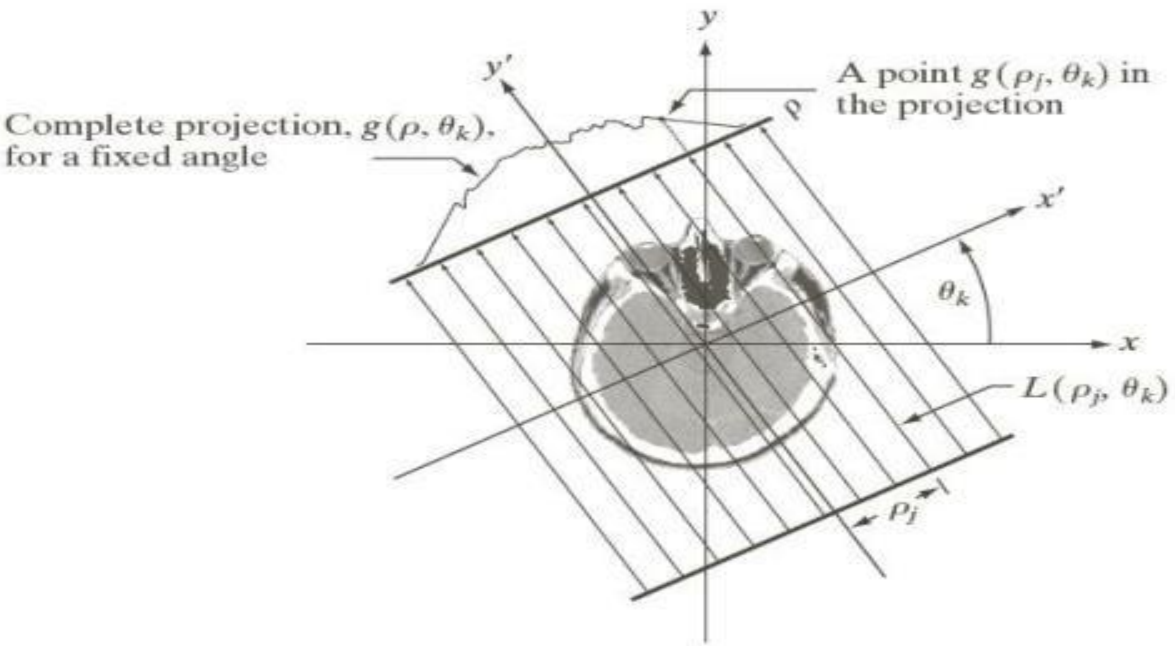
An arbitrary point (ρ_j, θ_k) in the projection signal is given by the ray-sum along the line $x\cos\theta_k + y\sin\theta_k = \rho_j$.



The Radon Transform (Cont…)

The ray-sum is a line integral:

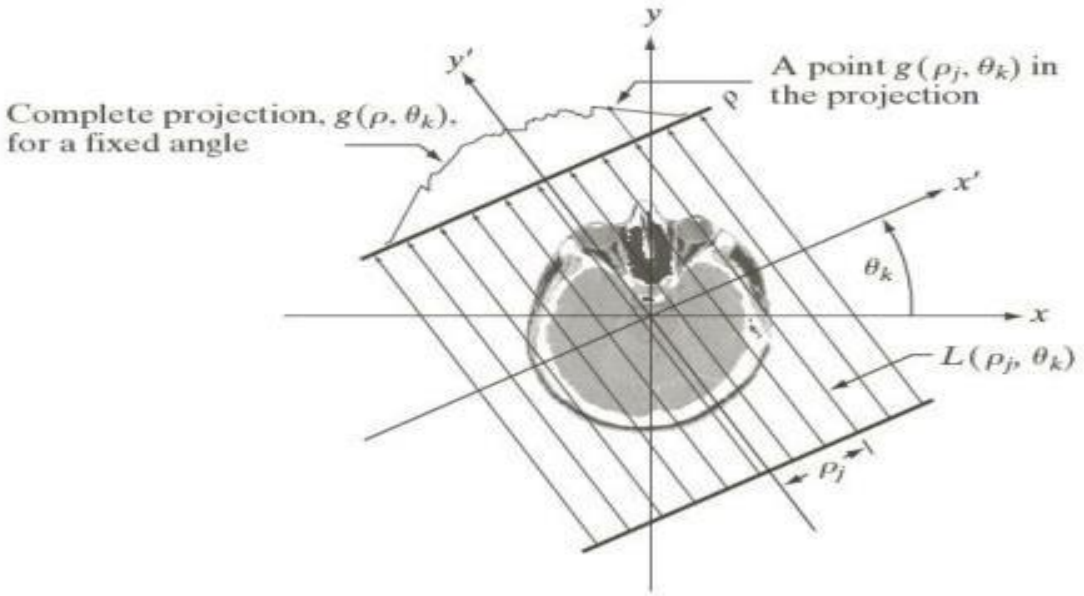
$$g(\rho_j,\theta_k)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x\cos\theta_k + y\sin\theta_k - \rho_j)dxdy$$



The Radon Transform (Cont...)

For all values of ρ and θ we obtain the Radon transform:

$$g(\rho,\theta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x\cos\theta+y\sin\theta-\rho)dxdy$$



The Radon Transform (Cont…)

The representation of the Radon transform $g(\rho,\theta)$ as an image with ρ and θ as coordinates is called a *sinogram*.

It is very difficult to interpret a sinogram.

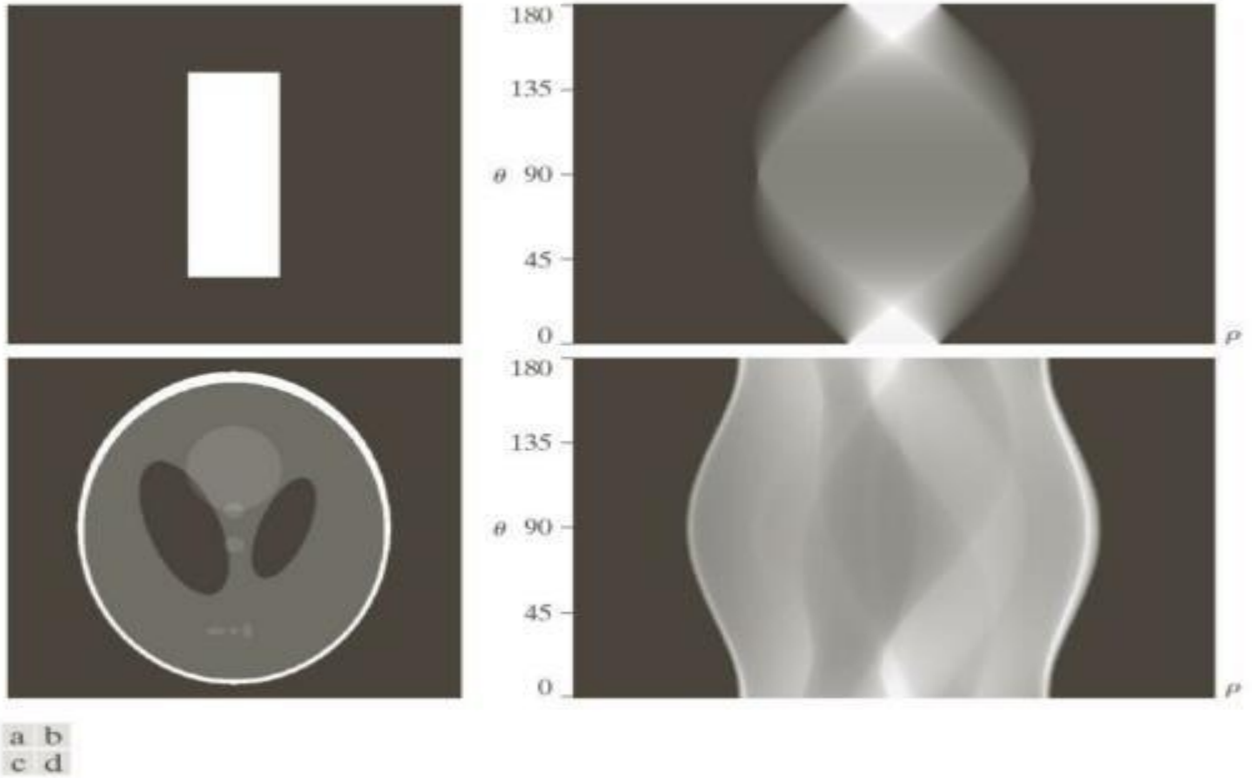


FIGURE 5.39 Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. Image (c) is called the *Shepp-Logan phantom*. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

The Radon Transform (Cont…)

The objective of CT is to obtain a 3D representation of a volume from its projections.

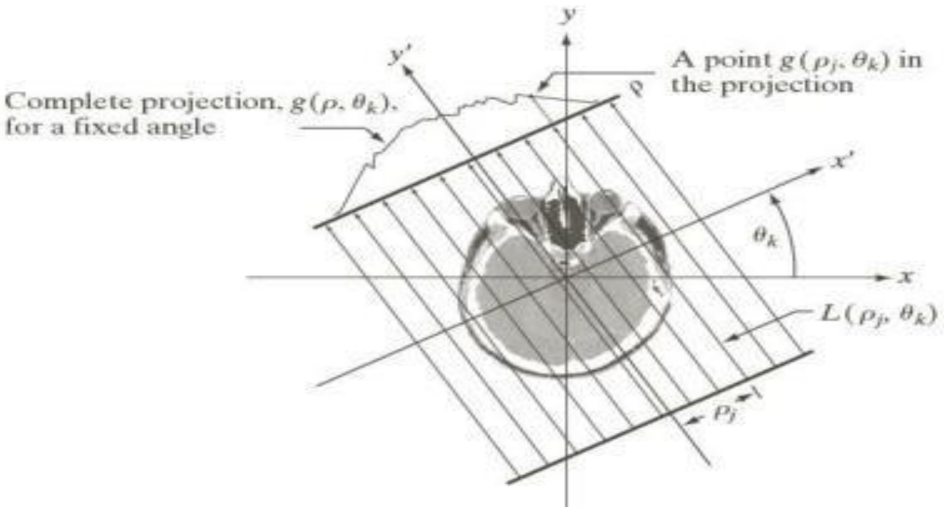
The approach is to back-project each projection and sum all the back-projections to generate a slice.

Stacking all the slices produces a 3D volume.

We will now describe the back-projection operation mathematically.

The Radon Transform (Cont…)

For a fixed rotation angle θ_k, and a fixed distance ρ_j, back-projecting the value of the projection $g(\rho_j, \theta_k)$ is equivalent to copying the value $g(\rho_j, \theta_k)$ to the image pixels belonging to the line $x\cos\theta_k + y\sin\theta_k = \rho_j$.



The Radon Transform (Cont…)

Repeating the process for all values of ρ_j, having a fixed angle θ_k results in the following expression for the image values:

$$f_{\theta_k}(x, y) = g(\rho, \theta_k) = g(x\cos\theta_k + y\sin\theta_k, \theta_k)$$

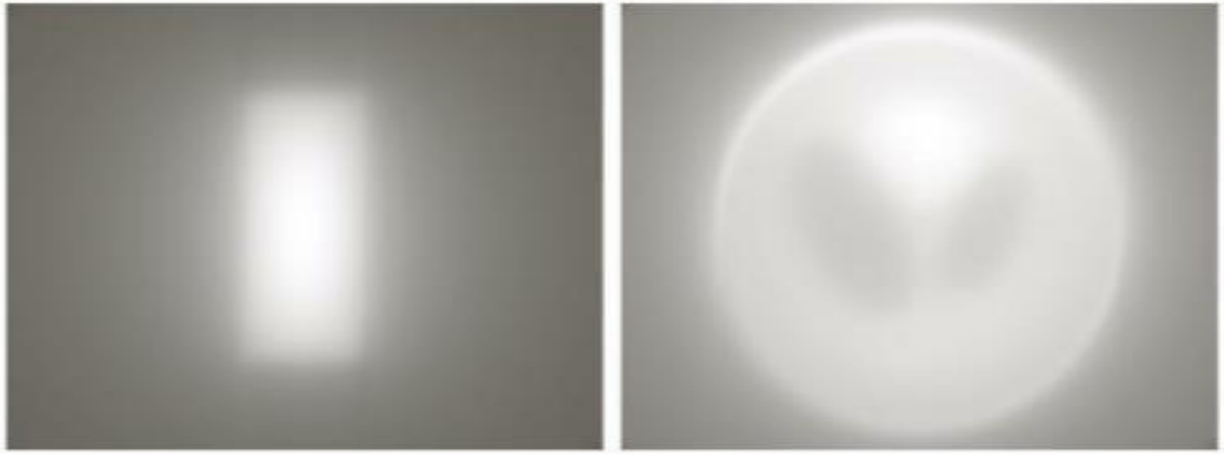
This equation holds for every angle θ:

$$f_{\theta}(x, y) = g(\rho, \theta) = g(x\cos\theta + y\sin\theta, \theta)$$

The Radon Transform (Cont…)

The final image is formed by integrating over all the back-projected images:

$$f(x, y) = \int_0^{\pi} f_\theta(x, y)\, d\theta$$



Back-projection provides blurred images. We will reformulate the process to eliminate blurring.

The Fourier Slice Theorem

The *Fourier-slice theorem* or the *central slice theorem* relates the 1D Fourier transform of a projection with the 2D Fourier transform of the region of the image from which the projection was obtained.

It is the basis of image reconstruction methods.

The Fourier Slice Theorem (Cont…)

Let the 1D F.T. of a projection with respect to ρ (at a given angle) be:

$$G(\omega,\theta)=\int_{-\infty}^{+\infty} g(\rho,\theta)\, e^{-j2\pi\omega\rho}\, d\rho$$

Substituting the projection $g(\rho,\theta)$ by the ray-sum:

$$G(\omega,\theta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\delta(x\cos\theta+y\sin\theta-\rho)dxdy\, e^{-j2\pi\omega\rho}\, d\rho$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\left[\int_{-\infty}^{+\infty}\delta(x\cos\theta+y\sin\theta-\rho)e^{-j2\pi\omega\rho}\, d\rho\right]dxdy$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi\omega(x\cos\theta+y\sin\theta)}\, dxdy$$

The Fourier Slice Theorem (Cont…)

$$G(\omega,\theta)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi\omega(x\cos\theta+y\sin\theta)}\,dxdy$$

Let now $u=\omega\cos\theta$ and $v=\omega\sin\theta$:

$$G(\omega,\theta)=\left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)e^{-j2\pi(ux+vy)}\,dxdy\right]_{u=\omega\cos\theta,v=\omega\sin\theta}$$

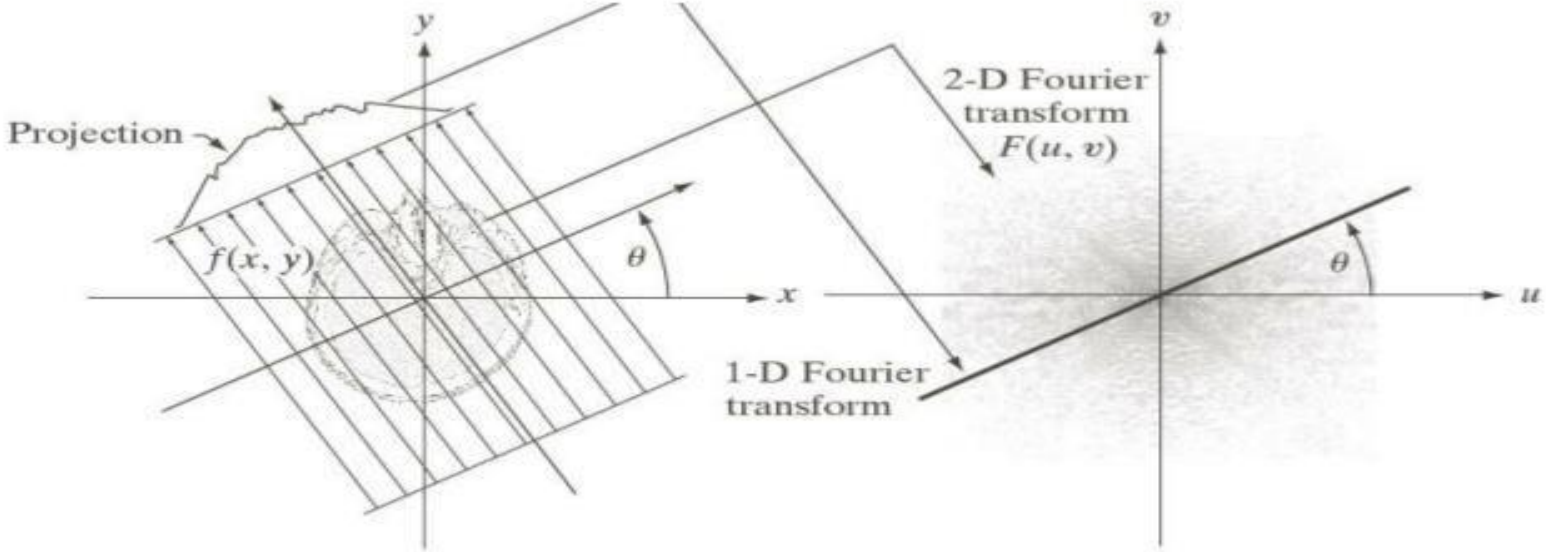
which is the 2D F.T. of the image $f(x,y)$ evaluated at the indicated frequencies u,v:

$$G(\omega,\theta)=\left[F(u,v)\right]_{u=\omega\cos\theta,v=\omega\sin\theta}=F(\omega\cos\theta,\omega\sin\theta)$$

The Fourier Slice Theorem (Cont…)

The resulting equation $G(\omega, \theta) = F(\omega\cos\theta, \omega\sin\theta)$ is known as the Fourier-slice theorem.

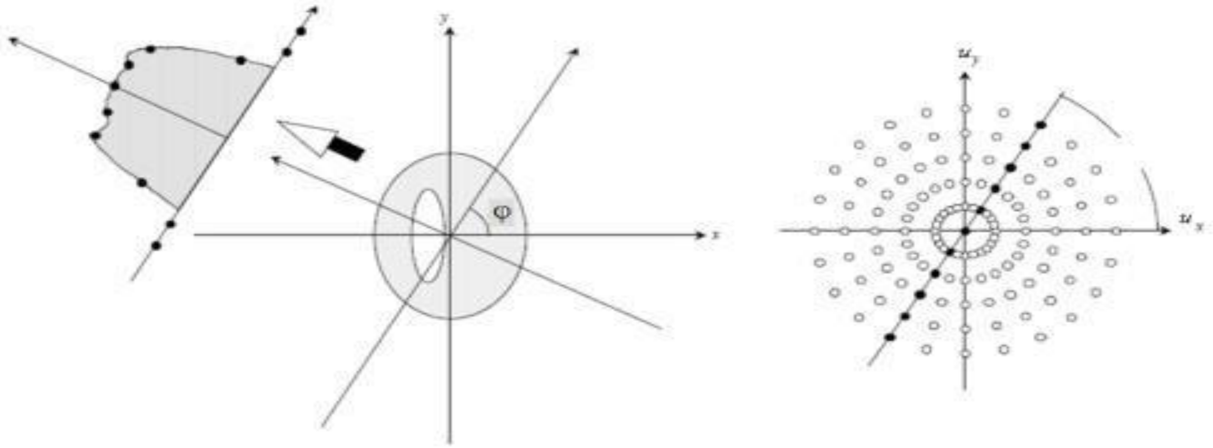
It states that the 1D F.T. of a projection (at a given angle θ) is a slice of the 2D F.T. of the image.



The Fourier Slice Theorem (Cont…)

We could obtain $f(x,y)$ by evaluating the F.T. of every projection and inverting them.

However, this procedure needs irregular interpolation which introduces inaccuracies.



Reconstruction using Parallel Beam filtered back projection

The 2D inverse Fourier transform of $F(u,v)$ is

$$f(x,y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u,v) e^{j2\pi(ux+vy)} \, du \, dv$$

Letting $u = \omega\cos\theta$ and $v = \omega\sin\theta$ then the differential

$$du \, dv = \omega \, d\omega \, d\theta$$

and

$$f(x,y) = \int_{0}^{2\pi} \int_{0}^{+\infty} F(\omega\cos\theta, \omega\sin\theta) e^{j2\pi\omega(x\cos\theta + y\sin\theta)} \omega \, d\omega \, d\theta$$

Reconstruction using Parallel Beam filtered back projection (Cont…)

Using the Fourier-slice theorem:

$$f(x,y)=\int_0^{2\pi}\int_0^{+\infty}G(\omega,\theta)e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega\,d\omega\,d\theta$$

With some manipulation

$$f(x,y)=\int_0^{\pi}\int_{-\infty}^{+\infty}|\omega|G(\omega,\theta)e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\,d\omega\,d\theta$$

The term $x\cos\theta+y\sin\theta=\rho$ and is independent of ω:

$$f(x,y)=\int_0^{\pi}\left[\int_{-\infty}^{+\infty}|\omega|G(\omega,\theta)e^{j2\pi\omega\rho}\,d\omega\right]_{\rho=x\cos\theta+y\sin\theta}d\theta$$

Reconstruction using Parallel Beam filtered back projection (Cont…)

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{+\infty} |\omega| G(\omega,\theta) e^{j2\pi\omega\rho} d\omega \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

For a given angle θ, the inner expression is the 1-D Fourier transform of the projection multiplied by a ramp filter $|\omega|$.

This is equivalent in filtering the projection with a high-pass filter with Fourier Transform $|\omega|$ before back-projection.

Reconstruction using Parallel Beam filtered back projection (Cont…)

$$f(x, y) = \int_0^{\pi} \left[\int_{-\infty}^{+\infty} |\omega| G(\omega, \theta) e^{j2\pi\omega\rho} d\omega \right]_{\rho = x\cos\theta + y\sin\theta} d\theta$$

Problem: the filter $H(\omega)=|\omega|$ is not integrable in the inverse Fourier transform as it extends to infinity in both directions.

It should be truncated in the frequency domain.

The simplest approach is to multiply it by a box filter in the frequency domain.

Ringing will be noticeable.

Windows with smoother transitions are used.

Reconstruction using Parallel Beam filtered back projection (Cont…)

An M-point discrete window function used frequently is

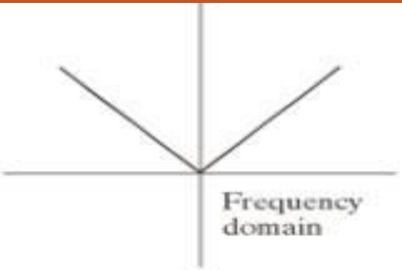
$$h(\omega) = \begin{cases} c + (c-1)\cos\dfrac{2\pi\omega}{M-1} & 0 \le \omega \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

When c=0.54, it is called the *Hamming* window.
When c=0.5, it is called the *Hann* window.

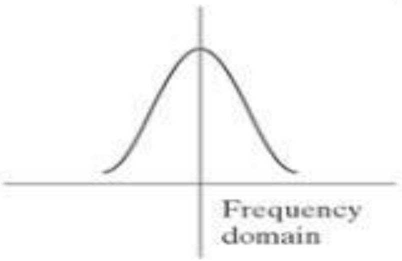
By these means ringing decreases.

Reconstruction using Parallel Beam filtered back projection (Cont…)

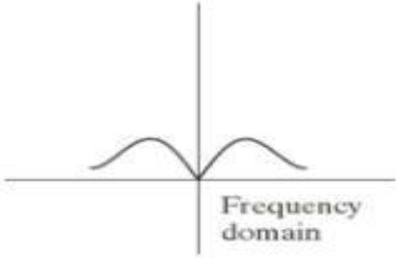
Ramp filter multiplied by a box window

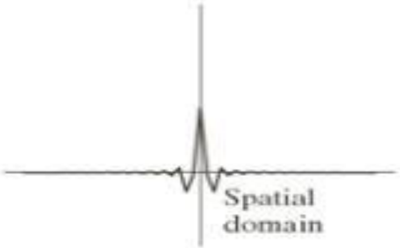


Hamming window



Ramp filter multiplied by a Hamming window





Reconstruction using Parallel Beam filtered back projection (Cont…)

The *complete* back-projection is obtained as follows:

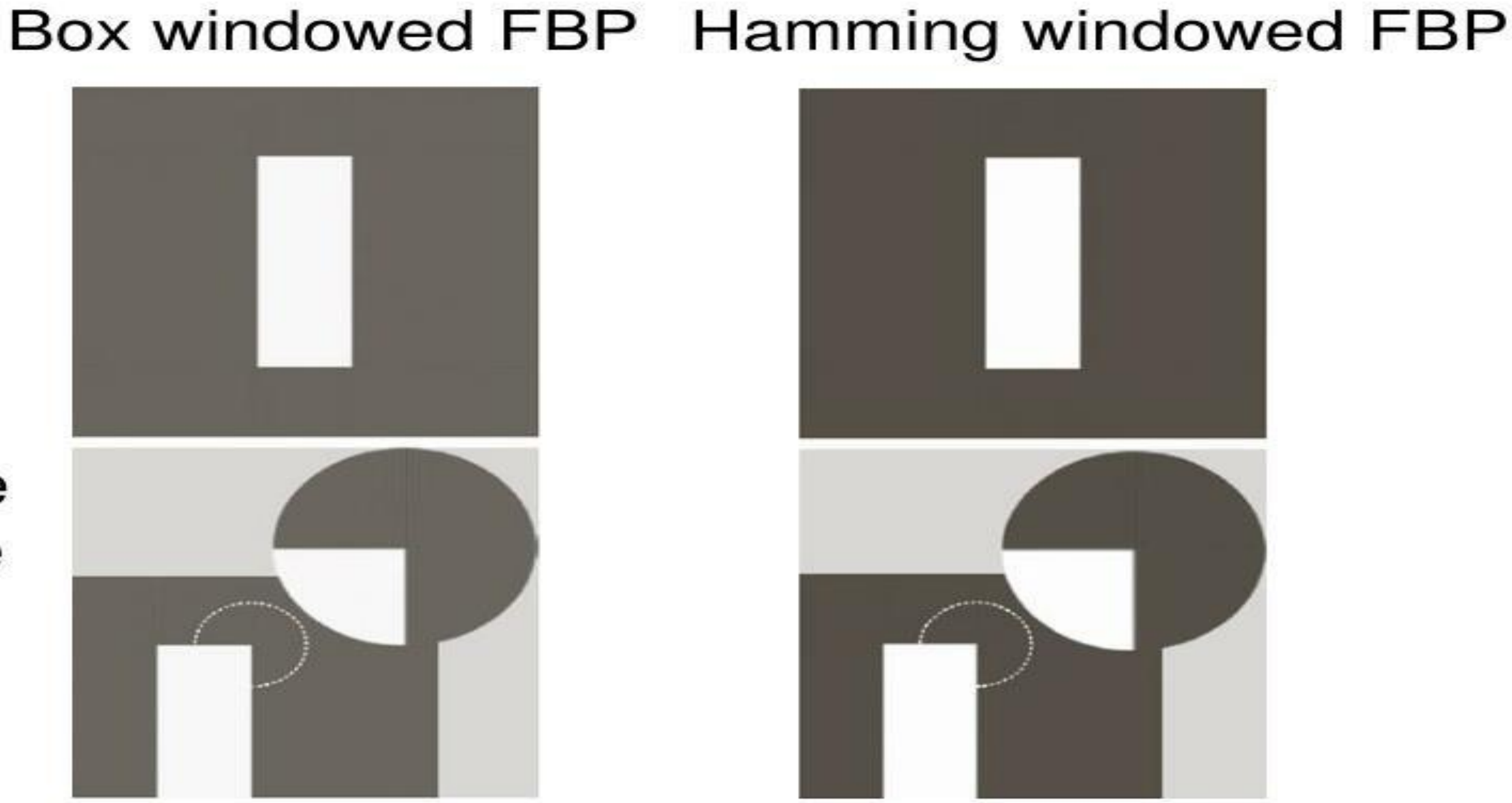
1. Compute the 1-D Fourier transform of each projection.
2. Multiply each Fourier transform by the filter function $|\omega|$ (multiplied by a suitable window, e.g. Hamming).
3. Obtain the inverse 1-D Fourier transform of each resulting filtered transform.
4. Back-project and integrate all the 1-D inverse transforms from step 3.

Reconstruction using Parallel Beam filtered back projection (Cont…)

Because of the filter function the reconstruction approach is called *filtered back-projection* (FBP).

- Sampling issues must be taken into account to prevent aliasing.
  - The number of rays per angle which determines the number of samples for each projection.
  - The number of rotation angles which determines the number of reconstructed images.

Reconstruction using Parallel Beam filtered back projection (Cont…)



Ringing is more pronounced in the Ramp FBP image

Reconstruction using Parallel Beam filtered back projection (Cont…)

Box windowed FBP

Reconstruction using Parallel Beam filtered back projection (Cont…)

Hamming windowed FBP

Reconstruction using Parallel Beam filtered back projection (Cont…)

Box windowed FBP

Hamming windowed FBP

There are no sharp transitions in the Shepp-Logan phantom and the two filters provide similar results.

Reconstruction using Parallel Beam filtered back projection (Cont…)

Back-projection Ramp FBP Hamming FBP

Notice the difference between the simple back-projection and the filtered back-projection.

Suggested Readings

- **Digital Image Processing by Rafel Gonzalez, Richard Woods, Pearson Education India, 2017.**

- **Fundamental of Digital image processing by A. K Jain, Pearson Education India, 2015.**

Thank you