— Moments of Random Variable :—

Given set of data $X_1, X_2, X_3, \ldots, X_N$ we know that the arithmetic ~~mean~~ average (or arithmetic mean) is given by:

$$\bar{X} = \frac{X_1 + X_2 + X_3 + \cdots + X_N}{N}$$

When above numbers occurs with different ~~frequency~~ frequencies, we usually assign weights $\omega_1, \omega_2, \omega_3, \ldots, \omega_N$ to them and so-called weighted arithmatic mean becomes:

$$\bar{X} = \frac{\omega_1 X_1 + \omega_2 X_2 + \cdots + \omega_N X_N}{\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_N}$$

$$\bar{X} = \frac{\omega_1}{\omega} X_1 + \frac{\omega_2}{\omega} X_2 + \frac{\omega_3}{\omega} X_3 + \cdots + \frac{\omega_N}{\omega} X_N \quad \text{— (A)}$$

where $\omega = \omega_1 + \omega_2 + \omega_3 + \cdots + \omega_N$. The average is a value that is representative or typical of a set of data and tends to lie centrally within a set of data that are arranged according to their magnitudes.

↳ Thus, it is called measure of central tendency

↳ observe (A) the relative weight $\frac{\omega_k}{\omega}$ is the ~~probability~~ probability of occurrence of the data point X_k in the set of observations

— The term expectation is used for the process of averaging when a random variable is involved

↳ It is a number used to locate the "center" of the distribution of a random variable.

~~Expectation of r.v. is linked with~~

Expectation of r.v. is linked with ~~to~~ the weighted arithmatic average.

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— Another measure of central ~~de~~ tendency of a r.v. is its variance, which measures the degree to which a r.v. is spread out relative to the average value

Expectation :- (Def^n) :- If X is a r.v., then the expectation (or expected value or mean) of X, denoted by $E[X]$ or $\bar{X}$, is defined by:

$$E[X] = \bar{X} = \begin{cases} \sum_k x_k p_X(x_k) & ; X \text{ is discrete} \\ \int_{-\infty}^{+\infty} x f_X(x) & ; X \text{ is continuous} \end{cases}$$

$p_X(\cdot) \equiv$ PMF, $f_X(\cdot)$: pdf

Thus, for a discrete random variable X, the expected value is the weighted average of the possible values that X can take, where each value is weighted by the probability that takes that value.

Note — Discrete case

$$E[X] = \sum_{X(\omega)} X(\omega)\, p(X(\omega))$$

if $X(\omega) = x_k$

$$= \sum_{x_k} x_k\, p_X(x_k)$$

$p(X(\omega)) \equiv p_X(x_k)$

$$\equiv \sum_k x_k\, p_X(x_k)$$

Contn case :

$$E[X] = \int_{-\infty}^{+\infty} X(\omega)\, f(X(\omega))\, d(X(\omega)) \quad ;$$

Denote $X(\omega) = x$

If X

$f(X(\omega)) = f_X(x)$

$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

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Now consider a function ψ of r.v. X i.e.

$$X : \Omega \longrightarrow \mathbb{R} \xrightarrow{\psi} \mathbb{R}$$

$$\psi(X(\omega)) \; ?$$

$$E[\psi(X(\omega))] = \int_{-\infty}^{+\infty} \psi(X(\omega))\, f(X(\omega))\, d(X(\omega))$$

$$E[\psi_X(x)] = \int_{-\infty}^{+\infty} \psi_X(x)\, f_X(x)\, dx$$

Here $\psi_X(x) \equiv \psi(X(\omega))$; $X(\omega) = x$

$f(X(\omega)) \equiv f_X(x)$: pdf of X.

<u>Discrete case</u> :-

$$E[\psi(X(\omega))] = \sum_{X(\omega)} \psi(X(\omega))\, p(X(\omega))$$

$X(\omega) = x_k$; $\psi(X(\omega)) \equiv \psi_X(x_k)$

$p(X(\omega)) \equiv p_X(x_k)$ → PMF

$$E[\psi_X(X(\omega))] \equiv \sum_{x_k} \psi_X(x_k)\, p_X(x_k)$$

$$E[\psi(X(\omega))] \equiv \sum_{k} \psi_X(x_k)\, p_X(x_k)$$

$$\boxed{E[\psi_X(x)] \equiv \sum_{k} \psi_X(x_k)\, p_X(x_k)}$$

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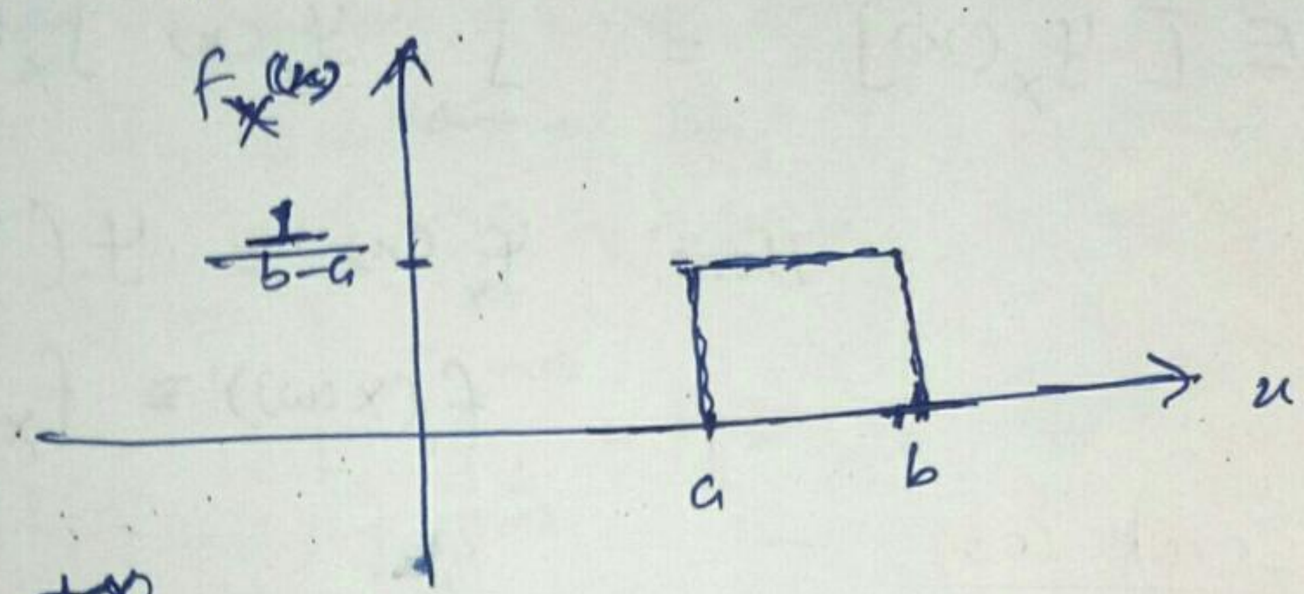
Ex:- Find the expected value of the r.v. X whose pdf is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

Solution:- Plot of pdf is



$$E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx$$

$$= \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$\therefore E[X] = \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{1}{\cancel{(b-a)}} \frac{\cancel{(b-a)}(b+a)}{2}$$

$$\boxed{E[X] = \frac{b+a}{2}}$$

Ex:- Find the expected value of the discrete r.v. X with the following PMF:

$$p_X(x) = \begin{cases} \frac{1}{3}; & x = 0 \\ \frac{2}{3}; & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Solution:-

$$E[X] = \sum_{x=-\infty}^{+\infty} x\, p_X(x) = 0 \cdot p_X(0) + 2 \cdot p_X(2)$$

$$= 0\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right)$$

$$= 4/3$$

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Example:- Find the expected value of the r.v. K with the following PMF:

$$p_K(k) = \frac{\lambda^k}{k!} e^{-\lambda} \; ; \quad k = 0, 1, 2, 3, \cdots$$

Solution

$$E[K] = \sum_{k=0}^{+\infty} k \, p_K(k) = \sum_{k=0}^{+\infty} k \frac{\lambda^k}{k!} e^{-\lambda}$$

$$E[K] = \sum_{k=1}^{+\infty} k \frac{\lambda^k}{(k-1)!} e^{-\lambda}$$

$$= \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!} \lambda e^{-\lambda}$$

$$E[K] = \lambda e^{-\lambda} \sum_{k=1}^{+\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \quad , m = k-1$$

Since $\sum_{m=0}^{\infty} \frac{a^m}{m!} = e^a$, we have that

$$E[K] = \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$\therefore \boxed{E[K] = \lambda}$$

Ex:- Find the expected value of the random variable X whose pdf is given by:

$$f_X(x) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} \; ; & x \geq 0, \; \theta > 0 \\ 0, & \text{otherwise i.e. } x < 0 \end{cases}$$

Solution:- The expected value of X is given by

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x) \, dx = \int_0^{+\infty} x \frac{1}{\theta} e^{-x/\theta} \, dx$$

$$E[X] = \int_0^{+\infty} \frac{x}{\theta} e^{-\frac{x}{\theta}} \, dx$$

$$\boxed{E[X] = \theta}$$

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Expectation of Nonnegative Random Variable:-

↳ Some random variable assume only nonnegative values.

e.g. the time X until component fails cannot be negative.

↳ Reliability function $R(t)$ of component as the probability that the component has not failed by time t. Thus, if the pdf of X is $f_X(x)$ and the CDF is $F_X(x)$, we can define the reliability function of the component by $R_X(t)$, which is related to the CDF and pdf as follows:

$$R_X(t) = P[X>t] = 1 - P[X \le t] = 1 - F_X(t)$$

$$\therefore \boxed{F_X(t) = 1 - R_X(t)}$$

Proposition ① For ~~non-ne~~ nonnegative r.v. X with CDF $F_X(x)$, the expected value is given by

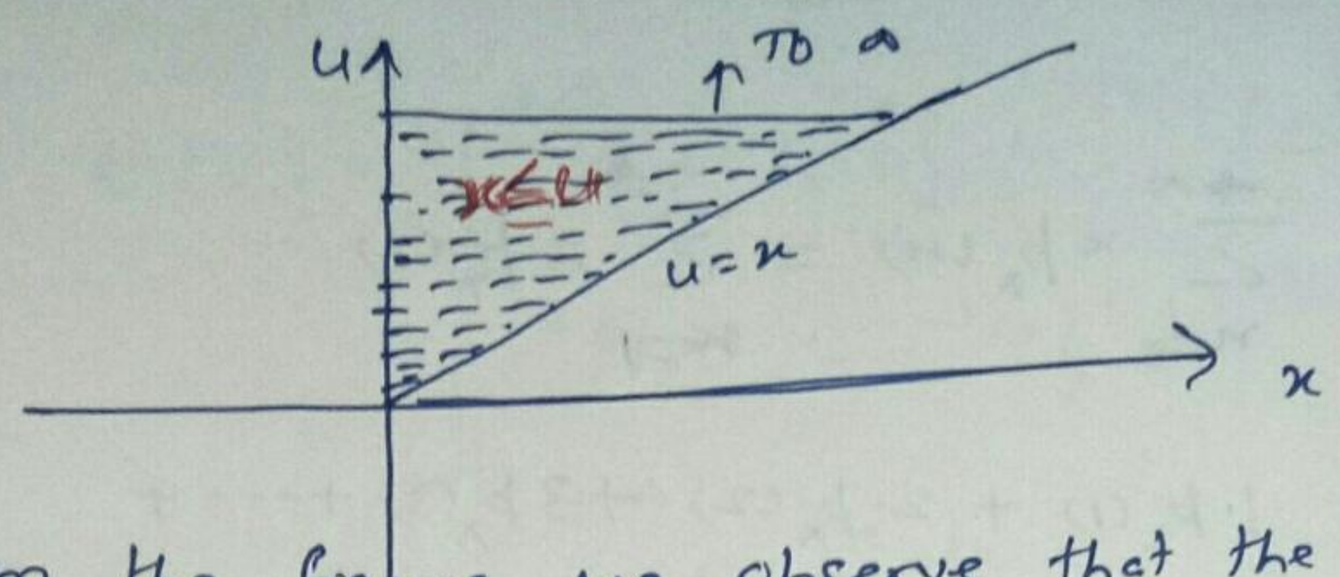
$$E[X] = \begin{cases} \int_0^{+\infty} P[X>x]\,dx = \int_0^{+\infty} \{1 - F_X(x)\}\,dx, & X \text{ continuous} \\ \sum_{x=0}^{\infty} P[X>x] = \sum_{x=0}^{+\infty} \{1 - F_X(x)\}; & X \text{ discrete.} \end{cases}$$

Proof:- Case 1: contn r.v.:-

$$P[X>x] = \int_x^{+\infty} f_X(u)\,du \qquad \int_0^{+\infty} P[X>x]\,dx = \int_0^{+\infty}\left[\int_x^{+\infty} f_X(u)\,du\right]dx$$

$$= \int_0^{\infty}\int_x^{\infty} f_X(u)\,du\,dx$$

The region of integration $\{(x,u) \mid 0 \le x < \infty;\ x \le u < \infty\}$ is as follows.



From the figure we observe that the region of integration can be transformed into $\{(x,u) \mid 0 \le x \le u;\ 0 \le u < \infty\}$, which gives:

$$\int_0^{+\infty} P[X > x]\, dx = \int_0^{+\infty} \int_x^{\infty} f_X(u)\, du\, dx$$

$$= \int_{u=0}^{+\infty} \left\{ \int_{x=0}^{u} dx \right\} f_X(u)\, du$$

$$= \int_{u=0}^{+\infty} u\, f_X(u)\, du$$

$$= E[X]$$

$$\Rightarrow \quad E[X] = \int_0^{+\infty} P[X > x]\, dx \quad \text{--- (1)}$$

Since $P[X > x] = 1 - P[X \le x]$

$$= 1 - F_X(x)$$

$\Rightarrow$ from (1)

$$\boxed{E[X] = \int_0^{+\infty} P[X > x]\, dx = \int_0^{+\infty} \{1 - F_X(x)\}\, dx}$$

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Discrete R.V :-

$$E[X] = \sum_{x=0}^{+\infty} x\, p_X(x) = \sum_{x=1}^{+\infty} x\, p_X(x)$$

$$= 1\cdot p_X(1) + 2\cdot p_X(2) + 3\, p_X(3) + \cdots +$$

$$= p_X(1)$$
$$+ p_X(2) + p_X(2)$$
$$+ p_X(3) + p_X(3) + p_X(3)$$
$$+ p_X(4) + p_X(4) + p_X(4) + p_X(4)$$
$$+ \cdots$$

$$\therefore E[X] = \sum_{x=1}^{\infty} p_X(x) + \sum_{x=2}^{\infty} p_X(x) + \sum_{x=3}^{\infty} p_X(x) + \cdots + \cdots$$

$$\therefore E[X] = \{1 - F_X(0)\} + \{1 - F_X(1)\} + \{1 - F_X(2)\} + \cdots + \cdots$$

$$= \sum_{x=0}^{\infty} \{1 - F_X(x)\}$$

$$\boxed{E[X] = \sum_{x=0}^{+\infty} P[X > x]}$$

Ex:- Use the above proposition result to find the expected value of exponential density.

$$f_X(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} ; & x \geq 0 \\ 0 , & x < 0 \end{cases}$$

Solution:- CDF

$$F_X(x) = 1 - e^{-\frac{x}{\theta}} , \quad x \geq 0$$
$$= 0 \quad ; \quad x < 0$$

$$\therefore \cancel{E[X] = \int_0^{+\infty} \{1 - e^{-x/\theta}\}\, dx}$$

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$$E[X] = \int_0^{+\infty} \{1 - F_X(x)\}\, dx$$

$$= \int_0^{+\infty} \{\cancel{1} - \cancel{1} + e^{-x/\theta}\}\, dx$$

$$= \int_0^{+\infty} e^{-x/\theta}\, dx$$

$$= \left[-\theta\, e^{-x/\theta}\right]_0^{+\infty}$$

$$= 0 - (-\theta\, e^{-\frac{0}{\theta}})$$

$$\boxed{E[X] = \theta}$$

Ex:- Find the expected value of the ~~uniform~~ r.v. K dist with following pmf; using cdf approach.

$$p_K(k) = \frac{\lambda^k}{k!} e^{-\lambda}; \quad k = 0, 1, 2, \cdots$$

$$\boxed{E[K] = \lambda.}$$

$$F_K(k) = \sum_{i=0}^{k} e^{-\lambda} \frac{\lambda^i}{i!}$$

$$F_K(x) = \sum_{i=0}^{k} e^{-\lambda} \frac{\lambda^i}{i!}$$

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Moments of Random Variables And The variance :

The nth moment of the r.v. X, denoted by $E[X^n] = \overline{X^n} = \mu_n$ is defined by:

$$E[X^n] = \overline{X^n} = \mu_n = \begin{cases} \sum_k x_k^n \, p_X(x_k), & X \text{ discrete} \\ \int_{-\infty}^{+\infty} x^n f_X(x)\, dx \,; & X \text{ continuous} \end{cases}$$

for $n = 1, 2, 3, \ldots$ The first moment (i.e., $n=1$), $E[X]$ is the expected value of X.

Central moment :- Central moment (moments about the mean) of a random variable is the moments of the difference between a random variable and its expected value. The nth central moment is defined by

$$E[(X-\bar{X})^n] = \overline{(X-\bar{X})^n} = \mu'_n = \begin{cases} \sum_k (x_k - \bar{X})^n \, p_X(x_k), & X \text{ is discrete} \\ \int_{-\infty}^{+\infty} (x-\bar{X})^n f_X(x)\, dx, & X \text{ continuous} \end{cases}$$

(For $n=2$) The central moment for the case $n=2$ is very imbatant and carries a special name, the Variance, which is usually denoted by σ_X^2. Thus

$$\sigma_X^2 = E[(X-\bar{X})^2] = \begin{cases} \sum_k (x_k - \bar{X})^2 \, p_X(x_k) \,; & X \text{ discrete} \\ \int_{-\infty}^{+\infty} (x-\bar{X})^2 f_X(x)\, dx \,; & X \text{ continuous} \end{cases}$$

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Proposition 2:- Let X be a r.v. with pdf, $f_X(x)$, and mean $E[X]$. Let a and b are constant. Then, if Y is random variable defined by $Y = aX + b$, the expected value of Y is given by $E[Y] = aE[X] + b$

Proof:- Since Y is a function of X, its expected value is given by

$$E[Y] = E[aX+b] = \int_{-\infty}^{+\infty} (ax+b) f_X(x)\, dx$$

$$E[Y] = a\int_{-\infty}^{+\infty} x f_X(x)\, dx + b\int_{-\infty}^{+\infty} f_X(x)\, dx$$

$$= a\,E[X] + b \cdot 1$$

→ From defn of Expectation.

→ since $f_X(x)$ is pdf

$$\Rightarrow \boxed{E[Y] = aE[X] + b}$$

Proposition 3:- Let X be a r.v. with pdf $f_X(x)$ and mean $E[X]$. Let $g_1(X)$, $g_2(X)$ be two fn of random variable X, and let $g_3(X)$ be defined by $g_3(X) = g_1(X) + g_2(X)$. The expected value of $g_3(X)$ is given by:

$$E[g_3(X)] = E[g_1(X)] + E[g_2(X)]$$

Proof:- Since $g_3(X)$ is function of X. Its expected value is given by

$$E[g_3(X)] = \int_{-\infty}^{+\infty} g_3(X) f_X(x)\, dx$$

$$= \int_{-\infty}^{+\infty} [g_1(x) + g_2(x)] f_X(x)\, dx$$

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$$E[g_3(x)] = \int_{-\infty}^{+\infty} g_1(x) \cdot f_x(x)\, dx + \int_{-\infty}^{+\infty} g_2(x)\, f_x(x)\, dx$$

$$\boxed{E[g_3(x)] = E[g_1(x)] + E[g_2(x)]}$$

~~Ato~~

Using these two propositions, and ~~nothing~~ noting that $\bar{x}$ is constant, we obtain following

$$\sigma_x^2 = E[(x-\bar{x})^2]$$

$$= E[x^2 - 2x\bar{x} + (\bar{x})^2]$$

$$= E[x^2] - 2E[x\bar{x}] + E[(\bar{x})^2]$$

$$\therefore \sigma_x^2 = E[x^2] - 2\bar{x}\, E[x] + (\bar{x})^2$$

$$\therefore \sigma_x^2 = E[x^2] - 2\bar{x}\,\bar{x} + (\bar{x})^2$$

$$\therefore \sigma_x^2 = E[x^2] - (\bar{x})^2$$

$$\rightarrow \boxed{\sigma_x^2 = E[x^2] - \{E[x]\}^2}$$

Standard deviation - square root of the variance is called the standard deviation, σ_x.

- Variance is measure of the "spread" of pdf or pmf about the mean / Expected value.

↳ If pdf or pmf ~~is~~ has concentrated pdf / pmf, it will have small variance.

Similarly, if it has a widely spread pdf or pmf, it will have a large variance.

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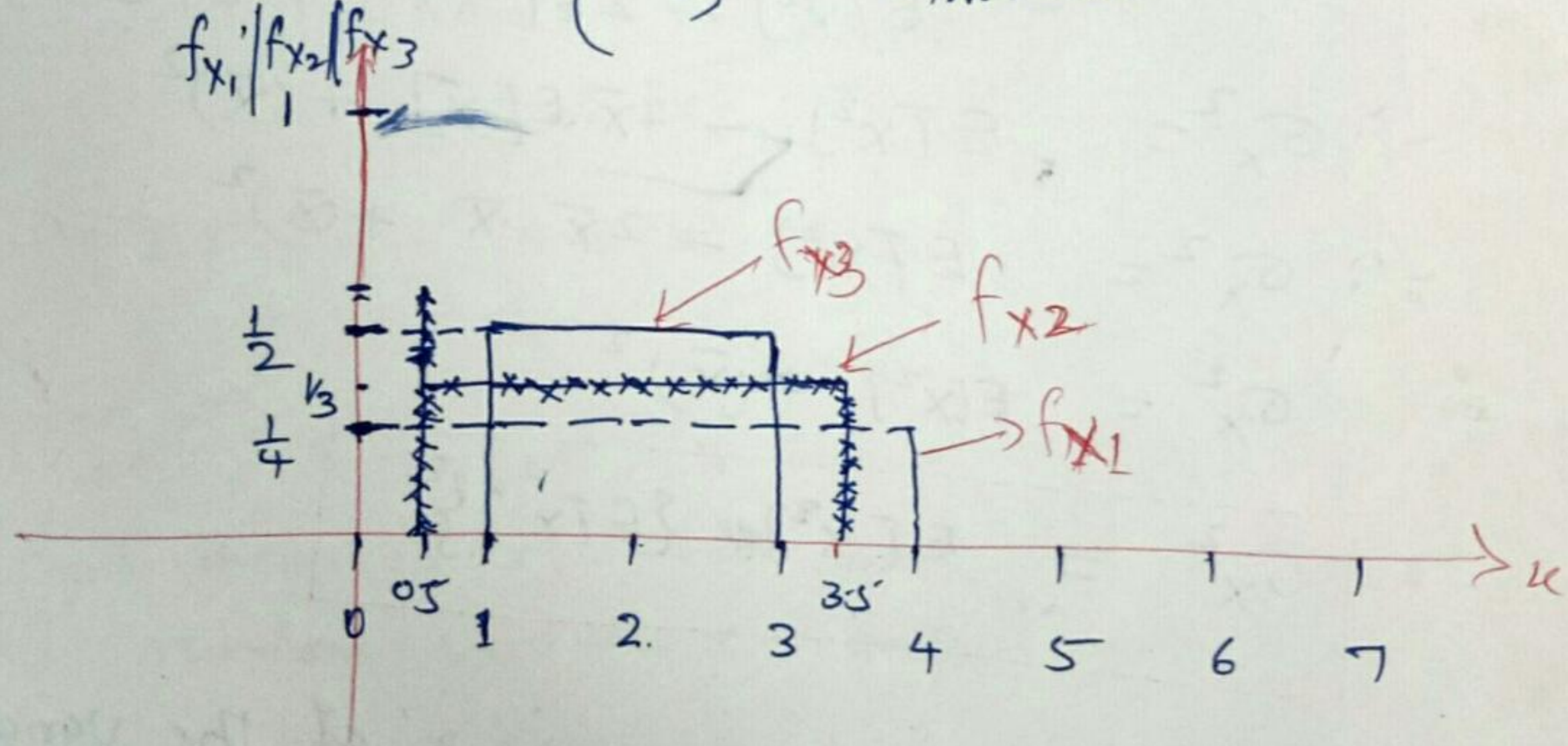
For example consider r.v. X_1, X_2 and X_3 with the following pdf:

$$f_{X_1}(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4 \\ 0; & \text{otherwise} \end{cases}$$

$$f_{X_2}(x) = \begin{cases} \frac{1}{3}, & 0.5 \le x \le 3.5 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X_3}(x) = \begin{cases} \frac{1}{2}; & 1 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$



Mean value of this class of distribution is $\frac{b+a}{2}$; b: upper limit, a lower limit

$\therefore$ $E[X_1] = E[X_2] = E[X_3] = 2$, However, their spreads ~~over~~ around mean value are different.

X_1 has largest spread largest spread about mean.

X_3 has the smallest spread about the mean.

$\hookrightarrow$ Values of X_3 tend to cluster around mean value the most.

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, While value of x_1 tend to cluster around mean value the least

In terms of variance we say that X_1 has largest variance, while X_3 has the ~~least~~ smallest variance.

Ex :- let X be a continuous random variable with the pdf:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{otherwise.} \end{cases}$$

Find the variance of X

Solution :- The expected value of r.v. X is given by

$$E[X] = \frac{b+a}{2}$$

The second moment X is given by

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = \int_a^b x^2 \frac{1}{b-a} \cdot dx$$

$$= \left[\frac{x^3}{3(b-a)} \right]_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$\therefore E[X^2] = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}}$$

$$\therefore E[X^2] = \frac{1}{3}(b^2+ab+a^2)$$

$\therefore$ To find variance we use the formula.

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2$$

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$$\sigma_x^2 = \frac{b^2+ab+a^2}{3} - \frac{b^2+a^2+2ab}{4}$$

$$\sigma_x^2 = \frac{4b^2+4ab+4a^2-3b^2-3a^2-6ab}{12}$$

$$\therefore \sigma_x^2 = \frac{b^2-2ab+a^2}{12}$$

$$\therefore \boxed{\sigma_x^2 = \frac{(b-a)^2}{12}}$$

— Now the variance of r.v. X_1, X_2 and X_3 given in example, we have:

$$\sigma_{x_1}^2 = \frac{(4-0)^2}{12} = \frac{\cancel{16}\,4}{\cancel{12}\,3} \; ; \quad \sigma_{x_1}^2 = \frac{4}{3} = 1.33.$$

$$\sigma_{x_2}^2 = \frac{(3.5-0.5)^2}{12} = \frac{(3)^2}{12} = \frac{\cancel{9}\,3}{\cancel{12}\,4} = \frac{3}{4} = 0.75$$

$$\sigma_{x_3}^2 = \frac{(3-1)^2}{12} = \frac{\cancel{4}\,1}{\cancel{12}\,3} = \frac{1}{3} = 0.33$$

Thus, while $E[X_1] = E[X_2] = E[X_3] = 2$, the variances are different with X_1 having the largest variance and X_3 having the smallest variance and hence smallest spread about the mean.

Example:— Find the variance of r.v. K with following pmf:

$$p(k) = \frac{\lambda^k}{k!} e^{-\lambda} \; ; \quad k = 0, 1, 2, 3, \ldots$$

Solution: Since we already calculated ~~$E[K] = \lambda$~~

$$E[K] = \lambda.$$

The second moment of K is given by:

$$E[K^2] = \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!} \quad \text{—— (1)}$$

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$$\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}$$

$$= \lambda e^{\lambda} \quad \text{——} \quad (B)$$

~~Using (B) in (A)~~

But $\frac{d}{d\lambda}\left[\sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}\right] = \sum_{k=1}^{\infty} \frac{d}{d\lambda}\left[\frac{\lambda^k}{(k-1)!}\right]$

$$= \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!}$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!} = \frac{d}{d\lambda}\{\lambda e^{\lambda}\} = (1+\lambda)e^{\lambda}$$

Thus Second moment of K is given by

$$E[K^2] = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}(1+\lambda)e^{\lambda}$$

$$\therefore E[K^2] = \lambda + \lambda^2.$$

$\therefore$ Variance of K.

$$\sigma_K^2 = E[K^2] - (E[K])^2$$

$$= \lambda + \lambda^2 - (\lambda)^2$$

$$= \lambda + \lambda^2 - \lambda^2$$

$$\boxed{\sigma_K^2 = \lambda}$$

$\hookrightarrow$ Expected value and variance of K are equal

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Ex:- A test engineer discovered that the CDF of the ~~life~~ lifetime of an equipment in years is given by

$$F_X(x) = \begin{cases} 0 \; ; & x < 0 \\ 1 - e^{-x/5} \; , & 0 \le x < \infty \end{cases}$$

(a) What is expected life time of the equipment?

(b) What is the variance of the lifetime of the equipment?

Solution:- From the definition of its CDF, we can see that X is a random variable that takes only nonnegative value. Thus, we proceed as follows:

(a) Expected value of equipment is given by

$$E[X] = \int_0^\infty P[X > x]\, dx = \int_0^\infty [1 - F_X(x)]\, dx$$

$$= \int_0^\infty e^{-x/5}\, dx = \left[-5e^{-x/5}\right]_0^\infty$$

$$= 0 - (-5e^{0/5})$$

$$\boxed{E[X] = 5}$$

(b) For finding variance we first get pdf from CDF

$$\therefore f_X(x) = \frac{d}{dx} F_X(x) = -\left(-\frac{1}{5}\right) e^{-x/5}$$

$$\therefore f_X(x) = \frac{1}{5} e^{-x/5} ; \quad x \ge 0$$

The second moment of X is given by

$$E[X^2] = \int_0^\infty x^2 f_X(x)\, dx = \int_0^{+\infty} x^2 \frac{1}{5} e^{-x/5}\, dx$$

$$E[X^2] = \frac{1}{5} \int_0^{+\infty} x^2 e^{-x/5}\, dx.$$

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~~take $u = x^2 \Rightarrow du = 2x\,dx$~~

~~let~~

Now do integration part

$$\int x^2 e^{-x/5} = x^2 \int e^{-x/5}\,dx - \int \left[\frac{d}{dx} x^2 \int e^{-x/5}\,dx\right] dx$$

$$= -5x^2 e^{-x/5} - \int 2x \times -5 e^{-x/5}\,dx$$

$$= -5x^2 e^{-x/5} + 10 \int x e^{-x/5}\,dx$$

$$\therefore \frac{1}{5}\int_0^{\infty} x^2 e^{-x/5}\,dx = \left[-\frac{5x^2 e^{-x/5}}{5}\right]_0^{\infty} + \frac{10}{5}\int_0^{+\infty} x e^{-x/5}\,dx$$

$$\therefore E[X^2] = \frac{1}{5}\int_0^{+\infty} x^2 e^{-x/5}\,dx = -0 - (0) + 2\int_0^{+\infty} x e^{-x/5}\,dx$$

$$\therefore E[X^2] = 2\int_0^{+\infty} x e^{-x/5}\,dx$$

Using By part integration:

$$\therefore E[X^2] = 2\left[-5x e^{-\frac{x}{5}}\right]_0^{+\infty} + 10\left[\int_0^{\infty} e^{-x/5}\,dx\right]$$

$$E[X^2] = 0 + 10\int_0^{\infty} e^{-x/5}\,dx$$

$$E[X^2] = 10\left[-5 e^{-x/5}\right]_0^{\infty}$$

$$= 10\left[0 - (-5 \times 1)\right]$$

$$= 10 \times 5$$

$$\therefore \boxed{E[X^2] = 50}$$

Finally the variance of X is given by:

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

$$= 50 - 25$$

$$\therefore \boxed{\sigma_X^2 = 25}$$

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Ex:- Find the variance of r.v. X whose pdf is given by

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Solution:- ~~The expected value of X, r.v. the~~

The expected value of X is given by $E[X] = \frac{1}{\lambda}$, The second moment of X is given by:

$$E[X^2] = \int_0^{\infty} x^2 f_X(x) = \int_0^{+\infty} x^2 \cdot \lambda e^{-\lambda x} dx$$

Let $u = x^2 \Rightarrow du = 2x\,dx$, and let $dv = \lambda e^{-\lambda x} dx$

$\Rightarrow v = -e^{-\lambda x}$. Thus, integrating by parts we obtain

$$E[X^2] = \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + 2\int_0^{+\infty} x e^{-\lambda x} dx$$

$$= 0 + 2\int_0^{+\infty} x e^{-\lambda x} dx$$

$$E[X^2] = 2\int_0^{+\infty} x e^{-\lambda x} dx$$

$$E[X^2] = \cancel{\frac{2[x e^{-\lambda x}}{\lambda}}$$

$$= 2\left[-\frac{x e^{-\lambda x}}{\lambda}\right]_0^{\infty} + \frac{2}{\lambda}\int_0^{+\infty} e^{-\lambda x} dx$$

$$E[X^2] = 0 + \frac{2}{\lambda}\int_0^{\infty} e^{-\lambda x} dx$$

$$E[X^2] = \frac{2}{\lambda} \times -\frac{1}{\lambda}\left[e^{-\lambda x}\right]_0^{+\infty}$$

$$E[X^2] = -\frac{2}{\lambda^2}[0 - 1]$$

$$\boxed{E[X^2] = \frac{2}{\lambda^2}}$$

Therefore, variance σ_X^2 is

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$

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$$\sigma_X^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{\therefore \sigma_X^2 = \frac{1}{\lambda^2}}$$

Example :— A shopping cart contains ten books whose weights are as follows: There are four books with a weight of 1.8 lbs. each, one book with a weight of 2lbs, two books with a weight of 2.5 lbs each, and three books with a weight of 3.2 lbs each.

(a) What is the mean weight of the books?

(b) What is the varience of the weights of the books

Solution :— The total number of books is 10. The fractions of books in each weight category are as follows:

— The fraction of books with weight 1.8 lbs is $4/10 = 0.4$

— The fraction of books with weight 2.0 lbs is $1/10 = 0.1$

— The fraction of books with weight 2.5 lbs is $2/10 = 0.2$

— The fraction of books with weight 3.2 lbs is $3/10 = 0.3$

Let Y be R.V. that denotes the weight of books. Since these fractions are essentially the probabilities of occurrence of these weight, we have that

$$E[Y] = (0.4 \times 1.8) + (0.1 \times 2.0) + (0.2 \times 2.5) + (0.3 \times 3.2)$$

$$\boxed{E[Y] = 2.38}$$

$$\sigma_Y^2 = \sum_{k=1}^{4} (y_k - E[Y])^2 p_Y(y_k)$$

$$= \{(1.8 - 2.38)^2 \times 0.4\} + \{(2.0 - 2.38)^2 \times 0.1\} + \{(2.5 - 2.38)^2 \times 0.2\} + \{(3.2 - 2.38)^2 \times 0.3\}$$

$$\boxed{\sigma_Y^2 = 0.3536}$$

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$\sigma_Y^2 = ?$ In order to find σ_Y^2 we first compute $E[Y^2]$ as follows:

$$E[Y^2] = (0.4 \times 1.8^2) + (0.1 \times 2.0^2) + (0.2 \times 2.5^2) + (0.3 \times 3.2^2)$$

$$\cancel{= 0.3536}$$

$$= 6.018$$

$\therefore$ Variance σ_Y^2 is given as:

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2$$

$$\sigma_Y^2 = 6.018 - (2.38)^2$$

$$= 6.018 - 5.6644$$

$$\therefore \boxed{\sigma_Y^2 = 0.3536}$$

Ex:- Find the variance of the discrete random variable X with the following pmf:

$$p_X(x) = \begin{cases} \frac{1}{3}; & x = 0 \\ \frac{2}{3}; & x = 2 \\ 0; & \text{otherwise} \end{cases}$$

Solution:-

$$E[X] = \sum_{x_k} x_k p_X(x_k)$$

$$= 0 \cdot p_X(0) + 2 \cdot p_X(2)$$

$$= 0 + 2 \times \frac{2}{3}$$

$$\boxed{E[X] = \frac{4}{3}}$$

The second order moment of X is given by

$$E[X^2] = \sum_{x_k} x_k^2 p_X(x_k)$$

$$= 0 \cdot p_X(0) + 2^2 p_X(2)$$

$$= 0 \times \frac{1}{3} + 4 \times \frac{2}{3}$$

$$= 0 + \frac{8}{3}$$

$$\therefore E[X^2] = 8/3$$

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Therefore, variance

$$\sigma_X^2 = E[X^2] - (E[X])^2$$

$$\sigma_X^2 = \left(\frac{8}{3}\right) - \left(\frac{4}{3}\right)^2$$

$$\sigma_X^2 = \frac{8}{3} - \frac{16}{9}$$

$$\boxed{\sigma_X^2 = \frac{8}{9}}$$

Example:— A student doing a summer internship in a company was asked to model the ~~life time~~ lifetime of a certain equipment that the company makes. After a series of tests the student proposed that the lifetime of the equipment can be modeled by a random variable X that has the pdf:

$$f_X(x) = \begin{cases} \dfrac{x e^{-x/10}}{100}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $f_X(x)$ is valid pdf.
(b) What is the probability that the lifetime of the equipment ~~exceed~~ exceeds 20?
(c) What is the expected value X?

Solution:— (a) For $f_X(x)$ to be a valid pdf, we must have that $\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$ and $f_X(x) \geq 0$ $\forall x$

Since $x \geq 0$; and exponential function is always positive ~~irr~~ irrespective to the value assign to x.

$$\therefore \quad \frac{x e^{-x/100}}{100} \geq 0 \quad \forall x.$$

$\therefore$ $f_X(x)$ is always positive; i.e.,

$$\therefore f_X(x) \geq 0, \quad \forall x \geq 0$$

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Now $\int_{-\infty}^{+\infty} f_X(x)\, dx = \int_0^{+\infty} \frac{x e^{-\frac{x}{10}}}{100}\, dx$

~~$u = x \Rightarrow du = dx$, and let~~

$$= \left[\frac{1}{100} x \times \frac{d}{dx} e^{-\frac{x}{10}}\right]_0^{+\infty} - \frac{1}{100}\left[\int\left\{\frac{d}{dx} x \int e^{-x/10} dx\right\} dx\right]_0^{+\infty}$$

$$= \left[-\frac{x e^{-x/10}}{100} \times 10\right]_0^{+\infty} - \frac{1}{100}\left[\int 1 \times -10\, e^{-x/10} dx\right]_0^{+\infty}$$

$$\therefore \int_{-\infty}^{+\infty} f_X(x)\, dx = \left[-\frac{x e^{-\frac{x}{10}}}{10}\right]_0^{\infty} - \left[\int \frac{e^{-x/10}}{10} dx\right]_0^{+\infty}$$

$$= 0 - \int_0^{+\infty} \frac{e^{-x/10}}{10} dx$$

$$= -\int_0^{+\infty} \frac{e^{-x/10}}{10} dx$$

$$= -\left[-\frac{\cancel{10} e^{-x/10}}{\cancel{10}}\right]_0^{+\infty}$$

$$\therefore \int_{-\infty}^{+\infty} f_X(x)\, dx = -\left[e^{-x/10}\right]_0^{+\infty}$$

$$= -[0 - 1]$$

$$= 1$$

$$\therefore \boxed{\int_{-\infty}^{+\infty} f_X(x)\, dx = 1}$$

Therefore $f_X(x)$ is valid density i.e. pdf.

(b) $P[X > 20] = \int_{20}^{+\infty} f_X(x)\, dx$

$$= \int_{20}^{+\infty} \frac{x e^{-x/10}}{100} dx$$

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$$P[X>20] = \frac{1}{100}\int_{20}^{+\infty} x e^{-x/10}\,dx$$

$$P[X>20] = \frac{1}{100}\left\{\left[-10xe^{-x/10}\right]_{20}^{\infty} + 10\int_{20}^{\infty} e^{-x/10}\,dx\right\}$$

$$= \frac{1}{100}\left\{0-(-10\times 20\times e^{-\frac{20}{10}}) + 10\left[-10\, e^{-x/10}\right]_{20}^{\infty}\right\}$$

$$P[X>20] = \frac{1}{100}\left\{200e^{-2} + 100\left(-10\times 0 + e^{-2}\right)\right\}$$

$$= \frac{1}{100}\left\{200e^{-2} + 100e^{-2}\right\}$$

$$P[X>20] = \frac{1}{100}\times 300\, e^{-2}$$

$$P[X>20] = 3e^{-2}$$

$$\boxed{P[X>20] = 0.406}$$

(e) The expected value of X is given by:

$$E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx$$

$$E[X] = \int_{-\infty}^{+\infty} x \frac{1}{100} x e^{-x/10}\,dx = \frac{1}{100}\int_{-\infty}^{+\infty} x^2 e^{-\frac{x}{10}}\,dx$$

$$= \frac{1}{100}\left[-10x^2 e^{-x/10}\right]_{0}^{+\infty} + 20\int_{0}^{+\infty}\frac{x e^{-x/10}}{100}\,dx$$

$$= 0 + 20$$

$$\boxed{E(X) = 20}$$

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Ex :- A high school student has a personal goal of scoring at least ~~1500~~ 1500 in the scholastic aptitude ~~te~~ test (SAT). He plans to keep taking the test until he acheives this goal. Time is not an issue for him. The probability that he scores 1500 or higher in any one SAT test is p. His performance in each SAT test is independent of his performance in other SAT tests. His friend who has just completed a course on probability informed him that if K is the number of times he has to take the test before getting a score of 1500 or higher, then the pmf of K is given by

$$p_K(k) = p(1-p)^{k-1}, \quad k = 1, 2 \cdots$$

(a) Prove that the function proposed by his friend is true pmf

(b) Given the above pmf, what is the probability that he takes the test more than 5 times

(c) What is expected number ~~tim~~ of times he takes the test to achieve his goal?

Solution:- (a) To prove that ~~then~~ the friend's proposed function is a true PMF, we need to show that it sums to one; that is:

$$\sum_{k=1}^{\infty} p_k(k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} = p \sum_{m=0}^{\infty} (1-p)^m$$

$$= \frac{p}{1-(1-p)} = 1$$

Thus, the proposed function is a true pmf.

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(b) The probability that he takes the test more than 5 times is:

$$P[K>5] = \sum_{k=6}^{\infty} p(1-p)^{k-1}$$

$$P[K>5] = p(1-p)^5 \sum_{k=6}^{\infty} (1-p)^{k-6}$$

Let $m = k-6$, since $(1-p)<1$, we get

$$P[K>5] = p(1-p)^5 \sum_{m=0}^{\infty} (1-p)^m$$

$$= \frac{p(1-p)^5}{1-(1-p)}$$

$$\therefore P[K>5] = (1-p)^5$$

↳ This is precisely the probability that he did not make the grade in the first 5 attempts.

(c) The expected number of ~~time~~ times he take the test is given by:

$$E[K] = \sum_{k=1}^{\infty} k\, p_k(k)$$

$$= \sum_{k=1}^{\infty} kp(1-p)^{k-1}$$

$$E[K] = p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

Let $G = \sum_{k=0}^{\infty} (1-p)^k = \frac{1}{p}$

$$\frac{dG}{dp} = \frac{d}{dp} \sum_{k=0}^{\infty} (1-p)^k = -\sum_{k=0}^{\infty} k(1-p)^{k-1}$$

$$\frac{dG}{dp} = -\sum_{k=1}^{\infty} k(1-p)^{k-1}$$

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$$E[K] = p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$E[K] = -p \frac{dG}{dp}$$

$$E[K] = -p \frac{d}{dp}\left(\frac{1}{p}\right)$$

$$\cancel{= -p \frac{d}{dp}(}$$

$$E[K] = -p\left(-\frac{1}{p^2}\right) = \frac{1}{p}$$

$$\therefore \boxed{E[K] = \frac{1}{p}}$$

↳ Thus smaller the value of p, the ~~more~~ more times one the average that he has to take the test

e.g. for $p = 1/5$, he will take test on average of 5 times, and when $p = \frac{1}{10}$, he will on the average take the test 10 times.

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Measure of Central Tendency :-

- → Mean / Expected value
- → Median
- → Mode

Median :- The idea of median value is that we are looking for a point M such that the probability of the random variable X falling below is equal to the probability of X falling above M.

↳ In this M is a middle point

↳ M may not be unique and there may be several points qualifying to be medians for a given X.

Def^n (Median) :- Median of a random variable X.

Let a M be a point such that

$$P_r\{X \le M\} \geq \frac{1}{2} \text{ i.e. } P_r\{X(\omega)=x \le M\} \geq \frac{1}{2}$$

and

$$P_r\{X \geq M\} \geq \frac{1}{2}, \text{ i.e. } P\{X(\omega)=x \geq M\} \geq \frac{1}{2}$$

All those points satisfying these two conditions are called median points.

Note :- In some cases we have a unique point M and in other cases we have several values of M.

Ex :-

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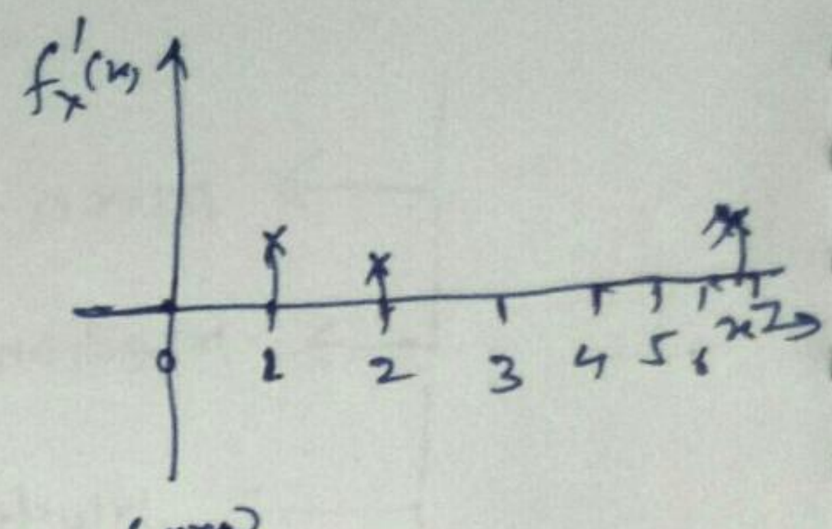
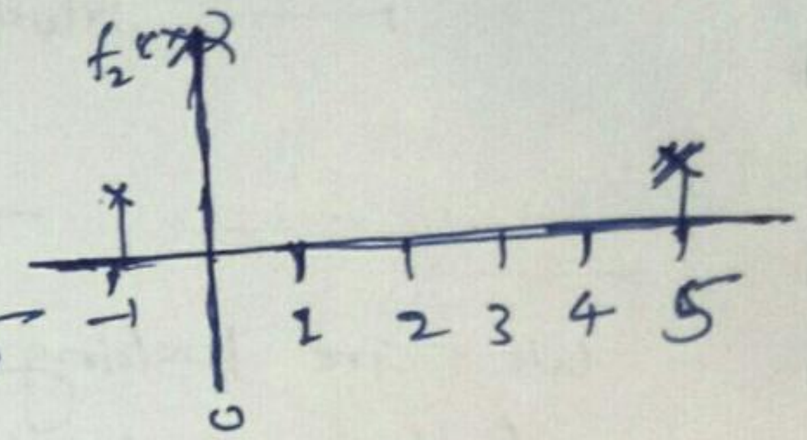
Ex :- Consider the following probability functions:

$$f_X^1(x) = \begin{cases} 0.4, & x=1, \\ 0.2, & x=2 \\ 0.4, & x=7 \\ 0, & \text{otherwise} \end{cases}$$



$$f_X^2(x) = \begin{cases} 0.5, & x=-1 \\ 0.5, & x=5 \end{cases}$$



Compute Median in ~~each~~ each case.

Solutn :- In $f_X^1(x)$ the point $x=2$ satisfy the both condition for Median

$$P_r\{X \le 2\} = 0.6 > 0.5$$
$$= p_X(1) + p_X(2) = 0.4 + 0.2 = 0.6$$

$$P_r\{X \ge 2\} = p_X(2) + p_X(7)$$
$$= 0.2 + 0.4$$
$$= 0.6 > 0.5$$

Thus $X=2$ is the unique median for ~~this X~~ this X.

- In $f_X^2(x)$ any point from -1 to 5 will qualify or $-1 \le M \le 5$.
- If unique point is preferred one can take the middle value of this ~~in~~ interval, namely, $X=2$ as the median.

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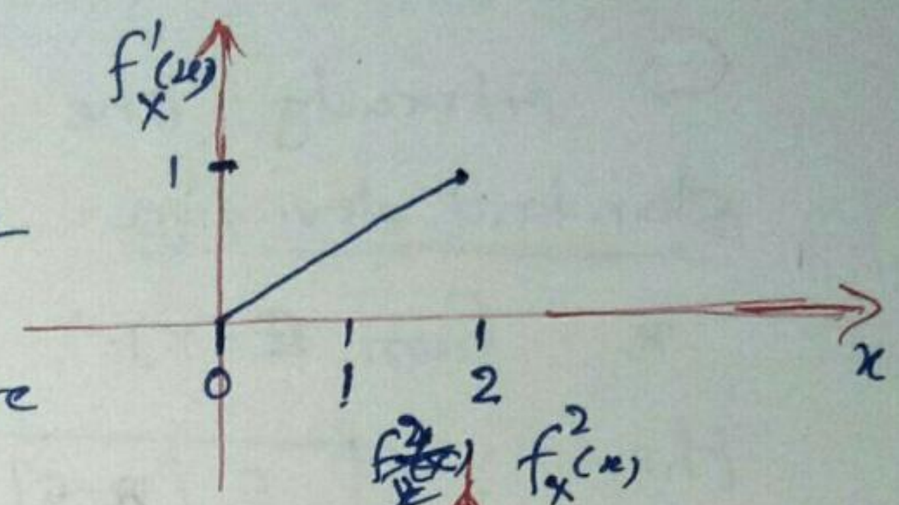
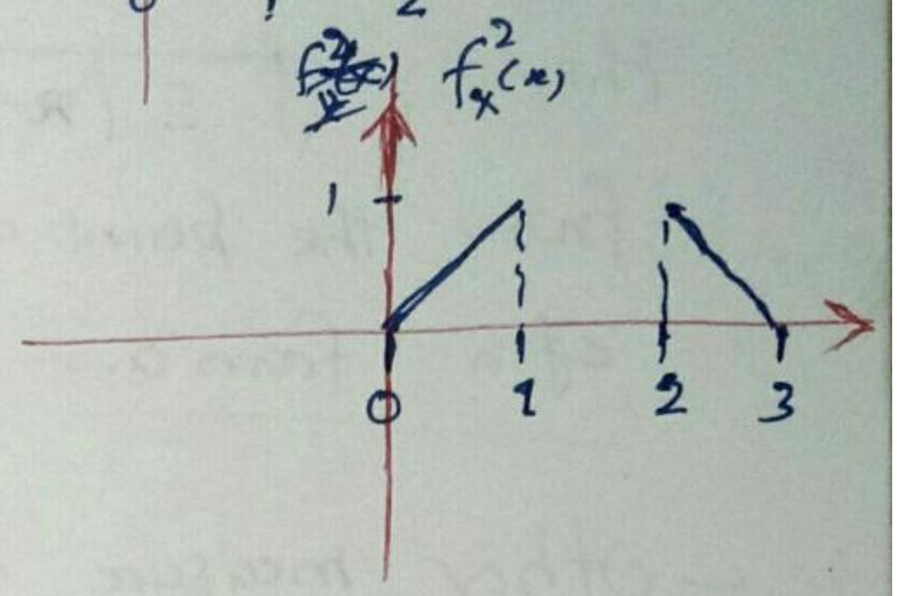
Ex:— compute the median point or points for the following densities

$$f_X^1(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le 2 \\ 0, & \text{elsewhere} \end{cases}$$



$$f_X^2(x) = \begin{cases} x, & 0 \le x \le 1 \\ 3-x, & 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$



Solution :- Case 1 :- $f_X^1(x)$:

$$Pr\{X \le M\} = \int_0^M f_X^1(x)\,dx$$

$$= \frac{1}{2}\int_0^M x\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^M$$

$$\Rightarrow Pr\{X \le M\} = \frac{M^2}{4}$$

We can equate this to $\frac{1}{2}$ because in this case

$$Pr\{X \le M\} = \frac{1}{2} = Pr\{X \ge M\}$$

Equating $\frac{M^2}{4}$ to $\frac{1}{2}$ we have $M = \sqrt{2}$

Case 2:— In this case $f_X^2(x)$:

Probabilities in two pieces where we do have non-zero probability

$$\int_0^1 x\,dx = \frac{1}{2}$$

and

$$\int_2^3 (3-x)\,dx = \frac{1}{2}$$

This means that at any point from 1 to 2, $1 \le M \le 2$ will qualify to a median point

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~~will qualify~~

Measure of Scatter or Dispersion

Already one measure of scatter, which is the standard deviation. This measure of the spread of x from $E(X)$. If an arbitrary point 'a' is taken then: $\sqrt{E[x-a]^2}$ is a measure of scatter of x from the point a, which is also called the mean deviation of x from a.

- Other measure of scatter of x from the point a, which is also called the mean deviation of x from a. Other measures of scatter from 'a' are the following:

$$M_r(a) = E[|x-a|^r]^{\frac{1}{r}}; \quad r=1,2,3,\ldots$$

$$M_1(a) = E[|x-a|] \equiv \text{mean absolute deviation from } a.$$

- All these qualify to ~~the~~ be "mathematical distances" of x from the point 'a' and ~~in this~~ these are measures of scatter of x from the point 'a'. ~~Out of these, the mean deviation~~

Out of these, the mean deviation from a, and mean absolute deviation from a are very important because they are very often used in statistical ~~because~~ decision making.

Result:- The mean diviation from 'a' is least when $a = E[X]$.

Proof:- Mean diviation $MD(a) = \sqrt{E(x-a)^2}$

since MD is positive function

Minimizing MD is equivalent to minimizing $MD^2(a)$

Let

$$J(a) = MD^2(a) = E(x-a)^2$$
$$= E[x^2] + a^2 - 2aE[x]$$

Now for minima / maxima

$$\frac{\partial J}{\partial a} = 0 \quad 2a - 2E[x] = 0$$

$$\Rightarrow \quad a = E[x]$$

$$\frac{\partial^2 J}{\partial a^2} = 2 > 0$$

Another way:

$$\cancel{E[x-a]^2 = E[x^2+a^2-2ax] = E[x^2] + E[a^2] - 2aE[x]}$$

$$\therefore \cancel{E[x-a]^2 = E[x^2] + a^2 - 2aE[x]}$$

$$E[x-a]^2 = E[x - E[x] + E[x] - a]^2$$
$$= E[[x-E[x]]^2 - 2E[x-E[x]][E[x]-a] + [E[x]-a]^2$$
$$= \sigma_x^2 - 2(E[x]-E[x])[E[x]-a] + [E[x]-a]^2$$
$$= \sigma_x^2 + [E[x]-a]^2$$

$$\therefore \boxed{E[x-a]^2 = \sigma_x^2 + [E[x]-a]^2}$$

$E[x]-a \rightarrow$ constant term

$E[x-E[x]]$
$= E[x] - E[E[x]]$
$= E[x] - E[x]$
$= 0$

R.H.S. both term is positive and the term $[E[x]-a]^2$ is only depend on a.

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Thus minimum is attain when the term containing a is zero, that is, $[E(x)-a]^2 = 0$

$$\Rightarrow \boxed{a = E[x]}$$

This result is very important for regression analysis and model building.

<u>Result</u>:- The mean absolute deviation is least when the deviations are taken from the ~~med~~ median or $E|x-a|$ is least when $a =$ median of X.

<u>Proof</u>:- left for ~~exer~~ exercise.

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Result :- Let $x = c$, where 'c' is constant then

$$\sigma_x^2 = 0$$

Proof :-

Now from definition

$$\sigma_x^2 = E[(x - E(x))^2]$$
$$= E[(c - E[c])^2]$$
$$= E[(c - c)^2]$$
$$= E[0]$$
$$\sigma_x^2 = 0$$

Result :- Let X and Y are R.V. and $Y = X + c$ where 'c' is content then

$$\sigma_Y^2 = \sigma_X^2$$

Proof :-

$$Y - E[Y] = Y - E[X + c]$$
$$= Y - E[X] - E[c]$$
$$= Y - E[X] - c$$
$$= X + \cancel{c} - E[X] - \cancel{c}$$
$$\therefore Y - E[Y] = X - E[X]$$

$$\therefore \sigma_Y^2 = E[(Y - E[Y])^2]$$
$$= E[(X - E[X])^2]$$
$$\boxed{\sigma_Y^2 = \sigma_X^2}$$

Result :- Let X and Y are R.V. and $Y = aX + b$, where a and b are constant, then $\sigma_Y^2 = a^2 \sigma_X^2$

Proof :-

$$\cancel{\sigma_Y^2 = E[(Y - E[Y])^2] = E[Y^2 + (E[Y])^2 - 2Y \cdot E[Y]]}$$

$$\sigma_Y^2 = \cancel{E[Y^2]} = E[(aX + b - E[aX + b])^2]$$

$$\cancel{\sigma_Y^2 = E[(aX + b - aE[X])^2]}$$
$$\cancel{= E[(a(X - E[X])) + b)^2] = E[a^2(X - E[X])^2 + b^2 + 2ab(X - E[X])]}$$

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$$\cancel{\sigma_Y^2 = E[a^2(X - E[X])^2 + b^2 + 2ab(X - E[X])]}$$

$$\cancel{\sigma_Y^2 = a^2 E[(X - E[X])^2] + E[b^2]}$$

$$\cancel{+ 2ab\, E[(X - E[X])]}$$

$$\sigma_Y^2 = a^2 \sigma_X^2 + b^2$$

$$\sigma_Y^2 = E[(Y - E[Y])^2]$$

$$= E[(aX + b - E[aX + b])^2]$$

$$\sigma_Y^2 = E[(aX + \cancel{b} - \cancel{a}E[X] - \cancel{b})^2]$$

$$\sigma_Y^2 = E[(aX - aE[X])^2]$$

$$\sigma_Y^2 = E[a^2(X - E[X])^2]$$

$$\sigma_Y^2 = a^2 E[(X - E[X])^2]$$

$$\therefore \boxed{\sigma_Y^2 = a^2 \sigma_X^2}$$

here 'a' is scaling factor.

~~a~~

and standard deviation

$$\boxed{\sigma_Y = |a| \sigma_X}$$

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Result :- (i) $\sigma_X^2 = E[X - E[X]]^2$ does not imply

$$\sigma_X = E[X - E[X]]$$

(ii) $\sigma_X^2 = E[X - E[X]]^2$ does not imply

$$\sigma_X = E[|X - E[X]|]$$

(iii) $\sigma_X^2 = 0$ iff X is degenerate random variable

(iv) $\sigma_X^2 > 0$, $\sigma_X > 0$ always except for the degenerate case where $\sigma_X^2 = 0$. That is, the variance, thereby the standard deviation, of r.v. is non-negative.

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Revisted Moment about origin and Central moment

- **Moment about origin:-** α-th moment or the α-th moment about origin of r.v. X is defined as:

$$E[X^\alpha] = \int_{-\infty}^{+\infty} x^\alpha f_X(x)\, dx, \text{ when } X \text{ is continuous}$$

$$= \sum_{-\infty < x < \infty} x^\alpha f_X(x), \text{ when } X \text{ is discrete}$$

and $E[X^\alpha]$ exists. Here α, could be any complex number. When α is positive integer, $\alpha = r = 1, 2, \ldots$ then α-th moment is denoted by μ_r' or

$$\mu_r' = E[X^r], \quad r = 1, 2, \ldots$$

Notation :- μ_r : rth ~~moment~~ central moment for $r = 1, 2, \ldots$

- **Central Moments:-** The α-th central moment is defined as $E[X - E[X]]^\alpha$ for any complex number α, whenever it exists, and when $\alpha = r = 1, 2, \ldots$ then the rth central moment is defined as:

$$\mu_r = E[X - E[X]]^r; \quad r = 1, 2, \ldots$$

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Defn :- Factorial moment :- The r-th factorial moment is defined by

$$\mu_{[r]} = E[X(X-1)(X-2)\cdots(X-r+1)]$$

whenever it exists.

Ex :- Factorial moment are usually easier to evaluate compared to moments about origin or central moment when factorial is involved in the denominator in some probability functions for discrete variables.

Ex :- Compute (i) $E[X^3]$; (ii) $E[X-E[X]]^4$, (iii) $E[|X|]$ (iv) $E[X^3-3X+4]$. for the following probability function

$$f_X(x) = \begin{cases} 0.5, & X=-2 \\ 0.5, & X=2 \\ 0, & \text{otherwise} \end{cases}$$

Solution :- (i) $E[X^3] = \sum_x x^3 f_X(x) = \sum$

$$= (-2)^3 f_X(-2) + (2)^3 f_X(2) + 0$$

$$= -8 \times 0.5 + 8 \times (0.5) + 0$$

$$= 0$$

It is easy to see that $E[X^r] = 0$ for $r = 1, 3, 5, \ldots$ or all odd moments are zero., hence the probability function is symmetric about $X=0$.

(ii) Since $E[X]=0$, here we have,

$$E[X-E[X]]^4 = E[X^4] = \sum_x x^4 f_X(x)$$

$$= (-2)^4(0.5) + (2)^4(0.5) + 0$$

$$= 16 \times 0.5 + 16 \times 0.5$$

$$= 8.0 + 8.0$$

$$= 16$$

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(iii)

$$E(|X|) = |(-2)| f_X(-2) + |2| \cdot f_X(2) + 0$$

$$= 2 \times 0.5 + 2 \times 0.5$$

$$E(|X|) = 2.0$$

(iv)

$$E[x^2 - 3x + 4] = E[x^2] - 3E[x] + E[4]$$

$$= E[x^2] - 3E[x] + 4$$

$$E[x] = 0$$

$$\therefore E[x^2 - 3x + 4] = E[x^2] + 4$$

$$E[x^2] = (-2)^2(0.5) + (2)^2(0.5) + 0$$

$$= 4 \times 0.5 + 4 \times 0.5$$

$$= 2.0 + 2.0$$

$$= 4.0$$

$$\therefore E[x^2] = 4.0$$

$$\therefore E[x^2 - 3x + 4] = 4.0 + 4.0$$

$$= 8.0.$$

Ex :- Answer the question in previous example for the following density

$$f_X(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2-x, & 1 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

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Ex:- Find the factorial moment of order two for the discrete random variable K, for which pmf is given as

$$p_K(k) = \begin{cases} \frac{\lambda^k}{k!} e^{-\lambda}, & k = 0, 1, 2, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

Solun:- The second factorial moment

$$M_{[2]} = E[K(K-1)]$$

$$= \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k}{k!} e^{-\lambda}$$

$$= \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} e^{-\lambda}$$

$$= \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!}$$

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$$E[K(K-1)] = \lambda^2 e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right]$$

$$= \lambda^2 e^{-\lambda} e^{\lambda}$$

$$\boxed{\therefore E[K(K-1)] = \lambda^2}$$

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Moment Generating function :—
(MGF)

MGF generates the moment about origin.

$M_X(t)$: Moment generating function of R.V. X.

Definition :- The moment generating function of r.v. X. ~~or the~~ having probability density / probability function $f_X(x)$ is given by

$$M_X(t) = \begin{cases} \int_{-\infty}^{+\infty} e^{tx} f_X(x)\, dx, & \text{when } X \text{ is cont.} \\ \sum_{-\infty \le x \le +\infty} e^{tx} f_X(x); & \text{when } X \text{ is discrete.} \end{cases}$$

whenever sum and integral exist

Why it is called moment generating f^n ?

expand $e^{tx} = \left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots + \frac{(tx)^r}{r!} + \cdots\right.$

$$e^{tx} = 1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots + \frac{t^rx^r}{r!} + \cdots$$

Therefore

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\, dx$$

$$M_X(t) = \int_{-\infty}^{+\infty} \left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots\right] f_X(x)\, dx$$

$$\therefore M_X(t) = \int_{-\infty}^{+\infty} f_X(x)\, dx + \frac{t}{1!}\int_{-\infty}^{+\infty} x f_X(x)\, dx + \frac{t^2}{2!}\int_{-\infty}^{+\infty} x^2 f_X(x)\, dx + \frac{t^3}{3!}\int_{-\infty}^{+\infty} x^3 f_X(x)\, dx + \cdots + \cdots$$

$$M_X(t) = 1 + \frac{t}{1!} E[X] + \frac{t^2}{2!} E[X^2] + \frac{t^3}{3!} E[X^3] + \cdots + \frac{t^r}{r!} E[X^r] + \cdots$$

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If term by term integration is possible

$$\therefore \quad M_X(t) = 1 + \frac{t}{1!}E[X] + \frac{t^2}{2!}E[X^2] + \cdots + \frac{t^r}{r!}E[X^r] + \cdots$$

Now r^{th} central moment

$$\cancel{\mu_r' = \frac{d^r}{dt^r}}$$

$$\mu_r' = E[X^r] = \frac{d^r M_X(t)}{dt^r}\Big|_{t=0}$$

$$= \text{coefficient of } \frac{t^r}{r!} \text{ in the series}$$

<u>Note</u> :- $M_X(t)$ may not exist for many of the probability / density functions

<u>Ex</u> :- Check whether the moment generating function $M_X(t)$ is ~~existt~~ exist for r.v. X, which have following density:

$$f_X(x) = \begin{cases} \frac{1}{x^2}, & 1 \le x < \infty \\ 0, & \text{otherwise} \end{cases}$$

<u>Solution</u> :-

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)\, dx$$

$$= \int_{-\infty}^{1} e^{tx} \times 0\, dx + \int_{1}^{+\infty} e^{tx} \frac{1}{x^2}\, dx$$

$$= 0 + \int_{1}^{\infty} \frac{1}{x^2} e^{tx}\, dx$$

$$\therefore \quad M_X(t) = \int_{1}^{\infty} \frac{1}{x^2} e^{tx}\, dx \quad \text{—— (1)}$$

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$\cancel{\int \frac{1}{x^2} e^{tx}}$

$$M_X(t) = \left[-\frac{1}{x}e^{tx}\right]_1^{\infty} + t\int_1^{\infty} \frac{1}{x} e^{tx}\,dx$$

$$\therefore M_X(t) = \left[-\frac{1}{x}e^{tx}\right]_1^{\infty} + t\left[\ln x \cdot e^{tx}\right]_1^{\infty} - t^2\int_1^{\infty} \ln x\, e^{tx}\,dx$$

$$\therefore M_X(t) = \underbrace{\left[-\frac{1}{x}e^{tx}\right]_1^{\infty}}_{I} + t\underbrace{\left[\ln x \cdot e^{tx}\right]_1^{\infty}}_{II} - t^2\underbrace{\int_1^{\infty} \ln x\, e^{tx}\,dx}_{III}.$$

I : I^{st} term $\to \infty$ for $t>0$. Thus integral does not exist. Thus $M_X(t)$ is not exist for $t>0$

$\hookrightarrow$ There ~~are~~ is another generating function which will exist always when $f_X(x)$ is density of a r.v. X. This is known as the characteristic function of the r.v. X.

Notation :- $\phi(t)$ = the characteristic function of X or $f_X(x)$

Definition :- Characteristic function. The characteristic function $\phi(t)$ is defined as:

$$E[e^{itx}] = \phi_X(t) = \int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx; \quad i=\sqrt{-1} \quad \text{when } x \text{ is continuous}$$

$$= \sum_{-\infty<x<+\infty} e^{itx} f_X(x); \quad \text{when } x \text{ is discrete}$$

Since $|e^{itx}| = |\cos tx + i\sin tx| = \sqrt{\cos^2 tx + \sin^2 tx} = 1$

$\therefore$ we have

$$|\phi_X(t)| = \left|\int_{-\infty}^{+\infty} e^{itx} f_X(x)\,dx\right| \le \int_{-\infty}^{+\infty} |e^{itx}|\, f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty} f_X(x)\,dx$$

$= 1$

Thus integral is always convergent

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~~Thus integral~~ : ~~Moments and Mellin Transform:~~

- $\phi_X(-it) = \int_{-\infty}^{+\infty} e^{i(-it)x} f_X(x)\,dx$

$$= \int_{-\infty}^{+\infty} e^{-i^2 tx} f_X(x)\,dx$$

$$= \int_{-\infty}^{+\infty} e^{tx} f_X(x)\,dx$$

$\therefore$ $\boxed{\phi_X(t) = M_X(t)}$

- $\phi_X(0) = 1$
- $|\phi_X(t)| \leq 1$
- kth order moment

$$E[X^k] = i^{-k}\,\varphi_X^{(k)}(0)$$

$$\Rightarrow \quad \varphi_k^{(k)}(0) = i^k E[X^k]$$

- $\boxed{Y = aX+b \;:\; \phi_Y(t) = e^{ib}\,\phi_X(at)}$

~~Moments Cont'~~

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Moments and Mellin Transforms :-

We have defined an arbitrary moments. One such moment is called the Mellin transform of a function. Consider a function $f_X(x)$ defined over $0 \leq X < \infty$ and consider an integral of the following type:

Notation 1.10 :- $M_f(s)$: Mellin transform of the function $f_X(x)$, with the Mellin parameter s

Definition :- Mellin transform :- The mellin transform of a function $f_X(x)$ is given by

$$M_f(s) = \int_0^{+\infty} x^{s-1} f_X(x)\, dx$$

provided the integral exists, where s is complex variable.

- When $f_X(x)$ is density function then it is nothing but the $(s-1)$th moment of x i.e $E[x^{s-1}]$
- Mellen transform is defined for $f_X(x)$ where $f_X(x)$ need not be a density.
- $E[x^{s-1}]$ is an arbitrary moment where s is a complex.
- for given $M_f(s)$ we can get $f_X(x)$ by evaluating integral

$$\boxed{f_X(x) = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} M_f(s)\, x^{-s}\, ds}$$

where ~~$i = \sqrt{-1}$~~ $i = \sqrt{-1}$. This integral is contour integral or an integral in the complex plane.

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Definition:- Fourier Transform:- Fourier transform of a function $f_X(x)$ with parameter 't' is defined as:

$$F_f(t) = \int_{-\infty}^{+\infty} e^{-itx} f_X(x)\, dx; \quad i=\sqrt{-1}$$

- The characteristic function $\phi(t)$ and the Fourier transform $F_f(t)$ when $f_X(x)$ is density function ~~$\phi(t)$~~ will generate the moments.

Notation $L_f(t)$: Laplace transform of the f^n $f_X(x)$ with the Laplace parameter t.

Definition:- Laplace transforms:- The laplace transform of a function $f_X(x)$ is defined as the following ~~int~~ integral:

$$L_f(t) = \int_0^{+\infty} e^{-tx} f_X(x)\, dx$$

whenever the integral is convergent, where 't' is called laplace parameter.

- ~~Thi~~

$$E[x^r] = (-1)^r \frac{d}{dt^r} L_f(t)\Big|_{t=0}, \text{ when } L_f(t) \text{ is differentiable}$$

$$= \text{coefficient of } \frac{(-t)^r}{r!} \text{ when } L_f(t) \text{ is expansible}$$

- This is use for r.v. having positive value.

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Conditional Expectation :-

- Some times we interested to compute the statistics of a subset of a population that shares the same property.

  eg – mean grade of those student who ~~passed~~ have passed an exam.

- Average age of the professor who have doctoral degrees

- The conditional expectation of a r.v. X given that an event A has occurred is given by

$$E[X|A] = \begin{cases} \sum_k x_k \, p_{X|A}(x_k|A) ; & X \text{ discrete} \\ \int_{-\infty}^{+\infty} x \, f_{X|A}(x|A) \, dx ; & X \text{ continuous} \end{cases}$$

~~where the conditional expectation of X given that an event A has occurred is given by:~~

$$\cancel{E[X|A] = \{}$$

where the conditional PMF $p_{X|A}(x|A)$ and the conditional pdf $f_{X|A}(x|A)$ are defined as follows

$$p_{X|A}(x|A) = \frac{p_X(x)}{P(A)} , \quad P[A] > 0$$

$$f_{X|A}(x|A) = \frac{f_X(x)}{P[A]} ; \quad P[A] > 0$$

where $x \in A$, and $P[A]$ is the probability that event A occurs.

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Ex:- Let A be the event: $A = \{X \le a\}$, $-\infty < a < +\infty$.

What is $E[X|A] = ?$

Solu^n:-

Cond^n pdf

$$f_{X|A}(x|A) = f_{X|A}(x|X \le a) = \begin{cases} \dfrac{f_X(x)}{P[X \le a]} = \dfrac{f_X(x)}{F_X(a)}, & x \le a \\ 0, & x > a \end{cases}$$

Thus

$$E[X|A] = \frac{\int_{-\infty}^{+\infty} x f_X(x)\,dx}{F_X(a)}$$

Ex:- Let assume that the random variable X has the PDF

$$f_X(x) = \begin{cases} \frac{1}{20}; & 40 \le x \le 60 \\ 0, & \text{otherwise} \end{cases}$$

and let a event $A = \{X \le a\}$, $-\infty < a < +\infty$

Find $E[X|A]$ when $a = 55$.

Solu^n:-

$$F_X(55) = \int_{40}^{55} f_X(x)\,dx = \left[\frac{x}{20}\right]_{40}^{55}$$

$$= \frac{55-40}{20}$$

$$= \frac{15}{20} = \frac{3}{4}$$

$$\therefore \boxed{F_X(55) = \frac{3}{4}}$$

Thus, $E[X|X \le 55]$ is given by:

$$E[X|X \le 55] = \frac{\int_{40}^{55} x f_X(x)\,dx}{F_X(55)}$$

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$$E[X \mid X \le 55] = \frac{\int_{40}^{55} \left(\frac{x}{20}\right) dx}{F_X(55)}$$

$$E[X \mid X \le 55] = \frac{4}{3}\left[\frac{x^2}{40}\right]_{40}^{55}$$

$$= \frac{4}{3}\left[\frac{3025-1600}{40}\right]$$

$$\therefore E[X \mid X \le 55] = \frac{4}{3} \times \frac{1425}{40}$$

$$\boxed{E[X \mid X \le 55] = 47.5}$$

- Expected value of r.v. that is uniformally distributed betwee a and b is $\frac{b+a}{2}$.

- In this example, given that $X \le 55$, the expected value is $\frac{55+40}{2} = 47.5$.

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The Markov Inequality :-

The markov inequality applies to random variables that take only non-negative values. If X is a r.v. that takes only nonnegative values, then for any value $a>0$;

$$P[X \geq a] \leq \frac{E[X]}{a}$$

Proof :-

$$E[X] = \int_0^{+\infty} x f_X(x)\, dx$$

$$= \int_0^{a} x f_X(x)\, dx + \int_a^{+\infty} x f_X(x)\, dx$$

$$\geq \int_a^{\infty} x f_X(x)\, dx$$

$$\therefore E[X] \geq \int_a^{\infty} x f_X(x)\, dx$$

$$\geq \int_a^{\infty} a f_X(x)\, dx = a \int_a^{\infty} f_X(x)\, dx$$

$$= a \cdot P[X \geq a]$$

$$\therefore \boxed{P[X \geq a] \leq \frac{E[X]}{a}}$$

Ex :- X is r.v. that takes value between 0 and 6. If $E[X] = 3.5$. Use the ~~mar~~ markov ~~Inequality~~ inequality to obtain an upper bound for $P[X \geq 5]$

Solu^n :- $P[X \geq 5] \leq \frac{3.5}{5} = 0.7$

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$P[x \geq 2] \leq \frac{3.5}{2} = 1.75$

↳ no usefull information since probability can not exceed 1.

Chebyshev Inequality :-

Let X is r.v. with finite mean $E[X] = \mu_x$ and variance σ_x^2. Then

$$\cancel{P[|X - E[X]| \geq}$$

$$\boxed{P[|X - E[X]| \geq a] \leq \frac{\sigma_x^2}{a^2}; \quad a > 0}$$

This is loose bound that can be obtained as follows:

$$\sigma_x^2 = \int_{-\infty}^{+\infty} (x - \mu_x)^2 f_x(x)\, dx$$

$$\cancel{\sigma_x^2 = \int_{-\infty}^{-(\mu_x - a)} (x - \mu_x)^2 f_x(x)\, dx + \int_{\mu_x - a}^{\infty} (x - \mu_x)^2 f_x(x)\, dx}$$

$$\cancel{= \int_{|x - \mu_x| \geq a} (x - \mu_x)^2 f_x(x)\, dx \geq \int_{|x - \mu_x| > a} a^2 f_x(x)\, dx}$$

$$\sigma_x^2 = \int_{-\infty}^{-(\mu_x - a)} + \int_{-(\mu_x - a)}^{\mu_x - a} + \int_{\mu_x - a}^{\infty}$$

omit the middle integral

$$\sigma_x^2 \geq \int_{-\infty}^{-(\mu_x - a)} + \int_{\mu_x - a}^{\infty} = \int_{|x - \mu_x| \geq a} (x - \mu_x)^2 f_x(x)\, dx$$

$\cancel{|\mu_x - a| \geq}$

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$$\geq \int_{|x-\mu_x| \geq a} a^2 f_x(x)\, dx$$

$$= a^2 \int_{|x-\mu_x| \geq a} f_x(x)\, dx$$

$$= a^2 P[|X-\mu_x| \geq a]$$

$$\therefore \sigma_x^2 \geq a^2 P[|X-\mu_x| \geq a]$$

$$\Rightarrow \boxed{P[|X-E[X]| \geq a] \leq \frac{\sigma_x^2}{a^2}, \quad a>0}$$

Ex:- A r.v. X has a mean 4 and variance 2. Use the chebyshev inequality to obtain an upper bound for $P[|X-4| \geq 3]$

Solution:- From the chebyshev inequality

$$P[|X-4| \geq 3] \leq \frac{\sigma_x^2}{3^2} = \frac{2}{9}$$

— chebyshev inequality is only an approximation that is some times uninformative, as in Markov inequality. For example, suppose we are required to use the data as in this example to obtain $P[|X-4| \geq 1] \leq \frac{\sigma_x^2}{1^2} = \frac{2}{1} = 2$

which provides no ~~usefull in~~ useful information

Note :- Chebyshev inequality ~~provides~~ tends to be more powerful than the Markov inequality, which means that it ~~prod~~ provides a more accurate bound than the Markov inequality, because in addition ~~to~~ the mean of random variable, it also uses information on the variance of the random variable.

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Special Probability Distributions

Commonly used probability distribution of a r.v. X encountered in different fields of engineering and science

(Basically depends on the experiments and its outcomes)

In this we describe the different distribution of r.v., with including their expected value ~~and~~, variance and moments.

Random Variables

- Discrete
  - Bernoulli
    - Geometric
    - Binomial
    - Negative Binomial
  - HyperGeometric
  - Poisson
- Continuous
  - Uniform
  - Normal
  - Exponential
    - Erlang

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Discrete Distributions :-

Bernoulli Trial :- Bernoulli trial is a random experiment that results two outcome. (success/failure).

e.g. – tossing a coin which results in heads or tail

– Writing an examination and if the final result is to be recorded as either a pass (P) or a failure (F). Then only one of the two possibilities occur.

~~Thus attempting~~

Then attempting each examination is a Bernoulli trial.

But if the final result is to be recorded as one of the grades A, B, C, D then there are four possibility in each outcome. Then this is not Bernoulli trial.

– A dice is rolled once and if we are looking for either an odd number O or an even number E, then there are only two possibilities. It is Bernoullitrial. And if we are looking only a number, then there are ~~only two~~ six possibilities, i.e., one of the numbers 1, 2, 3, 4, 5, 6 can appear or there are 6 possible ~~outcomes in~~ events in outcome. It is a multinomial trial. It is not Bernoulli trial.

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→ In Bernoulli trial let the possible events in ~~each~~ in each ~~outcome~~ trial be denoted by A and B. Then

$$A \cap B = \phi \text{ and } \Omega = A \cup B = \text{the sure event}$$

Let occurrence of A is called ~~"success~~ "a success", and the occurrence of B "a failure". Let

$$P[\text{success}] = P[A] = p, \quad 0 \le p \le$$

$$P[\text{failure}] = P[B] = \cancel{1-P(A^c)}\ P[A^c] = 1 - P[A] = 1-p = q$$

$$P[\text{success}] = P[A] = p, \quad 0 \le p \le 1$$

$$P[\text{failure}] = P[B] = 1-p = q.$$

→ We can associate a r.v. X with Bernoulli trial as follows

$$X(\omega) = \begin{cases} 1; & \text{if } \omega \text{ is success i.e } \omega = A \\ 0, & \text{if } \omega \text{ is failure i.e. } \omega = B. \end{cases}$$

This random variable is called ~~Ber~~ Bernoulli r.v.

Bernoulli Distribution:— Let X be Bernoulli r.v.

i.e.

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \equiv \text{success} \\ 0, & \text{if } \omega \text{ is failure} \end{cases}$$

Let probability of success is p and failure is q, then the probability mass function of this r.v. is

$$P_X(x) = \begin{cases} p^x q^{1-x}; & x = 0, 1 \\ 0, & \text{elsewhere} \end{cases}$$

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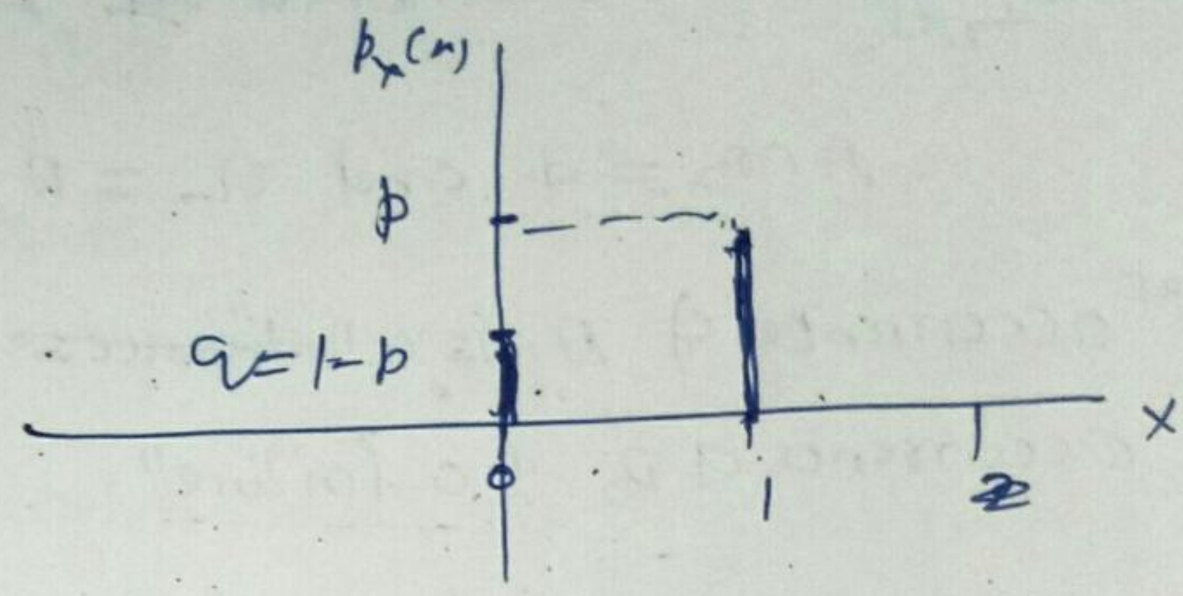
Here we have only one parameter p as since $q = 1-p$



The CDF is given by

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ 1-p, & 0 \leq x < 1 \\ 1, & x \geq 1 \end{cases}$$

Expected value of Bernoulli Random Variable :-

$$E[X] = \sum_x x\, p_X(x) = 0 \cdot p_X(0) + 1 \cdot p_X(1)$$
$$= 0\, p^0 q^1 + 1 \cdot p^1 q^0$$
$$= 0 + p.$$

$$\therefore \boxed{E[X] = p}$$

Variance :- Second moment.

$$E[X^2] = \sum_x x^2 p_X(x)$$
$$= \sum_x x^2 p^x q^{1-x}$$
$$= 0^2 p^0 q^1 + 1^2 p^1 q^0$$
$$\boxed{E[X^2] = p}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2$$
$$= p - p^2$$
$$= p(1-p)$$
$$\boxed{\sigma_X^2 = pq(1-p) = pq}$$

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Moment Generating Function :—

$$M_X(t) = \sum_x e^{tx} p_X(x)$$

$$= e^{t \cdot 0} p_X(0) + e^{t \cdot 1} p_X(1)$$

$$= 1 \cdot p^0 q^{1-0} + e^t p^1 q^{1-1}$$

$$= q + e^t p q^0$$

$$\boxed{M_X(t) = q + pe^t}$$

Assignment :- Find first moment, second moment and third moment of Bernoulli r.v. with the help of MGF.

Ex :- A gambler gets Rs 5 if the number 1 or 3 or 6 comes when balanced die is rolled once and he loses Rs 5 if the number 2 or 4 or 5 comes. How much money can he expect to win in one trial of rolling this die once?

Solution :- This is the expected value of a Bernoulli random variable with $p = \frac{1}{2}$, since the die is balanced.

$\cancel{E[X]}$

$$\cancel{X(x) = \begin{cases} 5, & x = 1, 3, 6 \\ -5, & x = 2, 4, 5 \end{cases}}$$

$$\cancel{X(x) = \begin{cases} 1, & x = \\ 0, & \end{cases}}$$

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Binomial Distribution | Binomial Probability Law

Suppose that a Bernoulli trial repeated ~~to~~ n-times under identical situations ~~or~~ or consider n identical independent Bernoulli trails. Represent the number of success in these n-trials by random variable X(n). Then this r.v. X(n) can ~~tasks~~ take value 0, 1, 2, 3, 4, --- n

- In each trail the probability of success is p. A success or failure in each trial does not depend upon what happend before

- A_1 is event of getting a failure (F) in first trial and A_2 is an event getting a success in the second trail (S), then.

$$P(A_1) = q;$$
$$P(A_2|A_1) = P(A_2) = p$$
$$P(A_1 \cap A_2) = P(A_1)P(A_2) = q \cdot p.$$

where $P(A_1 \cap A_2)$ is the probability of getting sequence " failure, success".

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"FS."

— Let first x trails resulted in successes and the remaining $n-x$ trials resulted in failures. Then the probability of getting sequence $\underbrace{SSSS\cdots S}_{x}\underbrace{FFF\cdots F}_{n-x}$, where S denotes the success and F: failure. is

$$\underbrace{p\cdot p\cdot p\cdots p}_{x\text{-times}}\ \underbrace{q\cdot q\cdot q\cdots q}_{n-x\text{ times}} = p^x q^{n-x}$$

— Suppose first three trials ~~are~~ were failure, the next x trails were success and remaining failure i.e. the sequence $\underset{3}{FFF}\underbrace{SSSS\cdots S}_{x}\underbrace{FFF\cdots F}_{n-x-3}$. Then the probability of getting this sequence:

$$\underbrace{qqq}_{3\text{-times}}\ \underbrace{ppp\cdots p}_{x\text{-times}}\ \underbrace{\cancel{fff\cdots f}\ qqq\cdots q}_{\cancel{3\text{ times}}\ n-x-3} = q^3 p^x q^{n-x-3} = p^x q^{n-x}$$

— Thus for any sequence of S and F, in any arrangement, the probability is $p^x q^{n-x}$

— How many such sequences are possible?

This is $\binom{n}{x} = \binom{n}{n-x}$

and these such sequence ~~are~~ represent a event and each are mutually exclusive and collectively exhaustive

— Hence ~~the~~ of the probability of getting exactly x success in n independent Bernoulli trial is given by:

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$$p_{X_n}(x) \equiv p_{X(n)}(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}; & x = 0, 1, 2, \ldots n \\ 0, & \text{elsewhere.} \end{cases}$$

Here $0 \le p \le 1$, $q = 1-p$, $n = 1, 2, 3, \ldots$

- Note Here n, and p are parameter, some time we use notation ~~B(n,p)~~ $X(n) \sim B(n,p)$ for ~~pm~~ pmf

- Shape of ~~B~~ $p_{X_n}(x)$ depends on the the parameter n and p.

To check this valid pmf

$$\sum_x p_{X_n}(x) = 0 + \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} \cancel{= (q+p)}$$

$$= \binom{n}{0} p^0 q^{n-0} + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \cdots + \binom{n}{n} p^n q^{n-n}$$

$$= (q+p)^n$$

$$= (1-p+p)^n$$

$$= 1^n$$

$$= 1$$

$$\Rightarrow \sum_{x=0}^{\infty} p_{X_n}(x) = 1$$

and $p_{X_n}(x_i) \ge 0 \quad \forall x_i = 0, 1, 2, \ldots n$

$\Rightarrow$ This is ~~valid~~ valid pmf.

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Expected Value :-

$$E[X_n] = E[X(n)] = \sum_{x=0}^{n} x\, p_{X_n}(x)$$

$$E[X(n)] = \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = \sum_{n=1}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x \frac{n!}{x(x-1)!(n-x)!} p^x q^{n-x}$$

$$E[X(n)] = \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x q^{n-x} \qquad [x \neq 0]$$

$$= \sum_{x=1}^{n} \frac{n(n-1)!}{(x-1)!(n-x)!} p\, p^{x-1} q^{n-x}$$

$$= \sum_{x=1}^{\infty} np \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x}$$

$$E[X(n)] = np \sum_{x=1}^{\infty} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x} \quad \text{—— (1)}$$

Take $m = x-1$, $\Rightarrow$ $x=1$, $m=0$ and when $x=n$, $m=n-1$

Thus $\cancel{E[X(n)] = np \sum_{m=0}^{n-1} \frac{(n-1)!}{(n-m}}$

$$E[X(n)] = np \sum_{m=0}^{n-1} \frac{(n-1)!}{m!(n-m-1)!} p^m q^{n-m-1}$$

$$\boxed{E[X(n)] = np[q+p]^{n-1} = np[1]^{n-1} = np}$$

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→ Variance:- $\sigma^2_{X(n)} = E[X(n)^2] - \{E[X(n)]\}^2$

For computing we use factorial moment of order 2

$$E[X(n)\{X(n)-1\}] = \sum_{x=0}^{n} x(x-1)\, p_{X(n)}(x)$$

$$= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \cancel{x(x-1)} \frac{n!}{\cancel{x(x-1)}(x-2)!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!} p^x q^{n-x}$$

$$E[X(n)\{X(n)-1\}] = \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!\,(n-x)!} p^2 p^{x-2} q^{n-x}$$

$$= p^2 n(n-1) \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!\,(x-2)!} p^{x-2} q^{n-x}$$

$$E[X(n)\{X(n)-1\}] = n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(n-x)!\,(x-2)!} p^{x-2} q^{n-x}$$

Let $m = x-2$, which means that when $x=2$, $m=0$, and when $x=n$, $m=n-2$. Thus we obtain

$$E[X(n)\{X(n)-1\}] = n(n-1)p^2 \sum_{m=0}^{n-2} \frac{(n-2)!}{(n-m-2)!\,m!} p^m q^{n-m-2}$$

$$\cancel{= n(n-1)p^2 [q+p]}$$

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$$E[X(n)\{X(n)-1\}] = n(n-1)p^2 \sum_{m=0}^{n-2} \binom{n-2}{m} p^m q^{n-m-2}$$

$$= n(n-1)p^2 [q+p]^{n-2}$$

$$= n(n-1)p^2 [1]^{n-2}$$

$$\therefore \boxed{E[X(n)\{X(n)-1\}] = n(n-1)p^2} \quad \text{--- (1)}$$

$$E[X^2(n) - X(n)] = n(n-1)p^2$$

$$\Rightarrow E[X^2(n)] - E[X(n)] = n(n-1)p^2$$

$$\therefore E[X^2(n)] = n(n-1)p^2 + E[X(n)]$$

$$E[X^2(n)] = n(n-1)p^2 + np$$

$$E[X^2(n)] = \cancel{n^2p^2 - np^2 + np}\ n(n-1)p^2 + np \quad \text{--- (2)}$$

Variance :-

$$\therefore \sigma^2_{X(n)} = E[X^2(n)] - \{E[X(n)]\}^2$$

$$= n(n-1)p^2 + np - \{np\}^2$$

$$\sigma^2_{X(n)} = n(n-1)p^2 + np - n^2p^2$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$\sigma^2_{X(n)} = np - np^2$$

$$= np(1-p)$$

$$\boxed{\sigma^2_{X(n)} = npq}$$

Moment Generating Function :-

$$M_{X_n}(t) = \sum_x e^{tx} p_{X_n}(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} e^{tx} p^x q^{n-x}$$

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$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$\boxed{M_{X(n)}(t) = (q + pe^t)^n}$$

CDF :- The CDF of $X(n)$ is given by:

$$F_{X(n)}(x) = P[X(n) \le x]$$
$$= \sum_{k=0}^{x} \binom{n}{k} p^k q^{n-k}, \quad x = 0, 1, 2, 3, \ldots n$$

Example :- Four fair coins are tossed. If outcomes are assumed to be independent, find the pmf of the number of heads obtained.

Solution :- Let $X(n)$ denote the number of heads that appear in the four tosses. Then $X[n]$ with $n = 4$, and $p = 1/2$. Thus pmf of $X(n)$ is

$$p_{X_4}(x) = \binom{4}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} = \binom{4}{x} \left(\frac{1}{2}\right)^4$$

$$p_{X_4}(0) = \binom{4}{0} \left(\frac{1}{2}\right)^4 = \frac{4!}{0!\,4!} = 1 \cdot \left(\frac{1}{2}\right)^4$$

$$\therefore p_X(0) = \frac{1}{16}$$

$$p_{X_4}(1) = \binom{4}{1} \left(\frac{1}{2}\right)^4 = \frac{4}{16}$$

$$p_{X_4}(2) = \binom{4}{2} \left(\frac{1}{2}\right)^4 = \frac{6}{16}$$

$$p_{X_4}(3) = \binom{4}{3} \left(\frac{1}{2}\right)^4 = \frac{4}{16}$$

$$p_{X_4}(4) = \binom{4}{4} \left(\frac{1}{2}\right)^4 = \frac{1}{16}.$$

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Ex :- 100 balls are tossed into 50 boxes. What is the expected number of balls in the tenth box?

Solution :- Balls are tossed ≡ Bernoulli trail

Success ≡ Getting ball in the tenth box.

$\therefore \; p = \frac{1}{50}$

$X \equiv$ the number of balls go into the 10th box.

100 balls are tossed ≡ 100 Bernoulli trial.

$n = 100$

$\therefore$ Therefore r.v. $X \sim B(100, \frac{1}{50})$

Thus $E[X] = n \cdot p = \cancel{100}^{2} \times \frac{1}{\cancel{50}}$

$\therefore \boxed{E[X] = 2}$

Ex :- A coin is tossed 10 times - Given that there are 6 heads in the 10 tosses, what is expected number of heads in the five (5) tosses?

Solution :- Since 6 heads and hence 4 tail in 10 tosses $\Rightarrow$ Link that a bag have 6 head and 4 tails $\Rightarrow$ Thus probability of head is

$p = \frac{6}{10} = 0.6$

Let Y denotes the no of heads in first five tosses of the coin. Then $Y \sim B(5, 0.6)$

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Thus Y is Binomial r.v. with parameter [5, 0.6]

$\therefore E[Y] = \cancel{5*}\ np = 5 \times 0.6$

$\therefore \boxed{E[Y] = 3.0}$

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Geometric Distribution / Geometric Probability Law

- The geometric r.v. is used to describe the number of Bernoulli trials until the first success occurs.
- Let X be r.v. that denotes the number of Bernoulli trials up to and including the trial that results in the first success.
- If first success occurs on the xth trial, then we know that the first $x-1$ trials resulted in failures

i.e. $\overbrace{\underbrace{F\ F\ F\ \cdots\ F}_{x-1 \text{ trial}}\ S}^{x \text{ trials}}$

Let each trial success probability is p and failure probability is q. Since trails are ~~inde~~ independent, the pmf of geometric random variable

$$p_X(x) = \begin{cases} q^{x-1}p\ ; & x = 1, 2, 3, \cdots \\ 0\ , & \text{otherwise.} \end{cases}$$

or.

$$p_X(x) = \begin{cases} \cancel{p(1-p)^{x-1}}\ (1-p)^{x-1}p\ ; & x = 1, 2, 3, 4, \cdots \\ 0, & \text{otherwise.} \end{cases}$$

Here p is only parameter.

Note:- The successive terms here are ~~p, pq, pq~~

$$p,\ qp,\ q^2p,\ q^3p \cdots\cdots \equiv p,\ pq,\ pq^2, \cdots$$

which are geometric progression and hence the distribution is called ~~for~~ Geometric distribution / Geometric law of probability

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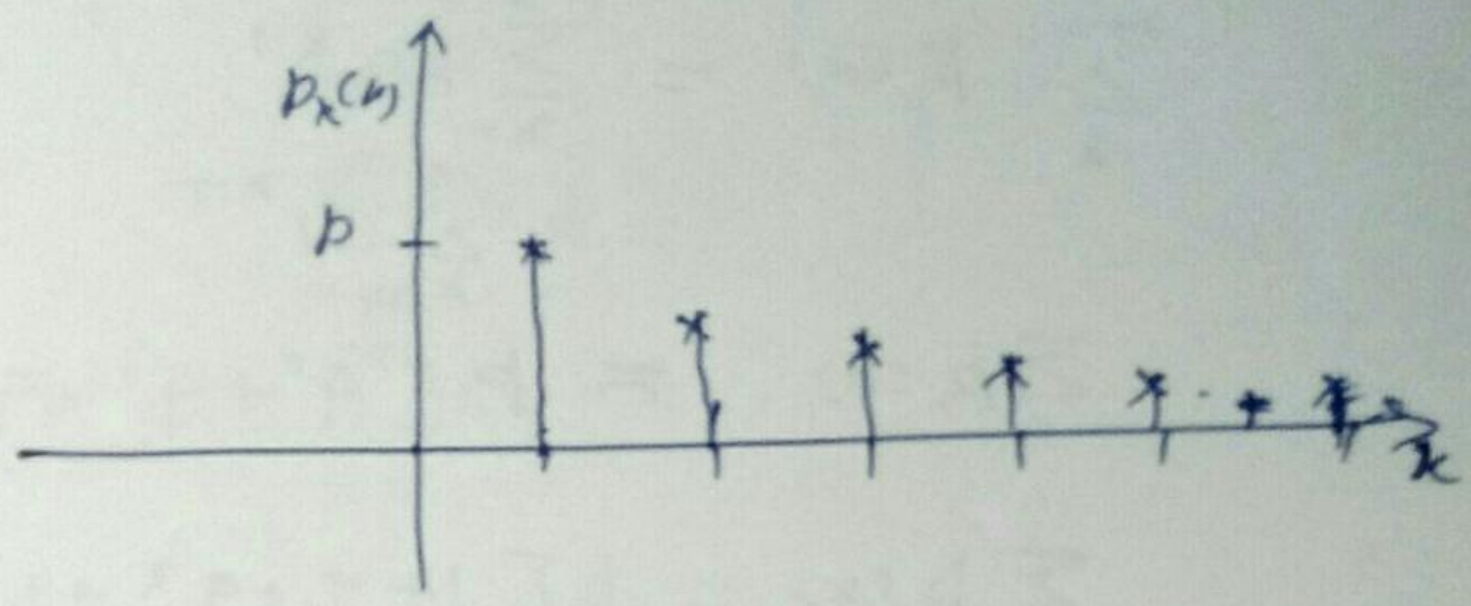
— $p_x(x)$ is exponentially decreasing function as x increases



Ex:- A bag contains 6 blue balls and 4 red balls Balls are randomelly drawn from the bag, one at a time, until a red ball is obtained. If we assume that each drawn ball is replaced before next one is drawn, what is the ~~prob~~ probability that exactly 5 balls are drawn.

Solution:- Let x denote the number of balls needed to select a red ball. Since balls are replaced after they are drawn from the bag

↳ The probability that a particular ball is drawn from the bag in any trial is constant

↳ Thus, each time red ball is drawn we have a success with probability $p = \frac{4}{4+6} = \frac{4}{10} = 0.4$

and when blue ball is drawn we have failure

$q = 1-p = 1-0.4 = 0.6$

↳ If exactly five (5) balls are drawn before the experiment is stoped, then we drew four blue balls ~~consequently~~ ~~consequ.~~ Consecutively [i.e. Four consecutive failure] and the fifth ball is red ball, which is success. Therefore desired probability is given by:

$$P[x=5] = p_x(5) = pq^{5-1} = p(1-p)^4 = 0.4 \times 0.6^4$$
$$= 0.05184$$

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Check This is valid pmf

$$\sum_x p_X(x) = \sum_{x=1}^{\infty} p q^{x-1}$$

$$= p \sum_{x=1}^{\infty} q^{x-1}$$

$$\sum_x p_X(x) = p[q^0 + q^1 + q^2 + q^3 + \cdots + \cdots]$$

$$\sum_x p_X(x) = p[1 + q + q^2 + q^3 + \cdots + \cdots]$$

$$= p\left[\frac{1}{1-q}\right] \quad \text{since } |q| \leq 1 \text{ Geometric series}$$

$$= \cancel{p} \cdot \frac{1}{\cancel{p}}$$

$\rightarrow \sum_x p_X(x) = 1$, since $p \geq 0$, $q \geq 0$

$\therefore p q^{x-1} \geq 0$ for $x = 1, 2, 3 \ldots$

$\rightarrow p_X(x) \geq 0 \quad x = 1, 2, 3 \ldots$

$\rightarrow$ Therefore $p_X(x) = \begin{cases} p q^{x-1}, & x = 1, 2, 3, \ldots \\ 0, & \text{Otherwise.} \end{cases}$

is valid pmf.

~~Expectall~~

Expected value :— Expected value of Geometric r.v. X is given by

$$E[X] = \sum_x x\, p_X(x) = \sum_{x=1}^{\infty} x \cdot p q^{x-1}$$

$$= p \sum_{x=1}^{\infty} x \cdot q^{x-1}$$

$$E[X] = p[1 + 2q + 3q^2 + 4q^3 + \cdots] \quad \text{—— (1)}$$

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Now consider

$$\frac{d}{dq}\sum_{x=0}^{\infty} q^x = \sum_{x=0}^{\infty}\frac{d}{dq}q^x$$

$$= \sum_{x=0}^{\infty} x q^{x-1}$$

$$\frac{d}{dq}\left[\sum_{x=0}^{\infty} q^x\right] = \sum_{x=1}^{\infty} x q^{x-1}$$

$$\therefore \frac{d}{dq}\left[\sum_{x=0}^{\infty} q^x\right] = 1+2q+3q^2+4q^3+ \text{---} \quad (2)$$

Now from (1) and (2)

$$E[X] = p\sum_{x=1}^{\infty} x q^{x-1} = p\cdot\frac{d}{dq}\left[\sum_{x=0}^{\infty} q^x\right]$$

$$= p\cdot\frac{d}{dq}\left[1+q+q^2+\text{--}q^n+\text{---}\right]$$

$$= p\frac{d}{dq}\left(\frac{1}{1-q}\right)$$

$$= p\times\frac{1}{(1-q)^2}\times 1$$

$$\therefore E[X] = +\frac{p}{(1-q)^2}$$

$$\therefore E[X] = \frac{p}{p^2}$$

$$\therefore \boxed{E[X] = \frac{1}{p}}$$

Variance $\sigma_x^2 = E[X^2] - (E[X])^2$

Now to find $E[X^2]$?

$$E[X^2] = \sum_{x=1}^{\infty} x^2 p_x(x) = \sum_{x=1}^{\infty} x^2 \cdot p\, q^{x-1}$$

$$E[X^2] = \sum_{x=1}^{\infty} x^2 q^{x-1} p \quad (3)$$

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Now use

$$x^2 q^{x-1} p = p \frac{d}{dq}\left[q \cdot \frac{d}{dq} q^x\right] \quad \text{—— (2)}$$

Now use (2) in (1) we have

$$E[X^2] = \sum_{x=1}^{\infty} x^2 q^{x-1} p = \sum_{x=1}^{\infty} p \cdot \frac{d}{dq}\left[q \cdot \frac{d}{dq} q^x\right]$$

$$\therefore E[X^2] = \sum_{x=1}^{\infty} p \cdot \frac{d}{dq}\left[q \frac{d}{dq} q^x\right]$$

$$= p \frac{d}{dq} q \frac{d}{dq} \sum_{x=1}^{\infty} q^x = p \cdot \frac{d}{dq} q \cdot \frac{d}{dq} q \sum_{x=1}^{\infty} q^{x-1}$$

$$= p \frac{d}{dq} q \cdot \frac{d}{dq}\left(\frac{q}{1-q}\right)$$

$$= p \cdot \frac{d}{dq} q \frac{d}{dq}\left[q(1-q)^{-1}\right]$$

$$= p \frac{d}{dq} q \left[+\frac{q}{(1-q)^2} + \frac{1}{(1-q)}\right]$$

$$= p \cdot \frac{d}{dq} q \left[\frac{q^2}{(1-q)^2} + \frac{q}{1-q}\right] \quad \checkmark$$

$$= p \left[\frac{d}{dq} \frac{q^2}{(1-q)^2} + \frac{d}{dq}\left(\frac{q}{1-q}\right)\right]$$

$$= p \left[\frac{(1-q)^2 \cdot 2q - q^2 \times 2(1-q) \times -1}{(1-q)^4} + \frac{(1-q) \times 1 - q \cdot \times -1}{(1-q)^2}\right]$$

$$= p \left[\frac{2q(1-q)^2 + 2q^2(1-q)}{(1-q)^4} + \frac{q(1-q)}{(1-q)^2}\right]$$

$$= p \left[\frac{2q(1-q)[1-q+q]}{(1-q)^4} + \frac{1}{(1-q)^2}\right] + \frac{1-q+q}{(1-q)^2}$$

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$$E[x^2] = p \cdot \frac{d}{dq}\left[q^2 (1-q)^{-2} + q(1-q)^{-1} \right]$$

$$= p \frac{d}{dq}\left(q(1-q)^{-1} + q^2 (1-q)^{-2} \right)$$

$$= p \left\{ \left[\frac{d}{dq} q (1-q)^{-1}\right] + \left[\frac{d}{dq} q^2 (1-q)^{-2}\right] \right\}$$

$$= p \left\{ \left[q \cdot \times -1 (1-q)^{-2} \times -1 + (1-q)^{-1} \right] + \left[q^2 \times -2 (1-q)^{-3} \times -1 + (1-q)^{-2} \times 2q \right] \right\}$$

$$= \cancel{p \left[\frac{q}{(1-q)^2} \right.}$$

$$= p \left[q(1-q)^{-2} + (1-q)^{-1} + 2q^2 (1-q)^{-3} + 2q(1-q)^{-2} \right]$$

$$= p \left[(1-q)^{-1} + 3q(1-q)^{-2} + 2q^2 (1-q)^{-3} \right]$$

$$= p \left[\frac{1}{p} + 3 \cdot \frac{q}{p^2} + 2 \frac{q^2}{p^3} \right]$$

$$= \left(1 + 3 \cdot \frac{q}{p} + 2 \cdot \frac{q^2}{p^2} \right)$$

$$\therefore E[x^2] = 1 + 3\frac{q}{p} + 2\frac{q^2}{p^2}$$

$$= \frac{p^2 + 3qp + 2q^2}{p^2}$$

$$= \frac{p^2 + 3(1-p)p + 2(1-p)^2}{p^2}$$

$$= \frac{p^2 + 3(p-p^2) + 2(1+p^2-2p)}{p^2}$$

$$= \frac{\cancel{p^2} + 3p - \cancel{3p^2} + 2 + \cancel{2p^2} - 4p}{p^2}$$

$$\boxed{E[x^2] = \frac{2-p}{p^2}}$$ — 3

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Therefore

$$\sigma_x^2 = E[x^2] - (E[x])^2$$

$$\sigma_x^2 = \frac{2-p}{p^2} - \frac{1}{p^2}$$

$$= \frac{2-p-1}{p^2}$$

$$\boxed{\therefore \sigma_x^2 = \frac{1-p}{p^2} = \frac{q}{p^2} = \frac{1}{p^2} - \frac{1}{p}}$$

Moment Generating Function :-

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} p q^{x-1} = p\{e^t + q e^{2t} + q^2 e^{3t} + \cdots\}$$

$$= p e^t \{1 + q e^t + q^2 e^{2t} + \cdots\}$$

$$= p e^t \frac{1}{1-qe^t} ; \quad |qe^t| < 1$$

$$\boxed{M_x(t) = p e^t (1-qe^t)^{-1}}$$

$$E[x] = \frac{d}{dt} M_x(t)\Big|_{t=0} = \{ p e^t \times -1 \times (1-qe^t)^{-2} \times -q e^t + (1-qe^t)^{-1} p e^t \}\Big|_{t=0}$$

$$= \{ p q e^{2t} (1-qe^t)^{-2} + p e^t (1-qe^t)^{-1} \}\Big|_{t=0}$$

$$E[x] = \{ p q e^0 (1-qe^0)^{-2} + p e^0 (1-qe^0)^{-1} \}$$

$$= \{ pq (1-q)^{-2} + p (1-q)^{-1} \}$$

$$E[x] = \frac{pq}{(1-q)^2} + p \cdot (1-q)^{-1}$$

$$\cancel{= \frac{pq}{p^2} + p\frac{1}{p} = \frac{q}{p} + 1 =} \quad q + p^2 q$$

$$= \frac{q(1+p^2)}{p}$$

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$$E[X] = \frac{pq}{p^2} + \frac{p}{p}$$

$$E[X] = 1 + \frac{q}{p}$$

$$= \frac{p+q}{p} = \frac{p+q}{p}$$

$$\boxed{\therefore E[X] = \frac{1}{p}}$$

$$\boxed{E[X^2] = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} = 1 + \frac{q}{p} + 2\frac{q^2}{p^2}}$$

CDF of Geometric Distribution

$$F_X(x) = P[X \le x] = \sum_{k=1}^{x} p(1-p)^{k-1}$$

$$= 1 - P[X > x]$$

$$\boxed{F_X(x) = 1 - (1-p)^x}$$

Another way;

$$F_X(x) = \sum_{k=1}^{x} p(1-p)^{k-1} = p \sum_{m=0}^{x-1} (1-p)^m$$

$$F_X(x) = p\left\{\frac{1-(1-p)^x}{1-(1-p)}\right\}$$

$$F_X(x) = \cancel{p}\left\{\frac{1-(1-p)^x}{\cancel{p}}\right\}$$

$$\therefore \boxed{F_X(x) = 1 - (1-p)^x}$$

$(1-p)^x$ = Probability that no success has occurred by the end of xth trial

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- The result indicates that $F_X(x)$ is the probability that at least one success has occurred in x trials of the experiment.

Modified Geometric Distribution

Number of trials until the first failure in series of Bernoulli trials. If we denote this by r.v. by Y, then Y is defined to be a modified geometric distribution and its PMF is given by:

$$P_Y(y) = (1-p)p^{y-1} = q\,p^{y-1}; \quad y = 1, 2, 3, \ldots$$

Note:- By changing role of p and q in "traditional" geometric distribution we obtain the results for the expected value and variance of Y as follows

$$E[Y] = \frac{1}{q} = \frac{1}{1-p}$$

$$\sigma_Y^2 = \frac{p}{q^2} = \frac{p}{(1-p)^2}$$

CDF, $F_Y(y) = P[Y \le y] = \sum_{k=1}^{y} p(1-p)^{k-1} = P[Y > y]$
$= 1 - p^y$

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Ex:- Balls are tossed at random into 50 boxes. Find the expected number of tosses required to get the first ball in the fourth box.

Solution:- Let X denote the number of tosses until the ~~first~~ first ball goes into the fourth box.

Then X follows the geometric distribution with $p = \frac{1}{50}$

$$\therefore \boxed{E[X] = \frac{1}{p} = 50}$$

Ex:- The probability that missile hits the target is p. If missiles are fired independently at a target until it is hit, what is the probability that it takes more than 3 missiles to get the target?

Solution:- p: probability of success. This problem can be stated as the number of attempts before a success is greater than 3. That is, there we have so far three consecutive failure. Since these attempts are independent. Thus required answer is $(1-p)^3$.

Ex:- A couple is planning to have a children in a community where the probability of having a boy is p.

(a) If they keep trying until they have a girl and then stop, what is expected number of children that they will have?

(b) Suppose they plan ~~to~~ have minimum four children. However, if the first four are all boys, they will continue until they get girl and then stop.

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If there is atleast one girl among the first 4 children, they will stop at 4. What is the expected number of children they have.?

Answer:- The births are Bernoulli trials, where the probability of success is $1-p$, the probability of having girl.

(a) Thus, the number of trials until the couple has a girl has the modified geometric distribution with mean $\frac{1}{1-p}$.

(b) The number of children Y that the couple will have depends on whether the first 4 are all boys or not.

↳ If first 4 all are boys, then the ~~number of~~ expected number of attempts until they have a girl after the four boys are born is $\frac{1}{1-p}$

↳ Thus expected number of children, given that the first four are all boys, is $4+\frac{1}{(1-p)}$.

- If atleast one of the first four is a girl, then the ~~expecte~~ expected number of children 4. Since ~~the probability that the first four is a girl, then the expected number of children is 4.~~

↳ probability that first four are all boys is p^4

↳ Thus probability that there is at least one girl among the four children is $1-p^4$. Thus the expected number of children that the couple will have is

$$E[Y] = p^4\left\{4+\frac{1}{1-p}\right\} + (1-p^4)(4)$$

$$\boxed{E[Y] = 4+\frac{p^4}{1-p}}$$

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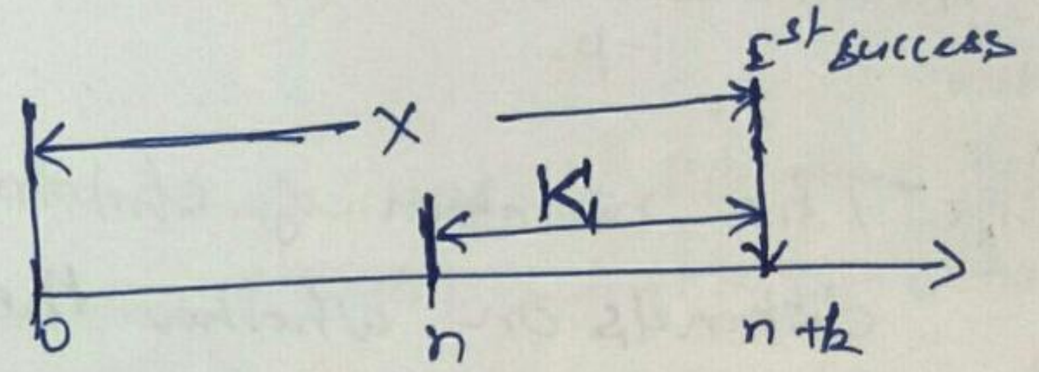
"Memoryless/Forgetfulness" Property of the Geometric Distribution

R.V. X: number of Bernoulli trials until the first success.

- Suppose observed a fixed number 'n' of these trials and they are all failures.
- K: number of additional trails until the first success

$\therefore K = X - n$



(Additional Trails Until success)

Compute the ~~cond~~ conditional probability mass function of K, given that $X > n$. We proceed as follows:

$$p_{K|X>n}(k|X>n) = P[K = k | X > n]$$

$$= P[X = n+k \mid X > n]$$

$$= \frac{P[\{X = n+k\} \cap \{X > n\}]}{P[X > n]}$$

Because $\{X = n+k\} \subset \{X > n\}$, we have that

$$\{X = k+n\} \cap \{X > n\} = \{X = n+k\}$$

~~Thus,~~

$$\cancel{p_{K|X>n}}$$

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Thus,

$$p_{K|X>n}(k|X>n) = \frac{P[X=n+k]}{P[X>n]}$$

$$= \frac{p(1-p)^{n+k-1}}{(1-p)^n}$$

$$p_{K|X>n}(k|X>n) = p(1-p)^{k-1} = p_X(k); \quad k=1,2,3\ldots$$

where the second equality follows from our earlier result that

$$P[X>n] = (1-p)^n$$

- The above result shows that the conditional probability that the number of trials remaining until the first success, given that no success occurred in the first n trials, has the same ~~pmf~~ PMF as X had originally

↳ This property is called the "forgetfulness or memoryless property" of the geometric distribution.

↳ ~~Sequence forgets~~ distribution "forgets" the past by not remembering how long the sequence has lasted if no success has already occurred. Thus, each time we want to know the number of trials until the first success has already occurred.

↳ Thus, each time ~~we~~ we want to know the ~~numb~~ number of trials until the first ~~st~~ success, given that the first success has not yet occurred, the process, "starts" from the ~~begin~~ beginning.

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Ex:- Tony is tossing balls randomly into 50 boxes and his goal is stop when he gets first ball into the ~~eight~~ eighth box. ~~box~~. Given that he has tossed ~~20~~ 20 balls without getting a ball into ~~the~~ the eighth box, what is expected number of additional tosses he needs to get a ball into the eighth box?

Solution:— With respect to eighth box, each toss is a Bernoulli trial with probability of success $p = \frac{1}{50}$. Let X ~~be~~ the random variable that denotes the number of tosses required to get a ball into the eighth box. ~~Then~~ Then the X has geometric distribution. Let K denote the number of additional tosses required to get a ball into the eighth box, given that no ball is in the box after 20 tosses. Then, because of the forgetfulness property of the geometric distribution, K has the same distribution as X. Thus

$$\boxed{E[K] = \frac{1}{5} = 50}$$

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Negative Binomial Distribution / Pascal Distribution

↳ Extenssion of geometric distribution. R.V. X_k denotes.

X_k = number of trials until the kth success

It is also known as: " k^{th} order interarrival time for a Bernoulli process"

: k^{th} ~~order~~ order Pascal

Let k^{th} success in a Bernoulli sequence of trials occurs at the nth trial.

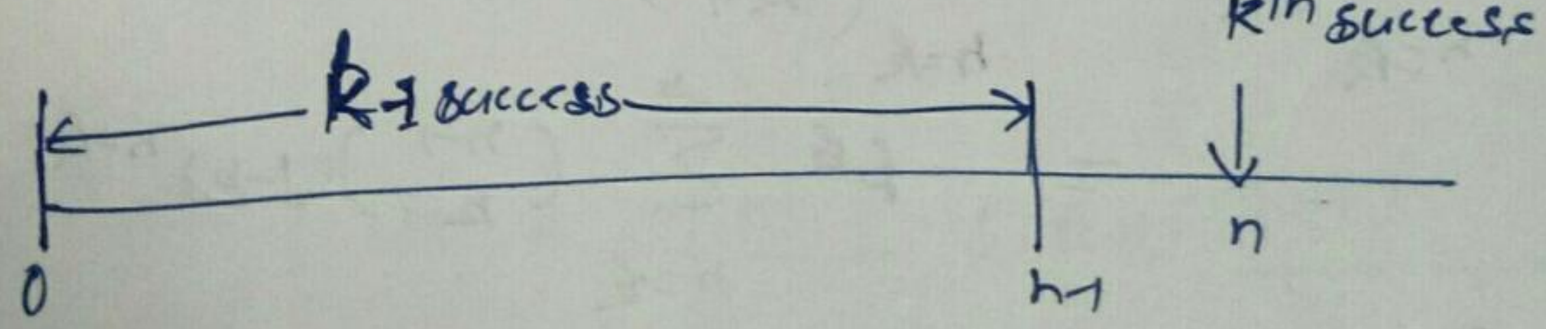
$$\underbrace{F\,F\,F \cdots S\,F\,F \cdots S \cdots F \cdots}_{n\text{-trial}}\ \overset{k^{th}\text{ success}}{S}$$

$\equiv$ $(k-1)$ success occured during the past $n-1$ trials.

From our knowledge binomial distribution, we know that the probability of this event is.

$$P_{X(n-1)}(k-1) = \begin{cases} \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} & n = k, k+1, \ldots \\ 0, & \text{otherwise} \end{cases}$$

where $X(n-1)$ is Binomial random variable with parameters $(n-1, p)$



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Thus if we define the Bernoulli random variable X_0 whose PMF we define earlier as $p_{X_0}(x) = p^x(1-p)^x$; where $x = 0$, or 1.

The PMF of kth order Pascal random variable, X_k, is obtained by knowing that X_k is defined by following two non-overlapping process:

(i) a $B(n-1, p)$ process that results in $(k-1)$ success.

(ii) A Bernouli process that results in a success.

Thus, we have that:

$$p_{X_k}(n) = P[\{X(n-1) = k-1\} \cap \{X_0 = 1\}]$$

$$= P[X(n-1) = k]\, P[X_0 = 1] \quad \text{[Independence]}$$

$$p_{X_k}(n) = \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}\, p$$

$$p_{X_k}(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}; \quad k = 1, 2, \ldots; n = k, k+1, \ldots$$

Since $p_{X_k}(n)$ is true PMF, we must have that

$$\sum_{n=k}^{\infty} p_{X_n}(n) = \sum_{n=k}^{\infty} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

$$= p^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} (1-p)^{n-k}$$

$$\sum_{n=k}^{\infty} p_{X_n}(n) = p^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k}$$

$$= p^k \left[\binom{k-1}{0} q^0 + \binom{k}{1} q + \binom{k+1}{2} q^2 + \cdots \right]$$

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$$= p^k \left[1 + k\frac{q}{1!} + k(k+1)\frac{q^2}{2!} + \cdots \right]$$

$$= p^k \left[(1-q)^{-k} \right]$$

$$= p^k \cdot p^{-k}$$

$$= 1$$

$$\therefore \sum_{n=k}^{\infty} p_{X_k}(n) = 1, \qquad p_{X_k}(n) \geq 0 \quad \forall n, k.$$

So it is valid PMF.

Note:- When $k=1$, we have the geometric probability law. Thus geometric ~~probabilit~~ probability law is a particular case of negative binomial probability law.

Moment Generating function:-

$$M_{X_k}(t) = p^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k} p^k e^{tn}$$

$$= p^k \sum_{n=k}^{\infty} \binom{n-1}{k-1} q^{n-k} e^{tn}$$

$$= p^k \left[\binom{k-1}{0} q^0 e^{kt} + \binom{k}{1} e^{(k+1)t} + \binom{k+1}{2} e^{(k+2)t} + \cdots \right]$$

$$= p^k e^{kt} \left[1 + k\frac{qe^t}{1!} + k(k+1)\frac{(qe^t)^2}{2!} + \cdots + \cdots \right]$$

$$= p^k e^{kt} \left(1 + k \cdot \frac{qe^t}{1!} + k(k+1)\frac{(qe^t)^2}{2!} + \cdots + \cdots \right)$$

$$M_{X_k}(t) = p^k e^{kt} (1 - qe^t)^{-k}; \quad qe^t < 1.$$

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This is a differentiable function. Hence

$$E[X_k] = \frac{d}{dt} M_{X_k}(t) \Big|_{t=0}$$

$$= p^k \left[k e^{kt}(1-qe^t)^{-k} + k e^{kt}(1-qe^t)^{-k-1} \right] \Big|_{t=0}$$

$$\therefore E[X_k] = k p^k \left\{ p^{-k} + q p^{-k-1} \right\}$$

$$E[X_k] = k \left\{ 1 + \frac{q}{p} \right\} = \frac{k}{p}$$

$$\therefore \boxed{E[X_k] = \frac{k}{p}}$$

$$E[X^2] = \frac{d^2}{dt^2} M_{X_k}(t) \Big|_{t=0}$$

$$= \frac{d}{dt} \left[\frac{d}{dt} M_{X_k}(t) \right] \Big|_{t=0}$$

$$= k p^k \left\{ k e^{kt} (1-qe^t)^{-k} + kq e^{(k+1)t} (1-qe^t)^{-(k+1)} \right.$$

$$\left. + q(k+1) e^{(k+1)t} (1-qe^t)^{-(k+1)} + (k+1) q^2 e^{(k+1)t} (1-qe^t)^{-(k+2)} \right\} \Big|_{t=0}$$

$$E[X^2] = k^2 + \frac{q}{p}(2k^2+k) + \frac{q^2}{p^2} k(k+1)$$

$$= \frac{k^2p^2 + qp(2k^2+k) + q^2 k(k+1)}{p^2}$$

$$= \frac{k^2p^2 + (p-p^2)(2k^2+k) + (1+p^2-2p)k(k+1)}{p^2}$$

$$= \frac{\cancel{k^2p^2} + 2pk^2 + pk - \cancel{2p^2k^2} - p^2k + k(k+1) + p^2k(k+1) - 2pk(k+1)}{p^2}$$

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$$= \frac{2pk^2 + pk - p^2k^2 - p^2k + k(k+1) + p^2k(k+1) - 2pk(k+1)}{p^2}$$

$$= \frac{\cancel{2pk^2} + pk - \cancel{p^2k^2} - p^2k + (k^2+k) + \cancel{p^2k^2} + p^2k - \cancel{2pk^2} - 2pk}{p^2}$$

$$= \frac{pk - \cancel{p^2k} + (k^2+k) + \cancel{p^2k} - 2pk}{p^2}$$

$$E[X^2] = \frac{-pk + (k^2+k)}{p^2}$$

$$= \frac{k^2 + k(1-p)}{p^2}$$

$$\therefore \boxed{E[X^2] = \frac{k^2 + k(1-p)}{p^2}}$$

Variance

Variance of Negative Binomial Distribution :-

$$\sigma_{X_k}^2 = E[X_k^2] - (E[X_k])^2$$

$$= \frac{k^2 + k(1-p)}{p^2} - \frac{k^2}{p^2}$$

$$= \frac{k(1-p)}{p^2}$$

$$\therefore \boxed{\sigma_{X_k}^2 = \frac{kq}{p^2} = \frac{k(1-p)}{p^2}}$$

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Example

Consider independent Bernoulli trials in which each trials results in a success with probability p. What is the probability that there are m success before r failures?

Answer: Since there are m success and r failure, Thus there are m+r trial.

- In order to mth success occur before r-failures, the mth success must occur no later than the (m+r-1)th trial.

(→) Since if mth success occurs before (m+r-1)th trial it would have occur ~~before~~ before the rth failure.

- Thus as long as mth success occurs any time from the mth trial up to the (m+r-1)th trial, the prescribed condition would have been ~~meet~~ met. Therefore, the desired probability is given by:

$$r = \sum_{n=m}^{m+r-1} \binom{n-1}{m-1} p^m (1-p)^{n-m}.$$

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Example 4.10 :-

A students summer job is to call the university alumni for support for the university's scholarship fund. Studies indicate that the probability that each of the student's call's is answered is $1/3$. What is probability that the second call to be answered on one particular day is the student's sixth call.?

Solution :- This is an example of the ~~negative bi~~ Negative Binomial distribution with $p = \frac{1}{3}$. Thus, the desired probability is given by :

$$p_{X_2}(6) = \binom{6-1}{2-1}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$$

$$= \binom{5}{1}\left(\frac{1}{9}\right)\cancel{\left(\frac{16}{81}\right)}\left(\frac{16}{81}\right)$$

$$= \frac{80}{729}$$

$$\boxed{p_{X_2}(6) = 0.1097}$$

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Example :—

The members of a Girl Scout troop are selling cookies from house to house in a suburban town. The probability that they sell a set of cookies (that is, one or more packs of cookies) at any house they visit ~~is~~ is 0.4.

(a) If they visited eight houses in one evening, what is the probability that they sold cookies in ~~of~~ ~~effective~~ exactly five of these houses?

(b) If they visited eight houses in one evening, what is the expected number of sets of cookies that they ~~sold~~ sold.

(c) What is the probability that they sold their third set of cookies in the sixth house they visited?

Solution:— Let X denote the number of sets of cookies that the troop sold:

(a) If they visited eight houses, then X is a binomial r.v., that is, $X \sim B(8, 0.4)$. Thus, $X = X(8)$ and its PMF is given by:

$$p_{X(8)}(x) = \binom{8}{x} (0.4)^x (0.6)^{8-x}; \quad x = 0, 1, 2, 3, \ldots, 8$$

~~the~~ Therefore the probability that they sold exactly five sets of cookies is the probability of five success in the eight attempts, which is,

$$p_{X(8)}(5) = \binom{8}{5} (0.4)^5 (0.6)^3 =$$

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$$p_{X(8)}(5) = \frac{8!}{5!\,3!}(0.4)^5(0.6)^3$$

$$\boxed{p_{X(8)}(5) = 0.1239.}$$

(b) The expected number of sets of cookies they sold after visiting eight houses is:

$$E[X(8)] = 8p = 8 \times 0.4 = 3.2$$

(c) Let X_k denote the number of houses they visited upto and including the house where they sold their third set cookies. Then X_k is a third-order Pascal random variable, that is, $X_k = X_3$, and

$$P[X_3 = 6] = p_{X_3}(6) = \binom{6-1}{3-1}(0.4)^3(0.6)^3$$

$$= \binom{5}{2}(0.4)^3(0.6)^3$$

$$= \binom{5}{2}(0.24)^3$$

$$P[X_3 = 6] = \frac{5!}{2!\,3!}(0.24)^3$$

$$\boxed{P[X_3 = 6] = 0.1382}$$

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Distinction Between

Binomial and Negative Binomial Distributions

- From, the PMF of the binomial random variable $X(n)$ is given by:

$$p_{X(n)}(x) = \binom{n}{x} p^x (1-p)^{n-x} ; \quad x = 0, 1, 2, \dots, n$$

- Similarly, ~~from (4.23)~~, the PMF of the negative Binomial random variable X_k is given by:

$$p_{X_k}(n) = \binom{n-1}{k-1} p^k (1-p)^{n-k} ; \quad k = 1, 2, 3, \dots, \quad n = k, k+1, k+2, \dots$$

- Structurally, the two PMFs look alike, the difference is primarily in the combinatorial term and their range of values.

— But they address two different situations:—

Binomial R.V.	Negative Binomial
Assumes that a fixed number of trials of an experiment have been completed before it ask for the number of successes in those trials.	→ Negative Binomial R.V. deals with a process that has a prescribed termination point. The experiment stops once the condition is met. ↳ This means that the negative random variable deals with an experiment that can continue forever and as a result the number of times that the experiment is performed can be unbounded
— Here experiments deals with finite ~~num~~ number of trials.	— This deals with cardinal numbers like the third, sixth etc.
— This deals with cardinal number like three, six etc.	

Hypergeometric Distribution :-

Suppose there are N objects $\begin{cases} \rightarrow N_1 < N \text{ are type 'A'} \\ \rightarrow N_2 = N - N_1 < N \text{ are type B.} \end{cases}$

Case I

Draw n sample objects with replacement:

The number of type A objects included in the sample is binomially distributed with parameters (N, p), where $p = \frac{N_1}{N}$.

[probability of selecting an objects from a particular group remains constant from trial to trial].

That is outcomes are independent and then PMF of r.v. X, which denotes the number of type A objects in the sample of size n, is given by

$$p_X(x) = \left\{ \binom{n}{x} \left(\frac{N_1}{N}\right)^x \left(\frac{N - N_1}{N}\right)^{n-x} \right. ; x = 0, 1, 2, 3, \ldots n \quad \text{--- (1)}$$

Case 2 :- Draw n sample from population without replacement

The outcomes are no longer independent and the Binomial distribution no longer applicable

Suppose the sample of size n is drawn without replacement. Let K_n be the random variable that denotes the number of type A objects in the sample of n objects drawn at a time. The random variable K_n is said to have a hypergeometric PMF that depends on n, N_1 and N. And this PMF is defined as follows.

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$$p_{K_n}(k) = \frac{\binom{N_1}{k}\binom{N-N_1}{n-k}}{\binom{N}{n}}, \quad k = 0, 1, 2, \ldots, \min(n, N_1) \quad \text{—(2)}$$

- PMF in (2) is obtained by using fundamental counting rule. ~~defined~~
- Since there are two types of objects, we can select an objects form each type independently, once the number of objects to be selected from each type. has been specified.
- Thus from given that we need k objects from type A and hence $n-k$ objects from type B, the selection process in each group becomes independent of that of the other group

↳ k objects ~~of~~ type A in the sample can be selected in $\binom{N_1}{k}$ ways.

↳ Similarly, the $n-k$ objects of type B in the sample can be selected in ~~$N-N_1$~~ $\binom{N-N_1}{n-k}$ ways.

↳ Thus total number of ~~possible~~ possible sample of size n that are specified by ~~contiga~~ configuration is product $\binom{N_1}{k} \cdot \binom{N-N_1}{n-k}$

→ Total number of possible sample of size n is $\binom{N}{n}$ in which we draw the sample.

↳ ~~were~~ unrestricted ways of choosing n objects from N objects.

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If the sample are drawn randomely, then each sample has the same probability of being selected, which accounts for above pmf.

Mean and Variance :-

$$E[K_n] = \frac{nN_1}{N} = np$$

$$\sigma^2_{K_n} = \frac{n N_1 (N-N_1)(N-n)}{N^2 (N-1)} = \frac{np(1-p)(N-n)}{N-1}$$

where $p = \frac{N_1}{N}$

Note :- As $N \longrightarrow \infty$ (or N becomes large compared to n), these equations, becomes identical to those of Binomial distribution; that is

$$\boxed{E[K_n] = np}$$

$$\boxed{\sigma^2_{K_n} = np(1-p)}$$

- The hypergeometric distribution is ~~used~~ widely used in quality control, ~~as the fol~~

Ex :- Suppose that there are $N_1 < N$ defective items in a box that contains N items. What is ~~prob~~ the probability that a random sample of n ~~#~~ items from the box contains k defective items?

Solution :- Since the sample was selected randomally, all samples of size n are equally likely. Therefore the total number of possible samples with unrestricted composition is

$$C(N, n) = \binom{N}{n}$$

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Since there are M_1 defective items in the Box, the total number of ~~possible~~ ways selecting k out of the items at a time is $C(M_1, k) = \binom{M_1}{k}$. Similarly, since there are $M - M_1$ non-defective items in the box,

— The total number of ways selecting $(n-k)$ of them at a time is $C(M-M_1, n-k) = \binom{M-M_1}{n-k}$

— Since these two selection are made independently, thus total number of ways of chosing k defective items and $(n-k)$ non-defective items is

$$C(M_1, k) \times C(M-M_1, n-k)$$

— Thus probability that the random sample contains k defective and $n-k$ ~~non-d~~ non-defective items is given by the ratio of ~~$C(M_1, k) \times C(M-M_1,$~~ $C(M_1, k) \times C(M-M_1, n-k)$ to the total number of ways of selecting sample of size n, $C(M, n)$.

Thus, we obtain the result:

$$p = \frac{C(M_1, k) \times C(M-M_1, n-k)}{C(M, n)}$$

$$p = \frac{\binom{M_1}{k}\binom{M-M_1}{n-k}}{\binom{M}{n}}$$

Ex 2:- A container has 100 items 5 of which the worker who packed the container knows are defective. A merchant wants to buy the container without knowing the information. However, he will randomally ~~picked~~ pick 20 items from the container and will accept the container as good ~~of~~ if there is ~~as m~~ atmost one bad

item in the selected sample. What is the probability that the merchant will declare the ~~conta~~ container to be good?

Solution:—

A: Event that there is no defective

B: Event that there is exactly one defective item in the selected sample.

q: Probability that merchant declare the container to be good

$A \equiv$ zero defective items and 20 non-defective item

$B \equiv$ 1 defective item and 19 non-defective items.

Since events A and B are mutually exclusive we obtain the following results:

$$P[A] = \frac{C(5,0)\,C(95,20)}{C(100,20)} = 0.3193.$$

$$P[B] = \frac{C(5,1)\,C(95,19)}{C(100,20)} = 0.421$$

$$q = P[A \cup B] = P[A] + P[B] = 0.7394$$

$$\therefore \boxed{q = 0.7394}$$

~~Exc~~ Example 3:— Same as Ex 2, ~~except~~ except test sample size is 50 instead of 20. As example 2, he will accept the container as good if ther is at most one bad item in the selected sample. What is the probability that the merchant will declare the container to be good?

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Solin :-

A : As previous; there is no defective item in the selected sample

B : Exactly one defective item in the selected sample.

~~A =~~ A ≡ zero defective items and 50 non-defective item

B ≡ 1 defective item and 49 non-defective items

$$P[A] = \frac{C(5,0)\,C(95,50)}{C(100,50)} = 0.0281$$

$$P[B] = \frac{C(5,1)\,C(95,49)}{C(100,50)} = 0.1530.$$

$$q = P[A \cup B] = P[A] + P[B] = 0.1811$$

$$\boxed{q = 0.1811}$$

From these example it is clear that sample size

↳ Bigger sample size, the merchant is more likely to make ~~bff~~ better decision.

↳ Because there is a greater probability that more of the defective items will be included in the sample ~~space~~.

e.g.

random sample size	q
10	0.9231
40	0.3316
60	0.0816

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~~Ex 4~~

Ex 4:— A certain library has a collection of 10 books on probability theory. Six of these books were written by American authors and four were written by foreign authors.

(a) If I randomally select one of these books, what is the probability that it was written by an American Author?

(b) If I select five of these books at random, what is the probability that two of them were written by American Authors and three of them written by foreign authors?

Solution :—

(a) There are $\binom{6}{1} = 6$ ways to choose a book written by an American Author and $\binom{10}{1} = 10$ ways to choose a book at random. Therefore, the probability that a book chosen at random was written by an ~~amen~~ American author is

$p = 6/10 = 0.6$

(b) This is an example of the hypergeometric distribution. Thus, the probability that of the five of these books selected at random

→ The probability that of the five of these books selected at random, two of them were written by American ~~Auth~~ Authors and three of them were written by foreign authors is given by:

$$p = \frac{\binom{6}{2}\binom{4}{3}}{\binom{10}{5}} = \frac{15 \times 4}{252} = 0.2381$$

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Poisson Probability Law

Poisson Distribution :-

(I) Limiting form Binomial :-

(I) As process satisfying some condition

(I) Limiting form of Binomial :-

(This is invented by French mathematician S. Poisson)

Now consider ~~the~~ a binomial probability law where the number of trial 'n' is very large and probability of success p is very small but $np = \lambda$ (Greek letter lambda), a finite quantity.

- This situation can be called a situation of rare ~~ev~~ events, the number of trials is very large and probability of success ~~is~~ in each trial is very small, something like lightning strike, earth quake at a particular place, traffic accidents on certain stretch of high way and so on.

- Let us see what happens if $n \to \infty, p \to 0, np \to \lambda$. Since we are assuming λ to be a finite quantity, we may replace one of n or p in terms of λ.

- let us substitute $p = \frac{\lambda}{n}$. Then

$$\lim_{n \to \infty} \binom{n}{x} p^x = \frac{1}{x!} \lim_{n \to \infty} n(n-1)\cdots(n-x+1)\left(\frac{\lambda}{n}\right)^x$$

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