Heap Data Structure

Heapsort, Priority Queue

Definition

- A heap can be viewed as a nearly complete binary tree.
- Each node in the heap satisfies the *heap property*
- A heap could be of two types: *min-heap and max-heap*
- For a max-heap, the heap property states that the value at each node is at most as large as its parent
- For min-heap, the heap property is satisfied if the value at each node is at least as large as its parent
- Heaps are used for sorting and for implementing priority queue

Types of Binary Trees

- Full Binary tree: each node is either of degree 0 or 2. There is no deg-1 node
- Complete Binary Tree: all leaves are at the same depth and all internal nodes are of degree 2
- Nearly Complete Binary tree: A full binary tree where each node is either of degree 0 or 2 accept possibly at the last-but-one level where the nodes might have deg-1 such that the positional information is preserved
- i.e. if the nodes are numbered from left to right then the numbering does not change in a full binary tree and a complete binary tree

Array elements as Heap

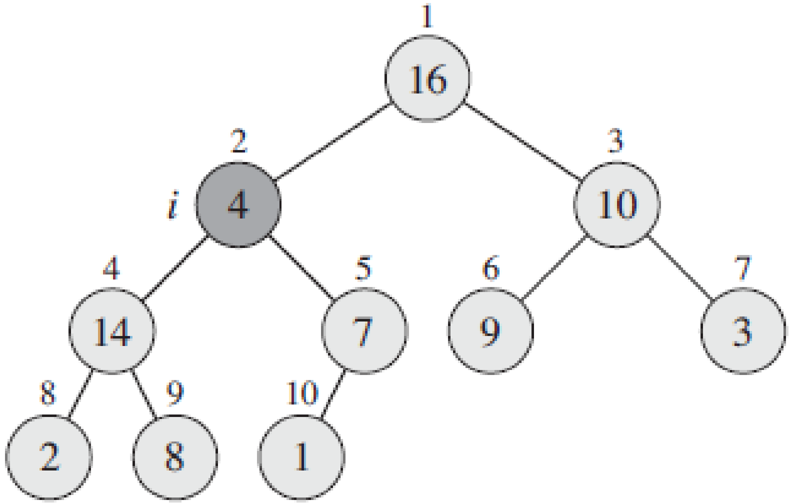
- Array elements can be represented in the form of a heap
- The array elements can be viewed as forming a nearly complete binary tree.
- For a given array element indexed at i we can compute the index of its parents and children as:
  - Parent(i) = $\lfloor i/2 \rfloor$
  - Left_child(i) = $2i$
  - Right_child(i) = $2i + 1$
- For an array A of size A.length, the size of the heap is defined as A.heapsize and we have $1 \leq A.heapsize \leq A.length$

Heap Procedures

- *Max-Heapify*: runs in O(lgn) time, used to maintain the heap property
- *Build-Max-Heap*: runs in O(n) time, produces a max-heap from an unordered array
- *Heapsort*: runs in O(nlgn) time, sorts an array *in place*
- *Max-Heap-Insert, Heap-Extract-Max, Heap-Increase-Key, Heap-Maximum* procedures, run in O(lgn) time, allow priority queue to be implemented using heap data structure

Maintaining the Heap property

```
MAX-HEAPIFY(A, i)
l = LEFT(i)
r = RIGHT(i)
if l ≤ A.heap-size and A[l] > A[i]
        largest = l
else largest = i
if r ≤ A.heap-size and A[r] > A[largest]
        largest = r
if largest ≠ i
        exchange A[i] with A[largest]
        MAX-HEAPIFY(A, largest)
```


1
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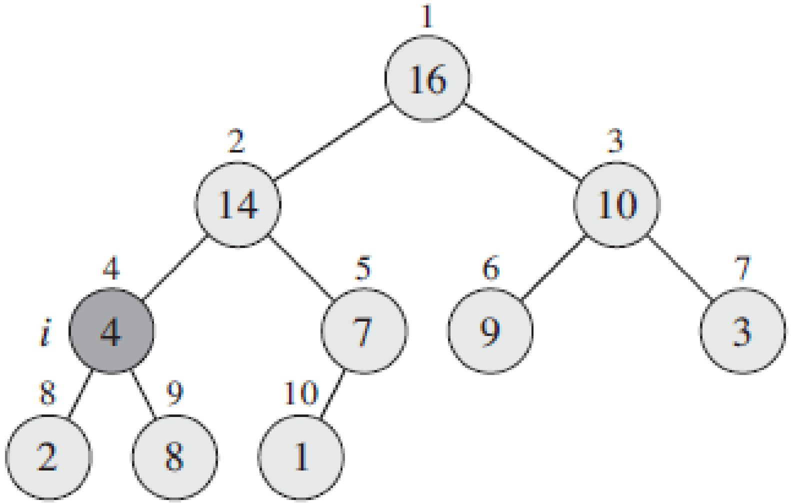
(a)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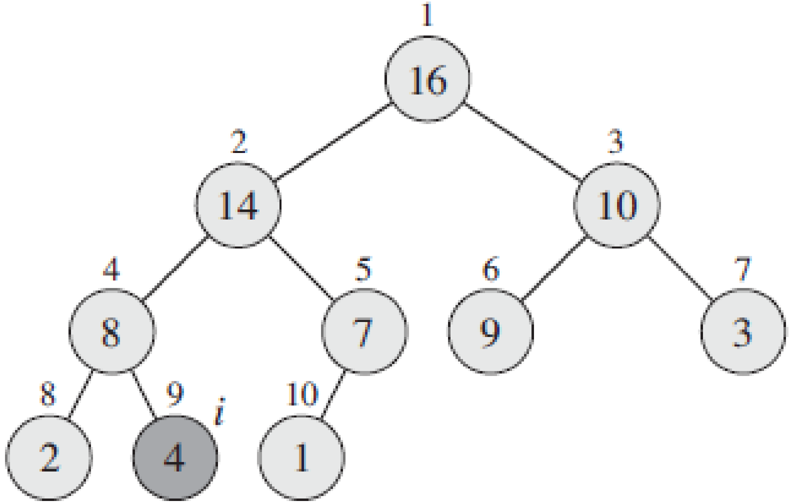
(b)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(c)

Complexity Analysis

- Steps 1 to 9 take constant time
- Step 10 is a recursive call to Max-Heapify
- In order to write the recurrence, we need to find the maximum size of a sub-tree
- Suppose the heap has n nodes in total. Since, heap is a nearly complete binary tree, the left and right-subtrees can have at most a difference of 1 in their heights

- Thus, we get the following expression:

$$1 + 2^{h+1} - 1 + 2^{h+2} - 1 = n$$

$$\Rightarrow 2^{h+1} \quad (2 + 1) - 1 = n$$

$$\Rightarrow 2^{h+1} = \frac{n+1}{3} \approx \frac{n}{3}$$

$$\Rightarrow 2^{h+2} \approx \frac{2n}{3}$$

- Thus, the maximum height of the subtree could be 2n/3

- We can write the recurrence as below:

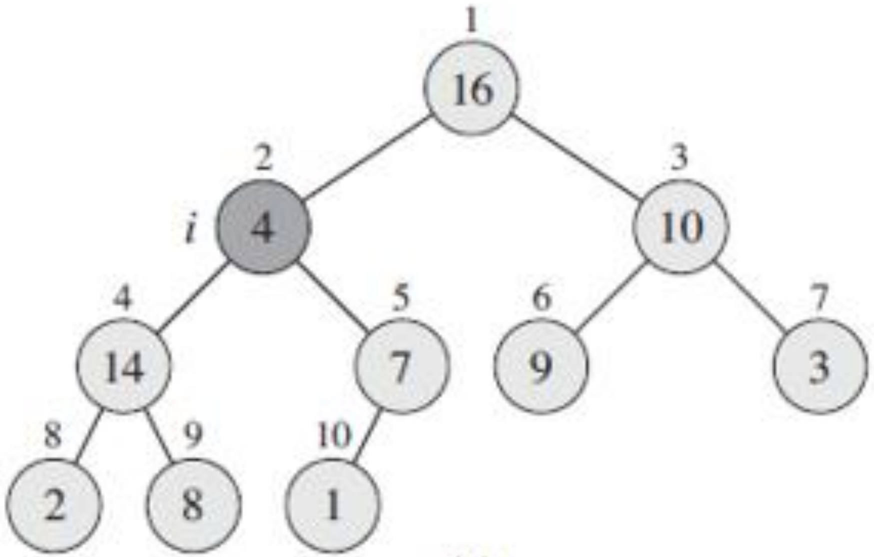
$$T(n) = T\left(\frac{2n}{3}\right) + \theta(1)$$

Using the case 2 of Master method, the solution to the above recurrence is $T(n) = O(lgn)$

Thus, the Heapify method runs in O(lgn) time

Exercise

- Show that, with the array representation for storing an n-element heap, the leaves are the nodes indexed by $\lfloor n/2 \rfloor + 1, \lfloor n/2 \rfloor + 2, \ldots n$



(a)

Exercise

- What is the effect of calling MAX-HEAPIFY (A,i) for i > A.*heap-size/*2?

Exercise

1. Show that the worst-case running time of MAX-HEAPIFY on a heap of size n is $\Omega(\lg n)$

- Worst case occurs when the Max-Heapify is called along the longest path from the root to the leaf
- In this case the running time will be bounded by the height of the heap i.e. $\Omega(\lg n)$

Exercise

- Show that there are at most $\lceil n/2^{h+1} \rceil$ nodes of height h in any n-element heap.

Proof: Maximum nodes with height h

- All the nodes from $\lfloor n/2 \rfloor + 1, \ldots n$ are leaf nodes
- Therefore, no. of nodes with height = 0 => $\lceil n/2 \rceil$
- No. of nodes with height = 1 will be half of this value => $\lceil n/2 \rceil \times \frac{1}{2}$
- No. of nodes with height = 2 will be half of this value => $\lceil n/2 \rceil \times \frac{1}{2^2}$

- Maximum no. of nodes with height h $= \left\lceil \frac{n}{2^{h+1}} \right\rceil$

Building a Heap

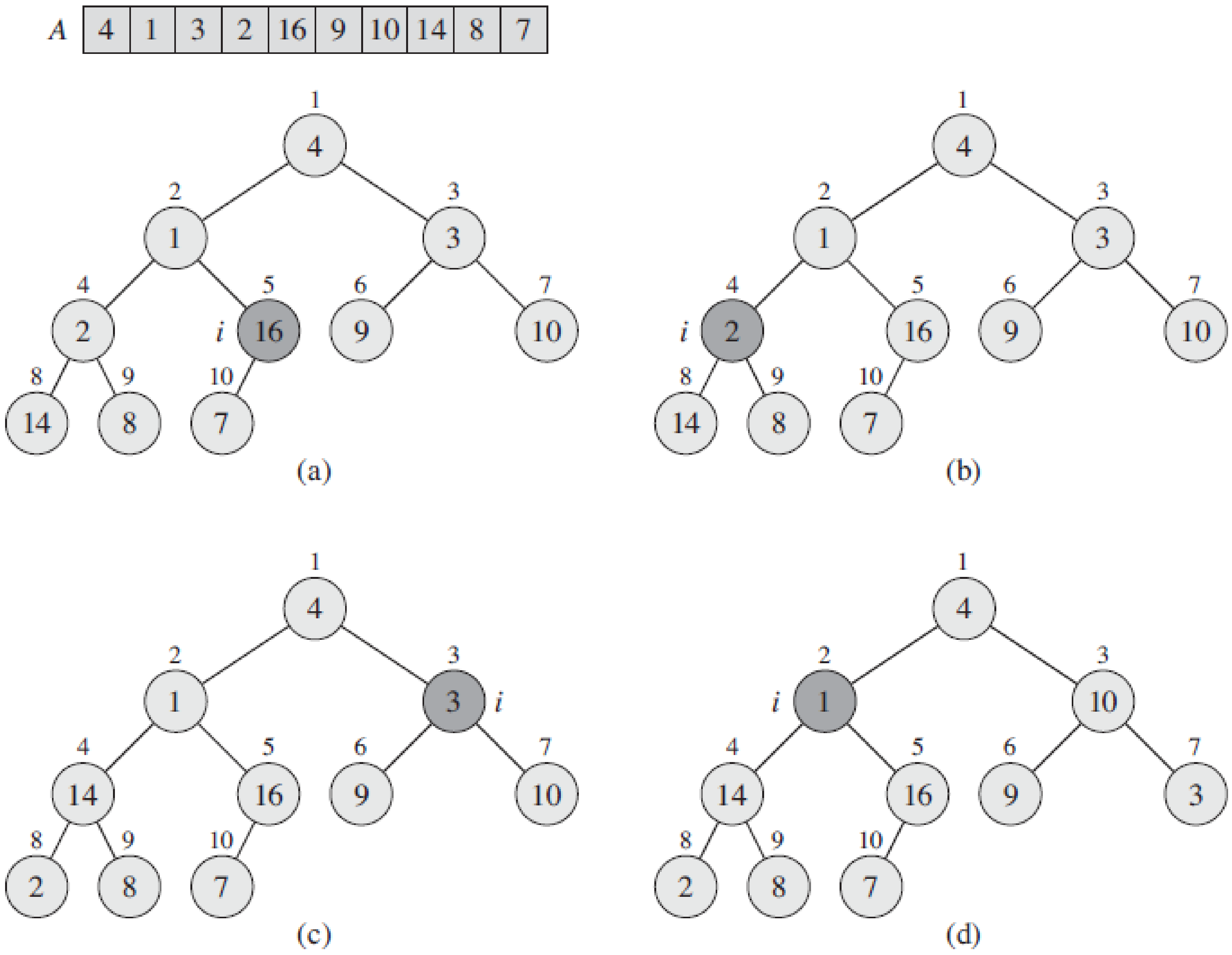
- Elements of an array can be built-up into a max-heap
- We use the procedure Build-max-heap for this
- All elements of the array from $\lfloor n/2 \rfloor + 1 \; upto \; n$ will comprise the leaf nodes and therefore each is a heap in itself
- In our algorithm, we will add higher level node one by one and use the heapify procedure to maintain the heap property as we add elements

Build-max-Heap Procedure

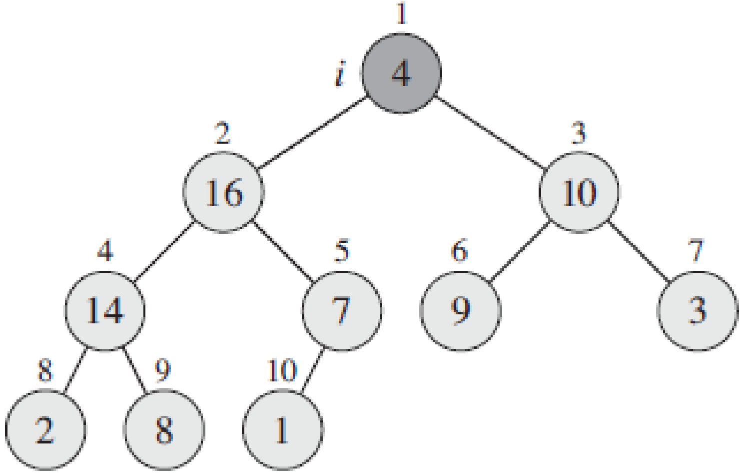
```
BUILD-MAX-HEAP(A)
A.heap-size = A.length
for i = ⌊A.length/2⌋ downto 1
    MAX-HEAPIFY(A, i)
```

Proof of Correctness

- Find the loop invariant
- Show that it is true prior to execution of the loop
- Show that it holds after execution of the loop
- Show that the loop invariant provides a useful property to show the correctness of the algorithm when loop terminates
- Loop invariant => all the elements from the initial value down to the highest value of i form a max-heap



A
4 1 3 2 16 9 10 14 8 7
i
(a)
(b)
(c)
(d)



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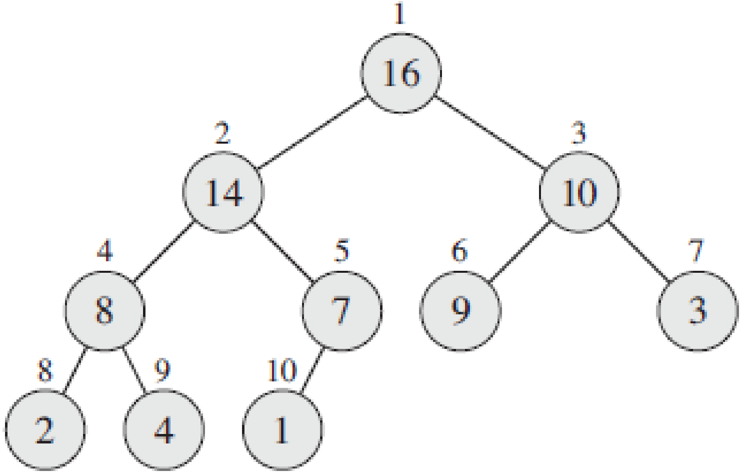


(f)

Complexity Analysis

- Each call to heapify takes at most O(lgn) time
- The procedure heapify is called for at most the total number of nodes in the heap i.e. n
- Therefore the upper bound on the running time of the algorithm is O(nlgn)
- This is not however asymptotically tight
- A tighter bound can be obtained by noting that an n-element heap has:
  - Maximum height = $\lfloor \lg n \rfloor$
  - Maximum number of nodes with height h = $\left\lceil \frac{n}{2^{h+1}} \right\rceil$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h) = O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}\right).$$

We evalaute the last summation by substituting $x = 1/2$ in the formula yielding

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

for $|x| < 1$

$$\sum_{h=0}^{\infty} \frac{h}{2^h} = \frac{1/2}{(1-1/2)^2} = 2.$$

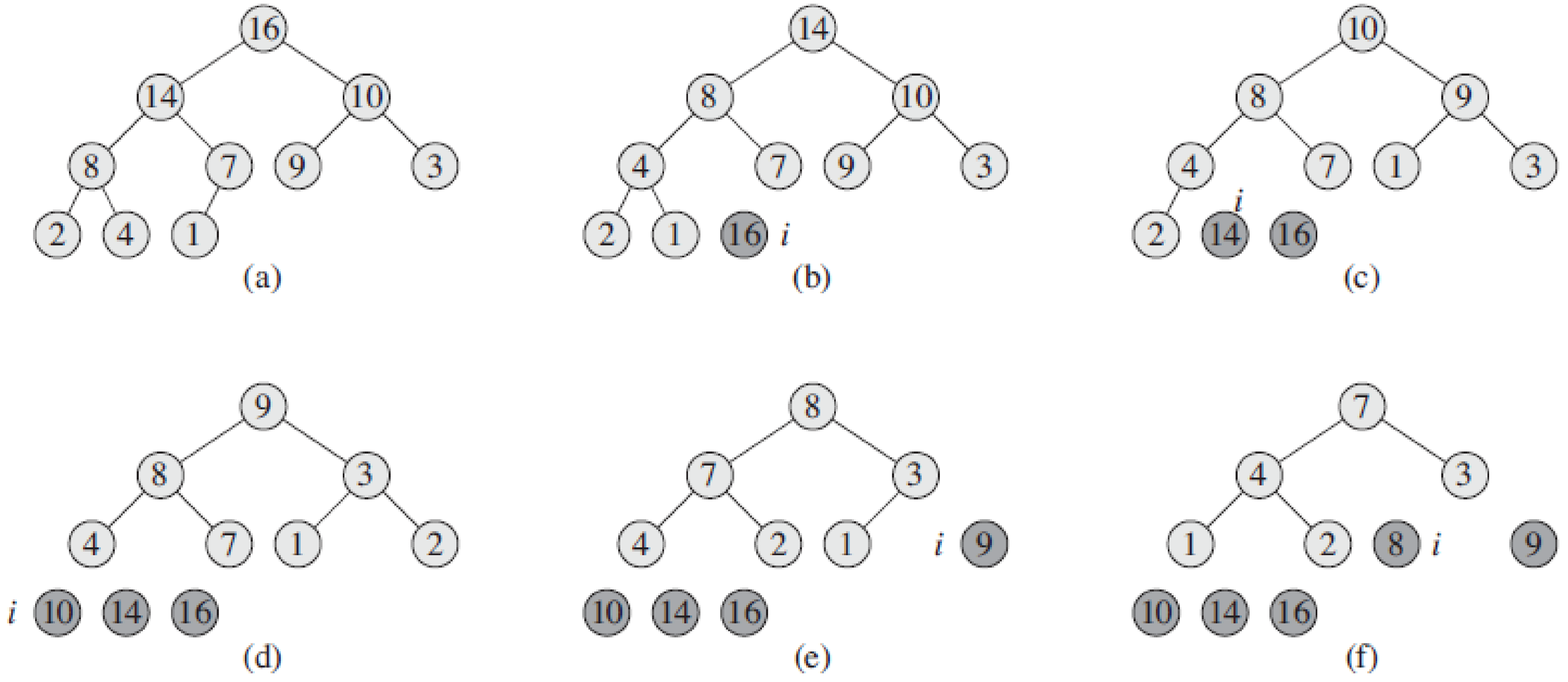
Thus, we can bound the running time of BUILD-MAX-HEAP as

$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}\right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h}\right) = O(n).$$

Heapsort

```
HEAPSORT(A)
BUILD-MAX-HEAP(A)
for i = A.length downto 2
    exchange A[1] with A[i]
    A.heap-size = A.heap-size − 1
    MAX-HEAPIFY(A, 1)
```

- Largest element is at the top and therefore can be placed at the end of the array
- This is done by exchanging the value of 1st element with the last element of the heap
- Since one element of the array is at its correct sorted position, we remove it from the heap by reducing the heap-size by 1
- After exchange, the root of the heap might not follow the heap property so we call Heapify with the root (1)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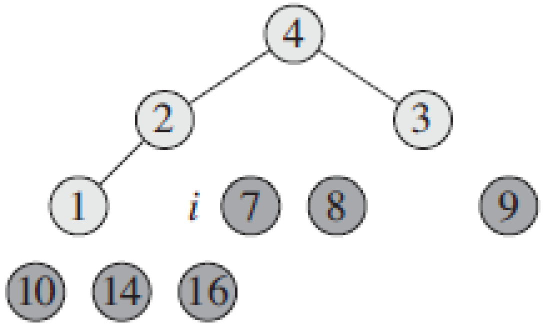

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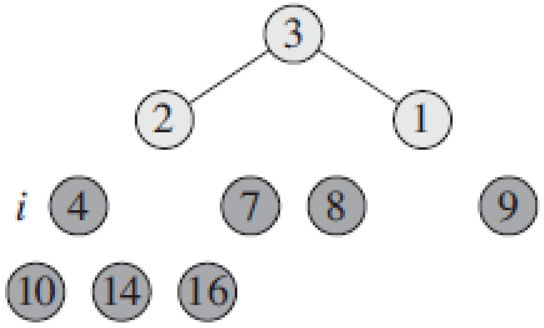

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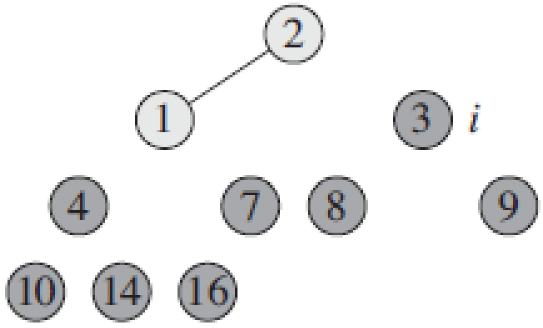

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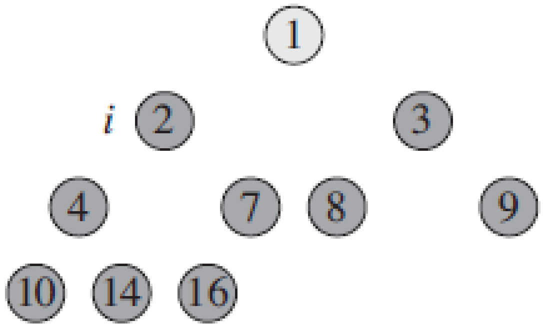


(j)

A	1	2	3	4	7	8	9	10	14	16

(k)

Complexity Analysis

- Build-heap takes O(n) time
- Each call to Heapify takes O(lgn) time
- The total running time is thus O(nlgn)

Priority Queue

- A data structure where the elements are accessed in the order of their priority
- For a set of elements S, each element is associated with a *key* value
- Priority queue is of two types: max-priority queue and min-priority queue
- Operations (Max-priority queue):
  - Insert(S,x)
  - Maximum(S)
  - Extract-Max(S)
  - Increase-Key(S,x,k)

Implementation of Priority Queue

- The priority queue designed for a particular application stores only the key values
- Each element of the priority queue is associated with an object of the application for which priority queue is designed
- Key values are nothing but the priorities of the associated objects
- All the manipulation is done using the key values only
- A handle is maintained that associates the key value with the corresponding object
- The program object stores the index value of the key and the key stores the handle of the program object (could be a pointer to the object)
- Each time a key is moved from its position, the handles are updated so as to maintain the correct index value

Heap-Maximum

```
HEAP-MAXIMUM(A)
    return A[1]
```

Heap-Extract-Max(A)

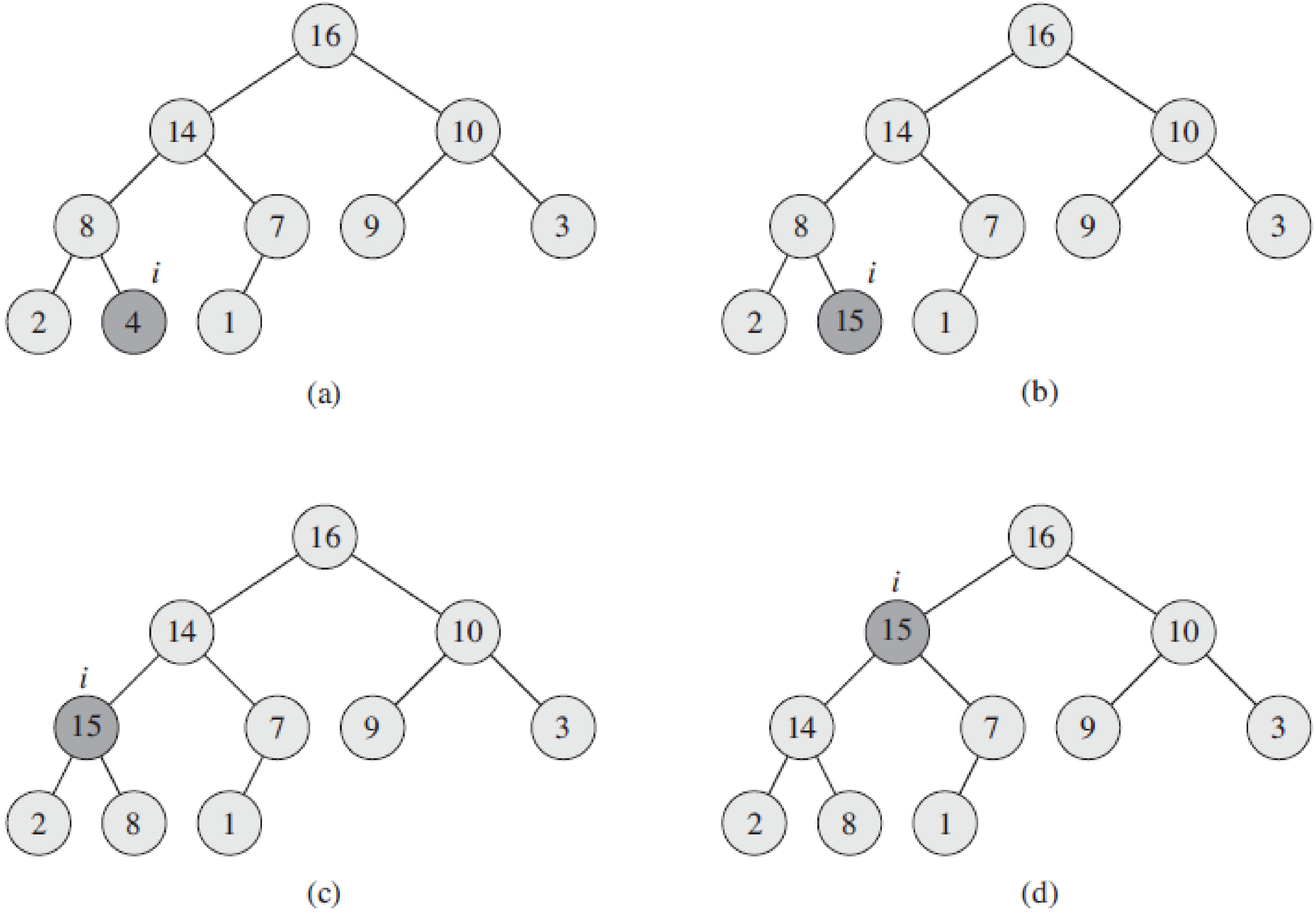
```
HEAP-EXTRACT-MAX(A)
if A.heap-size < 1
    error "heap underflow"
max = A[1]
A[1] = A[A.heap-size]
A.heap-size = A.heap-size - 1
MAX-HEAPIFY(A, 1)
return max
```

- Performs only constant amount of operations other than the call to Max-Heapify
- Running time is same as that of Max-Heapify i.e. O(lgn)

Heap-Increase-Key(A, i, key)

```
HEAP-INCREASE-KEY(A, i, key)
if key < A[i]
    error "new key is smaller than current key"
A[i] = key
while i > 1 and A[PARENT(i)] < A[i]
    exchange A[i] with A[PARENT(i)]
    i = PARENT(i)
```

- The while loop runs at most up to the complete height of the heap
- Running time of the algorithm is O(lgn)


(a)
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Max-Heap-Insert(A, key)

MAX-HEAP-INSERT(A, key)

1 $A.heap\text{-}size = A.heap\text{-}size + 1$
2 $A[A.heap\text{-}size] = -\infty$
3 HEAP-INCREASE-KEY$(A, A.heap\text{-}size, key)$

- Steps 1 and 2 add constant time
- Step 3 runs in O(lgn) time
- Therefore, runtime is bounded by O(lgn)