Divide and Conquer: Solving Recurrences in DAA

Master Theorem, Substitution Method, Recursion Tree

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Introduction

- **What is a Recurrence Relation?**
  A recurrence relation expresses the running time of a recursive algorithm.
- **Why Solve Recurrences?**
  To analyze the time complexity of recursive algorithms like Merge Sort, Quick Sort, and Binary Search.
- Agenda
  - Understand three key techniques for solving recurrences:
    - **Master Theorem**
    - **Substitution Method**
    - **Recursion Tree**

1. Substitution Method

- The Substitution Method solves a recurrence by guessing the solution and using mathematical induction to prove it.
- T(n) = { T(n/2) + c, if n>;

  { 1, if n = 1

  - Pros: Precise control
  - Cons: Guessing the bound can be tricky

2. Master Theorem

- The Master Theorem provides a shortcut to solve recurrences of the form T(n) = aT(n/b) + f(n) for divide-and-conquer algorithms.
- $a \geq 1, b > 1$
- Solution: $T(n) = n^{\log_b a} \cdot U(n)$

Where, U(n) depends on $h(n) = \frac{f(n)}{n^{\log_b a}}$

if $h(n)$	
$n^r, r > 0$	$O(n^r)$
$n, r < 0$	$O(1)$
$(\log n)^i, i >= 0$	$\frac{(\log_2 n)^{i+1}}{i+1}$

- Pros: Fast and Direction
- Cons: Doesn’t work for all recurrences

$T(n) = T(n/2) + c$

- $T(n) = T(n/2) + c$; $a = 1$, $b = 2$, $f(n) = C$
- $T(n) = n^{logba}\ U(n)$
- $T(n) = n^{log2\ 1}\ U(n) = n^0\ U(n) = U(n)$
- $U(n)$ depends on $h(n)$ $\quad h(n) = \dfrac{f(n)}{n^{\log_b a}} = C = (\log n)^0 \,.\, C$

if	$h(n)$	$U(n)$
	n^r, $r > 0$	$O(n^r)$
	n, $r < 0$	$O(1)$
	$(\log n)^i$, $i >= 0$	$\dfrac{(\log_2 n)^{i+1}}{i+1}$

answer: $O(\log_2 n)$

$T(n) = 8T(n/2) + n^2$

- Class Activity
- Answer: $O(n^3)$

3. Recursion Tree

- The Recursion Tree Method breaks down a recurrence into levels to visually sum the total work across recursive calls.
- Good for understanding how costs accumulate
- Helps form guesses for substitution
  - Pros: intuitive, helps with complex recursions
  - Cons: Can get messy for large recurrences

Thank You