Span and independence

We're looking at bases of vector spaces. Recall that a basis β of a vector space V is a set of vectors $\beta = (\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n)$ such that each vector $\mathbf{v}$ in V can be uniquely represented as a linear combination of vectors from β

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n.$$

How can a set S of vectors fail to be a basis for a vector space V? There are two ways. It might be that some vectors aren't linear combinations of S, that is, there aren't enough vectors to span all of V. It might be that some vectors can be expressed as linear combinations of S but in more than one way, that is, there are too many vectors in S. We'll study these two phenomena next.

The span of a set of vectors. Since vector spaces are closed under linear combinations, we should have a name for the set of all linear combinations of a given set of vectors, and that will be their *span*.

Definition 1. Let S be a set of vectors in a vector space V. The *span* of S, written $\operatorname{span}(S)$, is the set of all linear combinations of vectors in S. That is, $\operatorname{span}(S)$ consists of all vectors of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_k$$

where each c_i is a scalar and each $\mathbf{v}_i$ is a vector in S.

The proof of the following theorem is left for you to prove. It depends on showing that a linear combination of linear combinations is a linear combination.

Theorem 2. The span of a set S is a subspace of V.

You can also describe $\operatorname{span}(S)$ as the smallest subspace of V that contains all of S.

Theorem 3. The span of a set S is the intersection of all subspaces of V that contain S.

$$\operatorname{span}(S) = \bigcap\{W, \text{ a subspace of V} \mid S \subseteq W\}$$

Proof. First note that $\operatorname{span}(S)$ *is* a vector space that contains all of S, so it's one of spaces W in the intersection. Second, $\operatorname{span}(S)$ only has linear combinations of vectors in S, so every vector in $\operatorname{span}(S)$ has to be in every vector space W that contains all of S. Therefore $\operatorname{span}(S)$ is a subset of all the spaces W in the intersection, so it's the smallest one, and, therefore, equals the intersection of all of them. Q.E.D.

Some examples. A single nontrivial vector in $\mathbf{R}^n$ spans the line through the origin that contains it. Two vectors in $\mathbf{R}^3$ that don't both lie in the same line span a plane. The functions $\sin t$ and $\cos t$ span the solution space of the differential equation $y'' = -y$.

Definition 4. We say a set S of vectors in a vector space V spans V if $V = \operatorname{span}(S)$. Equivalently, every vector in V is a linear combination of vectors in S.

Note that this definition does not require that a vector can be a linear combination in only one way, just that there is at least one way. That's how this definition differs from the definition for basis of a vector space.

An example. The set $S = \{(1,3),(2,2),(3,1)\}$ spans the vector space $\mathbf{R}^2$, but it's not a basis of it. (Why not?)

The linear combination problem in Matlab. Consider the question whether a particular vector $\mathbf{v}$ is a linear combination of given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. This is the same question as: Is $\mathbf{v}$ in the span of the given vectors?

This question can be solved in MATLAB. After all, you're just looking to solve the vector equation

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_k$$

for the unknowns $c_1, c_2, \ldots, c_n$, and that's just a system of linear equations.

Here we'll determine if the vector $\mathbf{v} = (1,4,2,6)$ is a linear combination of the vectors $\mathbf{x} = (3,-1,1,0)$, $\mathbf{y} = (1,1,1,1)$, and $\mathbf{z} = (1,2,-1,3)$. Note that the system will have 4 equations (one for each coordinate) in three unknowns (being c_1, c_2, and c_3), so we don't expect it to have a solution. Treat all the vectors as column vectors, place the vectors as columns in an augmented matrix, and row reduce it using the function `rref`. (In fact, I'll enter them as rows, then transpose.)

```
>> aug=[3 -1 1 0;1 1 1 1;1 2 -1 3;1 4 2 6]'
aug =
     3     1     1     1
    -1     1     2     4
     1     1    -1     2
     0     1     3     6

>> rref(aug)
ans =
     1     0     0     0
     0     1     0     0
     0     0     1     0
     0     0     0     1
```

Thus the four equations are inconsistent since the last equation says $0 = 1$. Thus, $\mathbf{v}$ is not a linear combination of the others.

Linear independence. The question of spanning a vector space asks if you have enough vectors in a set S to get all other vectors in a space as a linear combination of the vectors in S. The question of independence asks if you have too many, that is, can you do without some of them because they're redundant.

Definition 5. A set

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

of vectors in a vector space V is said to be *linearly dependent* if there are scalars $c_1, c_2, \ldots, c_k$ not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}.$$

You can read this as saying that at least one of the vectors is a linear combination of the rest, for if $c_i \neq 0$, then $\mathbf{v}_i$ is a linear combination of the rest.

If the vectors aren't linearly dependent, then we say they're *linearly independent*. In other words, no vector in S is a linear combination of the others.

A logically equivalent statement is that S is linearly independent if the only way a linear combination of vectors in S can equal 0,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

is when each of the scalars $c_1, c_2, \ldots, c_k$ are all 0. In other words, $\mathbf{0}$ is not a nontrivial linear combination of the vectors in S.

How do you know whether the vectors in S are linearly dependent or independent? Just solve the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ for $c_1, c_2, \ldots, c_k$. This single vector equation is a system of homogeneous linear equations. If you only get the trivial solution, then the vectors in S are linearly independent. If you get any other solution, then they're dependent.

For $\mathbf{R}^n$, the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are linearly independent. You can see that each of them, $\mathbf{e}_i$, is the only one of them with a nonzero i^{th} coordinate, therefore it is not a linear combination of the rest. So they're all independent.

In general, two vectors $\mathbf{v}$ and $\mathbf{w}$ are linearly independent if and only if each is not a multiple of the other. Geometrically that means they do not lie on the same line through the origin $\mathbf{0}$.

Testing for linear independence using MATLAB. In order to tell if vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are independent, check to see if the homogeneous system $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$ has any nontrivial solutions. We can do that in MATLAB with the `rref` function.

For example, let's see see if the vectors $(1,3,5,7)$, $(2,0,1,3)$, $(-1,2,-1,0)$, and $(4,3,1,-5)$ are independent. Place them in four columns of a coefficient matrix, and row reduce the matrix.

```
>>A=[1 3 5 7;2 0 1 3; -1 2 -1 0;-5 17 9 15]'
A =
     1     2    -1    -5
     3     0     2    17
     5     1    -1     9
     7     3     0    15

>> rref(A)
ans =
     1     0     0     3
     0     1     0    -2
     0     0     1     4
     0     0     0     0
```

There are nontrivial solutions. The unknown c_4 can be chosen freely, and the general solution is $(c_1, c_2, c_3, c_4) = (-3c_4, 2c_4, -4c_4, c_4)$. Thus they are not independent. Taking $c_4 = -1$, we can write $\mathbf{0}$ as a nontrivial combination of the four vectors by

$$3\mathbf{v}_1 - 2\mathbf{v}_2 + 4\mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}.$$

A basis is a linearly independent spanning set. We defined basis first, then looked two aspects of a basis, that of span and that of independence. Now, we'll combine the two concepts together jointly mean basis.

Theorem 6. A subset S of a vector space V is a basis if and only if (1) S spans V, and (2) S is linearly independent.

Proof. Part I. Suppose S is a basis by the definition. Then every vector is a linear combination, so S spans V. Also, the vector $\mathbf{0}$ is uniquely a linear combination of elements of S, so S is linearly independent.

Part II. Suppose that S spans V and it's linearly independent. Since it spans V, every vector can be represented as some linear combination of elements of S. We have yet to show there's only one such linear combination. Suppose that a vector $\mathbf{v}$ can be represented in two ways:

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k \\ \mathbf{v} &= d_1\mathbf{v}_1 + d_2\mathbf{v}_2 + \cdots + d_k\mathbf{v}_k. \end{aligned}$$

Subtracting the second equation from the first yields the equation

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + (c_2 - d_2)\mathbf{v}_2 + \cdots + (c_k - d_k)\mathbf{v}_k.$$

But S is linearly independent, so $\mathbf{0}$ is only a trivial linear combination of the basis vectors, that is, $c_i - d_i = 0$ for each index i. Therefore, each $c_i = d_i$. Hence the two representations were the same. Q.E.D.

Theorem 7. Given a finite set of vectors spans a vector space, then it has a subset which is a basis for that vector space.

Proof. Let the finite set S span the vector space V. There are a couple of ways that you can find an independent subset of S that spans V.

One way is to throw out redundant vectors in S. If S is already independent, you're done. If not, one of the vectors $\mathbf{v}$ depends on the rest. Then $S' = S - \{\mathbf{v}\}$ also spans V since, as $\mathbf{v}$ is a linear combination of S', and every vector is a linear combination of $\mathbf{v}$ and the others, therefore every vector is a linear combination of just the others. Continue throwing out vectors until you're left with an independent subset that still spans V. Since there were only a finite number of vectors in S to begin with, the process will terminate.

The other way is to build up a basis. Go through the vectors in S one at a time. If the next one is dependent on the previous, then don't include it, otherwise do. When you're all done, you've got an independent subset S' of S, and every vector in S is dependent on it. Since S spanned V, so does S'. Q.E.D.

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