Chapter 5

DOUBLE AND TRIPLE INTEGRALS

5.1 Multiple-Integral Notation

Previously ordinary integrals of the form

$$\int_J f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \tag{5.1}$$

where $J = [a, b]$ is an interval on the real line, have been studied. Here we study *double integrals*

$$\int\int_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{5.2}$$

where Ω is some region in the xy-plane, and a little later we will study *triple integrals*

$$\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{5.3}$$

where T is a solid (volume) in the xyz-space.

5.2 Double Integrals

5.2.1 Properties

(1) Area property

$$\int\int_\Omega \mathrm{d}x\,\mathrm{d}y = \text{Area of } \Omega.$$

In particular if Ω is the rectangle $\Omega = [a, b] \times [c, d]$ then $\int\int_\Omega \mathrm{d}x\,\mathrm{d}y = (b-a)(d-c)$.

(2) Linearity

$$\int\int_\Omega [\alpha f(x, y) + \beta g(x, y)]\,\mathrm{d}x\,\mathrm{d}y = \alpha\int\int_\Omega f(x, y)\,\mathrm{d}x\,\mathrm{d}y + \beta\int\int_\Omega g(x, y)\,\mathrm{d}x\,\mathrm{d}y \tag{5.4}$$

where α and β are constants.

(3) Additivity

If Ω is broken up into a finite number of nonoverlapping basic regions $\Omega_1,\ldots,\Omega_n$, then

$$\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_{\Omega_1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \ldots + \int\int_{\Omega_n} f(x,y)\,\mathrm{d}x\,\mathrm{d}y. \tag{5.5}$$

5.2.2 Geometric Interpretation

The double integral over Ω gives the volume of the solid T whose upper boundary is the surface $z = f(x,y)$ and whose lower boundary is the region Ω in the xy-plane:

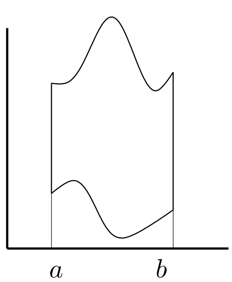
$$\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \text{volume of } T. \tag{5.6}$$

5.2.3 The Evaluation of Double Integrals by Repeated Integrals

If an ordinary integral $\int_a^b f(x)\,\mathrm{d}x$ proves difficult to evaluate, it is not because of the interval $[a,b]$ but because of the integrand f. Difficulty in evaluating a double integral $\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$ can come from two sources: from the integrand f or from the domain Ω. Even such a simple looking integral as $\int\int_{\Omega} 1\,\mathrm{d}x\,\mathrm{d}y$ is difficult to evaluate if Ω is complicated.

In this section we introduce a technique for evaluating double integrals over domains that have special shapes. The key idea is that double integrals over such special domains can be reduced to a pair of ordinary integrals.

Horizontally simple domain



The *projection* of the domain Ω onto the x-axis is a closed interval $[a,b]$ and Ω consists of all points (x,y) with

$$a \le x \le b, \text{ and } \phi_1(x) \le \ y \le \phi_2(x). \tag{5.7}$$

Then

$$\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x. \tag{5.8}$$

Here we first calculate $\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y$ by integrating $f(x,y)$ with respect to y from $y = \phi_1(x)$ to $y = \phi_2(x)$. The resulting expression is a function of x alone, which we then integrate with respect to x from $x = a$ to $x = b$.

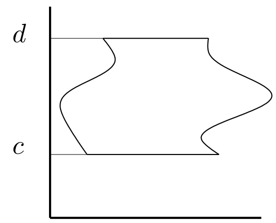
Example

Evaluate $\int\int_{\Omega}(x^4 - 2y)\,\mathrm{d}x\,\mathrm{d}y$, where the domain Ω consists of all points (x,y) with $-1 \le x \le 1$ and $-x^2 \le y \le x^2$.

Solution

$$\begin{aligned}\int\int_{\Omega}(x^4 - 2y)\,\mathrm{d}x\,\mathrm{d}y &= \int_{x=-1}^{x=1}\int_{y=-x^2}^{y=x^2}(x^4 - 2y)\,\mathrm{d}y\,\mathrm{d}x = \int_{x=-1}^{x=1}[x^4y - y^2]_{y=-x^2}^{y=x^2}\,\mathrm{d}x \\ &= \int_{x=-1}^{x=1} 2x^6\,\mathrm{d}x = [2x^7/7]_{x=-1}^{x=1} = 4/7\end{aligned}$$

Vertically simple domain



The *projection* of the domain Ω onto the y-axis is a closed interval $[c, d]$ and Ω consists of all points (x, y) with

$$c \leq y \leq d, \text{ and } \psi_1(y) \leq x \leq \psi_2(y). \tag{5.9}$$

Then

$$\int\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y. \tag{5.10}$$

Here we first calculate $\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$ by integrating $f(x,y)$ with respect to x from $x = \psi_1(y)$ to $x = \psi_2(y)$. The resulting expression is a function of y alone, which we then integrate with respect to y from $y = c$ to $y = d$.

The integrals in the right-hand sides of formulae (5.8) and (5.10) are called *repeated integrals*.

Remark 1

Sometimes a domain can be expressed both as a horizontally simple domain: $a \leq x \leq b$, $\phi_1(x) \leq y \leq \phi_2(x)$, and as a vertically simple domain: $c \leq y \leq d$, $\psi_1(y) \leq x \leq \psi_2(y)$. Then

$$\int\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \left(\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,\mathrm{d}y \right) \mathrm{d}x = \int_c^d \left(\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y. \tag{5.11}$$

Therefore we can, at least in theory, perform the integration in either order. However, there are situations where one order is preferable over the other.

Remark 2

Finally, if Ω, the domain of integration, is neither horizontally nor vertically simple, then it is usually possible to break it up into a finite number of domains, say $\Omega_1, \ldots, \Omega_n$, each of which is either horizontally or vertically simple. Then we can use the additivity property given by eq. (5.5).

5.2.4 Evaluating Double Integrals Using Polar Coordinates

Let Ω be a domain formed with all points (x, y) that have polar coordinates (r, θ) in the set

$$\Gamma : \alpha \leq \theta \leq \beta,\ \rho_1(\theta) \leq r \leq \rho_2(\theta) \tag{5.12}$$

where $\beta \leq \alpha + 2\pi$. Then

$$\begin{aligned} \int\int_\Omega f(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \int\int_\Gamma f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_\alpha^\beta \int_{\rho_1(\theta)}^{\rho_2(\theta)} f(r\cos(\theta), r\sin(\theta))r\,\mathrm{d}r\,\mathrm{d}\theta. \end{aligned} \tag{5.13}$$

5.3 Triple Integrals

5.3.1 Properties

(1) Volume property

$$\int\int\int_T \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \text{Volume of } T.$$

In particular if T is the box $T = [a,b]\times[c,d]\times[e,f]$ then $\int\int\int_T \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = (b-a)(d-c)(f-e)$.

(2) Linearity

$$\int\int\int_T [\alpha f(x,y,z) + \beta g(x,y,z)]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \alpha \int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \beta \int\int\int_T g(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad (5.14)$$

where α and β are constants.

(3) Additivity If T is broken up into a finite number of nonoverlapping basic regions $T_1,\ldots,T_n$, then

$$\int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int\int_{T_1} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \ldots + \int\int\int_{T_n} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \quad (5.15)$$

5.3.2 The Evaluation of Triple Integrals by Repeated Integrals

Let T be a solid whose projection onto the xy-plane is labelled Ω_{xy}. Then the solid T is the set of all points (x,y,z) satisfying

$$(x,y)\in\Omega_{xy}, \chi_1(x,y) \le z \le \chi_2(x,y). \quad (5.16)$$

The triple integral over T can be evaluated by setting

$$\int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int_{\Omega_{xy}} \left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y. \quad (5.17)$$

In eq. (5.17) we can evaluate the integration with respect to z first and then evaluate the double integral over the domain Ω_{xy} as studied for double integrals. In particular if Ω_{xy} is horizontally simple, say

$$a \le x \le b, \quad \phi_1(x) \le \ y \le \phi_2(x). \quad (5.18)$$

then the solid T itself is the set of all points (x,y,z) such that

$$a \le x \le b, \quad \phi_1(x) \le \ y \le \phi_2(x), \quad \chi_1(x,y) \le z \le \chi_2(x,y) \quad (5.19)$$

and the triple integral over T can be expressed by three ordinary integrals as:

$$\int\int\int_T f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} \left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x,y,z)\,\mathrm{d}z\right)\mathrm{d}y\right]\mathrm{d}x. \quad (5.20)$$

Here we first integrate with z [from $z=\chi_1(x,y)$ to $z=\chi_2(x,y)$], then with respect to y [from $y=\phi_1(x)$ to $y=\phi_2(x)$], and finally with respect to x [from $x=a$ to $x=b$].

There is nothing special about this order of integration. Other orders of integration are possible and in some cases more convenient. Suppose for example that the projection of T onto the xz-plane is a domain Ω_{xz} of the form

$$z_1 \leq z \leq z_2, \quad \phi_1(z) \leq x \leq \phi_2(z). \tag{5.21}$$

If T is the set of all (x, y, z) with

$$z_1 \leq z \leq z_2, \quad \phi_1(z) \leq \ x \leq \phi_2(z), \quad \psi_1(x, z) \leq y \leq \psi_2(x, z) \tag{5.22}$$

then

$$\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{z_1}^{z_2} \left[\int_{\phi_1(z)}^{\phi_2(z)} \left(\int_{\psi_1(x,z)}^{\psi_2(x,z)} f(x, y, z)\,\mathrm{d}y \right) \mathrm{d}x \right] \mathrm{d}z. \tag{5.23}$$

In this case we integrate first with respect to y, then with respect to x, and finally with respect to z. Still four other orders of integration are possible.

5.3.3 Evaluating Triple Integrals Using Cylindrical Coordinates

Let T be a solid whose projection onto the xy-plane is labelled Ω_{xy}. Then the solid T is the set of all points (x, y, z) satisfying

$$(x, y) \in \Omega_{xy}, \chi_1(x, y) \leq z \leq \chi_2(x, y). \tag{5.24}$$

The domain Ω_{xy} has polar coordinates in some set $\Omega_{r\theta}$ and then the solid T in cylindrical coordinates is some solid S satisfying

$$(r, \theta) \in \Omega_{r\theta}, \quad \chi_1(r\cos(\theta), r\sin(\theta)) \leq z \leq \chi_2(r\cos(\theta), r\sin(\theta)). \tag{5.25}$$

Then

$$\begin{aligned}\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int\int_{\Omega_{xy}} \left(\int_{\chi_1(x,y)}^{\chi_2(x,y)} f(x, y, z)\,\mathrm{d}z \right) \mathrm{d}x\,\mathrm{d}y \\ &= \int\int_{\Omega_{r\theta}} \left(\int_{\chi_1(r\cos(\theta),r\sin(\theta))}^{\chi_2(r\cos(\theta),r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\,\mathrm{d}z \right) r\,\mathrm{d}r\,\mathrm{d}\theta = \\ &\int\int\int_S f(r\cos(\theta), r\sin(\theta), z) r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z.\end{aligned} \tag{5.26}$$

5.3.4 Evaluating Triple Integrals Using Spherical Coordinates

Let T be a solid in xyz-space with spherical coordinates in the solid S of $\rho\theta\phi$-space. Then

$$\int\int\int_T f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int\int_S f(\rho\sin\theta\cos\phi, \rho\sin\theta\sin\phi, \rho\cos\theta)\,\rho^2\sin\theta\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{5.27}$$

5.4 Jacobians and changing variables in multiple integration

During the course of the last few sections you have met several formulae for changing variables in multiple integration: to polar coordinates, to cylindrical coordinates, to spherical coordinates. The purpose of this section is to bring some unity to that material and provide a general description for other changes of variable.

5.4.1 Change of variables for double integrals

Consider the change of variables $x = x(u,v)$ and $y = y(u,v)$, which maps the points (u,v) of some domain Γ into the points (x,y) of some other domain Ω. Then

$$\text{The area of } \Omega = \int\int_{\Gamma} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\,\mathrm{d}v. \tag{5.28}$$

Suppose now that we want to integrate some function $f(x,y)$ over Ω. If this proves difficult to do directly, then we can change variables (x,y) to (u,v) and try to integrate over Γ instead. Then

$$\int\int_{\Omega} f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int\int_{\Gamma} f(x(u,v),y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\,\mathrm{d}v. \tag{5.29}$$

5.4.2 Change of variables for triple integrals

Consider the change of variables $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$ which maps the points (u,v,w) of some solid S into the points (x,y,z) of some other solid T. Then

$$\text{The volume of } T = \int\int\int_{S} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w. \tag{5.30}$$

Suppose now that we want to integrate some function $f(x,y,z)$ over T. If this proves difficult to do directly, then we can change variables (x,y,z) to (u,v,w) and try to integrate over S instead. Then

$$\int\int\int_{T} f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int\int_{S} f(x(u,v,w),y(u,v,w),z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w. \tag{5.31}$$

Referring back to equations (5.26) and (5.27), and the Jacobians given at the end of §4.5, we can verify that this formula is correct for a change from Cartesian to cylindrical coordinates (Jacobian is r) and for a change from Cartesian to spherical coordinates (Jacobian is $\rho^2 \sin\theta$).

Chapter 6

LINE INTEGRALS AND SURFACE INTEGRALS

In this chapter we will study integration along curves and integration along surfaces. At the heart of this subject lie three great theorems: *Green's theorem*, *Gauss's theorem* (commonly known as the *divergence theorem*) and *Stokes's theorem*. All of these are ultimately based on the *fundamental theorem of integral calculus*, and all can be cast in the same general form: *An integral over a region* S = *An integral over the boundary of* S.

6.1 Line integrals

Let $\underline{h}(x,y,z) = (h_1(x,y,z), h_2(x,y,z), h_3(x,y,z))$ be a vector function that is continuous over a smooth curve C parametrised by $C : \underline{r}(u) = (x(u), y(u), z(u))$ with $u \in [a,b]$. The *line integral* of $\underline{h}$ over C is the number

$$\int_C \underline{h}(\underline{r}) \cdot \mathrm{d}\underline{r} = \int_a^b [\underline{h}(\underline{r}(u)) \cdot \underline{r}'(u)] \, \mathrm{d}u. \tag{6.1}$$

Although we stated this definition in terms of three-dimensional vectorial functions $\underline{h}(x,y,z)$ and curves in space $\underline{r}(u) = (x(u), y(u), z(u))$, it also includes the two-dimensional case: $\underline{h}(x,y)$ and plane curves $\underline{r}(u) = (x(u), y(u))$.

If the curve C is not smooth but is made up of a finite number of adjoining smooth pieces $C_1, \ldots, C_n$, i.e. it is *piecewise smooth*, then we define the integral over C as the sum of the integrals over C_i for $i = 1, \ldots, n$, that is $\int_C = \int_{C_1} + \cdots + \int_{C_n}$. All polygonal paths are piecewise smooth.

When we integrate over a parametrised curve, we integrate in the direction determined by the parametrisation. If we integrate in the opposite direction, our answer is altered by a factor of -1, that is $\int_{-C} = -\int_C$.

6.1.1 Another notation for line integrals

If $\underline{h}(x,y,z) = (h_1(x,y,z), h_2(x,y,z), h_3(x,y,z))$ then the line integral over a curve C can be written as

$$\int_C \underline{h}(\underline{r}) \cdot \mathrm{d}\underline{r} = \int_C \{h_1(x,y,z)\,\mathrm{d}x + h_2(x,y,z)\,\mathrm{d}y + h_3(x,y,z)\,\mathrm{d}z\}. \tag{6.2}$$

6.2 The Fundamental Theorem for Line Integrals

In general, if we integrate a vector function $\underline{h}$ from one point to another, the value of the line integral depends on the path chosen. There is, however, an important exception. If the vector function $\underline{h}$ is a *gradient*, i.e. there exists a scalar function f such that $\underline{h} = \underline{\nabla} f$, then the value of the line integral depends only on the endpoints of the path and not on the path itself. The details are spelled out in the following theorem.

Theorem

Let C, parametrised by $\underline{r} = \underline{r}(u)$ with $u \in [a, b]$, be a piecewise smooth curve that begins at $\underline{\alpha} = \underline{r}(a)$ and ends at $\underline{\beta} = \underline{r}(b)$. Then if the vector function $\underline{h}$ is a gradient, i.e. $\underline{h} = \underline{\nabla} f$, we have

$$\int_C \underline{h}(\underline{r}) \cdot \underline{\mathrm{d}r} = \int_C \underline{\nabla} f(\underline{r}) \cdot \underline{\mathrm{d}r} = f(\underline{\beta}) - f(\underline{\alpha}). \tag{6.3}$$

NOTE: It is important to see that this result is an extension of the fundamental theorem of integral calculus: $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.

Corollary

If the curve C is closed, i.e. $\underline{\alpha} = \underline{\beta}$, then $f(\underline{\alpha}) = f(\underline{\beta})$ and $\int_C \underline{\nabla} f(\underline{r}) \cdot \underline{\mathrm{d}r} = 0$.

6.3 Line integrals with respect to arc length

Suppose that f is a scalar function continuous on a piecewise smooth curve C parametrised by $\underline{r} = \underline{r}(u)$ with $u \in [a, b]$. If $s(u)$ is the length of the curve from the tip of $\underline{r}(a)$ to the tip of $\underline{r}(u)$, then, as we have seen in section 2.3, $s'(u) = ||\underline{r}'(u)||$. The integral of f over C *with respect to arc length* s is defined by setting

$$\int_C f(\underline{r})\,\mathrm{d}s = \int_a^b f(\underline{r}(u)) s'(u)\,\mathrm{d}u. \tag{6.4}$$

6.4 Green's Theorem

If $P(x, y)$ and $Q(x, y)$ are scalar functions defined over a domain Ω with piecewise smooth closed boundary C, then

$$\int\int_\Omega \left[\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y) \right] \mathrm{d}x\,\mathrm{d}y = \oint_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y \tag{6.5}$$

where the integral on the right is a line integral over C taken in the anticlockwise direction.

Remark As indicated, the symbol $\oint$ is used to denote the line integral over a simple closed curve C taken in the anticlockwise direction.

6.5 Parametrised Surfaces; Surface Area

We have seen that a space curve C can be parametrised by a vector function $\underline{r} = \underline{r}(u)$ where u ranges over some interval I of the u-axis. In an analogous manner, we can parametrise a surface S in space by a vector function $\underline{r} = \underline{r}(u, v)$ where (u, v) ranges over some domain Ω of the uv-plane.

Example (The graph of a function)

The graph of a function $y = f(x)$, $x \in [a, b]$ can be parametrised by setting $\underline{r}(u) = (u, f(u))$, $u \in [a, b]$.

Similarly, the graph of a function $z = f(x, y)$, $(x, y) \in \Omega$ can be parametrised by setting $\underline{r}(u, v) = (u, v, f(u, v))$, $(u, v) \in \Omega$.

Example (A plane)

If two vectors $\underline{a}$ and $\underline{b}$ are not parallel, then the set of all combinations $u\underline{a} + v\underline{b}$ generates a plane P_0 that passes through the origin. We can parametrise this plane by setting $\underline{r}(u, v) = u\underline{a} + v\underline{b}$, u, v real numbers.

The plane P that is parallel to P_0 and passes through the tip of a vector $\underline{c}$ can be parametrised by setting $\underline{r}(u, v) = u\underline{a} + v\underline{b} + \underline{c}$, u, v real numbers.

Example (A sphere)

The sphere of radius a centred at the origin can be parametrised by setting

$$\underline{r}(u, v) = (a\sin(u)\cos(v), a\sin(u)\sin(v), a\cos(u)), \quad (u, v) \in [0, \pi] \times [0, 2\pi). \tag{6.6}$$

6.5.1 The fundamental vector product

Let S be a surface parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$. The cross product

$$\underline{N} = \underline{r}'_u \times \underline{r}'_v \tag{6.7}$$

is called the *fundamental vector product* of the surface S.

The vector $\underline{N}(u, v)$ is perpendicular to the surface S at the point with position vector $\underline{r}(u, v)$ and, if different from zero, can be taken as the normal to the surface S at that point.

Example

For the plane $\underline{r}(u, v) = u\underline{a} + v\underline{b} + \underline{c}$, the vector $\underline{a} \times \underline{b}$ is normal to the plane.

Example

The fundamental vector product for a sphere is parallel to the radius vector $\underline{r}(u, v)$. (Using the parametrisation given above, $\underline{N} = a\sin(u)\underline{r}$.)

6.5.2 The area of a parametrised surface

The area of a surface S parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$, is given by

$$\text{Area of } S = \int\int_S ||\underline{N}(u, v)||\,\mathrm{d}u\,\mathrm{d}v. \tag{6.8}$$

Example (The surface area of a sphere)

Using the parametrisation given by equation (6.6), we had $\underline{N} = a\sin(u)\underline{r}$ so $||\underline{N}|| = a^2\sin(u)$ and the area is

$$\int\int_S ||\underline{N}(u, v)||\,\mathrm{d}u\,\mathrm{d}v = a^2\int_{v=0}^{2\pi}\int_{u=0}^{\pi}\sin(u)\,\mathrm{d}u\,\mathrm{d}v = 2\pi a^2[-\cos(v)]_0^{\pi} = 4\pi a^2.$$

Example (The area of a plane domain)

A plane domain may be parametrised as $\underline{r} = (u, v, 0)$ for $(u, v) \in \Omega$. Then $\underline{r}'_u = (1, 0, 0)$ and $\underline{r}'_v = (0, 1, 0)$ and so the fundamental vector product is $\underline{N} = (0, 0, 1)$ which has magnitude 1.

$$\int\int_\Omega 1\,\mathrm{d}u\,\mathrm{d}v = \text{Area of } \Omega.$$

6.5.3 The area of a surface $z = f(x, y)$

Let the surface S be the graph of the function $z = f(x, y)$ with $(x, y) \in \Omega$. Then

$$\text{Area of } S = \int\int_{\Omega} \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1\right]^{1/2} \mathrm{d}x\,\mathrm{d}y. \tag{6.9}$$

In this case the parametrisation of S is $\underline{r}(u, v) = (u, v, f(u, v))$, $(u, v) \in \Omega$ and so $\underline{N} = (-f_x, -f_y, 1)$. The unit vector $\underline{n} = \underline{N}/||\underline{N}||$ is called the *upper unit normal*.

6.6 Surface Integrals

Let $H(x, y, z)$ be a scalar function, continuous over a surface S parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$. The *surface integral* of H over S is the number

$$\int\int_S H(x, y, z)\,\mathrm{d}\sigma = \int\int_{\Omega} H(\underline{r}(u, v))||\underline{N}(u, v)||\,\mathrm{d}u\,\mathrm{d}v. \tag{6.10}$$

Taking $H \equiv 1$ and referring back to eq. (6.8) we get

$$\int\int_S \mathrm{d}\sigma = \text{Area of } S. \tag{6.11}$$

6.6.1 Flux of a vector function

Let $\underline{q}(x, y, z)$ be a vector function that is continuous over a smooth surface S parametrised by $\underline{r} = \underline{r}(u, v)$, $(u, v) \in \Omega$. The *flux* of $\underline{q}$ across S in the direction of the unit normal $\underline{n}$ to the surface S is the number

$$\int\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma \tag{6.12}$$

which can be calculated as

$$\int\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma = \int\int_{\Omega} \underline{q}(\underline{r}(u, v)) \cdot \underline{n}\,||\underline{N}||\,\mathrm{d}u\,\mathrm{d}v = \int\int_{\Omega} \underline{q}(\underline{r}(u, v)) \cdot \underline{N}\,\mathrm{d}u\,\mathrm{d}v. \tag{6.13}$$

Proposition

If S is the graph of a function $z = f(x, y)$ with $(x, y) \in \Omega$ and $\underline{n}$ is the upper unit normal, then the flux of the vector function $\underline{q} = (q_1(x, y, z), q_2(x, y, z), q_3(x, y, z))$ across S in the direction of $\underline{n}$ is

$$\int\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma = \int\int_{\Omega} (-q_1 f_x - q_2 f_y + q_3)\,\mathrm{d}x\,\mathrm{d}y. \tag{6.14}$$

Proof

We can parametrise the surface by $\underline{r} = (u, v, f(u, v))$ with $(u, v) \in \Omega$. Then the fundamental vector product is

$$\underline{N} = \underline{r}'_u \times \underline{r}'_v = (1, 0, f_u) \times (0, 1, f_v) = (-f_u, -f_v, 1)$$

and we have

$$\begin{aligned}
\int\int_S \underline{q} \cdot \underline{n}\,\mathrm{d}\sigma &= \int\int_{\Omega} (\underline{q} \cdot \underline{N})\,\mathrm{d}u\,\mathrm{d}v \\
&= \int\int_{\Omega} (-q_1 f_u - q_2 f_v + q_3)\,\mathrm{d}u\,\mathrm{d}v = \int\int_{\Omega} (-q_1 f_x - q_2 f_y + q_3)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

where we have simply changed the names of the variables at the end.

6.7 The Divergence (Gauss) Theorem

Recall that if $P(x, y)$ and $Q(x, y)$ are scalar functions defined over a domain Ω with piecewise smooth closed boundary C, then Green's theorem (section 6.4) allowed us to express a double integral over Ω as a line integral over C:

$$\int\int_{\Omega} \left[\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right] \mathrm{d}x\,\mathrm{d}y = \oint_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y. \tag{6.15}$$

This formula can be rewritten in vector terms (using $\underline{q} = (Q, -P)$) to give the *divergence theorem in two dimensions* as follows:

The divergence theorem in two dimensions

Let Ω be a two-dimensional domain bounded by a piecewise smooth closed curve C. Then for any (continuously differentiable) vector function $\underline{q}(x, y)$ we have that

$$\int\int_{\Omega} (\underline{\nabla} \cdot \underline{q})\,\mathrm{d}x\,\mathrm{d}y = \oint_C (\underline{q} \cdot \underline{n})\,\mathrm{d}s \tag{6.16}$$

where $\underline{n}$ is the outer unit normal and the integral on the right is taken with respect to arc length.

We can now give the three-dimensional analogue of the divergence (Gauss) theorem.

The divergence theorem in three dimensions

Let T be a three-dimensional solid bounded by a piecewise smooth closed surface S. Then for any (continuously differentiable) vector function $\underline{q}(x, y, z)$ we have that

$$\int\int\int_T (\underline{\nabla} \cdot \underline{q})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int_S (\underline{q} \cdot \underline{n})\,\mathrm{d}\sigma \tag{6.17}$$

where $\underline{n}$ is the outer unit normal.

6.7.1 Divergence as outward flux per unit volume

In eq. (6.17), the right-hand side $\int\int_S (\underline{q} \cdot \underline{n})\,\mathrm{d}\sigma$ represents the $\underline{q}$ across S in the direction of $\underline{n}$. In this sense, from eq. (6.17) we can say that *the divergence is the outward flux per unit volume*, as we discussed in section 4.4.1.

Points $(x, y, z) \in T$ for which

- $\underline{\nabla} \cdot \underline{q}(x, y, z) < 0$ are called *sinks*.
- $\underline{\nabla} \cdot \underline{q}(x, y, z) > 0$ are called *sources*.
- If $\underline{\nabla} \cdot \underline{q}(x, y, z) \equiv 0$ then $\underline{q}$ is called *solenoidal*.

6.8 Stokes's Theorem

We return to Green's theorem (section 6.4):

$$\int\int_{\Omega} \left[\frac{\partial Q}{\partial x}(x, y) - \frac{\partial P}{\partial y}(x, y)\right] \mathrm{d}x\,\mathrm{d}y = \oint_C P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y. \tag{6.18}$$

and this time setting $\underline{q} = (P, Q, R)$ a vector function, we have

$$(\underline{\nabla} \times \underline{q}) \cdot \underline{k} = det \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} \cdot \underline{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}. \tag{6.19}$$

Thus in vector terms Green's theorem can be written as

$$\int\int_\Omega [(\underline{\nabla} \times \underline{q}) \cdot \underline{k}] \,\mathrm{d}x \,\mathrm{d}y = \oint_C \underline{q}(\underline{r}) \cdot \underline{\mathrm{d}r}. \tag{6.20}$$

Since any plane can be coordinatised as the xy-plane, this result can be phrased in the following theorem

Stokes's theorem

Let S be a smooth surface with smooth bounding curve C. Then for any (continuously differentiable) vectorial function $\underline{q}(x, y, z)$ we have

$$\int\int_S [(\underline{\nabla} \times \underline{q}) \cdot \underline{n}] \,\mathrm{d}\sigma = \oint_C \underline{q}(\underline{r}) \cdot \underline{\mathrm{d}r} \tag{6.21}$$

where $\underline{n}$ is a unit normal that varies continuously on S, and the line integral $\oint_C$ is taken in the positive sense with respect to $\underline{n}$.

6.8.1 The normal component of $\underline{\nabla}\times\underline{q}$ as circulation per unit area; Irrotational flow

Interpret the vector function $\underline{q}(x, y, z)$ as the velocity of a fluid. In eq. (6.21), the right-hand side line integral $\oint_C \underline{q}(\underline{r}) \cdot \underline{\mathrm{d}r}$ is called the *circulation* of $\underline{q}$ around the curve C. In this sense, from eq. (6.21), we can say that $\underline{\nabla} \times \underline{q}$ in the direction $\underline{n}$ is the circulation of $\underline{q}$ per unit area, which relates to the rotation of the fluid as discussed in section 4.4.2.

If $\underline{\nabla} \times \underline{q} \equiv \underline{0}$ then there is no circulation and $\underline{q}$ is called *irrotational*, i.e. the fluid has no rotational tendency.