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→ Calculus [Continued: RON-Larso, - Bruce - H - · Edwards] 24/04/23

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24/04/2023

4.1 Antiderivatives and Indefinite Integration

- Write the general Solution of a differential equation.
- Use indefinite integral notation for antiderivatives.
- Use basic integration rules to find antiderivatives.
- Find a particular Solution of a differential equation.

Example $\boxed{1}$ Solving a Differential Equation

Find the general Solution of the differential equation $y' = 2$

Solution: To begin, we need to find a function whose derivative is 2. One Such function is

$$y = 2x \quad (2x \text{ is an antiderivative of } 2)$$

$$y = 2x + C \quad (\text{General Solution}); \ C \text{ is any constant}$$

▶ Notation for Antiderivatives

When solving a differential equation of the form

$$\frac{dy}{dx} = f(x)$$

it is convenient to write it in the equivalent differential form

$$dy = f(x)\,dx$$

: The operation of finding all Solutions of this equation is Called antidifferentiation (or indefinite integration) and is denoted by an integral Sign $\int$. The general Solution is denoted by

Variable of integration — Constant of integration

$$y = \int f(x)\,dx = F(x) + C$$

Integrand — An antiderivative of $f(x)$

The expression $\int f(x)dx$ is read as antiderivative of f w.r.t x.

$dx \Rightarrow$ differential, indefinite integral (Synonym antiderivative)

The differential dx serves to identify x as the variable of integration. The term indefinite integral is a synonym for anti-derivative.

Note:- In this text, the relation $\int f(x)\,dx = F(x)+C$ means that F is an anti derivative of f on an interval.

$\int F'(x)\,dx = F(x) + C$ Integration is "inverse" of differentiation

if $\int f(x)\,dx = F(x) + C$, then

$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ Differentiation is "inverse" of integration

Example $\boxed{2}$ Applying the Basic Integration Rules

Describe the anti derivatives of $3x$

Solution

$\int 3x\,dx = 3\int x\,dx$	Constant Multiple Rule
$= 3\int x^1\,dx$	Rewrite ~~x~~ as x^1
$= 3\left(\frac{x^{1+1}}{1+1}\right) + C$	Power Rule
$= \frac{3}{2}x^2 + C$	Simplify

So, the anti derivatives of $3x$ are of the form $\frac{3}{2}x^2+C$, where C is any constant.

$\boxed{\text{Unnecessary}}$

$\int 3x\,dx = 3\int x\,dx = 3\left(\frac{x^2}{2} + C\right) = \frac{3}{2}x^2 + 3C$

However C represents any constant it is unnecessary to write $3C$ as the constant of integration. So, $\frac{3}{2}x^2+3C$ is written in the simpler form $\frac{3}{2}x^2+C$

General Pattern of Integration

∴ Original integral ⇨ Rewrite ⇨ Integrate ⇨ Simplify

Example [3] Rewriting Before Integrating

Original Integral	Rewrite	Integrate	Simplify
a. $\int \frac{1}{x^3}\,dx$	$\int x^{-3}\,dx$	$\frac{x^{-2}}{-2} + C$	$\frac{-1}{2x^2} + C$
b. $\int \sqrt{x}\,dx$	$\int x^{\frac{1}{2}}\,dx$	$\frac{x^{3/2}}{3/2} + C$	$\frac{2}{3\sqrt[2]{x^3}} + C$
c. $\int 2\sin x\,dx$	$2\int \sin x\,dx$	$2(-\cos x) + C$	$-2\cos x + C$

Example [4] Integrating Polynomial Functions

a. $\int dx = \int 1\,dx$
$= x + C$ Integrand is understood to be 1.

b. $\int (x+2)\,dx = \int x\,dx + \int 2\,dx = \frac{x^{1+1}}{1+1} + C_1 + 2x + C_2$

$= \frac{x^2}{2} + 2x + C \qquad C = C_1 + C_2 + \dots C_n$

c. $\int (3x^4 - 5x^2 + x)\,dx = 3\int x^4 - 5\int x^2 + \int x$

$= 3\left(\frac{x^{4+1}}{4+1}\right) - 5\left(\frac{x^{2+1}}{2+1}\right) + \frac{x^{1+1}}{1+1}$

$= \frac{3}{5}x^5 - \frac{5}{3}x^3 + \frac{1}{2}x^2 + C$

Remember that you can check your answer by differentiating.
The antidifferentiation is the reverse process of differentiation is called integration and we write

$$\frac{dy}{dx} = x \implies y = \int x\,dx$$

Example $\boxed{5}$ Rewriting Before Integrating

$$\int \frac{x+1}{\sqrt{x}}\,dx = \int\left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right)dx$$

$$= \int\left(x^{\frac{1}{2}} + x^{-1/2}\right)dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2}{3}x^{3/2} + 2x^{1/2} + C$$

$$= \frac{2}{3}\sqrt[2]{x^3} + 2\sqrt{x} + C$$

Note: when integrating quotients, do not integrate the numerator and denominator separately.

Example $\boxed{6}$ Rewriting Before Integrating

$$\int \frac{\sin x}{\cos^2 x}\,dx = \int\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right)dx$$

$$= \int \sec x \tan x\,dx$$

$$\sec x + C$$

Example $\boxed{7}$ Finding a particular Solution

Find the general Solution of

$$F'(x) = \frac{1}{x^2}, \quad x > 0$$

and find the particular Solution that Satisfies the initial Condition $F(1) = 0$

Solution $F(x) = \int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{x^{-1}}{-1} + C = -\frac{1}{x} + C, \quad x > 0$

Use the initial Condition $F(1) = 0$, you can Solve for C as follows.

$$F(1) = -\frac{1}{1} + C = 0 \rightarrow \boxed{C = 1}$$

$$F(x) = \frac{-1}{x} + 1, \quad x > 0 \rightarrow \text{Particular Solution}$$

4.1 Exercises

1-4 Show that R.H.S = L.H.S

1. $\int \left(-\frac{6}{x^4}\right) dx = \frac{2}{x^3} + C$

:- $-6\int \frac{1}{x^4} dx = -6\int x^{-4} dx = -6\left(\frac{x^{-3}}{-3}\right) + C = \frac{2}{x^3} + C$

2. $\int \left(8x^3 + \frac{1}{2x^2}\right) dx = 2x^4 - \frac{1}{2x} + C$

: $8\int x^3 dx + \frac{1}{2}\int x^{-2} dx = 8\frac{(x^{3+1})}{3+1} + C_1 + \frac{1}{2}\frac{(x^{-2+1})}{-2+1} + C_2$

$= 2x^4 + C_1 + \frac{1}{2}\left(-\frac{1}{x}\right) + C_2 = 2x^4 - \frac{1}{2x} + C$

3. $\int (x-4)(x+4)\, dx = \frac{1}{3}x^3 - 16x + C$

:- $\int (x^2 + 4x - 4x - 16)\, dx = \int (x^2 - 16) = \int x^2 - 16\int 1$

$= \left(\frac{x^{2+1}}{2+1} + C_1\right) - 16\left(\frac{x^{0+1}}{0+1}\right) + C_2 = \frac{x^3}{3} - 16x + C$

4. $\int \frac{x^2 - 1}{x^{3/2}} dx = \frac{2(x^2+3)}{3\sqrt{x}} + C$

:- $\int \frac{x^2}{x^{3/2}} dx - \int \frac{1}{x^{3/2}} dx = \int x^{1/2} dx - \int x^{-3/2} dx = \left(\frac{x^{1/2+1}}{1/2+1}\right) - \left(\frac{x^{-3/2+1}}{-3/2+1}\right) + C$

$= \frac{x^{3/2}}{3/2} - \left(\frac{x^{-1/2}}{-1/2}\right) + C = \frac{2}{3}x^{3/2} + \frac{2}{\sqrt{x}} + C$

5-8 Find the general Solution of the differential equation and check the result by differentiation.

5. $\frac{dy}{dt} = 9t^2$

: $\int 9t^2 = 9\int t^2 dx = 9\left(\frac{t^{2+1}}{2+1}\right) + C = 9\left(\frac{t^3}{3}\right) + C = 3t^3 + C$

$\frac{dy}{dx}[3t^3 + C] = 3\frac{d}{dx}[t^3] + \frac{d}{dx}[C] = 3(3t^2) + 0 = 9t^2$

6) $\frac{dr}{d\theta} = \pi$: $\int \pi\, dx = \pi\int 1\, dx = \frac{\pi(x)^{0+1}}{0+1} = \pi x$

$\frac{dy}{dx}[\pi x] = \pi\frac{d}{dx}[x] = \pi(1) = \pi$

7) $\frac{dy}{dx} = x^{3/2}$:- $\int x^{3/2}\,dx = \frac{x^{3/2+1}}{3/2+1} + C = \frac{x^{5/2}}{5/2} + C = \frac{2}{5}x^{5/2} + C$

$\frac{dy}{dx}\left[\frac{2}{5}x^{5/2} + c\right] = \frac{2}{5}\frac{d}{dx}[x^{5/2}] + \frac{d}{dx}[c] = \frac{2}{5} \times \left[\frac{5}{2}x^{3/2}\right] + 0 = x^{3/2}$

8) $\frac{dy}{dx} = 2x^{-3}$:- $dy = 2x^{-3}(dx) \rightarrow \int dy = \int 2x^{-3}(dx)$

$y = \int 2x^{-3}(dx) = 2\int x^{-3}dx = 2\left(\frac{x^{-3+1}}{-3+1}\right) + C = -\frac{1}{x^2} + C$

$\frac{dy}{dx}\left[-\frac{1}{x^2} + C\right] = \frac{d}{dx}[-x^{-2}] + \frac{d}{dx}[c] = -1[-2x^{-2-1}] + C = 2x^{-3}$

- In Exercises 9–14 Complete the table.

Original Integral	Rewrite	Integrate	Simplify
9. $\int \sqrt[3]{x}\,dx$	$\int x^{1/3}\,dx$	$\frac{x^{1/3+1}}{1/3+1} + C$	$\frac{3}{4}x^{4/3} + C$
10. $\int \frac{1}{4x^2}\,dx$	$\frac{1}{4}\int x^{-2}\,dx$	$\frac{1}{4}\left(\frac{x^{-2+1}}{-2+1}\right) + C$	$-\frac{1}{4x} + C$
11. $\int \frac{1}{x\sqrt{x}}\,dx$	$\int x^{-3/2}\,dx$	$\left(\frac{x^{-3/2+1}}{-3/2+1}\right) + C$	$\frac{-2}{\sqrt{x}} + C$
12. $\int x(x^3+1)\,dx$	$\int (x^4+x)\,dx$	$\frac{x^{4+1}}{4+1} + \frac{x^{1+1}}{1+1} + C$	$\frac{x^5}{5} + \frac{x^2}{2} + C$
13. $\int \frac{1}{2x^3}\,dx$	$\frac{1}{2}\int x^{-3}\,dx$	$\frac{1}{2}\left(\frac{x^{-3+1}}{-3+1}\right) + C$	$-\frac{1}{4x^2} + C$

15–34, find the indefinite Integral and check the result by differentiation

15. $\int (x+7)\,dx$:- $y = \int x\,dx + 7\int 1\,dx = \frac{x^{1+1}}{1+1} + C_1 + 7\left(\frac{x^{0+1}}{0+1}\right) + C_2$

$= \frac{x^2}{2} + 7x + C$

$\frac{dy}{dx}\left[\frac{x^2}{2} + 7x + C\right] = \frac{1}{2}\frac{d}{dx}[x^2] + 7\frac{d}{dx}[x] + \frac{d}{dx}[C] = x^2 + 7 + 0$

$= \boxed{x^2 + 7}$

16) $\int (13-x)\,dx$:- $y = \int 13\,dx - \int x\,dx = 13\int 1\,dx - \int x\,dx$

$= \left(13\frac{(x^{0+1})}{0+1} + C_1\right) - \left(\frac{x^{1+1}}{1+1} + C_2\right) = 13x - \frac{1}{2}x^2 + C$

$\frac{d}{dx}\left[13x - \frac{1}{2}x^2 + C\right] = 13\frac{d}{dx}[x] - \frac{1}{2}\frac{d}{dx}[x^2] + \frac{d}{dx}[C] = 13(1) - \frac{1}{2}(2x) + 0$

$= 13 - x$

17) $\int (2x - 3x^2)\,dx$:- $y = 2\int x\,dx - 3\int x^2\,dx$

$= 2\left(\frac{x^{1+1}}{1+1}\right) - 3\left(\frac{x^{2+1}}{2+1}\right) + C = \boxed{x^2 - x^3 + C}$

18) $\int (8x^3 - 9x^2 + 4)\,dx$

:- $y = 8\int x^3\,dx - 9\int x^2\,dx + 4\int 1\,dx = 8\left(\frac{x^{3+1}}{3+1}\right) - 9\left(\frac{x^{2+1}}{2+1}\right) + 4\left(\frac{x^{0+1}}{0+1}\right) + C$

$= \boxed{2x^4 - 3x^3 + 4x + C}$

$\frac{d}{dx}[2x^4 - 3x^3 + 4x + C] = 8x^3 - 9x^2 + 4 + 0 = 8x^3 - 9x^2 + 4$

19) $\int (x^5 + 1)\,dx$

:- $y = \int x^5\,dx + 1\int 1\,dx = \frac{x^{5+1}}{5+1} + 1\left(\frac{x^{0+1}}{0+1}\right) + C = \boxed{\frac{x^6}{6} + x + C}$

$\frac{d}{dx}\left[\frac{x^6}{6} + x + C\right] = \frac{6x^5}{6} + 1 + 0 = \boxed{x^5 + 1}$

20) $\int (x^3 - 10x - 3)\,dx$

$y = \int x^3\,dx - 10\int x\,dx - 3\int 1\,dx = \left(\frac{x^{3+1}}{3+1}\right) - 10\left(\frac{x^{1+1}}{1+1}\right) - 3\left(\frac{x^{0+1}}{0+1}\right) + C$

$= \boxed{\frac{x^4}{4} - 5x^2 - 3x + C}$

$\frac{d}{dx}\left[\frac{x^4}{4} - 5x^2 - 3x + C\right] = \frac{4x^3}{4} - 10x - 3 + 0 = \boxed{x^3 - 10x - 3}$

21) $\int (x^{3/2} + 2x + 1)\,dx$

$y = \int x^{3/2}\,dx + 2\int x\,dx + 1\int 1\,dx = \left(\frac{x^{3/2+1}}{3/2+1}\right) + 2\left(\frac{x^{1+1}}{1+1}\right) + 1\left(\frac{x^{0+1}}{0+1}\right) + C$

$= \boxed{\frac{2}{5}x^{5/2} + x^2 + x + C}$

$\frac{d}{dx}\left[\frac{2}{5}x^{5/2} + x^2 + x + C\right] = \frac{2}{5} \times \frac{5}{2} \times x^{3/2} + 2x + 1 + 0 = \boxed{x^{3/2} + 2x + 1}$

8

22) $\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx$

:- $y = \int x^{1/2} dx + \frac{1}{2}\int x^{-1/2} dx = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + C = \boxed{\frac{2}{3}x^{3/2} + \frac{2}{\sqrt{x}} + C}$

23) $\int \sqrt[3]{x^2}\, dx$

$y = \int x^{2/3} dx = \frac{x^{2/3+1}}{2/3+1} + C = \frac{x^{5/3}}{5/3} + C = \frac{3}{5}x^{5/3} + C$

24) $\int (\sqrt[4]{x^3} + 1)\, dx$

$y = \int x^{3/4} + 1\int 1\, dx = \frac{x^{3/4+1}}{3/4+1} + 1\left(\frac{x^{0+1}}{0+1}\right) + C$

$= \boxed{\frac{4}{7}x^{7/4} + x + C}$

25) $\int \frac{1}{x^5} dx$

$y = \int x^{-5} dx = \frac{x^{-5+1}}{-5+1} + C = \frac{-1}{4x^4} + C$

26) $\int \frac{1}{x^6} dx$: $y = \int x^{-6} dx = \frac{x^{-6+1}}{-6+1} + C = \boxed{\frac{-1}{5x^5} + C}$

27) $\int \frac{x+6}{\sqrt{x}} dx$

$y = \int \frac{x}{\sqrt{x}} dx + \int \frac{6}{\sqrt{x}} = \int x^{1/2} dx + 6\int x^{-1/2} = \frac{x^{3/2}}{3/2} + \frac{6x^{1/2}}{1/2} + C$

$= \boxed{\frac{2}{3}x^{3/2} + 12\sqrt{x} + C}$

28) $\int \frac{x^2+2x-3}{x^4} dx$

$y = \int \frac{x^2}{x^4} dx + \int \frac{2x}{x^4} dx + \int \frac{-3}{x^4} dx = \int x^{-2} dx + 2\int x^{-3} dx - 3\int x^{-4} dx$

$= \left(\frac{x^{-1}}{-1}\right) + 2\left(\frac{x^{-2}}{-2}\right) - 3\left(\frac{x^{-3}}{-3}\right) + C = \frac{-1}{x} - \frac{2}{x^2} + \frac{1}{x^3}$

29) $\int (x+1)(3x-2)\,dx$

$y = \int (3x^2 - 2x + 3x - 2)\,dx = \int (3x^2 + x - 2)\,dx = 3\int x^2\,dx + 1\int x\,dx - 2\int 1\,dx$

$= 3\left(\frac{x^{2+1}}{2+1}\right) + 1\left(\frac{x^{1+1}}{1+1}\right) - 2\left(\frac{x^{0+1}}{0+1}\right) + C$

$= \boxed{x^3 + \frac{x^2}{2} - 2x + C}$

30) $\int (2t^2 - 1)^2\,dt$

$y = 2\int t^2\,dt - 1\int 1\,dt = 2\left(\frac{t^{2+1}}{2+1}\right) - 1\left(\frac{x^{0+1}}{0+1}\right) + C$

$= \boxed{\frac{2}{3}t^3 - x + C}$

31) $\int y^2\sqrt{y}\,dy$

$y = \int y^{5/2}\,dy = \left(\frac{y^{5/2+1}}{5/2+1}\right) + C = \frac{y^{7/2}}{7/2} + C = \boxed{\frac{2}{7}y^{7/2} + C}$

32) $\int (1+3t)t^2\,dt$

$y = \int (t^2 + 3t^3)\,dt = \int t^2\,dt + 3\int t^3\,dt = \left(\frac{t^{2+1}}{2+1}\right) + 3\left(\frac{t^{3+1}}{3+1}\right) + C$

$\boxed{y = \frac{t^3}{3} + \frac{3}{4}t^4 + C}$

33) $\int dx$:- $y = \int 1\,dx = 1\int 1\,dx = 1\left(\frac{x^{0+1}}{0+1}\right) + C = \boxed{x + C}$

34) $\int 14\,dt$:- $y = 14\int 1\,dt = \boxed{14x + C}$

▸ 35–44 find Indefinite Integral and check result by differentiation.

35) $\int (5\cos x + 4\sin x)\,dx$

$y = 5\int \cos x\,dx + 4\int \sin x\,dx = 5\sin x + 4(-\cos x)$

$= \boxed{5\sin x - 4\cos x}$

36) $\int (t^2 - \cos t)\, dt$

$y = \int t^2 dt - \int \cos t\, dt = \frac{t^3}{3} - (\sin t) = \boxed{\frac{t^3}{3} - \sin t + c}$

37) $\int (1 - \csc t \cot t)\, dt$

$y = 1\int 1\, dt - \int \csc t \cot t\, dt = t - (-\cot t) = \boxed{t + \cot t + c}$

38) $\int (\theta^2 - \sec^2(\theta))\, d\theta$

$y = \int \theta^2\, d\theta - \int \sec^2\theta\, dt = \frac{\theta^3}{2+1} - (\tan\theta) + c = \boxed{\frac{\theta^3}{3} - \tan\theta + c}$

39) $\int (\sec^2\theta - \sin\theta)\, d\theta$

$y = \int \sec^2\theta\, d\theta - \int \sin\theta\, d\theta = \tan\theta - (-\cos\theta) + c = \boxed{\tan\theta + \cos\theta + c}$

40) $\int \sec y (\tan y - \sec y)\, dy$

$y = \int \sec y \tan y\, dy - \int \sec^2 y\, dy = \boxed{\sec y - \tan y + c}$

41) $\int (\tan^2 y + 1)\, dy = \tan y + c$

42) $\int (4x - \csc^2 x)\, dx$

$y = 4\int x\, dx - \int \csc^2 x\, dx = 4\left(\frac{x^{1+1}}{1+1}\right) -$

43) $\int \frac{\cos x}{1-\cos^2 x}\, dx$

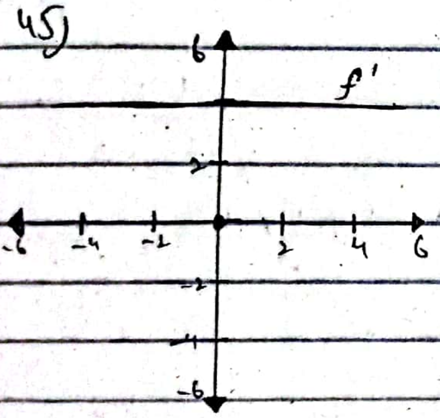
$\therefore y = \int \cos x\,(1-\cos^2 x)^{-1}\, dx = \int \left(\cos x - \cos x\,(\cos^{-2} x)\right) dx$

$= \int \cos x\, dx - \int \cos^{-1} x\, dx = \int \cos x\, dx - \int \sec x\, dx$

$= \sin x -$

45–48, the graph of the derivative of a function is given. Sketch the graphs of two functions that have the given derivative. (There is more than one correct answer)

45)

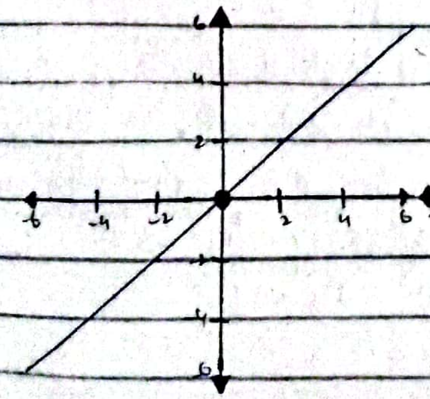


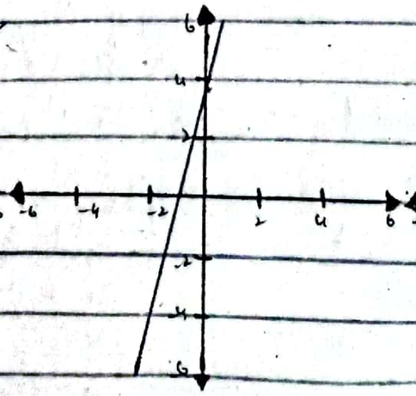
f' :- we can find the function of the the graph very easily, that is $y = 4$. The function we have $y = 4$, but this is derivative we have to find the original function and we can find original function by integration.

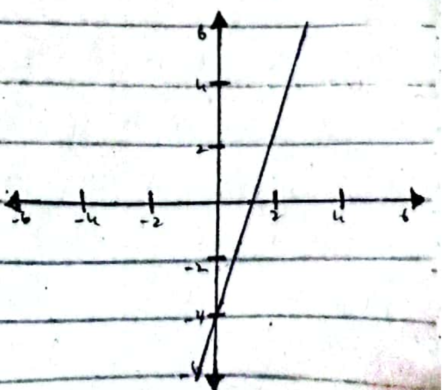
$\therefore y = \int 4\, dx = 4\int 1\, dx = 4\left(\frac{x^{0+1}}{0+1}\right) + C = \boxed{4x + C}$

The original function is $\boxed{y = 4x + C}$, three possible solutions

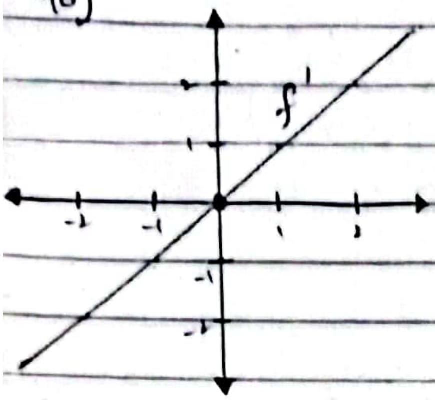
$C = 0$	$C = 4$	$C = -4$
$y = 4x$	$y = 4x + 4$	$y = 4x - 4$







46)



: one possible $f'(x) = x$, it means

$\boxed{y = x}$

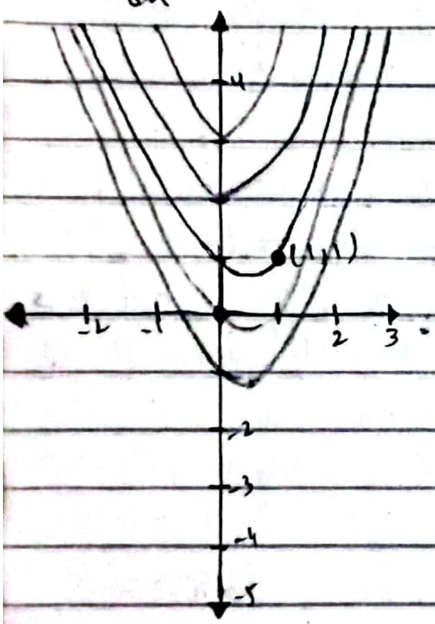
$y = \int x\,dx = \dfrac{x^{1+1}}{1+1} + C = \boxed{\dfrac{x^2}{2} + C}$

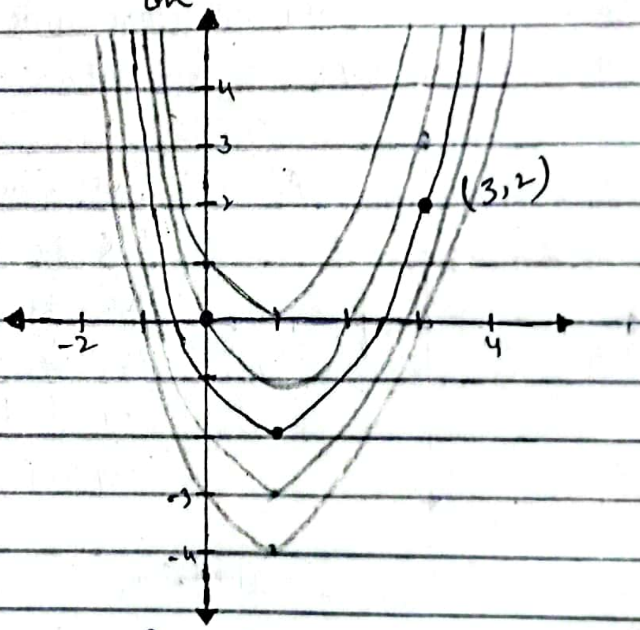
The graph will vary on C

In Exercises 49 and 50, find the equation of y, given the derivative and the indicated point on the curve.

49) $\dfrac{dy}{dx} = 2x - 1$

50) $\dfrac{dy}{dx} = 2(x-1)$





: $y = 2x - 1$

$y = \int(2x - 1)$

$y = 2\int x\,dx - 1\int 1\,dx$

$y = 2\left(\dfrac{x^2}{2}\right) - 1(x) + C$

$y = x^2 - x + C$

w.r.t given graph y-intercept is 1 where $x = 0$, so we can conclude that $C = 1$

$\boxed{y = x^2 - x + 1}$ → original function

$y = 2x - 2$

$y = 2\int x\,dx - 2\int 1\,dx$

$y = 2\left(\dfrac{x^2}{2}\right) - 2(x) + C$

$y = x^2 - 2x + C$

w.r.t above graph there is a y-intercept at $(0, -1)$, we can conclude that $C = -1$, so the original function of the above graph is:

$\boxed{y = x^2 - 2x - 1}$

In Exercises 57 – 64, Solve the differential equation.

57) $f'(x) = 6x$, $f(0) = 8$

:- $f'(x) = 6x$

$F(x) = 6\int x \, dx$

$F(x) = 6\left(\frac{x^2}{2}\right) + C$

$\boxed{F(x) = 3x^2 + C}$

$f(0) = 8$

$8 = 3(0)^2 + C$

$\boxed{C = 8}$

$\boxed{f(x) = 3x^2 + 8}$ → original function

58) $g'(x) = 6x^2$, $g(0) = -1$

:- $g'(x) = 6x^2$

$g(x) = 6\int x^2 \, dx$

$g(x) = 6\left(\frac{x^3}{3}\right) + C$

$\boxed{g(x) = 2x^3 + C}$

$g(0) = -1$

$-1 = 2(0)^3 + C$

$\boxed{C = -1}$

$\boxed{g(x) = 2x^3 - 1}$

59) $h'(t) = 8t^3 + 5$, $h(1) = -4$

:- $h'(t) = 8t^3 + 5$

$h(t) = \int (8t^3 + 5) \, dt$

$h(t) = 8\int t^3 \, dt + 5\int 1 \, dt$

$h(t) = 8\left(\frac{t^4}{4}\right) + 5(t) + C$

$\boxed{h(t) = 2t^4 + 5t + C}$

$h(1) = -4$

$-4 = 2(1)^4 + 5(1) + C$

$-4 = 2 + 5 + C$

$\boxed{-11 = C}$

$\boxed{h(t) = 2t^4 + 5t - 11}$ → original function

60) $f'(s) = 10s - 12s^3$, $f(3) = 2$

:- $f'(s) = 10s - 12s^3$

$f(s) = 10\int s \, ds - 12\int s^3 \, ds$

$f(s) = 10\left(\frac{s^2}{2}\right) - 12\left(\frac{s^4}{4}\right) + C$

$\boxed{f(s) = 5s^2 - 3s^4 + C}$

$f(3) = 2$

$2 = 5(3)^2 - 3(3)^4 + C$

$2 = 45 - 243 + C$

$\boxed{200 = C}$

original function ← $\boxed{f(s) = 5s^2 - 3s^4 + 200}$

61) $f''(x) = 2$, $f'(2) = 5$, $f(2) = 10$

Double derivative, means double integration I think, Wow!

$f''(x) = 2$

$f'(x) = \int 2 \, dx$

$f'(x) = 2\int 1 \, dx$

$f'(x) = 2(x) + C$

$\boxed{f'(x) = 2x + C}$

$5 = 2(2) + C$

$\boxed{C = 1}$

$f'(x) = 2x + 1$

$f(x) = \int (2x + 1) \, dx$

$f(x) = 2\int x \, dx + 1\int 1 \, dx$

$f(x) = 2\left(x^2/2\right) + 1(x) + C$

$\boxed{f(x) = x^2 + x + C}$

$10 = 2^2 + 2 + C$

$\boxed{C = 4}$

$\boxed{f(x) = x^2 + x + 4}$

62) $f''(x) = x^2, \quad f'(0) = 8, \quad f(0) = 4$

:- $f''(x) = x^2$

$f'(x) = \int x^2 \, dx$

$\boxed{f'(x) = \frac{x^3}{3} + C}$

$8 = \frac{0^3}{3} + C$

$\boxed{C = 8}$

$\boxed{f'(x) = \frac{x^3}{3} + 8}$

$f'(x) = \frac{x^3}{3} + 8$

$f'(x) = \int (x^3/3 + 8) \, dx$

$f'(x) = 1/3 \int x^3 \, dx + 8 \int 1 \, dx$

$\boxed{f'(x) = \frac{1}{3}\left(\frac{x^4}{4}\right) + 8x + C}$

$4 = 1/3 \left(0^4/4\right) + 8(0) + C$

$\boxed{C = 4}$

$\boxed{f(x) = \frac{x^4}{7} + 8x + 4}$

original function via Double integration

63) $f''(x) = x^{-3/2}, \quad f'(4) = 2, \quad f(0) = 0$

:- $f'(x) = \int x^{-3/2} \, dx$

$f'(x) = \left(\frac{x^{-3/2+1}}{-3/2+1}\right) + C$

$\boxed{f'(x) = -\frac{2}{\sqrt{x}} + C}$

$2 = \frac{-2}{\sqrt{4}} + C$

$\boxed{C = 3}$

$f'(x) = \frac{-2}{\sqrt{x}} + 3$

$f(x) = -2\int x^{-1/2} \, dx + 3 \int 1 \, dx$

$f(x) = -2\left(\frac{x^{1/2}}{1/2}\right) + 3x + C$

$f(x) = -4\sqrt{x} + 3x + C$

$\boxed{C = 0}$

original function

$\boxed{f(x) = -4\sqrt{x} + 3x}$

64) $f''(x) = \sin x, \quad f'(0) = 1, \quad f(0) = 6$

:- $f'(x) = \int \sin x \, dx$

$f'(x) = -\cos x + C$

$1 = -\cos(0) + C$

$1 = -(1) + C$

$\boxed{2 = C}$

$\boxed{f'(x) = -\cos x + 2}$

$f'(x) = -\cos x + 2$

$f(x) = \int -\cos x \, dx + 2 \int 1 \, dx$

$f(x) = -1 \int \cos x \, dx + 2 \int 1 (dx)$

$f(x) = -1 \sin x + 2x + C$

$\boxed{f(x) = -\sin x + 2x + C}$

$6 = -\sin(0) + 2(0) + C$

$\boxed{C = 6}$

$\boxed{f(x) = -\sin x + 2x + 6}$

4.2 Area

- Use sigma notation to write and evaluate a sum.
- Understand the concept of Area.
- Approximate the area of a plane region.
- Find the area of a plane region using limits.

Example [1] Examples of Sigma Notation

a) $\sum_{i=1}^{6} i = 1 + 2 + 3 + 4 + 5 + 6$

b) $\sum_{i=0}^{5} (i+1) = 1 + 2 + 3 + 4 + 5 + 6$

c) $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2$

d) $\sum_{k=1}^{n} \frac{1}{n}(k^2+1) = \frac{1}{n}(1^2+1) + \frac{1}{n}(2^2+1) + \cdots + \frac{1}{n}(n^2+1)$

e) $\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x$

Theorem $\rightarrow$ Summation Formulas

1) $\sum_{i=1}^{n} c = cn$

2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example [2] Approximating the Area of a plane region

Use 4 rectangles with left and Right endpoints to approximate the area of region between the graph of function and the x-axis over the interval $[0, 2]$, where $f(x) = 2x + 5$.

Solution

1: Divide the interval $[0, 2]$ into 4 equal sub intervals of length 0.5. (Because of 4 rectangles)

2: Determine x-coordinates for the left and right endpoints of each Sub interval.

Right endpoints: $\frac{x}{\text{num of rectangles}} \times i$ left endpoints $\frac{x}{\text{num of rects}} \times (i-1)$

$x \rightarrow$ Right endpoint of interval = 2 (original interval / super interval)

number of rectangles = 4 rectangles

$i \rightarrow$ from 1 to the number of rectangles

$\rightarrow i = 1, 2, 3, 4$

So, Right endpoints are

$\frac{x}{\text{nor}} i = \frac{2}{4}(1) = 0.5$, $\frac{2}{4}(2) = 1$, $\frac{2}{4}(3) = 1.5$, $\frac{2}{4}(4) = 2$

Right endpoints $\rightarrow$ 0.5, 1, 1.5, 2

1: Sub intervals $\rightarrow$ [0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]

2: Right endpoints $\rightarrow$ 0.5, 1, 1.5, 2

3: Calculate the height of each rectangle

For left endpoints $\rightarrow$ height of each rectangle evaluted at the left endpoint of Subintervial of the function $y = 2x+5$.

For Right endpoints $\rightarrow$ The height of each rectangle will be given by function $y = 2x+5$ evaluted at the right endpoint of Sub interval.

Heights

- For first rectangle (0 to 0.5), $y = 2(0.5)+5 = 6$
- For Second rectangle (0.5 to 1), $y = 2(1)+5 = 7$
- For third rectangle (1 to 1.5), $y = 2(1.5)+5 = 8$
- For fourth rectangle (1.5 to 2), $y = 2(2)+5 = 9$

4: Calculate the height of each rectangle.

For Right endpoint $\rightarrow$ the height of each rectangle will be given by the function $y = 2x+5$ evaluted at right endpoint of Subinterval.

Areas:
- Area of First rectangle $0.5 \times 6 = 3$
- Area of Second rectangle $0.5 \times 7 = 3.5$
- Area of third rectangle $0.5 \times 8 = 4$
- Area of fourth rectangle $0.5 \times 9 = 4.5$

5. Add up the areas of all the rectangles to get an approximate value for the area of the region between the graph $y = 2x+5$ and the x-axis over the interval $[0, 2]$

: Total approximate area using right endpoints and 4 rectangles:

$3 + 3.5 + 4 + 4.5 = 15$

FORMAL Solution

Using Left endpoints:

1. Subintervals:

$[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$

2. left end points: $0, 0.5, 1, 1.5$

3. Heights:

- For first rectangle $(0$ to $0.5)$, $y = 2(0) + 5 = 5$
- For Second rectangle $(0.5$ to $1)$, $y = 2(0.5) + 5 = 6$
- For third rectangle $(1$ to $1.5)$, $y = 2(1) + 5 = 7$
- For fourth rectangle $(1.5$ to $2)$, $y = 2(1.5) + 5 = 8$

4. Areas:

- Area of first rectangle $= 0.5 \times 5 = 2.5$
- Area of Second rectangle $= 0.5 \times 6 = 3$
- Area of third rectangle $= 0.5 \times 7 = 3.5$
- Area of fourth rectangle $= 0.5 \times 8 = 4$

5. Total approximate area using left endpoints and 4 rectangles:

$2.5 + 3 + 3.5 + 4 = 13$

Using Right endpoints:

1. Subintervals: $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$
2. Right endpoints: $0.5, 1, 1.5, 2$
3. Heights:

- For first rectangle $(0$ to $0.5)$, $y = 2(0.5) + 5 = 6$
- For Second rectangle $(0.5$ to $1)$, $y = 2(1) + 5 = 7$
- For third rectangle $(1$ to $1.5)$, $y = 2(1.5) + 5 = 8$
- For fourth rectangle $(1.5$ to $2)$, $y = 2(2) + 5 = 9$.

4. Areas:

- Area of first rectangle $= 0.5 \times 6 = 3$
- Area of Second rectangle $= 0.5 \times 7 = 3.5$
- Area of third rectangle $= 0.5 \times 8 = 4$
- Area of fourth rectangle $= 0.5 \times 9 = 4.5$

5. Total approximate area using right endpoints and 4 rectangles:

$3 + 3.5 + 4 + 4.5 = 15$

CHAPTER 20 Integration → Additional Mathematics

Hoo Soo Thung

■ 20.1 Integration as the reverse process of Differentiation and indefinite Integration.

Meaning of Integration: In the process of Differentiation, if $y = \frac{1}{2}x^2 + c$, where c is a constant, then $\frac{dy}{dx} = x$. This means that for any curve defined by $y = \frac{1}{2}x^2 + c$, we have, by the process of differentiation, the same gradient function $\frac{dy}{dx} = x$.

Conversely, if the gradient function $\frac{dy}{dx} = x$ then we know that the equation of the curve is of the form $y = \frac{1}{2}x^2 + c$. This process is the reverse of differentiation and is called Integration and we write:

$$\frac{dy}{dx} = x \Rightarrow y = \int x\,dx$$

Since $y = \frac{1}{2}x^2 + c$,

$$\frac{d}{dx}\left(\frac{1}{2}x^2 + c\right) = x \Rightarrow \int x\,dx = \frac{1}{2}x^2 + c.$$

■ Indefinite Integral

Since there is an arbitrary constant c in the expression $\frac{1}{2}x^2 + c$, we say that this expression is an indefinite integral. Similarly, we have

$$\frac{d}{dx}\left(\frac{1}{3}x^3 + c\right) = x^2 \Rightarrow \int x^2\,dx = \frac{1}{3}x^3 + c$$

which is an indefinite integral.

In general, if $n \neq -1$, a and n are constants,

$$\frac{d}{dx}\left(\frac{ax^{n+1}}{n+1} + c\right) = ax^n$$

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c,$$ where c is an arbitrary constant

$$\int (ax^n + bx^m)\, dx = \frac{ax^{n+1}}{n+1} + \frac{bx^{m+1}}{m+1} + C.$$

$$\int (ax^n + bx^m)\, dx = \int (ax^n)\, dx + \int (bx^m)\, dx + C$$

- Integration of $(ax+b)^n$, $a \neq 0$, $n \neq -1$

In the process of differentiation, we have

$$\frac{d}{dx}\left[\frac{(ax+b)^{n+1}}{a(n+1)} + c\right] = (ax+b)^n$$

In the reverse process, by integration:

$$\int (ax+b)^n\, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

Exercise 20.1

1) Find an expression for y if $\frac{dy}{dx}$ is each of the following:

(a) $2x^3$

$\int (2x^3)\,dx = 2\int x^3\,dx$

$= 2\int x^3 dx = 2\left(\frac{x^{3+1}}{3+1}\right) + c$

$= \frac{1}{2}x^4 + c$

(b) -5

$\int -5\,dx = -5\int (1)\,dx$

$= -5\left(\frac{x^{0+1}}{0+1}\right) + c = -5x + c$

$\boxed{y = -5x + c}$

(c) $\sqrt{x}$

$\int (x^{1/2})\,dx = \frac{x^{1/2+1}}{1/2+1} + c$

$= \frac{x^{3/2}}{3/2} + c = \frac{2}{3}x^{3/2} + c$

(d) $-\frac{1}{x^2}$

$\int (-x^{-2})\,dx = -1\int (x^{-2})\,dx$

$= -1\left(\frac{x^{-2+1}}{-2+1}\right) + c = -1\left(\frac{x^{-1}}{-1}\right) + c$

$\boxed{y = \frac{1}{x} + c}$

(e) $\frac{2}{\sqrt{x}}$

$2\int (x^{-1/2})\,dx = 2\left(\frac{x^{-1/2+1}}{-1/2+1}\right) + c$

$= 2\left(\frac{x^{1/2}}{1/2}\right) + c = 4\sqrt{x} + c$

$\boxed{y = 4\sqrt{x} + c}$

(f) $\frac{1}{2x^3}$

$1/2 \int (x)^{-3}\,dx = 1/2\left(\frac{x^{-3+1}}{-3+1}\right) + c$

$= \frac{1}{2}\left(\frac{x^{-2}}{-2}\right) + c = -\frac{1}{4x^2} + c$

$\boxed{y = \frac{-1}{4x^2} + c}$

2) Find an expression for y if $\frac{dy}{dx}$ is each of the following:

(a) $6x + 3$

$3(2x+1) = 3\int 2x\,dx + 3\int 1\,dx + c$

$= 6\left(\frac{x^{1+1}}{1+1}\right) + 3\left(\frac{x^{0+1}}{0+1}\right) + c$

$\boxed{y = 3x^2 + 3x + c}$

(b) 4

$\int (4)\,dx = 4\int (1)\,dx + c$

$= 4\left(\frac{x^{0+1}}{0+1}\right) + c$

$\boxed{y = 4x + c}$

(c) $3x(x+2)$

$3x^2 + 6x = 3\int x^2\,dx + 6\int x\,dx$

$= 3\left(\frac{x^{2+1}}{2+1}\right) + 6\left(\frac{x^{1+1}}{1+1}\right) + c$

$\boxed{y = x^3 + 3x^2 + c}$

(d) $(x-1)(x+2)$

$x^2 + x - 2 = \int (x^2+x-2)\,dx$

$= \int x^2\,dx + \int x\,dx + 2\int 1\,dx + c$

$\boxed{y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 2x + c}$

(e) $x\left(2+\frac{1}{x}\right)$

: $2x+1 = 2\int x\,dx + \int 1\,dx + C$

$= 2\left(\frac{x^{1+1}}{1+1}\right) + \left(\frac{x^{0+1}}{0+1}\right) + C$

$\boxed{y = x^2 + x + C}$

(f) $\frac{2x^2+3}{x^3} = \frac{2x^2}{x^3} + \frac{3}{x^3} = \frac{2}{x} + \frac{3}{x^3}$

$= 2\int (x^{-1})\,dx + 3\int (x^{-3})\,dx + C$

$= 2\left(\frac{x^{-1+1}}{-1+1}\right) + 3\left(\frac{x^{-3+1}}{-3+1}\right) + C$

$= 2\left(\frac{1}{0}\right) + 3\left(\frac{x^{-2}}{-2}\right) + C$

(f) $\frac{2x^2+3}{x^2} = \frac{2x^2}{x^2} + \frac{3}{x^2} = 2 + 3x^{-2}$

$\int (2+3x^{-2})\,dx = 2\int (1)\,dx + 3\int (x^{-2})\,dx + C$

$= 2\left(\frac{x^{0+1}}{0+1}\right) + 3\left(\frac{x^{-2+1}}{-2+1}\right) + C = 2x - 3x^{-1} + C \rightarrow \boxed{y = 2x - \frac{3}{x} + C}$

3) Integrate with respect to x.

(a) $x^2 + \frac{1}{x^2}$

: $\int (x^2)\,dx + \int (x^{-2})\,dx + C$

$= \left(\frac{x^{2+1}}{2+1}\right) + \left(\frac{x^{-2+1}}{-2+1}\right) + C$

$= \frac{x^3}{3} + \left(-x^{-1}\right) + C$

$\boxed{y = \frac{1}{3}x^3 - \frac{1}{x} + C}$

(b) $\frac{x^2+1}{2x^2} = \frac{x^2}{2x^2} + \frac{1}{2x^2} = \frac{1}{2} + \frac{1}{2}x^{-2}$

: $\int (1/2 + 1/2\,x^{-2})\,dx = \int (1/2)\,dx + \int (1/2\,x^{-2})\,dx + C$

$= 1/2\int (1)\,dx + 1/2\int (x^{-2})\,dx + C$

$= \frac{1}{2}\left(\frac{x^{0+1}}{0+1}\right) + \frac{1}{2}\left(\frac{x^{-2+1}}{-2+1}\right) + C$

$\boxed{y = \frac{1}{2}x - \frac{1}{2x} + C}$

(c) $3 - \sqrt{x}$

: $3\int (1)\,dx - \int (x^{1/2})\,dx + C$

$= 3\left(\frac{x^{0+1}}{0+1}\right) - \left(\frac{x^{1/2+1}}{1/2+1}\right) + C$

$= 3x - \left(\frac{2}{3}x^{3/2}\right) + C$

$\boxed{y = 3x - \frac{2}{3}x^{3/2} + C}$

(d) $\sqrt{x}\left(\sqrt{x}+3\right)$

: $x + 3\sqrt{x} = \int (x)\,dx + 3\int (x^{1/2})\,dx + C$

$= \left(\frac{x^{1+1}}{1+1}\right) + 3\left(\frac{x^{1/2+1}}{1/2+1}\right) + C$

$= \frac{1}{2}x^2 + 3\left(\frac{2}{3}x^{3/2}\right) + C$

$\boxed{y = \frac{1}{2}x^2 + 2x^{\frac{3}{2}} + C}$

4) Given that a and b are constants, integrate with respect to x.

(a) $ax + b$

$: a\int (x)\,dx + b\int (1)\,dx + c$

$= a\left(\frac{x^{1+1}}{1+1}\right) + b\left(\frac{x^{0+1}}{0+1}\right) + c$

$= \frac{ax^2}{2} + bx + c$

$\boxed{y = \frac{1}{2}ax^2 + bx + c}$

(b) $a - bx^2$

$: a\int (1)\,dx - b\int (x)^2\,dx + c$

$= a\left(\frac{x^{0+1}}{0+1}\right) - b\left(\frac{x^{2+1}}{2+1}\right) + c$

$\boxed{y = ax - \frac{1}{3}bx^3 + c}$

5) Find

(a) $\int (2 + 4x - 3x^2)\,dx$

$: 2\int (1)\,dx + 4\int (x)\,dx - 3\int (x)^2\,dx + c$

$= 2\left(\frac{x^{0+1}}{0+1}\right) + 4\left(\frac{x^{1+1}}{1+1}\right) - 3\left(\frac{x^{2+1}}{2+1}\right) + c$

$= 2x + 2x^2 - x^3 + c$

$\boxed{y = 2x + 2x^2 - x^3 + c}$

(b) $\int \left(x^4 - \frac{1}{x^2}\right) dx,$

$: \int (x^4)\,dx - \int (x)^{-2}\,dx + c$

$= \left(\frac{x^{4+1}}{4+1}\right) - \left(\frac{x^{-2+1}}{-2+1}\right) + c$

$\boxed{y = \frac{1}{5}x^5 + \frac{1}{x} + c}$

(c) $\int (2x - \sqrt{x})^2\,dx$

$: \int \left((2x)^2 - 2(2x)(\sqrt{x}) + (\sqrt{x})^2\right) dx$

$= \int (4x^2 - 4x^{3/2} + x)\,dx$

$= 4\int (x^2)\,dx - 4\int (x^{3/2})\,dx + \int (x)\,dx + c$

$= 4\left(\frac{x^{2+1}}{2+1}\right) - 4\left(\frac{x^{3/2+1}}{3/2+1}\right) + \left(\frac{x^{1+1}}{1+1}\right) + c$

$= \frac{4}{3}x^3 - 4\left(\frac{2}{5}x^{5/2}\right) + \frac{1}{2}x^2 + c$

$\boxed{y = \frac{4}{3}x^3 - \frac{8}{5}x^{5/2} + \frac{1}{2}x^2 + c}$

(d) $\int \frac{x+1}{\sqrt{x}}\,dx$

$: \int \left(\frac{x}{\sqrt{x}} + \frac{1}{\sqrt{x}}\right) dx$

$= \int (x^{1/2})\,dx + \int (x^{-1/2})\,dx + c$

$= \left(\frac{x^{1/2+1}}{1/2+1}\right) + \left(\frac{x^{-1/2+1}}{-1/2+1}\right) + c$

$\boxed{y = \frac{2}{3}x^{3/2} - \frac{2}{3}\sqrt{x} + c}$

6) Find the equation of the curve which passes through the point $(2,4)$ and for which $\frac{dy}{dx} = x(3x-1)$

1:- $\int x(3x-1)\,dx = \int (3x^2 - x)\,dx$

$y = 3\int (x^2)\,dx - \int (x)\,dx + C$

$y = 3\left(\frac{x^{2+1}}{2+1}\right) - \left(\frac{x^{1+1}}{1+1}\right) + C$

$$\boxed{y = x^3 - \frac{1}{2}x^2 + C}$$

2: Find the value of c, where $x = 2$ and $y = 4$

$4 = (2)^3 - \frac{1}{2}(2)^2 + c$

$4 = 8 - 2 + c$

$\boxed{c = -2}$

$\boxed{y = x^3 - \frac{1}{2}x^2 - 2}$ → equation of curve

7) Find the equation of the curve which passes through the points $(2,-2)$ and $(4,2)$ and for which $\frac{dy}{dx} = x^2(x-k)$ where k is a constant.

1: $\int x^2(x-k)\,dx = \int (x^3 - x^2 k)\,dx$

$= \int (x^3)\,dx - k\int (x^2)\,dx + C$

$y = \left(\frac{x^{3+1}}{3+1}\right) - k\left(\frac{x^{2+1}}{2+1}\right) + C$

$$\boxed{y = \frac{1}{4}x^4 - \frac{k}{3}x^3 + c}$$

2: Put $x = 2$ and $y = -2$, also $x = 4$ and $y = 2$, after this you will get equations interms of k and c, then solve them simultaneously to get the value of k and c.

$-2 = \frac{1}{4}(2)^4 - \frac{k}{3}(2)^3 + c$

$-2 = 4 - \frac{8k}{3} + c$

$-6 = 12 - 8k + 3c$

$\boxed{3c - 8k + 18 = 0}$

$3c = 8k - 18$

$c = \frac{8k - 18}{3}$

$2 = \frac{1}{4}(4)^4 - \frac{k}{3}(4)^3 + c$

$2 = 64 - \frac{64k}{3} + c$

$6 = 192 - 64k + 3c$

$\boxed{3c - 64k + 186 = 0}$

$3\left(\frac{8k - 18}{3}\right) - 64k + 186 = 0$

$-56k + 168 = 0$

$\frac{168}{56} = k$

$\boxed{k = 3}$

$c = \frac{8(3) - 18}{3}$

$\boxed{c = 2}$

$y = \frac{1}{4}x^4 - \frac{k}{3}x^3 + c$

$$\boxed{y = \frac{1}{4}x^4 - x^3 + 2}$$

8) Given that the gradient of curve is $2x + \frac{3}{x^2}$ and that curve passes through the point $(-1, 5)$, determine the equation of the curve.

1:- $\int (2x + 3x^{-2})\, dx$

$y = 2\int (x)\, dx + 3\int (x)^{-2} dx + c$

$y = 2\left(\frac{x^{1+1}}{1+1}\right) + 3\left(\frac{x^{-2+1}}{-2+1}\right) + c$

$y = x^2 - 3x^{-1} + c$

$\boxed{y = x^2 - \frac{3}{x} + c}$

2: Find the value of c, for which given that $x = -1$ and $y = 5$

$5 = (-1)^2 - \frac{3}{-1} + c$

$5 = 1 + 3 + c$

$\boxed{c = 1}$

$\boxed{y = x^2 - \frac{3}{x} + 1}$

9) Find the equation of the curve which passes through the points $(1, 3)$ and $(2, 9)$ and whose gradient is proportional to $x(2x^2 - 3)$.

1:- $\int x(2x^2 - 3)\, dx = \int (2x^3 - 3x)\, dx$

$y = 2\int (x)^3 dx - 3\int (x) dx + c$

$y = 2\left(\frac{x^{3+1}}{3+1}\right) - 3\left(\frac{x^{1+1}}{1+1}\right) + c$

$\boxed{y = \frac{1}{2}x^4 - \frac{3}{2}x^2 + c}$

2: $(1, 3)$ and $(2, 9)$ find C value(s)

$3 = \frac{1}{2}(1)^4 - \frac{3}{2}(1)^2 + c$; $9 = \frac{1}{2}(2)^4 - \frac{3}{2}(2)^2 + c$

$3 = \frac{1}{2} - \frac{3}{2} + c$ $\qquad$ $9 = 8 - 6 + c$

$\boxed{4 = c}$ $\qquad$ $\boxed{c = 7}$

10) Given that the gradient of a curve is $x(2 - 3x)$ and that the curve passes through the points $(1, 2)$ and $(-2, p)$, find value of p.

1:- $\int x(2 - 3x)\, dx = \int (2x - 3x^2)\, dx$

$y = 2\int (x)\, dx - 3\int (x)^2 dx + c$

$y = 2\left(\frac{x^{1+1}}{1+1}\right) - 3\left(\frac{x^{2+1}}{2+1}\right) + c$

$\boxed{y = x^2 - x^3 + c}$

2: put $x = 1$ and $y = 2$

$2 = (1)^2 - (1)^3 + c$

$2 = 1 - 1 + c$

$\boxed{c = 2}$

$\boxed{y = x^2 - x^3 + 2}$

3: put $x = -2$ and $y = p$

$y = x^2 - x^3 + 2$

$p = (-2)^2 - (-2)^3 + 2$

$p = 4 - (-8) + 2$

$p = 4 + 8 + 2$

$\boxed{p = 14}$

11) Given that the gradient of a curve is $an-3$ and that the curve passes through the points $(-1,8)$ and $(3,4)$, find the equation of the curve.

:— $\frac{dy}{dn} = an - 3$

$y = \int (an-3)\,dn$

$y = a\int (n)\,dn - 3\int (1)\,dn + c$

$y = a\left(\frac{n^{1+1}}{1+1}\right) - 3\left(\frac{n^{0+1}}{0+1}\right) + c$

$\boxed{y = \frac{1}{2}an^2 - 3n + c}$

Put the two points and you will get two equations then solve them simultaneously to get the value of both a and c.

$(-1, 8)$; $(3, 4)$

$y = \frac{+1}{2}an^2 - 3n + c$

$8 = (0.5)(a)(-1)^2 - 3(-1) + c$; $4 = (0.5)(a)(3)^2 - 3(3) + c$

$8 = (0.5)a + 3 + c$; $4 = (4.5)a - 9 + c$

$\boxed{10 = a + 2c}$; $\boxed{26 = 9a + 2c}$

$a = 10 - 2c$; $26 = 9(10 - 2c) + 2c$

$a = 10 - 2(4)$; $26 = 90 - 18c + 2c$

$a = 10 - 8$; $26 = 90 - 16c$

$\boxed{a = 2}$; $\boxed{c = 4}$

$y = \frac{1}{2}an^2 - 3n + c$

$\boxed{y = n^2 - 3n + 4}$

12) Given that $\frac{dy}{dn}$ is directly proportional to $n^2 - 1$, and that $y = 3$ and $\frac{dy}{dn} = 9$ when $n = 2$, find the value of y when $n = 3$.

1: directly proportional

$\frac{dy}{dn} \propto (n^2-1) \Rightarrow \frac{dy}{dn} = k(n^2-1)$

Inverse

$\frac{dy}{dn} \propto \frac{1}{(n^2-1)} \Rightarrow \frac{dy}{dn} = \frac{k}{n^2-1}$

2: Directly proportional

$\frac{dy}{dn} = k(n^2-1)$

$dy/dn = 9$, $n = 2$

$9 = k(2^2 - 1)$

$\boxed{k = 3}$

3: $\frac{dy}{dn} = 3(n^2-1)$

$y = 3n^2 - 3$

$y = \int (3n^2 - 3)\,dn$

$y = 3\int n^2\,dn - 3\int 1\,dn + c$

$\boxed{y = n^3 - 3n + c}$

4: $y = 3$, $n = 2$

$3 = 2^3 - 3(2) + c$

$3 = 8 - 6 + c$

$\boxed{c = 1}$

$y = n^3 - 3n + c$

5: $n = 3$, $y = ?$

$y = (3)^3 - 3(3) + 1$

$y = 27 - 9 + 1$

$y = 28 - 9$

$\boxed{y = 19}$

13) Find x as a function of t given that $\frac{dx}{dt} = 3t^2+2$ and that $x=1$ when $t=0$.

1: $\frac{dx}{dt} = 3t^2+2$

$dx = (3t^2+2)\,dt$

$\int dx = \int (3t^2+2)\,dt$

$x = \int (3t^2+2)\,dt$

$x = 3\int t^2 + 2\int 1\,dx + c$

$\boxed{x = t^3 + 2t + c}$

2: $x=1$, $t=0$

$1 = 0^3 + 2(0) + c$

$\boxed{c = 1}$

3: $x = t^3 + 2t + 1$

$\boxed{x = t^3 + 2t + 1}$

14) The rate of change of the Area, A cm^2, of a circle is $6t^2-2t+1$. Find A in terms of t if the area of the circle is 11 cm^2 when $t=2$.

1: $\frac{dA}{dt} = 6t^2-2t+1$

$dA = (6t^2-2t+1)\,dt$

$\int dA = \int (6t^2-2t+1)\,dt$

$A = 6\int t^2 - 2\int t + 1\int 1\,dt + c$

$A = 6\left(\frac{t^3}{3}\right) - 2\left(\frac{t^2}{2}\right) + t + c$

$\boxed{A = 2t^3 - t^2 + t + c}$

2: $A = 11$, $t = 2$

$11 = 2(2)^3 - (2)^2 + 2 + c$

$11 = 2(8) - 4 + 2 + c$

$11 = 16 - 2 + c$

$11 = 14 + c$

$\boxed{c = -3}$

$\boxed{A = 2t^3 - t^2 + t - 3}$

15) Integrate with respect to x.

(a) $(3x+1)^4$

$\frac{dy}{dx} = (3x+1)^4$

$dy = (3x+1)^4(dx)$

$\int dy = \int (3x+1)^4(dx)$

$y = \int (3x+1)^4\,dx$

$y = \frac{(3x+1)^{4+1}}{3(4+1)} + c$

$\boxed{y = \frac{(3x+1)^5}{15} + c}$

(b) $(1-x)^3$

$\frac{dy}{dx} = (1-x)^3$

$dy = (1-x)^3\,dx$

$\int dy = \int (1-x)^3\,dx$

$y = \int (1-x)^3\,dx$

$y = \frac{(1-x)^{3+1}}{-1(3+1)} + c$

$\boxed{y = \frac{(1-x)^4}{-4} + c}$

(c) $(2x+5)^{-3}$

$\frac{dy}{dx} = (2x+5)^{-3}$

$dy = (2x+5)^{-3}\,dx$

$\int dy = \int (2x+5)^{-3}\,dx$

$y = \int (2x+5)^{-3}\,dx$

$y = \frac{(2x+5)^{-3+1}}{2(-3+1)} + c$

$\boxed{y = \frac{(2x+5)^{-2}}{-4} + c}$

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d) $\sqrt{6x-1}$

$\therefore \frac{dy}{dx} = \sqrt{6x-1}$

$dy = (\sqrt{6x-1})\,dx$

$\int dy = \int (\sqrt{6x-1})\,dx$

$y = \int (6x-1)^{1/2}\,dx$

$y = \frac{(6x-1)^{\frac{1}{2}+1}}{6(1/2+1)} + c$

$\boxed{y = \frac{(6x-1)^{3/2}}{9} + c}$

(e) $\frac{2}{(2x-7)^2}$

$\therefore \frac{dy}{dx} = \frac{2}{(2x-7)^2}$

$dy = \frac{2}{(2x-7)^2}\,dx$

$y = \int \frac{2}{(2x-7)^2}\,dx$

$y = \frac{2(2x-7)^{-2+1}}{2(-2+1)} + c$

$y = -\frac{1}{2x-7} + c$

f) $\frac{1}{\sqrt{3-2x}}$

$\therefore \frac{dy}{dx} = (3-2x)^{-1/2}$

$\int dy = \int (3-2x)^{-1/2}\,dx$

$y = \frac{(3-2x)^{-\frac{1}{2}+1}}{-2(-1/2+1)} + c$

$y = \frac{(3-2x)^{1/2}}{} + c$

$\boxed{y = -\sqrt{3-2x} + c}$

(g) $\frac{3}{5(3x-1)^6}$

$\therefore \frac{dy}{dx} = \frac{3}{5}(3x-1)^{-6}$

$\int dy = \frac{3}{5}\int (3x-1)^{-6}\,dx$

$y = \frac{3}{5}\left(\frac{(3x-1)^{-6+1}}{3(-6+1)}\right) + c$

$y = \frac{(3x-1)^{-5}}{-25} + c$

$\boxed{y = -\frac{1}{25(3x-1)} + c}$

(h) $\frac{4}{3\sqrt{6x-1}}$

$\therefore \frac{dy}{dx} = 4(6x-1)^{-1/2}$

$\int dy = \frac{4}{3}\int (6x-1)^{-1/2}\,dx$

$y = \frac{4}{3}\left(\frac{(6x-1)^{-\frac{1}{2}+1}}{6(-1/2+1)}\right) + c$

$y = \frac{4}{9}\sqrt{6x-1} + c$

(i) $\left(\frac{4}{1-2x}\right)^2$

$\therefore \frac{dy}{dx} = \left(\frac{4}{1-2x}\right)^2$

$\int dy = \int \left(\frac{4}{1-2x}\right)^2 dx$

$y = \int \left(\frac{16}{(1-2x)^2}\right) dx$

$y = 16\int (1-2x)^{-2}\,dx$

$y = 16\left(\frac{(1-2x)^{-2+1}}{-2(-2+1)}\right) + c$

$y = 16\left(\frac{(1-2x)^{-1}}{2}\right) + c$

$y = \frac{8}{1-2x} + c$

16) Find

(a) $\int (1-x)^6 dx$

$y = \frac{(1-x)^{6+1}}{-1(6+1)} + c$

$y = \frac{(1-x)^7}{-7} + c$

$\boxed{y = \frac{-1}{7}(1-x)^7 + c}$

(b) $\int 3(2x-5)^2 dx$

$y = 3\int (2x-5)^2 dx$

$y = 3\left(\frac{(2x-5)^{2+1}}{2(2+1)}\right) + c$

$\boxed{y = \frac{(2x-5)^3}{2} + c}$

(c) $\int \frac{2}{(1-x)^2} dx$

:- $y = 2\int (1-x)^{-2} dx$

$y = 2\left(\frac{(1-x)^{-2+1}}{-1(-2+1)}\right) + c$

$y = 2\left(\frac{(1-x)^{-1}}{1}\right) + c$

$\boxed{y = \frac{2}{1-x} + c}$

(d) $\int \sqrt{4t-1}\, dt$

$y = \int (4t-1)^{1/2} dt$

$y = \frac{(4t-1)^{1/2+1}}{4(1/2+1)} + c$

$\boxed{y = \frac{(4t-1)^{3/2}}{6} + c}$

17) Given that $\frac{dy}{dx} = (3x-2)^2$ and that $y = 0$ when $x = 1$ Calculate the value of y when $x = 1.5$.

: $\frac{dy}{dx} = (3x-2)^2$

$dy = (3x-2)^2 dx$

$\int dy = \int (3x-2)^2 dx$

$y = \int (3x-2)^2 dx$

$y = \left(\frac{(3x-2)^{2+1}}{3(2+1)}\right) + c$

$\boxed{y = \frac{(3x-2)^3}{9} + c}$

$0 = \frac{(3(1)-2)^3}{9} + c$

$\boxed{c = -1}$

$y = \frac{(3x-2)^3}{9} - 1$

$y = \frac{(3(1.5)-2)^3}{9} - 1$

$y = \frac{(4.5-2)^3}{9} - 1 = \frac{(2.5)^3}{9} - 1$

$y = \frac{(2.5)^3 - 9}{9} = \frac{15.625 - 9}{9}$

$y = \frac{6.625}{9} = \frac{2.208}{3}$

18) The gradient of a curve is $6(4x-1)^2$ and the curve passes through the origin. Find the equation of curve

:- $\frac{dy}{dx} = 6(4x-1)^2$

$\therefore$ $dy = 6(4x-1)^2\,dx$

$\int dy = \int 6(4x-1)^2\,dx$

$y = \int 6(4x-1)^2\,dx$

$y = 6\int (4x-1)^2\,dx$

$y = 6\left(\frac{(4x-1)^{2+1}}{4(2+1)}\right) + c$

$y = 6\left(\frac{(4x-1)^3}{12}\right) + c$

$\boxed{y = \frac{(4x-1)^3}{2} + c}$

$\therefore$ origin $(0,0)$

$0 = \frac{(4(0)-1)^3}{2} + c$

$\boxed{c = 1/2}$

$\boxed{y = \frac{(4x-1)^3}{2} + \frac{1}{2}}$

19) Find A as a function of x given that $\frac{dA}{dx} = 3(x-1)^2$ and that $A = 10$ when $x = 3$.

:- $\frac{dA}{dx} = 3(x-1)^2$

$dA = 3(x-1)^2\,dx$

$\int dA = 3\int (x-1)^2\,dx$

$A = 3\left(\frac{(x-1)^{2+1}}{(2+1)}\right) + c$

$\boxed{A = (x-1)^3 + c}$

$A = (x-1)^3 + c$

$10 = (3-1)^3 + c$

$10 = (2)^3 + c$

$\boxed{2 = c}$

$\boxed{A = (x-1)^3 + 2}$

20) Find s as a function of t given that $\frac{ds}{dt} = 6(2t-1)^2 + 1$ and that $s = 4$ when $t = 1$

:- $\frac{ds}{dt} = 6(2t-1)^2 + 1$

$\int ds = 6\int (2t-1)^2\,dt + 1\int 1\,dt$

$s = 6\left(\frac{(2t-1)^3}{3}\right) + t + c$

$\boxed{s = 2(2t-1)^3 + t + c}$

$s = 2(2t-1)^3 + t + c$

$4 = 2(2(1)-1)^3 + 1 + c$

$4 = 2(1)^3 + 1 + c$

$\boxed{c = 1}$

$s = 2(2t-1)^3 + t + 1$

$$\frac{ds}{dt} = 6(2t-1)^2+1$$

$$ds = \left(6(2t-1)^2+1\right)dt$$

$$\int ds = 6\int (2t-1)^2\,dt + 2\int (1)\,dt$$

$$S = 6\left(\frac{(2t-1)^{2+1}}{2+1}\right) + \left(\frac{t^{0+1}}{0+1}\right) + C$$

$$S = 6\left(\frac{(2t-1)^3}{3}\right) + t + C$$

$$\boxed{S = 2(2t-1)^3 + t + C}$$

$$S = 2(2t-1)^3 + t + C$$

$$4 = 2(2(1)-1)^3 + 1 + C$$

$$4 = 2(2-1)^3 + C$$

$$4 = 2(1)^3 + C + 1$$

$$4 = 2 + C + 1$$

$$\boxed{C = 1}$$

$$\boxed{S = 2(2t-1)^3 + t + 1}$$

4.2 Exercise

In Exercises 1-6, Find the Sum.

1) $\sum_{i=1}^{6}(3i+2) = (3(1)+2)+(3(2)+2)+(3(3)+2)+(3(4)+2)+(3(5)+2)+(3(6)+2)$

$= 5+8+11+14+17+20 = 75$

2) $\sum_{k=5}^{8} k(k-4) = \cancel{5}(5-4)+\cancel{8}(6-4)+7(7-4)+8(8-4)$

$= 5+12+21+32$

$$\sum_{k=5}^{8} k(k-4) = 5(5-4)+6(6-4)+7(7-4)+8(8-4)$$

$$= 5(1) + 6(2) + 7(3) + 8(4)$$

$$= 5 + 12 + 21 + 32 = 70$$

3) $\sum_{k=0}^{4} \frac{1}{k^2+1} = \left(\frac{1}{0^2+1}\right)+\left(\frac{1}{1^2+1}\right)+\left(\frac{1}{2^2+1}\right)+\left(\frac{1}{3^2+1}\right)+\left(\frac{1}{4^2+1}\right)$

$= 1+0.5+0.2+0.10+0.058 = 1.85$

4) $\sum_{j=4}^{7} \frac{2}{j} = \left(\frac{2}{4}\right)+\left(\frac{2}{5}\right)+\left(\frac{2}{6}\right)+\left(\frac{2}{7}\right) = 0.5+0.4+0.33+0.28$

$= 1.51$

5) $\sum_{k=1}^{4} c = 1c+2c+3c+4c$ $\boxed{\text{WRONG}}$

Use Theorem $\rightarrow$ $\sum_{i=1}^{n} c = cn$

$\sum_{k=1}^{4} c = 4c$

6) $\sum_{i=1}^{4}\left[(i-1)^2+(i+1)^3\right]$

$= \left[(1-1)^2+(1+1)^3\right]+\left[(2-1)^2+(2+1)^3\right]+\left((3-1)^2+(3+1)^3\right]+\left[(4-1)^2+(4+1)^3\right]$

$= (8)+(28)+(68)+(134) = 208$

25) Consider the function $f(x) = 3x+2$

(a) Estimate the Area between the graph of f and the x-axis between $x=0$ and $x=3$ using six rectangles and right endpoints. sketch the graph and rectangles.

(b) Repeat (a) using left endpoints.

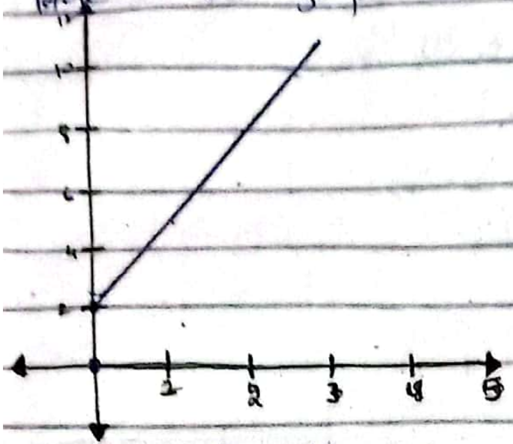
Solution

Follow the Pattern $\rightarrow$ Schat

1: Subintervals 2: endpoints (left & Right)

3: Heights 4: Areas 5: total Sum of Areas

Here is the graph of the function $f(x) = 3x+2$, $[0,3]$



Pattern → Schat

1: Subintervals

2: endpoints [left and Right]

3: Heights

4: Areas

5: Total. Sum of Areas.

1: Subintervals with respect to rectangles.

rectangles number = 6 ⇒ $\frac{0+3}{6} = 0.5$ ⇒ distance at each point

⇒ $\underset{1}{[0, 0.5]}, \underset{2}{[0.5, 1]}, \underset{3}{[1, 1.5]}, \underset{4}{[1.5, 2]}, \underset{5}{[2, 2.5]}, \underset{6}{[2.5, 3]}$

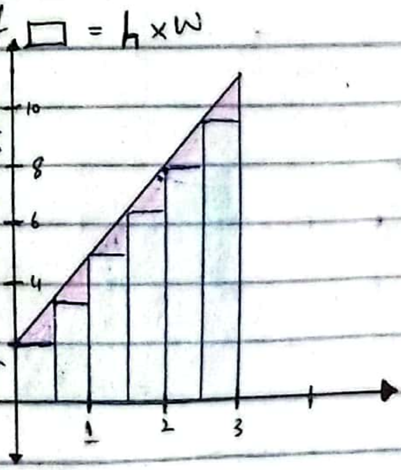
2: left end points: 0, 0.5, 1, 1.5, 2, 2.5

3: Heights w.r.t left endpoints (lower Sum | Inscribed)

- For First rectangle (0 to 0.5), $y = 3(0)+2 = 2$
- For Second rectangle (0.5 to 1), $y = 3(0.5)+2 = 3.5$
- For third rectangle (1 to 1.5), $y = 3(1)+2 = 5$
- For Fourth rectangle (1.5 to 2), $y = 3(1.5)+2 = 6.5$
- For fifth rectangle (2 to 2.5), $y = 3(2)+2 = 8$
- For Sixth rectangle (2.5 to 3), $y = 3(2.5)+2 = 9.5$

4: Areas : Calculate height of each rectangle

Height of each rectangle has been found in (3), where width of the each rectangle is 0.5 by (2). Area of □ = h × w



- Area for first rectangle = $(2) \times (0.5) = 1$
- Area for Second rectangle = $(3.5) \times (0.5) = 1.75$
- Area for third rectangle = $(5) \times (0.5) = 2.5$
- Area for fourth rectangle = $(6.5) \times (0.5) = 3.25$
- Area for fifth rectangle = $(8) \times (0.5) = 4$
- Area for Sixth rectangle = $(9.5) \times (0.5) = 4.75$

5: Total Sum of Areas

Area = $1 + 1.75 + 2.5 + 3.25 + 4 + 4.75 = 17.25$

(lower Sum | leftendpoints | Inscribed) Area ≈ 17.25

- Right end points

1: Subintervals:

[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]

2: Right end points: 0.5, 1, 1.5, 2, 2.5, 3

3: Heights:

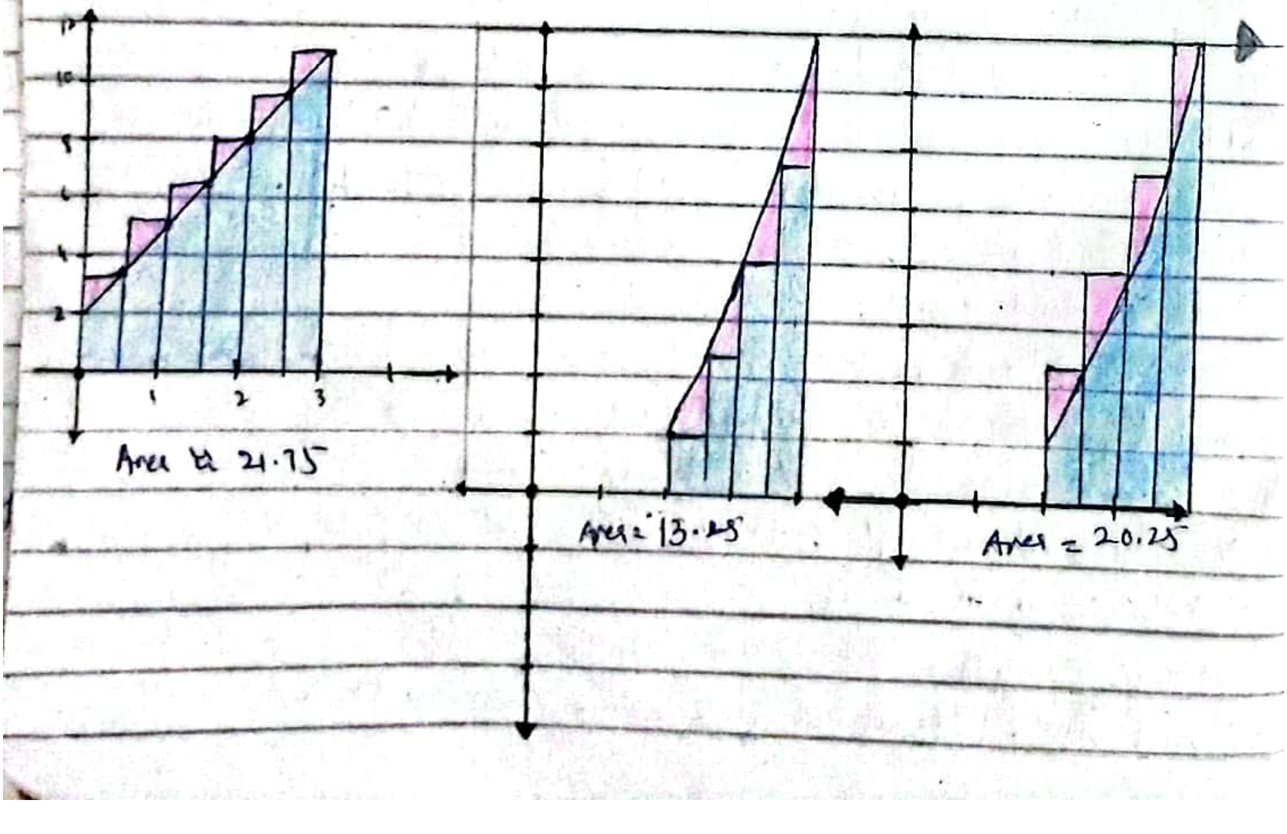
- First rectangle: $y = 3(0.5)+2 = 3.5$
- Second rectangle: $y = 3(1)+2 = 5$
- third rectangle: $y = 3(1.5)+2 = 6.5$
- fourth rectangle: $y = 3(2)+2 = 8$
- fifth rectangle: $y = 3(2.5)+2 = 9.5$
- Sixth rectangle: $y = 3(3)+2 = 11$

4: Areas: $\rightarrow h \times w$

- Area of first rectangle = $(3.5) \times (0.5) = 1.75$
- Area of Second rectangle = $(5) \times (0.5) = 2.5$
- Area of third rectangle = $(6.5) \times (0.5) = 3.25$
- Area of fourth rectangle = $(8) \times (0.5) = 4$
- Area of fifth rectangle = $(9.5) \times (0.5) = 4.75$
- Area of Sixth rectangle = $(11) \times (0.5) = 5.5$

5: Total Sum of Areas

Area = $1.75 + 2.5 + 3.25 + 4 + 4.75 + 5.5 = 21.75$ $\boxed{\text{Area} \approx 21.75}$



26: Consider the function $g(x) = x^2 + x - 4$. Estimate the area between the graph of g and the x-axis between $x=2$ and $x=4$ using four rectangles and both right and left end points, also sketch g.

Solution

$g(x) = x^2 + x - 4$, rectangles $= 4$, main interval $[2, 4]$;

Pattern Sehat

1: Sub intervals: Total 4, Common difference $\frac{4-2}{4} = \frac{2}{4} = 0.5$;
$[2, 2.5]$, $[2.5, 3]$, $[3, 3.5]$, $[3.5, 4]$

2: Left end points : $2, 2.5, 3, 3.5$

3: Heights : $y = x^2 + x - 4$

1st : $y = (2)^2 + (2) - 4 = 4 + 2 - 4 = 2$

2nd : $y = (2.5)^2 + (2.5) - 4 = 6.25 + 2.5 - 4 = 4.75$

3rd : $y = (3)^2 + (3) - 4 = 9 + 3 - 4 = 8$

4th : $y = (3.5)^2 + (3.5) - 4 = 12.25 + 3.5 - 4 = 11.75$

4: Areas: $h \times w$, width of each rectangle $= 0.5$ from (1)

1st : $(2) \times (0.5) = 1$, 2nd : $(4.75) \times (0.5) = 2.375$

3rd : $(8) \times (0.5) = 4$, 4th : $(11.75) \times (0.5) = 5.875$

5: Total Sum of Areas:

Area $= 1 + 2.375 + 4 + 5.875 = 13.25$, $\boxed{\text{Area} = 13.25}$

Right end points

1: Sub intervals : $[2, 2.5]$, $[2.5, 3]$, $[3, 3.5]$, $[3.5, 4]$

2: Right end points: $2.5, 3, 3.5, 4$

3: Height: $y = x^2 + x - 4$

1st : $y = (2.5)^2 + (2.5) - 4 = 4.75$

2nd : $y = (3)^2 + (3) - 4 = 8$

3rd : $y = (3.5)^2 + (3.5) - 4 = 11.75$

4th : $y = (4)^2 + (4) - 4 = 16$

4: Areas: width of each rectangle $= 0.5$ from (1)

1st : $(4.75) \times (0.5) = 2.375$ 2nd : $(8) \times (0.5) = 4$

3rd : $(11.75) \times (0.5) = 5.875$ 4th : $(16) \times (0.5) = 8$

5: Total Sum of Areas :

Area $= 2.375 + 4 + 5.875 + 8 = 20.25$, $\boxed{\text{Area} \approx 20.25}$

In Exercises 27 – 32, use left and right endpoints and the given number of rectangles to find two approximation of the area of the region between the graph of the function and the x-axis over the given interval.

27) $f(x) = 2x + 5$, $[0, 2]$, 4 rectangles

Solution → Sehal, left endpoints:

1: Sub intervals: $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$

2: Left endpoints: 0, 0.5, 1, 1.5

3: Heights:

1st: $f(0) = 2(0) + 5 = 5$; 2nd: $f(0.5) = 2(0.5) + 5 = 6$

3rd: $f(1) = 2(1) + 5 = 7$; 4th: $f(1.5) = 2(1.5) + 5 = 8$

4: Areas: $h \times w$, width of each rectangle: $\frac{2-0}{4} = \frac{2}{4} = 0.5$

1st: $(5) \times (0.5) = 2.5$; 2nd: $(6) \times (0.5) = 3$

3rd: $(7) \times (0.5) = 3.5$; 4th $(8) \times (0.5) = 4$

5: Total Sum of Areas:

Area $= 2.5 + 3 + 3.5 + 4 = 13$, Area ≈ 13

Right endpoints:

1: Sub intervals: $[0, 0.5]$, $[0.5, 1]$, $[1, 1.5]$, $[1.5, 2]$

2: Right endpoints: 0.5, 1, 1.5, 2

3: Heights:

1st: $y = 2(0.5) + 5 = 6$; 2nd: $y = 2(1) + 5 = 7$

3rd: $y = 2(1.5) + 5 = 8$; 4th: $y = 2(2) + 5 = 9$

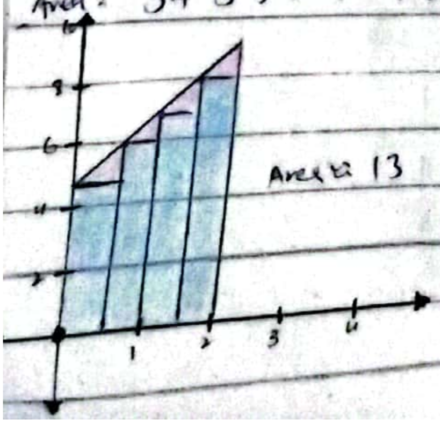
Areas: $h \times w$, width of each rectangle is 0.5

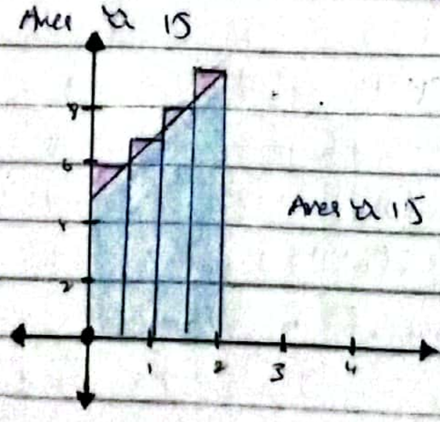
1st: $(6) \times (0.5) = 3$; 2nd: $(7) \times (0.5) = 3.5$

3rd: $(8) \times (0.5) = 4$; 4th $(9) \times (0.5) = 4.5$

5: Total Sum of Areas

Area $= 3 + 3.5 + 4 + 4.5 = 15$, Area ≈ 15





28) $f(x) = 9 - x$, $[2, 4]$, 6 rectangles

Solution → Sehal

1: Sub intervals: Common distance (width) of each rectangle $= \frac{4-2}{6} = \frac{2}{6} = 0.33$

2: Left end points to $[2, 2.33], [2.33, 2.66], [2.66, 3], [3, 3.33], [3.33, 3.66], [3.66, 4]$

→ $2, 2.33, 2.66, 3, 3.33, 3.66$

3: Heights:

1st: $f(2) = 9 - 2 = 7$; 2nd: $f(2.33) = 9 - 2.33 = 6.66$

3rd: $f(2.66) = 9 - 2.66 = 6.33$; 4th: $f(3) = 9 - 3 = 6$

5th: $f(3.33) = 9 - 3.33 = 5.66$; 6th: $f(3.66) = 9 - 3.66 = 5.33$

4: Areas: $h \times w$, width of each rectangle $= 0.33$ from (1)

1st: $(7) \times (0.33) = 2.33$ 2nd: $(6.66) \times (0.33) = 2.22$

3rd: $(6.33) \times (0.33) = 2.11$ 4th: $(6) \times (0.33) = 2$

5th: $(5.66) \times (0.33) = 1.87$ 6th: $(5.33) \times (0.33) = 1.76$

5: Total Sum of Areas:

Area $= 2.33 + 2.22 + 2.11 + 2 + 1.88 + 1.77 = 12.3$, $\boxed{\text{Area} \approx 12.3}$

• Right endpoints 1: Subintervals: $[2, 2.33], [2.33, 2.66], [2.66, 3], [3, 3.33], [3.33, 3.66], [3.66, 4]$

2: Right end points: $2.33, 2.66, 3, 3.33, 3.66, 4$

3: Heights:

1st: $f(2.33) = 9 - 2.33 = 6.66$; 2nd: $f(2.66) = 9 - 2.66 = 6.33$

3rd: $f(3) = 9 - 3 = 6$; 4th: $f(3.33) = 9 - 3.33 = 5.66$

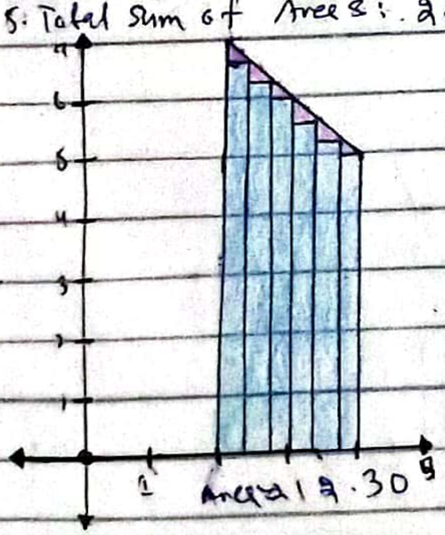
5th: $f(3.66) = 9 - 3.66 = 5.33$; 6th: $f(4) = 9 - 4 = 5$

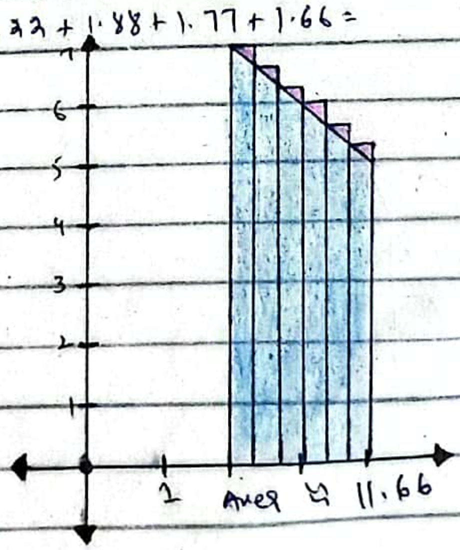
Areas:

1: $(6.66) \times (0.33) = 2.22$; 2: $(6.33) \times (0.33) = 2.11$; 3: $(6) \times (0.33) = 2$;

4: $(5.66) \times (0.33) = 1.88$; 5: $(5.33) \times (0.33) = 1.77$; 6: $(5) \times (0.33) = 1.66$

5: Total sum of Areas: $2 + 2.11 + 2.22 + 1.88 + 1.77 + 1.66 =$





28) $f(x) = 2x^2 - x - 1$, $[2, 5]$, 6 rectangles

<u>Solution Schal</u>

1: Subintervals: $[2, 2.5]$, $[2.5, 3]$, $[3, 3.5]$, $[3.5, 4]$, $[4, 4.5]$, $[4.5, 5]$

2: Endpoints: Right: 2.5, 3, 3.5, 4, 4.5, 5] Left: 2, 2.5, 3, 3.5, 4, 4.5

3: Heights: Left:

1: $y = 2(2)^2 - (2) - 1 = 5$; 2: $y = 2(2.5)^2 - (2.5) - 1 = 8$; 3: $y = 2(3)^2 - (3) - 1 = 14$

4: $y = 2(3.5)^2 - (3.5) - 1 = 20$; 5: $y = 2(4)^2 - (4) - 1 = 27$; 6: $y = 2(4.5)^2 - (4.5) - 1 = 35$

Right 1: $y = 2(2.5)^2 - (2.5) - 1 = 8$; 2: $y = 2(3)^2 - (3) - 1 = 14$; 3: $y = 2(3.5)^2 - (3.5) - 1 = 20$

4: $y = 2(4)^2 - (4) - 1 = 27$; 5: $y = 2(4.5) - (4.5) - 1 = 35$; 6: $y = 2(5)^2 - (5) - 1 = 44$

4: Areas: Left:

1: $(5) \times (0.5) = 2.5$; 2: $(9) \times (0.5) = 4.5$; 3: $(14) \times (0.5) = 7$

4: $(20) \times (0.5) = 10$; 5: $(27) \times (0.5) = 13.5$; 6: $(35) \times (0.5) = 17.5$

Right:

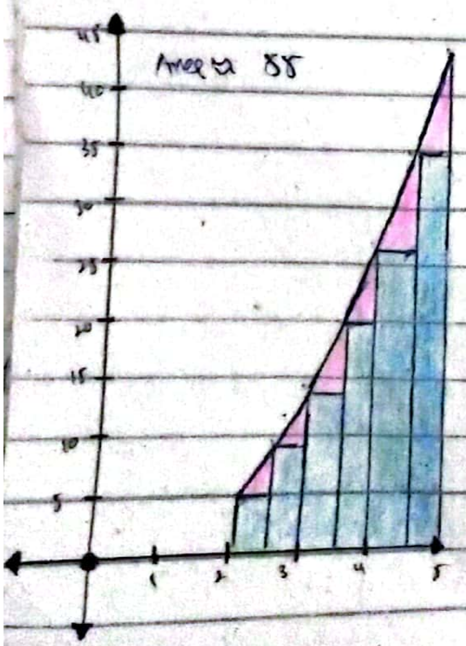
1: $(9) \times (0.5) = 4.5$; 2: $(14) \times (0.5) = 7$; 3: $(20) \times (0.5) = 10$

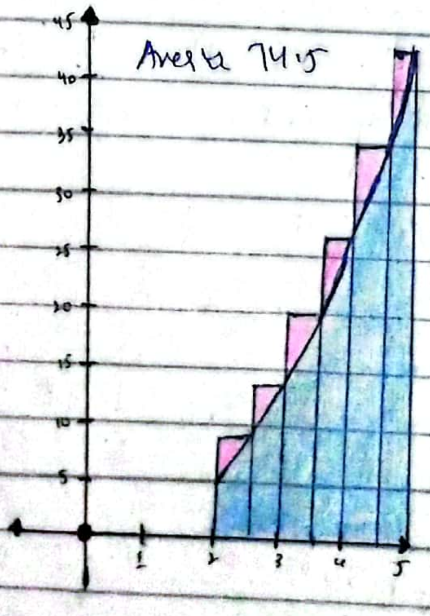
4: $(27) \times (0.5) = 13.5$; 5: $(35) \times (0.5) = 17.5$; 6: $(44) \times (0.5) = 22$

Total Sum

Left: $2.5 + 4.5 + 7 + 10 + 13.5 + 17.5 = 55$ $\boxed{\text{Area} \approx 55}$

Right: $4.5 + 7 + 10 + 13.5 + 17.5 + 22 = 74.5$ $\boxed{\text{Area} \approx 74.5}$





20.2 Definite Integrals [[illegible] theory]

Consider an expression $F(x)$ such that $\frac{d}{dx}(F(x)) = 3x^2$.
what is the value of $F(b) - F(a)$?

Since $\frac{d}{dx}(F(x)) = 3x^2$, $F(x) = \int 3x^2\,dx = \underline{x^3 + c}$

Then $F(b) - F(a) = (b^3 + c) - (a^3 + c) = b^3 - a^3$
(c is cancelled out)

or Simply $F(b) - F(a) = \left[x^3\right]_a^b$

which can be written as
$$F(b) - F(a) = \left[\int 3x^2\,dx\right]_a^b$$

Or $$F(b) - F(a) = \int_a^b 3x^2\,dx$$

- The integral $\int_a^b f(x)\,dx$ is known as a definite integral.

Ex: $$\frac{d}{dx}(3x^2 + 2x) = 6x + 2$$
$$\int_2^3 (6x + 2)\,dx = \left[3x^2 + 2x\right]_2^3$$
$$= \left[3(3)^2 + 2(3)\right] - \left[3(2)^2 + 2(2)\right]$$
$$= 17$$

Note that we ignore the constant c in the integral since it is cancelled out as mentioned earlier.

▸ Some Results Regarding Definite Integrals

(a) $$\int_a^a f(x)\,dx = 0$$

(b) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(c) $$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Proof:

(a) $\int_a^a f(x)\,dx = F(a) - F(a) = 0$

(b) $\int_a^b f(x)\,dx = F(b) - F(a) = -[F(a) - F(b)] = -\int_b^a f(x)\,dx$

c) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = [F(b) - F(a)] + [F(c) - F(b)]$

$= F(c) - F(a)$

$= \int_a^c f(x)\,dx$

20.2

1) Evalute the following integrals

(a) $\int_2^5 3x \, dx$

: $\left[\int_2^5 3x \, dx\right] = 3\int_2^5 x \, dx = 3\int_2^5 x \, dx = 3\left[\frac{x^{1+1}}{1+1}\right]_2^5 = \left[\frac{3}{2}x^2\right]_2^5$

$= \left[\frac{3}{2}(5)^2\right] - \left[\frac{3}{2}(2)^2\right] = \frac{75}{2} - 6 = 37.5 - 6 = 31.5$

(b) $\int_1^9 x^{\frac{1}{2}} \, dx$

: $\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^9 = \left[\frac{x^{3/2}}{3/2}\right]_1^9 = \left[\frac{(9)^{\frac{3}{2}}}{3/2}\right] - \left[\frac{(1)^{3/2}}{3/2}\right] = 18 - 0.66 = 17.33$

(c) $\int_1^8 \frac{1}{2} x^{-\frac{1}{3}} \, dx$

: $\frac{1}{2}\int_1^8 x^{-1/3} \, dx = \frac{1}{2}\left[\frac{x^{-1/3+1}}{-1/3+1}\right]_1^8 = \frac{1}{2}\left(\left[\frac{(8)^{2/3}}{2/3}\right] - \left[\frac{(1)^{2/3}}{2/3}\right]\right)$

$= \frac{1}{2}\left[\left(\frac{4}{2/3}\right) - \left(\frac{1}{2/3}\right)\right] = \frac{1}{2}\left(\frac{9}{2}\right) = \frac{9}{4} = 2.25$

(d) $\int_2^3 \frac{1}{3x^2} \, dx$

: $\frac{1}{3}\int_2^3 \frac{1}{x^2} \, dx = \frac{1}{3}\int_2^3 x^{-2} \, dx = \frac{1}{3}\left[\frac{x^{-2+1}}{-2+1}\right]_2^3 = \frac{1}{3}\left[\frac{x^{-1}}{-1}\right]_2^3 = \frac{1}{3}\left[-\frac{1}{x}\right]_2^3$

$= \frac{1}{3}\left[\left(-\frac{1}{3}\right) - \left(-\frac{1}{2}\right)\right] = \frac{1}{3}\left(\frac{1}{6}\right) = \frac{1}{18}$

e) $\int_4^9 \frac{1}{\sqrt{x}} \, dx$

: $\int_4^9 x^{-1/2} \, dx = \left[\frac{x^{-1/2+1}}{-1/2+1}\right]_4^9 = \left[\frac{x^{1/2}}{1/2}\right]_4^9 = \left[\left(\frac{(9)^{1/2}}{1/2}\right) - \left(\frac{(4)^{1/2}}{1/2}\right)\right]$

$= \left[\left(\frac{3}{1/2}\right) - \left(\frac{2}{1/2}\right)\right] = 6 - 4 = 2$

1/2

(f) $\int_1^4 x\sqrt{x}\,dx$

: $\int_1^4 x^{3/2}\,dx = \left[\frac{x^{3/2+1}}{3/2+1}\right]_1^4 = \left[\left(\frac{(4)^{5/2}}{5/2}\right) - \left(\frac{(1)^{5/2}}{5/2}\right)\right]$

$= \left[\left(\frac{32}{5/2}\right) - \left(\frac{1}{5/2}\right)\right] = \frac{31}{5/2} = \frac{62}{5} = 12.4$

2) Evalute the following definite integrals.

(a) $\int_{-1}^{1} (8x-4)\,dx$

: $8\int_{-1}^{1} x\,dx - 4\int_{-1}^{1} dx = 8\left[\frac{x^{1+1}}{1+1}\right]_{-1}^{1} - 4\left[\frac{x^{0+1}}{0+1}\right]_{-1}^{1}$

$= 8\left[\frac{x^2}{2}\right]_{-1}^{1} - 4\left[x\right]_{-1}^{1} = 8\left[\left(\frac{1^2}{2}\right) - \left(\frac{(-1)^2}{2}\right)\right] - 4\left[(1) - (-1)\right]$

$= 8\left[\frac{1}{2} - \frac{1}{2}\right] - 4\left[1+1\right] = 0 - 4[2] = -8$

(b) $\int_{-1}^{0} (3x^2 - 2x + 5)\,dx$

: $3\int_{-1}^{0} x^2 - 2\int_{-1}^{0} x + 5\int_{-1}^{0} dx = 3\left[\frac{x^{2+1}}{2+1}\right]_{-1}^{0} - 2\left[\frac{x^{1+1}}{1+1}\right]_{-1}^{0} + 5\left[\frac{x^{0+1}}{0+1}\right]_{-1}^{0}$

$= 3\left[\frac{x^3}{3}\right]_{-1}^{0} - 2\left[\frac{x^2}{2}\right]_{-1}^{0} + 5\left[x\right]_{-1}^{0}$

$= \left[x^3\right]_{-1}^{0} - \left[x^2\right]_{-1}^{0} + 5\left[x\right]_{-1}^{0}$

$= \left[(0)^3 - (-1)^3\right] - \left[(0)^2 - (-1)^2\right] + 5\left[0 - (-1)\right]$

$= \left[0 - (-1)\right] - \left[0 - (1)\right] + 5\left[0+1\right]$

$= [1] - [-1] + 5[1] = 2 + 5 = 7$ ✓

SECOND CHOICE

$\int_{-1}^{0} (3x^2 - 2x + 5)\,dx = 3\int_{-1}^{0} x^2\,dx - 2\int_{-1}^{0} x\,dx + 5\int_{-1}^{0} dx$

$= \left[3\left(\frac{x^3}{3}\right) - 2\left(\frac{x^2}{2}\right) + 5\left(\frac{x}{1}\right)\right]_{-1}^{0} = \left[x^3 - x^2 + 5x\right]_{-1}^{0}$

$= \left[(0)^3 - (0)^2 + 5(0)\right] - \left[(-1)^3 - (-1)^2 + 5(-1)\right]$

$= 0 - \left[-1 - 1 - 5\right] = -[-7] = 7$ ✓

c) $\int_1^4 (6x - 3\sqrt{x})dx$

: $\int_1^4 (6x - 3x^{1/2})\,dx = 6\int_1^4 x\,dx - 3\int_1^4 x^{1/2}\,dx$

$= \left[6\left(\frac{x^2}{2}\right) - 3\left(\frac{x^{3/2}}{3/2}\right)\right]_1^4 = \left[3x^2 - 2x^{3/2}\right]_1^4$

$= \left[3(4)^2 - 2(4)^{3/2}\right] - \left[3(1)^2 - 2(1)^{3/2}\right]$

$= [48 - 16] - [3 - 2] = 32 - 1 = 31$

d) $\int_1^4 \left(\sqrt{x} - \frac{2}{\sqrt{x}}\right) dx$

: $\int_1^4 \left(\frac{x - 2}{\sqrt{x}}\right) dx = \int_1^4 \frac{x}{\sqrt{x}} - 2\int_1^4 x^{-1/2} = \int_1^4 x^{1/2} - 2\int_1^4 x^{-1/2}$

$= \left[\left(\frac{x^{3/2}}{3/2}\right) - 2\left(\frac{x^{1/2}}{1/2}\right)\right]_1^4 = \left[\left(\frac{(4)^{3/2}}{3/2}\right) - 2\left(\frac{(4)^{1/2}}{1/2}\right)\right] - \left[\left(\frac{(1)^{3/2}}{3/2}\right) - 2\left(\frac{(1)^{1/2}}{1/2}\right)\right]$

$= \left[\left(\frac{8}{3/2}\right) - 8\right] - \left[\frac{1}{3/2} - 4\right] = [5.33 - 8] - [0.66 - 4]$

$= -2.66 + 3.33 = 0.666$

e) $\int_1^2 \left(x^2 - \frac{4}{x^2}\right) dx$

: $\int_1^2 \left(\frac{x^4 - 4}{x^2}\right) dx = \int_1^2 x^2\,dx - 4\int x^{-2}\,dx$

$= \left[\left(\frac{x^3}{3}\right) - 4\left(\frac{x^{-1}}{-1}\right)\right]_1^2 = \left[\frac{x^3}{3} + \frac{4}{x}\right]_1^2$

$= \left[\frac{(2)^3}{3} + \frac{4}{2}\right] - \left[\frac{(1)^3}{3} + \frac{4}{(1)}\right] = \left[\frac{8}{3} + 2\right] - \left[\frac{1}{3} + 4\right]$

$= [2.66 + 2] - [4.33] = 4.66 - 4.33 = 0.33$ OR $1/3$

(f) $\int_1^2 \left(8x^3 - 2 + \frac{1}{2x^2}\right) dx$

$: 8\int_1^2 x^3\, dx - 2\int_1^2 dx + \frac{1}{2}\int_1^2 x^{-2}\, dx$

$= \left[8\left(\frac{x^4}{4}\right) - 2\left(\frac{x}{1}\right) + \frac{1}{2}\left(\frac{x^{-1}}{-1}\right)\right]_1^2$

$= \left[2x^4 - 2x + \frac{1}{2x}\right]_1^2$

$= \left[2(2)^4 - 2(2) - \frac{1}{2(2)}\right] - \left[2(1)^4 - 2(1) - \frac{1}{2(1)}\right]$

$= \left[32 - 4 - \frac{1}{4}\right] - \left[2 - 2 - \frac{1}{2}\right] = [27.75] - [-1/2]$
$= 28.25$

3) Evalute the following definite Integrals

(a) $\int_0^2 x(x^2-2)\, dx$

$: \int_0^2 (x^3 - 2x)\, dx = \int_0^2 x^3\, dx - 2\int_0^2 x\, dx$

$= \left[\left(\frac{x^4}{4}\right) - 2\left(\frac{x^2}{2}\right)\right]_0^2 = \left[\frac{x^4}{4} - x^2\right]_0^2$

$= \left[\frac{(2)^4}{4} - (2)^2\right] - 0 = 4 - 4 = 0$

(b) $\int_1^2 (x+1)(x-2)\, dx$

$: \int_1^2 (x^2 - x - 2)\, dx = \int_1^2 x^2 - \int_1^2 x - 2\int_1^2 dx$

$= \left[\left(\frac{x^3}{3}\right) - \left(\frac{x^2}{2}\right) - 2x\right]_1^2 = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_1^2$

$= \left[\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2)\right] - \left[\frac{(1)^3}{3} - \frac{(1)^2}{2} - 2(1)\right]$

$= [2.66 - 2 - 4] - [0.33 - 0.5 - 2]$

$= [-3.33] - [-2.166] = -3.33 + 2.166$

$= -1.166$

c) $\int_{-1}^{0} x(x-2)(x+2)\,dx$

$: \int_{-1}^{0} x(x^2-4)\,dx = \int_{-1}^{0} (x^3-4x)\,dx = \int_{-1}^{0} x^3\,dx - 4\int_{-1}^{0} x\,dx$

$= \left[\left(\frac{x^4}{4}\right) - 4\left(\frac{x^2}{2}\right)\right]_{-1}^{0} = \left[\frac{x^4}{4} - 2x^2\right]_{-1}^{0}$

$= \left[\frac{(0)^4}{4} - 2(0)^2\right] - \left[\frac{(-1)^4}{4} - 2(-1)^2\right] = -\left[\frac{1}{4} - 2\right] = 1.75$

d) $\int_{0}^{4} \sqrt{x}\,(1-\sqrt{x})\,dx$

$: \int_{0}^{4} (\sqrt{x} - x)\,dx = \int_{0}^{4} \sqrt{x}\,dx - \int_{0}^{4} x\,dx$

$= \left[\left(\frac{x^{3/2}}{3/2}\right) - \left(\frac{x^2}{2}\right)\right]_{0}^{4} = \left[\frac{2x^{3/2}}{3} - \frac{x^2}{2}\right]_{0}^{4}$

$\left[\frac{2(4)^{3/2}}{3} - \left(\frac{(4)^2}{2}\right)\right] - 0 = \frac{16}{3} - 8 = 5.33 - 8 = -2.66$

e) $\int_{1}^{3} \frac{1}{x^2}(4x^2 - 8)\,dx$

$: \int_{1}^{3} \left(\frac{4x^2}{x^2} - \frac{8}{x^2}\right)dx = 4\int_{1}^{3} dx - 8\int_{1}^{3} x^{-2}\,dx$

$= \left[4x + \frac{8}{x}\right]_{1}^{3} = \left[4(3) + \frac{8}{(3)}\right] - \left[4(1) + \frac{8}{1}\right] = 2$

(f) $\int_{0}^{1} x^2(2 - 3\sqrt{x})\,dx$

$: \int_{0}^{1} (2x^2 - 3x^{5/2})\,dx = 2\int_{0}^{1} x^2\,dx - 3\int_{0}^{1} x^{5/2}\,dx$

$= \left[2\left(\frac{x^3}{3}\right) - 3\left(\frac{x^{7/2}}{7/2}\right)\right]_{0}^{1} = \left[2\left(\frac{(1)^3}{3}\right) - 3\left(\frac{(1)^{7/2}}{7/2}\right)\right] - 0$

$= [2/3 - 6/7] = \frac{2}{3} - \frac{6}{7} = \frac{14-18}{21} = \frac{-4}{21}$

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4) Evalute the following Definite Integrals

(a) $\int_1^4 \frac{x^2+1}{x^2} dx$

$= \int_1^4 \frac{x^2}{x^2} dx + \int_1^4 \frac{1}{x^2} dx = \int_1^4 dx + \int_1^4 x^{-2} dx$

$= \left[x\right]_1^4 + \left[\frac{-1}{x}\right]_1^4 = [4-1] + [-0.25+1]$

$= 3 + 0.75 = 3.75$

b) $\int_1^2 \frac{1-2x^3}{x^2} dx$

$= \int_1^2 \left(\frac{1}{x^2} - \frac{2x^3}{x^2}\right) = \int_1^2 \frac{1}{x^2} dx - \int_1^2 \frac{2x^3}{x^2} dx$

$= \int_1^2 x^{-2} dx - \int_1^2 2x\, dx = \left[\frac{-1}{x}\right]_1^2 - \left[\frac{2x^2}{2}\right]_1^2$

$= [-0.5-(-1)] - [4-1] = 0.5 - 3 = -2.5$

c) $\int_1^4 \frac{2x-1}{\sqrt{x}} dx$

$= \int_1^4 \frac{2x-1}{\sqrt{x}} dx = \int_1^4 \left(\frac{2x}{\sqrt{x}} - \frac{1}{\sqrt{x}}\right) = \int_1^4 \frac{2x}{\sqrt{x}} dx - \int_1^4 \frac{1}{\sqrt{x}} dx$

$= 2\int_1^4 x(x)^{-1/2} dx - \int_1^4 x^{-1/2} dx = 2\int_1^4 x^{1/2} dx - \int_1^4 x^{-1/2} dx$

$= \int_1^4 x^{1/2} dx - \int_1^4 x^{-1/2} dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4 - \left[\frac{x^{1/2}}{1/2}\right]_1^4$

$= \left[\frac{16}{3} - \frac{2}{3}\right] - \left[4-2\right] = \frac{14}{3} - 2 =$

d) $\int_1^9 \frac{3-2\sqrt{x}}{x^2} dx$

$= \int_1^9 \left(\frac{3}{x^2} - \frac{2\sqrt{x}}{x^2}\right) dx = \int_1^9 \frac{3}{x^2} dx - 2\int x^{-2+1/2} dx$

$= 3\int_1^9 x^{-2} dx - 2\int x^{-3/2} dx = 3\int_1^9 x^{-2} dx - 2\int_1^9 x^{-3/2} dx$

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$$= 3\left[\frac{x^{-2+1}}{-2+1}\right]_1^3 - 2\left[\frac{x^{-3/2+1}}{-3/2+1}\right]_1^9 = 3\left[\frac{x^{-1}}{-1}\right]_1^9 - 2\left[\frac{x^{-1/2}}{-1/2}\right]_1^9$$

$$= 3\left[-\frac{1}{x}\right]_1^9 - 2\left[\frac{-2}{\sqrt{x}}\right]_1^9 = 3\left[-\frac{1}{9}+\frac{1}{1}\right] - 2\left[-\frac{2}{3}+2\right]$$

$$= 3\left[\frac{8}{9}\right] - 2\left[\frac{4}{3}\right] = \frac{8}{3} - \frac{8}{3} = \boxed{0}$$

c) $\int_1^3 \frac{1-4x+x^3}{2x^3}\,dx$

$$\int_1^3\left(\frac{1}{2x^3} - \frac{4x}{2x^3} + \frac{x^3}{2x^3}\right)dx = \int_1^3 \frac{1}{2x^3}\,dx - \int_1^3 \frac{4x}{2x^3}\,dx + \int_1^3 \frac{x^3}{2x^3}$$

$$= \frac{1}{2}\int_1^3 x^{-3}\,dx - 2\int_1^3 x^{-2}\,dx + \frac{1}{2}\int_1^3 dx = \frac{1}{2}\left[\frac{x^{-3+1}}{-3+1}\right]_1^3 - 2\left[\frac{x^{-2+1}}{-2+1}\right]_1^3 + \frac{1}{2}\left[\frac{x^{0+1}}{0+1}\right]_1^3$$

$$= \frac{1}{2}\left[\frac{x^{-2}}{-2}\right]_1^3 - 2\left[\frac{x^{-1}}{-1}\right]_1^3 + \frac{1}{2}\left[x\right]_1^3 = \frac{1}{2}\left[\frac{1}{-2x^2}\right]_1^3 - 2\left[-\frac{1}{x}\right]_1^3 + \frac{1}{2}\left[x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-2(9)} + \frac{1}{2}\right] - 2\left[-\frac{1}{3} + 1\right] + \frac{1}{2}\left[3-1\right]$$

$$= \frac{1}{2}\left[\frac{2+(-18)}{-36}\right] - 2\left[+\frac{2}{3}\right] + \frac{\cancel{3}}{2}\left[2\right]$$

$$= \frac{1}{2} \times \frac{\cancel{-}16}{\cancel{-}36} + \frac{4}{3} + 1 = \frac{1}{\cancel{2}} \times \frac{\cancel{4}}{9} + \frac{4}{3} + 1 = \frac{1}{9} + \frac{4}{3} + 1$$

$$= \frac{38}{27} + \frac{4}{3} + 1 = \frac{38+36+27}{27} = \frac{102}{27} = \frac{34}{9}$$

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f) $\int_1^2 \frac{(x+2)(x-3)}{x^2}\,dx$

$= \int_1^2 \left(\frac{x^2-3x+2x-6}{x^2}\right) dx = \int_1^2 \left(\frac{x^2-x-6}{x^2}\right) dx = \int_1^2 \left(1 - \frac{x}{x^2} - \frac{6}{x^2}\right) dx$

$= \int_1^2 \left(1 - x^{-1} - \frac{6}{x^2}\right) dx = 1\int_1^2 dx - \int_1^2 x^{-1} - 6\int_1^2 x^{-2}\,dx$

$= \int_1^2 dx - \int_1^2 x^{-2}\,dx - 6\int_1^2 x^{-2}\,dx = [x]_1^2 - \left[\frac{x^{-1+1}}{-1+1}\right]_1^2 - 6\left[\frac{x^{-2+1}}{-2+1}\right]_1^2$

$= [2-1] - \left[-\frac{1}{2} + 1\right] - 6[$

$\Rightarrow \int_1^2 \frac{(x+2)(x-3)}{x^2}\,dx$

$\boxed{\int_1^2 \left(\frac{x+2}{\cancel{x}}\right) dx \times \int_1^2 \left(\frac{(x-3)}{x}\right) dx}$

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5) Evaluate

(a) $\int_0^2 (4t^2 - t)\,dt$

$= 4\int_0^2 t^2\,dt - \int_0^2 t\,dt = 4\left[\frac{t^{2+1}}{2+1}\right]_0^2 - \left[\frac{t^{1+1}}{1+1}\right]_0^2$

$= 4\left[\frac{t^3}{3}\right]_0^2 - \left[\frac{t^2}{2}\right]_0^2 = 4\left[\frac{8}{3} - 0\right] - \left[\frac{4}{2} - 0\right]$

$= 4\left[\frac{8}{3}\right] - 2 = \frac{32}{3} - \frac{2}{1} \quad \frac{32-6}{3} = \frac{26}{3} = 8.666$

(b) $\int_1^2 \frac{2t^2+1}{t^2}\,dt$

$= \int_1^2 \left(\frac{2t^2}{t^2} + \frac{1}{t^2}\right) dt = \int_1^2 \frac{2t^2}{t^2}\,dt + \int_1^2 \frac{1}{t^2}\,dt$

$= 2\int_1^2 dt + \int_1^2 t^{-2}\,dt = 2\left[t\right]_1^2 + \left[\frac{t^{-2+1}}{-2+1}\right]_1^2 = 2\left[t\right]_1^2 + \left[\frac{t^{-1}}{-1}\right]_1^2$

$= 2\left[t\right]_1^2 + \left[\frac{-1}{t}\right]_1^2 = 2\left[2-1\right] + \left[-\frac{1}{2} + 1\right]$

$= 2 + \frac{1}{2} = \boxed{2.5}$

c) $\int_1^3 \cancel{2r}\,(r-2)\,dr$

$: \int_1^3 (2r^2 - 4r)\,dr = 2\int_1^3 r^2\,dr - 4\int_1^3 r\,dr$

$= 2\left[\frac{r^{2+1}}{2+1}\right]_1^3 - 4\left[\frac{r^{1+1}}{1+1}\right]_1^3 = 2\left[\frac{r^3}{3}\right]_1^3 - 4\left[\frac{r^2}{2}\right]_1^3$

$= 2\left[\frac{27}{3} - \frac{1}{3}\right] - 4\left[\frac{9}{2} - \frac{1}{2}\right] = 2\left[\frac{26}{3}\right] - 4\left[\frac{8}{2}\right]$

$= \left[\frac{52}{3} - \frac{16}{1}\right] = \frac{52-48}{3} = \boxed{\frac{4}{3}}$

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(d) $\int_2^3 \frac{(r-1)(r+1)}{r^2}\, dr$

$$: \int_2^3 \left(\frac{r^2 + r - r + 1}{r^2} \right) dr = \int_2^3 \left(\frac{r^2 - 1}{r^2} \right) dr$$

$$= \int_2^3 \frac{r^2}{r^2}\, dr - \int_2^3 \frac{1}{r^2}\, dr = \int_2^3 1\, dr - \int_2^3 r^{-2}\, dr$$

$$= \int_2^3 dr - \int_2^3 r^{-2} dr = \Big[r \Big]_2^3 - \left[\frac{r^{-2+1}}{-2+1} \right]_2^3 = [3-2] - \left[\frac{-1}{3} + \frac{1}{2} \right]$$

$$= 1 - \frac{1}{6} = \boxed{\frac{5}{6}}$$

20.3 Integration of Trigonometric Functions 2/6/23

$\frac{d}{dx}(\sin x) = \cos x \Rightarrow \int \cos x \, dx = \sin x + c$

$\frac{d}{dx}(-\cos x) = \sin x \Rightarrow \int \sin x \, dx = -\cos x + c$

$\frac{d}{dx}(\tan x) = \sec^2 x \Rightarrow \int \sec^2 x \, dx = \tan x + c$

If $a \neq 0$, we have

$\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)\right] = \cos(ax+b) \Rightarrow \int \cos(ax+b) = \frac{1}{a}\sin(ax+b) + c$

$\frac{d}{dx}\left[-\frac{1}{a}\cos(ax+b)\right] = \sin(ax+b) \Rightarrow \int \sin(ax+b)\, dx = \frac{-1}{a}\cos(ax+b) + c$

$\frac{d}{dx}\left[\frac{1}{a}\tan(ax+b)\right] = \sec^2(ax+b) \Rightarrow \int \sec^2(ax+b)\, dx = \frac{1}{a}\tan(ax+b) + c$

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1) Integrate with respect to x.

(a) $\sin x + 2$

$= \int \sin x\,dx + 2\int dx$

$= \int \sin\,dx + 2\int dx$

$= -\cos x + 2x + C$

(b) $1 - 3\cos x$

$= 1\int dx - 3\int \cos x\,dx$

$= x - 3(\sin x) + C$

$= x - 3\sin x + C$

(c) $\cos x - \sin x$

$= \int \cos x\,dx - \int \sin x\,dx$

$= \sin x - (-\cos x) + C$

$= \sin x + \cos x + C$

(d) $\sec^2 x - 4\sin x$

$= \int \sec^2 x\,dx - 4\int \sin x\,dx$

$= \tan x - 4(-\cos x) + C$

$= \tan x + 4\cos x + C$

(e) $3\cos x - 2\sin x$

$= 3\int \cos x\,dx - 2\int \sin x\,dx$

$= 3\sin x - 2(-\cos x) + C$

$= 3\sin x + 2\cos x + C$

(f) $4\cos x + 3\sec^2 x$

$= 4\int \cos x\,dx + 3\int \sec^2 x$

$= 4(\sin x) + 3(\tan x) + C$

$= 4\sin x + 3\tan x + C$

2) Integrate w.r.t x.

(a) $\cos 2x$

$\because \int \cos(ax+b) = \frac{1}{a}\sin(ax+b) + C \Rightarrow \boxed{\frac{1}{2}\sin(2x) + C}$

Here $a = 2,\ b = 0$

(b) $\sin 3x$

$= -\frac{1}{3}\cos(3x) + C$

(c) $2\cos 4x$

$= 2\int \cos 4x$

$= 2\left[\frac{1}{4}\sin(4x) + C\right]$

$= \boxed{1/2\ \sin 4x + C}$

(d) $\cos \frac{1}{2}x$

$\because \int \cos \frac{1}{2}x$

$= +\frac{1}{1/2}\sin \frac{1}{2}x + C$

$= +2\sin 1/2x + C$

(e) $\frac{1}{2}\sin \frac{1}{4}x$

$= 1/2\int \sin 1/4x$

$= 1/2 \times \frac{-1}{1/4}\cos 1/4x + C$

$= \boxed{-2\cos \frac{1}{4}x + C}$

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(f) $\cos 3x$

: $\int \cos 3x$

$\frac{1}{3}\sin(3x) + c$

(g) $-3\sin \frac{1}{2}x$

$= -3\int \sin \frac{1}{2}x$

$= -3 \times -\frac{1}{1/2} \cos \frac{1}{2}x + c = +\frac{12}{2} \cos \frac{1}{2}x + c$

(h) $2\cos(1-x)$

$= 2\cos(-x+1)$

$= 2\int \cos(-x+1)$

$= 2\,\frac{1}{-1} \sin(-x+1) + c$

$= \boxed{-2\sin(-x+1) + c}$

(i) $6\sin(3x+2)$

$= 6\int \sin(3x+2)$

$= 6 \times -\frac{1}{3} \cos(3x+2) + c$

$= -2\cos(3x+2) + c$

(j) $\cos(1-2x)$

$= \int \cos(-2x+1)$

$\frac{1}{-2}\sin(-2x+1) + c$

(k) $-\sin(2x+1)$

$= -\int \sin(2x+1)$

$= -\left(-\frac{1}{2}\right)\cos(2x+1) + c$

$= \frac{1}{2}\cos(2x+1) + c$

(l) $3\sin(2-x)$

$= 3\int \sin(2-x)$

$= 3\int \sin(-x+2)$

$= 3\left(\frac{-1}{-1}\right)\cos(-x+2) + c$

$= \boxed{3\cos(-x+2) + c}$

(m) $\cos\left(2x + \frac{\pi}{4}\right)$

$= \cancel{2}\int \cos(2x + \pi/4)$

$= \frac{1}{2}\sin(2x + \pi/4) + c$

(n) $4\sin\left(x - \frac{\pi}{4}\right)$

$= 4\int \sin\left(x - \frac{\pi}{4}\right)$

$= 4 \times \frac{-1}{1}\cos\left(x - \frac{\pi}{4}\right) + c$

$= -4\cos\left(x - \frac{\pi}{4}\right) + c$

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3) Evalute the following definite Integrals.

(a) $\int_0^{\pi/6} \cos x \, dx$

: $\frac{1}{1} \sin x + c$

$\Rightarrow \left[\sin x\right]_0^{\pi/6}$

$= \left[\sin(\pi/6) - \sin(0)\right]$

$= \frac{1}{2} - 0 = \boxed{1/2}$

(b) $\int_0^{\pi/2} \sin x \, dx$

: $\left[-\cos x\right]_0^{\pi/2}$

$= -\cos(90) - (-\cos(0))$

$= -0 - (-1) = \boxed{1}$

(c) $\int_{-\pi/4}^{\pi/4} 2 \cos x \, dx$

: $2 \left[\sin x\right]_{-\pi/4}^{\pi/4}$

$= 2\left[\sin(\pi/4) - \sin(-\pi/4)\right]$

$= 2\left[\sin(45°) - \sin(-45°)\right]$

$= 2\left[1/\sqrt{2} - (-1/\sqrt{2})\right]$

$2\left[1/\sqrt{2} + 1/\sqrt{2}\right]$

$\cancel{2}\left[\frac{2\sqrt{2}}{\cancel{2}}\right] = \boxed{2\sqrt{2}}$

(d) $\int_0^{\pi/4} \sec^2 x \, dx$

: $\left[\tan x\right]_0^{\pi/4}$

$= \tan(\pi/4) - \tan(0)$

$= \tan(45) - \tan(0)$

$= 1 - 0$

$= \boxed{1}$

(e) $\int_0^{\pi/2} (1 - 2\sin x) \, dx$

$= \int_0^{\pi/2} (-2\sin + 1) \, dx$

$= \cancel{-\frac{1}{2}\cos}$

$= -2\int_0^{\pi/2} \sin x \, dx + \int_0^{\pi/2} dx$

$= -2\left[-\cos x\right]_0^{\pi/2} + \left[x\right]_0^{\pi/2}$

$= -2\left[-\cos(90) - (-\cos(0))\right] + \left[\pi/2 - 0\right]$

$= -2\left[-0 + 1\right] + \pi/2$

$= \boxed{-2 + \pi/2}$

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(f) $\int_0^{\pi/6} (3\cos x - 2)\,dx$

$= 3\int_0^{\pi/6} \cos x - 2\int_0^{\pi/6} dx = 3[\sin x]_0^{\pi/6} - 2[x]_0^{\pi/6}$

$= 3[\sin(30) - \sin(0)] - 2[(\pi/6) - 0]$

$= 3[1/2 - 0] - 2(\pi/6)$

$= \boxed{3/2 - \pi/3}$

(g) $\int_0^{\pi/4} \sin 2x\,dx$

$= \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/4}$

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4) Evalute $\int_0^{\pi/4} (1 + \tan^2 x)\, dx$

$\because 1 + \tan^2\theta = \sec^2\theta$

$= \int_0^{\pi/4} (1 + \tan^2 x)\, dx = \int_0^{\pi/4} (\sec^2 x)\, dx$

$= \int_0^{\pi/4} (\sec^2 x)\, dx = [\tan x]_0^{\pi/4} = [\tan(\pi/4) - \tan(0)]$

$= \tan(45)^\circ - \tan(0)^\circ = 1 - 0 = \boxed{1}$

5) Show that $\int_0^{\pi/4} \left(\frac{1 + \cos^2 x}{\cos^2 x}\right) dx = \frac{\pi}{4} + 1$

$= \int_0^{\pi/4} \frac{1}{\cos^2 x}\, dx + \int_0^{\pi/4} \frac{\cos^2 x}{\cos^2 x}\, dx$

$= \int_0^{\pi/4} \sec^2 x\, dx + \int_0^{\pi/4} dx$

$= [\tan x]_0^{\pi/4} + [x]_0^{\pi/4}$

$= [\tan(\pi/4) - \tan(0)] + [(\pi/4) - (0)]$

$= [1 - 0] + [\pi/4 - 0]$

$= \boxed{1 + \frac{\pi}{4}} \longrightarrow$ Hence prooved

30.4 Integration of Exponential Functions 2/6/23

For Exponential Functions, we have:

$$\frac{d}{dx}(e^x) = e^x \Rightarrow \int e^x\,dx = e^x + c$$

and

$$\frac{d}{dx}(-e^{-x}) = e^{-x} \Rightarrow \int e^{-x}\,dx = -e^{-x} + c$$

In general, if $a \neq 0$,

$$\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) = e^{ax+b} \Rightarrow \int e^{ax+b} = \frac{1}{a}e^{ax+b} + c$$

Examples:-

(a) $\int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1} + c$

(b) $\int e^{3-2x}\,dx = \frac{1}{-2}e^{3-2x} + c = -\frac{1}{2}e^{3-2x} + c$

(c) $\int_0^2 e^{2x}\,dx = \left[\frac{1}{2}e^{2x}\right]_0^2 = \frac{1}{2}(e^4 - e^0) = \frac{1}{2}(e^4 - 1)$

(d) $\int_1^{\ln 2} e^x\,dx = \left[e^x\right]_1^{\ln 2} = e^{\ln 2} - e^1 = 2 - e$

Note: $e^{\ln x} = x$ for $x > 0$

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Exercise 20.4

1) Integrate w.r.t x.

(a) $e^x + 1$

$: \int e^x dx + \int dx$

$= e^x + c + x$

(b) e^{2x}

$= \int e^{2x} dx$

$= \frac{1}{2} e^{2x} + c$

(c) $2e^{3x}$

$= 2\int e^{3x} dx$

$= 2 \times \frac{1}{3} e^{3x} + c$

(d) $e^{-x} - e^x$

$= \int e^{-x} dx - \int e^x dx$

$= -e^{-x} - e^x + c$

(e) e^{-2x}

$= \int e^{-2x} dx$

$= -\frac{1}{2} e^{-2x} + c$

(f) $2e^{\frac{1}{2}x}$

$= 2\int e^{\frac{1}{2}x} dx$

$= 2 \times \frac{1}{1/2} e^{1/2x} + c$

$= 4 e^{1/2x} + c$

(g) e^{2x+1}

$= \int e^{2x+1} dx$

$= \frac{1}{2} e^{2x+1} + c$

(h) $3e^{1-x}$

$= 3\int e^{-x+1} dx$

$= -3 e^{-x+1} + c$

(i) $\frac{1}{2} e^{3x+2}$

$= \frac{1}{2} \int e^{3x+2} dx$

$= \frac{1}{6} e^{3x+2} + c$

(j) $4e^{\frac{1}{2}(1-x)}$

$= 4 e^{\frac{1}{2} - \frac{x}{2}}$

$= 4\int e^{\frac{1}{2} - \frac{x}{2}} dx$

$= 4 \times \frac{-1}{1/2} e^{\frac{1}{2}(1-x)} + c$

$= \boxed{-8 e^{1/2(1-x)} + c}$

2) Evaluate the following, giving your answers in terms of e, where appropriate.

(a) $\int_0^2 e^x dx$

$: [e^x]_0^2 = [e^2 - e^0] = \boxed{e^2 - 1}$

(b) $\int_0^1 e^{2x} dx$

$= [e^{2x}]_0^1 = \frac{1}{2}(e^2 - e^0)$

$= \frac{1}{2}(e^2 - 1)$

(c) $\int_0^2 e^{-\frac{1}{2}x} dx$

$= \frac{1}{-1/2} [e^{-1/2x}]_0^2$

$= -2[e^{-1} - e^0]$

$= -2[\frac{1}{e} - 1]$

(d) $\int_1^2 e^{1-x} dx$

$= \frac{1}{-1} [e^{1-x}]_1^2$

$= -1[e^{1-2} - e^{1-1}]$

$= -1[\frac{1}{e} - e^0]$

$= -[\frac{1}{e} - 1]$

(e) $\int_0^{\ln 2} e^{3x} dx$

$= \frac{1}{3} [e^{3x}]_0^{\ln 2}$

$= \frac{1}{3} [e^{3(\ln 2)} - e^0]$

$= 1/3 [e^{3(\ln 2)} - e^0] = \frac{1}{3}[2^3 - 1]$

$= \frac{1}{3}[8 - 1] = \boxed{\frac{7}{3}}$

(f) $\int_0^{\ln 3} e^x \, dx$

$= [e^x]_0^{\ln 3}$

$= e^{\ln 3} - e^0$

$= 3 - 1$

$= \boxed{2}$

3) Find y as a function of x given that $\frac{dy}{dx} = 1 - 3e^x$ and $y = 4$ when $x = 0$.

Solution

$y = \int (1 - 3e^x) \, dx = x - 3e^x + c$

$y = 4$ when $x = 0 \Rightarrow 4 = 0 - 3e^0 + c$

$4 = -3 + c$

$\boxed{c = 7}$

$\boxed{y = x - 3e^x + 7}$

4) Express y in terms of x given that $\frac{dy}{dx} = 2e^{-x}$ and that $y = -1$ when $x = 0$

Solution

1: $\int 2e^{-x} \, dx = 2\int e^{-x} \, dx = -2e^{-x} + c$

$y = -2e^{-x} + c$, $y = -1$, $x = 0$

$-1 = -2e^{-0} + c$

$-1 = -2 + c$

$\boxed{c = 1}$

$\boxed{y = -2e^{-x} + 1}$

5) Find y as a function of x given that $\frac{dy}{dx} = e^{2x}$ and that $y = 6$ when $x = \ln 3$.

Solution

1: $\int e^{2x} = \frac{1}{2} e^{2x} + c \Rightarrow y = \frac{e^{2x}}{2} + c$

$y = 6$, $x = \ln 3$

$6 = \frac{e^{2(\ln 3)}}{2} + c$

$6 = \frac{3^2}{2} + \frac{c}{1}$

$12 = 9 + 2c$

$\boxed{3/2 = c}$

$y = \frac{e^{2x}}{2} + \frac{3}{2}$

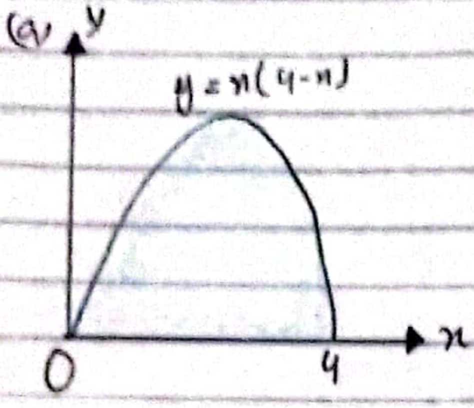
$\boxed{y = \frac{1}{2}(e^{x} + 3)}$

Chapter 21 Applications of Integration

21.1 Area between a curve and x-axis

1. For each of the following, find the shaded area.

(a)



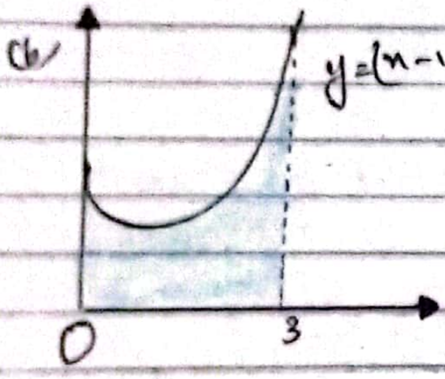
: $\int_0^4 4x - x^2 = 4\int_0^4 x - \int_0^4 x^2$

$4\left[\frac{x^2}{2}\right]_0^4 - \left[\frac{x^3}{3}\right]_0^4$

$= 2[16] - \frac{64}{3} = 32 - \frac{64}{3} = \frac{96-64}{3}$

$= \frac{32}{3} = \boxed{10.66}$

(b)



: $y = x^2 - 2x + 1 + 1 = x^2 - 2x + 2$

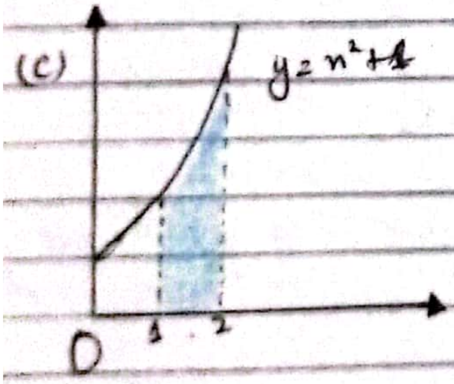
$\int_0^3 (x^2 - 2x + 2)\,dx$

$= \int_0^3 x^2 dx - 2\int_0^3 x\,dx + 2\int_0^3 dx$

$= \left[\frac{x^3}{3}\right]_0^3 - 2\left[\frac{x^2}{2}\right]_0^3 + 2[x]_0^3$

$= \frac{27}{3} - 9 + 6 = \frac{27}{3} - \frac{3}{1} = \frac{18}{3} = \boxed{6\ sq}$

(c)

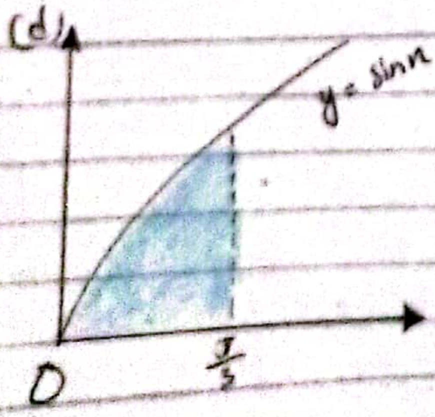


:- $\int_1^2 (x^2+1)\,dx = \int_1^2 x^2\,dx + \int_1^2 dx$

$= \left[\frac{x^3}{3}\right]_1^2 + [x]_1^2 =$

$= \left[\frac{8}{3} - \frac{1}{3}\right] + [2-1] = \frac{7}{3} + 1 = \frac{10}{3} = \boxed{3.33}$

(d)



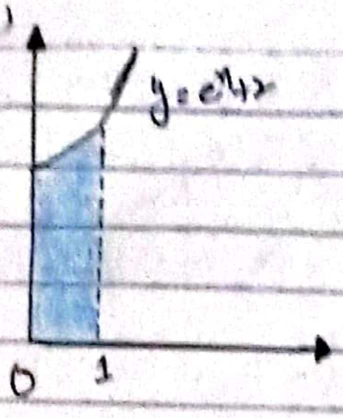
:- $\int_0^{\pi/3} \sin x\,dx = [-\cos x]_0^{\pi/3}$

$= [-\cos(\pi/3) - (-\cos(0))]$

$= [-\cos(60) + \cos(0)]$

$= [-\frac{1}{2} + 1] = \boxed{\frac{1}{2}}$

(e)



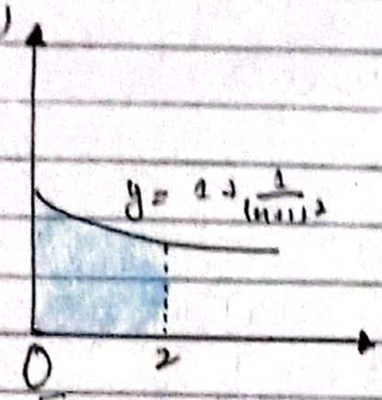
$: \int_0^1 (e^x+2)\,dx = \int_0^1 e^x\,dx + 2\int_0^1 dx$

$= [e^x]_0^1 + 2[x]_0^1$

$= [e^1 - e^0] + 2[1-0]$

$= e - 1 + 2 = \boxed{e+1}$

(f)



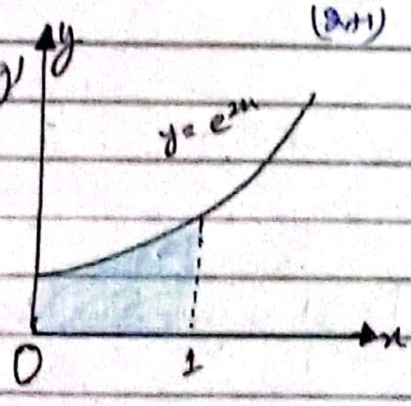
$:- \int_0^2 \left(1+\frac{1}{(x+1)^2}\right)dx$

$= \int_0^2 \left(\frac{(x+1)^2+1}{(x+1)^2}\right)dx = \int_0^2 dx + \int_0^2 (x+1)^{-2}\,dx$

$= [x]_0^2 + \left[-\frac{1}{(x+1)}\right]_0^2$

$= [2-0] + \left[\frac{-1}{(2+1)} + 1\right] = 2 + \frac{2}{3} = 2.66$

(g)

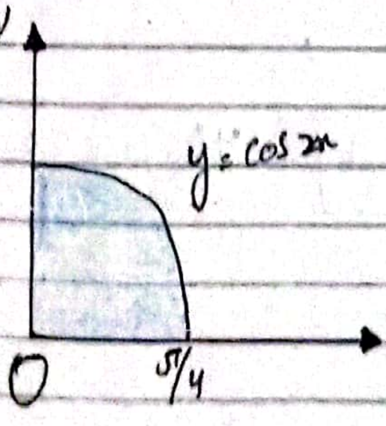


$: \int_0^1 e^{2x} = \frac{1}{2}\left[e^{2x}\right]_0^1$

$= \frac{1}{2}\left[e^2 - e^0\right] = \frac{1}{2}\left[e^2 - 1\right]$

$\boxed{\frac{1}{2}\left[e^2-1\right]}$

(h)



$: \int_0^{\pi/4} \cos 2x = \frac{1}{2}\left[\sin 2x\right]_0^{\pi/4}$

$= \frac{1}{2}\left[\sin 2(45)^\circ - \sin 2(0)^\circ\right]$

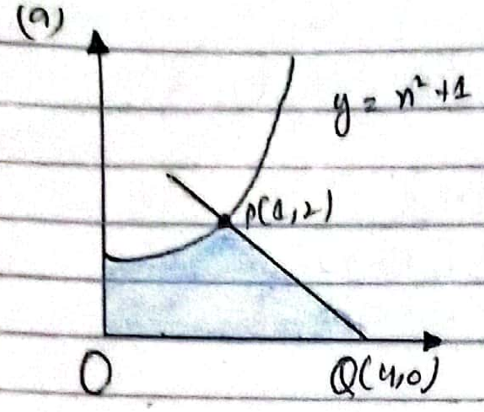
$= \frac{1}{2}\left[\sin 2(45)^\circ - 0\right]$

$= \frac{1}{2}\left[\frac{2}{\sqrt{2}}\right]$

6)

2) For each of the following, find the shaded area.

(a)



$y = x^2+1$: $\int_0^1 (x^2+1)dx + \int_1^4 \left(-\frac{2}{3}x + \frac{8}{3}\right)dx$

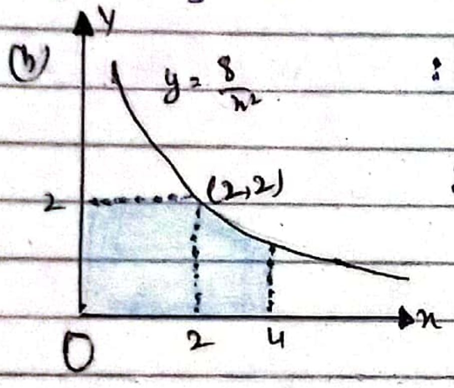
$= \int_0^1 x^2 + \int_0^1 dx - \frac{2}{3}\int_1^4 x\,dx + \frac{8}{3}\int dx$

$= \left[\frac{x^3}{3}\right]_0^1 + [x]_0^1 - \frac{2}{3}\left[\frac{x^2}{2}\right]_1^4 + \frac{8}{3}[x]_1^4$

$= \frac{1}{3} + 1 - \frac{2}{3}\left[8 - \frac{1}{2}\right] + \frac{8}{3}[4-1]$

$= \frac{1}{3} + 1 - \frac{2}{3}\left[\frac{15}{2}\right] + 8 = \frac{1}{3} + 1 - 5 + 8 = \boxed{4.33}$

(b)



: $y = 2$ and $y = \frac{8}{x^2}$

$2\int_0^2 dx + 8\int_2^4 x^{-2}\,dx = 2[x]_0^2 + 8\left[\frac{-1}{x}\right]_2^4$

$2[2-0] + 8\left[-\frac{1}{4} + \frac{1}{2}\right]$

$4 + 8\left[\frac{-2+4}{8}\right] = 4 + 1[2] = 4+2 = 6$

$\boxed{\text{Area} = 6}$

3) Find the area bounded by the following.

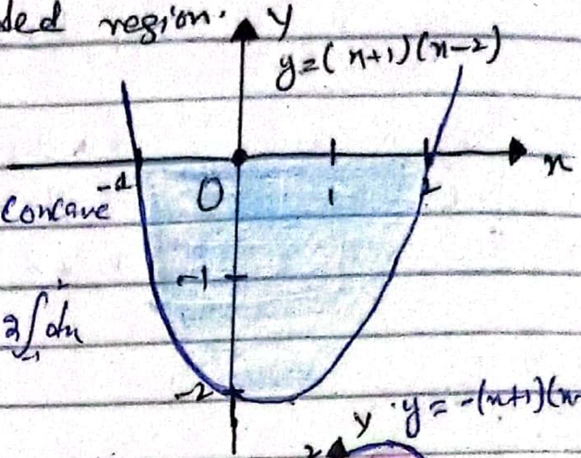
(a) $y = x^3$, $x = 2$, $x = 3$, x-axis

: $\int_2^3 x^3\,dx = \left[\frac{x^4}{4}\right]_2^3 = \left[\frac{3^4}{4} - \frac{2^4}{4}\right] = \frac{81}{4} - \frac{16}{4} = \frac{65}{4} = \boxed{16.25}$

(b) $y = \frac{4}{x^2}$; $x = 1$, $x = 3$, x-axis

: $4\int_1^3 x^{-2}\,dx = 4\left[-\frac{1}{x}\right]_1^3 = 4\left[-\frac{1}{3} + 1\right] = 4\left(\frac{2}{3}\right) = \frac{8}{3} = 2.66$

4) The diagram shows part of the curve $y = (x+1)(x-2)$. Find the area of the shaded region.



Solution

$y = x^2 - 2x + x - 2 = x^2 - x - 2$

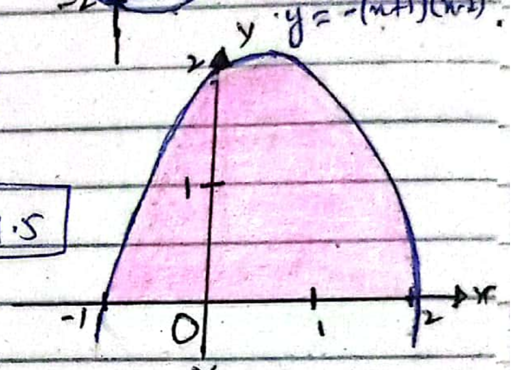
The very first step. to Reflex the Concave

$y = \int_{-1}^{2} -(x^2 - x - 2)\,dx = -\int_{-1}^{2} x^2 dx + \int_{-1}^{2} x\,dx + 2\int_{-1}^{2} dx$

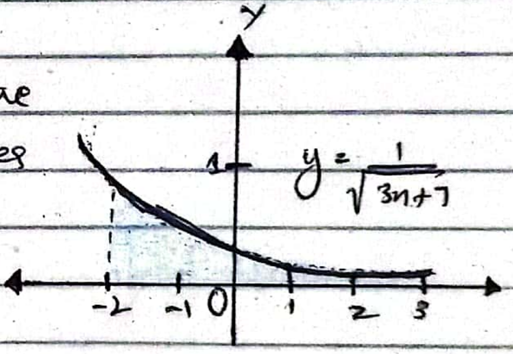
$y = -\left[\frac{x^3}{3}\right]_{-1}^{2} + \left[\frac{x^2}{2}\right]_{-1}^{2} + 2\left[x\right]_{-1}^{2}$

$\text{Area} = -\left[\frac{8}{3} - \left(\frac{-1}{3}\right)\right] + \left[\frac{4}{2} - \left(\frac{1}{2}\right)\right] + 2\left[2 - (-1)\right]$

$= -\left[\frac{8}{3} + \frac{1}{3}\right] + \left[\frac{3}{2}\right] + 6 = -3 + \frac{3}{2} + 6 = \boxed{4.5}$



5) The diagram shows part of the curve $y = \frac{1}{\sqrt{3x+7}}$, Find the area of the shaded region.



Solution

$y = \frac{1}{\sqrt{3x+7}} \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{3x+7}} \Rightarrow dy = \frac{1}{\sqrt{3x+7}}\,dx$

$\int dy = \int\left(\frac{1}{\sqrt{3x+7}}\right)dx \rightarrow$

$y = \int_{[-2,3]} (\sqrt{3x+7})^{-1}\,dx$

$y = \int_{-2}^{3} (3x+7)^{-1/2}\,dx$

power Rule of Integration

$y = \left[\frac{(3x+7)^{-1/2+1}}{3(-1/2+1)}\right]_{-2}^{3}$

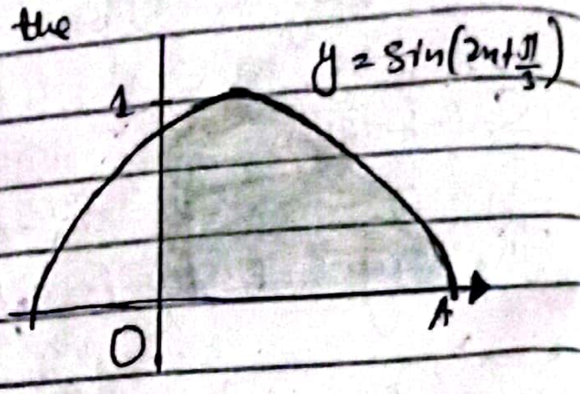
$y = \left[\frac{(3x+7)^{1/2}}{3/2}\right]_{-2}^{3}$

$y = \left[\frac{2\sqrt{3x+7}}{3}\right]_{-2}^{3}$

$= \left[\frac{2\sqrt{3(3)+7}}{3} - \frac{2\sqrt{3(-2)+7}}{3}\right]$

$= \frac{8}{3} - \frac{2}{3} = \frac{6}{3} = 2$ sq. unit.

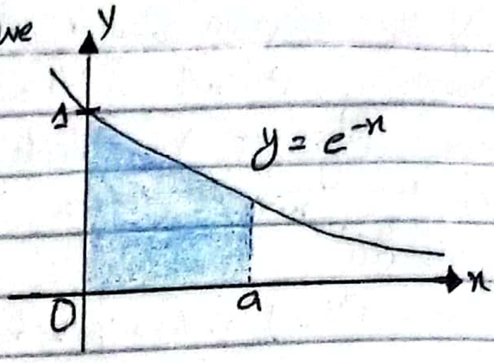
6) The diagram shows the part of the curve $y = \sin\left(2x + \frac{\pi}{3}\right)$. Find



(a) The x-coordinate of the point A where the curve cuts the x-axis,

(b) the area of the shaded region

7) The diagram shows part of curve $y = e^{-x}$



(a) Find, in terms of a, the area of the shaded region.
(b) Given that the area of the shaded region is 0.5 unit², find value of a.

Solution

a) $y = \int_0^a e^{-x} = \left[-e^{-x}\right]_0^a$

$= \left[-e^{-a} - (-e^{-0})\right]$

$= -e^{-a} + e^{-0}$

$= -e^{-a} + 1$

$\boxed{\text{Area} = -e^{-a} + 1}$

(b) value of a, where area = 0.5

Area $= -e^{-a} + 1$, Area $= 0.5$

$-e^{-a} + 1 = 0.5$

$\cancel{-}e^{-a} = \cancel{-}0.5$

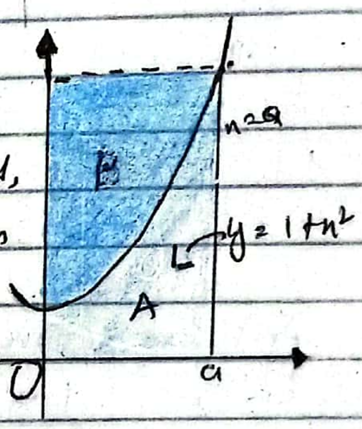
$e^{-a} = 0.5$

$a = -\ln(0.5)$

$\boxed{a = 0.693}$

also $e^a = b$
$a = \ln(b)$

8) The diagram shows part of the curve $y = 1 + x^2$ and of the line $x = a$. Find, in terms of a, the area of the shaded region A. Given that the area of the shaded regions A and B are equal, Find, to two decimal places, the value of a.



Solution

$y = \int (1 + x^2)\, dx$

$y = \int_0^a dx + \int_0^a x^2\, dx$

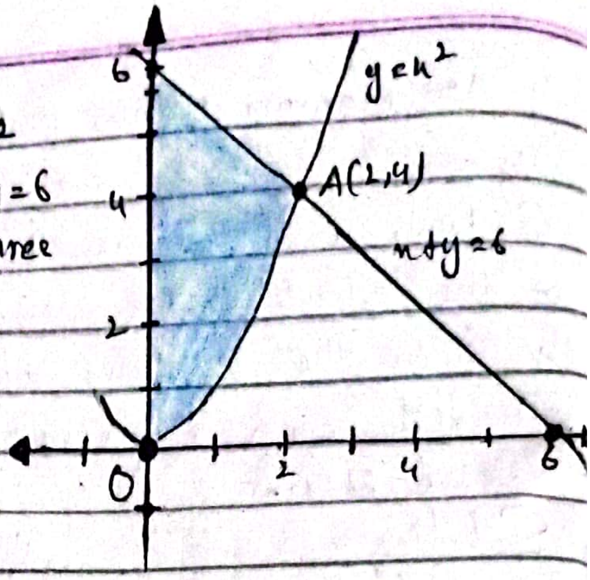
$y = [x]_0^a + \left[\frac{x^3}{3}\right]_0^a$

$y = [a - 0] + \left[\frac{a^3}{3} - 0\right]$

$y = a + \frac{a^3}{3}$

$\boxed{\text{Area} = a + \frac{a^3}{3}}$

8) The diagram shows part of the curve $y = x^2$ and of the line $x + y = 6$ intersecting at A. Calculate the area of the shaded region.



Solution

$y = x^2$, $[0,4]$; $y + x = 6$, $[4,6]$

1:

$y = \int_0^4 x^2 \, dy$? $\boxed{x = ?}$

$x^2 = y$

$\boxed{x = y^{1/2}}$ $\quad y = \int_0^4 y^{1/2} dy = \left[\frac{y^{3/2}}{3/2}\right]_0^4 = \left[\frac{4^{3/2}}{3/2}\right] = \frac{16}{3}$

2: Area of straight line

$y + x = 6$; Solve for x

$x = 6 - y$

$y = \int_4^6 (6-y)\, dy = 6\int_4^6 dy - \int_4^6 y \, dy$

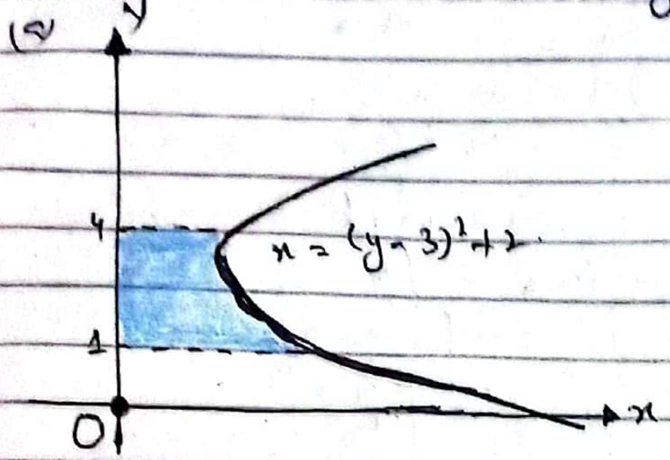
$= 6[y]_4^6 - \left[\frac{y^2}{2}\right]_4^6 = 6[6-4] - [18-8] = 12 - 10 = 2$

Area of curve $= 16/3$ + Area of straight line $= 2$

$A = \frac{16}{3} + 2 = \frac{16+6}{3} = \frac{22}{3} = \boxed{17.33}$

10) For each of the following, find the shade area

(a)



Solution

1: Solve for x

$x = (y-3)^2+2$

$x = y^2 - 6y + 9 + 2$

$\boxed{x = y^2 - 6y + 11}$

2: Integrate w.r.t y ; $[1, 4]$

$y = \int_1^4 (y^2 - 6y + 11)\, dy$

$y = \int_1^4 y^2 dy - 6\int_1^4 y\, dy + 11\int_1^4 dy$

$y = \left[\frac{y^3}{3}\right]_1^4 - 6\left[\frac{y^2}{2}\right]_1^4 + 11\left[\frac{y}{1}\right]_1^4$

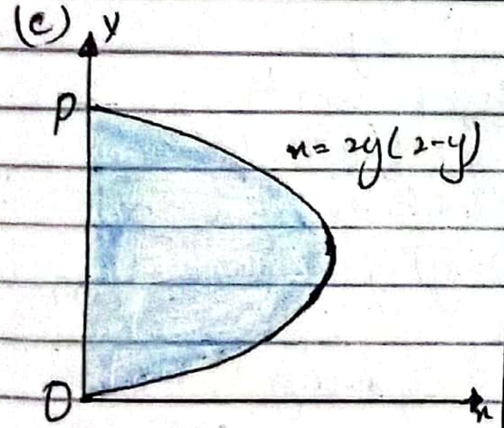
Area $= \left[\frac{64}{3} - \frac{1}{3}\right] - 6\left[8 - \frac{1}{2}\right] + 11\left[4 - 1\right]$

Area $= \frac{63}{3} - 45 + 33 = \frac{63 - 135 + 99}{3}$

Area $= \frac{162 - 135}{3} = \frac{62-35}{3} = \frac{27}{3} = 9$

$\boxed{\text{Area} = 9 \text{ sq units}}$

(c)



1: Solve for $x \rightarrow x = 4y - 2y^2$

2: Find value of P;

$x = 0$, $4y - 2y^2 = x$,

$4y - 2y^2 = 0$

$2y(2-y) = 0$

$\boxed{y=0}$ $\boxed{y=2}$

$O = 0$, $\boxed{P = 2}$

$[0, 2]$ → limit points

3: Integrate w.r.t y

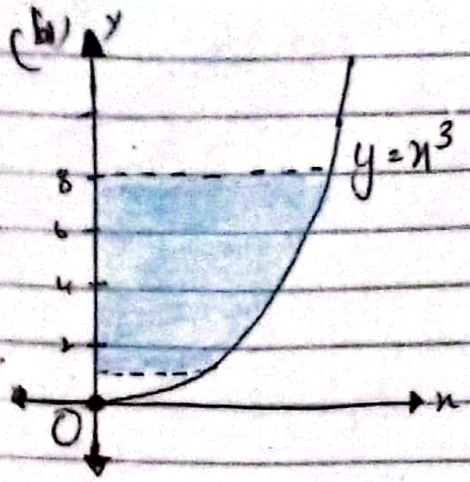
$y = \int_0^2 x\, dy = \int_0^2 (4y - 2y^2)\, dy$

$= 4\int_0^2 y\, dy - 2\int_0^2 y^2\, dy = 4\left[\frac{y^2}{2}\right]_0^2 - 2\left[\frac{y^3}{3}\right]_0^2$

$= 2[4] - 2[8/3] = \frac{8}{1} - \frac{16}{3} = \frac{24-16}{3} = \frac{8}{3}$

Area $= 8/3$ sq units ≈ 2.66

(b)



Solution

1: Solve for x

$x = y^{1/3}$

2: Integrate w.r.t y

$y = \int x \, dy = \int y^{1/3} \, dy$

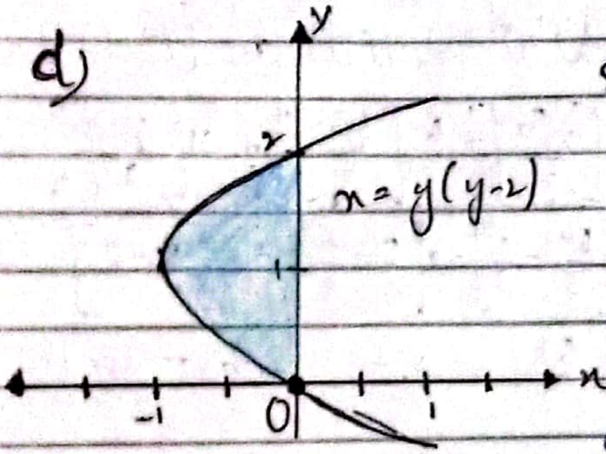
3: put limit points

$\int_1^8 y^{1/3} \, dy = \left[\frac{y^{4/3}}{4/3} \right]_1^8$

4: Upper limit − lower limit = Area

$A = \left[\frac{(8)^{4/3}}{4/3} - \frac{(1)^{4/3}}{4/3} \right] = \left[\frac{2^4}{4/3} - \frac{1}{4/3} \right] = \left[12 - \frac{3}{4} \right] = 11.25$ Sq units

d)



Solution

1: Solve for x

$x = y^2 - 2y$

2: Integrate w.r.t y

$y = \int x \, dy = \int (y^2 - 2y) \, dy$

3: put limit points

$y = \int_0^2 (y^2 - 2y) \, dy$

$y = \int_0^2 y^2 \, dy - 2\int_0^2 y \, dy$

$y = \left[\frac{y^3}{3} \right]_0^2 - 2\left[\frac{y^2}{2} \right]_0^2$

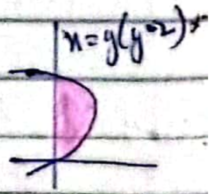
Area $= \left[\frac{8}{3} \right] - 2\left[\frac{4}{2} \right]$

Area $= \frac{8}{3} - \frac{4}{1} = \frac{8-12}{3} = \frac{-4}{3}$

$\boxed{\text{Area} = -4/3}$

WRONG PROCESS ?

Change the symmetry



$-y(y-2) = x$

$y = \int_0^2 (2y - y^2) \, dy = 2\int_0^2 y \, dy - \int_0^2 y^2 \, dy$

$= 2\left[\frac{y^2}{2} \right]_0^2 - \left[\frac{y^3}{3} \right]_0^2 = [y^2]_0^2 - \left[\frac{y^3}{3} \right]_0^2$

$= [4] - \left[\frac{8}{3} \right] = \frac{12-8}{3} = \boxed{\frac{4}{3}}$ Sq units

21.2 Area Bounded by Two Curves

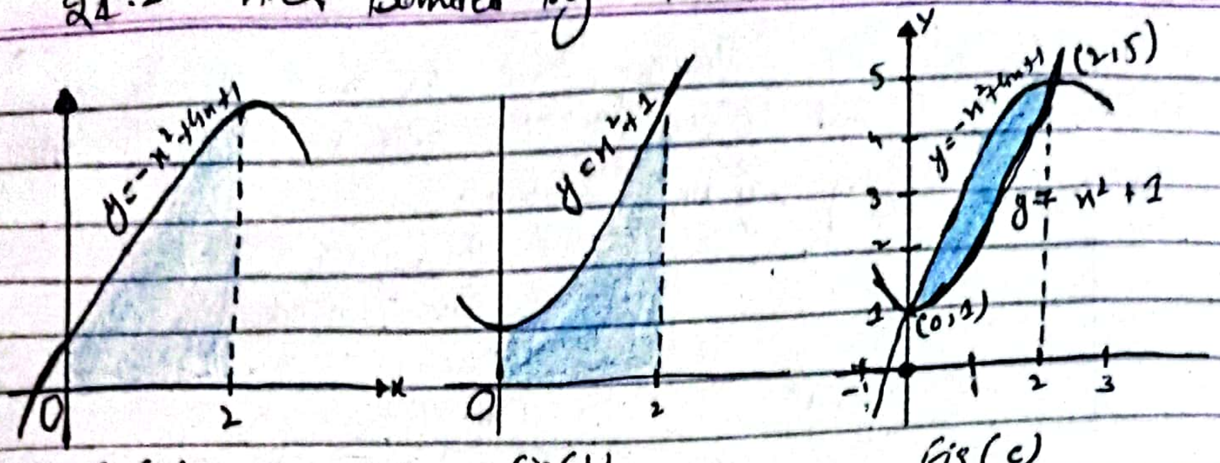


Fig (a) Fig (b) Fig (c)

Fig (c) shows the region B bounded by the two curves $y = x^2+1$ and $y = -x^2+4x+1$ which intersect at $(0,1)$ and $(2,5)$. Area bounded by the curves $y = -x^2+4x+1$ and $y = x^2+1$ is given by:

Area of Region = Area under upper curve − Area under lower curve

$$= \int_0^2 (-x^2+4x+1)\,dx - \int_0^2 (x^2+1)\,dx$$

$$= \int_0^2 [(-x^2+4x+1) - (x^2+1)]\,dx$$

$$= \int_0^2 [-x^2+4x \cancel{+1} - x^2 \cancel{-1}]\,dx$$

$$= \int_0^2 [-2x^2+4x]\,dx$$

$$= -2\int_0^2 x^2\,dx + 4\int_0^2 x\,dx$$

$$= -2\left[\frac{x^3}{3}\right]_0^2 + 4\left[\frac{x^2}{2}\right]_0^2$$

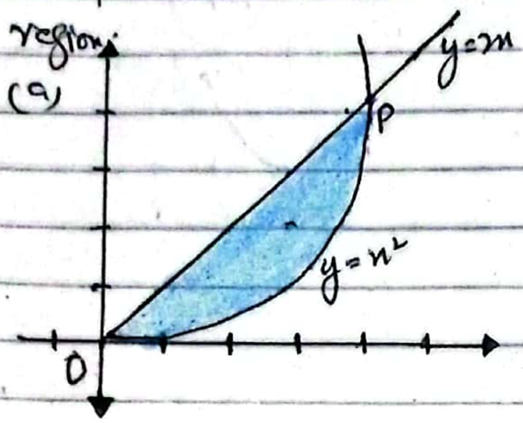
$$= -2\left[\frac{8}{3}-0\right] + 4\left[\frac{4}{2}-0\right] = -2\left[\frac{8}{3}\right] + 4\left[\frac{4}{2}\right]$$

$$= \frac{-16}{3} + 8 = \frac{-16+24}{3} = \frac{8}{3} = \boxed{2.66 \text{ sq units}}$$

Exercise 21.2

For each of the following, find the x-coordinate of the point P and calculate the area of the shaded region:

(a)



Solution

1: Find P, $y=2x$; $y=x^2$

$x^2=2x$

$x^2-2x=0$

$x(x-2)=0$

$\boxed{x=0}$ $\boxed{x=2}$

$y=2x$ or $y=x^2$; $x=2$

$\boxed{y=4}$ or $\boxed{y=4}$ $\boxed{P(2,4)}$

2: Interval → $[0, 2]$

, choose any x-value from $(0, 2)$

Let $\boxed{x=1}$ Put $x=1$ in both equations and find which curve is top curve and which curve is bottom

$x=1$

$y=2x$	$y=x^2$
$y=2(1)$	$y=(1)^2$
$\boxed{y=2}$	$y=1$

$\boxed{2>1}$

as we saw $2>1$, so the $\boxed{\text{Top curve is } y=2x}$

Note: Here we are given a graph, it is not possible in all conditions

3: Integrate w.r.t x and find the area between curve.

Area = Area Under Top Curve − Area under Bottom Curve

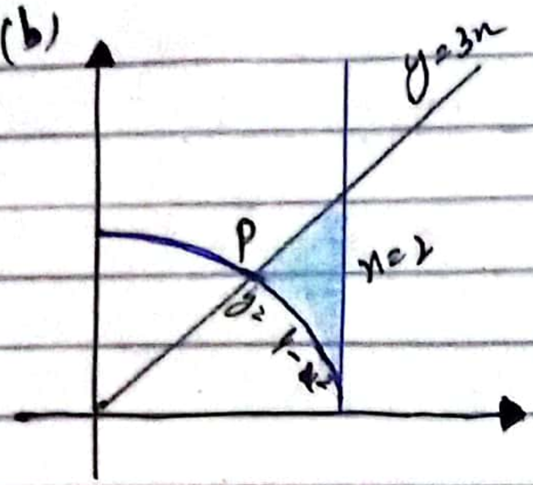
Limit points → $[0, 2]$

$$g=\int_0^2 (2x)\,dx-\int_0^2 (x^2)\,dx = 2\int_0^2 x\,dx-\int_0^2 x^2\,dx$$

$$=2\left[\frac{x^2}{2}\right]_0^2-\left[\frac{x^3}{3}\right]_0^2=[4-0]-\left[\frac{8}{3}\right]=4-\frac{8}{3}=\frac{4}{3}$$

$P=(0, 2)$ Area $=\frac{4}{3}$ Sq units

(b)



$y = 3x, \quad x = 1$

$\boxed{y = 3}$ $\quad P(1, 3)$

Solution

1: Find P, $y = 3x$; $y = 4 - x^2$

$4 - x^2 = 3x$

$x^2 + 3x - 4 = 0$

$x^2 + 4x - x - 4 = 0$

$x(x+4) - 1(x+4) = 0$

$(x-1)(x+4) = 0$

$\boxed{x = 1}$ ✓ $\quad \boxed{x = -4}$ ✗

Chapter 18 Derivative of Trigonometric Functions

In the following formulae, x is in radians.

1) $\frac{d}{dx}(\sin x) = \cos x$; $\frac{d}{dx}[\sin(ax+b)] = a\cos(ax+b)$

$\frac{d}{dx}(\cos x) = -\sin x$; $\frac{d}{dx}[\cos(ax+b)] = -a\sin(ax+b)$

$\frac{d}{dx}(\tan x) = \sec^2 x$; $\frac{d}{dx}[\tan(ax+b)] = a\sec^2(ax+b)$

where a and b are constants

2) Using $\frac{d}{dx}(u^n) = nu^{n-1}\frac{d}{dx}$

$\frac{d}{dx}(\sin^n x) = n\sin^{n-1}x\cos x$,

$\frac{d}{dx}(\cos^n x) = -n\cos^{n-1}x\sin x$

$\frac{d}{dx}(\tan^n x) = n\tan^{n-1}x\sec^2 x$

Exercise 18.1

1) Differentiate the following with respect to x.

(a) $4\sin x - 3$

$4\frac{d}{dx}(\sin x - 3)$

$4\frac{d}{dx}\sin x - \frac{d}{dx}(3)$

$4\cos x - 0$

$\boxed{4\cos x}$

(b) $x^2 - 5\cos x$

$: \frac{d}{dx}[x^2] - 5\frac{d}{dx}[\cos x]$

$\frac{dx}{dx}[2x] - 5\frac{dx}{dx}[-\sin x]$

$= 2x + 5\sin x$

(c) $2\sin x + 3\cos x$

$: 2\frac{d}{dx}[\sin x] + 3\frac{d}{dx}[\cos x]$

$2\frac{dx}{dx}[\cos x] + 3\frac{dx}{dx}[-\sin x]$

$2\cos x + 3(-\sin x)$

$\boxed{2\cos x - 3\sin x}$

(d) $4x^2 + 3\tan x$

$: 4\frac{d}{dx}[x^2] + 3\frac{d}{dx}[\tan x]$

$4\frac{dx}{dx}[2x] + 3\frac{dx}{dx}[\cancel{\tan x}\sec^2 x]$

$= 8x + 3\cancel{\tan}\sec^2 x$

$= \boxed{8x + 3\sec^2 x}$

2) Differentiate following with respect to x.

(a) $x^2\cos x$

: Product Rule

$\cos x \times \frac{d}{dx}[x^2] + x^2 \times \frac{d}{dx}[\cos x]$

$\cos x[2x] + x^2[-\sin x]$

$2x\cos x + x^2[-\sin x]$

$\boxed{2x\cos x - x^2\sin x}$

(b) $x\tan x$

: Product Rule

$\tan x\frac{d}{dx}[x] + x \times \frac{d}{dx}[\tan x]$

$\tan x[1] + x[\sec^2 x]$

$\boxed{\tan x + x\sec^2 x}$

(c) $(x+1)^2\sin x$

$: \sin x \times \frac{d}{dx}(x+1)^2 + (x+1)^2 \times \frac{d}{dx}(\sin x)$

$2\sin x(x+1) + (x+1)^2(\cos x)$

$\boxed{2(x+1)\sin x + (x+1)^2\cos x}$

(d) $\frac{1 - 2\sin x}{\cos x}$ → Quotient Rule

$: \frac{\cos x \times \frac{d}{dx}[1 - 2\sin x] - (1 - 2\sin x) \times \frac{d}{dx}\cos x}{\cos^2 x}$

$\frac{\cos x(-2\cos x) - (1 - 2\sin x) - \sin x}{\cos^2 x} = [-2\cos^2 x + (1 + 2\sin x)\sin x] / \cos^2 x$

$= [-2\cos^2 x + \sin x + 2\sin^2 x] / \cos^2 x$

$= [-2\cos^2 x - 2\sin^2 x + \sin x] / \cos^2 x$

$= \frac{-2(1) + \sin x}{\cos^2 x} = \frac{\sin x - 2}{\cos^2 x}$

4) Differentiate the following with respect to x. [Chain Rule]

(a) $(1-\cos x)^3$

: let $u = 1-\cos x$

$y = u^3$

$\frac{dy}{dx} = \frac{du}{dy} \times \frac{du}{dx}$

$\frac{dy}{dx} = 3u^2 \times \sin x$

$= 3(1-\cos x)^2 \sin x$

(b) $(3\sin x + 2)^2$

: let $u = 3\sin x + 2$; $y = u^2$

$\frac{dy}{dx} = \frac{du}{dy} \times \frac{du}{dx}$

$\frac{dy}{dx} = 2u \times 3\cos x = 2(3\sin x+2)3\cos x$

$\boxed{6(3\sin x+2)\cos x}$

(c) $\sqrt{2-\tan x}$

: let $u = 2-\tan x$; $y = \sqrt{u}$

$\frac{dy}{dx} = \frac{du}{dy} \times \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \times -\sec^2 x$

$= \frac{-\sec^2 x}{2\sqrt{2-\tan x}}$

(d) $\sqrt{\sin x + 2\cos x}$

: let $u = \sin x + 2\cos x$; $y = \sqrt{u}$

$\frac{dy}{dx} = \frac{du}{dy} \times \frac{du}{dx}$

$= +\cos x + 2(-\sin x) \times \frac{1}{2\sqrt{u}}$

$= \boxed{\frac{\cos x - 2\sin x}{2\sqrt{\sin x + 2\cos x}}}$

5) Differentiate the following with respect to x.

(a) $\sin 3x + \cos 4x$

: $\frac{d}{dx}[\sin(3x+0) + \cos(4x+0)]$

$= 3\cos x + 4(-\sin 4x)$

$= \boxed{3\cos x - 4\sin 4x}$

(b) $4\sin \frac{1}{2}x$

: $4 \times \frac{1}{2}(\cos \frac{1}{2}x)$

$\boxed{2\cos \frac{1}{2}x}$

(c) $\sin(2x-5)$

: $\frac{d}{dx}[\sin(2x-5)]$

$2\cos(2x-5)$

(d) $\cos\left(2x+\frac{\pi}{3}\right)$

: $\frac{d}{dx}\left[\cos\left(2x+\frac{\pi}{3}\right)\right]$

$= -2\sin\left(2x+\frac{\pi}{3}\right)$

(e) $3 \tan 2x$

$3 \frac{d}{dx}[\tan(2x)]$

$3 \times 2(\sec^2 2x)$

$\boxed{6 \sec^2 2x}$

(f) $6 \tan \frac{1}{2}x$

$: 6 \times \frac{1}{2}(\sec^2 \frac{1}{2}x)$

$\boxed{3 \sec^2 \frac{1}{2}x}$

(g) $2 \cos\left(\frac{\pi}{4} - x\right)$

$= 2 \frac{d}{dx} \cos\left(-x + \frac{\pi}{4}\right)$

$= 2 \times (-1) - \sin\left(-x + \frac{\pi}{4}\right)$

$2 \sin\left(\frac{\pi}{4} - x\right)$

(h) $8 \sin\left(\frac{3x - 5}{4}\right)$

$= 8 \sin\left(\frac{3x}{4} - \frac{5}{4}\right)$

$= 8 \times \frac{d}{dx} \sin\left(\frac{3}{4}x - \frac{5}{4}\right)$

$8 \times \frac{3}{4} \cos\left(\frac{3}{4}x - \frac{5}{4}\right)$

$\boxed{6 \cos(3x/4 - 5/4)}$

6) Differentiate the following with respect to x.

(a) $\sin x \cos 3x$

: Product Rule

$= \cos 3x \times \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\cos 3x)$

$= \cos 3x(\cos x) + \sin x(-\sin 3x)$

$\boxed{\cos x \cos 3x - \sin x \sin 3x}$

(b) $(1 + x^2) \tan 5x$

$: \tan 5x \times \frac{d}{dx}(1 + x^2) + (1 + x^2) \times \frac{d}{dx} \tan 5x$

$= \tan 5x(2x) + (1 + x^2)\sec^2 5x$

$= \boxed{2x \tan 5x + (1 + x^2)\sec^2 5x}$

(c) $\frac{x}{\cos 2x}$

$: \cos 2x \times \frac{d}{dx}[x] - x \times \frac{d}{dx}[\cos 2x] / (\cos 2x)^2$

$\cos 2x - x(-2\sin 2x) / \cos^2 2x$

$\cos 2x + 2x \sin 2x / \cos^2 2x$

$\frac{\cos 2x}{\cos 2x} \times \frac{1}{\cos 2x} + \frac{2x \sin 2x}{\cos 2x} \times \frac{1}{\cos 2x}$

$\sec 2x + 2x \tan 2x \sec 2x$

$\boxed{\sec 2x(1 + 2x \tan 2x)}$

(d) $\frac{\cos x}{\sin 3x}$

$: - \frac{\sin 3x \times \frac{d}{dx}[\cos x] - \cos x \times \frac{d}{dx}[\sin 3x]}{\sin^2 3x}$

$= \frac{\sin 3x(-\sin x) - \cos x\, 3 \cos 3x}{\sin^2 3x}$

$\boxed{\frac{-\sin x \sin 3x - 3\cos 3x \cos x}{\sin^2 3x}}$

7) Differentiate the following with respect to x.

(a) $2\sin^3 x$

$: 2\frac{d}{dx}[\sin^3 x]$

$= 2\times 3\sin^2 x \times \cos x$

$\boxed{6\sin^2 x\cos x}$

(b) $\sin 2x - 3\cos^4 x$

$= 2\cos 2x - 4\times 3(\cos^3 x)\times(-\sin x)$

$= 2\cos 2x - 12\cos^3 x(-\sin x)$

$= \boxed{2\cos 2x + 12\sin x\cos^3 x}$

(c) $\cos^2 3x$

$= 2(\cos 3x)\times\frac{d}{dx}(\cos 3x)$

$= 2\cos 3x(-3\sin 3x)$

$= -2\cos 3x\, 3\sin 3x$

$= \boxed{-6\cos 3x\sin 3x}$

(d) $x + 3\sin^5 2x$

$= \frac{d}{dx}[x] + 3\frac{d}{dx}[\sin^5 2x]$

$= 1 + 3\times 5[\sin^4 2x]\times\frac{d}{dx}[\sin 2x]$

$= 1 + 15\sin^4 2x(2\cos 2x)$

$= \boxed{1 + 30\sin^4 2x\cos 2x}$

(e) $4\tan^2 5x + 3$

$4\frac{d}{dx}[\tan^2 5x] + \frac{d}{dx}[3]$

$4[2\tan 5x]\times\frac{d}{dx}[\tan 5x] + 0$

$(8\tan 5x)(5\sec^2 5x)$

$\boxed{40\tan 5x\sec^2 5x}$

(f) $x\cos^7 2x$

$= \cos^7 2x\times\frac{d}{dx}[x] + x\times\frac{d}{dx}[\cos^7 2x]$

$= \cos^7 2x + x[7\cos^6 2x]\times\frac{d}{dx}[\cos 2x]$

$= \cos^7 2x + 7x\cos^6 2x(-2\sin 2x)$

$= \boxed{\cos^7 2x - 14x\cos^6 2x\sin 2x}$

(g) $(1-x)^2\sin^4 x$

$: \sin^4 x\times\frac{d}{dx}[1-x]^2 + (1-x)^2\times\frac{d}{dx}\sin^4 x$

$= \sin^4 x(2(1-x)) + (1-x)^2(4\sin^3 x)\times(\cos x)$

$2\sin^4 x(1-x) + (1-x)^2(4\sin^3 x)(\cos x)$

$\boxed{2\sin^4 x(1-x) + 4(1-x)^2(\sin^3 x\cos x)}$

(h) $\frac{\cos^2 x}{x}$

$= \frac{x\times\frac{d}{dx}[\cos^2 x] - \cos^2 x\times\frac{d}{dx}[x]}{x^2}$

$= [2x\cos x(-\sin x) - \cos^2 x]\div x^2$

$\boxed{\frac{-2x\cos x\sin x - \cos^2 x}{x^2}}$

1) Resolve the following into partial fraction:

$$\frac{2}{(x-1)(x+2)}$$

Solution

1: check the form of denominator

$$\frac{2}{(x-1)(x+2)} \longrightarrow \text{Linear form (2 linear forms}$$

$$\boxed{\frac{2}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}} \longrightarrow \text{eq (1)}$$

2: Multiply b/s by $(x-1)(x+2)$.

$$\frac{2\cancel{(x-1)}\cancel{(x+2)}}{\cancel{(x-1)}\cancel{(x+2)}} = \frac{\cancel{(x-1)}(x+2)A}{\cancel{(x-1)}} + \frac{(x-1)\cancel{(x+2)}B}{\cancel{(x+2)}}$$

$$\boxed{2 = A(x+2) + B(x-1)} \longrightarrow \text{eq (2)}$$

3: Use zero property and find the value of x

a) $x - 1 = 0$ $\boxed{x = 1}$

b) $x + 2 = 0$ $\boxed{x = -2}$

4: Put $x = 1$ and $x = -2$ in equation no (2)

$x = 1$

$2 = A(x+2) + B(x-1)$

$2 = A(1+2) + B(1-1)$

$2 = A(3) + B(0)$

$\boxed{A = 2/3}$

$x = -2$

$2 = A(x+2) + B(x-1)$

$2 = A(-2+2) + B(-2-1)$

$2 = A(0) + B(-3)$

$\boxed{B = -2/3}$

5: Put the values of A and B in equation (1)

$$\boxed{\frac{2}{(x-1)(x+2)} = \frac{2/3}{x-1} + \frac{(-2/3)}{x+2}}$$

Required partial Fraction

2) Resolve the following into partial fraction:

$$\frac{x+4}{x(x+1)}$$

Solution

1: form of denominator

$$\frac{x+4}{x(x+1)} \longrightarrow \text{Linear form}$$

$\Rightarrow$ Convert into partial fraction form

$$\boxed{\frac{x+4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}} \longrightarrow \text{eq (1)}$$

2: Rewrite / multiply b/s by $x(x+1)$ / LCM

$$\frac{x+4}{x\cancel{(x+1)}} = \frac{A(x+1) + Bx}{x\cancel{(x+1)}}$$

$$\boxed{x+4 = A(x+1) + Bx} \longrightarrow \text{eq (2)}$$

3: Find the value(s) of x

$$\frac{A}{x} \Rightarrow \boxed{x=0} \quad , \quad \frac{B}{x+1} \Rightarrow \boxed{x=-1}$$

4: Put $x=0$ and $x=-1$ in equation no (2)

$x=0$	$x=-1$
$x+4 = A(x+1) + Bx$	$x+4 = A(x+1) + Bx$
$0+4 = A(0+1) + B(0)$	$-1+4 = A(-1+1) + B(-1)$
$4 = A + 0$	$3 = A(0) - B$
$\boxed{\therefore A = 4}$	$3 = -B$
	$\boxed{B = -3}$

5: Put the values of A and B in equation (1)

$$\frac{x+4}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad , A = 4, \; B = -3$$

$$\Rightarrow \boxed{\frac{x+4}{x(x+1)} = \frac{4}{x} + \frac{-3}{x+1}}$$

Required partial fraction

3) Resolve the following into partial fraction

$$\frac{x-1}{(3x-5)(x-3)}$$

Solution

1: Form of Denominator

$$\frac{x-1}{(3x-5)(x-3)} \longrightarrow \text{linear form}$$

⇒ Convert into partial fraction form

$$\boxed{\frac{x-1}{(3x-5)(x-3)} = \frac{A}{3x-5} + \frac{B}{x-3}} \rightarrow eq\ (1)$$

2: Rewrite

$$\frac{x-1}{\cancel{(3x-5)(x-3)}} = \frac{A(x-3)+B(3x-5)}{\cancel{(3x-5)(x-3)}}$$

$$\boxed{x-1 = A(x-3)+B(3x-5)} \rightarrow eq\ (2)$$

3: Solve for x, then Solve for A and B.

$$\frac{A}{3x-5} \Rightarrow \boxed{x=5/3} \quad ; \quad \frac{B}{x-3} \Rightarrow \boxed{x=3}$$

Put $x=5/3$ and $x=3$ in equation No (2)

$x = 5/3$

$$x-1 = A(x-3)+B(3x-5)$$

$$\frac{5}{3}-1 = A\left(\frac{5}{3}-3\right)+B\left(3\left(\frac{5}{3}\right)-5\right)$$

$$\frac{5-3}{3} = A\left(\frac{5-9}{3}\right)+B(5-5)$$

$$\frac{2}{\cancel{3}} = A\left(\frac{-4}{\cancel{3}}\right)+0$$

$$2 = -4A$$

$$\boxed{A = -1/2}$$

$x = 3$

$$x-1 = A(x-3)+B(3x-5)$$

$$3-1 = A(3-3)+B(3(3)-5)$$

$$2 = A(0)+B(9-5)$$

$$2 = 0+4B$$

$$\boxed{B = 1/2}$$

4: Put $A=-1/2$ and $B=1/2$ in equation (1)

$$\frac{x-1}{(3x-5)(x-3)} = \frac{-1/2}{3x-5} + \frac{1/2}{x-3}$$

4) Resolve the following into Partial fraction

$$\frac{x}{x^2+x-2}$$

Solution

1: Check the form of Denominator

$$\frac{x}{x^2+x-2} \to \text{Quadratic} \to \text{factor out} \to \frac{x}{(x-1)(x+2)}$$

$$\frac{x}{(x-1)(x+2)} \xrightarrow{\text{Partial form}} \frac{A}{x-1} + \frac{B}{x+2}$$

$$\frac{x}{\boxed{(x-1)(x+2)}} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$\frac{x}{(x+2)(x-1)}$$

+ve -ve

$$\frac{x}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$\boxed{x = A(x+2)+B(x-1)}$$ By L.C.M

2: Get the value of x than Solve for A & B.

$x-1=0$

$\boxed{x=1}$

$x+2=0$

$\boxed{x=-2}$

$x=1$

$x = A(x+2)+B(x-1)$

$1 = A(1+2)+B(1-1)$

$1 = A(3)+B(0)$

$1 = 3A$

$\boxed{A = 1/3}$

$x=-2$

$x = A(x+2)+B(x-1)$

$-2 = A(-2+2)+B(-2-1)$

$-2 = A(0)+B(-3)$

$-2 = -3B$

$\boxed{B = 2/3}$

8.2 Integration By Parts

- Integration by Parts formula for indefinite Integral

$$\boxed{\int u\,dv = uv - \int v\,du}$$

- Integration by parts formula for Definite integral.

$$\int_a^b f(x)\,g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)\,g(x)\,dx$$

- formula for selection of $\boxed{u}$ and dv

L I A T E

1st	Logarithmic function
2nd	
3rd	Algebric function
4th	Trigonometric function
5th	Exponential function

Tabular Integration Can Simplify Repeated Integrations

- Tabular Integration

Ex: Evalute

$$\int x^2 e^x\,dx$$

Solution

$u = x^2$ and $dv = e^x$ OR

$f(x) = x^2$ and $g(x) = e^x$

u and its derivatives		dv/(g(x)) and its integrals
x^2	+	e^x
$2x$	−	e^x
2	+	e^x
0	← stop	e^x

$$\int x^2 e^x\,dx = x^2 e^x - 2x e^x + 2e^x + C$$

Ex ① Find $\int x\cos x\,dx$

Solution

Integration by parts

We use the formula $\int u\,dv = uv - \int v\,du$ with LIATE

$x \rightarrow$ Algebric function $\rightarrow$ 3rd

$\cos x \rightarrow$ Trigonometric function $\rightarrow$ 4th

$u = x$ and $dv = \cos x\,dx$

$du = 1\,dx$ and $v = \sin x$

$\hookrightarrow$ by derivative by Integration

$$\int x\cos x\,dx = (x)(\sin x) - \int (\sin x)(dx)$$
$$= x\sin x - \int \sin x\,dx$$
$$= x\sin x - (-\cos x)$$
$$\boxed{\int x\cos x\,dx = x\sin x + \cos x + c}$$

Ex ② Find $\int \ln x\,dx$

Solution

$u = \ln x$; $dv = dx$; $du = \frac{1}{x}\,dx$; $v = x$

$$\int \ln x\,dx = (\ln x)(x) - \int (x)\left(\frac{1}{x}\,dx\right)$$
$$= x\ln x - \int dx$$
$$\boxed{\int \ln x\,dx = x\ln x - x + c}$$

Ex ③ $\int x^2 e^x\,dx$

Solution

LIATE $\rightarrow u = x^2$; $dv = e^x\,dx$; $du = 2x\,dx$; $v = e^x$

(A: x^2, E: e^x)

$$\int u\,dv = uv - \int v\,du$$
$$\int x^2 e^x\,dx = (x^2)(e^x) - \int (e^x)(2x)\,dx$$
$$= x^2 e^x - 2\int x e^x\,dx \rightarrow \text{again integration}$$
$$= x^2 e^x - 2\left((x)(e^x) - \int (e^x)\,dx\right)$$
$$= x^2 e^x - 2(xe^x - e^x) + c$$
$$\boxed{\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + c}$$

Integration by parts formula for definite integrals

Ex (4) Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x=0$ to $x=4$

Solution

$$\int_0^4 xe^{-x}\,dx \rightarrow \int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

let $u = x$; $dv = e^{-x}dx$, $v = -e^{-x}$, and $du = dx$, then

$$\int_0^4 xe^{-x}\,dx = (x)(-e^{-x})\Big]_0^4 - \int_0^4 (dx)(-e^{-x})$$

$$= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx$$

$$= \left[-4e^{-4} - (-0\,e^{-0})\right] + \int_0^4 e^{-x}\,dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4$$

$$= -4e^{-4} - (e^{-4} - e^{-0})$$

$$= 1 - 5e^{-4} \approx 0.91 \qquad \left(\int_a^b f(x)g'(x) = uv\Big]_a^b - \int_a^b v\,du\right)$$

Ex (5)

Integration by parts formula for definite integrals

Ex ④ Find the area of the region bounded by the Curve $y = xe^{-x}$ and the x-axis from $x=0$ to $x=4$

Solution

$$\int_0^4 xe^{-x}\,dx \to \int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)\,g(x)\,dx$$

let $u = x$; $dv = e^{-x}dx$, $v = -e^{-x}$, and $du = dx$, then

$$\int_0^4 xe^{-x}dx = (x)(-e^{-x})\Big]_0^4 - \int_0^4 (dx)(-e^{-x})$$

$$= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx$$

= [redacted]

$$= \left[-4e^{-4} - (-0\,e^{-0})\right] + \int_0^4 e^{-x}dx$$

$$= -4e^{-4} - e^{-x}\Big]_0^4$$

$$= -4e^{-4} - (e^{-4} - e^{-0})$$

$$= 1 - 5e^{-4} \approx 0.91$$

$$\left(\int_a^b f(x)\,g'(x) = uv\Big]_a^b - \int_a^b v\,du\right)$$

Ex ⑤

Exercises 8.2

Integration by parts

Evalute the integrals in Exercises 1-24 using integration by parts.

1) $\int x \sin \frac{x}{2} \, dx$

:- LIATE $\rightarrow$ $u = x$; $dv = \sin\frac{x}{2}\,dx$; $du = dx$; $v = \frac{-1}{1/2}\cos\frac{x}{2}$

x $\sin\frac{x}{2}$

$v = -2\cos\frac{x}{2}$

$$\int u\,dv = uv - \int v\,du$$

$$\int x \sin\frac{x}{2}\,dx = (x)(-\cos\frac{x}{2}) - \int(-2\cos\frac{x}{2})(dx)$$

$$= -2x\cos\frac{x}{2} + 2\int \cos\frac{x}{2}\,dx$$

$$= -2x\cos\frac{x}{2} + 2\left(\frac{1}{1/2}\sin\frac{x}{2}\right) + c$$

$$= -2x\cos\frac{x}{2} + 4\sin\frac{x}{2} + c$$

$\therefore \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

2) $\int \theta \cos \pi\theta \, d\theta$

:-

40.5
38.5/
70.0

3) $\int t^2 \cos t \, dt$

:- $u = t^2$, $dv = \cos t \, dt$, $du = 2t \, dt$, $v = (\sin t)$

$$\int u \, dv = uv - \int v \, du$$

$$\int t^2 \cos t \, dt = (t^2)(\sin t) - \int (\sin t)(2t \, dt)$$

$$= t^2 \sin t - 2\int t \sin t \, dt$$

$$= t^2 \sin t - 2\left((t)(-\cos t) - \int (-\cos t) \, dt\right)$$

$$= t^2 \sin t - 2(-t\cos t + \sin t) + C$$

$$\boxed{\int t^2 \cos t \, dt = t^2 \sin t + 2t \cos t + 2\sin t + C}$$

4) $\int x^2 \sin x \, dx$

:- $u = x^2$, $dv = \sin x \, dx$, $du = 2x \, dx$, $v = -\cos x$

$$\int u \, dv = uv - \int v \, du$$

$$\int x^2 \sin x \, dx = (x^2)(-\cos x) - \int (-\cos x)(2x \, dx)$$

$$= -x^2 \cos x + 2\int x \cos x \, dx \rightarrow \text{repeated}$$

$$= -x^2 \cos x + 2\left((x)(\sin x) - \int \sin x \, dx\right)$$

$$= -x^2 \cos x + 2\left(x \sin x - (-\cos x)\right) + C$$

$$\boxed{\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2\cos x + C}$$

5) $\int_2^2 x \ln x \, dx$

$$\int_a^b f(x) g'(x) \, dx = f(x) g(x) \Big]_a^b - \int_a^b f'(x) g(x) \, dx$$

$u = x$; $dv = \ln x \, dx$; $du = dx$; $v =$

$u = \ln x$, $du = \frac{1}{x} dx$; $dv = x \, dx$; $v = \frac{x^2}{2}$

$$\int_1^2 x \ln x \, dx = \left[\frac{x^2}{2} \ln x\right]_1^2 - \int_1^2 \frac{x^2}{2} \frac{dx}{x}$$

$$= \left[(2)(0.693) - \frac{1}{2}(0)\right] - \frac{1}{2}\int_1^2 x \, dx$$

$$= [2\ln 2 - 0] - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^2 \qquad \because 0.693 = \ln 2$$

$\because 2\ln 2 = 4 \ln / \ln 4$ $\quad = 2\ln 2 - \frac{1}{4}[x^2]_1^2 = 4\ln - \frac{1}{4}[4-1]$

$$= 4\ln - \frac{1}{4}[3] = \boxed{4\ln - \frac{3}{4}}$$

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6) $\int_1^e x^3 \ln x \, dx$

$\boxed{\text{L I A T E}}$ → Principle

:- $u = \ln x$; $du = \frac{1}{x} dx$; $dv = x^3 dx$; $v = \frac{x^4}{4}$;

$$\int u\,dv = uv - \int v\,du$$

$$= (\ln x)\left(\frac{x^4}{4}\right)\Big]_1^e - \int_1^e \left(\frac{x^4}{4}\right)\left(\frac{1}{x}\right) dx$$

$$= \frac{x^4 \ln x}{4}\Big]_1^e - \frac{1}{4}\int_1^e x^3\, dx$$

$$= \frac{x^4 \ln x}{4}\Big]_1^e - \left[\frac{1}{4}\left(\frac{x^4}{4}\right)\right]_1^e \cancel{+c}$$

$$= \frac{x^4 \ln x}{4}\Big]_1^e - \left[\frac{x^4}{16}\right]_1^e + \ominus$$

$$= \left[\left(\frac{e^4 \ln e}{4}\right) - \frac{(1)^4 \ln(1)}{4}\right] - \left[\frac{e^4}{16} - \frac{(1)^4}{16}\right]$$

$$= \left[\frac{(e^4)(1)}{4} - \frac{(1)(0)}{4}\right] - \left[\frac{e^4 - 1}{16}\right] \quad \because \ln e = 1 \ \& \ \ln(1) = 0$$

$$= \frac{e^4}{4} - \frac{e^4 - 1}{16} = \frac{4e^4 - (e^4 - 1)}{16} = \frac{4e^4 - e^4 + 1}{16}$$

$$= \boxed{\frac{3e^4 + 1}{16}}$$

7) $\int x e^x \, dx$

: $u = x$; $du = dx$; $dv = e^x \, dx$; $v = e^x$;

$\int u \, dv = uv - \int v \, du$

$= (x)(e^x) - \int (e^x)(dx)$

$= \boxed{x e^x - e^x + c}$

8) $\int x e^{3x} \, dx$

: $u = x$; $du = dx$; $dv = e^{3x} \, dx$; $v = \frac{1}{3} e^{3x}$

$\int u \, dv = uv - \int v \, du$

$= (x)\left(\frac{1}{3} e^{3x}\right) - \int \left(\frac{1}{3} e^{3x}\right) (dx)$

$= \frac{x e^{3x}}{3} - \frac{1}{3} \int e^{3x} \, dx = \frac{x e^{3x}}{3} - \frac{1}{3} \left(\frac{1}{3} e^{3x}\right) + c$

$= \frac{x e^{3x}}{3} - \frac{e^{3x}}{9} + c$

9) $\int x^2 e^{-x} \, dx$

: Using Tabular integration, because when we integrate an exponential function then there occurs repeated integration.

$f(x) = x^2$ and $g(x) = e^{-x}$

derivatives of (x^2)		Integrations of (e^{-x})
x^2	$+$	e^{-x}
$2x$	$-$	$-e^{-x}$
2	$+$	e^{-x}
0	Stop	$-e^{-x}$

$+x^2(-e^{-x}) - 2x(+e^{-x}) + 2(-e^{-x}) + C$

$-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c$

$\boxed{-(x^2 + 2x + 2) e^{-x} + c}$

10) $\int (x^2 - 2x + 1)\, e^{2x}\, dx$

:- $u = (x^2 - 2x + 1)$; $dv = e^{2x}\, dx$

$f(x) = x^2 - 2x + 1$; $g(x) = e^{2x}\, dx$

Using Tabular Integration

derivatives of $f(x)$		Integrations of $g(x)$
$x^2 - 2x + 1$	+	e^{2x}
$2x - 2$	−	$\frac{1}{2} e^{2x}$
2	+	$\frac{1}{4} e^{2x}$
0	stop	$\frac{1}{8} e^{2x}$

$$\int (x^2 - 2x + 1)\, e^{2x}\, dx = \left((x^2 - 2x + 1)\frac{1}{2} e^{2x}\right) - \left((2x - 2)\frac{1}{4} e^{2x}\right) + \frac{2}{8} e^{2x} + C$$

$$= \left(\frac{(x^2 - 2x + 1)}{2} - \left(\frac{2x - 2}{4}\right) + \frac{1}{4}\right) e^{2x} + C$$

$$= \left(\frac{4x^2 - 8x + 4 - 4x + 4}{8} + \frac{1}{4}\right) e^{2x} + C$$

$$= \left(\frac{4x^2 - 12x + 10}{8}\right) e^{2x} + C$$

$$= \boxed{\left(\frac{x^2}{2} - \frac{3}{2}x + \frac{5}{4}\right) e^{2x} + C}$$

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(11) $\int \tan^{-1} y \, dy$

: $u = \tan^{-1} y$, $du = \frac{1}{1+y^2} dy$, $dv = dy$, $v = y$

$\int u \, dv = uv - \int v \, du$

$\int \tan^{-1} y \, dy = (\tan^{-1} y)(y) - \int y \left(\frac{1}{1+y^2} \right) dy$

$= y \tan^{-1} y - \frac{1}{2} \ln (1+y^2) + C$

$= y \tan^{-1} y - \ln \sqrt{1+y^2} + C$

12) $\int \sin^{-1} y \, dy$

: $u = \sin^{-1} y$, $du = \frac{1}{\sqrt{1-y^2}} dy$; $dv = dy$, $v = y$

$\int u \, dv = uv - \int v \, du$

$\int \sin^{-1} y \, dy = (\sin^{-1} y)(y) - \int (y) \left(\frac{1}{\sqrt{1-y^2}} \right) dy$

$= y \sin^{-1} y + \sqrt{1-y^2} + C$

13) $\int x \sec^2 x \, dx$

: $u = x$, $du = dx$; $dv = \sec^2 x \, dx$, $v = \tan x$

Integration by parts

$\int u \, dv = uv - \int v \, du$

$\int x \sec^2 x \, dx = (x)(\tan x) - \int (\tan x)(dx)$

$= x \tan x - \int \tan x \, dx$

$= x \tan x - \ln \sec x + C$

14) $\int 4x \sec^2 2x \, dx$

: $u = 4x$, $du = 4dx$; $\boxed{dv = \sec^2 2x \, dx, \ v = \tan 2x}$ X

$\boxed{\text{let } y = 2x \text{ and } dy = 2dx}$

$\int 4x \sec^2 2x \, dx = \boxed{\int y \sec^2 y \, dy} = \int (2x) \sec^2 2x \, (2\,dx)$

$u = y$, $du = dy$; $dv = \sec^2 y \, dy$, $v = \tan y$

Integration by parts

$\int y \sec^2 y \, dy = (y)(\tan y) - \int (dy)(\tan y)$

$= y \tan y - \int \tan y \, dy$

$= y \tan y - \ln \sec y + C$

Replace the y value(s)

$= (2x) \tan 2x - \ln \sec 2x + C$

$\boxed{\int y \sec^2 y \, dy = 2x \tan 2x - \ln \sec 2x + C}$

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15) $\int x^3 e^x \, dx$

: $u = x^3$, $du = 3x^2 \, dx$; $dv = e^x$, $v = e^x$

Integration by parts using tabular integration technique

$f(x) = x^2$; $g(x) = e^x$

derivatives of $f(x)$		Integrations of $g(x)$
x^3	+	e^x
$3x^2$	−	e^x
$6x$	+	e^x
6	Stop	e^x
0	Stop	e^x

$+x^3e^x - 3x^2e^x + 6xe^x + 6e^x + C$

$x^3e^x - 3x^2e^x + 6xe^x + 6e^x + C$

$\boxed{(x^3 - 3x^2 + 6x + 6)e^x + C}$

16) $\cancel{\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}}$ $\int p^4 e^{-p} \, dp$

: $f(x) = p^4$ $g(x) = e^{-p}$

derivatives of (p^4)		Integration of e^{-p}
p^4	+	e^{-p}
$4p^3$	−	$-e^{-p}$
$12p^2$	+	e^{-p}
$24p$	−	$-e^{-p}$
24	+	e^{-p}
0	Stop	$-e^{-p}$

$$\int p^4 e^{-p} dp = p^4(-e^{-p}) - 4p^3e^{-p} + 12p^2(-e^{-p}) - 24pe^{-p} + 24(-e^{-p})$$

$$= -p^4e^{-p} - 4p^3e^{-p} - 12p^2e^{-p} - 24pe^{-p} - 24e^{-p}$$

$$= (-p^4 - 4p^3 - 12p^2 - 24p - 24)e^{-p} + C$$

17) $\int (x^2 - 5x)e^x\,dx$

: Integration by parts & using Tabular Integration

$f(x) = x^2 - 5x$; $g(x) = e^x$

derivatives of $f(x)$		Integrations of $g(x)$
$x^2 - 5x$	+	e^x
$2x - 5$	−	e^x
2	+	e^x
0	stop	e^x

$$(x^2 - 5x)e^x - (2x-5)e^x + 2e^x + C$$

$$\int (x^2-5x)e^x\,dx = [x^2 - 5x - 2x + 5 + 2]e^x + C$$

$$= (x^2 - 7x + 7)e^x + C$$

18) $\int (r^2 + r + 1)e^r\,dr$

: Integration by parts & using tabular integration

$f(x) = r^2 + r + 1$ $g(x) = e^r$

derivatives of $f(x)$		Integrations of $g(x)$
$r^2 + r + 1$	+	e^r
$2r + 1$	−	e^r
2	+	e^r
0	stop	e^r

$$\int (r^2 + r + 1)e^r\,dr = (r^2 + r + 1)e^r - (2r+1)e^r + 2e^r + C$$

$$= [r^2 + r + \cancel{1} - 2r - \cancel{1} + 2]e^r + C$$

$$= [r^2 - r + 2]e^r + C$$

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19) $\int x^5 e^x dx$

: Integration by parts using Tabular integration

$f(x) = x^5$ $\quad$ $g(x) = e^x$

derivatives of $f(x)$		Integrations of $g(x)$
x^5	+	e^x
$5x^4$	−	e^x
$20x^3$	+	e^x
$60x^2$	−	e^x
$120x$	+	e^x
120	−	e^x
0	stop	e^x

$$\int x^5 e^x dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120x e^x - 120 e^x + C$$

$$= [x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120] e^x + C$$

20) $\int t^2 e^{4t} dt$

: Integration by parts & using Tabular Integration for repeated integrations of exponential function.

$f(x) = t^2$; $g(x) = e^{4t}$

derivatives of $f(x)$		Integrations of $g(x)$
t^2	+	e^{4t}
$2t$	−	$e^{4t}/4$
2	+	$e^{4t}/16$
0	stop	$e^{4t}/64$

$$\int t^2 e^{4t} dt = t^2 \frac{e^{4t}}{4} - 2t \frac{e^{4t}}{16} + 2 \frac{e^{4t}}{64} + C$$

$$= \frac{t^2 e^{4t}}{4} - \frac{t e^{4t}}{8} + \frac{e^{4t}}{32} + C$$

$$= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right) e^{4t} + C$$

☆

21) $\int e^{\theta} \sin\theta \, d\theta$

: Using Integration by parts only because we can't achieve the result using tabular integration when there exist the both Trigonometric & exponential terms in product form.

Integration by parts of indefinite integral

$$\int u\,dv = uv - \int v\,du$$

$u = \sin\theta$, $du = \cos\theta \, d\theta$; $dv = e^{\theta} d\theta$, $v = e^{\theta}$

$$\int e^{\theta}\sin\theta \, d\theta = (\sin\theta)(e^{\theta}) - \int (e^{\theta})(\cos\theta)\,d\theta$$

$$= \sin\theta e^{\theta} - \int e^{\theta}\cos\theta \, d\theta$$

$u = \cos\theta$, $du = -\sin\theta \, d\theta$; $dv = e^{\theta} d\theta$, $v = e^{\theta}$

$$= \sin\theta e^{\theta} - \left(uv - \int v\,du \right)$$

$$= \sin\theta e^{\theta} - \left((\cos\theta)(e^{\theta}) - \int (e^{\theta})(-\sin\theta)\,d\theta \right)$$

$$= \sin\theta e^{\theta} - \left(\cos\theta e^{\theta} + \int e^{\theta}\sin\theta \, d\theta \right)$$

$$= e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta \, d\theta$$

$$I = \int e^{\theta}\sin\theta \, d\theta$$

we got the original Question returned in our solution so here we use the 'I' substitution, in which we take the original Question equal to I.

$$\int e^{\theta}\sin\theta \, d\theta = e^{\theta}\sin\theta - e^{\theta}\cos\theta - \int e^{\theta}\sin\theta \, d\theta$$

$$I = e^{\theta}\sin\theta - e^{\theta}\cos\theta - I + c'$$

$$2I = e^{\theta}\sin\theta - e^{\theta}\cos\theta + c'$$

$$I = \frac{1}{2}\left(e^{\theta}\sin\theta - e^{\theta}\cos\theta\right) + c$$

where $c = \frac{c'}{2}$ is another arbitrary constant

22) $\int e^{-y} \cos y \, dy$

$u = \cos y$, $du = -\sin y \, dy$; $dv = e^{-y} dy$, $v = -e^{-y}$

<u>Integration by parts</u>

$\int u \, dv = uv - \int v \, du$

$$\int e^{-y} \cos y \, dy = (\cos y)(-e^{-y}) - \int (-e^{-y})(-\sin y) \, dy$$

$$= -e^{-y} \cos y - \int e^{-y} \sin y \, dy$$

$u = \sin y$, $du = \cos y \, dy$; $dv = e^{-y}$, $v = -e^{-y}$

$$= -e^{-y} \cos y - \left((\sin y)(-e^{-y}) - \int (-e^{-y})(\cos y) \, dy \right)$$

$$= -e^{-y} \cos y + e^{-y} \sin y + \int (-e^{-y})(\cos y) \, dy$$

$$= -e^{-y} \cos y + e^{-y} \sin y - \int e^{-y} \cos y \, dy$$

let $I = \int e^{-y} \cos y \, dy$

$$\int e^{-y} \cos y \, dy = -e^{-y} \cos y + e^{-y} \sin y - \int e^{-y} \cos y \, dy$$

$$I = -e^{-y} \cos y + e^{-y} \sin y - I + c'$$

$$2I = -e^{-y} \cos y + e^{-y} \sin y + c'$$

$$I = \frac{1}{2} (-\cos y + \sin y) e^{-y} + c$$

where $c = \frac{c'}{2}$

23) $\int e^{2x} \cos 3x \, dx$

: Integration by parts & using I substitution

$$I = \int e^{2x} \cos 3x \, dx$$

$u = \cos 3x$, $du = -3\cos 3x \, dx$; $dv = e^{2x} dx$, $v = \frac{e^{2x}}{2}$

$$\int u \, dv = uv - \int v \, du$$

$$I = (\cos 3x)\left(\frac{e^{2x}}{2}\right) - \int \left(\frac{e^{2x}}{2}\right)(-3\cos 3x) \, dx$$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{1}{2} \int e^{2x} 3\cos 3x \, dx$$

$u = 3\cos 3x$, $du = -9\sin 3x \, dx$; $dv = e^{2x} dx$, $v = \frac{e^{2x}}{2}$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{1}{4}\left((3\cos 3x)(e^{2x}) - \int (e^{2x})(-9\sin 3x) \, dx\right)$$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{4} \cos 3x \, e^{2x} + \frac{9}{4} \int e^{2x} \sin 3x \, dx$$

$u = \sin 3x$, $du = 3\cos 3x \, dx$; $dv = e^{2x} dx$, $v = \frac{e^{2x}}{2}$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{4} \cos 3x \, e^{x} + \frac{9}{4}\left((\sin 3x)(e^{2x}) - \int (e^{2x})(3\cos 3x) \, dx\right)$$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{4} \cos 3x \, e^{2x} + \frac{9}{4}\left(e^{2x} \sin 3x - \int e^{2x} 3\cos 3x \, dx\right.$$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{2} e^{2x} \cos 3x + \frac{9}{4} e^{2x} \sin 3x - \frac{27}{4} \underline{\int e^{2x} \cos 3x \, dx}$$

$$I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{4} e^{2x} \cos 3x + \frac{9}{4} e^{2x} \sin 3x - \frac{27}{4} I + c'$$

$$I + \frac{27}{4} I = \frac{e^{2x}}{2} \cos 3x + \frac{3}{4} e^{2x} \cos 3x + \frac{9}{4} e^{2x} \sin 3x + c'$$

X

23) $\int e^{2x}\cos 3x\,dx$

: Integration by parts

$u=\cos 3x$, $du=-3\sin 3x\,dx$; $dv=e^{2x}\,dx$, $v=\frac{1}{2}e^{2x}$

$I=\int e^{2x}\cos 3x\,dx$

$I=(\cos 3x)(\frac{1}{2}e^{2x})-\int(\frac{1}{2}e^{2x})(-3\sin 3x)dx$

$I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\int e^{2x}\sin 3x\,dx$

$u=\sin 3x$, $du=3\cos 3x$; $dv=e^{2x}dx$, $v=\frac{1}{2}e^{2x}$

$I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\left((\sin 3x)(\frac{1}{2}e^{2x})-\int(\frac{1}{2}e^{2x})(3\cos 3x)\,dx\right.$

$I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{2}\left(\frac{1}{2}e^{2x}\sin 3x-\frac{3}{2}\int e^{2x}\cos 3x\,dx\right)$

$I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}\underline{\int e^{2x}\cos 3x\,dx}$

$I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x-\frac{9}{4}I+C'$

$I+\frac{9}{4}I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x+C'$

$\frac{13}{4}I=\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x+C'$

$I=\frac{4}{13}\left(\frac{1}{2}e^{2x}\cos 3x+\frac{3}{4}e^{2x}\sin 3x\right)+C$, $C=\frac{4C'}{13}$

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24) $\int e^{-2x} \sin 2x\,dx$

: $u = \sin 2x$, $du = 2\cos 2x\,dx$; $dv = e^{-2x}dx$, $v = -\frac{1}{2}e^{-2x}$

Integration by parts

$I = \int e^{-2x} \sin 2x\,dx$

$$\int e^{-2x}\sin 2x\,dx = (\sin 2x)\left(-\frac{1}{2}e^{-2x}\right) - \int \left(-\frac{1}{2}e^{-2x}\right)(2\cos 2x)\,dx$$

$$I = -\frac{1}{2}e^{-2x}\sin 2x + \int e^{-2x}\cos 2x\,dx$$

$u = \cos 2x\,dx$, $du = -2\sin 2x\,dx$; $dv = e^{-2x}dx$, $v = -\frac{1}{2}e^{-2x}$

$$I = -\frac{1}{2}e^{-2x}\sin 2x + \left((\cos 2x)\left(-\frac{1}{2}e^{-2x}\right) - \int \left(-\frac{1}{2}e^{-2x}\right)(-2\sin 2x\,dx)\right)$$

$$I = -\frac{1}{2}e^{-2x}\sin 2x + \left(\left(-\frac{1}{2}e^{-2x}\cos 2x\right) + \int e^{-2x}\sin 2x\,dx\right)$$

$$I = -\frac{1}{2}e^{-2x}\sin 2x - \frac{1}{2}e^{-2x}\cos 2x - I + c'$$

$$2I = -\frac{1}{2}e^{-2x}\sin 2x - \frac{1}{2}e^{-2x}\cos 2x + c'$$

$$I = -\frac{1}{4}e^{-2x}\sin 2x - \frac{1}{4}e^{-2x}\cos 2x + c \quad , \quad c = \frac{c'}{2}$$

8.3 Trigonometric Integrals

Products of Powers of Sines and Cosines

we begin with integrals of the form

$$\int \sin^m x \cos^n x \, dx$$

where m and n are nonnegative integers (positive or zero). we can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If m is odd, we write m as $2k+1$ and use the Identity $\sin^2 = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x \quad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$ we write n as $2k+1$ and use the Identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

we then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$

Case 3 if both m and n are even in $\int \sin^m x \cos^n x \, dx$ we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Single double

$$\boxed{\sin^2 2x = \frac{1 - \cos 4x}{2}} \to \text{always}$$

Example 1 Evalute

$$\int \sin^3 x \cos^2 x \, dx$$

: Case 1

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x)(\cos^2 x)(-d(\cos x)) \qquad \sin x \, dx = -d\cos x$$

$u = \cos x$

$$= \int (1 - u^2)(u^2)(-du)$$

$$= \int (u^2 - u^4)(-du)$$

$$= \int (u^4 - u^2) \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

Example 2 Evalute

$$\int \cos^5 x \, dx$$

Case 2

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x)$$

$u = \sin x$

$$= \int (1 - u^2)^2 \, du$$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$$

Example 3 Evaluate

$$\int \sin^2 x \cos^4 x \, dx$$

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+2\cos 2x+\cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1+\cos 2x-\cos^2 2x-\cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x+\frac{1}{2}\sin 2x-\int (\cos^2 2x+\cos^3 2x)\, dx\right]$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2}\int (1+\cos 4x)\, dx$$

$$= \frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)$$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1-\sin^2 2x)\cos 2x \, dx$$

$u = \sin 2x$

$du = 2\cos 2x \, dx$

$$= \frac{1}{2}\int (1-u^2)\, du = \frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16}\left(x-\frac{1}{4}\sin 4x+\frac{1}{3}\sin^3 2x\right)+C$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = \frac{(1+\cos 2\theta)}{2}$ to eliminate a square root.

Example 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$$

Solution

To eliminate the square root we use the identity

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \quad \text{OR} \quad 1+\cos^2\theta = 2\cos^2\theta$$

with $\theta = 2x$, this becomes

$$1+\cos 4x = 2\cos^2 2x$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} \sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1-0] = \frac{\sqrt{2}}{2}$$

Integrals of Power x and $\sec x$

we know how to integrate the tangent and secant and their squares. To integrate higher powers, we use the identities $\tan^2 = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

Example 5 Evalute

$$\int \tan^4 x \, dx$$

Solution

$$\int \tan^4 x dx = \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x(\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x dx - \int \tan^2 x \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx$$

In the first integral we let

$$u = \tan x, \quad du = \sec^2 x \, dx$$

$$\int u^2 \, du = \frac{1}{3} u^3 + C_2$$

The remaining integrals are standard form, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

Example 6 Evalute

$$\int \sec^3 x \, dx$$

Solution we integrates by parts using

$u = \sec x, \quad dv = \sec^2 x \, dx, \quad v = \tan x, \quad du = \sec x \tan x \, dx$

$$\int \sec^3 x \, dx = \sec x \tan x - \int (\tan x)(\sec x \tan x \, dx)$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan + \frac{1}{2} \ln|\sec x + \tan x| + C$$

Example 7 Evalute

$$\int \tan^4 x \sec^4 x \, dx$$

$$\int (\tan^4 x)(\sec^4 x)\, dx = \int (\tan^4 x)(1 + \tan^2 x)(\sec^2 x)\, dx$$

$$= \int (\tan^4 x + \tan^6 x)(\sec^2 x)\, dx$$

$$= \int (\tan^4 x)(\sec^2 x)\, dx + \int (\tan^6 x)(\sec^2 x)\, dx$$

$u = \tan x, \quad du = \sec^2 x \, dx$

$$= \int u^4 du + \int u^6 du = \frac{u^5}{5} + \frac{u^7}{7} + C$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \int \sin mx \cos nx \, dx, \int \cos mx \cos nx \, dx$$

arise in many application involving periodic functions. We can evalute these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities:

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \quad (3)$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x] \quad (4)$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x] \quad (5)$$

These identities come from the angle sum formulas for the sine and cosine functions

Example 8

$$\int \sin 3x \cos 5x \, dx$$

Solution From equation (4) with $m=3$ and $n=5$, we get

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2}\int [\sin(-2x) + \sin 8x] dx$$

$$= \frac{1}{2}\int (\sin 8x - \sin 2x) \, dx$$

$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C$$

Exercises 8.3

Powers of Sines and Cosines

Evalute the integrals in Exercises 1-22.

1) $\int \cos 2x \, dx = \frac{1}{2} \sin 2x + C$

2) $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$

$= 3\int_0^{\pi} \sin \frac{x}{3} \, dx = 3\left[\frac{1}{1/3} - \cos \frac{x}{3}\right]_0^{\pi} = 3\left[3 - \cos \frac{x}{3}\right]_0^{\pi}$

$= 3\left[\left(3 - \cos \frac{\pi}{3}\right) - \left(3 - \cos \frac{0}{3}\right)\right] = 3\left[\left(3 - \frac{\sqrt{3}}{2}\right) - (3 - 1)\right]$

$= 3\left[\frac{6 - \sqrt{3}}{2} - 2\right]$

2) $\int_0^{\pi} 3 \sin \frac{x}{3} \, dx$

$: 3\int_0^{\pi} \sin \frac{x}{3} \, dx = 3\left[-\frac{1}{1/3} \cos \frac{x}{3}\right]_0^{\pi} = 3\left[-3 \cos \frac{x}{3}\right]_0^{\pi}$

$= 9\left[-\cos \frac{\pi}{3} - \left(-\cos \frac{0}{3}\right)\right] = 9\left[-\frac{1}{2} + 1\right] = \boxed{\frac{9}{2}}$

3) $\int \cos^3 x \sin x \, dx$

$: \int \cos^2 x \sin x \cos x \, dx = \int (1 - \sin^2 x) \sin x \cos x \, dx$

$u = \sin x$, $du = \cos dx$

$= \int (1 - u^2) u \, du = \int (u - u^3) \, du = \int u \, du - \int u^3 \, du$

$= \int u \, du - \int u^3 \, du = \frac{u^2}{2} - \frac{u^4}{4} + C = \frac{\sin^2 x}{2} - \frac{\sin^4 x}{4} + C$

This Theorem can't be applied because both m & n are odd.

5) $\int \sin^3 x \, dx$

: Case 2 m is even and n is odd

$\int \sin^3 dx = \int \sin^2 x \sin x \, dx = \int (1-\cos^2 x)\sin x \, dx$

let $u = \cos x$, then $du = \sin x \, dx$

$\int (1-\cos^2 x)\sin x \, dx = \int (1-u^2) \, du$

$= \int du - \int u^2 du$

$= u - \frac{u^3}{3} + c = \boxed{\cos x - \frac{\cos^3 u}{3} + c}$

6) $\int \cos^3 4x \, dx$

: Case 2 m is even and n is odd

$\int \cos^3 4x \, dx = \int \cos^2 4x \cos 4x \, dx = \int \left(\frac{1+\cos 8x}{2}\right) \cos 4x \, dx$

$\because$ $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$

But here we use the Identity defined by pythagorus

$\boxed{\sin^2 x + \cos^2 x = 1}$

$\int \cos^3 4x \, dx = \int \cos^2 4x \cos 4x \, dx$

we put $\cos^2 4x = 1 - \sin^2 4x$

$= \int (1-\sin^2 4x)\cos 4x \, dx$

$\int \cos^3 4x \, dx = \int \cos 4x \, dx - \int \sin^2 4x \cos 4x \, dx$

let $u = \sin 4x$; $du = 4\cos 4x \, dx \rightarrow \frac{du}{4} = \cos 4x \, dx$

$\int \cos^3 4x \, dx = \int \frac{du}{4} - \int (u^2)\left(\frac{du}{4}\right)$

$\int \cos^3 4x \, dx = \frac{1}{4}\int du - \frac{1}{4}\int u^2 du = \frac{1}{4}[u] - \frac{1}{4}\left[\frac{u^3}{3}\right] + c$

$\int \cos^3 4x \, dx = \frac{1}{4}\sin 4x - \frac{1}{4\times 3}\sin^3 4x + c$

$\boxed{\int \cos^3 4x \, dx = \frac{1}{4}\sin 4x - \frac{1}{12}\sin^3 4x + c}$

7) $\int \sin^5 x \, dx$

: Case 2 m is even and n is odd

$$\int \sin^5 x \, dx = \int \sin^4 x \sin x \, dx = \int (\sin^2 x)^2 (\sin x) \, dx$$

put $\sin^2 x = 1 - \cos^2 x$

$$= \int (1 - \cos^2 x)^2 \sin x \, dx$$
$$= \int (1 - 2\cos^2 x + \cos^4 x) \sin x \, dx$$
$$= \int \sin x \, dx - 2\int \cos^2 x \sin x \, dx + \int \cos^4 x \sin x \, dx$$

$u = \cos x$, $du = -\sin x \, dx \rightarrow -du = \sin x \, dx$

$$= \int -du - 2\int (u^2)(-du) + \int (u^4)(-du)$$
$$= -\int du + 2\int u^2 \, du - \int u^4 \, du$$
$$= -u + \frac{2}{3}u^3 - \frac{u^5}{5} + C$$

$$\boxed{\int \sin^5 x \, dx = -\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C}$$

8) $\int_0^{\pi} \sin^5 \frac{x}{2} \, dx$

:

Case 2 m is even and n is odd

$$\int_0^{\pi} \sin^5 \frac{x}{2} \, dx = \int_0^{\pi} \sin^4 \frac{x}{2} \sin \frac{x}{2} \, dx = \int_0^{\pi} \left(\sin^2 \frac{x}{2}\right)^2 \left(\sin \frac{x}{2}\right) dx$$

$$= \int_0^{\pi} \left(1 - \cos^2 \frac{x}{2}\right)^2 \left(\sin \frac{x}{2}\right) dx \Rightarrow u = \cos \frac{x}{2} \; ; \; du = -2 \sin \frac{x}{2}$$

$-2du = \sin x/2$

$$= \int_0^{\pi} (1 - u^2)^2 (-2du) = -2\int_0^{\pi} (1 - u^2)^2 \, du$$

$$= -2\int_0^{\pi} (1 - 2u^2 + u^4) \, du = -2\left[u - \frac{2}{3}u^3 + \frac{u^5}{5}\right]_0^{\pi}$$

$$= -2\left[\cos \frac{x}{2} - \frac{2}{3}\cos^3 \frac{x}{2} + \frac{1}{5}\cos^5 \frac{x}{2}\right]_0^{\pi}$$

$$= -2\left[\left(\cos \frac{\pi}{2} - \frac{2}{3}\cos^3 \frac{\pi}{2} + \frac{1}{5}\cos^5 \frac{\pi}{2}\right) - \left(\cos \frac{0}{2} - \frac{2}{3}\cos^3 \frac{0}{2} + \frac{1}{5}\cos^5 \frac{0}{2}\right)\right]$$

$$= -2\left[\left(0 - \frac{2}{3}(0) + 0\right) - \left(1 - \frac{2}{3} + \frac{1}{5}\right)\right] = +2\left(\frac{15 - 10 + 3}{15}\right) = 2\left(\frac{8}{15}\right) = \boxed{\frac{16}{15}}$$

8) $\int \cos^3 x \, dx$

: Case 2 m is even and n is odd

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1-\sin^2 x) \cos x \, dx$$

$u = \sin x$, $du = \cos x \, dx$

$$= \int (1-u^2) \, du$$

$$= \int du - \int u^2 du = u - \frac{u^3}{3} + c$$

$$\int \cos^3 x \, dx = \boxed{\sin x - \frac{\sin^3 x}{3} + c}$$

10) $\int_0^{\pi/6} 3 \cos^5 3x \, dx$

: Case 2. m is even & n is odd

$$3\int_0^{\pi/6} \cos^5 3x \, dx = 3\int \cos^4 3x \cos 3x \, dx = 3\int (\cos^2 3x)^2 (\cos 3x) \, dx$$

$$= 3\int (1-\sin^2 3x)^2 (\cos 3x) \, dx$$

$u = \sin 3x$, $du = 3\cos 3x \, dx$; $\frac{du}{3} = \cos 3x \, dx$

$$= 3\int (1-u^2)^2 \left(\frac{du}{3}\right)$$

$$= \int (1-u^2)^2 du$$

$$= \int (1-2u^2+u^4) \, du$$

$$= \int du - 2\int u^2 du + \int u^4 du$$

$$= \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^{\pi/6}$$

$$= \left[\sin 3x - \frac{2}{3}\sin^3 3x + \frac{1}{5}\sin^5 3x\right]_0^{\pi/6}$$

$$= \left[\left(\sin\frac{\pi}{2} - \frac{2}{3}\sin^3\frac{\pi}{2} + \frac{1}{5}\sin^5\frac{\pi}{2}\right) - \left(\sin(0) - \frac{2}{3}\sin^3(0) + \frac{1}{5}\sin^5 3(0)\right)\right]$$

$$= \left[\left(\sin 90^\circ - \frac{2}{3}\sin^3 90^\circ + \frac{1}{5}\sin^5 90^\circ\right)\right]$$

$$= \left[1 - \frac{2}{3}(1) + \frac{1}{5}(1)^2\right] = \left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{15-10+3}{15} = \frac{18-10}{15} = \boxed{\frac{8}{15}}$$

11) $\int \sin^3 x \cos^3 x \, dx$

: m and n both are odd

$$\int \sin^3 x \cos^3 x \, dx = \int \sin^3 x \cos^2 x \cos x \, dx$$

$$= \int \sin^3 x (1 - \sin^2 x) \cos x \, dx$$

$$= \int \sin^3 x \cos x \, dx - \int \sin^5 x \cos x \, dx$$

$u = \sin x$, $du = \cos x \, dx$

$$\int \sin^3 x \cos^3 x \, dx = \int u^3 du - \int u^5 du$$

$$= \frac{1}{4} u^4 - \frac{1}{6} u^6 + C$$

$$\int \sin^3 x \cos^3 x \, dx = \boxed{\frac{\sin^4 x}{4} - \frac{\sin^6 x}{6} + C}$$

12) $\int \cos^3 2x \sin^5 2x \, dx$

: $$\int \sin^5 2x \cos^3 2x \, dx = \int \sin^5 2x \cos^2 2x \cos 2x \, dx$$

$$= \int \sin^5 2x (1 - \sin^2 2x) \cos 2x \, dx$$

$$= \int (\sin^5 2x \cos 2x - \sin^7 2x \cos 2x) \, dx$$

$u = \sin 2x$, $du = 2 \cos 2x \, dx$

$$= \int \left((u^5) \left(\frac{du}{2} \right) - (u^7) \left(\frac{du}{2} \right) \right)$$

$$= \frac{1}{2} \int u^5 du - \frac{1}{2} \int u^7 du$$

$$= \frac{1}{2} \times \frac{1}{6} u^6 - \frac{1}{2} \times \frac{1}{8} u^8 + C$$

$$= \boxed{\frac{1}{12} \sin^6 2x - \frac{1}{16} \sin^8 2x + C}$$

13) $\int \cos^2 x \, dx$

Case 3 both m and n are even

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\int \cos^2 x \, dx = \int \left(\frac{1 + \cos 2x}{2} \right) dx = \frac{1}{2} \int (1 + \cos 2x) \, dx$$

$$= \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx = \frac{x}{2} + \frac{1}{2}\left(\frac{1}{2} \sin 2x\right) + C$$

$$= \boxed{\frac{x}{2} + \frac{1}{4} \sin 2x + C}$$

14) $\int_0^{\pi/2} \sin^2 x \, dx$

: Case 3 both m and n are even

$$\int_0^{\pi/2} \sin^2 x \, dx = \int_0^{\pi/2} \left(\frac{1 - \cos 2x}{2} \right) dx = \frac{1}{2} \int_0^{\pi/2} (1 - \cos 2x) \, dx$$

$$= \frac{1}{2} \int_0^{\pi/2} dx - \frac{1}{2} \int_0^{\pi/2} \cos 2x \, dx = \frac{1}{2} [x]_0^{\pi/2} - \frac{1}{2} \left[\frac{1}{2} \sin 2x \right]_0^{\pi/2}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right] + \frac{1}{4} [\sin \pi - \sin 0]$$

$$= \frac{\pi}{4} + \frac{1}{4}(0 - 0) = \boxed{\frac{\pi}{4}}$$

15) $\int_0^{\pi/2} \sin^7 y \, dy$

: $\int_0^{\pi/2} \sin^6 y \sin y \, dy = \int_0^{\pi/2} (\sin^2 y)^3 \sin y \, dy = \int_0^{\pi/2} (1 - \cos^2 y)^3 \sin y \, dy$

$u = \cos y$, $du = -\sin y \, dy$

$$= \int_0^{\pi/2} (1 - u^2)^3 (-du) = \int_0^{\pi/2} (u^2 - 1)^3 du$$

$$= \int_0^{\pi/2} ((u^2)^3 - 3(u^2)^2 + 3(u^2) - 1) \, du = \int_0^{\pi/2} (u^6 - 3u^4 + 3u^2 - 1) \, du$$

$$= \int_0^{\pi/2} (u^6 - 3u^4 + 3u^2 - 1) \, du = \left[\frac{u^7}{7} - \frac{3u^5}{5} + \frac{3u^3}{3} - u \right]_0^{\pi/2}$$

$$= \left[\frac{\cos^7 y}{7} - \frac{3 \cos^5 y}{5} + \cos^3 y - \cos y \right]_0^{\pi/2}$$

$$= \left[\left(\frac{1}{7}\cos^7(90) - \frac{3}{5}\cos^5(90) + \cos^3(90) - \cos(90)\right) - \left(\frac{1}{7}\cos^7(0) - \frac{3}{5}\cos^5(0) + \cos^3(0) - \cos(0)\right)\right]$$

$$= \left[(0 - 0 + 0 - 0) - \left(\frac{1}{7}(1) - \frac{3}{5}(1) + \cancel{(1)} - \cancel{1}\right)\right]$$

$$= -\left(\frac{1}{7} - \frac{3}{5}\right) = -\left(\frac{-16}{35}\right) = \boxed{\frac{16}{35}}$$

16) $\int 7\cos^7 t \, dt$

Case 2 m is even and n is odd

$$: 7\int \cos^6 t \cos t \, dt = 7\int (\cos^2 t)^3 \cos t \, dt$$
$$= 7\int (1 - \sin^2 t)^3 \cos t \, dt$$

let $u = \sin t$, $du = \cos t \, dt$

$$= 7\int (1-u^2)^3 du = 7\int \left((1)^3 - 3(1)^2(u^2) + 3(1)(u^2)^2 - (u^2)^3\right) du$$
$$= 7\int (1 - 3u^2 + 3u^4 - u^6) du$$
$$= 7\left[u - \frac{3}{3}u^3 + \frac{3}{5}u^5 - \frac{1}{7}u^7\right] + C$$

$$= 7\left[\sin t - \sin^3 t + \frac{3}{5}\sin^5 t - \frac{\sin^7 t}{7}\right] + C$$

$$= 7\sin t - 7\sin^3 t + \frac{21}{5}\sin^5 t - \sin^7 t + C$$

17) $\int_0^{\pi} 8\sin^4 x \, dx$

Case 3 both m and n are even

$$: 8\int_0^{\pi} \sin^4 x \, dx = 8\int_0^{\pi} \sin^2 x \sin^2 x \, dx = 8\int (\sin^2 x)^2 dx$$
$$= 8\int_0^{\pi} \left(\frac{1 - \cos 2x}{4}\right)^2 dx$$

$$= \frac{8}{4}\int_0^{\pi} \left((1 - \cos 2x)\right)^2 dx = 2\int_0^{\pi} (1 - \cos 2x)^2 dx$$

$$= 2\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x) \, dx = 2\left[x - \sin 2x\right]_0^{\pi} + 2\int_0^{\pi} \cos^2 2x \, dx$$
$$= 2\left[x - \sin 2x\right]_0^{\pi} + 2\int_0^{\pi} \left(\frac{1 + \cos 4x}{2}\right) dx$$

$$= 2[x - \sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x$$
$$= 2[x - \sin 2x]_0^{\pi} + [x]_0^{\pi} + \left[\frac{\sin 4x}{4}\right]_0^{\pi}$$
$$= 2[(\pi - \sin 2\pi) - (0 - \sin(0))] + [\pi - 0] + \left[\frac{\sin 4\pi}{4} - \frac{\sin(0)}{4}\right]$$
$$= 2\pi + \pi + 0 = \boxed{3\pi}$$

18) $\int 8\cos^4 2\pi x\,dx$

Case 3 both m and n are even

$$8\int \cos^4 2\pi x\,dx = 8\int (\cos^2 2\pi x)^2\,dx = 8\int \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx$$

$$= 2\int (1+\cos 4\pi x)^2\,dx = 2\int (1+2\cos 4\pi x + (\cos 4\pi x)^2)\,dx$$

$$= 2\int dx + 4\int \cos 4\pi x\,dx + 2\int (\cos 4\pi x)^2\,dx$$

$$= 2\int dx + 4\int \cos 4\pi x\,dx + 2\int \left(\frac{1+\cos 8\pi x}{2}\right) dx$$

$$= 2x + \sin 4\pi x + x + \frac{1}{8\pi}\sin 8\pi x + C$$

$$= \boxed{3x + \sin 4\pi x + \frac{1}{8\pi}\sin 8\pi x + C}$$

19) $\int 16\sin^2 x\cos^2 x\,dx$

: Case 3 both m and n are even

$$16\int \sin^2 x\cos^2 x\,dx = 16\int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx$$

$$= 4\int (1-\cos 2x)(1+\cos 2x)\,dx = 4\int (1-\cos^2 2x)\,dx$$

$$= 4\int dx - 4\int \cos^2 2x\,dx = 4\int dx - 4\int \left(\frac{1+\cos 4x}{2}\right) dx$$

$$= 4x - 2x - \frac{2}{4}\sin 4x + C = \boxed{2x - \frac{1}{2}\sin 4x + C}$$

20) $\int_0^{\pi} 8 \sin^4 y \cos^2 y \, dy$

: case 3 both m and n are even

$8\int_0^{\pi} \sin^4 y \cos^2 y \, dy = 8\int_0^{\pi} (\sin^2 y)^2 \cos^2 y \, dy$

$= 8\int_0^{\pi} \left(\frac{1-\cos y}{2}\right)^2 \left(\frac{1+\cos y}{2}\right) dy = \frac{8}{8}\int_0^{\pi} (1-\cos y)^2 (1+\cos y) \, dy$

$= \int_0^{\pi} (1 - 2\cos y + \cos^2 y)(1+\cos y) \, dy$

$= \int_0^{\pi} (1 - 2\cos y + \cos^2 y) + (\cos y - 2\cos^2 y + \cos^3 y) \, dy$

$= \int_0^{\pi} dy - \int_0^{\pi} \cos y \, dy - \int_0^{\pi} \cos^2 y \, dy + \int_0^{\pi} \cos^3 y \, dy$

$= [y]_0^{\pi} - [\sin y]_0^{\pi} - \int_0^{\pi} \left(\frac{1+\cos y}{2}\right) dy + \int_0^{\pi} \cos^3 y \, dy$

$= \pi - [0-0] - \frac{1}{2}\int_0^{\pi} (1+\cos y) \, dy + \int_0^{\pi} \cos^2 y \cos y \, dy$

$= \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi} \cos y \, dy + \int_0^{\pi} (1-\sin^2 y)\cos y \, dy$

$= \pi - \frac{1}{2}[y]_0^{\pi} - \frac{1}{2}[\sin y]_0^{\pi} + \int_0^{\pi} (1-\sin^2 y)\cos y \, dy$

$u = \sin y, \; du = \cos y \, dy$

$= \pi - \frac{1}{2}[\pi - 0] - \frac{1}{2}[0-0] + \int_0^{\pi} (1-u^2) \, du$

$= \pi - \frac{\pi}{2} + \int_0^{\pi} (1-u^2) \, du = \frac{2\pi - \pi}{2} + [u]_0^{\pi} - \left[\frac{u^3}{3}\right]_0^{\pi}$

$= \frac{\pi}{2} + [\sin y]_0^{\pi} - \frac{1}{3}[\sin^3 y]_0^{\pi}$

$= \frac{\pi}{2} + [0-0] - \frac{1}{3}[0-0] = \boxed{\frac{\pi}{2}}$

21) $\int 8\cos^3 2\theta \sin 2\theta \, d\theta$

: both m and n are odd

$= 8\int \cos^3 2\theta \sin 2\theta \, d\theta = 8\int \cos^2 2\theta \cos 2\theta \sin 2\theta \, d\theta$

$= 8\int (1-\sin^2 2\theta)\cos 2\theta \sin 2\theta \, d\theta$

$u = \sin 2\theta$, $du = 2\cos 2\theta \, d\theta \rightarrow \frac{du}{2} = \cos 2\theta \, d\theta$

$= 8\int (1-u^2)u\left(\frac{du}{2}\right) = 4\int (1-u^2)u \, du$

$= 4\int (u-u^3)\, du = 4\left[\frac{u^2}{2}\right] - 4\left[\frac{u^4}{4}\right] + C$

$= 2u^2 - u^4 + C = 2(\sin 2\theta)^2 - \sin^4 2\theta + C$

22) $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta$

: $\int_0^{\pi/2} \sin^2 2\theta \cos^3 2\theta \, d\theta = \int_0^{\pi/2} \sin^2 2\theta \cos^2 2\theta \cos 2\theta \, d\theta$

$= \int_0^{\pi/2} \sin^2 2\theta (1-\sin^2 2\theta)\cos 2\theta \, d\theta$

$= \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta \, d\theta - \int_0^{\pi/2} \sin^4 2\theta \cos 2\theta \, d\theta$

$= \int_0^{\pi/2} \sin^2 2\theta \cos 2\theta \, d\theta - \int_0^{\pi/2} \sin^4 2\theta \cos 2\theta \, d\theta$

let $u = \sin 2\theta$, $du = 2\cos 2\theta \, d\theta \rightarrow \frac{du}{2} = \cos 2\theta \, d\theta$

$= \int_0^{\pi/2} (u^2)\left(\frac{du}{2}\right) - \int_0^{\pi/2} (u^4)\left(\frac{du}{2}\right)$

$= \frac{1}{2}\int_0^{\pi/2} u^2 \, du - \frac{1}{2}\int_0^{\pi/2} u^4 \, du$

$= \left[\frac{1}{2}\times\frac{1}{3}u^3 - \frac{1}{2}\times\frac{1}{5}u^5\right]_0^{\pi/2} = \left[\frac{1}{6}\sin^3 2\theta - \frac{1}{10}\sin^5 2\theta\right]_0^{\pi/2}$

$\left[\left(\frac{1}{6}\sin^3 \pi - \frac{\sin^5 \pi}{10}\right) - \left(\frac{\sin^3(0)}{6} - \frac{\sin^5(0)}{10}\right)\right] = \boxed{0}$

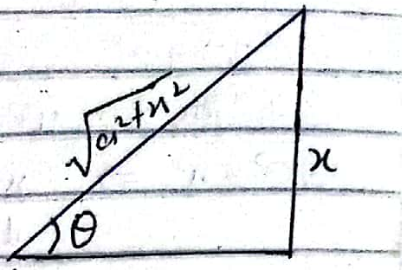
8.4 Trigonometric Substitution

Trigonometric Substitutions occur when we replace the variable of integration by a trigonometric function.

$\sqrt{a^2+x^2}$

$x = a\tan\theta$

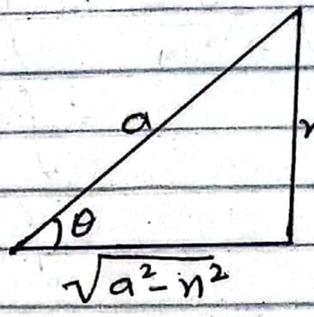
$\sqrt{a^2+x^2} = \boxed{a|\sec\theta|}$



$\sqrt{a^2-x^2}$

$x = a\sin\theta$

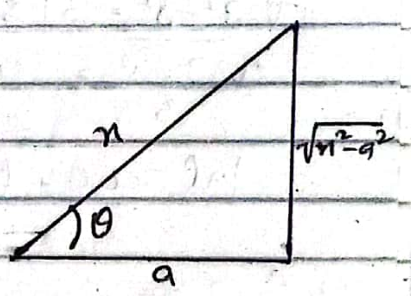
$\sqrt{a^2-x^2} = \boxed{a|\cos\theta|}$



$\sqrt{x^2-a^2}$

$x = a\sec\theta$

$\sqrt{x^2-a^2} = \boxed{a|\tan\theta|}$



$\sqrt{a^2+x^2}$	$\rightarrow$	$x = a\tan\theta$
$\sqrt{a^2-x^2}$	$\rightarrow$	$x = a\sin\theta$
$\sqrt{x^2-a^2}$	$\rightarrow$	$x = a\sec\theta$

Procedure for a trigonometric substitution

1) write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2) Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3) Integrate the trigonometric integral, keeping in mind the restrictions on the angle θ for reversibility.
4) Drawn an appropriate reference triangle to reverse the Substitution in the integration result and convert it back to the original variable x.

Exercises 8.4

Using Trigonometric Substitutions

Evalute the integrals in 1–14

1) $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$

Solution using trigonometric substitution

$\sqrt{a^2+x^2} \Rightarrow \quad x = a\tan\theta \quad = 3\tan\theta$

$dx = a\sec^2\theta \, d\theta = 3\sec^2\theta \, d\theta$

also $\sqrt{a^2+x^2} = a|\sec\theta| \quad = 3|\sec\theta|$

$$\int \frac{dx}{\sqrt{3^2+x^2}} = \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9+(3\tan\theta)^2}} = \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9+9\tan^2\theta}}$$

$$= \int \frac{3\sec^2\theta \, d\theta}{\sqrt{9(1+\tan^2\theta)}} = \int \frac{3\sec^2\theta \, d\theta}{3\sqrt{\sec^2\theta}}$$

$$= \int \frac{\cancel{3}\sec^2\theta \, d\theta}{\cancel{3}\sec\theta} = \int \sec\theta \, d\theta = \boxed{\ln(\sec\theta+\tan\theta)+c}$$

$\sqrt{3^2+x^2} = 3|\sec\theta| \; ; \quad x = 3\tan\theta$

$\dfrac{\sqrt{3^2+x^2}}{3} = \sec\theta \qquad \dfrac{x}{3} = \tan\theta$

$$\ln(\sec\theta+\tan\theta)+c = \boxed{\ln\left(\frac{\sqrt{9+x^2}}{3}+\frac{x}{3}\right)+c}$$

2) $\displaystyle\int \frac{3\,dx}{\sqrt{1+9x^2}} = \int \frac{3\,dx}{\sqrt{1+(3x)^2}}$

let $u = 3x$, $du = 3\,dx$, $du/3 = dx$

$$\int \frac{3\,dx}{\sqrt{1+(3x)^2}} = 3\int \frac{du/3}{\sqrt{1+u^2}} = \frac{3}{3}\int \frac{du}{\sqrt{1+u^2}}$$

$$= \int \frac{du}{\sqrt{1+u^2}} = \int \frac{du}{\sqrt{1^2+u^2}} = \boxed{\tan^{-1}u + c}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)+C$$

$$\int \frac{du}{\sqrt{1^2+u^2}} = \ln\left(u+\sqrt{u^2+1^2}\right)+c = \ln\left(3x+\sqrt{9x^2+1}\right)+c$$

$$\boxed{\int \frac{3\,dx}{\sqrt{1+9x^2}} = \ln\left(3x+\sqrt{9x^2+1}\right)+C}$$

3) $\int_{-2}^{2} \frac{dx}{4+x^2}$

Solution using trigonometric Substitution

$a^2+x^2 \Rightarrow x = a\tan\theta = 2\tan\theta$

$dx = a\sec^2\theta\, d\theta = 2\sec^2\theta\, d\theta$

also $4+x^2 = 2|\sec\theta|$

$$\int_{-2}^{2} \frac{dx}{4+x^2} = \int_{-2}^{2} \frac{2\sec^2\theta\, d\theta}{4+(2\tan\theta)^2} = \int_{-2}^{2} \frac{2\sec^2\theta\, d\theta}{4+4\tan^2\theta}$$

$$= \int_{-2}^{2} \frac{2\sec^2\theta\, d\theta}{4(1+\tan^2\theta)} = \int_{-2}^{2} \frac{2\sec^2\theta\, d\theta}{4\sec^2\theta}$$

$$= \frac{1}{2}\int_{-2}^{2} d\theta$$

$$\Rightarrow \int_{-2}^{2} \frac{dx}{4+x^2} = \frac{1}{2}\tan^{-1}\frac{x}{2}\Big]_{-2}^{2} = \frac{1}{2}\left[\tan^{-1}(1) - \tan^{-1}(-1)\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4}+\frac{\pi}{4}\right] = \frac{1}{2}\left[\frac{2\pi}{4}\right] = \boxed{\frac{\pi}{4}}$$

4) $\int_{0}^{2} \frac{dx}{8+2x^2} = \int_{0}^{2} \frac{dx}{2(4+x^2)} = \frac{1}{2}\int_{0}^{2} \frac{dx}{4+x^2}$

$$\boxed{\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c}$$

$$= \frac{1}{2}\int_{0}^{2} \frac{dx}{2^2+x^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{0}^{2}$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left[\tan^{-1}\frac{x}{2}\right]_{0}^{2} = \frac{1}{4}\left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{4} - 0\right] = \boxed{\frac{\pi}{16}}$$

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5) $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}}$

<u>Solution</u> using trigonometric substitution

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3}\Big]_0^{3/2} = \left(\sin^{-1}\left(\frac{3}{2}\right)\left(\frac{1}{3}\right) - \sin^{-1}\frac{0}{3}\right)$$

$$= \sin^{-1}\left(\frac{1}{2}\right) - 0 = \sin^{-1}(30^\circ) = \boxed{\frac{\pi}{6}}$$

6) $\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$ → trigonometric substitution

∴ Let $u = 2x$, $du = 2\,dx$ → $du/2 = dx$

$$\int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}} = \int_0^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-(2x)^2}} = \int_0^{1/2\sqrt{2}} \frac{du}{\sqrt{1-u^2}}$$

$$= \int_0^{1/2\sqrt{2}} \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u\Big]_0^{1/2\sqrt{2}} = \left[\sin^{-1} 2x\right]_0^{1/2\sqrt{2}}$$

$$= \left[\left(\sin^{-1}\frac{(2)}{2\sqrt{2}}\right) - \sin^{-1} 2(0)\right] = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0$$

$$= \sin(45^\circ) = \boxed{\frac{\pi}{4}}$$

7) $\int \sqrt{25-t^2}\,dt$

:

$$\int \sqrt{a^2-t^2}\,dt = \frac{1}{2}t\sqrt{a^2-t^2} + \frac{1}{2}a^2\sin^{-1}\frac{t}{a} + C$$

$$\int \sqrt{25-t^2}\,dt = \frac{1}{2}t\sqrt{25-t^2} + \frac{1}{2}25\sin^{-1}\frac{t}{5} + C$$

$$= \boxed{\frac{t\sqrt{25-t^2}}{2} + \frac{25}{2}\sin^{-1}\frac{t}{5} + C}$$

Alternative Solution process of Question No (7)

7) $\int \sqrt{25-t^2}\, dt$

Solution using trigonometric substitution

$\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta = 5\sin\theta = t$

$dx = a\cos\theta\, d\theta = 5\cos\theta\, d\theta = dt$

also $\sqrt{a^2-x^2} = a|\cos\theta| = 5|\cos\theta|$

$\int \sqrt{25-t^2}\, dt = \int \sqrt{25-(5\sin\theta)^2}\, dt$

$= \int \sqrt{25-25\sin^2\theta}\,(5\cos\theta\, d\theta)$

$= \int \sqrt{25(1-\sin^2\theta)}\,(5\cos\theta\, d\theta)$

$= 5\int \sqrt{1-\sin^2\theta}\; 5\cos\theta\, d\theta$

$= 5\int \sqrt{\cos^2\theta}\; 5\cos\theta\, d\theta$

$= 25\int \cos\theta\cos\theta\, d\theta$

$= 25\int \cos^2\theta\, d\theta$

$\boxed{\cos^2 x = \frac{1+\cos 2x}{2}}$

$= 25\int\left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{25}{2}\int d\theta + \frac{25}{2}\int \cos 2\theta\, d\theta$

$= \frac{25}{2}\theta + \frac{25}{2}\frac{\sin 2\theta}{2} = \frac{25}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + c$

$\boxed{\text{double angle formula } \sin 2\theta = 2\sin\theta\cos\theta}$

$= \frac{25}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + c = \frac{25}{2}(\theta + \sin\theta\cos\theta) + c$

Replace θ values

$t = 5\sin\theta$, $5\sin\theta = t$ $\quad 5|\cos\theta| = \sqrt{25-t^2}$

$\boxed{\sin^{-1}\frac{t}{5} = \theta}$ $\quad \boxed{\sin\theta = \frac{t}{5}}$ $\quad \cos\theta = \frac{\sqrt{25-t^2}}{5}$

$\frac{25}{2}(\theta + \sin\theta\cos\theta) + c = \frac{25}{2}\left(\sin^{-1}\frac{t}{5} + \frac{t}{5}\frac{\sqrt{25-t^2}}{5}\right) + c$

$= \frac{25}{2}\left(\sin^{-1}\frac{t}{5} + \frac{t\sqrt{25-t^2}}{25}\right) + c = \frac{25}{2}\sin^{-1}\frac{t}{5} + \frac{25}{2}\frac{t\sqrt{25-t^2}}{25} + c$

$= \boxed{\frac{25}{2}\sin^{-1}\frac{t}{5} + \frac{t\sqrt{25-t^2}}{2} + c}$

8) $\int \sqrt{1-9t^2}\, dt$

Solution using Trigonometric Substitution

let $u=3t$, $du=3\,dt \rightarrow \frac{du}{3}=dt$

Note: we can also integrate it by using $\int \sqrt{a^2-x^2} = \frac{x\sqrt{a^2-x^2}}{2} + \frac{1}{2}a^2 \sin^{-1}\frac{x}{a} + c$

$\int \sqrt{1-u^2}\, dt$

$= \int \sqrt{1-u^2}\, \frac{du}{3}$

$\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta$

$dx = a\cos\theta\, d\theta$

also $\sqrt{a^2-x^2} = a|\cos\theta|$

$u = a\sin\theta$

$du = a\cos\theta . d\theta$

also $\sqrt{1-u^2} = a|\cos\theta|$

$\int \sqrt{1-u^2}\, dt$

$= \int \sqrt{1-(\sin\theta)^2}\, dt$

$= \int \sqrt{1-\sin^2\theta}\, \left(\frac{du}{3}\right)$

$= \int \frac{1}{3}\sqrt{\cos^2\theta}\, du$

$= \int \frac{1}{3}\sqrt{\cos^2\theta}\, (\cos\theta\, d\theta)$

$= \int \frac{1}{3} \cos\theta \cos\theta\, d\theta$

$= \int \frac{1}{3} \cos^2\theta\, d\theta$

$= \frac{1}{3}\int \cos^2\theta\, d\theta$

$\Rightarrow \boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$

$= \frac{1}{3}\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{6}\int (1+\cos 2\theta)\, d\theta$

$= \frac{1}{6}\int d\theta + \frac{1}{6}\int \cos 2\theta$

$= \frac{1}{6}(\theta) + \frac{1}{6}\left(\frac{1}{2}\sin 2\theta\right)$

$= \frac{\theta}{6} + \frac{1}{12}(\sin 2\theta)$

Replace substituted values

$\sin\theta = u$ $\quad$ $\sin\theta = 3t$

$\sin\theta = 3t$ $\quad$ $\cos\theta = \sqrt{1-9t^2}$

$\boxed{\theta = \sin^{-1}3t}$

$\theta\ \sin^{-1}3t$, $\sin\theta = 3t$, $\cos\theta = \sqrt{1-9t^2}$

$\frac{\theta}{6} + \frac{1}{12}(\sin 2\theta) = \frac{\theta}{6} + \frac{1}{6}(2\sin\theta\cos\theta)$

$\frac{\theta}{6} - \frac{1}{12}(2\sin\theta\cos\theta) = \frac{\theta}{6} + \frac{1}{3}(\sin\theta\cos\theta) + c$

$\frac{\theta}{6} - \frac{\sin\theta\cos\theta}{6} = \frac{\theta}{6} + \frac{\sin\theta\cos\theta}{3} + c$

$\frac{\sin^{-1}3t}{6} + \frac{3t(\sqrt{1-9t^2})}{6} = \frac{\sin^{-1}3t}{6} + \frac{3t(\sqrt{1-9t^2})}{3}$

$\boxed{\frac{\sin^{-1}3t}{6} + \frac{t\sqrt{1-9t^2}}{2}}$

$\frac{\theta}{6} + \frac{1}{6}(2\sin\theta\cos\theta) \rightarrow$ double angle

$\frac{\theta}{6} + \frac{1}{3}(\sin\theta\cos\theta) + C$

$\frac{\sin^{-1}3t}{6} + \frac{1}{3}(3t)(\sqrt{1-9t^2}) + C$

$\frac{\sin^{-1}3t}{6} +$

8) $\int \sqrt{1-9t^2}\, dt$

Solution using trigonometric Substitution

let $u = 3t$ and $du = 3\,dt \Rightarrow \frac{du}{3} = dt$

The format can be used from trigonometric references.

$\sqrt{a^2-x^2} \Rightarrow a\sin\theta$,

$x = a\sin\theta$ $\quad dx = a\cos\theta\, d\theta$, also $\sqrt{a^2-x^2} = a|\cos\theta|$

$\int \sqrt{1-9t^2}\,dt = \int \sqrt{1-(3t)^2}\,dt$, put $3t = u$ & $dt = \frac{du}{3}$

$= \int \sqrt{1-u^2}\,\frac{du}{3} = \frac{1}{3}\int \sqrt{1-u^2}\,du$, put $u = \sin\theta$, $du = \cos\theta\, d\theta$

$= \frac{1}{3}\int \sqrt{1-(\sin\theta)^2}\,du = \frac{1}{3}\int \sqrt{1-\sin^2\theta}\,\cos\theta\, d\theta$

$= \frac{1}{3}\int \sqrt{\cos^2\theta}\,\cos\theta\, d\theta = \frac{1}{3}\int \cos\theta\cos\theta\, d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta$

put $\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$

$= \frac{1}{3}\int \sqrt{\cos^2\theta}\,\cos\theta\, d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta = \frac{1}{3}\int\left(\frac{1+\cos 2\theta}{2}\right) d\theta$

$= \frac{1}{3}\times\frac{1}{2}\int (1+\cos\theta)\, d\theta = \frac{1}{6}\int (1+\cos\theta)\, d\theta = \frac{1}{6}\int d\theta + \frac{1}{6}\int \cos 2\theta\, d\theta$

$= \frac{1}{6}(\theta) + \frac{1}{6}\frac{(\sin 2\theta)}{2} + C = \boxed{\frac{\theta}{6} + \frac{\sin 2\theta}{12} + C}$

Replace values of θ and $\sin\theta$, but we have $\sin 2\theta$

So we will use the double angle formula.

$\frac{\theta}{6} + \frac{\sin 2\theta}{12} + C = \frac{\theta}{6} + \frac{2\sin\theta\cos\theta}{12} + C = \boxed{\frac{\theta}{6} + \frac{\sin\theta\cos\theta}{6} + C}$

$\boxed{\sin 2\theta = 2\sin\theta\cos\theta} \rightarrow$ double angle formula

$x = u = a\sin\theta = \sin\theta = 3t$

$\sin\theta = 3t$, bcz $u = 3t$ $\quad u = \sin\theta$

$\theta = \frac{3t}{\sin} = \boxed{3t\sin^{-1}}$ $\quad \sin^{-1}3t$

$\boxed{\sin\theta = 3t}$

$\cos\theta = \sqrt{1-9t^2}$

bcz $a|\cos\theta| = \sqrt{a^2-x^2}$

$\frac{\theta}{6} + \frac{\sin\theta\cos\theta}{6} + C = \frac{3t\sin^{-1}}{6} + \frac{(3t)(\sqrt{1-9t^2})}{6} + C$

$= \frac{3t\sin^{-1}}{6} + \frac{t\sqrt{1-9t^2}}{2} + C$

$= \frac{\sin^{-1}3t}{6} + \frac{t\sqrt{1-9t^2}}{2} + C$

8) $\int \frac{dx}{\sqrt{4x^2-49}}$, $x > \frac{7}{2}$

Solution using trigonometric substitution

let $u = 2x$, $du = 2\,dx$,

$\sqrt{x^2 - a^2} \Rightarrow x = a|\sec\theta|$, $dx = a\sec\theta\tan\theta\,d\theta$, $\boxed{a = 7}$

also $\sqrt{x^2-a^2} = a|\tan\theta|$

$$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{du/2}{\sqrt{u^2-49}} = \int \frac{du}{2} \times \frac{1}{\sqrt{u^2-49}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2-49}}$$

put $u = \sec\theta$ and dx is already plugged in

$$\frac{1}{2}\int \frac{du}{\sqrt{(7\sec\theta)^2-49}} = \frac{1}{2}\int \frac{du}{\sqrt{(7\sec\theta)^2-49}} = \frac{1}{2}\int \frac{du}{\sqrt{49\sec^2\theta-49}}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{49(\sec^2\theta-1)}} = \frac{1}{2}\int \frac{du}{7\sqrt{\sec^2\theta-1}} = \frac{1}{2}\int \frac{du}{7\sqrt{\tan^2\theta}}$$

put $du = 7\sec\theta\tan\theta\,d\theta$

$$= \frac{1}{2}\int \frac{\cancel{7}\sec\theta\tan\theta\,d\theta}{\cancel{7}\sqrt{\tan^2\theta}} = \frac{1}{2}\int \frac{\sec\theta\tan\theta\,d\theta}{\tan\theta} = \frac{1}{2}\int \sec\theta\,d\theta$$

$$= \frac{1}{2}\int \sec\theta\,d\theta = \boxed{\frac{1}{2}\ln(\sec\theta+\tan\theta)+c}$$

Replace $\sec\theta$ and $\tan\theta$ values

$u = 7\sec\theta = 2x$, $a\tan\theta = \sqrt{x^2-a^2} = 7\tan\theta = \sqrt{4x^2-49}$

$\sec\theta = \frac{2x}{7}$ $\qquad$ $\tan\theta = \frac{\sqrt{4x^2-49}}{7}$

$$\boxed{\frac{1}{2}\left(\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right)+c}$$

10) $\int \frac{5\,dx}{\sqrt{25x^2-9}}$, $x > \frac{3}{5}$

Solution using trigonometric Substitution

let $u = 5x$, $du = 5\,dx \Rightarrow \frac{du}{5} = dx$

$\sqrt{x^2-a^2} \Rightarrow x = a|\sec\theta|$, $dx = a(\sec\theta\tan\theta)$, $\boxed{a=3}$

also $\sqrt{x^2-a^2} = \sqrt{25x^2-9} = a|\tan\theta| = 3|\tan\theta|$

$\tan\theta = \frac{\sqrt{25x^2-9}}{3}$, $\frac{5x}{3} = \sec\theta$

$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\,dx}{\sqrt{(5x)^2-9}}$, put $u = 5x$, & $dx = \frac{du}{5}$

$= \int \frac{5(\frac{du}{5})}{\sqrt{u^2-9}} = \int \frac{du}{\sqrt{u^2-9}} = \int \frac{du}{\sqrt{u^2-9}}$, put $u = 3\sec\theta$

$du = 3\sec\theta\tan\theta\,d\theta$

$= \int \frac{3\sec\theta\tan\theta\,d\theta}{\sqrt{(3\sec\theta)^2-9}} = \int \frac{3\sec\theta\tan\theta\,d\theta}{\sqrt{9\sec^2\theta-9}} = \int \frac{3\sec\theta\tan\theta\,d\theta}{\sqrt{9(\sec^2\theta-1)}}$

$= \int \frac{\cancel{3}\sec\theta\tan\theta\,d\theta}{\cancel{3}\sqrt{\tan^2\theta}} = \int \frac{\sec\theta\cancel{\tan\theta}\,d\theta}{\cancel{\tan\theta}} = \int \sec\theta\,d\theta$

$= \int \sec\theta\,d\theta = \ln(\sec\theta + \tan\theta) + C$

put $\sec\theta = \frac{5x}{3}$ & $\tan\theta = \frac{\sqrt{25x^2-9}}{3}$

$\ln\left(\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right)$

☆

iii) $\int \frac{\sqrt{y^2-49}}{y}\, dy$

<u>Solution</u> using trigonometric substitution

$\sqrt{x^2-a^2} \Rightarrow x = a|\sec\theta|$, $dx = a\sec\theta\tan\theta\, d\theta$, $\boxed{a=7}$

also $\sqrt{x^2-a^2} = a|\tan\theta|$

$\int \frac{\sqrt{y^2-49}}{y}\, dy = \int \frac{\sqrt{y^2-7^2}}{y}\, dy$. put $y = 7\sec\theta$

$dy = 7\sec\theta\tan\theta\, d\theta$

$= \int \frac{\sqrt{(7\sec\theta)^2-49}\,(7\sec\theta\tan\theta\, d\theta)}{7\sec\theta}$

$= \int \frac{\sqrt{49\sec^2\theta-49}\,(7\sec\theta\tan\theta\, d\theta)}{7\sec\theta} = \int \frac{\cancel{7}\sqrt{\sec^2\theta-1}\,(7\sec\theta\tan\theta)\, d\theta}{\cancel{7}\sec\theta}$

$= \int \frac{\sqrt{\tan^2\theta}\,(7\sec\theta\tan\theta)\, d\theta}{\sec\theta} = 7\int \frac{\tan\theta\sec\theta\tan\theta\, d\theta}{\sec\theta}$

$= 7\int \frac{\tan^2\theta\sec\theta\, d\theta}{\sec\theta} = 7\int \tan^2\theta\, d\theta$

put $\boxed{\tan^2\theta = \sec^2\theta - 1}$

$7\int (\sec^2\theta - 1)\, d\theta = 7\int \sec^2\theta\, d\theta - 7\int d\theta$

$= 7(\tan\theta) - 7\theta + c = \boxed{7(\tan\theta - \theta) + C}$

$\sqrt{x^2-a^2} = a|\tan\theta|$ $\qquad x = a|\sec\theta|$

$\sqrt{y^2-49} = 7\tan\theta$ $\qquad y = 7\sec\theta$

$\boxed{\frac{\sqrt{y^2-49}}{7} = \tan\theta}$ $\qquad \boxed{\sec^{-1}\frac{y}{7} = \theta}$

$7(\tan\theta - \theta) + c = \boxed{7\left(\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\frac{y}{7}\right) + c}$

12) $\int \frac{\sqrt{y^2-25}}{y^3}\, dy,\ y>5$

Solution using trigonometric substitution

$\sqrt{x^2-a^2} \Rightarrow x = a|\sec\theta|$, $dx = a(\sec\theta\tan\theta)\, d\theta$

also $\sqrt{x^2-a^2} = a|\tan\theta|$

$\sqrt{y^2-25} \Rightarrow y = 5|\sec\theta|$, $dy = 5(\sec\theta\tan\theta)\, d\theta$

$\frac{y}{5} = \sec\theta$

$\sqrt{y^2-25} = 5|\tan\theta| \Rightarrow \sqrt{y^2-25}/5 = \tan\theta$

$$\int \frac{\sqrt{y^2-25}}{y^3}\, dy = \int \frac{\sqrt{(5\sec\theta)^2-25}}{(5\sec\theta)^3}(5\sec\theta\tan\theta)\, d\theta$$

$$= \int \frac{\sqrt{25\sec^2\theta-25}}{125\sec^3\theta}(5\sec\theta\tan\theta)\, d\theta = \int \frac{5\sqrt{\sec^2\theta-1}}{125\sec^3\theta}(5\sec\theta\tan\theta)\, d\theta$$

$$= \int \frac{25\sqrt{\tan^2\theta}\,(\sec\theta\tan\theta)\, d\theta}{125\sec^3\theta} = \int \frac{\tan\theta\,(\sec\theta\tan\theta)\, d\theta}{5\sec^3\theta}$$

$$= \int \frac{\tan^2\theta}{5\sec^2\theta}\, d\theta = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\, d\theta$$

$\boxed{\text{put } \tan^2\theta = \sec^2\theta - 1}$

$$\frac{1}{5}\int \frac{(\sec^2\theta - 1)}{\sec^2\theta}\, d\theta = \frac{1}{5}\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta - \int \frac{1}{\sec^2\theta}\, d\theta$$

$$= \frac{1}{5}\int d\theta - \int \cos^2\theta\, d\theta = \frac{1}{5}\left(\int d\theta - \int (\sec^2\theta - 1)\, d\theta\right)$$

$$= \frac{1}{5}\left(\int d\theta - \int \sec^2\theta\, d\theta - \int d\theta\right) = \frac{1}{5}(\theta - \tan\theta - \theta) + c$$

$$= \frac{1}{5}(-\tan\theta) + c = -\frac{1}{5}(\tan\theta) + c, \text{ put } \tan\theta = \frac{\sqrt{y^2-25}}{5}$$

$$= -\frac{1}{5}\frac{\sqrt{y^2-25}}{5} + c$$

$$\boxed{= \frac{1}{5}\int d\theta - \frac{1}{5}\int \cos^2\theta\, d\theta = \frac{\theta}{5} - \frac{1}{5}\int\left(\frac{1+\cos 2\theta}{2}\right) d\theta \ldots}$$

Exercise 8.5 Integration of Rational function by partial fraction

- **Case I:** when the factors of the denominator $D(x)$ are all linear and distinct.
- **Case II:** When all the factors of the denominator $D(x)$ are linear but some are repeated.
- **Case III:** Where $D(x)$ has non-repeated irreducible quadratic factors.
- **Case IV:** When $D(x)$ has repeated irreducible quadratic factors.

Case I Example: Resolve into partial fraction

$$\frac{x^2+2x+3}{x^3-x}$$

Solution

$$\frac{x^2+2x+3}{x^3-x} = \frac{x^2+2x+3}{x(x^2-1)} = \frac{x^2+2x+3}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$\frac{x^2+2x+3}{x(x-1)(x+1)} = \frac{A\cancel{x}(x-1)(x+1)}{\cancel{x}} + \frac{Bx\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} + \frac{Cx(x-1)\cancel{(x+1)}}{\cancel{(x+1)}}$$

$$\frac{x^2+2x+3}{\cancel{x(x-1)(x+1)}} = \frac{A(x-1)(x+1) + Bx(x+1) + Cx(x-1)}{\cancel{x(x-1)(x+1)}}$$

$$x^2+2x+3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$$

$$\frac{A}{x},\ x=0,\quad \frac{B}{x-1},\ x=1,\quad \frac{C}{x+1},\ x=-1$$

- $x=0$, $x^2+2x+3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

$$0^2+2(0)+3 = A(-1)(1) + 0(1) + 0(-1)$$

$$3 = -A \rightarrow \boxed{A=-3}$$

- $x=1$, $x^2+2x+3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

$$1+2+3 = A(0)(2) + B(2) + C(0)$$

$$6 = 2B \rightarrow \boxed{B=3}$$

- $x=-1$, $x^2+2x+3 = A(x-1)(x+1) + Bx(x+1) + Cx(x-1)$

$$1-2+3 = A(-2)(0) + B(-1)(0) + C(-1)(-2)$$

$$2 = 2C \rightarrow \boxed{C=1}$$

$$\frac{x^2+2x+3}{x(x-1)(x+1)} = -\frac{3}{x} + \frac{3}{x-1} + \frac{1}{x+1}$$

Case II Example: Resolve

$$\frac{x+5}{x^3-3x+2}$$

Solution

Factorize x^3-3x+2

Using Synthetic division method

$x=1,\ x^3-3x+2 = (1)^3-3(1)+2 = 1-3+2=0$ ✓

$x^3-3x+2,\ x=1$

1	1	0	-3	2
		1	1	-2
	1	1	-2	$\boxed{0}$

$x^2+x-2=0$

$x^2+2x-x-2=0$

$x(x+2)-1(x+2)=0$

$(x-1)(x+2)=0$

$$\frac{x+5}{x^3-3x+2} = \frac{x+5}{(x-1)(x-1)(x+2)} = \frac{x+5}{(x-1)^2(x+2)}$$

$$\frac{x+5}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$$\frac{x+5}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$$

$\boxed{\text{Multiply Both sides by } (x-1)^2(x+2)}$

$$\frac{(x+5)\cancel{(x-1)^2}\cancel{(x+2)}}{\cancel{(x-1)^2}\cancel{(x+2)}} = \frac{A\cancel{(x-1)^2}(x+2)}{\cancel{(x-1)}} + \frac{B\cancel{(x-1)^2}(x+2)}{\cancel{(x-1)^2}} + \frac{C(x-1)^2\cancel{(x+2)}}{\cancel{(x+2)}}$$

$\boxed{x+5 = A(x-1)(x+2) + B(x+2) + C(x-1)^2} \rightarrow$ eq. ①

$x+5 = A(x^2+x-2) + Bx + 2B + C(x^2-2x+1)$

$x+5 = Ax^2 + Ax - 2A + Bx + 2B + Cx^2 - 2Cx + C$

$x+5 = Ax^2 + Cx^2 + Ax + Bx - 2Cx - 2A + 2B + C$

$x+5 = x^2(A+C) + x(A+B-2C) - 2A + 2B + C$

$0x^2 + x + 5 = x^2(A+C) + x(A+B-2C) - 2A + 2B + C$

$0x^2 = x^2(A+C)$, $x = x(A+B-2C)$, $+5 = 2B-2A+C$

$\boxed{0 = A+C}$, $0 = A+B-2C$, $5 = 2B-2A+C$

eq. ① put $x=1,\ x=-2$

$x=1:\ 6 = A(0)(3) + B(3) + C(0)^2 \rightarrow B = 2$

$x=-2:\ 3 = A(-3)(0) + B(0) + C(-3)^2 \rightarrow C = 1/3$

$A = -C$

$A = -1/3$

Case III Example : Resolve $\frac{3x^2+x-2}{(x-1)(x^2+1)}$ into partial fractions

Solution

$$\frac{3x^2+x-2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

Multiply both sides by $(x-1)(x^2+1)$

$$\frac{(3x^2+x-2)\cancel{(x-1)(x^2+1)}}{\cancel{(x-1)(x^2+1)}} = \frac{A\cancel{(x-1)}(x^2+1)}{\cancel{x-1}} + \frac{(Bx+C)(x-1)\cancel{(x^2+1)}}{\cancel{(x^2+1)}}$$

$$\boxed{3x^2+x-2 = A(x^2+1) + (Bx+C)(x-1)} \rightarrow eq \; 1$$

$\frac{A}{x-1}$, $x=1$, $\frac{Bx+C}{x^2+1}$,, $x=\sqrt{-1} \rightarrow x=i$

Put $x=1$ in equation no 1

$x=1$: $3x^2+x-2 = A(x^2+1) + (Bx+C)(x-1)$

$3(1)^2+(1)-2 = A(1+1) + (B+C)(0)$

$3+1-2 = 2A + 0 \rightarrow \boxed{A=1}$

for B, C $\rightarrow 3x^2+x-2 = Ax^2+A + Bx^2 - Bx + Cx - C$

$3x^2+x-2 = Ax^2+Bx^2 + Cx - Bx + A - C$

$3x^2+x-2 = x^2(A+B) + x(C-B) + A-C$

$3x^2 = x^2(A+B)$ $\qquad x = x(C-B)$, $\qquad -2 = A-C$

$3 = A+B$ $\qquad 1 = C-2$ $\qquad -2 = -1-C$

$3 = 1+B \rightarrow \boxed{B=2}$ $\qquad \boxed{C=3}$ $\qquad \boxed{C=3}$

$$\frac{3x^2+x-2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\boxed{\frac{3x^2+x-2}{(x-1)(x^2+1)} = \frac{1}{(x-1)} + \frac{2x+3}{x^2+1}}$$

Case IV Example: Resolve $\dfrac{x^2}{(1-x)(1+x^2)^2}$

Solution

$$\frac{x^2}{(1-x)(1+x^2)^2} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$$

multiply Both Sides by $(1-x)(1+x^2)^2$

$$\frac{x^2\cancel{(1-x)}\cancel{(1+x^2)^2}}{\cancel{(1-x)}\cancel{(1+x^2)^2}} = \frac{A\cancel{(1-x)}(1+x^2)^2}{\cancel{(1-x)}} + \frac{(Bx+C)(1-x)\cancel{(1+x^2)^2}}{\cancel{(1+x^2)}} + \frac{(Dx+E)(1-x)\cancel{(1+x^2)^2}}{\cancel{(1+x^2)^2}}$$

$$\boxed{x^2 = A(1+x^2)^2 + (Bx+C)(1-x)(1+x^2) + (Dx+E)(1-x)}$$

$\dfrac{A}{1-x}$, $\boxed{x=1}$

$x=1$: $x^2 = A(1+x^2)^2 + (Bx+C)(1-x)(1+x^2) + (Dx+E)(1-x)$

$1 = A(4) + (B+C)(0)(2) + (D+E)(0)$

$1 = 4A + 0 + 0 \rightarrow A = \boxed{1/4}$

$\Rightarrow x^2 = A(1+2x^2+x^4) + (Bx+C)(1+x^2-x-x^3) + Dx - Dx^2 + E - Ex$

$x^2 = A + 2Ax^2 + Ax^4 + Bx + Bx^3 - Bx^2 - Bx^4 + C + Cx^2 - Cx - Cx^3 + Dx - Dx^2 + E - Ex$

$x^2 = Ax^4 - Bx^4 + Bx^3 + 2Ax^2 - Bx^2 + Cx^2 - Dx^2 + Bx - Cx + Dx - Ex + A + C + E - Cx^3$

$x^2 = Ax^4 - Bx^4 + Bx^3 - Cx^3 + 2Ax^2 - Bx^2 + Cx^2 - Dx^2 + Bx - Cx + Dx - Ex + A + C + E - 0$

$x^2 = x^4(A-B) + x^3(B-C) + x^2(2A-B+C-D) + x(B-C+D-E) + A + C + E$

$0x^4 + 0x^3 + x^2 + 0x + 0 = x^4(A-B) + x^3(B-C) + x^2(2A-B+C-D) + x(B-C+D-E) + A + C + E$

$0x^4 = x^4(A-B)$, $0x^3 = x^3(B-C)$, $x^2 = x^2(2A-B+C-D)$, $0 = A + C + E$

$0 = A - B$ $\boxed{B = 1/4}$

$0 = B - C$ $\boxed{C = 1/4}$

$1 = \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}} - D$

$D = \frac{1}{2} - 1$

$\boxed{D = -1/2}$

$0 = \frac{1}{4} + \frac{1}{4} + E$

$0 = \frac{2}{4} + E$

$\boxed{E = -1/2}$

$$\frac{x^2}{(1-x)(1+x^2)^2} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$$

$$\frac{x^2}{(1-x)(1+x^2)^2} = \frac{1/4}{1-x} + \frac{x/4 + 1/4}{1+x^2} + \frac{-x/2 + 1/2}{(1+x^2)^2}$$

$$= \frac{1}{4(1-x)} + \frac{x+1}{4(1+x^2)} - \frac{x+1}{2(1+x^2)^2}$$

14.3

■ Calculating First-Order Partial Derivatives

▶ In Exercises 1–22, find $\partial f/\partial x$ and $\partial f/\partial y$.

1) $f(x,y) = 2x^2 - 3y - 4$

Solution

⇒ Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = 2\frac{\partial}{\partial x}[x^2] - 3\frac{\partial}{\partial x}[y] - \frac{\partial}{\partial x}[4]$$

$$\frac{\partial f}{\partial x} = 4x - 3(0) - 0 = 4x \qquad \boxed{\frac{\partial f}{\partial x} = 4x}$$

⇒ Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = 2\frac{\partial}{\partial y}[x^2] - 3\frac{\partial}{\partial y}[y] - \frac{\partial}{\partial y}[4]$$

$$\frac{\partial f}{\partial y} = 2(0) - 3(1) - 0 = -3 \qquad \boxed{\frac{\partial f}{\partial y} = -3}$$

2) $f(x,y) = x^2 - xy + y^2$

Solution

⇒ Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2] - y\frac{\partial}{\partial x}[x] + \frac{\partial}{\partial x}[y^2]$$

$$\frac{\partial f}{\partial x} = 2x - y(1) + 0 = 2x - y \qquad \boxed{\frac{\partial f}{\partial x} = 2x - y}$$

⇒ Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2] - x\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}[y^2]$$

$$\frac{\partial f}{\partial y} = 0 - x(1) + 2y = 2y - x \qquad \boxed{\frac{\partial f}{\partial y} = 2y - x}$$

3) $f(x,y) = (x^2-1)(y+2)$ → Product Rule

Solution $(x^2-1)(y+2) = x^2y + 2x - y - 2$

$$\frac{\partial f}{\partial x} = y\frac{\partial}{\partial x}[x^2] + 2\frac{\partial}{\partial x}[x] - \frac{\partial}{\partial x}[y] - \frac{\partial}{\partial x}[2]$$

$$\frac{\partial f}{\partial x} = 2xy + 2 - 0 - 0 = 2xy + 2 \qquad \boxed{\frac{\partial f}{\partial x} = 2xy + 2} \quad \times$$

⇒ Derivative (partial) w.r.t "x" $\partial f/\partial x$ → Product Rule

$$\frac{\partial f}{\partial x} = (y+2) \times \frac{\partial}{\partial x}(x^2-1) + (x^2-1) \times \frac{\partial}{\partial x}(y+2)$$

$$\frac{\partial f}{\partial x} = (y+2)(2x) + (x^2-1)(0) = \boxed{2x(y+2)}$$

⇒ Derivative w.r.t "y" $\partial f/\partial y$ → Product Rule

$$\frac{\partial f}{\partial y} = (y+2) \times \frac{\partial}{\partial y}(x^2-1) + (x^2-1) \times \frac{\partial}{\partial y}(y+2)$$

$$\frac{\partial f}{\partial y} = (y+2)(0) + (x^2-1)(1) = \boxed{x^2-1}$$

4) $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

Solution

$$\frac{\partial f}{\partial x} = 5y\frac{\partial}{\partial x}[x] - 7\frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[y^2] + 3\frac{\partial}{\partial x}[x] - 6\frac{\partial}{\partial x}[y] + \frac{\partial}{\partial x}[2]$$

$$\frac{\partial f}{\partial x} = 5y - 14x - 0 + 3 - 0 + 0 = \boxed{5y - 14x + 3}$$

⇒ Derivative w.r.t "y"

$$\frac{\partial f}{\partial y} = 5x\frac{\partial}{\partial y}[y] - 7\frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[y^2] + 3\frac{\partial}{\partial y}[x] - 6\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}[2]$$

$$\frac{\partial f}{\partial y} = 5x - 0 - 2y + 0 - 6 + 0 = \boxed{5x - 2y - 6}$$

5) $f(x,y) = (xy-1)^2$

Solution

⇒ Derivative w.r.t "x" → chain Rule

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy-1)^2 = 2(xy-1) \times \frac{\partial}{\partial x}(xy-1) = 2(xy-1) \times \frac{\partial}{\partial x}[xy] - \frac{\partial}{\partial x}[1]$$

$$\frac{\partial f}{\partial x} = 2(xy-1)(y) - 0 = \boxed{2y(xy-1)}$$

⇒ Derivative w.r.t "y"

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy-1)^2 = 2(xy-1) \times \frac{\partial}{\partial y}(xy-1) = \boxed{2x(xy-1)}$$

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6) $f(x,y) = (2x-3y)^3$

Solution

$$\frac{\partial f}{\partial x} = 3(2x-3y)^2 \times \frac{\partial}{\partial x}(2x-3y) = 3(2x-3y)^2(2) = \boxed{6(2x-3y)^2}$$

$$\frac{\partial f}{\partial y} = 3(2x-3y)^2 \times \frac{\partial}{\partial y}(2x-3y) = 3(2x-3y)^2(-3) = \boxed{-9(2x-3y)^2}$$

7) $f(x,y) = \sqrt{x^2+y^2}$

Solution

$\Rightarrow$ Partial Derivative w.r.t "x"

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-1/2} \times \frac{\partial}{\partial x}(x^2+y^2)$$

$$\frac{\partial f}{\partial x} = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \times (\cancel{2}x) = \boxed{\frac{x}{\sqrt{x^2+y^2}}}$$

$\Rightarrow$ Partial Derivative w.r.t "y"

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-1/2} \times \frac{\partial}{\partial y}(x^2+y^2)$$

$$\frac{\partial f}{\partial y} = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \times (\cancel{2}y) = \boxed{\frac{y}{\sqrt{x^2+y^2}}}$$

8) $f(x,y) = (x^2+(y/2))^{2/3}$

Solution

$\Rightarrow$ Partial Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+(y/2))^{2/3} = \frac{2}{3}(x^2+(y/2))^{-1/3} \times \frac{\partial}{\partial x}(x^2+(y/2))$$

$$\frac{\partial f}{\partial x} = \frac{2}{3}(x^2+(y/2))^{-1/3} \times (2x) = \boxed{\frac{4x}{3\sqrt[3]{x^2+(y/2)}}}$$

$\Rightarrow$ Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+(y/2))^{2/3} = \frac{2}{3}(x^2+(y/2))^{-1/3} \times \frac{\partial}{\partial y}(x^2+y/2)$$

$$\frac{\partial f}{\partial y} = \frac{\cancel{2}}{3}(x^2+(y/2))^{-1/3} \times \left(\frac{1}{\cancel{2}}\right) = \boxed{\frac{1}{3\sqrt[3]{x^2+(y/2)}}}$$

9) $f(x,y) = 1/(x+y)$

Solution Quotient Rule

=> Partial Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \left[(x+y) \times \frac{\partial}{\partial x}[1] - [1] \times \frac{\partial}{\partial x}[x+y]\right] \div (x+y)^2$$

$$\frac{\partial f}{\partial x} = \left[(x+y) \times (0) - (1)(1) + 0\right] \div (x+y)^2$$

$$\boxed{\frac{\partial f}{\partial x} = \frac{-1}{(x+y)^2}}$$

=> Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \left[(x+y) \times \frac{\partial}{\partial y}[1] - [1] \times \frac{\partial}{\partial y}[x+y]\right] \div [x+y]^2$$

$$\frac{\partial f}{\partial y} = \left[(x+y)(0) - (1)(0) + (1)(1)\right] \div (x+y)^2 = \boxed{\frac{-1}{(x+y)^2}}$$

10) $f(x,y) = \dfrac{x}{(x^2+y^2)}$

Solution Quotient Rule

=> Partial Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \left[(x^2+y^2) \times \frac{\partial}{\partial x}[x] - (x) \times \frac{\partial}{\partial x}[x^2+y^2]\right] \div (x^2+y^2)^2$$

$$\frac{\partial f}{\partial x} = \left[(x^2+y^2) - (x)(2x)\right] \div (x^2+y^2) = \boxed{\frac{y^2 - x^2}{(x^2+y^2)^2}}$$

=> Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \left[(x^2+y^2) \times \frac{\partial}{\partial y}[x] - [x] \times \frac{\partial}{\partial y}[x^2+y^2]\right] \div (x^2+y^2)^2$$

$$\frac{\partial f}{\partial y} = \left[(x^2+y^2)(0) - (x)(2y)\right] \div (x^2+y^2)^2 = \boxed{\frac{-2xy}{(x^2+y^2)^2}}$$

11) $f(x,y) = (x+y)/(xy-1)$

Solution Quotient Rule $\frac{u}{v} = (vu' - uv')/v^2, \quad v^2 \neq 0$

$$\frac{\partial f}{\partial x} = \left[(xy-1) \times \frac{\partial}{\partial x}[x+y] - (x+y) \times \frac{\partial}{\partial x}[xy-1] \right] \div (xy-1)^2$$

$$\frac{\partial f}{\partial x} = \left[(xy-1)(1) - (x+y)(y) \right] \div (xy-1)^2$$

$$\frac{\partial f}{\partial x} = \frac{\cancel{xy} - 1 - \cancel{xy} - y^2}{(xy-1)^2} = \boxed{\frac{-1 - y^2}{(xy-1)^2}}$$

12) $f(x,y) = \tan^{-1}(y/x)$

Solution

⇒ Partial Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}(y/x)\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \frac{\partial}{\partial x}\left(\frac{y}{x}\right) \quad \because \tan^{-1}x = \frac{1}{1+x^2}$$

Angle's Derivative Quotient Rule

$$\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2} \times \frac{-y}{x^2}$$

$$\frac{\partial f}{\partial x} = \frac{-y}{x^2[1+(y/x)^2]} = \frac{-y}{x^2 + \frac{\cancel{x^2} y^2}{\cancel{x^2}}} = \boxed{\frac{-y}{x^2+y^2}}$$

⇒ Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\tan^{-1}(y/x)\right)$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2} \times \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{1+(y/x)^2} \times \frac{1}{x}\frac{\partial}{\partial y}(y) = \frac{1}{x + y^2/x}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x + y^2/x} = \frac{1}{\frac{x^2+y^2}{x}} = \boxed{\frac{x}{x^2+y^2}}$$

13) $f(x,y) = e^{(x+y+1)}$

Solution

⇒ Partial Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(e^{(x+y+1)}\right) = \boxed{e^{x+y+1}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(e^{x+y+1}\right) = \boxed{e^{x+y+1}}$$

14) $f(x,y) = e^{-x}\sin(x+y)$

Solution

consist x in both terms

⇒ Partial Derivative w.r.t "x" $\partial f/\partial x$ (Product Rule)

$$\frac{\partial f}{\partial x} = \left[\sin(x+y) \times \frac{\partial}{\partial x}(e^{-x}) + (e^{-x}) \times \frac{\partial}{\partial x}(\sin(x+y))\right]$$

$$\frac{\partial f}{\partial x} = \left[\sin(x+y)(-e^{-x}) + (e^{-x})(\cos(x+y))\right]$$

$$\boxed{\frac{\partial f}{\partial x} = \left[-e^{-x}\sin(x+y) + e^{-x}\cos(x+y)\right]}$$

⇒ Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = e^{-x} \times \frac{\partial \sin(x+y)}{\partial y}$$

Scalar

Because it doesn't have any "y" term/variable

$$\frac{\partial f}{\partial y} = (e^{-x})(1) = \boxed{e^{-x}(\cos(x+y))}$$

15) $f(x,y) = \ln(x+y)$

Solution

⇒ Partial Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\ln(x+y)) = \boxed{\frac{1}{x+y}} \qquad * \frac{d}{dx}(\ln x) = \frac{1}{x}$$

⇒ Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\ln(x+y) = \boxed{\frac{1}{x+y}}$$

16) $f(x,y) = e^{xy}\ln y$

Solution

$$\frac{\partial f}{\partial x} = \ln y \times \frac{\partial}{\partial x}(e^{xy}) = \ln y \times \left(\frac{y e^{xy}}{y}\right) = \boxed{y e^{xy}\ln y}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{xy}\ln y\right] = \ln y \times \frac{\partial}{\partial y}(e^{xy}) + (e^{xy}) \times \frac{\partial}{\partial y}\ln y\,]$$

$$\frac{\partial f}{\partial y} = \ln y(x e^{xy}) + (e^{xy})\left(\frac{1}{y}\right) = \boxed{x e^{xy}\ln y + \frac{e^{xy}}{y}}$$

17) $f(x,y) = \sin^2(x-3y)$

Solution

⇒ Partial Derivative w.r.t "x" $\partial f/\partial x$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\sin^2(x-3y)] = 2\sin(x-3y) \times \frac{\partial}{\partial x}[\sin(x-3y)]$

$\frac{\partial f}{\partial x} = 2\sin(x-3y) \times (\cos(x-3y)) \times \frac{\partial}{\partial x}(x-3y)$

$\boxed{\frac{\partial f}{\partial x} = 2\sin(x-3y)\cos(x-3y)}$

⇒ Partial Derivative w.r.t "y" $\partial f/\partial y$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\sin^2(x-3y) = 2\sin(x-3y) \times \frac{\partial}{\partial y}[\sin(x-3y)]$

$\frac{\partial f}{\partial y} = 2\sin(x-3y)\cos(x-3y) \times \frac{\partial}{\partial y}(x-3y)$

$\frac{\partial f}{\partial y} = 2\sin(x-3y)\cos(x-3y)(-3) = \boxed{-6\sin(x-3y)\cos(x-3y)}$

(18) $f(x,y) = \cos^2(3x-y^2)$

Solution

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\cos^2(3x-y^2)] = 2\cos(3x-y^2) \times \frac{\partial}{\partial x}\cos(3x-y^2)$

$\frac{\partial f}{\partial x} = 2\cos(3x-y^2)(-\sin(3x-y^2)) \times \frac{\partial}{\partial x}(3x-y^2)$

$\frac{\partial f}{\partial x} = -2\cos(3x-y^2)\sin(3x-y^2)(3) = \boxed{-6\cos(3x-y^2)\sin(3x-y^2)}$

⇒ Partial Derivative w.r.t "y". $\partial f/\partial y$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\cos^2(3x-y^2)] = 2\cos(3x-y^2) \times \frac{\partial}{\partial y}(\cos(3x-y^2))$

$\frac{\partial f}{\partial y} = 2\cos(3x-y^2)(-\sin(3x-y^2)) \times \frac{\partial}{\partial y}(3x-y^2)$

$\frac{\partial f}{\partial y} = -2\cos(3x-y^2)(\sin(3x-y^2))(-2y) = \boxed{4y\cos(3x-y^2)\sin(3x-y^2)}$

19) $f(x, y) = x^y$

Solution

⇒ Partial Derivative with respect to "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^y] = \boxed{yx^{y-1}}$$

⇒ Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^y] = \boxed{x^y \ln x} \rightarrow \frac{d}{dx}a^x = a^x \ln a$$

20) $f(x, y) = \log_y x$

Solution

⇒ Partial Derivative w.r.t "x" $\partial f/\partial x$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[\log_y x] = \boxed{\frac{1}{x \ln y}}, \boxed{OR}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{\ln x}{\ln y}\right] = \boxed{\frac{1}{x \ln y}}$$

⇒ Partial Derivative w.r.t "y" $\partial f/\partial y$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\log_y x]$$

In Exercises 23-34, find f_x, f_y, and f_z

23) $f(x,y,z) = 1 + xy^2 - 2z^2$

Solution

$\Rightarrow f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[1] + y^2\frac{\partial}{\partial x}[x] - 2\frac{\partial}{\partial x}[z^2]$

$f_x = \frac{\partial f}{\partial x} = 0 + y^2 - 0 = \boxed{y^2}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[1] + x\frac{\partial}{\partial y}[y^2] - 2\frac{\partial}{\partial y}[z^2] = 0 + 2xy - 0 = \boxed{2xy}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[1] + \frac{\partial}{\partial z}[xy^2] - 2\frac{\partial}{\partial z}[z^2] = 0 + 0 - 4z^1 = \boxed{-4z^1}$

24) $f(x,y,z) = xy + yz + xz$

Solution

$\Rightarrow f_x = \frac{\partial f}{\partial x} = y + 0 + z = \boxed{y+z}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = x + z + 0 = \boxed{x+z}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = 0 + y + x = \boxed{y+x}$

25) $f(x,y,z) = x - \sqrt{y^2+z^2}$

Solution

$\Rightarrow f_x = \frac{\partial f}{\partial x} = 1 - 0 = \boxed{1}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = 0 - \frac{1}{2}(y^2+z^2)^{-1/2} \times \frac{\partial}{\partial y}(y^2+z^2)$

$= -\frac{1}{\cancel{2}}(y^2+z^2)^{-1/2}(\cancel{2}y) = \boxed{\frac{-y}{\sqrt{y^2+z^2}}}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = 0 - \frac{1}{2}(y^2+z^2)^{-1/2} \times \frac{\partial}{\partial z}(y^2+z^2)$

$= 0 - \frac{1}{\cancel{2}}(y^2+z^2)^{-1/2} \times (\cancel{2}z) = \boxed{\frac{-z}{\sqrt{y^2+z^2}}}$

26) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Solution

$\Rightarrow f_x = \frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times \frac{\partial}{\partial x}(x^2+y^2+z^2)$

$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times (2x) = \boxed{\frac{-x}{(x^2+y^2+z^2)^{3/2}}}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times \frac{\partial}{\partial y}(x^2+y^2+z^2)$

$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times (2y) = \boxed{\frac{-y}{(x^2+y^2+z^2)^{3/2}}}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times \frac{\partial}{\partial z}(x^2+y^2+z^2)$

$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times (2z) = \boxed{\frac{-z}{(x^2+y^2+z^2)^{3/2}}}$

27) $f(x,y,z) = \sin^{-1}(xyz)$

Solution

$\Rightarrow f_x = \frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(xyz)^2}} \times \frac{\partial}{\partial x}(xyz) = \boxed{\frac{yz}{\sqrt{1-(xyz)^2}}}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(xyz)^2}} \times \frac{\partial}{\partial y}(xyz) = \boxed{\frac{xz}{\sqrt{1-(xyz)^2}}}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = \frac{1}{\sqrt{1-(xyz)^2}} \times \frac{\partial}{\partial z}(xyz) = \boxed{\frac{xy}{\sqrt{1-(xyz)^2}}}$

28) $f(x,y,z) = \sec^{-1}(x+yz)$

Solution

$f_x = \frac{\partial f}{\partial x} = \frac{1}{(x+yz)\sqrt{(x+yz)^2-1}} \times \frac{\partial}{\partial x}(x+yz) = \boxed{\frac{1}{(x+yz)\sqrt{(x+yz)^2-1}}}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}} \times \frac{\partial}{\partial y}(x+yz) = \frac{z}{(x+yz)\sqrt{(x+yz)^2-1}}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = \frac{1}{|x+yz|\sqrt{(x+yz)^2-1}} \times \frac{\partial}{\partial z}(x+yz) = \frac{y}{|x+yz)\sqrt{(x+yz)^2-1}}$

29) $f(x,y,z) = \ln(x+2y+3z)$

Solution

$f_x = \frac{\partial f}{\partial x} = \frac{1}{x+2y+3z} \times \frac{\partial}{\partial x}(x+2y+3z) = \frac{1}{x+2y+3z}$

$f_y = \frac{\partial f}{\partial y} = \frac{1}{x+2y+3z} \times \frac{\partial}{\partial y}(x+2y+3z) = \frac{2}{x+2y+3z}$

$f_z = \frac{\partial f}{\partial z} = \frac{1}{x+2y+3z} \times \frac{\partial}{\partial z}(x+2y+3z) = \frac{3}{x+2y+3z}$

30) $f(x,y,z) = yz\ln(xy)$

Solution

$\Rightarrow f_x = \frac{\partial f}{\partial x} = yz \times \frac{\partial}{\partial x}\ln(xy) = yz\left(\frac{1}{xy}\right) \times \frac{\partial}{\partial x}(xy)$

$= yz\left(\frac{1}{xy}\right) \times (y) = \boxed{\frac{yz}{x}}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = \ln(xy) \times \frac{\partial}{\partial y} z(y) + yz \times \frac{\partial}{\partial y}\ln(xy)$

$= \ln(xy) \times z + (yz)\left(\frac{1}{xy}\right) \times \frac{\partial}{\partial y}(xy)$

$= z\ln xy + (yz)\left(\frac{1}{xy}\right) \times (x) = \boxed{z\ln xy + z}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = \ln(xy) \times \frac{\partial}{\partial z}(yz) = \ln(xy) \times y\frac{\partial}{\partial z}(z)$

$= \ln(xy)(y) = \boxed{y\ln(xy)}$

31) $f(x,y,z) = e^{-(x^2+y^2+z^2)}$

Solution

$\Rightarrow f_x = \frac{\partial f}{\partial x} = e^{-(x^2+y^2+z^2)} \times \frac{\partial}{\partial x} - (x^2+y^2+z^2) = \boxed{-2xe^{-(x^2+y^2+z^2)}}$

$\Rightarrow f_y = \frac{\partial f}{\partial y} = e^{-(x^2+y^2+z^2)} \times \frac{\partial}{\partial y} - (x^2+y^2+z^2) = \boxed{-2y\, e^{-(x^2+y^2+z^2)}}$

$\Rightarrow f_z = \frac{\partial f}{\partial z} = e^{-(x^2+y^2+z^2)} \times \frac{\partial}{\partial z} - (x^2+y^2+z^2) = \boxed{-2z\left(e^{-(x^2+y^2+z^2)}\right)}$

32) $f(x,y,z) = e^{-xyz}$

Solution

$f_x = \frac{\partial f}{\partial x} = e^{-xyz} \times \frac{\partial}{\partial x} -(xyz) = e^{-xyz} \times (-yz) = \boxed{-yz\, e^{-xyz}}$

$f_y = \frac{\partial f}{\partial y} = e^{-xyz} \times \frac{\partial}{\partial y} -(xyz) = e^{-xyz} \times (-xz) = \boxed{-xz\, e^{-xyz}}$

$f_z = \frac{\partial f}{\partial z} = e^{-xyz} \times \frac{\partial}{\partial z} -(xyz) = e^{-xyz} \times (-xy) = \boxed{-xy\, e^{-xyz}}$

■ Calculating Second Order Partial Derivatives

Find all the second-order Partial Derivatives of the functions in Exercises 41 – 50.

41) $f(x,y) = x + y + xy$

Solution

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x} \rightarrow$ 6 Derivatives

1: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial x}[y] + y\frac{\partial}{\partial x}[x] = 1 + 0 + y = \boxed{1+y}$

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial y}[y] + x\frac{\partial}{\partial y}[y] = 0 + 1 + x = \boxed{1+x}$

3: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[1+y] = \frac{\partial}{\partial x}[1] + \frac{\partial}{\partial x}[y] = 0 + 0 = \boxed{0}$

4: $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial y}[1] = 0 + 0 = \boxed{0}$

5: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left[1 + x\right] = 0 + 1 = \boxed{1}$

6: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}[1 + y] = 0 + 1 = \boxed{1}$

146 $\boxed{\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}}$

42) $f(x,y) = \sin xy$

Solution

6 Derivatives: $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$

1: $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}[\sin xy] = \cos(xy) \times \dfrac{\partial}{\partial x}(xy) = \boxed{y \cos xy}$

2: $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}[\sin xy] = (\cos xy) \times \dfrac{\partial}{\partial y}(xy) = \boxed{x \cos xy}$

3: $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}(y\cos xy) = -y \sin xy \times y = \boxed{-y^2 \sin xy}$

4: $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}[x \cos xy] = -x \sin xy \times \dfrac{\partial}{\partial y}(xy) = \boxed{-x^2 \sin xy}$

5: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial x}[x\cos xy] = \cos xy + x(-\sin xy)y = \boxed{\cos xy - xy \sin xy}$

6: $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}(y\cos xy) = \cos xy + y(-\sin xy)(x) = \boxed{\cos xy - xy \sin xy}$

43) $g(x,y) = x^2 y + \cos y + y \sin x$

Solution

1: $\dfrac{\partial f}{\partial x} = 2xy + 0 + y\cos x = \boxed{2xy + y\cos x}$

2: $\dfrac{\partial f}{\partial y} = x^2 + (-\sin y) + \sin x = \boxed{x^2 - \sin y + \sin x}$

3: $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}[2xy + y\cos x] = 2y + y(-\sin x) = \boxed{2y - y\sin x}$

4: $\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}[x^2 - \sin y + \sin x] = 0 - \cos y + 0 = \boxed{-\cos y}$

5: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left[\dfrac{\partial f}{\partial y}\right] = \dfrac{\partial}{\partial x}[x^2 - \sin y + \sin x] = \boxed{2x + \cos x}$

6: $\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial y}[2xy + y\cos x] = \boxed{2x + \cos x}$

44) $h(x, y) = xe^y + y + 1$

Solution

1: $\frac{\partial f}{\partial x} = e^y + 0 + 0 = \boxed{e^y}$

2: $\frac{\partial f}{\partial y} = xe^y + 1 + 0 = \boxed{xe^y + 1}$

3: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(e^y\right) = \boxed{0}$

4: $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(xe^y + 1\right) = xe^y + 0 = \boxed{xe^y}$

5: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(xe^y + 1\right) = 1e^y + 0 = \boxed{e^y}$

6: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^y\right) = e^y \times \frac{\partial}{\partial y}(y) = \boxed{e^y}$

45) $r(x, y) = \ln(x + y)$

Solution

1: $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\ln(x+y)\right) = \frac{1}{x+y}$

2: $\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\ln(x+y)\right) = \frac{1}{x+y}$

3: $\frac{\partial^2 r}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{1}{x+y}\right] = \frac{(x+y)(0) + (1)(1+0)}{(x+y)^2} = 0 + 1 = \frac{-1}{(x+y)^2}$

4: $\frac{\partial^2 r}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial y}\left[\frac{1}{x+y}\right] = \frac{(x+y)(0) + (1)(0+1)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

5: $\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{1}{x+y}\right] = \frac{(x+y)(0) + (1)(1+0)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

6: $\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right) = \frac{(x+y)(0) + (1)(0+1)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

46) $S(x,y) = \tan^{-1}(y/x)$

Solution

1: $\dfrac{\partial S}{\partial x} = \dfrac{1}{1+(y/x)^2} \times \dfrac{\partial}{\partial x}\left(\dfrac{y}{x}\right) = \dfrac{1}{1+(y/x)^2} \times y\dfrac{((x)(0) - (1)(1))}{x^2} = \dfrac{-y}{x^2(1+(y/x)^2)}$

2: $\dfrac{\partial S}{\partial y} = \dfrac{1}{1+(y/x)^2} \times \dfrac{\partial}{\partial y}(y/x) = \dfrac{1}{1+(y/x)^2} \times \dfrac{1}{x} = \dfrac{1}{x(1+(y/x)^2)}$

3: $\dfrac{\partial^2 S}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{-y}{x^2(1+(y/x)^2)}\right] = \dfrac{x^2(1-(y/x)^2)(-0) - (-y)(2x)}{(x^2(1+(y/x)^2))^2} = \dfrac{2xy}{(x^2+y^2)^2}$

4: $\dfrac{\partial^2 S}{\partial y^2} = \dfrac{\partial}{\partial y}\left[\dfrac{x}{x^2+y^2}\right] = \dfrac{(x^2+y^2)(0) - (x)(2y)}{(x^2+y^2)^2} = \dfrac{-2xy}{(x^2+y^2)^2}$

5: $\dfrac{\partial^2 S}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial S}{\partial y}\right) = \dfrac{\partial}{\partial x}\left[\dfrac{x}{x^2+y^2}\right] = \dfrac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2} = \dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2}$

6: $\dfrac{\partial^2 S}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial S}{\partial x}\right) = \dfrac{\partial}{\partial y}\left[\dfrac{-y}{x^2+y^2}\right] = \dfrac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} = \dfrac{2y^2 - (x^2+y^2)}{(x^2+y^2)^2}$

47) $w = x^2\tan(xy)$

Solution

1: $\dfrac{\partial w}{\partial x} = \tan(xy)(2x) + (x^2)\left(\sec^2(xy) \times \dfrac{\partial}{\partial x}(xy)\right) = \boxed{2x\tan xy + x^2y\sec^2 xy}$

2: $\dfrac{\partial w}{\partial y} = x^2 \times \dfrac{\partial}{\partial y}\tan(xy) = x^2\sec^2(xy) \times (x) = \boxed{x^3\sec^2 xy}$

3: $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial}{\partial x}\left[2x\tan xy + x^2y\sec^2 xy\right] = \tan xy(2) + (2x)y\sec^2 xy + \sec^2 xy(2xy) + (x^2y)\tan xy$

4: $\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial y}\left[x^3\sec^2 xy\right] = x^3(2\sec xy \times \sec xy\tan xy \times (x))$
$= \boxed{2x^4\sec^2 xy\tan xy}$

5: $\dfrac{\partial^2 w}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial y}\right) = \dfrac{\partial}{\partial x}\left[x^3\sec^2 xy\right] = \sec^2 xy(3x^2) + x^3(2\sec xy\sec xy\tan xy \cdot y)$
$= \boxed{3x^2\sec^2 xy + 2x^3y\sec^2 xy\tan xy}$

6: $\dfrac{\partial^2 w}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial w}{\partial x}\right) = \dfrac{\partial}{\partial y}\left[x^3\sec^2 xy\right] =$

6: $\dfrac{\partial^2 w}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial w}{\partial x}\right) = \dfrac{\partial}{\partial y}\left[2x\tan xy + x^2y\sec^2 xy\right] = 2x^2\sec^2 xy + \sec^2 xy(x^2) + (x^2y)\,2\sec xy\sec xy\tan xy\,(x)$

$= 2x^2\sec^2 xy + \sec^2 xy(x^2) + 2(x^3y)\sec^2 xy\tan xy$

$= \sec^2 xy(2x^2 + x^2) + 2x^3y\sec^2 xy\tan xy = \boxed{3x^2\sec^2 xy + 2x^3y\sec^2 xy\tan xy}$

48) $w = ye^{x^2-y}$

Solution

1: $\frac{\partial w}{\partial x} = ye^{x^2-y} \times \frac{\partial}{\partial x}(x^2-y) = ye^{x^2-y} \times (2x) = \boxed{2xye^{x^2-y}}$

2: $\frac{\partial w}{\partial y} = (e^{x^2-y})(1) + (y)(e^{x^2-y})(-1) = \boxed{e^{x^2-y} - ye^{x^2-y}}$

3: $\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}\left(2xye^{x^2-y}\right) = (e^{x^2-y})(2y) + (2xy)(e^{x^2-y})(2x)$

$= \boxed{2ye^{x^2-y} + 4x^2ye^{x^2-y}}$

4: $\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left(e^{x^2-y} - ye^{x^2-y}\right) = -e^{x^2-y} - e^{x^2-y}(1) - y(e^{x^2-y})(-1)$

$= -e^{x^2-y} - e^{x^2-y} + ye^{x^2-y}$

$= \boxed{-2e^{x^2-y} + ye^{x^2-y}}$

5: $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y}\right) = \frac{\partial}{\partial x}\left(e^{x^2-y} - ye^{x^2-y}\right) = \boxed{2xe^{x^2-y} - 2xye^{x^2-y}}$

6: $\frac{\partial^2 w}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial y}\left[2xye^{x^2-y}\right] = 2xe^{x^2-y} + 2xy(e^{x^2-y})(-1)$

$= \boxed{2xe^{x^2-y} - 2xye^{x^2-y}}$

49) $w = x\sin(x^2y)$

Solution

1: $\frac{\partial w}{\partial x} = \sin(x^2y) + (x)(\cos(x^2y))(2xy) = \boxed{\sin(x^2y) + 2x^2y\cos(x^2y)}$

2: $\frac{\partial w}{\partial y} = x \times \frac{\partial}{\partial y}\sin(x^2y) = x\cos(x^2y)(x^2) = \boxed{x^3\cos(x^2y)}$

3: $\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\sin(x^2y) + 2x^2y\cos(x^2y)\right] = \cos(x^2y)(2xy) + \cos(x^2y)(4xy) + 2x^2y \times (-\sin(x^2y)) \times (2xy)$

$= 2xy\cos(x^2y) + 4xy\cos(x^2y) + 4x^3y^2\sin(x^2y)$

$= \boxed{6xy\cos(x^2y) - 4x^3y^2\sin(x^2y)}$

4: $\frac{\partial^2 w}{\partial y^2} = \frac{\partial}{\partial y}\left[x^3\cos(x^2y)\right] = \boxed{x^3(-\sin(x^2y))(x^2) = -x^5\sin(x^2y)}$

5: $\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial w}{\partial y}\right] = \frac{\partial}{\partial x}\left[x^3\cos(x^2y)\right] = \cos(x^2y)(3x^2) + (x^3)(-\sin(x^2y))(2xy)$

$= \boxed{3x^2\cos(x^2y) - 2x^4y\sin(x^2y)}$

$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x} = 3x^2\cos(x^2y) - 2x^4y\sin(x^2y)$

80) $w = \dfrac{x-y}{x^2+y}$

Solution

1: $\dfrac{\partial w}{\partial x} = \dfrac{(x^2+y)(1)-(x-y)(2x)}{(x^2+y)^2} = \dfrac{x^2+y-2x^2+2xy}{(x^2+y)^2} = \dfrac{2xy+y-x^2}{(x^2+y)^2}$

2: $\dfrac{\partial w}{\partial y} = \dfrac{(x^2+y)(-1)-(x-y)(1)}{(x^2+y)^2} = \dfrac{-x^2-\cancel{y}-x+\cancel{y}}{(x^2+y)^2} = \dfrac{-x^2-x}{(x^2+y)^2}$

3: $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial}{\partial x}\left[\dfrac{2xy+y-x^2}{(x^2+y)^2}\right] = \left[\dfrac{(x^2+y)^2(2y-2x)-(2xy+y-x^2)(2(x^2+y))(2x)}{(x^2+y)^4}\right]$

$= \dfrac{(x^2+y)^2(2y-2x)-4x(x^2+y)(2xy+y-x^2)}{(x^2+y)^4}$

4: $\dfrac{\partial^2 w}{\partial y^2} = \dfrac{\partial}{\partial y}\left[\dfrac{-x^2-x}{(x^2+y)^2}\right] = \left[\dfrac{(x^2+y)^2(0)-(-x^2-x)(2(x^2+y))}{(x^2+y)^4}\right] = \dfrac{2(x^2+x)(x^2+y)}{(x^2+y)^4}$

5: $\dfrac{\partial^2 w}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial w}{\partial y}\right) = \dfrac{\partial}{\partial x}\left[\dfrac{-x^2-x}{(x^2+y)^2}\right] = \left[\dfrac{(x^2+y)^2(-2x-1)-(-x^2-x)(2(x^2+y)2x)}{(x^2+y)^4}\right]$

$= \dfrac{(x^2+y)^2(-2x-1)+4x(x^2+x)(x^2+y)}{(x^2+y)^4}$

Differentiating Implicitly

65) Find the value of $\partial z/\partial x$ at the point $(1, 1, 1)$ if the equation $xy + z^3x - 2yz = 0$

Solution

here y is considered as constant.

$$\frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}[z^3x] - 2y\frac{\partial}{\partial x}[z] = 0$$

$$y + (x)(3z^2)\frac{\partial z}{\partial x} + z^3 - 2y\frac{\partial z}{\partial x} = 0$$

$$y + z^3 + \frac{\partial z}{\partial x}[3xz^2 - 2y] = 0$$

$$\frac{\partial z}{\partial x} = \frac{-y - z^3}{3xz^2 - 2y}, \quad (1,1,1)$$

$$\frac{\partial z}{\partial x} = \frac{-(1)-(1)^3}{3(1)(1)^2 - 2(1)} = \frac{-1-1}{3-2} = \frac{-2}{1} = \boxed{-2}$$

$\Rightarrow$ Let's solve with another way $\rightarrow \frac{dy}{dx} = -\frac{F_x}{F_y}$

So here we have $\frac{\partial z}{\partial x} = -\frac{F_{\cancel{y}}}{F_{\cancel{x}}} = -\frac{F_x}{F_z}$

1: Derivative of f w.r.t "z" $\rightarrow F_z$

$$\frac{\partial}{\partial z}[xy] + x\frac{\partial}{\partial z}[z^3] - 2y\frac{\partial}{\partial z}[z] = 0$$

$$0 + 3xz^2 - 2y = 0 \quad \rightarrow \quad \boxed{3xz^2 - 2y = 0} \rightarrow F_z$$

2: Derivative of f w.r.t "x" $\rightarrow F_x$

$$\frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}[xz^3] - \frac{\partial}{\partial x}[2yz] = 0$$

$$y + z^3 - 0 = 0 \quad \boxed{y + z^3 = 0} \rightarrow F_x$$

$\boxed{\text{put } (1,1,1)}$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{(y+z^3)}{(3xz^2 - 2y)} = \frac{-(1+(1)^3)}{(3(1)(1)^2 - 2(1))} = \frac{-2}{3-2} = \boxed{-2}$$

66) Find the value of $\partial x/\partial z$ at the point $(1,-1,-3)$ if the equation $xz + y\ln x - x^2 + 4 = 0$ defines x as a function of the two independent variables y and z and the partial derivative exists.

Solution

1 $$\frac{\partial}{\partial z}[xz] + y\frac{\partial}{\partial z}(\ln x) - \frac{\partial}{\partial z}[x^2] + \frac{\partial}{\partial z}[4] = 0$$

2 $$(z)\frac{\partial x}{\partial z} + x + \frac{y}{x}\frac{\partial x}{\partial z} - 2x\frac{\partial x}{\partial z} + 0 = 0$$

3 $$x + \frac{\partial x}{\partial z}\left(z + \frac{y}{x} - 2x\right) = 0$$

Put $x=1$, $y=-1$, and $z=-3$

$$\frac{\partial x}{\partial z} = \frac{-x}{z + \frac{y}{x} - 2x} = \frac{-1}{(-3) + \frac{(-1)}{1} - 2(1)} = \frac{-1}{-6} = \boxed{\frac{1}{6}} \checkmark$$

■ Let's try with second format

$$\frac{\partial x}{\partial z} = \frac{-F_z}{F_x}$$

1: $$F_z = \frac{\partial}{\partial z}[xz] + \frac{\partial}{\partial z}[y\ln x] - \frac{\partial}{\partial z}[x^2] + \frac{\partial}{\partial z}[4] = 0$$

$$F_z = x + 0 - 0 + 0 = \boxed{x} \rightarrow F_z$$

2: $$F_x = \frac{\partial}{\partial x}[xz] + \frac{\partial}{\partial x}[y\ln x] - \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial x}[4] = 0$$

$$F_x = z + \frac{y}{x} - 2x + 0 = \boxed{z + \frac{y}{x} - 2x} \rightarrow F_x$$

$(1,-1,-3)$

4 $$\frac{-F_z}{F_x} = \frac{-(x)}{z + \frac{y}{x} - 2x} = \frac{-(1)}{-3 + \frac{(-1)}{1} - 2(1)} = \frac{-1}{-3-1-2} = \frac{-1}{-6} = \boxed{\frac{1}{6}} \checkmark$$

▶ Show that each function in Exercises 73-80 satisfies a Laplace equation.

73) $f(x, y, z) = x^2 + y^2 - 2z^2$

Solution

$\frac{\partial f}{\partial x} 1: \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial x}[y^2] - 2\frac{\partial}{\partial x}[z^2] = \boxed{2x}$

$\frac{\partial^2 f}{\partial x^2} : \frac{\partial}{\partial x}[2x] = \boxed{2}$

2: First Derivative w.r.t "y"

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2 + y^2 - 2z^2] = \boxed{2y}$

• Second derivative of f w.r.t "y"

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[2y] = \boxed{2}$

3: First & Second derivative of f w.r.t "z"

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x^2 + y^2 - 2z^2] = \boxed{-4z}$

$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}[-4z] = \boxed{-4}$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \rightarrow$ Laplace equation's satisfication

$(2) + (2) + (-4) = 4 - 4 = \boxed{0}$ ✓

74) $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$

Solution

1: $\frac{\partial}{\partial x}[2z^3 - 3(x^2+y^2)z] = 0 - 6xz + 0 = \boxed{-6xz}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[-6xz] = \boxed{-6z}$ ✓

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[2z^3 - 3z(x^2+y^2)] = \boxed{-6zy}$, $\frac{\partial^2 f}{\partial y^2} = \boxed{-6z}$ ✓

3: $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[2z^3 - 3z(x^2+y^2)] = \boxed{6z^2 - 3(x^2+y^2)}$, $\frac{\partial^2 f}{\partial z^2} = \boxed{12z}$ ✓

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (-6z) + (-6z) + (12z) = \boxed{0}$ ✓

Satisfied

75) $f(x,y) = e^{-2y}\cos 2x$

Solution

1: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[e^{-2y}\cos 2x] = -2e^{-2y}\sin 2x$, $\frac{\partial^2 f}{\partial x^2} = \boxed{(-2e^{-2y}\cos 2x)2}$

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[e^{-2y}\cos 2x] = -2e^{-2y}\cos 2x$, $\frac{\partial^2 f}{\partial y^2} = \boxed{(-2e^{-2y}\cos 2x)(-2)}$

3: $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[e^{-2y}\cos 2x] = \dots$ in two variables

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (-4e^{-2y}\cos 2x) + (4e^{-2y}\cos 2x) = \boxed{0}$ ✓

76) $f(x,y) = \ln\sqrt{x^2+y^2}$

1: $\frac{\partial f}{\partial x} = \frac{1}{2}\ln(x^2+y^2) = \frac{\partial}{\partial x}\frac{1}{2}(\ln(x^2+y^2)) = \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\times\frac{\partial}{\partial x}(x^2+y^2)$

$= \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\times(2x) = \frac{x}{x^2+y^2}$, $\frac{\partial^2 f}{\partial x^2} = \boxed{\frac{y^2-x^2}{(x^2+y^2)^2}}$ ✓

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\ln\sqrt{x^2+y^2}] = \frac{\partial}{\partial y}[\ln(x^2+y^2)^{1/2}] = \frac{\partial}{\partial y}[\frac{1}{2}\ln(x^2+y^2)]$

$= \frac{1}{2}\left(\frac{1}{x^2+y^2}\right)\times(2y) = \frac{y}{x^2+y^2}$, $\frac{\partial^2 f}{\partial y^2} = \boxed{\frac{x^2-y^2}{(x^2+y^2)^2}}$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \boxed{0}$ ✓

77) $3x + 2y - 4 = f(x, y)$

Solution

1: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[3x + 2y - 4] = \boxed{3}$, $\frac{\partial^2 f}{\partial x^2} = \boxed{0}$

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[3x + 2y - 4] = \boxed{2}$, $\frac{\partial^2 f}{\partial y^2} = \boxed{0}$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 + 0 = \boxed{0} \checkmark$$

78) $f(x, y) = \tan^{-1} \frac{x}{y}$

Solution

1: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}\frac{x}{y}\right] = \left(\frac{1}{1+(x/y)^2}\right) \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{1+(x/y)^2} \times \frac{1}{y}$

$$\frac{1}{y\left(1 + \frac{x^2}{y^2}\right)} = \frac{1}{y + \frac{x^2}{y}} = 1 \div y + \frac{x^2}{y} = 1 \div \frac{y^2+x^2}{y} = \boxed{\frac{y}{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{y}{x^2+y^2}\right] = \frac{(x^2+y^2)(0) - (y)(2x)}{(x^2+y^2)^2} = \boxed{\frac{-2xy}{(x^2+y^2)^2}}$$

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = \frac{1}{1+\left(\frac{x}{y}\right)^2} \times \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{\frac{y^2+x^2}{y^2}} \times \frac{-x}{y^2}$

$$\frac{\partial f}{\partial y} = \boxed{\frac{-x}{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2+y^2)(0) - (-x)(2y)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = \boxed{0} \checkmark$$

78) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Solution

1: $\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left[x^2+y^2+z^2\right]^{-1/2} = -\dfrac{1}{2}\left[x^2+y^2+z^2\right]^{-3/2} \times \dfrac{\partial}{\partial x}\left[x^2+y^2+z^2\right]$

$= -\dfrac{1}{\cancel{2}}\left[x^2+y^2+z^2\right]^{-3/2} \times \left[\cancel{2}x\right] = \boxed{-x\left[x^2+y^2+z^2\right]^{-3/2}}$

$\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x} -\left[x^2+y^2+z^2\right]^{-3/2} = +\dfrac{3}{2}\left[x^2+y^2+z^2\right]^{-5/2} \times \left[2x\right] = \boxed{3x\left[x^2+y^2+z^2\right]^{-5/2}}$

2: $\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[x^2+y^2+z^2\right]^{-1/2} = -\dfrac{1}{2}(x^2+y^2+z^2)^{-3/2} \times (2y) = \boxed{-y(x^2+y^2+z^2)^{-3/2}}$

$\dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial}{\partial y}\left[-y(x^2+y^2+z^2)^{-3/2}\right] = (x^2+y^2+z^2)^{-3/2}(-1) + (-y)\left(-\dfrac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}(\cancel{2}y)\right)$

$= \boxed{-(x^2+y^2+z^2)^{-3/2} - 3y^2(x^2+y^2+z^2)^{-5/2}}$

3: $\dfrac{\partial f}{\partial z} = \dfrac{\partial}{\partial z}\left[x^2+y^2+z^2\right]^{-1/2} = -\dfrac{1}{2}\left[x^2+y^2+z^2\right]^{-3/2}(2z) = \boxed{-z\left[x^2+y^2+z^2\right]^{-3/2}}$

$\dfrac{\partial^2 f}{\partial z^2} = \dfrac{\partial}{\partial z}\left[-z(x^2+y^2+z^2)^{-3/2}\right] = (x^2+y^2+z^2)^{-3/2}(-1) + (-z)\left(-\dfrac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}(2z)\right)$

$= \boxed{-(x^2+y^2+z^2)^{-3/2} - z^2(x^2+y^2+z^2)^{-5/2}}$

$\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} + \dfrac{\partial^2 f}{\partial z^2} = 3x\left[x^2+y^2+z^2\right]^{-5/2} + (x^2+y^2+x^2)^{-3/2} - 3y^2(x^2+y^2+z^2)^{-5/2}$
$- (x^2+y^2+z^2)^{-3/2} - 3z^2(x^2+y^2+z^2)^{-5/2}$

$(x^2+y^2+z^2)^{-5/2}(3x - 3y^2 - 3z^2) - (x^2+y^2+z^2)^{-3/2} - (x^2+y^2+z^2)^{-3/2}$

78) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Solution

1: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2+y^2+z^2]^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x) = \boxed{-x[x^2+y^2+z^2]^{-3/2}}$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[-x(x^2+y^2+z^2)^{-3/2}] = (x^2+y^2+z^2)^{-3/2}(-1)+(-x)(-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2})\times(\cancel{2}x)$

$= \boxed{-(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}}$

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2+y^2+z^2]^{-1/2} = -\frac{1}{2}[x^2+y^2+z^2]^{-3/2}\times(2y) = \boxed{-y[x^2+y^2+z^2]^{-3/2}}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[-y(x^2+y^2+z^2)^{-3/2}] = (x^2+y^2+z^2)^{-3/2}(-1) + (-y)(-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}\times(\cancel{2}y))$

$= \boxed{-(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}}$

3: $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x^2+y^2+z^2]^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\times(2z) = \boxed{-z[x^2+y^2+z^2]^{-3/2}}$

$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}[-z(x^2+y^2+z^2)^{-3/2}] = (x^2+y^2+z^2)^{-3/2}(-1) + (-z)(-\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-5/2}\times(\cancel{2}z))$

$= \boxed{-(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}}$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}$
$+ 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$

$-(x^2+y^2+z^2)^{-3/2} - (x^2+y^2+z^2)^{-3/2} - (x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2} + 3y^2(x^2+y^2+z^2)^{-5/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$

$-3(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-5/2}(3x^2+3y^2+3z^2)$

$\boxed{-3(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-5/2}(3x^2+3y^2+3z^2) = 0}$ ✓

80) $f(x,y,z) = e^{3x+4y}\cos 5z$

Solution

1: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{3x+4y}\cos 5z\right] = 3e^{3x+4y}\cos 5z$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}3\left[e^{3x+4y}\cos 5z\right] = \boxed{9e^{3x+4y}\cos 5z}$

2: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{3x+4y}\cos 5z\right] = 4e^{3x+4y}\cos 5z$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}4\left[e^{3x+4y}\cos 5z\right] = \boxed{16e^{3x+4y}\cos 5z}$

3: $\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}16\left[e^{3x+4y}\cos 5z\right] = 5e^{3x+4y}\sin 5z$

$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}5\left[e^{3x+4y}\sin 5z\right] = -5\left[5e^{3x+4y}\cos 5z\right] = \boxed{-25e^{3x+4y}\cos 5z}$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 9e^{3x+4y}\cos 5z + 16e^{3x+4y}\cos 5z - 25e^{3x+4y}\cos 5z = \boxed{0}$

The wave Equation

- one-dimensional equation

$$\frac{\partial^2 w}{\partial t^2} = c^2\frac{\partial^2 w}{\partial x^2}$$

Show that the functions in Exercises 81–87 are all solutions of the wave equation.

81) $w = \sin(x+ct)$

Solution

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x+ct)\right] = \cos(x+ct)\times\frac{\partial}{\partial x}(x+ct) = \boxed{\cos(x+ct)}$

$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct)\right] = -\sin(x+ct)\times\frac{\partial}{\partial x}(x+ct) = \boxed{-\sin(x+ct)}$

2: $\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct)\right] = \cos(x+ct)\times\frac{\partial}{\partial t}(x+ct) = \boxed{c\cos(x+ct)}$

$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}c\left[\cos(x+ct)\right] = (c)(-\sin(x+ct))\times(c) = \boxed{-c^2\sin(x+ct)}$

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

$$\frac{\partial^2 w}{\partial t^2} = (c^2)\left(-\sin(x+ct)\right)$$

$$\boxed{\frac{\partial^2 w}{\partial t} = -c^2(\sin(x+ct))}$$ showed

82) $w = \cos(2x+2ct)$

Solution

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}[\cos(2x+2ct)] = \boxed{-\sin(2x+2ct)(2)}$

$\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x} 2[-\sin(2x+2ct)] = -(2)(+\cos(2x+2ct))(2) = \boxed{-4\cos(2x+2ct)}$

2: $\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}[\cos(2x+2ct)] = -\sin(2x+2ct)(2c)$

$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t} 2c[-\sin(2x+2ct)] = (-2c)(\cos(2x+2ct))(2c)$
$= \boxed{-4c^2\cos(2x+2ct)}$

$\frac{\partial^2 w}{\partial x^2} = -4\cos(2x+2ct)$; $\frac{\partial^2 w}{\partial t^2} = -4c^2\cos(2x+2ct)$,

$\frac{\partial w}{\partial t^2} = (c^2)(\underline{-4\cos(2x+2ct)})$

$$\boxed{\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}}$$ ✓

83) $w = \sin(x + ct) + \cos(2x + 2ct)$

Solution

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\sin(x + ct) + \cos(2x + 2ct)\right]$

$= \cos(x + ct) + \sin(2x + 2ct)(2) = \boxed{\cos(x+ct) - 2\sin(2x + 2ct)}$

2: $\frac{\partial^2 w}{\partial x^2} = \frac{\partial}{\partial x}\left[\cos(x+ct) - 2\sin(2x + 2ct)\right]$

$= \boxed{-\sin(x+ct) - 4\cos(2x+2ct)}$

3: $\frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left[\sin(x+ct) + \cos(2x+2ct)\right]$

$= \cos(x+ct)(c) - \sin(2x+2ct)(2c) = \boxed{\cos(x+ct) - 2\sin(2x+2ct)}$

4: $\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial t}\left[(c)\cos(x+ct) - 2c\sin(2x+2ct)\right]$

$= \left[-\sin(x+ct)(c)(c) - (2c)\cos(2x+2ct)(2c)\right]$

$= \left[-\sin(x+ct)c^2 - 4c^2\cos(2x+2ct)\right]$

$= \boxed{c^2\left[(-\sin(x+ct) + 4\cos(2x+2ct)\right]}$

$\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - 4\cos(2x+2ct)$

$\frac{\partial^2 w}{\partial t^2} = c^2\left(-\sin(x+ct) - 4\cos(2x+2ct)\right)$

$\boxed{\frac{\partial^2 w}{\partial t} = c^2 \frac{\partial^2 w}{\partial x^2}}$ ✓

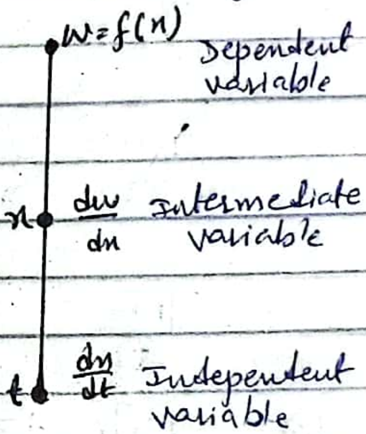
14.4 Chain Rule

■ **Function of two variables**

The chain Rule formula for a differentiable function $w = f(x,y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

▸ **Chain Rule For Function of One Independent Variable and Two intermediate Variables:**

If $w = f(x,y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then composite $w = f(x(t), y(t))$ is a differentiable function of t and



$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

rewritten,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

■ **Function of Three Variables**

▸ **Chain Rule for Functions of One independent Variable and Three intermediate variables** If $w = f(x,y,z)$ is differentiable and $x, y,$ and z are differentiable functions of t, then w is a differentiable function of t and

Dependent variable

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

Intermediate variables

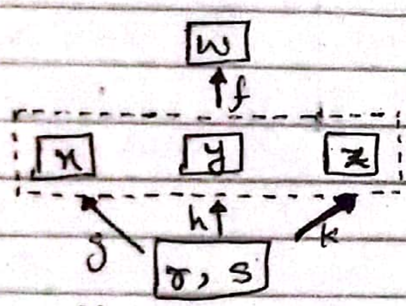
Independent variable

■ Functions Defined on Surfaces

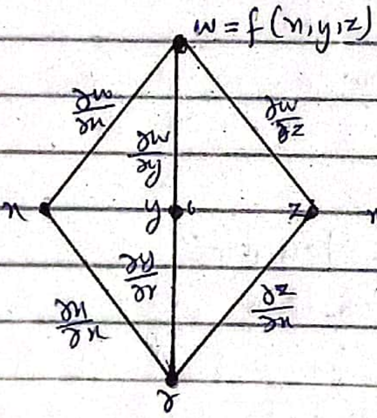
▶ **Chain Rule for Two Independent Variables and Three Intermediate Variables** Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas:

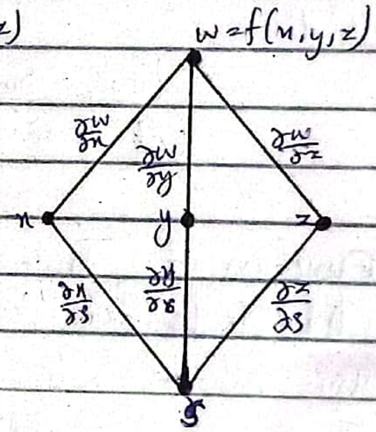
$$\boxed{\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}}$$

$$\boxed{\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}}$$



$w = f(g(r,s), h(r,s), k(r,s))$



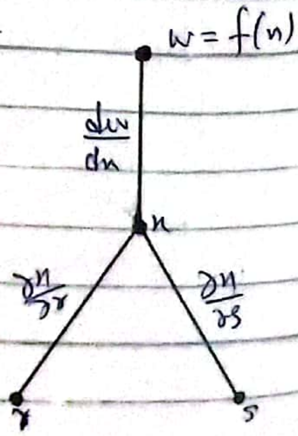


• If $w = f(x, y)$, $x = g(r, s)$, and $y = h(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s}$$

• If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$



■ Formula For implicit differentiation Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y)=0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = \frac{-F_x}{F_y}, \quad F_y \neq 0$$

14.4 Exercises

■ Chain Rule: One Independent variable

In Exercises 1–6, (a) express dw/dt as a function of t, both by using the chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then (b) evalute dw/dt at the given value of t.

1) $w = x^2 + y^2$, $x = \cos t$, $y = \sin t$, $t = \pi$

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

$$\frac{dw}{dt} = \frac{\partial}{\partial x}[x^2+y^2] \times \frac{d}{dt}[\cos t] + \frac{\partial}{\partial y}[x^2+y^2] \times \frac{d}{dt}[\sin t]$$

$$\frac{dw}{dt} = (2x)(-\sin t) + (2y)(\cos t) \quad , \quad t = \pi, \ x = \cos t, \ y = \sin t$$

$$\frac{dw}{dt} = 2(\cos(\pi))(-\sin \pi) + 2(\sin(\pi))(\cos(\pi))$$

$$\frac{dw}{dt} = 2(-1)(0) + 2(0)(-1) = 0 \ ; \quad \boxed{\frac{dw}{dt} = 0}$$

2) $w = x^2 + y^2$; $x = \cos t + \sin t$, $y = \cos t - \sin t$; $t = 0$

Solution

$$\frac{dw}{dt} = \frac{\partial}{\partial x}[x^2+y^2]\frac{d}{dt}[\cos t + \sin t] + \frac{\partial}{\partial y}[x^2+y^2]\frac{d}{dt}[\cos t - \sin t]$$

$$\frac{dw}{dt} = (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t)$$

$$\frac{dw}{dt} = 2(\cos t + \sin t)(-\sin t + \cos t) + 2(\cos t - \sin t)(-\sin t - \cos t)$$

$$\frac{dw}{dt} = 2(\cos(0) + \sin(0))(-\sin(0) + \cos(0)) + 2(\cos(0) - \sin(0))(-\sin(0) - \cos(0))$$

$$\frac{dw}{dt} = 2(1+0)(-0+1) + 2(1-0)(-0-1) = 2 + (-2) = 0$$

$$\boxed{\frac{dw}{dt} = 0}$$

3) $w = \frac{x}{z} + \frac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = \frac{1}{t}$; $t = 3$

Solution $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$\frac{dw}{dt} = \frac{\partial}{\partial x}\left[\frac{x}{z} + \frac{y}{z}\right]\frac{d}{dt}[\cos^2 t] + \frac{\partial}{\partial y}\left[\frac{x}{z} + \frac{y}{z}\right]\frac{d}{dt}[\sin^2 t]$

$\frac{dw}{dt} = \left[\frac{1}{z}\right][-2\cos t \sin t] + \left[\frac{1}{z}\right][2\sin t \cos t]$

$\frac{dw}{dt} = \frac{-2\cos t \sin t + 2\sin t \cos t}{z} = \frac{0}{z} = 0$

3 - variables

(a) $\frac{dw}{dt} = \frac{\partial}{\partial x}\left[\frac{x}{z} + \frac{y}{z}\right]\frac{d}{dt}[\cos^2 t] + \frac{\partial}{\partial y}\left[\frac{x}{z} + \frac{y}{z}\right]\frac{d}{dt}[\sin^2 t]$

$\frac{dw}{dt} = \left(\frac{1}{z}\right)(-2\cos t \sin t) + \left(\frac{1}{z}\right)(2\sin t \cos t)$

$\frac{dw}{dt} = \frac{-2\cos t \sin t}{z} + \frac{2\sin t \cos t}{z} = \frac{-2\cos t \sin t}{1/t} + \frac{2\sin t \cos t}{1/t}$

$\Rightarrow \boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$

$\frac{dw}{dt} = \frac{\partial}{\partial x}\left[\frac{x}{z} + \frac{y}{z}\right]\frac{d}{dt}[\cos^2 t] + \frac{\partial}{\partial y}\left[\frac{x}{z} + \frac{y}{z}\right]\frac{d}{dt}[\sin^2 t] + \frac{\partial}{\partial z}\left[\frac{x}{z} + \frac{y}{z}\right]\frac{d}{dt}\left[\frac{1}{t}\right]$

$\frac{dw}{dt} = \left(\frac{1}{z}\right)(-2\cos t \sin t) + \left(\frac{1}{z}\right)(2\sin t \cos t) + \left(\frac{z(0) - (1)(1)}{z^2} + \frac{z(0) - (1)(1)}{z^2}\right)\left(\frac{t(0) - (1)(1)}{t^2}\right)$

$\frac{dw}{dt} = \frac{-2\cos t \sin t}{z} + \frac{2\sin t \cos t}{z} + \left(-\frac{x}{z^2} - \frac{y}{z^2}\right)\left(-\frac{1}{t^2}\right)$

$\frac{dw}{dt} = \frac{-2\cos t \sin t}{z} + \frac{2\sin t \cos t}{z} + \frac{x}{(zt)^2} + \frac{y}{(zt)^2}$, $z = 1/t$, $t = 3$

$\frac{dw}{dt} = \frac{-2\cos t \sin t}{1/t} + \frac{2\sin t \cos t}{1/t} + \frac{x}{(\frac{t}{t})^2} + \frac{y}{(\frac{t}{t})^2}$

$\frac{dw}{dt} = \cancel{t(-2\cos t \sin t) + t(2\sin t \cos t)} + x + y$

$\frac{dw}{dt} = x + y = \cos^2 t + \sin^2 t = \boxed{1}$

4) $w = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, $t = 3$

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = \frac{\partial}{\partial x}\left[\ln(x^2+y^2+z^2)\right]\frac{d}{dt}[\cos t] + \frac{\partial}{\partial y}\left[\ln(x^2+y^2+z^2)\right]\frac{d}{dt}[\sin t] + \frac{\partial}{\partial z}\left[\ln(x^2+y^2+z^2)\right]\frac{d}{dt}[4\sqrt{t}]$$

$$\frac{dw}{dt} = \left(\frac{1}{x^2}\right)(-\sin t) + \left(\frac{1}{y^2}\right)(\cos t) + \left(\frac{1}{z^2}\right)\left(\frac{2}{\sqrt{t}}\right)$$

$$\frac{dw}{dt} = \frac{-\sin t}{x^2} + \frac{\cos t}{y^2} + \frac{2}{z^2\sqrt{t}} = \frac{-\sin t}{\cos^2 t} + \frac{\cos t}{\sin^2 t} + \frac{2}{(4\sqrt{t})^2\sqrt{t}}$$

$$\frac{dw}{dt} = \frac{-\sin t}{\cos^2 t} + \frac{\cos t}{\sin t} + \frac{2}{16t\sqrt{t}}$$

4) $$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = \frac{\partial}{\partial x}\left[\ln(x^2+y^2+z^2)\right]\frac{d}{dt}[\cos t] + \frac{\partial}{\partial y}\left[\ln(x^2+y^2+z^2)\right]\frac{d}{dt}[\sin t] + \frac{\partial}{\partial z}\left[\ln(x^2+y^2+z^2)\right]\frac{d}{dt}[4\sqrt{t}]$$

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\ln(x^2+y^2+z^2)\right] = \left[\frac{1}{x^2+y^2+z^2}\right] \times \frac{\partial}{\partial x}\left[(x^2+y^2+z^2)\right] = \boxed{\frac{2x}{x^2+y^2+z^2}}$

→ Derivative of Base then Derivative of power

2: $\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left[\ln(x^2+y^2+z^2)\right] = \left(\frac{1}{x^2+y^2+z^2}\right) \times \frac{\partial}{\partial y}(x^2+y^2+z^2) = \boxed{\frac{2y}{x^2+y^2+z^2}}$

3: $\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left[\ln(x^2+y^2+z^2)\right] = \left(\frac{1}{x^2+y^2+z^2}\right)\left(\frac{\partial}{\partial z}(x^2+y^2+z^2)\right) = \boxed{\frac{2z}{x^2+y^2+z^2}}$

4: $\frac{dx}{dt} = \frac{d}{dt}[\cos t] = \boxed{-\sin t}$; 5: $\frac{dy}{dt} = \frac{d}{dt}[\sin t] = \boxed{\cos t}$

6: $\frac{dz}{dt} = \frac{d}{dt}[4\sqrt{t}] = \boxed{\frac{2}{\sqrt{t}}}$

$$\frac{dw}{dt} = \left(\frac{2x}{x^2+y^2+z^2}\right)(-\sin t) + \left(\frac{2y}{x^2+y^2+z^2}\right)(\cos t) + \left(\frac{2z}{x^2+y^2+z^2}\right)\left(\frac{2}{\sqrt{t}}\right)$$

$$\frac{dw}{dt} = \frac{-2x\sin t}{x^2+y^2+z^2} + \frac{2y\cos t}{x^2+y^2+z^2} + \frac{4z}{\sqrt{t}(x^2+y^2+z^2)}$$

$$\frac{dw}{dt} = \frac{-2x\sin t + 2y\cos t + 4zt^{-1/2}}{x^2+y^2+z^2} = \frac{-2(\cos t)\sin t + 2\sin t\cos t + 16}{\cos^2 t + \sin^2 t + 16t}$$

$$\frac{dw}{dt} = \frac{-2\cos t\cancel{\sin t} + 2\cancel{\sin t}\cos t + 16}{1+16t} = \frac{16}{1+16t} = \frac{16}{1+16(3)} = \boxed{\frac{16}{48}}$$

5) $w = 2ye^x - \ln z$, $x = \ln(t^2+1)$, $y = \tan^{-1}t$, $z = e^t$; $t = 1$

Solution

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}[2ye^x - \ln z] = 2ye^x - 0 = \boxed{2ye^x}$

2: $\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}[2ye^x - \ln z] = 2e^x - 0 = \boxed{2e^x}$

3: $\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}[2ye^x - \ln z] = 0 - \frac{1}{z} = \boxed{\frac{-1}{z}}$

4: $\frac{dx}{dt} = \frac{d}{dt}[\ln(t^2+1)] = \frac{1}{t^2+1} \times \frac{d}{dt}(t^2+1) = \boxed{\frac{2t}{t^2+1}}$

5: $\frac{dy}{dt} = \frac{d}{dt}[\tan^{-1}t] = \left(\frac{1}{1+t^2}\right) \times \frac{d}{dt}(t) = \boxed{\frac{1}{1+t^2}}$

6: $\frac{dz}{dt} = \frac{d}{dt}[e^t] = \boxed{e^t}$

$$\frac{dw}{dt} = (2ye^x)\left(\frac{2t}{t^2+1}\right) + (2e^x)\left(\frac{1}{1+t^2}\right) + \left(\frac{-1}{z}\right)(e^t)$$

$$\frac{dw}{dt} = \frac{4ye^x t}{t^2+1} + \frac{2e^x}{1+t^2} - \frac{e^t}{z}$$

$$\frac{dw}{dt} = \frac{4(\tan^{-1}t)e^{\ln(t^2+1)}}{2} + \frac{2e^{\ln(t^2+1)}}{2} - \frac{e}{e^t}$$

$$\frac{dw}{dt} = \frac{4(\tan^{-1}(1))e^{\ln(2)}}{2} + \frac{2e^{\ln(2)}}{2} - \frac{e}{e}$$

base $\to e$

$\ln_e$

$$\frac{dw}{dt} = 2(\tan^{-1}(t))e^{\ln(2)} + e^{\ln(2)} - 1 = 2(\tan^{-1}t)(2) + 2 - 1$$

$$= (2)2\tan^{-1}t + 1$$

$$= \boxed{4\tan^{-1}t + 1}$$

6) $w = z - \sin xy$, $x = t$, $y = \ln t$, $z = e^{t-1}$; $t = 1$

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}[z - \sin xy] = (0 - \cos xy) \times \frac{\partial}{\partial x}(xy) = \boxed{-y\cos xy}$

2: $\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}[z - \sin xy] = [0 - \cos xy] \times \frac{\partial}{\partial y}(xy) = \boxed{-x\cos xy}$

3: $\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}[z - \sin xy] = [1 - 0] = \boxed{1}$

4: $\frac{dx}{dt} = \frac{d}{dt}[t] = 1$; 5: $\frac{dy}{dt} = \frac{d}{dt}[\ln t] = \boxed{\frac{1}{x}}$

6: $\frac{dz}{dt} = \frac{d}{dt}[e^{t-1}] = (e^{t-1}) \times \frac{d}{dt}[t-1] = \boxed{e^{t-1}}$

$$\frac{dw}{dt} = (-y\cos xy)(1) + (-x\cos xy)\left(\frac{1}{x}\right) + (1)(e^{t-1})$$

$$\frac{dw}{dt} = -y\cos xy - \frac{x\cos xy}{x} + 0 = \boxed{-y\cos xy - \cos xy} + e^{t-1}$$

$$\frac{dw}{dt} = -y\cos xy - \cos xy + e^{t-1}, \quad t = 1$$

$$\frac{dw}{dt} = -\cos xy\,(y+1) = -\cos(t\ln t)(\ln t + 1) \qquad , \boxed{\ln 1 = 0}$$

$$= -\cos(\ln 1)(\ln 1 + 1) = -\cos(0)(0+1) = -1$$

$$\frac{dw}{dt} = -y\cos xy - \cos xy + e^{t-1}$$

6) $w = z - \sin xy$, $x = t$, $y = \ln t$, $z = e^{t-1}$; $t = 1$

Solution

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}[z - \sin xy] = (0 - \cos xy) \times \frac{\partial}{\partial x}(xy) = \boxed{-y\cos xy}$

2: $\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}[z - \sin xy] = (0 - \cos xy) \times \frac{\partial}{\partial y}(xy) = \boxed{-x\cos xy}$

3: $\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}[z - \sin xy] = (1 - 0) = \boxed{1}$

4: $\frac{dx}{dt} = \frac{d}{dt}[t] = \boxed{1}$; 5: $\frac{dy}{dt} = \frac{d}{dt}[\ln t] = \boxed{\frac{1}{t}}$

6: $\frac{dz}{dt} = \frac{d}{dt}[e^{t-1}] = e^{t-1} \times \frac{d}{dt}(t-1) = \boxed{e^{t-1}}$

$$\frac{dw}{dt} = (-y\cos xy)(1) + (-x\cos xy)\left(\frac{1}{t}\right) + (1)(e^{t-1})$$

$$\frac{dw}{dt} = -y\cos xy - \frac{x\cos xy}{t} + e^{t-1}$$

$$\frac{dw}{dt} = -(\ln t)\cos(t\ln t) - \frac{(\cancel{t})\cos(t\ln t)}{\cancel{t}} + e^{t-1}$$

$$\boxed{\frac{dw}{dt} = -\ln t\cos(t\ln t) - \cos(t\ln t) + e^{t-1}}, \quad t = 1$$

$$\frac{dw}{dt} = -\ln(1)\cos(1\ln 1) - \cos(1\ln 1) + e^{1-1}$$

$$\frac{dw}{dt} = -(0)\cos(0) - \cos(0) + e^{0} = 0 - 1 + 1 = \boxed{0}$$

Chain Rule : Two and Three Independent variables

In Exercises 7 and 8, (a) express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the chain Rule and by expressing z directly in terms of u and v before differentiating. Then (b) evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

7) $z = 4e^x \ln y$, $x = \ln(u\cos v)$, $y = u\sin v$; $(u,v) = (2, \pi/4)$

Solution

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \quad ; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}$$

$$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

1: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[4e^x \ln y] = 4e^x \ln y$; $\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}[\ln(u\cos v)] = \frac{1}{u\cos v} \times \cos v = \frac{\cos v}{u\cos v}$

2: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[4e^x \ln y] = \frac{4e^x}{y}$; $\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}[u\sin v] = \sin v$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4e^x \ln y)\left(\frac{\cancel{\cos v}}{u\cancel{\cos v}}\right) + \left(\frac{4e^x}{y}\right)(\sin v)$$

$$= \frac{4e^x \ln y}{u} + \frac{4e^x \sin v}{y} = \frac{4e^{\ln(u\cos v)}\ln(u\sin v)}{u} + \frac{4e^{\ln(u\cos v)}\sin v}{u\sin v}$$

$$= \frac{4(\cancel{u}\cos v)\ln(u\sin v)}{\cancel{u}} + \frac{4(\cancel{u}\cos v)\cancel{\sin v}}{\cancel{u}\cancel{\sin v}} = \boxed{4\cos v\ln(u\sin v) + 4\cos v}$$

$$\Rightarrow \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

1: $\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[4e^x \ln y] = 4e^x \ln y$; $\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}[\ln(u\cos v)] = \frac{1}{u\cos v} \times -u\sin v = \frac{-u\sin v}{u\cos v}$

2: $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[4e^x \ln y] = \frac{4e^x}{y}$; $\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}[u\sin v] = u\cos v$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} = (4e^x \ln y)\left(\frac{-u\sin v}{u\cos v}\right) + \left(\frac{4e^x}{y}\right)(u\cos v)$$

$$= (4e^x \ln y)\left(\frac{-\sin v}{\cos v}\right) + \frac{4e^x u\cos v}{y} = \left(4e^{\ln(u\cos v)}\ln(u\sin v)\right)\frac{-\sin v}{\cos v} + \frac{4e^{\ln(u\cos v)}u\cos v}{u\sin v}$$

$$= 4(u\cos v)\ln(u\sin v)\left(\frac{-\sin v}{\cos v}\right) + \frac{4(u\cos v)u\cos v}{u\sin v}$$

$$= 4(u\cos v)\ln(u\sin v)\left(-\frac{\sin v}{\cos v}\right) + \frac{4(u\cos v)(u\cos v)}{u\sin v}$$

$$= \boxed{-4u\sin v\ln(u\sin v) + \frac{4u\cos^2 v}{\sin v}} \qquad (u,v) = (2, \pi/4)$$

$$\boxed{\frac{\partial z}{\partial v}} = -4(2)\sin\left(\frac{\pi}{4}\right)\ln\left(2\sin\left(\frac{\pi}{4}\right)\right) + 4(2)\cos^2\left(\frac{\pi}{4}\right) \div \sin(\pi/4)$$

$$= -8\left(\frac{1}{\sqrt2}\right)\ln\left(2\times\frac{1}{\sqrt2}\right) + 8\left(\frac{1}{\sqrt2}\right)^2 \Big/ \frac{1}{\sqrt2} = \frac{-8}{\sqrt2}\ln\left(\frac{2}{\sqrt2}\right) + 4\sqrt2$$

$$= \frac{-8}{\sqrt2}\ln\left(\frac{2}{\sqrt2}\right) + 4\sqrt2 = -4\left(\frac{2}{\sqrt2}\right)\ln\left(\frac{2}{\sqrt2}\right) + 4\sqrt2 = \boxed{-4\sqrt2\ln\sqrt2 + 4\sqrt2}$$

$$\because \frac{2}{\sqrt2} = \sqrt2$$

$$\boxed{\frac{\partial z}{\partial u}} = 4\cos v\ln(u\sin v) + 4\cos v \qquad (u,v) = (2, \pi/4)$$

$$= 4\cos\left(\frac{\pi}{4}\right)\ln(2\sin(\pi/4)) + 4\cos(\pi/4)$$

$$= 4\times\frac{1}{\sqrt2}\ln\left(2\,\frac{1}{\sqrt2}\right) + 4\left(\frac{1}{\sqrt2}\right) = \frac{2}{\sqrt2}\ln\left(\frac{2}{\sqrt2}\right) + \frac{4}{\sqrt2}$$

$$= 2\sqrt2\ln\sqrt2 + 2\sqrt2 = \boxed{2\sqrt2(\ln\sqrt2 + 1)}$$

8) $z = \tan^{-1}(x/y)$, $x = u\cos v$, $y = u\sin v$; $(u,v) = (1.3, \pi/6)$

Solution

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\ ; \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$\Rightarrow \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

1: $$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\tan^{-1}(x/y)] = \frac{1}{1+(x/y)^2}\times\frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y(1+(x/y)^2)} = \boxed{\frac{y}{y^2+x^2}}$$

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}[u\cos v] = \boxed{\cos v}$$

2: $$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[\tan^{-1}(x/y)] = \frac{1}{1+(x/y)^2}\times\frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{1}{1+(x/y)^2}\times\frac{((y)(0)-(1)(1))}{y^2} = \boxed{\frac{-x}{y^2(1+(x/y)^2)}}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}[u\sin v] = \boxed{\sin v}$$

$$\Rightarrow \frac{\partial z}{\partial u} = \left(\frac{y}{y^2+x^2}\right)(\cos v) + \left(\frac{-x}{y^2(1+(x/y)^2)}\right)(\sin v) = \frac{y\cos v}{y^2+x^2} - \frac{x\sin v}{y^2+x^2}$$

$$= \frac{y\cos v}{x^2+y^2} - \frac{\sin v}{x^2+y^2} \ldots$$

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$$\frac{\partial z}{\partial u} = \frac{y\cos v}{x^2+y^2} - \frac{x\sin v}{x^2+y^2} \Rightarrow (u,v) = (1.3,\ \pi/6)$$

$$\frac{\partial z}{\partial x} = \frac{u\sin v\cos v}{(u\cos v)^2+(u\sin v)^2} - \frac{(u\cos v)\sin v}{(u\cos v)^2+(u\sin v)^2}$$

$$\frac{\partial z}{\partial x} = \frac{u\sin v\cos v - u\cos v\sin v}{(u\cos v)^2+(u\sin v)^2} = \frac{0}{(u\cos v)^2+(u\sin v)^2} = \boxed{0}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

$$1: \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}(x/y)\right] = \frac{1}{1+(x/y)^2} \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y(1+(x/y)^2)} = \boxed{\frac{y}{y^2+x^2}}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}[u\cos v] = \boxed{-u\sin v}$$

$$2: \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\tan^{-1}(x/y)\right] = \frac{1}{1+(x/y)^2} \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{-x}{y^2(1+x^2/y^2)} = \boxed{\frac{-x}{y^2+x^2}}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}[u\sin v] = \boxed{u\cos v}$$

$$\frac{\partial z}{\partial v} = \left(\frac{y}{y^2+x^2}\right)\left(-u\sin v\right) + \left(\frac{-x}{y^2+x^2}\right)(u\cos v)$$

$$= \frac{-yu\sin v}{y^2+x^2} - \frac{xu\cos v}{y^2+x^2} = \frac{-(u\sin v)\sin v - (u\cos v)\cos v}{y^2+x^2}$$

$$= \frac{-u\sin^2 v - u\cos^2 v}{(u\sin v)^2+(u\cos v)^2} = \frac{-((u\sin v)^2+(u\cos v)^2)}{(u\sin v)^2+(u\cos v)^2} = \boxed{-1}$$

$\partial w/\partial u$ and $\partial w/\partial v$

8) $w = xy + yz + xz$, $x = u+v$, $y = u-v$, $z = uv$; $(u,v) = (1/2, 1)$

Solution

$\Rightarrow \dfrac{\partial w}{\partial u} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial u} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial u}$

1: $\dfrac{\partial w}{\partial x} = \dfrac{\partial}{\partial x}[xy + yz + xz] = [y + 0 + z] = \boxed{y+z}$

$\dfrac{\partial x}{\partial u} = \dfrac{\partial}{\partial u}[u+v] = \boxed{1}$

2: $\dfrac{\partial w}{\partial y} = \dfrac{\partial}{\partial y}[xy + yz + xz] = [x + z] = \boxed{x+z}$

$\dfrac{\partial y}{\partial u} = \dfrac{\partial}{\partial u}[u-v] = \boxed{1}$

3: $\dfrac{\partial w}{\partial z} = \dfrac{\partial}{\partial z}[xy + yz + xz] = [0 + y + x] = \boxed{x+y}$

$\dfrac{\partial z}{\partial u} = \dfrac{\partial}{\partial u}[uv] = \boxed{v}$

$\dfrac{\partial w}{\partial u} = (y+z)(1) + (x+z)(1) + (x+y)(v) = y+z+x+z+(x+y)v$

$= x+y+2z+(x+y)v = u+\cancel{v}+u-\cancel{v}+2uv+(u+\cancel{v}+u-\cancel{v})v$

$= 2u+2uv+(2u)v = 2u+2uv+2uv = \boxed{2u+4uv}$

$\boxed{\dfrac{\partial w}{\partial u} = 2u+4uv}$ $\quad (u,v) = (1/2, 1)$

$\dfrac{\partial w}{\partial u} = 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{2}\right)(1) = 1 + 2 = \boxed{3}$

$\Rightarrow \dfrac{\partial w}{\partial v} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial v} + \dfrac{\partial w}{\partial z}\dfrac{\partial z}{\partial v}$

1: $\dfrac{\partial w}{\partial x} = \underline{y+z}$; $\dfrac{\partial x}{\partial v} = \underline{1}$, 2: $\dfrac{\partial w}{\partial y} = \underline{x+z}$, $\dfrac{\partial y}{\partial v} = \underline{-1}$

3: $\dfrac{\partial w}{\partial z} = \underline{x+y}$, $\dfrac{\partial z}{\partial v} = \underline{u}$

$\dfrac{\partial w}{\partial v} = y+\cancel{z}-x-\cancel{z}+ux+uy = y-x+ux+uy = \cancel{u}-v-\cancel{u}-v+u^2+\cancel{uv}+u^2-\cancel{uv}$

$= \boxed{-2v+2u^2} \longrightarrow -2(1)+2\left(\frac{1}{2}\right)^2 = -2+\frac{2}{4} = -2+\frac{1}{2} = \boxed{-\frac{3}{2}}$

10) $w = \ln(x^2+y^2+z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$

$(u,v) = (-2,0)$

Solution

① $\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$

1: $\frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left[\ln(x^2+y^2+z^2)\right] = \frac{1}{x^2+y^2+z^2} \times \frac{\partial}{\partial x}(x^2+y^2+z^2) = \boxed{\frac{2x}{x^2+y^2+z^2}}$

$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left[ue^v \sin u\right] = \sin u(e^v) + (ue^v)(\cos u) = \boxed{e^v \sin u + ue^v \cos u}$

2: $\frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left[\ln(x^2+y^2+z^2)\right] = \frac{1}{x^2+y^2+z^2} \times \frac{\partial}{\partial y}(x^2+y^2+z^2) = \boxed{\frac{2y}{x^2+y^2+z^2}}$

$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left[ue^v \cos u\right] = \cos u(e^v) + (ue^v)(-\sin u) = \boxed{e^v \cos u - ue^v \sin u}$

3: $\frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left[\ln(x^2+y^2+z^2)\right] = \frac{1}{x^2+y^2+z^2} \times \frac{\partial}{\partial z}(x^2+y^2+z^2) = \boxed{\frac{2z}{x^2+y^2+z^2}}$

$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\left[ue^v\right] = \boxed{e^v}$

$$\frac{\partial w}{\partial u} = \left(\frac{2x}{x^2+y^2+z^2}\right)(e^v \sin u + ue^v \cos u) + \frac{2y(e^v \cos u - ue^v \sin u)}{x^2+y^2+z^2} + \frac{2z(e^v)}{x^2+y^2+z^2}$$

$$\frac{\partial w}{\partial u} = \frac{2x(e^v \sin u + ue^v \cos u) + 2y(e^v \cos u - ue^v \sin u) + 2ze^v}{x^2+y^2+z^2}$$

$$= \frac{2(ue^v \sin u)(e^v \sin u + ue^v \cos u) + 2(ue^v \cos u)(e^v \cos u - ue^v \sin u) + 2ue^v e^v}{(ue^v \sin u)^2 + (ue^v \cos u)^2 + (ue^v)^2}$$

$$= \frac{2ue^{2v}\sin^2 u + \cancel{2u^2e^{2v}\sin u \cos u} + 2ue^{2v}\cos^2 u - \cancel{2u^2e^{2v}\cos u \sin u} + 2ue^{2v}}{(ue^v \sin u)^2 + (ue^v \cos u)^2 + (ue^v)^2}$$

$$= \frac{2ue^{2v}\sin^2 u + 2ue^{2v}\cos^2 u + 2ue^{2v}}{(ue^v \sin u)^2 + (ue^v \cos u)^2 + (ue^v)^2}$$

$$= \frac{2u\left(\cancel{(e^v \sin u)^2 + (e^v \cos u)^2 + (e^v)^2}\right)}{u^2\left(\cancel{(e^v \sin u)^2 + (e^v \cos u)^2 + (e^v)^2}\right)} = \frac{2u}{u^2} = \boxed{\frac{2}{u}}$$

$$\frac{\partial w}{\partial u} = \frac{2}{u} = \frac{2}{-2} = \boxed{-1}$$

10) $\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$

1: $\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2}$, $\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}[ue^v \sin u] = \boxed{ue^v \sin u}$

2: $\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}$, $\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}[ue^v \cos u] = \boxed{ue^v \cos u}$

3: $\frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$, $\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}[ue^v] = \boxed{ue^v}$

$\frac{\partial w}{\partial v} = \frac{(2x)(ue^v \sin u)}{x^2+y^2+z^2} + \frac{(2y)(ue^v \cos u)}{x^2+y^2+z^2} + \frac{(2z)(ue^v)}{x^2+y^2+z^2}$

$= \frac{(2ue^v \sin u)(ue^v \sin u) + 2(ue^v \cos u)(ue^v \cos u) + 2(ue^v)(ue^v)}{(ue^v \sin u)^2 + (ue^v \cos u)^2 + (ue^v)^2}$

$= \frac{2u^2e^{2v}\sin^2 u + 2u^2e^{2v}\cos^2 u + 2u^2e^{2v}}{(ue^v \sin u)^2 + (ue^v \cos u)^2 + (ue^v)^2}$

$= \frac{2(ue^v \sin u)^2 + 2(ue^v \cos u)^2 + 2(ue^v)^2}{(ue^v \sin u)^2 + (ue^v \cos u)^2 + (ue^v)^2}$

$= \frac{2[\cancel{(ue^v \sin u)^2} + \cancel{(ue^v \cos u)^2} + \cancel{(ue^v)^2}]}{\cancel{(ue^v \sin u)^2} + \cancel{(ue^v \cos u)^2} + \cancel{(ue^v)^2}} = \boxed{2}$

• In Exercises 11 and 12, (a) express $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ as functions of x, y, and z both by using the Chain Rule and by expressing u directly in terms of x, y, and z before differentiating. Then (b) evaluate $\partial u/\partial x$, $\partial u/\partial y$, and $\partial u/\partial z$ at the given point (x, y, z)

11) $u = \frac{p-q}{q-r}$, $p = x+y+z$, $q = x-y+z$, $r = x+y-z$, $(x, y, z) = (\sqrt{3}, 2, 1)$

Solution

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial x}$

1: $\frac{\partial u}{\partial p} = \frac{\partial}{\partial p}\left[\frac{p-q}{q-r}\right] = \frac{1}{q-r}[p-q] = \boxed{\frac{1}{q-r}}$

$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}[x+y+z] = \boxed{1}$

$$2: \frac{\partial u}{\partial q} = \frac{\partial}{\partial q}\left[\frac{p-q}{q-r}\right] = \left[\frac{(q-r)(-1)-(p-q)(1)}{(q-r)^2}\right] = \frac{(r-q)+(q-p)}{(q-r)^2}$$

$$\frac{\partial q}{\partial x} = \frac{\partial}{\partial x}[x-y+z] = \boxed{1}$$

$$3: \frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left[\frac{p-q}{q-r}\right] = (p-q)\left[\frac{1}{q-r}\right] = (p-q)\left[\frac{(q-r)(0)-(1)(-1)}{(q-r)^2}\right]$$

$$= \boxed{\frac{p-q+0}{(q-r)^2}}$$

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}[x+y+z] = \boxed{1}$$

$$\frac{\partial u}{\partial x} = \left(\frac{1}{q-r}\right)(1) + \left(\frac{(r-q)+(q-p)}{(q-r)^2}\right)(1) + \left(\frac{p-q}{(q-r)^2}\right)(1)$$

$$= \frac{1}{q-r} + \frac{r-p}{(q-r)^2} + \frac{p-q}{(q-r)^2} = \frac{\cancel{q}-\cancel{r}+\cancel{r}-\cancel{p}+\cancel{p}-\cancel{q}}{(q-r)^2} = \boxed{0}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$1: \frac{\partial u}{\partial p} = \frac{\partial}{\partial p}\left[\frac{p-q}{q-r}\right] = \frac{1}{q-r}\frac{\partial}{\partial p}[p-q] = \boxed{\frac{1}{q-r}}$$

$$\frac{\partial p}{\partial y} = \frac{\partial}{\partial y}[x+y+z] = \boxed{1}$$

$$2: \frac{\partial u}{\partial q} = \frac{\partial}{\partial q}\left[\frac{p-q}{q-r}\right] = \frac{(q-r)(-1)-(p-q)(1)}{(q-r)^2} = \frac{(r-\cancel{q})-p+\cancel{q}}{(q-r)^2} = \boxed{\frac{r-p}{(q-r)^2}}$$

$$\frac{\partial q}{\partial y} = \frac{\partial}{\partial y}[x-y+z] = \boxed{-1}$$

$$3: \frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left[\frac{p-q}{q-r}\right] = (p-q)\frac{\partial}{\partial r}\left[\frac{1}{q-r}\right] = (p-q)\left[\frac{(q-r)(0)-(1)(-1)}{(q-r)^2}\right] = \boxed{\frac{p-q}{(q-r)^2}}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}[x+y+z] = \boxed{1}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{1}{(q-r)}(1) + \left(\frac{r-p}{(q-r)^2}\right)(-1) + \left(\frac{p-q}{(q-r)^2}\right)(1) = \boxed{\cancel{\frac{q-r+r-p+p-q}{(q-r)^2} = 0}}$$

$$\frac{\partial u}{\partial y} = \frac{\cancel{q}-r-r+p+p-\cancel{q}}{(q-r)^2} = \boxed{\frac{2p-2r}{(q-r)^2}}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

1: $\frac{\partial u}{\partial p} = \frac{\partial}{\partial p}\left[\frac{p-q}{q-r}\right] = \boxed{\frac{1}{q-r}}$; $\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}[x+y+z] = \boxed{1}$

2: $\frac{\partial u}{\partial q} = \boxed{\frac{r-p}{(q-r)^2}}$; $\frac{\partial q}{\partial z} = \frac{\partial}{\partial z}[x-y+z] = \boxed{1}$

3: $\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}\left[\frac{p-q}{q-r}\right] = \boxed{\frac{p-q}{(q-r)^2}}$; $\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}[x+y-z] = \boxed{-1}$

$$\Rightarrow \frac{\partial u}{\partial z} = \left(\frac{1}{q-r}\right)(1) + \left(\frac{r-p}{(q-r)^2}\right)(1) + \left(\frac{p-q}{(q-r)^2}\right)(-1)$$

$$\frac{\partial u}{\partial z} = \frac{q-r+r-p+(p-q)(-1)}{(q-r)^2} = \frac{q-r+r-p-p+q}{(q-r)^2} = \boxed{\frac{2q-2p}{(q-r)^2}}$$

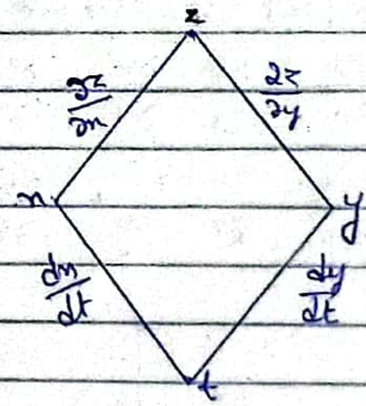
$$\frac{\partial u}{\partial x} = 0 \;,\quad \frac{\partial u}{\partial y} = \frac{2p \cancel{-2r}}{(q-r)^2} \;,\quad \frac{\partial u}{\partial z} = \frac{2q-2p}{(q-r)^2}$$

■ Using a branch Diagram

In Exercises 13–24, draw a branch Diagram and write a chain Rule formula for each derivative.

13/ $\frac{dz}{dt}$ for $z = f(x, y)$, $x = g(t)$, $y = h(t)$

Solution $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

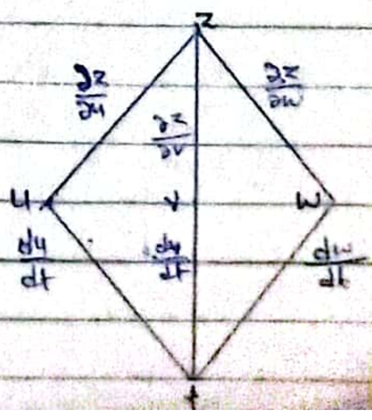


14) $\frac{dz}{dt}$ for $z = f(u, v, w)$, $u = g(t)$, $v = h(t)$, $w = k(t)$

Solution

ordinary derivative of z w.r.t "t".

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\frac{du}{dt} + \frac{\partial z}{\partial v}\frac{dv}{dt} + \frac{\partial z}{\partial w}\frac{dw}{dt}$$



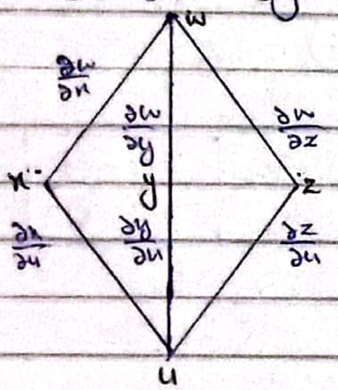
15) $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = h(x,y,z), x = f(u,v), y = g(u,v), z = k(u,v)$

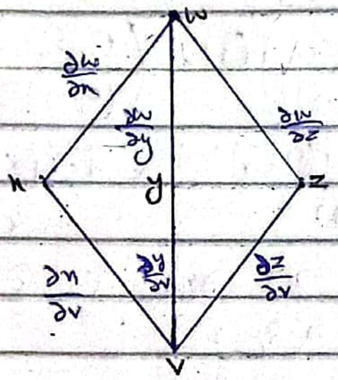
Solution

Partial derivative → Two independent and Three Intermediate variables

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u};$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v};$$





16) $\frac{\partial w}{\partial x}$ and $\frac{\partial w}{\partial y}$ for $w = f(r,s,t), r = g(x,y), s = h(x,y), t = k(x,y)$

Solution

Partial Derivative → Two independent (x,y) 3 Intermediate variables

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial x};$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y};$$