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(REVISED EDITION)

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PREFACE TO FIRST EDITION

About thirty years ago we wrote a series of books on different subjects of Engineering Mathematics. The series was named "FARKALEET SERIES". FARKELEET means MUHAMMAD (S.A.W.S). This name of our Prophet (S.A.W.S) is mentioned in the BIBLE.

The series though was hand written but became very much popular among our students at Mehran University of Engineering and Technology, Jamshoro, Pakistan. The series also gained popularity in the students of other universities and colleges of Sindh as well. Many teachers of different universities and colleges also appreciated such efforts of mine. The reason is that each topic was clearly explained and that every student as well as teacher enjoyed reading without any difficulty.

The book "DIFFERENTIAL EQUATIONS, FOURIER SERIES AND LAPLACE TRANSFORMS " is in your hands. Each topic is explained in well manner and in detail as well. We shall be looking forward to hear from the readers any critic that will improve the standard of the book.

We are indebted to Dr. Sania Nizamani who had edited the book and has tried to remove any king of typographical mistakes or otherwise.

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PREFACE TO SECOND EDITION

In the first edition, the book was containing some typo-graphical mistakes as well as mathematical errors. These mistakes/errors have been now removed from the first edition. We hope that, students will be much comfortable with the new revised edition. However, if still any mistake is found, we will appreciate if the same may be conveyed to the authors.
Moreover, new chapters on Laplace Transforms have been included as per requirement of the syllabus of MUET and QUEST.

Prof. (R) Muhammad Urs Shaikh

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CHAPTER ONE

INTRODUCTION TO DIFFERENTIAL EQUATIONS

INTRODUCTION

Differential equations are of fundamental importance not only in engineering disciplines but in almost every field of applied sciences as many physical laws and relations appear mathematically in the form of differential equations. Various laws of nature can be translated into differential equations. When a physical problem consisting of rate of change of dependent variable with respect to one or more independent variables is transformed into a mathematical model a differential equation is formed. More often we obtain a system of differential equations along with certain conditions from such situations. The solution of such differential equation/system of differential equations provides solution to the original problem. In the coming sections, we shall consider various physical and geometrical problems that lead to differential equations and we shall explain the most important standard methods for solving such problems. Before explaining the physical problems, we will learn some important terms given below:

Definition: An equation involving derivatives or differentials of one dependent variable with respect to one or more independent variables is called a **differential equation**.

For example, equations listed below are differential equations:

$$\frac{dy}{dx} = x\log x \qquad (1) \qquad\qquad \frac{d^4y}{dt^4} + \frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^5 + \sin y = e^t \qquad (2)$$

$$y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{dy/dx} \qquad (3) \qquad\qquad k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \qquad (4)$$

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2 \qquad (5) \qquad\qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad (6)$$

Ordinary Differential Equation

A differential equation involving derivatives with respect to a single independent variable is called an ***ordinary differential equation***. Equations (1), (2), (3) and (4) as shown above are ordinary differential equations.

Partial Differential Equation

Differential equations that involve partial derivatives with respect to two or more independent variables is called a ***partial differential equation***. Equation (5) and (6) as shown above are partial differential equations.

Order and Degree of a Differential Equation

Order of a differential equation is the order of highest derivative involved in the differential equation.

For equations shown above, equation (2) is of order four, equations (1) and (3) are of order one, equations (4) and (6) are of second order and equation (5) is of order three.

Degree of a differential equation is the power/exponent of highest derivative that occurs in it. In the above given equations, (1), (2) and (6) are of first degree. Equation (3) may be re-written as:

$$y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$$

It shows that this equation is of degree two. Again if we square both sides of (4) to make it free from radicals, then by definition equations (4) and (5) are of degree two.

Linear and Non–linear Differential Equations

A differential equation is called ***linear*** if it satisfies the following three conditions:

(i) The dependent variable and its derivative(s) in the equation occur in first degree only.

(ii) There is no term in the equation that contains the product of dependent variable and/or its derivative.

(iii) No transcendental function with dependent variable as its argument occurs in the equation.

A differential equation that is not linear is called a ***non–linear differential equation***. Equations (1) and (6) shown above are linear and all other equations are non-linear.

Solution of a Differential Equation

Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution of the differential equation. To understand this, let us consider few examples.

Example 01: Show that $y = c\,e^{2x}$ is solution of differential equation $y'' - 5y' + 6y = 0$

Solution: Since $y = c\,e^{2x}$ or $y' = 2c\,e^{2x}$ and $y'' = 4c\,e^{2x}$. Substituting these in given differential equation, we get: $4c\,e^{2x} - 10c\,e^{2x} + 6c\,e^{2x} = 10c\,e^{2x} - 10c\,e^{2x} = 0$

Since, given differential equation is satisfied hence, we say that $y = ce^{2x}$ is solution of differential equation $y'' - 5y' + 6y = 0$. Observe that $y = ce^{2x}$ is solution of given differential equation for any real `c`. This constant `c` is known as ***arbitrary constant***.

Example 02: Show that $y = c/x + d$ is solution of $y'' + 2y'/x = 0$ and $x^2 + 4y = 0$ is a solution of $(y')^2 + x\,y' - y = 0$

Solution: **(i)** The given differential equation is $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\cdot\dfrac{dy}{dx} = 0$ (1)

The given function is: $y = \dfrac{c}{x} + d$ (2)

Differentiating (2), with respect to x, we get $\dfrac{dy}{dx} = -\dfrac{c}{x^2}$ (3)

Differentiating again with respect to x, we get: $\dfrac{d^2y}{dx^2} = \dfrac{2c}{x^3}$ (4)

Substituting for dy/dx and d^2y/dx^2 in (1), we get: $\dfrac{2c}{x^3} - \dfrac{2c}{x^3} = 0 \Rightarrow 0 = 0$

This is true, therefore, (2) is the solution of equation (1).

(ii) Given differential equation is

$$\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} - y = 0 \qquad (1)$$

And given function is $x^2 + 4y = 0$ (2)

Differentiating (2) with respect to x, we get

$$2x + 4\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2} \qquad (3)$$

Substituting for y and dy/dx into (1), we get

$$\left(\frac{-x}{2}\right)^2 + x\left(\frac{-x}{2}\right) - \left(\frac{-x^2}{2}\right) = 0 \quad \Rightarrow \frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{4} = 0 \quad \Rightarrow 0 = 0$$

This is true, therefore, (2) is a solution of (1).

Types of Solutions

There are five types of solutions of any differential equation. They are:

i. Explicit Solution: When the solution of differential equation is expressed in the form y = f(x), we call such solution as explicit solution of given differential equation. For instance, $y = ce^{2x}$ is explicit solution of differential equation y`` – 5y` + 6y = 0 as shown in Example 1.

ii. Implicit Solution: An implicit solution of differential equation is of form f(x, y) = 0. For example, the solution of differential equation x + 3yy` = 0 is $x^2 + 3y^2 = 4$. [Students may verify this]

iii. General Solution: **S**olution of a differential equation in which the number of arbitrary constants is equal to order of the differential equation is called *general solution or complete solution* or *primitive* or *integral*. For instance, in Example 1, we see that given differential equation is of order two and we proved that $y = ce^{2x}$ satisfies the equation. As y involves only one arbitrary constant thus according to definition, $y = ce^{2x}$ does not form a general solution of this differential equation. However, $y = c_1e^{2x} + c_2e^{3x}$ forms a general solution of this differential equation because y contains two arbitrary constants and given differential equation is also of order two. Students may verify this by finding y` and y`` and then substituting the values of y, y` and y`` in the equation y`` – 5y` + 6y = 0. Similarly, Differential equation in Example 2(i) is of second order and the solution contains two arbitrary constants, that is; c and d. Hence this is the general solution.

iv. Particular Solution: A solution of differential equation that does not contain an arbitrary constant is called particular solution. In Example 2(ii) the solution contains no arbitrary constant hence it is a particular solution of given differential equation.

v. Singular Solution: A solution of a differential equation that cannot be obtained from its general solution by assigning any particular values to the arbitrary constants is called a singular solution. For example, consider the differential equation yy`–x(y`)^2 = 1, its general solution is y = cx + 1/c. If we put c = 1 we get y = x + 1. This is a particular solution. Equation $y^2 = 4x$ also satisfies given differential equation but cannot be obtained from the general solution by assigning any value to c. Hence, this is singular solution.

Let us see whether or not the equation $y^2 = 4x$ satisfies differential equation $y\,y' - x(y')^2 = 1$. Since $y^2 = 4x$ or $2yy' = 4$ or $yy' = 2$ giving $y' = 2/y$.

Now substituting $yy' = 2$ and $y' = 2/y$ in the equation $yy' - x(y')^2 = 1$, we obtain

$$2 - x\,(2/y)^2 = 1 \text{ or } 2y^2 - 4x = y^2 \text{ or } y^2 = 4x$$

This shows that solution of differential equation $y\,y' - x(y')^2 = 1$ is $y^2 = 4x$.

Remark: **(i)** Some authors are of the opinion that it is not necessary that a general solution must contain arbitrary constants but this is not true. For instance, the equation $(y')^2 + y^2 = 0$ has only one solution $y = 0$ which contains no arbitrary constant.

(ii) We normally expect that a differential equation will have a solution. But this not true! For instance, the equation $(y')^2 + 4 = 0$ has no real-valued solution.

FORMATION OF DIFFERENTIAL EQUATIONS

In this section, we shall discuss how differential equations arise in different situations and how they are modeled. It may be noted that a differential can be obtained in three different situations:

Differential Equations from Physical Phenomenon

The problems that involve rate of change of one variable with respect to time always give rise to differential equation. We shall provide couple of examples that will help you to understand how differential equations arise in such situations.

Example 01: The rate of growth of a population in a small town is proportional to population present. Find the differential equation.

Solution: Let P be the population at any time. As given:

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP \Rightarrow \frac{dP}{dt} - kP = 0$$

This is required differential equation. Here, k is the constant of proportionality.

Example 02: The rate of decrease in a radio active material (say Sodium) is proportional to the amount present. Find the differential equation that governs this problem.

Solution: Let A be the amount of radioactive material at any time. Then as given:

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = -kA \Rightarrow \qquad \frac{dA}{dt} + kA = 0$$

This is the required differential equation. Here `k` is the constant of proportionality and the negative sign indicates that the amount of radioactive material decreases.

Example 03: A man jump from an airplane using parachute. If m is the mass of a man with parachutes, find the equation of his motion in the form of differential equation.

Solution: By Newton's second law of motion: $\quad m\,a = F \qquad (1)$

Here, F is composed of two forces, F_U and F_D, that is $F = F_U + F_D$

F_D is the downward force that is equal to weight of the body. Thus

$$F_D = w = mg$$

F_U is the upward force that is proportional to the velocity of body when it falls down. Thus

$$F_U = \alpha\, v \text{ or } F_U = -\,kv$$

Here, `k` is the constant of proportionality and negative sign shows that body faces the air resistance. Also, `a` is the acceleration of the body and is given by $a = dv/dt$. Thus equation (1) becomes:

$$m\frac{dv}{dt} = mg - kv \quad \Rightarrow \quad m\frac{dv}{dt} + kv = mg \,.$$

This is the differential equation that governs the motion of falling object. If we put v = ds/dt, above differential equation becomes: $m\frac{d^2s}{dt^2} + k\frac{ds}{dt} = mg$

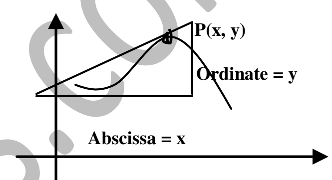
This is another form of differential equation that governs the motion of a falling body.

Differential Equations from Geometrical Phenomenon

Problems that involve slope of a function of one variable with respect to another variable, always give rise to the differential equations. The following example shows formation of differential equation using the concept of slope.

Example 04: The slope of a curve at any point P (x, y) is given by ratio of an ordinate to twice its abscissa. Find the differential equation that governs this problem.

Solution: We know that slope of a curve at any point P (x, y) is given by dy/dx. Now as per condition



$$\frac{dy}{dt} = \frac{\text{Ordinate}}{\text{Twice the abscissa}} = \frac{y}{2x}$$

$\Rightarrow \quad 2x\frac{dy}{dt} = y \;\Rightarrow\; 2x\frac{dy}{dt} - y = 0$. This is differential equation that arises from the situation explained above.

Differential Equations by Eliminating an Arbitrary Constant(s)

If an equation involves a dependent variable, an independent variable and some arbitrary constants, we can obtain a differential equation by eliminating the arbitrary constants. Following steps will help to find required equation.

- Write down given equation and differentiate with respect to x successively as many times as the number of arbitrary constants.
- Eliminate arbitrary constants from the equations obtained.
- The resulting equation is required differential equation.

Example 05: Form the differential equation by eliminating arbitrary constant(s)

(i) $y = (x^3 + c)\, e^{-3x}$ **(ii)** y = A sin x + B cos x

Solution: **(i)** Given equation is $y = \left(x^3 + c\right) e^{-3x}$ (1)

Differentiating (1) with respect to x, we get: $\frac{dy}{dx} = \left(x^3 + c\right)\left(-3e^{-3x}\right) + e^{-3x}\left(3x^2\right)$

$$= 3x^2e^{-3x} - 3\left(x^3 + c\right)e^{-3x} \Rightarrow \frac{dy}{dx} = 3x^2e^{-3x} - 3y \qquad \text{[Using (1)]}$$

or $$\frac{dy}{dx} + 3y = 3x^2e^{-3x}$$

(ii) Given equation is $y = A\sin x + B\cos x$ (1)

Differentiating (1) with respect to x, we get: $\frac{dy}{dx} = A\cos x - B\sin x$ (2)

Differentiating again w.r.t x, we get

$$\frac{d^2y}{dx^2} = -A\sin x - B\cos x \Rightarrow \frac{d^2y}{dx^2} = -(A\sin x + B\cos x) \Rightarrow \frac{d^2y}{dx^2} = -y \Rightarrow \frac{d^2y}{dx^2} + y = 0 \quad [\text{using (1)}]$$

Example 06: Form the differential equation of simple harmonic motion given by $x = A\cos(nt + a)$ where A and a are arbitrary constants.

Solution: The given equation is: $x = A\cos(nt + \alpha)$ (1)

To eliminate constants A and α we differentiate (1) twice w.r.t t and get

$$\frac{dx}{dt} = -A\sin(nt+\alpha)(n) \Rightarrow \frac{dx}{dt} = -nA\sin(nt+\alpha)$$

Differentiating again w.r.t t, we get: $\frac{d^2x}{dt^2} = -nA\cos(nt+\alpha)(n) \Rightarrow \frac{d^2x}{dt^2} = -n^2A\cos(nt+\alpha)$

$$\Rightarrow \frac{d^2x}{dt^2} = -n^2x \Rightarrow \frac{d^2x}{dt^2} + n^2x = 0 \quad [\text{using (1)}]$$

Example 07: Obtain differential equation of all circles of radius r and centered at (h, k).

Solution: Equation of all circles centered at (h, k) and of radius r is:

$$(x-h)^2 + (y-k)^2 = a^2 \quad (1)$$

where h and k are the coordinates of center and are arbitrary constants.

Differentiating equation (1) with respect to x, we get

$$2(x-h) + 2(y-k)\frac{dy}{dx} = 0 \Rightarrow (x-h) + (y-k)\frac{dy}{dx} = 0 \quad (2)$$

Again differentiating, we get

$$1 + (y-k)\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(\frac{dy}{dx}\right) = 0 \Rightarrow 1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \quad (3)$$

From (3) , we have $y - k = -\frac{\left[1+(dy/dx)^2\right]}{d^2y/dx^2}$ and from (2), we have

$$x - h = -(y-k)\frac{dy}{dx} = \frac{\left[1+(dy/dx)^2\right]\frac{dy}{dx}}{d^2y/dx^2}$$

Substituting the values of x– h and y – k in (1), we get

$$\frac{\left[1+(y')^2\right]^2(y')^2}{(y'')^2} + \frac{\left[1+(y')^2\right]^2}{(y'')^2} = r^2 \Rightarrow \left[1+(y')^2\right]^2(y')^2 + \left[1+(y')^2\right]^2 = r^2(y'')^2$$

$$\left[1+(y')^2\right]^2\left[1+(y')^2\right] = r^2(y'')^2 \quad \Rightarrow \frac{\left[1+(y')^2\right]^3}{(y'')^2} = r^2 \Rightarrow \frac{\left[1+(y')^2\right]^{3/2}}{(y'')} = r$$

Example 08: Obtain differential equations by eliminating an arbitrary constant(s)
From the following equations,

(i) $y = x + c\,e^{-x}$

Solution: Given equation contains only one arbitrary constant, hence we differentiate it once to get: $y` = 1 - ce^{-x}$

Now, $y = x + c\,e^{-x}$

Adding, we get $y` + y = x + 1$

(ii) $a x + \ln y = y + b$

Solution: Given equation is: $a x + \ln y = y + b$. This contains two arbitrary constants, hence differentiate it twice, we get $\quad 1 + y'/y = y'$

Differentiate once again, we get: $\quad 0 + (y y'' - y' y')/y^2 = y''$ or $(y y'' - y' y') = y'' y^2$

or $\quad y y'' - (y')^2 = y^2 y''$ $\quad$ or $\quad (y - y^2) y'' - (y')^2 = 0$

(iii) $y = a e^x + b \ln x + cx + d$

Solution: Given equation contains four arbitrary constant, hence we differentiate it four times.

$y' = ae^x + (b/x) + c \qquad (1) \qquad y'' = ae^x - b/x^2 \qquad (2)$

$y''' = ae^x + 2b/x^3 \qquad (3) \qquad y^{iv} = ae^x - 6b/x^4 \qquad (4)$

Subtracting (2) from (3), we get

$y''' - y'' = (2b/x^3) + (b/x^2) = b(2 + x)/x^3$ $\quad$ giving $b = x^3(y - y'')/(2 + x) \qquad (5)$

Subtracting (3) from (4), we get

$y^{\mathbf{iv}} - y''' = (-6b/x^4) - (2b/x^3) = b(-6 - 3x)/x^4$ or $b = x^4(y^{\mathbf{iv}} - y''')/(-6 - 3x) \qquad (6)$

Equating (5) and (6), we get

$$\frac{x^4(y^{iv} - y''')}{(-6-3x)} = \frac{x^3(y'''-y'')}{(2+x)} \Rightarrow \frac{x(y^{iv} - y''')}{-3(2+x)} = \frac{(y'''-y'')}{(2+x)}$$

or, $\quad x(2 + x)(y^{\mathbf{iv}} - y''') = -3(2 + x)(y''' - y'')$

or $x(y^{\mathbf{iv}} - y''') = -3(y''' - y'')$ or $x(y^{\mathbf{iv}} - y''') + 3(y''' - y'') = 0$

(iv) $y = ax^2 + b x$

Solution: Given equation contains two arbitrary constants hence we differentiate it twice. Now,

$y = ax^2 + b x \quad (1) \quad y' = 2ax + b \quad (2) \quad y'' = 2a \qquad (3)$

From equation (3), $\quad a = y''/2$. Put this is (2), we get

$y' = xy'' + b$ $\quad$ giving $b = y' - xy''$

Now substituting the values of a and b in (1), we obtain

$y = x^2(y''/2) + x(y' - xy'')$ or $2y = x^2 y'' + 2xy' - 2x^2y''$

or $\quad x^2 y'' - 2xy' + 2y = 0$

(v) $y = A e^{2x} + B e^{-3x}$

Solution: This equation contains two arbitrary constants hence we differentiate it twice. Now,

$y = A e^{2x} + B e^{-3x} \quad (1) \quad y' = 2 A e^{2x} - 3 B e^{-3x} \quad (2) \quad y'' = 4 A e^{2x} + 9 B e^{-3x} \quad (3)$

Multiply (1) by 2 and subtracting it from (2), we obtain:

$$y' - 2 y = -5 B e^{-3x} \qquad (4)$$

Now multiply (1) by 4 and subtracting it from (3), we get

$$y'' - 4 y = 5 B e^{-3x} \qquad (5)$$

Adding (4) and (5), we get: $\quad y'' + y' - 6 y = 0$

(vi) $y = e^x (A \cos 2x + B \sin 2x)$

Solution: Given differential equation contains two arbitrary constants, hence we differentiate it twice. Now, $\quad y = e^x (A \cos 2x + B \sin 2x) \qquad (1)$

or $\quad y' = e^x(-2 A \sin 2x + 2 B \cos 2x) + e^x(A \cos 2x + B \sin 2x)$

or $y` = e^x(-2A\sin 2x + 2B\cos 2x) + y$ [Using equation (1)]

or $y` - y = e^x(-2A\sin 2x + 2B\cos 2x)$ (2)

Differentiate (2) again, we get

$$y`` - y` = e^x(-4A\sin 2x - 4B\cos 2x) + (-2A\sin 2x + 2B\cos 2x)e^x$$
$$= -4e^x(A\sin 2x + B\cos 2x) + y` - y \quad \text{[Using equation (2)]}$$

or $y`` - y` = -4y + y` - y$ or $y`` - 2y` + 5y = 0$

(vii) $y = ax + a^2$

Solution: Given; $y = ax + a^2$ (1)

On differentiating, we obtain $y` = a$.

Substituting this value of a in equation (1), we get: $y = xy` + (y`)^2$

(viii) $y^2 = ax^2 + bx + c$

Solution: Given equation is; $y^2 = ax^2 + bx + c$

On differentiating, we get $2yy` = 2ax + b$

Again differentiating, we obtain $2yy`` + 2(y`)^2 = 2a$ or $yy`` + (y`)^2 = a$

On differentiating again, we get $yy``` + y`y`` + 2y`y`` = 0$

or $yy``` + 3y`y`` = 0$

(ix) $y = ae^x + bx\,e^x + cx^2\,e^x$

Solution: Given $y = ae^x + bx\,e^x + cx^2\,e^x$

Differentiating both sides, we get

$$y` = ae^x + bx\,e^x + b\,e^x + cx^2\,e^x + 2cx\,e^x$$
$$= (ae^x + bx\,e^x + cx^2\,e^x) + (be^x + 2cx\,e^x) = y + (b\,e^x + 2cx\,e^x)$$

or $y` - y = (be^x + 2cx\,e^x)$ (1)

Again differentiate both sides, we get

$$y`` - y` = be^x + 2c\,e^x + 2cx\,e^x = (be^x + 2cx\,e^x) + 2c\,e^x$$
$$= y` - y + 2c\,e^x \quad \text{[Using equation (1)]}$$

or $y`` - 2y` + y = 2c\,e^x$ (2)

Finally, on differentiating again, we obtain

$$y``` - 2y`` + y` = 2c\,e^x = y`` - 2y` + y \quad \text{[Using equation (2)]}$$

or $y``` - 3y`` + 3y` - y = 0$

INITIAL and BOUNDARY CONDITIONS

The arbitrary constants in general solution of a differential equation can often be determined by giving additional conditions on the unknown function and/or its derivatives. If these conditions are specified at the same value of an independent variable, we call them ***initial conditions***. If the conditions are given at different values of independent variable, we call them ***boundary conditions***. A differential equation together with its initial conditions is called an ***initial value problem*** and a differential equation together with its boundary conditions is called a ***boundary value problem***.

The general solution of a first order differential equation contains one arbitrary constant and so requires one additional condition to determine this arbitrary constant. Therefore, first order differential equations always present initial value problems. Boundary value problems involve differential equations that are at least of second order. For example, the differential equation

$y''' + 3y'' - y' + y = \sin x$ with initial conditions $y(0) = 0$, $y'(0) = 3$, $y''(0) = 5$ is an initial value problem because all conditions are given at $x = 0$. A general solution of given differential equation will contain three arbitrary constants, which can be determined by three conditions.

Again consider $y'' + 5\,xy = \cos x$, $y(0) = 0$, $y'(\pi) = 3$.This is a boundary value problem where two conditions are given at $x = 0$ and $x = \pi$ which will determine two arbitrary constants in the general solution of given differential equation.

Example 01: Solve the following initial value problem $y'' - y' - 12y = 0$, $y(0) = -2$, $y'(0) = 6$ where it is given that $y = A\,e^{4x} + B\,e^{-3x}$ is the general solution of given equation.

Solution: We have: $y = Ae^{4x} + Be^{-3x}$ (1)

Differentiating (1) with respect to x, we get: $y' = 4Ae^{4x} - 3Be^{-3x}$ (2)

Since $y = -2$ when $x = 0$, then (1) gives $-2 = A + B$ (3)

Also, $y = 6$ when $x = 0$ then (2) gives: $6 = 4A - 3B$ (4)

Multiplying equation (3) by 3 and adding to (4), we get $0 = 7A$ giving $A = 0$

Substituting it into (3), we get $B = -2$.

Hence from (1), we have $y = -2e^{-3x}$ as particular solution of given differential equation.

WORKSHEET 01

1. Find the orders and degrees of following differential equations. Also state whether given differential equations are linear or non-linear and that they are ordinary or partial differential equations.

(a) $e^x\,dx + e^y\,dy = 0$ (b) $y'' + n^2 y = 0$ (c) $y = x\,y' + x/y'$

(d) $[1 + (y')^2]^{3/2} = c^2\,(y'')^2$ (e) $y = x\,y' + a\,[1 + (y')^2]$

(f) $y''' + 4y'' - 6y' + y = \cos x$ (g) $U_{xx} + U_{yy} = 0$ (h) $(y')^2 = (y'' + y)^{3/2}$

2. Verify that:

(a) $y = a + be^{-2x} + e^x/3$ is the general solution of equation $y'' + 2y' = e^x$

(b) $y^2 = (x + 5)^3$ is general solution of equation $27y - 8(y')^3 = 0$

3. Obtain the differential equations by eliminating an arbitrary constant(s) from the following equations.

(a) $y = ax^3 + bx^2$ (b) $xy = A\,e^x + B\,e^{-x}$ (c) $y^2 = 4a\,(x + a)$

(d) $y = a\,e^{2x} + b\,e^{-3x} + c\,e^x$ (e) $A\,x^2 + B\,y^2 = 1$

(f) $y = a\sin(x + 3)$ (g) $y = \sqrt{6x + c} \Rightarrow y^2 = 6x + c$ (h) $y = x + c\,e^x$

(i) $y = (x^3 + c)\,e^{-3x}$ (j) $a\,x + \ln y = y + b,\ y > 0$ (k) $y^2 - 2a\,y + x^2 = a^2$

(*l*) $y = ae^x + b\ln x + cx + d$ (m) $y = a\sinh 2x + b\cosh 2x$

(n) $y = a\sin x + b\cos x + x\sin x$

4. Solve following initial/boundary problems:

(a) $y' = -x/y$, $y(3) = 4$ given that $x^2 + y^2 = r^2$ is the solution of given differential equation.

(b) $y'' - y' - 12y = 0$, $y(0)= -2$, $y'(0) = 6$, given that $y = a\,e^{4x} + b\,e^{-3x}$ is the solution of given differential equations.

(c) $y'' + y = 0$, $y(0) = 1$, $y'(\pi) = 1$, given that $y = a\sin x + b\cos x$ is the solution of given differential equations.

(d) $y'' - 4y' + 3y = 0$, $y(0) = 1$, $y(1) = 1$, given that $y = a\,e^x + b\,e^{3x}$ is the solution of given differential equations.

CHAPTER TWO

FIRST ORDER DIFFERENTIAL EQUATIONS

SOLUTION OF FIRST ORDER DIFFERENTAI EQUATIONS

In chapter one, we discussed the solution of differential equation. By solution of a differential equation, we mean the value of a dependent variable (in explicit form y = f(x) or any relation between x and y in implicit form f(x, y) = 0) that satisfies given differential equation. In this chapter, we shall study/learn various methods to solve the differential equations of first and first degree because not every first order differential equation can be solved exactly in the same manner or same method. They can be solved if they belong to standard forms discussed in the following sections.

Before we discuss these methods it may be noted that most general form of differential equation of first order and first degree is:

$$\frac{dy}{dx} = f(x,y) \quad \text{or} \quad M(x,y)\,dx + N(x,y)\,dy = 0 \qquad \text{(I)}$$

To find a solution of this differential equation, various methods have been developed depending upon the type of the functions M and N as mentioned earlier. We shall discuss these methods now.

Separable Variable Method

If equation (I) as shown above is expressed as: $\frac{dy}{dx} = \frac{M(x)}{N(y)}$

where M(x) is a function of x only and N(y) is a function of y only, then we may rewrite given differential equation as: $N(y)dy = M(x)dx$

This equation is in separable variable form. Integrating both sides, we obtain

$$\int N(y)dy = \int M(x)dx + c$$

where c is an arbitrary constant known as constant of integration. Solving the integrals will produce the solution of given differential equation.

Remarks:

- **i.** Never forget to add an arbitrary constant on any side of solution but it is preferred to be placed on right side.
- **ii.** The nature of arbitrary constant depends upon the nature of the solution.
- **iii.** The solution of a differential equation must be put in a form as simple as possible.

Example 01: Find the solution of differential equation y y` = x

Solution: Given that $y\,dy/dx = x \Rightarrow ydy = xdx$

Integrating both sides, we get

$$\int y\,dy = \int x\,dx + c \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + c \qquad (1)$$

Multiplying equation (1) by 2, we get $y^2 = x^2 + r^2$ [Assuming $2c = r^2$]
This is general solution of given differential equation.

Example 02: Solve the following differential equations:

(i) $\mathbf{y` = e^{x-y} + x^2 e^{-y}}$ **(ii)** $\mathbf{y` + y^2 \sin x = 0}$ **(iii)** $\mathbf{x e^{x^2+y} dx = y\,dy}$

Solution: **(i)** Given that $y` = e^{x-y} + x^2 e^{-y} = e^{-y}(e^x + x^2)$ (1)

Separating the variables, (1) becomes: $e^y dy = \left(e^x + x^2\right)dx$

Integrating, $\int e^y dy = \int \left(e^x + x^2\right)dx \Rightarrow e^y = e^x + x^3/3 + c$

This is the solution of given differential equation.

(ii) We have $y` + y^2 \sin x = 0$ or $y` = -y^2 \sin x$ (2)

Separating the variables, (2) become: $dy/y^2 = -\sin x\,dx$

Integrating, $\int y^{-2} dy = \int -\sin x\,dx + c \Rightarrow -1/y = +\cos x + c \Rightarrow y\cos x + cy + 1 = 0$

(iii) We have $x e^{x^2+y} dx = y\,dy \Rightarrow x e^{x^2} e^y dx = y\,dy$ (1)

Separating the variables, (1) becomes: $x e^{x^2} dx = y e^{-y} dy$

Integrating both sides, we get $\int x e^{x^2} dx = \int y e^{-y} dy + c`$ (2)

Left side of (2) is solved using integration by substitution as follows:

Let $z = x^2$ or $dz = 2x\,dx$ or $dz/2 = x\,dx$. Thus

$$\int x e^{x^2} dx = \frac{1}{2}\int e^z dz = e^z/2 = e^{x^2}/2$$

Right side of (2) will be solved using integration by parts as follows:

$$\int y e^{-y} dy = y\left(-e^{-y}\right) + \int e^{-y} dy = -y e^{-y} - e^{-y} = -e^{-y}(y+1)$$

Thus, (2) becomes $\dfrac{e^{x^2}}{2} = -e^{-y}(y+1) + c` \Rightarrow e^{x^2} = -2e^{-y}(y+1) + 2c`$

$\Rightarrow e^{x^2} = -2e^{-y}(y+1) + c$ [assuming $2c` = c$]

[**NOTE:** Integration by Parts formula is $\int u\,v\,dx = u\int v\,dx - \int [u' \int v\,dx]dx$]

Example 03: Solve the following initial value problem

$$(3x + 8)(y^2 + 4)\,dx - 4y(x^2 + 5x + 6) = 0,\ y(1) = 2$$

Solution: Given $(3x + 8)(y^2 + 4)\,dx - 4y(x^2 + 5x + 6) = 0$ (1)

or $(3x + 8)(y^2 + 4)\,dx = 4y(x + 5)(x + 3)$

Separating the variables, we get $\dfrac{(3x+8)}{(x+2)(x+3)}dx = \dfrac{4y}{y^2+4}dy$

Integrating both sides, we get $\displaystyle\int \frac{(3x+8)}{(x+2)(x+3)}dx = \int \frac{4y}{y^2+4}dy + c_1$ (2)

Left side of (2) will be solved using integration by partial fractions.

$$\frac{3x+8}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \qquad (3)$$

Multiplying both sides by (x + 2) (x + 3), we get

$$3x + 8 = A(x + 3) + B(x + 2) \qquad (4)$$

Put x + 3 = 0 or x = – 3 then (4) gives B = 1

Put x + 2 = 0 or x = –2 then (4) gives A = 2

Thus equation (3) becomes: $\dfrac{3x+8}{(x+2)(x+3)} = \dfrac{2}{x+2} + \dfrac{1}{x+3}$

Integrating, we get:

$$\int \frac{(3x+8)}{(x+2)(x+3)}dx = 2\int \frac{1}{x+2}dx + \int \frac{1}{x+3}dx = 2\ln(x+2) + \ln(x+3) = \ln(x+2)^2(x+3)$$

Now the right side of (2) is: $2\int \dfrac{2y}{y^2+4}dy = 2\ln(y^2+4) = \ln(y^2+4)^2$

Hence, equation (2) becomes: $\ln(x + 2)^2 (x + 3) = \ln(y^2 + 4)^2 + \ln c = \ln c(y^2 + 4)^2$

or $$(x + 2)^2 (x + 3) = c (y^2 + 4)^2 \qquad (5)$$

Applying initial conditions, that is, put y = 2 and x = 1, we obtain

$$(1 + 2)^2 (1 + 3) = c (4 + 4)^2 \text{ or } c = 9/16$$

Substituting in equation (5), we get

$(x + 2)^2 (x + 3) = 9(y^2 + 4)/16$ or $16 (x + 2)^2 (x + 3) = 9(y^2 + 4)$

This is a particular solution of given differential equation.

Example 04: Solve the following differential equations

(i) $dy/dx = x^2/[y(1 + x^3)]$

Solution: Given differential equation is $dy/dx = x^2/[y(1 + x^3)]$

Separating the variables and integrating, we get

$$\int y\, dy = \int \frac{x^2}{(1+x^3)}dx \Rightarrow \frac{y^2}{2} = \frac{1}{3}\int \frac{3x^2}{(1+x^3)}dx + C \Rightarrow \frac{y^2}{2} = \frac{1}{3}\ln(1+x^3) + C$$

(ii) $dy/dx = 1 + x + y^2 + x\,y^2$

Solution: Given differential equation is $\dfrac{dy}{dx} = (1+x) + y^2(1+x) \quad \Rightarrow \dfrac{dy}{dx} = (1+x)(1+y^2)$

Separating the variables and integrating, we get

$$\int \frac{1}{1+y^2}dy = \int (1+x)dx + C \Rightarrow \tan^{-1} y = x + \frac{x^2}{2} + C$$

(iii) $(xy + 2x + y + 2)\, dx + (x^2 + 2x)\, dy = 0$

Solution: Given differential equation is:

$(x^2 + 2x)\, dy = -[x(y + 2) + 1 (y + 2)]\, dx$ or $(x^2 + 2x)\, dy = -(y + 2)(x + 1)$

Separating the variables and integrating, we get

$$\int \frac{1}{y+2}dy = \int \frac{(1+x)}{(x^2+2x)}dx + C_1 \Rightarrow \ln(y+2) = \frac{1}{2}\int \frac{2x+2}{x^2+2x}dx + \ln C$$

$$\Rightarrow \ln(y+2) = \frac{1}{2}\ln(x^2+2x) + \ln C \Rightarrow \ln(y+2) = \ln C\sqrt{x^2+2x}$$

$$\Rightarrow y + 2 = C\sqrt{x^2+2x} \text{ or } y = C\sqrt{x^2+2x} - 2$$

(iv) $y` = 2x^2 + y - x^2y + xy - 2x - 2$

Solution: Given that: $y` = 2x^2 + y - x^2y + xy - 2x - 2$. Rearranging the terms on right side, we get: $y` = -x^2(y-2) - x(y-2) + (y-2) = (y-2)(-x^2 - x + 1)$

Separating the variables and integrating, we get

$$\int \frac{1}{y-2}dy = \int (-x^2 - x + 1)dx + C \qquad \Rightarrow \ln(y-2) = -\frac{x^3}{3} - \frac{x^2}{2} + x + C$$

(v) cosec y dx + sec x dy = 0

Solution: Rewriting the given equation as: sec x dy = - cosec y dx

Now separating the variables and integrating, we get

$$\int \frac{1}{-\cos ecy}dy = \int \frac{1}{\sec x}dx + C \quad \Rightarrow -\int \sin y\, dy = \int \cos x\, dx + C \quad \Rightarrow \cos y = \sin x + C$$

(vi) y (1 + x) dx + x (1 + y) dy = 0

Solution: Rearranging the terms, we get: x (1 + y) dy = – y (1 + x) dx

Separating the variables and integrating, we get

$$\int \frac{1+y}{y}dy = -\int \frac{1+x}{x}dx + C \quad \Rightarrow \int \left(\frac{1}{y} + 1\right)dy + \int \left(\frac{1}{x} + 1\right)dx = C$$

$$\Rightarrow \ln y + y + \ln x + x = C \text{ or } \ln xy = C - x - y$$

(vii) $\mathbf{y\sqrt{1+x^2}dx + x\sqrt{1+y^2}dy = 0}$

Solution: Re-arranging the terms, separating the variables and integrating, we get

$$\int \frac{\sqrt{1+y^2}}{y}dy = -\int \frac{\sqrt{1+x^2}}{x}dx \quad \Rightarrow \int \frac{\sqrt{1+y^2}}{y}dy + \int \frac{\sqrt{1+x^2}}{x}dx = C \qquad (1)$$

Let us consider, $\int \frac{\sqrt{1+x^2}}{x}dx$. Put $z = \sqrt{1+x^2} \Rightarrow z^2 = 1 + x^2 \Rightarrow 2z\,dz = 2x\,dx$

or $z\,dz/x = dx$. Thus

$$\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{z}{x}.\frac{z}{x}dz = \int \frac{z^2}{x^2}dz = \int \frac{z^2}{z^2-1}dz = \int \frac{(z^2-1)+1}{z^2-1}dz = \int \left(1 + \frac{1}{z^2-1}\right)dz$$

$$= z + \frac{1}{2}\ln\left(\frac{z-1}{z+1}\right) = \sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)$$

Similarly, $\int \frac{\sqrt{1+y^2}}{y}dy = \sqrt{1+y^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right)$. Thus equation (1) becomes:

$$\sqrt{1+y^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right) + \sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) = C$$

This is the general solution of given differential equation.

(viii) $(e^x + 1)$ y dy = (y + 1) e^x dx

Solution: Separating the variables and integrating, we get

$$\int \frac{y}{y+1}dy = \int \frac{e^x}{e^x+1}dx + C \quad \Rightarrow \int \frac{(y+1)-1}{(y+1)}dy = \ln(e^x + 1) + C$$

$$\Rightarrow \int \left(1-\frac{1}{y+1}\right)dy = \ln(e^x+1)+C \quad \Rightarrow y+\ln(y+1)=\ln(e^x+1)+C$$

(ix) $\frac{dy}{dx}=\frac{y^3+2y}{x^2+3x}$

Solution: Separating the variables and integrating, we get

$$\int \frac{1}{y(y^2+2)}dy = \int \frac{1}{x(x+3)}dx + \ln C \qquad (1)$$

Consider, $\frac{1}{y(y^2+2)}=\frac{A}{y}+\frac{By+C}{y^2+2}=\frac{1/2}{y}+\frac{-1/2y+0}{y^2+2}$ [Note: This is by partial fractions]

Thus, $\int \frac{1}{y(y^2+2)}dy=\frac{1}{2}\left(\int \frac{1}{y}dy-\frac{1}{2}\int \frac{2y}{y^2+2}dy\right)=\frac{1}{2}\left(\ln y-\frac{1}{2}\ln(y^2+2)\right)=\frac{1}{2}\ln\frac{y}{\sqrt{y^2+2}}$

Also $\frac{1}{x(x+3)}=\frac{A}{x}+\frac{B}{x+3}=\frac{1/3}{x}+\frac{-1/3}{x+3}$ [Note: This is by partial fractions]

Thus, $\int \frac{1}{x(x+3)}dx=\frac{1}{3}\left(\int \frac{1}{x}dx-\int \frac{1}{x+3}dx\right)=\frac{1}{3}\left(\ln x-\ln(x+3)\right)=\frac{1}{3}\ln\left(\frac{x}{x+3}\right)$

Hence equation (1) becomes: $\frac{1}{2}\ln\frac{y}{\sqrt{y^2+2}}=\frac{1}{3}\ln\left(\frac{x}{x+3}\right)+\ln C$

(x) (sinx + cos x) dy + (cos x – sin x) dx = 0

Solution: Separating the variables and integrating, we get:

$$\int 1.dy=-\int \frac{\cos x-\sin x}{\sin x+\cos x}dx+\ln C \quad \Rightarrow y=-\ln(\sin x+\cos x)+\ln C=\ln\left(\frac{C}{\sin x+\cos x}\right)$$

(xi) $e^x(1+y`)=x e^{-y}$

Solution: Rearranging the terms, we get:

$$1+\frac{dy}{dx}=\frac{x}{e^x e^y}=\frac{x}{e^{x+y}} \qquad (1)$$

Put x + y = z or 1 + y` = z`. Thus equation (1) becomes:

$\frac{dz}{dx}=\frac{x}{e^z} \Rightarrow e^z\,dz = x\,dx.$ Integrating, we get: $\int e^z\,dz=\int x\,dx+C_1 \Rightarrow e^z=\frac{x^2}{2}+C_1$

Substituting the value of z, we get:

$$e^{x+y}=\frac{x^2}{2}+C \Rightarrow 2e^{x+y}=x^2+2C_1 \quad \Rightarrow 2e^{x+y}=x^2+C,\ \text{where } 2C_1=C$$

(xii) $x\,e^{x^2+y}\,dx = y\,dy$

Solution: Rearranging, we get: $x\,e^{x^2}\,e^y\,dx = y\,dy$. Separating the variables and integrating, we get:

$$\int y e^{-y}\,dy=\int x\,e^{x^2}\,dx+C \qquad (1)$$

Consider, $\int y e^{-y}dy=-ye^{-y}+\int e^{-y}\,dy=-e^{-y}(y+1)$

Now consider, $\int x e^{x^2}\,dx$. Putting $z=x^2 \Rightarrow dz=2\,x\,dx$

Thus $\int x e^{x^2}\,dx=\frac{1}{2}\int e^z dz=\frac{1}{2}e^z=\frac{1}{2}e^{x^2}$

Thus equation (1) becomes: $-e^{-y}(y+1)=\frac{1}{2}e^{x^2}+C$

(xiii) $2x \cos y\, dx + x^2 (\sec y - \sin y)\, dy = 0$

Solution: Rearranging the terms and integrating, we get

$x^2 (\sec y - \sin y)\, dy = -2x \cos y\, dx$ or $\int \frac{\sec y - \sin y}{\cos y} dy = -2\int \frac{1}{x} dx + C$

or $\int (\sec^2 y - \tan y)\, dy = -2\ln x + C \quad \Rightarrow \tan y - \ln \sec y = -2\ln x + C$

(xiv) $\mathbf{(x+y)^2 (xy`+y) = xy(1+y`)}$

Solution: Given $(x+y)^2 (xy`+y) = xy(1+y`)$

or $\frac{(xy`+y)}{xy} = \frac{(1+y`)}{(x+y)^2}$. Integrating, we obtain : $\int \frac{(xy`+y)}{xy} dx = \int \frac{(1+y`)}{(x+y)^2} dx$

or $\ln xy = -1/(x + y) + C$

Remark: In the first integral, we have used the formula: $\int \frac{f`(x)}{f(x)} dx = \ln f(x)$

And in the second integral, we have used the formula: $\int [f(x)]^n f`(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}, \; n \neq -1$

Equations Reducible To Separable Variable Form

Equations of the form $\frac{dy}{dx} = f(ax+by)$ or $\frac{dy}{dx} = f(ax+by+c)$ can be reduced to separable variables form on substituting $ax + by = z$ or $ax + by + c = z$. On differentiating with respect to x, we get

$$a + b\frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

Then given equations becomes: $\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z)$ or $\frac{dz}{dx} - a = bf(z)$ or $\frac{dz}{dx} = a + bf(z)$

Separating the variables, we get: $\frac{dz}{a + bf(z)} = dx$

This can be integrated easily to get the required solution.

Example 05: Solve the differential equation $y` = (4x + y + 1)^2$

Solution: Given that

$$y` = (4x + y + 1)^2 \qquad (1)$$

Put $4x + y + 1 = z$. Differentiating with respect to x, we get

$$4 + \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dz}{dx} - 4$$

Therefore equation (1) becomes: $\frac{dz}{dx} - 4 = z^2$ or $\frac{dz}{dx} = 4 + z^2 \Rightarrow \frac{dz}{z^2+4} = dx$

Integrating, we get

$\int \frac{dz}{z^2+4} = \int 1.dx + c \Rightarrow \frac{1}{2}\tan^{-1}\frac{z}{2} = x + c.$ $\quad \left(\text{Note}: \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right)$

Replacing z by $4x + y + 1$, we get: $\frac{1}{2}\tan^{-1}\frac{4x+y+1}{2} = x + c$

This is the solution of given equation.

Example 06: Solve the differential equation $\frac{dy}{dx} = \cos(x+y)$

Solution: Put $z = x + y$ or $dz/dx = dy/dx + 1$ or $dy/dx = dz/dx - 1$

Then given differential equation becomes:

$$\frac{dz}{dx}-1=\cos z \quad \Rightarrow \frac{dz}{dx}=1+\cos z$$

Separating the variables and integrating, we get:

$$\int\frac{dz}{1+\cos z}=\int 1.dx+c \Rightarrow \int\frac{dz}{2\cos^2 z/2}=x+c \quad \Rightarrow \quad \frac{1}{2}\int \sec^2(z/2)\,dz=x+c$$

$$\Rightarrow \frac{1}{2}\frac{\tan z/2}{1/2}=x+c \quad \Rightarrow \tan\left(\frac{x+y}{2}\right)=x+c$$

Homogeneous Equations

Definition: A function f(x, y) is said to be ***homogeneous function*** of degree n if it can be expressed in the form:

$$f(x,y)=x^n f\left(\frac{y}{x}\right) \text{ or } f(tx,ty)=t^n f\left(x,y\right),\ t\neq 0$$

For instance, let $f\left(x,y\right)=\dfrac{x^3+y^3}{x-y}\cdot$ Replacing x by tx and y by ty, we get

$$f\left(tx,ty\right)=\frac{t^3x^3+t^3y^3}{tx-ty}=\frac{t^3\left(x^3+y^3\right)}{t\left(x-y\right)}=t^2\frac{x^3+y^3}{x-y}=t^2 f\left(x,y\right)$$

Thus, f(x, y) is a homogeneous function of degree 2.

Definition: A differential equation $\dfrac{dy}{dx}=\dfrac{M\left(x,y\right)}{N\left(x,y\right)}$ where M(x, y) and N(x, y) are homogeneous functions of the same degree in x, y is called a homogeneous equation.

Since M(x, y) and N(x, y) are homogeneous functions of same degree in x, y.

Hence, $M\left(x,y\right)=x^n g\left(\dfrac{y}{x}\right)$ and $N\left(x,y\right)=x^n h\left(\dfrac{y}{x}\right)$

Then given differential equation becomes:

$$\frac{dy}{dx}=\frac{x^n g\left(\frac{y}{x}\right)}{x^n h\left(\frac{y}{x}\right)}=\frac{g\left(\frac{y}{x}\right)}{h\left(\frac{y}{x}\right)}=f\left(\frac{y}{x}\right) \quad \left(\text{say}\right) \qquad (1)$$

Put y = vx or $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}\cdot$ Then (1) becomes,

$$v+x\frac{dv}{dx}=f\left(v\right) \text{ or } x\frac{dv}{dx}=f\left(v\right)-v$$

Separating the variables, we have $\dfrac{dv}{f\left(v\right)-v}=\dfrac{1}{x}dx$

This can now easily be integrated. Finally, put v = y/x we get the required solution.

Here we produce the tips that will help to solve the homogeneous equation as given above.

- Put y = vx then $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$
- Put the above values of y and dy/dx in given equation.
- Separate the variables and integrate.
- Replace v by y/x to get the required solution.

Example 01: Solve the differential equation $(x^2 + y^2)\,dx + 2xy\,dy = 0$

Solution: Give that $(x^2 + y^2)\,dx + 2xy\,dy = 0$

This can be re – written as: $\dfrac{dy}{dx} = -\dfrac{x^2+y^2}{2xy}$ (1)

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}\cdot$ Then (1) becomes:

$$x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v \Rightarrow x\frac{dv}{dx} = \frac{-1-v^2-2v^2}{2v} \Rightarrow x\frac{dv}{dx} = -\frac{1+3v^2}{2v}$$

Separating the variables, we get

$$\frac{2v}{1+3v^2}dv = -\frac{1}{x}dx \text{ or } \frac{1}{3}\left(\frac{6v}{1+3v^2}\right)dv = -\frac{1}{x}dx$$

Integrating both the sides, we get: $\dfrac{1}{3}\displaystyle\int\left(\frac{6v}{1+3v^2}\right)dv = -\int\frac{1}{x}dx + \ln C`$

$$\frac{1}{3}\ln\left(1+3v^2\right) = -\ln x + \ln C` \Rightarrow \ln\left(1+3v^2\right) = -3\ln x + 3\ln C` = \ln\frac{C}{x^3}$$

Substituting $v = y/x$ and notice that we have put $3\ln C` = \ln C$.

$$\ln\left(\frac{x^2+3y^2}{x^2}\right) = \ln\frac{C}{x^3} \Rightarrow \left(\frac{x^2+3y^2}{x^2}\right) = \frac{C}{x^3} \Rightarrow x\left(\frac{x^2+3y^2}{x^2}\right) = C \Rightarrow x^2+3y^2 = Cx$$

This is the solution of given differential equation.

Example 02: Solve the differential equation $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0$

Solution: Given that $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0$

This gives $\left(1+e^{x/y}\right)dx = -e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0 \Rightarrow \dfrac{dx}{dy} = \dfrac{-e^{x/y}\left(1-\dfrac{x}{y}\right)}{\left(1+e^{x/y}\right)}$ (1)

Put $\dfrac{x}{y} = v \Rightarrow x = vy,$ then $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$ [See the change in the problem]

Therefore from (1), we have $v + y\dfrac{dv}{dy} = \dfrac{-e^v(1-v)}{1+e^v} \Rightarrow y\dfrac{dv}{dy} = \dfrac{-e^v(1-v)}{1+e^v} - v$

Or, $$y\frac{dv}{dy} = \frac{-\left(e^v - e^v v + v + e^v v\right)}{1+e^v} \Rightarrow y\frac{dv}{dy} = -\frac{v+e^v}{1+e^v}$$

Separating the variables, we have: $\dfrac{1+e^v}{v+e^v}dv = -\dfrac{dy}{y}$

Integrating both sides, we have

$$\int\frac{1+e^v}{v+e^v}dv = -\int\frac{1}{y}dy + \ln c \Rightarrow \ln\left(v+e^v\right) = -\ln y + \ln c = \ln\frac{c}{y}$$

or $v + e^v = c/y$. Substituting $v = x/y$, we get: $\dfrac{x}{y} + e^{x/y} = \dfrac{c}{y} \Rightarrow x + ye^{x/y} = c$

Example 03: Solve the differential equation $x\sin\left(\frac{y}{x}\right)dy = \left[y\sin\left(\frac{y}{x}\right) - x\right]dx$

Solution: The given equation is: $x\sin\left(\frac{y}{x}\right)dy = \left[y\sin\left(\frac{y}{x}\right) - x\right]dx$

Rewrite this as $\frac{dy}{dx} = \frac{\left[y\sin\left(\frac{y}{x}\right) - x\right]}{x\sin\left(\frac{y}{x}\right)} = \frac{x\left[\frac{y}{x}\sin\left(\frac{y}{x}\right) - 1\right]}{x\sin\left(\frac{y}{x}\right)} \Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}\sin\left(\frac{y}{x}\right) - 1}{\sin\left(\frac{y}{x}\right)}$ (1)

Put y = vx then $\frac{dy}{dx} = v + x\frac{dv}{dx}\cdot$ Therefore from (1), we have

$$v + x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} \Rightarrow x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v = \frac{v\sin v - 1 - v\sin v}{\sin v} \Rightarrow x\frac{dv}{dx} = -\frac{1}{\sin v}$$

Separating the variables, we get

$$-\sin v\, dv = \frac{1}{x}dx \Rightarrow -\int \sin v\, dv = \int \frac{1}{x}dx + c \Rightarrow -(-\cos v) = \ln x + c$$

$\cos v = \ln x + c \Rightarrow \cos\left(\frac{y}{x}\right) = \ln x + c$ the solution of given differential equation.

Example 4: Solve the following homogeneous equations:

(i) (x – y) dx + (x + y) dy = 0

Solution: Rearranging the terms, we get: $\frac{dy}{dx} = \frac{y - x}{y + x}$ (1)

Put y = vx or y` = v + x dv/dx. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = \frac{vx - x}{vx + x} = \frac{x(v-1)}{x(v+1)} = \frac{v-1}{v+1} \Rightarrow x\frac{dv}{dx} = \frac{v-1}{v+1} - v = \frac{v - 1 - v^2 - v}{v+1} = -\frac{v^2+1}{v+1}$$

Separating the variables and integrating, we get $\int \frac{v+1}{v^2+1}dv = -\int \frac{1}{x}dx + \ln C$

$$\Rightarrow \frac{1}{2}\int \frac{2v}{v^2+1}dv + \int \frac{1}{v^2+1}dv = -\int \frac{1}{x}dx + \ln C \Rightarrow \ln\sqrt{v^2+1} + \tan^{-1}v = -\ln x + \ln C$$

$\tan^{-1}v = \ln\frac{C}{x\sqrt{v^2+1}}$. Substituting $v = y/x$, we get: $\tan^{-1}\left(\frac{y}{x}\right) = \ln\frac{C}{\sqrt{y^2+x^2}}$

(ii) $(y^2 + 2xy)\,dx + x^2\,dy = 0$

Solution: Rearranging the terms, we get: $\frac{dy}{dx} = -\frac{y^2+2xy}{x^2}$ (1)

Put y = vx or y` = v . 1 + x dv/dx. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = -\frac{v^2x^2 + 2x.vx}{x^2} = -\frac{x^2(v^2+2v)}{x^2} = -(v^2+2v)$$

$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v - v = -(v^2+3v)$. Separating the variables and integrating:

$$\int \frac{1}{v(v+3)}dv = -\int \frac{1}{x}dx + \ln C \qquad (2)$$

Using partial fractions, we have: $\frac{1}{v(v+3)} = \frac{1}{3}\left(\frac{1}{v} - \frac{1}{v+3}\right)$

Thus equation (2) becomes:

$$\Rightarrow \frac{1}{3}\left(\int \frac{1}{v}\, dv - \int \frac{1}{v+3}\, dv\right)dv = -\int \frac{1}{x}dx + \ln C \Rightarrow \ln v - \ln(v+3) = 3[-\ln x + \ln C]$$

$$\Rightarrow \ln \frac{v}{v+3} = 3\ln \frac{C}{x} = \ln\left(\frac{C}{x}\right)^3 \Rightarrow \frac{v}{v+3} = \left(\frac{C}{x}\right)^3$$. Substituting $v = y/x$ and simplifying

we get, $\left(\dfrac{y}{y+3x}\right) = \left(\dfrac{C}{x}\right)^3 \Rightarrow yx^3 = C^3(y+3x)$

(iii) $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

Solution: Rearranging the terms, we get: $\dfrac{dy}{dx} = \dfrac{3y^2 - x^2}{2xy}$ (1)

Put y = vx or y\` = v . 1 + x dv/dx. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = \frac{3v^2x^2 - x^2}{2vx^2} = \frac{x^2(3v^2-1)}{2vx^2} = \frac{(3v^2-1)}{2v} \Rightarrow x\frac{dv}{dx} = \frac{(3v^2-1)}{2v} - v = \frac{3v^2-1-2v^2}{2v} = \frac{(v^2-1)}{2v}$$

Separating the variables and integrating, we get:

$$\int \frac{2v}{v^2-1}dv = \int \frac{1}{x}dx + \ln C \Rightarrow \ln(v^2-1) = \ln Cx \Rightarrow v^2 - 1 = Cx$$

Substituting v = y/x, and simplifying, we get: $y^2 - x^2 = C\,x^3$

(iv) $(x^2 + xy + y^2)\,dx - x^2\,dy = 0$

Solution: Rearranging th terms, we get: $\dfrac{dy}{dx} = \dfrac{x^2 + xy + y^2}{x^2}$ (1)

Put y = vx or y\` = v . 1 + x dv/dx. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = \frac{x^2 + x^2v + x^2v^2}{x^2} = \frac{x^2(1+v+v^2)}{x^2} = 1 + v + v^2 \Rightarrow x\frac{dv}{dx} = 1 + v + v^2 - v = 1 + v^2$$

Separating the variables and integrating, we get: $\displaystyle\int \frac{1}{1+v^2}dv = \int \frac{1}{x}dx + \ln C \Rightarrow \tan^{-1} v = \ln Cx$

Substituting v = y/x, and simplifying, we get: $\tan^{-1}\left(\dfrac{y}{x}\right) = \ln Cx$

(v) $(x^2 + 3\,xy + y^2)\,dx - x^2\,dy = 0$

Solution: Rearranging the terms, we get: $\dfrac{dy}{dx} = \dfrac{x^2 + 3xy + y^2}{x^2}$ (1)

Put y = vx or y\` = v + x dv/dx. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = \frac{x^2 + 3x^2v + x^2v^2}{x^2} = \frac{x^2(1+3v+v^2)}{x^2} = 1 + 3v + v^2 \Rightarrow x\frac{dv}{dx} = 1 + 3v + v^2 - v = 1 + 2v + v^2$$

Separating the variables and integrating, we get:

$$\int \frac{1}{(1+v)^2}dv = \int \frac{1}{x}dx + \ln C \Rightarrow -\frac{1}{(1+v)} = \ln Cx$$

Substituting v = y/x, and simplifying, we get $-\left(\dfrac{x}{x+y}\right) = \ln Cx$

(vi) dy/dx = (4y – 3x)/(2x – y)

Solution: Put y = vx or y\` = v + x dv/dx. Thus given equation becomes

$$v + x\frac{dv}{dx} = \frac{4vx - 3x}{2x - vx} = \frac{x(4v-3)}{x(2-v)} = \frac{(4v-3)}{(2-v)} \Rightarrow x\frac{dv}{dx} = \frac{(4v-3)}{(2-v)} - v = \frac{4v - 3 - 2v + v^2}{(2-v)}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2+2v-3}{-(v-2)}=\frac{(v+3)(v-1)}{-(v-2)}$$

Separating the variables and integrating, we get:

$$\int\frac{v-2}{(v+3)(v-1)}dv=-\int\frac{1}{x}dx+\ln C \quad\Rightarrow\int\frac{v-2}{(v+3)(v-1)}dv=-\int\frac{1}{x}dx+\ln C$$

$$\int\frac{5/4}{(v+3)}dv+\int\frac{-1/4}{(v-1)}dv=-\int\frac{1}{x}dx+\ln C \quad\Rightarrow\frac{1}{4}\big(5\ln(v+3)-\ln(v-1)\big)=\ln\frac{C}{x}$$

Substituting v = y/x, and simplifying, we get: $\frac{1}{4}\left(5\ln\left[\frac{y+3x}{x}\right]-\ln\left[\frac{y-x}{x}\right]\right)=\ln\frac{C}{x}$

(vii) $\left(\mathbf{x^3+y^2\sqrt{x^2+y^2}}\right)\mathbf{dx-xy\sqrt{x^2+y^2}dy=0}$

Solution: Rewriting the equation, we get: $\frac{dy}{dx}=\frac{\left(x^3+y^2\sqrt{x^2+y^2}\right)}{xy\sqrt{x^2+y^2}}$ (1)

Substituting y = vx so that y\` = v + x v\`. Thus equation (1) becomes

$$v+x\frac{dv}{dx}=\frac{x^3+v^2x^2\sqrt{x^2+v^2x^2}}{x(vx)\sqrt{x^2+v^2x^2}}=\frac{x^3(1+v^2\sqrt{1+v^2})}{x^3v\sqrt{1+v^2}}=\frac{(1+v^2\sqrt{1+v^2})}{v\sqrt{1+v^2}}$$

$$x\frac{dv}{dx}=\frac{(1+v^2\sqrt{1+v^2})}{v\sqrt{1+v^2}}-v \quad\Rightarrow x\frac{dv}{dx}=\frac{(1+v^2\sqrt{1+v^2}-v^2\sqrt{1+v^2})}{v\sqrt{1+v^2}}=\frac{1}{v\sqrt{1+v^2}}$$

Separating the variables and integrating, we get:

$$\int v\sqrt{1+v^2}\,dv=\int\frac{1}{x}dx+\ln C\Rightarrow\frac{1}{2}\int\sqrt{1+v^2}\,(2v)dv=\int\frac{1}{x}dx+\ln C$$

$$\frac{2}{3}(1+v^2)^{3/2}=\ln Cx\Rightarrow 2(1+v^2)^{3/2}=3\ln Cx.$$

Substituting v = y/x, we get: $2\left(x^2+y^2\right)^{3/2}=3x^{3/2}\ln Cx$

(viii) $\left(\mathbf{\sqrt{x+y}+\sqrt{x-y}}\right)\mathbf{dx}-\left(\mathbf{\sqrt{x+y}-\sqrt{x-y}}\right)\mathbf{dy=0}$

Solution: Rearranging the equation, we get: $\frac{dy}{dx}=\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$ (1)

Substituting y = vx so that y\` = v + x v\`. Thus equation (1) becomes:

$$v+x\frac{dv}{dx}=\frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\Rightarrow x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}-v$$

$$x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}-v\sqrt{1+v}+v\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}=\frac{\sqrt{1+v}(1-v)+\sqrt{1-v}(1+v)}{\sqrt{1+v}-\sqrt{1-v}}$$

$$=\frac{\sqrt{1+v}\sqrt{1-v}\sqrt{1-v}+\sqrt{1-v}\sqrt{1+v}\sqrt{1+v}}{\sqrt{1+v}-\sqrt{1-v}}=\frac{\sqrt{1+v}\sqrt{1-v}\left(\sqrt{1-v}+\sqrt{1+v}\right)}{\sqrt{1+v}-\sqrt{1-v}}$$

Rationalizing, we get

$$x\frac{dv}{dx}=\frac{\sqrt{1+v}\sqrt{1-v}\left(\sqrt{1-v}+\sqrt{1+v}\right)}{\sqrt{1+v}-\sqrt{1-v}}\times\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}=\frac{\sqrt{1-v^2}\left(\sqrt{1-v}+\sqrt{1+v}\right)^2}{(1+v)-(1-v)}$$

$$=\frac{\sqrt{1-v^2}\left(1-v+1+v-2\sqrt{1-v^2}\right)}{1+v-1+v}=\frac{2\sqrt{1-v^2}\left(1-\sqrt{1-v^2}\right)}{2v}=\frac{\sqrt{1-v^2}\left(1-\sqrt{1-v^2}\right)}{v}$$

Separating the variables and integrating we get

$$\int \frac{v}{\sqrt{1-v^2}\left(1-\sqrt{1-v^2}\right)}dv = \int \frac{1}{x}dx + \ln C = \ln Cx \qquad (2)$$

Substituting $v = \sin\theta$ or $dv = \cos\theta\, d\theta$. Thus left side of equation (2) becomes:

$$\int \frac{\sin\theta}{\cos\theta\,(1-\cos\theta)}\cos\theta\, d\theta = \int \frac{\sin\theta}{(1-\cos\theta)}d\theta = \int \frac{2\sin\theta/2.\cos\theta/2}{2\sin^2\theta/2}d\theta = \int \cot\frac{\theta}{2}\, d\theta$$

Thus equation (2) becomes:

$$\int \cot\frac{\theta}{2}\, d\theta = \ln Cx \Rightarrow 2\ln(\sin\theta/2) = \ln Cx \Rightarrow \ln(\sin\theta/2)^2 = \ln Cx \Rightarrow \sin^2\frac{\theta}{2} = Cx$$

$$\frac{1-\cos\theta}{2} = Cx \Rightarrow 1-\sqrt{1-\sin^2\theta} = 2Cx \Rightarrow 1-\sqrt{1-v^2} = 2Cx$$

Substituting y = vx, we get: $x-\sqrt{x^2-y^2} = 2Cx^2 \Rightarrow x-\sqrt{x^2-y^2} = C_1x^2$ $[\text{NOTE: } C_1 = 2C]$

(ix) $\mathbf{\dfrac{dy}{dx} = \dfrac{x+y}{x}, y(1) = 1}$

Solution: The given equation is homogeneous hence we put y = v x so that y\` = v + x v\`. Thus given differential equation becomes:

$$v + x\frac{dv}{dx} = \frac{x+vx}{x} = 1+v \Rightarrow x\frac{dv}{dx} = 1+v-v = 1$$

Separating the variables and integrating, we get:

$$\int 1.dv = \int \frac{1}{x}dx + \ln C \Rightarrow v = \ln Cx \Rightarrow y/x = \ln Cx \Rightarrow y = x\ln Cx \qquad (1)$$

Now using the initial condition that is put x = 1 and y = 1, we obtain:

1 = ln C, giving C = e^1 = e.

Thus equation (1) becomes: y = x ln ex or y = x [ln x +ln e]

or y = x[ln x + 1]. This is the particular solution. [NOTE: ln e = 1]

Equations Reducible to Homogeneous

The differential equation of the form: $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is not homogeneous.

Method of solving such differential equation depends on the coefficients a_1, b_1, a_2 and b_2. We shall consider two cases:

Case I: Given $\dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ (1)

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$,then it can be reduced to homogeneous form. This is explained as follows: Substituting x = X + h and y = Y + k (2)

where X and Y are new variables and h, k are constants to be chosen so that resulting equation in terms of X and Y becomes homogeneous.

From (2), we have $dx = dX,\ dy = dY \Rightarrow dy/dx = dY/dX$ (3)

Using (2) and (3), equation (1) becomes

$$\frac{dY}{dX} = \frac{a_1(X+h)+b_1(Y+k)+c_1}{a_2(X+h)+b_2(Y+k)+c_2} = \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)} \qquad (4)$$

Now (4) becomes homogeneous if we put:

$$a_1h + b_1k + c_1 = 0 \ \text{ and } \ a_2h + b_2k + c_2 = 0 \qquad (5)$$

Solving (5), we get $h = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$ and $k = \dfrac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}$ (6)

Given that $a_1b_2 - a_2b_1 \neq 0,$ hence, h and k given by (6) are meaningful, that is; h and k exist.

Now h and k are known, so in view of (5), equation (4) reduces to

$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is homogeneous equation in X and Y and can be solved by putting Y/X = v as usual. After getting solution in terms of X and Y, substituting X = x – h and Y = y – k and obtain solution in terms of original variables x and y. This is depicted in the following example.

Example 05: Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$

Solution: The given equation is: $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$ (1)

$\left[\text{Here } a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 1 \Rightarrow \ a_1b_2 - \ a_2b_1 = -3 \neq 0\right]$

Put x = X + h and y = Y + k, therefore dx = dX and dy = dY, then equation (1) becomes

$$\frac{dY}{dX} = \frac{X + h + 2Y + 2k - 3}{2X + 2h + Y + k - 3} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)} \qquad (2)$$

Choose h and k such that $h + 2k - 3 = 0$ and $2h + k - 3 = 0 \cdot$ Multiplying first equation with 2 and subtracting second equation from the new equation, we get

$$3k - 3 = 0 \Rightarrow k = 1$$

Substituting this value of k in the first equation, we get h = 1.

Thus, $k = h = 1$ (3)

Now, from equation (2) we have $\dfrac{dY}{dX} = \dfrac{X + 2Y}{2X + Y}$ (4)

This is homogeneous equation. so putting $Y = vX,$ and $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

Therefore, equation (4) becomes

$$v + X\frac{dv}{dX} = \frac{X + 2vX}{2X + vX} = \frac{1 + 2v}{2 + v} \Rightarrow X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 + 2v - 2v - v^2}{2 + v} = \frac{1 - v^2}{2 + v}$$

Separating the variables, we get: $\dfrac{2 + v}{(1 - v)(1 + v)}dv = \dfrac{1}{X}dX$

Integrating, $\displaystyle\int \frac{2 + v}{(1 - v)(1 + v)}dv = \int \frac{1}{X}dX + c_1$ (5)

Left side of (5) will be solved by resolving into partial fractions.

$$\frac{2 + v}{(1 - v)(1 + v)} = \frac{A}{1 - v} + \frac{B}{1 + v} \qquad (6)$$

$$2 + v = A(1 + v) + B(1 - v) \qquad (7)$$

Put $1 + v = 0 \Rightarrow v = -1$ into (7), we get $B = 1/2$

Put $1 - v = 0 \Rightarrow v = 1$ into (7), we get A = 3/2. Thus, equation (6) becomes

$$\frac{2+v}{(1-v)(1+v)}=\frac{3}{2(1-v)}+\frac{1}{2(1+v)}$$

Integrating, $\int\frac{2+v}{(1-v)(1+v)}dv=-\frac{3}{2}\int\frac{-1}{1-v}dv+\frac{1}{2}\int\frac{1}{1+v}dv=-\frac{3}{2}\ln(1-v)+\frac{1}{2}\ln(1+v)$

Equation (5) becomes, $\ln\left[\frac{1+v}{(1-v)^3}\right]=\ln(c_1X)^2\Rightarrow\left[\frac{1+v}{(1-v)^3}\right]=cX^2\quad[(c_1)^2=c]$

Put $v=\frac{Y}{X}$, we get $\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3}=cX^2\Rightarrow\frac{\frac{X+Y}{X}}{\frac{(X-Y)^3}{X^3}}=cX^2\Rightarrow\frac{X+Y}{(X-Y)^3}=c\Rightarrow X+Y=c(X-Y)^3$

Substituting $X=x-1$ and $Y=y-1$, we get

$$x-1+y-1=c(x-1-y+1)^3\Rightarrow x+y-2=c(x-y)^3$$

This is the solution of given differential equation.

Case II. Given $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, where $\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}=a_1b_2-a_2b_1=0.$

This means either $a_1=a_2$ and $b_1=b_2$ or they are multiple of each other. In this case we substitute $z=a_1x+b_1y$. This will change above equation directly in separable variable form. The procedure is shown in the following example.

Example 06: Solve the differential equation $\frac{dy}{dx}=\frac{x+2y+1}{2x+4y+3}$

Solution: The given differential equation is

$$\frac{dy}{dx}=\frac{x+2y+1}{2x+4y+3}\text{ or }\frac{dy}{dx}=\frac{x+2y+1}{2(x+2y)+3}\qquad(1)$$

Put $x+2y=z$, then $1+2\frac{dy}{dx}=\frac{dz}{dx}$ or $\frac{dy}{dx}=\frac{1}{2}\left(\frac{dz}{dx}-1\right)$. Thus, equation (1) becomes

$$\frac{1}{2}\left(\frac{dz}{dx}-1\right)=\frac{z+1}{2z+3}\Rightarrow\frac{dz}{dx}-1=\frac{2z+2}{2z+3}\Rightarrow\frac{dz}{dx}=\frac{2z+2}{2z+3}+1\Rightarrow\frac{dz}{dx}=\frac{2z+2+2z+3}{2z+3}=\frac{4z+5}{2z+3}$$

Separating the variables, we get $\frac{2z+3}{4z+5}dz=dx$ or $\left[\frac{1}{2}+\frac{1}{2(4z+5)}\right]dz=dx$

Integrating, $\int\left[\frac{1}{2}+\frac{1}{2(4z+5)}\right]dz=\int 1dx+c_1\Rightarrow\frac{1}{2}z+\frac{1}{8}\ln(4z+5)=x+c_1$

$\Rightarrow 4z+\ln(4z+5)=8x+8c_1\Rightarrow 4(x+2y)+\ln(4x+8y+5)=8x+c\;\;(8c_1=c)$

or $4x+8y-8x+\ln(4x+8y+5)=c\Rightarrow 4(2y-x)+\ln(4x+8y+5)=c$

This is the solution of given differential equation.

Example 07: Solve the differential equation $\frac{dy}{dx}=\frac{x-2y+3}{2x-4y+5}$

Solution: The given differential equation is

$$\frac{dy}{dx}=\frac{x-2y+3}{2x-4y+5}\text{ or }\frac{dy}{dx}=\frac{(x-2y)+3}{2(x-2y)+5}\qquad(1)$$

Putting $x-2y=z$, then $1-2\frac{dy}{dx}=\frac{dz}{dx}$ or $\frac{dy}{dx}=-\frac{1}{2}\left(\frac{dz}{dx}-1\right)$

From equation (1), we have

$$\frac{1}{2}\left(1-\frac{dz}{dx}\right)=\frac{z+3}{2z+5}\Rightarrow\frac{dz}{dx}=1-\left(\frac{2z+6}{2z+5}\right)\Rightarrow\frac{dz}{dx}=\frac{2z+5-2z-6}{2z+5}\Rightarrow\frac{dz}{dx}=-\frac{1}{2z+5}$$

Separating the variables, we have

$$(2z+5)dz=-dx\Rightarrow\int(2z+5)dz=-\int 1dx\Rightarrow z^2+5z=-x+c\Rightarrow z^2+5z+x=c$$

Replacing z by x – 2y, we get

$$(x-2y)^2+5(x-2y)+x=c\Rightarrow x^2-4xy+4y^2+5x-10y+x=c$$

or $$x^2-4xy+4y^2+6x-10y=c$$

Exact Differential Equations

We know that general form of first order differential equation is

$$\frac{dy}{dx}=f(x,y)=\frac{M(x,y)}{N(x,y)}$$

This may also be expressed as: $M(x, y)\,dx + N(x, y)\,dy = 0$ (1)

Consider a function f(x, y) of two variables, x and y, and f(x, y) = c. Then ***total or exact differential*** of f(x, y) = c is defined as

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0 \qquad (2)$$

Comparing (1) and (2), we see that, $\frac{\partial f}{\partial x}=M(x,y)$ and $\frac{\partial f}{\partial y}=N(x,y)\Rightarrow\frac{\partial M}{\partial y}=\frac{\partial^2 f}{\partial y\partial x},\ \frac{\partial N}{\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

Now we know that if first partial derivatives of f exist then: $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

$$\Rightarrow \qquad \frac{\partial M}{\partial x}=\frac{\partial N}{\partial y}$$

Therefore, equation (1) is total or exact differential equation, if and only if,

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\quad\Rightarrow M_y=N_x \qquad (3)$$

Equation (3) is the test for differential equation (1) to be exact.

Method of Solving Exact Differential Equation

As mentioned above that a differential equation $M(x, y)\,dx + N(x, y)\,dy = 0$ is said to be exact if $M_y = N_x$. The following example will explain you the procedure to find the solution of exact differential equation.

Example 01: Solve the differential equation $\left(2x^3-6x^2y+3xy^2\right)dx-\left(2x^3-3x^2y+y^3\right)dy=0$

Solution: Given that $\left(2x^3-6x^2y+3xy^2\right)dx-\left(2x^3-3x^2y+y^3\right)dy=0$ (1)

Here $M=2x^3-6x^2y+3xy^2$ and $N=-\left(2x^3-3x^2y+y^3\right)$

Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y} = -6x^2 + 6xy \quad \text{and} \quad \frac{\partial N}{\partial x} = -6x^2 + 6xy$$

Observe that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, which implies that (1) is exact.

Now, $$\frac{\partial f}{\partial x} = M = 2x^3 - 6x^2y + 3xy^2 \tag{2}$$

And $$\frac{\partial f}{\partial y} = N = -2x^3 + 3x^2y - y^3 \tag{3}$$

We try to find f(x, y) from equations (2) and (3). Integrating (2) with respect to x, holding y constant, we obtain $f(x,y) = \int M dx = \int \left(2x^3 - 6x^2y + 3xy^2\right) dx + g(y)$

$$\Rightarrow \quad f(x,y) = \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 + g(y)$$

Here g(y) is constant of integration to be determined.

Note: Since, we integrate equation (1) partially w.r.t x hence constant of integration is a function of y so g(y) in fact is a constant of integration.

To find g(y), we use the fact that the function f must also satisfy (3). Hence,

$$\frac{\partial f}{\partial y} = -2x^3 + 3x^2y + \frac{dg}{dy} = N = -2x^3 + 3x^2y - y^3$$

or $\frac{dg}{dy} = -y^3$. Integrating, we get $g(y) = -\frac{1}{4}y^4$

Thus, $$f(x,y) = \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 - \frac{y^4}{4}$$

Therefore, general solution of the given differential equation is

$$\frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 - \frac{y^4}{4} = c$$

An Alternative Method of Solving Exact Differential Equation

An alternative method which is more quick and easy to apply is presented here. For this the following three steps will help you to find the solution of exact differential equation Mdx + Ndy = 0.

- **I.** First integrate M with respect to x holding y constant.
- **II.** Integrate with respect to y those terms in N which do not contain x.
- **III.** The sum of the expressions so obtained in steps I and II be equated to an arbitrary constant c. This will provide you the solution of given exact differential equation.

Let us solve above differential equation by using this technique.

I. Integrate M w. r. t x keeping y constant

$$\int M dx = \int (2x^3 - 6x^2y + 3xy^2) dx = 2\frac{x^4}{4} - 6y\frac{x^3}{3} + 3y^2\frac{x^2}{2} = \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2$$

II. Integrate those terms of N w.r.t y which do not contain x. This gives

$$-\int y^3 dy = -\frac{y^4}{4}$$

III. Adding the above results and equate to a constant, we get

$$\frac{x^4}{2}-2x^3y+\frac{3}{2}x^2y^2-\frac{y^4}{4}=c$$

This is the same solution as obtained above.

Remark: The above three steps can be summarized as follows to get the solution of Exact Equation.

$$\int Mdx+\int(\text{term in N not containing x})dy=\text{constant}$$

Alternatively, $\int Ndy+\int(\text{term in M not containing y})dx=\text{constant}$

Example 02: Solve $\left(xy\sin xy-\cos xy-e^{2x}\right)dx+\left(y^2-x^2\sin xy\right)dy=0$

Solution: Here $M=xy\sin xy-\cos xy-e^{2x}$ and $N=y^2-x^2\sin xy$

Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y}=x\left(y\cos xy\times x+\sin xy\right)+\sin xy\left(x\right)-0\Rightarrow\frac{\partial M}{\partial y}=x^2y\cos xy+2x\sin xy$$

$$\frac{\partial N}{\partial x}=x^2\cos xy\left(y\right)+\sin xy\left(2x\right)\Rightarrow\frac{\partial N}{\partial x}=x^2y\cos xy+2x\sin xy$$

We see that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. This shows that given differential equation is exact.

Using the alternative method the solution of given differential equation is

$$\int Mdx+\int(\text{term in N not containing x})\,dy=\text{constant}$$

$$\int\left(xy\sin xy-\cos xy-e^{2x}\right)dx+\int\left(-y^2\right)dy=c\ \text{ or }\ y\int x\sin xy\,dx-\int\cos xy\,dx-\int e^{2x}dx-\int y^2dy=c$$

$$y\left[\left(-x\frac{\cos xy}{y}\right)+\frac{1}{y}\int\cos xy\,dx\right]-\int\cos xy\,dx-\frac{1}{2}e^{2x}-\frac{1}{3}y^3=c\quad[\text{using by parts formula}]$$

$$-x\cos\,xy-\frac{1}{2}e^{2x}-\frac{1}{2}y^3=c\Rightarrow x\cos\,xy+\frac{1}{2}e^{2x}+\frac{1}{3}y^3=c_1,\quad(c_1=-c)$$

Alternatively

$$\int N\,dy+\int(\text{term in M not containing y})\,dx=\text{constant}$$

$$\int\left(-y^2+x^2\sin xy\right)dy+\int\left(-e^{2x}\right)dx=\text{constant}\Rightarrow-\frac{1}{3}y^3+x^2\left(-\frac{\cos xy}{x}\right)-\frac{e^{2x}}{2}=c$$

$$-\frac{1}{3}y^3-x\cos\,xy-\frac{1}{2}e^{2x}=c\ \text{ or }\ x\cos\,xy+\frac{1}{2}e^{2x}+\frac{1}{3}y^3=c_1;\ \text{as before.}$$

Example 03: Solve $\left(y\sec^2x+\sec x\tan x\right)dx+\left(\tan x+2y\right)dy=0$

Solution: Here $M=y\sec^2x+\sec x\tan x$ and $N=\tan x+2y$

Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y}=\sec^2x\ ,\ \frac{\partial N}{\partial x}=\sec^2x$$

Thus, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Thus, given differential equation is exact.

Now its solution is $\int M\,dx + \int \left(\text{term in N not containing x}\right) dy = \text{constant}$

$$\int \left(y\sec^2 x + \sec x \tan x \right) dx + \int (2y)\, dy = c \Rightarrow y\tan x + \sec x + y^2 = c$$

Example 04: Solve the following initial value problem

$$\left(2x\cos y + 3x^2 y\right)dx + \left(x^3 - x^2 \sin y - y\right)dy,\ y(0) = 2$$

Solution: Here $M = 2x\cos y + 3x^2 y$ and $N = x^3 - x^2 \sin y - y$

Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y} = -2x\sin y + 3x^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 3x^2 - 2x\sin y$$

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ so the given differential equation is exact.

Now the solution is: $\int M dx + \int \left(\text{term in N not containing x}\right) dy = \text{constant}$

$$\Rightarrow \int \left(2x\cos y + 3x^2 y\right)dx + \int (-y)dy = c \Rightarrow x^2 \cos y + x^3 y - \frac{1}{2}y^2 = c$$

Using the initial conditions, we have: $(0)\cos(2) + (0)(2) - 2 = c \Rightarrow c = -2$

Thus particular solution of given equation is $x^2 \cos y + x^3 y - y^2/2 = -2$

Example 05: Solve the following exact differential equations:

(i) $(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$

Solution: Here $M = (3x^2 + 4xy)$ and $N = (2x^2 + 2y)$. Now,

$\frac{\partial M}{\partial y} = 4x$ and $\frac{\partial N}{\partial x} = 4x \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact. Its solution is:

$$\int M\,dx + \int (\text{terms of N not containing x})\,dy = C$$

$$\int (3x^2 + 4xy)dx + \int 2y\,dy = C \Rightarrow x^3 + 2x^2 y + y^2 = C$$

(ii) $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

Solution: Here $M = (2xy + y - \tan y)$ and $N = (x^2 - x\tan^2 y + \sec^2 y)$. Now,

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2x - \sec^2 y \quad \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence given equation is exact. Its solution is:

$$\int M\,dx + \int (\text{terms of N not containing x})\,dy = C$$

$$\int (2xy + y - \tan y)dx + \int \sec^2 y\,dy = C \Rightarrow x^2 y + xy - x\tan y + \tan y = C$$

(iii) $\frac{x+y}{y-1}dx - \frac{1}{2}\left(\frac{x+1}{y-1}\right)^2 dy = 0$

Solution: Here $M = \frac{x+y}{y-1}$ and $N = -\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2$. Now

$\frac{\partial M}{\partial y} = -\frac{x+1}{(y-1)^2}$ and $\frac{\partial N}{\partial x} = -\frac{x+1}{(y-1)^2} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M\,dx + \int (\text{terms of N not containing x})\,dy = C$

or or $\int \frac{x+y}{y-1}dx - \frac{1}{2}\int \frac{1}{(y-1)^2}dy = C \Rightarrow \frac{1}{y-1}\left(\frac{x^2}{2}+xy\right)+\frac{1}{2(y-1)} = C$

(iv) $\mathbf{\frac{dy}{dx} = -\frac{ax+hy}{hx+by}}$

Solution: Re-writing the given differential equation as:

(ax + hy) dx + (hx + by) dy = 0, we see that M = ax + hy and N = hx + by. Now

$\frac{\partial M}{\partial y} = h$ and $\frac{\partial N}{\partial x} = h \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact. Its solution is:

$\int M\,dx + \int (\text{terms of N not containing x})\,dy = C$

or $\int (ax+hy)\,dx + \int by\,dy = C \Rightarrow \left(\frac{ax^2}{2}+h\,xy\right)+\frac{by}{2} = C$

(v) (1 + ln xy)dx + (1 +x/y) dy = 0

Solution: Here M = (1 + ln xy) = 1 + ln x + ln y and N = 1 + x/y. Now

$\frac{\partial M}{\partial y} = 1/y$ and $\frac{\partial N}{\partial x} = 1/y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact. Its solution is:

$\int M\,dx + \int (\text{terms of N not containing x})\,dy = C$

or $\int (1+\ln x+\ln y)\,dx + \int 1\,dy = C \Rightarrow x + x\ln x - x + y = C \Rightarrow x\ln x - y = C$

(vi) $\mathbf{\frac{ydx+xdy}{1-x^2y^2} + xdx = 0}$

Solution: Re-writing given differential equation as:

$\left(\frac{y}{1-x^2y^2}+x\right)dx + \frac{x}{1-x^2y^2}dy = 0$. Here $M = \left(\frac{y}{1-x^2y^2}+x\right)$ and $N = \frac{x}{1-x^2y^2}$. Now,

$\frac{\partial M}{\partial y} = \left(1+x^2y^2\right)/\left(1-x^2y^2\right)^2, \frac{\partial N}{\partial x} = \left(1+x^2y^2\right)/\left(1-x^2y^2\right)^2$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M\,dx + \int (\text{terms of N not containing x})\,dy = C.$

Remark: Since N has no term that is free from x, hence we skip this step.

or $\int \frac{y}{\left(1-x^2y^2\right)}dx + \int x\,dx = C \Rightarrow \frac{y}{y^2}\int \frac{1}{\left((1/y)^2-x^2\right)}dx + \frac{x^2}{2} = C$

$\frac{1}{y}\frac{1}{(2/y^2)}\ln\left(\frac{(1/y)+x}{(1/y)-x}\right)+\frac{x^2}{2} = C \Rightarrow y\ln\left(\frac{1+xy}{1-xy}\right)+x^2y = 2C.$

NOTE: $\int \frac{1}{(a^2-x^2)}dx = \frac{1}{2a}\ln\frac{(a+x)}{(a-x)}$

(vii) (6xy + 2y^2 – 5) dx + (3x^2 + 4xy – 6) dy = 0

Solution: Here M = (6xy + 2y^2 – 5) and N = (3x^2 + 4xy – 6). Thus,

$\dfrac{\partial M}{\partial y} = 6x + 4y$ and $\dfrac{\partial N}{\partial x} = 6x + 4y \Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

or $\int (6xy + 2y^2 - 5)\,dx + \int -6\, dy = C \quad \Rightarrow 3x^2y + 2xy^2 - 5x - 6y = C$

(viii) (y cos x + 2x e^y) dx + (sin x + x^2 e^y – 1) dy = 0

Solution: Here M = (y cos x + 2x e^y) and N = (sinx + x^2 e^y – 1). Thus,

$\dfrac{\partial M}{\partial y} = \cos x + 2x e^y$ and $\dfrac{\partial N}{\partial x} = \cos x + 2x e^y \Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

or $\int (6xy + 2y^2 - 5)\,dx + \int -6\, dy = C \quad \Rightarrow 3x^2y + 2xy^2 - 5x - 6y = C$

(ix) (y e^{xy} cos 2x – 2 e^{xy} sin 2x + 2x) dx + (x e^{xy} cos 2x – 3) dy = 0

Solution: M = (y e^{xy} cos 2x – 2 e^{xy} sin 2x + 2x) and N = (x e^{xy} cos 2x – 3) dy. Thus,

$\dfrac{\partial M}{\partial y} = e^{xy}[\cos 2x + xy \cos 2x - 2x \sin 2x]$ and $\dfrac{\partial N}{\partial x} = e^{xy}[\cos 2x + xy \cos 2x - 2x \sin 2x]$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, hence given equation is exact. Its solution is

$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

or $\int (y e^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x)\,dx + \int -3\, dy = C$

or $y \int y e^{xy} \cos 2x\, dx - 2\int e^{xy} \sin 2x\, dx + 2\int x\, dx - 3\int 1\, dy = C$

$$y \frac{e^{xy}}{y^2+4}[y \cos 2x + 2 \sin 2x] - 2\frac{e^{xy}}{y^2+4}[y \sin 2x - 2\cos 2x] + x^2 - 3y = C$$

NOTE: The following TWO formulae are used here.

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{(a^2+b^2)}[a \sin bx - b \cos bx] \;\&\; \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{(a^2+b^2)}[a \cos bx + b \sin bx]$$

Non–Exact Differential Equations

If equation: $\quad M(x, y)dx + N(x, y)dy = 0,$

is not exact, it may be possible to multiply it by a function $\mu(x, y)$ so that the resulting equation $\mu\, M(x, y)dx + \mu\, N(x, y)dy = 0$ is exact. Such a function $\mu(x, y)$ is called an ***integrating factor*** and we find the solution of the original equation by solving the new (exact) equation.

For instance, the equation $\quad (3x + 2y)\, dx + \left(x^2 + x + xy\right) dy = 0 \qquad (1)$

is not exact because, for $M = 3x + 2y, N = x^2 + x + xy,$ we have

$$\frac{\partial M}{\partial y} = 2 \text{ and } \frac{\partial N}{\partial x} = 2x + 1 + y \quad \Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

If we multiply given equation by $\mu(x, y) = xe^y$, we get

$$\left(3x^2 + 2xy\right)e^y dx + \left(x^3 + x^2 + x^2y\right)e^y dy = 0 \qquad (2)$$

Now, for equation (2), we have $M = \left(3x^2 + 2xy\right) e^y,\ N = \left(x^3 + x^2 + x^2y\right) e^y$

$$\frac{\partial M}{\partial y}=\left(3x^2+2xy\right)e^y+e^y\left(2x\right)=\left(3x^2+2x+2xy\right)e^y,\ \frac{\partial N}{\partial x}=\left(3x^2+2x+2xy\right)e^y$$

$$\Rightarrow \qquad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Thus, equation (2) is exact and $\mu\left(x,y\right)=xe^y$ is an integrating factor.

We find solution of equation (2) by considering:

$$\int M dx+\int\left(\text{term in N not containing x}\right)dy=\text{constant}$$

$$\int\left(3x^2+2xy\right)e^y+\int 0\, dy=c \Rightarrow x^3e^y+x^2ye^y=c.$$ This is the general solution.

Rules for Finding Integrating Factor

The problem of finding an integrating factor $\mu\left(x,y\right)$ for the equation $Mdx+Ndy=0$ is generally difficult. However, in some special cases, it is easy to determine $\mu\left(x,y\right)\cdot$

RULE 1: If $\dfrac{M_y-N_x}{N}$ is a function of x alone, say f(x) then $e^{\int f(x)dx}$ is an integrating factor of equation $M\left(x,y\right)dx+N\left(x,y\right)dy=0\cdot$

Example 01: Solve $\left(x^2+y^2\right)dx-2xy\,dy=0$

Solution: The given equation is: $\left(x^2+y^2\right)dx-2xy\,dy=0$ (1)

Here $M=x^2+y^2, N=-2xy \Rightarrow \dfrac{\partial M}{\partial y}=2y, \dfrac{\partial N}{\partial x}=-2y \Rightarrow \dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

This implies that (1) is not exact.

Now by **RULE 1,** $\dfrac{M_y-N_x}{N}=\dfrac{2y+2y}{-2xy}=\dfrac{4y}{-2xy}=-\dfrac{2}{x}=f\left(x\right)$

Therefore, $I.F.=e^{\int f(x)dx}=e^{-2\int\frac{1}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}=\dfrac{1}{x^2}$

Multiplying (1) by this I.F., we get $\left(1+\dfrac{y^2}{x^2}\right)dx-\dfrac{2y}{x}dy=0$ (2)

For the new equation, we have: $M=1+\dfrac{y^2}{x^2}, N=-\dfrac{2y}{x}\cdot$

Therefore, $\dfrac{\partial M}{\partial y}=\dfrac{2y}{x^2}, \dfrac{\partial N}{\partial x}=\dfrac{2y}{x^2} \Rightarrow \dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$. This shows that equation (2) is exact.

Now the solution is found by using the formula:

$$\int M dx+\int\left(\text{term in N not containing x}\right)dy=\text{constant}$$

$$\int\left(1+\frac{y^2}{x^2}\right)dx+\int 0\,dy=c \Rightarrow x+y^2\left(-\frac{1}{x}\right)=c \Rightarrow x^2-y^2=cx$$

RULE 2: If $\dfrac{N_x-M_y}{M}$ is a function of y alone, say g(y) then $e^{\int g(y)dy}$ is an integrating factor of the equation $M\left(x,y\right)dx+N\left(x,y\right)dy=0\cdot$

Example 02: Solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$

Solution: The given equation is

$$\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0 \qquad (1)$$

Here, $M=y^4+2y,\ N=xy^3+2y^4-4x$

$\dfrac{\partial M}{\partial y}=4y^3+2,\ \dfrac{\partial N}{\partial x}=y^3-4\Rightarrow\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$. Therefore, equation (1) is not exact.

Now by **RULE 2,** $\dfrac{N_x-M_y}{M}=\dfrac{y^3-4-4y^3-2}{y^4+2y}=\dfrac{-3\left(y^3+2\right)}{y\left(y^3+2\right)}=-\dfrac{3}{y}$

Therefore, $\text{I.F.}=e^{-3\int\frac{1}{y}dy}=e^{-3\ln y}=e^{\ln y^{-3}}=y^{-3}=\dfrac{1}{y^3}$.

Multiplying (1) by I.F., we get $\left(y+\dfrac{2}{y^2}\right)dx+\left(x+2y-\dfrac{4x}{y^3}\right)dy=0 \qquad (2)$

For the new equation, we have $M=y+\dfrac{2}{y^2},N=x+2y-\dfrac{4x}{y^3}$

$\dfrac{\partial M}{\partial y}=1-\dfrac{4}{y^3},\dfrac{\partial N}{\partial x}=1-\dfrac{4}{y^3}\Rightarrow\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ thus, equation (2) is exact. Now the solution is:

$$\int Mdx+\int\left(\text{term in N not containing x}\right)dy=\text{constant}$$

$$\int\left(y+\frac{2}{y^2}\right)dx+\int 2ydy=c\Rightarrow xy+\frac{2x}{y^2}+y^2=c$$

This is solution of given differential equation.

RULE 3: If and the equation $Mdx+Ndy=0$ has the form $yf\left(xy\right)dx+xg\left(xy\right)dy=0,$ then

$\text{I.F}=\dfrac{1}{xM-yN}$, provided $xM-yN\neq 0.$

Example 03: Solve $\left(3y+4xy^2\right)dx+\left(2x+3x^2y\right)dy=0$

Solution: The given differential equation is

$$\left(3y+4xy^2\right)dx+\left(2x+3x^2y\right)dy=0 \qquad (1)$$

Here, $M=3y+4xy^2,\ N=2x+3x^2y\Rightarrow\dfrac{\partial M}{\partial y}=3+8xy,\ \dfrac{\partial N}{\partial x}=2+6xy\Rightarrow\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$

Thus, equation (2) is not exact.

Now, equation (1) can be re – written as: $y\left(3+4xy\right)dx+x\left(2+3xy\right)dy=0$

This if of the form $y\,f(x\,y)\,dx+x\,g(x\,y)\,dy=0$

Now, $xM-yN=3xy+4x^2y^2-2xy-3x^2y^2=xy+x^2y^2\neq 0.$

Thus by **Rule 3,** $\dfrac{1}{xy+x^2y^2}$ is an I.F. Multiplying equation (1) by I.F., we get

$$\frac{\left(3+4xy\right)y}{\left(1+xy\right)xy}dx+\frac{\left(2+3xy\right)x}{\left(1+xy\right)xy}dy=0\Rightarrow\frac{\left(3+4xy\right)}{\left(1+xy\right)x}dx+\frac{\left(2+3xy\right)}{\left(1+xy\right)y}dy=0 \qquad (2)$$

Equation (2) is an exact differential equation. It may now be written as

$$\frac{2+2xy+1+2xy}{\left(1+xy\right)x}dx+\frac{2+2xy+xy}{\left(1+xy\right)y}dy=0$$

$$\Rightarrow \left[\frac{2(1+xy)}{(1+xy)x}+\frac{1+2xy}{(1+xy)x}\right]dx+\left[\frac{2(1+xy)}{(1+xy)y}+\frac{xy}{(1+xy)y}\right]dy=0$$

or $$\left[\frac{2}{x}+\frac{1+xy}{(1+xy)x}+\frac{xy}{(1+xy)x}\right]dx+\left[\frac{2}{y}+\frac{x}{(1+xy)}\right]dy=0$$

or $$\left[\frac{2}{x}+\frac{1}{x}+\frac{y}{(1+xy)}\right]dx+\left[\frac{2}{y}+\frac{x}{(1+xy)}\right]dy=0 \qquad (3)$$

Now the solution is: $\int Mdx+\int(\text{term in N not containing x})dy=c$

$$\int\left[\frac{2}{x}+\frac{1}{x}+\frac{y}{(1+xy)}\right]dx+\int\frac{2}{y}dy=c\Rightarrow 3\int\frac{1}{x}dx+\int\frac{y}{(1+xy)}dx+2\int\frac{1}{y}dy=c$$

$$3\ln x+\ln(1+xy)+2\ln y=c\Rightarrow \ln x^3+\ln y^2+\ln(1+xy)=\ln k \quad (c=\ln k)$$

Or $\ln x^3 y^2 (1+xy)=\ln k$ or $x^3 y^2 (1+xy)=k$ is the general solution.

RULE 4: If in equation Mdx + Ndy = 0, M and N are homogeneous functions of same degree in x and y, then $\frac{1}{(xM+yN)}$ is an I.F provided $(xM+yN)\neq 0$.

Example 04: Solve $(3xy+y^2)dx+(x^2+xy)dy=0$

Solution: The given equation is $(3xy+y^2)dx+(x^2+xy)dy=0 \qquad (1)$

Now, $M=3xy+y^2, N=x^2+xy$ and $\frac{\partial M}{\partial y}=3x+2y, \frac{\partial N}{\partial x}=2x+y\Rightarrow\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ so (1) is not exact.

Observe that both M and N are homogeneous functions of degree 2. Therefore

$$\text{I.F.}=\frac{1}{xM+yN}=\frac{1}{3x^2y+xy^2+x^2y+xy^2}=\frac{1}{4x^2y+2xy^2}=\frac{1}{2xy(2x+y)}$$

Multiplying (1) by I.F, we get

$$\frac{(3xy+y^2)}{2xy(2x+y)}dx+\frac{(x^2+xy)}{2xy(2x+y)}dy=0\Rightarrow\frac{3x+y}{2x(2x+y)}dx+\frac{x+y}{2y(2x+y)}dy=0 \qquad (2)$$

This is an exact equation. Now, equation (2) may be written as:

$$\frac{2x+y+x}{2x(2x+y)}dx+\frac{2x+y-x}{2y(2x+y)}dy=0$$

$$\Rightarrow\left[\frac{2x+y}{2x(2x+y)}+\frac{x}{2x(2x+y)}\right]dx+\left[\frac{2x+y}{2y(2x+y)}-\frac{x}{2y(2x+y)}\right]dy=0$$

$$\Rightarrow\left[\frac{1}{2x}+\frac{1}{2(2x+y)}\right]dx+\left[\frac{1}{2y}-\frac{x}{2y(2x+y)}\right]dy=0 \qquad (3)$$

Now the solution is $\int Mdx+\int(\text{term in N not containing x})dy=c$

$$\Rightarrow\int\left[\frac{1}{2x}+\frac{1}{2(2x+y)}\right]dx+\int\frac{1}{2y}dy=c\Rightarrow\frac{1}{2}\int\frac{1}{x}dx+\frac{1}{2}\cdot\frac{1}{2}\int\frac{2}{2x+y}dx+\frac{1}{2}\int\frac{1}{y}dy=c$$

or $$\frac{1}{2}\ln x+\frac{1}{4}\ln(2x+y)+\frac{1}{2}\ln y=c\Rightarrow 2\ln x+\ln(2x+y)+2\ln y=4c$$

or $\ln x^2+\ln(2x+y)+\ln y^2=\ln k$ or $\ln[x^2(2x+y)y^2]=\ln k$ giving $x^2+y^2(2x+y)=k$.

This is the solution of given equation.

RULE 5: For appropriate values of m and n, $x^m y^n$ may be an integrating factor of equation $Mdx + Ndy = 0\cdot$ The procedure of finding m and n is illustrated in the following examples.

Example 05: Solve $\left(y^3 - 2yx^2\right) dx + \left(2xy^2 - x^3\right) dy = 0$

Solution: The given equation is

$$\left(y^3 - 2yx^2\right) dx + \left(2xy^2 - x^3\right) dy = 0 \qquad (1)$$

$M = y^3 - 2yx^2,\ N = 2xy^2 - x^3 \Rightarrow \dfrac{\partial M}{\partial y} = 3y^2 - 2x^2,\ \dfrac{\partial N}{\partial x} = 2y^2 - 3x^2$ so equation (1) is not exact.

Let $x^m y^n$ be an integrating factor of (1). Then equation (1) becomes

$x^m y^n \left(y^3 - 2yx^2\right) dx + x^m y^n \left(2xy^2 - x^3\right) dy = 0$ which is assumed to be exact.

Hence, $\dfrac{\partial}{\partial y}\left[x^m y^n \left(y^3 - 2yx^2\right)\right] = \dfrac{\partial}{\partial x}\left[x^m y^n \left(2xy^2 - x^3\right)\right]$

or $x^m \left\{y^n \left(3y^2 - 2x^2\right) + \left(y^3 - 2yx^2\right) ny^{n-1}\right\} = y^n \left\{x^m \left(2y^2 - 3x^2\right) + \left(2xy^2 - x^3\right) mx^{m-1}\right\}$

$\Rightarrow x^m \left(3y^{n+2} - 2x^2 y^n + 2y^{n+2} - 2ny^n x^2\right) = y^n \left(2x^m y^2 - 3x^{m+2} + 2mx^m y^2 - mx^{m+2}\right)$

or $(n+3)\, x^m y^{n+2} - 2(n+1)\, x^{m+2} y^n = 2(m+1)\, x^m y^{n+2} - (m+3)\, x^{m+2} y^n$

Equating the coefficients of like powers of x and y, we obtain

$$n + 3 = 2(m+1) \quad \text{and} \quad 2(n+1) = m + 3$$

Solving these equations simultaneously, we get m = n = 1.

Thus, the integrating factor is xy. Therefore equation (1) when multiplied by xy gives:

$$\left(xy^4 - 2y^2x^3\right) dx + \left(2x^2y^3 - x^4y\right) dy = 0 \qquad (2)$$

which is exact, because $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 4xy^3 - 4x^3y$. The solution of (2) is, therefore,

$$\int Mdx + \int (\text{term in N not containing x})dy = \text{constant}$$

$$\int \left(xy^4 - 2y^2x^3\right)dx + \int 0dy = c \Rightarrow \frac{1}{2}x^2y^4 - \frac{2}{4}x^4y^2 + c_1 = c$$

$$x^2y^4 - x^4y^2 = k \Rightarrow x^2y^2\left(y^2 - x^2\right) = k\cdot \ \left[2(c - c_1) = k\right]$$

Example 06: Solve the following differential equations by finding appropriate Integrating Factor.

(i) (x y² + y) dx – x dy = 0 (1)

Solution: Here M = $(x^2\, y + y)$ and N = – x. Thus,

$\dfrac{\partial M}{\partial y} = 2xy + 1$ and $\dfrac{\partial N}{\partial x} = -1$. We see that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ so given equatuion is not exact.

To find an I.F we apply RULE I: $\dfrac{M_y - N_x}{N} = \dfrac{2xy + 1 + 1}{-x} \neq P(x)$ hence, Rule I fails.

RULE II: $\dfrac{N_x - M_y}{M} = \dfrac{-1 - 2xy - 1}{y(xy+1)} = \dfrac{-2(xy+1)}{y(xy+1)} = \dfrac{-2}{y} = P(y)$. Thus

$I.F = e^{\int P(y)dy} = e^{-2\int dy/y} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = 1/y^2$

Multiply (1) by I.F, we get: $\frac{1}{y^2}\left(xy^2+y\right)dx - \frac{x}{y^2}dy = 0$ (2)

Equation (2) is now exact. Its solution is:

$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$\int (x+1/y)\,dx = C \Rightarrow \frac{x^2}{2}+\frac{x}{y} = C \Rightarrow x^2y + 2x = 2Cy$ NOTE: N contains no term free of x.

(ii) x dy – y dx = (x^2 + y^2) dx

Solution: Given differential equation may be rewritten as: $\frac{xdy - ydx}{x^2+y^2} = dx$

This is equivalent to: $d\left(\tan^{-1}\frac{x}{y}\right) = dx$. Now integrating both sides, we obtain: $\tan^{-1}\frac{x}{y} = x + c$.

This is the solution of given differential equation.

(iii) (x^2 + x – y) dx + x dy = 0 (1)

Solution: Here M = (x^2 + x - y) and N = x. Thus

$\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Hence given equatuion is not exact.

To find an I.F we apply: RULE I: $\frac{M_y - N_x}{N} = \frac{-1-1}{x} = \frac{-2}{x} = P(x)$. Thus

$I.F = e^{\int P(x)dx} = e^{-2\int dx/x} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = 1/x^2$

Multiply (1) by I.F, we get: $\frac{1}{x^2}\left(x^2+x-y\right)dx + \frac{1}{x}dy = 0$ (2)

Equation (2) is now exact. Its solution is $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$\int \left(1+(1/x) - yx^{-2}\right)dx = C \Rightarrow x + \ln x + \frac{y}{x} = C \Rightarrow x^2 + x\ln x + y = Cx$

NOTE: Here, N contains no term which is free of x.

(iv) $\mathbf{dy + \frac{y - \sin x}{x}dx = 0}$ (1)

Solution: Here M = (y/x) – (sin x/x) and N = 1. Thus

$\frac{\partial M}{\partial y} = 1/x$ and $\frac{\partial N}{\partial x} = 0$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Hence given equatuion is not exact.

To find an I.F we apply RULE I: $\frac{M_y - N_x}{N} = \frac{1/x - 0}{1} = \frac{1}{x} = P(x)$.

Thus, $I.F = e^{\int P(x)dx} = e^{\int dx/x} = e^{\ln x} = e^{\ln x} = x$. Multiply (1) by I.F, we get

$\left(y - \sin x\right)dx + x\,dy = 0$ (2)

Equation (2) is now exact. Its solution is: $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$y\int 1.dx - \int \sin x\,dx = C \quad \Rightarrow yx + \cos x = C$ [NOTE: N contains no term which is free of x]

(v) y (2xy + e^x) dx – e^x dy = 0 (1)

Solution: Here M = $2xy^2 + y\,e^x$ and N = – e^x. Thus

$\dfrac{\partial M}{\partial y} = 4xy + e^x$ and $\dfrac{\partial N}{\partial x} = -e^x$. We see that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Hence given equatuion is not exact.

To find an I.F we apply RULE I: $\dfrac{M_y - N_x}{N} = \dfrac{4xy + e^x + e^x}{-e^x} \neq P(x)$. Hence RULE I fails.

RULE II: $\dfrac{N_x - M_y}{M} = \dfrac{-e^x - 4xy - e^x}{y(2xy + e^x)} = -2\dfrac{(2xy + e^x)}{y(2xy + e^x)} = -2/y = P(y)$. Thus,

$\text{I. F} = e^{\int P(y)\,dy = -\frac{1}{2}\int \frac{1}{y} dy = -\frac{1}{2}\ln y} = e^{\ln y^{-2} = y^{-2} = \frac{1}{y^2}}.$

Multiply equation (1) by I.F, we get $\dfrac{1}{y^2}\left[y\ (2xy\ +\ e^x)\ dx\ -\ e^x dy\ \right] = 0$

Or $\left(2x + \dfrac{e^x}{y^2}\right)dx - \dfrac{e^x}{y^2}dy = 0$. Its solution is: $\int M\,dx + \int(\text{terms of } N \text{ not containing } x)\ dy = C$

$\Rightarrow 2\int x.dx + y^{-2}\int e^x\,dx = C \quad \Rightarrow x^2 + e^x y^{-2} = C \Rightarrow x^2y^2 + e^x = Cy^2$

NOTE: N contains no term which is free from x.

(vi) $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$ (1)

Solution: Here $M = x^2 + y^2 + 2x$ and $N = 2y$. Thus

$\dfrac{\partial M}{\partial y} = 2y$ and $\dfrac{\partial N}{\partial x} = 0$. We see that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

RULE I: $\dfrac{M_y - N_x}{N} = \dfrac{2y - 0}{2y} = 1 = P(x)$. Thus, $\text{I.F} = e^{\int 1.dx} = e^x$.

Multiply (1) by I.F, we get: $e^x(x^2 + y^2 + 2x)dx + 2e^x y\,dy = 0$ (2)

Equation (2) is now exact. Its solution is $\int M\,dx + \int(\text{terms of } N \text{ not containing } x)\ dy = C$

$\int(x^2 + y^2 + 2x)e^x\,dx = C$. Integrating by parts, we get $(x^2 + y^2 + 2x)e^x - \int(2x + 0 + 2)e^x dx = C$

or $(x^2 + y^2 + 2x)e^x - 2\int(x+1)e^x dx = C$. Integrating by parts once again, we get

$(x^2 + y^2 + 2x)e^x - 2\left[(x+1)e^x - \int 1.e^x dx\right] = C$ or $(x^2 + y^2 + 2x)e^x - 2\left[(x+1)e^x - e^x\right] = C$

$\Rightarrow (x^2 + y^2 + 2x)e^x - 2xe^x = C$ or $(x^2 + y^2 + 2x - 2x)e^x = C$ giving $(x^2 + y^2)e^x = C$

Thus solution of given differential equation is $(x^2 + y^2)e^x = C$

$(x^2 + y^2)e^x = C$NOTE: N contains no term free from x.

(vii) $(4x + 3y^2)\,dx + 2xy\,dy = 0$ (1)

Solution: Here $M = 4x + 3y^2$ and $N = 2xy$. Thus

$\dfrac{\partial M}{\partial y} = 6y$ and $\dfrac{\partial N}{\partial x} = 2y$. We see that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I. F, we apply

RULE I: $\dfrac{M_y - N_x}{N} = \dfrac{6y - 2y}{2xy} = 2/x = P(x)$. Thus, $\text{I.F} = e^{2\int 1/x\,.dx} = e^{2\ln x} = e^{\ln x^2} = x^2$.

Multiply (1) by I.F, we get:

$(4x^3 + 3x^2y^2)\,dx + 2x^3y\,dy = 0$ (2)

Equation (2) is now exact. Its solution is $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$4\int x^3 dx + 3y^2 \int x^2\,dx = C \;\Rightarrow x^4 + x^3y^2 = C$ [NOTE: N contains no term free of x]

(viii) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$ (1)

Solution: Here $M = (3x^2y^4 + 2xy)$ and $N = (2x^3y^3 - x^2)$. Thus

$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$ and $\frac{\partial N}{\partial x} = 6x^2y^3 - 2x$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply

$$\text{RULE I}: \frac{M_y - N_x}{N} = \frac{12x^2y^3 + 2x - 6x^2y^3 + 2x}{x^2(2xy^3 - 1)} = \frac{6x^2y^3 + 4x}{x^2(2xy^3 - 1)} \neq P(x).\text{[RULE I fails]}$$

$$\text{RULE II}: \frac{N_x - M_y}{M} = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{xy(3xy^3 + 2)} = \frac{-6x^2y^3 - 4x}{xy(3xy^3 + 2)}$$

$$= \frac{-2x(3x^2y^3 + 2)}{xy(3xy^3 - 1)} = \frac{-2}{y} = P(y)$$

Thus, $\text{I.F} = e^{-2\int 1/y\,.dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2}$. Multiply (1) by I.F, we get:

$(3x^2y^2 + 2xy^{-1})\,dx + (2x^3y - x^2y^{-2})\,dy = 0$ (2)

Equation (2) is now exact. Its solution is $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$\int (3x^2y^2 + 2xy^{-1})\,dx = C \;\Rightarrow x^3y^2 + \frac{x^2}{y} = C$ [NOTE: N contains no term free of x]

(ix) y – x y` = x + y y`

Solution: Re-writing the given differential equation, we obtain:

$$(x - y)\,dx + (x + y)\,dy = 0 \qquad (1)$$

This is a homogeneous equation where $M = x - y$ and $N = x + y$. Now

$\frac{\partial M}{\partial y} = -1$ and $\frac{\partial N}{\partial x} = 1$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equation is not exact. To find an I.F we have

$$\text{RULE III}: \text{I.F} = \frac{1}{xM + yN} = \frac{1}{x^2 - xy + xy + y^2} = \frac{1}{x^2 + y^2}$$

Now multiply equation (1) by an I.F, we obtain:

$$\frac{x - y}{x^2 + y^2}\,dx + \frac{x + y}{x^2 + y^2}\,dy = 0 \qquad (2)$$

Equation (2) is now exact. Its solution is obtained as follows:

$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$$\Rightarrow \int \frac{x}{x^2 + y^2}\,dx - \int \frac{y}{x^2 + y^2}\,dx = C \;\Rightarrow \frac{1}{2}\int \frac{2x}{x^2 + y^2}\,dx - y.\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right) = C$$

$$\frac{1}{2}\ln(x^2 + y^2) - \tan^{-1}\left(\frac{x}{y}\right) = C.$$

NOTE: Here in solving the integral, y is treated as constant. Function N contains no term of x and finally the formula $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$ is used.

(x) y` = e^{2x} + y – 1 = 0 (1)

Solution: Given equation may be re-written as: $dy = (e^{2x} + y - 1)\,dx$

Or $(1 - y - e^{2x})\,dx - dy = 0$ (1)

Here $M = (e^{2x} + y - 1)$ and $N = -1$. Thus, $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = -1$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply

$$\text{RULE I}: \frac{M_y - N_x}{N} = \frac{1+1}{(-1)} = -2 = P(x).\ \text{Thus, I.F} = e^{\int P(x)\,dx} = e^{-2\int 1\,.dx} = e^{-2x}.$$

Multiply (1) by I.F, we get: $e^{-2x}(1 - y - e^{2x})\,dx - e^{-2x}\,dy = 0$ (2)

Equation (2) is now exact. Its solution is:

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$\int e^{-2x}dx - y\int e^{-2x}dx - \int 1\,dx = C \Rightarrow -\frac{e^{-2x}}{2}(1-y) - x = C$$

NOTE: N contains no term free from x.

(xi) (y^2 + xy) dx – x^2 dy = 0 (1)

Solution: Given equation is a homogeneous equation where $M = (y^2 + xy)$ and $N = -x^2$. Now,

$\frac{\partial M}{\partial y} = 2y + x$ and $\frac{\partial N}{\partial x} = 2x$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

$$\text{RULE III}: \text{I.F} = \frac{1}{xM + yN} = \frac{1}{xy^2 + x^2y - yx^2} = \frac{1}{xy^2}$$

Now multiply equation (1) by an I.F, we obtain:

$$\left(\frac{1}{x} + \frac{1}{y}\right)dx - \frac{x}{y}dy = 0 \qquad (2)$$

Equation (2) is now exact. Its solution is obtained as follows:

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$\Rightarrow \int \frac{1}{x}dx + \frac{1}{y}\int dx = C \Rightarrow \ln x + \frac{x}{y} = C \quad [\text{NOTE: N contains no term free of } x]$$

(xii) ($3x^2y$ + 2xy + y^3) dx + (x^2 + y^2) dy = 0 (1)

Solution: Here $M = (3x^2y + 2xy + y^3)$ and $N = (x^2 + y^2)$. Thus,

$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2$ and $\frac{\partial N}{\partial x} = 2x$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply

$$\text{RULE I}: \frac{M_y - N_x}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = \frac{3(x^2 + y^2)}{(x^2 + y^2)} = 3 = P(x)$$

Thus, $I.F = e^{\int P(x)\,dx} = e^{3x}$. Multiply eq(1) by I.F, we get:

$$e^{3x}(3x^2y + 2xy + y^2)\,dx + e^{3x}(x^2 + y^2)\,dy = 0 \qquad (2)$$

Equation (2) is now exact. Its solution is:

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$3y\int e^{3x}x^2dx + 2y\int xe^{3x}\,dx + y^2\int e^{3x}dx = C \quad [\text{NOTE: N contains no term free of x}]$$

Integrating by parts and simplifying, we obtain the solution: $e^{3x}(x^2y + y^3/3) = C$

(xiii) y dx + (2xy – e^-2y^) dy = 0 (1)

Solution: Here M = y and N = $(2xy - e^{-2y})$. Thus

$$\frac{\partial M}{\partial y} = 1 \text{ and } \frac{\partial N}{\partial x} = 2y. \text{ We see that } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence given equatuion is not exact.To find an I.F we apply

$$\text{RULE I}: \frac{M_y - N_x}{N} = \frac{1-2y}{2xy - e^{-2y}} \neq P(x).\text{RULE I fails.}$$

$$\text{RULE II}: \frac{N_x - M_y}{M} = \frac{2y-1}{y} = (2 - 1/y) = P(y)$$

$$\text{Thus, } I.F = e^{\int P(y)\,dy} = e^{2\int dy - \int 1/y\,dy} = e^{2y - \ln y} = e^{2y}.e^{-\ln y} = y^{-1}e^{2y}.$$

Multiply eq(1) by I.F, we get: $e^{2y}\,dx + (2xe^{2y} - y^{-1})\,dy = 0$ (2)

Equation (2) is now exact. Its solution is: $\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$

$$e^{2y}\int 1.dx - \int 1/y\;dy = C \;\Rightarrow\; xe^{2y} - \ln y = C$$

(xiv) e^x dx + (e^xcot y + 2y cosec y) dy = 0 (1)

Solution: Here M = e^x and N = (e^xcot y + 2y cosec y). Thus

$$\frac{\partial M}{\partial y} = 0 \text{ and } \frac{\partial N}{\partial x} = e^x \cot y. \text{ We see that } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}. \text{ Hence given equation is not exact.}$$

To find an I.F we apply: $\text{RULE I}: \dfrac{M_y - N_x}{N} = \dfrac{0 - e^x \cot y}{e^x \cot y + 2y\,\text{cosec}\,y} \neq P(x)$. [RULE I fails]

$$\text{RULE II}: \frac{N_x - M_y}{M} = \frac{e^x \cot y - 0}{e^x} = \cot y = P(y)$$

$$\text{Thus, } I.F = e^{\int P(y)\,dy} = e^{\int \cot y\,dy} = e^{\ln \sin y} = \sin y.$$

Multiply (1) by I.F, we get: $e^x \sin y\,dx + (e^x \cos y + 2y)\,dy = 0$ (2)

Equation (2) is now exact. Its solution is:

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C \Rightarrow \sin y\int e^x dx + 2\int y\;dy = C \;\Rightarrow \sin y e^x + y^2 = C$$

(xv) (x + 2) sin y dx + x cos y dy = 0 (1)

Solution: Here M = (x + 2) sin y and N = x cos y. Thus

$$\frac{\partial M}{\partial y} = (x+2)\cos y \text{ and } \frac{\partial N}{\partial x} = \cos y. \text{ We see that } \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Hence given equatuion is not exact.To find an I.F we apply

RULE I: $\dfrac{M_y - N_x}{N} = \dfrac{(x+2)\cos y - \cos y}{x\cos y} = \dfrac{\cos y(x+1)}{x\cos y} = \dfrac{x+1}{x} = P(x).$

Thus, $\text{I.F} = e^{\int P(x)dx} = e^{\int (1+1/x)\,dx} = e^{x+\ln x} = e^x e^{\ln x} = xe^x$

Multiply (1) by I.F, we get: $xe^x(x+2)\sin y\,dx + x^2 e^x \cos y\,dy = 0$ (2)

Equation (2) is now exact. Its solution is:

$\int M\,dx + \int(\text{terms of } N \text{ not containing } x)\,dy = C$ or $\sin y\int e^x dx + 2\int y\,dy = C \Rightarrow \sin y e^x + y^2 = C$

Linear Differential Equations

A first order differential equation is called linear, if it is of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \qquad (1)$$

Equation (1) is known as linear because, y and its derivative dy/dx appear in first degree. Dividing equation (1) by $a_1(x)$, we obtain

$$\frac{dy}{dx} + P(x)y = Q(x) \qquad (2)$$

where $P(x) = \dfrac{a_0(x)}{a_1(x)}$ and $Q(x) = \dfrac{f(x)}{a_1(x)}$.

We call (2) the standard form of the first order linear differential equation. To solve equation (2), we write it in the form

$$dy + P(x)y\,dx = Q(x)\,dx \quad \text{or} \quad \{P(x)y - Q(x)\}\,dx + dy = 0$$

Apply the exactness test to find that it is an exact differential or not, comparing it with $Mdx + Ndy = 0$, we have $M = P(x)y - Q(x)$ and $N = 1$. Now $M_y = P(x)$ and $N_x = 0$. Since $M_y \neq N_x$ hence equation (2) is not exact. To find an integrating factor, we apply RULE I , we see that $\dfrac{M_y - N_x}{N} = \dfrac{P(x) - 0}{1} = P(x)$

Hence, $e^{\int P(x)dx}$ is an integrating factor of equation (2). Let us multiply (2) by integrating factor to obtain:

$$e^{\int P(x)\,dx}\frac{dy}{dx} + e^{\int P(x)\,dx}P(x)y = Q(x)e^{\int P(x)\,dx} \qquad (3)$$

The left member of equation (3) is the derivative of the product $y\,e^{\int P(x)\,dx}$. Thus, equation (3) is equivalent to

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right) = Q(x)e^{\int P(x)\,dx}$$

Integrating, we get

$$\left(ye^{\int P(x)\,dx}\right) = \int Q(x)e^{\int P(x)\,dx}\,dx$$

Solving this equation we get the solution of equation (1) in explicit form:

$$y = f(x) + c$$

Here we summarize the steps involved in finding the solution of linear differential equation of first order.

Step 1: Put the given equation into standard form: $\dfrac{dy}{dx} + P(x)y = Q(x)$

Step 2: Obtain the integrating factor $e^{\int P(x)\,dx}$

Step 3: Multiply the equation in step 1 by the integrating factor to get

$$e^{\int P(x)\,dx}\frac{dy}{dx}+e^{\int P(x)\,dx}P(x)\,y=Q(x)e^{\int P(x)\,dx}$$

Step 4: The left side of equation in step 3 is the derivative of the product of the dependent variable and integrating factor, that is: $\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right)=Q(x)e^{\int P(x)\,dx}$

Step 5: Integrate both sides of equation in the step 4 to get the solution.

Remark: Sometimes an equation that is not linear in dependent variable y but it can be made linear in x by interchanging the roles of dependent and independent variables

Example 01: Solve the linear differential equation $\frac{dy}{dx}+\frac{1}{x\ln x}y=\frac{3x^2}{\ln x}$

Solution: The equation in standard form is $\frac{dy}{dx}+\frac{1}{x\ln x}y=\frac{3x^2}{\ln x}$ (1)

Here the integrating factor is:

$$e^{\int\frac{1}{x\ln x}dx}=e^{\int\frac{1}{\ln x}\left(\frac{1}{x}\right)dx}=e^{\ln(\ln x)}=e^{\ln x}\quad [\text{Formula}: \int\frac{f`(x)}{f(x)}dx=\ln\left(f(x)\right)]$$

Multiplying (1) by integrating factor, we have $\ln x\frac{dy}{dx}+\frac{1}{x}y=3x^2\Rightarrow\frac{d}{dx}\left(y\ln x\right)=3x^2$

Integrating, $y\ln x=x^3+c\quad\Rightarrow y=\left(x^3+c\right)/\ln x$

Example 02: Solve the linear differential equation $x\frac{dy}{dx}+\left(1+x\cot x\right)y=x$

Solution: Dividing given equation by x, we get: $\frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$ (1)

Here the integrating factor is:

$$e^{\int\frac{1}{x\ln x}dx}=e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\int\frac{1}{x}dx+\int\cot x\,dx}=e^{\ln x+\ln\sin x}=e^{\ln(x\sin x)}x\sin x$$

Multiplying (1) by the integrating factor, we have

$$x\sin x\frac{dy}{dx}+x\sin x\left(\frac{1}{x}+\cot x\right)y=x\sin x\Rightarrow x\sin x\frac{dy}{dx}+y\sin x+xy\cos x=x\sin x$$

$$\frac{d}{dx}\left(xy\sin x\right)=x\sin x\Rightarrow\int\frac{d}{dx}\left(xy\sin x\right)dx=\int x\sin x\,dx\Rightarrow xy\sin x=\int x\sin x\,dx$$

Now, $\int x\sin x\,dx=-x\cos x+\int\cos x\,dx=-x\cos x+\sin x$

Thus solution of given differential equation is:

$$xy\sin x=-x\cos x+\sin x+c\Rightarrow y=\frac{-x\cos x+\sin x+c}{x\sin x}\Rightarrow y=-\cot x+\frac{1}{x}+\frac{c}{x}\csc x$$

Example 03: Solve (x + y) dy + dx = 0

Solution: Given equation may be put in the form: $\frac{dy}{dx}+\frac{1}{x+y}=0$

This is not linear in y. However, we may write it as

$$\frac{dy}{dx}=-\frac{1}{x+y}\Rightarrow\frac{dx}{dy}=-x-y\Rightarrow\frac{dx}{dy}+x=-y\qquad(1)$$

This is linear differential equation in x. Thus, reversing the roles of x and y, P(y) = 1. So integrating factor is $e^{\int 1dy}=e^y$. Equation (1) after multiplication by e^y becomes

$$e^{y}\frac{dy}{dx}+e^{y}x=-ye^{y}\ \Rightarrow \frac{d}{dy}\left(xe^{y}\right)=-ye^{y}\ \Rightarrow xe^{y}=-\int ye^{y}\,dy+c \quad \text{[Integrating w.r.t y]}$$

$$\Rightarrow\ xe^{y}=(-y+1)e^{y}+c\ \Rightarrow x=(1-y)+ce^{-y} \qquad \text{[Integrating by parts]}$$

This is solution of given differential equation.

Example 04: Solve the following initial value problem

$$e^{x}\left\{y-3\left(e^{x}+1\right)^{2}\right\}dx+\left(e^{x}+1\right)dy=0,\ y(0)=4$$

Solution: Given differential equation is: $e^{x}\left\{y-3\left(e^{x}+1\right)^{2}\right\}dx+\left(e^{x}+1\right)dy=0$

$$\frac{dy}{dx}=-\frac{e^{x}\left\{y-3\left(e^{x}+1\right)^{2}\right\}}{\left(e^{x}+1\right)}\ \Rightarrow \frac{dy}{dx}+\frac{e^{x}}{\left(e^{x}+1\right)}y=3e^{x}\left(e^{x}+1\right) \quad (1)$$

which is linear differential equation.

Here the integrating factor is: $e^{\int\frac{e^{x}}{e^{x}+1}dx}=e^{\ln(e^{x}+1)}=\left(e^{x}+1\right)$

Multiplying (1) by integrating factor, we have

$$\left(e^{x}+1\right)\frac{dy}{dx}+e^{x}y=3e^{x}\left(e^{x}+1\right)^{2}\Rightarrow\frac{d}{dx}\left\{y\left(e^{x}+1\right)\right\}=3e^{x}\left(e^{x}+1\right)^{2}$$

$$\left(e^{x}+1\right)\frac{dy}{dx}+e^{x}\ y=3e^{x}\left(e^{x}+1\right)^{2}\Rightarrow\frac{d}{dx}\left\{y\left(e^{x}+1\right)\right\}=3e^{x}\left(e^{x}+1\right)^{2}$$

Integrating, $\left\{y\left(e^{x}+1\right)\right\}=\left(e^{x}+1\right)^{3}+c\ \Rightarrow y=\left(e^{x}+1\right)^{2}+c\left(e^{x}+1\right)^{-1} \qquad (2)$

Applying the initial conditions, that is, put x = 0 and y = 4, we get

$$4=\left(e^{0}+1\right)^{2}+c\left(e^{0}+1\right)^{-1}\Rightarrow 4=4+\ c/2\Rightarrow c=0$$

Hence the equation (2) becomes: $y = (e^{x} + 1)^{2}$. This is a particular solution of given differential equation.

Example 05: Solve the following differential equations:

(i) $\mathbf{\dfrac{dy}{dx}+\left(\dfrac{2x+1}{x}\right)y=e^{-2x}}$ (1)

Solution: Here P(x) = (2x + 1)/x

$$\Rightarrow I.F=e^{\int\left(\frac{2x+1}{x}\right)dx}=e^{2\int 1.dx+\int 1/x\ dx}=e^{2x+\ln x}=e^{2x}e^{\ln x}=x\,e^{2x}.$$

Multiplying equation (1) by an I. F, we get

$$x\,e^{2x}\left[\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y\right]=x\,e^{2x}e^{-2x}\ \Rightarrow\frac{d}{dx}\left[y\times I.F\right]=x$$

Integrating both sides and placing the value of an I.F we obtain: $y\ x\ e^{2x} = x^{2}/2 + C$.

This is the solution of given differential equation.

(ii) $\mathbf{\dfrac{dy}{dx}+\dfrac{3}{x}y=6x^{2}}$ (1)

Solution: Here P(x) = 3/x $\Rightarrow I.F=e^{3\int 1/x\ dx}=e^{3\ln x}=e^{\ln x^{3}}=x^{3}$

Multiplying equation (1) by an I. F, we get $x^{3}\left[\dfrac{dy}{dx}+\dfrac{3}{x}y\right]=6x^{3}x^{2}\ \Rightarrow\dfrac{d}{dx}\left[y\times I.F\right]=6x^{5}$

Integrating both sides and placing the value of an I.F we obtain: $y\,x^3 = x^6 + C$.
This is the solution of given differential equation.

(iii) $\mathbf{\frac{dy}{dx} + 3y = 3x^2e^{-3x}}$ (1)

Solution: Here P(x) = 3 $\Rightarrow \text{I.F} = e^{3\int 1\,dx} = e^{3x}$

Multiplying equation (1) by an I. F, we get $e^{3x}\left[\frac{dy}{dx} + 3y\right] = 3x^2e^{-3x}.e^{3x} \Rightarrow \frac{d}{dx}\left[y\times \text{I.F}\right] = 3x^2$

Integrating both sides and placing the value of an I.F we obtain: $y\,e^{3x} = x^3 + C$.
This is the solution of given differential equation.

(iv) $\mathbf{\cos^3 x\frac{dy}{dx} + y\cos x = \sin x}$ (1)

Solution: Dividing both sides of equation (1) by $\cos^3 x$, we obtain:

$$y` + \sec^2 x\,.\,y = \sin x/\cos^3 x = \tan x\sec^2 x \quad (2)$$

Here P(x) = $\sec^2 x \Rightarrow \text{I.F} = e^{\int \sec^2 x\,dx} = e^{\tan x}$. Multiplying equation (1) by an I. F, we get

$$e^{\tan x}\left[\frac{dy}{dx} + \sec^2 x.y\right] = e^{\tan x}\tan x\sec^2 x \Rightarrow \frac{d}{dx}\left[y\times \text{I.F}\right] = e^{\tan x}\tan x\sec^2 x$$

Integrating both sides and placing the value of an I.F we obtain:

$$y.e^{\tan x} = \int e^{\tan x}\tan x\sec^2 x\,dx + C$$

Substituting z = tan x or dz = $\sec^2 x$ dx, we obtain

$$y.e^{\tan x} = \int z.e^z dz + C = z.e^z - \int e^z dz + C = ze^z - e^z + C = e^z(z-1) + C$$

Substituting z = tan x, get: $y\,e^{\tan x} = e^{\tan x}(\tan x - 1) + C$. Dividing both sides by $e^{\tan x}$, we get $y = (\tan x - 1) + C.\,e^{-\tan x}$. This is the solution of given differential equation.

(v) $\mathbf{(x+1)\frac{dy}{dx} - ny = e^x(x+1)^{n+1}}$ (1)

Solution: Dividing both sides of equation (1) by (x + 1), we obtain:

$$y` - n/(x+1)\,.\,y = e^x(x+1)^n \quad (2)$$

Here P(x) = $-n/(x+1) \Rightarrow \text{I.F} = e^{-n\int 1/(x+1)\,dx} = e^{-n\ln(x+1)} = e^{\ln(x+1)-n} = (x+1)^{-n}$

Multiplying equation (2) by an I. F, we get $(x+1)^{-n}\left[\frac{dy}{dx} - \frac{n}{x+1}y\right] = e^x \Rightarrow \frac{d}{dx}\left[y\times \text{I.F}\right] = e^x$

Integrating both sides and placing the value of an I.F we obtain

$$y.(x+1)^{-n} = \int e^x\,dx + C = e^x + C \Rightarrow y = (x+1)^n[e^x + C]$$

This is the solution of given differential equation.

(vi) $\mathbf{(x^2+1)\frac{dy}{dx} + 2xy = 4x^2}$ (1)

Solution: Dividing both sides of equation (1) by $(x^2 + 1)$, we obtain:

$$y` + 2xy/(x^2+1) = 4x^2/(x^2+1) \quad (2)$$

Here P(x) = $2x/(x^2+1) \Rightarrow \text{I.F} = e^{\int 2x/(x^2+1)\,dx} = e^{\ln(x^2+1)} = (x^2+1)$

Multiplying equation (2) by an I. F, we get

$$(x^2+1)\left[\frac{dy}{dx} + \frac{2x}{x^2+1}y\right] = 4x^2 \Rightarrow \frac{d}{dx}\left[y\times \text{I.F}\right] = 4x^2$$

Integrating both sides and placing the value of an I.F, we obtain

$$y.(x^2+1)=4\int x^2\,dx+C=\frac{4x^3}{3}+C \quad\Rightarrow\quad (x^2+1)\,y=\frac{4x^3}{3}+C$$

This is the solution of given differential equation.

(vii) $\mathbf{x\frac{dy}{dx}+2y=\sin x}$ (1)

Solution: Dividing both sides of equation (1) by x, we obtain:

$$y` + 2y/x = \sin x\,/x \qquad (2)$$

Here P(x) = 2/x $\Rightarrow \text{I.F}=e^{2\int 1/x\ dx}=e^{2\ln x}=e^{\ln x^2}=x^2$

Multiplying equation (2) by an I. F, we get $x^2\left[\frac{dy}{dx}+2y\right]=x^2\sin x \;\Rightarrow \frac{d}{dx}\left[y\times \text{I.F}\right]=x^2\sin x$

Integrating both sides and placing the value of an I.F, we obtain

$$y.x^2=\int x^2\sin x\,dx+C=x^2(-\cos x)+2\int x\cos x\ dx+C$$

$$=-x^2\cos x+2[x\sin x-\int\sin x\,dx]+C \quad\Rightarrow x^2y=-x^2\cos x+2x\sin x+2\cos x+C$$

This is the solution of given differential equation.

(viii) $\mathbf{(1+x^2)\frac{dy}{dx}+4xy=\frac{1}{(1+x^2)^2}}$ (1)

Solution: Dividing both sides of equation (1) by $(1 + x^2)$, we obtain

$$\frac{dy}{dx}+\frac{4x}{(1+x^2)}y=\frac{1}{(1+x^2)^3} \qquad (2)$$

Here P(x) = $4x/(1 + x^2)$ $\Rightarrow \text{I.F}=e^{2\int 2x/(1+x^2)\ dx}=e^{2\ln(1+x^2)}=e^{\ln(1+x^2)^2}=(1+x^2)^2$

Multiplying equation (2) by an I. F, we get

$$(1+x^2)^2\left[\frac{dy}{dx}+\frac{4x}{(1+x^2)}y\right]=\frac{1}{(1+x^2)} \;\Rightarrow \frac{d}{dx}\left[y\times \text{I.F}\right]=\frac{1}{(1+x^2)}$$

Integrating both sides and placing the value of an I.F we obtain:

$$y.(1+x^2)^2=\int\frac{1}{(1+x^2)}dx+C=\tan^{-1}x+C \;\text{ or }\; y.(1+x^2)^2=\tan^{-1}x+C$$

This is the solution of given differential equation.

(ix) $\mathbf{\frac{dy}{dx}=\frac{1}{(e^y-x)}}$

Solution: This equation is not linear in y. If we write it in the form

$$\frac{dx}{dy}=(e^y-x) \;\Rightarrow \frac{dx}{dy}+x=e^y \qquad (1)$$

Equation (1) is linear in x. Here P(y) = 1 $\Rightarrow \text{I.F}=e^{\int 1\,dy}=e^y$.

Multiplying equation (1) by an I. F, we get $e^y\left[\frac{dx}{dy}+x\right]=ye^y \;\Rightarrow \frac{d}{dy}\left[x\times \text{I.F}\right]=ye^y$

Integrating both sides and placing the value of an I.F, we obtain

$$xe^y=\int ye^y dy+C=ye^y-e^y+C=e^y(y-1)+C \Rightarrow x=(y-1)+Ce^{-y}$$

This is the solution of given differential equation.

(x) $\mathbf{(x+2y^3)\frac{dy}{dx}=y}$

Solution: This equation is not linear in y. If we write it in the form

$$\frac{dy}{dx}=\frac{y}{(x+2y^3)} \Rightarrow \frac{dx}{dy}=\frac{(x+2y^3)}{y} \quad \Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y^2 \qquad (1)$$

Equation (1) is linear in x. Here P(y) = 1 $\Rightarrow \text{I.F}=e^{-\int 1/y\, dy}=e^{-\ln y}=y^{-1}$

Multiplying equation (1) by an I. F, we get

$$y^{-1}\left[\frac{dx}{dy}-\frac{1}{y}x\right]=2y \quad \Rightarrow \frac{d}{dy}\left[x\times \text{I.F}\right]=2y$$

Integrating both sides and placing the value of an I.F we obtain:

$$xy^{-1}=2\int y\,dy + C = y^2 + C = \quad \Rightarrow x = y(y^2+C)$$

This is the solution of given differential equation.

(xi) $\mathbf{x\frac{dy}{dx}-2x^2y=y\ln y}$

Solution: Dividing both sides by x, we get: $\frac{dy}{dx}-2xy=y\ln y/x$ (1)

Substituting z = ln y or $y=e^z$. Differentiating w.r.t x, we get, dy/dx = e^z dz/dx. Thus equation (1) becomes $e^z\frac{dz}{dx}-2x\,e^z=\frac{z.e^z}{x}$. Dividing by e^z, we obtain : $\frac{dz}{dx}-2x=\frac{z}{x}$

or $$\frac{dz}{dx}-\frac{1}{x}z=2x \qquad (2)$$

Equation (1) is Here P(x) = -1/x $\Rightarrow \text{I.F}=e^{-\int 1/x\, dx}=e^{-\ln x}=x^{-1}$.

Multiplying equation (1) by an I. F, we get $x^{-1}\left[\frac{dz}{dx}-\frac{1}{x}z\right]=2 \quad \Rightarrow \frac{d}{dx}\left[z\times \text{I.F}\right]=2$

Integrating both sides and placing the value of an I.F we obtain

$$zx^{-1}=2\int 1dx + C = 2x + C \quad \Rightarrow z = 2x^2+Cx \text{ or } \ln y = 2x^2 + Cx$$

This is the solution of given differential equation.

Bernoulli's Differential Equation

A well-known equation that is not linear but can be transformed to a linear equation is

$$\frac{dy}{dx}+P(x)y=Q(x)y^n \qquad (1)$$

This is known as ***Bernoulli's differential equation***. If n = 0 or n = 1 then (1) is linear. So we consider the cases where $n\neq 0$ and $n\neq 1\cdot$ Dividing both sides of (1) by y^n we get

$$y^{-n}\frac{dy}{dx}+P(x)y^{1-n}=Q(x) \qquad (2)$$

Substituting $y^{1-n}=z \Rightarrow (1-n)y`=z` \Rightarrow y^{-n}y`=z`/(1-n)$. Therefore (2) becomes

$$\frac{1}{1-n}\frac{dz}{dx}+P(x)z=Q(x) \text{ or } \frac{dz}{dx}+(1-n)P(x)z=(1-n)Q(x) \qquad (3)$$

Equation (3) is a linear differential equation in standard form and can be solved for z and substitution y^{n-1} = z gives the solution of (1).

Example 06: Solve Bernoulli differential equation $x\frac{dy}{dx}+y=y^2\ln x$

Solution: Dividing the given equation by x, we have $\frac{dy}{dx}+\frac{1}{x}y=\frac{\ln x}{x}y^2$ (1)

This is Bernoulli's differential equation where n = 2. Dividing by y^2, we get

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=\frac{\ln x}{x} \qquad (2)$$

Let $z=y^{-1}\Rightarrow\frac{dz}{dx}=-y^{-2}\frac{dy}{dx}\Rightarrow-\frac{dz}{dx}=y^{-2}\frac{dy}{dx}\cdot$ Then equation (2) becomes

$$-\frac{dz}{dx}+\frac{1}{x}z=\frac{\ln x}{x} \text{ or } \frac{dz}{dx}-\frac{1}{x}z=-\frac{\ln x}{x} \qquad (3)$$

This is linear differential equation in z. Therefore, $\text{I.F}=e^{\int P(x)\,dx}=e^{-\int 1/x\,dx}=e^{-\ln x}=e^{\ln x^{-1}}=x^{-1}$

Multiplying (3) by I.F, we get $\frac{1}{x}\frac{dz}{dx}-\frac{1}{x^2}z=-\frac{\ln x}{x^2}\Rightarrow\frac{d}{dx}\left(z\times\frac{1}{x}\right)=-\frac{\ln x}{x^2}$

Integrating both sides, we get $\frac{z}{x}=-\int\frac{\ln x}{x^2}dx+c$ (4)

Now using integration by parts, we get $\int\frac{\ln x}{x^2}dx=-\frac{\ln x}{x}+\int\frac{1}{x^2}dx=-\frac{\ln x}{x}-\frac{1}{x}$. Thus (4) becomes

$$\frac{z}{x}=-\left(-\frac{\ln x}{x}-\frac{1}{x}\right)+c\Rightarrow\frac{z}{x}=\frac{\ln x}{x}+\frac{1}{x}\Rightarrow z=\ln x+1+cx\Rightarrow\frac{1}{y}=\ln x+cx+1$$

This is the solution of given differential equation.

Example 07: Solve $y`+y=xy^3$

Solution: We have $y`+y=xy^3$ (1)

This is a Bernoulli's differential equation with n = 3. Dividing both sides of (1) by y^3, we get

$$y^{-3}\frac{dy}{dx}+y^{-2}=x \qquad (2)$$

Let $z=y^{-2}\Rightarrow\frac{dz}{dx}=-2y^{-3}\frac{dy}{dx}\Rightarrow-\frac{1}{2}\frac{dz}{dx}=y^{-3}\frac{dy}{dx}$

Thus equation (2) becomes: $-\frac{1}{2}\frac{dz}{dx}+z=x \quad\Rightarrow\frac{dz}{dx}-2z=-2x$ (3)

This is linear in z. This implies that $\text{I.F}=e^{\int P(x)\,dx}=e^{-\int 2\,dx}=e^{-2x}$. Multiplying (3) by I. F, we get: $e^{-2x}\frac{dz}{dx}-2e^{-2x}z=-2xe^{-2x} \quad\Rightarrow\frac{d}{dx}\left(ze^{-2x}\right)=-2xe^{-2x}$

Integrating both sides, we get: $ze^{-2x}=-2\int xe^{-2x}dx+c$ (4)

The right side of (4) will be solved using integration by parts.

$\int xe^{-2x}dx=-\frac{1}{2}xe^{-2x}+\frac{1}{2}\int e^{-2x}dx=-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}$. Now, equation (4) becomes

$ze^{-2x}=-2\left(-\frac{1}{2}xe^{-2x}-\frac{1}{4}e^{-2x}\right)+c \Rightarrow z=x+ce^{2x}+\frac{1}{2}$. Put $z=y^{-2}$, we get : $y^{-2}=x+ce^{2x}+\frac{1}{2}$.

This is solution of given differential equation.

Example 07: Solve the following initial value problem $\frac{dy}{dx}+\frac{y}{2x}=\frac{x}{y^3},\ y(1)=2$

Solution: Given equation is $\frac{dy}{dx}+\frac{y}{2x}=\frac{x}{y^3}\ \Rightarrow\ \frac{dy}{dx}+\frac{1}{2x}y=xy^{-3}$ (1)

This is a Bernoulli's differential equation with n = – 3. Dividing both sides of (1) by y^{-3}, we get

$$y^3\frac{dy}{dx}+\frac{1}{2x}y^4=x \tag{2}$$

Let $z=y^4\Rightarrow\frac{dz}{dx}=4y^3\frac{dy}{dx}\Rightarrow\frac{1}{4}\frac{dz}{dx}=y^3\frac{dy}{dx}$. Thus equation (2) becomes:

$$\frac{1}{4}\frac{dz}{dx}+\frac{1}{2x}z=x\Rightarrow\frac{dz}{dx}+\frac{2}{x}z=4x \tag{3}$$

$$\text{I.F.}=e^{\int(2/x)\,dx}=e^{2\ln x}=e^{\ln x^2}=x^2$$

Multiplying (3) by I. F, we get $x^2\frac{dz}{dx}+2xz=4x^3\Rightarrow\frac{d}{dx}\left(zx^2\right)=4x^3$

Integrating both sides, we get $x^2z=x^4+c\Rightarrow x^2y^4=x^4+c$ (4)

Applying the initial condition, we get $(1)^2(2)^4=(1)^4+c\Rightarrow c=15$

Equation (4) becomes $x^2y^4=x^4+15$

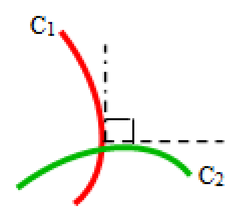
This is the particular solution of given differential equation.

ORTHOGONAL TRAJECTORIES

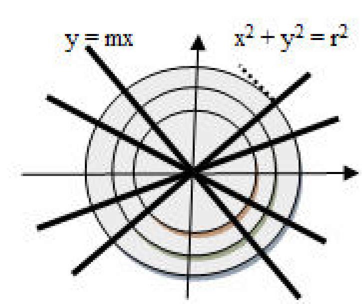
Definition: An n–parameter family of curves is a set of relations of the form:

$$\{(x,y):f(x,y,c_1,c_2,\ldots,c_n)=0\}$$

Here \`f\` is a real valued function of x, y and each c_i $(i=1,2,\ldots,n)$ ranges over an interval of real values. For example, the set of concentric circles defined by: $x^2+y^2=c$ has one non-negative parameter c. Again the set of circles, defined by $(x-c_1)^2+(y-c_2)^2=c_3$ has three–parameter family where c_1, c_2 are real and c_3 is non–negative real.



Definition: An angle between two curves is defined as angle between their tangents at the point of their intersection. If these tangents are perpendicular to each other we say that curves are normal/perpendicular/orthogonal to each other. This is shown in the adjacent figure.



Definition: If a family of curves cuts every member of other family of curves at right angle, the two families of curves are known as ***Orthogonal/ Perpendicular/Normal Trajectories*** of each. If the angle is not right angle the trajectories are known as ***oblique trajectories***. As an example, consider two family of curves y = mx and $x^2+y^2=r^2$, where m and r are parameters.

We find from the above figure that every line given by $y = mx$ through the origin is orthogonal to every circle given by $x^2 + y^2 = r^2$.

We shall discuss the "Orthogonal Trajectories" in two systems of coordinates.

(a) **Cartesian coordinate system**

(b) **Polar coordinate system**

Before we present the working rule of finding an O.T, the following may be noted:

(i) If two line are parallel their slopes are equal, that is; if $L_1 \parallel L_2$ then $m_1 = m_2$

(ii) If two lines are perpendicular then the product of their slopes is equal to -1, that is; if $L_1 \perp L_2$ then $m_1 . m_2 = -1$ then $m_2 = -1/m_1$

Orthogonal Trajectories in Cartesian Coordinates

Working Rule

Step1: Given an equation $f(x, y, c) = 0$ (1)

Differentiate (1) and eliminate an arbitrary constant and get

$$dy/dx = F(x, y) \qquad (2)$$

This is the slope of equation (1).

Step 2: Replace dy/dx by $-dx/dy$ to get

$$-dx/dy = F(x, y) \qquad (3)$$

This is the slope of O.T.

Step 3: Solving (3) to get required O.T of family of curves (1).

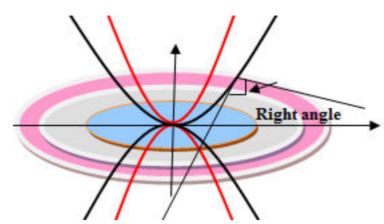
Remark: Orthogonal trajectories are important in various fields of applied sciences, for example, in hydrodynamics and heat conduction.

Example 01: Find orthogonal trajectories of the family of circles $x^2 + y^2 = r^2$

Solution: Differentiate w. r. t x, we get

$2x + 2yy' =$ or $dy/dx = -x/y$. This is the slope of given family of circles. Hence, the slope of the family of O.T is

$$-dx/dy = -x/y \text{ or equivalently, } dy/dx = + y/x$$



Separating the variables and integrating, we have:

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx + \ln c \Rightarrow \ln y = \ln x + \ln c \quad \Rightarrow \ln y = \ln cx \quad \Rightarrow y = mx$$

This is the **"Orthogonal Trajectories"** of family of circles. The figure of these "Orthogonal Trajectories" is shown above.

Example 02: Find the orthogonal trajectories of family of curves $y = cx^2$

Solution: We are given $y = cx^2$ (1)

The family (1) consists of parabolas symmetric about the y – axis with vertices at the origin. Differentiating equation (1) with respect to x, we get

$$dy/dx = 2cx \qquad (2)$$

To eliminate c, we observe from equation (1) that $c = y/x^2$. Put this in (2)

$$\frac{dy}{dx} = 2\left(\frac{y}{x^2}\right)x \Rightarrow \frac{dy}{dx} = \frac{2y}{x} \qquad (3)$$

This is the slope of (1). Hence slope of O.T is $-\frac{dx}{dy}=\frac{2y}{x}\Rightarrow -xdx=2ydy$. Integrating,

$$-\frac{1}{2}x^2=y^2+a\Rightarrow -\left(\frac{1}{2}x^2+y^2\right)=a\Rightarrow x^2+2y^2=k,\quad (k=-2a)$$

which is the required orthogonal trajectories, k being a parameter. These orthogonal trajectories are ellipses. Both parabolas and ellipses are shown in the above figure. Note that each ellipse intersects each parabola at right angle.

Example 03: Find the orthogonal trajectories of family of curves $x^2 + y^2 = cx$.

Solution: Equation of curve is $x^2 + y^2 = cx$. (1)

Differentiating with respect to x, we get: $2x+2y\frac{dy}{dx}=c$ (2)

From (1), we have $c = (x^2 + y^2)/x$

Substituting this into (2), we get

$$2x+2y\frac{dy}{dx}=\frac{x^2+y^2}{x}\Rightarrow x+y\frac{dy}{dx}=\frac{x^2+y^2}{2x}\Rightarrow y\frac{dy}{dx}=\frac{x^2+y^2}{2x}-x$$

$$\Rightarrow y\frac{dy}{dx}=\frac{x^2+y^2-2x^2}{2x}\Rightarrow y\frac{dy}{dx}=\frac{y^2-x^2}{2x}\Rightarrow \frac{dy}{dx}=\frac{y^2-x^2}{2xy}.$$

Replacing dy/dx by –dx/dy, we obtain

$$-\frac{dx}{dy}=\frac{y^2-x^2}{2xy}\Rightarrow \frac{dy}{dx}=\frac{2xy}{x^2-y^2}\Rightarrow \frac{dy}{dx}=\frac{2(y/x)}{1-(y/x)^2} \tag{3}$$

Equation (3) is a homogeneous differential equation. Put

$\frac{y}{x}=v$ or $y=vx$, then $\frac{dy}{dx}=v+x\frac{dv}{dx}$. Then (3) becomes

$$v+x\frac{dv}{dx}=\frac{2v}{1-v^2}\Rightarrow x\frac{dv}{dx}=\frac{2v}{1-v^2}-v\Rightarrow x\frac{dv}{dx}=\frac{2v-v+v^3}{1-v^2}\Rightarrow x\frac{dv}{dx}=\frac{v+v^3}{1-v^2}$$

Separating the variables, we have $\frac{1-v^2}{v+v^3}dv=\frac{1}{x}dx$ or $\frac{1-v^2}{v\left(1+v^2\right)}dv=\frac{1}{x}dx$

Integrating, $$\int\frac{1-v^2}{v\left(1+v^2\right)}dv=\int\frac{1}{x}dx+d \tag{4}$$

Left side of (4) will be solved using integration by partial fractions.

$$\frac{1-v^2}{v\left(1+v^2\right)}=\frac{A}{v}+\frac{Bv+C}{1+v^2} \tag{5}$$

$$1-v^2=A\left(1+v^2\right)+\left(Bv+C\right)v$$

Put v = 0, we get 1 = A. Now, $1 - v^2 = A(1 + v^2) + B v^2 + C v$.

Comparing the coefficients of v^2 and v on both sides, we have

$$A+B=-1\Rightarrow B=-2 \text{ and } C=0\cdot[\text{Note}: A=1]$$

Thus, (5) becomes $$\frac{1-v^2}{v\left(1+v^2\right)}=\frac{1}{v}-\frac{2v}{1+v^2}\Rightarrow \int\frac{1-v^2}{v\left(1+v^2\right)}dv=\int\frac{1}{v}dv-\int\frac{2v}{1+v^2}dv$$

$$= \ln v - \ln\left(1+v^2\right) = \ln\frac{v}{1+v^2}.$$

Hence equation (4) becomes after taking antilog:

$$\frac{v}{1+v^2} = xd_1 \Rightarrow \frac{y/x}{\left(x^2+y^2\right)/x^2} = xd_1 = \frac{y}{x}\times\frac{x^2}{x^2+y^2} = xd_1 \Rightarrow \frac{y}{x^2+y^2} = d_1$$

$\Rightarrow x^2+y^2 = ky\ \left(k = 1/d_1\right)$, where $d_1 = \ln d$

This solution gives the orthogonal trajectories of the equation (1).

Example 04: Find an equation of orthogonal trajectories of the curves of the family $y^2 = x^2 + cx$.

Solution: Given curve is $y^2 = x^2 + cx$ (1)

Differentiating (1) with respect to x, we get: $2y\,dy/dx = 2x + c$ (2)

From (1), we have: $c = \left(y^2 - x^2\right)/x$

Put this in (2), we get

$$2y\frac{dy}{dx} = 2x + \frac{y^2-x^2}{x} \Rightarrow 2y\frac{dy}{dx} = \frac{2x^2+y^2-x^2}{x} \Rightarrow 2xy\frac{dy}{dx} = x^2+y^2 \Rightarrow \frac{dy}{dx} = \frac{x^2+y^2}{2xy}$$

Replacing dy/dx by –dx/dy, we get $-\frac{dx}{dy} = \frac{x^2+y^2}{2xy}$ or $\frac{dy}{dx} = \frac{-2xy}{x^2+y^2}$

This is homogeneous equation and can be written as $\frac{dy}{dx} = \frac{-2\left(y/x\right)}{1+\left(y/x\right)^2}$ (3)

Putting: $y/x = v$ or $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$. Thus (3) becomes

$$v + x\frac{dv}{dx} = \frac{-2v}{1+v^2} \Rightarrow x\frac{dv}{dx} = -\left(\frac{2v}{1+v^2}+v\right)$$

$$\Rightarrow x\frac{dv}{dx} = -\left(\frac{2v+v+v^3}{1+v^2}\right) \Rightarrow x\frac{dv}{dx} = -\left(\frac{3v+v^3}{1+v^2}\right) \qquad (4)$$

Separating the variables, we get

$$-\frac{1+v^2}{3v+v^3}dv = \frac{1}{x}dx \Rightarrow -\int\frac{1+v^2}{3v+v^3}dv = \int\frac{1}{x}dx + d \Rightarrow -\frac{1}{3}\int\frac{3+3v^2}{3v+v^3}dv = \int\frac{1}{x}dx + d$$

$$-\frac{1}{3}\ln\left(3v+v^3\right) = \ln x + \ln d_1 \Rightarrow \ln\left(3v+v^3\right) = -3\ln\left(xd_1\right) \Rightarrow \ln\left(3v+v^3\right) = \ln\left(xd_1\right)^{-3}$$

$$3v+v^3 = \left(xd_1\right)^{-3} \Rightarrow 3\frac{y}{x} + \left(\frac{y}{x}\right)^3 = \frac{1}{\left(xd_1\right)^3} \Rightarrow 3x^2y + y^3 = k$$

This is the equation of the orthogonal trajectories of the given family.

Definition: A family of curves which is orthogonal to itself is called self – orthogonal**.**

Remark: From the above definition, it follows that if the differential equation of the family of curves is identical with the differential equation of its orthogonal trajectories, then such family of curves is self–orthogonal.

Example 05: Show that the orthogonal trajectories of the system of parabolas $y^2 = 4c\ (x + c)$ belong to the system itself, c being parameter.

Solution: We have $y^2 = 4c\ (x + c)$ (1)

Differentiating (1) with respect to x, we get $2y\frac{dy}{dx}=4c(1)\Rightarrow y\frac{dy}{dx}=2c\Rightarrow c=\frac{1}{2}y\frac{dy}{dx}$

Substituting this value of c into (1), we get

$$y^2=4\left(\frac{1}{2}y\frac{dy}{dx}\right)\left(x+\frac{1}{2}y\frac{dy}{dx}\right)\Rightarrow y^2=2xy\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2\Rightarrow y=2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2 \qquad (2)$$

Replacing dy/dx by –dx/dy, we get $y=2x\left(-\frac{dx}{dy}\right)+y\left(\frac{dx}{dy}\right)^2\Rightarrow y=\frac{-2x}{dy/dx}+\frac{y}{(dy/dx)^2}$

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2=-2x\frac{dy}{dx}+y\Rightarrow y=2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2 \qquad (3)$$

Equation (2) and (3) are identical. Hence the system of parabolas (1) is itself orthogonal, that is; each member of given family of parabolas intersects its own member orthogonally.

Example 06: Find an Orthogonal Trajectories for the following curves

(i) $x^2-y^2=c$

Solution: Differentiate w.r.t x, we get: $2x-2y\,y`=0$ or $y`=x/y$

This is the slope of equation (1). Hence slope of an O.T is $dy/dx=-y/x$

Separating the variables and integrating, we get:

$$\int\frac{1}{y}dy=-\int\frac{1}{x}dx+\ln C \quad \Rightarrow \ln y=-\ln x+\ln C=\ln C/x$$

$\Rightarrow y=C/x \quad \Rightarrow xy=C.$ This is the O.T of given family.

(ii) $x=c\,y^2$

Solution: Differentiate w.r.t x, we get: $1=2\,cy\,y`$ or $y`=1/2cy$

Substituting the value of c from equation (1), we get: $dy/dx=y/2x$.

This is the slope of equation (1). Hence slope of an O.T is, $dy/dx=-2x/y$

Separating the variables and integrating, we get:

$$\int y\,dy=-2\int x\,dx+C \quad \Rightarrow y^2/2=-2x^2/2+C \quad \Rightarrow y^2+2x^2=K,\ \text{where } 2C=K$$

This is O.T of a given family.

(iii) $y=e^{c\,x}$ (1)

Solution: Differentiate w.r.t x, we get: $y`=c\,e^{c\,x}=c\,y$ (from 1) (2)

Now from equation (1), $\ln y=cx$ giving, $c=\ln y/x$.

Substituting this in equation (2), we obtain: $dy/dx=y\ln y/x$.

This is slope of equation (1). Hence slope of an O.T is: $dy/dx=-x/y.\ln y$

Separating the variables and integrating, we get:

$$\int y\ln y\,dy=-\int x\,dx+C \quad \Rightarrow \frac{y^2\ln y}{2}-\int\frac{1}{y}.\frac{y^2}{2}dy=-\left(x^2/2\right)+C\Rightarrow \frac{y^2\ln y}{2}-\frac{1}{2}\int y\,dy=-\frac{x^2}{2}+C$$

$$\frac{y^2\ln y}{2}-\frac{y^2}{4}=-\frac{x^2}{2}+C \quad \Rightarrow 2y^2\ln y-y^2+2x^2=K,\quad \text{where } K=4C$$

This is an O.T of a family.

(iv) $y=x-1+c\,e^{-x}$ (1)

Solution: Differentiate w.r.t x, we get: $y`=1-c\,e^{-x}$ (2)

Adding (1) and (2), we obtain: $y` + y = x$ or $y` = x - y$. This is the slope of equation (1).

Hence slope of an O.T is: $dy/dx = -1/(x - y)$ (3)

Equation (3) may be written as: $dx/dy = -(x - y)$ or $dx/dy + x = y$ (4)

We see that equation (4) is a linear differential equation in x with P(y) = 1.

The integrating factor is: $e^{\int P(y)dy} = e^{\int 1.dy} = e^y$.

Multiplying (4) by an I.F, we get: $e^y\left(\frac{dx}{dy} + x\right) = ye^y \Rightarrow \frac{d}{dy}(x \times I.F) = ye^y$

Integrating both sides w.r.t y to get: $x \times I.F = \int ye^y dy + C \Rightarrow xe^y = e^y(y-1) + c$

Dividing both sides by e^y, we get: $x = (y - 1) + C e^{-y}$. This is an O.T of given family.

(v) $\mathbf{x = \frac{y^2}{4} + -\frac{c}{y^2}}$ (1)

Solution: Differentiate both sides w.r.t x, we get

$$1 = \frac{2y}{4}y` - \frac{2c}{y^3}y` = \left(\frac{y}{2} - \frac{2c}{y^3}\right)y` = \left(\frac{y^2-4c}{2y^3}\right)y` \Rightarrow \frac{dy}{dx} = \left(\frac{2y^3}{y^2-4c}\right) \quad (2)$$

From (1), $c = (4xy^2 - y^4)/4$. Substituting this value of c in (2) and simplifying, we get

$$\frac{dy}{dx} = \left(\frac{y}{y^2 - 2x}\right)$$

This is a slope of equation (1). Hence the slope of an O. T is:

$$\frac{dy}{dx} = -\frac{y^2 - 2x}{y} = \Rightarrow y\frac{dy}{dx} = -y^2 + 2x \Rightarrow y\frac{dy}{dx} + y^2 = 2x \quad (3)$$

Equation (3) is a Bernoulli differential equation. Substituting $z = y^2$ and $dz/dx = 2y\, dy/dx$, we get

$$\frac{dz}{dx} + 2z = 4x \quad (4)$$

This is a linear differential equation in z. Here P = 2, hence an I.F = $e^{\int 2.dx} = e^{2x}$. Multiplying (4) by an I. F, we get: $e^{2x}\left(\frac{dz}{dx} + 2z\right) = 4xe^{2x} \Rightarrow \frac{d}{dx}(z \times I.F) = 4xe^{2x}$

Integrating both sides, we get: $ze^{2x} = 4\int xe^{2x}dx + C = 2xe^{2x} - e^{2x} + C$.

Dividing both sides by e^{2x} and substituting the value of $z = y^2$, we get:

$$y^2 = (2x - 1) + C e^{-2x}$$

This is the orthogonal trajectory of given family.

(vi) $\mathbf{y = (x - c)^2}$ (1)

Solution: From equation (1), we have $\sqrt{y} = x - c$

Differentiate both sides w.r.t x, we get: $\frac{1}{2\sqrt{y}}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = 2\sqrt{y}$.

This is a slope of equation (1). Hence the slope of an O.T is: $\frac{dy}{dx} = \frac{1}{2\sqrt{y}}$

Separating the variables we get: $2\int\sqrt{y}dy = \int 1\, dx + C \Rightarrow 4\frac{y^{3/2}}{3} = x + C$.

This is the required O.T.

(vii) $\mathbf{x^2 + y^2 = 1 + 2cy}$ (1)

Solution: Differentiate w.r.t x, we get: $2x + 2yy` = 0 + 2cy`$

or $x + y y` = c y`$ (2)

From equation (1), $c = (x^2 + y^2 - 1)/2y$. Substituting in (2) and simplifying, we get

$x + yy` = (x^2 + y^2 - 1)y`/2y$ or $yy` = (x^2 + y^2 - 1)y`/2y - x$

or $yy` = [(x^2 + y^2 - 1) y` - 2xy]/2y$ giving $2y^2 y` = (x^2 + y^2 - 1)y` - 2xy$

or $(2y^2 - x^2 - y^2 + 1)y` = - 2xy$ or $(y^2 - x^2 + 1)y` = -2xy$

giving $y` = -2xy/(y^2 - x^2 + 1)$. This is the slope of equation (1).

Hence slope of O.T is: $y` = + (y^2 - x^2 + 1)/2xy$

or $$\frac{dy}{dx} = \frac{y^2}{2xy} + \frac{1-x^2}{2xy} \Rightarrow y\frac{dy}{dx} - \frac{1}{2x}y^2 = \frac{1-x^2}{2x} \qquad (3)$$

Equation (3) is a Bernoulli differential equation.

Substituting $y^2 = z$ or $2y\, y` = z`$ or $y\, y` = z`/2$ in equation (3) and after simplifying, we get

$$\frac{dz}{dx} - \frac{1}{x}z = \frac{1-x^2}{x} \qquad (4)$$

Equation (4) is a linear differential equation with $P(x) = -1/x$. Thus its I.F is:

$$e^{-\int 1/x\, dx} = e^{-\ln x} = x^{-1} = 1/x$$

Multiplying equation (4) by an I.F, we get:

$$\frac{1}{x}\left(\frac{dz}{dx} - \frac{1}{x}z\right) = \frac{1-x^2}{x^2} \Rightarrow \frac{d}{dx}\left(z \times \text{I.F}\right) = \frac{1-x^2}{x^2}$$

Integrating both sides and then substituting $z = y^2$, we get

$$z \times \text{I.F} = \int (x^{-2} - 1)\, dx + C \quad \Rightarrow y^2.\frac{1}{x} = -x^{-1} - 1 + C = -\left(\frac{1+x}{x}\right) + C$$

$\Rightarrow y^2 = -(1+x) + Cx$. This is an O.T of given family (1).

Orthogonal Trajectories in Polar Coordinates

If equation of a curve is given in polar form $f(r, \theta, c) = 0$, the following working rule will help to find the orthogonal trajectories.

Step 1. Differentiate given equation $f(r, \theta, c) = 0$ w.r.t r, eliminate an arbitrary constant and find $r\frac{d\theta}{dr} = F(r, \theta)$. This is slope of given family $f(r, \theta, c) = 0$.

Step 2. The slope of O.T is obtained by replacing the right side by $-1/ F(r, \theta)$, that is,

$$r\frac{d\theta}{dr} = -1/F(r, \theta)$$

Step 3. Solving this equation to obtain required O.T.

Example 06: Prove that the orthogonal trajectories of $r^n \cos n\theta = a^n$ is $r^n \sin n\theta = c^n$.

Solution: Given equation of family of curves is: $r^n \cos n\theta = a^n$ (1)

Taking logarithm of both sides, we get

$$\ln r^n \cos n\theta = \ln a^n \Rightarrow \ln r^n + \ln\cos n\theta = \ln a^n \Rightarrow n\ln r + \ln\cos n\theta = \ln a^n \qquad (2)$$

Differentiating (2) with respect to r,we get

$$n\left(\frac{1}{r}\right) + \frac{1}{\cos n\theta}\left(-\sin n\theta\right)\left(n\right)\frac{d\theta}{dr} = 0 \Rightarrow -\tan n\theta\frac{d\theta}{dr} = -\frac{1}{r} \Rightarrow r\frac{d\theta}{dr} = \cot n\theta \quad (3)$$

This is slope given family of curves (1). Hence slope of O.T is $r\frac{d\theta}{dr} = -\tan n\theta$

Separating the variables, we get $-\frac{1}{\tan n\theta}d\theta = \frac{1}{r}dr \Rightarrow -\cot n\theta d\theta = \frac{1}{r}dr$

Integrating,

$$-\int \cot n\theta d\theta = \int \frac{1}{r}dr + \ln c_1 \Rightarrow -\frac{\ln \sin n\theta}{n} = \ln r + \ln c_1 \Rightarrow \frac{1}{n}\ln \sin n\theta + \ln r = \ln c \quad (-\ln c_1 = \ln c)$$

$$\ln \sin n\theta + n\ln r = n\ln c \Rightarrow \ln \sin n\theta + \ln r^n = \ln c^n \Rightarrow \ln r^n \sin n\theta = \ln c^n \Rightarrow r^n \sin n\theta = c^n$$

This is an O.T of given family (1).

Example 07: Find an orthogonal trajectory of

(i) r = a (1 + sin θ) (1)

Solution: From (1) we have a = r/(1 + sin θ). Now differentiating equation (1) w.r.t r to get:

$$1 = a\,(0 + \cos\theta)\, d\theta/dr .$$

Substituting the value of a, we get: $1 = \frac{r\cos\theta}{(1+\sin\theta)}.\frac{d\theta}{dr} \Rightarrow r\frac{d\theta}{dr} = \frac{1+\sin\theta}{\cos\theta}$

This is the slope of equation (1). Hence the slope of O.T is: $r\frac{d\theta}{dr} = -\frac{\cos\theta}{1+\sin\theta}$

Separating the variables and integrating, we get:

$$\int \frac{1+\sin\theta}{\cos\theta}\,d\theta = -\int \frac{1}{r}dr + \ln C \Rightarrow \int \sec\theta\, d\theta + \int \tan\theta\, d\theta = -\ln r + \ln C$$

$$\ln(\sec\theta + \tan\theta) + \ln\sec\theta = \ln C/r \Rightarrow \ln\sec\theta(\sec\theta + \tan\theta) = \ln C/r$$

$\Rightarrow \sec\theta(\sec\theta + \tan\theta) = \frac{C}{r} \Rightarrow r\sec\theta(\sec\theta + \tan\theta) = C$. This is the required O.T

(ii) r^2 = a sin 2θ (1)

Solution: From (1) we have a = r^2/sin 2θ. Now differentiating equation (1) w.r.t r to get:

$$2r = a\,(2\cos 2\theta)\, d\theta/dr \text{ or } r = a\,(\cos 2\theta)\, d\theta/dr$$

Substituting the value of a, we get $r = \frac{r^2\cos 2\theta}{\sin 2\theta}.\frac{d\theta}{dr} \Rightarrow r\frac{d\theta}{dr} = \frac{\sin 2\theta}{\cos 2\theta}$

This is the slope of equation (1). Hence the slope of O.T is $r\frac{d\theta}{dr} = -\frac{\cos 2\theta}{\sin 2\theta}$

Separating the variables and integrating, we get:

$$\int \frac{\sin 2\theta}{\cos 2\theta}\,d\theta = -\int \frac{1}{r}dr + \ln C` \Rightarrow \int \tan 2\theta\, d\theta = -\ln r + \ln C` \Rightarrow \frac{1}{2}\ln\sec 2\theta = \ln\frac{C`}{r}$$

$$\ln\sec 2\theta = 2\ln(C`/r) \Rightarrow \ln\sec 2\theta = \ln(C`/r)^2 \Rightarrow \sec 2\theta = (C`/r)^2$$

$\Rightarrow r^2\sec 2\theta = C \quad [C`^2 = C]$. This is the required O.T

(iii) $r^n = a^n \cos n\theta$ (1)

Solution: From (1) we have $a^n = r^n/\cos n\theta$. Now differentiating equation (1) w.r.t r to get:

$$nr^{n-1} = a^n\left(-n\sin n\theta\right)\frac{d\theta}{dr} \text{ or } r^n.\,r^{-1} = a^n\left(-\sin n\theta\right)\frac{d\theta}{dr}$$

Substituting the value of a^n, we get: $r^n.r^{-1} = \frac{r^n(-\sin\theta)}{\cos n\theta}.\frac{d\theta}{dr} \Rightarrow r\frac{d\theta}{dr} = -\frac{\cos n\theta}{\sin n\theta}$

This is the slope of equation (1). Hence the slope of O.T is $r\frac{d\theta}{dr} = +\frac{\sin n\theta}{\cos n\theta}$

Separating the variables and integrating, we get:

$$\int \frac{\cos n\theta}{\sin n\theta}\, d\theta = \int \frac{1}{r} dr + \ln C` \Rightarrow \int \cot n\theta\, d\theta = \ln r + \ln C \quad \Rightarrow \frac{1}{2} \ln \sin n\theta = \ln Cr$$

$$\ln \sin n\theta = 2\ln(Cr) \quad \Rightarrow \ln \sin n\theta = \ln(Cr)^2 \Rightarrow \quad \sin n\theta = (Cr)^2 \Rightarrow \sin n\theta = Kr^2 \quad [\ K = C^2]$$

This is the required O.T.

(iv) r = a/(2 + cos θ) (1)

Solution: From (1) we have: $r(2 + \cos\theta) = a$.

Now differentiating equation (1) w.r.t r to get:

$$r(-\sin\theta)\frac{d\theta}{dr} + (2 + \cos\theta) = 0 \Rightarrow r\frac{d\theta}{dr} = \frac{2+\cos\theta}{\sin\theta}$$

This is the slope of equation (1). Hence the slope of O.T is: $r\frac{d\theta}{dr} = -\frac{\sin\theta}{2+\cos\theta}$

Separating the variables and integrating, we get:

$$\int \frac{2+\cos\theta}{\sin\theta}\, d\theta = -\int \frac{1}{r} dr + \ln C` \Rightarrow 2\int \operatorname{cosec}\theta\, d\theta + \int \cot\theta\, d\theta = -\ln r + \ln C$$

$$2\ln(\operatorname{cosec}\theta - \cot\theta) + \ln\sin\theta = \ln(C/r) \quad \Rightarrow \ln \sin\theta(\operatorname{cosec}\theta - \cot\theta)^2 = \ln(C/r)$$

$$\Rightarrow \quad \sin\theta(\operatorname{cosec}\theta - \cot\theta)^2 = C/r \qquad \Rightarrow r\sin\theta(\operatorname{cosec}\theta - \cot\theta)^2 = C$$

This is the required O.T.

(v) r^n = a sin nθ (1)

Solution: From (1) we have: $a = r^n / \sin n\theta$.

Now differentiating equation (1) w.r.t r to get $\quad n\, r^{n-1} = a\, (n \cos n\theta)\, \frac{d\theta}{dr}$

Substituting the value of a, we get $\quad r^n r^{-1} = \frac{r^n(\cos n\theta)}{\sin n\theta}.\frac{d\theta}{dr} \Rightarrow r\frac{d\theta}{dr} = \frac{\sin n\theta}{\cos n\theta} = \tan n\theta$

This is the slope of equation (1). Hence the slope of O.T is $\ r\frac{d\theta}{dr} = -\cot n\theta$

Separating the variables and integrating, we get:

$$\int \frac{1}{\cot n\theta}\, d\theta = -\int \frac{1}{r} dr + \ln C \Rightarrow \int \tan n\theta\, d\theta = -\ln r + \ln C \quad \Rightarrow \frac{1}{n}\ln \sec n\theta = \ln\frac{C}{r}$$

$$\ln \sec n\theta = n\ln(C/r) \quad \Rightarrow \ln \sec n\theta = \ln(C/r)^n \quad \Rightarrow \quad \sec n\theta = (C/r)^n$$

$$\Rightarrow \quad r^n \sec n\theta = K \quad [C^n = K]$$

This is the required O.T.

WORKSHEET 02

1. Solve the following differential equations by separating the variables method.

(a) $y - x\,y` = (y^2 + ay`)$

(b) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

(c) $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

(d) $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$

(e) $(x - xy^2)\, dx + (y - yx^2)\, dy = 0$

(f) $x(1 + y^2)\, dx + y\,(1 + x^2)\, dy = 0$

(h) $y`\tan y = \sin(x + y) + \sin(x - y)$

(i) $\ln(y`) = a\,x + by$

(j) $(x + y)^2\, y` = a^2$

(k) $y` = x^2(1 + y^2)$; when x = 0 and y = π/4

(*l*) $\cos u\, du - \sin v\, dv = 0$; $(u, v) = (\pi/2, \pi/2)$

(m) $y` = 1 + \tan(y - x)$

(n) $(4x + y)\, y' = 1$

(o) $y' = x(2 \ln x + 1)/(\sin y + y \cos y)$

(p) $y' = [\sin x + \ln x/x]/[\cos y - \sec^2 y]$

(q) $y' = e^{x-y} + x^2 e^{-y}$

(r) $y' = e^{x-y} + 1$

(s) $y' = (x + y)^2$

(t) $dx + e^x y^2\, dy = 0$; when $x = 0$ and $y = 1$

(u) $y' = \sin(x + y)$

2. Solve the following homogeneous differential equations

(a) $(x^2 - y^2)\, dx + 2xy\, dy = 0$

(b) $(x^2 + y^2)\, dx - 2xy\, dy = 0$

(c) $x^2 y\, dx - (x^3 + y^3)\, dy = 0$

(d) $xy' - y = (x^2 + y^2)^{1/2}$

(e) $y' = (y/x) + \sin(y/x)$

(f) $y' = (y/x) + \tan(y/x)$

(g) $(1 + e^{x/y})\, dx + e^{x/y}(1 - x/y)\, dy = 0$

(h) $[x \cos(y/x) + y \sin(y/x)]y = x\, [y \sin(y/x) - x \cos(y/x)]\, y'$

(i) $x y' = y[\ln y - \ln x + 1]$ (NOTE: $\ln a - \ln b = \ln(a/b)$)

3. Solve the following differential equations (Reducible to homogeneous)

(a) $y' = [(x + 2y - 3)/(2x + y - 3)]$

(b) $(2x + 3y - 5)\, dy + (3x + 2y - 5)\, dx = 0$

(c) $(4x + 6y + 3)\, dx = (6x + 9y + 2)\, dy$

(d) $(2x + 3y + 4)\, dx - (4x + 6y + 5)\, dy = 0$

(e) $(x - y)\, dy = (x + y + 1)\, dx$

(f) $y' = (x + y)/x$, $y(1) = 1$

(g) $(x + 6y)\, dx + (4x - y)\, dy = 0$; $y(-1) = 6$

(h) $\left(y + \sqrt{x^2 + y^2}\right)dx - x\, dy = 0;\ y(1) = 0$

4. Solve the following exact differential equations

(a) $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0$

(b) $(a^2 - 2x - y)dx - (x - y)\, dy = 0$

(c) $\left(3x^2 + 6xy^2\right)dx + 6\left(1 + x^2 y\right)dy = 0$

(d) $(x^2 - a y)\, dx = (a x - y^2)\, dy$

(e) $\sec 2x \tan y\, dx + \sec 2y \tan x\, dy = 0$

(f) $(2y \sin 2x)\, dx - (y^2 + \cos 2x)\, dy = 0$

(g) $(x^4 - 2xy^2 + y^4)\, dx - (2x^2 y - 4xy^3 + \sin y)\, dy = 0$

(h) $\cos x\, (\cos x - \sin a \sin y)\, dx + \cos y\, (\cos y - \sin a \sin x)\, dy = 0$

(i) $(2y \sin x \cos x + y^2 \sin x)\, dx + (\sin^2 x - 2y \cos x)\, dy = 0$; $y(0) = 3$

(j) $(2xy - 3)\, dx + \left(x^2 + 4y\right) dy = 0;\ y(1) = 2$

(k) $\left(3x^2y^2 - y^3 + 2x\right) dx + \left(2x^3y - 3xy^2 + 1\right) dy = 0;\ y(-2) = 1$

(*l*) $\dfrac{3 - y}{x^2}dx + \dfrac{y^2 - 2x}{xy^2}dy = 0;\ y(-1) = 2$

(m) $\left(4x^3 e^{x+y} + x^4 e^{x+y} + 2x\right) dx + \left(x^4 e^{x+y} + 2y\right) dy = 0;\ y(0) = 1$

5. Find an appropriate Integral Factor and hence solve the following differential equations.

(a) $\left(x^2 + y^2 + 2x\right) dx + 2y\, dy = 0$

(b) $\left(x y^2 - e^{1/x^3}\right)dx - x^2 y\, dy = 0$

(c) $\left(y - 2x^3\right) dx - x(1 - xy)\, dy = 0$

(d) $\left(y^2 + 2xy\right)dx + \left(2x^2 - xy\right) dy = 0$

(e) $\left(y + y^3/3 + x^2/2\right) dx + 0.25\left(x + x y^2\right) dy = 0$

(f) $\left(x - y^2\right) dx + 2xy\, dy = 0$

(g) $(y^4 + 2y)\, dx + (x y^3 + 2y^4 - 4x)\, dy = 0$

(h) $\left(2x^2y^2 + 6y\right)dx + \left(x^3y + 3x\right)dy = 0$

(i) $\left(3x^2 y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0$

(j) $\left(x^2 y - 2xy^2\right)dx - \left(x^3 - 3x^2 y\right) dy = 0$

(k) $\left(3xy^2 - y^3\right) dx - \left(2x^2y - x y^2\right) dy = 0$

(*l*) $\left(y^2x + 2x^2y^3\right) dx - \left(x^2 y - x^3y^2\right) dy = 0$

(m) $\left(xy \sin xy + \cos xy\right) y\, dx + \left(xy \sin xy - \cos xy\right) x\, dy = 0$

(n) $\left(y + 2xy^2\right) dx + \left(x - x^2 y\right) dy = 0$ (o) $\left(x^2 y - y^3\right) dx + \left(x^3 + x y^2\right) dy = 0$

(p) $y(1 + xy)\, dx + x(1 - xy)\, dy = 0$ (q) $y \ln y\, dx + \left(x - \ln y\right) dy = 0$

(r) $y\left(1 + xy\right) dx + x\left(1 + xy + x^2 y^2\right) dy = 0$

(s) $(y^3 - 3xy^2)dx + \left(2x^2y - xy^2\right) dy = 0$

6. Solve the following Linear/Bernoulli differential equations

(a) $\left(1 + x^2\right) y' + 2xy = 4x^2$ (b) $y' + y \sec x = \tan x$

(c) $y' + y/x = x^2$ (d) $y' - y \tan x = -2 \sin x$

(e) $\sec x\, y' = y + \sin x$ (f) $y' + y \tan x = \sec x$

(g) $\left(1 + x^2\right) y' + 2xy = \cos x$ (h) $x \ln x\, y' + y = 2 \ln x$

(i) $\left(x + 2y^3\right) y' = y$ (j) $\left(1 + y^2\right) dx = \left(\tan^{-1} y - x\right) dy$

(k) $\left(2x - 10y^3\right) y' + y = 0$ (m) $y' = \left(y/x\right) + y^2$

(n) $\dfrac{dy}{dx} = \dfrac{x + y + 1}{x + 1}$ (o) $\dfrac{dy}{dx} - \dfrac{1}{2x} y = \dfrac{y^2}{2x^2}$

(p) $y' = x^3y^3 - xy$ (q) $\dfrac{dr}{d\theta} + r \tan\theta = \cos^2\theta;\ r\left(\dfrac{\pi}{4}\right) = 1$

(r) $\left(x^2 + 1\right) y' + 4xy = x;\ y(2) = 1$

(s) $x(2 + x)\, y' + 2(1 + x)\, y = 1 + 3x^2;\ y(-1) = 1$

(t) $y' + 2xy = 2x^3;\ y(0) = 1$ (u) $\dfrac{dy}{dx} + \dfrac{2}{x} y = -x^9y^5;\ y(-1) = 2$

(v) $y' - y/x = y^2 \sin x$ (w) $xy - y' = y^3 e^{-x^2}$

7. Find the orthogonal trajectories for the following curves:

(a) $xy = c$ (b) $y^2 = 4ax$ (c) $x^{2/3} + y^{2/3} = a^{2/3}$

(d) $a y^2 = x^3$ (e) $r = a\left(1 - \cos\theta\right)$ (f) $r = a\left(1 + \cos\theta\right)$

(g) $r = a^{\theta}$ (h) $r^n \sin n\theta = a^n$ (i) $r^2 = c \sin 2\theta$

8. Find the member of the orthogonal trajectories for $x + y = ce^y$ that passes through (0, 5)

CHAPTER THREE

APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

INTRODUCTION

Linear first order differential equations have various applications in Physical sciences and Engineering. To solve physical problems one needs the techniques of differential equations. If a physical problem is translated into a mathematical model then quite often we obtain a set of differential equations along with conditions and the solution of differential equations provides the solution of the physical problem. In this section we shall discuss applications of differential equations to physical and engineering problems.

Growth and Decay Problems

Through experiments scientists have learnt that population of bacterial culture, population of people etc; grows or decays at a rate proportional to the population present or otherwise. That is, if A(t) denotes the amount of substance that is either growing or decaying, then we define dA/dt as the rate of change of this amount of substance with respect to time and according to the given law: $\frac{dA}{dt} \propto A$ or $\frac{dA}{dt} = kA$ or $\frac{dA}{dt} - kA = 0$

where k is constant of proportionality. We shall now discus few problems regarding such applications.

Example 01: The population of a town was 1000 a year ago and the present population is 2000. What will be the population after the end of 4^{th} year if the rate of increase in population is proportional to present population?

Solution: Let P be the population at time t. The initial population was 1000 so we can say that P = 1000 when t = 0. If unit of time is year then at t = 1, P = 2000. Since rate of growth is proportional to P, therefore, $\frac{dP}{dt} \propto P$ or $\frac{dP}{dt} = kP$ (1)

where k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore $\frac{dP}{P} = k\,dt \Rightarrow \int \frac{1}{P} dP = k\int 1dt + c \Rightarrow \ln P = kt + c$

$$\Rightarrow P = e^{kt+c} \Rightarrow P = e^c e^{kt} \Rightarrow P = me^{kt}, \quad \left(e^c = m\right) \qquad (2)$$

This is general solution of differential equation (1).

Now N = 1000 when t = 0, therefore, 1000 = m e^0 or m = 1000

Thus (2) becomes $P = 1000\, e^{kt}$ (3)

Also, N = 2000 when t = 1 and so 2000 = 1000 e^k or $e^k = 2$

Thus, from (3) we have: $P=1000(e^k)^t \Rightarrow P = 1000(2)^t$

Thus at the end of 4^{th} year the population of the town will be P = 1000 $(2)^4$ = 16, 000.

Example 02: A mold grows at a rate proportional to itself. At the beginning, the amount was 2 grams. In 2 days the amount increased to 3 grams. Find the amount after 8 days?

Solution: Let A be the amount of mold in grams at any time t, and unit of time be a day at time t.

Then according to the law of growth, we have $\frac{dA}{dt} \propto A$ or $\frac{dA}{dt} = kA$ (1)

where k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore, $\frac{dA}{A} = kdt \Rightarrow \int \frac{1}{A} dA = k\int 1dt + c \Rightarrow \ln A = kt + c$

$$\Rightarrow A = e^{kt+c} \Rightarrow A = e^c e^{kt} \Rightarrow A = me^{kt}, \text{ where } e^c = m. \quad (2)$$

This is general solution of differential equation (1).

Now, at t = 0, A = 2. Then from (2), we have $2 = m\, e^{k(0)}$ or m = 2.

Thus, (2) becomes: $A = 2\, e^{kt}$ (3)

Now, when t = 2, A = 2 then from (3), we have $3 = 2e^{2k} \Rightarrow e^{2k} = 3/2 \Rightarrow e^k = (3/2)^{1/2}$

Thus, (3) becomes $A = 2(3/2)^{t/2}$. When t = 8, $A = 2(3/2)^4 = \frac{1}{2^3}(3)^4 = \frac{81}{8} \Rightarrow A \approx 10$ gms

Hence the amount of mold will be 10 grams (approximately) after 8 days.

Example 03: Bacteria grow in a nutrient solution at a rate proportional to the amount present. Initially, there are 250 strands of bacteria in the solution that grows to 800 strands after seven hours. Find (a) an expression for the approximate number of strands in the culture at any time t (b) the time needed for the bacteria to grow to 1600 strands.

Solution: **(a)** Let N(t) denote the number of bacteria strands in the culture at any time t. Then according to law of growth, we have $\frac{dN}{dt} \propto N$ or $\frac{dN}{dt} = kN$ (1)

where k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and $\frac{dN}{N} = kdt \Rightarrow \int \frac{1}{N} dN = k\int 1dt + c \Rightarrow \ln N = kt + c$

$$\Rightarrow N = e^{kt+c} \Rightarrow N = e^c e^{kt} \Rightarrow N = me^{kt}, \text{ where } e^c = m \quad (2)$$

This is general solution of differential equation (1).

(b) At t = 0, N = 250, hence 250 = m $e^{k(0)}$ or m = 250. Thus, (2) becomes: N = 250 e^{kt}

Now, when t = 7, N = 800, hence 800 = 250 e^{7k} or e^{7k} = 3.2 or 7k = ln 3.2 giving k = 0. 166

Thus, above equation becomes $N = 250\, e^{0.166\, t}$ (3)

This is the required expression for the approximate number of strands in the culture at any time t.

We require t when N = 1600. Substituting the N in (3), we get

$$1600 = 250e^{0.166t} \Rightarrow e^{0.166t} = 6.4 \Rightarrow 0.166t = \ln(6.4) \Rightarrow t \approx 11.2 \text{ hours.}$$

Hence 11.2 hours are needed for bacteria to grow to 1600 strands.

Example 04: The population of a certain country has grown at a rate proportional to number of people in the country. At present, the country has 80 million inhabitants. Ten years ago it had 70 million. Assuming that this trend continues, find (a) an expression for the approximate number of people living in the country at time t (taking t = 0 to be the present time) and (b) the approximate number of people at the end of the next ten year period.

Solution: **(a)** Let P(t) denote the number of people in the country at any time t. Then according to the law of growth, we have $\frac{dP}{dt} \propto P$ or $\frac{dP}{dt} = kP$ (1)

Here, k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore $\frac{dP}{P} = kdt \Rightarrow \int \frac{1}{P} dP = k\int 1dt + c \Rightarrow \ln P = kt + c$

$$\Rightarrow P = e^{kt+c} \Rightarrow P = e^c e^{kt} \Rightarrow P = me^{kt} \quad \text{where } e^c = m \qquad (2)$$

This is the general solution of the differential equation (1).

(b) At t = 0; P = 80 million which gives; $80 = m\, e^{k(0)}$ giving 80 = m.

Thus, (2) becomes: $P = 80\, e^{kt}$ (3)

At t = –10, P = 70 million which gives $70 = 80\, e^{-10k}$ or $e^{-10k} = 7/8$ or $-10k = \ln(7/8)$ giving k = 0.01335.

Thus, (3) becomes $P = 80\, e^{0.01335\, t}$ million

Now, when t = 10, $P = 80\, e^{0.01335(10)} = 91.43$ million.

Example 05: The population of a certain state is known to grow at a rate proportional to the number of people presently living in the state. If after 10 years the population has trebled and if after 20 years the population is 150,000, find the number of people initially living in the state.

Solution: Let P(t) denote the population in the state at any time t. Then according to the law of growth, we have $\frac{dP}{dt} \propto P$ or $\frac{dP}{dt} = kP$ (1)

where k is constant of proportionality. Equation (1) is clearly a separable first order differential equation and $\frac{dP}{P} = kdt \Rightarrow \int \frac{1}{P} dP = k\int 1dt + c \Rightarrow \ln P = kt + c$

$$\Rightarrow \quad P = e^{kt+c} \Rightarrow P = e^c e^{kt} \quad \Rightarrow \quad P = me^{kt} \quad \text{where } e^c = m \qquad (2)$$

This is the general solution of differential equation (1).

At t = 10, $P = 3P_0$ or $3P_0 = m\, e^{10k}$ (3)

At t = 20, N = 150, 000 or $150,000 = m\, e^{20k}$ (4)

At t = 0, let $P = P_0$. Thus, (2) becomes $P_0 = m\, e^0$ giving $m = P_0$

Substituting in (2), we have

$3P_0 = P_0\, e^{10k}$ or $3 = e^{10k}$ or $10k = \ln 3$ giving k = 0.11

Thus equation (4) becomes $150,000 = me^{20(0.11)} \Rightarrow m = \frac{150,000}{e^{2.2}} \Rightarrow m \approx 16,620$

Thus, the number of people initially living in the state is m = P0 =16,620.

Example 06: A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 milligrams of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find (a) an expression for the mass of the material remaining at any time t, (b) the mass of the material after four hours, and (c) the time at which the material has decayed to one half of its initial mass.

Solution: **(a)** Let A denote the amount of material present at any time t. Then accordingly, we have

$$\frac{dA}{dt} \propto A \quad \text{or} \quad \frac{dA}{dt} = kA \qquad (1)$$

where k is constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore: $\frac{dA}{A} = kdt \Rightarrow \int \frac{1}{A} dA = k\int 1dt + c \Rightarrow \ln A = kt + c$

$$\Rightarrow A = e^{kt+c} \Rightarrow A = e^c e^{kt} \Rightarrow A = me^{kt} \quad \text{where } e^c = m \qquad (2)$$

This is the general solution of the differential equation (1).

(b) At t = 0, we are given that A = 50. Therefore, from (2), we have

$$50 = m\, e^{k(0)} \quad \text{or } m = 50$$

Thus, (2), becomes: $A = 50\, e^{kt}$ (3)

At t = 2, 10 percent of the original mass of 50 mg, that is, 5mg, is lost/decayed. Hence, at t = 2, A = 50 – 5 = 45 mg. Substituting these values into (3), we have

$$45 = 50e^{2k} \Rightarrow e^{2k} = 0.9 \Rightarrow 2k = \ln(0.9) \Rightarrow k \approx -0.053$$

Hence, equation (3), becomes: $A = 50\, e^{-0.053t}$ (4)

We require A at t = 4. Substituting it into (4), we obtain

$$A = 50e^{-0.053(4)} \Rightarrow A = 50e^{-0.212} \Rightarrow A \approx 40.5\text{mg}$$

(c) We require t when A = 50/2 = 25 Substituting it into (4), we obtain

$$25 = 50e^{-0.053t} \Rightarrow -0.053t = \ln(1/2) \Rightarrow t = \frac{-0.693}{-0.053} \Rightarrow t \approx 13\,\text{hours}$$

Note: The time required reducing a decaying material to one half its original mass is called the ***half – life*** of the material. For this problem, the half – life is 13 hours.

Investments

The growth of capital compounded continuously is similar to the growth of population, and first order differential equations are helpful in finding the compounded capital.

Example 07: If Rs.10, 000 are invested with annual interest of 10% compounded continuously. What will be the total amount after 5 years?

Solution: Let A denote the amount after t years. Then the rate of growth

$$\frac{dA}{dt} = 10\% \text{ of } A \quad \text{or} \quad \frac{dA}{dt} = 0.1A$$

After separating the variables and integrating, we get $A = me^{0.1t}$ (1)

Initially, A(0) = 10, 000, therefore, $10{,}000 = me^{0.1(0)} \Rightarrow m = 10{,}000$

Equation (1) becomes $A = 10{,}000e^{0.1t}$ (2)

Substituting t = 5 in (2), we find the balance after 5 years to be

$$A = 10{,}000e^{0.1(5)} \Rightarrow A \approx 16{,}487$$

Thus, after 5 years the capital grows to approximately Rs.16, 487.

Example 08: A depositor places \$5000 in an account established for a child at birth. Assuming no additional deposits or withdrawals, how much will the children have upon reaching the age of 21 if the bank pays 5 percent interest per annum compounded continuously for the entire time period?

Solution: Let A denote the amount at any time t.

Then the rate of growth $\frac{dA}{dt} = 5\%A$ or $\frac{dA}{dt} = 0.05N$ (1)

Separating the variables and integrating, we get: $A = me^{0.05t}$

At t = 0; A = 5000, therefore, $5000 = me^{0.05(0)} \Rightarrow m = 5000$

Therefore (1) becomes: $A = 5000e^{0.05t}$ (2)

Substituting t = 21 in (2), we find the balance after twenty-one years, that is,

$$A(21) = 5000e^{0.05(2)} \Rightarrow A \approx 14,288$$

Thus, the child will have approximately \$14, 288 reaching at the age of 21 years.

Example 09: How long will it take a bank deposit to triple in value if interest is compounded continuously at a rate of 21/4 percent per annum?

Solution: Let A denote the amount at any time t. Then the rate of growth, that is,

$$\frac{dA}{dt} = 21/4\%A \text{ or } \frac{dA}{dt} = 0.0525\,A \qquad (1)$$

Separating the variables and integrating, we get: $N = e^{0.0525\,t}$

At t = 0; A = A_0, therefore, $A_0 = me^{0.0525(0)} \Rightarrow m = A_0$

Therefore (1) becomes $A = A_0e^{0.0525t}$ (2)

Now, A = 3 A_0, then $3A_0 = A_0e^{0.0525t} \Rightarrow 3 = e^{0.0525t} \Rightarrow t = \frac{\ln(3)}{0.0525} \Rightarrow t \approx 21$ years

Thus, bank will take approximately 21 years to make the deposit tripled.

Example 10: A depositor currently has \$6000 and plans to invest it in an account that accrues interest continuously. What interest rate must the bank pay if the depositor needs to have \$10,000 in four years?

Solution: Let A denote the amount invested in the account at any time t. Then the rate of growth

$\frac{dA}{dt} = kA$. Separating variables and Integrating, we get $A = me^{kt}$ (1)

At t = 0, A = 6000, therefore, $6000 = me^{k(0)} \Rightarrow m = 6000$

Therefore (1) becomes $N = 6000e^{kt}$ (2)

At t = 4, A = 10, 000, then $1000 = 6000e^{4k} \Rightarrow k = \frac{\ln(5/3)}{4} \Rightarrow k \approx 12.5$

Thus, bank should pay approximately 12.5% interest.

Example 11: A depositor currently has \$8000 and plans to invest it in an account that accrues interest continuously at the rate of 25/4 percent. How long will it take for the account to grow to \$13,500?

Solution: Let A denote the amount invested in the account at any time t. Then the rate of growth

$$\frac{dA}{dt} = 25/4\%A \text{ or } \frac{dA}{dt} = 0.0625\,A \Rightarrow A = me^{0.0625\,t} \quad (1)$$

At t = 0, A = 8000, therefore, $8000 = Ae^{0.0625(0)} \Rightarrow A = 8000$

Therefore (1) becomes: A = 8000 $e^{0.0625\ t}$ (2)

When, A = 13, 500 then we have

$$13500 = 8000e^{0.0625t} \Rightarrow e^{0.0625t} = 1.6875 \Rightarrow 0.0625t = \ln(1.6875) \Rightarrow t \approx 8.4$$

Thus, approximately 8.4years will take for the amount to grow to $13500.

Newton's Law of Cooling

A hot body cools at a rate proportional to the difference of temperature of the body and temperature of its surroundings. This law also translates to a first order differential equation.

Example 12: A cup of coffee was initially boiling at 120 C^o. It was then placed in air at 20 C^o. After 10 minutes the temperature was 80 C^o. What will be the temperature of coffee after another 10 minutes?

Solution: Let T be the temperature of the cup of coffee at any time t and T_m be the temperature of the surroundings (air). Then according to the ***"Newton's Law of Cooling"***, we have

$$\frac{dT}{dt} \propto (T - T_m) \text{ or } \frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{dt} = k(T - 20) \quad (1)$$

From (1), we have $\frac{1}{T-20}dT = kdt \Rightarrow \int \frac{1}{T-20}dT = \int kdt + c$

$$\Rightarrow \ln(T-20) = kt + c \Rightarrow T - 20 = e^{kt+c} \Rightarrow T - 20 = e^{kt}e^{c} \Rightarrow T - 20 = me^{kt} \quad (2)$$

Substituting t = 0, T = 120, we have

$$120 - 20 = me^{k(0)} \Rightarrow m = 100$$

Now (2) becomes $T - 20 = 100e^{kt}$ (3)

At t = 0, T = 80, we have: $60 = 100e^{10k} \Rightarrow e^{10k} = \frac{3}{5} \Rightarrow k = \frac{\ln(3/5)}{10} \Rightarrow k \approx -0.0511$

Thus, (3) becomes $T - 20 = 100e^{-0.0511t}$ (4)

At t = 20, we have, $T - 20 = 100e^{-0.0511(20)} \Rightarrow T = 100e^{-1.022} + 20 \Rightarrow T \approx 56C^{\circ}$

Hence, the temperature of the coffee after another 10 minutes will be approximately 56 C^o.

Example 13: A body was heated to $100\,C^{o}$ and then placed in a freezer at $0\,C^{o}$. After 30 minutes its temperature was $80\,C^{o}$. How much additional time is required for it to cool to 50 C^o.

Solution: Let T be the temperature of the body at any time t. Then according to the Newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T - T_m) \text{ or } \frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{dt} = k(T - 0) \Rightarrow \frac{dT}{dt} = kT \quad (1)$$

which has the solution $T = me^{kt}$ (2)

Since at t = 0, T = 100, we have $100 = me^{k(0)} \Rightarrow m = 100$

Therefore, equation (2) becomes: $T = 100e^{kt}$ (3)

Now at t = 30, T = 80. Thus we have

$$80 = 100e^{30k} \Rightarrow e^{30k} = \frac{4}{5} \Rightarrow 30k = \ln(4/5) \Rightarrow k \approx -0.0074$$

Thus, (3) becomes $T = 100e^{-0.00074t}$ (4)

Now when T = 50, we have

$$50 = 100e^{-0.00074t} \Rightarrow e^{-0.00074t} = 0.5 \Rightarrow -0.0074t = \ln(0.5) \Rightarrow t \approx 94$$

Thus, additional time required for the body to cool down to 50 C^o is approximately $(94-30) = 64$ minutes.

Example 14: A body at a temperature of 0^o F is placed in a room whose temperature is kept at $100\,F^o$. If after 10 minutes the temperature of the body is 25 F^o, find (i) the time required for the body to reach a temperature of 50 F^o and (ii) the temperature of body after 20 minutes.

Solution: Let T be the temperature of the body at any time t. Then according to the Newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T - T_m) \text{ or } \frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{dt} = k(T - 100) \Rightarrow \frac{1}{T-100}dT = kdt$$

Integrating, $\int \frac{1}{T-100}dT = k\int 1dt \Rightarrow \ln(T-100) = kt + c \Rightarrow T - 100 = e^{kt+c}$

$\Rightarrow$ $T - 100 = me^{kt}$ $\Rightarrow$ $T = me^{kt} + 100$ (1)

At t = 0, T = 0 F^o, therefore, $0 = me^{k(0)} + 100 \Rightarrow m = -100$.

Therefore (1) becomes $T = -100e^{kt} + 100$ (2)

At t = 0, T = 25 F^o, therefore: $25 = -100e^{10k} + 100 \Rightarrow e^{10k} = \frac{3}{4} \Rightarrow k = \frac{\ln(3/4)}{10} \Rightarrow k \approx -0.029$

Equation (2) becomes $T = -100\,e^{-0.029\,t}$ (3)

(i) When $T = 50F^o$: $50 = -100e^{-0.029t} + 100 \Rightarrow e^{-0.029t} = 0.5 \Rightarrow t = \frac{\ln(0.5)}{-0.029} \approx 24$ minutes

(ii) When t = 20 minute: $T = -100e^{-0.029(20)} + 100 \Rightarrow T \approx 44^o F$

Thus, the temperature of the body after 20 minutes will be approximately 44 F^o.

Falling Body Problems

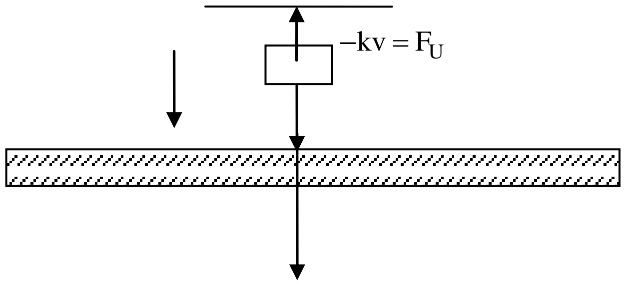
Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction.

Newton's Second Law of Motion: The net force acting on a body is equal to the time rate of change of the momentum of the body; or, for constant mass,

$$F = ma = m\frac{dv}{dt} \quad (1)$$

where F is the net force on the body and v is the velocity of the body and a the acceleration at any time t. For the problem at hand, there are two forces acting on the body:

- **(i)** the force due to gravity given by the weight w of the body equals mg.
- **(ii)** the force due to air resistance given by -kv where $k \geq 0$ is a constant of proportionality. The minus sign is required because this force opposes the velocity; that is, it acts in the up-ward, or negative direction. (See the figure):



The net force F on the body is, therefore, $F = F_D + F_U = mg + (-kv) \Rightarrow F = mg - kv$

Substituting this in (1) we obtain

$$mg - kv = m\frac{dv}{dt} \Rightarrow g - \frac{kv}{m} = \frac{dv}{dt} \Rightarrow \frac{dv}{dt} + \frac{kv}{m} = g \qquad (2)$$

as the equation of motion for falling body.

If air resistance is negligible or non existing, then k = 0 and (2) simplifies to

$$dv / dt = g \qquad (3)$$

When k > 0, the limiting velocity v_l is defined by

$$v_l = mg/k \qquad (4)$$

Caution: Equations (2), (3) and (4) are valid only if the given conditions are satisfied. These equations are not valid if, for example, air resistance is not proportional to velocity but to the velocity squared, or if the upward direction is taken to be as positive direction.

Example 15: A body of mass 3 slugs is dropped from a height of 500 feet with zero velocity. Assuming no air resistance, find **(i)** an expression for the velocity of the body at any time t **(ii)** an expression for the position of body at any time t and **(iii)** the time required for the body to hit the ground?

Solution: **(i)** Since, there is no resistance, therefore $dv / dt = g$, which; is the governing equation.

Integrating gives $\int 1dv = g\int 1dt + c \Rightarrow v = gt + c$ (1)

When t = 0, v = 0, hence $0 = g(0) + c \Rightarrow c = 0$

Thus, (1) becomes: v = gt. Assuming g = 32 ft/sec^2, we have v = 32t (2)

(ii) Recall that velocity is the time rate of change of displacement, designated here by x. Hence, v = dx/dt and (2) becomes:

$$\frac{dx}{dt} = 32t \Rightarrow dx = 32tdt \Rightarrow \int 1dx = 32\int tdt + c \Rightarrow x = 16t^2 + c \qquad (3)$$

But at t = 0, x = 0, therefore, $0 = 16(0)^2 + c \Rightarrow c = 0$

Substituting this value in (3), we obtain: $x = 16t^2$ (4)

(iii) We require t when x = 500, from (4), we have

$$500 = 16t^2 \Rightarrow t = \sqrt{500/16} \Rightarrow t \approx 5.6\sec\cdot$$

Example 16: A body of mass 2 slugs is dropped from a height of 450 feet with an initial velocity of 10 ft/sec. Assuming no air resistance, find (i) an expression for the velocity of the body at any time t and (ii) the time required for the body to hit the ground.

Solution: **(i)** Since, there is no air resistance, therefore dv/dt = g which is the governing equation.

Therefore, $dv = gdt \Rightarrow \int 1dv = g\int 1dt + c \Rightarrow v = gt + c$ (1)

We are given that at t = 0, v = 0, then from (1), 10 = g(0) + c or c = 10.

Put this in (1),we obtain: $v = 32t + 10$, assuming $g = 32 \text{ft/sec}^2$

(ii) Since v = dx/dt, therefore above equation becomes

$$\frac{dx}{dt} = 32t + 10 \Rightarrow dx = (32t + 10)\,dt \Rightarrow \int 1dx = \int (32t + 10)\,dt + c \Rightarrow x = 16t^2 + 10t + c$$

But at $t = 0, x = 0,$ therefore, $0 = 16(0)^2 + 10(0) + c \Rightarrow c = 0$ (2)

Therefore equation (2) becomes $x = 16t^2 + 10\,t$ (3)

We require t when x = 450, therefore, from (3), we have

$$450 = 16t^2 + 10t \Rightarrow 16t^2 + 10t - 450 = 0 \Rightarrow 8t^2 + 5t - 225 = 0$$

Using quadratic formula, we have

$$t = \frac{-5 \pm \sqrt{25 + 7200}}{16} = \frac{-5 \pm 85}{16} = \frac{-5 - 85}{16}, \frac{-5 + 85}{16} \Rightarrow t = -5.625, 5$$

Neglecting negative sign, we get t = 5.6 sec.

Example 17: A ball is propelled straight up with an initial velocity of 250 ft/sec in a vacuum with no air resistance. How high will it go?

Solution: Since the direction is upward, therefore v = –25 ft/sec. Because of no air resistance, we have dv/dt = g which is the governing equation.

or $dv = gdt \Rightarrow \int 1dv = g\int 1dt + c \Rightarrow v = gt + c$ (1)

At t = 0, v = –250, therefore, $-250 = 32(0) + c \Rightarrow c = -250$ where, $g = 32\text{ft/sec}^2$

Equation (1) becomes: v = 32 t – 250 (2)

Since v = dx/dt, therefore, $\frac{dx}{dt} = 32t - 250 \Rightarrow dx = (32t - 250)\,dt \Rightarrow \int 1\,dx = \int (32t - 250)\,dt + c$

$\Rightarrow x = 16t^2 - 250t + c$. At t = 0, x = 0, therefore $0 = 16(0)^2 - 250(0) + c \Rightarrow c = 0$

or $x = 16\,t^2 - 250\,t$ (3)

Thus at the maximum height, v = 0.Therefore from (2), we have $0 = 32t - 250 \Rightarrow t = 7.8125$

Substituting this value in (3), we get: $x = 16(7.8125)^2 - 250(7.8125)$

$$\Rightarrow x = -976.5625 \text{ or } x = 976.5625 \quad (\text{ignoring the negativ sign})$$

Example 18: A body of mass 10 slugs is dropped from a height of 100 feet with no initial velocity. The body encounters an air resistance proportional to its velocity. If the limiting velocity is known to be 320 ft/sec, find (i) an expression for the velocity of the body at any time t, (ii) an expression for the position of the body at any time t and (iii) the time required for the body to attain a velocity of 160 ft/sec.

Solution: **(i)** The limiting velocity is defined to be

$$v_l = mg / k \Rightarrow 320 \text{ or } 10 \times 32 = 320k \Rightarrow k = 1$$

Equation of motion of the body is

$$\frac{dv}{dt}+\frac{k}{m}v=g\Rightarrow\frac{dv}{dt}+\frac{1}{10}v=32\Rightarrow\frac{dv}{dt}+0.1v=32 \qquad (1)$$

which is a linear differential equation in v. Therefore, $I.F=e^{\int 0.1dt}=e^{0.1t}$

Thus, (1) becomes:

$$e^{0.1t}\frac{dv}{dt}+0.1ve^{0.1t}=32e^{0.1t}\ \Rightarrow\frac{d}{dt}\left(e^{0.1t}v\right)=32e^{0.1t}\ \Rightarrow e^{0.1t}v=32\int e^{0.1t}dt+c$$

or $$e^{0.1t}v=32\frac{e^{0.1t}}{0.1}+c\ \Rightarrow e^{0.1t}v=32e^{0.1t}+c\ \Rightarrow v=ce^{-0.1t}+320 \qquad (2)$$

At t = 0, we are given that v = 0. Substituting these values into (2) we get c = 320. The velocity at any time is therefore given by: $v=320e^{-0.1t}+320$ (3)

(ii) Since, v = dx/dt, therefore

$$\frac{dx}{dt}=-320e^{-0.1t}+320\ \Rightarrow\ dx=\left(-320e^{-0.1t}+320\right)dt\ \Rightarrow\int 1dx=\int\left(-320e^{-0.1t}+320\right)dt$$

or $x=-\frac{320}{-0.1}e^{-0.1t}+320t+c\ \Rightarrow x=3200e^{-01.t}+320t+c$

When t = 0, x = 0, therefore $0=3200e^{0}+320(0)+c\Rightarrow c=-3200$

Thus, $x=3200\ e^{-0.1t}+320\ t-3200$

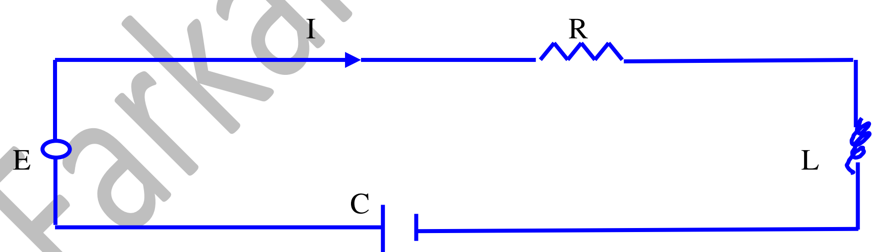
(iii) Since $v=160$ ft/sec^2, therefore, from (3), we have

$160=-320e^{-0.1t}+320\Rightarrow-160=-320e^{-0.1t}\ \Rightarrow e^{-0.1t}=0.5\ \Rightarrow-0.1t=\ln(0.5)\ \Rightarrow t\approx 6.93$ sec.

Electrical Circuits

In a simple electrical circuit like the one shown in following figure:

E is the source of power (known as electromotive force), measured in volts, I is the current produced, measured in amperes, R is resistance of a resistor, measured in ohms, L is inductance of an inductor, measured in henries, and C is capacitance of a capacitor which stores the charge Q, measured in farads.



According to Ohm's Law, V_R = IR where V_R is voltage drop in emf at the resistor. Similarly V_L is the voltage drop in emf at inductor and is given by, V_L = L. dI/dt. Voltage drop in emf at the capacitor is V_C = Q/C.

Now ***Kirchoff's law*** states that the algebraic sum of voltage drops around a closed circuit is equal to total voltage in the circuit. Therefore,

$$E=V_R+V_L+V_C\Rightarrow E=IR+L\cdot\frac{dI}{dt}+\frac{1}{C}Q \qquad (1)$$

At present we consider two types of electric circuits, first, in which there is no capacitor. In this case equation (1) becomes $E=IR+L\cdot\frac{dI}{dt}\Rightarrow L\cdot\frac{dI}{dt}+IR=E$ (2)

The second case when there is no inductor. In this case equation (1) becomes:

$$E = IR + \frac{1}{C}Q \Rightarrow R\cdot\frac{dQ}{dt} + \frac{1}{C}Q = E \qquad (3)$$

Equations (2) and (3) are linear first order differential equations. It may be noted that I = dQ/dt.

Example 19: An RL circuit has an e.m.f of 5 volts, a resistance of 50 ohms, an inductance of 1henry, and no initial current. Find the current in the circuit at any time t.

Solution: Given that $E = 5$ volts, $R = 50$ ohms, $L = 1$henry$\cdot$ The governing equation is:

$$L\cdot\frac{dI}{dt} + IR = E$$

Substituting the given values, we get

$$\frac{dI}{dt} + 50I = 5 \Rightarrow \frac{dI}{dt} + 50I = 5 \qquad (1)$$

This is linear equation; therefore its integrating factor is

$$I.F = e^{\int 50\,dt} = e^{50t}$$

Multiplying equation (1) by I.F, we get $e^{50t}\frac{dI}{dt} + 50e^{50t}I = 5e^{50t} \Rightarrow \frac{d}{dt}\left(e^{50t}I\right) = 5e^{50t}$

Integrating, we obtain $e^{50t}I = 5\int e^{50t}\,dt + c \Rightarrow e^{50t}I = \frac{1}{10}e^{50t} + c \Rightarrow I = \frac{1}{10} + ce^{-50t}$ (2)

At $t = 0, I = 0,$ then (2) becomes, $0 = \frac{1}{10} + ce^{-50(0)} \Rightarrow c = -\frac{1}{10}$

Thus equation (2) becomes, $I = \frac{1}{10} - \frac{1}{10}e^{-50t} \Rightarrow I = \frac{1}{10}\left(1 - e^{-50t}\right)$

Example 20: An RL circuit has an e.m.f of 9 volts, a resistance of 10 ohms, an inductance of 1.5 Henry, and an initial current of 6amperes. Find the current in the circuit at any time t.

Solution: Given that $E = 9$ volts, $R = 10$ ohms, $L = 9$henries$\cdot$ The governing equation is:

$$L\cdot\frac{dI}{dt} + IR = E.$$

Substituting the given values, we get: $(1.5)\frac{dI}{dt} + 10I = 9 \Rightarrow \frac{dI}{dt} + \frac{20}{3}I = 6$ (1)

This is linear equation; therefore, its integrating factor is: $I.F = e^{\int \frac{20}{3}t\,dt} = e^{\frac{20}{3}t}$

Multiplying equation (1) by I.F: $e^{\frac{20}{3}t}\frac{dI}{dt} + \frac{20}{3}e^{\frac{20}{3}t}I = 6e^{\frac{20}{3}t} \Rightarrow \frac{d}{dt}\left(e^{\frac{20}{3}t}I\right) = 6e^{\frac{20}{3}t}$.

Integrating, we obtain

$$e^{\frac{20}{3}t}I = 6\int e^{\frac{20}{3}t}\,dt + c \Rightarrow e^{\frac{20}{3}t}I = 0.9e^{\frac{20}{3}t} + c \Rightarrow I = 0.9 + ce^{-\frac{20}{3}t} \qquad (2)$$

At t = 0, I = 6 then (2) becomes, $6 = 0.9 + ce^{-(20/3)(0)} = 0.9 + c \Rightarrow c = 6 - 0.9 = 5.1$

Put this in equation (2), we get $I = 0.9 + 5.1e^{-(20t/3)}$

Example 21: An RL circuit has an e.m.f given by 4sin t (in volts) a resistance of 100 ohms, an inductance of 4henries and no initial current. Find the current at any time t.

Solution: Here we have $E = 4\sin t, R = 100, L = 4$. The governing equation is thus

$$L\cdot\frac{dI}{dt} + IR = E$$

Substituting the given values, we get

$$4\frac{dI}{dt}+100I=4\sin t\Rightarrow\frac{dI}{dt}+25I=\sin t \qquad (1)$$

This equation is linear; therefore we find its integrating factor: $I.F.=e^{\int 25\,dt}=e^{25t}$

Multiply equation (1) by integrating factor, we obtain

$$e^{25t}\frac{dI}{dt}+25e^{25t}I=e^{25t}\sin t\Rightarrow\frac{d}{dt}\left(e^{25t}I\right)=e^{25t}\sin t\Rightarrow e^{25t}I=\int e^{25t}\sin t\,dt+c \qquad (2)$$

Use integration by parts on right side of (2), we get

$$e^{25t}I=\frac{25}{226}e^{25t}\left(\sin t-\frac{1}{25}\cos t\right)+c\Rightarrow I=\frac{25}{226}\sin t-\frac{1}{226}\cos t+ce^{-25t} \qquad (3)$$

At $t=0, I=0,$ then (4) becomes, $0=-\frac{1}{226}+c\Rightarrow c=\frac{1}{226}$

Hence (3) becomes, $I=\frac{25}{226}\sin t-\frac{1}{226}\cos t+\frac{1}{226}e^{-25t}\Rightarrow I=\frac{1}{226}\left(25\sin t-\cos t+e^{-25t}\right)$

Example 22: An RC circuit has an emf given (in volts) by 400 cos 2t a resistance of 100 ohms and a capacitance of 10^{-2} Farad. Initially there is no charge on the capacitor. Find the current in the circuit at any time t.

Solution: Here we have $E=400\cos 2t, R=100, C=10^{-2}$. The governing equation is therefore

$$R\cdot\frac{dQ}{dt}+\frac{1}{C}Q=E$$

Substituting the given values, we get

$$(100)\frac{dQ}{dt}+\frac{1}{10^{-2}}Q=400\cos 2t\Rightarrow\frac{dQ}{dt}+Q=4\cos 2t \qquad (1)$$

This is linear equation and its solution is given by:

$$Q=\frac{8}{5}\sin 2t+\frac{4}{5}\cos 2t+ce^{-t} \qquad (2)$$

[Readers are advised to solve this differential equation (1) for Q.

At $t=0, Q=0,$ then (2) becomes, $0=(0)+\frac{4}{5}+c\Rightarrow c=-\frac{4}{5}$

Thus, from (2): $Q=\frac{8}{5}\sin 2t+\frac{4}{5}\cos 2t-\frac{4}{5}e^{-t}$

Differentiate w.r.t t: $I=\frac{dQ}{dt}=\frac{d}{dt}\left(\frac{8}{5}\sin 2t+\frac{4}{5}\cos 2t-\frac{4}{5}e^{-t}\right)$

$$I=\frac{8}{5}\cos 2t(2)+\frac{4}{5}(-\sin 2t)(2)-\frac{4}{5}e^{-t}(-1)\Rightarrow I=\frac{2}{5}\left(8\cos 2t-4\sin 2t+2e^{-t}\right)$$

Example 23: An RC circuit has an emf of 100 volts, a resistance of 5 ohms, a capacitance of 0.02 farad, and an initial charge on the capacitor of 5 coulombs. Find (i) the expression for the charge on the capacitor at any time t and (ii) the current in the circuit at any time t.

Solution: **(i)** Here we have $E=100, R=5, C=0.02$. The governing equation is therefore:

$$R\cdot\frac{dQ}{dt}+\frac{1}{C}Q=E$$

Substituting the given values, we get

$$5\frac{dQ}{dt}+\frac{1}{0.02}Q=100\Rightarrow\frac{dQ}{dt}+10Q=20 \qquad (1)$$

This is linear equation; therefore its integrating factor: $I.F=e^{\int 10\,dt}$

Therefore equation (1) becomes

$$e^{10t}\frac{dQ}{dt}+10e^{10t}Q=20e^{10t}\Rightarrow\frac{d}{dt}\left(e^{10t}Q\right)=20e^{10t}\Rightarrow e^{10t}Q=20\int e^{10t}dt+c$$

$$\Rightarrow e^{10t}Q=20e^{10t}\left(\frac{1}{10}\right)+c\Rightarrow e^{10t}Q=2e^{10t}+c\Rightarrow Q=2+ce^{-10t} \qquad (2)$$

At $t=0, Q=5$, then (2) becomes $\qquad 5=2+ce^{0}\Rightarrow c=3$

Thus, from (2) we have $\qquad Q=2+3e^{-10t}$

(ii) Differentiate w.r.t. t, we get: $I=\frac{dQ}{dt}=\frac{d}{dt}\left(2+3e^{-10t}\right)=-30e^{-10t}$

WORKSHEET 04

1. The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after 2 years population has been doubled and after 3 years the population is 20,000, find the number of people initially living in the country.
2. A certain culture of bacteria grows at a rate that is proportional to the number present. If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?
3. The radioactive isotope thorium 234 disintegrates at a rate proportional to the amount present. It is found that in one week 17.96% of this material has disintegrated. Determine how long will it take for one half of this material to disintegrate?
4. A certain radioactive material is known to decay at a rate proportional to the amount present. If initially, there are 100 milligrams of the material present and if after two years it is observed that 5 percent of the original mass has decayed, find (a) an expression for the mass at any time t and (b) the time necessary for 10 percent of the original mass to have decayed.
5. A man currently has Rs. 12,000 and plans to invest it in an account that accrues interest continuously. What interest rate must he receive, if his goal is to have Rs. 16,000 in 3 years?
6. How long will it take a bank deposit to double if interest is compounded continuously at a constant rate of 8 percent pr annum?
7. A depositor places Rs. 100,000 in a certificate of deposit account which pays 7 percent interest per annum, compounded continuously. How much will be in the account after 2 years?
8. A man places \$700 in an account that accrues interest continuously. Assuming no additional deposits and no withdraws, how much will be in the account after 10 years if the interest rate is a constant 7.5 percent for first 6 years and a constant 8.25 percent for the last 4 years?
9. A body at a temperature of 50 F^o is placed outdoors where the temperature is 100 F^o. If after 5 minutes the temperature of the body is 60 F^o, determine the temperature of the body after 20 minutes.
10. A substance cools in air from 100 C^o to 70 C^o in 15 minutes. If the temperature of air is 30 C^o, find when the temperature will be 40 C^o.

11. One liter of ice cream at a temperature of -15 C^o is removed from the deep freezer and placed in a room where the temperature is 20 C^o. If after 15 minutes the temperature of the ice cream is -10 C^o how long will it take the ice cream to reach a temperature of 0 C^o.

12. The temperature of a machine, when it is first shut down after operating, is 220^{o}C and temperature of the surrounding air is 30^{o}C. After 20 minutes, the temperature of the machine is 160 C^o. Find the temperature of the machine 30 minutes after it is shut down?

13. A copper ball is heated to a temperature of 100 C^o. Then at time t = 0 it is placed in water which is maintained at a temperature of 30 C^o. At the end of 3 minutes the temperature of the ball is reduced to 70 C^o. Find the time at which the temperature of the ball is reduced to 31 C^o.

14. A body is dropped from a height of 300 feet with an initial velocity of 30 ft/sec. Assuming, no air resistance, find (a) an expression for the velocity of the body at any time t and (b) the time required for the body to hit the ground?

15. The magnitude of the velocity (in meters per second) of a particle moving along the t-axis is given by v = t/4,. When t = 0, the particle is 2 meters to the right of the origin. Determine the position of the particle when t = 3 seconds.

16. A particle initially at rest, moves from a fixed point in a straight line so that its acceleration is given by sin t + $[1/(t+1)]^2$. What is its distance at the end of π seconds from the start?

17. A particle free to move along a straight line, becomes subject to an acceleration a = cos pt. If initially, the particle is at rest at the origin, what is its distance at any instant?

18. A train starting from rest is accelerated that is given by 10/(v + 1) ft/sec^2, where v is the velocity in ft/sec. Find the distance in which the train attains a velocity of 44 ft/sec.

19. An RC circuit has an emf of 5 volts, a resistance of 10 ohms, a capacitance of 10^{-2} Farad, and initially a charge of 5 coulombs on the capacitor. Find (a) the transient current and (b) the steady – state current.

20. An RC circuit has an emf of 300 cos 2t volts, a resistance of 150 ohms, a capacitance of $\frac{1}{6}\times 10^{-2}$ Farad, and an initial charge on the capacitor of 5 coulombs. Find (a) the charge on the capacitor at any time t and (b) the steady – state current.

21. An RL circuit has a resistance of 10 ohms, an inductance of 1.5 henries, an applied emf of 9 volts, and an initial current of 6 amperes. Find (a) the current in the circuit at any time t and (b) its transient component.

CHAPTER FOUR

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

INTRODUCTION

Higher order differential equations are extensively applied in finding the solution of physical problems. It is not easy, in general to solve a higher order differential equation but particular types of differential equations known as, linear differential equations, are easier to deal with. Although quite a few differential equations that appear in physical problems are linear, nevertheless, learning the solution of linear differential equations is very much important as they provide a foundation for the solution of non-linear differential equations. In this chapter only linear differential equations are studied and discussed.

Definition: A linear differential equation of order n is of the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\ldots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x) \qquad (1)$$

where $a_0(x), a_1(x), \ldots, a_{n-1}(x), a_n(x)$ and $f(x)$ are functions of independent variable x only and $a_0(x)$ is not zero. If $a_0, a_1, \ldots, a_{n-1}, a_n$ are real constants then we have

$$a_0 y^{(n)}+a_1 y^{(n-1)}+\ldots+a_{n-1}y'+a_n y=f(x) \qquad (2)$$

Equation (1) is known as differential equation with variable coefficients while (2) is known as differential equation with constant coefficients. We shall first discuss the solution of differential equation (2). If f(x) = 0, we get

$$a_0 y^{(n)}+a_1 y^{(n-1)}+\ldots+a_{n-1}y'+a_n y=0 \qquad (3)$$

Differential equation (3) is called ***homogeneous linear differential equation*** of order n. If f(x) is not identically zero then (2) is called ***non-homogeneous linear homogeneous differential equation***.

D-Operator

Differential coefficient d/dx sometimes is expressed as an operator D = d/dx. In this case, we write

$$\frac{dy}{dx}=Dy, \frac{d^2y}{dx^2}=D^2y, \ldots, \frac{d^n y}{dx^n}=D^n y$$

Then equation (2) may be expressed as:

$$a_0 D^n y+a_1 D^{n-1}y+\ldots+a_{n-1}Dy+a_n y=f(x)$$

Or, $$\left(a_0 D^n+a_1 D^{n-1}+\ldots+a_{n-1}D+a_n\right)y=f(x)$$

We may write $F(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n$.

In this case above equation may be expressed as F(D) y = f(x) and F(D) is regarded as single operator that operates on y.

Definition: If $y_1, y_2, ..., y_n$ are n functions of an independent variable x and $c_1, c_2, ..., c_n$ are constants, then $c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is called ***linear combination*** of $y_1, y_2, ..., y_n$.

Functions $y_1, y_2, ..., y_n$ are called ***linearly dependent*** if and only if, there exist constants $c_1, c_2, ..., c_n$ where at least one of which is nonzero, such that

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n = 0$$

Functions $y_1, y_2, ..., y_n$ are called ***linearly independent*** if and only if, they are not linearly dependent, that is;

$$c_1 y_1 + c_2 y_2 + ... + c_n y_n = 0 \quad \text{or} \quad c_1 = c_2 = ... = c_n = 0$$

Complementary Function and Particular Integral

Before we talk about the solution of (2) we state the following facts:

- If $y_1, y_2, ..., y_n$ are solutions of (3) then any linear combination $c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also a solution of (3) where $c_1, c_2, ..., c_n$ are arbitrary constants.
- Every homogeneous linear n^{th} order differential equation

  $$a_0 y^{(n)} + a_1 y^{(n-1)} + ... + a_{n-1} y' + a_n y = 0$$

  has n linearly independent solutions $y_1, y_2, ..., y_n$.
- If $y_1, y_2, ..., y_n$ are n linearly independent solutions of (3) then any linear combination $c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also its solution and this solution is called ***complementary function*** of (2). $c_1, c_2, ..., c_n$ being arbitrary constants.
- Equation (2) has two solutions y_c called ***complementary function*** and y_p the ***particular integral***. The complete solution of (2) is therefore $y = y_c + y_p$.
- The particular integral y_p is obtained by solving the equation $y_p = \frac{1}{F(D)} f(x)$.

These statements are also true for equation (1) with variable coefficients.

HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Consider the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + ... + a_{n-1} \frac{dy}{dx} + a_n y = 0 \qquad (1)$$

Using D-operator, equation (1) may be expressed as:

$$(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n)\, y = 0 \qquad (1a)$$

where $a_0, a_1, ..., a_{n-1}, a_n$ are real constants. To find a solution of (1), let us consider a simple case of (1) by taking n = 1. This gives: $a_0 y` + a_1 y = 0$ (1b)

$a_0 \frac{dy}{dx} + a_1 y = 0 \quad \Rightarrow \frac{dy}{dx} = my, \quad$ where $m = -a_1 / a_0$. Separating the variables,

$$\frac{dy}{y} = m\,dx \quad \Rightarrow \int \frac{1}{y} dy = m \int 1.dx + c` \quad \Rightarrow \ln y = mx + c` \quad \Rightarrow y = e^{mx + c`} = ce^{mx}$$

Here $c = e^{c'}$. This shows that $y = c\,e^{mx}$ is solution of (1b). Now equation (1b) is the special case of (1). This suggests that solution of (1) is also of the form $y = c\,e^{mx}$.

Now, $\frac{dy}{dx} = cme^{mx}, \frac{d^2y}{dx^2} = cm^2e^{mx}, ..., \frac{d^{n-1}y}{dx^{n-1}} = cm^{n-1}e^{mx}, \frac{d^ny}{dx^n} = cm^ne^{mx}$

Substituting into (1), we have $\quad ce^{mx}\left(a_0m^n + a_1m^{n-1} + ... + a_{n-1}m + a_n\right) = 0$

Since $ce^{mx} \neq 0$, we have $\quad a_0m^n + a_1m^{n-1} + ... + a_{n-1}m + a_n = 0 \qquad (2)$

Equation (2) is called ***characteristic or auxiliary equation*** of differential equation (1). Observe that (2) can be obtained directly from (1) by merely replacing the k^{th} derivative in (1) by $m^k \ (k = 0,1,2,...,n)\cdot$ Three cases arise according to as the roots of (2) are Real and Distinct or Real and Repeated or Complex.

CASE I: When the auxiliary equation has real and distinct roots

Let $m_1, m_2, ..., m_n$ be n distinct real roots of (2), then $e^{m_1x}, e^{m_2x}, ..., e^{m_nx}$ are n distinct solutions of (1). These n solutions are linearly independent. Therefore, the general solution of (2) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + ... + c_ne^{m_nx}$$

where $c_1, c_2, ..., c_n$ are arbitrary constants.

Example 01: Solve the following differential equations:

(i) $(D^3 - 4D^2 + D + 6)y = 0$

(ii) $(D^3 - 6D^2 + 11D - 6)y = 0;\ y(0) = 0 = y'(0),\ y''(0) = 2$

Solution: **(i)** Auxiliary equation is $\quad m^3 - 4m^2 + m + 6 = 0 \qquad (1)$

Using hit-and-trial method, if we put $m = -1$ in (1), we get

$$(-1)^3 - 4(-1)^2 + (-1) + 6 = 0 \Rightarrow -1 - 4 - 1 + 6 = 0 \Rightarrow 0 = 0$$

This implies that $m = -1$ is a root of the auxiliary equation (1). This means $(m + 1)$ is one of the factors of (1). Now, to find the other roots, we use the *synthetic division method* or otherwise,

$$m^3 - 4m^2 + m + 6 = (m+1)(m^2 - 5m + 6) = (m+1)(m-2)(m-3) = 0 \Rightarrow m = -1,\ 2,\ 3$$

We observe that roots of auxiliary equation are real and distinct, therefore, general solution of given differential equation is: $y = c_1e^{-x} + c_2e^{2x} + c_3e^{3x}$.

(ii) Auxiliary equation is $\quad m^3 - 6m^2 + 11m - 6 = 0 \qquad (1)$

Put $m = 1$ into (1) to get $\quad (1)^3 - 6(1)^2 + 11(1) - 6 = 0 \Rightarrow 1 - 6 + 11 - 6 = 0 \Rightarrow 0 = 0$

It implies that $(m - 1)$ is one of the factors of (1) or $m = 1$ is a root of auxiliary equation $m^3 - 6m^2 + 11m - 6 = (m-1)(m^2 - 5m + 6) = (m-1)(m-2)(m-3) = 0 \Rightarrow m = 1,\ 2,\ 3$

Other method to find the remaining roots of auxiliary equation (1) is by using the method of "***Synthetic Division***" which is shown below. [*Students are advised to ask their tutors to help them to understand the method*]

1	1	–6	11	–6
		1	–5	6
	1	–5	6	0

Thus, $m^3 - 6m^2 + 11m - 6 = (m-1)\left(m^2 - 5m + 6\right) = (m-1)(m-2)(m-3) = 0 \Rightarrow m = 1, 2, 3$.

The roots are real and distinct roots, therefore, *general solution* is given by

$$y = c_1e^{x} + c_2e^{2x} + c_3e^{3x} \tag{2}$$

Now,
$$y' = c_1e^{x} + 2c_2e^{2x} + 3c_3e^{3x} \tag{3}$$

$$y'' = c_1e^{x} + 4c_2e^{2x} + 9c_3e^{3x} \tag{4}$$

Now applying the given initial conditions, we get

$$0 = c_1 + c_2 + c_3 \tag{5}$$

$$0 = c_1 + 2c_2 + 3c_3 \tag{6}$$

$$2 = c_1 + 4c_2 + 9c_3 \tag{7}$$

Solving equations (5), (6) and (7), we get: $c_1 = 1,\ c_2 = -2,$ and $c_3 = 1$

Now substituting these values in equation (2), we get a *particular solution* of given differential equation as: $y = e^{x} - 2e^{2x} + e^{3x}$

CASE II: When auxiliary equation has real and repeated roots

Given $(a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n)\,y = 0$ (1)

The corresponding auxiliary equation is

$$F(m) = (a_0m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n) = 0 \tag{2}$$

If $m_1, m_2, \ldots, m_n$ are the roots of $F(m) = 0$ then (1) may be written as

$$(D - m_1)(D - m_2)\ldots(D - m_n)\,y = 0$$

If the root m_1 is repeated twice say $m_1 = m_2$, then general solution of (1) is

$$y = c_1e^{m_1x} + c_2e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx} \Rightarrow y = (c_1 + c_2)e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}$$

$$\Rightarrow y = c_0e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx},\ c_1 + c_2 = c_0$$

This solution contains only $(n - 1)$ arbitrary constants and therefore, it is not the general solution of (1). To obtain general solution we proceed as follows:

The part of general solution of (1) corresponding to twice-repeated root m_1 of equation (2) is the solution of
$$(D - m_1)^2\,y = 0 \ \text{ or } \ (D - m_1)(D - m_1)\,y = 0 \tag{3}$$

Let $(D - m_1)\,y = v$. Then (3) becomes

$$(D - m_1)\,v = 0 \Rightarrow \left(\frac{d}{dx} - m_1\right)v = 0 \Rightarrow \frac{dv}{dx} - m_1v = 0$$

Separating the variables and integrating, we get

$$\int \frac{1}{v}\,dv = m_1\int 1\,dx + k \Rightarrow \ln v = m_1x + k \text{ or } v = e^{(m_1x + k)} \Rightarrow v = e^{k}e^{m_1x} \Rightarrow v = c_2e^{m_1x}$$

Replacing v by $(D - m_1)\,y$, we obtain

$$(D - m_1)\,y = c_2e^{m_1x} \Rightarrow \frac{dy}{dx} - m_1y = c_2e^{m_1x} \tag{4}$$

This is the linear first order differential equation with $\text{I.F.} = e^{\int(-m_1)dx} = e^{-m_1x}$

Multiplying (4) by the integrating factor, we get

$$e^{m_1x}y` - m_1e^{m_1x}y = c_2 \quad \Rightarrow \frac{d}{dx}\left(e^{m_1x}y\right) = c_2$$

Integrating, we get: $\quad e^{-m_1x}y = c_2x + c_1 \quad \Rightarrow y = \left(c_2x + c_1\right)e^{m_1x}$

This is the part of general solution corresponding to repeated root m_1. The general solution of (1) is, therefore, $\quad y = \left(c_1 + c_2x\right)e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}$

Similarly, if auxiliary equation (2) has the roots which repeat three times, the corresponding part of the general solution of (1) is the solution of $\left(D - m_1\right)^3 y = 0 \cdot$ Proceeding as before, we can easily find $\quad y = (c_1 + c_2x + c_3x^2)\ e^{m_1x}$

as the part of general solution corresponding to this triple root m_1. If the auxiliary equation (2) has the real root m_1 occurring k times, then part of the general solution of (1) corresponding to the k times repeated root m_1 is $\quad y = \left(c_1 + c_2x + c_3x^2 + \ldots + c_kx^{k-1}\right)e^{m_1x}$

Example 02: Solve the following differential equations

(i) $(4D^4 - 4D^3 - 3D^2 + 4D - 1)y = 0$ (ii) $(D^2 + 6D + 9)y = 0$, $y(0) = 2$, $y`(0) = -3$

Solution: **(i)** Auxiliary equation is

$$4m^4 - 4m^3 - 3m^2 + 4m - 1 = 0 \qquad (1)$$

Put m = 1 in (1), we get: $4(1)^4 - 4(1)^3 - 3(1)^2 + 4(1) - 1 = 0 \Rightarrow 0 = 0$

It implies that m = 1 is one of the roots of equation (1). To find the other roots, we use the synthetic division method, that is,

$$\begin{array}{c|ccccc}
1 & 4 & -4 & -3 & 4 & -1 \\
 & & 4 & 0 & -3 & 1 \\
\hline
 & 4 & 0 & -3 & 1 & \multicolumn{1}{|c}{0}
\end{array}$$

Thus, $4m^4 - 4m^3 - 3m^2 + 4m - 1 = (m-1)\left(4m^3 - 3m + 1\right) = 0$.

Again, by hit and trial method, if we put m = –1, we see that:

$4m^3 - 3m + 1 = -4 + 3 + 1 = 0$. This shows that m = –1 is also the root of auxiliary equation. Thus, again by synthetic division method,

$$\begin{array}{c|cccc}
-1 & 4 & 0 & -3 & 1 \\
 & & -4 & 4 & -1 \\
\hline
 & 4 & -4 & 1 & \multicolumn{1}{|c}{0}
\end{array}$$

or $4m^3 - 3m + 1 = (m+1)\left(4m^2 - 4m + 1\right) = (m+1)(2m-1)(2m-1)$

or $4m^4 - 4m^3 - 3m^2 + 4m - 1 = (m-1)(m+1)(2m-1)(2m-1) = 0$

or $\quad m = 1, -1, 1/2, 1/2.$

Two roots of auxiliary equation are real, distinct and two are real and repeated, therefore, the general solution of given differential equation is given by

$$y = c_1e^x + c_2e^{-x} + \left(c_3 + c_4x\right)e^{x/2}$$

(ii) The auxiliary equation is $m^2+6m+9=0$ (1)

$$m^2+3m+3m+9=0 \Rightarrow m(m+3)+3(m+3)=0 \Rightarrow (m+3)(m+3)=0 \Rightarrow m=-3,-3$$

The roots are real and repeated, therefore the general solution of given equation is

$$y=(c_1+c_2x)e^{-3x} \quad (2)$$

Now $$y'=(c_1+c_2x)e^{-3x}(-3)+e^{-3x}(c_2) \quad (3)$$

Applying the given initial conditions, y(0) = 2 and y`(0) = –3, we get

$$2=c_1 \text{ and } -3=-3c_1+c_2 \Rightarrow c_2=3$$

Substituting these values in (2), we obtain particular solution of given differential equation as:

$$y=(2+3x)e^{-3x}$$

CASE III: When the auxiliary equation has complex roots

Suppose that auxiliary equation has complex number $a+ib$ as a non–repeated root. Since coefficients of auxiliary equation are real and is a polynomial, then the conjugate complex number $a-ib$ is also its non–repeated root. The corresponding part of general solution is:

$$y=k_1e^{(a+ib)x}+k_2e^{(a-ib)x}, \text{ where } k_1 \text{ and } k_2 \text{ are arbitrary constants.}$$

Or $$y=k_1e^{ax}e^{ibx}+k_2e^{ax}e^{-ibx}=e^{ax}\left(k_1e^{ibx}+k_2e^{-ibx}\right)$$

$$=e^{ax}\{k_1(\cos bx+i\sin bx)+k_2(\cos bx-i\sin bx)\}$$

$$=e^{ax}\{i(k_1+k_2)\sin bx+(k_1+k_2)\cos bx\} \qquad [\text{Note: } e^{i\alpha}=\cos\alpha+i\sin\alpha]$$

$$\Rightarrow \quad y=e^{ax}(c_1\sin bx+c_2\cos bx) \quad [c_1=i(k_1+k_2), c_2=(k_1+k_2)]$$

It may be noted that if $a+ib$ and $a-ib$ are conjugate complex roots and each one is repeated k times, then the corresponding part of general solution of (1) may be written as

$$y=e^{ax}\left\{\left(c_1+c_2x+c_3x^2+\ldots+c_kx^{k-1}\right)\sin bx+\left(c_{k+1}+c_{k+2}x+\ldots+c_{2k}x^{k-1}\right)\cos bx\right\}$$

Example 03: Solve the following differential equations

(i) $(75D^2 + 50D + 12)y = 0$

(ii) $(D^4 + 2D^3 - 2D^2 - 6D + 5)y = 0$

(iii) $(D^2 + 6D + 13)y = 0$; $y(0) = 3$, $y`(0) = -1$

Solution: **(i)** Auxiliary equation is: $75m^2+50m+12=0$ (1)

Using quadratic formula, we get

$$m=\frac{-50\pm\sqrt{(50)^2-(4)(75)(12)}}{2(75)}=\frac{-50\pm\sqrt{2500-3600}}{150}=\frac{-50\pm\sqrt{-1100}}{150}$$

$$=\frac{-50\pm10i\sqrt{11}}{150}=\frac{-5\pm i\sqrt{11}}{15}=\frac{-1}{3}\pm\frac{\sqrt{11}}{15}i$$

The roots of auxiliary equation are complex, therefore general solution is:

$$y=e^{-x/3}\left(c_1\sin\frac{\sqrt{11}}{15}x+c_2\cos\frac{\sqrt{11}}{15}x\right)$$

(ii) Auxiliary equation is $m^4+2m^3-2m^2-6m+5=0$ (1)

Put m = 1in (1), we get $(1)^4+2(1)^3-2(1)^2-6(1)+5=0 \Rightarrow 1+2-2-6+5=0 \Rightarrow 0=0$

This implies that $(m-1)$ is one of the factors of (1).

Now, $m^4+2m^3-2m^2-6m+5=0$

or $(m-1)(m^3+3m^2+m-5)=(m-1)(m-1)(m^2+4m+5)=0$

Using quadratic formula for $(m^2+4m+5)=0$, we get

$$m=\frac{-4\pm\sqrt{16-20}}{2}=\frac{-4\pm\sqrt{-4}}{2}=\frac{-4\pm 2i}{2}=-2\pm i$$

Hence the roots of (1) are $m=1,1,-2\pm i$. Thus general solution of given differential equation is:

$$y=(c_1+c_2x)e^x+e^{-2x}(c_3\sin x+c_4\cos x)$$

(iii) Auxiliary equation is: $m^2+6m+13=0$ (1)

Using quadratic formula, we get $m=\frac{-6\pm\sqrt{36-52}}{2}=\frac{-6\pm\sqrt{-16}}{2}=\frac{-6\pm 4i}{2}=-3\pm 2i$

The general solution of given equation is: $y=e^{-3x}(c_1\sin 2x+c_2\cos 2x)$ (2)

$$y'=e^{-3x}(2c_1\cos 2x-2c_2\sin 2x)-3e^{-3x}(c_1\sin 2x+c_2\cos 2x) \quad (3)$$

Applying the given initial conditions, y(0) = 3 and y`(0) = –1, we get

$$3=c_2 \text{ and } -1=(2c_1)-3(c_2)\Rightarrow c_1=4$$

Thus, particular solution of given differential equation is

$$y=e^{-3x}(4\sin 2x+3\cos 2x)$$

Example 04: Solve the following differential equations

(i) $(D^3 - 2D^2 + 4D - 8)\, y = 0$ (ii) $(D^4 - 4D^3 + 8D^2 - 8D + 4)\, y = 0$

Solution: **(i)** Given $(D^3 - 2D^2 + 4D - 8)\, y = 0$. Auxiliary equation is

$m^3 - 2m^2 + 4m - 8 = 0$ or $m = 2, \pm i2$ [Students may verify this]

Thus general solution is: $y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$.

(ii) Given differential equation is $(D^4 - 4D^3 + 8D^2 - 8D + 4)\, y = 0$

Auxiliary equation is: $m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$

or $(m^2 - 2m + 2)^2 = 0$ or $(m^2 - 2m + 2)(m^2 - 2m + 2) = 0$

This gives, $(m^2 - 2m + 2) = 0$ and $(m^2 - 2m + 2) = 0$. Solving these equations, we get

$m = 1 \pm i$ and $m = 1 \pm i$. Thus roots are complex and repeated as well. Hence the general solution is $y = e^x[(c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x]$

Example 05: Solve the following differential equations:

(i) $(9D^2 - 12D + 4)\, y = 0$

Solution: Auxiliary equation (A. E.) is: $9m^2 - 12m + 4 = 0$ giving m = 2/3, 2/3.

Since the roots are real and equal, hence the general solution is:

$$y = (C_1 + C_2 x)\, e^{2/3\, x}$$

(ii) $(D^3 + D^2 + D + 1)\, y = 0$

Solution: Auxiliary equation is: $m^3 + m^2 + m + 1 = 0$ or $m^2 (m + 1) + 1 (m + 1) = 0$

or $(m + 1) (m^2 + 1) = 0$ or $m + 1 = 0$ or $m^2 + 1 = 0$ giving $m = –1$ or $m = \pm i$.

Since one root of auxiliary equation is real and other two are complex hence, general solution of given differential equation is: $y = C_1 e^{-x} + (C_2 \cos x + C_3 \sin x)$

(iii) $(D^3 –6D^2 + 3D + 10)\, y = 0$

Solution: Auxiliary equation is: $m^3 – 6m^2 + 3m + 10 = 0$

It may be noted that in problem 2, it was easy to find the factors whereas in this problem, we can not do so. Thus we employ "Hit & Trial" method.

Put m = 1, we get: $1 – 6 + 3 + 10 \neq 0$

Put m = –1, we get: $–1 – 6 – 3 + 10 = 0$

This shows that m = –1 is a root of Auxiliary Equation. Now to find other two roots, we use the method of *"Synthetic Division"*.

–1	1	–6	3	10
		–1	7	–10
	1	–7	10	0

Thus, $m^3 – 6m^2 + 3m + 10 = (m + 1)(m^2 – 7 m + 10) = 0$ giving $(m + 1) (m – 2) (m – 5) = 0$.

Thus roots of Auxiliary Equation are m = –1, 2 and 5 which are all real and distinct. Hence the general solution of given differential equation is: $y = C_1 e^{-x} + C_2 e^{2x} + C_3 e^{5x}$

(iv) $(D^3 – 27)\, y = 0$

Solution: Auxiliary equation is: $m^3 – 27 = 0$

or $(m^3 – 3^3) = 0$ or $(m – 3) (m^2 + 3m + 9) = 0$ or $(m – 3) = 0$ or $(m^2 + 3m + 9) = 0$

Giving, m = 3, or $m = \frac{3}{2} \pm i\frac{3\sqrt{3}}{2}$. Thus one roots of A. E is real and other two are complex.

Hence the general solution of given differential equation is:

$$y = C_1 e^{3x} + e^{\frac{3}{2}x}\left[C_2 \cos\frac{3\sqrt{3}}{2}x + C_3 \sin\frac{3\sqrt{3}}{2}x\right]$$

(v) $(D^4 – 5D^3 + 6D^2 + 4 D – 8)\, y = 0$

Solution: Auxiliary equation is: $m^4 – 5m^3 + 6m^2 + 4m – 8 = 0$

Using "Hit & Trial" method, putting:

m = 1, we get: $1 – 5 + 6 + 4 – 8 \neq 0$ and if m = –1, we get: $1 + 5 + 6 – 4 – 8 = 0$

This shows that m = –1 is a root of A. E. Now to find other three roots, we use the method of *"Synthetic Division"* once again

–1	1	–5	6	4	–8
		–1	6	–12	8
	1	–6	12	–8	0

or $(m^4 – 5m^3 + 6m^2 + 4m – 8) = (m + 1) (4m^3 – 6m^2 + 12m – 8) = 0$. Now consider cubic equation: $4m^3 – 6m^2 + 12m – 8 = 0$. Putting

$m = -1$, we get: $-1 - 6 + 12 - 8 \neq 0$

$m = 2$, we get: $8 - 24 + 24 - 8 = 0$

This shows that $m = 2$ is also a root of A. E. To find the remaining two roots, we once again use *"Synthetic Division"*.

2	1	–6	12	–8
		2	–8	8
	1	– 4	4	0

or $m^2 - 4m + 4 = 0$ giving $m = 2, 2$. Thus roots of A. Eq. are $m = -1, 2, 2, 2$. All roots are real where one root is distinct and other three roots are repeated. Thus, general solution of given differential equation is $y = C_1 e^{-x} + (C_2 + C_3 x + C_4 x^2) e^{2x}$

vi. $(D^4 - 4D^3 - 7D^2 + 22D + 24) y = 0$

Solution: Auxiliary equation is: $m^4 - 4m^3 - 7m^2 + 22m + 24 = 0$

By "Hit & Trial" method, put:

$m = 1$, we get: $1 - 4 - 7 + 22 + 24 \neq 0$

$m = -1$, we get: $1 + 4 - 7 - 22 + 24 = 0$

This shows that $m = -1$ is a root of A. E. Now to find other three roots, we use *"Synthetic Division"* again.

–1	1	–4	–7	22	24
		–1	5	2	–24
	1	–5	–2	24	0

or $m^3 - 5m^2 - 2m + 24 = 0$. This is a cubic equation. To find its roots, we put

$m = -1$, we get: $-1 - 5 + 2 + 24 \neq 0$

$m = 2$, we get: $8 - 20 - 4 + 24 \neq 0$

$m = -2$, we get: $-8 - 20 + 4 + 24 = 0$

This shows that $m = -2$ is also a root of A. E. To find the remaining two roots, we use *"Synthetic Division"*.

–2	1	–5	–2	24
		–2	14	–24
	1	– 7	12	0

or $m^2 - 7m + 12 = 0$ giving $m = 3, 4$. Thus roots of A. Eq. are $m = -1, -2, 3, 4$. All roots are real and distinct. Thus general solution of given differential equation is

$$y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x} + C_4 e^{4x}$$

(vii) $(D^4 + 4) y = 0$

Solution: A. Eq. is: $m^4 + 4 = 0$

or $m^4 + 2^2 = 0$ or $(m^2)^2 + 2.m^2.2 + 2^2 = 2.m^2.2$ or $(m^2 + 2)^2 = 4m^2$

Taking square root on both sides, we get: $m^2 + 2 = \pm 2m$

or $m^2 + 2 = 2m$ or $m^2 + 2 = -2m$

or $m^2 - 2m + 2 = 0$ or $m^2 + 2m + 2 = 0$

This gives, $m = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ or $m = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

The roots of A. Eq. are complex hence the general solution of given differential equation is:

$$y = e^{x}\,(C_1 \cos x + C_2 \sin x) + e^{-x}\,(C_3 \cos x + C_4 \sin x)$$

(viii) $(D^4 - D^3 - 3D^2 + D + 2)\,y = 0$

Solution: Auxiliary equation is: $m^4 - m^3 - 3m^2 + m + 2 = 0$

By "Hit & Trial" method, put:

$m = 1$, we get: $1 - 1 - 3 + 1 + 2 = 0$

This shows that $m = 1$ is a root of A. E. Now to find other three roots, we use *"Synthetic Division"*.

$$\begin{array}{c|ccccc}
1 & 1 & -1 & -3 & 1 & 2 \\
 & & 1 & 0 & -3 & -2 \\
\hline
 & 1 & 0 & -3 & -2 & \multicolumn{1}{|c}{0}
\end{array}$$

or $m^3 + 0\,m^2 - 3m - 2 = 0$. This is a cubic equation. By hit and trial method put:

$m = 1$, we get: $1 + 0 - 3 - 2 \neq 0$. Put $m = 2$, we get: $-8 + 6 + 2 = 0$

This shows that $m = -2$ is also a root of A. E. To find the remaining two roots, we use once again the *"Synthetic Division"*.

$$\begin{array}{c|cccc}
2 & 1 & 0 & -3 & -2 \\
 & & 2 & 4 & 2 \\
\hline
 & 1 & 2 & 1 & \multicolumn{1}{|c}{0}
\end{array}$$

or $m^2 + 2m + 1 = 0$ or $m = -1, -1$. Thus roots of A. E. are $m = 1, 2, -1, -1$. All roots are real but two of them are repeated. Thus general solution of given differential equation is:

$$y = C_1 e^{x} + C_2 e^{2x} + (C_3 + C_4 x)\,e^{-x}$$

ix. $(16D^6 + 8D^4 + D^2)\,y = 0$

Solution: Auxiliary equation is: $16m^6 + 8m^4 + m^2 = 0$

or $m^2\,(16m^4 + 8m^2 + 1) = 0$ or $m^2 = 0$ or $16m^4 + 8m^2 + 1 = 0$

Now $m^2 = 0$ gives $m = 0, 0$. To solve $16m^4 + 8m^2 + 1 = 0$, we let $m^2 = k$, hence given equation becomes: $16k^2 + 8k + 1 = 0$

or $k = 1/4, 1/4$ or $m^2 = 1/4, 1/4$ giving $m = \pm 1/2, \pm 1/2$

The roots of A. E. are: 0, 0, 1/2, 1/2, –1/2, –1/2. All six roots are real but pair wise repeated. Thus general solution of given differential equation is:

$$y = (C_1 + C_2 x)e^{0x} + (C_3 + C_4 x)\,e^{x/2} + (C_5 + C_6 x)\,e^{-x/2}$$

(x) $(D^4 + 6D^3 + 15D^2 + 20D + 12)\,y = 0$

Solution: Auxiliary equation is: $m^4 + 6m^3 + 15m^2 + 20m + 12 = 0$

By "Hit & Trial" method, put:

$m = -1$, we get: $1 - 6 + 15 - 20 + 12 \neq 0$

$m = -2$, we get: $16 - 48 + 60 - 40 + 12 = 0$

This shows that $m = -2$ is a root of A. Eq. Now to find other three roots, we use *"Synthetic Division"*.

–2	1	6	15	20	12
		–2	–8	–14	–12
	1	4	7	6	0

or $m^3 + 4m^2 + 7m + 6 = 0$. This is a cubic equation. To find its roots, we put:
$m = -2$, we get: $-8 + 16 - 14 + 6 = 0$. This shows that $m = -2$ is also a root of A. Eq. To find the remaining two roots, we use once again the *"Synthetic Division"*.

–2	1	4	7	6
		–2	–4	–6
	1	2	3	0

or $m^2 + 2m + 3 = 0$ giving $m = -1 \pm i\sqrt{2}$. Thus roots of A. E. are $m = -2, -2, m = -1 \pm i\sqrt{2}$. Two of the roots are real and repeated and two are complex. Thus general solution of given differential equation is: $y = (C_1 + C_2 x)\, e^{-2x} + e^{-x}\left(C_3 \cos\sqrt{2}\,x + C_4 \sin\sqrt{2}\,x\right)$

NON–HOMOGENEOUS LINEAR DIFFERENTIAL EQUATION

In this section we shall discuss the solution of non-homogeneous linear differential equation:

$$\left(a_0 D^n + a_1 D^{n-1} + \ldots + a_{n-1} D + a_n\right) y = f(x) \qquad (1)$$

This differential equation has two solutions: $y = y_c$ and $y = y_p$. Here, y_c is called complementary function and is obtained by solving $F(D)\, y = 0$ as shown in the previous section and y_p is found by solving $y_p = \frac{1}{F(D)} f(x)$. Then complete solution of (1) is $y = y_c + y_p$.

We shall now discuss how to find y_p depending upon the nature of function f(x).

Working Rule For Finding Particular Integral (P.I)

- ***When $f(x) = e^{ax}$***

CASE-I: $y_p = \frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}$, provided $F(a) \neq 0$

Proof: Consider,
$D\, e^{ax} = a\, e^{ax}$, $\quad D^2 e^{ax} = a^2 e^{ax}$, $\quad D^3 e^{ax} = a^3 e^{ax}$,
or $D^n e^{ax} = a^n e^{ax}$ or $\frac{1}{D^n} e^{ax} = \frac{1}{a^n} e^{ax}$ $\Rightarrow$, $\frac{1}{F(D)} e^{ax} = \frac{1}{F(a)} e^{ax}$, provided $F(a) \neq 0$

In case $F(a) = 0$ then an other approach is employed. But before we show you this, we prove the following theorem.

Prove that $\frac{1}{(D-a)} f(x) = e^{ax}\int e^{-ax} f(x)\,dx$

Proof: Let $\frac{1}{(D-a)} f(x) = y \Rightarrow f(x) = (D-a)y = Dy - ay$

$\therefore y' - ay = f(x)$ which is linear differential equation.

I.F = e^{-ax}. Multiplying by I. F and solving, we get:

$ye^{-ax} = \int e^{-ax} f(x)\,dx \Rightarrow y = e^{ax}\int e^{-ax} f(x)\,dx$. Thus, $y = \frac{1}{(D-a)} f(x) = e^{ax}\int e^{-ax} f(x)\,dx$

CASE-II: Above formula for computing y_p is valid provided that $F(a) \neq 0$.

If F(a) = 0 then it implies that 'a' is a root of equation $a_0m^n + a_1m^{n-1} + ... + a_{n-1}m + a_n = 0$.

This means that (m – a) is a factor of $a_0m^n + a_1m^{n-1} + ... + a_{n-1}m + a_n = 0$.

In this case F(D) = (D – a) φ (D). Thus, $\frac{1}{F(D)}e^{ax} = \frac{1}{(D-a)\phi(D)}e^{ax} = \frac{1}{\phi(D)}.\frac{1}{(D-a)}e^{ax}$

$= \frac{1}{\phi(D)}e^{ax}\int e^{ax}.e^{-ax}dx$ **[See the above theorem with, $f(x) = e^{ax}$]**

$= \frac{1}{\phi(D)}e^{ax}\int 1dx = \frac{1}{\phi(D)}xe^{ax}$

In general, if 'a' is a repeated root of $a_0m^n + a_1m^{n-1} + ... + a_{n-1}m + a_n = 0$, we have

$$\frac{1}{(D-a)^k\varphi(D)}e^{ax} = \frac{x^ke^{ax}}{k!\varphi(a)},\ F(a) = 0,\ \text{where } F(D) = (D-a)^k\phi(D)$$

Another simple method to compute y_p in case F(a) = 0 is as follows:

$$\frac{1}{F(D)}e^{ax} = \frac{x}{F`(a)}e^{ax} \text{ provided } F`(a) \neq 0$$

$$= \frac{x^2}{F``(a)}e^{ax} \text{ provided } F``(a) \neq 0 \text{ ,etc.}$$

AN IMPORTANT REMARK: After computation of complementary function y_c you must see whether or not the right side function f(x) contains any function that is present in y_c. If there exists such function in f(x) then F(D) will become zero and therefore we use appropriate case.

Example 01: Find the general solution of

(i) $(D^2 + 3D - 4)y = 15e^{2x}$ (ii) $(D^2 - 3D + 2)y = e^x + e^{2x}$

Solution: **(i)** The auxiliary equation is $m^2 + 3m - 4 = 0 \Rightarrow m = 1, -4$

The Complementary Function is: $y_c = c_1e^x + c_2e^{-4x}$. Now,

$$y_p = \frac{1}{(D^2+3D-4)}(15e^{2x}) = 15\frac{1}{2^2+3(2)-4}e^{2x} = \frac{5}{2}e^{2x} \text{ [Use Case I]}$$

Thus general solution of given differential equation is $y = y_c + y_p = c_1e^x + c_2e^{-4x} + 5e^{2x}/2$

(ii) The auxiliary equation is: $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$

Thus, complementary function is: $y_c = c_1e^x + c_2e^{2x}$

Now $y_p = \frac{1}{(D^2-3D+2)}(e^x + e^{2x}) = \frac{1}{(D^2-3D+2)}e^x + \frac{1}{(D^2-3D+2)}e^{2x}$

$$= \frac{x}{2D-3}e^x + \frac{x}{2D-3}e^{2x} = \frac{xe^x}{(2-3)} + \frac{xe^{2x}}{(4-3)} = -xe^x + xe^{2x} \qquad \text{[Use Case II]}$$

Thus, the general solution of the given equation is $y = y_c + y_p = c_1e^x + c_2e^{2x} - xe^x + xe^{2x}$

Remark: Observe that function f(x) contains both e^x and e^{2x} which are already in y_c. Thus we use Case II. Students are advised to check this for every problem before finding y_p.

Example 02: Solve the following differential equations

(i) $(D^3 - 2D^2 - 5D + 6)\, y = e^{3x}$ (ii) $(D^3 - D^2 + 4D - 6)\, y = e^x$

(iii) $(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$ (iv) $(D^3 - 3D - 2)\, y = e^{2x} + e^x$

(v) $(D^2 - D - 6)\, y = e^x \cosh 2x$

Solution: **(i)** Given differential equation is $(D^3 - 2D^2 - 5D + 6)\, y = e^{3x}$

A.E is $m^3 - 2m^2 - 5m + 6 = 0$ or $m = 1, -2, 3$. Thus, $y_c = A\, e^x + B\, e^{-2x} + C\, e^{3x}$

$$y_p = \frac{1}{(D^3 - 2D^2 - 5D + 2)} e^{3x} = \frac{x}{(3D^2 - 4D - 5)} e^{3x} = \frac{x}{(27 - 12 - 5)} e^{3x} = \frac{xe^{3x}}{10}$$ [Use Case II]. Thus

general solution is $y = y_c + y_p = A\, e^x + B\, e^{-2x} + C\, e^{3x} + (xe^{3x})/10$

(ii) Given equation is $(D^3 - D^2 + 4D - 6)\, y = e^x$

A.E is $m^3 - m^2 + 4m - 6 = 0$ or $m = 1, \pm 2i$. Thus, $y_c = Ae^x + B\cos 2x + C\sin 2x$

$$y_p = \frac{1}{(D^3 - D^2 + 4D - 6)} e^x = \frac{x}{(3D^2 - 2D + 4)} e^x = \frac{x}{(3 - 2 + 4)} e^x = \frac{xe^x}{5}$$ [Use Case II]

Thus general solution is: $y = y_c + y_p = Ae^x + B\cos 2x + C\sin 2x + (xe^x)/5$

(iii) Given equation is $(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$

A.E is $m^3 + 3m^2 + 3m + 1 = 0$ or $(m + 1)^3 = 0$ giving $m = -1, -1, -1$

Thus, $y_c = (A + Bx + Cx^2)\, e^{-x}$

$$y_p = \frac{1}{(D^3 + 3D^2 + 3D + 1)} e^{-x} = \frac{x}{(3D^2 + 6D + 3)} e^{-x} = \frac{x.x}{(6D + 6)} e^{-x} = \frac{x.x.x}{6} e^{-x} = \frac{x^3 e^{-x}}{6}$$ [Case II]

Thus general solution is: $y = y_c + y_p = (A + Bx + Cx^2)\, e^{-x} + (x^3\, e^{-x})/6$

(iv) Given equation is $(D^3 - 3D - 2)\, y = e^{2x} + e^x$

A.E is: $m^3 - 3m - 2 = 0$ giving $m = 2, -1, -1$

Thus, $y_c = Ae^{2x} + (B + Cx)\, e^{-x}$

$$y_p = \frac{1}{(D^3 - 3D - 2)}(e^{2x} + e^x) = \frac{1}{(D^3 - 3D - 2)} e^{2x} + \frac{1}{(D^3 - 3D - 2)} e^x$$

$$= \frac{x}{(3D^2 - 3)} e^{2x} + \frac{1}{(1 - 3 - 2)} e^x = \frac{x}{(3.4 - 3)} e^{2x} + \frac{1}{(-4)} e^x = \frac{xe^{2x}}{9} - \frac{e^x}{4} \left[\textbf{By Cases I \& II}\right]$$

Thus general solution is: $y = y_c + y_p = Ae^{2x} + (B + Cx)\, e^{-x} + (x\, e^{2x})/9 - e^x/4$

(v) Given equation is $(D^2 - D - 6)\, y = e^x \cosh 2x$

A.E is: $m^2 - m - 6 = 0$ giving $m = 3, -2$

Thus, $y_c = A\, e^{3x} + B\, e^{-2x}$

Now $e^x \cosh 2x = e^x\, (e^{2x} + e^{-2x})/2 = (e^{3x} + e^{-x})/2$. Thus,

$$y_p = \frac{1}{(D^2 - D - 6)} e^x \cosh 2x = \frac{1}{(D^2 - D - 6)}\left(\frac{e^{3x} + e^{-x}}{2}\right)$$

$$= \frac{1}{2(D^2 - D - 6)} e^{3x} + \frac{1}{2(D^2 - D - 6)} e^{-x}$$

$$= \frac{x}{2(2D - 1)} e^{3x} + \frac{1}{2(1 + 1 - 6)} e^{-x} = \frac{xe^{3x}}{2(2.3 - 1)} + \frac{1}{2(-4)} e^{-x} = \frac{xe^{3x}}{10} - \frac{e^{-x}}{8} \left[\text{By Cases I \& II}\right]$$

Thus general solution is $y = y_c + y_p = A\, e^{3x} + B\, e^{-2x} + (xe^{3x})/10 - e^{-x}/8$

- ***When f(x) = sin ax or cos ax***

CASE-I: $\dfrac{1}{F(D^2)}\sin ax = \dfrac{1}{F(-a^2)}\sin ax$ and $\dfrac{1}{F(D^2)}\cos ax = \dfrac{1}{F(-a^2)}\cos ax$, provided, $F(-a^2) \neq 0$.

Note: $F(-a^2) \neq F(-a)^2$

Proof: To find y_p when f(x) is either sin ax or cos ax, we proceed as under:

$$D(\sin ax) = a\cos ax,$$

$$D^2(\sin ax) = D(D(\sin ax)) = D(a\cos ax) = a(-a\sin ax) = -a^2\sin ax$$

or
$$\frac{1}{D^2}\sin ax = \frac{1}{-a^2}\sin ax \Rightarrow \frac{1}{F(D^2)}\sin ax = \frac{1}{F(-a^2)}\sin ax$$

Similarly, $\dfrac{1}{F(D^2)}\cos ax = \dfrac{1}{F(-a^2)}\cos ax$, provided, $F(-a^2) \neq 0$

CASE II: If $F(-a^2) = 0$ then $\dfrac{1}{F(D)}\sin ax = \dfrac{x}{F'(-a^2)}\sin ax$

Remark: If F(D) contains D^3 and higher terms then following may be noted:

Write $D^3 = D^2 \cdot D$ and $D^4 = D^2 \cdot D^2$ and so on. Then replace D^2 by $-(a^2) = (-a^2)$.

In case, if F(D) reduces to linear factor such as $(D + \alpha)$ after using above identities, we proceed as under

$$\frac{1}{D+\alpha}\sin ax = \frac{D-\alpha}{(D+\alpha)(D-\alpha)}\sin ax = \frac{D-\alpha}{D^2-\alpha^2}\sin ax$$

$$= \frac{D-\alpha}{-a^2-\alpha^2}\sin ax, \ \left(\text{putting } -a^2 \text{ for } D^2 \text{ in the denominator}\right)$$

$$= -\frac{1}{a^2+\alpha^2}(D-\alpha)\sin ax = -\frac{1}{a^2+\alpha^2}\left\{\frac{d}{dx}(\sin ax) - \alpha\sin ax\right\}$$

Therefore,
$$\frac{1}{D+\alpha}\sin ax = -\frac{1}{a^2+\alpha^2}(a\cos ax - \alpha\sin ax)\cdot$$

The same steps may be taken when f(x) = cos ax. This process of finding y_p is valid if $F(-a^2) \neq 0$.

In case if $F(-a^2) = 0$ then $\dfrac{1}{F(D)}\sin ax = \dfrac{x}{F'(-a^2)}\sin ax$ as we have seen above when $f(x) = e^{ax}$.

Example 03: **Find the general solution of (i) $(D^2 - 5D + 6)y = \sin 2x$**
(ii) $(D^3 - D^2 + D - 1)\,y = 4\sin x$ (iii) $(D^3 - 3D^2 + 4D - 2)\,y = e^x + \cos x$

Solution: **(i)** Auxiliary equation is $m^2 - 5m + 6 = 0$ giving m = 2, 3. Thus

$$y_c = A\,e^{2x} + B\,e^{3x}$$

Now, $y_p = \dfrac{1}{D^2-5D+6}\sin 2x = \dfrac{1}{-2^2-5D+6}\sin 2x = \dfrac{1}{2-5D}\times\dfrac{2+5D}{2+5D}\sin 2x = \dfrac{2+5D}{4-25D^2}\sin 2x$

$$= \frac{2+5D}{4-25(-2^2)}\sin 2x = \frac{2\sin 2x + 5D\sin 2x}{104} = \frac{2\sin 2x + 10\cos 2x}{104} = \frac{\sin 2x + 5\cos 2x}{52}$$

Thus general solution is: $y = y_c + y_p = Ae^{2x} + Be^{3x} + \dfrac{\sin 2x + 5\cos 2x}{52}$

(ii) Auxiliary equation is: $m^3 - m^2 + m - 1 = 0 \Rightarrow (m-1)(m^2+1) = 0 \Rightarrow m = 1, \pm i$

Thus Complementary Function is

$$y_c = c_1e^x + e^{0x}(c_2\sin x + c_3\cos x) = c_1e^x + c_2\sin x + c_3\cos x$$

$$y_p = \frac{1}{\left(D^3 - D^2 + D - 1\right)}\left(4\sin x\right) = 4\frac{x}{3D^2 - 2D + 1}\sin x = 4\frac{x}{3(-1^2) - 2D + 1}\sin x$$

$$= 4\frac{x}{-2(1+D)} \times \frac{(1-D)}{(1-D)}\sin x = -2\frac{x(1-D)}{(1-D^2)}\sin x = -2\frac{x(1-D)}{1-(-1^2)}\sin x$$

$$= -2\frac{x(\sin x - D\sin x)}{2} = -x(\sin x - \cos x) \qquad [\text{Note}: \sin x \text{ is present in } y_c]$$

Thus general solution is $y = y_c + y_p = c_1e^x + c_2\sin x + c_3\cos x - x(\sin x - \cos x)$

(iii) Auxiliary equation is $m^3 - 3m^2 + 4m - 2 = 0 \Rightarrow m = 1,\ 1 \pm i$

Thus Complementary Function is: $y_c = c_1e^x + e^x\left(c_2\sin x + c_3\cos x\right)$

$$y_p = \frac{1}{D^3 - 3D^2 + 4D - 2}e^x + \frac{1}{D^3 - 3D^2 + 4D - 2}\cos x$$

$$= \frac{x}{3D^2 - 6D + 4}e^x + \frac{1}{D^2.D - 3D^2 + 4D - 2}\cos x$$

$$= \frac{xe^x}{3 - 6 + 4} + \frac{1}{(-1^2)D - 3(-1^2) + 4D - 2}\cos x$$

$$= xe^x + \frac{1}{3D+1}\cos x = xe^x + \frac{1}{3D+1} \times \frac{3D-1}{3D-1}\cos x$$

$$= xe^x + \frac{3D-1}{9D^2-1}\cos x = xe^x + \frac{3D-1}{9(-1^2)-1}\cos x = xe^x - \frac{1}{10}(3D\cos x - \cos x)$$

$$= xe^x - (-3\sin x - \cos x)/10 = xe^x + (3\sin x + \cos x)/10$$

Thus general solution is: $y = y_c + y_p = c_1e^x + e^x(c_2\sin x + c_3\cos x) + (3\sin x + \cos x)/10$

Example 04: Solve the initial value problem $y'' - 4y' + 13y = 8\sin 3x$; $y(0) = 1$, $y'(0) = 2$

Solution: Given equation can be written as:

$$D^2y - 4Dy + 13y = 8\sin 3x \Rightarrow \left(D^2 - 4D + 13\right)y = 8\sin 3x$$

Auxiliary equation is: $m^2 - 4m + 13 = 0$

Using quadratic formula, we get: $m = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$

Thus Complementary Function is: $y_c = e^{2x}\left(c_1\sin 3x + c_2\cos 3x\right)\cdot$Now

$$y_p = \frac{1}{D^2 - 4D + 13}\left(8\sin 3x\right) = 8\frac{1}{-9 - 4D + 13}\sin 3x, \ [\text{Replacing } D^2 \text{by} - 3^2]$$

$$= 8\frac{1}{4 - 4D}\sin 3x = -2\frac{1}{D-1}\sin 3x = \frac{-2\left(D+1\right)}{\left(D-1\right)\left(D+1\right)}\sin 3x = -2\frac{1}{D^2-1}\left(D+1\right)\sin 3x$$

$$\Rightarrow \qquad y_p = \frac{-2}{-9-1}\left\{D\sin 3x + \sin 3x\right\} = \frac{1}{5}\left(3\cos 3x + \sin 3x\right)$$

Hence, the general solution is $y = y_c + y_p = e^{2x}\left(c_1\sin 3x + c_2\cos 3x\right) + \frac{1}{5}\left(3\cos 3x + \sin 3x\right)$

Now, $y' = e^{2x}\left(3c_1\cos 3x - 3c_2\sin 3x\right) + 2e^{2x}\left(c_1\sin 3x + c_2\cos 3x\right) + \frac{1}{5}\left(-9\sin 3x + 3\cos 3x\right)\cdot$

Applying the given initial conditions, we get $\qquad 1 = \left(c_1 \times 0 + c_2\right) + \frac{1}{5}\left(3 + 0\right) \Rightarrow c_2 = \frac{2}{5}$

$$2=(3c_1-0)+2(0+c_2)+\frac{1}{5}(0+3)\Rightarrow 2-\frac{4}{5}-\frac{3}{5}=3c_1\Rightarrow c_1=\frac{1}{5}$$

Hence, $y=e^{2x}\left(\frac{1}{5}\sin 3x+\frac{2}{5}\cos 3x\right)+\frac{1}{5}(3\cos 3x+\sin 3x)$

or $y=\frac{1}{5}\left\{e^{2x}(\sin 3x+2\cos 3x)+3\cos 3x+\sin 3x\right\}$ is a particular solution.

Example 05: Solve $(D^2 + 4)\ y = \sin^2 x$

Solution: Auxiliary equation is $m^2 + 4 = 0$ giving $m = \pm 2i$. Thus,

$$y_c = c_1 \sin 2x + c_2 \cos 2x$$

$$y_p=\frac{1}{D^2+4}\sin^2 x=\frac{1}{D^2+4}\left(\frac{1-\cos 2x}{2}\right)=\frac{1}{2}\left\{\frac{1}{D^2+4}e^{0x}-\frac{1}{D^2+4}\cos 2x\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{0^2+4}e^{0x}-\frac{x}{2D}\cos 2x\right\}$$

$$=\frac{1}{8}-\frac{x}{4}\int\cos 2x\,dx=\frac{1}{8}-\frac{x}{4}.\frac{\sin 2x}{2}=\frac{1}{8}[1-x\sin 2x].$$

$$\text{NOTE}: Df(x)=\frac{df}{dx}\Rightarrow\frac{1}{D}f(x)=D^{-1}f(x)=\int f(x)\ dx$$

Thus, $y = y_c + y_p = c_1\sin 2x + c_2\cos 2x + \frac{1}{4}\left[\frac{1}{2}-x\sin 2x\right]$

- ***When $f(x) = e^{ax} g(x)$***

If $f(x) = e^{ax} g(x)$, then $\frac{1}{F(D)}e^{ax}g(x)=e^{ax}\frac{1}{F(D+a)}g(x)$

This is known as *Exponential shift*.

Proof: Consider, $D(e^{ax} u) = e^{ax} Du + a ue^{ax} = e^{ax} (D + a) u$. Here u is a function of x. Similarly,

$$D^2 (e^{ax} u) = D[e^{ax} Du + a ue^{ax}] = e^{ax} D^2u + Du\, a e^{ax} + a.\ u\ a.\ e^{ax} + a e^{ax} Du$$
$$= e^{ax} (D^2 + 2a D + a^2) = e^{ax} (D + a)^2$$

Generalizing, we get $D^n (e^{ax} u) = e^{ax} (D + a)^n$ or $F(D) (e^{ax} u) = e^{ax} F(D + a)$

or $\frac{1}{F(D)}e^{ax}u=e^{ax}\frac{1}{F(D+a)}u$. Put $u = g(x)$, we get: $\frac{1}{F(D)}e^{ax}g(x)=e^{ax}\frac{1}{F(D+a)}g(x)$

Example 06: Find the general solution of

(i) $(D^2 - 2D + 4)\ y = e^x \cos x$ **(ii) $(D^3 - D^2 + 3D + 5)\ y = e^x \cos 2x$**

Solution: **(i)** Auxiliary equation is: $m^2-2m+4=0$

Using quadratic equation, we get: $m=\frac{2\pm\sqrt{4-16}}{2}=\frac{2\pm 2i\sqrt{3}}{2}=1\pm i\sqrt{3}$

The complementary function is: $y_c=e^x\left(c_1\sin\sqrt{3}x+c_2\cos\sqrt{3}x\right)$

Now using shift property,

$$y_p=\frac{1}{D^2-2D+4}\left(e^x\cos x\right)=e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x=e^x\frac{1}{D^2+3}\cos x$$

$$=e^x\frac{1}{-1^2+3}\cos x=\frac{e^x\cos x}{2}$$

Hence, the general solution of the given differential equation is

$$y = y_c + y_p = e^x(c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x) + (e^x \cos x)/2$$

(ii) A. E is $m^3 - m^2 + 3m + 5 = 0$ giving $m = -1,\ 1 \pm 2i$ (verify by synthetic division)

Thus, $y_c = Ae^{-x} + e^x(B\cos 2x + C\sin 2x)$

You may observe that $e^x \cos 2x$ is a function on right side of differential equation. Thus,

$$y_p = \frac{1}{D^3 - D^2 + 3D + 5} e^x \cos 2x = \frac{x}{3D^2 - 2D + 3} e^x \cos 2x. \text{ Apply shift property, we get:}$$

$$= e^x \frac{x}{3(D+1)^2 - 2(D+1) + 3} \cos 2x = e^x \frac{x}{3D^2 + 4D + 4} \cos 2x$$

$$= e^x \frac{x}{3(-2^2) + 4D + 4} \cos 2x = e^x \frac{x}{4D - 8} \cos 2x = \frac{e^x}{4} \frac{x}{D - 2} \cos 2x$$

$$= \frac{e^x}{4} \frac{x}{D-2} \times \frac{D+2}{D+2} \cos 2x = \frac{xe^x(D+2)}{4(D^2 - 4)} \cos 2x = \frac{xe^x(D+2)}{4(-2^2 - 4)} \cos 2x$$

$$= \frac{xe^x(D\cos 2x + 2\cos 2x)}{-32} = \frac{2xe^x(-\sin 2x + \cos 2x)}{-32} = -\frac{xe^x(\cos 2x - \sin 2x)}{16}$$

Thus general solution is

$$y = y_c + y_p = Ae^{-x} + e^x\left(B\cos 2x + C\sin 2x\right) - xe^x(\cos 2x - \sin 2x)/16$$

- ***When $f(x) = x^m$***

To evaluate $\frac{1}{F(D)} f(x)$, where f(x) is a polynomial of degree m.

- Take out lowest degree term common from F(D) to make the first term unity. The remaining term will contain $\{1 + \varphi(D)\}$ or $\{1 - \varphi(D)\}$
- Take this factor in the numerator where it takes the form $\{1 + \varphi(D)\}^{-1}$ or $\{1 - \varphi(D)\}^{-1}$
- Expand by binomial theorem, using either of the results:

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots \text{ or } (1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

The expansion to be carried up to the term D^m, since f(x) is a polynomial of degree m and hence, $D^{m+1} f(x) = 0,\ D^{m+2} f(x) = 0,$ all higher derivatives of f(x) vanish.

Example 07: Solve the initial value problem $y'' - 4y = (2 - 8x)$; $y(0) = 0$, $y'(0) = 5$

Solution: A.E is $m^2 - 4 = 0$ giving $m = -2, 2$. Thus $y_c = c_1 e^{2x} + c_2 e^{-2x}$

$$y_p = \frac{1}{D^2 - 4}(2 - 8x) = -\frac{1}{4\left(1 - D^2/4\right)}\left(2 - 8x\right) = \frac{-1}{4}\left(D^2 - 1/4\right)^{-1}(2 - 8x)$$

Expanding by Binomial Theorem, and keeping only the term contains D asf(x) = (2 – 8 x) is a polynomial of degree 1, hence $D^2(2 - 8x) = 0$. Thus,

$$y_p = -\frac{1}{4}\left(1 + \frac{D^2}{4} + \ldots\right)\left(2 - 8x\right) = -\frac{1}{4}\left(2 - 8x\right) = 2x - \frac{1}{2}$$

Hence, general solution is $y = y_c + y_p = c_1 e^{2x} + c_2 e^{-2x} + 2x - 1/2$

Now $\qquad y' = 2c_1e^{2x} - 2c_2e^{-2x} + 2$

Applying given initial conditions, we get

$$0 = c_1 + c_2 - \frac{1}{2} \Rightarrow 1 = 2c_1 + 2c_2 \qquad (1)$$

$$5 = 2c_1 - 2c_2 + 2 \Rightarrow 3 = 2c_1 - 2c_2 \qquad (2)$$

Adding (1) and (2), we get: $c_1 = 1 \cdot$ Put this in (1), we get $c_2 = -1/2 \cdot$

Hence $y = e^{2x} - \frac{e^{-2x}}{2} + 2x - \frac{1}{2}$ is particular solution of given differential equation.

- ***When f(x) = x^n sin ax or f(x) = x^n cos ax***

$$\frac{1}{F(D)} x^n (\cos ax + i \sin ax) = \frac{1}{F(D)} x^n e^{iax} = e^{iax} \frac{1}{F(D+ia)} x^n. \text{ Thus,}$$

$$\frac{1}{F(D)} x^n \cos ax = \text{Real part of} \left\{ e^{iax} \frac{1}{F(D+ia)} x^n \right\} \text{ and}$$

$$\frac{1}{F(D)} x^n \sin ax = \text{Imaginary part of} \left\{ e^{iax} \frac{1}{F(D+ia)} x^n \right\}$$

Example 08: Solve $(D^2 - 2D + 1)\, y = x \sin x$

Solution: A. E is $m^2 - 2m + 1 = 0$ giving m = 1, 1. Thus $y_c = (A + Bx)e^x$

$$y_p = \text{Im}\left\{ \frac{1}{D^2 - 2D + 1} x e^{ix} \right\} = \text{Im}\left\{ e^{ix} \frac{1}{(D+i)^2 - 2(D+i) + 1} x \right\} \text{[By shift property]}$$

$$= \text{Im}\left\{ e^{ix} \frac{1}{D^2 + 2iD - 1 - 2D - 2i + 1} x \right\}$$

$$= \text{Im}\left\{ e^{ix} \frac{1}{D^2 - 2(1-i)D - 2i} x \right\} = \text{Im}\left\{ e^{ix} \frac{1}{-2(1-i)D - 2i} \right\} x. \qquad \left[\text{Neglecting } D^2\right]$$

$$= \text{Im} \frac{1}{-2i} \left\{ e^{ix} \frac{1}{1 - (1+i)D} x \right\} = \text{Im} \frac{1}{-2i} e^{ix} \left[1 - (1+i)D\right]^{-1} x = \text{Im} \frac{1}{-2i} e^{ix} \left[1 + (1+i)D\right] x$$

$$= \text{Im} \frac{1}{-2i} (\cos x + i \sin x)(x + x + iDx) = \text{Im} \frac{1}{2} (i \cos x - \sin x)\left[2x + i\right]$$

$$= \text{Im} \frac{1}{2} \left\{ \left[-2x \sin x - \cos x\right] + i\left[2x \cos x - \sin x\right] \right\} = \frac{1}{2}\left[2x \cos x - \sin x\right]$$

NOTE: $-2(1-i)D - 2i = 2i^2(1-i)D - 2i = -2i[-i(1-i)D + 1] = -2i\,[1 - i(1-i)D]$

$= -2i\,[1 - (i - i^2)\,D] = -2i\,[1 - (i + 1)D] = -2i\,[1 - (1 + i)D]$

Thus complete solution is $y = (A + Bx)e^x + (2x \cos x - \sin x)/2$ (Note: $-1/i = i$)

Example 09: Solve the following differential equations

(i) $(D^2 - 2D - 3)y = 2e^x - 10 \sin x$

Solution: Auxiliary equation is $m^2 - 2m - 3 = 0$ giving m = –1, 3. Thus, $y_c = C_1 e^{-x} + C_2 e^{3x}$

$$y_p = \frac{1}{(D^2 - 2D - 3)}[2e^x - 10 \sin x] = 2\frac{1}{(D^2 - 2D - 3)} e^x - 10 \frac{1}{(D^2 - 2D - 3)} \sin x$$

$$= 2\frac{1}{1 - 2 - 3} e^x - 10 \frac{1}{-1^2 - 2D - 3} \sin x = -\frac{e^x}{2} + \frac{10}{2(D+2)} \times \frac{(D-2)}{(D-2)} \sin x$$

$$= -\frac{e^x}{2} + 5\frac{(D-2)}{(D^2-4)}\sin x = -\frac{e^x}{2} + 5\frac{(D-2)}{(-1^2-4)}\sin x = -\frac{e^x}{2} + 5\frac{(D\sin x - 2\sin x)}{-5}$$

$$= -\frac{e^x}{2} - (\cos x - 2\sin x)$$

Thus, general solution of given differential equation is $y = y_c + y_p$

This gives, $$y = C_1e^{-x} + C_2e^{3x} - \frac{e^x}{2} - (\cos x - 2\sin x)$$

(ii) $(D^3 - 2D^2 - 3D + 10)\, y = 40 \cos x$

Solution: Auxiliary equation is $m^3 - 2m^2 - 3m + 10 = 0$. Its roots are $m = -2, 2 \pm i$. Thus,

$$y_c = C_1 e^{-2x} + e^{2x}(C_2 \cos x + C_3 \sin x)$$

$$y_p = 40\frac{1}{(D^3-2D^2-3D+10)}\cos x = 40\frac{1}{(D^2.D-2D^2-3D+10)}\cos x$$

$$= 40\frac{1}{(D^2.D-2D^2-3D+10)}\cos x = 40\frac{1}{[(-1^2).D-2(-1^2)-3D+10]}\cos x$$

$$= 40\frac{1}{(-.D+2-3D+10)}\cos x = 40\frac{1}{(12-4D)}\cos x = \frac{40}{4}\frac{(3+D)}{(3-D)(3+D)}\cos x$$

$$=10\frac{(3+D)}{(9-D^2)}\cos x = 10\frac{(3+D)}{(9-(-1^2)}\cos x = 10\frac{(3\cos x + D\cos x)}{10} = 3\cos x - \sin x$$

Thus, general solution of given differential equation is $y = y_c + y_p$

or $$y = C_1 e^{-2x} + e^{2x}(C_2 \cos x + C_3 \sin x) + 3 \cos x - \sin x$$

(iii) $(D^3 + D)\, y = 2x^2 + 4\sin x$

Solution: Auxiliary equation is $m^3 + m = 0$ or $m(m^2 + 1) = 0$ giving $m = 0, \pm i$.

Thus, $$y_c = C_1 e^{0x} + C_2 \cos x + C_3 \sin x = C_1 + C_2 \cos x + C_3 \sin x$$

$$y_p = \frac{1}{(D^3+D)}(2x^2 + 4\sin x) = 2\frac{1}{D(D^2+1)}x^2 + 4\frac{1}{(D^3+D)}\sin x$$

$$= 2\frac{(1+D^2)^{-1}}{D}x^2 + 4\frac{x}{(3D^2+D)}\sin x = \frac{2}{D}\left(1 - D^2 + D^4 - \ldots\right)x^2 + 4\frac{x}{[3(-1^2)+D]}\sin x$$

$$= \frac{2}{D}\left(x^2 - D^2x^2\right) + 4\frac{x(D+3)}{(D-3)(D+3)}\sin x = \frac{2}{D}\left(x^2 - 2x\right) + 4\frac{x(D+3)}{(D^2-9)}\sin x$$

$$= 2\int (x^2 - 2x)\,dx + 4\frac{x(D+3)}{[(-1^2)-9]}\sin x = 2\left(\frac{x^3}{3} - 2\frac{x^2}{2}\right) + 4\frac{x(D\sin x + 3\sin x)}{-10}$$

$$= 2\left(\frac{x^3}{3} - x^2\right) - 2\frac{x(\cos x + 3\sin x)}{5}$$

Thus, the general solution of given differential equation is $y = y_c + y_p$

or $$y = C_1 + C_2 \cos x + C_3 \sin x + 2\left(\frac{x^3}{3} - x^2\right) - \frac{2}{5}x(\cos x + 3\sin x)$$

(iv) $(D^4 + D^2)\, y = 3x^2 + 4\sin x - 2\cos x$

Solution: Auxiliary equation is $m^4 + m^2 = 0$ or $m^2(m^2 + 1) = 0$ giving $m = 0, 0, \pm i$.

Thus, $y_c = (C_1 + C_2 x) e^{0x} + C_3 \cos x + C_4 \sin x = (C_1 + C_2 x) + C_3 \cos x + C_4 \sin x$

$$y_p = \frac{1}{(D^4+D^2)}(3x^2 + 4\sin x - 2\cos x) = 3\frac{1}{D^2(1+D^2)}x^2 + 4\frac{1}{(D^4+D^2)}\sin x - 2\frac{1}{(D^4+D^2)}\cos x$$

$$= 3\frac{(1+D^2)^{-1}}{D^2}x^2 + 4\frac{x}{(4D^3+2D)}\sin x - 2\frac{x}{(4D^3+2D)}\cos x$$

$$= 3\frac{1}{D^2}\left(1 - D^2 + D^4 - \ldots\right)x^2 + 4\frac{x}{2D(2D^2+1)}\sin x - 2\frac{x}{2D(2D^2+1)}\cos x$$

$$= 3\int\int(x^2 - 2)dx\ dx + 2\frac{x}{D[2(-1^2)+1]}\sin x - \frac{x}{D[2(-1^2)+1]}\cos x$$

$$= 3\int\left(\frac{x^3}{3} - 2x)\right)dx - 2x\frac{1}{D}\sin x + x\frac{1}{D}\cos x = \frac{x^4}{4} - 3x^2 - 2x\int\sin x\,dx + x\int\cos x\,dx$$

$$= \left(x^4/4\right) - 3x^2 + 2x\ \cos x + x\sin x$$

Note: Both sin x and cos x are in y_c.

Now the general solution of given differential equation is $y = y_c + y_p$

or $y = (C_1 + C_2\, x) + C_3 \cos x + C_4 \sin x + \left(x^4/4\right) - 3x^2 + 2x\ \cos x + x\sin x$

(v) $(D^3 - 2D + 4)\, y = e^x \cos x$

Solution: Auxiliary equation is $m^3 - 2m + 4 = 0$ giving $m = -2, 1 \pm i$. Thus,

$$y_c = C_1\, e^{-2x} + e^x\,(C_2 \cos x + C_3 \sin x)$$

$$y_p = \frac{1}{(D^3 - 2D + 4)}e^x \cos\ x = \frac{x}{3D^2 - 2}e^x \cos x = xe^x\frac{1}{3(D+1)^2 - 2}\cos x\ \text{[by shift theorem]}$$

$$= xe^x\,\frac{1}{3D^2 + 6D + 3 - 2}\cos x = xe^x\frac{1}{3(-1^2) + 6D + 1}\cos x = xe^x\,\frac{1}{6D - 2}\cos x$$

$$= xe^x\,\frac{1}{2(3D-1)}\frac{(3D+1)}{(3D+1)}\cos x = \frac{xe^x}{2}\frac{(3D+1)}{(9D^2 - 1)}\cos x = \frac{xe^x}{2}\frac{(3D+1)}{[9(-1^2) - 1]}\cos x$$

$$= -\frac{xe^x}{20}(3D\cos x + \cos x) = -\frac{xe^x}{20}(-3\sin x + \cos x) = \frac{xe^x}{20}(3\sin x - \cos x)$$

Hence the general solution is:

$y = y_c + y_p = C_1\, e^{-2x} + e^x\,(C_2 \cos x + C_3 \sin x) +\ xe^x(3\sin x - \cos x)/20$

(vi) $(D^3 - D^2 + 3D + 5)\, y = e^x \sin 2x$

Solution: The auxiliary equation is $m^3 - m^2 + 3m + 5 = 0$ giving, $m = -1, 1 \pm 2i$.

Thus, $y_c = C_1\, e^{-x} + e^x\,(C_2 \cos 2x + C_3 \sin 2x)$

$$y_p = \frac{1}{(D^3 - D^2 + 3D + 5)}e^x \sin 2x = \frac{x}{3D^2 - 2D + 3}e^x \sin 2x = xe^x\,\frac{1}{3(D+1)^2 - 2(D+1) + 3}\cos x$$

$$= xe^x\,\frac{1}{3D^2 + 6D - 2D + 3}\sin 2x = xe^x\,\frac{1}{3(-2^2) + 4D + 3}\sin 2x = xe^x\,\frac{1}{4D - 9}\sin 2x$$

$$= xe^x\,\frac{1}{(4D-9)}\frac{(4D+9)}{(4D+9)}\sin 2x = xe^x\,\frac{(4D+9)}{(16D^2 - 81)}\sin 2x = xe^x\,\frac{(4D+9)}{[16(-2^2) - 81]}\sin 2x$$

$$= -\frac{xe^x}{145}(4D\sin 2x + 9\sin 2x) = -\frac{xe^x}{145}(8\cos 2x + 9\sin 2x) = -\frac{xe^x}{145}(8\cos 2x + 9\sin 2x)$$

Note: Above we have used the shift theorem. Hence general solution is

$y = y_c + y_p = C_1 e^{-x} + e^{x} (C_2 \cos 2x + C_3 \sin 2x) - xe^{x}(8\cos 2x + 9\sin 2x)/145$

(vii) $(D^3 - 7D - 6)\, y = e^{2x}\,(1 + x)$

Solution: Auxiliary equation is $m^3 - 7m - 6 = 0$ or $m = -1, -2, 3$. Thus

$$y_c = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x}$$

$$y_p = \frac{1}{(D^3 - 7D - 6)} e^{2x}(1+x) = e^{2x}\frac{1}{(D+2)^3 - 7(D+2) - 6}(x+1) \text{ [by shift theorem]}$$

$$= e^{2x}\frac{1}{D^3 + 6D^2 + 12D + 8 - 7D - 14 - 6}(x+1)$$

$$= e^{2x}\frac{1}{5D - 12}(x+1) \quad \text{[Neglecting } D^2 \text{ \& high power terms]}$$

$$= e^{2x}\frac{1}{-12(1 - 5D/12)}(x+1) = -\frac{e^{2x}}{12}\left[1 - \frac{5}{12}D\right]^{-1}(x+1) = -\frac{e^{2x}}{12}\left[1 + \frac{5}{12}D\right](x+1)$$

$$= -\frac{e^{2x}}{12}\left[(x+1) - \frac{5}{12}D(x+1)\right] = -\frac{e^{2x}}{12}\left[(x+1) - \frac{5}{12}\right] = -\frac{e^{2x}}{144}[12x + 7]$$

Hence general solution is $\quad y = y_c + y_p = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x} - e^{2x}[12x + 7]/144$

(viii) $(D^4 + 3D^2 - 4)\, y = \sinh x - \cos^2 x$

Solution: Auxiliary equation is $m^4 + 3m^2 - 4 = 0$ giving $m = \pm 1, \pm 2i$. Thus,

$$y_c = C_1 e^{-x} + C_2 e^{x} + C_3 \cos 2x + C_4 \sin 2x$$

NOTE: $\sinh x = (e^x - e^{-x})/2$ and $\cos^2 x = (1 + \cos 2x)/2$. Hence,

$$y_p = \frac{1}{(D^4 + 3D^2 - 4)}\left(\frac{e^x - e^{-x} - 1 - \cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^4 + 3D^2 - 4)}e^{x} - \frac{1}{(D^4 + 3D^2 - 4)}e^{-x} - \frac{1}{(D^4 + 3D^2 - 4)}e^{0x} - \frac{1}{(D^4 + 3D^2 - 4)}\cos 2x\right]$$

$$= \frac{1}{2}\left(\frac{x}{(4D^3 + 6D)}e^{x} - \frac{x}{(4D^3 + 6D)}e^{-x} - \frac{1}{(0^4 + 3.0^2 - 4)}e^{0x} - \frac{x}{(4D^3 + 6D)}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{x}{(4.1^3 + 6.1)}e^{x} - \frac{x}{(4.(-1^3) + 6(-1))}e^{-x} - \frac{1}{(-4)} - \frac{1}{(D^2)(D) + 6D}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{x}{10}e^{x} + \frac{x}{10}e^{-x} + \frac{1}{4} - \frac{1}{(-2^2.D + 6D)}\cos 2x\right) = \frac{1}{2}\left(\frac{x}{10}(e^{x} + e^{-x}) + \frac{1}{4} - \frac{1}{2D}\cos 2x\right)$$

$$= \frac{x}{10}\frac{(e^{x} + e^{-x})}{2} + \frac{1}{8} - \frac{1}{4}\int \cos 2x \, dx = \frac{x}{10}\cosh x - \frac{1}{8}\sin 2x + \frac{1}{8}$$

$$= \frac{x}{10}\frac{(e^{x} + e^{-x})}{2} + \frac{1}{8} - \frac{1}{4}\int \cos 2x \, dx = \frac{x}{10}\cosh x - \frac{1}{8}\sin 2x + \frac{1}{8}$$

Hence the general solution is:

$$y = y_c + y_p = C_1 e^{-x} + C_2 e^{x} + C_3 \cos 2x + C_4 \sin 2x + \frac{x}{10}\cosh x - \frac{1}{8}\sin 2x + \frac{1}{8}$$

CAUCHY-EULER DIFFERENTIAL EQUATION

A differential equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+...+a_{n-1}x\frac{dy}{dx}+a_ny=F(x) \qquad (1)$$

- where $a_0,a_1,a_2,...,a_{n-1},a_n$ are constants, is known as *Cauchy – Euler Equation*, or *equi-dimensional equation*. The unique characteristic of this type of differential equation is that **degree** of each monomial coefficient matches the **order** of differential equation. The equation can be reduced to a linear differential equation with constants coefficients by transformation:

$$x=e^t \text{ or } t=\ln x$$

Using Chain Rule, we can write

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\frac{d}{dx}(\ln x)=\frac{dy}{dt}\cdot\left(\frac{1}{x}\right)\Rightarrow\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dt} \qquad (2)$$

Differentiating (2), we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)=\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{dy}{dt}\right)+\frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)\frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)+\frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2}=\frac{1}{x}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right)-\frac{1}{x^2}\frac{dy}{dt}=\frac{1}{x^2}\frac{d^2y}{dt^2}-\frac{1}{x^2}\frac{dy}{dt}\Rightarrow\frac{d^2y}{dx^2}=\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right) \qquad (3)$$

Differentiating (3), we obtain

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\left\{\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right\}=\frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$\frac{d^3y}{dx^3}=\frac{1}{x^2}\left\{\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)\left(\frac{dt}{dx}\right)-\frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)\right\}-\frac{2}{x^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3}=\frac{1}{x^2}\left(\frac{d^3y}{dt^3}\frac{1}{x}-\frac{d^2y}{dt^2}\frac{1}{x}\right)-\frac{2}{x^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)=\frac{1}{x^3}\left(\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}-2\frac{d^2y}{dt^2}+2\frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3}=\frac{1}{x^3}\left(\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+2\frac{dy}{dt}\right) \qquad (4)$$

Now if we write $D=d/dx$ and $\Delta=d/dt$, then

from (2), $xDy=\Delta y$,

from (3), $x^2D^2y=(\Delta^2-\Delta)y=\Delta(\Delta-1)y$

from (4), $x^3D^3y=(\Delta^3-3\Delta^2+2\Delta)y=\Delta(\Delta-1)(\Delta-2)y$

In general, $x^nD^ny=\Delta(\Delta-1)(\Delta-2)\{\Delta-(n-1)\}y=\Delta(\Delta-1)(\Delta-2)(\Delta-n+1)y\cdot$

Substituting these values of $xD, x^2D^2, x^3D^3, ..., x^nD^n$ in (1), we obtain an equation of n^{th} order with constant coefficients having t as independent variable. This new equation may be solved by methods discussed in previous sections.

Example 01: Solve: $x^2 y`` - 3xy` + 5y = x^2 + \sin(\ln x)$

Solution: Given that $\quad x^2y``-3xy`+5y = x^2 \sin(\ln x) \qquad (1)$

Let $x = e^t$ and $t = \ln x\cdot$ Also substitute $xD = \Delta$ and $x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta\cdot$

Equation (1) becomes

$$\left(\Delta^2 - \Delta\right)y - 3\Delta y + 5y = e^{2t}\sin t \Rightarrow \left(\Delta^2 - 4\Delta + 5\right)y = e^{2t}\sin t \qquad (2)$$

Auxiliary equation is: $m^2 - 4m + 5 = 0 \Rightarrow m = \dfrac{-(-4) \pm \sqrt{16-20}}{2} = \dfrac{4 \pm 2i}{2} = 2 \pm i\cdot$

Thus Complementary Function is: $y_c = e^{2t}\left(c_1 \sin t + c_2 \cos t\right)\cdot$

Particular Integral of (2) is

$$y_p = \frac{1}{\Delta^2 - 4\Delta + 5}\left(e^{2t}\sin t\right) \Rightarrow y_p = e^{2t}\frac{1}{[(\Delta+2)^2 - 4(\Delta+2) + 5]}\sin t, \text{[By shift theorem]}$$

$$y_p = e^{2t}\frac{1}{\Delta^2+1}\sin t = e^{2t}\frac{t}{2\Delta}\sin t = \frac{te^{2t}}{2}\int \sin t\, dt = -\frac{te^{2t}}{2}\cos t \text{ [Observe that } e^{2t}\sin t \text{ is in } y_c].$$

Thus, general solution of the given equation is

$$y = y_c + y_p = e^{2t}\left(c_1 \sin t + c_2 \cos t\right) - te^{2t}\cos t / 2$$

Now replacing t by ln x and e^t by x we get:

$$y = x^2\left\{c_1 \sin(\ln x) + c_2 \cos(\ln x)\right\} - \frac{x^2 \ln x \cos(\ln x)}{2}.$$

Example 02: Solve: $x^3 y``` + 4x^2 y`` - 5xy` - 15y = x^4$

Solution: We have $\quad x^3y```+4x^2y``-5xy`-15y = x^4 \qquad (1)$

Let $x = e^t$ and $t = \ln x\cdot$ Then, $xD = \Delta,\ x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta,$ and

$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta\cdot$ Thus equation (1) becomes

$$\left(\Delta^3 - 3\Delta^2 + 2\Delta\right)y + 4\left(\Delta^2 - \Delta\right)y - 5\Delta y - 15y = e^{4t}$$

$$\Rightarrow \left(\Delta^3 - 3\Delta^2 + 2\Delta + 4\Delta^2 - 4\Delta - 5\Delta - 15\right)y = e^{4t}$$

or $$\left(\Delta^3 + \Delta^2 - 7\Delta - 15\right)y = e^{4t} \qquad (2)$$

Auxiliary equation is: $m^3 + m^2 - 7m - 15 = 0$ or $(m-3)(m^2 + 4m + 5) = 0$

Giving, $m = 3,\ -2 \pm i\cdot$

The roots of the auxiliary equation are: $m = 3, -2 \pm i\cdot$

Thus Complementary Function is: $y_c = c_1e^{3t} + e^{-2t}\left(c_2 \sin t + c_3 \cos t\right)\cdot$

Particular Integral of (2) is

$$y_p = \frac{1}{\Delta^3 + \Delta^2 - 7\Delta - 15} e^{4t} = \frac{e^t}{4^3 + 4^2 - 7(4) - 15} = \frac{e^t}{47}$$

Thus, general solution of given equation is

$$y = y_c + y_p = c_1e^{3t} + e^{-2t}\left(c_2 \sin t + c_3 \cos t\right) + \frac{1}{47}e^{4t}\cdot$$

Replacing t by lnx and e^t by x, we get:

$$y = y_c + y_p = c_1x^3 + x^{-2}\left[c_2 \sin\left(\ln x\right) + c_3 \cos\left(\ln x\right)\right] + \frac{1}{47}x^4\cdot$$

Example 03: Solve, (2x + 1) y`` - 6(2x + 1) y` + 16y = $8(2x + 1)^2$

Solution: Given that (2x + 1) y`` - 6(2x + 1) y` + 16y = $8(2x + 1)^2$ (1)

Let $2x + 1 = e^t \Rightarrow t = \ln\left(2x+1\right)\cdot$ Now

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\left\{\frac{d}{dx}\ln\left(2x+1\right)\right\} = \frac{dy}{dt}\cdot\left(\frac{1}{2x+1}(2)\right) = \frac{2}{2x+1}\frac{dy}{dt}$$

$$\left(2x+1\right)\frac{dy}{dx} = 2\frac{dy}{dt} \Rightarrow \left(2x+1\right)D = 2\Delta\cdot$$

Again differentiating, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{d}{dx}\left\{\left(2x+1\right)^{-1}\frac{dy}{dt}\right\} = 2\left\{\left(2x+1\right)^{-1}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(2x+1\right)^{-1}\right\}$$

$$\frac{d^2y}{dx^2} = 2\left\{\frac{1}{2x+1}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{dy}{dt}\left(\frac{-2}{\left(2x+1\right)^2}\right)\right\} = 2\left\{\frac{1}{2x+1}\frac{d^2y}{dt^2}\left(\frac{2}{2x+1}\right) - \frac{2}{\left(2x+1\right)^2}\frac{dy}{dt}\right\}$$

$$\frac{d^2y}{dx^2} = \frac{4}{\left(2x+1\right)^2}\frac{d^2y}{dt^2} - \frac{4}{\left(2x+1\right)^2}\frac{dy}{dt} = \frac{4\frac{d^2y}{dt^2} - 4\frac{dy}{dt}}{\left(2x+1\right)^2}$$

or $\left(2x+1\right)^2 D^2y = 4\Delta^2y - 4\Delta y \Rightarrow \left(2x+1\right)^2 D^2 = 4\Delta^2 - 4\Delta\cdot$ Thus, equation (1) becomes

$$\left\{4\Delta^2 - 4\Delta - 6\left(2\Delta\right) + 16\right\}y = 8e^{2t} \Rightarrow \left(4\Delta^2 - 16\Delta + 16\right)y = 8e^{2t}$$

$$\Rightarrow \left(\Delta^2 - 4\Delta + 4\right)y = 2e^{2t} \qquad (2)$$

Auxiliary equation is: $m^2 - 4m + 4 = 0$ giving m = 2, 2

Hence, complementary function is: $y_c = \left(c_1 + c_2t\right)e^{2t}\cdot$

The particular Integral

$$y_p = \frac{1}{\Delta^2 - 4\Delta + 4}\left(2e^{2t}\right) = \frac{2t}{2\Delta - 4}e^{2t} = \frac{2t^2}{2}e^{2t} = t^2e^{2t}$$

Thus, general solution is: $y = y_c + y_p = (c_1 + c_2 t)e^{2t} + t^2 e^{2t}\cdot$

Now replacing t by $\ln(2x+1)$ and e^t by (2x + 1), we get

$$y = \{c_1 + c_2 \ln(2x+1)\}(2x+1)^2 + \{\ln(2x+1)\}^2 (2x+1)^2$$

Example 04: Solve: $x^2 y'' - 2xy' + 2y = x \ln x$; $y(1) = 1$, $y'(1) = 0$

Solution: Given that $x^2 y'' - 2xy' + 2y = x\ln x$ or $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x\ln x$ (1)

Let $x = e^t$ and $t = \ln x\cdot$ Then, $xD = \Delta$ and $x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

Equation (1) becomes: $\left(\Delta^2 - 3\Delta + 2\right)y = te^t$ (2)

Auxiliary equation is: $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$ or $y_c = c_1 e^t + c_2 e^{2t}\cdot$

$$y_p = \frac{1}{\Delta^2 - 3\Delta + 2}\left(te^t\right) = \frac{1}{(\Delta-1)(\Delta-2)}\left(te^t\right) = \frac{1}{(\Delta-1)(\Delta-1-1)}\left(te^t\right) \text{[By shift Prop]}$$

$$y_p = e^t \frac{1}{\Delta(\Delta-1)} t = -e^t \frac{1}{\Delta(1-\Delta)} t = -e^t \frac{1}{\Delta}(1-\Delta)^{-1}(t) = -e^t \frac{1}{\Delta}\left(1 + \Delta + \Delta^2 + \ldots\right)(t)$$

$$y_p = -e^t \frac{1}{\Delta}(1+\Delta)(t) = -e^t \frac{1}{\Delta}(t+1) = -e^t \int (t+1)\Delta t = -e^t\left(\frac{t^2}{2} + t\right)\cdot$$

Thus, general solution is: $y = y_c + y_p = c_1 e^t + c_2 e^{2t} - e^t\left(\dfrac{t^2}{2} + t\right)\cdot$

Now replacing t by lnx and e^t by x, we get

$$y = c_1 x + c_2 x^2 - x\left\{\frac{(\ln x)^2}{2} + \ln x\right\} = c_1 x + c_2 x^2 - \frac{1}{2}x(\ln x)^2 - x\ln x \quad (3)$$

or $$y' = c_1 + 2c_2 x - \frac{1}{2}\left\{x(2\ln x)\frac{1}{x} + (\ln x)^2(1)\right\} - \left(x\frac{1}{x} + \ln x\right)$$

$$= c_1 + 2c_2 x - \ln x - \ln x - 1 - \ln x = c_1 + 2c_2 x - 3\ln x - 1\cdot$$

Applying the given initial conditions, we get: $1 = c_1 + c_2$ and $0 = c_1 + 2c_2 - 1\cdot$

Subtracting first equation from second, we get $c_2 = 0\cdot$

Putting it into first, we get $c_1 = 1\cdot$

Thus, (3) becomes: $y = x - x\ln x - \dfrac{x(\ln x)^2}{2}\cdot$

This is particular solution of given differential equation.

Example 05: Solve the following Cauchy-Euler differential Equations

(i) $(x^2D^2 + 7\,xD + 5)\,y = x^5$

Solution: Substituting $x = e^t$, we get: $x\,D = \Delta$ and $x^2\,D^2 = \Delta\,(\Delta - 1)$. Thus given differential equation becomes:

$[\Delta(\Delta-1)+7\Delta+5]\, y = e^{5t}$ or $(\Delta^2+6\Delta+5)\, y = e^{5t}$.

The auxiliary equation is

$m^2 + 6m + 5 = 0$ giving $m = -1, -5$.

Thus, $y_c = C_1 e^{-t} + C_2 e^{-5t}$

Now, $y_p = \dfrac{1}{(\Delta^2+5\Delta+6)} e^{5t} = \dfrac{1}{5^2+5.5+6} e^{5t} = \dfrac{1}{56} e^{5t}$.

Thus general solution of given differential equation is

$y = y_c + y_p = C_1 e^{-t} + C_2 e^{-5t} + \dfrac{1}{56} e^{5t} = C_1 x^{-1} + C_2 x^{-5} + x^5/56$ [Substituting $e^t = x$]

(ii) $[x^2 D^2 - (2m-1)\, x D + (m^2+n^2)]\, y = n^2 x^m \ln x$

Solution: Substituting $x = e^t$ and $\ln x = t$ we get: $x D = \Delta$ and $x^2 D^2 = \Delta(\Delta - 1)$. Thus given differential equation becomes:

$[\Delta(\Delta-1) - (2m-1)\Delta + (m^2+n^2)]\, y = n^2 e^{mt}\, t$

or $[\Delta^2 - 2m\Delta + (m^2+n^2)]\, y = n^2 e^{mt}\, t$

Auxiliary equation is: $k^2 - 2mk + (m^2+n^2) = 0$ or $k = m \pm n\, i$.

Thus, $y_c = e^{mt}(C_1 \cos nt + C_2 \sin nt)$

Now,

$$y_p = n^2 \frac{1}{[\Delta^2 - 2m\Delta + (m^2+n^2)]} e^{mt} t = n^2 e^{mt} \frac{1}{[(\Delta+m)^2 - 2m(\Delta+m) + (m^2+n^2)]} t$$

$$= n^2 e^{mt} \frac{1}{\Delta^2+n^2} t = n^2 e^{mt} \frac{1}{n^2(1+\Delta^2/n^2)} t = e^{mt}\left(1+\frac{\Delta^2}{n^2}\right)^{-1} t = e^{mt}\left(1-\frac{\Delta^2}{n^2}+\ldots\right) t.$$

$$= e^{mt}\left(t - \frac{\Delta^2}{n^2} t\right) = e^{mt} t$$

Thus general solution of given differential equation is

$y = y_c + y_p = e^{mt}(C_1 \cos nt + C_2 \sin nt) + e^{mt}\, t$

$= x^m [C_1 \cos(n \ln x) + C_2 \sin(n \ln x)] + x^m \ln x$ [Substituting $e^t = x$ & $t = \ln x$]

(iii) $(4x^2D^2 - 4xD + 3)\, y = \sin \ln(-x)$, $x < 0$

Solution: Since x is negative hence, we let $x = -u$ where $u > 0$. Then $x^2 = u^2$ and $du/dx = -1$. Thus,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{dy}{du} = -Dy. \text{ Also } \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-\frac{dy}{du}\right) = -\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}$$

$$= -\frac{d^2y}{du^2}(-1) = +\frac{d^2y}{du^2} = D^2y. \qquad [\text{NOTE: We define } Dy = dy/du]$$

Thus given differential equation becomes:

$[4u^2D^2 - 4(-u)(-D) + 3]y = \sin(\ln u)$ or $(4u^2D^2 - 4uD - 3)\, y = \sin(\ln u)$.

Now substituting $u = e^t$ and $\ln u = t$ and $u D = \Delta$, $u^2 D^2 = \Delta(\Delta - 1)$, above differential equation becomes:

$$(4\,\Delta(\Delta - 1) - 4\,\Delta - 3)\,y = \sin t \quad \text{or} \quad (4\,\Delta^2 - 8\,\Delta - 3)\,y = \sin t$$

The auxiliary equation is

$4\,m^2 - 8\,m - 3 = 0$ giving $m = 3/2, 1/2$. Thus, $y_c = C_1\,e^{t/2} + C_2\,e^{3t/2}$. Now,

$$y_p = \frac{1}{4\Delta^2 - 8\Delta - 3}\sin t = \frac{1}{4(-1^2) - 8\Delta - 3}\sin t = \frac{1}{-(8\Delta + 7)}\sin t = -\frac{1}{(8\Delta + 7)}\frac{(8\Delta - 7)}{(8\Delta - 7)}\sin t$$

$$= -\frac{(8\Delta - 7)}{(64\Delta^2 - 49)}\sin t = -\frac{(8\Delta - 7)}{64(-1^2) - 49}\sin t = +\frac{1}{113}(8\Delta\sin t - 7\sin t) = \frac{1}{113}(8\cos t - 7\sin t)$$

Thus general solution is:

$y = y_c + y_p = C_1\,e^{t/2} + C_2\,e^{3t/2} + (8\cos t - 7\sin t)/113$

$= C_1 u^{1/2} + C_2\,u^{3/2} + [8\cos(\ln u) - 7\sin(\ln u)]/113$, where $u = -x$.

(iv) $(x^4 D^3 + 2\,x^3 D^2 - x^2\,D + x\,)y = 1$

Solution: Dividing both sides by x, we obtain: $(x^3\,D^3 + 2\,x^2\,D^2 - x\,D + 1)\,y = 1/x$

Now put $x = e^t$ and $x\,D = \Delta$, $x^2\,D^2 = \Delta\,(\Delta - 1)$, $x^3\,D^3 = \Delta\,(\Delta - 1)\,(\Delta - 2)$, we obtain

$(\Delta\,(\Delta - 1)\,(\Delta - 2) + 2\,\Delta\,(\Delta - 1) - \Delta + 1)\,y = e^{-t}$ or $(\Delta^3 - \Delta^2 - \Delta + 1)\,y = e^{-t}$

Auxiliary equation is: $m^3 - m^2 - m + 1 = 0$ giving $m = -1, 1, 1$.

Thus, $y_c = C_1\,e^{-t} + (C_2 + C_3\,t)\,e^t$

$$y_p = \frac{1}{\Delta^3 - \Delta^2 - \Delta + 1}e^{-t} = \frac{t}{3\Delta^2 - 2\Delta - 1}e^{-t} = \frac{t^2}{6\Delta - 2}e^{-t} = \frac{t^2}{6(-1) - 2}e^{-t} = -\frac{t^2 e^{-t}}{8}$$

Thus general solution of given differential equation is:

$y = y_c + y_p = C_1\,e^{-t} + (C_2 + C_3\,t)\,e^t - t^2 e^{-t}/8$

$= C_1\,x^{-1} + (C_2 + C_3\ln x) - \dfrac{x^{-1}(\ln x)^2}{8}$ [Substituting $x = e^t$ and $\ln x = t$]

METHOD OF VARIATION OF PARAMETERS

The most general for of linear non-homogeneous differential equation of order two is:

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)\,y = F(x) \tag{1}$$

The solution of such equation may be determined by a procedure known as *method of variation of parameters*. This method can be applied even to equations of higher orders, but we shall restrict to second order differential equations. However, we shall present one example of order three to make the readers acquainted with such problems.

Suppose that linearly independent solutions of

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \tag{2}$$

are given by $y = y_1$ and $y = y_2$. Hence

$$\frac{d^2y_1}{d^2x} + P(x)\frac{dy_1}{dx} + Q(x)y_1 = 0 \text{ and } \frac{d^2y_2}{d^2x} + P(x)\frac{dy_2}{dx} + Q(x)y_2 = 0.$$

Then complementary function is: $y_c = c_1y_1 + c_2y_2$,

where c_1 and c_2 are arbitrary constants. We replace arbitrary constants c_1 and c_2 by unknown functions $u_1(x)$ and $u_2(x)$ and require that

$$y_p = u_1y_1 + u_2y_2, \tag{3}$$

be a particular solution of (1).

Note. The arbitrary constants that occur in the former case are changed into functions of the independent variables. For this reason, the method is known as *variation of parameters*.

In order to determine two functions u_1 and u_2, we need two conditions. One condition is that (3) must satisfy (1). A second condition can be imposed arbitrarily.

Differentiating (3) with respect to x, we get

$$y'_p = u_1y'_1 + y_1u'_1 + u_2y'_2 + y_2u'_2 \Rightarrow y'_p = (u'_1y_1 + u'_2y_2) + (u_1y'_1 + u_2y'_2).$$

If we differentiate above equation, y''_p will contain u''_1 and u''_2. To avoid second derivatives of u_1 and u_2, we set: $u'_1y_1 + u'_2y_2 = 0$ (4)

With this condition, we have: $y'_p = u_1y'_1 + u_2y'_2$

so that, $y''_p = u_1y''_1 + y'_1u'_1 + u_2y''_2 + y'_2u'_2 \Rightarrow y''_p = u'_1y'_1 + u'_2y'_2 + u_1y''_1 + u_2y''_2$.

Substituting y_p, y'_p, y''_p into equation (1), we obtain

$$(u'_1y'_1 + u'_2y'_2 + u_1y''_1 + u_2y''_2) + P(u_1y'_1 + u_2y'_2) + Q(u_1y_1 + u_2y_2) = F(x)$$

or $$u_1(y''_1 + Py'_1 + Qy_1) + u_2(y''_2 + Py'_2 + Qy_2) + u'_1y'_1 + u'_2y'_2 = F(x)$$

Expressions in the parenthesis are zero since y_1 and y_2 are solutions of (2). Hence

$$u'_1y'_1 + u'_2y'_2 = F(x) \tag{5}$$

Taking (4) and (5) together, we have two equations in the two unknowns u'_1 and u'_2. This is:

$u`_1\, y_1 + u`_2\, y_2 = 0$ and $u`_1\, y`_1 + u`_2\, y`_2 = F(x)$

Multiplying (4) by y'_2 and (5) by y_2, we get

$$u'_1y_1y'_2 + u'_2y_2y'_2 = 0 \tag{6}$$

$$u'_1y'_1y_2 + u'_2y_2y'_2 = y_2F(x) \tag{7}$$

Subtracting (7) from (6), we have

$$u'_1y_1y'_2 - u'_1y'_1y_2 = -y_2F(x) \Rightarrow u'_1(y_1y'_2 - y'_1y_2) = -y_2F(x) \Rightarrow u'_1 = \frac{-y_2F(x)}{(y_1y'_2 - y'_1y_2)}.$$

Similarly, $u_2' = \dfrac{y_1F(x)}{(y_1y_2' - y_1'y_2)}$.

Thus,
$$\left.\begin{aligned} u_1' &= \frac{-y_2F(x)}{(y_1y_2' - y_1'y_2)} \\ u_2' &= \frac{y_1F(x)}{(y_1y_2' - y_1'y_2)} \end{aligned}\right\} \qquad (8)$$

In (8) $y_1y_2' - y_1'y_2 \neq 0$, since y_1, y_2 are linearly independent solutions of (2)

Integrating (8), we find u_1 and u_2 as

$$\left.\begin{aligned} u_1 &= \int \frac{-y_2F(x)}{(y_1y_2' - y_1'y_2)}dx \\ u_2 &= \int \frac{y_1F(x)}{(y_1y_2' - y_1'y_2)}dx \end{aligned}\right\} \qquad (9)$$

Thus, y_p is completely determined.

In numerical problems, instead of performing the complete process, formulas (9) will directly be applied to evaluate u_1 and u_2.

It may be noted that expression in the denominator of (9) is known as WRONSKIN and is some times denoted by W = $\begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix}$ = $y_1y`_2 - y`_1y_2$.

Example 01: Find a general solution of y`` + y = tan x sec x

Solution: Given that $y`` + y = \tan x \sec x$ (1)

$$\Rightarrow (D^2 + 1)y = \tan x \sec x.$$

Auxiliary equation is: $m^2 + 1 = 0 \Rightarrow m = \pm i$ or $y_c = c_1 \sin x + c_2 \cos x$.

Let $y_p = u_1 \sin x + u_2 \cos x$ (2)

Here $y_1 = \sin x, y_2 = \cos x, F(x) = \tan x \sec x, y_1' = \cos x, y_2' = -\sin x$.

Also W = $y_1y_2' - y_1'y_2 = \sin x(-\sin x) - \cos x(\cos x) = -\sin^2 x - \cos^2 x = -1$.

By formulas (9), we have

$$\left.\begin{aligned} u_1 &= \int \frac{-y_2F(x)}{(y_1y_2' - y_1'y_2)}dx \\ u_2 &= \int \frac{y_1F(x)}{(y_1y_2' - y_1'y_2)}dx \end{aligned}\right\}$$

or $u_1 = \int \dfrac{-\cos x \tan x \sec x}{-1}dx = \int \tan x dx = \ln(\sec x)$.

$$u_2 = \int \frac{\sin x \tan x \sec x}{-1} dx = -\int \frac{\sin^2 x}{\cos^2 x} dx = -\int \tan^2 x dx$$

$$= -\int \left(\sec^2 x - 1\right) dx = \int 1 dx - \int \sec^2 x dx = x - \tan x \cdot$$

Thus, (2) becomes

$$y_p = \ln\left(\sec x\right) \sin x + \left(x - \tan x\right) \cos x = \sin x \ln\left(\sec x\right) + x \cos x - \sin x \cdot$$

Thus general solution of given differential equation is: $y = y_c + y_p$

$\Rightarrow$ $$y = c_1 \sin x + c_2 \cos x + \sin x \ln\left(\sec x\right) + x \cos x - \sin x \cdot$$

Example 02: Solve the following differential equations by method of variation of parameters

(i) $y'' + 4y = \sec 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0$ or $m = \pm 2i$. Thus

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = \cos 2x$, $y_2 = \sin 2x$

and

$$u_1 = -\int \frac{y_2 f(x)}{W} dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W} dx$$

Here,

$W = y_1 y'_2 - y_2 y'_1 = \cos 2x\,(2 \cos 2x) - \sin 2x\,(-2 \sin 2x) = 2\,(\cos^2 2x + \sin^2 2x) = 2.$

Thus, $u_1 = -\int \frac{\sin 2x \sec 2x}{2} dx = -\frac{1}{2}\int \tan 2x\, dx = -\frac{1}{2}\left(\frac{\ln \sec 2x}{2}\right) = -\frac{\ln \sec 2x}{4}$ and

$$u_2 = \int \frac{\cos 2x \sec 2x}{2} dx = \frac{1}{2}\int 1.dx = \frac{x}{2}$$

Therefore, $y_p = -\frac{\ln \sec 2x}{4}.\cos 2x + \frac{x}{2}.\sin 2x$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x - \frac{\ln \sec 2x}{4} + \frac{x}{2}.\sin 2x$$

(ii) $y'' - 3y' + 2y = (1 + e^{-x})^{-1}$

Solution: The auxiliary equation is: $m^2 - 3m + 2 = 0$ giving $m = 1, 2$. Thus

$$y_c = C_1 e^x + C_2 e^{2x}$$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^x$, $y_2 = e^{2x}$

Here, $$u_1 = -\int \frac{y_2 f(x)}{W} dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W} dx \quad .$$

Here, $$W = y_1 y'_2 - y_2 y'_1 = e^x (2 e^{2x}) - e^{2x} (e^x) = e^{3x}.$$

Thus, $$u_1 = -\int \frac{e^{2x}(1+e^{-x})^{-1}}{e^{3x}} dx = -\int \frac{1}{(e^x + 1)} dx.$$

Now put $e^x = z \Rightarrow e^x dx = dz \Rightarrow zdx = dz \Rightarrow dx = dz/z$

$$\therefore u_1 = -\int \frac{1}{z(z+1)} dz = -\int \left(\frac{1}{z} - \frac{1}{z+1}\right) dz = -\left(\ln z - \ln(z+1)\right) = \ln \frac{z+1}{z} = \ln\left(\frac{e^x+1}{e^x}\right)$$

Also $u_2 = \int \frac{e^x (1+e^{-x})^{-1}}{e^{3x}} dx = \int \frac{1}{e^x(e^x+1)} dx.$

Now put $e^x = z \Rightarrow e^x dx = dz \Rightarrow zdx = dz \Rightarrow dx = dz/z$

$$\therefore u_2 = \int \frac{1}{z^2(z+1)} du = \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{z+1}\right) du = \left(-\frac{1}{z} - \ln z + \ln(z+1)\right)$$

$$= \ln \frac{z+1}{z} - \frac{1}{z} = \ln\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x}$$

Therefore, $y_p = e^x . \ln\left(\frac{e^x+1}{e^x}\right) + e^{2x}\left[\ln\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x}\right]$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = C_1 e^x + C_2 e^{2x} + e^x \ln\left(\frac{e^x+1}{e^x}\right) + e^{2x}\left[\ln\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x}\right]$$

(iii) $y`` + 4 y` + 5 y = e^{-2x} \sec x$

Solution: Auxiliary equation is $m^2 + 4 m + 5 = 0$ giving $m = -2 \pm i$.

Thus $y_c = e^{-2x} (C_1 \cos x + C_2 \sin x)$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^{-2x} \cos x$, $y_2 = e^{-2x} \sin x$

Here,

$$u_1 = -\int \frac{y_2 f(x)}{W} dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W} dx .$$

Also, $W = y_1 y`_2 - y_2 y`_1 = e^{-4x}$.

Thus, $u_1 = -\int \frac{e^{-2x} \sin x . e^{-2x} \sec x}{e^{-4x}} dx = -\int \sec x \, dx = -\ln(\sec x + \tan x)$

Also, $u_2 = \int \frac{e^{-2x} \cos x \, e^{-2x} \sec x}{e^{-4x}} dx = \int 1 \, dx. = x$

Therefore, $y_p = e^{-2x}[-\cos x \ln(\sec x + \tan x) + x \sin x]$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = e^{-2x} (C_1 \cos x + C_2 \sin x) + e^{-2x}[-\cos x \ln(\sec x + \tan x) + x \sin x]$$

(iv) $y`` - 4 y` + 4 y = e^{2x}/(1 + x)$

Solution: Auxiliary equation is $m^2 - 4m + 4 = 0$ giving $m = 2, 2$.

Thus, $y_c = (C_1 + C_2 x) e^{2x}$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^{2x}$ and $y_2 = x e^{2x}$

and $\quad u_1 = -\int \frac{y_2 f(x)}{W} dx$ and $u_2 = \int \frac{y_1 f(x)}{W} dx$.

Here, $\quad W = y_1\, y`_2 - y_2\, y`_1 = e^{4x}$.

Thus

$$u_1 = -\int \frac{xe^{2x}.\, e^{2x}}{e^{4x}(1+x)} dx = -\int \frac{x}{x+1}\, dx = -\int \frac{(x+1)-1}{x+1} dx = -\int \left(1 - \frac{1}{x+1}\right) dx = \ln(x+1) - x$$

Also, $u_2 = \int \frac{e^{2x}.\, e^{2x}}{e^{4x}(1+x)} dx = \int \frac{1}{x+1}\, dx = \ln(x+1)$

Therefore, $\quad y_p = e^{2x}[\ln(x+1) - x + x\ln(x+1)] = e^{2x}[\ln(x+1)(x+1) - x]$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = (C_1 + C_2 x)\, e^{2x} + e^{2x}[\ln(x+1)(x+1) - x]$$

(v) $y`` - 2y` + y = e^x \sin^{-1} x$

Solution: Auxiliary equation is $\quad m^2 - 2m + 1 = 0 \quad$ giving $\quad m = 1, 1$.

Thus, $\quad y_c = (C_1 + C_2 x)\, e^x$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^x$, $y_2 = x e^x$

and $\quad u_1 = -\int \frac{y_2 f(x)}{W} dx$ and $u_2 = \int \frac{y_1 f(x)}{W} dx$

Here, $\quad W = y_1\, y`_2 - y_2\, y`_1 = e^{2x}$.

Thus

$$u_1 = -\int \frac{xe^x.\, e^x \sin^{-1} x}{e^{2x}} dx = -\int x.\sin^{-1} x\, dx = -\sin^{-1} x.\frac{x^2}{2} + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{-x^2}{\sqrt{1-x^2}}\, dx = -\frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{(1-x^2)+1}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{(1-x^2)}{\sqrt{1-x^2}}\, dx - \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\, dx = -\frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sqrt{1-x^2}\, dx - \frac{1}{2}\sin^{-1} x$$

$$= -\frac{(1+x^2)}{2}\sin^{-1} x - \frac{1}{2}\int \sqrt{1-x^2}\, dx = -\frac{(1+x^2)}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{x\sqrt{1-x^2}}{2} - \frac{1}{2}\sin^{-1} x\right)$$

$$= -\frac{(2x^2 + \sin^{-1} x)}{4} - \frac{x\sqrt{1-x^2}}{4} = -\frac{1}{4}\left(2x^2 + \sin^{-1} x + x\sqrt{1-x^2}\right)$$

Also, $u_2 = \int \frac{e^x.\, e^x \sin^{-1} x}{e^{2x}} dx = \int \sin^{-1} x\, dx = \int \sin^{-1} x.1\, dx = \sin^{-1} x.x - \int \frac{1}{\sqrt{1-x^2}}.x\, dx$

$$= x\sin^{-1} x + \frac{1}{2}\int \frac{2x}{\sqrt{1-x^2}}\, dx = x\sin^{-1} x - \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\, dx$$

$= x\ \sin^{-1} x - \frac{1}{2}\frac{\sqrt{1-x^2}}{1/2} = x\ \sin^{-1} x - \sqrt{1-x^2}$

Therefore, $y_p = -\frac{1}{4}\left(2x^2 + \sin^{-1} x + x\sqrt{1-x^2}\right)e^x + x\,e^x[\,x\ \sin^{-1} x - \sqrt{1-x^2}\,]$

Hence, general solution of given differential equation is:

$y = (C_1 + C_2\,x)\,e^x - \frac{1}{4}\left(2x^2 + \sin^{-1} x + x\sqrt{1-x^2}\right)e^x + x\,e^x\,[\,x\ \sin^{-1} x - \sqrt{1-x^2}\,]$

(vi) $y'' - 2y' + 5y = e^x \tan 2x$

Solution: Auxiliary equation is $m^2 - 2m + 5 = 0$ giving, $m = 1 \pm 2\,i$.

Thus, $y_c = e^x\,(C_1 \cos 2x + C_2 \sin 2x)$

To find y_p, we know that $y_p = u_1\,y_1 + u_2\,y_2$, where: $y_1 = e^{x} \cos 2x$, $y_2 = e^{x} \sin 2x$

Now,

$$u_1 = -\int \frac{y_2 f(x)}{W}dx \ \text{ and } \ u_2 = \int \frac{y_1 f(x)}{W}dx\ .$$

Here, $W = y_1\,y'_2 - y_2\,y'_1 = 2\,e^{2x}$.

Thus, $u_1 = -\int \frac{e^x \sin 2x\,.\,e^x \tan 2x}{2e^{2x}}dx = -\frac{1}{2}\int \frac{\sin 2x.\sin 2x}{\cos 2x}\ dx = -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x}\ dx$

$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\ dx = -\frac{1}{2}\int (\sec 2x - \cos 2x)\ dx = -\frac{1}{4}\ln(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x$

$= \frac{1}{4}\left[\sin 2x - \ln(\sec 2x + \tan 2x)\right]$

Also, $u_2 = \int \frac{e^x \cos 2x\,.e^x \tan 2x}{2e^{2x}}dx = \frac{1}{2}\int \sin 2x\ dx = -\frac{1}{4}\cos 2x$

Therefore, $y_p = \frac{1}{4}\left[\sin 2x - \ln(\sec 2x + \tan 2x)\right]e^x \cos 2x - \frac{1}{4}\cos 2x\ .e^x \sin 2x$

Hence the general solution of given differential equation is:

$y = e^x\,(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{4}e^x \cos 2x\left[\sin 2x - \ln(\sec 2x + \tan 2x) - \sin 2x\right]$

(vii) $y'' + 2y' + y = e^{-x} \ln x$

Solution: Auxiliary equation is $m^2 + 2m + 1 = 0$ giving, $m = -1, -1$.

Thus, $y_c = (C_1 + C_2\,x)\,e^{-x}$

To find y_p, we know that $y_p = u_1\,y_1 + u_2\,y_2$, where: $y_1 = e^{-x}$, $y_2 = xe^{-x}$

Now,

$$u_1 = -\int \frac{y_2 f(x)}{W}dx \ \text{ and } \ u_2 = \int \frac{y_1 f(x)}{W}dx\ .$$

Here, $W = y_1\,y'_2 - y_2\,y'_1 = e^{-2x}$.

Thus,

$$u_1 = -\int \frac{xe^{-x}.e^{-x}\ln x}{e^{-2x}}dx = -\int x\ln x\ dx = \frac{x^2}{4}[1-\ln x^2] \quad \left[\text{integrating by parts}\right]$$

$$\text{Also, } u_2 = \int \frac{e^{-x}.\,e^{-x}\ln x}{e^{-2x}}dx = \int \ln x\ dx = x[\ln x - 1] \qquad [\text{integrating by parts}]$$

Therefore, $y_p = e^{-x}\ \dfrac{x^2}{4}[1-\ln x^2] + x\,e^{-x}\,.\,x[\ln x - 1]$

Hence, general solution of given differential equation is:

$y = y_c + y_p = (C_1 + C_2\,x)\,e^{-x} + e^{-x}\ \dfrac{x^2}{4}[1-\ln x^2] + x^2\,e^{-x}\,[\ln x - 1]$

(viii) $y`` + 2\,y` + 2\,y = 2\,e^{-x}\tan^2 x$

Solution: Auxiliary equation is $m^2 + 2m + 2 = 0$ giving, $m = -1 \pm i$.

Thus, $y_c = e^{-x}\,(C_1 \cos x + C_2 \sin x)$

To find y_p, we know that $y_p = u_1\,y_1 + u_2\,y_2$, where: $y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$

Now, $u_1 = -\int \dfrac{y_2 f(x)}{W}dx$ and $u_2 = \int \dfrac{y_1 f(x)}{W}dx$

Here, $W = y_1\,y`_2 - y_2\,y`_1 = e^{-2x}$

Thus

$$u_1 = -\int \frac{e^{-x}\sin x\,.\,2e^{-x}\tan^2 x}{e^{-2x}}dx = -2\int \frac{\sin x.\sin^2 x}{\cos^2 x}dx = 2\int \frac{1-\cos^2 x}{\cos^2 x}(-\sin x)\,dx$$

Putting $z = \cos x$ so that $dz = -\sin x\,dx$

$$\therefore u_1 = 2\int \frac{1-z^2}{z^2}dz = 2\int \left(\frac{1}{z^2}-1\right)dz = 2\int (z^{-2}-1)\,dz = 2(-z^{-1}-z)$$

$$= -2\left(\frac{1}{z}+z\right) = -2(\sec x + \cos x)$$

$$\text{Also, } u_2 = \int \frac{e^{-x}\cos x\,.e^{-x}.2e^{-x}\tan^2 x}{e^{-2x}}dx = 2\int \cos x\,\frac{\sin^2 x}{\cos^2 x}dx = 2\int \frac{(1-\cos^2 x)}{\cos x}dx$$

$$= 2\int (\sec x - \cos x)dx = 2[\ln(\sec x + \tan x) - \sin x]$$

Therefore, $y_p = -2\,e^{-x}\cos x\,(\sec x + \cos x) + 2e^{-x}\sin x\,[\ln(\sec x + \tan x) - \sin x]$

Hence, general solution of given differential equation is:

$y = y_c + y_p = e^{-x}\,(C_1 \cos x + C_2 \sin x)$ $-2\,e^{-x}$ {cos x (sec x + cos x)

APPLICATIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS

Although, applications of higher order differential equations are not restricted to only Electrical Engineering. Looking at the limitations and size of the book, we restrict ourselves to applications pertaining to only Electrical Simple RLC Circuits.

Here, we use two laws.

1. Ohm's law and

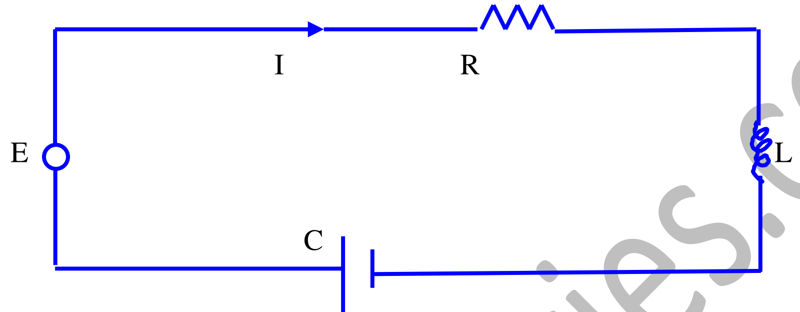
2. Kirchhoff's voltage law.

Ohm's law states that voltage is directly proportional to the current, that is,

$$V \propto I \text{ or } V = I R,$$

where R is the resistance.

Kirchhoff's law states that sum of the voltage drops across the elements of a closed circuit are equal to total electromotive force E in the circuit. The voltage drop across a resistor is IR, across a coil of inductance is $L \cdot \frac{dI}{dt}$, and across a condenser of capacitance is $\frac{1}{C}Q \cdot$ Here, current I and the charge Q are related by $I = \frac{dQ}{dt}$, we will consider R, L, and C as constants.



The differential equation of an electric circuit containing an inductance L, a resistor R, a condenser of capacitance C, and an electromotive force $E(t)$ is therefore

$$L\frac{dI}{dt} + IR + \frac{1}{C}Q = E \qquad (1)$$

or, since $I = \frac{dQ}{dt}$ and $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$, then (1) becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E, \qquad (2)$$

Example 01: If the resistor R, capacitor C and inductor L are connected with an emf E in series, find the charge and current at any time when R = 2 ohms, L = 1 Henry, C = 1 Farad and E = 5 sin t.

Solution: We have $$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E \cdot$$

Substituting the given values, we get

$$\frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + \frac{1}{1}Q = 5\sin t \Rightarrow \frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + Q = 5\sin t \Rightarrow \left(D^2 + 2D + 1\right)Q = 5\sin t$$

Auxiliary equation is: $D^2 + 2D + 1 = 0 \Rightarrow (D+1)^2 = 0 \Rightarrow D = -1, -1 \cdot$

The Complementary Function is: $Q_c = (c_1 + c_2 t)e^{-t} \cdot$

Now the Particular Integral is

$$Q_p = \frac{1}{(D+1)^2}(5\sin t) = 5\,\text{Im}\left[\frac{1}{(D+1)^2}\left(e^{it}\right)\right] = 5\,\text{Im}\left[\frac{e^{it}}{(i+1)^2}\right] = \text{Im}\left[\frac{5e^{it}}{2i}\right] = \text{Im}\left[\frac{5e^{it}}{2i}\times\frac{i}{i}\right]$$

$$= -\frac{5}{2}\text{Im}\left[i(\cos t + i\sin t)\right] = -\frac{5}{2}\text{Im}\left[(i\cos t - \sin t)\right] = -\frac{5}{2}\cos t$$

Thus, $Q = Q_c + Q_p = (c_1 + c_2 t)e^{-t} - 5\cos t/2$

Also, $I = \frac{dQ}{dt} = -e^{-t}(c_1 + c_2 t) + c_2 e^{-t} + \frac{5}{2}\sin t \cdot$

Example 02: Solve the differential equation: Q`` + R Q` + Q/C = E for charge Q and current I, where L = 1, R = 2, C = 1 and E = 5, given that Q(0) = 0 and I(0) = 0 at t = 0.

Solution: We have $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E \cdot$

Substituting the given values, we get

$$\frac{d^2Q}{dt^2} + 2\frac{dQ}{dt} + Q = 5 \Rightarrow \left(D^2 + 2D + 1\right)Q = 5 \cdot$$

Auxiliary equation is: $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1 \cdot$

Complementary Function is: $Q_c = (c_1 + c_2 t)e^{-t} \cdot$

Now, Particular Integral is: $Q_p = \frac{1}{(D+1)^2}\left(5e^{0t}\right) = \frac{5}{(0+1)} = 5 \cdot$

Thus, $Q = Q_c + Q_p = (c_1 + c_2 t)e^{-t} + 5$ (1)

Also, $I = Q` = -e^{-t}(c_1 + c_2 t) + c_2 e^{-t}$ (2)

Applying the given conditions, that is, put t = 0, Q = 0 and I = 0, we get: $0 = c_1 + 5 \Rightarrow c_1 = -5 \cdot$

And $0 = -e^{-10}(c_1 + 10c_2) + c_2 e^{-10} \Rightarrow 0 = -e^{-10}(-5 + 10c_2) + c_2 e^{-10} \Rightarrow c_2 = 5/9 \cdot$

Hence, $Q = e^{-t}\left(-5 + \frac{5}{9}t\right) + 5,\ I = -e^{-t}\left(-5 + \frac{5}{9}t\right) + \frac{5}{9}e^{-t} \cdot$

Example 03: Find the charge on the capacitor and the current in the given LRC series circuit, where L = 5/3 H, R = 10 ohms, C = 1/30 F and E = 300 V and Q(0) = 0, I(0) = 0.

Solution: We have $L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E \cdot$

Substituting the given values, we get

$$\frac{5}{3}\frac{d^2Q}{dt^2} + 10\frac{dQ}{dt} + 30Q = 300 \Rightarrow \frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 18Q = 180 \Rightarrow \left(D^2 + 6D + 18\right)Q = 180 \cdot$$

The auxiliary equation is: $m^2 + 6m + 18 = 0 \Rightarrow m = \frac{-6 \pm \sqrt{36 - 72}}{2} = \frac{-6 \pm 6i}{2} = -3 \pm 3i \cdot$

The Complementary Function is: $Q_c = e^{-3t}\left(c_1 \sin 3t + c_2 \cos 3t\right)\cdot$

Now the Particular Integral is: $Q_p = \dfrac{1}{D^2+6D+18}(180)e^{0t} = \dfrac{180}{18} = 10\cdot$

Thus, $Q = Q_c + Q_p = e^{-3t}\left(c_1 \sin 3t + c_2 \cos 3t\right) + 10\cdot$

Also, $I = \dfrac{dQ}{dt} = -3e^{-3t}\left(c_1 \sin 3t + c_2 \cos 3t\right) + e^{-3t}\left(3c_1 \cos 3t - 3c_2 \sin 3t\right)\cdot$

Applying the given conditions, we get

$$0 = c_2 + 10 \Rightarrow c_2 = -10 \text{ and } 0 = -3(-10) + 3c_1 \Rightarrow c_1 = -10\cdot$$

Hence $Q = e^{-3t}\left(-10\sin 3t - 10\cos 3t\right) + 10\cdot$

$$I = -3e^{-3t}\left(-10\sin 3t - 10\cos 3t\right) + e^{-3t}\left(-30\cos 3t + 30\sin 3t\right)$$

$$I = e^{-3t}\left(30\sin 3t + 30\cos 3t - 30\cos 3t + 30\sin 3t\right) = 60e^{-3t}\sin 3t\cdot$$

Example 04: A circuit consists of an inductance of 0.05 Henry, a resistance of 20 ohms, a condenser of capacitance 100 microfarads, and an emf of E = 100 V. Find I and Q, given the initial conditions Q = 0, I = 0 when t = 0.

Solution: We have $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E\cdot$

Substituting the given values, we get

$$(0.05)\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + \frac{Q}{100\times 10^{-6}} = 100 \Rightarrow (0.05)\frac{d^2Q}{dt^2} + 20\frac{dQ}{dt} + 10000Q = 100$$

$$\frac{d^2Q}{dt^2} + 400\frac{dQ}{dt} + 2,00000Q = 2000 \Rightarrow \left(D^2 + 400D + 2,00000\right)Q = 2000\cdot$$

The auxiliary equation is: $m^2 + 400m + 2,00000 = 0\cdot$

$$m = \frac{-400 \pm \sqrt{160,000 - 8,00000}}{2} = \frac{-400 \pm 800i}{2} = -200 \pm 400i\cdot$$

The Complementary Function is: $Q_c = e^{-200t}\left(c_1 \sin 400t + c_2 \cos 400t\right)\cdot$

Now the Particular Integral is: $Q_p = \dfrac{1}{D^2 + 400D + 2,00000}(2000) = \dfrac{2000}{2,00000} = 0.01\cdot$

Thus, $Q = Q_c + Q_p = e^{-200t}\left(c_1 \sin 400t + c_2 \cos 400t\right) + 0.01\cdot$

Also,

$$I = \frac{dQ}{dt} = -200e^{-200t}\left(c_1 \sin 400t + c_2 \cos 400t\right) + e^{-200t}\left(400c_1 \cos 400t - 400c_2 \sin 400t\right)$$

$$I = -e^{-200t}\left(200c_1 \sin 400t + 200c_2 \cos 400t - 400c_1 \cos 400t + 400c_2 \sin 400t\right)\cdot$$

Applying the given conditions, we get

$$0 = c_2 + 0.01 \Rightarrow c_2 = -0.01 \text{ and } 0 = -\left(200c_2 - 400c_1\right) \Rightarrow c_1 = -0.005 \cdot$$

Hence, $I = -e^{-200t}\left(-\sin 400t - 2\cos 400t + 2\cos 400t - 4\sin 400t\right) = 5e^{-200t}\sin 400t \cdot$

This is the particular solution.

WORKSHEET 04

1. Solve the following homogeneous linear differential equations:

(a) $(D^2 - 3D - 4)\, y = 0$

(b) $(D^3 - 7D - 6)\, y = 0$

(c) $(D^3 - 9D^2 + 23D - 15)\, y = 0$

(d) $[D^2 + (a + b)D + ab]\, y = 0$

(e) $(D^3 - 2D^2 + 4D - 8)\, y = 0$

(f) $(D^4 - 5D^2 + 4)\, y = 0$

(g) $(D^2 - 4D + 1)\, y = 0$

(h) $(D^4 + k^4)\, y = 0$

(i) $(D^3 - D^2 - D - 2)\, y = 0$

(j) $(D^4 + 2D^3 + 3D^2 + 2D + 1)\, y = 0$

(k) $(D^3 + 3D^2 + 3D + 1)\, y = 0$

(*l*) $(D^4 - 7D^3 + 18D^2 - 20D + 8)\, y = 0$

(m) $(D^4 - 8D^2 + 16)\, y = 0$

(n) $(D^4 - k^4)\, y = 0$

(o) $(D^6 - k^6)\, y = 0$

(p) $(D^4 + 2D^3 - 3D^2 - 4D + 4)\, y = 0$

(q) $(D^2 + 4D + 3)\, y = 0,\ y(0) = 0,\ y'(0) = 12$

(r) $(D^4 + D^2)\, y = 0;\ y(0) = y'(0) = y''(0) = 0,\ y'''(0) = 1$

(s) $(D^3 + 6D^2 + 12D + 8)\, y = 0;\ y(0) = y'(0) = 0,\ y''(0) = 2$

(t) $(D^3 + D^2 + 4D + 4)\, y = 0;\ y(0) = y'(0) = 0,\ y''(0) = -5$

(u) $(D^3 - 6D^2 - 12D + 8)\, y = 0;\ y(0) = y'(0) = 0,\ y''(0) = 2$

2. Solve the following non-homogeneous linear differential equations:

(a) $(D^2 + D + 1)\, y = e^{-x}$

(b) $(D^2 - 3D + 2)\, y = e^{5x}$

(c) $(D^2 - 5D + 6)\, y = \sinh 2x$

(d) $(D^3 + 1)\, y = 5e^x - \cosh x$

(e) $(4D^2 + 4D - 3)\, y = e^{2x}$

(f) $(D^2 + D + 1)\, y = \sin 2x$

(g) $(D^4 - 1)\, y = \cos x$

(h) $(D^2 - 5D + 6)\, y = \sin 3x$

(i) $(D^4 - 2D^2 + 1)\, y = \cos x$

(j) $(D^2 + D - 6)\, y = xe^{2x}$

(k) $(D^3 - 3D - 2)\, y = x^2$

(*l*) $(D^3 - 13D + 12)\, y = x$

(m) $(D^2 - 4)\, y = x^2$

(n) $(D^2 - 2D + 4)\, y = e^x \cos x$

(o) $(D^2 - 5D + 6)\, y = xe^{4x}$

(p) $(D^2 + 1)\, y = x \sin 2x$

(q) $(D - 1)^3\, y = 16e^x$

(r) $(D^3 + 2D^2 - D - 2)\, y = e^x$

(s) $(D^3 - 2D^2 - 5D + 6)y = e^{3x}$

(t) $(D^2 - D - 6)y = e^x \sinh 3x$

(u) $(D^4 + 1)y = x^4$

(v) $(5D^3 - D^2 - 6D)y = x^2$

(w) $(D^3 + 1)y = 2\cos^2 x$

(x) $(D^2 - 3D + 2)y = x^2 e^{4x}$

(y) $(D^3 - 4D^2 + 3D)y = \cos 2x$

(z) $(D^2 - 7D + 10)y = e^{2x}\sin x + xe^x$

3. Solve the following Cauchy-Euler differential equations

(a) $x^2 y'' - xy' + y = \ln x$

(b) $x^2 y'' - 4x y' + 6 y = x^2$

(c) $x^2 y'' - 2x y' - 4 y = x^4$

(d) $x^2 y'' - 3x y' + 4 y = (x + 1)^2 = x^2 + 2x + 1$

(e) $x^2 y`` - 2y = x^2 + (1/x)$ (f) $x^3 y``` + 2x^2 y`` + 2y = 10[x + (1/x)]$

(g) $x^2 y`` + x y` + y = \ln x \sin(\ln x)$ (h) $(2x + 3)^2 y`` - (2x + 3) y` - 12 y = 6x$

(i) $(x + 1)^2 y`` + (x + 1) y` + y = 4 \cos [\ln (x + 1)]$

(j) $(x + 1)^2 y`` + (x + 1) y` + y = \sin 2[\ln (x + 1)]$

(k) $x^2 y``` + 3xy`` + y` = x^2 \ln x$ [Hint: Multiply both sides by x]

(*l*) $x^3 y``` + 3x^2 y`` + xy` + y = x + \ln x$

(m) $y`` + y`/x = 12 \ln x/x^2$ [Hint: Multiply by x^2]

(n) $x^2 y`` - 2x y` + 2y = x^2 + \sin(5\ln x)$

(o) $x^3 y`` + 3x^2 y` + xy = \sin(\ln x)$ [Hint: Divide by x]

4. Solve the following differential equations by the method of variation of parameters:

(a) $y`` + y = \operatorname{cosec} x$ (b) $y`` + a^2 = \sec ax$

(c) $y`` + y = \tan x$ (d) $y`` + y = x \sin x$

(e) $y`` - 6 y` + 9 y = e^{3x}/x^2$ (f) $y`` - 2 y` + 2y = e^x \tan x$

(g) $y`` - y = e^{-2x} \sin (e^{-x})$ (h) $y`` + y = \sin x$

(i) $y`` - 3y` + 2y = \sin x$ (j) $y`` + y = \sec x \tan x$

(k) $y`` + 2y` + 2y = e^{-x} \sec^3 x$ (*l*). $y`` + 4y = \tan 2x$

(m) $y`` - 2y` + 2y = e^x \tan x$ (n) $y`` + y = x - \cot x$

CHAPTER FIVE

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

INTRODUCTION

Like ordinary differential equations, partial differential equations are equations to be solved in which the unknown element is a function of several variables. Again, one generally looks for qualitative statements about the solution. For example, in many cases, solutions exist only if some of the parameters lie in a specific set (say, the set of integers). Various broad families of PDE's admit general statements about the behavior of their solutions. This area has a long-standing close relationship with the physical sciences, especially physics, thermodynamics, and quantum mechanics: for many of the topics in the field, the origins of the problem and the qualitative nature of the solutions are best understood by describing the corresponding result in physics, as we shall do below.

Roughly, corresponding to the initial values in an ODE problem, PDEs are usually solved in the presence of *boundary conditions*. For example, the Dirichlet problem (actually introduced by Riemann) ask for the solution of the Laplace condition on an open subset D of the plane, with the added condition that the value of u on the boundary of D is some prescribed function f. (Physically this corresponds to asking, for example, for the steady-state distribution of electrical charge within D when prescribed voltages are applied around the boundary.) It is a nontrivial task to determine how much boundary information is appropriate for a given PDE.

Linear differential equations occur, perhaps most frequently in applications (in settings in which a superposition principle is appropriate.) When these differential equations are first-order, they share many features with ordinary differential equations. (More precisely, they correspond to *families* of ODEs, in which considerable attention must be focused on the dependence of the solutions on the parameters.)

Historically, three equations were of fundamental interest and exhibit distinctive behavior. These led to the clarification of three types of second-order linear differential equations.

The Laplace Equation $u_{xx} + u_{yy} = 0$ applies to potential energy functions for a conservative force field in the plane. These types of equations are known as ***elliptic* PDEs.**

The Heat Equation $u_t = u_{xx} + u_{yy}$ applies to the temperature distribution u(x, y) in the plane when heat is allowed to flow from warm areas to cool ones. These types of equations are known as parabolic PDEs.

The Wave Equation $u_{tt} = u_{xx} + u_{yy}$ applies to the heights u(x, y) of vibrating membranes and other wave functions. These types of equations are known as ***hyperbolic*** **PDEs**.

The analysis of these three types of equations is quite distinct in character. Allowing non-constant coefficients, we see that solution of a general second-order linear PDE may change character from point to point. These behaviors generalize to nonlinear PDEs as well.

Definition: A differential equation that contains dependent, independent variables and one or more partial derivatives of the dependent variable is called a PDE. In general, it has the form:

$$f\left(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy} \ldots\right) = 0$$

where f is any general function with x an y as independent variables, u is unknown function $u_x, u_y, \ldots u_{xx}, u_{xy}$ are partial derivatives of u and subscripts on dependent variables denote differentiations , e.g.; $u_x = \partial u / \partial x,\ u_y = \partial u / \partial y,\ u_{xx} = \partial^2 u / \partial x^2$ etc.

Thus, (i) $u_{xx} + 2yu_{xy} + 3xu_{yy} = 4\sin x$ (ii) $\left(u_x\right)^2 + \left(u_y\right)^2 = 0$ are examples of PDEs.

Formation of Partial Differential Equations

We know that ordinary differential equations are formed by eliminating arbitrary constants. In contrast, the partial differential equations are formed either by eliminating arbitrary constants or by eliminating arbitrary functions from a relation involving three or more variables.

Example 01: Form the Partial differential equation from $2z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

Solution: Given that $$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \qquad (1)$$

Differentiate (1) w.r.t x partially, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow \frac{\partial z}{\partial x} = \frac{x}{a^2} \Rightarrow \frac{1}{x}\frac{\partial z}{\partial x} = \frac{1}{a^2}$$

Similarly, differentiate (1) w.r.t y partially, to get: $\dfrac{1}{y}\dfrac{\partial z}{\partial y} = \dfrac{1}{b^2}$

Now substituting the values of $1/a^2$ and $1/b^2$ in (1) ,we get

$$2z = \frac{1}{x}\frac{\partial z}{\partial x}x^2 + \frac{1}{y}\frac{\partial z}{\partial y}y^2 \quad \text{or} \quad 2z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

This is required partial differential equation.

Example 02: Form the Partial differential equation by eliminating arbitrary function from: $z = f\left(x^2 + y^2\right)$

Solution: Given that $z = f\left(x^2 + y^2\right)$ (1)

Differentiate (1) w.r.t x partially .we get

$$\frac{\partial z}{\partial x} = f`\left(x^2 + y^2\right).2x \Rightarrow f`\left(x^2 + y^2\right) = \frac{z_x}{2x} \qquad (2)$$

Similarly differentiate (1) w.r.t y partially .we get

$$\frac{\partial z}{\partial y} = f`\left(x^2 + y^2\right).2y \Rightarrow f`\left(x^2 + y^2\right) = \frac{z_y}{2y} \qquad (3)$$

Equating (2) and (3) we obtain: $$\frac{z_x}{2x} = \frac{z_y}{2y} \Rightarrow yz_x - xz_y = 0$$

This is required partial differential equation.

Example 03: Form the partial differential equation from $u = f(x, y, z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$ where, $(x, y, z) \neq 0$.

Solution: Given that $u = f(x,y,z) = \left(x^2 + y^2 + z^2\right)^{-1/2}$ (1)

Differentiate (1) w.r.t x partially, we have

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-3/2}.2x = -x\left(x^2 + y^2 + z^2\right)^{-3/2}$$

Again differentiate w.r.t x partially, we get

$$\frac{\partial^2 u}{\partial x^2} = -\left[\left(x^2 + y^2 + z^2\right)^{-3/2} - \frac{3}{2}x.\left(x^2 + y^2 + z^2\right)^{-5/2}.2x\right]$$

$$\frac{\partial^2 u}{\partial x^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3x^2\left(x^2 + y^2 + z^2\right)^{-5/2} \quad (2)$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3y^2\left(x^2 + y^2 + z^2\right)^{-5/2} \quad (3)$$

And $$\frac{\partial^2 u}{\partial z^2} = -\left(x^2 + y^2 + z^2\right)^{-3/2} + 3z^2\left(x^2 + y^2 + z^2\right)^{-5/2} \quad (4)$$

Now adding (2), (3) and (4) we get

$$u_{xx} + u_{yy} + u_{zz} = -3\left(x^2 + y^2 + z^2\right)^{-3/2} + 3\left(x^2 + y^2 + z^2\right)\left(x^2 + y^2 + z^2\right)^{-5/2}$$

$$= -3\left(x^2 + y^2 + z^2\right)^{-3/2} + 3\left(x^2 + y^2 + z^2\right)^{-3/2}$$

$$\Rightarrow \quad u_{xx} + u_{yy} + u_{zz} = 0$$

Example 04: Form the differential equation from $x^2 + y^2 + (z - c)^2 = a^2$

Solution: Given equation contains two arbitrary constants hence we differentiate it twice. Notice that here z is dependent variable. Now differentiate w.r.t x partially, we get:

$$2x + 0 + 2\,(z - c)\,z_z = 0 \qquad \text{or } (z - c) = -x/z_x \quad (1)$$

Similarly, differentiate partially w.r.t y, we get

$0 + 2y + 2\,(z - c)\,z_y = 0$ or $(z - c) = -y/z_y$ (2)

Equating (1) and (2), we get $-x/\,z_x = -y/z_y$

$$yz_x - xz_y = 0$$

Example 05: Form a partial differential equation by eliminating h and k from

$$(x - h)^2 + (y - k)^2 + z^2 = c^2$$

Solution: Given equation is $(x - h)^2 + (y - k)^2 + z^2 = c^2$ (1)

Differentiate partially w.r.t x and y we get $2(x - h) + 2\,zz_x = 0$ giving $(x - h) = -\,zz_x$

$2(y - k) + 2zz_y = 0$ or $(y - k) = -zz_y$

Substituting the values of $(x - h)$ and $(y - k)$ in equation (1), we get

$(-zz_x)^2 + (-zz_y)^2 + z^2 = c^2$ or $z^2(z_x + z_y + 1) = c^2$

This is the required differential equation.

Example 06: Form a partial differential equation by eliminating an arbitrary function f from the equation: $z = y^2 + 2f(x^{-1} + \ln y)$

Solution: Given equation is $z = y^2 + 2f(x^{-1} + \ln y)$. Differentiate w.r.t x and y, we have

$z_x = 2f'(x^{-1} + \ln y).\,(-x^{-2})$ or $2f'(x^{-1} + \ln y) = -x^2\,z_x$ (1)

$z_y = 2y + 2f'(x^{-1} + \ln y)\,(1/y)$ or $2f'(x^{-1} + \ln y) = yz_x - 2y^2$ (2)

Equating (1) and (2), we get

$-x^2 z_x = yz_x - 2y^2$ or $x^2z_x + yz_x = 2y^2$

This is the required partial differential equation.

Example 07: Form a partial differential equation by eliminating the function from $z = e^{ny} f(x - y)$

Solution: Given equation is $z = e^{ny} f(x - y)$. Differentiate partially w.r.t x and y, we get

$$z_x = e^{ny} f'(x - y) \qquad (1)$$

$$z_y = ne^{ny} f(x - y) + e^{ny} f'(x - y)(-1)$$

or $$z_y - n e^{ny} f(x - y) = - e^{ny} f'(x - y)$$

or $$z_y - nz = -e^{ny} f'(x - y) \qquad (2)$$

Adding (1) and (2) we get: $z_x = z_y - nz$ or $z_x - z_y - nz = 0$

Example 08: Form a partial differential equation by eliminating the functions f and g from $z = f(x + iy) + g(x - iy)$.

Solution: Given equation is $z = f(x + iy) + g(x - iy)$. Differentiate partially w.r.t x twice, we get

$$z_x = f'(x + iy) + g'(x - iy) \text{ and } z_{xx} = f''(x + iy) + g''(x - iy) \qquad (1)$$

Differentiate given equation w.r.t y twice, we get

$$z_y = i.f'(x + iy) - i.g'(x - iy) \text{ and or } z_{yy} = -[f''(x + iy) + g''(x - iy) \qquad (2)$$

Note: $i^2 = -1$. Adding (1) and (2), we get: $z_{xx} + z_{yy} = 0$

This is the required partial differential equation.

CLASSIFICATION OF 2ND ORDER PARTIAL DIFFERENTIAL EQUATIONS

The most general form of second order partial differential equation is:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$$

Here A, B, C are linear functions of independent variable x and y or constants. This equation is:

Elliptic if $B^2 - AC < 0$; Parabolic if $B^2 - AC = 0$; Hyperbolic if $B^2 - AC > 0$

Example 01: Classify the following partial differential equation as Parabolic, Elliptic or hyperbolic.

(i) $\mathbf{u_{xx} + u_{yy} = 0}$ **(ii)** $\mathbf{u_{xx} = a^2 u_t}$ **(iii)** $\mathbf{c^2 u_{xx} = u_{tt}}$

Solution: **(i)** Here A = 1, B = 0 and C = 1. Using $B^2 - 4 A C = 0^2 - 1.1 = -1 < 0$. Therefore, given differential equation is Elliptical equation.

(ii) Writing the given equation in standard form, we get $u_{xx} - a^2 u_t = 0$

Here A = 1 , B = 0 and C = 0. Using $B^2 - 4 A C = 0^2 - 1.0 = 0$. Therefore, given differential equation is parabolic equation.

(iii) Writing the given equation in standard form, we get $u_{xx} - (1/c^2)u_{tt} = 0$

Here A =1, B = 0 and C =$-1/c^2$. Using $B^2 - 4 A C = 0^2 - 1.(-1/c^2) = 1/c^2 > 0$

Therefore, given differential equation is hyperbolic equation.

SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

From above examples, we have observed that partial differential equations can be obtained both by eliminating an arbitrary constants or an arbitrary function.

In other words function $u(x, y, \ldots)$ which satisfy $f(x, y, u, u_x, u_y, u_{xx}, u_{xy} \ldots) = 0$ identically in a suitable domain D of n-dimensional space in n-independent variables x , y ,…. (if exists) is called solution. For instance, consider a partial differential equation

$$u_{xx} - u_{yy} = 0 \qquad (1)$$

We shall show that the functions: $u = (x + y)^3$ and $u = \sin (x + y)$ form the solutions of (1).

For consider, $u = (x + y)^3$ giving $u_x = 3(x + y)^2$ and $u_{xx} = 6(x + y)$

Similarly, $u_y = 3(x + y)^2$ and $u_{yy} = 6(x + y)$.

Therefore, $u_{xx} - u_{yy}= 6(x + y) - 6(x + y) = 0$

This shows that $u = (x + y)^3$ is a solution of partial differential equation (1)

Now consider, $u = \sin (x + y)$

or $u_x = \cos (x + y)$ and $u_{xx} = -\sin (x + y)$

Similarly, $u_y = \cos(x + y)$ and $u_{yy} = - \sin (x + y)$

Thus, $u_{xx} - u_{yy} = - \sin(x + y) + \sin (x + y) = 0$

Since $u = \sin(x + y)$ satisfies equation (1) hence it is also a solution of (1).

This shows that a partial differential equation may possess more than one solution in contrast with the solution of ordinary differential equation which always possesses a unique solution if it exists.

In this section we shall learn how a solution of partial differential equation is computed. There are two methods to find the solution of partial differential equations. They are:

- **(i) Direct Method**
- **(ii) Method of Separable Variables**

Direct Method

Example 01: Solve $z_{xx} = xy$

Solution: Given $z_{xx} = xy$, Integrate w.r.t *x*, keeping y constant, we have

$$z_x = x^2y/2 + f(y)$$

Integrate again integrate w.r.t x, keeping y constant, we have

$$z = x^3y/6 + xf(y) + g(y)$$

This is the solution of given differential equation.

Remark: It may be noted that while finding the solutions of ordinary differential equations we used A, B or c_1, c_2 etc as constants of integration. But in case of partial differential equations, these constant of integrations may be functions of y or x according to as we integrate w.r.t x or y respectively.

Example 02: Solve the PDE $z_{yy} = z$ subject to conditions $y = 0$, $z = e^x$, $z_y = e^{-x}$

Solution: Given that $z_{yy} = z$ or, $z_{yy} - z = 0$

Using D–operator, we get: $(D^2 - 1) z = 0$

Auxiliary equation is: $m^2 - 1 = 0$ giving, $m = -1, +1$. Hence the solution of given PDE is

$$z = f(x) e^y + g(x) e^{-y} \qquad (1)$$

Using $y = 0$ and $z = e^x$, (1) gives: $e^x = f(x) + g(x) \qquad (2)$

Now differentiate (1) w.r.t y partially, we get

$$z_y = f(x) e^y - g(x) e^{-y}$$

Putting $y = 0$ and $z_y = e^{-x}$, we get $e^{-x} = f(x) - g(x) \qquad (3)$

Adding (2) and (3) ,we get: $e^x + e^{-x} = 2 f(x)$ or $f(x) = (e^x + e^{-x})/2 = \cosh x$

Subtracting (2) and (3), we get $e^x - e^{-x} = 2 g(x)$ or $g(x) = (e^x - e^{-x})/2 = \sinh x$

Substituting f (x) and g(x) in (1), we obtain $z = \cosh x\, e^{y} + \sinh x\, e^{-y}$

This is the required solution.

Example 03: Solve the PDE $z_{xx} = a^2 z$ subject to conditions $x = 0$, $z_x = a \sin y$ and $z_y = 0$.

Solution: Given $z_{xx} - a^2 z = 0$ or $(D^2 - a^2) z = 0$. A. E is $m^2 - a^2 = 0$ giving $m = \pm a$. Thus general solution is

$$z = h(y)\, e^{ax} + g(y)\, e^{-ax} \qquad (1)$$

or

$$z_x = ah(y)\, e^{ax} - a\, g(y)\, e^{-ax} \qquad (2)$$

and

$$z_y = h'(y)\, e^{ax} + g'(y)\, e^{-ax} \qquad (3)$$

Putting $x = 0$ and $z_x = a \sin y$, (2) becomes: $\sin y = ah(y) - a\, g(y)$

or $\sin y = h(y) - g(y)$ or $\cos y = h'(y) - g'(y)$ (i)

Putting $x = 0$ and $z_y = 0$, we get

$$0 = h'(y) + g'(y) \qquad \text{(ii)}$$

Solving (i) and (ii) simultaneously, we get $h(y) = g(y) = (1/2) \sin y$. Thus solution of given differential equation becomes: $z = \sin y\, [e^{ax} + e^{-ax}]/2 = \sin y \cosh x$

Example 04: Solve $z_{xxy} = \cos(2x + 3y)$

Solution: Integrate w.r.t x we get: $z_{xy} = \sin(2x + 3y)/2 + f(y)$

Integrate w.r.t x again, we get: $z_y = -\cos(2x + 3y)/4 + xf(y) + g(y)$

Finally integrate w.r.t y, we get: $z = -\sin(2x + 3y)/12 + x \int f(y)\, dy + \int g(y)\, dy$

Or, $z = --\sin(2x + 3y)/12 + x F(y) + G(y)$

Example 05: Solve $z_{xy} = x^2 y$, subject to conditions $z(x, 0) = x^2$ and $z(1, y) = \cos y$

Solution: Integrate w.r.t x we get: $z_y = x^3y/3 + f(y)$

Now, integrate w.r.t y, we get: $z = x^3y^2/6 + \int f(y)\, dy + g(x) = x^3y^2/6 + F(y) + g(x)$ (1)

Put $z = x^2$ and $y = 0$, we get: $x^2 = 0 + F(0) + g(x)$ or $g(x) = x^2 - F(0)$

Put this in (1) we get: $z = x^3y_2/6 + F(y) + x^2 - F(0)$ (2)

Now putting $z = \cos y$ and $x = 1$ in (2), we get: $\cos y = y^2/6 + F(y) + 1 - F(0)$

or $F(y) = \cos y - y^2/6 - 1 + F(0)$

Put this in (2), we get: $z = x^3 y^2/6 + \cos y - y^2/6 - 1 + F(0) + x^2 - F(0)$

or $z = x^3y^2/6 + \cos y - y^2/6 - 1 + x^2$

Separable Variable Method

Separation of variables is one of the oldest and most efficient method for a certain class of partial differential equations. The general idea is one of the oldest techniques learned when studying first order ODEs .When applied to partial differentials, this method is similar in some respects to ODEs but eventually take on a whole new form. The theme here is "To reduce given partial differential equation into number of ODEs and then construct the solutions of the governing ODEs."

If u be the dependent variable and x, y two independent variables in the differential equation, then we assume the solution to be the product of independent variables, one of them is a function of x and other one is a function of y alone. In this way, the solution of differential equation is converted into the solution of ordinary differential equations.

Example 05: Solve $z_{xx} - 2z_x + z_y = 0$ by separation of variable method.

Solution: Here we see that z is dependent and x, y are independent variables, so we assume the solution is: $z = X(x)\, Y(y)$ (1)

where X is function of x alone and Y is function of y alone. Differentiate (1) twice w.r.t x partially, we get: $z_x = X`Y$ and $z_{xx} = X``\, Y$

Now differentiate (1) w.r.t y partially, we get: $z_y = XY`$

Substituting these in the given differential equation, we get

$$X``\, Y - 2\, X`Y + XY` = 0 \quad \text{or} \quad (X`` - 2X`)\, Y = -XY`$$

Separating the variables, we get: $(X`` - 2X`)/X = -\, Y`/Y$ (2)

Since x and y are independent variables, therefore (2) can be true if each side is equal to same constant, k . That is; $(X`` - 2X`)/X = -\, Y`/Y = k$

or $(X`` - 2X`)/X = k$ (3)

and $-\, Y`/Y = k$ (4)

Now consider, $(X`` - 2X`) - kX` = 0$. Using D operator, we get

$$(D^2 - 2D - k)\, x = 0$$

The auxiliary equation is $m^2 - 2m - k = 0$ or $m = 1 \pm \sqrt{1+2k}$

Hence the solution of (3) is $X = c_1 e^{\left(1+\sqrt{1+2k}\right)x} + c_2 e^{\left(1-\sqrt{1+2k}\right)x}$

Now from (4) $-Y`/Y = k$

Integrating w.r.t y, we get: $\ln Y = -\, ky + c$ or $Y = e^{-ky + c} = e^c\, e^{-ky} = c_3\, e^{-ky}$

Substituting values of X and Y in (1) ,we get

$$z = \left[c_1 e^{\left(1+\sqrt{1+2k}\right)x} + c_2 e^{\left(1-\sqrt{1+2k}\right)x}\right].c_3 e^{-ky} = \left[c_1 c_3 e^{\left(1+\sqrt{1+2k}\right)x} + c_2 c_3 e^{\left(1-\sqrt{1+2k}\right)x}\right] e^{-ky}$$

$$z = \left[p e^{\left(1+\sqrt{1+2k}\right)x} + q e^{\left(1-\sqrt{1+2k}\right)x}\right] e^{-ky}, \text{ where } p = c_1 c_3 \text{ and } q = c_2 c_3$$

This is the required solution.

Example 06: Use separation of variable method to solve $z_x = 2z_t + z$, given the condition $z(x, 0) = 6e^{-3x}$.

Solution: Here we see that z is dependent and x, t are independent variables, so we assume the solution: $z = X(x)\, T(t)$ (1)

where X is function of x alone and T is function of t alone.

Differentiate (1) w.r.t x and y partially, we get: $z_x = X`T$ and $z_{\mathbf{t}} = XT`$

Substituting these in given differential equation, we get

$$X`T = 2XT` + XT \text{ or } (X` - X)T = 2XT`$$

Separating the variables, we get $(X` - X)/2X = T`/T$ (2)

Since x and t are independent variables, therefore (2) can be true if each side is equal to same constant k.

$$(X` - X)/2X = k = T`/T$$

or $X` - X = 2kX$ or $X` - (1 + 2k)X = 0$ (3)

and $T` = Tk$ (4)

From (3) we have $X`/X = (1 + 2k)$

Integrating both sides, we get $\ln X = (1 + 2k)x + c$ or $X = c_1 e^{(1+2k)x}$ $(c_1 = e^c)$.

From (4) $T` / T = k$

Integrating both side, we obtain: $\ln T = kt + d$ or $T = c_2 e^{kt}$ $(c_2 = e^d)$

Substituting values of X and T in (1) ,we get

$$z = XY = c_1 e^{(1+2k)x} . c_2 e^{kt} = pe^{(1+2k)x+kt} \text{ where } p = c_1 c_2. \qquad (5)$$

Now using the condition $z(x, 0) = 6e^{-3x}$ i.e, put $z = 6e^{-3x}$ and $t = 0$, we get

$$6e^{-3x} = p\, e^{(1 + 2k)\, x}$$

Equating both sides, we get $p = 6$ and $1 + 2k = –3$ or $2k = –4$ giving $k = –2$.

Putting these values in (5), we get: $z = 6\, e^{-(3x + 2t)}$

This is the required solution of given differential equation.

Equation of Vibrating String (Wave Equation)

Consider an elastic string tightly stretched between two points O and A. Let O be the origin and OA as x–axis. On giving a small displacement to the string, perpendicular to its length (parallel to the y-axis), let u be the displacement at the point P(x, y) at any time.

The resulting equation is the Wave equation. (Fig. below).

$$u_{tt} = c^2\, u_{xx}$$

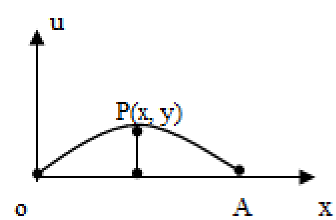
Solution: Assuming the solution $u = X(x)\, T(t)$ where X is function of x alone and T is a Function of t alone. Then $u_{xx} = X``T$ and $u_{tt} = XT``$. Putting these in given differential equation, we get: $XT`` = c^2\, X``\, T$. Separating the variables giving $X``/X = T``/c^2T$.

Since x and t are independent variables, therefore, (2) can be true if each side is equal to same constant k.

Thus, $X``/X = k$ and $T``/Tc^2 = k$

or $X`` – kX = 0$ (1)

and $T`` – kc^2T = 0$ (2)



Now k may be positive, negative or zero. Thus we shall consider three cases:

Case I: Let k be positive, so that $k = p^2$, then equations (1) and (2) become:

$$X`` – p^2\, X = 0 \text{ and } T`` – p^2c^2\, T = 0$$

The auxiliary equations for D.E $X`` – p^2\, X = 0$ is $m^2 – p^2 = 0$

The auxiliary equation for D.E $T`` – p^2c^2\, T = 0$ is $m^2 – p^2\, c^2 = 0$

This gives: $m = \pm p$ and $m = \pm cp$

The respective solutions are therefore, $X = (c_1\, e^{px} + c_2 e^{-px})$ and $Y = (c_3 e^{cpt} + c_4\, e^{-cpt})$

Hence the solution of given wave equation is: $u = XT = (c_1\, e^{px} + c_2 e^{-px})\, (c_3 e^{cpt} + c_4\, e^{-cpt})$

Case II: Let k be negative, so that $k = –p^2$, then equations (1) and (2) become:

$$X`` + p^2\, X = 0 \text{ and } T`` + p^2\, c^2\, T = 0$$

The auxiliary equation for D.E $X`` + p^2\, X = 0$ is $m^2 + p^2 = 0$ or $m = \pm\, ip$

And auxiliary equation for D.E $T`` + p^2\, c^2\, T = 0$ is $m^2 + p^2\, c^2 = 0$ or $m = \pm\, icp$

This gives $X = (c_1 \cos px + c_2 \sin px)$ and $T = (c_3 \cos cpt + c_4 \sin cpt)$

Thus, solution of wave equation is: $u = XT = (c_1 \cos px + c_2 \sin px)\, (c_3 \cos cpt + c_4 \sin cpt)$

Case III: If k is zero, then equations (3) and (4) become: $X'' = 0$ and $Y'' = 0$

Integrating twice both equations, we get $u = XY = (c_1 + c_2x)(c_3 + c_4t)$

Hence the various possible solutions of wave equation are

$$u = (c_1 e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4 e^{-cpt}) \qquad (3)$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3\cos cpt + c_4 \sin cpt) \qquad (4)$$

$$u = (c_1 + c_2x)(c_3 + c_4t) \qquad (5)$$

From these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. In dealing with the problems on vibrations, u must be a periodic in x and t. Hence this solution must involve sinusoid function. The solution given by (4) is, therefore, a proper solution of given wave equation.

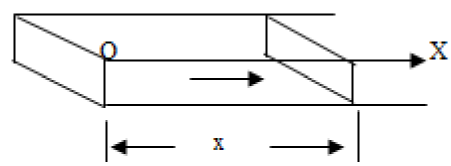
Remark: Readers may find the derivation of this equation in any advanced book on partial differential equations.

One Dimensional Heat Flow

Let heat flow along a bar of uniform cross – section, in the direction perpendicular to the cross–section. Take one end of the bar as origin and the direction of heat flow along x–axis. Let the temperature of the bar at any time t at point x distance from the origin be u(x, t). Then the equation of one dimensional heat flow is $u_t = c^2 u_{xx}$

Remark: The derivation of this equation can be found in any advanced book on PDEs.

Let us assume that u = XT, where X is a function of x alone and T is a function of t alone. Therefore,



$$u_t = XT' \text{ and } u_{xx} = X''T$$

Thus given equation becomes: $XT' = c^2 TX'' = k$

Thus, $$X'' - kX = 0 \qquad (1)$$

And $$T' - kc^2T = 0 \qquad (2)$$

Now k may be positive, negative or zero. Thus we shall consider three cases:

Case I: Let k be positive, so that $k = p^2$, then equations (1) and (2) become:

$$X'' - p^2 X = 0 \text{ and } T' - p^2c^2 T = 0$$

The auxiliary equations for D.E: $X'' - p^2 X = 0$ is $m^2 - p^2 = 0$ giving $m = \pm p$

The auxiliary equations for D.E: $T' - p^2 c^2 T = 0$ is $m - p^2 c^2 = 0$ giving $m = p^2c^2$

Thus, $$X = (c_1e^{px} + c_2e^{-px}) \text{ and } T = \left(c_3e^{c^2p^2t}\right)$$

Hence the solution of given wave equation is: $u = XT = (c_1e^{px} + c_2e^{-px})\left(c_3e^{c^2p^2t}\right)$

Case II: Let k be negative, so that $k = -p^2$, then equations (1) and (2) become:

$$X'' + p^2 X = 0 \text{ and } T' + p^2c^2 T = 0$$

The auxiliary equations for D.E: $X'' + p^2 X = 0$ is $m^2 + p^2 = 0$ giving $m = \pm ip$

The auxiliary equations for D.E: $T' + p^2 c^2 T = 0$ is $m + p^2 c^2 = 0$ giving $m = -c^2p^2$

The solutions of the respective differential are therefore,

Hence the solution of given wave equation is: $u = XT = (c_1 \cos px + c_2 \sin px)\ c_3e^{-c^2p^2t}$

Case III: If k is zero, then equations (1) and (2) become: $X`` = 0$ and $T` = 0$

Integrating two times first equation and one time the second equation, we get the solution as:

$$u = XY = (c_1 + c_2x)\,(c_3) = (A + Bx);\ \text{where } A = c_1c_3 \text{ and } B = c_2c_3$$

Hence the various possible solutions of wave equation are

$$u = (c_1e^{px} + c_2e^{-px})\left(c_3e^{c^2p^2t}\right) \quad (3)$$

$$u = (c_1\cos px + c_2 \sin px)\ c_3e^{-c^2p^2t} \quad (4)$$

$$u = (A + Bx) \quad (5)$$

From these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. In dealing with the problems on one dimensional heat flow we consider that solution where the temperature u rises or drops as the time increases or decreases. Thus (3) is the solution when the body is initially heated and solution (4) is considered when the heat source is removed away from the body. Solution (5) is of no use.

Remark: Readers may find the derivation of this equation in any advanced book on partial differential equations.

Example 07: Use the method of separation of variables to solve the equation $u_{xx} = u_t$ given that $u = 0$ at $x = 0$ and $x = l$.

Solution: We see that given partial differential equation is one dimensional heat flow equation with $c^2 = 1$. Putting $u = XT$ where $X = X(x)$ and $T = T(t)$, we get: $u_{xx} = X``T$ and $u_t = XT`$.

Thus given differential equation becomes $X``T = XT`$. Separating the variables, we get

$$X`` / X = T`/T = k = -p^2$$

or $$X``/X = -p^2 \quad \text{and} \quad T`/T = -p^2$$

or $$X`` + p^2 X = 0 \quad \text{and } T` + p^2 T = 0$$

Notice that we have considered k as negative as the given equation is ***Heat Flow Equation.*** The respective auxiliary equations are:

For D.E: $X`` + p^2 X = 0$. The A. E is $m^2 + p^2 = 0$ and the roots are $m = \pm\, ip$

For D.E: $T`` + p^2 T = 0$. The A. E is $m + p^2 = 0$ and the root is $m = -\,p^2$

Thus solution is: $$u = (c_1\cos px + c_2\sin px)\left(c_3e^{-p^2t}\right)$$

Or $$u = e^{-p^2t}\ (A \cos px + B \sin px) \quad (1)$$

[NOTE: $A = c_1.c_3$ and $B = c_2.c_3$]

Now let us use the initial conditions: Putting u = 0 and x = 0 in (2), we get:

$$A\ e^{-p^2t} = 0 \quad \text{or} \quad A = 0 \quad (2)$$

Thus $$u = B\ e^{-p^2t} \sin px \quad (3)$$

Now put $x = l$ and $u = 0$, we get: $0 = B\, e^{-p^2t} \sin pl$ or $\sin pl = 0$ or $pl = n\pi$ giving $p = n\pi/l$.

Thus (3) becomes: $$u = B\ e^{-n^2\pi^2t/l} \sin(n\pi/l)\ x \quad (4)$$

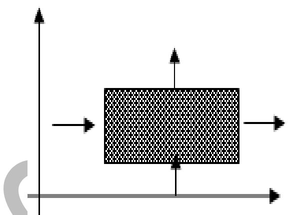
Equation (4) is the solution of given one-dimensional heat equation for all n = 1, 2, 3, … Hence the most general solution is

$$u = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t/l} \sin\frac{n\pi x}{l}$$

Two Dimensional Heat Flow

Consider the heat flow in a metal plate of uniform thickness, in the directions parallel to length and breadth of the plate assuming that there is no heat flow along the normal to the plane of the rectangle. Let u(x, y) be the temperature at any point (x, y) of plate at time t. (See figure below). The equation governing this problem is given by:

$$u_t = c^2(u_{xx} + u_{yy}) \qquad (1)$$



In the steady state, u does not change with t. Therefore, $\partial u / \partial t = 0$.

Thus equation (1) becomes: $\quad u_{xx} + u_{yy} = 0 \qquad (2)$

This is called ***Laplace equation*** in two dimensions.

To solve this, Let us assume that u = XY be the solution of (2) where X is a function of x alone and Y is a function of y alone. Therefore, $u_{xx} = X''Y$ and $u_{yy} = XY''$. Substituting in (2) and separating the variables, we get:

$$X''/X = -Y''/Y = k$$

Thus, $\quad X'' - kX = 0 \qquad (3)$

And $\quad Y'' + kY = 0 \qquad (4)$

Now k may be positive, negative or zero. Thus we shall consider three cases:

Case I: Let k be positive, so that $k = p^2$, then equations (3) and (4) become:

$$X'' - p^2 X = 0 \text{ and } T'' + p^2 T = 0$$

The auxiliary equation for D.E: $X'' - p^2 X = 0$ is $m^2 - p^2 = 0$ giving $m = \pm p$

The auxiliary equation for D.E: $Y'' + p^2 T = 0$ is $m^2 + p^2 = 0$ giving $m = \pm ip$

Thus, solutions respectively are $X = (c_1 e^{px} + c_2 e^{-px})$ and $Y = (c_3 \cos py + c_4 \sin py)$

Hence the solution of given wave equation is: $\quad u = XY = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py)$

Case II: Let k be negative, so that $k = -p^2$, then equations (3) and (4) become:

The auxiliary equation for D.E: $X'' + p^2 X = 0$ is $m^2 + p^2 = 0$ giving $m = \pm ip$

The auxiliary equation for D.E: $Y'' - p^2 T = 0$ is $m^2 - p^2 = 0$ giving $m = \pm p$

Thus, solutions respectively are $X = (c_1 \cos py + c_2 \sin py)$ and $Y = (c_3 e^{px} + c_4 e^{-px})$

Hence the solution of given wave equation is: $\quad u = XY = (c_1 \cos py + c_2 \sin py)(c_3 e^{px} + c_4 e^{-px})$

Case III: If k is zero, then equations (3) and (4) become:

$$X'' = 0 \qquad \text{and} \qquad T'' = 0$$

Integrating twice both equations, we get $\quad u = XY = (c_1 + c_2 x)(c_3 + c_4 y)$

Hence the various possible solutions of wave equation are

$$u = (c_1 e^{px} + c_2 e^{-px})(c_3 \cos py + c_4 \sin py) \qquad (5)$$

$$u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py}) \qquad (6)$$

$$u = (c_1 + c_2 x)(c_3 + c_4 y) \qquad (7)$$

From these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. Thus (5) and (6) are the required solutions. Solution (7) is of no use.

Remark: Readers may find the derivation of this equation in any advanced book on partial differential equations.

WORKSHEET 05

1. Form the partial differential equations by eliminating arbitrary constants.

NOTE: a, b and c are arbitrary constants.

(a) $z = ax + by + a^2 + b^2$

(b) $z = (a + x)(b + y)$

(c) $(x - a)^2 + (y - b)^2 + z^2 = c^2$

(d) $2z = (ax + y)^2 + b$

(e) $ax^2 + by^2 + z^2 = 1$

(f) $x^2 + y^2 = (z - c)^2 \tan^2 a$

(g) $z = ax + by + ab$

(h) $z = a(x + y) + b$

(i) $z = ax + a^2 y^2 + b$

(j) $z = ax e^y + (1/2) a^2 e^{2y} + b$

(k) $z = a e^{bx} \sin by$

(*l*) $az + b = a^2 x + y$

2. Form the partial differential equation by eliminating arbitrary functions.

NOTE: f and g are arbitrary functions.

(a) $z = f(x^2 - y^2)$

(b) $z = x f(y) + y g(x)$

(c) $z = f(x/y)$

(d) $z = f(x \cos a + y \sin a - a)$

3. Classify the following partial differential equations

(a) $u_{xx} + u_{xy} + u_{yy} = 0$

(b) $xu_{xx} + y u_{xy} + u_{yy} = 0$

(c) $-u_{xx} + x^2 u_{yy} + u = 0$

(d) $xu_{xx} + 2u_{xy} + yu_{yy} + u_x = 0$

(e) $z_{xx} + z_y = 0$

(f) $a^2 z_{xx} = z_y$

4. Solve the following partial differential equations by direct method

(a) $z_{xx} = xy$

(b) $z_{xxy} = -18xy^2 - \sin(2x - y)$

(c) $z_{xy} = e^y \cos x$

(d) $z_{xy} = y/x + 2$

(e) $z_{xx} = a^2 z$ given that $z_x = a \sin y$ and $z_y = 0$ when $x = 0$

(f) $z_{yy} = z$ subject to conditions $z = e^x$ and $z_y = e^{-x}$ when $y = 0$

(g) $z_{xy} = \sin x \sin y$ subject to $z_y = -2 \sin y$ and $z = \cos y$ when $x = 0$

5. Solve the following partial differential equations by the method of separable variables.

(a) $z_x y^3 + z_y x^2 = 0$

(b) $4z_x + z_y = 0$, $z(0, y) = 3e^{-y}$

(c) $3z_x + 2z_y = 0$ given that $z(x, 0) = 3e^{-x}$

(d) $u_x + u = u_t$ if $u = 4e^{-3x}$ when $t = 0$

(e) $u_x + u_y = 2(x + y)u$

(f) $u_t = 4u_{xx}$

(g) $u_{xx} = 2u_t$

(h) $u_{xx} = u_t$

(i) $u_t = u_{xx}$ if $u(x, 0) = \sin \pi x$

(j) $z_{xx} - 2z_x - z_y = 0$

(k) $x^2 u_{xx} + 3y^2 u_y = 0$

(*l*) $u_{tt} = u_{xx}$

CHAPTER SIX

SOLUTIONS IN SERIES

INTRODUCTION

There are many differential equations which may not be solved by the methods that have been discussed in earlier. In such situations we must seek other methods for the solutions of differential equations. One such method is furnished by infinite series representation. The present chapter is devoted to a method of obtaining solutions in series form. The solution so obtained would be called as series solution of given differential equation. The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's polynomial, Laguerre's polynomial, Hermit's polynomial, Chebyshev polynomials and so on. These special functions have many applications in engineering.

Power Series Solutions of Differential Equations

In chapter three we learnt that solution of differential equation $y'' - y = 0$ is:

$$y = A\,e^{x} + B\,e^{-x} \qquad (1)$$

We also know that $e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ and $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

Hence (1) becomes:

$$y = A\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) + B\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right) \qquad (2)$$

Solution (1) of differential equation $y'' - y = 0$ is in function form where as (2) is known as series solution of this differential equation.

Consider another differential equation $y'' + y = 0$. Its solution is

$$y = A \cos x + B \sin x \qquad (3)$$

Now we know that: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$ and $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$

Thus (3) may also be expressed as:

$$y = A\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right) + B\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right) \qquad (4)$$

Equation (4) is series solution of differential equation $y'' + y = 0$ where as the solution (3) is the functional form.

In fact solutions of differential equations discussed in chapter three are all in functional form and each function presented in such solution can be converted into infinite series as shown above in two examples. But it is preferable to have a solution in functional form. However, when it is not possible to get the solution in functional form because of the nature of differential equation the only way is to find the solution in ***"series solution"*** form.

REMARK: In this section we shall only discuss the solution of homogeneous linear differential equations of second order.

Power Series Method

The most general form of second order homogeneous linear differential equation is:

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad (1)$$

Where P_0, P_1 and, P_2 are polynomials in x. Now before we discuss the procedure of finding the series solution, we must understand the following definitions which are very much important for our work.

1. Ordinary Point: In (1), if $P_0(x)$ does not vanish for x = 0 then x = 0 is called an ordinary point.

2. Singular Point: In (1) if $P_0(x)$ vanishes for x = 0 then x = 0 is a singular point. Now there are two types of singularities.

Regular Singular Point and Irregular Singular Point

In equation (1) assume that at least one of the functions P_1/P_0 and P_2/P_0 is not analytic, that is; $P_1/P_0 = \infty$ or $P_2/P_0 = \infty$ at x = 0. Also let $Q_1(x) = xP_1/P_0$, $Q_2(x) = x^2P_2/P_0$. If Q_1 and Q_2 are analytic (not ∞) at x = 0 then x = 0 is a ***Regular Singular Point***, otherwise, it is called ***Irregular or Essential Singular Point***.

Example 01: Find the nature of singularity for the differential equation:

$$2x^2 y'' + 3x y' + (x^2 - 4) y = 0 \qquad (1)$$

Solution: Here $P_0 = 2x^2$ which is zero when x = 0. Hence, x = 0 is a singular point. Now dividing both sides by $2x^2$, we get $y'' + \frac{3}{2x}y' + \frac{x^2-4}{2x^2}y = 0$

Here $P_1/P_0 = 3/2x$ and $P_2/P_0 = (x^2-4)/2x^2$ are both infinite at x = 0, thus P_1/P_0 and P_2/P_0 are not analytic at x = 0. Now, consider

$$Q_1 = x(P_1/P_0) = x(3/2x) = 3/2, \text{ and } Q_2 = x^2(P_2/P_0) = (x^2-4)/2$$

Both Q_1 and Q_2 are analytic $(Q_1 \neq \infty, Q_2 \neq \infty)$ at x = 0. Hence, x = 0 is a regular singular point of equation (1).

Example 02: Find the nature of singularity for the differential equation:

$$x^2 (x-2)^2 y'' + 2(x-2) y' + (x+3) y = 0$$

Solution: Here $P_0 = x^2(x-2)^2$ which is zero when x = 0. Hence, x = 0 is a singular point. Now dividing both sides by $x^2(x-2)^2$ we get

$$y'' + \frac{2(x-2)}{x^2(x-2)^2}y' + \frac{x+3}{x^2(x-2)^2}y = 0 \Rightarrow y'' + \frac{2}{x^2(x-2)}y' + \frac{x+3}{x^2(x-2)^2}y = 0$$

Here, $\frac{P_1}{P_o} = \frac{2}{x^2(x-2)}$ and $\frac{P_2}{P_o} = \frac{x+3}{x^2(x-2)^2}$

At x = 0 and at x = 2, $P_1 = \infty$ and $P_2 = \infty$. Thus P_1 and P_2 are not analytic at x = 0 and x = 2.

Now consider the point x = 0: $Q_1 = xP_1/P_0 = 2/x(x-2)$

And $Q_2 = x^2 P_2/P_0 = (x+3)/(x-2)^2$

Since Q_1 is not analytic ($Q_1 = \infty$) at $x = 0$ hence, $x = 0$ is irregular singular point.

Consider the point $x = 2$; $Q_1 = (x - 2)\, P_1/P_0 = 2/x^2$

And $Q_2 = (x - 2)^2\, P_2/P_0 = (x + 3)/x^2$

Since Q_1 and Q_2 are analytic ($Q_1 \neq \infty$, $Q_2 \neq \infty$) at $x = 2$ hence, $x = 2$ is regular singular point.

Remark: In what follows, we shall consider only those differential equations of order 2 where $x = 0$ is either an ordinary point or a regular singular point.

Solution Series at Ordinary Point

When $x = 0$ is an ordinary point, the solution series method involves the following steps:

Step-I Consider $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_k x^k + \ldots$ or $y = \sum_{k=0}^{\infty} a_k x^k$ be the solution of the given differential equation.

Step-II Find y\` and y\`\`, where $y` = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + k a_k x^{k-1} + \ldots = \sum_{k=1}^{\infty} k a_k x^k$

$$y`` = 2a_2 + 6a_3 x + \ldots + k(k-1) a_k x^{k-2} + \ldots = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

Step-III Substituting the values of y, y\` and y\`\` in the given differential equation.

Step-IV Calculate $a_2, a_3, a_4, \ldots$ in terms of a_0 and a_1 by equating the coefficients similar powers of x to zero.

Step-V Substitute the values of a_2, a_3, a_4, … in the series for y. This will give a series solution of given differential equation.

Example 03: Determine a series solution for the following differential equation y\`\` + y = 0.

Solution: Notice that here $P_0(x) = 1$ and so $x = 0$ is an ordinary point. We will be looking for a solution of the form,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \cdot$$

or

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}, \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Substituting these in given equation, we obtain $\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$

The next step is to combine every term into a single series. To do this we require that both series should start at the same point and that the exponent on the x be the same in both series. It is better to get the exponent to be an n. The second series already has the proper exponent and the first series will need to be shifted up by 2 in order to get the exponent up to an n. Shifting the first power series by 2, we get,

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

or

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} + a_n \right] x^n = 0$$

Equating the coefficient to zero, we get: $(n+2)(n+1) a_{n+2} + a_n = 0, \; n = 0, 1, 2, \ldots$

$$a_{n+2} = -\frac{a_0}{(n+2)(n+1)}, \qquad n = 0, 1, 2, \ldots$$

This is called the recurrence relation. To find a_2, a_3, etc; we get

Put n = 0, we get: $a_2 = \frac{-a_0}{2.1}$. Put n = 1, we get: $a_3 = \frac{-a_1}{3.2}$,

Put n = 2, we get: $a_4 = \frac{-a_2}{4.3} = \frac{a_0}{4.3.2.1}$ Put n = 3, we get: $a_5 = \frac{-a_3}{5.4} = \frac{a_1}{5.4.3.2}$

Put n = 4, we get: $a_6 = \frac{-a_4}{6.5} = \frac{-a_0}{6.5.4.3.2.1}$ Put n = 5, we get: $a_7 = \frac{-a_5}{7.6} = \frac{-a_1}{7.6.5.4.3.2}, \ldots$

Continuing the process, we get

For $n = 2k,\ a_{2k} = \frac{(-1)^k a_0}{(2k)!},\ k = 1, 2, \ldots$ and $n = 2k+1,\ a_{2k+1} = \frac{(-1)^k a_1}{(2k+1)!},\ k = 1, 2, \ldots$

Notice that at each step we always plugged back the values of a_2, a_3, a_4, … etc in the previous results. In this case, when the subscript is even we write a_n in terms of a_0 and when the subscript is odd we write a_n in term of a_1. Do not get excited about a_0 and a_1. As you will see, we actually need these to be in the problem to get the correct solution. Now substituting the values of a_2, a_3, etc in the series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_{2k} x^{2k} + a_{2k+1} x^{2k+1} + \ldots$$

$$= a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \ldots + \frac{(-1)^k a_0}{(2k)!} x^{2k} + \frac{(-1)^k a_1}{(2k+1)!} x^{2k+1} + \ldots$$

The next step is to collect all the terms with the same coefficient and then factor out the coefficients.

$$y(x) = a_0 \left\{ 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^k}{(2k)!} x^{2k} + \ldots \right\} + a_1 \left\{ x - \frac{x^3}{3!} + \ldots + \frac{(-1)^k}{(2k+1)!} x^{2k+1} + \ldots \right\}$$

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = a_0 \cos x + a_1 \sin x$$

In the last step the solution in series is expressed in general form.

You may observe that last two series are series of functions cos x and sin x respectively. Thus general solution of given differential equation is: $y = a_0 \cos x + a_1 \sin x$. Now let's work on an example with non constant coefficients which; will prove that why solutions in series are useful.

Example 04: Find a series solution around for the following differential equation $y'' + xy = 0$.

Solution: We may observe that differential equation of this type is never discussed before. As there is no method to solve it, the only way to find its solution is to use series solution method. Now, $P_0(x) = 1$ and so $x = 0$ is an ordinary point. Thus we assume that the solution of differential equation is:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_n x^n + \ldots \qquad (1)$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n a_n x^{n-1} + \ldots$$

$$y'' = 2a_2 + 3 \cdot 2 a_3 x + 4 \cdot 3 a_4 x^2 + \ldots + n(n-1) a_n x^{n-2} + \ldots$$

Substituting these in (1), we get, $\left\{2a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+...+n(n-1)a_nx^{n-2}+...\right\}$

$$+x\left\{a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+...+a_nx^n+...\right\}=0$$

$$\Rightarrow \left\{2\cdot 1a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+...+n(n-1)a_nx^{n-2}+...\right\}$$

$$+\left\{a_0x+a_1x^2+a_2x^3+a_3x^4+a_4x^5+...+a_nx^{n+1}+...\right\}=0$$

or $2\cdot 1a_2+(3\cdot 2a_3+a_0)x+(4\cdot 3a_4+a_1)x^2+(5\cdot 4a_5+a_2)x^3+...\left\{(n+1)(n+1)a_{n+2}+a_{n-1}\right\}x^n+...=0$

Equating to zero the coefficients of various powers of x, we get

$2\cdot 1a_2=0\Rightarrow a_2=0,\ 3\cdot 2a_3+a_0=0\Rightarrow a_3=-a_0/6,\ 4\cdot 3a_4+a_1=0\Rightarrow a_4=-a_1/12,$

$5\cdot 4a_5+a_2=0\Rightarrow a_5=0$ $(\text{Since } a_2=0)$ and so on.

In general, $(n+1)(n+1)a_{n+2}+a_{n-1}=0 \Rightarrow a_{n+2}=-\dfrac{a_{n-1}}{(n+1)(n+2)}$

This is called recurrence relation.

For n = 4, $a_6=-a_3/5.6=a_0/180$. For n = 5 $a_7=-a_4/6.7=a_1/504$

For n = 6 $a_8=-\dfrac{a_5}{7.8}=0$ (Since $a_5=0$) and so on. Substituting these values in (1), we get

$$y=a_0+a_1x-\frac{1}{6}a_0x^3-\frac{1}{12}a_1x^4+\frac{1}{180}a_0x^6+\frac{1}{504}a_1x^7+...$$

$$y=a_0\left(1-\frac{1}{6}x^3+\frac{1}{180}x^6+...\right)+a_1\left(x-\frac{1}{12}x^4+\frac{1}{504}x^7+...\right).$$

This the solution of given differential equation in series form.

Example 05: Find the of the series solution of differential equation $(1 + x^2)\, y`` + xy` - y = 0$

Solution: Here, $P_0(x)=1+x^2$ and $P_0(0)=1\neq 0$. So x = 0 is an ordinary point for this differential equation. Therefore, the solution is of the form

$$y=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+...+a_nx^n+... \qquad (1)$$

$$y`=a_1+2a_2x+3a_3x^2+4a_4x^3+...+na_nx^{n-1}+...$$

$$y``=2a_2+3\cdot 2a_3x+4\cdot 3a_4x^2+...+n(n-1)a_nx^{n-2}+...$$

Substituting y, y` and y`` in given differential equation, we obtain,

$$(1+x^2)\left(2a_2+6a_3x+12a_4x^2+20a_5x^3+...\right)$$

$$+x\left(a_1+2a_2x+3a_3x^2+4a_4x^3+...\right)-\left(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+...\right)=0$$

or $2a_2+6a_3x+12a_4x^2+20a_5x^3+2a_2x^2+6a_3x^3+12a_4x^4+20a_5x^5$

$$+a_1x+2a_2x^2+3a_3x^3+4a_4x^4-a_0x-a_1x^2-a_2x^3-a_3x^4-a_4x^5-...=0$$

$$\Rightarrow (2a_2-a_0)+(6a_3+a_1-a_1)x+(12a_4+2a_2+2a_2-a_2)x^2$$

$$+(20a_5+6a_3+3a_3-a_3)x^3+...=0$$

or $(2a_2-a_0)+(6a_3)x+(12a_4+3a_2)x^2+(20a_5+8a_3)x^3+...=0$

Equating the coefficients of various powers of x to zero, we obtain

Coefficient of $x^0 : 2a_2 - a_0 = 0 \Rightarrow a_2 = a_0 / 2$ Coefficient of $x : 6a_3 = 0 \Rightarrow a_3 = 0$

Coefficient of $x^2 : 12a_4 + 3a_2 = 0 \Rightarrow 4a_4 = -a_2 \Rightarrow a_4 = -a_0 / 8$

Coefficient of $x^3 : 20a_5 + 8a_3 = 0 \Rightarrow 20a_5 = -8a_3 = 0 \ \because a_3 = 0$

Thus, $y = a_0 + a_1x + \frac{1}{2}a_0x^2 - \frac{1}{8}a_0x^4 + ... = a_0\left(1 + \frac{x^2}{2} - \frac{x^4}{8} + ...\right) + a_1x$

This is series solution of given differential equation.

Example 06: Solve $y'' + (x-1)^2 y' - 4(x-1)y = 0$ in the series about $x = 1$

Solution: Putting $z = x - 1 \Rightarrow \frac{dz}{dx} = 1$. Now, $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx} = \frac{dy}{dz}.1 = \frac{dy}{dz}$

Also, $\frac{d^2y}{dx^2} = \frac{d}{dx}.\frac{dy}{dx} = \frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right).\frac{dz}{dx} = \frac{d^2y}{dz^2}.1 = \frac{d^2y}{dz^2}$.

Therefore, $\frac{dy}{dx} = \frac{dy}{dz}$ and $\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2}$

Thus given equation becomes: $y'' + z^2 y' - 4z\, y = 0$ where $y' = dy/dz$.

Notice the change in the problem and technique being used to solve it.

$$\text{Let } y(z) = \sum_{n=0}^{\infty} a_n z^n \Rightarrow y'(z) = \sum_{n=1}^{\infty} na_n z^{n-1} \text{ and } y''(z) = \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}$$

Substituting these in given equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=1}^{\infty} na_n z^{n+1} - 4\sum_{n=0}^{\infty} a_n z^{n+1} = 0$$

$$\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} + \sum_{n=0}^{\infty} (n-4)a_n z^{n+1} = 0 \qquad (1)$$

Replace n by n + 3 in the first summation to get same degree of z in each summation.

Thus we get: $\sum_{n=0}^{\infty} (n+3)(n+2)a_{n+3}z^{n+1} + \sum_{n=0}^{\infty} (n-4)a_n z^{n+1} = 0$

or $(n + 2)(n + 3)\, a_{n+3} + (n - 4)\, a_n = 0$

This gives, $a_{n+3} = -\frac{(n-4)}{(n+3)(n+2)}a_n$, n = 0, 1, 2, … (2)

Before we find a_3, a_4, … we must find out a_2. To do this we equate the coefficients of the lowest degree term to zero, by putting n = 2 in the first summation of (1). If we put n = 0 and n = 1 in the second summation of (1) the degree of z will be – 2 and – 1 which are not required for our solution. Thus, we put n = 2 only in its summation to get $a_2(2)(2 - 1) = 0$ giving $a_2 = 0$

Now in (2), put n = 0: $a_3 = 2a_0/3$ put n = 1: $a_4 = a_1/4$

put n = 2: $a_5 = a_2/10 = 0$ put n = 3: $a_6 = -a_3/30 = -a_0/45$

Note: $a_2 = 0$ hence $a_5 = 0$. Similarly, $a_8 = 0$, etc.

Also $a_6 = -a_3/30 = -2a_0/3(30) = -2a_0/45$, etc.

Now, solution of $y'' + z^2 y' - 4zy = 0$ is:

$y = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + a_5 z^5 + a_6 z^6 + \dots$ Substituting the values of a's and z, we get

$y = a_0 + a_1 (x - 1) + 0 + 2a_0 (x - 1)^3/3 + a_1 (x - 1)^4/3 + 0 - a_0 (x - 1)^6/45 + \dots$

$= a_0[1 + 2(x-1)^3/3 - (x-1)^6/45 + \ldots] + a_1[(x-1) + (x-1)^4/4 + \ldots]$

This is the required solution of given differential equation.

Solution Series at Regular Singular Point (Frobenius Method)

If x = 0 is a regular singular point of the following equation:

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \qquad (1)$$

then the series solution is of the form $y = x^m\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots\right) = \sum_{k=0}^{\infty} a_k x^{m+k}$

The value of m will be determined by substituting the expressions for y, y` and y`` in (1) and upon equating the coefficient of lowest power of x to zero, a quadratic equation **(known as Indicial Equation)** in m is obtained. Thus, two values of m are found. The series solution of (1) will depend on the nature of the roots of the indicial equation. This method is attributed to German mathematician F. G Frobenius (1849 – 1917) who is known for his active contributions to the theory of matrices and groups.

CASE I: When the roots of indicial equation are distinct and do not differ by an integer. If the roots of an indicial equation are distinct and not differing by an integer then, complete solution of differential equation is: $y = c_1(y)_{m_1} + c_2(y)_{m_2}$

Example 07: Find solution in series of differential equations:

(i) 3x y`` + 2y` + y = 0** **(ii) 9x(1 – x) y`` – 12 y` + 4y = 0

Solution: (i) Given equation is 3x y`` + 2y` + y = 0 (1)

Since x = 0 is a regular singular point, the solution is of the form: $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

$$\Rightarrow \frac{dy}{dx} = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$3x\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2} + 2\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} 3\{a_k(m+k)(m+k-1)\}x^{m+k-1} + \sum_{k=0}^{\infty} 2\{a_k(m+k)\}x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

or $$\sum_{k=0}^{\infty} a_k\{3(m+k)(m+k-1) + 2(m+k)\}x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero. The indicial equation will be:

$$a_0\{3m(m-1) + 2m\} = 0 \Rightarrow a_0\left(3m^2 - m\right) = 0 \Rightarrow a_0 m(3m-1) = 0$$

Since $a_0 \neq 0$, therefore, m = 0 and m = 1/3 are the roots of indicial equation. We see that roots of indicial equation are distinct and do not differ by an integer.

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$a_{k+1}\left\{3(m+k+1)(m+k)+2(m+k+1)\right\}+a_k=0 \quad \Rightarrow \quad a_{k+1}(m+k+1)(3m+3k+2)+a_k=0$$

or
$$a_{k+1}=-\frac{a_k}{(m+k+1)(3m+3k+2)}$$

Now, for $k=0$: $a_1=-\dfrac{a_0}{(m+1)(3m+2)}$

For $k=1$: $a_2=-\dfrac{a_1}{(m+2)(3m+5)}=\dfrac{a_0}{(m+1)(m+2)(3m+2)(3m+5)}$

For $k=2$: $a_3=-\dfrac{a_2}{(m+3)(3m+8)}=-\dfrac{a_0}{(m+1)(m+2)(m+3)(3m+2)(3m+5)(3m+8)}$

and so on. Now put m = 0 the first value of root from indicial equation:

$$a_1=-\frac{a_0}{2}, a_2=\frac{a_0}{20}, a_3=-\frac{a_0}{480}.$$

Hence,
$$y_1=a_0\left(1-\frac{1}{2}x+\frac{1}{20}x^2-\frac{1}{480}x^3+\ldots\right)$$

For m = 1/3 the second root of indicial equation: $a_1=-\dfrac{a_0}{4}, a_2=\dfrac{a_0}{56}, a_3=-\dfrac{a_0}{1680}$

Hence,
$$y_2=a_0\left(x^{1/3}-\frac{1}{4}x^{4/3}+\frac{1}{56}x^{7/3}-\frac{1}{1680}x^{10/3}+\ldots\right)$$

Thus the complete solution is: $y=c_1(y)_{m_1}+c_2(y)_{m_2}$

$$y=c_1\left\{a_0\left(1-\frac{1}{2}x+\frac{1}{20}x^2-\frac{1}{480}x^3+\ldots\right)\right\}+c_2\left\{a_0\left(x^{1/3}-\frac{1}{4}x^{4/3}+\frac{1}{56}x^{7/3}-\frac{1}{1680}x^{10/3}+\ldots\right)\right\}$$

$$y=A\left(1-\frac{1}{2}x+\frac{1}{20}x^2-\frac{1}{480}x^3+\ldots\right)+Bx^{1/3}\left(1-\frac{1}{4}x+\frac{1}{56}x^2-\frac{1}{1680}x^3+\ldots\right)$$

(ii) Given differential equation is $9x(1-x)y'' - 12y' + 4y = 0$ (1)

Here x = 0 is a regular singular point hence we assume the solution to be:

$$y=\sum_{k=0}^{\infty}a_k x^{m+k}$$

$$\Rightarrow \frac{dy}{dx}=\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1} \text{ and } \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$9x(1-x)\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^{m+k-2}-12\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1}+4\sum_{k=0}^{\infty}a_k x^{m+k}=0$$

or
$$\sum_{k=0}^{\infty}9\left\{a_k(m+k)(m+k-1)\right\}x^{m+k-1}-\sum_{k=0}^{\infty}9\left\{a_k(m+k)(m+k-1)\right\}x^{m+k}$$
$$-\sum_{k=0}^{\infty}12\left\{a_k(m+k)\right\}x^{m+k-1}+\sum_{k=0}^{\infty}4a_k x^{m+k}=0$$

$$\Rightarrow 3\sum_{k=0}^{\infty}a_k(m+k)(3m+3k-7)x^{m+k-1}-\sum_{k=0}^{\infty}a_k\left[9(m+k)^2-9(m+k)-4\right]x^{m+k}=0$$

$$\Rightarrow 3\sum_{k=0}^{\infty}a_k(m+k)(3m+3k-7)x^{m+k-1}-\sum_{k=0}^{\infty}a_k\left[(3m+3k-4)(3m+3k+1)\right]x^{m+k}=0$$

NOTE: $9(m + k)^2 - 9(m + k) - 4 = 9p^2 - 9p - 4$ [Assume $(m + k) = p$]

$= 9p^2 - 12p + 3p - 4 = (3p - 4)(3p + 1) = (3m + 3k - 4)(3m + 3k + 1)$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero. The indicial equation will be:

$$3a_0\{m(3m-7)\} = 0 \Rightarrow m = 0,\ m = 7/3$$

These are the roots of indicial equation. We see that roots are distinct and do not differ by an integer. Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$3a_{k+1}\{(m+k+1)(3m+3k-4)\} - a_k\{(3m+3k-4)(3m+3k+1)\} = 0$$

$$\Rightarrow a_{k+1} = \frac{(3m+3k-4)(3m+3k+1)}{3(m+k+1)(3m+3k-4)}a_k = \frac{(3m+3k+1)}{(3m+3k+3)}a_k$$

For $k = 0$: $a_1 = \dfrac{(3m+1)}{(3m+3)}a_0$

For $k = 1$: $a_2 = \dfrac{(3m+4)}{(3m+6)}a_1 = \dfrac{(3m+1)(3m+4)}{(3m+3)(3m+6)}a_0$

For $k = 2$: $a_3 = \dfrac{(3m+7)a_2}{(3m+9)} = \dfrac{(3m+1)(3m+4)(3m+7)}{(3m+3)(3m+6)(3m+9)}a_0$ and so on.

Now put m = 0 the first value of root from indicial equation: $a_1 = \dfrac{a_0}{3}, a_2 = \dfrac{2a_0}{9}, a_3 = \dfrac{14a_0}{81}$

Hence, $$y_1 = a_0\left(1 + \frac{1}{3}x + \frac{2}{9}x^2 + \frac{14}{81}x^3 + \ldots\right)$$

For m = 7/3, the second root of indicial equation: $a_1 = \dfrac{4a_0}{5}, a_2 = \dfrac{44a_0}{65}, a_3 = \dfrac{77a_0}{130}$

Hence, $$y_2 = a_0x^{7/3}\left(1 + \frac{4}{5}x + \frac{44}{65}x^2 + \frac{77}{130}x^3 + \ldots\right)$$

Thus the complete solution is: $y = c_1y_1 + c_2y_2$

Or $y = A\left(1 + \dfrac{1}{3}x + \dfrac{2}{9}x^2 + \dfrac{14}{81}x^3 + \ldots\right) + Bx^{7/3}\left(1 + \dfrac{4}{5}x + \dfrac{44}{65}x^2 + \dfrac{77}{130}x^3 + \ldots\right)$

Where, $A = a_0c_1$ and $B = a_0\,c_2$.

CASE II: When the roots of indicial equation are equal

If the roots of indicial equation are equal then, solution of differential is of the form:

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}$$

Example 08: Solve the differential equation x(x – 1)y`` + (3x – 1)y` + y = 0 by solution in series method.

Solution: Since x = 0 is a regular singular point, we assume the solution

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

This gives $\dfrac{dy}{dx} = \displaystyle\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1}$ and $\dfrac{d^2y}{dx^2} = \displaystyle\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$

Substituting these in the given differential equation, we get

$$x(x-1)\sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-2} + (3x-1)\sum_{k=0}^{\infty} a_k (m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k} - \sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-1}$$

$$+\sum_{k=0}^{\infty} 3a_k (m+k)x^{m+k} - \sum_{k=0}^{\infty} 3a_k (m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} a_k \{(m+k)(m+k-1)+3(m+k)+1\}x^{m+k} - \sum_{k=0}^{\infty} a_k \{(m+k)(m+k-1)+(m+k)\}x^{m+k-1} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k \{(m+k)(m+k+2)+1\}x^{m+k} - \sum_{k=0}^{\infty} a_k \{(m+k)(m+k)\}x^{m+k-1} = 0$$

$$\text{or} \sum_{k=0}^{\infty} a_k \{(m+k)(m+k+2)+1\}x^{m+k} - \sum_{k=0}^{\infty} a_k (m+k)^2 x^{m+k-1} = 0 \qquad (1)$$

The coefficient of lowest degree term x^{m-1} in (1) is obtained by putting k = 0 in the second summation only of (1) and equating it to zero. This gives an indicial equation:

$$a_0 (m+0)^2 = 0 \Rightarrow a_0 m^2 = 0 \Rightarrow m = 0, 0 \qquad [\text{Since } a_0 \neq 0]$$

Thus roots of indicial equation are equal.

Equating the coefficient of x^{m+k} to zero, the recurrence relation is given by

$$a_k \{(m+k)(m+k+2)+1\} - a_{k+1}(m+k+1)^2 = 0$$

$$\Rightarrow a_k \{(m+k)(m+k+1+1)+1\} = a_{k+1}(m+k+1)^2$$

$$\Rightarrow a_k \{(m+k)(m+k+1)+(m+k)+1\} = a_{k+1}(m+k+1)^2$$

$$\Rightarrow a_k \{(m+k)(m+k+1)+(m+k+1)\} = a_{k+1}(m+k+1)^2$$

$$\Rightarrow a_k (m+k+1)^2 = a_{k+1}(m+k+1)^2 \text{ or } a_{k+1} = a_k$$

For k = 0; $a_1 = a_0$: For k = 1; $a_2 = a_1 = a_0$: For k = 2; $a_3 = a_2 = a_0$; etc.

Thus first solution of given differential equation is:

$$y = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right) = a_0 \left(1 + x + x^2 + x^3 + ...\right) \qquad [\text{Since } m = 0]$$

Since the roots of indicial equation are repeated, the second solution is obtained as follows:

$$y = x^m \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right) = a_0 x^m \left(1 + x + x^2 + x^3 + ...\right)$$

$$\therefore \left(\frac{\partial y}{\partial m}\right)_{m_1} = \left[a_0 x^m \ln x \left(1 + x + x^2 + x^3 + ...\right)\right]_{m=0} = a_0 \ln x \left(1 + x + x^2 + x^3 + ...\right)$$

NOTE: If $y = a^x$ then $y` = a^x \ln a$. Similary if $y = x^m f(x)$ then $\left(\frac{\partial y}{\partial m}\right) = x^m . \ln x . f(x)$

Here m is variable and x is constant.

Hence the general solution is $y = c_1 (y)_{m_1} + c_2 \left(\frac{\partial y}{\partial m}\right)_{m_1}$

$$y = c_1 \left\{a_0 \left(1 + x + x^2 + x^3 + ...\right)\right\} + c_2 \left\{a_0 \ln x \left(1 + x + x^2 + x^3 + ...\right)\right\}$$

Or $y = A\left(1 + x + x^2 + x^3 + ...\right) + B\ln x\left(1 + x + x^2 + x^3 + ...\right)\cdot$

CASE III: When the roots of indicial equation are distinct and differ by an integer

If the roots of indicial equation differ by an integer the solution of given differential equation will be:

$$y = c_1\left(y\right)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m2}$$

Example 09: Solve the equation $x^2y`` + xy` + (x^2 - 4)y = 0$ by solution series method.

Solution: You may first check that x = 0 is regular singular point. Hence we assume the solution to be: $y = \sum_{k=0}^{\infty} a_k x^{m+k} \Rightarrow \frac{dy}{dx} = \sum_{k=0}^{\infty} a_k\left(m+k\right)x^{m+k-1}$ and $\frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}$

Substituting these in the given differential equation, we get

$$x^2\sum_{k=0}^{\infty} a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2} + x\sum_{k=0}^{\infty} a_k\left(m+k\right)x^{m+k-1} + \left(x^2 - 4\right)\sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k\left[(m+k)(m+k-1) + (m+k) - 4\right]x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

Or $$\sum_{k=0}^{\infty} a_k(m+k+2)(m+k-2)x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0 \qquad (1)$$

The coefficient of lowest degree term x^m in (1) is obtained by putting k = 0 in the first summation only of (1) and equating it to zero. This gives an indicial equation:

$$a_0(m+2)(m-2) = 0 \Rightarrow m = 2, -2 \qquad [\because a_0 \neq 0]$$

Thus roots of indicial equation are distinct but have difference 2 – (–2) = 4 which is an integer.

The coefficient of next lowest term x^{m+1} in (1) is obtained by putting k = 1 in the first summation only and equating it to zero, we get: $a_1\,(m + 3)\,(m - 1) = 0$ or $a_1 = 0$

Equating the coefficient of x^{m+k+2} to zero, the recurrence relation is given by

$$a_{k+2}\left\{\left(m+k+4\right)\left(m+k\right)\right\} + a_k = 0 \quad \Rightarrow a_{k+2} = -\frac{a_k}{(m+k+4)(m+k)}$$

Since $a_1 = 0$ therefore, $a_3 = a_5 = a_7 = \ldots = 0$. This is due to recurrence relation.

For k = 0: $a_2 = -\frac{1}{m(m+4)}a_0,$

For k = 2: $a_4 = -\frac{1}{(m+2)(m+6)}a_2 = \frac{1}{m(m+2)(m+4)(m+6)}a_0$

For k = 4: $a_6 = -\frac{a_4}{(m+4)(m+8)} = -\frac{1}{m(m+2)(m+4)^2(m+6)(m+8)}a_0,$

and so on. Thus solution of given differential equation is:

$$y = a_0x^m\left(1 - \frac{x^2}{m(m+4)} + \frac{x^4}{m(m+2)(m+4)(m+6)} - \frac{x^6}{m(m+2)(m+4)^2(m+6)(m+8)} + ...\right)$$

Putting m = 2, we get $$y = a_0x^2\left(1 - \frac{x^2}{12} + \frac{x^4}{384} - \frac{x^6}{23040} + ...\right) = y_1 \qquad (2)$$

Remark: It may be noted that coefficients of x^4, x^6 … become infinite if we put m = – 2.

To overcome this problem, we put $a_0 = b_0 (m + 2)$ in (2) and get:

$$y = b_0 x^m \left((m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \ldots \right)$$

$$\therefore \left(\frac{\partial y}{\partial m}\right) = b_0 \left(x^m \ln x\right)\left((m+2) - \frac{(m+2)x^2}{m(m+4)} + \frac{x^4}{m(m+4)(m+6)} - \frac{x^6}{m(m+4)^2(m+6)(m+8)} + \ldots \right)$$

$$+b_0 x^m \left[1 - \frac{(m+2)x^2}{m(m+4)}\left(\frac{1}{m+2} - \frac{1}{m} - \frac{1}{m+4}\right) + \frac{x^4}{m(m+4)(m+6)}\left(-\frac{1}{m} - \frac{1}{m+4} - \frac{1}{m+6}\right) + \ldots \right]$$

Putting m = -2, and simplifying, we get:

$$\therefore \left(\frac{\partial y}{\partial m}\right)_{m=-2} = b_0 x^2 \ln x \left(-\frac{1}{2^2.4} + \frac{x^2}{2^3.4.6} - \frac{x^4}{2^3.4^2.6.8} + \ldots \right) + b_0 x^{-2}\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \ldots \right) = y_2$$

Hence the general solution is: $y = c_1 y_1 + c_2 \left(\frac{\partial y}{\partial m}\right)_{m2} = c_1 y_1 + c_2 y_2$

Where, y_1 and y_2 are given as above.

BESSEL'S DIFFERENTIAL EQUATION

The differential equation $x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - n^2\right) y = 0$ (1)

is known as ***Bessel's differential equation*** and particular solutions of this equation are called ***Bessel's functions*** of order n. Many physical problems involving oscillatory motion of hanging chain, the study of planetary motion, the propagation of waves, the elasticity, the fluid motion, the potential energy, heat conduction in cylindrical regions give rise to this equation. Bessel's functions are also known as cylindrical or spherical functions. To find the solution of Bessel's Differential Equation, let us assumed its solution is $y = \sum_{r=0}^{\infty} a_r x^{m+r}$ (2)

so that, $\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r) x^{m+r-1}$ and $\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1) x^{m+r-2}$

Substituting these values in the given differential equation, we get

$$x^2 \sum a_r (m+r)(m+r-1) x^{m+r-2} + x\sum a_r (m+r) x^{m+r-1} + \left(x^2 - n^2\right)\sum a_r x^{m+r} = 0$$

$$\sum a_r (m+r)(m+r-1) x^{m+r} + \sum a_r (m+r) x^{m+r} + \sum a_r x^{m+r+2} - n^2 \sum a_r x^{m+r} = 0$$

$$\sum a_r \left\{(m+r)(m+r-1) + (m+r) - n^2\right\} x^{m+r} + \sum a_r x^{m+r+2} = 0$$

Equating the lowest degree term x^m to zero, we obtain the indicial equation

$$a_0 \left\{(m+0)(m-1) + m - n^2\right\} = 0$$

$$\Rightarrow a_0 \left(m^2 - n^2\right) = 0 \Rightarrow m^2 - n^2 = 0, \text{ (since } a_0 \neq 0) \quad \Rightarrow m^2 = n^2 \Rightarrow m = n, -n \cdot$$

Equating the coefficient of x^{m+1} to zero, we get

$$a_1 \left\{(m+1)m + (m+1) - n^2\right\} = 0 \Rightarrow a_1 \left\{(m+1)^2 - n^2\right\} = 0 \Rightarrow a_1 = 0, \ \because (m+1)^2 - n^2 \neq 0$$

Equating the coefficient of x^{m+r+2} to zero to find the recurrence relation, we have

$$a_{r+2}\left\{(m+r+2)(m+r+1)+(m+r+2)-n^2\right\}+a_r=0$$

$$a_{r+2}\left\{(m+r+2)^2-n^2\right\}+a_r=0 \Rightarrow a_{r+2}=-\frac{a_r}{(m+r+2)^2-n^2}$$

Now, $a_3=a_5=a_7=a_9=...=0$ because $a_1=0$ and the nature of recurrence relations. Now:

If r = 0: $a_2=-\dfrac{a_0}{(m+2)^2-n^2}$,

If r = 2: $a_4=-\dfrac{a_2}{(m+4)^2-n^2}=\dfrac{a_0}{\left[(m+2)^2-n^2\right]\left[(m+4)^2-n^2\right]}$, and so on.

On substituting the values of coefficients in (2), we have

$$y=a_0x^m-\frac{a_0}{(m+2)^2-n^2}x^{m+2}+\frac{a_0}{\left[(m+2)^2-n^2\right]\left[(m+4)^2-n^2\right]}x^{m+4}-...$$

$$y=a_0x^m\left\{1-\frac{1}{(m+2)^2-n^2}x^2+\frac{1}{\left[(m+2)^2-n^2\right]\left[(m+4)^2-n^2\right]}x^4-...\right\}$$

For m = n, $$y=a_0x^n\left\{1-\frac{1}{4(n+1)}x^2+\frac{1}{4^2\cdot 2!(n+1)(n+2)}x^4-...\right\} \qquad (3)$$

For m = -n, $$y=a_0x^{-n}\left\{1-\frac{1}{4(-n+1)}x^2+\frac{1}{4^2\cdot 2!(-n+1)(-n+2)}x^4-...\right\} \qquad (4)$$

where a_0 is an arbitrary constant. It is customary to take $a_0=\dfrac{1}{2^n\Gamma(n+1)}$

Then (3) becomes: $y=\dfrac{1}{2^n\Gamma(n+1)}x^n\left\{1-\dfrac{1}{4(n+1)}x^2+\dfrac{1}{4^2\cdot 2!(n+1)(n+2)}x^4-...\right\}$

$$y=\frac{1}{2^n\Gamma(n+1)}x^n\left\{1-\frac{1}{1!.2^2(n+1)}x^2+\frac{1}{2!.2^4(n+1)(n+2)}x^4-...\right\}$$

This solution of Bessel equation is known as Bessel's Function of first kind.

Its order is n and is usually denoted by $J_n(x)$. It may also be expressed in simplified form as:

$$J_n(x)=\left(\frac{x}{2}\right)^n\left\{\frac{1}{\Gamma(n+1)}-\frac{1}{1!\Gamma(n+2)}\left(\frac{x}{2}\right)^2+\frac{1}{2!\Gamma(n+3)}\left(\frac{x}{2}\right)^4-...\right\}$$

Or, $$J_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} \qquad (5)$$

NOTE: $\Gamma(n+1)=n.\Gamma(n)$

Similarly corresponding to (4), we have

$$J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r} \qquad (6)$$

which is called the Bessel's function of the first kind of order (– n).

Hence the complete solution of the Bessel's differential equation (1) may be expressed in the following form: $y=AJ_n(x)+BJ_{-n}(x)$

Bessel functions are also known as ***"Cylinder functions"*** or ***"Cylindrical harmonics"*** because they are found in the solution to Laplace's equation in cylindrical coordinates.

$J_0(x)$ and $J_1(x)$ Function

If we substitute n = 0 in (5), we get

$$J_0(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(r+1)}\left(\frac{x}{2}\right)^{2r}=1-\frac{1}{1!\ \Gamma(2)}\left(\frac{x}{2}\right)^2-\frac{1}{2!\ \Gamma(3)}\left(\frac{x}{2}\right)^4+\frac{1}{3!\ \Gamma(4)}\left(\frac{x}{2}\right)^6-\dots$$

$$=1-\frac{x^2}{1!\ 1.2^2}-\frac{x^4}{2!.2.2^4}+\frac{x^6}{3!.3!.2^6}-\dots\quad=1-\frac{x^2}{2^2}-\frac{x^4}{4.16}+\frac{x^6}{6.6.64}-\dots$$

$$=1-\frac{x^2}{2^2}-\frac{x^4}{2^2.4^2}+\frac{x^6}{6.6.4.16}-\dots\qquad=1-\frac{x^2}{2^2}-\frac{x^4}{2^2.4^2}+\frac{x^6}{2^2.4^2.6^2}-\dots$$

$\text{NOTE}:\Gamma(1)=1,\Gamma(n+1)=n!$

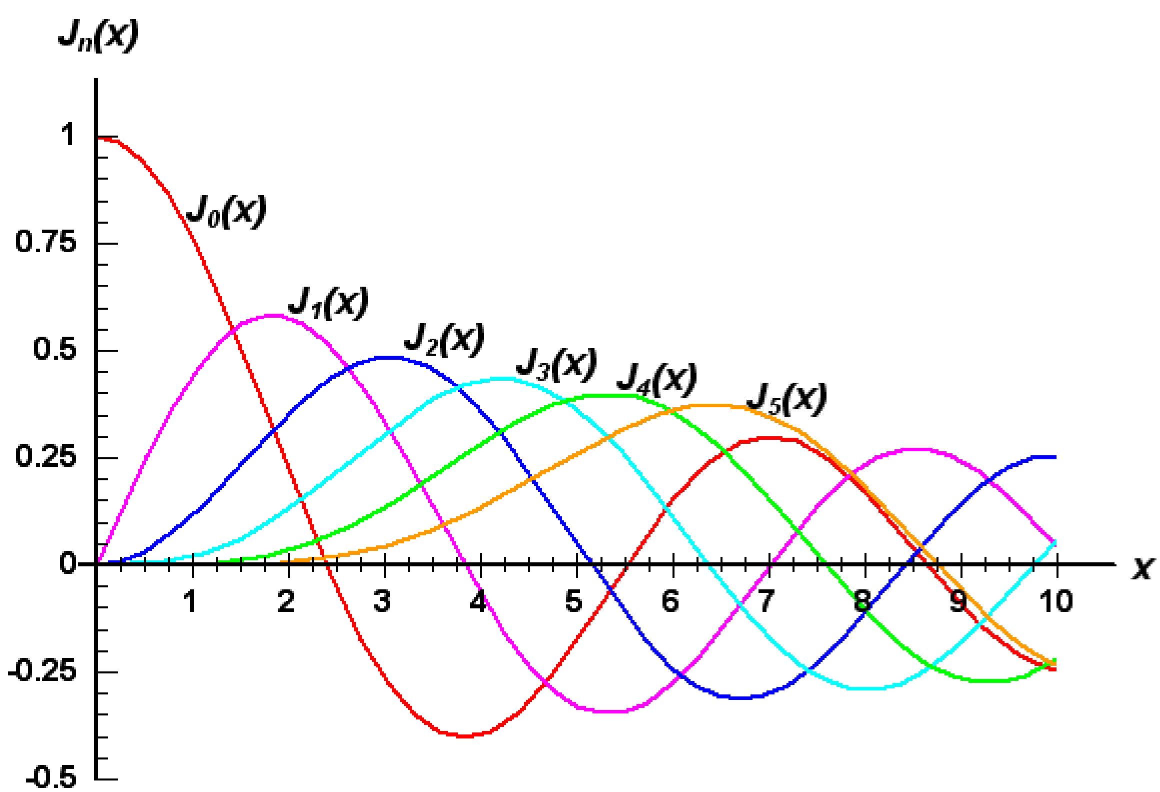
If we substitute n = 1 in (5), we get

$$J_1(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(r+2)}\left(\frac{x}{2}\right)^{2r+1}=\frac{1}{0!\ \Gamma(2)}\left(\frac{x}{2}\right)-\frac{1}{1!\ \Gamma(3)}\left(\frac{x}{2}\right)^3+\frac{1}{2!\ \Gamma(4)}\left(\frac{x}{2}\right)^5-\dots$$

$$=\frac{1}{0!\ \Gamma(2)}\left(\frac{x}{2}\right)-\frac{1}{1!\ \Gamma(3)}\left(\frac{x}{2}\right)^3+\frac{1}{2!\ \Gamma(4)}\left(\frac{x}{2}\right)^5-\dots=\frac{x}{2}-\frac{x^3}{2.2^3}+\frac{x^5}{2.6.2^5}$$

$$=\frac{x}{2}-\frac{x^3}{2^2.4}+\frac{x^5}{2^2.4^2.6}-\dots$$

Similarly we may find Bessel's functions of higher order. The graphs of some of these functions are shown below:



Example 01: Prove that $J_{-n}(x) = (-1)^n J_n(x)$

Solution: By definition $J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$

Substituting $-n+r=k$ or $r=k+n$. Thus

$$J_{-n}(x)=\sum_{r=0}^{\infty}(-1)^{k+n}\frac{1}{(n+k)!\Gamma(k+1)}\left(\frac{x}{2}\right)^{-n+2k+2n}$$

$$=(-1)^n\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}=(-1)^n J_n(x)$$

$\text{NOTE}:\Gamma(k+1)=k!,\text{and }(n+k)!=\Gamma(n+k+1)$

Example 02: Show that (i) $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$ (ii) $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

Solution: **(i)** Substituting $n=1/2$ in (5), we get

$$J_{1/2}(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(r+3/2)}\left(\frac{x}{2}\right)^{(1/2)+2r}$$

$$=\left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{\Gamma(3/2)}-\frac{1}{1!\Gamma(5/2)}\left(\frac{x}{2}\right)^2+\frac{1}{2!\Gamma(7/2)}\left(\frac{x}{2}\right)^4-\ldots\right\}$$

$$=\left(\frac{x}{2}\right)^{1/2}\left\{\frac{1}{1/2\,\Gamma(1/2)}-\frac{1}{3/2\cdot1/2\,\Gamma(1/2)}\left(\frac{x}{2}\right)^2+\frac{1}{2\cdot5/2\cdot3/2\cdot1/2\,\Gamma(1/2)}\left(\frac{x}{2}\right)^4-\ldots\right\}$$

$$=\left(\frac{x}{2}\right)^{1/2}\left\{\frac{2}{\sqrt{\pi}}-\frac{4}{3\sqrt{\pi}}\left(\frac{x}{2}\right)^2+\frac{4}{15\sqrt{\pi}}\left(\frac{x}{2}\right)^4-\ldots\right\}=\sqrt{\frac{x}{2}}\left(\frac{2}{\sqrt{\pi}}\right)\left\{1-\frac{2}{3}\left(\frac{x}{2}\right)^2+\frac{2}{15}\left(\frac{x}{2}\right)^4-\ldots\right\}$$

Multiply and divide by x, we get $\sqrt{\frac{2}{\pi x}}\left\{x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\ldots\right\}\Rightarrow J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$

(ii) Substituting n = -1/2 in (6), we get

$$J_{-1/2}(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(r+1/2)}\left(\frac{x}{2}\right)^{-\frac{1}{2}+2r}=\left(\frac{x}{2}\right)^{-\frac{1}{2}}\left\{\frac{1}{\Gamma(1/2)}-\frac{1}{1!\Gamma(3/2)}\left(\frac{x}{2}\right)^2+\frac{1}{2!\Gamma(5/2)}\left(\frac{x}{2}\right)^4-\ldots\right\}$$

$$=\left(\frac{2}{x}\right)^{\frac{1}{2}}\left\{\frac{1}{\sqrt{\pi}}-\frac{1}{1/2\sqrt{\pi}}\left(\frac{x}{2}\right)^2+\frac{1}{2\cdot3/2\cdot1/2\sqrt{\pi}}\left(\frac{x}{2}\right)^4-\ldots\right\}=\left(\frac{2}{x}\right)^{\frac{1}{2}}\left(\frac{1}{\sqrt{\pi}}\right)\left\{1-\frac{2}{\sqrt{\pi}}\left(\frac{x}{2}\right)^2+\frac{2}{3\sqrt{\pi}}\left(\frac{x}{2}\right)^4-\ldots\right\}$$

$$=\sqrt{\frac{2}{\pi x}}\left\{1-\frac{1}{2}x^2+\frac{1}{4\cdot3\cdot2}x^4-\ldots\right\}=\sqrt{\frac{2}{\pi x}}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\Rightarrow J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$$

Recurrence Formulae For $J_n(x)$

The following recurrence formulae can easily be derived from the series expansion for $J_n(x)$.

(1) $\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$ (2) $\frac{d}{dx}\left[x^{-n}J_n(x)\right]=-x^{-n}J_{n+1}(x)$

(3) $J_n(x)=\frac{x}{2n}\{J_{n-1}(x)+J_{n+1}(x)\}$ (4) $J_n'(x)=\frac{1}{2}\{J_{n-1}(x)-J_{n+1}(x)\}$

(5) $J_n'(x)=\frac{n}{x}J_n(x)-J_{n+1}(x)$ (6) $J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$

Proof: **(1)** We know that: $J_n(x)=\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$\Rightarrow x^nJ_n(x)=x^n\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\Gamma(n+r+1)}=\sum_{r=0}^{\infty}\frac{(-1)^r x^{2n+2r}}{2^{n+2r}r!\Gamma(n+r+1)}=\sum_{r=0}^{\infty}\frac{(-1)^r x^{2(n+r)}}{2^{n+2r}r!\Gamma(n+r+1)}$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\left\{x^nJ_n(x)\right\}=\sum_{r=0}^{\infty}\frac{(-1)^r 2(n+r)x^{2(n+r)-1}}{2^{n+2r}r!\Gamma(n+r+1)}=x^n\sum_{r=0}^{\infty}\frac{(-1)^r(n+r)x^{n-1+2r}}{2^{n-1+2r}r!\Gamma(n+r+1)}$$

$$=x^n\sum_{r=0}^{\infty}\frac{(-1)^r(n+r)x^{n-1+2r}}{2^{n-1+2r}r!(n+r)\Gamma(n+r+1-1)}$$

$$=x^n\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{n-1+2r}}{r!\Gamma(n-1+r+1)}=x^nJ_{n-1}(x)\qquad\Rightarrow\frac{d}{dx}\left\{x^nJ_n(x)\right\}=x^nJ_{n-1}(x)$$

(2) Multiply $J_n(x)$ by x^{-n} we obtain

$$x^{-n}J_n(x)=x^{-n}\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\Gamma(n+r+1)}=\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{2^{n+2r}r!\Gamma(n+r+1)}$$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}\left\{x^{-n}J_n(x)\right\}=\sum_{r=0}^{\infty}\frac{(-1)^r 2r\,x^{2r-1}}{2^{n+2r}r(r-1)!\Gamma(n+r+1)}=-x^{-n}\sum_{r=1}^{\infty}\frac{(-1)^{r-1}x^{n-1+2r}}{2^{n+1+2(r-1)}(r-1)!\Gamma(n+r+1)}$$

Replacing r – 1 by k, we get $\frac{d}{dx}\left\{x^{-n}J_n(x)\right\}=-x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(n+r+1+1)}\left(\frac{x}{2}\right)^{n+1+2k}=-x^{-n}J_{n+1}(x)$

$$\Rightarrow\qquad\frac{d}{dx}\left\{x^{-n}J_n(x)\right\}=-x^{-n}J_{n+1}(x).$$ NOTE: Put n = 0, we get

$$\frac{d}{dx}\left\{x^0J_0(x)\right\}=-x^0J_{0+1}(x)\ \Rightarrow\frac{d}{dx}\left\{J_0(x)\right\}=-J_1(x)$$

(3) From (1), we have $x^nJ`_n(x)+n\,x^{n-1}J_n(x)=x^nJ_{n-1}(x)$

Dividing by x^n, we get $J`_n(x)+n\,x^{-1}J_n(x)=J_{n-1}(x)$ (i)

Similarly, from (2), we get after differentiating, and dividing by x^{-n},

$$-J`_n(x)+n\,x^{-1}J_n(x)=J_{n+1}(x)\qquad\text{(ii)}$$

Adding (i) and (ii), we obtain

$$\frac{2n}{x}J_n(x)=J_{n-1}(x)+J_{n+1}(x)\ \Rightarrow\ J_n(x)=\frac{x}{2n}\left[J_{n-1}(x)+J_{n+1}(x)\right]$$

This establishes (3).

Further to note that from (i); $J`_n(x)+n\,x^{-1}J_n(x)=J_{n-1}(x)$.

Multiplying by x, we get: $xJ`_n(x)+n\,J_n(x)=xJ_{n-1}(x)\ \Rightarrow xJ`_n(x)=-n\,J_n(x)+xJ_{n-1}(x)$

From (ii), $-J`_n(x)+n\,x^{-1}J_n(x)=J_{n+1}(x)$. Multiply both sides by – x, we get:

$$xJ`_n(x)-n\,J_n(x)=-x\,J_{n+1}(x)\qquad\Rightarrow\ xJ`_n(x)=n\,J_n(x)-x\,J_{n+1}(x)$$

(4) Subtracting (ii) from (i), we get:

$$2J`_n(x) = J_{n-1}(x) + J_{n+1}(x) \qquad \Rightarrow J`_n(x) = \frac{1}{2}\left[J_{n-1}(x) + J_{n+1}(x)\right].$$ This establishes (4).

(5) From (ii), $-J`_n(x) + n\,x^{-1}J_n(x) = J_{n+1}(x) \;\Rightarrow\; J`_n(x) = n\,x^{-1}J_n(x) + J_{n+1}(x)$

(6) From (3), we have $J_n(x) = \frac{x}{2n}\left[J_{n-1}(x) + J_{n+1}(x)\right]$.

Rearranging, we obtain: $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

Example 03: Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$

Solution: We know that: $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$

Putting n =1, we get: $J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$

Putting n =2, we get:

$$J_3(x) = \frac{4}{x}J_2(x) - J_1(x) = \frac{4}{x}\left[\frac{2}{x}J_1(x) - J_0(x)\right] - J_1(x) = \left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)$$

Putting n =3, we get:

$$J_4(x) = \frac{6}{x}J_3(x) - J_2(x) = \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right)J_1(x) - \frac{4}{x}J_0(x)\right] - \left(\frac{2}{x}J_1(x) - J_0(x)\right)$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right)J_1(x) - \left(1 - \frac{24}{x^2}\right)J_0(x)$$

Finally putting n = 4 and simplifying as above we get:

$$J_5(x) = \left(\frac{384}{x^4} - \frac{72}{x^2} - 1\right)J_1(x) - \left(\frac{12}{x} - \frac{192}{x^3}\right)J_0(x)$$

Example 04: Prove that $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3 - x^2}{x^2}\right)\sin x - \frac{3}{2}\cos x\right]$

Solution: Using the recurrence formula $J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$ and substituting n = 1/2, we get: $J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$. Now put n = 3/2 in the given recurrence relation, we get

$$J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x) = \frac{3}{x}\left[\frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)\right] - J_{1/2} = \left(\frac{3}{x^2} - 1\right)J_{1/2}(x) - \frac{3}{x}J_{-1/2}(x)$$

Substituting the values of $J_{1/2}(x)$ and $J_{-1/2}(x)$, we get

$$J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{3 - x^2}{x^2}\right)\sin x - \frac{3}{x}\cos x\right]$$

Example 05: Show that $\sqrt{\frac{\pi x}{2}}J_{3/2}(x) = \frac{\sin x}{x} - \cos x$

Solution: We know that $J_{3/2}(x) = \frac{1}{x}J_{1/2}(x) - J_{-1/2}(x)$

Substituting the values of $J_{1/2}(x)$ and $J_{-1/2}(x)$, we get

$$J_{3/2}(x)=\frac{1}{x}\sqrt{\frac{2}{\pi x}}\sin x-\sqrt{\frac{2}{\pi x}}\cos x \;\Rightarrow\; \sqrt{\frac{\pi x}{2}}J_{3/2}(x)=\frac{\sin x}{x}-\cos x \text{ . Proved}$$

Example 06: Prove that $J_2'(x)=\left(\frac{x^2-4}{x^2}\right)J_1(x)+\frac{2}{x}J_0(x)$

Solution: We know that $xJ'_n(x) = -n\,J_n(x) + xJ_{n-1}$. Putting n = 2, we obtain:

$$xJ'_2(x) = -2\,J_2(x) + xJ_1$$

Dividing by x, we get: $J'_2(x) = -(2/x)\,J_2(x) + J_1(x)$ (1)

We also know that: $J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)$. Putting n = 1, we obtain

$$J_2(x)=\frac{2}{x}J_1(x)-J_0(x)$$

Substituting this in (1), we get:

$$J'_2(x)=-\frac{2}{x}\left[\frac{2}{x}J_1(x)-J_0(x)\right]+J_1(x)=\left[1-\frac{4}{x^2}\right]J_1(x)+\frac{2}{x}J_0(x)=\left[\frac{x^2-4}{x^2}\right]J_1(x)+\frac{2}{x}J_0(x)$$

Example 07: Show that $4J_n'' = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

Solution: We know that: $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ (1)

Differentiating w.r.t x: $2J_n'' = J'_{n-1} - J'_{n+1}$ (2)

Replacing n by n – 1 in (1): $2J'_{n-1} = J_{n-2} - J_n$

Or $J'_{n-1} = (J_{n-2} - J_n)/2$

Replacing n by n+ 1 in (1): $2J'_{n+1} = J_n - J_{n+2}$

This gives, $J'_{n+1} = (J_n - J_{n+2})/2$

Substituting these in (2), we get: $2J_n'' = (J_{n-2} - J_n)/2 - (J_n - J_{n+2})/2$

Multiplying by 2, we obtain: $4J_n'' = J_{n-2} - J_n - J_n - J_{n+2}$

or $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$

Example 08: Show that $J_3 + 3J_0' + 4J_0''' = 0$

Solution: We know that: $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ (1)

And $4J_n'' = J_{n-2} - 2J_n + J_{n+2}$ (2)

Differentiate (2) and multiplying by 2, we get

$$8J_n''' = 2J'_{n-2} - 2.2J'_n + 2J'_{n+2}$$

We know that: $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

Or $8\,J_n''' = [J_{n-3} - J_{n-1}] - 2[J_{n-1} - J_{n+1}] + [J_{n+1} - J_{n+3}]$

$$= J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}$$

Putting n = 0, we get: $8\,J_0''' = J_{-3} - 3J_{-1} + 3J_1 - J_3$ (3)

But we know that: $J_{-n}(x) = (-1)^n\,J_n(x)$

Thus (3) becomes: $8J_0''' = (-1)^3 J_3 - 3(-1)J_1 + 3J_1 - J_3$

$$= -J_3 + 3J_1 + 3J_1 - J_3 = 6J_1 - 2J_3$$

Dividing by 2 and rearranging the terms, we get

$$J_3 - 3J_1 + 4J_0''' = 0 \qquad (4)$$

But $J_1 = -J'_0$. Thus (4) becomes: $J_3 + 3J_0' + 4J_0''' = 0$ Proved.

Example 09: Show that $\frac{d}{dx}\left[xJ_nJ_{n+1}\right] = x\left[\left(J_n\right)^2 - \left(J_{n+1}\right)^2\right]$

Solution: Before we prove this result we may notice that:

(i) $x J'_n = n J_n - x J_{n+1}$

(ii) $x J'_n = -n J_n + x J_{n-1}$. Replacing n by (n + 1) in (ii), we get

(iii) $x J'_{n+1} = -(n+1) J_{n+1} + x J_n$

Now, $\frac{d}{dx}\left[x J_n J_{n+1}\right] = x\frac{d}{dx}\left[J_n J_{n+1}\right] + J_n J_{n+1}\frac{d}{dx}(x) = x\left[J_n J'_{n+1} + J'_n J_{n+1}\right] + J_n J_{n+1}$

$$= J_n\left[xJ'_{n+1}\right] + J_{n+1}\left[xJ'_n\right] + J_n J_{n+1}$$

Using (i) and (iii), we get:

$$\frac{d}{dx}\left[x J_n J_{n+1}\right] = J_n\left[-(n+1)J_{n+1} + xJ_n\right] + J_{n+1}\left[nJ_n - xJ_{n+1}\right] + J_n J_{n+1}$$

$$= -nJ_nJ_{n+1} - J_nJ_{n+1} + x(J_n)^2 + nJ_nJ_{n+1} - x(J_{n+1})^2 + J_nJ_{n+1}$$

$$\Rightarrow \frac{d}{dx}\left[x J_n J_{n+1}\right] = x\left[\left(J_n\right)^2 - \left(J_n + 1\right)^2\right] \qquad \text{Proved.}$$

Example 10: Show that $\frac{2}{x}J_1 + J_2 + \int J_3(x)\,dx = 0$

Solution: We know that

$\frac{d}{dx}\left[x^{-n}J_n\right] = -x^{-n}J_{n+1}(x)$. Integrating, we obtain $\int x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x)$

Putting $n = 2$, we get: $\int x^{-2}J_3(x)\,dx = -x^{-2}J_2(x)$ (1)

Let us assume that $v = x^{-2}J_2(x) \Rightarrow \int v\,dx = -x^{-2}J_3(x)$

Now consider $\int J_3(x)\,dx = \int x^2\left(x^{-2}J_3(x)\right)dx = \int u.v\,dx$, where $u = x^2$ and $v = x^{-2}J_3(x)$

Now $\int u.v\,dx = u\int v\,dx - \int\left[u'\int v\,dx\right]dx = x^2\left(-x^{-2}J_2(x)\right) - \int 2x\left(-x^{-2}J_2(x)\right)dx$

$$= -J_2(x) + 2\int x^{-1}J_2(x)\,dx = -J_2(x) + 2\left(-x^{-1}J_1(x)\right) \qquad \text{[Using (1)]}$$

Thus $\int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x) \quad \Rightarrow \quad \frac{2}{x}J_1(x) + J_2(x) + \int J_3(x)\,dx = 0.$ Proved

Example 11: Show that $\int_0^x x^{n+1}J_n(x)\,dx = x^{n+1}J_{n+1}(x)\,dx$

Solution: We know that $\frac{d}{dx}\left[x^nJ_n(x)\right] = x^nJ_{n-1}(x)$. Replacing n by (n + 1), we get:

$\dfrac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right]=x^{n+1}J_n(x)$. Integrating from 0 to x, we get

$\int_0^x x^{n+1}J_n(x)\,dx = x^{n+1}J_{n+1}(x)$ Proved.

LEGENDRE'S DIFFERENTIAL EQUATION

Another differential equation of practical importance in Applied Mathematics, particularly in Strum–Liouville Boundary Value problems for spheres, is ***Legendre's Differential Equation***. This differential equation is frequently encountered in physics and other technical fields. In particular, it occurs when solving Laplace's equation (and related partial differential equations) in spherical coordinates. The Legendre's differential equation is defined as:

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n\left(n+1\right)y=0 \qquad (1)$$

where n is a real number. But in most applications only integral values of n are required. Then solutions of such differential equation are called ***Legendre Functions*** of order n.

When n is non – negative integer, that is; n = 0, 1, 2, 3, …, Legendre Functions are often referred to as ***Legendre Polynomials*** and are denoted by $P_n(x)$. Each Legendre's polynomial $P_n(x)$ is a polynomial of degree n.

To find its solution, we substitute: $y=a_0x^m+a_1x^{m+1}+a_2x^{m+2}+\ldots$ $(a_0\neq 0)$ and its derivatives in (1) to get:

$$a_0m(m-1)x^{m-2}+a_1(m+1)mx^{m-1}+\ldots$$
$$+\left[a_{r+2}(m+r+2)(m+r+1)-a_r\left\{(m+r)(m+r+1)-n(n+1)\right\}\right]x^{m+r}+\ldots=0$$

Equating to zero the coefficient of the lowest power of x, i.e. of x^{m-2}, we obtain

$$a_0\, m\,(m-1)=0 \quad\text{or}\quad m=0,\,1 \qquad (\text{Since } a_0\neq 0)$$

Equating the coefficients of x^{m-1} and x^{m-r} to zero, we get

$$a_1(m+1)m=0 \qquad (2)$$

$$\left[a_{r+2}(m+r+2)(m+r+1)-a_r\left\{(m+r)(m+r+1)-n(n+1)\right\}\right]=0 \qquad (3)$$

When m = 0 then (2) is satisfied and hence $a_1\neq 0$. Then (3) gives, taking r = 1, 2, 3, …

$$a_2=-\frac{n(n+1)}{2!}a_0, \qquad a_3=-\frac{(n-1)(n+2)}{3!}a_1,$$

$$a_4=-\frac{(n-2)(n+3)}{4.3}a_2=\frac{n(n-2)(n+1)(n+3)}{4!}a_0,$$

$$a_5=-\frac{(n-3)(n+4)}{5.4}a_3=\frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_1,\text{etc.}$$

Hence, for m = 0, there are two independent solutions of Legendre equation (1). They are:

$$y_1=a_0\left[1-\frac{n(n+1)}{2!}x^2+\frac{(n-2)n(n+1)(n+3)}{4!}x^4-\ldots\right] \qquad (4)$$

$$y_2=a_1\left[x-\frac{(n-1)(n+2)}{3!}x^3+\frac{(n-3)(n-1)(n+2)(n+5)}{5!}x^5-\ldots\right] \qquad (5)$$

For m = 1, equation (2) shows that $a_1 = 0$. Therefore, (3) gives:

$$a_3 = a_5 = a_7 = \ldots = 0$$

and $$a_2 - \frac{(n-1)(n+2)}{3!}a_0, \quad a_4 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!}a_0, \text{ etc.}$$

This gives y_2 as in (5). Thus $y = c_1\, y_1 + c_2\, y_2$ is the general solution of (1).

It may be noted that if n is a positive even integer, the series (4) terminates at the term in x^n and y_1 becomes a polynomial of degree n. Similarly if n is an odd integer, (5) becomes a polynomial of degree n. Thus, whenever n is a positive integer, the general solution of Legendre equation consists of a polynomial solution and an infinite series solution.

These polynomials with a_0 or a_1 are so chosen that the value of polynomial is 1 for x = 1, are called ***Legendre Polynomials of degree n*** and are usually denoted by $P_n(x)$.The infinite series solution with a_0 and a_1 properly chosen are called ***Legendre Function of second kind*** and are denoted by $Q_n(x)$.

Rodriguez's Formula

The Legendre's polynomials $P_n(x)$ can be expressed by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n \text{ where } n = 0, 1, 2, 3, \cdots \tag{1}$$

Proof: Let $v = (x^2 - 1)^n$. Then $v_1 = dv/dx = 2n\, x(x^2 - 1)^{n-1} = 2n\, x(x^2 - 1)^n/(x^2 - 1)$

or $v_1\,(x^2 - 1) = 2n\, xv$ or $-v_1(1 - x^2) - 2n\, xv = 0$

This gives, $$(1 - x^2)\, v_1 + 2n\, xv = 0 \tag{2}$$

Differentiate (2), (n + 1) times by using Leibniz's theorem, we get

$$(1-x^2)v_{n+2} + (n+1)(-2x)v_{n+1} + (n+1)n(-2)v_n / 2! + 2n[xv_{n+1} + (n+1)v_n] = 0$$

$$\Rightarrow (1-x^2)\frac{d^2}{dx^2}(v_n) - 2x\frac{d}{dx}(v_n) + n(n+1)v_n = 0$$

This is a Legendre's equation in v_n. Thus its solution must be $y = cv_n$ where c is an arbitrary constant. We have also discussed in the previous section that a finite series solution of Legendre's equation is $y = P_n(x)$, hence $$P_n(x) = cv_n = c\frac{d^n}{dx^n}(x^2-1) \tag{3}$$

To determine c, put x = 1 in (3), we get

$$P_n(1) = 1 = c\frac{d^n}{dx^n}\left[(x-1)^n(x+1)^n\right]_{x=1}$$

$$= c\left[n!(x+1)^n + \text{terms containing } (x-1) \text{ and its powers}\right]_{x=1}$$

$$\Rightarrow \quad 1 = c.n!2^n \quad \text{or} \quad c = 1/n!2^n$$

Substituting the value of c in (3), we get the required result, that is,

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

Legendre's Polynomials

Using the Rodriguez's formula $P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$ where $n = 0, 1, 2, 3, \cdots$

If $n=0$, $P_0(x)=\frac{1}{2^0\cdot 0!}\frac{d^0}{dx^0}\left(x^2-1\right)^0=1$

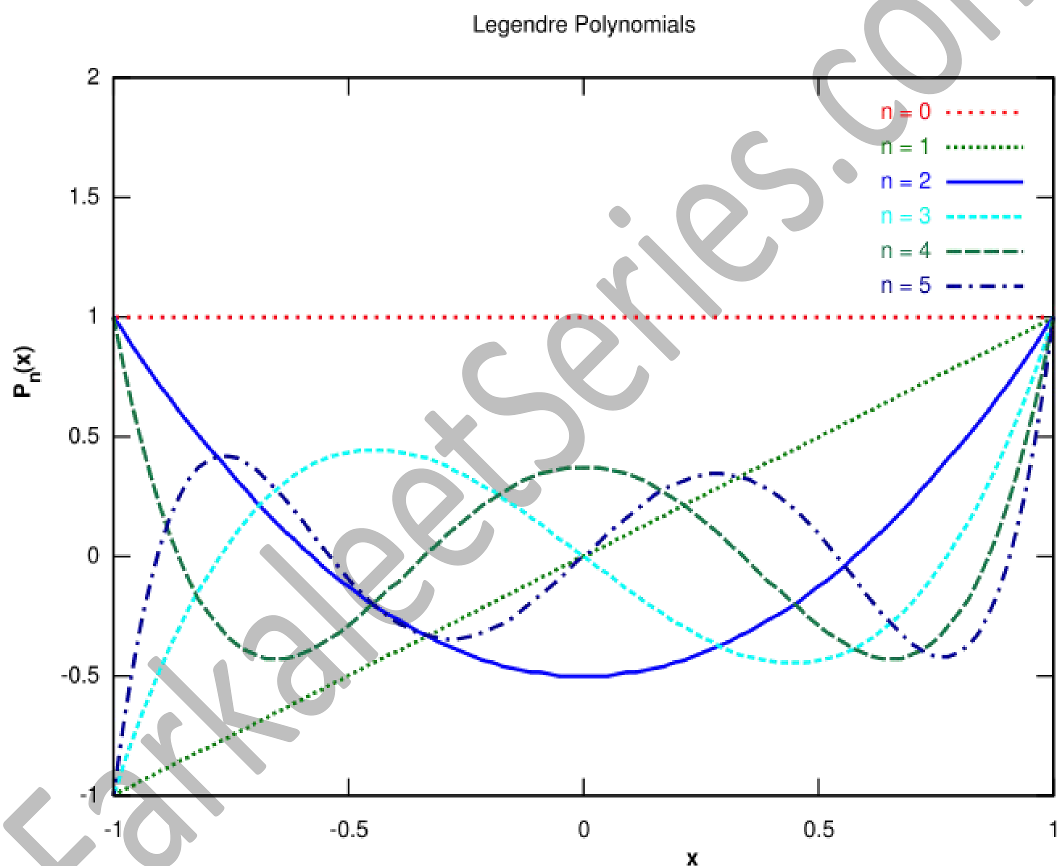
If $n=1$, $P_1(x)=\frac{1}{2^1\cdot 1!}\frac{d}{dx}\left(x^2-1\right)=\frac{1}{2}(2x)=x$

If $n=2: P_2(x)=\frac{1}{2^2\cdot 2!}\frac{d^2}{dx^2}\left(x^2-1\right)^2=\frac{1}{8}\frac{d}{dx}\left[2\left(x^2-1\right)(2x)\right]=\frac{1}{2}\frac{d}{dx}\left(x^3-x\right)=\frac{1}{2}\left(3x^2-1\right)$

Similarly, $P_3(x)=\frac{1}{2}\left(5x^3-3x\right)$, $P_4(x)=\frac{1}{8}\left(35x^4-30x^2+3\right)$, $P_5(x)=\frac{1}{8}\left(63x^5-70x^3+15x\right)$

$P_6(x)=\frac{1}{16}\left(231x^6-315x^4+105x^2-5\right)$, etc.

The graphs of these polynomials (up to n = 5) are shown below:



Example 01: Prove that $P_n(1) = 1$

Solution: By Rodrigue formula: $P_n(x)=\frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n=\frac{1}{2^n\cdot n!}\left[\frac{d^n}{dx^n}\left\{(x-1)^n(x+1)^n\right\}\right]$

Differentiate right side n times using Leibniz theorem, we get:

$$P_n(x)=\frac{1}{2^n\cdot n!}\left\{(x+1)^n\frac{d^n}{dx^n}(x-1)^n+\binom{n}{1}n(x+1)^{n-1}\frac{d^{n-1}}{dx^{n-1}}(x-1)^n+\ldots+(x-1)^n\frac{d^n}{dx^n}(x+1)^n\right\}$$

Substituting $x=1$ and notice that $\frac{d^n}{dx^n}(x-1)^n=n!$, we obtain

$$P_n(1)=\frac{1}{2^n\cdot n!}\left\{(2)^n .n!+0+0+...+0\right\}=\frac{2^n n!}{2^n n!}=1$$

This formula has a geometrical meaning that value of each Legendre polynomial at x = 1 is unity. This is also clear from the above figure which shows that each Legendre polynomial passes through the point (1, 1).

Example 02: Express $f(x) = x^3 + 2x^2 - 5x + 8$ in terms of Legendre's polynomials.

Solution: Let $x^3+2x^2-5x+8=aP_3(x)+bP_2(x)+cP_1(x)+dP_0(x)$ (1)

$$x^3+2x^2-5x+8=a\left(\frac{5x^3}{2}-\frac{3x}{2}\right)+b\left(\frac{3x^2}{2}-\frac{1}{2}\right)+c(x)+d(1)$$

$$x^3+2x^2-5x+8=\frac{5a}{2}x^3-\frac{3a}{2}x+\frac{3b}{2}x^2-\frac{b}{2}+cx+d$$

$$x^3+2x^2-5x+8=\frac{5a}{2}x^3+\frac{3b}{2}x^2+\left(c-\frac{3a}{2}\right)x+\left(d-\frac{b}{2}\right)$$

Equating the coefficients of like powers of x from both sides, we obtain

$$\frac{5a}{2}=1\Rightarrow a=\frac{2}{5},\quad \frac{3b}{2}=2\Rightarrow b=\frac{4}{3},\ c-\frac{3a}{2}=-5\Rightarrow c-\frac{3}{2}\left(\frac{2}{5}\right)=-5\Rightarrow c=\frac{3}{5}-5\Rightarrow c=-\frac{22}{5},$$

$$d-\frac{b}{2}=8\Rightarrow d-\frac{1}{2}\left(\frac{4}{3}\right)=8\Rightarrow d=8+\frac{2}{3}\Rightarrow d=\frac{26}{3}\cdot$$

Substituting these in (1), we get $x^3+2x^2-5x+8=\frac{2}{5}P_3(x)+\frac{4}{3}P_2(x)-\frac{22}{5}P_1(x)+\frac{26}{3}P_0(x)$

Example 03: Show that $\begin{cases}\int_{-1}^{1}P_n(x)\,dx=0, \text{ if } n\neq 0\\ \qquad\qquad\quad =2, \text{ if } n=0\end{cases}$

Solution: By Rodrigue formula: $P_n(x)=\frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$.

Integrating between –1 and + 1, we get

$$\int_{-1}^{1}P_n(x)dx=\frac{1}{2^n\cdot n!}\int_{-1}^{1}\frac{d^n}{dx^n}\left(x^2-1\right)^n dx=\frac{1}{2^n\cdot n!}\left[\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]_{-1}^{1}$$

$$=\frac{1}{2^n\cdot n!}[0-0]=0,\ n\neq 0$$

If n = 0, we have: $\int_{-1}^{1}P_0(x)dx=\frac{1}{2^0\cdot 0!}\int_{-1}^{1}\frac{d^0}{dx^0}\left(x^2-1\right)^0 dx=\frac{1}{1\cdot 1}\int_{-1}^{1}1\,dx=[x]_{-1}^{1}=1-(-1)=1+1=2$

Example 04: If f(x) possesses n continuous derivatives then prove that

$$\int_{-1}^{1}f(x)P_n(x)dx=\frac{(-1)^n}{2^n\cdot n!}\int_{-1}^{1}\left(x^2-1\right)^n f^n(x)\,dx$$

Solution : $\int_{-1}^{1}f(x)P_n(x)dx=\frac{1}{2^n\cdot n!}\int_{-1}^{1}f(x)\frac{d^n}{dx^n}\left(x^2-1\right)^n dx$. Integrating by parts, we get

$$=\frac{1}{2^n\cdot n!}\left\{\left[f(x)\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]_{-1}^{1}-\int_{-1}^{1}f`(x)\left[\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]dx\right\}$$

$$=\frac{1}{2^n\cdot n!}\left\{\left[f(x)(0-0)\right]-\int_{-1}^{1}f`(x)\left[\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]dx\right\}$$

$$=\frac{-1}{2^n\cdot n!}\int_{-1}^{1}f`(x)\left[\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\right]dx$$

Continue this way, n-times, we get the required result:

$$\int_{-1}^{1}f(x)P_n(x)dx=\frac{(-1)^n}{2^n\cdot n!}\int_{-1}^{1}\left(x^2-1\right)^n f^n(x)\,dx$$

Example 05: Show that $\int_{-1}^{1}P_m(x)P_n(x)dx=0$ where $m\neq n$.

Solution: Since, $P_n(x)$ is a solution of $(1-x^2)\,y``-2x\,y`+n(n+1)\,y=0$ (1)

Then $P_m(x)$ will be the solution of $(1-x^2)\,z``-2x\,z`+m(m+1)\,z=0$ (2)

Multiplying (1) by z and (2) by y and subtracting, we get

$(1-x^2)[zy``-yz``]-2x\,[zy`-yz`]+[n(n+1)-m(m-1)]\,yz=0$

$(1-x^2)\,[zy``+z`y`-z`y`-yz``]-2x\,[zy`-yz`]+[n(n+1)-m(m-1)]\,yz=0$

$(1-x^2)\,[\{zy``+z`y`\}-\{z`y`+yz``\}]-2x\,[zy`-yz`]+[n(n+1)-m(m-1)]\,yz=0$

or $\frac{d}{dx}\left[\left(1-x^2\right)\left(zy`-yz`\right)\right]+\left(n-m\right)\left(n+m+1\right)yz=0$. Integrating from –1 to +1, we get

$$\left[\left(1-x^2\right)\left(zy`-yz`\right)\right]_{-1}^{1}+\left(n-m\right)\left(n+m+1\right)\int_{-1}^{1}yz\,dx=0$$

$$\Rightarrow(0-0)+\left(n-m\right)\left(n+m+1\right)\int_{-1}^{1}yz\,dx=0\ \Rightarrow\left(n-m\right)\left(n+m+1\right)\int_{-1}^{1}yz\,dx=0.$$

Dividing by $(n-m)(n+m+1)$ and putting $y=P_n(x)\ \&\ z=P_m(x)$, we obtain:

$$\int_{-1}^{1}P_n(x)P_m(x)\,dx=0,\qquad \text{provided } m\neq n.\qquad \text{Proved.}$$

The above property of Legendre polynomials is known as Orthogonally Property.

Remark: $n(n+1)-m(m-1)=n^2+n-m^2-m=(n^2-m^2)+(n-m)$

$=(n-m)(n+m)+(n-m)=(n-m)(n+m+1)$

If n = m then: $P_n(x)\,P_m(x)=[P_n(x)]^2$. In this case we have the following:

Example 06: Show that $\int_{-1}^{1}\left[P_n(x)\right]^2dx=\frac{2}{2n+1}$

Solution: From Rodrigue formula: $P_n(x)=\frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$

$$\left(2^n\cdot n!\right)P_n(x)=\frac{d^n}{dx^n}\left(x^2-1\right)^n\quad\Rightarrow\quad\left(2^n\cdot n!\right)^2\left[P_n(x)\right]^2=\left[D^n\left(x^2-1\right)^n\right]^2,\ D=d/dx$$

Integrating both sides from -1 to + 1, we get:

$$\left(2^n \cdot n!\right)^2 \int_{-1}^{1} \left[P_n(x)\right]^2 dx = \int_{-1}^{1} \left[D^n\left(x^2-1\right)^n\right].\left[D^n\left(x^2-1\right)^n\right] dx$$

Integrating by parts the R.H.S taking $u = D^n(x^2 - 1)^n$ and $v = D^n(x^2 - 1)^n$. Thus,

$$\left(2^n \cdot n!\right)^2 \int_{-1}^{1} \left[P_n(x)\right]^2 dx = \left[D^n\left(x^2-1\right)^n\right]\left[D^{n-1}\left(x^2-1\right)^n\right]_{-1}^{1} - \int_{-1}^{1} \left[D^{n+1}\left(x^2-1\right)^n\right].\left[D^{n-1}\left(x^2-1\right)^n\right] dx$$

Now $D^{n-1}(x^2 - 1)^n$ will contain $(x^2 - 1)$ after differentiating it $(n - 1)$ times as the power of $(x^2 - 1)$ is n. Hence upon putting the limits, this term becomes zero. Thus

$$\left(2^n \cdot n!\right)^2 \int_{-1}^{1} \left[P_n(x)\right]^2 dx = -\int_{-1}^{1} \left[D^{n+1}\left(x^2-1\right)^n\right].\left[D^{n-1}\left(x^2-1\right)^n\right] dx$$

Integrating right side by parts further $(n - 1)$ times and use the similar argument as above, we get

$$\left(2^n \cdot n!\right)^2 \int_{-1}^{1} \left[P_n(x)\right]^2 dx = (-1)^n \int_{-1}^{1} \left[D^{2n}\left(x^2-1\right)^n\right].\left[\left(x^2-1\right)^n\right] dx = (-1)^n \int_{-1}^{1} (2n)!\left[\left(x^2-1\right)^n\right] dx$$

$$= (-1)^n\, 2.(2n!) \int_{0}^{1} \left[\left(x^2-1\right)^n\right] dx = (-1)^n (-1)^n\, 2.(2n!) \int_{0}^{1} \left(1-x^2\right)^n dx$$

$$= +2(2n!) \int_{0}^{1} \left(1-x^2\right)^n dx$$

Note: a. $(x^2 - 1)$ is even function hence; $\int_{-1}^{1} (x^2-1)\,dx = 2\int_{0}^{1} (x^2-1)\,dx$

b. $(x^2-1)^n = (-1)^n(1-x^2)^n$ c. $(-1)^n(-1)^n = \left[(-1)^2\right]^n = 1$

d. Refer to proof of Rodrigue formula where it is shown that $D^{2n}\left(x^2-1\right)^n = (2n)!$

Now substituting $x = \sin\theta$, $dx = \cos\theta\, d\theta$. Also the limits of θ will be 0 to $\pi/2$. Thus,

$$\left(2^n \cdot n!\right)^2 \int_{-1}^{1} \left[P_n(x)\right]^2 dx = 2(2n!) \int_{0}^{\pi/2} \left(1-\sin^2\theta\right)^n .\cos\theta\, d\theta = 2(2!) \int_{0}^{\pi/2} \left(\cos^2\theta\right)^n .\cos\theta\, d\theta$$

$$= 2(2n!) \int_{0}^{\pi/2} \cos^{2n+1}\theta\, d\theta$$

Apply the reduction formula $\int_{0}^{\pi/2} \cos^m\theta\, d\theta = \frac{m-1}{m}\int_{0}^{\pi/2} \cos^{m-2}\theta\, d\theta$ repeatedly, we get

$$\left(2^n \cdot n!\right)^2 \int_{-1}^{1} \left[P_n(x)\right]^2 dx = 2(2n!) \int_{0}^{\pi/2} \cos^{2n+1}\theta\, d\theta = 2(2n!) \frac{2n(2n-2)...4.2}{(2n+1)(2n-1)...3.1} \int_{0}^{\pi/2} \cos\theta\, d\theta$$

$$= 2(2n!) \frac{2n(2n-2)...4.2}{(2n+1)(2n-1)...2.1} \times \frac{2n(2n-2)...4.2}{2n(2n-2)...3.2} \left[\sin\theta\right]_0^{\pi/2}$$

$$= 2(2n!) \frac{[2n(2n-2)...4.2]\,[2n(2n-2)...4.2]}{(2n+1)2n(2n-1)(2n-2)...4.3.2.1} \left[\sin\pi/2 - \sin 0\right]_0^{\pi/2}$$

$$= 2(2n!) \frac{\left[2^n\left(n.(n-1)...2.1\right)\right]\left[2^n\left(n.(n-1)...2.1\right)\right]}{(2n+1)!} = \frac{2.(2n!)\left(2^n\, n!\right)\left(2^n\, n!\right)}{(2n+1)!}$$

$$= \frac{2.(2n!)\left(2^n\, n!\right)^2}{(2n+1)(2n!)} = \frac{2.\left(2^n\, n!\right)^2}{(2n+1)}$$. Dividing by $\left(2^n\, n!\right)^2$ we get:

$$\int_{-1}^{1} \left[P_n(x)\right]^2 dx = \frac{2}{2n+1}$$ Proved.

Using the above results, for instance,

$$\int_{-1}^{1} \left[P_7(x)\, P_4(x)\right] dx = 0 \quad \text{and} \quad \int_{-1}^{1} \left[P_7(x)\, P_7(x)\right] dx = \int_{-1}^{1} \left[P_7(x)\right]^2 dx = \frac{2}{2.7+1} = \frac{2}{15}$$

WORKSHEET 06

1. Solve the following differential equations using solution in series technique.

(a) $y`` + x^2 y = 0$ (b) $y`` + xy` + y = 0$

(c) $y`` + xy = 0$ (d) $y`` + xy` + x^2 y = 0$

(e) $(x^2 + 1) y`` + xy` - xy = 0$ (f) $y`` - y = 0$

2. Solve the following differential equations in power series by Frobenius method.

(a) $2x^2 y`` - xy` + (1 - x^2) y = 0$ (b) $4x y`` + 2y` + y = 0$

(c) $x y`` + y` - y = 0$ (d) $x y`` + y` + x^2 y = 0$

(e) $x^2 y`` + 6 xy` + (6 - x^2) y = 0$ (f) $2x^2 y`` - xy` + (1 + x^2) y = 0$

(g) $2x(1 - x)y`` - 7xy` - 3y = 0$ (h) $2x(1 - x)y`` + (1 - x) y` + 3y = 0$

(i) $xy`` + 3y` + 4x^3 y = 0$ (j) $xy`` + (x - 1)y` - y = 0$

(k) $xy`` + y` + xy = 0$ (*l*) $x(1 - x) y`` + 4y` + 2y = 0$

3. Show that $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$

4. Show that:

(a) $J_{1/2}(x) = J_{-1/2}(x) \cot x$ (b) $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}} \left(\frac{3}{2} \sin x + \frac{3 - x^2}{x^2} \cos x\right)$

5. Show that: (a) $J_2 = J_0`` - x^{-1} J_0`$ (b) $J_1''(x) = \left(\frac{2}{x^2} - 1\right) J_1(x) - \frac{1}{x} J_0(x)$

6. Evaluate $\int_0^{\pi} x^{n+1} J_n(x)\, dx$ [Hint: Use Example 11 in the above section]

7. If $J_0(2) =$ a, $J_1(2) =$ b find $J_1`(2)$ and $J_2`(2)$ in terms of a and b.

8. Show that $P_n(-x) = (-1)^n P_n(x)$

9. Express $x^3 + 2x^2 - x - 3$ in terms of Legendre polynomials.

10. Find $\int_{-1}^{1} \left[P_9(x)\, P_5(x)\right] dx$ and $\int_{-1}^{1} \left[P_9(x)\right]^2 dx$

11. Show that $x^5 = \frac{8}{63}\left[P_5(x) + \frac{7}{2} P_3(x) + \frac{27}{8} P_1(x)\right]$ & $x^3 = \left[\frac{2}{5} P_3(x) + \frac{3}{5} P_1(x)\right]$

12. Evaluate the following integrals using problem 11 and orthogonally property of Legendre polynomials (a) $\int_{-1}^{1} x^3 P_4(x)\, dx$ (b) $\int_{-1}^{1} x^3 P_3(x)\, dx$

CHAPTER SEVEN

FOURIER SERIES

INTRODUCTION

In many engineering problems, especially in the study of periodic phenomena in conduction of heat, electro-dynamics and acoustics, it is necessary to express a function in a series of sines and cosines. It may be noted that a function f(x) is said periodic if it satisfies the condition f(x + T) = f(x), where T is the period of the function. It may further be noted that both sine and cosine are the periodic functions with period 2π because sin (x + 2π) = sin x and cos (x + 2π) = cos x.

Most of the functions that are periodic in their nature can be expressed in infinite series of sines and cosines of the form:

$$f(x) = \frac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \ldots + b_1\sin x + b_2\sin 2x + \ldots$$

$$= \frac{1}{2}a_0 + \sum_1^{\infty} a_n\cos nx + \sum_1^{\infty} b_n\sin nx$$

within a desired range of values of a variable. Such a series is known as Fourier series.

Before we develop a Fourier series for a periodic function f(x), we prove some important results.

1. $$\int_{\alpha}^{\alpha+2\pi}\cos nx\,dx = \frac{\sin nx}{n}\bigg|_{\alpha}^{\alpha+2\pi} = \frac{\sin n(\alpha+2\pi)-\sin n\alpha}{n}$$

$$= \frac{\sin n\alpha\cos 2n\pi + \cos n\alpha\sin 2n\pi - \sin n\alpha}{n} = \frac{\sin n\alpha - \sin n\alpha}{n} = 0$$

REMARK: **(1)** sinn $2n\pi$ = 0, cos $2n\pi$ = 1, and $\int_{\alpha}^{\alpha+2\pi}\sin nx\,dx = 0,\quad \int_{\alpha}^{\alpha+2\pi}\cos nx\,dx = 0$

(2) cos(a + b)= cos a cos b – sin a sin b, sin(a + b) = sin a cos b + cos a sin b

(3) $$\int_{\alpha}^{\alpha+2\pi}\sin nx\cos mx\,dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi}2\sin nx\cos mx\,dx$$

$$= \frac{1}{2}\int_{\alpha}^{\alpha+2\pi}\left[\sin(n+m)x + \cos(n-m)x\,dx\right] = \frac{1}{2}\left[\int_{\alpha}^{\alpha+2\pi}\sin px\,dx + \int_{\alpha}^{\alpha+2\pi}\sin qx\,dx\right] = \frac{1}{2}[0+0] = 0$$

[Using (1) and (2) and supposing that (m + n) = p and (m – n) = q]

(4) $$\int_{\alpha}^{\alpha+2\pi}\cos nx\cos mx\,dx = \frac{1}{2}\int_{\alpha}^{\alpha+2\pi}2\cos nx\cos mx\,dx$$

$$= \frac{1}{2}\int_{\alpha}^{\alpha+2\pi}\cos(n+m)x + \cos(n-m)x\,dx = 0, \text{if } m \neq n$$

(5) In the above result, if m = n, then

$$\int_{\alpha}^{\alpha+2\pi} \cos^2 nx\ dx = \int_{\alpha}^{\alpha+2\pi} \frac{(1+\cos 2nx)}{2} = \frac{1}{2}[x]_{\alpha}^{\alpha+2\pi} + \int_{\alpha}^{\alpha+2\pi} \frac{\cos 2nx dx}{2} = \frac{1}{2}(\alpha+2\pi-\alpha) - 0 = \pi$$

(6) $\int_{\alpha}^{\alpha+2\pi} \sin nx \sin mx\, dx = 0, \text{if } m \neq n$ **(7)** If m = n, then $\int_{\alpha}^{\alpha+2\pi} \sin^2 nx\ dx = \pi$

It may be noted that the coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ are known as ***Euler's coefficients*** and are derived now. We are given that:

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots.$$

$$= \frac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos nx + \sum_{1}^{\infty} b_n \sin nx \qquad \textbf{(I)}$$

Integrating (I) from α to α + 2π, and using formulae (1) and (2), we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)dx = \frac{1}{2}a_0 \int_{\alpha}^{\alpha+2\pi} 1.dx + 0 + 0 = a_0\pi \ \Rightarrow a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx \qquad (1)$$

Now multiplying (I) by cos nx and using the results in Remarks (1) and (5), we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\, dx = 0 + a_n \int_{\alpha}^{\alpha+2\pi} \cos^2 nx\, dx + 0 = a_n\pi \ \Rightarrow a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\ dx \qquad (2)$$

Finally, multiplying (I) by sin nx and using the results in equations (2) and (7), we get

$$\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\, dx = 0 + 0 + b_n \int_{\alpha}^{\alpha+2\pi} \sin^2 nx\, dx + 0 = b_n\pi \ \Rightarrow b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\ dx \qquad (3)$$

We now have THREE important equations:

$$a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx\ dx, \quad b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x) \sin nx\ dx$$

REMARK: If α = 0, then above equations become:

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \cos nx\ dx, \quad b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \sin nx\ dx$$

If α = – π, then above equations become:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\ dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\ dx$$

These coefficients will help to find the Fourier series of a periodic function according to the conditions imposed on the function's interval.

Even and Odd Functions

A function f(x) is such that f(–x) = f(x) then, f(x) is called an even function and if f(–x) = – f(x) then, f(x) is called an odd function.

It may be noted that cos x is even function because cos(–x) = cos x and sin x is an odd function because sin(–x) = – sin x. It may also be noted that:

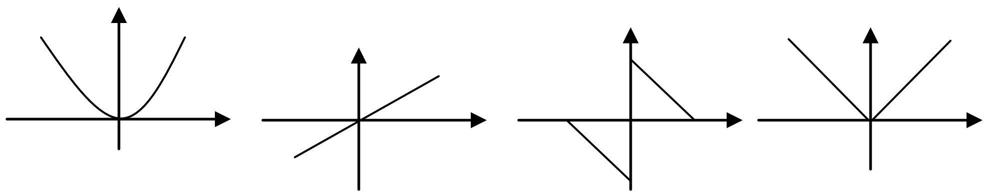
1. If f(x) is an even function, then $\int_{-\pi}^{\pi} f(x)dx = 2\int_{0}^{\pi} f(x)dx$
2. If f(x) is an odd of unction, then $\int_{-\pi}^{\pi} f(x)dx = 0$

REMARK: If you are given a graph of a function f(x) then how do you recognize it as a graph of even or odd function? The following simple tips will help you to do this.

(i) If the graph of a function is symmetric about y-axis then function is even.

(ii) If the graph of a function is symmetric about the origin then function is odd.

This is shown below:



Even f^n **Odd f^n** **Odd f^n** **Even f^n**

REMARK: Throughout our work, we shall frequently use the following results.

$\sin n\pi = \sin 0 = \sin 2n\pi = 0$, $\cos 0 = \cos 2n\pi = 1$ and $\cos \pi = (-1)^n$.

Also $\sin(-\alpha) = -\sin\alpha$, $\cos(-\alpha) = \cos\alpha$.

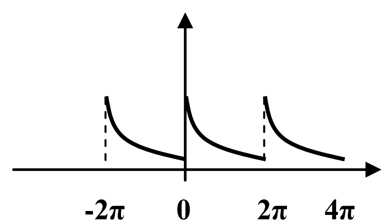
FOURIER SERIES OF A FUNCTION WITH PERIOD 2π

In many engineering other practical problems, we are often asked to find the Fourier series of a periodic function with period 2π. In this section, we shall present variety of problems to show how the periodic function is expressed into the Fourier series.

Example 01: Express in Fourier series the periodic function $f(x) = e^{-x}$ with period 2π in the interval (0, 2π).

Solution: The Fourier series of f(x) is given by:

$$f(x) = \frac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos nx + \sum_{1}^{\infty} b_n \sin nx$$



The periodic extension of f(x) is shown here.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{-1}{\pi} e^{-x}\Big|_0^{2\pi} = -\frac{e^{-2\pi} - e^0}{\pi} = -\frac{e^{-2\pi} - 1}{\pi} = \frac{1 - e^{-2\pi}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\;dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\;\cos nx\;dx = \frac{1}{\pi(n^2 - 1)}\left| e^{-x}(-\cos nx + n\sin nx)\right|_0^{2\pi}$$

$$\therefore a_n = \frac{1}{(n^2 - 1)}\left(\frac{1 - e^{-2\pi}}{\pi}\right) \qquad \text{NOTE}: \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{(a^2 + b^2)}[a\sin bx - b\cos bx]$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\;\sin nx\;dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\;\sin nx\;dx = \frac{1}{\pi(n^2 - 1)}\left| e^{-x}(-\sin nx - n\cos nx)\right|_0^{2\pi}$$

$$\therefore b_n = \frac{n}{(n^2 - 1)}\left(\frac{1 - e^{-2\pi}}{\pi}\right) \qquad \text{NOTE}: \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{(a^2 + b^2)}[a\sin bx - b\cos bx]$$

$$e^{-x} = \frac{1}{2}\left(\frac{1 - e^{-2\pi}}{\pi}\right) + \left(\frac{1 - e^{-2\pi}}{\pi}\right)\sum_{n=1}^{\infty}\frac{1}{n^2 + 1}\cos nx + \left(\frac{1 - e^{-2\pi}}{\pi}\right)\sum_{n=1}^{\infty}\frac{n}{n^2 + 1}\sin nx$$

$$\Rightarrow\; e^{-x} = \left(\frac{1 - e^{-2\pi}}{\pi}\right)\left\{\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2 + 1}\cos nx + \sum_{n=1}^{\infty}\frac{n}{n^2 + 1}\sin nx\right\}$$

This is the Fourier series of a periodic function with period 2π.

Example 02: Find a Fourier series representing a function f(x) = x – x^2 from π to π.

Solution: The Fourier series of a function is given by:

$$f(x)=\frac{1}{2}a_0+\sum_1^{\infty}a_n\cos nx+\sum_1^{\infty}b_n\sin nx \text{. Here}$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}(x-x^2)\,dx=\frac{1}{\pi}\left|\frac{x^2}{2}-\frac{x^3}{3}\right|_{-\pi}^{\pi}=-\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x-x^2\right)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\left(x-x^2\right)\frac{\sin nx}{n}\bigg|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(1-2x)\frac{\sin nx}{n}dx\right]\ \text{[Use integration by parts formula]}$$

$$=\frac{1}{\pi}\left[0-0\left|-(1-2x)\left(\frac{-\cos nx}{n^2}\right)\right|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}-2x.\frac{-\cos nx}{n^2}\right]\ \text{NOTE}: \sin n\pi=\sin(-n\pi)=0$$

$$=\frac{1}{\pi}\left[\frac{(1-2\pi)\cos n\pi-(1+2\pi)\cos n\pi}{n^2}+\frac{2}{n^2}\int_{-\pi}^{\pi}x\cos nx\,dx\right]\ \text{NOTE}: \int_{-\pi}^{\pi}x\cos nx\,dx=0$$

$$=\frac{1}{\pi}\left[\frac{\cos n\pi-2\pi\cos n\pi-\cos n\pi-2\pi\cos n\pi}{n^2}+0\right]=\frac{-4\pi\cos n\pi}{n^2\pi}$$

$$\therefore a_n=\frac{-4(-1)^n}{n^2}=\frac{4(-1)^{n+1}}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x-x^2\right)\sin nx\,dx$$

$$=\frac{1}{n\pi}\left[\left(-\pi+\pi^2-\pi-\pi^2\right)\cos n\pi+0+0\right]\ \text{NOTE}: \int_{-\pi}^{\pi}x\cos nx\,dx=0\ \text{ and }\ \sin n\pi=0$$

$$=\frac{1}{n\pi}\left[-2\pi(-1)^n\right]\quad\Rightarrow\quad b_n=\frac{2(-1)^{n+1}}{n}$$

Thus Fourier series of given function is given by:

$$x-x^2=-\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}\cos nx+\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

REMARK: If we put x = 0 and noticing that cos 0 = 1 and sin 0 = 0, we get:

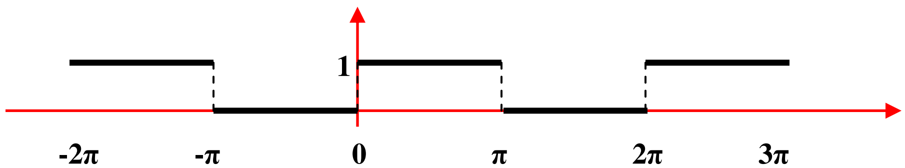
$$0=-\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2}=4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...\right]\Rightarrow\frac{\pi^2}{3}=4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...\right]$$

$$\Rightarrow\frac{\pi^2}{12}=\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-...\right]$$

REMARK: It is proved that if a function f(x) is periodic with period 2π and is piecewise continuous in the interval (–π, π) then it's Fourier series converges to its periodic extension or it converges to the average value [f(x^+) + f(x^-)]/2 where it has a jump discontinuity. This is known as ***Drichlet's property***. Here, f(x^+), f(x^-) are the left and right hand limits at x = 0.

Example 03: Find the Fourier series of the function f(x) defined as under

$$f(x)=\begin{cases}0, & -\pi<x<0\\ 1 & 0<x<\pi\end{cases}$$



Solution: Here f(x) is periodic function of period 2π. The graph of f(x) with periodic extension is shown above. It may be noted that here f(x) is piecewise continuous in (–π, π) and has a jump discontinuity at x = 0. Now,

$$a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\,dx+\int_{0}^{\pi}1\,dx\right]=0+\frac{1}{\pi}x\Big|_0^{\pi}=\frac{1}{\pi}[\pi]=1$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\,dx+\int_{0}^{\pi}1.\cos nx\,dx\right]=\frac{1}{\pi}\frac{\sin nx}{n}\Big|_0^{\pi}=\frac{1}{n\pi}[\sin n\pi-\sin 0]=0$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\,dx+\int_{0}^{\pi}1.\sin nx\,dx\right]=\frac{1}{\pi}\frac{-\cos nx}{n}\Big|_0^{\pi}=\frac{-1}{n\pi}[\cos n\pi-\cos 0]=\frac{-1}{n\pi}\left[(-1)^n-1\right]$$

$$\Rightarrow\ b_n=\begin{cases}0, & \text{if } n \text{ is even}\\ 2/n\pi, & \text{if } n \text{ is odd}\end{cases}$$

Thus Fourier series of given function is given by:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=\frac{1}{2}(1)+\sum_{n=1}^{\infty}(0)\cos nx+\frac{2}{n\pi}\sum_{n=1,3,5..}^{\infty}\sin nx$$

Thus the Fourier series of given function f(x) is given by:

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 2x}{2}+\frac{\sin 3x}{3}+...\right] \qquad (1)$$

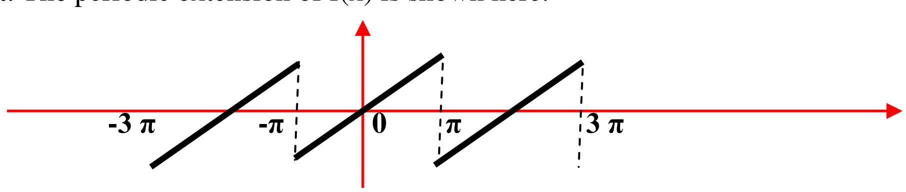
If we put x = π/2 and notice that sin π/2 = sin 5π/2 …= 1 and sin 3π/2 = sin 7 π/2 … = -1, we get:

$$f\left(\frac{\pi}{2}\right)=\frac{1}{2}+\frac{2}{\pi}\left[1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-...\right]$$. But f(π/2) = 1 [See the above figure]

$$1=\frac{1}{2}+\frac{2}{\pi}\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...\right]\quad\Rightarrow 1-\frac{1}{2}=\frac{2}{\pi}\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...\right]\quad\text{or}\quad\frac{\pi}{4}=\left[1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+...\right]$$

REMARK: The above series may be used to compute the value of π.

Example 04: Find the Fourier series of f(x) = x, –π < x < π where f(x) is periodic function of period 2π. The periodic extension of f(x) is shown here.



Solution: We have already mentioned that if f(x) is odd function, its integration is zero if the limits are taken from – a to + a, where a may be any real number. Now we know that x is an odd function and cos nx is an even function, hence, x con nx is an odd function. Thus, $a_0 = a_n = 0$.

Now, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x.\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} x.\sin nx\,dx = \frac{2}{\pi}\left[x\left[\frac{-\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi} 1.\left(\frac{-\cos nx}{n}\right)dx\right]$

Using integration by parts formula and noticing that (x sin nx) is an even function, then $\int_{-\pi}^{\pi} x.\sin nx\,dx = 2\int_0^{\pi} x.\sin nx\,dx$

$$\therefore b_n = \frac{2}{n\pi}\left[-\pi\cos n\pi + 0.\cos 0\right] + \frac{2}{n\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{2}{n\pi}\left[-\pi(-1)^n + 0 + \frac{\sin n\pi - \sin 0}{n}\right]$$

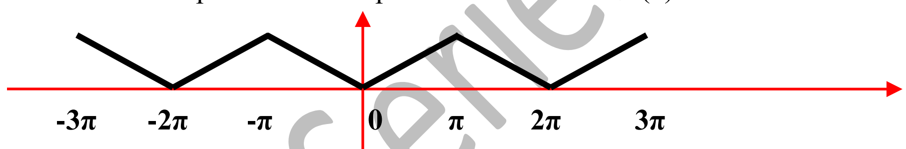
$$= \frac{2}{n\pi}(-1)^{n+1}\pi = \frac{2(-1)^{n+1}}{n}$$

Thus Fourier series of f(x) is:

$$x = \sum_{n=1}^{\infty} b_n \sin nx = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right]$$

Example 05: Find the Fourier series of: $f(x) = |x|, -\pi < x < \pi$

where f(x) is periodic function of period 2π. The periodic extension of f(x) is shown here.



Solution: We know that $f(x) = |x|$ is an even function of x, hence $b_n = 0$.

Now, $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \frac{1}{\pi}\left[\pi^2\right] = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x.\cos nx\,dx = \frac{2}{\pi}\left[x\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1.\frac{\sin nx}{n}dx\right] = \frac{2}{n\pi}\left[\pi\sin n\pi - \sin 0 - \frac{-\cos nx}{n}\Big|_0^{\pi}\right]$$

$$= \frac{2}{n\pi}\left[0 - 0 + \frac{\cos n\pi - \cos 0}{n}\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

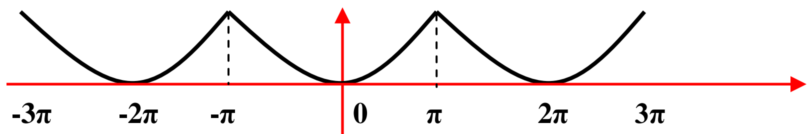
$$\therefore a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -4/n^2\pi, & \text{if } n \text{ is odd} \end{cases}$$

Thus Fourier series of given function is given by $|x| = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{\cos nx}{n^2}$

If we put x = 0, we obtain: $0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2}\ldots\right] \Rightarrow -\frac{\pi}{2} = -\frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2}\ldots\right] \Rightarrow \frac{\pi^2}{8} = \left[1 + \frac{1}{3^2} + \frac{1}{5^2}\ldots\right]$

Example 06: Find the Fourier series of: $f(x) = x^2, -\pi < x < \pi$ where, f(x) is periodic function of period 2π. (**Note:** Here f(x) is even function hence; $b_0 = 0$)

Solution: Now, $a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\frac{x^3}{3}\Big|_0^{\pi} = \frac{2}{3\pi}\left[\pi^3\right] = \frac{2\pi^2}{3}$



$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2.\cos nx\,dx = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x.\frac{\sin nx}{n}dx\right] \text{(Use integration by parts formula)}$$

$$= \frac{2}{n\pi}\left[\pi^2 \sin n\pi - 0.\sin 0 - 2\int_0^{\pi} x \sin nx\,dx\right]$$

$$= \frac{2}{n\pi}\left[0 - 0 - 2\left(x\frac{-\cos nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1.\frac{-\cos nx}{n}dx\right)\right] \qquad \sin n\pi = \sin 0 = 0$$

$$= \frac{2}{n^2\pi}\left[2(\pi\cos n\pi - 0.\cos 0) - \frac{\sin nx}{n}\Big|_0^{\pi}\right] \qquad \cos n\pi = (-1)^n$$

$$= \frac{2}{n^2\pi}\left[2\pi(-1)^n - \frac{\sin n\pi - \sin 0}{n}\right] = \frac{2}{n^2\pi}\left[2\pi(-1)^n - \frac{0-0}{n}\right] = \frac{4}{n^2}(-1)^n$$

Thus Fourier series of given function is given by:

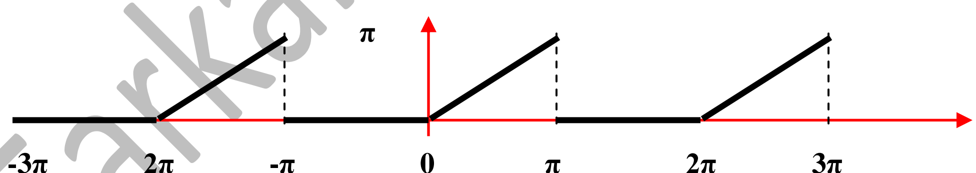
$$x^2 = \frac{\pi^2}{3} + 4\sum_1^{\infty}(-1)^n\frac{\cos nx}{n^2} = \frac{\pi^2}{3} - 4\left[\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right]$$

Putting x = 0 and using the fact that cos 0 = 1, we get

$$0 = \frac{\pi^2}{3} - 4\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right] \qquad \Rightarrow \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Example 07: Find the Fourier series of the function f(x) defined as under where; f(x) is periodic function of period 2π. The graph of f(x) with periodic extension is shown as under.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$



Solution: Consider, $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x\,dx\right] = \frac{1}{\pi}\left[0 + \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi^2}{2\pi} = \frac{\pi}{2}$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x.\cos nx\,dx\right] = \frac{1}{\pi}\left[0 + x\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1.\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{n\pi}\left[(\pi \sin n\pi - 0) + \frac{\cos nx}{n}\Big|_0^{\pi}\right] = \frac{1}{n^2\pi}(\cos n\pi - \cos 0) = \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\text{Thus, } a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -2/n^2\pi, & n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x.\sin nx\,dx\right] = \frac{1}{\pi}\left[0 - x\frac{\cos nx}{n}\Big|_0^\pi - \int_0^\pi 1.\frac{-\cos nx}{n}dx\right]$$

$$= \frac{1}{n\pi}\left[-(\pi\cos n\pi - 0) + \frac{\sin nx}{n}\Big|_0^\pi\right] = \frac{1}{n\pi}\left(-(-1)^n\pi + \frac{\sin n\pi - \sin 0}{n}\right) = \frac{1}{n\pi}\left[(-1)^{n+1}\pi + 0\right]$$

Thus, $b_n = \dfrac{(-1)^{n+1}}{n}$. Now, the Fourier series of given function is given by

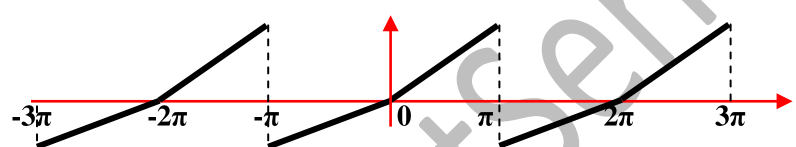
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} a_n \sin nx$$

$$f(x) = \frac{1}{2}\frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1,3,5..}^{\infty}\frac{\cos nx}{n^2} + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}\sin nx$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right]$$

Example 08: Find the Fourier series of the function f(x) defined as under where f(x) is periodic function of period 2π.

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ 2x & 0 < x < \pi \end{cases}$$



Solution: $a_0 = \dfrac{1}{\pi}\left[\int_{-\pi}^{0} x\,dx + 2\int_0^\pi x\,dx\right] = \dfrac{1}{\pi}\left[\dfrac{x^2}{2}\Big|_{-\pi}^0 + 2\dfrac{x^2}{2}\Big|_0^\pi\right] = \dfrac{1}{\pi}\left[\dfrac{0-\pi^2}{2} + \pi^2 - 0\right] = \dfrac{\pi}{2}$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\,dx + 2\int_0^\pi x\cos nx\ dx\right]$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n}\Big|_{-\pi}^0 - \int_{-\pi}^0 1.\frac{\sin nx}{n}dx + 2x\frac{\sin nx}{n}\Big|_0^\pi - \int_0^\pi 2\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{\pi}\left[(0-0) + \frac{\cos nx}{n^2}\Big|_{-\pi}^0 + (0-0) + 2\frac{\cos nx}{n^2}\Big|_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos 0 - \cos(-n\pi)}{n^2} + 2\frac{\cos(n\pi) - \cos 0}{n^2}\right] = \frac{1}{\pi}\left[\frac{1 - (-1)^n + 2(-1)^n - 2}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right] \qquad \Rightarrow a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -2/n^2\pi, & \text{if } n \text{ is odd}\end{cases}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^0 x\sin nx\,dx + 2\int_0^\pi x\sin nx\ dx\right]$$

$$= \frac{1}{\pi}\left[x\frac{-\cos nx}{n}\Big|_{-\pi}^0 - \int_{-\pi}^0 1.\frac{-\cos nx}{n}dx + 2x\frac{-\cos nx}{n}\Big|_0^\pi - \int_0^\pi 2\frac{-\cos nx}{n}dx\right]$$

$$=\frac{1}{\pi}\left[-\frac{0-\pi\cos(-n\pi)}{n}+0-2\frac{\pi\cos nx-0}{n}+0\right] \qquad \text{NOTE}: \int_0^{\pi}\cos nx\,dx=0$$

$$=\frac{1}{\pi}\left[\frac{\pi(-1)^n}{n}-2\frac{\pi(-1)^n}{n}\right]=\frac{\pi}{n\pi}\left[(-1)^n-2(-1)^n\right]=\frac{1}{n}\left[-(-1)^n\right] \Rightarrow b_n=\frac{(-1)^{n+1}}{n}$$

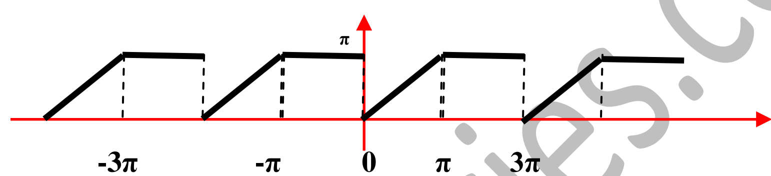
Thus Fourier series of given function is given by:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx=\frac{1}{2}\cdot\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\cos nx}{n^2}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$$

$$=\frac{\pi}{4}+\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\right]+\left[\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\dots\right]$$

Example 09: Find the Fourier series of the function f(x) defined as

$$f(x)=\begin{cases}\pi, & -\pi<x<0\\ x & 0<x<\pi\end{cases}$$



Solution: The Fourier series of a periodic function is given by:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}\pi\,dx+\int_0^{\pi}x\,dx\right]=\frac{1}{\pi}\left[\pi x\Big|_{-\pi}^{0}+\frac{x^2}{2}\Big|_0^{\pi}\right]=\frac{1}{\pi}\left[\pi\big(0-(-\pi)\big)+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left[\frac{2\pi^2+\pi^2}{2}\right]=\frac{3\pi}{2}$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}\pi\cos nx\,dx+\int_0^{\pi}x\cos nx\,dx\right]=\frac{1}{\pi}\left[\pi\frac{\sin nx}{n}\Big|_{-\pi}^{0}+x\frac{\sin nx}{n}\Big|_0^{\pi}-\int_0^{\pi}1.\frac{\sin nx}{n}\right]$$

$$=\frac{1}{n\pi}\left((0-0)+(0-0)-\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right)=\frac{1}{n^2\pi}\big(\cos n\pi-\cos 0\big)=\frac{1}{n^2\pi}\big((-1)^{-n}-1\big)$$

$$\therefore a_n=\begin{cases}0, & \text{if } n \text{ is even}\\ -2/n^2\pi, & \text{if } n \text{ is odd}\end{cases}$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}\pi\sin nx\,dx+\int_0^{\pi}x\sin nx\,dx\right]=\frac{1}{\pi}\left[-\pi\frac{\cos nx}{n}\Big|_{-\pi}^{0}+x\frac{-\cos nx}{n}\Big|_0^{\pi}-\int_0^{\pi}1.\frac{-\cos nx}{n}\right]$$

$$=\frac{1}{n\pi}\left(-\pi\big(\cos 0-\cos[-n\pi]\big)-(\pi\cos n\pi-0)+\left[\frac{\sin nx}{n}\right]_0^{\pi}\right)$$

$$=\frac{1}{n\pi}\left(\pi[-1+(-1)^n]-\pi(-1)^n+\frac{0-0}{n}\right)=\frac{\pi}{n\pi}\left[-1+(-1)^n-(-1)^n\right]=\frac{-1}{n}$$

Thus Fourier series of given function is given by:

$$f(x)=\frac{1}{2}a_0-\frac{2}{\pi}\sum_{n=1,3,5\dots}^{\infty}\frac{\cos nx}{n^2}-\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

Example 10: Find a Fourier series representing the function f(x) = x from 0 to 2 π

Solution: The Fourier series is; $f(x)=\frac{1}{2}a_0+\sum_1^\infty a_n\cos nx+\sum_1^\infty b_n\sin nx$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{2\pi}x\,dx=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}=\frac{1}{2\pi}\left(4\pi^2\right)=2\pi$$

$a_n=\frac{1}{\pi}\int_0^{2\pi}x\cos nx\ dx.$ Integrating by parts, we get

$$a_n=\frac{1}{n^2\pi}\left[(-1)^n-1\right]\qquad\Rightarrow a_n=\begin{cases}0, & \text{if } n \text{ is even}\\ -2/n^2\pi, & \text{if } n \text{ is odd}\end{cases}$$

$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\ dx=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\ dx.$ Integrating by parts, we get

$$=\frac{1}{\pi}\left[x\frac{-\cos nx}{n}\Big|_0^{2\pi}-\int_0^{2\pi}1.\frac{-\cos nx}{n}dx\right]=\frac{1}{n\pi}\left[-\left(2\pi\cos 2n\pi-0\cos 0\right)+\frac{\sin nx}{n}\Big|_0^{2\pi}\right]$$

$$=\frac{1}{n\pi}\left[-\left(2\pi-0\right)+\frac{\sin 2n\pi-\sin 0}{n}\right]=\frac{1}{n\pi}\left[-2\pi+(0-0)\right]=\frac{-2\pi}{n\pi}=\frac{-2}{n}$$

Hence Fourier series of given function is:

$$f(x)=\frac{1}{2}(2\pi)+\sum_{n=1,3,5...}^{\infty}\left(-\frac{2}{n^2\pi}\right)\cos nx+\sum_1^\infty\left(-\frac{2}{n}\right)\sin nx$$

$$x=\pi-\frac{2}{\pi}\sum_{n=1,3,5...}^{\infty}\frac{\cos nx}{n^2}-2\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

Example 11: Find the Fourier series of the function f(x) defined as under.

$$f(x)=\begin{cases}0, & -\pi\le x\le 0\\ \pi, & 0\le x\le\pi\end{cases}$$

Solution: The Fourier series is; $f(x)=\frac{1}{2}a_0+\sum_1^\infty a_n\cos nx+\sum_1^\infty b_n\sin nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}0dx+\int_0^{\pi}\pi dx\right]=\frac{\pi}{\pi}\int_0^{\pi}1\ dx=\left[x\right]_0^{\pi}=\left(\pi-0\right)=\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\cos nxdx+\int_0^{\pi}\pi\cos nxdx\right]=\int_0^{\pi}\cos nxdx$$

$$=\left[\frac{\sin nx}{n}\right]_0^{\pi}=\frac{1}{n}\left(\sin n\pi-\sin 0\right)\ \Rightarrow a_n=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}0\sin nxdx+\int_0^{\pi}\pi\sin nxdx\right]=\int_0^{\pi}\sin nxdx$$

$$=\left[-\frac{\cos nx}{n}\right]_0^{\pi}=-\frac{1}{n}\left(\cos n\pi-\cos 0\right)=-\frac{1}{n}\left[\left(-1\right)^n-1\right]=\begin{cases}\frac{2}{n}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

Hence the Fourier series of the function is given by:

$$f(x)=\frac{\pi}{2}+2\sum_{n=1,3,5...}^{\infty}\frac{\sin nx}{n}=\frac{\pi}{2}+2\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+...\right]$$

Example 12: Find the Fourier series of the function f(x) defined as under.

$$f(x)=\begin{cases}-a, & -\pi\le x\le 0\\ +a, & 0\le x\le \pi\end{cases}$$

Solution: The Fourier series is; $f(x)=\frac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos nx+\sum_{1}^{\infty}b_n\sin nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-a)dx+\int_{0}^{\pi}a\ dx\right]=\frac{a}{\pi}\left[-\int_{-\pi}^{0}1\ dx+\int_{0}^{\pi}1\ dx\right]$$

$$=\frac{a}{\pi}\left[-x\Big|_{-\pi}^{0}+x\Big|_{0}^{\pi}\right]=\frac{a}{\pi}\left[-(0-(-\pi))+(\pi-0)\right]=\frac{a}{\pi}\left[-\pi+\pi\right]\quad\Rightarrow a_0=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-a)\cos nx\ dx+\int_{0}^{\pi}a\cos nx\ dx\right]=\frac{a}{\pi}\left[-\int_{-\pi}^{0}\cos nx\ dx+\int_{0}^{\pi}\cos nx\ dx\right]$$

$$=\frac{a}{\pi}\left[-\frac{\sin nx}{n}\Big|_{-\pi}^{0}+\frac{\sin nx}{n}\Big|_{0}^{\pi}\right]=\frac{a}{n\pi}\left[-(\sin 0-\sin(-n\pi))+(\sin n\pi-\sin 0)\right]=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nxdx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-a)\sin nx\ dx+\int_{0}^{\pi}a\sin nx\ dx\right]=\frac{a}{\pi}\left[-\int_{-\pi}^{0}\sin nx\ dx+\int_{0}^{\pi}\sin nx\ dx\right]$$

$$=\frac{a}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{0}-\frac{\cos nx}{n}\Big|_{0}^{\pi}\right]=\frac{a}{n\pi}\left[\cos 0-\cos n\pi-(\cos n\pi-\cos 0)\right]$$

$$=\frac{a}{n\pi}(1-2\cos n\pi+1)=\frac{a}{n\pi}\left(2-2(-1)^n\right)=\frac{2a}{n\pi}\left[1-(-1)^n\right]=\begin{cases}\frac{4a}{n\pi}, & \text{if } n \text{ is odd}\\ 0, & \text{if } n \text{ is even}\end{cases}$$

Hence the Fourier series of given function is:

$$f(x)=\frac{4a}{\pi}\sum_{1}^{\infty}\frac{\sin(2n-1)x}{2n-1}=\frac{4a}{\pi}\left[\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\cdots\right]$$

Example 13: Find the Fourier series representing the function $f(x)=e^x,\ 0<x<2\pi$

Solution: The Fourier series is; $f(x)=\frac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos nx+\sum_{1}^{\infty}b_n\sin nx$

$$a_0=\frac{1}{\pi}\int_{0}^{2\pi}f(x)dx=\frac{1}{\pi}\int_{0}^{2\pi}e^x dx=\frac{1}{\pi}\left[e^x\right]_0^{2\pi}=\frac{1}{\pi}\left[e^{2\pi}-1\right]$$

$$a_n=\frac{1}{\pi}\int_{0}^{2\pi}f(x)\cos nx\ dx=\frac{1}{\pi}\int_{0}^{2\pi}e^x\cos nx\ dx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx+n\sin nx)\right]_0^{2\pi}\qquad \text{NB}:\int e^{ax}\cos bx\ dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$$

$$=\frac{1}{\pi(n^2+1)}\left[e^{2\pi}(\cos 2n\pi+n\sin 2n\pi)-1\right]\Rightarrow a_n=\frac{1}{\pi(n^2+1)}\left(e^{2\pi}-1\right)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\ dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi} \quad \text{NB}: \int e^{ax}\sin bx\ dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= \frac{1}{\pi(n^2+1)}\left[e^{2\pi}(\sin 2n\pi - n\cos 2n\pi) + n\right] \Rightarrow b_n = \frac{-n(e^{2\pi}-1)}{\pi(n^2+1)}$$

Hence the required Fourier series is given by:

$$f(x) = \frac{1}{2\pi}(e^{2\pi}-1) + \sum_1^{\infty}\frac{e^{2\pi}-1}{\pi(n^2+1)}\cos nx + \sum_1^{\infty}\frac{-n(e^{2\pi}-1)}{\pi(n^2+1)}\sin nx$$

$$\Rightarrow f(x) = \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_1^{\infty}\left(\frac{\cos nx}{n^2+1} - \frac{n\sin nx}{n^2+1}\right)\right]$$

Example 14: Find the Fourier series of the function defined by

$$f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ \sin x, & 0 \le x \le \pi \end{cases}$$

Solution: The Fourier series is: $f(x) = \frac{1}{2}a_0 + \sum_1^{\infty} a_n \cos nx + \sum_1^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi}\sin x\,dx\right] = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = -\frac{1}{\pi}(\cos\pi - 1) = \frac{2}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\cos nx\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right] = \frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} = \frac{1}{2\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ \frac{1}{2\pi}\left(\frac{2}{n+1} - \frac{2}{n-1}\right) = \frac{-2}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\sin nx\,dx + \int_0^{\pi}\sin x\sin nx\,dx\right] = \frac{1}{\pi}\int_0^{\pi}\sin x\sin nx\,dx \qquad (1)$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right]dx \quad \text{NB}: 2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi} = \frac{1}{2\pi}\left[\frac{\sin(n-1)\pi}{n-1} - \frac{\sin(n+1)\pi}{n+1}\right] \Rightarrow b_n = 0, \text{ for } n > 1$$

Now substituting $n=1$ in (1), we have

$$b_1=\frac{1}{\pi}\int_0^{\pi}\sin^2 x dx=\frac{1}{\pi}\int_0^{\pi}\frac{1-\cos 2x}{2}dx=\frac{1}{2\pi}\left[x-\frac{\sin 2x}{2}\right]_0^{\pi}=\frac{1}{2\pi}\left(\pi-\frac{\sin 2\pi}{2}-0+0\right)=\frac{1}{2}$$

Thus the required Fourier series is: $f(x)=\frac{1}{\pi}-\frac{2}{\pi}\sum_1^{\infty}\frac{\cos 2nx}{4n^2-1}+\frac{1}{2}\sin x$

FOURIER SERIES WITH ARBITRARY PERIOD

In many engineering problems, the period of the function f (x) required to be expanded into Fourier series is not 2π but some other interval, say 2L. The Fourier series of such functions is expressed as under:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right) \qquad \textbf{(I)}$$

To show this we have, let the Fourier series of a periodic function f(t) with period 2π is given by:

$$f(t)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nt+\sum_{n=1}^{\infty}b_n\sin nt$$

Substituting $t = \pi x/L$. Now if $t = -\pi$, then $x = -L$ and if $t = \pi$ then $x = L$. This shows that if f(x) is a periodic function with period 2L, its Fourier representation is given by (I).

Example 01: Find the Fourier series of the function defined as $f(x)=\begin{cases}0, & -2<x<0\\ 1, & 0<x<2\end{cases}$

Solution: Here 2L = 4 so that L = 2. Thus,

$$a_0=\frac{1}{2}\left[\int_{-2}^{0}0dx+\int_0^2 1dx\right]=\frac{1}{2}\left[0+x\Big|_0^2\right]=\frac{1}{2}[2-0]=1$$

$$a_n=\frac{1}{2}\left[\int_{-2}^{0}0.\cos\left(\frac{n\pi x}{2}\right)dx+\int_0^2 1.\cos\left(\frac{n\pi x}{2}\right)dx\right]=\frac{1}{2}\left[0+\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)_0^2\right]=\frac{2}{2n\pi}[\sin n\pi-\sin 0]=0$$

$$b_n=\frac{1}{2}\left[\int_{-2}^{0}0.\sin\left(\frac{n\pi x}{2}\right)dx+\int_0^2 1.\sin\left(\frac{n\pi x}{2}\right)dx\right]=\frac{1}{2}\left[0-\frac{2}{n\pi}\text{cod}\left(\frac{n\pi x}{2}\right)_0^2\right]$$

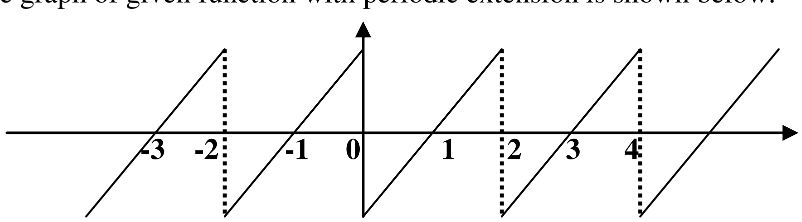
$$=\frac{-2}{2n\pi}[\cos n\pi-\cos 0]=\frac{-1}{n\pi}\left[(-1)^n-1\right]. \text{ Thus,} \quad b_n=\begin{cases}0, & \text{if } n \text{ is even}\\ -2/n\pi, & \text{if } n \text{ is odd}\end{cases}$$

Hence the Fourier series of given function is $\quad f(x)=\frac{1}{2}-\frac{2}{\pi}\sum_{n=1,3,5...}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{2}$

Example 02: Let a function f(x) is defined as follows. Find the Fourier series of f(x).

$$f(x)=\begin{cases}x+1, & -1<x<0\\ x-1, & 0<x<1\end{cases}$$

Solution: The graph of given function with periodic extension is shown below.



We see that f(x) is an odd function, hence a_0 and a_n are zero. Thus we are required to find b_n only. Here 2L = 2 hence L = 1. Therefore,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\frac{\sin n\pi x}{L}dx = \left[\int_{-1}^{0}(x+1).\sin n\pi x\,dx + \int_{0}^{1}(x-1).\sin n\pi x\,dx\right] \text{NOTE}: L=1$$

$$= \left[(x+1)\frac{-\cos n\pi x}{n\pi}\Big|_{-1}^{0} - \int_{-1}^{0}1.\frac{-\cos n\pi x}{n\pi}dx + (x-1)\frac{-\cos n\pi x}{n\pi}\Big|_{0}^{1} - \int_{-1}^{0}1.\frac{-\cos n\pi x}{n\pi}dx\right]$$

$$= \frac{1}{n\pi}\left[-(\cos 0 - 0.\cos(-n\pi)) + \int_{-1}^{0}\cos n\pi x\,dx - (0\cos(n\pi) - (-\cos 0)) + \int_{0}^{1}\cos n\pi x\,dx\right]$$

$$= \frac{1}{n\pi}\left[-1 + \frac{\sin n\pi x}{n\pi}\Big|_{-1}^{0} - 1 + \frac{\sin n\pi x}{n\pi}\Big|_{0}^{1}\right] = \frac{1}{n\pi}\left[-1+(0-0)-1+(0-0)\right] = -\frac{2}{n\pi}$$

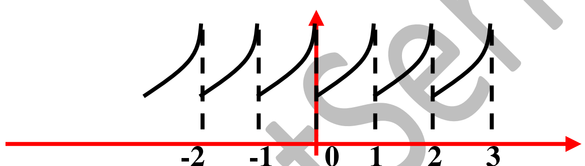
$$\text{NOTE}: \sin 0 = \sin n\pi = 0$$

Thus Fourier series of given function is given by

$$f(x) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n} = -\frac{2}{\pi}\left[\frac{\sin \pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \dots\right]$$

Example 03: Find the Fourier series of f(x) = x^2 for 1< x < 2, where f(x) is a periodic function.

Solution: Here 2L = 1 or L = 1/2. Thus graph of f(x) with periodic extension is shown as under.



$$a_0 = \frac{1}{L}\int_{1}^{2} f(x)\,dx = \frac{1}{(1/2)}\int_{1}^{2} x^2\,dx = 2\frac{x^3}{3}\Big|_{0}^{2} = \frac{2}{3}[8-1] = \frac{14}{3}$$

$$a_n = \frac{1}{L}\int_{1}^{2} x^2\cos\frac{n\pi x}{L}dx = 2\int_{1}^{2} x^2\cos 2n\pi x\,dx = 2\left[x^2\frac{\sin 2n\pi x}{2n\pi}\Big|_{1}^{2} - \int_{1}^{2} 2x\frac{\sin 2n\pi x}{2n\pi}dx\right]$$

$$= \frac{2}{2n\pi}\left[0 - 0 - 2\int_{1}^{2} x\sin 2n\pi x\,dx\right] \quad = \frac{-2}{n\pi}\left[\int_{1}^{2} x\sin 2n\pi x\,dx\right]$$

$$= \frac{-2}{n\pi}\left[x\frac{-\cos 2n\pi x}{2n\pi}\Big|_{1}^{2} - \int_{1}^{2} 1.\frac{-\cos 2n\pi x}{2n\pi}dx\right]$$

$$= \frac{-2}{2n^2\pi^2}\left[-(2\cos 4n\pi - \cos 2n\pi) + \int_{1}^{2}\cos 2n\pi x\;dx\right] = \frac{2}{2n^2\pi^2}\left[-(2-1) - \frac{\sin 2n\pi x}{2n\pi}\Big|_{1}^{2}\right]$$

$$= \frac{1}{n^2\pi^2}\left[-1 - (\sin 4n\pi - \sin 2n\pi)\right] = \frac{1}{n^2\pi^2}\left[-1-(0-0)\right] = \frac{-1}{n^2\pi^2}$$

$$b_n = \frac{1}{L}\int_{1}^{2} x^2\sin\frac{n\pi x}{L}dx = 2\int_{1}^{2} x^2\sin 2n\pi x\,dx = 2\left[x^2\frac{-\cos 2n\pi x}{2n\pi}\Big|_{1}^{2} - \int_{1}^{2} 2x\frac{-\cos 2n\pi x}{2n\pi}dx\right]$$

$$= \frac{2}{2n\pi}\left[-(4\cos 4n\pi - \cos 2n\pi) + 2\int_{1}^{2} x\cos 2n\pi x\,dx\right] \quad = \frac{1}{n\pi}\left[-(4-1) + 2\int_{1}^{2} x\cos 2n\pi x\,dx\right]$$

$$=\frac{1}{n\pi}\left[-3+2x\frac{\sin 2n\pi x}{2n\pi}\Bigg|_1^2-2\int_1^2 1.\frac{\sin 2n\pi x}{2n\pi}dx\right]=\frac{1}{n\pi}\left[-3+2(0-0)+2\left(\frac{\cos 2n\pi}{4n^2\pi^2}\right)\Bigg|_1^2\right]$$

$$=\frac{1}{n\pi}\left[-3+\frac{1}{2n^2\pi^2}(\cos 4n\pi-\cos 2n\pi)\right]=\frac{1}{n\pi}\left[-3+\frac{1}{2n^2\pi^2}(1-1)\right]=\frac{-3}{n\pi}$$

Thus the Fourier series of given function is:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}=\frac{7}{3}+\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}-\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n}$$

Example 04: Find the Fourier series of the function f defined by $f(x)=\begin{cases}1, & -2<x<0\\ -1, & 0<x<2\end{cases}$

Solution: The Fourier series is: $f(x)=\frac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$.

Here, the period $=2L=4\Rightarrow L=2\cdot$

$$a_0=\frac{1}{2}\int_{-2}^{2}f(x)dx=\frac{1}{2}\left[\int_{-2}^{0}1dx+\int_{0}^{2}-1dx\right]=\frac{1}{2}\left[x|_{-2}^{0}-x|_0^2\right]=\frac{1}{2}\left[(0+2)-(2-0)\right]=0$$

$$a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\left(\frac{n\pi x}{2}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}\cos\left(\frac{n\pi x}{2}\right)dx+\int_{0}^{2}-\cos\left(\frac{n\pi x}{2}\right)dx\right]$$

$$=\frac{1}{2}\left[\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right)\Bigg|_{-2}^{0}-\left(\frac{2}{n\pi}\right)\sin\left(\frac{n\pi x}{2}\right)\Bigg|_{0}^{2}\right]=\frac{1}{n\pi}\left[(0+\sin n\pi)-(\sin n\pi-0)\right]=0$$

$$b_n=\frac{1}{2}\int_{-2}^{2}f(x)\sin\left(\frac{n\pi x}{2}\right)dx=\frac{1}{2}\left[\int_{-2}^{0}\sin\left(\frac{n\pi x}{2}\right)dx+\int_{0}^{2}-\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$=\frac{1}{2}\left[-\left(\frac{2}{n\pi}\right)\cos\left(\frac{n\pi x}{2}\right)\Bigg|_{-2}^{0}+\left(\frac{2}{n\pi}\right)\cos\left(\frac{n\pi x}{2}\right)\Bigg|_{0}^{2}\right]=-\frac{1}{n\pi}\left[(1-\cos n\pi)-(\cos n\pi-1)\right]$$

$$=-\frac{1}{n\pi}\left[2-2\cos n\pi\right]=-\frac{2}{n\pi}\left[1-(-1)^n\right]\Rightarrow b_n=\begin{cases}0, & \text{if } n \text{ is even}\\ -\frac{4}{n\pi}, & \text{if } n \text{ is odd}\end{cases}$$

Thus the required Fourier series is: $f(x)=-\frac{4}{\pi}\sum_{1}^{\infty}\frac{1}{2n-1}\sin\left[\frac{(2n-1)\pi x}{2}\right]$

Example 05: Expand into a Fourier series the function $f(x)=\begin{cases}0, & -2<x<0\\ x, & 0<x<2\end{cases}$

Solution: The Fourier series of given function is

$$f(x)=\frac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$$

Here, the period $=2L=4\Rightarrow L=2\cdot$ Therefore,

$$a_0=\frac{1}{2}\int_{-2}^{2}f(x)dx=\frac{1}{2}\int_{0}^{2}xdx=\frac{1}{4}\left[x^2\right]_0^2=1$$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)dx.$. Integration by parts, we get

$$a_n = \frac{1}{2}\left[\frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)+\left(\frac{2}{n\pi}\right)\left(\frac{2}{n\pi}\right)\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 = \frac{1}{n\pi}\left[\left(2\sin n\pi+\frac{2}{n\pi}\cdot\cos n\pi\right)-\left(0+\frac{2}{n\pi}\right)\right]$$

$$= \frac{1}{n\pi}\left[\left(0+\frac{2}{n\pi}(-1)^n\right)-\left(\frac{2}{n\pi}\right)\right] = \frac{2}{n^2\pi^2}\left[(-1)^n-1\right].$$ Therefore,

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ -\frac{4}{n^2\pi^2}, & \text{if } n \text{ is odd} \end{cases}$$

Now, $b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx = \frac{1}{2}\int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)dx.$ Using integration by parts, we get

$$= -\frac{1}{n\pi}\left[\left(2\cos n\pi-\frac{2}{n\pi}\cdot\sin n\pi\right)-(0-0)\right] \Rightarrow b_n = -\frac{2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

Thus the required Fourier series is:

$$f(x) = \frac{1}{2}-\frac{4}{\pi^2}\sum_{1}^{\infty}\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)\pi x}{2}\right]+\frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{2}\right)$$

Example 06: Find a Fourier series of $f(x) = 2x + 1,\ 0 < x < 2$

Solution: The Fourier series is: $f(x) = \frac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$

Here, the period $= 2L = 2 \Rightarrow L = 1\cdot$

$$a_0 = \int_0^2 f(x)dx = \int_0^2 (2x+1)dx = \left[x^2+x\right]_0^2 = 6$$

$$a_n = \int_0^2 f(x)\cos n\pi x\,dx = \int_0^2 (2x+1)\cos n\pi x\,dx = 2\int_0^2 x\cos n\pi x\,dx + \int_0^2 \cos n\pi x\,dx$$

$$= 2\left[\frac{x\sin n\pi x}{n\pi}+\frac{\cos n\pi x}{n^2\pi^2}\right]_0^2+\left[\frac{\sin n\pi x}{n\pi}\right]_0^2$$

$$= 2\left[\frac{2\sin 2n\pi}{n\pi}+\frac{\cos 2n\pi}{n^2\pi^2}-\frac{1}{n^2\pi^2}\right]+\frac{1}{n\pi}(\sin 2n\pi-0) \Rightarrow a_n = 2\left(\frac{1}{n^2\pi^2}-\frac{1}{n^2\pi^2}\right) = 0$$

$$b_n = \int_0^2 f(x)\sin n\pi x dx = \int_0^2 (2x+1)\sin n\pi x dx = 2\int_0^2 x\sin n\pi x dx + \int_0^2 \sin n\pi x dx$$

$$= \left[-\frac{x\cos n\pi x}{n\pi}+\frac{\sin n\pi x}{n^2\pi^2}\right]_0^2-\left[\frac{\cos n\pi x}{n\pi}\right]_0^2$$

$$= -\frac{2}{n\pi}\left[2\cos 2n\pi-\frac{1}{n\pi}\sin 2n\pi-0+0\right]-\frac{1}{n\pi}(\cos 2n\pi-1) \Rightarrow b_n = -\frac{4}{n\pi}$$

Thus the required Fourier series is:

$$f(x) = \frac{1}{2}(6)+\sum_{1}^{\infty}(0)\cos n\pi x+\sum_{1}^{\infty}\left(-\frac{4}{n\pi}\right)\sin n\pi x = 3-\frac{4}{\pi}+\sum_{1}^{\infty}\frac{\sin n\pi x}{n}$$

Example 07: Expand into Fourier series the function

$$f(x) = x^2, -1 < x < 1$$

Solution: The Fourier series of f(x) is: $f(x) = \frac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

Here, the period $= 2L = 2 \Rightarrow L = 1 \cdot$ Now,

$$a_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{1} x^2 dx = 2\int_{0}^{1} x^2 dx = \frac{2}{3}\left[x^3\right]_0^1 = \frac{2}{3}$$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\, dx = \int_{-1}^{1} x^2 \cos n\pi x\, dx = 2\int_{0}^{1} x^2 \cos n\pi x\, dx$$

Applying the formula: $\int uv dx = u\int v dx - \int \left[u' \int v\, dx\right]dx$, we have

$$a_n = 2\left[\frac{x^2 \sin n\pi x}{n\pi} - (2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1 = \frac{2}{n\pi}\left[x^2 \sin n\pi x + \frac{2x\cos n\pi x}{n\pi} - \frac{2\sin n\pi x}{n^2\pi^2}\right]_0^1$$

$$= \frac{2}{n\pi}\left[\sin n\pi + \frac{2x\cos n\pi}{n\pi} - \frac{2\sin n\pi}{n^2\pi^2}\right] \Rightarrow a_n = \frac{4}{n^2\pi^2}(-1)^n$$

$b_n = \int_{-1}^{1} f(x)\sin n\pi x\, dx = \int_{-1}^{1} x^2 \sin n\pi x\, dx = 0$,since, $x^2 \sin n\pi x$ is an odd function.

Thus the required Fourier series is:

$$f(x) = \frac{1}{2}\left(\frac{2}{3}\right) + \sum_{1}^{\infty} \frac{4}{n^2\pi^2}(-1)^n \cos n\pi x + \sum_{1}^{\infty}(0)\sin n\pi x = \frac{1}{3} + \frac{4}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x$$

Example 08: Represent the following function by Fourier series f(x) = sin x, $0 < x < \pi$

Solution: The Fourier series is: $f(x) = \frac{1}{2}a_0 + \sum_{1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

Here $L = \pi$. Now, we produce here an alternative method. Readers are advised to understand the steps taken towards solving the problem.

Substituting $x = \frac{\pi}{2\pi} t \Rightarrow x = \frac{t}{2}, \quad 0 \le t \le 2\pi \cdot$ Then, $g(t) = \sin\frac{t}{2}, \quad 0 \le t \le 2\pi$

Suppose $$g(t) = \frac{1}{2}a_0 + \sum_{1}^{\infty}(a_n \cos nt + b_n \sin nt)$$

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} g(t)dt = \frac{1}{\pi}\int_{0}^{2\pi} \sin\frac{t}{2}dt = \frac{1}{\pi}\left[-\frac{\cos t/2}{1/2}\right]_0^{2\pi} = -\frac{2}{\pi}\left[\cos\frac{t}{2}\right]_0^{2\pi} = -\frac{2}{\pi}(\cos\pi - 1) = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} g(t)\cos nt\, dt = \frac{1}{\pi}\int_{0}^{2\pi} \sin\frac{t}{2}\cos nt\, dt$$

Applying the formula $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$, we have

$$= \frac{1}{2\pi}\int_{0}^{2\pi}\left[\sin\left(\frac{1}{2}+n\right)t + \sin\left(\frac{1}{2}-n\right)t\right]dt = \frac{1}{2\pi}\left[-\frac{\cos\left(n+\frac{1}{2}\right)t}{n+\frac{1}{2}} + \frac{\cos\left(n-\frac{1}{2}\right)t}{n-\frac{1}{2}}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[-\frac{\cos\left(n+\frac{1}{2}\right)2\pi}{n+\frac{1}{2}}+\frac{\cos\left(n-\frac{1}{2}\right)2\pi}{n-\frac{1}{2}}+\frac{1}{n+\frac{1}{2}}-\frac{1}{n-\frac{1}{2}}\right]\quad \text{NOTE}: \cos(\theta-\pi)=-\cos\theta$$

$$=\frac{1}{2\pi}\left[-\frac{\cos(2n\pi+\pi)}{n+\frac{1}{2}}+\frac{\cos(2n\pi-\pi)}{n-\frac{1}{2}}+\frac{1}{n+\frac{1}{2}}-\frac{1}{n-\frac{1}{2}}\right]\quad \text{and}\ \ \cos(\theta+\pi)=-\cos\theta$$

$$=\frac{1}{2\pi}\left[\frac{\cos(2n\pi)}{n+\frac{1}{2}}-\frac{\cos(2n\pi)}{n-\frac{1}{2}}+\frac{1}{n+\frac{1}{2}}-\frac{1}{n-\frac{1}{2}}\right]=\frac{1}{2\pi}\left[\frac{2}{n+\frac{1}{2}}-\frac{2}{n-\frac{1}{2}}\right]=\frac{1}{\pi}\left[\frac{2}{2n+1}-\frac{2}{2n-1}\right]$$

$$=\frac{2}{\pi}\left[\frac{2n-1-2n-1}{4n^2-1}\right]\qquad \Rightarrow a_n=-\frac{4}{\pi\left(4n^2-1\right)}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} g(t)\sin nt\,dt=\frac{1}{\pi}\int_0^{2\pi}\sin\frac{t}{2}\sin nt\,dt$$

Applying the formula: $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$, we have

$$=\frac{1}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{1}{2}-n\right)t-\cos\left(\frac{1}{2}+n\right)t\right]dt=\frac{1}{2\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)t-\cos\left(n+\frac{1}{2}\right)t\right]dt$$

$$=\frac{1}{2\pi}\left[\frac{\sin(2n\pi-\pi)}{\frac{2n-1}{2}}-\frac{\sin(2n\pi+\pi)}{\frac{2n+1}{2}}\right]\Rightarrow b_n=\frac{1}{\pi}\left[-\frac{\sin 2n\pi}{2n-1}+\frac{\sin 2n\pi}{2n+1}\right]=0$$

REMARK: $\sin(\theta-\pi)=-\sin\theta,\ \sin(\theta+\pi)=-\sin\theta$

Thus the required Fourier series is: $g(t)=\frac{1}{2}\left(\frac{4}{\pi}\right)+\sum_1^{\infty}\frac{-4}{\pi\left(4n^2-1\right)}\cos nt=\frac{2}{\pi}-\frac{4}{\pi}\sum_1^{\infty}\frac{\cos nt}{\left(4n^2-1\right)}$

or $$f(x)=\frac{2}{\pi}-\frac{4}{\pi}\sum_1^{\infty}\frac{\cos 2nx}{\left(4n^2-1\right)}$$

Example 09: Expand into Fourier series the function

$$f(x) = \cos x,\quad -\pi/2 < x < \pi/2$$

Solution: The Fourier series is: $f(x)=\frac{1}{2}a_0+\sum_1^{\infty}a_n\cos\left(\frac{n\pi x}{p}\right)+\sum_1^{\infty}b_n\sin\left(\frac{n\pi x}{p}\right)$

Here, the period $=2L=\pi\Rightarrow L=\pi/2$. Now,

$$a_0=\frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2}f(x)dx=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos x dx=\frac{4}{\pi}\int_0^{\pi/2}\cos x dx=\frac{4}{\pi}\left[\sin x\right]_0^{\pi/2}=\frac{4}{\pi}\left(\sin\frac{\pi}{2}-0\right)=\frac{4}{\pi}$$

$$a_n=\frac{1}{\pi/2}\int_{-\pi/2}^{\pi/2}f(x)\cos 2nx\,dx=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\cos 2nx\,dx=\frac{4}{\pi}\int_0^{\pi/2}\cos x\cos 2nx\,dx$$

Applying the formula: $2s\cos\alpha\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta)$, we have

$$a_n=\frac{2}{\pi}\int_0^{\pi/2}\left[\cos(1+2n)x+\cos(1-2n)x\right]dx=\frac{2}{\pi}\int_0^{\pi/2}\left[\cos(2n+1)x+\cos(2n-1)x\right]dx$$

$$=\frac{2}{\pi}\left[\frac{\sin(2n+1)x}{2n+1}+\frac{\sin(2n-1)x}{2n-1}\right]_0^{\pi/2}=\frac{2}{\pi}\left[\frac{\sin(2n+1)\pi/2}{2n+1}+\frac{\sin(2n-1)\pi/2}{2n-1}\right]_0^{\pi/2}$$

$$=\frac{2}{\pi}\left[\frac{\sin(n\pi+\pi/2)}{2n+1}+\frac{\sin(n\pi-\pi/2)}{2n-1}\right]=\frac{2}{\pi}\left[\frac{\cos n\pi}{2n+1}-\frac{\cos n\pi}{2n-1}\right]$$

$$=\frac{2}{\pi}(-1)^n\left[\frac{2n-1-2n-1}{4n^2-1}\right]\qquad\Rightarrow a_n=\frac{4(-1)^{n+1}}{\pi\left(4n^2-1\right)}$$

Now, $b_n=\dfrac{1}{\pi/2}\displaystyle\int_{-\pi/2}^{\pi/2}f(x)\sin 2nx dx=\frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos x\sin 2nx dx=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}2\cos x\sin 2nx\,dx$

$$=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left[\sin(2n+1)x+\sin(2n-1)x\right]dx=-\frac{1}{\pi}\left[\frac{\cos(2n+1)x}{2n+1}+\frac{\cos(2n-1)x}{2n-1}\right]_{-\pi/2}^{\pi/2}=0$$

Notice that cosine odd multiple of $\pi/2$ is always zero and $(2n+1)$ and $(2n-1)$ are odd.

Thus the required Fourier series is

$$f(x)=\frac{1}{2}\left(\frac{4}{\pi}\right)+\sum_1^\infty\frac{4(-1)^{n+1}}{\pi\left(4n^2-1\right)}\cos 2nx=\frac{2}{\pi}+\frac{4}{\pi}\sum_1^\infty\frac{(-1)^{n+1}}{\left(4n^2-1\right)}\cos 2nx$$

Example 10: Find the Fourier series of the function f(x) defined as under.

$$f(x)=\begin{cases}x, & 0<x<1\\ 1, & 1<x<2\end{cases}$$

Solution: The Fourier series is: $f(x)=\dfrac{1}{2}a_0+\displaystyle\sum_1^\infty a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_1^\infty b_n\sin\left(\frac{n\pi x}{L}\right)$

Here, the period $=2L=2\Rightarrow L=1\cdot$ Then,

$$a_0=\int_0^2 f(x)dx=\int_0^1 xdx+\int_1^2 1dx=\left.\frac{x^2}{2}\right|_0^1+\left.x\right|_1^2=\frac{1}{2}+1=\frac{3}{2}$$

$$a_n=\int_0^2 f(x)\cos n\pi x\,dx=\int_0^1 x\cos n\pi x\,dx+\int_1^2\cos n\pi x\,dx=\left[\frac{x\sin n\pi x}{n\pi}+\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1+\left.\frac{\sin n\pi x}{n\pi}\right|_1^2$$

$$=\left[\frac{\sin n\pi}{n\pi}+\frac{\cos n\pi}{n^2\pi^2}-0-\frac{1}{n^2\pi^2}\right]+\frac{1}{n\pi}(\sin 2n\pi-\sin n\pi)=\frac{1}{n^2\pi^2}\left[(-1)^n-1\right]$$

Thus, $a_n=0$, if n is even and $a_n=-\dfrac{2}{n^2\pi^2}$, if n is odd.

Now, $b_n=\displaystyle\int_0^2 f(x)\sin n\pi x\,dx=\int_0^1 x\sin n\pi x\,dx+\int_1^2\sin n\pi x\,dx=\left[-\frac{x\cos n\pi x}{n\pi}+\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1-\left.\frac{\cos n\pi x}{n\pi}\right|_1^2$

$$=\left[-\frac{\cos n\pi}{n\pi}+\frac{\sin n\pi}{n^2\pi^2}\right]-\frac{1}{n\pi}(\cos 2n\pi-\cos n\pi)=-\frac{(-1)^n}{n\pi}-\frac{1}{n\pi}+\frac{(-1)^n}{n\pi}\quad\Rightarrow b_n=-\frac{1}{n\pi}$$

Thus the required Fourier series is:

$$f(x)=\frac{1}{2}\left(\frac{3}{2}\right)+\sum_{1}^{\infty}\left(-\frac{2}{\pi^2}\right)\frac{\cos(2n-1)\pi x}{(2n-1)^2}+\sum_{1}^{\infty}\left(-\frac{1}{n\pi}\right)\sin n\pi x=\frac{3}{4}-\frac{2}{\pi^2}\sum_{1}^{\infty}\frac{\cos(2n-1)\pi x}{(2n-1)^2}-\frac{1}{\pi}\sum_{1}^{\infty}\frac{\sin n\pi x}{n}$$

Example 11: Find the Fourier series of the function f(x) defined as under.

$$f(x)=\begin{cases}0, & -2<x<-1\\ k, & -1<x<1\\ 0, & 1<x<2\end{cases}$$

Solution: The Fourier series is: $f(x)=\frac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)$

Here, the period $2L=4\Rightarrow L=2\cdot$ Now,

$$a_0=\frac{1}{2}\int_{-2}^{2}f(x)dx=\frac{1}{2}\int_{-1}^{1}kdx=\frac{k}{2}[x]_{-1}^{1}=\frac{k}{2}(1+1)=k$$

$$a_n=\frac{1}{2}\int_{-2}^{2}f(x)\cos\left(\frac{n\pi x}{2}\right)dx=\frac{1}{2}\int_{-1}^{1}k\cos\left(\frac{n\pi x}{2}\right)dx=\frac{k}{2}\int_{-1}^{1}\cos\left(\frac{n\pi x}{2}\right)dx=k\int_{0}^{1}\cos\left(\frac{n\pi x}{2}\right)dx$$

$$=k\frac{\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1}{(n\pi/2)}=\frac{2k}{n\pi}\left[\sin\left(\frac{n\pi x}{2}\right)\right]_0^1=\frac{2k}{n\pi}\left[\sin\frac{n\pi}{2}-\sin 0\right]=\frac{2k}{n\pi}\sin\frac{n\pi}{2}.$$ Therefore,

$$a_n=\begin{cases}0, & \text{if n is even}\\ \frac{2k}{\pi(2n-1)}\sin(2n-1)\frac{\pi}{2}, & \text{if n is odd}\end{cases}\quad\text{or}\quad\Rightarrow a_n=\frac{2k(-1)^{2n-1}}{\pi(2n-1)},\text{ n being odd.}$$

$$b_n=\frac{1}{2}\int_{-2}^{2}f(x)\sin\left(\frac{n\pi x}{2}\right)dx=\frac{1}{2}\int_{-1}^{1}k\sin\left(\frac{n\pi x}{2}\right)dx=\frac{k}{2}\int_{-1}^{1}\sin\left(\frac{n\pi x}{2}\right)dx=0$$

[Since; sine is an odd function hence; its integration from -1 to $+1$ is zero].

Thus, Fourier series of given function $f(x)$ is: $f(x)=\frac{k}{2}+\frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{2n-1}}{(2n-1)}\cos(2n-1)\frac{\pi x}{2}$

Example 12: Expand $f(x) = e^x$, $-p < x < p$ into Fourier series.

Solution: The Fourier series is: $f(x)=\frac{1}{2}a_0+\sum_{1}^{\infty}a_n\cos\left(\frac{n\pi x}{p}\right)+\sum_{1}^{\infty}b_n\sin\left(\frac{n\pi x}{p}\right)$

$$a_0=\frac{1}{p}\int_{-p}^{p}f(x)dx=\frac{1}{p}\int_{-p}^{p}e^xdx=\frac{1}{p}\left[e^x\right]_{-p}^{p}=\frac{1}{p}\left(e^p-e^{-p}\right)$$

$$a_n=\frac{1}{p}\int_{-p}^{p}f(x)\cos\left(\frac{n\pi x}{p}\right)dx=\frac{1}{p}\int_{-p}^{p}e^x\cos\left(\frac{n\pi x}{p}\right)dx\qquad \text{NB}:\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\cos bx+b\sin bx\right)$$

$$a_n=\frac{1}{p}\left[\frac{e^x}{1+(n\pi/p)^2}\left\{\cos\left(\frac{n\pi x}{p}\right)+\frac{n\pi}{p}\sin\left(\frac{n\pi x}{p}\right)\right\}\right]_{-p}^{p}=\frac{1}{p}\left[\frac{p^2e^x}{p^2+n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right)+\frac{n\pi}{p}\sin\left(\frac{n\pi x}{p}\right)\right]_{-p}^{p}$$

$$=\frac{1}{p}\left[\frac{p^2e^p}{p^2+n^2\pi^2}\left(\cos n\pi+\frac{n\pi}{p}\sin n\pi\right)-\frac{p^2e^{-p}}{p^2+n^2\pi^2}\left(\cos n\pi-\frac{n\pi}{p}\sin n\pi\right)\right]$$

$$= \frac{p}{p^2+n^2\pi^2}\left[e^{p}(-1)^n - e^{-p}(-1)^n\right] \qquad \Rightarrow\ a_n = \frac{p(-1)^n}{p^2+n^2\pi^2}\left(e^{p}-e^{-p}\right)$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)dx = \frac{1}{p}\int_{-p}^{p} e^{x}\sin\left(\frac{n\pi x}{p}\right)dx \quad \text{NB}: \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

$$b_n = \frac{1}{p}\left[\frac{e^{x}}{1+\left(\frac{n\pi}{p}\right)^2}\left\{\sin\left(\frac{n\pi x}{p}\right) - \frac{n\pi}{p}\cos\left(\frac{n\pi x}{p}\right)\right\}\right]_{-p}^{p} = \frac{1}{p}\left[\frac{e^{x}}{\frac{p^2+n^2\pi^2}{p^2}}\sin\left(\frac{n\pi x}{p}\right) - \frac{n\pi}{p}\cos\left(\frac{n\pi x}{p}\right)\right]_{-p}^{p}$$

$$= \frac{1}{p}\left[\frac{p^2e^{p}}{p^2+n^2\pi^2}\left(\sin n\pi - \frac{n\pi}{p}\cos n\pi\right) - \frac{p^2e^{-p}}{p^2+n^2\pi^2}\left(\sin n\pi - \frac{n\pi}{p}\cos n\pi\right)\right]$$

$$= \frac{-n\pi(-1)^n}{p^2+n^2\pi^2}\left(e^{p}-e^{-p}\right) \Rightarrow b_n = \frac{n\pi(-1)^{n+1}}{p^2+n^2\pi^2}\left(e^{p}-e^{-p}\right)$$

$$= \frac{p}{p^2+n^2\pi^2}\left[e^{p}\left\{-\frac{n\pi}{p}(-1)^n\right\} + e^{-p}\left\{\frac{n\pi}{p}(-1)^n\right\}\right] = \frac{n\pi(-1)^n}{p^2+n^2\pi^2}\left(-e^{p}+e^{-p}\right)$$

Thus the required Fourier series is:

$$f(x) = \frac{1}{2p}\left(e^{p}-e^{-p}\right) + \sum_{1}^{\infty}\frac{p(-1)^n}{p^2+n^2\pi^2}\left(e^{p}-e^{-p}\right)\cos\left(\frac{n\pi x}{p}\right) + \sum_{1}^{\infty}\frac{n\pi(-1)^{n+1}}{p^2+n^2\pi^2}\left(e^{p}-e^{-p}\right)\sin\left(\frac{n\pi x}{p}\right)$$

$$f(x) = \left(e^{p}-e^{-p}\right)\left[\frac{1}{2p} + p\sum_{1}^{\infty}\frac{(-1)^n}{p^2+n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right) + \pi\sum_{1}^{\infty}\frac{n(-1)^{n+1}}{p^2+n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\right]$$

HALF RANGE FOURIER SERIES

Sometimes it is required to obtain a Fourier series of a function f(x) for the range (0, L) which is half the period of the Fourier series. As it is of no importance whatever the function may be outside the interval (0, L), we extend the function to cover the range (–L, L) so that the new function may be odd or even. The Fourier expansion of such a function with half period, therefore, consists of sine or cosine terms only. In such cases the graphs for the values of x in (0, L) are the same but outside (0, L) are different for odd and even functions.

That is why we get different forms of series for the same function as will be clear from the examples 1 and 2 produced below.

Half Range Fourier Sine and Cosine Series

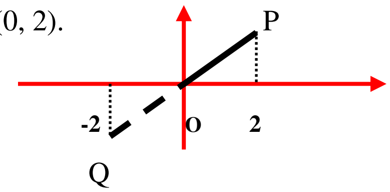
If it is required to expand f(x) as a sine series in (0, L); then we extend the function reflecting it in the origin, so that f(–x) = –f(x). In this case the extended function is odd in (–L, L) and the expansion will give the desired Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \text{ where } b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

If it is required to express f(x) as a cosine series on (0, L), we extend the function reflecting it in the y-axis, so that f(–x) = f(x). Then the extended function is even in (–L, L) and its expansion will give the required Fourier cosine series:

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}, \text{ where } a_0 = \frac{2}{L}\int_{0}^{L} f(x)dx \text{ and } a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

Example 01: Express f(x) = x as a half range sine series in (0, 2).



Solution: The graph of f(x) = x in 0 < x < 2 is the line OP. Let us extend the function in the interval –2 < x < 0 shown by the line OQ so that the new function is symmetrical about the origin and, therefore, represents an odd function in (–2, 2). Thus the Fourier series for f(x) over the full period (–2, 2) will contain only sine terms given by:

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \quad \text{where}$$

$$b_n=\frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}dx=\left\{\left[-x\frac{\cos(n\pi x)/2}{(n\pi)/2}\right]_0^2-\int_0^2 1.\frac{\cos(n\pi x)/2}{(n\pi)/2}dx\right\}$$

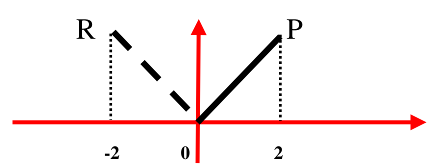
$$=\frac{2}{n\pi}\left\{\left[-\left(2\cos n\pi-0\right)+\frac{\sin(n\pi x)/2}{(n\pi)/2}\right]_0^2\right\}=\frac{2}{n\pi}\left[-2(-1)^n+(0-0)\right]=\frac{4}{n\pi}(-1)^{n+1}$$

Thus Fourier sine series of given function is: $f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}=\frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\sin\frac{n\pi x}{2}$

Example 02: Express f(x) = x as a half range cosine series in (0, 2).

Solution: The graph of f(x) = x in 0 < x < 2 is the line OP. Let us extend the function in the interval – 2 < x < 0 shown by the line OR so that the new function is symmetrical about the y-axis and, therefore, represents an even function in (–2, 2). Thus the Fourier series for f(x) over the full period (–2, 2) will contain only cosine terms given by:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L},$$



$$\text{where } a_0=\frac{2}{L}\int_0^L f(x)dx \text{ and } a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$$

$$\text{Here, } a_0=\frac{2}{2}\int_0^2 x\,dx=\left[\frac{x^2}{2}\right]_0^2=\frac{4-0}{2}=2$$

$$a_n=\frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx=\left\{\left[x\frac{\sin(n\pi x)/2}{(n\pi)/2}\right]_0^2-\int_0^2 1.\frac{\sin(n\pi x)/2}{(n\pi)/2}dx\right\}$$

$$=\frac{2}{n\pi}\left\{\left[(2\sin n\pi-0)-\frac{-\cos(n\pi x)/2}{(n\pi)/2}\right]_0^2\right\}=\frac{4}{n^2\pi^2}\left[(0-0)+(\cos n\pi-\cos 0)\right]$$

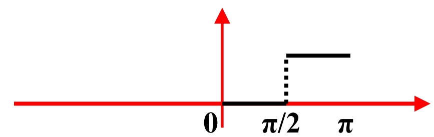
$$=\frac{4}{n^2\pi^2}\left[(-1)^n-1\right] \quad\Rightarrow\quad a_n=\begin{cases}0, & \text{if } n \text{ is even}\\ -8/n^2\pi^2, & \text{if } n \text{ is odd}\end{cases}$$

Thus Fourier sine series of given function is:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}=1-\frac{8}{\pi^2}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{2}$$

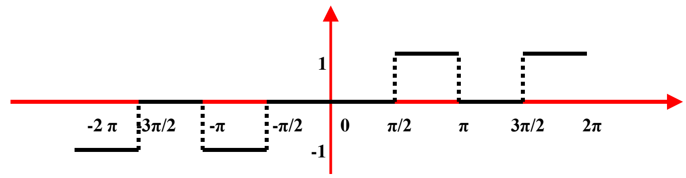
Example 03: Let f(x) be defined as

$$f(x)=\begin{cases}0, & 0<x<\pi/2\\ 1, & \pi/2<x<\pi\end{cases}$$

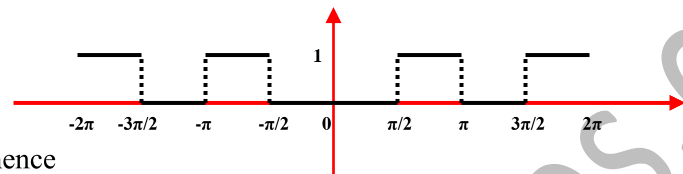


Find the Fourier sine and cosine series of f(x).

Solution: For Fourier sine series, we sketch the odd extension of f(x). This is shown in the following figure. The odd extension of f(x) is shown as under:



The even extension of f(x) is shown asunder:



Here L = π/2, hence

$$a_0=\frac{2}{\pi}\int_0^{\pi} f(x)\,dx=\frac{2}{\pi}\left[\int_0^{\pi/2} 0\,dx+\int_{-\pi/2}^{\pi} 1\,dx\right]=\frac{2}{\pi}\left[0+x\right]_{\pi/2}^{\pi}=\frac{2}{\pi}\left[\pi-\frac{\pi}{2}\right]=\frac{2}{\pi}.\frac{\pi}{2}=1$$

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx=\frac{2}{\pi}\left[\int_0^{\pi/2} 0.\cos nx\,dx+\int_{\pi/2}^{\pi} 1.\cos nx\,dx\right]=\frac{2}{\pi}\left[0+\frac{\sin nx}{n}\right]_{\pi/2}^{\pi}$$

$$=\frac{2}{n\pi}\left[\sin n\pi-\sin\frac{n\pi}{2}\right]=\frac{2}{n\pi}\left[0-\sin\frac{n\pi}{2}\right]\quad\Rightarrow\quad a_n=\begin{cases}0, & \text{if } n \text{ is even}\\ \dfrac{2}{n\pi}(-1)^{n+1}, & \text{if } n \text{ is odd}\end{cases}$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\left[\int_0^{\pi/2} 0.\sin nx\,dx+\int_{\pi/2}^{\pi} 1.\sin nx\,dx\right]=\frac{2}{\pi}\left[0-\frac{\cos nx}{n}\right]_{\pi/2}^{\pi}$$

$$=-\frac{2}{n\pi}\left[\cos n\pi-\cos(n\pi/2)\right].\ \text{Now there occurs two cases.}$$

Case 1: If n is even say $n=2k$, then

$$b_n=b_{2k}=-\frac{2}{2k\pi}\left[\cos 2k\pi-\cos k\pi\right]=\frac{-1}{k\pi}\left[1-(-1)^k\right]$$

Case 2: If n is odd say $n=2k-1$, then

$$b_n=b_{2k-1}=\frac{-2}{(2k-1)\pi}\left[\cos(2k-1)\pi-0\right]=\frac{2(-1)^{k-1+1}}{(2k-1)\pi}=\frac{2(-1)^k}{(2k-1)\pi}$$

Thus, Fourier cosine series of f(x) is given by:

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty} b_n\cos nx=\frac{1}{2}+\sum_{n=1,3,5...}^{\infty}\frac{2}{n\pi}(-1)^{n+1}\cos 2n\pi x$$

Similarly, Fourier sine series of f(x) is given by:

$$f(x)=\sum_{n=1}^{\infty} b_n\sin nx=\sum_{k=1}^{\infty}\frac{1}{k\pi}\left[(-1)^k-1\right]\sin 2kx+\sum_{k=1}^{\infty}\frac{1}{(2k-1)\pi}(-1)^k\sin 2kx$$

Example 04: Represent the function f(x) = 2p, 0 < x< p into

(i) Fourier Sine Series (ii) Fourier Cosine Series.

Solution: **(i) Fourier Sine Series**

For the Fourier Sine Series representation of f(x) the periodic extension of f(x) having period 2p is given as:
$$f(x)=\begin{cases}-2p, & -p\le x<0\\ 2p, & 0\le x\le p\end{cases}$$

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx=\frac{2}{p}\int_0^p(2p)\sin\left(\frac{n\pi x}{p}\right)dx=4\int_0^p\sin\left(\frac{n\pi x}{p}\right)dx$$

$$=4\left[-\frac{p}{n\pi}\cos\left(\frac{n\pi x}{p}\right)\right]_0^p=-\frac{4p}{n\pi}\left[\cos(n\pi)-\cos(0)\right]=\frac{4p}{n\pi}\left[1-(-1)^n\right]\Rightarrow b_n=\begin{cases}0, & \text{if n is even}\\ 8p/n\pi, & \text{if n is odd}\end{cases}$$

Thus, Fourier Sine series is given by:

$$f(x)=\sum_1^\infty b_n\sin\left[\frac{(2n-1)\pi x}{p}\right]=\frac{8p}{\pi}\sum_1^\infty\frac{1}{n}\sin\left[\frac{(2n-1)\pi x}{p}\right]$$

(ii) Fourier Cosine Series

For the Fourier Cosine Series representation of f(x) the periodic extension of f(x) having period 2p is given as: $f(x)=\begin{cases}2p, & -p\le x<0\\ 2p, & 0\le x\le p\end{cases}$

$$a_0=\frac{2}{p}\int_0^p f(x)dx=\frac{2}{p}\int_0^p(2p)dx=4\int_0^p 1dx=4\left[x\right]_0^p=4p$$

$$a_n=\frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx=\frac{2}{p}\int_0^p(2p)\cos\left(\frac{n\pi x}{p}\right)dx=4\int_0^p\cos\left(\frac{n\pi x}{p}\right)dx$$

$$=4\left[\frac{p}{n\pi}\sin\left(\frac{n\pi x}{p}\right)\right]_0^p=\frac{4p}{n\pi}\left[\sin(n\pi)-\sin(0)\right]\Rightarrow a_n=0$$

Thus the required Fourier Cosine series is given by

$$f(x)=\frac{a_0}{2}+\sum_1^\infty a_n\cos\left(\frac{n\pi x}{p}\right)=\frac{1}{2}(4p)=2p$$

Example 05: Represent the following function into (i) Fourier Sine Series (ii) Fourier Cosine Series.

$$f(x)=\begin{cases}1, & 0<x<p/2\\ 0, & p/2<x<p\end{cases}$$

Solution: **(i) Fourier Sine Series**

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx=\frac{2}{p}\left[\int_0^{p/2}\sin\left(\frac{n\pi x}{p}\right)dx+\int_{p/2}^p(0)\sin\left(\frac{n\pi x}{p}\right)dx\right]=\frac{2}{p}\int_0^{p/2}\sin\left(\frac{n\pi x}{p}\right)dx$$

$$=\frac{2}{p}\left[-\frac{p}{n\pi}\cos\left(\frac{n\pi x}{p}\right)\right]_0^{p/2}=-\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right)-\cos(0)\right]=\frac{2}{n\pi}\left[1-\cos\left(\frac{n\pi}{2}\right)\right]\Rightarrow b_n=\frac{2}{\pi}\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}$$

Thus Fourier Sine series is given by:

$$f(x)=\sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos\left(\frac{n\pi}{2}\right)}{n}\sin\left(\frac{n\pi x}{p}\right)$$

(ii) Fourier Cosine Series

$$a_0=\frac{2}{p}\int_0^p f(x)dx=\frac{2}{p}\int_0^{p/2} 1dx=\frac{2}{p}[x]_0^{p/2}=\frac{2}{p}\left(\frac{p}{2}\right)=1$$

$$a_n=\frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx=\frac{2}{p}\left[\int_0^{p/2}\cos\left(\frac{n\pi x}{p}\right)dx+\int_{p/2}^{p}(0)\cos\left(\frac{n\pi x}{p}\right)dx\right]=\frac{2}{p}\int_0^{p/2}\cos\left(\frac{n\pi x}{p}\right)dx$$

$$=\frac{2}{p}\left[\frac{p}{n\pi}\sin\left(\frac{n\pi x}{p}\right)\right]_0^{p/2}=\frac{2}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right)-\sin(0)\right]=\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Now $a_n=0,$ if n is even but when n is odd, that is, $n=2m-1$ (say) then

$$a_{2m-1}=\frac{2}{(2m-1)\pi}\sin\left[(2m-1)\frac{\pi}{2}\right]=\frac{2}{(2m-1)\pi}\sin\left(m\pi-\frac{\pi}{2}\right)$$

$$=\frac{2}{(2m-1)\pi}(-\cos m\pi)=\frac{2}{(2m-1)\pi}\left[-(-1)^m\right]=\frac{2}{(2m-1)\pi}(-1)^{m+1}$$

Hence the required Fourier Cosine series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{p}\right)=\frac{1}{2}+\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)}\cos\left(\frac{(2m-1)\pi x}{p}\right)$$

Example 06: Represent the following function into (i) Fourier Sine Series (ii) Fourier Cosine Series.

$$f(x)=\begin{cases}0, & 0<x<p/2\\ p^2, & p/2<x<p\end{cases}$$

Solution: **(i) Fourier Sine Series**

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx=\frac{2}{p}\int_{p/2}^{p}p^2\sin\left(\frac{n\pi x}{p}\right)dx=2p\left[-\frac{p}{n\pi}\cos\left(\frac{n\pi x}{p}\right)\right]_{p/2}^{p}$$

$$=-\frac{2p^2}{n\pi}\left[\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right]\Rightarrow b_n=\frac{2p^2}{\pi}\left[\frac{\cos(n\pi/2)-\cos(n\pi)}{n}\right]$$

Thus the required Fourier Sine Series is:

$$f(x)=\sum_{1}^{\infty}b_n\sin\left(\frac{n\pi x}{p}\right)=\frac{2p^2}{\pi}\sum_{n=1}^{\infty}\frac{\cos\left(\frac{n\pi}{2}\right)-\cos(n\pi)}{n}\sin\left(\frac{n\pi x}{p}\right)$$

(ii) Fourier Cosine Series

$$a_0=\frac{2}{p}\int_0^p f(x)dx=\frac{2}{p}\int_{p/2}^{p}p^2dx=2p[x]_{p/2}^{p}=2p\left(p-\frac{p}{2}\right)=2p^2-p^2=p^2$$

$$a_n=\frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx=\frac{2}{p}\int_{p/2}^{p}p^2\cos\left(\frac{n\pi x}{p}\right)dx=2p\left[\frac{p}{n\pi}\sin\left(\frac{n\pi x}{p}\right)\right]_{p/2}^{p}$$

$$=\frac{2p^2}{n\pi}\left[\sin(n\pi)-\sin\left(\frac{n\pi}{2}\right)\right]\Rightarrow a_n=-\frac{2p^2}{\pi}\sin\left(\frac{n\pi}{2}\right)$$

Hence the required Fourier Cosine series is

$$f(x)=\frac{a_0}{2}+\sum_1^\infty a_n\cos\left(\frac{n\pi x}{p}\right)=\frac{p^2}{2}-\frac{2p^2}{\pi}\sum_1^\infty\frac{\sin\left(\frac{n\pi}{2}\right)}{n}\cos\left(\frac{n\pi x}{p}\right)$$

Example 07: Represent the following function into (i) Fourier Sine Series (ii) Fourier Cosine Series.

$$f(x)=\begin{cases}0, & 0<x<p/2\\ p-x, & p/2<x<p\end{cases}$$

Solution: **(i) Fourier Sine Series**

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx=\frac{2}{p}\left[\int_0^{p/2}x\sin\left(\frac{n\pi x}{p}\right)dx+\int_{p/2}^{p}(p-x)\sin\left(\frac{n\pi x}{p}\right)dx\right]$$

$$=\frac{2}{p}\left[\int_0^{p/2}x\sin\left(\frac{n\pi x}{p}\right)dx+\int_{p/2}^{p}p\sin\left(\frac{n\pi x}{p}\right)dx+\int_{p/2}^{p}x\sin\left(\frac{n\pi x}{p}\right)dx\right]$$

$$=\frac{2}{p}\left[\begin{array}{l}-\frac{p}{n\pi}x\cos\left(\frac{n\pi x}{p}\right)+\frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\Big|_0^{p/2}-\frac{p^2}{n\pi}\cos\left(\frac{n\pi x}{p}\right)\Big|_{p/2}^{p}\\ -\left\{-\frac{p}{n\pi}x\cos\left(\frac{n\pi x}{p}\right)+\frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\Big|_{p/2}^{p}\right\}\end{array}\right]$$

$$=\frac{2}{p}\left[\begin{array}{l}-\frac{p}{n\pi}\left(\frac{p}{2}\right)\cos\left(\frac{n\pi}{2}\right)+\frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)-\frac{p^2}{n\pi}\left\{\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right\}\\ -\left\{-\frac{p}{n\pi}(p)\cos(n\pi)+\frac{p^2}{n^2\pi^2}\sin(n\pi)+\frac{p}{2}\left(\frac{p}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right)-\frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right\}\end{array}\right]$$

$$=\frac{2}{p}\left[\begin{array}{l}-\frac{p^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)-\frac{p^2}{n\pi}\cos(n\pi)+\frac{p^2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\\ +\frac{p^2}{n\pi}\cos(n\pi)-\frac{p^2}{n^2\pi^2}\sin(n\pi)-\frac{p^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\end{array}\right]$$

$$=\frac{2}{p}\left[\frac{2p^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]=\frac{4p}{\pi^2}\frac{\sin\left(\frac{n\pi}{2}\right)}{n^2}$$

Now $b_n=0$, if n is even but when n is odd, that is, $2n-1$ (say) then

$$b_{2n-1}=\frac{4p}{\pi^2}\frac{\sin(2n-1)\frac{\pi}{2}}{(2n-1)^2}=\frac{4p}{\pi^2}\frac{\sin\left(n\pi-\frac{\pi}{2}\right)}{(2n-1)^2}=\frac{4p}{\pi^2}\frac{(-\cos n\pi)}{(2n-1)^2}=\frac{4p}{\pi^2}\frac{(-1)^{n+1}}{(2n-1)^2}$$

Thus the required Fourier Sine series is

$$f(x)=\sum_1^\infty b_n\sin\left(\frac{n\pi x}{p}\right)=\frac{4p}{\pi^2}\sum_1^\infty\frac{(-1)^{n+1}}{(2n-1)^2}\sin\left(\frac{(2n-1)\pi x}{p}\right)$$

(ii) Fourier Cosine Series

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx = \frac{2}{p}\left[\int_0^{p/2} x\,dx + \int_{p/2}^{0}(p-x)\,dx\right] = \frac{2}{p}\left[\frac{1}{2}x^2\Big|_0^{p/2} + px\Big|_{p/2}^{p} - \frac{1}{2}x^2\Big|_{p/2}^{p}\right]$$

$$= \frac{2}{p}\left[\frac{1}{2}\left(\frac{p^2}{4}\right) + p\left(p-\frac{p}{2}\right) - \frac{1}{2}\left(p^2 - \frac{p^2}{4}\right)\right] = \frac{2}{p}\left(\frac{p^2}{8} + p^2 - \frac{p^2}{2} - \frac{p^2}{2} + \frac{p^2}{8}\right) \quad \Rightarrow \; a_0 = \frac{p}{2}$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx = \frac{2}{p}\left[\int_0^{p/2} x\cos\left(\frac{n\pi x}{p}\right)dx + \int_{p/2}^{p}(p-x)\cos\left(\frac{n\pi x}{p}\right)dx\right]$$

$$= \frac{2}{p}\left[\int_0^{p/2} x\cos\left(\frac{n\pi x}{p}\right)dx + p\int_{p/2}^{p}\cos\left(\frac{n\pi x}{p}\right)dx - \int_{p/2}^{p} x\cos\left(\frac{n\pi x}{p}\right)dx\right]$$

$$= \frac{2}{p}\left[\begin{array}{l} \left.\frac{p}{n\pi}x\sin\left(\frac{n\pi x}{p}\right) + \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right)\right|_0^{p/2} + \left.\frac{p^2}{n\pi}\sin\left(\frac{n\pi x}{p}\right)\right|_{p/2}^{p} \\ -\left\{\left.\frac{p}{n\pi}x\sin\left(\frac{n\pi x}{p}\right) + \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right)\right|_{p/2}^{p}\right\} \end{array}\right]$$

$$= \frac{2}{p}\left[\begin{array}{l} \frac{p}{n\pi}\left(\frac{p}{2}\right)\sin\left(\frac{n\pi}{2}\right) + \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{p^2}{n^2\pi^2} + \frac{p^2}{n\pi}\left\{\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right\} \\ -\left\{\frac{p}{n\pi}\sin(n\pi) + \frac{p^2}{n^2\pi^2}\cos(n\pi) - \frac{p}{2}\left(\frac{p}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right) + \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right\} \end{array}\right]$$

$$= \frac{2}{p}\left[\begin{array}{l} \frac{p^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) - \frac{p^2}{n^2\pi^2} - \frac{p^2}{n\pi}\sin\left(\frac{n\pi}{2}\right) \\ -\frac{p^2}{n^2\pi^2}\cos(n\pi) + \frac{p^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right) \end{array}\right]$$

$$= \frac{2}{p}\left[-\frac{p^2}{n^2\pi^2} - \frac{p^2}{n^2\pi^2}\cos(n\pi)\right] = -\frac{2p^2}{pn^2\pi^2}\left[1+(-1)^n\right] = -\frac{2p}{n^2\pi^2}\left[1+(-1)^n\right]$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\dfrac{4p}{4n^2\pi^2}, & \text{if } n \text{ is even} \end{cases}$$

Hence the required Fourier Cosine series is given by:

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n\cos\left(\frac{2n\pi x}{p}\right) = \frac{p}{4} - \frac{p}{\pi^2}\sum_1^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{p}\right)$$

Example 08: Represent the following function into (i) Fourier Sine Series (ii) Fourier Cosine Series: $f(x) = 4x, \quad 0 < x < p$.

Solution: **(i) Fourier Sine Series**

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)dx = \frac{2}{p}\int_0^p 4x\sin\left(\frac{n\pi x}{p}\right)dx = \frac{8}{p}\int_0^p x\sin\left(\frac{n\pi x}{p}\right)dx$$

$$= \frac{8}{p}\left[-\frac{p}{n\pi}x\cos\left(\frac{n\pi x}{p}\right) + \frac{p^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{p}\right)\right]_0^p = \frac{8}{p}\left[-\frac{p^2}{n\pi}\cos(n\pi) + \frac{p^2}{n^2\pi^2}\sin(n\pi)\right]$$

$$= -\frac{8p}{n\pi}\cos(n\pi) = -\frac{8p}{n\pi}(-1)^n \Rightarrow b_n = \frac{8p}{n\pi}(-1)^{n+1}$$

Thus the required Fourier Sine series is:

$$f(x) = \sum_{1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right) = \sum_{1}^{\infty} \frac{8p}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{p}\right) = \frac{8p}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{p}\right)$$

(ii) Fourier Cosine Series: $a_0 = \frac{2}{p}\int_0^p f(x)\,dx = \frac{2}{p}\int_0^p 4x\,dx = \frac{8}{p}\left[\frac{x^2}{2}\right]_0^p = \frac{4}{p}\left(p^2 - 0\right) = 4p$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)dx = \frac{2}{p}\int_0^p 4x\cos\left(\frac{n\pi x}{p}\right)dx = \frac{8}{p}\int_0^p x\cos\left(\frac{n\pi x}{p}\right)dx$$

$$= \frac{8}{p}\left[\frac{p}{n\pi}x\sin\left(\frac{n\pi x}{p}\right) + \frac{p^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{p}\right)\right]_0^p = \frac{8}{p}\left[\frac{p^2}{n\pi}\sin(n\pi) + \frac{p^2}{n^2\pi^2}\cos(n\pi) - \frac{p^2}{n^2\pi^2}\right]$$

$$= \frac{8}{p}\left[\frac{p^2}{n^2\pi^2}(-1)^n - \frac{p^2}{n^2\pi^2}\right] = \frac{8p}{n^2\pi^2}\left[(-1)^n - 1\right] \Rightarrow a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-16p}{\pi^2(2n-1)^2}, & \text{if } n \text{ is odd} \end{cases}$$

Thus the required Fourier Cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n\cos\left(\frac{n\pi x}{p}\right) = 2p - \frac{16p}{\pi^2}\sum_{1}^{\infty}\frac{\cos\left(\frac{(2n-1)\pi x}{p}\right)}{(2n-1)^2}$$

WORKSHEET 07

1. Find a Fourier series to represent $f(x) = \pi - x$ for $0 < x < \pi$.

2. If $f(x) = [(\pi - x)/2]^2$ in the range 0 to 2π, show that $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$.

3. Given $f(x) = x + x^2$ for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x = \pm\pi\cdot$ Expand f(x) in Fourier series and show that $f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{n^2}\cos nx - \frac{2}{n}\sin nx\right)$

4. Expand $f(x) = x\sin x,\ 0 < x < 2\pi$, in a Fourier series.

5. If $f(x) = |\cos x|$, expand f(x) as a Fourier series in the interval $(-\pi, \pi)\cdot$

6. Show that for $-\pi < x < \pi$,

$$x\cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{1\cdot 3}\sin 2x - \frac{3}{2\cdot 4}\sin 3x + \frac{4}{3\cdot 5}\sin 4x - \ldots\right)$$

7. Find the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1 & \text{for} \quad \frac{\pi}{2} < x < \pi \end{cases}$$

8. Find the Fourier series of the function

$$f(x)=\begin{cases}-1 & \text{for} \quad -\pi<x<0 \\ +1 & \text{for} \quad 0<x<\pi\end{cases} \text{ where } f(x+2\pi)=f(x)$$

9. Find the Fourier series of the function defined as

$$f(x)=\begin{cases}x+\pi & \text{for} \quad 0<x<\pi \\ -x-\pi & \text{for} \quad -\pi<x<0\end{cases}, \text{ where } f(x+2\pi)=f(x)$$

10. Find the Fourier series for the function

$$f(x)=\begin{cases}-\dfrac{\pi}{4} & \text{for} \quad -\pi<x<0 \\ +\dfrac{\pi}{4} & \text{for} \quad 0<x<\pi\end{cases}$$

and $f(-\pi)=f(0)=f(\pi)=0,\ f(x)=f(x+2\pi)$ for all x. Deduce that

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots$$

11. Find the Fourier series for f(x) if

$$f(x)=\begin{cases}-\pi & \text{for} \quad -\pi<x\le 0 \\ x & \text{for} \quad 0<x<\pi \\ -\pi/2 & \text{for} \quad x=0\end{cases}. \text{ Deduce that } \frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots$$

12. Expand f(x) as a Fourier series, the function defined as

$$f(x)=\begin{cases}\pi+x & \text{for} \quad -\pi<x<-\dfrac{\pi}{2} \\ \pi/2 & \text{for} \quad -\dfrac{\pi}{2}<x<\dfrac{\pi}{2} \\ \pi-x & \text{for} \quad \dfrac{\pi}{2}<x<\pi\end{cases}$$

13. Obtain a Fourier series to represent the function

$$f(x)=|\sin x| \quad \text{for} \quad -\pi<x<\pi, \quad \begin{cases}\text{Hint } f(x)=-\sin x & \text{for} \quad -\pi<x<0 \\ \qquad\quad\ \ = \sin x & \text{for} \quad 0<x<\pi\end{cases}$$

14. A function is defined as follows:

$$f(x)=\begin{cases}-x & \text{when} \quad -\pi<x<0 \\ x & \text{when} \quad 0<x<\pi\end{cases}$$

Show that $f(x)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\ldots\right)$. Deduce that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

15. Find the Fourier cosine series for the function

$$f(x)=\begin{cases}1 & \text{for} \quad 0<x<\dfrac{\pi}{2} \\ 0 & \text{for} \quad \dfrac{\pi}{2}<x<\pi\end{cases}$$

16. Find the Fourier sine series for the function:

$f(x) = e^{ax}$ for $0 < x < \pi$ where a is constant.

17. Find a series of cosine of multiples of x which will represent $f(x)$ in $(0,\pi)$ where

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 < x < \dfrac{\pi}{2} \\ \dfrac{\pi}{2} & \text{for} \quad \dfrac{\pi}{2} < x < \pi \end{cases}$$. Deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

18. Express f(x) = x as a sine series in $0 < x < \pi$

19. Find the cosine series for f(x) = π – x in the interval $0 < x < \pi$

20. If $f(x) = \begin{cases} x, & \text{for} \quad 0 < x < \dfrac{\pi}{2} \\ \pi - x, & \text{for} \quad \dfrac{\pi}{2} < x < \pi \end{cases}$ then show that:

(i) $f(x) = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right)$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right)$

21. A periodic function of period 4 is defined as: f(x) = |x|, –2 < x < 2
Find its Fourier series expansion.

22. Find the Fourier series expansion of the periodic function of period 1:

$$f(x) = \begin{cases} \dfrac{1}{2} + x, & \text{for} \quad -\dfrac{1}{2} < x \le 0 \\ \dfrac{1}{2} - x, & \text{for} \quad 0 < x < \dfrac{1}{2} \end{cases}$$

23. Obtain the half – range cosine series for f(x) = x^2 in 0 < x < π

24. Express f(x) = x as a cosine, half range series in 0 < x < 2

25. Find the Fourier series of the function:

$$f(x) = \begin{cases} -2 & \text{for} \quad -4 < x < -2 \\ x & \text{for} \quad -2 < x < 2 \\ 2 & \text{for} \quad 2 < x < 4 \end{cases}$$

26. Expand f(x) = 1 – |x|, –1 < x < 1 into Fourier series.

27. Represent f(x) = sin x, 0 < x < π into Fourier sine and cosine series.

CHAPTER EIGHT

LAPLACE TRANSFORMS

INTRODUCTION

Laplace transform is a mathematical tool or mechanism that can be used to solve several problems in science and engineering. This was first introduced by French mathematician *Laplace* in the year 1790, in his work on probability theory. This technique became popular when *Heaviside* (An Electrical Engineer) applied it to the solution of ordinary differential equations in electrical engineering. It has become an essential part of mathematical background of engineers, physicists, mathematicians and many scientists.

Laplace Transform has got wider applications in physics, engineering, mechanics, heat flow etc. The methods of Laplace transforms are very simple, and they give solutions of differential equations satisfying given initial/boundary conditions without the use of the general solutions. Since these particular solutions are usually required in physics, mechanics, chemistry and various fields of practical research, Laplace transform is highly important.

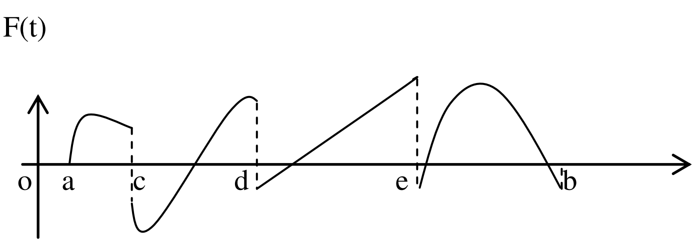
The basic question as to why one should learn Laplace transforms techniques when other techniques are available, the answer is very simple .Transforms are used to accomplish the solution of certain problems with less effort and in a simple routine way.

For illustration, consider the problem of finding the value of x from the equation $x^{1.85} = 3$. It is extremely difficult task to solve this problem algebraically. However, taking logarithms on both sides, we have the transformed equation as 1.85 ln x = ln3. In this transformed equation, the algebraic operation and exponentiation have been changed to multiplication which immediately gives ln x = ln 3/(1.85). To get the required result, it is enough if we take antilogarithm on both sides of the above equation, which yields:

$$x = \ln^{-1}[\ln 3/(1.85)] = 1.3785.$$

Piece-wise or Sectionally Continuous Function

A function F(t) is said to be piecewise or sectionally continuous in any interval [a, b] if it is continuous in every sub-intervals a < c < d < e < … < b. This is show as under.



You may observe that in each sub-interval the left hand and right hand limits of F(t) are finite. Thus piecewise function is continuous function except for a finite number of jump discontinuities.

Function of Exponential Order

A function F(t) defined on $[0, \infty)$ is said to be of exponential order a as $t \to \infty$ if

$$\lim_{t\to\infty} e^{-at} F(t) = \text{finite quantity}$$

This means that for a given integer a > 0 there exists a real number M > 0 such that

$$|e^{-at} F(t)| < M \text{ OR } |F(t)| \le M e^{at} \text{ whenever, } t \ge a.$$

For example, the function $F(t) = t^n$ is of exponential order for

$$\lim_{t\to\infty} e^{-at} t^n = \lim_{t\to\infty} \frac{t^n}{e^{at}} = \lim_{t\to\infty} \frac{n!}{a^n e^{at}} = 0 \le M. \quad \text{[Use L`Hospital Rule]}$$

*Remark***:** A function F(t) is said to be of class A if it is piecewise continuous and it is of exponential order.

Integral Transform

An improper integral of the form

$$\int_{-\infty}^{\infty} K(s,t)\, F(t)\, dt = f(s) \qquad (1)$$

is called *integral transform* of F(t) provided that the integral exists/converges. It is usually denoted by T[(F(t)] or f(s). The function K(s, t) appearing in the integrand is called *kernel* of the transform. It may be noted that `s` is a parameter which may be real or complex. It may further be noted that `s` is known as *Frequency Domain* variable. Throughout this book, we shall consider `s` as a real number. If we take

$$K(s,t) = \begin{cases} e^{-st} & ,t \ge 0 \\ 0 & ,t < 0 \end{cases}$$

Then (1) becomes: $L[F(t)] = \int_0^{\infty} e^{-st}\, F(t)\, dt = f(s)$

This is known as *Laplace Transform* of a function F(t).

If we take $K(s,t) = e^{-ist}$ for $-\infty < t < \infty$

Then (1) becomes: $\mathcal{F}[F(t)] = \int_{-\infty}^{\infty} e^{-ist}\, F(t)\, dt$

This is known as *Fourier Transform*.

Remark: It may be noted that in *Control Engineering Theory* there is an extensive use of Laplace transformation where F(t) is considered as *Input Signal* or *Object function* and f(s) as an *Output Signal*.

Laplace Transforms of Some Elementary Functions

In this section we shall represent the Laplace transforms of some elementary functions that would help us in our further working.

1. L[1]: By definition, $L[1] = \int_0^{\infty} e^{-st}(1)\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty} = -\frac{1}{s}\left(e^{-\infty} - e^{0}\right) = -\frac{1}{s}(0-1) = \frac{1}{s}, s>0$

2. L[t]: By definition,

$$L[t] = \int_0^\infty e^{-st}t\,dt.$$ Now integrating by parts taking $u = t$ and $v = e^{-st}$

$$= \lim_{p\to\infty}\left[t\,\frac{1}{-se^{st}}\right]_0^p - \int_0^\infty 1.\frac{e^{-st}}{-s}dt = 0 + \frac{1}{s}\int_0^\infty (1)\,e^{-st}dt = \frac{1}{s}L(1) = \frac{1}{s}.\frac{1}{s} = \frac{1}{s^2}, s>0$$

$[\text{NOTE}: \lim_{p\to\infty}(1/e^{sp}) = 0]$

3. $L[t^n]$: By definition,

$L[t^n]$ $$L[t^n] = \int_0^\infty t^n e^{-st}\,dt.$$ Now integrating by parts taking $u = t^n$ and $v = e^{-st}$, we get

$$L\left[t^n\right] = \lim_{p\to\infty}\left[t^n\,\frac{e^{-st}}{-s}\right]_0^p - \int_0^\infty n\,t^{n-1}.\frac{e^{-st}}{-s}dt = \lim_{p\to\infty}\left[t^n\,\frac{1}{-se^{st}}\right]_0^p + \frac{n}{s}\int_0^\infty t^{n-1}\,e^{-st}dt = 0 + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

Thus, $$L\left[t^n\right] = \frac{n}{s}L\left(t^{n-1}\right)$$

This is a recurrence formula. If we repeatedly apply this formula, we get

$$L\left[t^n\right] = \frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}...\frac{3}{s}.\frac{2}{s}.\frac{1}{s}L\left[t^0\right] = \frac{n!}{s^n}L(1) = \frac{n!}{s^n}.\frac{1}{s} = \frac{n!}{s^{n+1}}$$

Thus, $$L[t^n] = \frac{n!}{s^{n+1}}, s>0$$

We know that gamma function for a positive integer n is given by

$$\Gamma(n+1) = \int_0^\infty e^{-u}u^n du = n!$$

Also notice that $\Gamma(n+1) = n\Gamma(n)$. Hence above formula can also be expressed as

$$L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}, s>0 \qquad \text{NOTE: } \left[\Gamma(1/2) = \sqrt{\pi}\right]$$

For instance, $L[t^2] = \dfrac{2!}{s^3}, s>0$

$$L\left[\left(\sqrt{t}\right)\right] = \frac{\Gamma(1/2+1)}{s^{1/2+1}} = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{1}{2s^{3/2}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2s^{3/2}}, s>0$$

$$L\left[t^{-1/2}\right] = \left[\frac{\Gamma(-1/2+1)}{s^{-1/2+1}}\right] = \Gamma\left(\frac{1}{2}\right)\frac{1}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}, s>0$$

4. $L(e^{at})$: By definition

$$L\left[e^{at}\right] = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-st+at}dt = \int_0^\infty e^{-t(s-a)}dt = \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty = \frac{-1}{(s-a)}\left(e^{-\infty} - e^0\right) = \frac{1}{(s-a)}$$

Thus, $L[e^{at}] = \dfrac{1}{(s-a)}, s>a$ and $L[e^{-at}] = \dfrac{1}{(s+a)}, s>-a$

For instance, $L\left[e^{3t}\right] = \dfrac{1}{(s-3)}$ and $L\left[e^{-3t}\right] = \dfrac{1}{(s+3)}$.

5. L[sin at]: Before we find the Laplace transform of $\sin at$ and $\cos at$, the following formulas may kindly be noted:

(i) $\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$ and

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$

Now consider, $L\left[\sin at\right] = \int_0^\infty e^{-st}\sin at\,dt = \left[\frac{e^{-st}}{s^2+a^2}\left(-s\sin at - a\cos at\right)\right]_0^\infty$

$$= \frac{1}{s^2+a^2}\left(e^{-\infty} - e^0(0-a)\right) = \frac{a}{s^2+a^2} \qquad \left(\because\ \sin 0 = 0, \cos 0 = 1, \text{and}\, e^{-\infty} = 0\right)$$

Thus, $L\left[\sin at\right] = a/(s^2+a^2)$

6. L[cos at]: By definition

$$L\left[\cos at\right] = \int_0^\infty e^{-st}\cos at\,dt = \frac{1}{s^2+a^2}\left[e^{-st}\left(-s\cos at + a\sin at\right)\right]_0^\infty$$

$$= \frac{1}{s^2+a^2}\left(e^{-\infty} + e^0(s-0)\right) = \frac{s}{s^2+a^2} \qquad \left(\because \sin 0 = 0, \cos 0 = 1, \text{and}\, e^{-\infty} = 0\right)$$

Thus, $L\left[\cos at\right] = s/(s^2+a^2)$

For example, $L\left[\sin 3t\right] = \frac{3}{s^2+9}$ and $L\left[\cos 3t\right] = \frac{s}{s^2+9}$

7. L[sinh at]: We know that sinh at = $(e^{at} - e^{-at})/2$, therefore

$L[\sinh at] = (1/2)\, L[e^{at} - e^{-at}] = (1/2)[L(e^{at}) - L(e^{-at})]$

Using formula (4), we get

$$L[\sinh at] = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a-(s-a)}{(s-a)(s+a)}\right] = \frac{a}{s^2-a^2}$$

Thus, $L\left[\sinh at\right] = a/(s^2-a^2)$, $s > |a|$

8. L[cosh at]: We know that cosh at = $(e^{at} + e^{-at})/2$, therefore

$$L[\cosh at] = (1/2)L[e^{at} + e^{-at}] = (1/2)[L(e^{at}) + L(e^{-at})]$$

Using formula (4), we get

$$L[\cosh at] = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{1}{2}\left[\frac{s+a+(s-a)}{(s-a)(s+a)}\right] = \frac{s}{s^2-a^2}$$

Thus, $L\left[\cosh at\right] = s/(s^2-a^2)$, $s > |a|$.

For example, $L\left[\sinh 3t\right] = \frac{3}{(s^2-9)}$ and $L\left[\cosh 3t\right] = \frac{s}{(s^2-9)}$

The following table summarizes the Laplace transforms of these elementary functions.

F(t)	L[F(t)] = f(s)	F(t)	L[F(t)] = f(s)
1	1/s, $s > 0$	e^{-at}	$1/(s + a)$, $s > a$
t	$1/s^2$, $s > 0$	sin at	$a/(s^2 + a^2)$, $s > a$
t^n	$n!/(s^{n+1})$, $s > 0$, $n\ \varepsilon\ N$	cos at	$s/(s^2 + a^2)$, $s > a$
t^n	$\Gamma(n + 1)/s^{n+1}$, $s > 0$, n is real	sinh t	$a/(s^2 - a^2)$, $s > \lvert a\rvert$
e^{at}	$1/(s - a)$, $s > a$	cosh t	$s/(s^2 - a^2)$, $s > \lvert a\rvert$

PROPERTIES OF LAPLACE TRANSFORMS

In this section, we shall learn the most important properties of Laplace transforms which, are frequently used in our further study.

Linearity Property

If $c_1, c_2, ..., c_n$ are constants and $F_1, F_2, ..., F_n$ are functions of t, whose Laplace transforms exist, then $L[c_1F_1(t) + c_2 F_2(t) + ... + c_nF_n(t)] = c_1f_1(s) + c_2 f_2(s)] + ... + c_n f_n(s)$

Proof: By definition,

$$L\left[c_1F_1(t) + c_2 F_2(t) + ... + c_nF_n(t)\right] = \int_0^\infty e^{-st}\left[c_1F_1(t) + c_2 F_2(t) + ... + c_nF_n(t)\right]dt$$

$$= c_1\int_0^\infty e^{-st}F_1(t)dt + c_2\int_0^\infty e^{-st}F_2(t)dt + ... + c_n\int_0^\infty e^{-st}F_n(t)dt$$

$$= c_1LF_1(t) + c_2LF_2(t) + ... + c_nLF_n(t) = c_1f_1(s) + c_2f_2(s) + ... + c_nf_n(s)$$

Example 01: Evaluate $L[3e^{-4t} + \cosh 2t - 2\sin 3t + t^3]$

Solution: By linearity property,

$$L\left[3e^{-4t} + \cosh 2t - 2\sin 3t + t^3\right] = 3L[e^{-4t}] + L[\cosh 2t] - 2L[\sin 3t] + L[t^3]$$

$$= \frac{3}{(s+4)} + \frac{s}{(s^2-4)} - 2\frac{3}{(s^2+9)} + \frac{3!}{s^4} = \frac{3}{(s+4)} + \frac{s}{(s^2-4)} - \frac{6}{(s^2+9)} + \frac{6}{s^4}$$

Example 02: Evaluate $L[e^{-4t}\cosh 2t]$

Solution: We know that

$\text{Cosh } 2t = \dfrac{e^{2t}+e^{-2t}}{2}$ then $\left[e^{-4t}\cosh 2t\right] = e^{-4t}\left(\dfrac{e^{2t}+e^{-2t}}{2}\right) = \dfrac{e^{-2t}+e^{-6t}}{2}$

Thus, $L\left[e^{-4t}\cosh 2t\right] = L\left[\dfrac{e^{-2t}+e^{-6t}}{2}\right] = \dfrac{1}{2}L[e^{-2t}] + \dfrac{1}{2}L[e^{-6t}] = \dfrac{1}{2}\left[\dfrac{1}{s+2} + \dfrac{1}{s+6}\right] = \dfrac{s+4}{(s+2)(s+6)}$

Example 03: Evaluate $L[\sin 2t \sin 3t]$

Solution: We know that $\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

$\therefore \sin 2t\sin 3t = \frac{1}{2}\left(\cos(2t-3t) - \cos(2t+3t)\right) = \frac{1}{2}\left(\cos t - \cos 5t\right)$ [Note: cos (–a) = cos a]

$$\therefore L\left[\sin 2t\sin 3t\right] = L\left[\frac{1}{2}(\cos t - \cos 5t)\right] = \frac{1}{2}\left[L(\cos t) - L(\cos 5t)\right] = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right]$$

$$= \frac{1}{2}\left[\frac{s(s^2+25) - s(s^2+1)}{(s^2+1)(s^2+25)}\right] = \frac{1}{2}\left[\frac{s^3+25s-s^3-s}{(s^2+1)(s^2+25)}\right]$$

Example 04: Find $L\left\{\sin\sqrt{t}\right\}$

Solution: We know that $\sin x = x - x^3/3! + x^5/5! - .....$

$$\therefore \sin\sqrt{t} = \sqrt{t} - \left(\sqrt{t}\right)^3/3! + \left(\sqrt{t}\right)^5/5! - ... = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{3/2}}{5!} - ...$$

Taking Laplace transform on both sides, we have

$$L\left\{\sin\sqrt{t}\right\} = L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{3/2}}{5!} - ...\right\} = L\left\{t^{1/2}\right\} - L\left\{\frac{t^{3/2}}{3!}\right\} + L\left\{\frac{t^{5/2}}{5!}\right\}...$$

Using the formula: $L\left\{t^n\right\} = \dfrac{\Gamma(n+1)}{s^{n+1}} \Rightarrow L\left\{\sin\sqrt{t}\right\} = \dfrac{\Gamma(3/2)}{s^{3/2}} - \dfrac{1}{3!}\dfrac{\Gamma(5/2)}{s^{5/2}} + \dfrac{1}{5!}\dfrac{\Gamma(7/2)}{s^{7/2}} - ...$

Using the recurrence relation of gamma function, that is, $\Gamma(n+1) = n\Gamma(n)$ and notice that:

$$\Gamma(1/2)=\sqrt{\pi}\Rightarrow\Gamma(3/2)=\frac{1}{2}\Gamma(1/2)=\frac{\sqrt{\pi}}{2},\Gamma(5/2)=\frac{3}{2}.\frac{1}{2}\Gamma(1/2)=\frac{1.3}{2.2}\sqrt{\pi},$$

$$\Gamma(7/2)=\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma(1/2)=\frac{1.3.5}{2.2.2}\sqrt{\pi},\text{ etc; }\Rightarrow\sqrt{\sin t}\ =\frac{\sqrt{\pi}}{2s^{3/2}}-\frac{3\sqrt{\pi}}{24s^{5/2}}+\frac{15\sqrt{\pi}}{120.8s^{5/2}}-\ldots$$

$$=\frac{\sqrt{\pi}}{2s^{3/2}}\left[1-\frac{(1/2^2s)}{1!}+\frac{(1/2^2s)^2}{2!}-\frac{(1/2^2s)^3}{3!}\ldots\right] \qquad (1)$$

Now, we know that $e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$ (2)

Comparing (1) and (2), we see that $L\{\sin\sqrt{t}\}=\frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/2^2 s}=\frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/4s}$

Example 05: Find (i) $L[\cos^2 3t]$ (ii) $L[\cosh^2 3t]$

Solution: (i) We know that $\cos^2 a = (1 + \cos 2a)/2$. Therefore,

$L[\cos^2 3t] = L[(1 + \cos 6t)/2] = 0.5\ \{L(1) + L(\cos 6t)\}$

$$= =\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2+36}\right]$$

Note: $\sin^2 t = (1 - \cos 2t)/2$

(ii) We know that $\cosh^2 a = (1 + \cosh 2a)/2$. Therefore,

$L[\cosh^2 3t] = L[(1 + \cosh 6t)/2] = 0.5\ \{L(1) + L(\cosh 6t)\} = \frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-36}\right]$

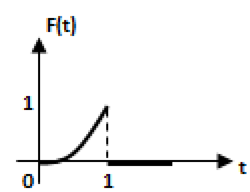
Note: $\sinh^2 t = (\cosh 2t - 1)/2$

Laplace Transform of Discontinuous Functions

The Laplace transform of object function F(t) also exists if it is piecewise discontinuous and provided it is of class `A`.

Example 06: Find the Laplace transform of a function defined as $F(t)=\begin{cases}t^2 & 0<t<1\\ 0 & \text{Elsewhere}\end{cases}$

Solution: The graph of the given function is shown. By definition,



$$L[F(t)]=\int_0^\infty e^{-st}F(t)\,dt=\int_0^1 e^{-st}t^2\,dt+\int_1^\infty e^{-st}(0)\,dt=\int_0^1 e^{-st}t^2\,dt$$

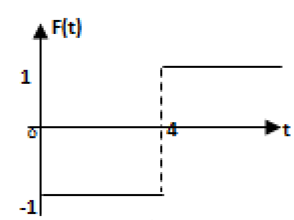
Now integrating by parts, we get

$$L[F(t)]=\left[t^2.\frac{e^{-st}}{-s}\right]_0^1-2\int_0^1 t\frac{e^{-st}}{-s}dt=-\frac{e^{-s}}{s}+\frac{2}{s}\int_0^1 t\,e^{-st}\,dt$$

Integrating by parts once again, we obtain

$$L[F(t)]=-\frac{e^{-s}}{s}+\frac{2}{s}\left\{\left[t\frac{e^{-st}}{-s}\right]_0^1-\int_0^1 1.\frac{e^{-st}}{-s}dt\right\}=-\frac{e^{-s}}{s}-\frac{2e^{-s}}{s^2}+\frac{2}{s^2}\int_0^1 e^{-st}\,dt$$

$$=-\frac{e^{-s}}{s}-\frac{2e^{-s}}{s^2}+\frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^1=-\frac{e^{-s}}{s}-\frac{2e^{-s}}{s^2}-\frac{2}{s^3}[e^{-s}-1]$$



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Example 07: Find the Laplace transform of a function

$$\begin{aligned}F(t)&=-1 \quad 0\le t\le 4\\ &=1 \qquad t>4\end{aligned}$$

Solution: The graph of F(t) is shown here. By definition,

$$L\left[F(t)\right]=\int_0^\infty e^{-st}F(t)\,dt=\int_0^4 e^{-st}F(t)\,dt+\int_4^\infty e^{-st}F(t)\,dt$$

$$=\int_0^4 e^{-st}(-1)\,dt+\int_4^\infty e^{-st}(1)\,dt=-\left[\frac{e^{-st}}{-s}\right]_0^4+\int_4^\infty e^{-st}dt$$

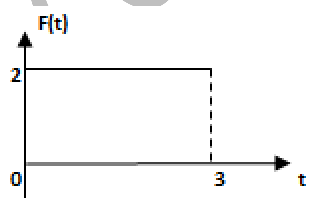
$$=\frac{1}{s}\left(e^{-4s}-e^{0}\right)+\lim_{p\to\infty}\left[\frac{e^{-st}}{-s}\right]_4^p=\frac{1}{s}\left(e^{-4s}-1\right)+\lim_{p\to\infty}\left(\frac{e^{-ps}}{-s}+\frac{e^{-4s}}{s}\right)$$

$$=\frac{1}{s}\left(e^{-4s}-1\right)+0+\frac{e^{-4s}}{s}\qquad\left(\text{Note: If } p\to\infty \text{ then } e^{-ps}\to 0\right)$$

$$=\left(\frac{e^{-4s}}{s}-\frac{1}{s}+\frac{e^{-4s}}{s}\right)=\left(\frac{2e^{-4s}}{s}-\frac{1}{s}\right),\quad s>0$$

Example 08: Find the Laplace transform of the function shown below.
Solution: The function shown in the graph is

$$F(t)=2\quad 0\le t\le 3$$
$$=0\quad t>3$$



$$\text{Now, } L\{F(t)\}=\int_0^\infty e^{-st}F(t)\,dt$$

$$=\int_0^3 e^{-st}F(t)\,dt+\int_3^\infty e^{-st}F(t)\,dt=\int_0^3 e^{-st}(2)\,dt+\int_3^\infty e^{-st}(0)\,dt=2\int_0^3 e^{-st}dt$$

$$=2\left|\frac{e^{-st}}{-s}\right|_0^3=\frac{-2}{s}\left(e^{-3s}-e^{0}\right)=\frac{-2}{s}\left(e^{-3s}-1\right)=\frac{2}{s}\left(1-e^{-3s}\right)\quad\text{or}\quad L\{F(t)\}=\frac{2}{s}\left(1-e^{-3s}\right)$$

Example 09: Find Laplace transform of function F(t) = 1/t, t > 0.

Solution: By definition, $L\left\{\frac{1}{t}\right\}=\int_0^\infty \frac{e^{-st}}{t}dt=\int_0^1 \frac{e^{-st}}{t}dt+\int_1^\infty \frac{e^{-st}}{t}dt$

For, $0\le t\le 1$ and $s>0$, we have $e^{-st}\ge e^{-s}$.

Therefore, $L\left\{\frac{1}{t}\right\}=\int_0^\infty \frac{e^{-st}}{t}dt=\int_0^1 \frac{e^{-st}}{t}dt+\int_1^\infty \frac{e^{-st}}{t}dt$

But, $\int_0^1 \frac{e^{-st}}{t}dt\ge e^{-s}\int_0^1 \frac{1}{t}dt=e^{-s}\lim_{p\to 0}\int_p^1 \frac{1}{t}dt=e^{-s}\lim_{p\to 0}\left[\ln t\right]_0^p\Rightarrow =e^{-s}\lim_{p\to 0}\left(\ln 1-\ln p\right)=e^{-s}(\infty)=\infty$

Hence, $\int_0^1 \frac{e^{-st}}{t}dt$ diverges and consequently $\int_0^\infty \frac{e^{-st}}{t}dt$ diverges.

Therefore, $L[1/t]$ does not exist.

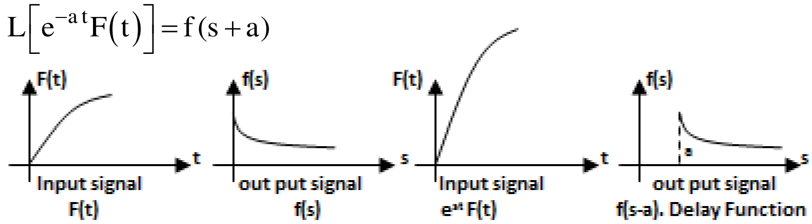
First Shifting Property of Laplace Transform

Statement: If L[F(t) = f(s) ,then $L\left[e^{at}F(t)\right]=f(s-a)$

Proof: By definition,

$$L\left[F(t)\right]=\int_0^\infty e^{-st}F(t)\,dt=f(s)\Rightarrow L\left[e^{at}F(t)\right]=\int_0^\infty e^{-st}e^{at}F(t)\,dt=\int_0^\infty e^{-st+at}F(t)\,dt=\int_0^\infty e^{-(s-a)t}F(t)\,dt=f(s-a)$$

Similarly, $L\left[e^{-at}F(t)\right] = f(s+a)$



This property of Laplace transform has an important physical implication. It helps to delay an output signal and to release it after the required destination. This is done by multiplying the given function F(t) by the factor e^{at} or e^{-at} respectively. This is shown in the above figures.

Example 10: Find (i) $L\left[t^3e^{-3t}\right]$ (ii) $L\left[\sinh 2t \sin 3t\right]$

Solution: (i) Here $F(t) = t^3$,and $L[t^3] = \dfrac{3!}{s^{3+1}} = \dfrac{6}{s^4}$

Thus by first shifting property, $L\left[t^3e^{-3t}\right] = \dfrac{6}{(s+3)^4}$

(ii) Here $L\left[\sinh 2t \sin 3t\right] = L\left[\dfrac{e^{2t}-e^{-2t}}{2}\right]\sin 3t$. Now $L(\sin 3t) = \dfrac{3}{s^2+9}$. Thus

$L\left[\sinh 2t \sin 3t\right] = \dfrac{1}{2}\left\{L\left(e^{2t}\sin 3t\right) - L\left(e^{-2t}\sin 3t\right)\right\}$.By shift property, we have

$$L\left[\sinh 2t \sin 3t\right] = \frac{1}{2}\left\{\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right\}$$

Example 11: Find $L\left[(t+2)^2 e^t\right]$

Solution: $L\left[(t+2)^2 e^t\right] = L\left[\left(t^2+4t+4\right)e^t\right] = L\left(t^2e^t\right) + 4L\left(te^t\right) + 4L(1)$ (1)

Now $L[t] = \dfrac{1}{s^2}$, $L[t^2] = \dfrac{2}{s^3}$ and $L[1] = \dfrac{1}{s}$

Using the first shifting property, we get: $L[te^t] = \dfrac{1}{(s-1)^2}$ and $L[t^2e^t] = \dfrac{2}{(s-1)^3}$ and thus (1)

becomes: $$L\left[(t+2)^2 e^t\right] = \frac{2}{(s-1)^3} + \frac{4}{(s-1)^2} + \frac{4}{(s-1)}$$

Example 12: Find $L\left[e^{-3t}\left(2\cos 5t - 3\sin 5t\right)\right]$

Solution: $L\left[e^{-3t}\left(2\cos 5t - 3\sin 5t\right)\right] = 2L\left[e^{-3t}\cos 5t\right] - 3L\left[e^{-3t}\sin 5t\right]$ (1)

Now, $L(\cos 5t) = \dfrac{s}{s^2+25}$ and $L(\sin 5t) = \dfrac{5}{s^2+25}$

Thus using first shifting property, we get

$L\left[e^{-3t}\cos 5t\right] = \dfrac{s+3}{(s+3)^2+25}$ and $L\left[e^{-3t}\sin 5t\right] = \dfrac{5}{(s+3)^2+25}$

Thus (1) becomes, $L\left[e^{-3t}\left(2\cos 5t - 3\sin 5t\right)\right] = \dfrac{2s-9}{s^2+6s+34}$

Example 13: Evaluate $L(1 + t\,e^{-t})^3$

Solution: $L\left(1 + t\,e^{-t}\right)^3 = L\left[\left(1+3te^{-t}+3t^2e^{-2t}+t^3e^{-3t}\right)\right]$

$$= L[1] + 3L\left[te^{-t}\right] + 3L\left[t^2e^{-2t}\right] + L\left[t^3e^{-3t}\right] \qquad (1)$$

Now, $L[1] = \frac{1}{s}, L[t] = \frac{1}{s^2}, L[t^2] = \frac{2}{s^3}$ and $L[t^3] = \frac{6}{s^4}$. Now by 1st shifting property

$L\left[te^{-t}\right] = \frac{1}{(s+1)^2}$, $L\left[t^2e^{-2t}\right] = \frac{2}{(s+2)^3}$ and $L\left[t^3e^{-3t}\right] = \frac{6}{(s+3)^4}$

Hence equation (1) becomes

$$L\left[\left(1+te^{-t}\right)^3\right] = \frac{1}{s} + 3.\frac{1}{(s+1)^2} + 3.\frac{2}{(s+2)^3} + \frac{6}{(s+3)^4} = \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Second Shifting Property of Laplace Transform

Statement: If $L[F(t) = f(s)$ and $G_a(t) = F(t-a)$ for $t > a$ and $G_a(t) = 0$ for $t < a$, then $L\,[G_a(t)\,] = e^{-as} f(s)$

Proof: By definition, $L\left[G_a(t)\right] = \int_0^\infty e^{-st} G_a(t)\,dt$

$$= \int_0^a e^{-st} G_a(t)\,dt + \int_a^\infty e^{-st} G_a(t)dt = \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st} F(t-a)dt = \int_a^\infty e^{-st} F(t-a)dt$$

Substituting $t - a = z \;\therefore\; dt = dz$. Now as $t \to a,\ z \to 0$ and as $t \to \infty, z \to \infty$

Thus $L\left[G_a(t)\right] = \int_a^\infty e^{-s(z+a)} F(z)dz = \int_a^\infty e^{-sz} e^{-as} F(z)dz$

or $\quad L\left[G_a(t)\right] = e^{-as} \int_a^\infty e^{-sz} F(z)dz = e^{-as} L[F(z)] = e^{-as} f(s)$

This property has also an important physical implication. It states that if an input signal is to be delayed for some time $t = a$, the effect of this delay on the output signal is to multiply Laplace Transform of F(t) that is, f(s) by the factor e^{-as}. This is depicted in the following figure.

Example 14: Evaluate $L[\cos(t-2)]$

Solution: We know that $L \cos t = s/(s^2 + a^2)$. Using second shifting property, we get

$$L[\cos(t-2)] = e^{-2s}\, s/(s^2 + a^2)$$

Third Shifting Property or Change of Scale Property

Statement: If $L[F(t) = f(s)$, then $L[F(at)] = (1/a)\, f\,(s/a)$ for some constant `a`.

Proof: By definition, $L\left[F(t)\right] = \int_0^\infty e^{-st} F(t)\,dt$ then $L\left[F(at)\right] = \int_0^\infty e^{-st} F(at)\,dt$

Let $at = z$ then $t = z/a$ and therefore, $dt = dz/a$

Now as $t \to 0$ then $z \to 0$ and as $t \to \infty$ then $z \to \infty$. Hence,

$$L\left[F(at)\right]=\int_0^\infty e^{-s(z/a)}F(z)\frac{1}{a}dz=\frac{1}{a}\int_0^\infty e^{-sz/a}F(z)dz=\frac{1}{a}\int_0^\infty e^{-(s/a)z}F(z)dz=\frac{1}{a}f\left(\frac{s}{a}\right)$$

Example 15: If $L\left[F(t)\right]=\dfrac{s^2-s+1}{(2s+1)^2(s-1)}$,find $L\left[e^{-t}F(2t)\right]$

Solution: In this problem, we have to use two properties. First property that we shall use is the change of scale property and the second one is the first shifting property. According to change of scale property, if L[F(t) = f(s), then L[F(at)] = (1/a) f(s/a)

Now given that $L\left[F(t)\right]=\dfrac{s^2-s+1}{(2s+1)^2(s-1)}=f(s)$

Using the above formula with a = 2, we get

$$L\left[F(2t)\right]=\frac{1}{2}\left\{\frac{\left(\frac{s}{2}\right)^2-\frac{s}{2}+1}{\left(2\frac{s}{2}+1\right)^2\left(\frac{s}{2}-1\right)}\right\}=\frac{1}{2}\left\{\frac{\frac{s^2}{4}-\frac{s}{2}+1}{(s+1)^2\left(\frac{s-2}{2}\right)}\right\}=\frac{1}{2}\left\{\frac{s^2-2s+4}{4}\cdot\frac{2}{(s+1)^2(s-2)}\right\}$$

$$=\frac{1}{4}\left\{\frac{s^2-2s+4}{(s+1)^2(s-2)}\right\}$$

Now using the first shifting property, we have

$$L\left[e^{-t}F(2t)\right]=\frac{1}{4}\left\{\frac{(s+1)^2-2(s+1)+4}{(s+1+1)^2(s+1-2)}\right\}=\frac{1}{4}\frac{s^2+3}{(s+2)^2(s-1)}$$

Multiplication by t^n Property

Statement: If L[F(t) = f(s) ,then L[t^n F(t)] = $(-1)^n f^n(s)$, where $f^n(s)$ is the n^{th} derivative of f (s).

Proof: By definition, $L\left[F(t)\right]=f(s)=\int_0^\infty e^{-st}F(t)\,dt$

Differentiate both sides w.r.t s and using Leibniz Theorem of integration under differentiation, that is: $\dfrac{d}{dx}\int f(x,y)dy=\int\dfrac{\partial}{\partial x}f(x,y)dy,$

we get $\dfrac{d}{ds}f(s)=\dfrac{d}{ds}\int_0^\infty e^{-st}F(t)\,dt=\int_0^\infty\dfrac{\partial}{\partial s}\left(e^{-st}\right)F(t)dt=\int_0^\infty -t\,e^{-st}F(t)dt$

or $(-1)\ \dfrac{d}{ds}f(s)\ =\int_0^\infty e^{-st}\,t\,F(t)dt=L\left[t\,F(t)\right]$

Differentiating again w.r.t s and using Leibniz Theorem, we obtain

$(-1)^2\ \dfrac{d^2}{ds^2}f(s)\ =\int_0^\infty e^{-st}\,t^2F(t)dt=L\left[t^2F(t)\right]$. Generalizing, we get

$$(-1)^n\ \frac{d^n}{ds^n}f(s)\ =\int_0^\infty e^{-st}\,t^nF(t)dt\ =L\left[t^nF(t)dt\right]\Rightarrow L\left[t^nF(t)dt\right]=\ (-1)^n\ \frac{d^n}{ds^n}f(s)\ =\ (-1)^n f^n(s)$$

Example 16: Find L [t^2 e^{2t}]

Solution: We know that $L[e^{2t}]=1/(s-2)=f(s)$. Now using multiplication by t^n property, we get

$$L(t^2e^{at})=(-1)^2\ \frac{d^2}{ds^2}\cdot\frac{1}{s-2}$$

Now differentiate R.H.S twice w.r.t s, we have

$$L\left[t^2e^{at}\right]=(-1)^2\ \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{1}{s-2}\right)\right]=\frac{d}{ds}\left[\frac{-1}{(s-2)^2}\right]=\frac{d}{ds}\left[-(s-2)^{-2}\right]=2(s-2)^{-3}=\frac{2}{(s-2)^3}$$

Example 17: Find L(t e^{-4t} sin 3t)

Solution: $L\left[te^{-4t}\sin 3t\right]=L\left[e^{-4t}(t\sin 3t)\right]$ (1)

Now using multiplication by t property, we obtain

$$L[t\sin 3t]=-\frac{d}{ds}L(\sin 3t)=-\frac{d}{ds}\frac{3}{s^2+9}=-3\frac{d}{ds}\frac{1}{s^2+9}=\frac{6s}{(s^2+9)^2}$$

Now using first shifting property on (1), we have

$$L\left[e^{-4t}(t\sin 3t)\right]=\frac{6(s+4)}{[(s+4)^2+9]^{22}}=\frac{6(s+4)}{(s^2+8s+25)^2}$$

Example 18: Use multiplication by t^n property to evaluate $\int_0^\infty e^{-3t}\ t\sin 3t\,dt$

Solution: Using the fact that $L[t\sin 3t]=-\frac{d}{ds}L(\sin 3t)=-\frac{d}{ds}\frac{3}{s^2+9}=-3\frac{d}{ds}\frac{1}{s^2+9}=\frac{6s}{(s^2+9)^2}$

Now, by definition, $L[t\sin 3t]=\int_0^\infty e^{-st}t\sin 3t\,dt=6s/(s^2+9)$

Substituting s = 3, we obtain $\int_0^\infty e^{-3t}t\sin 3t\,dt=\frac{18}{(18)^2}=\frac{1}{18}$

Laplace Transform of Derivatives

Statement: If L[F(t) = f(s) , then L[F$'$(t)] = s f(s) – F(0). Hence show that

L[$F^{(n)}$ (t)] = s^n f(s) – s^{-1} F(0) –s^{n-2} F$'$(0) - …– s F^{n-2}(0) – F^{n-1}(0)

Proof: We know that, $L\left[F(t)\right]=f(s)=\int_0^\infty e^{-st}F(t)dt$ then $L\left[F'(t)\right]=\int_0^\infty e^{-st}F'(t)dt$

Integrating by parts, taking u = e^{-st} and v = $F'(t)$ we get,

$$L\left[F'(t)\right]=\left[e^{-st}F(t)\right]_0^\infty-\int_0^\infty -se^{-st}F(t)dt=\left[e^{-\infty}-e^0F(0)\right]+s\int_0^\infty e^{-st}F(t)dt$$

$$=\left[0-F(0)+s\int_0^\infty e^{-st}F(t)dt\right]=-F(0)+sf(s)$$

Hence, $L\left[F'(t)\right]$ = $sf(s)-F(0)$. Similarly,

$$L\left[F''(t)\right]=\left[e^{-st}F'(t)\right]_0^\infty-\int_0^\infty -se^{-st}F'(t)dt=\left[e^{-\infty}-e^0F'(0)\right]+s\int_0^\infty e^{-st}F'(t)dt$$

$$=-F'(0)+sL(F'(t))=-F'(0)+s[sf(s)-F(0)]=s^2f(s)-sF(0)-F'(0)$$

In general, $L\left[F^{(n)}(t)\right]=s^nf(s)-s^{n-1}F(0)-s^{n-2}F'(0)-...-sF^{n-2}(0)-F^{n-1}(0)$

This property is useful for solving differential equations.

Example 19: (i) Evaluate $L\left(\sin\sqrt{t}\right)$, hence find $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)$

(ii) Show that $L\left\{2\sqrt{t/\pi}\right\}=1/s^{3/2}$.Hence show that $L\left\{1/\sqrt{\pi t}\right\}=1/\sqrt{s}$

Solution: (i) Let F(t) = $\sin\sqrt{t}$ or $F'(t)=\dfrac{\cos\sqrt{t}}{2\sqrt{t}}$ and F(0) = sin (0) = 0

Now using property of Laplace transform of derivative, that is,

$L\{F'(t)\}=sL\{F(t)\}-F(0)$, we have

$L\{F'(t)\}=L\{\cos\sqrt{t}/2\sqrt{t}\}=sL\{\sin\sqrt{t}\}-0$ or $L\{\cos\sqrt{t}/2\sqrt{t}\}=\sqrt{\pi}se^{-1/4s}/2s^{3/2}$

since, $L(\sin\sqrt{t})=\sqrt{\pi}e^{-1/4s}/2s^{3/2}$ (See Example 4)

$$\therefore L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\}=\frac{1}{2}L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}=\frac{\sqrt{\pi}}{2s^{1/2}}e^{-1/4s} \text{ or } L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}=\sqrt{\left(\frac{\pi}{s}\right)}e^{-1/4s}$$

(ii) $L\left\{2\sqrt{\frac{t}{\pi}}\right\}=\frac{2}{\sqrt{\pi}}L\sqrt{t}=\frac{2}{\sqrt{\pi}}\left(\frac{\Gamma(3/2)}{s^{3/2}}\right)=\frac{2}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2}.\frac{1}{s^{3/2}}=\frac{1}{s^{3/2}}$

Let $F(t)=\left\{2\sqrt{\frac{t}{\pi}}\right\}$. Then $F`(t)=\frac{1}{\sqrt{\pi t}}$ and $F(0)=0$

Now LF `(t) = sf(s) – F(0) = s L[F(t)] – 0 = s L[F(t)].

Substituting the values of F `(t) and F(t), we obtain: $L\left\{\sqrt{\frac{1}{\pi t}}\right\}=s.\frac{1}{s^{3/2}}=1/\sqrt{s}$

Division by t Property

Statement: If $L[F(t)]=f(s)$, then $L\left[\frac{F(t)}{t}\right]=\int_s^\infty f(u)du$, provided $\lim_{t\to 0}\frac{F(t)}{t}$ exists.

Proof: We know that $L[F(t)]=f(s)=\int_0^\infty e^{-st}F(t)dt$

Let G(t) = F(t)/t then F(t) = t G(t). Taking Laplace transform on both sides, we have

$$L[F(t)]=L[tG(t)]$$

Using the multiplication by t property, we get

$$f(s) = -g`(s) \quad \text{or} \quad g`(s) = -f(s)$$

Now integrating both sides from ∞ to s, we have

$$g(s)=-\int_\infty^s f(u)du \text{ or } g(s)=\int_s^\infty f(u)du$$

Note: Here we have changed the variable from s to u. But g(s) = L[G(t)] hence,

$L[G(t)]=\int_s^\infty f(u)du$. Putting G(t) = F(t)/t, we obtain: $L\left[\frac{F(t)}{t}\right]=\int_s^\infty f(u)du$

Example 20: Evaluate (i) $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$ (ii) $L\left[\frac{\cos 4t-\cos 6t}{t}\right]$

Use the results to evaluate $\int_0^\infty e^{-2t}\frac{\cos 4t-\cos 6t}{t}dt$ and $\int_0^\infty\frac{\cos 4t-\cos 6t}{t}dt$

Solution: We know that $L\left[\frac{F(t)}{t}\right]=\int_s^\infty f(u)du$. Here F(t) = $e^{-at}-e^{-bt}$

(i) Thus, $L[F(t)]=L\left[e^{-at}-e^{-bt}\right]=\frac{1}{s+a}-\frac{1}{s+b}$

Hence, $L\left[\frac{F(t)}{t}\right]=L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\int_s^{\infty}\left(\frac{1}{u+a}-\frac{1}{u+b}\right)du$ [By using the division by t property]

Integrating R.H.S from s to ∞, we have

$$L\left[\frac{F(t)}{t}\right]=\left[\ln(u+a)-\ln(u+b)\right]_s^{\infty}=\left[\ln\frac{(u+a)}{(u+b)}\right]_s^{\infty}=\left[\ln\frac{u(1+a/u)}{u(1+b/u)}\right]_s^{\infty}$$

$$=\left[\ln\frac{(1+a/u)}{(1+b/u)}\right]_s^{\infty}=\ln\frac{1-0}{1-0}-\ln\frac{(1+a/s)}{(1+b/s)}=\ln 1-\ln\frac{(s+a)}{(s+b)}=0+\ln\frac{(s+b)}{(s+a)}$$

Therefore, $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\ln\frac{(s+b)}{(s+a)}$

(ii) $L(\cos 4t)=\left[\frac{s}{s^2+16}\right]$ or $L\left(\frac{\cos 4t}{t}\right)=\int_s^{\infty}\frac{u}{u^2+16}du=\frac{1}{2}\int_s^{\infty}\frac{2u}{u^2+16}du$

(ii) $=\frac{1}{2}\ln\left[s^2+16\right]_s^{\infty}=\frac{1}{2}\lim_{P\to\infty}\ln\left[s^2+16\right]_s^{P}=\frac{1}{2}\left[\lim_{P\to\infty}\ln\left(P^2+16\right)-\ln\left(s^2+16\right)\right]$

Similarly, $L\left(\frac{\cos 6t}{t}\right)=\frac{1}{2}\left[\lim_{P\to\infty}\ln\left(P^2+36\right)-\ln\left(s^2+36\right)\right]$. Subtracting, we get

$$L\left[\frac{\cos 4t-\cos 6t}{t}\right]=L\left(\frac{\cos 6t}{t}\right)=\frac{1}{2}\left[\lim_{P\to\infty}\ln\left(\frac{P^2+16}{P^2+36}\right)+\ln\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$=\frac{1}{2}\left[\lim_{P\to\infty}\ln\left(\frac{P^2\left(1+16/P^2\right)}{P^2\left(1+36/P^2\right)}\right)+\ln\left(\frac{s^2+36}{s^2+16}\right)\right]=\frac{1}{2}\left[\lim_{P\to\infty}\ln\left(\frac{\left(1+\frac{16}{P^2}\right)}{\left(1+\frac{36}{P^2}\right)}\right)+\ln\left(\frac{s^2+36}{s^2+16}\right)\right]$$

$$=\frac{1}{2}\left[\lim_{P\to\infty}\ln 1+\ln\left(\frac{s^2+36}{s^2+16}\right)\right]=\frac{1}{2}\left[0+\ln\left(\frac{s^2+36}{s^2+16}\right)\right]\Rightarrow L\left[\frac{\cos 4t-\cos 6t}{t}\right]=\frac{1}{2}\ln\left(\frac{s^2+36}{s^2+16}\right)$$

Now by definition:

$$L[F(t)]=\int_0^{\infty}e^{-st}F(t)\,dt \quad\text{or}\quad L\left[\frac{\cos 4t-\cos 6t}{t}\right]=\int_0^{\infty}e^{-st}\,\frac{\cos 4t-\cos 6t}{t}dt=\frac{1}{2}\ln\left(\frac{s^2+36}{s^2+16}\right)$$

Put $s=2$, we get $\int_0^{\infty}e^{-2t}\,\frac{\cos 4t-\cos 6t}{t}dt=\frac{1}{2}\ln\left(\frac{4+36}{4+16}\right)=\frac{1}{2}\ln 2=\ln 2^{1/2}=\ln\sqrt{2}$

Put $s=0$, we get $\int_0^{\infty}\frac{\cos 4t-\cos 6t}{t}dt=\frac{1}{2}\ln\left(\frac{36}{16}\right)=\ln\sqrt{\frac{36}{16}}=\ln\left(\frac{6}{4}\right)=\ln\left(\frac{3}{2}\right)$

Example 21: Show that $L\left(\frac{\sin^2 t}{t}\right)=\frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$. Hence evaluate $\int_0^{\infty}\frac{e^{-t}\sin^2 t}{t}dt$

Solution: Let $F(t)=\sin^2 t=\frac{1-\cos 2t}{2}$

Then, $L\left[\sin^2 t\right]=L\left[\frac{1-\cos 2t}{2}\right]=\frac{1}{2}\left[L[1]-L[\cos 2t]\right]=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$

Using the division by t property of Laplace transform, we get

$$L\left[\frac{F(t)}{t}\right]=L\left[\frac{\sin^2 t}{t}\right]=\frac{1}{2}\int_s^\infty\left[\frac{1}{u}-\frac{u}{u^2+4}\right]du=\frac{1}{2}\int_s^\infty\left[\frac{1}{u}-\frac{2u}{2(u^2+4)}\right]du$$

$$=\frac{1}{2}\left[\ln u-\frac{1}{2}\ln\left(u^2+4\right)\right]_s^\infty=\frac{1}{2}\ln\frac{u}{(u^2+4)^{1/2}}\Bigg|_s^\infty=\frac{1}{2}\left(\ln 1-\ln\frac{s}{(s^2+4)^{1/2}}\right)$$

$$=\frac{1}{2}\left(0-\ln\frac{s}{(s^2+4)^{1/2}}\right)=-\frac{1}{2}\left(\ln s-\ln(s^2+4)^{1/2}\right)=\frac{1}{2}.\frac{1}{2}\ln\left(s^2+4\right)-\frac{1}{2}\ln s$$

$$=\frac{1}{4}\ln\left(s^2+4\right)-\frac{1}{4}\ln s^2=\frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$$

Since, $L\left(\frac{\sin^2 t}{t}\right)=\frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$. Then by definition

$L\left(\frac{\sin^2 t}{t}\right)=\int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t}\right)=\frac{1}{4}\ln\left(\frac{s^2+4}{s^2}\right)$. Substituting s = 1, we get

$$\int_0^\infty e^{-t}\left(\frac{\sin^2 t}{t}\right)=\frac{1}{4}\ln\left(\frac{1^2+4}{1^2}\right)=\frac{1}{4}\ln 5$$

Example 22: Show that $\int_0^\infty \frac{e^{-t}\sin t}{t}dt=\frac{\pi}{4}$

Solution: We know that L[sin t] = $1/(s^2 + 1)$. Using division by t property, we get

$$L\left[\frac{\sin t}{t}\right]=\int_s^\infty\frac{1}{u^2+1}du=\left[\tan^{-1}u\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}s=\frac{\pi}{2}-\tan^{-1}s$$

$\int_s^\infty\frac{1}{u^2+1}du=\left[\tan^{-1}u\right]_s^\infty=\tan^{-1}\infty-\tan^{-1}s=\frac{\pi}{2}-\tan^{-1}s.$ Thus, $\int_0^\infty e^{-st}\frac{\sin t}{t}dt=\frac{\pi}{2}-\tan^{-1}s$

Now substituting s = 1, we get: $\int_0^\infty e^{-t}\frac{\sin t}{t}dt=\frac{\pi}{2}-\tan^{-1}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

Example 23: Show that $\int_0^\infty\frac{\sin^2 t}{t^2}dt=\frac{\pi}{2}$

Proof: Let $F(t) = \sin^2 t=\left(\frac{1-\cos 2t}{2}\right)$,then

$$L\{F(t)\}=\frac{1}{2}\left[L\{1\}-L\{\cos 2t\}\right]=\frac{1}{2}\left(\frac{1}{s}-\frac{s}{s^2+4}\right)=f(s)$$

Now, $L\left\{\frac{F(t)}{t}\right\}=\int_s^\infty\frac{\sin^2 t}{t}dt=\frac{1}{2}\int_s^\infty\left(\frac{1}{u}-\frac{u}{u^2+4}\right)du=\frac{1}{2}\lim_{t\to\infty}\left(\ln u-\frac{1}{2}\ln\left(u^2+4\right)\right)_s^t$

$$=\frac{1}{2}\lim_{t\to\infty}\left(\frac{1}{2}\ln u^2-\frac{1}{2}\ln\left(u^2+4\right)\right)_s^t=\frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{u^2}{u^2+4}\right)_s^t=\frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{t^2}{t^2+4}-\ln\frac{s^2}{s^2+4}\right)$$

$$=\frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{t^2}{t^2(1+4/t^2)}-\lim_{t\to\infty}\ln\frac{s^2}{s^2+4}\right)$$

$$=\frac{1}{4}\lim_{t\to\infty}\left(\ln\frac{1}{1+0}-\ln\frac{s^2}{s^2+4}\right)=-\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right)\quad[\ln 1=0]\Rightarrow L\left\{\frac{F(t)}{t}\right\}=-\frac{1}{4}\ln\left(\frac{s^2}{s^2+4}\right)=f(s)$$

Now using division by t property once again, we obtain

$$L\left\{\frac{F(t)}{t^2}\right\}=-\frac{1}{4}\int_s^\infty \ln\left(\frac{u^2}{u^2+4}\right)du=-\frac{1}{4}\int_s^\infty \ln\left(\frac{u^2}{u^2+4}\right).1du$$

Now integrate by parts, gives

$$L\left\{\frac{F(t)}{t^2}\right\}=-\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right)-\int\frac{u^2+4}{u^2}.\left(\frac{2u.(u^2+4)-2u.u^2}{(u^2+4)^2}\right).u\ du\right]_s^\infty$$

$$=-\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right)-\int\frac{u^2+4}{u}.\left(\frac{2u^3+8u-2u^3}{(u^2+4)^2}\right)du\right]_s^\infty$$

$$=-\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right)-\int\frac{8u(u^2+4)}{u(u^2+4)^2}du\right]_s^\infty=-\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right)-8\int\frac{1}{(u^2+4)}du\right]_s^\infty$$

$$=-\frac{1}{4}\left[u\ln\left(\frac{u^2}{u^2+4}\right)-\frac{8}{2}\tan^{-1}\left(\frac{u}{2}\right)\right]_s^\infty=-\frac{1}{4}\left[0-4\frac{\pi}{2}-s\ln\left(\frac{s^2}{s^2+4}\right)+4\tan^{-1}\left(\frac{s}{2}\right)\right]$$

Thus $\int_0^\infty e^{-st}\left(\frac{\sin^2 t}{t^2}\right)dt=-\frac{1}{4}\left[-2\pi-s\ln\left(\frac{s^2}{s^2+4}\right)+4\tan^{-1}\left(\frac{s}{2}\right)\right]$

Now substituting s = 0, we get $\int_0^\infty\left(\frac{\sin^2 t}{t^2}\right)dt=\frac{\pi}{2}$ [Note: $\tan^{-1} 0 = 0$ and $\ln 1 = 0$]

Laplace Transform of Integrals

Statement: If $L[F(t)]=f(s)$, then $L\int_0^t F(u)du=\frac{f(s)}{s}$

Proof: Let $G(t)=\int_0^t F(u)\ du$ or $G(0)=\int_0^0 F(u)\ du=0$

Differentiate both sides w.r.t t, we have: $G'(t)=\frac{d}{dt}\int_0^t F(u)\ du=F(t)$

Taking Laplace transform on both sides, we obtain

$$L(F(t))=L(G'(t))=s\ g(s)-G(0)=s\ g(s)\quad [\because G(0)=0]$$

This gives, $f(s)=sg(s)$ or $g(s)=f(s)/s$. But $g(s)=L[G(t)]$

Thus, $L[G(t)]=\frac{f(s)}{s}$

Example 24: Prove that $L\left[\int_0^t \sin u\ du\right]=\frac{1}{s(s^2+1)}$

Solution: *Method-I* $\int_0^t \sin u\ du=-[\cos u]_0^t=-\{\cos t-\cos 0\}=1-\cos t$. Thus

$$L\left[\int_0^t \sin u\ du\right]=L[1-\cos t]=L[1]-L[\cos t]=\frac{1}{s}-\frac{s}{s^2+1}=\frac{s^2+1-s^2}{s(s^2+1)}=\frac{1}{s(s^2+1)}$$

Method-II L[sin t] = $1/(s^2 + 1)$, hence by Laplace transforms of integral property,

$$L\left[\int_0^t \sin u\, du\right] = \frac{1}{s\left(s^2+1\right)}$$

Example 25: Prove that $L\left[\int_0^t \frac{\sin u}{u} du\right] = \frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right)$

Solution: Consider, $\int_0^t \frac{\sin u}{u} du$. Let u = t v then du = t dv.

Now, if $u \to 0$ then $v \to 0$ and similarly if $u \to t$ then $v \to 1$.

Hence $\left[\int_0^t \frac{\sin u}{u} du\right] = \int_0^1 \frac{\sin tv}{v} dv$. Now taking the Laplace transform on both sides, we have

$$L\left[\int_0^t \frac{\sin u}{u} du\right] = L\left[\int_0^1 \frac{\sin tv}{v} dv\right] = \int_0^\infty e^{-st}\left\{\int_0^1 \frac{\sin tv}{v} dv\right\} dt = \int_0^1 \frac{1}{v}\left\{\int_0^\infty e^{-st} \sin vt\, dt\right\} dv$$

$$= \int_0^1 \frac{1}{v}\left\{L\left[\sin vt\right]\right\} dv = \int_0^1 \frac{1}{v}\left\{\frac{v}{v^2+s^2}\right\} dv = \int_0^1 \left\{\frac{1}{v^2+s^2}\right\} dv = \frac{1}{s}\tan^{-1}\left[\frac{v}{s}\right]_0^1 = \frac{1}{s}\left[\tan^{-1}\left(\frac{1}{s}\right) - \tan^{-1}\left(\frac{0}{s}\right)\right]$$

$$\frac{1}{s}\tan^{-1}\left(\frac{1}{s}\right) \qquad \left[\text{Note}: \tan^{-1} 0 = 0\right]$$

Laplace Transform of Periodic Functions

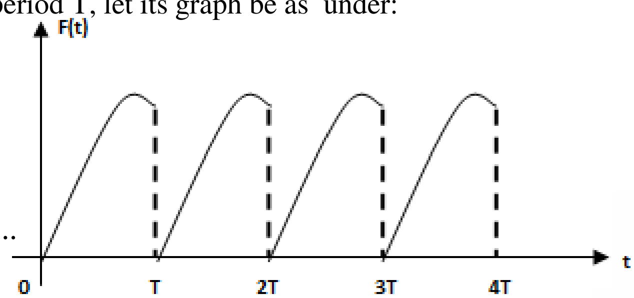
A function F(t) is said to be periodic of period T if it satisfies the condition:

$$F(t \pm T) = F(t) \text{ for all } T > 0.$$

Theorem: If F(t) is periodic function of period T ,then

$$L\left[F(t)\right] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt$$

Proof: Since F(t) is a periodic function of period T, let its graph be as under:



Now by definition:

$$L\left[F(t)\right] = \int_0^\infty e^{-st}F(t)\,dt = \int_0^T e^{-st}F(t)\,dt + \int_T^{2T} e^{-st}F(t)\,dt + \int_{2T}^{3T} e^{-st}F(t)\,dt + \dots$$

Substituting:

t = u in the 1st integral $\to$ dt = du. Also if t = 0 $\to$ u = 0 and if t = T $\to$ u = T

t = u + T in the 2nd integral $\to$ dt = du. Also if t = T $\to$ u = 0 and if t = 2T $\to$ u = T

t = u + 2T in the 3rd integral $\to$ dt = du. Also if t = 2T $\to$ u = 0 and if t = 3T $\to$ u = T. Thus,

$$L\left[F(t)\right] = \int_0^T e^{-su}F(u)\,du + \int_0^T e^{-s(u+T)}F(u+t)\,du + \int_0^T e^{-s(u+2T)}F(u+2t)\,du + \dots$$

$$= \int_0^T e^{-su}F(u)\,du + e^{-sT}\int_0^T e^{-su}F(u)\,du + e^{-2sT}\int_0^T e^{-su}F(u)\,du + \dots$$

Since, F(T) is periodic function hence $F(u) = F(U+T) = F(u+2T) = \dots$ Thus,

$$L\left[F(t)\right]=(1+e^{-sT}+e^{-2sT}+\ldots)\int_0^T e^{-su}F(u)\,du=(1-e^{-sT})^{-1}\int_0^T e^{-su}F(u)\,du=\frac{1}{(1-e^{-sT})}\int_0^T e^{-st}F(t)\,dt$$

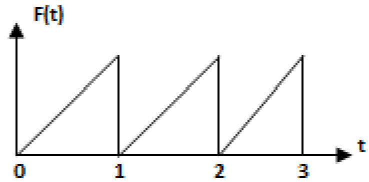
Remarks: (i) It may be noted that here we have used an infinite binomial series

$$1 + x + x^2 + \ldots = (1 - x)^{-1}$$

(ii) Variable u is changed with variable s in the last integral. Such change has no effect on the result.

Example 26: Obtain the Laplace transform of saw-tooth wave function given by F(t) = t/k, 0 < t < 1.

Solution: The graph of the periodic saw-tooth wave function is shown below. Here F(t) is a periodic function with period T = 1. Hence, by definition



$$L\left[F(t)\right]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt=\frac{1}{1-e^{-s}}\int_0^1 e^{-st}\,\frac{t}{k}\,dt$$

Integration by parts, gives

$$L\left[F(t)\right]=\frac{1}{k\left(1-e^{-s}\right)}\left\{\left[t\frac{e^{-st}}{-s}\right]_0^1-\int_0^1 1.\frac{e^{-st}}{-s}dt\right\}$$

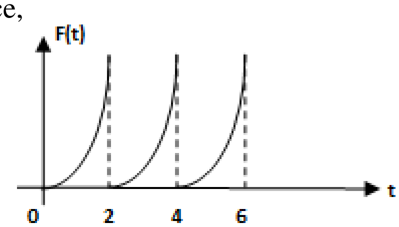
$$=\frac{1}{k\left(1-e^{-s}\right)}\left\{\left[\frac{e^{-s}}{-s}-0\right]+\frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^1\right\}=\frac{1}{k\left(1-e^{-s}\right)}\left\{\left[\frac{e^{-s}}{-s}\right]-\frac{1}{s^2}\left[e^{-s}-e^{0}\right]\right\}$$

$$=\frac{1}{k\left(1-e^{-s}\right)}\left\{\left[\frac{e^{-s}}{-s}\right]+\frac{1}{s^2}\left[1-e^{-s}\right]\right\}=-\frac{e^{-s}}{sk\left(1-e^{-s}\right)}+\frac{1}{ks^2}=\frac{1}{sk}\left[\frac{1}{s}-\frac{e^{-s}}{\left(1-e^{-s}\right)}\right]$$

Example 27: Find the Laplace transform of the function F(t) defined as

$$F(t)=t^2,\ 0<t<2 \text{ and } F(t+2)=F(t)$$

Solution: The function F(t) is periodic function of period 2 hence,



$$L\{F(t)\}=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}F(t)\,dt\,.$$

$$\therefore L\{F(t)\}=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}F(t)\,dt=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}t^2\,dt\,.$$

Integrating by parts gives:

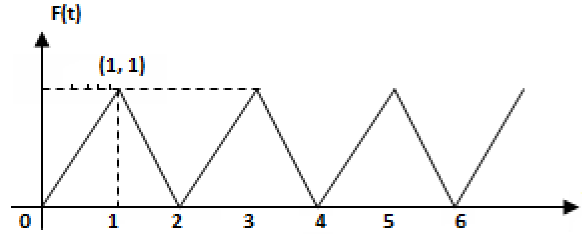
$$\therefore L[F(t)]=\frac{1}{\left(1-e^{-2s}\right)}\left[\left[t^2\frac{e^{-st}}{-s}\right]_0^2-2\int_0^2 t\frac{e^{-st}}{-s}dt\right]=\frac{1}{\left(1-e^{-2s}\right)}\left[\left[\frac{4e^{-2s}-0}{-s}\right]+\frac{2}{s}\int_0^2 te^{-st}\,dt\right]$$

$$=\frac{1}{(1-e^{-2s})}\left[-\frac{4e^{-2s}}{s}+\frac{2}{s}\left\{\left[\frac{te^{-st}}{-s}\right]_0^2-\int_0^2\frac{e^{-st}}{-s}dt\right\}\right]=\frac{1}{(1-e^{-2s})}\left[-\frac{4e^{-2s}}{s}+\frac{2}{s}\left\{\left[\frac{2e^{-2s}}{-s}\right]+\frac{1}{s}\int_0^2 e^{-st}\,dt\right\}\right]$$

$$=\frac{1}{(1-e^{-2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}+\frac{2}{s^2}\int_0^2 e^{-st}\,dt\right]=\frac{1}{(1-e^{-2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}+\frac{2}{s^2}\left[\frac{e^{-st}}{-s}\right]_0^2\right]$$

$$=\frac{1}{(1-e^{-2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}-\frac{2}{s^3}\left[e^{-2s}-1\right]\right]=\frac{1}{(1-e^{-2s})}\left[-\frac{4e^{-2s}}{s}-\frac{4e^{-2s}}{s^2}-\frac{2e^{-2s}}{s^3}+\frac{2}{s^3}\right]$$

Example 28: Show that the function F(t) whose graph is the triangular wave given in the figure below, has the Laplace transform $\frac{1}{s^2}\tanh\left(\frac{s}{2}\right)$



Solution: Here given function is periodic function with period T = 2. To find F(t), using the formula of a straight line passing through two points, we have, equation of line through (1, 1) and (2, 0) is by using formula

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)\Rightarrow y-1=\frac{0-1}{2-1}(x-1)\rightarrow y-1=-1(x-1) \text{ or } y=2-x$$

Similarly, equation of line through two points (0, 0) and (1, 1) is y = t.

$$\therefore F(t)=t \qquad 0\le t\le 1$$
$$=2-t \quad 1\le t\le 2$$

Now using the Laplace transform of periodic function:

$$L\left[F(t)\right]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)dt, \text{ we have}$$

$$=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}F(t)dt+\int_1^2 e^{-st}F(t)dt\right]=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}tdt+\int_1^2 e^{-st}(2-t)dt\right]$$

$$=\frac{1}{1-e^{-2s}}\left[\left(\frac{te^{-st}}{-s}-\frac{e^{-st}}{(-s)(-s)}\right)_0^1+\left(\frac{e^{-st}(2-t)}{-s}-\frac{e^{-st}.-1}{(-s)(-s)}\right)_1^2\right]$$

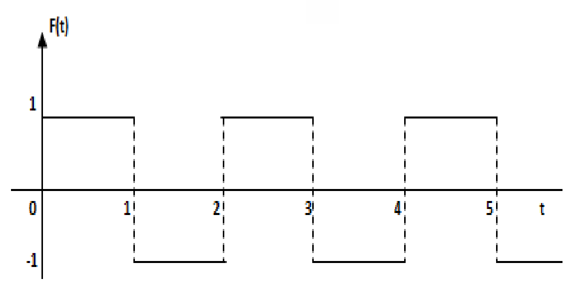
$$=\frac{1}{1-e^{-2s}}\left[\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}\right)-\left(0-\frac{1}{s^2}\right)+\left(0+\frac{e^{-2s}}{s^2}-\left(\frac{e^{-s}}{-s}+\frac{e^{-s}}{s^2}\right)\right)\right]$$

$$=\frac{1}{1-e^{-2s}}\left[\left(\frac{e^{-s}}{-s}-\frac{e^{-s}}{s^2}+\frac{1}{s^2}\right)+\left(\frac{e^{-2s}}{s^2}+\frac{e^{-s}}{s}-\frac{e^{-s}}{s^2}\right)\right]=\frac{1}{1-e^{-2s}}\left[\left(\frac{-se^{-s}-e^{-s}+1+e^{-2s}+se^{-s}-e^{-s}}{s^2}\right)\right]$$

$$=\frac{1}{s^2\left(1-e^{-2s}\right)}\left[e^{-2s}-2e^{-s}+1\right]=\frac{\left(1-e^{-s}\right)^2}{s^2\left(1-e^{-s}\right)\left(1+e^{-s}\right)}=\frac{\left(1-e^{-s}\right)}{s^2\left(1+e^{-s}\right)}.$$ Multiplying & dividing by $e^{s/2}$,

$$\text{we get}: L[F(t)]=\frac{\left(1-e^{-s}\right)}{s^2\left(1+e^{-s}\right)}.\frac{e^{s/2}}{e^{s/2}}=\frac{\left(e^{s/2}-e^{-s/2}\right)}{s^2\left(e^{s/2}+e^{-s/2}\right)}=\frac{1}{s^2}\tanh\left(\frac{s}{2}\right)$$

Example 29: Find the Laplace transform of the following periodic function



Solution: Here given function is periodic function with period T = 2 and F(t) is defined as

$$F(t)=1 \qquad 0\le t\le 1$$
$$=-1 \quad 1\le t\le 2$$

Now using the Laplace transform of periodic function, we have

$$L\left[F(t)\right]=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)dt$$

$$=\frac{1}{1-e^{-2s}}\left[\int_0^1 e^{-st}.1dt+\int_1^2 e^{-st}(-1)dt\right]=\frac{1}{1-e^{-2s}}\left[\left.\frac{e^{-st}}{-s}\right|_0^1-\left.\frac{e^{-st}}{-s}\right|_1^2\right]$$

$$=\frac{1}{-s\left(1-e^{-2s}\right)}\left[\left(e^{-s}-e^{0}\right)-\left(e^{-2s}-e^{-s}\right)\right]=\frac{-1}{s\left(1-e^{-2s}\right)}\left[-1+2e^{-s}-e^{-2s}\right]$$

$$=\frac{1}{s\left(1-e^{-2s}\right)}\left[1-2e^{-s}+e^{-2s}\right]=\frac{\left(1-e^{-s}\right)^2}{s\left(1-e^{-s}\right)\left(1+e^{-s}\right)}=\frac{\left(1-e^{-s}\right)}{s\left(1+e^{-s}\right)}=\frac{1}{s}\tanh^{-1}\left(\frac{s}{2}\right)$$

LAPLACE TRANSFIRMS OF SPECIAL FUNCTIONS

In this section we shall study the Laplace transforms of some special functions that are frequently used in engineering applications.

Laplace Transforms of Bessel Functions

We know that Bessel function of order n is defined as:

$$J_n(t)=\sum_{r=0}^{\infty}\frac{(-1)^r\left(\frac{t}{2}\right)^{n+2r}}{r!\Gamma(n+r+1)} \quad \text{or} \quad J_n(t)=\frac{t^n}{2^n\Gamma(n+1)}\left\{1-\frac{t^2}{2(2n+2)}+\frac{t^4}{2.4(2n+2)(2n+4)}+\ldots\right\} \qquad (1)$$

As an special case if we put n = 0 in (1) ,we get

$$J_0(t)=\frac{t^0}{2^0\Gamma(0+1)}\left\{1-\frac{t^2}{2(2(0)+2)}+\frac{t^4}{2.4(2(0)+2)(2(0)+4)}+\ldots\right\}$$

$$=\frac{1}{\Gamma(1)}\left\{1-\frac{t^2}{2(2)}+\frac{t^4}{2.4(2)(4)}+\ldots\right\}=\left\{1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}+\ldots\right\}\left(\because \Gamma(1)=1\right)$$

Taking Laplace transform on both sides, we get

$$L\{J_0(t)\}=L\left\{1-\frac{t^2}{2^2}+\frac{t^4}{2^2.4^2}+\ldots\right\}=L\left\{J_{0(t)}\right\}=L\{1\}-\frac{1}{2^2}L\left\{t^2\right\}+\frac{1}{2^2 4^2}L\left\{t^4\right\}+\ldots$$

$$=\frac{1}{s}-\frac{1}{2^2}\frac{2!}{s^3}+\frac{1}{2^2 4^2}\frac{4!}{s^5}-\ldots \qquad \left(\because L\left\{t^n\right\}=\frac{n!}{s^{n+1}}, s>0\right)$$

$$=\frac{1}{s}-\frac{1}{2s^3}+\frac{3}{8s^5}-\ldots=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1.3}{2.4}\left(\frac{1}{s^4}\right)-\frac{1.3.5}{2.4.6}\left(\frac{1}{s^6}\right)+\ldots\right]$$

$$\left(\text{Use the formula}: (1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\ldots\right)$$

$$=\frac{1}{s}\left\{1+\frac{1}{s^2}\right\}^{-1/2}=\frac{1}{s}\left\{\frac{s^2+1}{s^2}\right\}^{-1/2}=\frac{1}{s}\frac{\left(s^2+1\right)^{-1/2}}{\left(s^2\right)^{-1/2}}$$

$$=\frac{1}{s}\frac{\left(s^2+1\right)^{-1/2}}{s^{-1}}=\frac{1}{\sqrt{s^2+1}} \qquad \text{or} \quad L\{J_0(t)\}=\frac{1}{\sqrt{s^2+1}}=f(s)$$

$$\therefore L\{J_0(at)\}=\frac{1}{a}\frac{1}{\sqrt{(s/a)^2+1}}=\frac{1}{a}\frac{a}{\sqrt{s^2+a^2}}=\frac{1}{\sqrt{s^2+a^2}} \left(\text{use change of scale property}\right)$$

Example 01: Find $L\{J_1(t)\}$, where $J_1(t)$ is the Bessel function of order 1.

Solution: We know from recurrence formula of Bessel function that

$$\frac{d}{dx}\{J_0(t)\} = -J_1(t) \quad \text{or} \quad J(t)_1 = -J'_0(t)$$

Taking Laplace transform on both sides, we have: $L\{J_1(t)\} = -L\{J'_0(t)\}$

Using Laplace transform of derivatives property, that is, $L\{F'(t)\} = sf(s) - F(0)$

Thus, we have: $L\{J_1(t)\} = -[sL\{J_0(t)\} - J_0(0)]\backslash$

$$\text{or } L\{J_1(t)\} = -\left[s\frac{1}{\sqrt{s^2+1}} - 1\right] = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}} \qquad \left(\because L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}, J_0(0) = 1\right)$$

Example 02: Evaluate $L\{t\,J_0(t)\}$. Hence or otherwise find $\int_0^\infty e^{-2t}\,t\,J_0(t)\,dt$

Solution: We know that

$$L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}} \Rightarrow L\{t\,J_0(t)\} = (-1)\frac{d}{ds}\frac{1}{\sqrt{s^2+1}} = (-1)\left[-\frac{1}{2}\right](s^2+1)^{-3/2}(2s) = \frac{s}{(s^2+1)^{3/2}}$$

$$\text{Then, } L\{t\,J_0(t)\} = \int_0^\infty e^{-st}\,t\,J_0(t)\,dt = \frac{s}{(s^2+1)^{3/2}} \;\Rightarrow\; \int_0^\infty e^{-2t}\,t\,J_0(t) = \frac{2}{(4+1)^{3/2}} = \frac{2}{5\sqrt{5}}$$

Unit Step Function and Its Laplace Transform

Sometimes we come across functions whose inverse Laplace transforms cannot be determined from the formulas so for derived in the previous sections. In order to cover such cases, we introduce the unit step function also known as Heaviside's unit function named after the British Electrical Engineer Oliver Heaviside (1850-1925) who widely used this function in his work. The unit step function is denoted by $U(t-a)$ and is defined as:

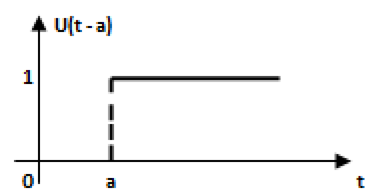


Fig. 1

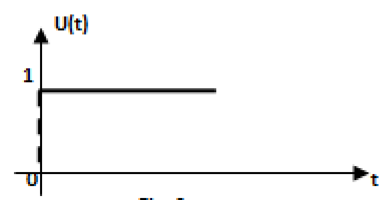


Fig. 2

$U(t-a) = 0, \quad 0 \le t \le a$

$\qquad\quad = 1, \quad t > a,$ where a being a positive number.

The graph of unit step function is shown in figure 1.

By definition,

$$L\{U(t-a)\} = \int_a^\infty e^{-st}dt = \frac{-1}{s}\left[e^{-st}\right]_a^\infty = \frac{-1}{s}(e^{-\infty} - e^{-as}) = \frac{-1}{s}(0 - e^{-as}) = \frac{e^{-as}}{s}$$

If a = 0, we get

$$L\{U(t)\} = \int_0^\infty e^{-st}dt = \frac{-1}{s}\left[e^{-st}\right]_0^\infty = \frac{-1}{s}(e^{-\infty} - e^{0}) = \frac{-1}{s}(0-1) = \frac{1}{s} = L(1)$$

The figure 2 shows the unit step function with a = 0.

Readers have studied second shifting property. We now, use unit step function to study second shift property and connect them to solve the problems that otherwise would be solved with great efforts. It may be noted that second shifting property can be expressed as:

$$L\{G_a(t)\} = L\{F(t-a)\,U(t-a)] = e^{-as}\,L[F(t)] \qquad (1)$$

Example 03: Find the Laplace transform of:

(i) $F(t)=\begin{cases} t-1, & 1<t<2 \\ 3-t, & 2<t<3 \end{cases}$ (ii) $F(t)=e^{-t}[1-U(t-2)]$

Solution: (i) Given function can be expressed using unit step function as under:

$F(t) = (t-1)\,[U(t-1) - U(t-2)] + (3-t)\,[U(t-2) - U(t-3)]$

$= (t-1)U(t-1) - (t-1)U(t-2) - (t-3)U(t-2) + (t-3)U(t-3)$

$= (t-1)U(t-1) - (t-2+1)U(t-2) - (t-2-1)U(t-2) + (t-3)U(t-3)$

$= (t-1)U(t-1) - (t-2)U(t-2) + U(t-2) - (t-2)U(t-2) - U(t-2) + (t-3)U(t-3)$

$= (t-1)U(t-1) - (t-2)U(t-2) - (t-2)U(t-2) + (t-3)U(t-3)$

$= (t-1)U(t-1) - 2(t-2)U(t-2) + (t-3)U(t-3)$

Thus using equation (1) and note that $L(t) = 1/s^2$, we see that:

$L\{F(t)\} = L(t-1)U(t-1) - 2L(t-2)U(t-2) + L(t-3)U(t-3)$

$$=\frac{e^{-s}}{s^2}-2\frac{e^{-2s}}{s^2}+\frac{e^{-3s}}{s^2}=\frac{1}{s^2}\left[e^{-s}-2e^{-2s}+e^{-3s}\right]$$

The readers must have realized that how we have used unit step function and second shifting property to compute the Laplace transform without involving the integration.

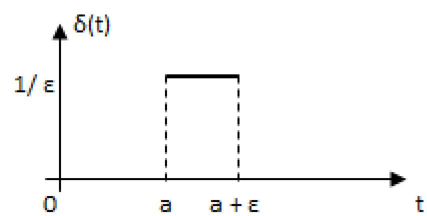
(ii) $L\{e^{-t}[1-U(t-2)]\} = L\{e^{-t}\} - L\{e^{-t}\,U(t-2)\} = L\{e^{-t}\} - L\{e^{-t+2-2}\,U(t-2)\}$

$= L\{e^{-t}\} - e^{-2}\,L\{e^{-(t-2)}\,U(t-2)\} = L\{e^{-t}\} - e^{-2}\,e^{-2s}\,L\{e^{-t}\}$

$= [1 - e^{-2(s+1)}]\,L\{e^{-t}\} = [1 - e^{-2(s+1)}]/(s+1)$

Impulse Function and Its Laplace Transform with Properties

In many engineering applications such as "Electrical and Mechanical Engineering" an idea of very large force acting for a very short time is of frequent occurrence. To deal with such and similar problems, we introduce the Unit Impulse Function (also known as Dirac Delta function, after the English physicist Paul Dirac (1902 – 1984) who was awarded the Nobel prize in 1933 for his work in Quantum Mechanics.



The unit impulse function is considered as limiting form of the function

$\delta(t-a) = 1/\varepsilon, \quad a \le t \le a+\varepsilon$

$\quad = 0, \quad$ elsewhere

The graph of Dirac delta function is shown here.

One may observe that as ε approaches zero,

δ(t – a) indefinitely getting large and the width decreases in such a way that area under the rectangle is always unity.

This shows that δ(t – a) tends to infinity as t approaches a and that δ(t – a) = 0 for t ≠ a such that:

$$\int_0^\infty \delta(t-a)\,dt = 1 \quad (a \ge 0)$$

$$\therefore L[\delta(t-a)] = \int_0^\infty e^{-st}\delta(t-a)\,dt = \int_a^{a+\varepsilon} e^{-st}\frac{1}{\varepsilon}\,dt = \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon} = \frac{-1}{\varepsilon s}\left[e^{-as-\varepsilon s}-e^{-as}\right] = \frac{-e^{-as}}{s}\left[\frac{e^{-\varepsilon s}-1}{\varepsilon}\right]$$

Now, if $\varepsilon \to 0$ then by L'Hopital rule,

$$\lim_{\varepsilon\to 0}\frac{(e^{-\varepsilon s}-1)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{-s\,e^{-\varepsilon s}}{1}=\left(-se^{0}\right)=-s\,.$$

Thus, $L[\delta(t-a)]=\left(\dfrac{-e^{-as}}{s}\right)(-s)=e^{-as}$. Setting $a=0$, we obtain : $L[\delta(t)]=1$

Property I: $\displaystyle\int_0^{\infty} f(t)\,\delta(t-a)\,dt=f(a)$

Proof: By definition, $\displaystyle L[\delta(t-a)]=\int_0^{\infty} e^{-st}\,\delta(t-a)\,dt=e^{-as}$

$$\Rightarrow \qquad \int_0^{\infty} f(t)\,\delta(t-a)\,dt=f(a)\ \left[\text{keeping in mind that } e^{-st}=f(t) \text{ then } e^{-sa}=f(a)\right]$$

For example, $\displaystyle\int_0^{\infty} e^{-5t}\,\delta(t-3)\,dt=e^{-5(3)}=e^{-15}$ and $\displaystyle\int_0^{\infty}(t^2+1)\,\delta(t-2)\,dt=\left(2^2+1\right)=5$

Property II: $\displaystyle\int_0^{\infty} f(t)\,\delta'(t-a)\,dt=-f'(a)$

Proof: Integrating by parts, we get

$$\int_0^{\infty} f(t)\,\delta'(t-a)\,dt=f(t)\,\delta(t-a)\Big|_0^{\infty}-\int_0^{\infty} f'(t)\,\delta(t-a)\,dt=\left[0-0-f'(a)\right]=-f'(a)$$

NOTE: By definition $\delta(\infty)=\delta(-\infty)=0$ and using Property-I.

WORKSHEET 08

Find the Laplace transforms of the following functions:

(1) $e^{3t}+t^4-2\sin 4t$ (2) $2+2\sqrt{t}+1/\sqrt{t}$ (3) $\cos(2t+b)$

(4) $\left(\sin t-\cos t\right)^2$ (5) $\sin 2t\cos 4t$ (6) $\sin^2 4t+3\cosh 4t$

(7) $\sin 2t\,\cosh 2t$ (8) t^2e^{-4t} (9) $e^{-2t}\cos^2 t$

(10.a) $F(t)=\begin{cases} e^t, & 0<t<1 \\ 0, & t>1\end{cases}$ (10.b) $F(t)=\begin{cases} 2t, & 0<t<3 \\ 1, & t>3\end{cases}$ (11) $F(t)=\begin{cases} \sin t, & 0<t<\pi \\ 0, & t>\pi\end{cases}$

12. $F(t)=|t-1|+|t+1|,\ t\ge 0$ $\displaystyle\left[\text{NOTE}: L[F(t)]=\int_0^{1}-(t-1)e^{-st}dt+\int_1^{\infty}(t-1)e^{-st}dt+\int_0^{\infty}(t+1)e^{-st}dt\right]$

13. (a) $F(t)=\begin{cases}\sin\left(t-\pi/3\right), & t>\pi/3 \\ 0, & t<\pi/3\end{cases}$ (b) $F(t)=\begin{cases}\left(t-2\right), & t>2 \\ 0, & t<2\end{cases}$

14. If $L\left(\cos^2 t\right)=\dfrac{s^2+2}{s\left(s^2+4\right)}$, find $L\left(\cos^2 4t\right)$

15. $\displaystyle\int_0^{t} e^{-t}\cos t\,dt$ 16. $t\sin^2 t$ 17. $t^2\cos at$ 18. $t\,e^{-2t}\sin 2t$

19. $(e^{-4t}-e^{-6t})/t$ 20. $(\cos 2t-\cos 3t)/t$ 21. $(1-\cos 2t)/t$ 22. $(1-\cos t)/t^2$

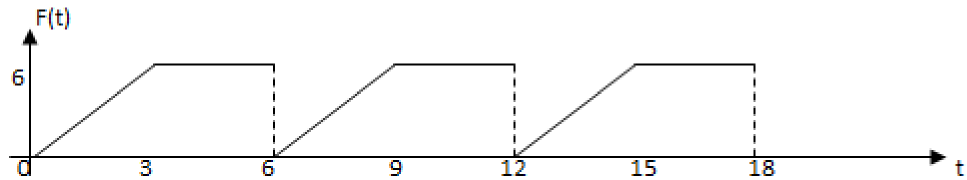
Using an appropriate property of Laplace transform, evaluate the following:

23. $\displaystyle\int_0^{\infty} t\,e^{-3t}\sin t\,dt$ 24. $\displaystyle\int_0^{\infty}\frac{e^{-t}-e^{-3t}}{t}dt$ 25. $\displaystyle L\left(\int_0^{t}\frac{\sin u}{u}dt\right)$ 26. $\displaystyle L\left(\int_0^{t} e^{-t}\cos t\,dt\right)$

Find the Laplace transform of the following periodic functions:

27. $F(t)=\begin{cases}1, & \text{when } 0<t<2\\ -1, & \text{when } 2<t<4\end{cases}$

28.



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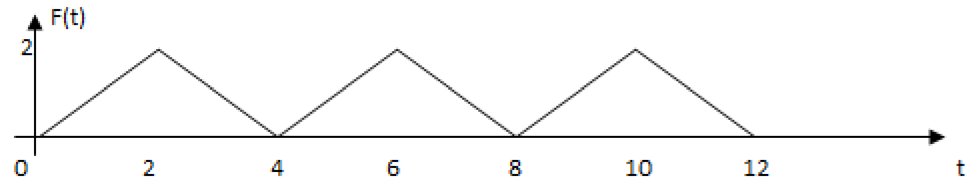

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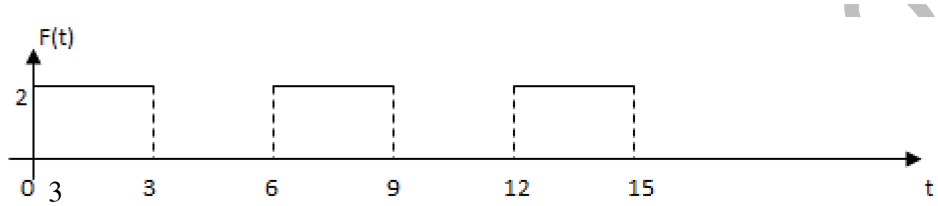

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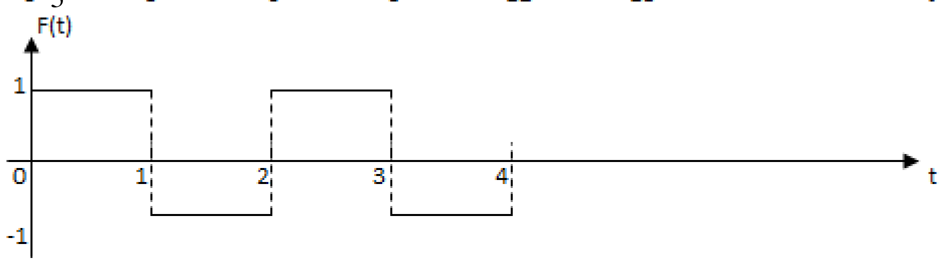


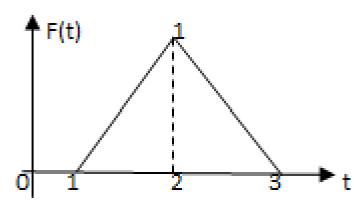
32.. Show that $\int_0^{\infty} e^{-\sqrt{2}\,t}\,\frac{\sinh t \sin t}{t}\,dt=\frac{\pi}{8}$

33. Find the Laplace transforms of (i) $\int_0^t e^{-t}\cos t\,dt$ (ii) $\int_0^t e^{t}\,\frac{\sin t}{t}\,dt$

34. Find the Laplace transforms of the following functions:

(i) $(t-1)^2\,U(t-1)$ (ii) $\sin t\,U(t-\pi)$ (iii) $e^{-3t}\,U(t-2)$

35. Express the function shown in the figure below in terms of unit step function and then find its Laplace transform.



36. Evaluate $\int_0^{\infty} e^{-4t}\,\delta(t-3)\,dt$

37. Find the Laplace transforms of function $t^3\delta(t-5)$

CHAPTER NINE

INVERSE LAPLACE TRANSFORMS

INTRODUCTION

In chapter one, we discussed Laplace transforms of elementary function and some of important properties used to solve certain improper integrals which could not be evaluated otherwise by well known methods of integration. In this chapter we shall study the inverse Laplace transform and some of its important properties as well. These properties will also help to solve some of typical integrals.

Definition: If Laplace transform of A function F(t) is f(s) that is; if L{F(t)} = f(s) then F(t) is called an inverse Laplace transform of f(s). Symbolically we may express it as:

$$L^{-1}\{f(s)\} = F(t)$$

Here, L^{-1} denotes the inverse Laplace transformation operator.

Inverse Laplace Transforms of Some Elementary Functions

The following table shows a list of Laplace and inverse Laplace transforms of an elementary functions that have been discussed earlier.

S. No.	F(t)	L F(t) = f(s)	$L^{-1}f(s) = F(t)$
1	t	$\frac{1}{s}$	$L^{-1}\left\{\frac{1}{s}\right\} = t$
2.	t^n n being a positive integer	$\frac{n!}{s^{n+1}}$	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$
3.	t^n n being a real number	$\frac{\Gamma(n+1)}{s^{n+1}}$	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$
4.	e^{at}	$1/(s-a), s > a$	$L^{-1}\{1/(s-a)\} = e^{at}$
5.	sin at	$\frac{a}{(s^2+a^2)}$	$L^{-1}\frac{1}{(s^2+a^2)} = \frac{\sin at}{a}$
6.	cos at	$\frac{s}{(s^2+a^2)}$	$L^{-1}\frac{s}{(s^2+a^2)} = \frac{\cos at}{a}$
7.	sinh at	$\frac{a}{(s^2-a^2)}$	$L^{-1}\frac{1}{(s^2-a^2)} = \frac{\sinh at}{a}$
8.	cosh at	$\frac{s}{(s^2-a^2)}$	$L^{-1}\frac{s}{(s^2-a^2)} = \frac{\cosh at}{a}$

PROPERTIES OF INVERSE LAPLACE TRANSFORMS

This section will introduce you some of the important properties of *Inverse Laplace Transforms* with proofs.

Linearity Property

If $c_1, c_2, \dots c_n$ are any constants while $f_1(s), f_2(s), \dots f_n(s)$ are the inverse Laplace transforms of $F_1(t), F_2(t), \dots F_n(t)$ respectively, then

$$L^{-1}\{c_1f_1(s)+c_2f_2(s)+\dots+c_nf_n(s)\}=c_1L^{-1}\{f_1(s)\}+c_2L^{-1}\{f_2(s)\}+\dots+c_nL^{-1}\{f_n(s)\}$$

Proof: According to linearity property of Laplace transform

$$L\{c_1F_1(t)+c_2F_2(t)+\dots+c_nF_n(t)\}=c_1L\{F_1(t)\}+c_2L\{F_2(t)\}+\dots+c_nL\{F_n(t)\}$$

$$=c_1f_1(s)+c_2f_2(s)+\dots+c_nf_n(s)$$

or $L^{-1}\{c_1f_1(s)+c_2f_2(s)+\dots+c_nf_n(s)\}=c_1F_1(t)+c_2F_2(t)+\dots+c_nF_n(t)$

$$=c_1L^{-1}\{f_1(s)\}+c_2L^{-1}\{f_2(s)\}+\dots+c_nL^{-1}\{f_n(s)\}$$

Example 01: Find $L^{-1}\left(\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}\right)$

Solution: By using linearity property of inverse Laplace transform, we have

$$L^{-1}\left(\frac{4}{s-2}-\frac{3s}{s^2+16}+\frac{5}{s^2+4}\right)=4L^{-1}\left\{\frac{1}{s-2}\right\}-3L^{-1}\left\{\frac{s}{s^2+16}\right\}+5L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=4L^{-1}\left\{\frac{1}{s-2}\right\}-3L^{-1}\left\{\frac{s}{s^2+4^2}\right\}+\frac{5}{2}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}=4e^{2t}-3\cos 4t+\frac{5}{2}\sin 2t$$

Example 02: Find $L^{-1}\left(\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right)$

Solution: Using the linearity property of inverse Laplace transform, we have

$$L^{-1}\left(\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right)=L^{-1}\left(\frac{5}{s^2}+\frac{4}{s^3}-\frac{2s}{s^2+9}+\frac{18}{s^2+9}+\frac{24}{s^4}-\frac{30}{s^{\frac{7}{2}}}\right)$$

$$=5L^{-1}\left\{\frac{1}{s^2}\right\}+4L^{-1}\left\{\frac{1}{s^3}\right\}-2L^{-1}\left\{\frac{s}{s^2+3^2}\right\}+6L^{-1}\left\{\frac{3}{s^2+3^2}\right\}+24L^{-1}\left\{\frac{1}{s^4}\right\}-30L^{-1}\left\{\frac{1}{s^{7/2}}\right\}$$

$$=5t+4\frac{t^2}{2!}-2\cos 3t+6\sin 3t+4t^3-30\frac{t^{5/2}}{\Gamma(7/2)}$$

$$=5t+2t^2-2\cos 3t+6\sin 3t+4t^3-\frac{16}{\sqrt{\pi}}t^{5/2} \qquad \text{NOTE}: \Gamma(7/2)=\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma(1/2)=\frac{15}{8}\sqrt{\pi}$$

Example 03: Find $L^{-1}\left(\frac{\sqrt{s}-1}{s}\right)^2$

Solution: Using the linearity property of inverse Laplace transforms, we have

$$L^{-1}\left(\frac{\sqrt{s}-1}{s}\right)^2=L^{-1}\left\{\frac{s-2\sqrt{s}+1}{s^2}\right\}=L^{-1}\left\{\frac{1}{s}\right\}-2L^{-1}\left\{\frac{1}{s^{3/2}}\right\}+L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$=\left(1-2\frac{t^{1/2}}{\Gamma(3/2)}+t\right)=\left(1+t-\frac{4\sqrt{t}}{\sqrt{\pi}}\right)$$

First Shifting Property

If $L^{-1}f(s)=F(t)$, then $L^{-1}f(s-a)=e^{at}F(t)$

Proof: According to the first shifting property of Laplace transformation, we know that if

$$L\{F(t)\} = f(s) \text{ then } L\{e^{at}F(t)\} = f(s-a)$$

Now taking inverse Laplace transform on each side, we have

$$L^{-1}\left[L\{e^{at}F(t)\}\right] = L^{-1}\{f(s-a)\} \text{ or } L^{-1}\{f(s-a)\} = e^{at}F(t)$$

Remark: The following Inverse Laplace transform results will frequently be used in our problems. These results are obtained by using shifting properties of ILT.

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} \quad \text{thus } L^{-1}\frac{b}{(s-a)^2+b^2} = e^{at}\sin bt$$

$$L\{e^{at}\cos bt\} = \frac{s}{(s-a)^2+b^2} \quad \text{thus } L^{-1}\frac{s}{(s-a)^2+b^2} = e^{at}\cos bt$$

$$L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2} \quad \text{thus } L^{-1}\frac{b}{(s-a)^2-b^2} = e^{at}\sinh bt$$

$$L\{e^{at}\cosh bt\} = \frac{s}{(s-a)^2-b^2} \quad \text{thus } L^{-1}\frac{s}{(s-a)^2-b^2} = e^{at}\cosh bt$$

Example 04: Find $L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\}$

Solution: $L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\} = L^{-1}\left\{\frac{8s-6}{s^2-4s+20+4-4}\right\} = L^{-1}\left\{\frac{8s-6}{(s-2)^2+16}\right\}$

$$= L^{-1}\left\{\frac{8s-16+10}{(s-2)^2+16}\right\} = L^{-1}\left\{\frac{8(s-2)+10}{(s-2)^2+16}\right\}$$

Now using the linearity property, we have

$$L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\} = 8L^{-1}\left\{\frac{(s-2)}{(s-2)^2+4^2}\right\} + \frac{10}{4}L^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

Now using the first shifting property of inverse Laplace transform, we have

$$L^{-1}\left\{\frac{8s-6}{s^2-4s+20}\right\} = 8e^{2t}\cos 4t + \frac{5}{2}e^{2t}\sin 4t$$

Remark: Above method of finding ILT is known as *Completing the Squares Method*.

Example 05: Find $L^{-1}\{1/\sqrt{2s+3}\}$

Solution: $L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\} = L^{-1}\left\{\frac{1}{\sqrt{2(s+3/2)}}\right\} = \frac{1}{\sqrt{2}}L^{-1}\left\{\frac{1}{\sqrt{s+3/2}}\right\} = \frac{1}{\sqrt{2}}L^{-1}\left\{\frac{1}{(s+3/2)^{\frac{1}{2}}}\right\}$

Using the shifting property of inverse Laplace transforms, we have

$$L^{-1}\left\{\frac{1}{\sqrt{2s+3}}\right\} = \frac{1}{\sqrt{2}}e^{\frac{-3t}{2}}\frac{t^{-1/2}}{\Gamma(1/2)} = \frac{e^{-3t/2}}{\sqrt{2\pi t}} \qquad \left(\because L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}, \Gamma(1/2) = \sqrt{\pi}\right)$$

Second Shifting Property

If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & \text{if } t>a \\ 0 & t<a \end{cases}$

Proof: We know that

$L\{F(t)\} = \int_0^\infty e^{-st}F(t)dt = f(s)$. Multiplying each side by e^{-as}, we obtain

$\int_0^\infty e^{-as}e^{-st}F(t)dt = \int_0^\infty e^{-(t+a)s}F(t)dt = e^{-as}f(s)$. Substituting u = t + a or dt = du.

Now if $t \to 0$ then $u \to a$ and if $t \to \infty$ then $u \to \infty$. Thus,

$\int_a^\infty e^{-us}F(u-a)du = e^{-as}f(s)$. Replacing u by t, we obtain

$$\int_a^\infty e^{-st}F(t-a)dt = e^{-as}f(s) \qquad (1)$$

$$\text{Now, } \int_0^\infty e^{-st}F(t-a)dt = \int_0^a e^{-su}(0)dt + \int_a^\infty e^{-st}F(t-a)dt = L\{F(t-a)\} \text{ provided } t>a \qquad (2)$$

From (1) and (2) we deduce that L{F(t – a)}= e^{-as}f(s), provided t > a. Thus,

$$L^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & \text{if } t>a \\ 0 & \text{if } t<a \end{cases}$$

Remark: Above function may be expressed in terms of Heaviside unit step function as

$$L^{-1}\{e^{-as}f(s)\} = F(t)\,U(t-a)$$

It may further be noted that this property is used for finding the inverse Laplace transform of product of two functions where one function must be multiple of e^{-as}.

Example 06: Find $L^{-1}\left(\dfrac{s\ e^{-4\pi s/5}}{s^2+25}\right)$

Solution: $L^{-1}\left\{\dfrac{s\ e^{-4\pi s/5}}{s^2+25}\right\} = L^{-1}\left\{\dfrac{s}{s^2+25}e^{-4\pi s/5}\right\}$

We see that given function is product of two functions where one of the function is $e^{-4\pi s/5}$. Now, $L^{-1}\left\{\dfrac{s}{s^2+5^2}\right\} = \cos 5t = F(t)$. Thus by second shift property,

$$L^{-1}\left[e^{-as}f(s)\right] = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$$

Therefore, $$L^{-1}\left\{\frac{se^{-4\pi s/5}}{s^2+5^2}\right\} = \begin{cases} \cos 5\left(t-\dfrac{4\pi}{5}\right), & t>4\pi/5 \\ 0, & t<4\pi/5 \end{cases}$$

Thus, $L^{-1}\left\{\dfrac{se^{-4\pi s/5}}{s^2+5^2}\right\} = \cos 5\left(t-\dfrac{4\pi}{5}\right) = \cos(5t-4\pi)$ provided $t>4\pi/5$

$$= \cos 5t \quad (\because \cos(\theta-4\pi) = \cos\theta)$$

Example 07: Find $L^{-1}\left\{\dfrac{e^{-3s}}{(s-2)^4}\right\}$

Solution: $L^{-1}\left\{\dfrac{e^{-3s}}{(s-2)^4}\right\} = L^{-1}\left\{\dfrac{1}{(s-2)^4}e^{-3s}\right\}$

We know that $L^{-1}\left\{\dfrac{1}{s^4}\right\} = \dfrac{t^3}{3!} = \dfrac{t^3}{6}$. Therefore, using first shifting property of inverse Laplace, we have $L^{-1}\dfrac{1}{(s-2)^4} = \dfrac{1}{6}e^{2t}t^3$ .Now using the 2nd shifting property of Laplace transform, we have

$$L^{-1}\left[\frac{e^{-3s}}{(s-2)^4}\right]=\begin{cases}\frac{1}{6}(t-3)^3 e^{2(t-3)}, & t>3\\ 0, & t<3\end{cases}$$

Change of Scale Property

If $L^{-1}\{f(s)\}=F(t)$,then $L^{-1}\{f(ks)\}=\frac{1}{k}F\left(\frac{t}{k}\right)$

Proof: We know that $L\{F(t)\}=\int_0^\infty e^{-st}F(t)\,dt=f(s)$.

Therefore, $$f(ks)=\int_0^\infty e^{-skt}F(t)\,dt=\int_0^\infty e^{-(kt)s}F(t)\,dt \qquad (1)$$

Let $u=k\,t \Rightarrow t=u/k \Rightarrow dt=du/k$ (2)

Now from (2) we see that if $t\to 0$ then $u\to 0$ and if $t\to\infty \Rightarrow u\to\infty$.

Thus equation (1) becomes,

$$f(ks)=\int_0^\infty e^{-su}F\left(\frac{u}{k}\right)\frac{du}{k}=\frac{1}{k}\int_0^\infty e^{-su}F\left(\frac{u}{k}\right)du=\frac{1}{k}L\left\{F\left(\frac{t}{k}\right)\right\}$$ (Replacing u by t)

or $$L^{-1}\{f(ks)\}=\frac{1}{k}F\left(\frac{t}{k}\right)$$

Example 08: Find $L^{-1}\left\{\frac{s}{4s^2+16}\right\}$

Solution: $L^{-1}\left\{\frac{s}{4s^2+16}\right\}=L^{-1}\left\{\frac{s}{(2s)^2+4^2}\right\}=\frac{1}{2}L^{-1}\left\{\frac{2s}{(2s)^2+4^2}\right\}$

We know that $L^{-1}\left\{\frac{s}{s^2+16}\right\}=\cos 4t=F(t)$.

Using change of scale property of ILT, we obtain

$\frac{1}{2}L^{-1}\left\{\frac{2s}{(2s)^2+4^2}\right\}=\frac{1}{2}\cos\left(\frac{4t}{2}\right)$. Thus $L^{-1}\left\{\frac{s}{4s^2+16}\right\}=\frac{1}{2}\cos 2t$

Inverse Laplace Transform of Derivatives

If $L^{-1}\{f(s)\}=F(t)$ then $L^{-1}\left\{f^{(n)}(s)\right\}=L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=(-1)^n t^n F(t)$

Proof: We know that by derivatives property of Laplace transforms that if

$L\{F(t)\}=f(s)$ then $L\{t^nF(t)\}=(-1)^n\frac{d^n}{ds^n}f(s)$

Taking inverse Laplace transform on both sides, we have

$L^{-1}L\{t^nF(t)\}=L^{-1}\left\{(-1)^n\frac{d^n}{ds^n}f(s)\right\}$ or $\{t^nF(t)\}=L^{-1}\left\{(-1)^n\frac{d^n}{ds^n}f(s)\right\}$

or $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=L^{-1}\left\{f^{(n)}(s)\right\}=(-1)^n t^n F(t)$

Example 09: Find $L^{-1}\left\{s/(s^2+1)^2\right\}$

Solution: Consider, $\frac{d}{ds}.\frac{1}{s^2+1}=\frac{-2s}{(s^2+1)^2}$ or $\frac{s}{(s^2+1)^2}=\frac{-1}{2}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

Taking inverse Laplace transform on both sides, and using the inverse Laplace transform property of derivatives, we get

$$L^{-1}\left\{s/\left(s^2+1\right)^2\right\}=\frac{-1}{2}L^{-1}\frac{d}{ds}\left(\frac{1}{s^2+1}\right)=\frac{-1}{2}(-1)t\sin t \qquad \left(\because L^{-1}\left(\frac{1}{s^2+1}\right)=\sin t\right)$$

Thus, $$L^{-1}\left\{s/\left(s^2+1\right)^2\right\}=\frac{1}{2}t\sin t$$

Example 10: Find $L^{-1}\left\{\ln\left(\frac{s+2}{s+1}\right)\right\}$

Solution: Let f (s) = $\ln\left(\frac{s+2}{s+1}\right)=\ln(s+2)-\ln(s+1)$

Differentiate w.r.t s, we get: $f'(s)=\frac{1}{s+2}-\frac{1}{s+1}$

Taking inverse Laplace transform, we obtain: $L^{-1}\left\{f'(s)\right\}=L^{-1}\left\{\frac{1}{s+2}-\frac{1}{s+1}\right\}$

Using the derivative property, that is, $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=(-1)^n t^n F(t)$, we get

$$(-1)tL^{-1}\left\{f(s)\right\}=L^{-1}\left(\frac{1}{s+2}\right)-L^{-1}\left(\frac{1}{s+1}\right) \text{ or } -tF(t)=e^{-2t}-e^{-t}$$

$$\text{or } F(t)=-\left(\frac{e^{-2t}-e^{-t}}{t}\right)=L^{-1}f(s) \text{ or } L^{-1}\left\{\ln\left(\frac{s+2}{s+1}\right)\right\}=\left(\frac{e^{-t}-e^{-2t}}{t}\right)$$

Example 11: Find $L^{-1}\left\{\ln\left(\frac{s^2+1}{(s-1)^2}\right)\right\}$

Solution: Let $f(s)=\ln\frac{s^2+1}{(s-1)^2}$

Now, we know that $L^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\}=L^{-1}\left\{f^{(n)}(s)\right\}=(-1)^n t^n F(t)$.

Substituting n = 1, we get,

$$L^{-1}\left\{f`(s)\right\}=-tF(t) \text{ or } F(t)=-\frac{1}{t}L^{-1}f`(s)=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left(\ln\frac{s^2+1}{(s-1)^2}\right)\right\}$$

$$=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left(\ln\left(s^2+1\right)-2\ln\ln(s-1)\right)\right\} \quad \left(\because \ln a^b=b\ln a\right)$$

$$=\frac{-1}{t}L^{-1}\left\{\frac{2s}{s^2+1}-\frac{2}{s-1}\right\}=\frac{-1}{t}\left[2L^{-1}\left\{\frac{s}{s^2+1}\right\}-2L^{-1}\left\{\frac{1}{s-1}\right\}\right]=\frac{-1}{t}\left[2\cos t-2e^t\right]$$

Hence, $$L^{-1}\left\{\ln\left(\frac{s^2+1}{(s-1)^2}\right)\right\}=\frac{-2\cos t}{t}+\frac{2e^t}{t}$$

Example 12: Evaluate $L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}$

Solution: Let $f(s)=\tan^{-1}\left(\frac{a}{s}\right)$

We know that $L\left[tF(t)\right]=(-1)f`(s) \Rightarrow L^{-1}\left\{f`(s)\right\}=(-1)\,t\,F(t)$

or $F(t)=-\frac{1}{t}L^{-1}f`(s)=\frac{-1}{t}L^{-1}\left\{\frac{d}{ds}\left(\tan^{-1}\frac{a}{s}\right)\right\}$

$$=\frac{-1}{t}L^{-1}\left\{\frac{1}{1+\frac{a^2}{s^2}}.\frac{-a}{s^2}\right\}=\frac{-1}{t}L^{-1}\left\{\frac{s^2}{s^2+a^2}.\frac{-a}{s^2}\right\}=\frac{-1}{t}L^{-1}\left\{\frac{-a}{s^2+a^2}\right\}=\frac{1}{t}L^{-1}\left\{\frac{a}{s^2+a^2}\right\}=\frac{1}{t}(\sin at)$$

Therefore, F(t) = (sin at)/t. But, F(t) = L^{-1} f(s). Thus,

$F(t)=L^{-1}\left\{f(s)\right\}=\frac{\sin at}{t}$ or $L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}=\frac{\sin at}{t}$ [substituting the value of f (s)]

Division by s^n Property

If $L^{-1}\left\{f(s)\right\}=F(t)$, then $L^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)du$

Proof: By Laplace transform of integrals property, we know that $L\left\{\int_0^t F(u)du\right\}=\frac{f(s)}{s}$.

Taking Inverse Laplace Transform on each side, we get $L^{-1}\left\{\frac{f(s)}{s}\right\}=\int_0^t F(u)du$.

Example 13: Find $L^{-1}\frac{1}{s^2(s+2)}$

Solution: We know that $L\left\{\frac{1}{s+2}\right\}=e^{2t}$

Now using division by s property, we get $L^{-1}\left\{\frac{1}{s(s+2)}\right\}=L^{-1}\left\{\frac{1/(s+2)}{s}\right\}=\int_0^t e^{-2u}du$

$$=\left[\frac{e^{-2u}}{-2}\right]_0^t=\frac{-1}{2}\left(e^{-2t}-e^0\right)=\frac{-1}{2}\left(e^{-2t}-1\right)=\frac{(1-e^{-2t})}{2}$$

or $L^{-1}\left\{\frac{1}{s^2(s+2)}\right\}=\frac{1}{2}\int_0^t\left(1-e^{-2u}\right)du=\frac{1}{2}\left[u-\frac{e^{-2u}}{-2}\right]_0^t=\frac{1}{2}\left[u+\frac{e^{-2u}}{2}\right]_0^t$

$=\frac{1}{2}\left(t+\frac{e^{-2t}}{2}-\left(0+\frac{e^0}{2}\right)\right)=\frac{1}{2}\left(t+\frac{e^{-2t}}{2}-\frac{1}{2}\right)=\frac{1}{2}\left(\frac{e^{-2t}+2t-1}{2}\right)=\left(\frac{e^{-2t}+2t-1}{4}\right)$. Thus,

$$L^{-1}\left\{\frac{1}{s^2(s+2)}\right\}=\left(\frac{e^{-2t}+2t-1}{4}\right)$$

Example 14: Find $L^{-1}\left\{\frac{1}{s}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

Solution: Let us first evaluate $L^{-1}\left\{\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$

Let, $f(s)=\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)=\ln\left(s^2+a^2\right)-\ln\left(s^2+b^2\right)$

Differentiate both sides w.r.t s, we have

$f'(s)=\dfrac{2s}{s^2+a^2}-\dfrac{2s}{s^2+b^2}$. Taking inverse Laplace transform on both sides

$$L^{-1}\left\{f'(s)\right\}=L^{-1}\left\{\frac{2s}{s^2+a^2}\right\}-L^{-1}\left\{\frac{2s}{s^2+b^2}\right\}$$

Now by inverse Laplace transform of derivative property, we have

$$(-1)tL^{-1}\{f(s)\}=2L^{-1}\left\{\frac{s}{s^2+a^2}\right\}-2L^{-1}\left\{\frac{s}{s^2+b^2}\right\}\Rightarrow L^{-1}\{f(s)\}=2\left(\frac{\cos bt-\cos at}{t}\right)$$

Now by division by s property i-e $L^{-1}\left\{\dfrac{f(s)}{s}\right\}=\int_0^t F(u)du$

$$\therefore L^{-1}\left\{\frac{1}{s}\ln\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}=2\int_0^t\left(\frac{\cos bu-\cos au}{u}\right)du$$

Inverse Laplace Transforms by Partial Fractions

Inverse Laplace transform of a function f(s) = P(s)/Q(s), where P(s) and Q(s) are polynomials such that degree of P(s) is lower than the degree of Q(s) can be evaluated by using the techniques of *Partial Fractions*. There are three cases of partial fractions.

CASE-I: When Q(s) contains linear and non-repeated factors.

CASE-II: When Q(s) contains linear and repeated factors.

CASE-III: When Q(s) contains non-factorizable quadratic factors.

The following examples will help you to understand the method of finding inverse Laplace transforms by the method of partial fractions.

Example 15: Use partial fraction method to find $L^{-1}\left\{\left(\dfrac{3s+16}{(s-3)(s+2)}\right)\right\}$

Solution: Here we see that Q(s) contains the factors that are linear and non-repeated. Hence we are having Case I.. Consider

$$\frac{3s+16}{(s-3)(s+2)}=\frac{A}{s-3}+\frac{B}{s+2} \qquad (1)$$

$$\text{or } 3s+16=A(s+2)+B(s-3) \qquad (2)$$

To find A and B, we proceed as under:

Put s - 3 = 0 or s = 3 in (2) we get

$$3(3)+16=A(3+2)+B(3-3).\ \text{This gives } A=5$$

Now put s + 2 = 0 or s = – 2 in (2) we get

$$3(-2)+16=A(-2+2)+B(-2-3).\ \text{This gives } B=-2$$

Thus equation (1) becomes: $\dfrac{3s+16}{(s-3)(s+2)}=\dfrac{5}{s-3}+\dfrac{-2}{s+2}$ (3)

Now taking the inverse Laplace transform on both sides, we get

$$L^{-1}\left\{\frac{3s+16}{(s-3)(s+2)}\right\}=5L^{-1}\left\{\frac{1}{s-3}\right\}-2L^{-1}\left\{\frac{1}{s+2}\right\}=5e^{3t}-2e^{-2t}$$

Example 16: Find $L^{-1}\left\{(3s+16)/(s^2-s-6)\right\}$

Solution: We see that $\dfrac{3s+16}{(s^2-s-6)}=\dfrac{3s+16}{(s-3)(s+2)}$. This function is same as in Example 15.

Thus,

$$L^{-1}\left\{\frac{3s+16}{(s^2-s-6)}\right\}=L^{-1}\left\{\frac{3s+16}{(s-3)(s-2)}\right\}=5e^{3t}-2e^{-2t}$$

Example 17: Evaluate $L^{-1}\left(\dfrac{4s+5}{(s-1)^2(s+2)}\right)$

Solution: Since Q(s) contains linear factors but one of these factors is repeated, hence this is an example of partial fractions Case II. The method of resolving into partial fractions is shown as under. Consider,

$$\frac{4s+5}{(s-1)^2(s+2)}=\frac{A}{(s-1)^2}+\frac{B}{(s-1)}+\frac{C}{(s+2)} \qquad (1)$$

This gives $\quad 4s+5=A(s+2)+B(s-1)(s+2)+C(s-1)^2 \qquad (2)$

Now put s – 1 = 0 or s = 1 in (2), we get: 9 = 3 A or A = 3.

Now put s + 2 = 0 or s = -2 in (2), we get: –3 = 9 C or C = –1/3.

To find B, we re-write equation (2) to get:

$$4s+5=A(s+2)+B(s^2+s-2)+C(s^2-2s+1)$$

Now comparing the coefficients of

s^2: 0 = B + C or B = – C = – (–1/3) = 1/3

Thus equation (1) becomes: $\dfrac{4s+5}{(s-1)^2(s+2)}=\dfrac{3}{(s-1)^2}+\dfrac{1/3}{(s-1)}+\dfrac{-1/3}{(s+2)}$.

Taking the inverse Laplace transform on each side to get

$$L^{-1}\left\{\left(\frac{4s+5}{(s-1)^2(s+2)}\right)\right\}=3L^{-1}\frac{1}{(s-1)^2}+\frac{1}{3}L^{-1}\frac{1}{(s-1)}-\frac{1}{3}L^{-1}\frac{1}{(s+2)}=3te^t+\frac{1}{3}e^t-\frac{1}{3}e^{-2t}$$

Example 18: Find $L^{-1}\dfrac{1}{(s-1)(s^2+4)}$

Solution: Here Q(s) contains a non-factorizable quadratic factor hence this is an example of partial fractions Case III. Thus

$$\frac{1}{(s-1)(s^2+4)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+4} \qquad (1)$$

Or $1=A(s^2+4)+(s-1)(Bs+C) \qquad (2)$

To find A, put s = 1 in (2), we get

$1=A(1^2+1)+(1-1)(b.1+C)\Rightarrow A=1/5$

To compute B and C, we rewrite equation (2) in simplified form as

$1=As^2+4A+Bs^2-Bs+Cs-C$

Now comparing the coefficients of same powers of

s^2: $A+B=0$ or B = – A or B = –1/5

s: $C-B=0$ or C = B or C= –1/5

Thus equation (1) becomes: $\dfrac{1}{(s-1)(s^2+4)}=\dfrac{1}{5(s-1)}-\dfrac{s+1}{5(s^2+4)}$

Taking inverse Laplace transform on both sides, we have

$$L^{-1}\left\{\frac{1}{(s-1)(s^2+4)}\right\}=\frac{1}{5}L^{-1}\left\{\frac{1}{(s-1)}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s+1}{(s^2+4)}\right\}=\frac{1}{5}\left\{\frac{1}{(s-1)}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{1}{5}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=\frac{1}{5}\left\{\frac{1}{(s-1)}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s}{s^2+2^2}\right\}-\frac{1}{10}L^{-1}\left\{\frac{2}{s^2+2^2}\right\}=\frac{1}{5}e^t-\frac{1}{5}\cos 2t-\frac{1}{10}\sin 2t$$

Example 19: Evaluate $L^{-1}\left\{\frac{s^3+16s-24}{s^4+20s^2+64}\right\}$

Solution: We see that: $s^4+20s^2+64=(s^2+16)(s^2+4)$. Hence

$L^{-1}\left\{\frac{s^3+16s-24}{s^4+20s^2+64}\right\}=L^{-1}\left\{\frac{s^3+16s-24}{(s^2+16)(s^2+4)}\right\}$. Now consider,

$$\frac{s^3+16s-24}{(s^2+16)(s^2+4)}=\frac{As+B}{(s^2+16)}+\frac{Cs+D}{(s^2+4)} \qquad (1)$$

Or, $s^3+16s-24=(As+B)(s^2+4)+(Cs+D)(s^2+16)$. Simplifying we get

$s^3+16s-24=As^3+4As+Bs^2+4B+Cs^3+16Cs+Ds^2+16D$

or, $s^3+16s-24=(A+C)s^3+(B+D)s^2+(4A+16C)s+(4B+16D)$

Now comparing the coefficients of same powers of s, we have

$A+C=1$ (i) $B+D=0$ (ii) $A+4C=4$ (iii) and $B+4D=-6$ (iv)

Solving equations (i) through (iv) simultaneously, we obtain:

A = 0, B = 2, C = 1 and D = –2. Substituting these values in equation (1) and taking the inverse Laplace transform, we get

$$L^{-1}\left\{\frac{s^3+16s-24}{s^4+20s^2+64}\right\}=2L^{-1}\left\{\frac{1}{s^2+16}\right\}+L^{-1}\left\{\frac{s}{s^2+4}\right\}-2L^{-1}\left\{\frac{1}{s^2+4}\right\}=2\sin 4t+\cos 2t-\sin 2t$$

Inverse Laplace Transform by Convolution Method

Here we present another method of finding inverse Laplace transforms. Although this method involves simple integration of well known functions like e^{at} cos (b + ct), e^{at} sin (b + ct), sin at sin bt, sin at cos bt or cos at cos bt, it is limited to certain functions. The method is described as under.

Let , $L^{-1}\{f(s)\}=F(t)$ and $L^{-1}\{g(s)\}=G(t)$, then $L^{-1}\{f(s)g(s)\}=\int_0^t F(u)G(t-u)du$

This property is known as convolution property and is very much useful in finding the Inverse Laplace Transform of product of two functions whose inverse Laplace transforms are known.

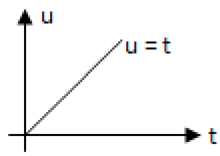
Proof: Since $L^{-1}\{f(s)g(s)\}=\int_0^t F(u)G(t-u)du$

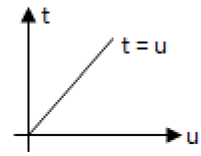
Or $$\{f(s)g(s)\}=L\int_0^t F(u)G(t-u)du \qquad (1)$$

Now, $$L\int_0^t F(u)G(t-u)du=\int_0^\infty\int_0^t e^{-st}F(u)\,G(t-u)\,du\,dt \qquad (2)$$

Now to evaluate (2) we have to change the order of integration shown as under.

Limits of u are from 0 to t, that is u = 0 and u = t. u = t is a straight line in ut-plane where t is dependent variable. This is shown in fig. 1. Now if u = t then t = u. Here t depends on u and the equation t = u is a straight line in tu-plane. See fig 2.





Thus if we change the order of integration the limits of u will be from 0 to ∞ and limits of t will be from u to ∞. Thus above integration becomes:

$$\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st}F(u)\,G(t-u)\,du\,dt = \int_{u=0}^{\infty}\int_{t=u}^{\infty} e^{-st}F(u)\,G(t-u)\,dt\,du = \int_{u=0}^{\infty} F(u)\,du\int_{t=u}^{\infty} e^{-st}\,G(t-u)\,dt$$

Substituting $t-u=z$ or $dt=dz$ (u being constant for the second integral).
Also the limits of z will become 0 to ∞. Thus,

$$\int_{t=0}^{\infty}\int_{u=0}^{t} e^{-st}F(u)\,G(t-u)\,du\,dt = \int_{u=0}^{\infty} F(u)\,du\int_{z=0}^{\infty} e^{-s(z+u)}\,G(z)\,dz$$

$$= \int_{u=0}^{\infty} e^{-su}F(u)\,du\int_{z=0}^{\infty} e^{-sz}\,G(z)\,dz = L\left[F(u)\right].L\left[G(z)\right] = f(s).g(s)$$

Thus from (2), we have: $L\int_0^t F(u)G(t-u)du = \int_0^\infty\int_0^t e^{-st}F(u)\,G(t-u)\,du\,dt = f(s).g(s)$

Taking inverse Laplace transform on each side, we get $L^{-1}\left[f(s)g(s)\right] = \int_0^t F(u)\,G(t-u)\,du$

Example 20: Find $L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\}$ using convolution property.

Solution: Let $f(s) = 1/(s+3)$ and $g(s) = 1/(s-1)$

Thus, $L^{-1}\{f(s)\} = L^{-1}\frac{1}{s+3} = e^{3t} = F(t)$ and $L^{-1}\{g(s)\} = L^{-1}\frac{1}{s-1} = e^{t} = G(t)$

Now using convolution property, we obtain

$$L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\} = \int_0^t e^{-3u}e^{t-u}\,du = \int_0^t e^{-3u}e^{t}e^{-u}\,du = e^{t}\int_0^t e^{-4u}\,du = e^{t}\left[\frac{e^{-4u}}{-4}\right]_0^t$$

$$= -\frac{e^t}{4}(e^{-4t}-e^{0}) = -\frac{e^t}{4}(e^{-4t}-1)\left(e^{-4t}-1\right) \Rightarrow L^{-1}\left\{\frac{1}{(s+3)(s-1)}\right\} = -\frac{1}{4}(e^{-3t}-e^{t}) = \frac{e^{t}-e^{-3t}}{4}$$

Example 21: Find $L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}$

Solution: Let us assume that $f(s) = \frac{1}{s-2}$ and $g(s) = \frac{1}{(s+2)^2}$

Let, $L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} = F(t)$ and $L^{-1}\{g(s)\} = L^{-1}\left\{\frac{1}{(s+2)^{22}}\right\} = t\,e^{-2t}$

Or

$$\left(\because L^{-1}\frac{1}{s^2} = t \Rightarrow L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t}\left(\text{by first shifting property}\right)\right)$$

Now by using convolution property, we have

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\} = \int_0^t e^{2u}\left(t-u\right)e^{-2(t-u)}du = \int_0^t e^{2u}e^{-2t}e^{2u}\left(t-u\right)du = e^{-2t}\int_0^t e^{4u}\left(t-u\right)du$$

Integration by parts, we get

$$L^{-1}\left\{\frac{1}{(s+2)^2(s-2)}\right\}=e^{-2t}\left[\frac{(t-u)e^{4u}}{4}+\frac{4e^{u}}{16}\right]_0^t=e^{-2t}\left[0+\frac{e^{4t}}{16}+\frac{4e^{u}}{16}-\left(\frac{t}{4}+\frac{1}{16}\right)\right]$$

$$=e^{-2t}\left[\frac{e^{4t}}{16}-\frac{t}{4}-\frac{1}{16}\right]=\frac{1}{16}e^{-2t}\left[e^{4t}-4t-1\right]=\frac{1}{16}\left[e^{2t}-4te^{-2t}-e^{-2t}\right]$$

Example 22: Using Convolution theorem verify that $\int_0^t \sin u\cos(t-u)\,du=\frac{1}{2}t\sin t$

Solution: By convolution theorem, $L^{-1}\{f(s)g(s)\}=\int_0^t F(u)G(t-u)\,du$ (1)

Now given integral is: $\int_0^t \sin u\cos(t-u)\,du$ (2)

Comparing the two integrals, we see that: F(u) = sin u ,G(t – u) = cos(t – u)
or $F(t)=\sin t$ and $G(t)=\cos t$

Now $L^{-1}\{f(s)\}=F(t)$ and $L^{-1}\{g(s)\}=G(t)$

or $f(s)=L\{F(t)\}$ and $g(s)=L\{G(t)\}\Rightarrow$ $L\{\sin t\}=\frac{1}{s^2+1}=f(s)$, $L\{\cos t\}=\frac{s}{s^2+1}=g(s)$

Now, $L^{-1}\{f(s)g(s)\}=L^{-1}\left\{\left(\frac{1}{s^2+1}\right)\left(\frac{s}{s^2+1}\right)\right\}=L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}$

$=\frac{-1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right\}$ NOTE: $\frac{d}{ds}\left(\frac{-1}{2(s^2+1)}\right)=\frac{s}{(s^2+1)^2}$

Now using inverse Laplace transform property of derivatives, we get

$L^{-1}\{f(s)g(s)\}=\frac{-1}{2}\left(-t\,L^{-1}\left(\frac{1}{s^2+1}\right)\right)=\frac{t}{2}\sin t$. Thus, $\int_0^t \sin u\cos(t-u)\,du=\frac{t}{2}\sin t$

Example 23: Evaluate $L^{-1}\frac{1}{(s+3)(s^2+4)}$

Solution: Let $f(s)=\frac{1}{(s+3)}$ and $g(s)=\frac{1}{(s^2+4)}$ or $F(t)=e^{-3t}$, $G(t)=\sin 2t/2$

$$\therefore L^{-1}\left\{\frac{1}{(s+3)(s^2+4)}\right\}=\frac{1}{2}\int_0^t e^{-3u}\sin(2t-2u)\,du$$

Using the formula: $\int e^{ax}\sin(bx+c)\,dx=\frac{e^{ax}}{(a^2+b^2)}\left[a\sin(bx+c)-b\cos(bx+c)\right]$

Remark: Integration is w.r.t u thus taking a = -3, b = -2 and using the above formula, we get:

$$L^{-1}\left\{\frac{1}{(s+3)(s^2+4)}\right\}=\frac{1}{2}\left[\frac{e^{-3u}}{(9+4)}\{-3\sin(2t-2u)+2\cos(2t-2u)\}\right]_0^t$$

$$=\frac{1}{2.13}\left[e^{-3t}\{-3\sin(2t-2t)+2\cos(2t-2t)\}-e^{0}\{-3\sin(2t-0)+2\cos(2t-0)\}\right]$$

$$=\frac{1}{26}\left[2e^{-3t}+3\sin 2t-2\cos 2t\right]\qquad\left[\text{Note: } \sin 0=0,\ \cos 0=1,\ e^0=1\right]$$

Example 24: Find $L^{-1}\left\{\dfrac{s^2}{(s^2+9)(s^2+1)}\right\}$

Solution: Let $f(s)=\dfrac{s}{(s^2+9)}$, $g(s)=\dfrac{s}{(s^2+4)}$ giving $F(t)=\cos 3t,\ G(t)=\cos 2t$

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+9)(s^2+4)}\right\}=\int_0^t \cos 3u\ \cos(2t-2u)\,du=\frac{1}{2}\int_0^t 2\cos 3u\ \cos(2t-2u)\,du$$

Using the formula: $2\cos a\cos b=\cos(a+b)+\cos(a-b)$, we get

$$L^{-1}\left\{\frac{1}{(s^2+9)(s^2+4)}\right\}=\frac{1}{2}\int_0^t\left[\cos\left(u+2t\right)+\cos(5u-2t)\right]du=\frac{1}{2}\left[\sin(u+2t)+\sin(5u-2t)/5\right]_0^t$$

$$=\frac{1}{2}\left[\sin 3t-\sin 2t+\sin 3t/5-\sin(-2t)/5\right]=\frac{1}{10}\left[6\sin 3t-4\sin 2t\right]=\frac{1}{5}\left[3\sin 3t-2\sin 2t\right]$$

Inverse Laplace Transform by Heaviside Expansion Formula

Let P(s) and Q(s) be polynomials, where P(s) has degree less than that of Q(s) and suppose that Q(s) is a polynomial with n distinct zeros $\alpha_1,\alpha_2,\alpha_3,...$ then

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=\frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t}+\frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t}+\frac{P(\alpha_3)}{Q'(\alpha_3)}e^{\alpha_3 t}+...+\frac{P(\alpha_n)}{Q'(\alpha_n)}e^{\alpha_n t}=\sum_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k t}$$

This formula is known as Heaviside Expansion Formula.

Example 25: Use Heaviside expansion to find $L^{-1}\left\{\dfrac{1}{(s+3)(s-1)}\right\}$

Solution: Here P(s) = 1 and Q(s) = (s + 3) (s – 1). Since Q(s) has two zeros, that is, $\alpha_1=3$, $\alpha_2=1$, hence P(α_1) = P(–3) = 1 and P(α_2)= P(1) = 1.

Also Q(s) = (s + 3)(s – 1) = s^2 + 2s + 2 or Q`(s) = 2s +2

Hence, Q`( $\alpha_1$ ) = Q`(–3)= 2(–3) + 2 = – 4 and Q`( $\alpha_2$ ) = Q`(1) = 2(1) + 2 = 4

Now using Heaviside expansion formula, we get

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=L^{-1}\left[\frac{1}{(s+3)(s-1)}\right]=\frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t}+\frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t}$$

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=L^{-1}\left[\frac{1}{(s+3)(s-1)}\right]=-\frac{1}{4}e^{-3t}+\frac{1}{4}e^{t}$$

Example 26: Use Heaviside expansion to find $L^{-1}\left[\dfrac{s+5}{(s+1)(s^2+1)}\right]$

Solution: Here P(s) = s + 5 and Q(s) = (s + 1) (s^2 + 1) = s^3 + s^2 + s +1 and Q`(s) = $3s^2$ + 2s +1

Since s^2 + 1 = (s + i) (s – i), hence zeros of Q(s) are $\alpha_1=-1,\ \alpha_2=-i$ and $\alpha_3=i$

Now using Heaviside's expansion formula

$$L^{-1}\left[\frac{P(s)}{Q(s)}\right]=\frac{P(\alpha_1)}{Q'(\alpha_1)}e^{\alpha_1 t}+\frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t}+\frac{P(\alpha_3)}{Q'(\alpha_3)}e^{\alpha_3 t}=\frac{P(-1)}{Q'(1)}e^{-1.t}+\frac{P(-i)}{Q'(-i)}e^{-it}+\frac{P(i)}{Q'(i)}e^{it}$$

$$=\frac{4}{2}e^{-t}+\frac{5-i}{-2-2i}e^{-it}+\frac{5+i}{-2+2i}e^{it}=\frac{4}{2}e^{-t}+\frac{5-i}{-2-2i}.\frac{-2+2i}{-2+2i}e^{-it}+\frac{5+i}{-2+2i}.\frac{-2-2i}{-2-2i}e^{it}$$

$$=2e^{-t}+\frac{12i-8}{8}.e^{-it}+\frac{-8-12i}{8}.e^{it}=2e^{-t}+\left(-1+\frac{3i}{2}\right).e^{-it}+\left(-1-\frac{3i}{2}\right).e^{it}$$

$$=2e^{-t}+\left(-1+\frac{3i}{2}\right)\left(\cos t-i\sin t\right)+\left(-1-\frac{3i}{2}\right)\left(\cos t+i\sin t\right)\quad\left(\text{NOTE: } e^{\pm i\theta}=\cos\theta\pm i\sin\theta\right)$$

$$\therefore L^{-1}\left[\frac{s+5}{(s+1)(s^2+1)}\right]=\left(2e^{-t}-\cos t+\frac{3}{2}i\cos t+i\sin t+\frac{3}{2}\sin t-\cos t-\frac{3}{2}i\cos t-i\sin t+\frac{3}{2}\sin t\right)$$

$$=\left(2e^{-t}-2\cos t+\frac{6\sin t}{2}\right)=\left(2e^{-t}-2\cos t+3\sin t\right)$$

Remark: Heaviside expansion formula is preferable when denominator possesses linear factors, otherwise calculation is little difficult.

Evaluation of Improper Integrals

Laplace transform technique is often useful in evaluating some definite integrals. The method is procedure is illustrated in the following examples.

Example 27: Evaluate $\int_0^{\infty}\frac{\sin x}{x}\,dx$

Solution: Consider, $G(t)=\int_0^{\infty}\frac{\sin tx}{x}\,dx$.

Taking Laplace transform we get: $L\left[G(t)\right]=L\left[\int_0^{\infty}\frac{\sin tx}{x}\,dx\right]=\int_0^{\infty}e^{-st}\left[\int_0^{\infty}\frac{\sin tx}{x}\,dx\right]dt$.

Changing the order of integration, we get

$$L\left[G(t)\right]=\int_0^{\infty}\frac{1}{x}dx\left[\int_0^{\infty}e^{-st}\sin tx\;dt\right]=\int_0^{\infty}\frac{1}{x}dx\left[L(\sin tx)\right]=\int_0^{\infty}\frac{1}{x}\frac{x}{s^2+x^2}dx$$

$$L\left[G(t)\right]=\int_0^{\infty}\frac{1}{s^2+x^2}dx=\frac{1}{s}\tan^{-1}\left[\frac{x}{s}\right]_0^{\infty}=\frac{1}{s}\left(\tan^{-1}(\infty)-\tan^{-1}(0)\right)=\frac{\pi}{2s}$$

Taking the inverse Laplace transform on both sides, we obtain

$\left[G(t)\right]=L^{-1}\left(\frac{\pi}{2s}\right)=\frac{\pi}{2}L^{-1}\left(\frac{1}{s}\right)=\frac{\pi}{2}(1)=\frac{\pi}{2}$. Now,

$G(t)=\int_0^{\infty}\frac{\sin tx}{x}\,dx=\frac{\pi}{2}$. Substituting $t=1$, we get $\int_0^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}$

Example 28: Evaluate $\int_0^{\infty}\cos x^2\,dx$

Solution: Consider, $G(t)=\int_0^{\infty}\cos tx^2\;dx$.

Taking Laplace transform we get: $L\left[G(t)\right]=L\left[\int_0^{\infty}\cos tx^2\;dx\right]=\int_0^{\infty}e^{-st}\left[\int_0^{\infty}\cos tx^2\;dx\right]dt$

Changing the order of integration, we get

$$L\left[G(t)\right]=\int_0^{\infty}dx\left[\int_0^{\infty}e^{-st}\cos tx^2\;dt\right]=\int_0^{\infty}dx\left[L\left(\cos tx^2\right)\right]=\int_0^{\infty}\frac{s}{s^2+x^4}dx=s\int_0^{\infty}\frac{1}{x^4+s^2}dx$$

Substituting $x^2 = s\tan\theta$ or $2x\,dx = s\sec^2\theta\,d\theta$ or $dx = s\sec^2\theta\,d\theta/2x$.

Also if $x = 0$ then $\theta = 0$ and if $x = \infty$ then $x = \pi/2$. Thus

$$L\left[G(t)\right]=s\int_0^{\pi/2}\frac{s.\sec^2\theta}{s^2\tan^2\theta+s^2}\frac{d\theta}{2x}=\int_0^{\pi/2}\frac{\sec^2\theta}{\tan^2\theta+1}.\frac{d\theta}{2\sqrt{s\tan\theta}}=\int_0^{\pi/2}\frac{\sec^2\theta}{\sec^2\theta}.\frac{d\theta}{2\sqrt{s\tan\theta}}$$

$$=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\frac{1}{\sqrt{\tan\theta}}\,d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta=\frac{1}{2\sqrt{s}}\int_0^{\pi/2}\cos^{1/2}\theta.\sin^{-1/2}\theta\,d\theta \qquad (1)$$

Notice that *Beta function* is define as: $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta.\cos^{2n-1}\theta\,d\theta \qquad (2)$

Also relation between Beta and Gamma Functions is: $\beta(m,n)=\frac{\Gamma(m).\Gamma(n)}{\Gamma(m+n)}$

Comparing (1) and (2), we see that 2m – 1 = –1/2 or m = 1/4 and 2n – 1 = 1/2 or n = 3/4. Thus equation (1) becomes:

$$L[G(t)]=\frac{1}{2\sqrt{s}}\frac{\beta\left(\frac{1}{4},\frac{3}{4}\right)}{2}=\frac{1}{4\sqrt{s}}\frac{\Gamma(1/4).\Gamma(3/4)}{\Gamma(1/4+3/4)}=\frac{1}{4\sqrt{s}}\frac{\Gamma(1/4).\Gamma(3/4)}{\Gamma(1)}=\frac{\pi\sqrt{2}}{4\sqrt{s}}$$

Note: $\Gamma(1)=1$ and $\Gamma(1/4).\Gamma(3/4)=\sqrt{2}\pi.$

This identity can be found in advance calculus book.

Taking the inverse Laplacetransform on both sides, we obtain

$$[G(t)]=L^{-1}\frac{\pi\sqrt{2}}{4\sqrt{s}}=\frac{\sqrt{2}\pi}{4}L^{-1}\left(\frac{1}{\sqrt{s}}\right)=\frac{\sqrt{2}\pi}{4}\frac{t^{-1/2}}{\Gamma(1/2)}=\frac{\sqrt{2}\pi}{4\sqrt{\pi t}}.$$ Thus,

$G(t)=\int_0^{\infty}\cos^2 tx\,dx=\frac{\sqrt{2}\pi}{4\sqrt{\pi t}}$. Substituting $t=1$, we get $\int_0^{\infty}\cos x^2\,dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}$

WORKSHEET 09

1. Use linear property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\frac{6}{2s-3}-\frac{3+4s}{9s^2-16}+\frac{8-6s}{16s^2+9}\right\}$ (ii) $L^{-1}\left\{\frac{3\left(s^2-1\right)^2}{2s^5}+-\frac{4s-18}{9-s^2}+\frac{(s+1)\left(2-\sqrt{s}\right)}{s^{5/2}}\right\}$

(iii) $L^{-1}\left\{\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}+\frac{24-30\sqrt{s}}{s^4}\right\}$ (iv) $L^{-1}\left\{\frac{s+1}{s^{4/3}}\right\}$ (v) $L^{-1}\left\{\frac{5s+10}{9s^2-16}\right\}$

2. Use first shifting property of inverse Laplace evaluate the following:

(i) $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$ (ii) $L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}$ (iii) $L^{-1}\left\{\frac{1}{s^2-2s+5}\right\}$ (iv) $L^{-1}\left\{\frac{s}{(s+1)^{5/2}}\right\}$

(v) $L^{-1}\left\{\frac{5s-12}{3s^2+4s+8}\right\}$ (vi) $L^{-1}\left\{\frac{1}{\sqrt[3]{8s-27}}\right\}$ (vii) $L^{-1}\left\{\frac{3s+1}{s^2-6s+18}\right\}$

3. Use second shifting property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\frac{e^{-3s}}{s^2-2s+5}\right\}$ (ii) $L^{-1}\left\{\frac{se^{-2s}}{s^2+3s+2}\right\}$ (iii) $L^{-1}\left\{\frac{e^{-\pi s/3}}{s^2+1}\right\}$ (iv) $L^{-1}\left\{\frac{e^{-2s}}{s^3}\right\}$

4. Use change of scale property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\frac{s}{4s^2+16}\right\}$ (ii) If $L^{-1}\left\{\frac{e^{-1/s}}{s^{1/2}}\right\}=\cos\frac{2\sqrt{t}}{\sqrt{\pi t}}$ find $L^{-1}\left\{\frac{e^{-2/s}}{s^{1/2}}\right\}$

5. Using derivative property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{s/\left(s^2+1\right)^2\right\}$ (ii) $L^{-1}\left\{s/\left(s^2-1\right)^2\right\}$ (iii) $L^{-1}\left\{\ln\left(\frac{s+2}{s+1}\right)\right\}$

6. If $F(t) = L^{-1}\{f(s)\}$ prove that:

(i) $L^{-1}\left\{sf^{'}(s)\right\} = -tF^{'}(t) - F(t)$ (ii) $L^{-1}\left\{sf^{''}(s)\right\} = t^2F^{'}(t) + 2tF(t)$

(iii) $L^{-1}\left\{s^2f^{''}(s)\right\} = t^2F^{''}(t) + 4tF^{'}(t) + 2F(t)$

7. Use division by s property of inverse Laplace transforms evaluate the following:

(i) $L^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}$ (ii) $L^{-1}\left\{\dfrac{1}{s^3(s^2+1)}\right\}$ (iii) $L^{-1}\left\{\dfrac{1}{s(s+1)^3}\right\}$

(iv) $L^{-1}\left\{\dfrac{1}{s}\ln\left(1+\dfrac{1}{s^2}\right)\right\}$ (v) $L^{-1}\left\{\ln\left(\dfrac{s+2}{s+1}\right)\right\}$

8. Use partial fraction method to evaluate the following:

(i) $L^{-1}\left\{\dfrac{3s+7}{(s^2-2s-3)}\right\}$ (ii) $L^{-1}\left\{\dfrac{2s^2-4}{(s-2)(s-3)(s+1)}\right\}$ (iii) $L^{-1}\left\{\dfrac{2s-1}{(s^3-s)}\right\}$

(iv) $L^{-1}\left\{\dfrac{s^2-2s+3}{(s-1)^2(s+1)}\right\}$ (v) $L^{-1}\left\{\dfrac{s^3+16s-24}{s^4+20s^2+64}\right\}$ (vi) $L^{-1}\left\{\dfrac{2s^3+10s^2+8s+40}{s^2(s^2+9)}\right\}$

(vii) $L^{-1}\left\{\dfrac{2s-3}{(2s^3+3s^2-2s-3)}\right\}$ (viii) $L^{-1}\left\{\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$

9. Use convolution theorem to evaluate the following:

(i) $L^{-1}\left\{\dfrac{1}{(s+2)^2(s-2)}\right\}$ (ii) $L^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}$ (iii) $L^{-1}\left\{\dfrac{s^2}{(s^2+4)^2}\right\}$

(iv) $L^{-1}\left\{\dfrac{1}{s^2(s+1)^2}\right\}$ (v) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

10. Using Heaviside Expansion to evaluate the following:

(i) $L^{-1}\left\{\dfrac{(19s+37)}{(s-2)(s+1)(s+3)}\right\}$ (ii) $L^{-1}\left\{\dfrac{(s+5)}{(s+1)(s^2+1)}\right\}$ (iii) $L^{-1}\left\{\dfrac{(3s+16)}{(s^2-s-6)}\right\}$

(i)

(iv) $L^{-1}\left\{\dfrac{(2s-1)}{(s^3-s)}\right\}$ (v) $L^{-1}\left\{\dfrac{(s+1)}{(6s^2+7s+2)}\right\}$

11. Show that (i) $\int_0^\infty \sin x^2 \, dx = \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$ (ii) $\int_0^\infty \dfrac{x \sin tx}{1+x^2} \, dx = \dfrac{\pi}{2}e^{-t} \quad t > 0$

CHAPTER TEN

APPLICATIONS of LAPLACE TRANSFORMS

INTRODUCTION

Readers may recall that when higher order non-homogeneous linear differential equations are to be solved the first step is to find the complementary function y_C and particular integral y_P. The particular solution is then found by using the initial and/or boundary conditions in the general solution $y = y_C + y_P$. The Laplace transform method for solving differential equations yields particular solution without finding the general solution. It makes use of initial conditions at the beginning of process. This method is shorter than the methods that have been discussed so far in the subject of *Solution of Differential Equations of Higher Orders*. This method is also useful in solving differential equations with variable coefficients as well as the system of differential equations.

Solutions of Ordinary Linear Differential Equations with Constant Coefficients

In this section, we shall discuss the solutions of ordinary differential equations with constant coefficients. The procedure is explained by solving some problems.

Example 01: Solve the initial value problem

$$Y''(t) + Y(t) = t\,, \quad Y(0) = 1, Y'(0) = -2$$

Solution: Taking Laplace transform on both sides of given differential equation, we get

$$L\{Y''(t)\} + L\{Y(t)\} = L\{t\} \Rightarrow s^2 y(s) - sY(0) - Y'(0) + y(s) = 1/s^2$$

Using initial conditions, we get

$$s^2 y(s) - s(1) - (-2) + y(s) = \frac{1}{s^2} \quad \text{or} \quad (s^2+1)y(s) = \frac{1}{s^2} + s - 2$$

$$\Rightarrow y(s) = \frac{1}{s^2(s^2+1)} + \frac{s}{(s^2+1)} - \frac{2}{(s^2+1)} = \frac{s^2+1-s^2}{s^2(s^2+1)} + \frac{s}{(s^2+1)} - \frac{2}{(s^2+1)}$$

$$= \frac{(s^2+1)}{s^2(s^2+1)} - \frac{s^2}{s^2(s^2+1)} + \frac{s}{(s^2+1)} - \frac{2}{(s^2+1)} = \frac{1}{s^2} - \frac{1}{(s^2+1)} + \frac{s}{(s^2+1)} - \frac{2}{(s^2+1)}$$

Taking inverse Laplace transform on both sides, we obtain

$$L^{-1}\{y(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} - L^{-1}\left\{\frac{3}{(s^2+1)}\right\} + L^{-1}\left\{\frac{s}{(s^2+1)}\right\} \ (\text{By linear property})$$

Thus particular solution is $\quad Y(t) = t - 3\sin t + \cos t$

Example 02: Solve the initial value problem

$$Y''(t) - 3Y'(t) + 2Y(t) = 4e^{2t}\,, \quad Y(0) = -3, Y'(0) = 5$$

Solution: Given $Y''(t) - 3Y'(t) + 2Y(t) = 4e^{2t}$

Taking the Laplace transform on both sides, we get

$$L\left\{Y''(t)\right\}-3L\left\{Y'(t)\right\}+2L\left\{Y(t)\right\}=4L\left\{e^{2t}\right\}$$

$$\text{or}\quad \left[s^2y(s)-sY(0)-Y'(0)\right]-3\left[sy(s)-Y(0)\right]+2y(s)=\frac{4}{s-2}$$

Using the given conditions, we get

$$\left[s^2y(s)-s(-3)-5\right]-3\left[s\,y(s)-(-3)\right]+2y(s)=\frac{4}{s-2}$$

$$\text{or}\quad \left(s^2-3s+2\right)y(s)=\frac{4}{s-2}-3s+14$$

$$\text{or}\quad y(s)=\frac{4}{(s-2)\left(s^2-3s+2\right)}+\frac{14-3s}{\left(s^2-3s+2\right)}\quad \text{or}\quad y(s)=\frac{4+14s-28-3s^2+6s}{(s-2)\left(s^2-3s+2\right)}$$

$$=\frac{-3s^2+20s-24}{(s-2)(s-2)(s-1)}\qquad \left(\because s^2-3s+2=(s-2)(s-1)\right)$$

$$\text{Thus,}\quad y(s)=\frac{-3s^2+20s-24}{(s-2)^2(s-1)}$$

Now resolving into partial fractions, we get

$$y(s)=\frac{-3s^2+20s-24}{(s-1)(s-2)^2}=\frac{A}{s-1}+\frac{B}{(s-2)^2}+\frac{C}{s-2}\tag{1}$$

$$\text{Or}\quad -3s^2+20s-24=\quad A(s-2)^2+B(s-1)+C(s-1)(s-2)\tag{2}$$

Now put s = 1 in (2), we get

$$-3(1)^2+20(1)-24=\quad A(1-2)^2+B(1-1)+C(1-1)(1-2)\quad \text{or}\quad A=-7$$

Now put s = 2 in (2), we obtain

$$-3(2)^2+20(2)-24=\quad A(2-2)^2+B(2-1)+C(2-1)(2-2)\ \text{ or }\ B=4$$

To find C, we rewrite equation (2) to get

$$-3s^2+20s-24=\quad (A+C)s^2-(B-4A-3C)s+4A-B+2C$$

Now comparing the coefficients of s^2 on both sided, we get

$A + C = -3$ or $-7 + C = -3$ or $C = 4$. Thus equation (1) becomes

$$y(s)=\frac{-3s^2+20s-24}{(s-1)(s-2)^2}=\frac{-7}{s-1}+\frac{4}{(s-2)^2}+\frac{4}{s-2}$$

Now taking the inverse Laplace transform on both sides, we get

$$L^{-1}\left\{y(s)\right\}=L^{-1}\left\{\frac{-3s^2+20s-24}{(s-1)(s-2)^2}\right\}=-7L^{-1}\left\{\frac{1}{s-1}\right\}+4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}+4L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$\therefore\ Y(t)=-7e^t+4te^{2t}+4e^{2t}\ \text{ or }\ Y(t)=-7e^t+4e^{2t}(t+1)$$

Example 03: Solve the differential equation

$$Y'''(t)-3Y''(t)++3Y'(t)-Y(t)=t^2e^t,\quad Y(0)=1,Y'(0)=0,Y''(0)=-2$$

using Laplace transform method.

Solution: Given differential equation is $Y'''(t)-3Y''(t)++3Y'(t)-Y(t)=t^2e^t$

Taking Laplace transform on both sides, we have

$\left\{Y'''(t)\right\}-3L\left\{Y''(t)\right\}++3L\left\{Y'(t)\right\}-L\left\{Y(t)\right\}=L\left\{t^2e^t\right\}$. This gives

$$\left\{s^3y(s)-s^2Y(0)-sY'(0)-Y''(0)\right\}-3\left\{s^2y(s)-sY(0)-Y'(0)\right\}+3\left\{sy(s)-1\right\}-y(s)=\frac{2}{(s-3)^3}$$

Now using given conditions, we get

$$\left\{s^3y(s)-s^2(1)-s(0)-(-2)\right\}-3\left\{s^2y(s)-s(1)-0\right\}+3\{sy(s)-1\}-y(s)=\frac{2}{(s-1)^3}$$

$$\text{or } y(s)=\frac{2}{(s-1)^3\left(s^3-3s^2+3s-1\right)}+\frac{s^2-3s+1}{\left(s^3-3s^2+3s-1\right)}$$

$$y(s)=\frac{2}{(s-1)^3(s-1)^3}+\frac{s^2-3s+1}{(s-1)^3} \qquad \text{NOTE}: (a-b)^3=a^3-3a^2b+3ab^2-b^3$$

$$=\frac{2}{(s-1)^6}+\frac{s^2-3s+1}{(s-1)^3}=\frac{s^2-2s+1-s+1-1}{(s-1)^3}+\frac{2}{(s-1)^6}$$

$$=\frac{(s-1)^2}{(s-1)^3}-\frac{s-1}{(s-1)^3}-\frac{1}{(s-1)^3}+\frac{2}{(s-1)^6}=\frac{1}{(s-1)}-\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3}+\frac{2}{(s-1)^6}$$

Now taking the inverseLaplace transform on both sides, we have

$$L^{-1}\{y(s)\}=L^{-1}\left\{\frac{1}{(s-1)}\right\}-L^{-1}\left\{\frac{1}{(s-1)^2}\right\}-L^{-1}\left\{\frac{1}{(s-1)^3}\right\}+2L^{-1}\left\{\frac{1}{(s-1)^6}\right\}$$

$$\text{or } Y(t)=e^t-te^t-\frac{t^2}{2}e^t+2\frac{t^5}{5!}e^t=e^t-te^t-\frac{t^2}{2}e^t+\frac{t^5}{60}e^t$$

Example 04: Solve the diff. equation $Y''(t)+2Y'(t)+5Y=e^{-t}\sin t \quad ,Y(0)=0,Y'(0)=1$

Solution: Given $Y''(t)+2Y'(t)+5Y=e^{-t}\sin t$

Taking Laplace transform on both sides, we obtain

$$L\left\{Y''(t)\right\}Y''(t)+2L\left\{Y'(t)\right\}+5L\{Y\}=L\left\{e^{-t}\sin t\right\}$$

$$\text{or } \left\{s^2y(s)-sY(0)-Y'(0)\right\}+2\{sy(s)-Y(0)\}+5y(s)=\frac{1}{(s+1)^2+1}$$

Using the given conditions, that is; $Y(0)=0,\ Y'(0)\ =\ 1$, we obtain

$$\left\{s^2y(s)-s(0)-1\right\}+2\{sy(s)-0\}+5y(s)=\frac{1}{s^2+2s+2}$$

$$s^2y(s)-1+2sy(s)+5y(s)=\frac{1}{s^2+2s+2} \quad \text{or } \ y(s)=\frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}$$

Taking the inverseLaplace transform on both sides, we have

$$L^{-1}\{y(s)\}=L^{-1}\left\{\frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}\right\}$$

Now consider, $\dfrac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}=\dfrac{As+B}{\left(s^2+2s+2\right)}+\dfrac{Cs+D}{\left(s^2+2s+5\right)}$ (1)

$$\text{or } s^2+2s+3=(As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)$$

$$\text{or } s^2+2s+3=(A+C)s^3+(2A+B+2C+D)s^2+(5A+2B+2C+2D)s+5B+2D$$

Now by comparing the coefficients of same powers of:

$s^3:\ (A+C)=0$ (i)

$s^2:\ 2A+B+2C+D=1$ (ii)

$s\ :\ (5A+2B+2C+2D)=2$ (iii)

$s^0:\ 5B+2D=3$ (iv)

Now subtracting (ii) from (iii), we have

$3A+B+D=1$ (v)

Now subtract (v) from (ii) ,we have

$-A+2C=0$ (vi)

Now from (i) A= - C. Put in (vi), we get

$C+2C=0$ or $C=0$ Put C = 0 in (vi), we have $A=0$

Put A = 0 in (v), we get $B+D=1$ or $2B+2D=2$ (vii)

Adding (iv) and (vii) ,we have: $B=1/3$

Put A = 0 and B = 1/3 in (v), we get D = 2/3

Thus equation (1) becomes after taking inverse Laplace transform

$$L^{-1}(y(s))=L^{-1}\left\{\frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)}\right\}=L^{-1}\left\{\frac{1/3}{s^2+2s+2}\right\}+L^{-1}\left\{\frac{2/3}{s^2+2s+5}\right\}$$

$$=\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}+\frac{2}{6}L^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}=\frac{1}{3}e^{-t}\sin t+\frac{2}{6}e^{-t}\sin 2t$$

or $$Y(t)=\frac{1}{3}e^{-t}\left(\sin t+\sin 2t\right)$$

Example 05: Solve the differential equation Y`` + 9Y = 18t, Y(0) = 0, Y(π/2) = = 2

Solution: Taking Laplace transform of given differential equation, we get

L[Y``] + 9 L[Y] = 18 L[t] or s^2 y(s) – sY(0) – Y`(0) + 9 y(s) = 18 /s^2 (1)

Here we see that only one condition Y(0) = 0 is given where as the second condition is not. Thus we assume that Y`(0) = c and so equation (1) becomes

$$(s^2 + 9)\ y(s) = c + 18/s^2$$

Or $y(s)=\dfrac{cs^2+18}{s^2\left(s^2+9\right)}=\dfrac{A}{s^2}+\dfrac{B}{s}+\dfrac{Cs+D}{s^2+9}$ (By PF Case III) (2)

Or $cs^2 + 18 = A(s^2 + 9) + Bs(s^2 + 9) + Cs^3 + Ds^2$ (3)

Put s = 0, we get: 18 = 9A or A = 2.

To find B, C and D, we compare the coefficients of different terms of s in equation (3). Now comparing the coefficients of

s^3: 0 = B + C (i)

s^2: c = A + D or D = c – A = c - 2

s: 0 = 9B or B = 0 then C = 0 [using (i)]

Thus equation (2) becomes $y(s)=\dfrac{cs^2+18}{s^2(s^2+9)}=\dfrac{2}{s^2}+\dfrac{c-2}{s^2+9}$

Taking inverse Laplace transform on both sides, we obtain

$$L^{-1}\left[y(s)\right]=Y(t)=2L^{-1}\frac{1}{s^2}+(c-2)L^{-1}\frac{1}{s^2+9}=2t+\frac{(c-2)\sin 3t}{3} \quad (4)$$

To find c we use the remaining condition that Y(π/2) = 2, that put t = π/2 and Y = 2.

or 2 = 0 + (c – 2) (–1)/3 or c – 2 = –6 or c = –4.

Thus equation (4) becomes: $Y(t) = 2t + \dfrac{(-4-2)\sin 3t}{3} = 2t - 2\sin 3t$

Solutions of Ordinary Differential Equations with Variable Coefficients

This section is devoted towards the solution of ordinary differential equations with variable coefficients. The method is best explained by solving the following few problems.

Example 06: Solve the differential equation

$$t\,Y''(t) + Y'(t) + 4t\,Y(t) = 0\,, \quad Y(t) = 3, Y'(0) = 0$$

Solution: Taking the Laplace transform on both sides, we have

$$L\{tY''(t)\} + L\{Y'(t)\} + 4L\{tY(t)\} = L\{0\} \qquad (1)$$

We know that $L\{t^n F(t)\} = (-1)^n f^n(s)$ (2)

Now, $L\{Y''(t)\} = \{s^2 y(s) - sY(0) - Y'(0)\}$. Thus using (2), we get

$$L\{tY''(t)\} = \frac{-d}{ds}\{s^2 y(s) - sY(0) - Y'(0)\}$$

Thus equation (1) becomes

$$\frac{-d}{ds}\{s^2 y(s) - sY(0) - Y'(0)\} + \{sy - Y(0)\} - 4\frac{dy}{ds} = 0$$

Using given conditions, we get

$$\frac{-d}{ds}\{s^2 y - s(3) - 0\} + \{sy - 3\} - 4\frac{dy}{ds} = 0 \text{ or } 3\frac{d}{ds}.s - \frac{d}{ds}\{s^2 y\} + \{sy - 3\} - 4\frac{dy}{ds} = 0$$

$$3.1 - \left\{2sy + s^2\frac{dy}{ds}\right\} + \{sy - 3\} - 4\frac{dy}{ds} = 0$$

$$\text{or } (s - 2s)y - (s^2 + 4)\frac{dy}{ds} = 0 \text{ or } sy + (s^2 + 4)\frac{dy}{ds} = 0$$

This differential equation in y can be solved by separable variable method. Thus separating the variables, we get $\dfrac{dy}{y} = \dfrac{-s}{s^2 + 4}ds$

Integrating both sides, we obtain

$$\int\frac{dy}{y} = -\int\frac{s}{s^2+4}ds \quad \text{or} \quad \ln y = -\frac{1}{2}\ln(s^2+4) + \ln c \quad \text{or} \quad y = \frac{c}{\sqrt{(s^2+4)}}$$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}\{y(s)\} = cL^{-1}\left\{\frac{1}{\sqrt{(s^2+4)}}\right\}. \quad \text{Thus, } Y(t) = c\,J_0(2t) \qquad (3)$$

$$\text{Note:} \quad L^{-1}\left(\frac{1}{\sqrt{s^2 + a^2}}\right) = J_0(at)$$

In order to compute c, we use the condition Y(0) = 3, that is; we put $Y = 3, t = 0$.

$$\therefore \quad Y(0) = 3 = c\,J_0(0) \quad \text{or} \quad 3 = c \qquad (\because J_0(0) = 1)$$

Thus equation (3) becomes $Y(t) = 3J_0(2t)$. This is the solution of given differential equation.

Example 07: Solve the differential equation

$$tY''(t) + 2Y'(t) + tY(t) = 0\,, \quad Y(0) = 1, Y(\pi) = 0$$

Solution: Taking Laplace transform on both sides of given differential equation, we get

$$L\left\{tY''(t)\right\}+2L\left\{Y'(t)\right\}+L\left\{tY(t)\right\}=L\{0\}$$

or $$\frac{-d}{ds}\left(s^2y-sY(0)-Y'(0)\right)+2\left(sy-Y(0)\right)-\frac{d}{ds}y=0$$

Using the condition Y (0) = 1 and suppose that Y'(0) = c, so that above equation becomes:

$$\frac{-d}{ds}\left(s^2y-s(1)-c\right)+2\left(sy-1\right)-\frac{dy}{ds}=0$$

or $1-2sy-s^2\frac{dy}{ds}+2sy-2-\frac{dy}{ds}=0$ or $-1-\left(s^2+1\right)\frac{dy}{ds}=0$ or $\left(s^2+1\right)\frac{dy}{ds}=-1$

Separating the variables and integrating, we get

$$\int dy=\int\frac{-1}{(s^2+1)}ds \quad \text{or} \quad y=-\tan^{-1}s+c \qquad (1)$$

It may be noted that by the definition of inverseLaplace transform:

When $s\to\infty$, $y\to 0$. This gives $0=\tan^{-1}\infty+c=(\pi/2)+c$ or $c=\pi/2$

Thus, equation (1) becomes $y=-\tan^{-1}s+\frac{\pi}{2}=\tan^{-1}\left(\frac{1}{s}\right)$ (2) (see below)

Now taking inverse Laplace transform on both sides of (2), we have

$$L^{-1}\left(y(s)\right)=L^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right) \quad \text{or} \quad Y(t)=\frac{\sin t}{t} \qquad \left(\because L^{-1}\left(\frac{\sin t}{t}\right)=\tan^{-1}\left(\frac{1}{s}\right)\right)$$

This is the required solution of given differential equation.

NOTE: $\tan^{-1}a-\tan^{-1}b=\tan^{-1}\left(\frac{a-b}{1+ab}\right)=\tan^{-1}\left[\frac{a\left(1-\frac{b}{a}\right)}{a\left(\frac{1}{a}+b\right)}\right]=\tan^{-1}\left[\frac{\left(1-\frac{b}{a}\right)}{\left(\frac{1}{a}+b\right)}\right]$

Now let $a\to\infty$ then above equation becomes: $\tan^{-1}\infty-\tan^{-1}b=\tan^{-1}\left(\frac{1}{b}\right)$.

But $\tan^{-1}\infty=\pi/2$. Thus, $\frac{\pi}{2}-\tan^{-1}b=\tan^{-1}\left(\frac{1}{b}\right)$. This is same as equation (2).

Solutions of Simultaneous Ordinary Differential Equations

In this section we shall discuss the solution of system of ordinary differential equations. The method is best explained by solving the following problems.

Example 08: Solve the system of differential equations

$$\frac{dX}{dt}=2X-3Y, \quad \frac{dY}{dt}=Y-2X \quad ; \text{ given that } X(0)=8, Y(0)=3$$

Here, dX/dt = X `, dY/dt = Y `

Solution: Taking the Laplace transform of given equations, we get

$$L\left\{X'\right\}=2L\{X\}-3L\{Y\},\ L\left\{Y'\right\}=L\{Y\}-2L\{X\}$$

$$\Rightarrow sx-X(0)=2x-3y,\ \ sy-Y(0)=y-2x$$

Now using given conditions X(0) = 8 and Y(0) =3, we have

$$sx-8=2x-3y \qquad\qquad sy-3=y-2x$$

$$(s-2)x+3y=8 \quad \text{(i)} \qquad\qquad (s-1)y+2x=3 \quad \text{(ii)}$$

Now solving (i) and (ii) simultaneously using Cramer's rule or otherwise, we get

$$x=\frac{\begin{vmatrix}8 & 3\\ 3 & s-1\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}} \quad \text{and} \quad y=\frac{\begin{vmatrix}s-2 & 8\\ 2 & 3\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}} \Rightarrow x=\frac{8s-17}{s^2-3s-4}, \quad =\frac{3s-22}{s^2-3s-4}$$

or $\quad x=\dfrac{8s-17}{(s+1)(s-4)}$ and $y=\dfrac{3s-22}{(s+1)(s-4)}$. Now by partial fractions,

$$x=\frac{8(-1)-17}{(s+1)(-1-4)}+\frac{8(4)-17}{(4+1)(s-4)} \quad \text{and} \quad y=\frac{3(-1)-22}{(s+1)(-1-4)}+\frac{3(4)-22}{(4+1)(s-4)}$$

$$=\frac{-8-17}{-5(s+1)}+\frac{32-17}{5(s-4)} \quad \text{and} \quad =\frac{-3-22}{-5(s+1)}+\frac{12-22}{5(s-4)}$$

$$\text{or } x=\frac{5}{(s+1)}+\frac{3}{(s-4)} \quad \text{and} \quad y=\frac{5}{(s+1)}+\frac{-2}{(s-4)}$$

Now taking inverse Laplace transforms on both sides, we obtain

$$L^{-1}\{x(s)\}=5L^{-1}\left\{\frac{1}{(s+1)}\right\}+3L^{-1}\left\{\frac{1}{(s-4)}\right\}, \quad L^{-1}\{y(s)\}=L^{-1}\left\{\frac{5}{(s+1)}\right\}-L^{-1}\left\{\frac{2}{(s-4)}\right\}$$

$$\Rightarrow X(t)=5e^{-t}+3e^{4t} \quad \text{and} \quad Y(t)=5e^{-t}-2e^{4t}$$

This is the solution of given system of linear equations.

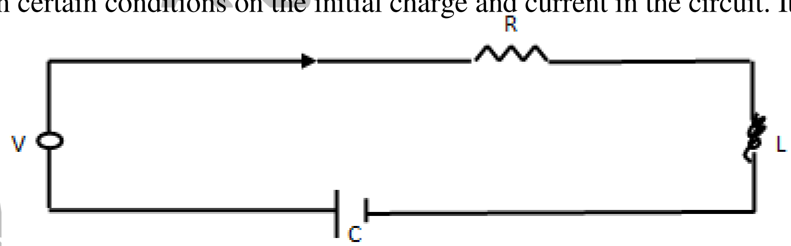
APPLICATIONS OF LAPLACE TRANSFORMS

Applications to Electrical Circuits

Readers are familiar with RLC circuit. The differential equation that governs this circuit is given by

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

Here we are also given certain conditions on the initial charge and current in the circuit. It may be noted that Q`(t) = I



Example 01: At time t = a constant voltage E is applied to RLC-circuit. The current and the charge initially are zero. Find the current in the circuit at ant time t > 0 given that R = 2 ohms, L = 1 H and C = 1 F.

Solution: Substituting the values of L, R, C and E = k in the differential equation, we get

$$Q''+2Q'+Q=k$$

Taking Laplace transform, we get: $L(Q'')+2L(Q')+L(Q)=kL(1)$

or $\quad s^2 q(s) - s\,Q`(0) - Q(0) + 2s\,q(s) - 2Q(0) + q(s) = k/s.$

Now Q(0) = Q`(0) = 0. Thus above equation becomes:

$$(s^2+2s+1)\,q(s)=k/s$$

$$\text{or } q(s)=\frac{ks}{(s+1)^2}=k\frac{s+1-1}{(s+1)^2}=k\left[\frac{s+1}{(s+1)^2}-\frac{1}{(s+1)^2}\right]=k\left[\frac{1}{(s+1)}-\frac{1}{(s+1)^2}\right]$$

Taking inverse Laplace transform, we obtain

$$L^{-1}q(s) = k\left[L^{-1}\frac{1}{(s+1)} - L^{-1}\frac{1}{(s+1)^2}\right] \text{ or } Q(t) = k\left(e^{-t} - te^{-t}\right)$$

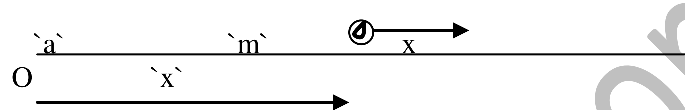
This is the charge in the circuit at any time. Now differentiate both sides w.r.t time t, we get:

$$I = Q'(t) = k\left(-e^{-t} + t e^{-t} - e^{-t}\right) = ke^{-t}\left(t-2\right)$$

Applications to Mechanics

Example 02: A mass m moves along the x-axis under the influence of a force which is proportional to its instantaneous speed and in a direction opposite to the direction of motion. Assuming that at t = 0 the particle is located at x = a and moving to the right with speed v_0, find the position where the mass comes to rest.

Solution:



According Newton's 2nd law of motion: ma = F. Here `a` is acceleration and is given by a = d^2x/dt^2. The force F is proportional to speed and is acting in the opposite direction of F. Thus equation governing this problem is

$$mX'' = -k X' \quad \text{with initial conditions } X(0) = a \text{ and } X`(0) = v_0$$

Taking Laplace transform on each side, we get

$$m LX'' = -kL X' \text{ or } m\left(s^2x - sX(0) - X'(0)\right) + k\left(s x - X'(0)\right) = 0$$

$$m\left(s^2x - a.s - v_0\right) + k\left(s.x - v_0\right) = 0 \qquad \text{or } s\left(m.s + k\right)x = m.a.s + mv_0 + kv_0$$

$$\Rightarrow x = \frac{m.a.s + v_0\left(m+k\right)}{s\left(m.s+k\right)} = \frac{A}{s} + \frac{B}{ms+k} \text{ or } m.a.s + v_0\left(m+k\right) = A(m.s+k) + Bs$$

Now put s = 0, we get : A = $\left(mv_0 + ak\right)/k$. Put ms + k = 0, we get B $= -m^2v_0/k$

$$x = \frac{\left(mv_0 + ak\right)}{ks} - \frac{m^2v_0}{ms+k} = \frac{\left(mv_0+ak\right)}{ks} - \frac{mv_0}{(s+k/m)}$$

Thus, taking inverseLaplace transform, we get :

$$L^{-1}(x) = X(t) = \frac{\left(mv_0 + ak\right)}{k}L^{-1}\left(\frac{1}{s}\right) - mv_0L^{-1}\frac{1}{(s+k/m)}$$

Thus $X(t) = \dfrac{\left(mv_0 + ak\right)}{k} mv_0 e^{-\left(\frac{k}{m}\right)t}$ is the general solution.

Solution of Partial Differential Equations by Laplace Transform

Example 03: Solve the following PDE using Laplace Transform method.

$$\frac{\partial Y}{\partial t} = 3\frac{\partial^2 Y}{\partial x^2}, \; Y\left(\pi/2, t\right) = 0 = \left(\frac{\partial Y}{\partial x}\right)_{x=0} \text{ and } Y\left(x,0\right) = 30\cos 5x$$

Solution: Taking the Laplace transform of both sides, we get

$$L\left(\frac{\partial Y}{\partial t}\right) = 3L\left(\frac{\partial^2 Y}{\partial x^2}\right) \Rightarrow s\,y - Y\left(x,0\right) = 3\left[\frac{\partial^2 L(Y)}{\partial x^2}\right] \Rightarrow s\,y - Y\left(x,0\right) = 3\left[\frac{d^2y}{dx^2}\right]$$

Note the difference of using Lapalce transform on each side. Here we have used Liebnit'z theorem of integration under derivatives on RHS.

Further to note that $L\left(Y\right) = y$. Also notice that $Y = Y(x,t)$ and $y = y(x,s)$.

Using the initial condition $Y(x,0)=30\cos 5x$, we get

$3\dfrac{d^2y}{dx^2}-s\,y=-30\cos 5x.$ Using D – operator, and dividing by 3, we get

$\left(D^2-s/3\right)y=-10\cos 5x.$

The auxiliary equation is $m^2-s/3=0 \;\Rightarrow m=\pm\sqrt{s/3}.$

Thus, complementary function is $\qquad y_c=C_1e^{x\sqrt{s/3}}+C_2e^{-x\sqrt{s/3}}$

For particular integral, we have

$$y_p=\frac{-10}{D^2-(s/3)}\cos 5x=\frac{-10}{\left(-5^2\right)-(s/3)}\cos 5x=\frac{30\cos 5x}{75+s}$$

$$\text{Now, } y=y_c+y_p=C_1e^{x\sqrt{s/3}}+C_2e^{-x\sqrt{s/3}}+\frac{30\cos 5x}{75+s} \qquad (1)$$

Let us use the conditions now.

Given $\dfrac{\partial Y}{\partial x}=0$ at $x=0 \Rightarrow L\left(\dfrac{\partial Y}{\partial x}\right)=0$ at $x=0$. Thus, $\dfrac{\partial L(Y)}{\partial x}=\dfrac{dy}{dx}=0$ when $x=0$.

From (1), $\dfrac{dy}{dx}=\sqrt{\dfrac{s}{3}}\left(C_1e^{x\sqrt{s/3}}-C_2e^{-x\sqrt{s/3}}\right)$. Putting $x=0$ and $\dfrac{dy}{dx}=0$, we get

$$C_1-C_2=0 \quad\text{or}\quad C_1=C_2$$

$$\text{Thus,}\qquad y=C_1\left(e^{x\sqrt{s/3}}+e^{-x\sqrt{s/3}}\right)+\frac{30\cos 5x}{75+s} \qquad (2)$$

Now using the second condition $y\left(\dfrac{\pi}{2},s\right)=0$ in equation (2), we get

$$C_1\left(e^{\frac{\pi}{2}\sqrt{\frac{s}{3}}}+e^{-\frac{\pi}{2}\sqrt{\frac{s}{3}}}\right)+\frac{30\cos(5\pi/2)}{75+s}=0. \qquad \left[\text{NOTE}:\cos(5\pi/2)=0\right]$$

$$\text{Or,}\quad C_1\left(e^{\frac{\pi}{2}\sqrt{\frac{s}{3}}}+e^{-\frac{\pi}{2}\sqrt{\frac{s}{3}}}\right)=2C_1\cosh\left(\frac{\pi}{2}\sqrt{\frac{s}{3}}\right)=0 \;\Rightarrow\; C_1=0.$$

Thus, equation (1) becomes: $y=\dfrac{30\cos 5x}{75+s}$. Now, taking inverse Laplace transform

$$L^{-1}(y)=Y=L^{-1}\left(\frac{30\cos 5x}{75+s}\right)=30\cos 5x\;L^{-1}\left(\frac{1}{s+75}\right)$$

$$\text{Thus,}\qquad Y=30e^{-75t}\cos 5x$$

WORKSHEET 10

Solve the following differential equations by using Laplace transforms

1. $Y''+Y=t \quad y(0)=1,\, y'(0)=-2$
2. $Y''-3Y'+2Y=4e^{2t}t \quad Y(0)=-3,\, Y'(0)=5$
3. $Y''+2Y'+5Y=e^{-t}\sin t \quad Y(0)=0,\, Y'(0)=1$
4. $Y'''-3Y''+3Y'-Y=t^2e^t \quad Y(0)=1,\, Y'(0)=0,\, Y''(0)=-2$

5. $Y'' + Y = 8\cos t \quad y(0) = 1, y'(0) = -1$

6. $Y'' + 4Y = \cos 2t \quad y(\pi) = 0 = y'(\pi)$

7. $Y'' + 2Y' + 2Y = 4e^{-t} t \sin t \quad Y(0) = 2, Y'(0) = 3$

8. $tY'' + Y' + 4tY = 0 \quad Y(0) = 3, Y'(0) = 0$

9. $Y'' - tY' + Y = 1 \quad Y(0) = 1, Y'(0) = 2$

10. $X' - Y' - 2X - Y = e^t, 2X' + Y' - 3X - 3Y = 6e^{2t}, \quad X(0) = 3, Y(0) = 0$

11. $X' + Y' = T, X'' - Y = e^{-t}, \quad X(0) = 3, X'(0) = -2, Y(0) = 0$

12. Solve the differential equation: $LQ'' + RQ' + \frac{1}{C}Q = E(t)$ where L = 1 H, R = 2 ohms, C = 1 F and E = 5V with Q(0) = 0 and I(0) = 0.

13. Find the charge on the capacitor and the current in the given LRC series circuit, where L = 5/3 H, R = 10 ohms, C = 1/30 F and E = 300 V and Q(0) = 0, I(0) = 0.

14. A circuit consists of an inductance of 0.05 Henry, a resistance of 20 ohms, a condenser of capacitance 100 microfarads, and an emf of E = 100V. Find I and Q, given the initial conditions Q = 0, I = 0 at t = 0.

15. A particle of mass 2 grams on the x-axis is attracted towards origin with a force equal to 8X. If it is initially at rest at X = 10, find its position at any time.

16. Solve the following PDE using Laplace transform method.

(i) $\frac{\partial Y}{\partial t} = \frac{\partial^2 Y}{\partial x^2}, \; Y(x,t) = \sin \pi x$

(ii) $\frac{\partial Y}{\partial t} = 4\frac{\partial^2 Y}{\partial x^2}, \; Y(x,0) = 4x - 0.5x^2$

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