SUKKUR IBA UNIVERSITY

Lecture #07

Student's Name: Newton's Law of Cooling Date.

Class: CMS ID. Course:

Instructor: No.

A hot body cools at a rate proportional to the difference temperature of the body and temperature of its surroundings.

Ex: A cup of coffee was initially boiling at 120°C. It was placed in air at 20°C after 10 minutes the temp: was 80°C, what will be the temp: of coffee after an other 10 minutes.

$$\frac{dT}{dt} \propto (T - T_m)$$

$$\frac{dT}{dt} = k(T - T_m)$$

$$\frac{dT}{dt} = k(T - 20)$$

$$\int \frac{dT}{T-20} = \int k\,dt$$

$$\ln(T-20) = kt + c$$

$$T - 20 = e^{kt+c}$$

$$T(t) = 20 + c_1 e^{kt}$$

$t = 0 \quad T = 120$

$$T(0) = 20 + c_1 e^{k(0)}$$

$$120 - 20 = c_1$$

$$\boxed{c_1 = 100}$$

$$T(10) = 20 + 100e^{10k}$$

$$80 - 20 = 100e^{10k}$$

$$\frac{\cancel{60}^{3}}{\cancel{100}_{5}} = e^{10k}$$

$$\ln(3/5) = 10k$$

$$k = -0.0511$$

$$T(t) = 20 + 100e^{k(t)}$$

$$\boxed{T(t) = 20 + 100e^{-0.0511t}}$$

$$T(20) = 20 + 100e^{-0.0511(20)}$$

$$= 20 +$$

$$T(20) \approx 56°C$$

Ex: A dead body is found at 12 PM in a room that is maintained at 72°F. If the body is 82F when it is found, and has cooled to 80°F at 1 PM estimate the time of death. (98.6°F → normal body temp.)

$T(0) = 82$ $\qquad T_{room} = 72$

$T(1) = 80$

$$\frac{dT}{dt} \propto (T - T_{room})$$

due to decrease

$$\frac{dT}{dt} = -k\,(T - T_{room})$$

$$\int \frac{dT}{T-72} = -k\int dt$$

$$\ln(T-72) = -kt + c$$

$$T(t) = 72 + e^{-kt+c}$$

$$T(t) = 72 + c_1 e^{-kt}$$

① A body of mass 3 slugs is dropped from a height of 500 feet with zero velocity. Assuming no air resistance, find:

(a) an expression for the velocity of body at any time t

(b) an expression for the position of body at any time t.

(c) the time required for the body to hit the ground?

(a) no air resistance + velocity of body: at any time t

$$\boxed{\frac{dv}{dt} = g} \Rightarrow dv = g\,dt \Rightarrow \int dv = g\int dt$$

$$v = gt + c \quad \text{put} \quad \boxed{\because v=0,\ t=0}$$

$$\boxed{v = gt} \quad \text{General Form} \quad \text{put} \quad \boxed{\because \text{Assume: } g = 32\ ft/sec^2}$$

Hence, expression for velocity of body at any time t:

$$\boxed{v = 32 \cdot t}$$

(b) Position of the body at any time t

$$\frac{dx}{dt} = v \Rightarrow \frac{dx}{dt} = 32t$$

$$= \int dx = 32\int t\,dt$$

$$x = \overset{16}{\cancel{32}}\frac{t^2}{\cancel{2}} + c$$

$$x = 16t^2 + c$$

$$\boxed{x = 16t^2} \quad \text{height} = 500$$

(c)

$$500 = 16t^2$$

$$t = \sqrt{\frac{500}{16}}$$

$$\boxed{t = 5.6\ sec}$$

(c) The time required for body to hit the ground.

We have

$x = 500$ and $t = ?$

$500 = 16t^2 \Rightarrow t = \sqrt{\frac{500}{16}}$

$$\boxed{t = 5.6 \text{ sec}}$$

② A body of mass 2 slugs is dropped from a height of 450 feet with an initial velocity of 10 ft/sec. Assuming no air resistance, find

(a) an expression for the velocity of the body at any time t, and

(b) the time required for the body to hit the ground.

(a) Velocity of body at time t

$$\frac{dv}{dt} = g$$

$$\frac{dv}{dt} = g$$

$$dv = g\,dt$$

integrate

$$\int dv = g \int dt$$

$$\boxed{v = gt + c}$$

given: $t = 0$, $v = 0$

then

$V_{initial} = gt + c$

$10 = g(0) + c$

$$\boxed{c = 10}$$

$$\boxed{V = 32t + 10}$$

$\because$ Assuming $g = 32 \text{ ft/sec}^2$

(b) Since: $\frac{dx}{dt} = v$, then equations becomes:

$$\frac{dx}{dt} = 32t + 10$$

$$dx = (32t + 10)\,dt$$

Integrate both sides,

$$\int dx = \int (32t + 10t)\,dt$$

$$\boxed{x = 16t^2 + 10t + c}$$

But at $t=0$, $x=0$

$$0 = 16(0)^2 + 10(0) + c$$

$$c = 0$$

Hence:

$$\boxed{x = 16t^2 + 10t}$$

We require t when $x = 450$

so, we have:

$$450 = 16t^2 + 10t$$

using quadratic formula,

$$t = \frac{-5 \pm \sqrt{25 + 7200}}{16}$$

$t = -5.625$ Neglect b/c time (-ve)

$$\boxed{t = 5.6 \text{ sec}}$$

③ A ball is propelled straight up with an intial velocity of 250 ft/sec in a vaccum with no air resistance. How high will it go?

$$\frac{dv}{dt} = g$$

$$dv = g\,dt$$

$$\int dv = g \int dt$$

$$v = gt + c$$

At $t=0$, $v = -250$

Therefore:

$$-250 = 32(0) + c$$

$$\boxed{c = -250}$$

$$g = 32 \text{ ft/sec}^2$$

so, $v = 32t - 250$ since $v = \frac{dx}{dt}$

$$\frac{dx}{dt} = 32t - 250$$

$$\int dx = \int (32t - 250)\,dt$$

$$\boxed{x = 16t^2 - 250t + c}$$

At $t=0$, $x=0$ then $c=0$

$$x = 16t^2 - 250t$$

At maximum height, $v=0$ therefore:

$0 = 32t - 250 \Rightarrow \boxed{t = 7.8125}$

substitute values we get:

$$x = -976.5625$$

or $\boxed{x = 976.5625}$

③

④ A body of mass 10 slugs is dropped from a height of 100 feet with no initial velocity. The body encounters an air resistance proportional to its velocity. If the limiting velocity is known to be 320 ft/sec, find,

(a) an expression for velocity of the body at any time t

(b) an expression for the position of the body at any time t

(c) the time required for the body to attain a velocity of 160 ft/sec.

(a) The limiting velocity is defined to be:

$$v_l = mg/k \Rightarrow 320 \text{ or } 10 \times 32 = 320k \Rightarrow k = 1$$

Equation of motion of body is

$$\frac{dv}{dt} + \frac{k}{m}v = g \Rightarrow \frac{dv}{dt} + \frac{1}{10}v = 32$$

$$\boxed{\frac{dv}{dt} + 0.1v = 32} \longrightarrow \text{This is Linear D.E hence I.F} = e^{\int 0.1 dx}$$

$$\text{I.F} = e^{\int 0.1\, dv} \Rightarrow \boxed{e^{0.1t}}$$

$$e^{0.1t}\frac{dv}{dt} + 0.1 v e^{0.1t} = 32e^{0.1t} \Rightarrow e^{0.1t}v = 32\int e^{0.1t}dt + c$$

At $t = 0$, we are given that $v = 0$. Substituting these values we get $\boxed{c = 320}$, hence velocity at any time is

$$\boxed{v = 320e^{-0.1t} + 320}$$

(b) Since $v = dx/dt$, therefore

$$\frac{dx}{dt} = -320e^{-0.1t} + 320$$

$$\int dx = \int(-320e^{-0.1t} + 320)\,dt$$

$$\boxed{x = 3200e^{-0.1t} + 320t + c}$$

when $t = 0$, $x = 0$ so, $c = -3200$

Thus

$$\boxed{x = 3200e^{-0.1t} + 320t - 3200}$$

(c) Since, $v = 160 \text{ ft/sec}^2$

$$160 = -320e^{-0.1t} + 320$$

$$-160 = -320e^{-0.1t}$$

$$e^{-0.1t} = 0.5$$

$$0.1t = \ln(0.5)$$

$$\boxed{t \approx 6.93 \text{ sec}}$$

④

Ex:. A body was heated to 100°c and then placed in a freezer at 0°c. after 30 mint: its temp: was 80°c. How much additional time is required for it to cool to 50°c.

$$\frac{dT}{dt} \propto (T-T_s)$$

$$\frac{dT}{dt} = k(T-T_s)$$

$$\int \frac{dT}{T-T_s} = \int k\,dt$$

$$\ln(T-T_s) = kt+c$$

$$T-T_s = e^{kt+c}$$

$$T(t) = T_s + c_1 e^{kt}$$

$$T(t) = 0 + c_1 e^{kt}$$

$$T(t) = 100 e^{kt}$$

$$T(0) = c_1 e^{0}$$

$$\boxed{100 = c_1}$$

$$T(30) = 100 e^{k(30)}$$

$$\frac{\cancel{80}^{\,4}}{\cancel{100}_{\,5}} = e^{30k}$$

$$\ln(4/5) = 30k$$

$$k = \frac{\ln(4/5)}{30} \approx -0.0074$$

$$50 = 100 e^{-0.0074t}$$

$$\frac{\cancel{50}}{\cancel{100}_{\,2}} = e^{-0.0074t}$$

$$\ln(1/2) = -0.0074t$$

$$-\frac{\ln(1/2)}{0.0074} = t$$

$$\boxed{94 = t}$$

Thus $(94-30) = 64$ mint: approximately for the body to cool to 50°c.