Graphs

Computer Science E-22
Harvard Extension School

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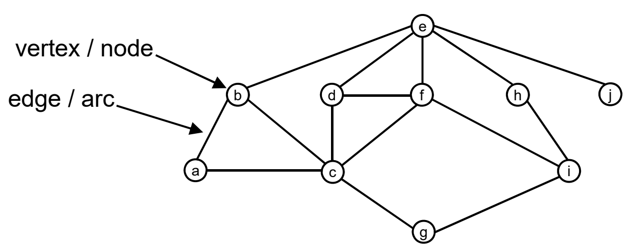
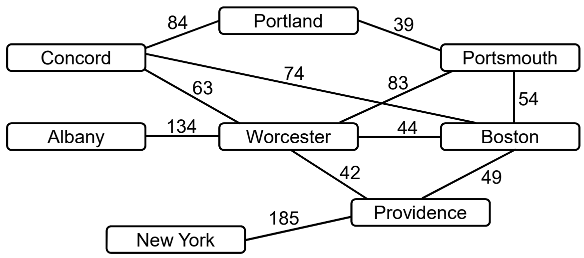
What is a Graph?



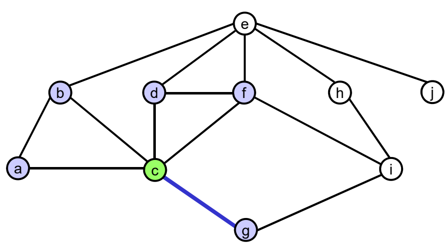
- A graph consists of:
  - a set of *vertices* (also known as *nodes*)
  - a set of *edges* (also known as *arcs*), each of which connects a pair of vertices

Example: A Highway Graph



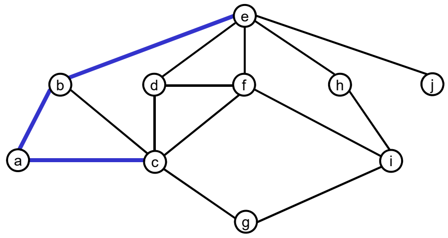
- Vertices represent cities.
- Edges represent highways.
- This is a *weighted* graph, with a *cost* associated with each edge.
  - in this example, the costs denote mileage
- We'll use graph algorithms to answer questions like "What is the shortest route from Portland to Providence?"

Relationships Among Vertices

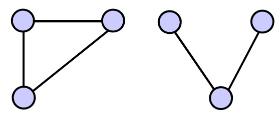


- Two vertices are *adjacent* if they are connected by a single edge.
  - *ex:* c and g are adjacent, but c and i are not
- The collection of vertices that are adjacent to a vertex v are referred to as v's *neighbors.*
  - *ex:* c's neighbors are a, b, d, f, and g

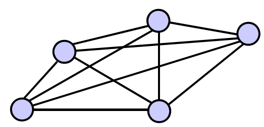
Paths in a Graph



- A *path* is a sequence of edges that connects two vertices.
- A graph is *connected* if there is a path between any two vertices.
  - *ex:* the six vertices at right are part of a graph that is *not* connected

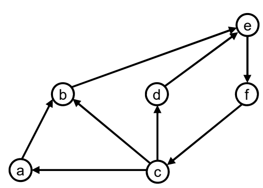


- A graph is *complete* if there is an edge between every pair of vertices.
  - *ex:* the graph at right *is* complete



Directed Graphs

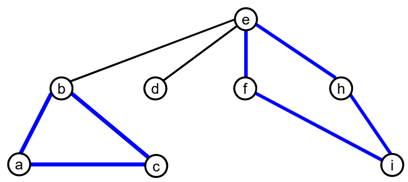
- A *directed* graph has a direction associated with each edge, which is depicted using an arrow:



- Edges in a directed graph are often represented as ordered pairs of the form (start vertex, end vertex).
  - *ex:* (a, b) is an edge in the graph above, but (b, a) is not.
- In a path in a directed graph, the end vertex of edge i must be the same as the start vertex of edge i + 1.
  - *ex:* { (a, **b**), (**b**, **e**), (**e**, f) } is a valid path.
    { (a, **b**), (**c**, **b**), (**c**, a) } is not.

Cycles in a Graph

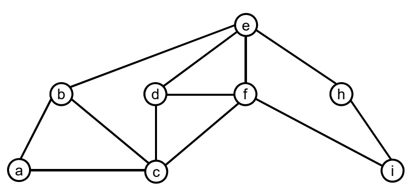
- A *cycle* is a path that:
  - leaves a given vertex using one edge
  - returns to that same vertex using a different edge
- Examples: the highlighted paths below



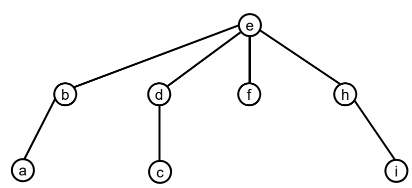
- An *acyclic* graph has no cycles.

Trees vs. Graphs

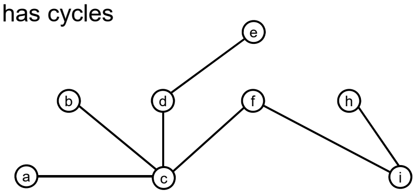
- A tree is a special type of graph.
  - connected, undirected, and acyclic
  - we usually single out one of the vertices to be the root, but graph theory does *not* require this



a graph that is *not* a tree, because it has cycles



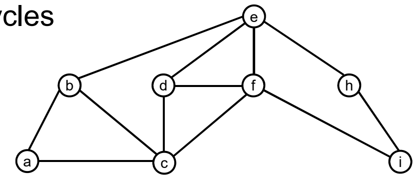
a tree using the same nodes



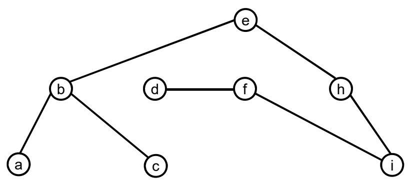
another tree using the same nodes

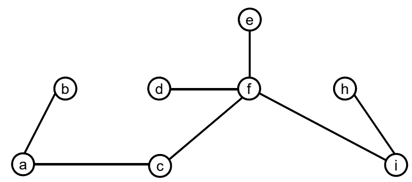
Spanning Trees

- A spanning tree is a subset of a connected graph that contains:
  - all of the vertices
  - a subset of the edges that form a tree
- Recall this graph with cycles from the previous slide:



- The trees on that slide were spanning trees for this graph. Here are two others:

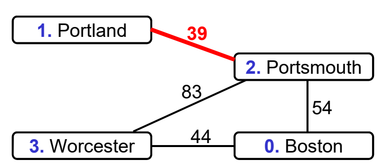




Representing a Graph: Option 1

- Use an *adjacency matrix* – a two-dimensional array in which element [r][c] = the cost of going from vertex r to vertex c
- Example:

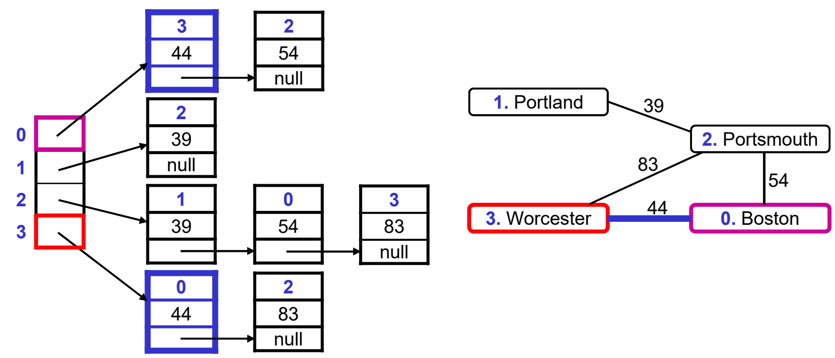
	0	1	2	3
0			54	44
1			39	
2	54	39		83
3	44		83	



- Use a special value to indicate there's no edge from r to c
  - shown as a shaded cell above
  - can't use 0, because an edge may have an actual cost of 0
- This representation:
  - wastes memory if a graph is *sparse* (few edges per vertex)
  - is memory-efficient if a graph is *dense* (many edges per vertex)

Representing a Graph: Option 2

- Use one *adjacency list* for each vertex.
  - a linked list with info on the edges coming from that vertex



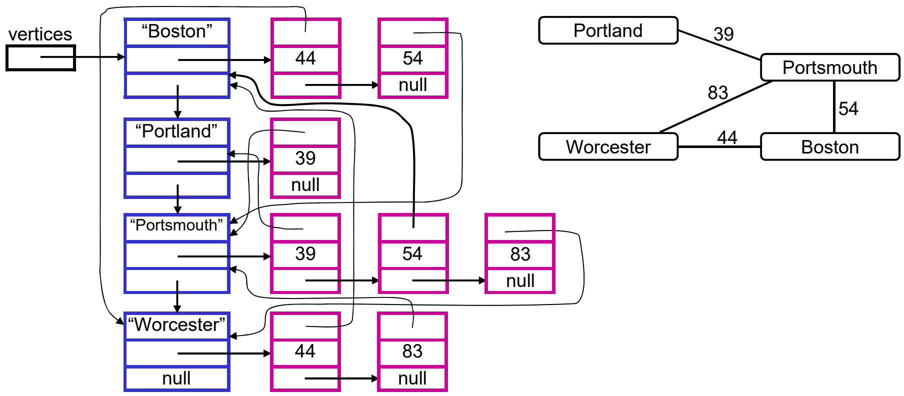
- This representation uses less memory if a graph is sparse.
- It uses more memory if a graph is dense.
  - because of the references linking the nodes

Graph Class

```
public class Graph {
    private class Vertex {
        private String id;
        private Edge edges;               // adjacency list
        private Vertex next;
        private boolean encountered;
        private boolean done;
        private Vertex parent;
        private double cost;
        …
    }
    private class Edge {
        private Vertex start;
        private Vertex end;
        private double cost;
        private Edge next;
        …
    }
    private Vertex vertices;
    …
}
```

The highlighted fields are shown in the diagram on the previous page.

Our Graph Representation



- Each `Vertex` object (shown in blue) stores info. about a vertex.
  - including an adjacency list of `Edge` objects (the purple ones)
- A `Graph` object has a single field called `vertices`
  - a reference to a linked list of `Vertex` objects
  - a linked list of linked lists!

Traversing a Graph

- Traversing a graph involves starting at some vertex and visiting all vertices that can be reached from that vertex.
  - visiting a vertex = processing its data in some way
  - if the graph is connected, all of its vertices will be visited
- We will consider two types of traversals:
  - **depth-first**: proceed as far as possible along a given path before backing up
  - **breadth-first**: visit a vertex
visit all of its neighbors
visit all unvisited vertices 2 edges away
visit all unvisited vertices 3 edges away, etc.
- Applications:
  - determining the vertices that can be reached from some vertex
  - web crawler (vertices = pages, edges = links)

Depth-First Traversal

- Visit a vertex, then make recursive calls on all of its yet-to-be-visited neighbors:

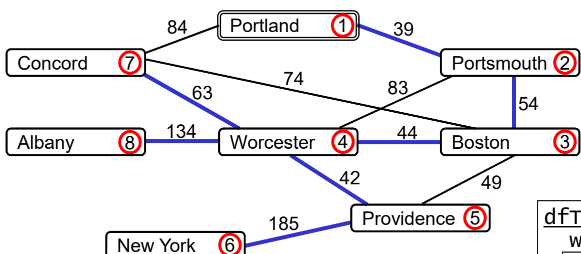
```
dfTrav(v, parent)
    visit v and mark it as visited
    v.parent = parent
    for each vertex w in v's neighbors
        if (w has not been visited)
            dfTrav(w, v)
```

- Java method:

```
private static void dfTrav(Vertex v, Vertex parent) {
    System.out.println(v.id);    // visit v
    v.done = true;
    v.parent = parent;

    Edge e = v.edges;
    while (e != null) {
        Vertex w = e.end;
        if (!w.done)
            dfTrav(w, v);
        e = e.next;
    }
}
```

Example: Depth-First Traversal from Portland

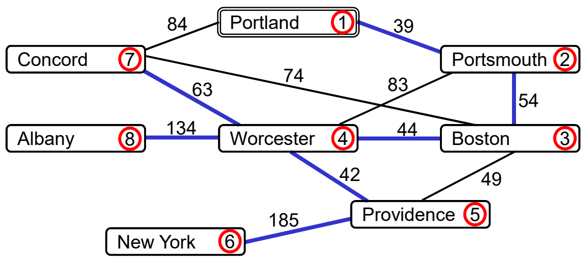


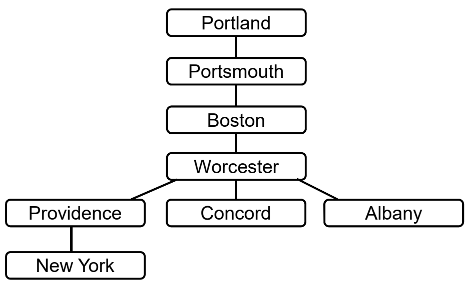
For the examples, we'll assume that the edges in each vertex's adjacency list are sorted by increasing edge cost.

```
void dfTrav(Vertex v, Vertex parent) {
    System.out.println(v.id);
    v.done = true;
    v.parent = parent;
    Edge e = v.edges;
    while (e != null) {
        Vertex w = e.end;
        if (!w.done)
            dfTrav(w, v);
        e = e.next;
    }
}
```

```
dfTrav(Ptl, null)
  w = Pts
  dfTrav(Pts, Ptl)
    w = Ptl, Bos
    dfTrav(Bos, Pts)
      w = Wor
      dfTrav(Wor, Bos)
        w = Pro
        dfTrav(Pro, Wor)
          w = Wor, Bos, NY
          dfTrav(NY, Pro)
            w = Pro
            return
          no more neighbors
          return
        w = Bos, Con
        dfTrav(Con, Wor)
          …
```

Depth-First Spanning Tree



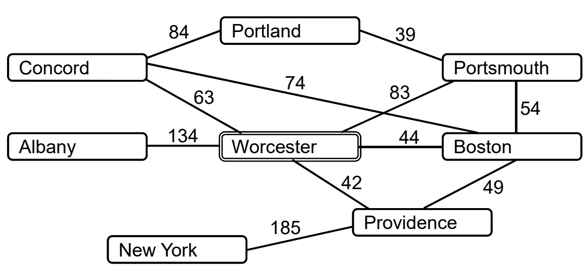


The edges obtained by following the `parent` references form a spanning tree with the origin of the traversal as its root.

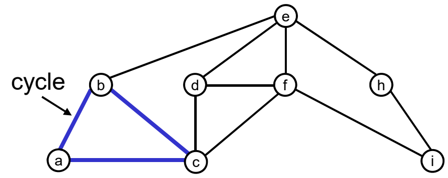
From any city, we can get to the origin by following the roads in the spanning tree.

Another Example: Depth-First Traversal from Worcester

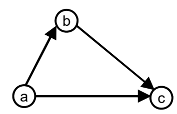
- In what order will the cities be visited?
- Which edges will be in the resulting spanning tree?



Checking for Cycles in an Undirected Graph



- To discover a cycle in an undirected graph, we can:
  - perform a depth-first traversal, marking the vertices as visited
  - when considering neighbors of a visited vertex, if we discover one already marked as visited, there must be a cycle
- If no cycles found during the traversal, the graph is acyclic.
- This doesn't work for directed graphs:
  - c is a neighbor of both a and b
  - there is no cycle



Breadth-First Traversal

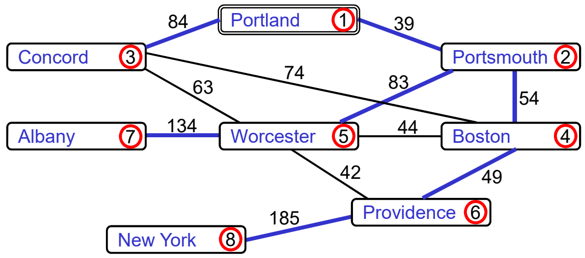
- Use a queue to store vertices we've seen but not yet visited:

```
private static void bfTrav(Vertex origin) {
    origin.encountered = true;
    origin.parent = null;
    Queue<Vertex> q = new LLQueue<Vertex>();
    q.insert(origin);

    while (!q.isEmpty()) {
        Vertex v = q.remove();
        System.out.println(v.id);         // Visit v.

        // Add v's unencountered neighbors to the queue.
        Edge e = v.edges;
        while (e != null) {
            Vertex w = e.end;
            if (!w.encountered) {
                w.encountered = true;
                w.parent = v;
                q.insert(w);
            }
            e = e.next;
        }
    }
}
```

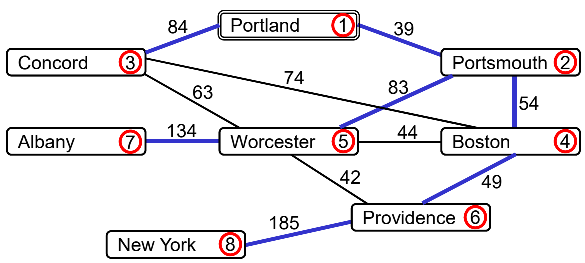
Example: Breadth-First Traversal from Portland

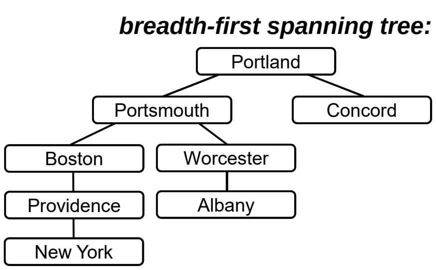


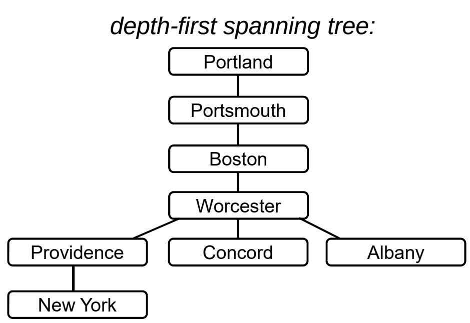
Evolution of the queue:

<u>remove</u>	<u>insert</u>	<u>queue contents</u>
	Portland	Portland
Portland	Portsmouth, Concord	Portsmouth, Concord
Portsmouth	Boston, Worcester	Concord, Boston, Worcester
Concord	*none*	Boston, Worcester
Boston	Providence	Worcester, Providence
Worcester	Albany	Providence, Albany
Providence	New York	Albany, New York
Albany	*none*	New York
New York	*none*	*empty*

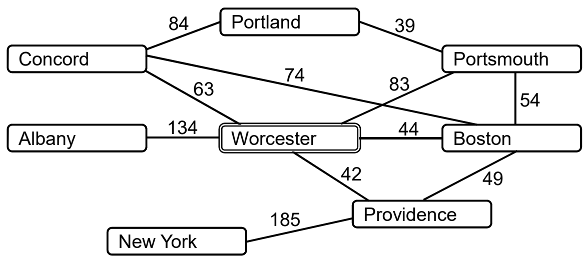
Breadth-First Spanning Tree







Another Example: Breadth-First Traversal from Worcester



Evolution of the queue:

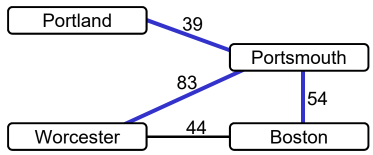
<u>remove</u> <u>insert</u> <u>queue contents</u>

Time Complexity of Graph Traversals

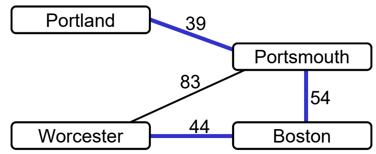
- let V = number of vertices in the graph
  E = number of edges
- If we use an adjacency matrix, a traversal requires $O(V^2)$ steps.
  - why?
- If we use adjacency lists, a traversal requires $O(V + E)$ steps.
  - visit each vertex once
  - traverse each vertex's adjacency list at most once
    - the total length of the adjacency lists is at most $2E = O(E)$
  - for a sparse graph, $O(V + E)$ is better than $O(V^2)$
  - for a dense graph, $E = O(V^2)$, so both representations are $O(V^2)$
- In the remaining notes, we'll assume an adjacency-list implementation.

Minimum Spanning Tree

- A *minimum spanning tree* (MST) has the smallest total cost among all possible spanning trees.
  - *example:*



one possible spanning tree
(total cost = 39 + 83 + 54 = 176)



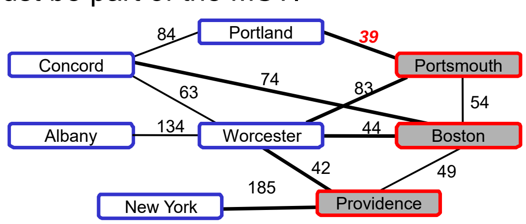
the minimal-cost spanning tree
(total cost = 39 + 54 + 44 = 137)

- If all edges have unique costs, there is only one MST.
  If some edges have the same cost, there may be more than one.

- Example applications:
  - determining the shortest highway system for a set of cities
  - calculating the smallest length of cable needed to connect a network of computers

Building a Minimum Spanning Tree

- *Claim:* If you divide the vertices into two disjoint subsets A and B, the lowest-cost edge (v_a, v_b) joining a vertex in A to a vertex in B must be part of the MST.



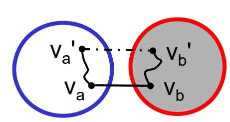
example:
subset A = unshaded
subset B = shaded

The 6 bold edges each join a vertex in A to a vertex in B.

The one with the lowest cost (Portland to Portsmouth) must be in the MST.

Proof by contradiction:

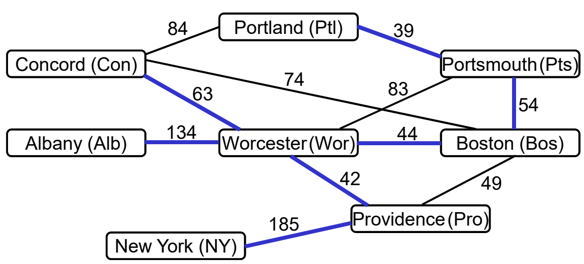
1. Assume we can create an MST (call it T) that doesn't include (v_a, v_b).
2. T must include a path from v_a to v_b, so it must include one of the other edges (v_a', v_b') that span A and B, such that (v_a', v_b') is part of the path from v_a to v_b.
3. Adding (v_a, v_b) to T introduces a cycle.
4. Removing (v_a', v_b') gives a spanning tree with a lower total cost, which contradicts the original assumption.



Prim's MST Algorithm

- Begin with the following subsets:
  - A = any one of the vertices
  - B = all of the other vertices
- Repeatedly do the following:
  - select the lowest-cost edge (v_a, v_b) connecting a vertex in A to a vertex in B
  - add (v_a, v_b) to the spanning tree
  - move vertex v_b from set B to set A
- Continue until set A contains all of the vertices.

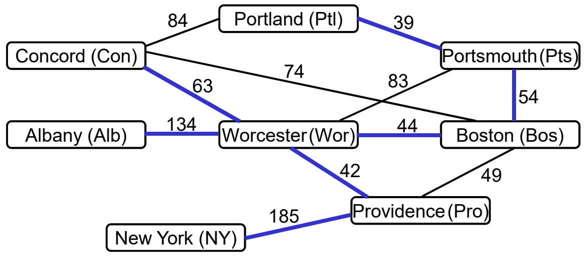
Example: Prim's Starting from Concord



- Tracing the algorithm:

edge added	set A	set B
	{Con}	{Alb, Bos, NY, Ptl, Pts, Pro, Wor}
(Con, Wor)	{Con, Wor}	{Alb, Bos, NY, Ptl, Pts, Pro}
(Wor, Pro)	{Con, Wor, Pro}	{Alb, Bos, NY, Ptl, Pts}
(Wor, Bos)	{Con, Wor, Pro, Bos}	{Alb, NY, Ptl, Pts}
(Bos, Pts)	{Con, Wor, Pro, Bos, Pts}	{Alb, NY, Ptl}
(Pts, Ptl)	{Con, Wor, Pro, Bos, Pts, Ptl}	{Alb, NY}
(Wor, Alb)	{Con, Wor, Pro, Bos, Pts, Ptl, Alb}	{NY}
(Pro, NY)	{Con, Wor, Pro, Bos, Pts, Ptl, Alb, NY}	{}

MST May Not Give Shortest Paths



- The MST is the spanning tree with the minimal *total* edge cost.
- It does <u>not</u> necessarily include the minimal cost path between a pair of vertices.
- Example: shortest path from Boston to Providence is along the single edge connecting them
  - that edge is not in the MST

Implementing Prim's Algorithm

- We use the `done` field to keep track of the sets.
  - if `v.done == true`, v is in set A
  - if `v.done == false`, v is in set B
- We repeatedly scan through the lists of vertices and edges to find the next edge to add.

  ➔ *O*(EV)
- We can do better!
  - use a heap-based priority queue to store the vertices in set B
  - priority of a vertex x = –1 * cost of the lowest-cost edge connecting x to a vertex in set A
    - why multiply by –1?
  - somewhat tricky: need to update the priorities over time

  ➔ *O*(E log V)

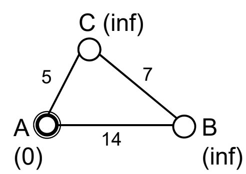
The Shortest-Path Problem

- It's often useful to know the shortest path from one vertex to another – i.e., the one with the minimal total cost
  - example application: routing traffic in the Internet
- For an *unweighted* graph, we can simply do the following:
  - start a breadth-first traversal from the origin, v
  - stop the traversal when you reach the other vertex, w
  - the path from v to w in the resulting (possibly partial) spanning tree is a shortest path
- A breadth-first traversal works for an unweighted graph because:
  - the shortest path is simply one with the fewest edges
  - a breadth-first traversal visits cities in order according to the number of edges they are from the origin.
- Why might this approach fail to work for a *weighted* graph?

Dijkstra's Algorithm

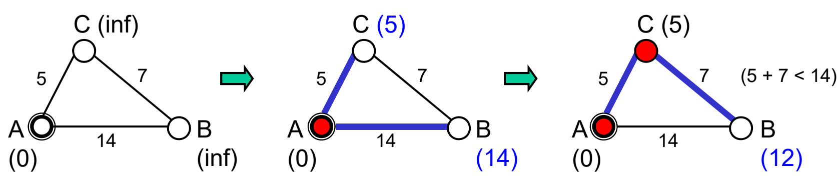
- One algorithm for solving the shortest-path problem for weighted graphs was developed by E.W. Dijkstra.
- It allows us to find the shortest path from a vertex v (the origin) to *all other vertices* that can be reached from v.
- Basic idea:
  - maintain estimates of the shortest paths from the origin to every vertex (along with their costs)
  - gradually refine these estimates as we traverse the graph
- Initial estimates:

	path	cost
the origin itself:	stay put!	0
all other vertices:	unknown	infinity



Dijkstra's Algorithm (cont.)

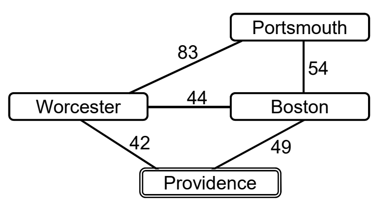
- We say that a vertex w is *finalized* if we have found the shortest path from v to w.
- We repeatedly do the following:
  - find the unfinalized vertex w with the lowest cost estimate
  - mark w as finalized (shown as a filled circle below)
  - examine each unfinalized neighbor x of w to see if there is a shorter path to x that passes through w
    - if there is, update the shortest-path estimate for x
- Example:



Another Example: Shortest Paths from Providence

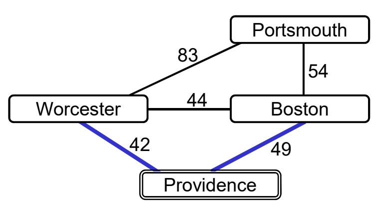
- Initial estimates:

Boston	infinity
Worcester	infinity
Portsmouth	infinity
Providence	***0***



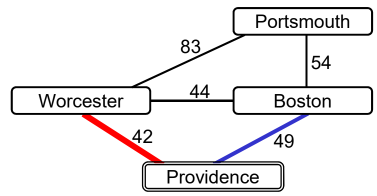
- Providence has the smallest unfinalized estimate, so we finalize it.
- We update our estimates for its neighbors:

Boston	*49* (< infinity)
Worcester	*42* (< infinity)
Portsmouth	infinity
Providence	0



Shortest Paths from Providence (cont.)

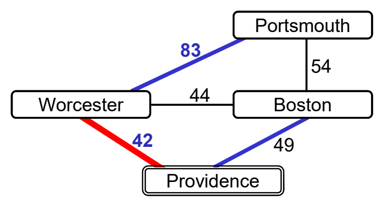
Boston	49
Worcester	***42***
Portsmouth	infinity
Providence	0



- Worcester has the smallest unfinalized estimate, so we finalize it.
  - any other route from Prov. to Worc. would need to go via Boston, and since (Prov → Worc) < (Prov → Bos), we can't do better.

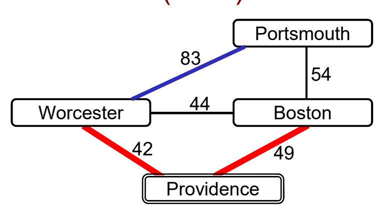
- We update our estimates for Worcester's unfinalized neighbors:

Boston	49 (no change)
Worcester	42
Portsmouth	*125* (42 + 83 < infinity)
Providence	0



Shortest Paths from Providence (cont.)

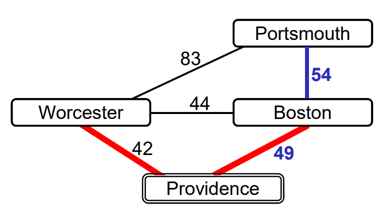
Boston	***49***
Worcester	42
Portsmouth	125
Providence	0



- Boston has the smallest unfinalized estimate, so we finalize it.
  - we'll see later why we can safely do this!

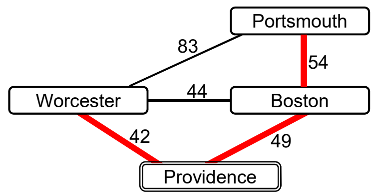
- We update our estimates for Boston's unfinalized neighbors:

Boston	49
Worcester	42
Portsmouth	*103* (49 + 54 < 125)
Providence	0



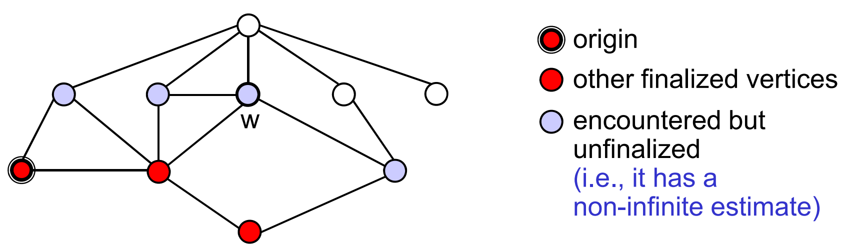
Shortest Paths from Providence (cont.)

Boston	49
Worcester	42
Portsmouth	***103***
Providence	0



- Only Portsmouth is left, so we finalize it.

Finalizing a Vertex



- Let w be the unfinalized vertex with the smallest cost estimate. Why can we finalize w, before seeing the rest of the graph?
- We know that w's current estimate is for the shortest path to w that passes through only *finalized* vertices.
- Any shorter path to w would have to pass through one of the other encountered-but-unfinalized vertices, but they are all further away from the origin than w is!
  - their cost estimates may decrease in subsequent stages, but they can't drop below w's current estimate!

Pseudocode for Dijkstra's Algorithm

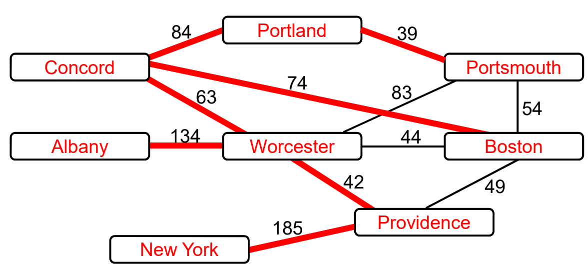
```
dijkstra(origin)
   origin.cost = 0
   for each other vertex v
        v.cost = infinity;

   while there are still unfinalized vertices with cost < infinity
        find the unfinalized vertex w with the minimal cost
        mark w as finalized

        for each unfinalized vertex x adjacent to w
            cost_via_w = w.cost + edge_cost(w, x)
            if (cost_via_w < x.cost)
                x.cost = cost_via_w
                x.parent = w
```

- At the conclusion of the algorithm, for each vertex v:
  - v.cost is the cost of the shortest path from the origin to v
  - if v.cost is infinity, there is no path from the origin to v
  - starting at v and following the parent references yields the shortest path

Example: Shortest Paths from Concord

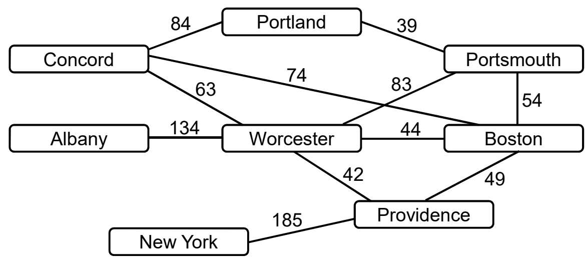


Evolution of the cost estimates (costs in bold have been finalized):

Albany	inf	inf	197	197	197	197	***197***	
Boston	inf	74	***74***					
Concord	***0***							
New York	inf	inf	inf	inf	inf	290	290	***290***
Portland	inf	84	84	***84***				
Portsmouth	inf	inf	146	128	123	***123***		
Providence	inf	inf	105	105	***105***			
Worcester	inf	***63***						

Note that the Portsmouth estimate was improved three times!

Another Example: Shortest Paths from Worcester



Evolution of the cost estimates (costs in bold have been finalized):

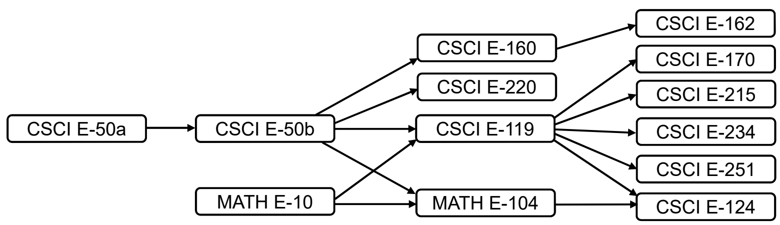
Albany								
Boston								
Concord								
New York								
Portland								
Portsmouth								
Providence								
Worcester								

Implementing Dijkstra's Algorithm

- Similar to the implementation of Prim's algorithm.
- Use a heap-based priority queue to store the unfinalized vertices.
  - priority = ?
- Need to update a vertex's priority whenever we update its shortest-path estimate.
- Time complexity = $O(E \log V)$

Topological Sort

- Used to order the vertices in a directed acyclic graph (a DAG).
- Topological order: an ordering of the vertices such that, if there is directed edge from a to b, a comes before b.
- Example application: ordering courses according to prerequisites



  - a directed edge from a to b indicates that a is a prereq of b
- There may be more than one topological ordering.

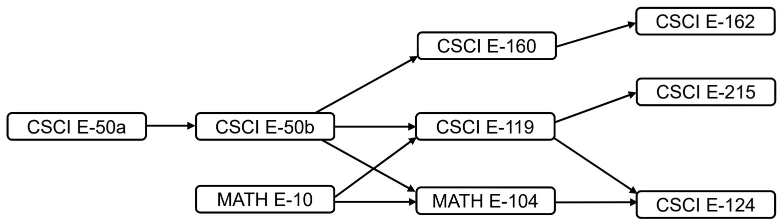
Topological Sort Algorithm

- A *successor* of a vertex v in a directed graph = a vertex w such that (v, w) is an edge in the graph $(v \bullet\!\!\longrightarrow\!\!\bullet w)$
- Basic idea: find vertices with no successors and work backward.
  - there must be at least one such vertex. why?
- Pseudocode for one possible approach:

```
topolSort
    S = a stack to hold the vertices as they are visited
    while there are still unvisited vertices
        find a vertex v with no unvisited successors
        mark v as visited
        S.push(v)
    return S
```

- Popping the vertices off the resulting stack gives one possible topological ordering.

Topological Sort Example

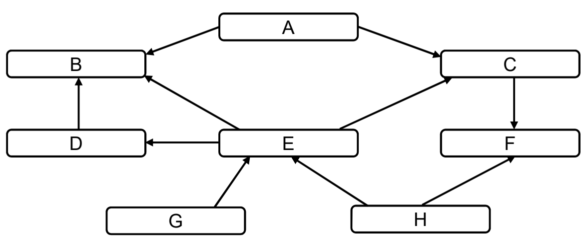


Evolution of the stack:

push	stack contents (top to bottom)
E-124	E-124
E-162	E-162, E-124
E-215	E-215, E-162, E-124
E-104	E-104, E-215, E-162, E-124
E-119	E-119, E-104, E-215, E-162, E-124
E-160	E-160, E-119, E-104, E-215, E-162, E-124
E-10	E-10, E-160, E-119, E-104, E-215, E-162, E-124
E-50b	E-50b, E-10, E-160, E-119, E-104, E-215, E-162, E-124
E-50a	**E-50a, E-50b, E-10, E-160, E-119, E-104, E-215, E-162, E-124**

one possible topological ordering

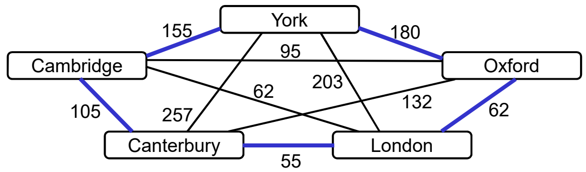
Another Topological Sort Example



Evolution of the stack:

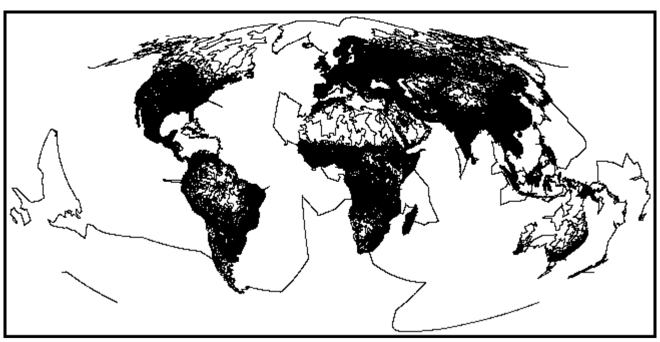
push	stack contents (top to bottom)

Traveling Salesperson Problem (TSP)



- A salesperson needs to travel to a number of cities to visit clients, and wants to do so as efficiently as possible.
- A *tour* is a path that:
  - begins at some starting vertex
  - passes through every other vertex *once and only once*
  - returns to the starting vertex
- TSP: find the tour with the lowest total cost

TSP for Santa Claus



source: http://www.tsp.gatech.edu/world/pictures.html

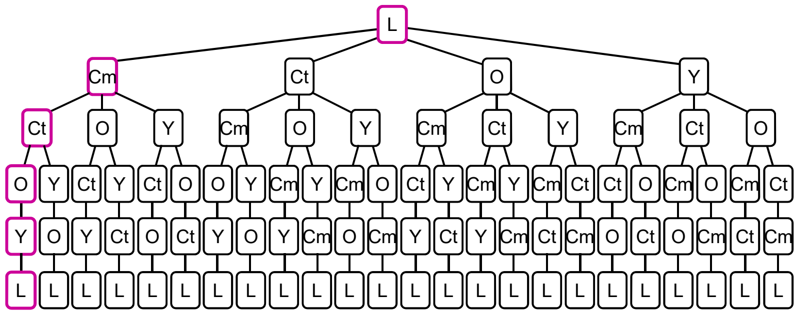
A "world TSP" with 1,904,711 cities.

The figure at right shows a tour with a total cost of 7,516,353,779 meters – which is at most 0.068% longer than the optimal tour.

- Other applications:
  - coin collection from phone booths
  - routes for school buses or garbage trucks
  - minimizing the movements of machines in automated manufacturing processes
  - many others

Solving a TSP: Brute-Force Approach

- Perform an exhaustive search of all possible tours.
  - represent the set of all possible tours as a tree



- The leaf nodes correspond to possible solutions.
  - for n cities, there are (n – 1)! leaf nodes in the tree.
  - half are redundant (e.g., L-Cm-Ct-O-Y-L = L-Y-O-Ct-Cm-L)
- Problem: exhaustive search is intractable for all but small n.
  - example: when n = 14, ((n – 1)!) / 2 = over 3 billion

Solving a TSP: Informed Search

- Focus on the most promising paths through the tree of possible tours.
  - use a function that estimates how good a given path is
- Better than brute force, but still exponential space and time.

Algorithm Analysis Revisited

- Recall that we can group algorithms into classes (n = problem size):

name	example expressions	big-O notation
constant time	1, 7, 10	$O(1)$
logarithmic time	$3\log_{10}n$, $\log_2 n + 5$	$O(\log n)$
linear time	$5n$, $10n - 2\log_2 n$	$O(n)$
n log n time	$4n \log_2 n$, $n \log_2 n + n$	$O(n \log n)$
quadratic time	$2n^2 + 3n$, $n^2 - 1$	$O(n^2)$
n^c ($c > 2$)	$n^3 - 5n$, $2n^5 + 5n^2$	$O(n^c)$
exponential time	2^n, $5e^n + 2n^2$	$O(c^n)$
factorial time	$(n - 1)!/2$, $3n!$	$O(n!)$

- Algorithms that fall into one of the classes above the dotted line are referred to as *polynomial-time* algorithms.
- The term *exponential-time algorithm* is sometimes used to include *all* algorithms that fall below the dotted line.
  - algorithms whose running time grows as fast or faster than c^n

Classifying Problems

- Problems that can be solved using a polynomial-time algorithm are considered "easy" problems.
  - we can solve large problem instances in a reasonable amount of time
- Problems that don't have a polynomial-time solution algorithm are considered "hard" or "intractable" problems.
  - they can only be solved exactly for small values of n
- Increasing the CPU speed doesn't help much for intractable problems:

	CPU 1	CPU 2 (1000x faster)
max problem size for $O(n)$ alg:	N	1000N
$O(n^2)$ alg:	N	31.6 N
$O(2^n)$ alg:	N	**N + 9.97**

Dealing With Intractable Problems

- When faced with an intractable problem, we resort to techniques that quickly find solutions that are "good enough".
- Such techniques are often referred to as *heuristic* techniques.
  - heuristic = rule of thumb
  - there's no guarantee these techniques will produce the optimal solution, but they typically work well

Take-Home Lessons

- Object-oriented programming allows us to capture the abstractions in the programs that we write.
  - creates reusable building blocks
  - key concepts: encapsulation, inheritance, polymorphism
- Abstract data types allow us to organize and manipulate collections of data.
  - a given ADT can be implemented in different ways
  - fundamental building blocks: arrays, linked nodes
- Efficiency matters when dealing with large collections of data.
  - some solutions can be *much* faster or more space efficient
  - what's the best data structure/algorithm for *your* workload?
    - example: sorting an almost sorted collection

Take-Home Lessons (cont.)

- Use the tools in your toolbox!
  - interfaces, generic data structures
  - lists/stacks/queues, trees, heaps, hash tables
  - recursion, recursive backtracking, divide-and-conquer

- Use built-in/provided collections/interfaces:
  - `java.util.ArrayList<T>` (implements `List<T>`)
  - `java.util.LinkedList<T>` (implements `List<T>` and `Queue<T>`)
  - `java.util.Stack<T>`
  - `java.util.TreeMap<K, V>` (a balanced search tree)
  - `java.util.HashMap<K, V>` (a hash table)

    } implement `Map<K, V>`
  - `java.util.PriorityQueue<T>` (a heap)

- But use them intelligently!
  - ex: `LinkedList` maintains a reference to the last node in the list
  - `list.add(item, n)` will add `item` to the end in O(n) time
  - `list.addLast(item)` will add `item` to the end in O(1) time!