Higher Order Differential Equations

The general form of Linear Differential Equation with constant Coefficient of order n is $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots a_n y = Q(x)$

Then above equation can be written as

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = Q(x)$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = Q(x) \rightarrow f(D) y = Q(x)$$

where $f(D)$ is a polynomial in D of degree n or,

$f(D) =$ function of D; $Q(x) =$ function of x

The general solution for equation (1) is $\boxed{y = C.F + P.I}$

$C.F =$ Complementary function

$P.I =$ Particular Integral, where $P.I = \frac{1}{f(D)} Q(x)$

Complementary Function

Standard form of D.E is $f(D) \cdot y = Q(x)$

- Method to find the complementary function

Consider the equation $\boxed{f(D) y = 0}$ form the auxilary equation by putting $D = m$, $f(m) = 0$. Solve the equation $f(m) = 0$ and find the roots.

- Types of roots

(1) Real and Different (m_1, m_2)

(2) Real and Equal $(m_1 = m_2 = m)$

(3) Imaginary and different $(\alpha \pm i\beta)$

(4) Imaginary and equal $(\alpha \pm \beta, \alpha \pm \beta)$

→ Particular Integral

* Standard form of higher order D.E: $f(D)y = Q(x)$
* method to find particular Integral – $P.I = \frac{1}{f(D)} Q(x)$

* Now depending on the form of the function $Q(x)$, there are 6 different cases for $Q(x)$.

→ Complementary Functions

i) Real & Different

$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

ii) Real and Equal

$C.F = (c_1 + c_2 x) e^{mx}$

$C.F = (c_1 + c_2 x + c_3 x^2) e^{mx}$

iii) Imaginary & different

$C.F = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

iv) Imaginary and equal

$C.F = e^{\alpha x}[(c_1 + c_2)\cos\beta x + (c_3 + c_4)\sin\beta x]$

v) $Q(x) = x \cdot V(x)$

$V(x)$: sin ax or cos ax

$$P.I = \left(x - \frac{f'(D)}{f(D)}\right)\frac{1}{f(D)} V(x)$$

$e^{i\theta} = \underbrace{\cos\theta}_{R.P = \text{Real part}} + \underbrace{i\sin\theta}_{I.P = \text{imaginary part}}$

→ Particular Integral

i) $Q(x) = e^{ax}$

$P.I = \frac{1}{f(D)} e^{ax}$, put $D = a$; If it result in zero, then multiply x with numerator & derivative of $f(D)$ in den.

ii) $Q(x) = \sin ax$ OR $\cos ax$

$P.I = \frac{1}{f(D)} \sin ax$ OR $\cos ax$, put $D^2 = -a^2$

Provided that denominator $\neq 0$

iii) $Q(x) = x^m$

$P.I = \frac{1}{f(D)} x^m$

Take least common from $f(D)$ and use binomial Expansion method.

iv) $Q(x) = e^{ax} \cdot V(x)$

$V(x)$: sin ax, cos ax, or x^m

$$P.I = \frac{1}{f(D)} e^{ax} \cdot V(x) = e^{ax} \frac{1}{f(D+a)} V(x)$$

vi) $Q(x) = x^n \cdot V(x)$

$V(x)$: sin ax or cos ax

(1) $P.I = \frac{1}{f(D)} x^n \sin ax$

$= \frac{1}{f(D)} x^n \text{ I.P of } e^{iax}$

$= \text{I.P of } \frac{1}{f(D)} e^{iax} \cdot x^n$

(2) $P.I = \frac{1}{f(D)} x^n \cos ax$

$= \frac{1}{f(D)} x^n \text{ R.P of } e^{iax}$

$= \text{R.P of } \frac{1}{f(D)} e^{iax} \cdot x^n$

① $(D^2-3D+2)y = e^{3x}$ $\qquad$ $G.S = c_1e^{x} + c_2e^{2x} + \frac{e^{3x}}{2}$

Step1: Auxiliary equation; Step2: $C.F = c_1e^{m_1x} + c_2e^{m_2x}$

$D^2-3D+2=0$, $D=m$ $\qquad$ $C.F = c_1e^{2x} + c_2e^{x}$

$m^2-3m+2=0$ $\qquad$ Step3: $P.I = \frac{1}{f(D)}\,Q(x) = \frac{1}{D^2-3D+2} \times e^{3x}$; put $D=3$

$m = \frac{3 \pm \sqrt{9-4(2)}}{2}$ $\qquad$ $P.I = \frac{1}{9-9+2}e^{3x} = \frac{e^{3x}}{2}$

$m = \frac{3\pm 1}{2} = \boxed{2, 1}$ Real & distinct

$$\boxed{G.S = C.F + P.I = c_1e^{2x} + c_2e^{x} + \frac{e^{3x}}{2}}$$

② $(D^2-4D+3)y = e^{2x}$ $\qquad$ $G.S = c_1e^{x} + c_2e^{3x} - e^{2x}$

Step1: A.E $\qquad$ Step2: $C.F = c_1e^{x} + c_2e^{3x}$

$D^2-4D+3=0$ $\qquad$ Step3: $P.I = \frac{1}{D^2-4D+3} \times e^{2x} = \frac{1}{4-8+3}e^{2x} = -e^{2x}$

$D = \frac{4 \pm \sqrt{16-4(3)}}{2}$ $\qquad$ $\boxed{G.S = c_1e^{x} + c_2e^{3x} - e^{2x}}$

$D = \frac{4\pm 2}{2} = \boxed{3, 1}$

③ $\frac{d^2y}{dx^2} - 4y = e^{2x} + 3e^{-2x} + 7e^{x}$; $\quad y = c_1e^{2x} + c_2e^{-2x} + \frac{xe^{2x}}{4} - \frac{3xe^{-2x}}{4} - \frac{7e^{x}}{3}$

Step1: A.E $\qquad$ Step2: $C.F = c_1e^{2x} + c_2e^{-2x}$

$D = \frac{d}{dx}$ $\qquad$ Step3: $P.I = \frac{1}{D^2-4}(e^{2x} + 3e^{-2x} + 7e^{x})$ $\qquad$ Denominator $=0$, then derivative & multiply x with Nr

$D^2y - 4y = 0$ $\qquad$ $P.I = \frac{e^{2x}}{D^2-4} + \frac{3e^{-2x}}{D^2-4} + \frac{7e^{x}}{D^2-4} = \frac{xe^{2x}}{2D} + \frac{3xe^{-2x}}{2D} + \frac{7e^{x}}{1^2-4}$

$D^2-4=0$

$\boxed{D=\pm 2}$ $\qquad$ $P.I = \frac{xe^{2x}}{4} + \frac{3xe^{-2x}}{-4} + \frac{7e^{x}}{-3} = \frac{xe^{2x}}{4} - \frac{3xe^{-2x}}{4} - \frac{7e^{x}}{3}$

$$\boxed{G.S = c_1e^{2x} + c_2e^{-2x} + \frac{xe^{2x}}{4} - \frac{3xe^{-2x}}{4} - \frac{7e^{x}}{3}}$$

④ $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x - 3e^{-2x}$; G.S $= (c_1 + c_2x + c_3x^2)e^x + \frac{e^x x^3}{6} + \frac{e^{-2x}}{9}$

Step1: A.E

$D^3y - 3D^2y + 3Dy - y = 0$

$D^3 - 3D^2 + 3D - 1 = 0$, $\boxed{D=1}$

$$\begin{array}{c|cccc} & 1 & -3 & 3 & -1 \\ 1 & & 1 & -2 & 1 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array}$$

$a=1,\ b=-2,\ c=1$

$D = \frac{2 \pm \sqrt{4-4(1)}}{2} = \frac{2}{2} = 1$

$\boxed{D = 1, 1, 1}$ real & equal

Step2: $C.F = e^x(c_1 + c_2x + c_3x^2)$

Step3: $P.I = \frac{e^x}{D^3 - 3D^2 + 3D - 1} - \frac{3e^{-2x}}{D^3 - 3D^2 + 3D - 1}$

$P.I = \frac{xe^x}{3D^2 - 6D + 3} - \frac{3e^{-2x}}{-8 - 12 - 6 - 1}$

$P.I = \frac{x^2e^x}{6D - 6} + \frac{3e^{-2x}}{27} = \frac{x^3e^x}{6} + \frac{e^{-2x}}{9}$

$$\boxed{G.S = e^x(c_1 + c_2x + c_3x^2) + \frac{x^3e^x}{6} + \frac{e^{-2x}}{9}}$$

⑤ $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = e^{3x} - 4e^{-x} + 5$;

Step1: A.E

$D^2y + 2Dy + 2y = 0$

$D^2 + 2D + 2 = 0$

$D = \frac{-2 \pm \sqrt{4 - 4(2)}}{2}$

$D = \frac{-2 \pm 2i}{2} = \boxed{-1 \pm i}$

imaginary & distinct

Step2: $C.F = e^{\alpha x}[c_1\cos\beta x + c_2\sin\beta x] = e^{-x}[c_1\cos x + c_2\sin x]$

Step3: $P.I = \frac{1}{D^2 + 2D + 2} \times e^{3x} - 4e^{-x} + 5$

$P.I = \frac{e^{3x}}{D^2 + 2D + 2} - \frac{4e^{-x}}{D^2 + 2D + 2} + \frac{5e^{0x}}{D^2 + 2D + 2} = \frac{e^{3x}}{17} - 4e^{-x} + \frac{5}{2}$

$$\boxed{G.S = e^{-x}[c_1\cos x + c_2\sin x] + \frac{e^{3x}}{17} - 4e^{-x} + \frac{5}{2}}$$

⑥ $(D^3 + 8)y = 4 + e^{-2x}$

Step1: A.E

$D^3 + 8 = 0$, $D = -2$

$$\begin{array}{c|cccc} & 1 & 0 & 0 & 8 \\ -2 & & -2 & 4 & -8 \\ \hline & 1 & -2 & 4 & \boxed{0} \end{array}$$

$D = \frac{2 \pm \sqrt{4 - 4(4)}}{2} = \frac{2 \pm 2\sqrt{3}i}{2}$

$\boxed{D = -2}$ $\boxed{D = 1 \pm \sqrt{3}i}$

Step2: $c_1e^{-2x} + e^x[c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x]$

Step3: $P.I = \frac{4}{D^3 + 8} + \frac{e^{-2x}}{D^3 + 8} = \frac{1}{2} + \frac{xe^{-2x}}{3D^2}$

$P.I = \frac{1}{2} + \frac{xe^{-2x}}{12}$

$$\boxed{G.S = c_1e^{-2x} + e^x[c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x] + \frac{1}{2} + \frac{xe^{-2x}}{12}}$$

(7) $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Step 1: A.E

$(D+2)(D-1)^2 = 0$

$\boxed{D=-2}$ $\boxed{D=1,1}$

D.H Equal

Step 2: C.F

$C.F = c_1 e^{-2x} + e^{x}[c_2 + c_3 x]$

Step 3: $P.I = \frac{e^{-2x}}{(D+2)(D-1)^2} + \frac{2}{(D+2)(D-1)^2} \times \frac{e^{x}-e^{-x}}{2}$; $\sinh x = \frac{e^{x}-e^{-x}}{2}$

$P.I = \frac{e^{-2x}}{(D+2)(D^2-2D+1)} + \frac{e^{x}}{D^3-2D^2+D+2D^2-4D+2} - \frac{e^{-x}}{D^3-2D^2+D+2D^2-4D+2}$

$= \frac{e^{-2x}}{0 \leftarrow (-2)^3-3(-2)+2} + \frac{xe^{x}}{3D^2-3} - \frac{e^{-x}}{-1+3+2}$

$= \frac{xe^{-2x}}{3D^2-3} + \frac{x^2e^{x}}{6D} - \frac{e^{-x}}{4} = \frac{xe^{-2x}}{9} + \frac{x^2e^{x}}{6} - \frac{e^{-x}}{4}$

$$\boxed{C.S = c_1 e^{-2x} + e^{x}[c_2 + c_3 x] + \frac{xe^{-2x}}{9} + \frac{x^2e^{x}}{6} - \frac{e^{-x}}{4}}$$

(8) $(D^2-4)y = \sin x$

Step 1: A.E

$D^2-4=0$

$\boxed{D=\pm 2}$

Step 2: C.F

$C.F = c_1 e^{2x} + c_2 e^{-2x}$

Step 3: $P.I = \frac{\sin x}{D^2-4}$, $a=1$, $D^2 = -(a)^2 = -1$

$P.I = \frac{\sin x}{-1-4} = \frac{\sin x}{-5}$

$$\boxed{C.S = c_1 e^{2x} + c_2 e^{-2x} - \frac{\sin x}{5}}$$

(9) $(D^2+4)y = \cos 2x$

Step 1: A.E

$D^2+4=0$

$D = \frac{\pm\sqrt{0-4(4)}}{2} = \frac{\pm 4i}{2}$

$\boxed{D = 0 \pm 2i}$ $\alpha=0, \beta=2$

Imaginary & distinct

Step 2: $C.F = c_1\cos 2x + c_2 \sin 2x$

Step 3: $P.I = \frac{\cos 2x}{D^2+4}$; $a=2$, $D^2=-4$

$P.I = \frac{x\cos 2x}{2D}$; $D = \frac{d}{dx}$; $\Rightarrow \frac{1}{D} = \int dx$

$P.I = \frac{x}{2}\int \cos 2x = \frac{x}{2}\frac{\sin(2x)}{2} = \frac{x}{4}\sin 2x$

$$\boxed{C.S = c_1\cos 2x + c_2 \sin 2x + \frac{x\sin 2x}{4}}$$

(10) $(D^4+3D^2-4)y = 5\sin 2x - e^{-2x}$

Step 1: A.E

$D^4-3D^2-4=0$, $D=2$

$$\begin{array}{c|cccc} & 1 & 0 & -3 & -4 \\ 2 & & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & \boxed{-2} \end{array}$$

$D^4-3D^2-4=0$, let $D^2=m$

$m^2-3m-4=0$

$m = \frac{3 \pm \sqrt{9-4(-4)}}{2} = \frac{3\pm 5}{2} = 4, -1$

$D^2+2D+1=-2$

$D^2+2D+3=0$

$D = \frac{-2\pm\sqrt{4-4(3)}}{2} = \frac{-2\pm\sqrt{-8}}{2}$

$\boxed{D=-1\pm 2i}$

$D^2=4 = \pm 2$; $D^2=-1 = \pm i$

$\boxed{D=\pm 2}$ and $\boxed{D=\pm i}$

Real & D imaginary & D

$C.F = c_1e^{2x}+c_2e^{-2x}+c_3\cos x + c_4\sin x$

Step 3: $P.I = \frac{5\sin 2x}{D^4-3D^2-4} - \frac{e^{-2x}}{D^4-3D^2-4} = \frac{5\sin 2x}{(-4)(-4)-3(-4)-4} - \frac{xe^{-2x}}{4D^3-6D}$

$= \frac{5\sin 2x}{16+12-4} - \frac{xe^{-2x}}{-32+12} = \frac{5\sin 2x}{24} + \frac{xe^{-2x}}{20}$

$$\boxed{G.S = c_1e^{2x}+c_2e^{-2x}+c_3\cos x+c_4\sin x + \frac{5\sin 2x}{24} + \frac{xe^{-2x}}{20}}$$

(11) $(D^2+2D+1)y = 4\sin 2x$

Step 1: A.E

$D^2+2D+1=0$

$D = \frac{-2\pm\sqrt{4-4(1)}}{2} = -1,-1$

Step 2: $C.F = e^{-x}(c_1+c_2x)$

Step 3: $P.I = \frac{4\sin 2x}{D^2+2D+1} = \frac{4\sin 2x}{-4+2D+1} = \frac{4\sin 2x}{2D-3}$

$= \frac{4\sin 2x}{2D-3} \times \frac{(2D+3)}{(2D+3)} = \frac{(8\sin 2x)D + 12\sin 2x}{(2D-3)(2D+3)}$ $\frac{d}{dx}$

$= \frac{8\cos 2x \times 2 + 12\sin 2x}{4D^2-9} = \frac{16\cos 2x + 12\sin 2x}{4(-4)-9}$

$= \frac{16\cos 2x + 12\sin 2x}{-25}$

$$\boxed{G.S = e^{-x}(c_1+c_2x) - \frac{4(4\cos 2x + 3\sin 2x)}{25}}$$

(12) $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

Step 1: A.E

$D^3y + 4Dy = \sin 2x$

$D^3 + 4D = 0$

$D(D^2+4) = 0$

$\boxed{D=0,\ D^2=\pm 2i}$

Step 2: $C.F = c_1 + c_2\cos 2x + c_3 \sin 2x$

Step 3: $P.I = \frac{\sin 2x}{D(D^2+4)} = \frac{x \sin 2x}{3D^2+4} = \frac{x\sin 2x}{-8}$

$$\boxed{G.S = c_1 + c_2\cos 2x + c_3 \sin 2x - \frac{x\sin 2x}{8}}$$

(13) $(D^2+1)y = \sin x \cdot \sin 2x$

Step 1: A.E

$D^2+1=0$

$D = \pm\sqrt{-1} = \pm i$

Step 2: $\boxed{C.F = c_1\cos x + c_2 \sin x}$

Step 3: $P.I = \frac{\sin x \cdot \sin 2x}{D^2+1}$; $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\cos(-x) = \cos(x)$

$P.I = \frac{\sin x \cdot \sin 2x}{D^2+1} = \frac{\cos(-x) - \cos(3x)}{2D^2+2} = \frac{+\cos x}{2D^2+2} - \frac{\cos(3x)}{2D^2+2}$

$= \frac{+x\cos x}{4D} - \frac{\cos(3x)}{2D^2+2} = \frac{+x\cos x}{4} - \frac{\cos(3x)}{2(-9)+2} = \boxed{\frac{x\cos x}{4} + \frac{\cos(3x)}{16}}$

$$\boxed{G.S = c_1\cos x + c_2 \sin x + \frac{x\cos x}{4} + \frac{\cos(3x)}{16}}$$

(14) $\frac{d^4y}{dx^4} - 3\frac{d^2y}{dx^2} - 4y = 5\sin 2x - e^{-2x}$

Step 1: A.E

$D^4y - 3D^2y - 4y = 0$

$D^4 - 3D^2 - 4 = 0$

Let $m = D^2$

$m^2 - 3m - 4 = 0$

$m = \frac{3 \pm \sqrt{9-4(-4)}}{2}$

$m = \frac{3 \pm \sqrt{25}}{2} = \frac{3\pm 5}{2}$

$m = 4, -1$, $D^2 = 4$, $D^2 = -1$

$\boxed{D = \pm 2,\ D = \pm i}$

Step 2: $C.F = c_1e^{2x} + c_2e^{-2x} + c_3\cos x + c_4 \sin x$

Step 3: $P.I = \frac{5\sin 2x}{D^4-3D^2-4} - \frac{e^{-2x}}{D^4-3D^2-4}$, $D^2 = -4$

$P.I = \frac{5\sin 2x}{(-4)(-4)-3(-4)-4} - \frac{e^{-2x}}{(-2)^4-3(-2)^2-4}$

$P.I = \frac{5\sin 2x}{16+12-4} - \frac{e^{-2x}}{16-12-4=0}$

$= \frac{5\sin 2x}{24} - \frac{xe^{-2x}}{4D^3-6D} = \frac{5\sin 2x}{24} + \frac{xe^{-2x}}{20}$

$$\boxed{G.S = c_1e^{2x} + c_2e^{-2x} + c_3\cos x + c_4 \sin x + \frac{5\sin 2x}{24} + \frac{xe^{-2x}}{20}}$$

(15) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = \sinh x + \sin(\sqrt{2}\,x)$

Step 1: A.E

$D^2y - 2Dy + 2y = 0$

$D^2 - 2D + 2 = 0$

$D = \frac{2 \pm \sqrt{4-4(2)}}{2}$

$D = \frac{2 \pm 2i}{2} = 1 \pm i$

Step 2: $C.F = e^x[c_1\cos x + c_2\sin x]$

Step 3: $P.I = \frac{\sinh x}{D^2-2D+2} + \frac{\sin(\sqrt{2}\,x)}{D^2-2D+2}$ $\rightarrow D^2=-2$

$= \frac{e^x - e^{-x}}{2D^2-4D+4} + \frac{\sin(x\sqrt{2})}{-2-2D+2} = \frac{e^x}{2D^2-4D+4} - \frac{e^{-x}}{2D^2-4D+4} + \frac{\sin(x\sqrt{2})}{2D}$

$= \frac{e^x}{2-4+4} - \frac{e^{-x}}{2+4+4} - \frac{1}{2}\int \sin(x\sqrt{2})\,dx$

$= \frac{e^x}{2} - \frac{e^{-x}}{10} - \frac{1}{2}(-\cos(x\sqrt{2}) \times \sqrt{2} = \frac{e^x}{2} - \frac{e^{-x}}{10} + \frac{\cos(x\sqrt{2})}{2\sqrt{2}}$

$\boxed{C.S = e^x[c_1\cos x + c_2\sin x] + \frac{e^x}{2} - \frac{e^{-x}}{10} + \frac{\cos(x\sqrt{2})}{2\sqrt{2}}}$

(16) $(D^4 - 3D^2 - 4)y = 24\sin 2x - 40e^{-2x}$

Step 1: A.E

$D^4 - 3D^2 - 4 = 0,\ D^2 = m$

$m^2 - 3m - 4 = 0$

$m = \frac{3 \pm \sqrt{9-4(-4)}}{2} = \frac{3 \pm 5}{2}$

$m = 4, -1;\ \boxed{D = \pm 2,\ D = \pm i}$

Step 2: $C.F = c_1e^{2x} + c_2e^{-2x} + c_3\cos x + c_4\sin x$

Step 3: $P.I = \frac{24\sin 2x}{D^4-3D^2-4} - \frac{40e^{-2x}}{D^4-3D^2-4}$

$= \frac{24\sin 2x}{16+12-4} - \frac{40xe^{-2x}}{4D^3-6D} = \frac{24\sin 2x}{24} - \frac{40xe^{-2x}}{-32+12}$

$= \frac{\cancel{24}\sin 2x}{\cancel{24}} + \frac{40xe^{-2x}}{20} = \boxed{\sin 2x + 2xe^{-2x}}$

$\therefore C.S = c_1e^{2x} + c_2e^{-2x} + c_3\cos x + c_4\sin x + \sin 2x + 2xe^{-2x}$

(17) $(D^3+1)y = \cos^2\left(\frac{x}{2}\right) + e^{-x}$

Step 1: A.E

$D^3 + 1 = 0;\ D = -1$

$$\begin{array}{r|rrrr} & 1 & 0 & 0 & 1 \\ -1 & & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

$D = \frac{1 \pm \sqrt{1-4(1)}}{2} = \frac{1 \pm \sqrt{3}i}{2}$

$D = -1,\ \frac{1 \pm \sqrt{3}\,i}{2}$

Step 2: $c_1e^{-x} + e^{\frac{x}{2}}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$

Step 3: $P.I = \cos^2\left(\frac{x}{2}\right) \rightarrow \cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$

$P.I = \frac{1+\cos x}{2D^3+2} + \frac{e^{-x}}{D^3+2} = \frac{1}{2D^3+2} + \frac{\cos x}{2D^3+2} + \frac{xe^{-x}}{3D^2}$

$= \frac{e^{0x}}{2(0)+2} + \frac{\cos x}{2D(-1)+2} + \frac{xe^{-x}}{3}$

$$= \frac{e^{0x}}{2} + \frac{\cos x}{2-2D} + \frac{xe^{-x}}{3} = \frac{1}{2} + \frac{\cos x}{(2-2D)} \times \frac{(2+2D)}{(2+2D)} + \frac{xe^{-x}}{3}$$

$$= \frac{1}{2} + \frac{2\cos x + 2D\cos x}{4-4D^2} + \frac{xe^{-x}}{3} = \frac{1}{2} + \frac{2\cos + 2(-\sin x)}{4-4(-1)} + \frac{xe^{-x}}{3}$$

$$= \frac{1}{2} + \frac{2\cos x - 2\sin x}{8} + \frac{xe^{-x}}{3} = \frac{1}{2} + \frac{\cos x - \sin x}{4} + \frac{xe^{-x}}{3}$$

$$\boxed{G.S = c_1 e^{-x} + e^{x/2}\left[c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right] + \frac{1}{2} + \frac{\cos x - \sin x}{4} + \frac{xe^{-x}}{3}}$$

(18) $\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + \frac{dy}{dt} - 5y = \cosh t + \sin t$

(1) $D^3y + 3D^2y + Dy - 5y = 0$

$D^3 + 3D^2 + D - 5 = 0$, $D = 1$

	1	3	1	-5
1		1	4	5
	1	4	5	0

$D = \frac{-4 \pm \sqrt{16-4(5)}}{2} = \frac{-4 \pm 2i}{2}$

$\boxed{D = -2 \pm i}$ $\boxed{D = 1}$

Step 2: $C.F = c_1 e^{x} + e^{-2x}[c_1 \cos x + c_2 \sin x]$

Step 3: $P.I \Rightarrow \cosh t = \frac{e^{t} + e^{-t}}{2}$

$$P.I = \frac{e^{t}}{2(D^3+3D^2+D-5)} + \frac{e^{-t}}{2(D^3+3D^2+D-5)} + \frac{\sin t}{D^3+3D^2+D-5}$$

$$= \frac{te^{t}}{2(3D^2+6D+1)} + \frac{e^{-t}}{2(-1+3-1-5)} + \frac{\sin t}{-D-3+D-5} \quad \sin t \to D^2 = -1$$

$$= \frac{te^{t}}{20} + \frac{e^{-t}}{-8} + \frac{\sin t}{-8}$$

$$\boxed{C.S = c_1 e^{x} + e^{-2x}[c_1 \cos x + c_2 \sin x] + \frac{te^{t}}{20} - \frac{e^{-t}}{8} - \frac{\sin t}{8}}$$

(19) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \sin 2x \cos 3x$

(1) $D^2y + 2Dy + 2y = 0$

$D^2 + 2D + 2 = 0$

$D = \frac{-2 \pm \sqrt{4-4(2)}}{2}$

$D = \frac{-2 \pm 2i}{2} = \boxed{-1 \pm i}$

(2) $C.F = e^{-x}[c_1 \cos x + c_2 \sin x]$

(3) $P.I = \sin 2x \cos 3x$

$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

$\sin(-\theta) = -\sin(\theta)$

$$P.I = \frac{\sin 2x \cos 3x}{D^2+2D+2} = \frac{1}{2} \times \frac{[\sin(2x+3x) + \sin(2x-3x)]}{D^2+2D+2} \quad D^2 = -25 \quad D^2 = -1$$

$$= \frac{\sin(5x)}{2D^2+4D+4} + \frac{\sin(-x)}{2D^2+4D+4} = \frac{\sin(5x)}{2D^2+4D+4} - \frac{\sin(x)}{2D^2+4D+4}$$

$$= \frac{\sin(5x)}{2(-25)+4D+4} - \frac{\sin(x)}{2(-1)+4D+4} = \frac{\sin(5x)}{4D-46} - \frac{\sin(x)}{4D-2}$$

$$= \frac{\sin(5x)}{4D-46} \times \frac{4D+46}{4D+46} - \frac{\sin(x)}{4D-2} \times \frac{4D+2}{4D+2}$$

$$= \frac{4D\sin(5x) + 46\sin(5x)}{16D^2 - 2116} - \frac{4D\sin(x) - 2\sin(x)}{16D^2-4}$$

$$= \frac{4\cos(5x)\times 5 + 46\sin(5x)}{16(-25) - 2116} - \frac{4\cos(x) - 2\sin(x)}{16(-1)-4}$$

$$= \frac{\overset{10}{\cancel{20}}\cos(5x) + \overset{23}{\cancel{46}}\sin(5x)}{-2516 \;\; 1258} - \frac{\overset{2}{\cancel{4}}\cos(x) - \overset{1}{\cancel{2}}\sin(x)}{-\cancel{20}\,10}$$

$$= \frac{10\cos(5x) + 23\sin(5x)}{-1258} + \frac{2\cos(x) - \sin(x)}{10}$$

$$\boxed{G.S = e^{-x}[c_1\cos x + c_2 \sin x] + \frac{2\cos(x) - \sin(x)}{10} - \frac{10\cos(5x)+23\sin(5x)}{1258}}$$

(26) $(D^2+2)y = x^3$

(1) $D^2+2=0$ (2) $C.F = c_1\cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

$D = \pm\sqrt{2}i$ (3) $P.I = \frac{x^3}{D^2+2} = \frac{x^3}{2(\frac{D^2}{2}+1)} = \frac{1}{2}\left(\frac{D^2}{2}+1\right)^{-1} x^3$

Binomial Theorem

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 \dots x^n$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots x^n$

Stop

$$P.I = \frac{1}{2}\left[\frac{D^2}{2}+1\right]^{-1} x^3 = \frac{1}{2}\left[1 - \left(\frac{D^2}{2}\right) + \left(\frac{D^2}{2}\right)^2 - \left(\frac{D^2}{2}\right)^3 + \dots\right] x^3$$

$$\frac{1}{2}\left[x^3 - \frac{D^2}{2}x^3 + \frac{D^4}{4}x^3 - \frac{D^6}{8}x^3\right] = \frac{1}{2}\left[x^3 - \frac{D^2}{2}x^3 + 0 - 0\right]$$

4- times $\frac{d}{dx}$ of $x^3 = 0$ 6- times

$$\frac{1}{2}\left[x^3 - \frac{D^2}{2}x^3\right] = \frac{x^3}{2} - \frac{6x}{24} = \frac{x^3}{2} - \frac{3x}{2} = \boxed{\frac{x(x^2-3)}{2}}$$

$\rightarrow$ 2 times $\frac{d}{dx^2}(x^3)$

$$\boxed{G.S = c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x + \frac{x(x^2-3)}{2}}$$

(21) $\frac{d^4y}{dx^4} + 4y = x^3$

(1) $D^4 + 4 = 0$, $D^2 = m$

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm 2i$

$D^4 + 4D^2 + 4 - 4D^2 = 0$

$[(D^2)^2 + 2(2)(D^2) + (2)^2] - 4D^2 = 0$

$[D^2+2]^2 - 4D^2 = 0$

$\underbrace{(D^2+2)^2}_{a} - \underbrace{(2D)^2}_{b} = 0 \rightarrow a^2 - b^2 = (a-b)(a+b)$

$[(D^2+2)-(2D)][(D^2+2)+(2D)] = D^2 - 2D + 2$ — (i)

$D^2 + 2D + 2$ — (ii)

(i) $D^2 - 2D + 2 = 0$

$D = \frac{2 \pm \sqrt{4-4(2)}}{2} = \frac{2 \pm 2i}{2} = \boxed{1 \pm i}$;

(ii) $D^2 + 2D + 2$

$D = \frac{-2 \pm \sqrt{4-4(2)}}{2} = \frac{-2 \pm 2i}{2} = \boxed{-1 \pm i}$

$\boxed{D = 1 \pm i,\ -1 \pm i}$

(2) $C.F = e^{x}[c_1\cos x + c_2\sin x] + e^{-x}[c_3\cos x + c_4\sin x]$

(3) $P.I = \frac{1}{D^4+4}x^3 = \frac{1}{4\left[\frac{D^4}{4}+1\right]}x^3 = \frac{1}{4}\left[1 + \frac{D^4}{4}\right]^{-1}x^3$

$\hookrightarrow$ ultimately $\rightarrow 0$

$P.I = \frac{1}{4}[1+0]^{-1}x^3$

$= \frac{1}{4}[1]x^3 = \frac{x^3}{4}$

$$\boxed{G.S = e^{x}[c_1\cos x + c_2\sin x] + e^{-x}[c_3\cos x + c_4\sin x] + \frac{x^3}{4}}$$

(22) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 + 2x + 4$

(1) $D^2y + Dy = x^2 + 2x + 4$ (2) $C.F = C_1 + C_2 e^{-x}$

$D^2 + D = 0$ (3) $P.I = \frac{1}{D^2+D} \times x^2 + 2x + 4 = \frac{1}{D(D+1)} \times x^2 + 2x + 4$

$D(D+1) = 0$

$\boxed{D=0}$ $\boxed{D=-1}$ $= \frac{1}{D}\left[D+1\right]^{-1} x^2 + 2x + 4$; $(x+1)^{-1} = [1 - x + x^2 - x^3 \dots]$

$= \frac{1}{D}\left[1 - D + D^2 + \dots\right] x^2 + 2x + 4 = \frac{1}{D}\left[(x^2+2x+4) - D(x^2+2x+4) + D^2(x^2+2x+4)\right]$

$= \frac{1}{D}\left[x^2+2x+4\right] - \frac{1}{D}\left[D(x^2+2x+4)\right] + \frac{1}{D}\left[D^2(x^2+2x+4)\right]$

$= \int (x^2+2x+4)\,dx - \int (2x+2)\,dx + \int (2)\,dx$

$= \frac{x^3}{3} + \cancel{x^2} + 4x - \cancel{x^2} - \cancel{2x} + \cancel{2x} = \boxed{\frac{x^3}{3} + 4x}$

$$\boxed{G.S = C_1 + C_2 e^{-x} + \frac{x^3}{3} + 4x}$$

(23) $(D^2 - 4D + 1)y = \cos 2x + x$

(1) $D^2 - 4D + 1 = 0$ (2) $C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x}$

$D = \frac{4 \pm \sqrt{16 - 4(1)}}{2}$ (3) $P.I = \frac{\cos 2x}{D^2 - 4D + 1} + \frac{x}{D^2 - 4D + 1} = \frac{\cos 2x}{-4 - 4D + 1} + [1 + (D^2 - 4D)]^{-1} x$

$D = \frac{4 \pm 2\sqrt{3}}{2} = \boxed{2 \pm \sqrt{3}}$ $= \frac{-\cos 2x}{4D+3} \times \frac{(4D-3)}{(4D-3)} + \left[1 - (D^2-4D)^1 + (D^2-4D)^2 - (D^2-4D)^3 \dots\right] x$

$= \frac{-\cos 2x\,(4D-3)}{16D^2 - 9} + \left[1 - (D^2 - 4D)\right] x = \frac{-4D\cos 2x + 3\cos 2x}{16(-4) - 9} + [x - 0 + 4]$

$= \frac{3\cos 2x - 4(-\sin 2x) \times 2}{-64 - 9} + [x + 4] = \boxed{\frac{3\cos 2x + 8\sin 2x}{-73} + x + 4}$

$$\boxed{G.S = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x} + \frac{3\cos 2x + 8\sin 2x}{-73} + x + 4}$$

(24) $\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = x^3 - 3x^2 + 1$

(1) $D^2y - Dy + y = x^3 - 3x^2 + 1$

$D^2 - D + 1 = 0$

$D = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$

(2) $C.F = e^{x/2}\left[c_1 \cos \sqrt{3}/_2 x + \sin \sqrt{3}/_2 x\right]$

(3) $P.I = \left[(D^2 - D) + 1\right]^{-1} (x^3 - 3x^2 + 1)$

$= \left[1 - (D^2 - D) + (D^2 - D)^2 \cdots\right] (x^3 - 3x^2 + 1)$

$= \left[1 - (D^2 - D) + (D^2 - D)^2 - (D^2 - D)^3\right] (x^3 - 3x^2 + 1)$

$= \left[1 - D^2 + D + \cancel{D^4} - 2D^3 + \cancel{D^2} ((D^2)^3 - 3D^4 D + 3D^2 D^2 - D^3)\right]$

$= \left[1 - 2D^3 + D^2 + D - D^6 + 3D^5 - 3D^4 + D^3\right] x^3 - 3x^2 + 1$

$= \left[1 - 2D^3 + D^2 + D - 0 + 0 - 0 + D^3\right] x^3 - 3x^2 + 1$

$= \left[1 - D^3 + \cancel{D^2} + D\right] (x^3 - 3x^2 + 1)$

$= (x^3 - 3x^2 + 1) - D^3(x^3 - 3x^2 + 1) + \cancel{D^2}(x^3 - 3x^2 + 1) + D(x^3 - 3x^2 + 1)$

$= (x^3 - 3x^2 + 1) - (6) + (6x - 6) + (3x^2 - 6x)$

$= (x^3 - 3x^2 + 1) - 6 + (6x - 6) + 3x^2 + 6x$

$= x^3 - 3x^2 + 1 - 6 + (6x - 6) + 3x^2 + 6x = x^3 - 6x^2 + 12x - 11 = x^3$

$= x^3 - \cancel{3x^2} + 1 - 6 + \cancel{3x^2} + 6x$

$= \boxed{x^3 + 6x - 5}$

(25) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \sin 2x$

① $D^2y - 2Dy + y = 0$

$D^2 - 2D + 1 = 0$

$D = \frac{2 \pm \sqrt{4 - 4(1)}}{2} = 1, 1$

② $C.F = (c_1 + c_2 x)e^x$

③ $P.I = \frac{e^x \sin 2x}{D^2 - 2D + 1} = e^x \times \frac{1}{(D+1)^2 - 2(D+1) + 1} \times \sin 2x$

$= e^x \times \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1} \sin 2x = e^x \times \frac{1}{D^2} \sin 2x \quad ; D^2 = -4$

$= \frac{e^x}{-4} \times \sin 2x = \boxed{\frac{e^x \sin 2x}{-4}}$

$$\boxed{G.S = (c_1 + c_2 x)e^x + \frac{e^x \sin 2x}{-4}}$$

(26) $\frac{d^2y}{dx^2} - y = e^x + x^2 e^x$

① $D^2y - y = e^x + x^2e^x$

$D^2 - 1 = 0$

$\boxed{D = 1, -1}$

② $C.F = c_1 e^x + c_2 e^{-x}$

③ $P.I = \frac{e^x}{D^2 - 1} + \frac{x^2 e^x}{D^2 - 1} = \frac{xe^x}{2D} + e^x \times \frac{1}{(D+1)^2 - 1} x^2$

$= \frac{xe^x}{2} + e^x \times \frac{1}{D^2 + 2D + 1 - 1} \times x^2 = \frac{xe^x}{2} + e^x \times \frac{1}{D^2 + 2D} \times x^2$

$= \frac{xe^x}{2} + e^x \times \frac{1}{2D\left(\frac{D}{2} + 1\right)} \times x^2 = \frac{xe^x}{2} + \frac{e^x}{2D}\left[\frac{D}{2} + 1\right]^{-1} x^2$

$= \frac{xe^x}{2} + \frac{e^x}{2D}\left[1 - \left(\frac{D}{2}\right) + \left(\frac{D}{2}\right)^2 \dots\right] x^2$

$= \frac{xe^x}{2} + \frac{e^x}{2D}\left[x^2 - \frac{D}{2}x^2 + \frac{D^2}{4}x^2\right]$

$= \frac{xe^x}{2} + \frac{e^x}{2D}\left[x^2 - x + \frac{1}{2}\right] = \frac{xe^x}{2} + \frac{e^x}{2}\left[\int x^2\,dx - \int x\,dx + \frac{1}{2}\int dx\right]$

$= \frac{xe^x}{2} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right]$

$$\boxed{G.S = c_1 e^x + c_2 e^{-x} + \frac{xe^x}{2} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right]}$$

(28) $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x}\sin x$

(1) $D^3y - 7D^2y + 10Dy = e^{2x}\sin x$ (2) C.F $= c_1 + c_2e^{5x} + c_3e^{2x}$

$D^3 - 7D^2 + 10D = 0,\ D = 2$

	1	-7	10
2		2	-10
	1	-5	$\boxed{0}$

$D = \frac{5 \pm \sqrt{25 - 4(0)}}{2} = \frac{5 \pm 5}{2}$

$D^2 - 5D = D(D-5)$

$\boxed{D=0}\ \boxed{D=5}\ \boxed{D=2}$

(3) P.I $= \frac{e^{2x}\sin x}{D^3 - 7D^2 + 10D} = e^{2x} \times \frac{1}{(D+2)^3 - 7(D+2)^2 + 10(D+2)} \times \sin x$

$= e^{2x} \times \frac{1}{D^3 + 3D^2(2) + 3D(2)^2 + 8 - 7(D^2 + 4D + 4) + 10D + 20} \times \sin x$

$= e^{2x} \times \frac{1}{D^3 + 6D^2 + 12D + 8 - 7D^2 - 28D - 28 + 10D + 20} \times \sin x$

$= e^{2x} \times \frac{1}{D^3 - D^2 - 6D + 0} \times \sin x$

$= e^{2x} \times \frac{1}{D \cdot D^2 - D^2 - 6D} \times \sin x, \quad D^2 = -1$

$= e^{2x} \times \frac{1}{-D + 1 - 6D} \sin x = e^{2x} \times \frac{1}{1 - 7D} \sin x = e^{2x} \times \frac{\sin x}{(1-7D)} \times \frac{(1+7D)}{(1+7D)}$

$= e^{2x} \times \frac{\sin x\,(1+7D)}{1 - 49D^2} = e^{2x} \times \frac{\sin x + 7D\sin x}{1 - 49(-1)} = \boxed{e^{2x} \times \frac{\sin x + 7\cos x}{50}}$

$\boxed{G.S = c_1 + c_2e^{5x} + c_3e^{2x} + \frac{e^{2x}}{50}(\sin x + 7\cos x)}$

(27) $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + xe^x$

(1) $D^3 - 3D^2 + 3D - 1 = 0$

$D = 1$

	1	-3	3	-1
1		1	-2	1
	1	-2	1	$\lfloor$

$D = \frac{2 \pm \sqrt{4 - 4(1)}}{2} = 1, 1$

$\boxed{D = 1, 1, 1}$

(3) P.I $= \frac{e^x}{D^3 - 3D^2 + 3D - 1} + e^x \times \frac{1}{(D+1)^3 - 3(D+1)^2 + 3(D+1) - 1} \times x$

P.I $= \frac{xe^x}{3D^2 - 6D + 3} + e^x \frac{1}{(D+1)^3 - 3D^2 - 6D - \cancel{3} + 3D + \cancel{3} - 1} \times x$

$= \frac{x^2e^x}{6D - 6} + e^x \frac{1}{D^3 + \cancel{3D^2} + \cancel{3D} + \cancel{1} - \cancel{3D^2} - \cancel{3D} - \cancel{1}} \times x$

$= \frac{x^3e^x}{6} + e^x \frac{1}{D^3} x = \frac{x^3e^x}{6} + e^x \left[\iiint x\, dx\, dx\, dx\right.$

(2) C.F $= (c_1 + c_2x + c_3x^2)e^x$

$\boxed{G.S = \frac{x^3e^x}{6} + \frac{e^x x^4}{24} + (c_1 + c_2x + c_3x^2)e^x}$

(25) $(D^4+4)y = e^x \sin^2 x$; $\sin^2 x = \frac{1-\cos 2x}{2}$

(1) $D^4+4=0$

$D^4+4D^2+4-4D^2=0$

$[(D^2)^2+2(D^2)(2)+2^2]-4D^2=0$

$\underbrace{(D^2+2)^2-(2D)^2}_{a^2-b^2}=0$

$[(D^2+2)+(2D)][(D^2+2)-(2D)]$

D^2+2D+2 — (i)

D^2+2D+2 — (ii)

(i) D^2+2D+2

$D=\frac{-2\pm\sqrt{4-4(2)}}{2} = \frac{-2\pm 2i}{2} = \boxed{-1\pm i}$; (ii) $D^2+2D+2=0$; $D=\frac{2\pm\sqrt{4-4(2)}}{2} = \frac{2\pm 2i}{2} = \boxed{1\pm i}$

$D=-1\pm i$ and $D=1\pm i$

(2) $C.F = e^{-x}[c_1\cos x + c_2\sin x] + e^{x}[c_3\cos x + c_4\sin x]$

(3) $P.I = \frac{e^x\sin^2 x}{D^4+4} = e^x \times \frac{1}{(D+1)^4+4}\sin^2 x = e^x \times \frac{1}{(D+1)^4+4}\times\left(\frac{1-\cos 2x}{2}\right)$

$= e^x \times \frac{1-\cos 2x}{2[(D+1)^4+4]} = e^x\left[\frac{1}{2[(D+1)^4+4]} - \frac{\cos 2x}{2[(D+1)^4+4]}\right]$

$= e^x \times\left[\frac{e^{0x}}{2[(D+1)^4+4]} - \frac{\cos 2x}{2[D^4+4D^3+6D^2+4D+1+4]}\right]$

$= e^x \times\left[\frac{e^{0x}}{2[1+4]} - \frac{\cos 2x}{2[D^2\cdot D^2+4D\cdot D^2+6D^2+4D+5]}\right] \to D^2=-4$

$= \frac{e^x}{2}\times\left[\frac{e^{0x}}{5} - \frac{\cos 2x}{(-4)(-4)+4D(-4)+6(-4)+4D+5}\right]$

$= \frac{e^x}{2}\times\left[\frac{1}{5} - \frac{\cos 2x}{16-16D-24+4D+5}\right]$

$= \frac{e^x}{2}\times\left[\frac{1}{5} - \frac{\cos 2x}{-12D-3}\right] = \frac{e^x}{2}\left[\frac{1}{5} - \frac{\cos 2x}{-(12D+3)}\right]$

$= \frac{e^x}{2}\left[\frac{1}{5} + \frac{\cos 2x}{12D+3}\times\frac{(12D+3)}{(12D+3)}\right] = \frac{e^x}{2}\left[\frac{1}{5} + \frac{12D\cos 2x + 3\cos 2x}{144D^2+9}\right]$

$= \frac{e^x}{2}\left[\frac{1}{5} + \frac{12\sin 2x \times 2 + 3\cos 2x}{144(-4)+9}\right] = \frac{e^x}{2}\left[\frac{1}{5} + \frac{24\sin 2x + 3\cos 2x}{}\right]$

$= \frac{e^x}{2}\left[\frac{1}{5} + \frac{24\sin 2x - 3\cos 2x}{-585}\right] = \frac{e^x}{2}\left[\frac{1}{5} + \frac{24\sin 2x - 3\cos 2x}{-585}\right]$

(31) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 7y = e^{-2x}\sin 2x + x^2$

① $D^2y + 5Dy + 7y = 0$

$D^2 + 5D + 7 = 0$

$D = \frac{-5 \pm \sqrt{25-4(7)}}{2}$

$D = \frac{-5 \pm \sqrt{3}i}{2}$

② C.F $= e^{-5/2 x}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right]$

③ P.I $= \frac{e^{-2x}\sin 2x}{D^2+5D+7} + \frac{x^2}{D^2+5D+7}$

$$= e^{-2x} \times \frac{1}{(D-2)^2+5(D-2)+7} \times \sin 2x + \frac{x^2}{7\left[1+\left(\frac{D^2+5D}{7}\right)\right]}$$

$$= e^{-2x} \times \frac{1}{D^2-4D+4+5D-10+7} \times \sin 2x + \frac{1}{7}\left[1+\left(\frac{D^2+5D}{7}\right)\right]^{-1} x^2$$

$$= e^{-2x} \times \frac{1}{D^2+D+1} \times \sin 2x + \frac{1}{7}\left[1-\left(\frac{D^2+5D}{7}\right)+\left(\frac{D^2+5D}{7}\right)^2 \ldots\right] x^2$$

$$= e^{-2x} \times \frac{1}{-4+D+1} \times \sin 2x + \frac{1}{7}\left[x^2-\left(\frac{D^2+5D}{7}\right)x^2+\left(\frac{D^2+5D}{7}\right)^2 x^2\right]$$

$$= e^{-2x} \times \frac{1}{D-3} \times \sin 2x + \frac{1}{7}\left[x^2-\frac{D^2}{7}x^2-\frac{5D}{7}x^2+\left(\frac{D^4+2(D^2)(5D)+25D^2}{49}\right)x^2\right]$$

$$= e^{-2x} \times \frac{\sin 2x}{(D-3)} \times \frac{(D+3)}{(D+3)} + \frac{1}{7}\left[x^2-\frac{2}{7}-\frac{10x}{7}+\frac{D^4+10D^3+25D^2}{49} \times x^2\right]$$

$$= e^{-2x} \times \frac{\sin 2x\,(D+3)}{(D-3)(D+3)} + \frac{1}{7}\left[x^2-\frac{2}{7}-\frac{10x}{7}+\frac{25D^2}{49}x^2\right]$$

$$= e^{-2x} \times \frac{D\sin 2x + 3\sin 2x}{D^2-9} + \frac{1}{7}\left[x^2-\frac{2}{7}-\frac{10x}{7}+\frac{50}{49}\right]$$

$$= e^{-2x} \times \frac{2\cos 2x + 3\sin 2x}{-13} + \frac{1}{7}\left[x^2-\frac{10x}{7}-\frac{2}{7}+\frac{50}{49}\right]$$

$$= e^{-2x} \times \frac{2\cos 2x + 3\sin 2x}{-13} + \frac{1}{7}\left[x^2-\frac{10x}{7}-\frac{14+50}{49}\right]$$

$$= e^{-2x} \times \frac{2\cos 2x + 3\sin 2x}{-13} + \frac{1}{7}\left[x^2-\frac{10x}{7}-\frac{36}{49}\right]$$

$$\boxed{G.S = e^{-5/2x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] - \frac{e^{-2x}\left(2\cos 2x+3\sin 2x\right)}{13} + \frac{1}{7}\left[x^2-\frac{10x}{7}-\frac{36}{49}\right]}$$

(33) $\frac{d^2x}{dx^2} - y = x\sin x$

(1) $D^2y - y = 0$

$D^2 - 1 = 0$

$(D-1)(D+1) = 0$

$\boxed{D = 1, -1}$

(2) $C.F = c_1e^{x} + c_2e^{-x}$

(3) $P.I = \frac{x\sin x}{D^2-1} = \sin x \times \frac{1}{D^2-1}\, x$

case-v

$D^2 = -1$

If $Q(x) = x \cdot V(x)$

$V(x)$: $\sin ax$ or $\cos ax$, then $P.I = \left(x - \frac{f'(D)}{f(D)}\right)\frac{1}{f(D)}\, V(x)$

$P.I = \frac{x\sin x}{D^2-1} = \left(x - \frac{2D}{D^2-1}\right) \times \frac{1}{D^2-1} \cdot \sin x$

$= x \times \frac{1}{D^2-1} \times \sin x - \frac{2D}{(D^2-1)^2} \times \sin x$

$= x \times \frac{1}{-1-1} \times \sin x - \frac{2D\sin x}{(D^2-1)^2} = \frac{x\sin x}{-2} - \frac{2\cos x}{(-1-1)^2}$

$= \frac{x\sin x}{-2} - \frac{2\cos x}{4}$

(34) $D^2y - 2Dy + y = xe^x \sin x$

Step 1: A.E

$m^2 - 2m + 1 = 0$

$m^2 - m - m + 1 = 0$

$m(m-1) - 1(m-1) = 0$

$\boxed{m = 1, 1}$

Step 2: C.F

$C.F = e^x[c_1 + c_2 x]$

Step 3: $P.I = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{D^2 - 2D + 1} x e^x \sin x$ ↳ EAT: Exp. Algebraic Trigonometric

$P.I = e^x \times \frac{1}{(D+1)^2 - 2(D+1) + 1} x \sin x = e^x \times \frac{1}{D^2 + 2D + 1 - 2D - 2 + 1} \times x \sin x$

$= e^x \cdot \frac{1}{D^2} x \sin x$ Either integrate 2 times OR apply case (v) of P.I

Applying case (v): $Q(x) = x \cdot V(x)$; $V(x)$: $\sin x$ OR $\cos x$

$P.I = \left(x - \frac{f'(D)}{f(D)}\right) \frac{1}{f(D)} V(x)$

$P.I = e^x \left[\left(x - \frac{2D}{D^2}\right) \frac{1}{D^2} \sin x\right]$

$P.I = e^x \left[x \cdot \frac{1}{D^2} \sin x - \frac{2D}{D^4} \sin x\right] = e^x \left[x \cdot \frac{1}{D}(-\cos x) - \frac{2}{D^2 \cdot D} \sin x\right]$; $D^2 = -1$

$= e^x \left[x \cdot (-\sin x) - \frac{2}{(-1) \cdot D} \sin x\right] = e^x \left[-x \sin x + \frac{2}{D} \sin x\right]$

$= e^x \left[-x \sin x + 2(-\cos x)\right] = e^x[-x \sin x - 2\cos x] = -e^x[x \sin x + 2\cos x]$

$\boxed{G.S = e^x[c_1 + c_2 x] - e^x[x \sin x + 2\cos x]}$

(35) $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 20y = 40 \sin 4x$

Step 1: A.E

$D^2y - 8Dy + 20y = 40 \sin 4x$

$m^2 - 8m + 20 = 0$

$m^2 - 10m + 2m + 20 = 0$

$m = \frac{8 \pm \sqrt{64 - 80}}{2}$

$m = \frac{8 \pm 4i}{2}$, $\boxed{4 \pm 2i}$

$\boxed{C.F = e^{4x}[c_1 \cos 2x + c_2 \sin 2x]}$

Step 3: $P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 8D + 20} 40 \sin 4x$, $D^2 = -16$

$P.I = \frac{40 \sin 4x}{-8D + 4} = \frac{20 \sin 4x}{(2 - 4D)} \times \frac{(2+4D)}{(2+4D)}$

$= \frac{40 \sin 4x + 80 D \sin 4x}{4 - 16D^2} = \frac{40 \sin 4x + 80 \cos 4x \cdot (4)}{4 - 16(-16)}$

$= \frac{40 \sin 4x + 320 \cos 4x}{260} = \frac{4\cancel{40} \sin 4x + \overset{16}{\cancel{32}} \cos 4x}{\cancel{26}\,13}$

$\boxed{G.S = e^{4x}[c_1 \cos 2x + c_2 \sin 2x] + \frac{2 \sin 4x + 16 \cos 4x}{13}}$

(36) $(D^2+2D+5)y = 3^x$ $\qquad a^c = b \,;\ \log_a b = c$

Step 1: A.E $\qquad$ Step 2: $C.F = e^{-x}\left[c_1\cos 2x + c_2\sin 2x\right]$

$D^2+2D+5=0$ $\qquad$ Step 3: $P.I = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{D^2+2D+5} 3^x$

$m^2+2D+5=0$

$m = \frac{-2 \pm \sqrt{4-20}}{2}$ $\qquad e^{\ln} = 1 \,;\ 1.3^x = e^{\ln 3^x} = e^{x\ln 3}$

$m = \frac{-2 \pm 4i}{2} = \boxed{-1 \pm 2i}$ $\qquad e^{ax} = e^{x\ln 3} \,,\ \boxed{a = \ln 3}$

$P.I = \frac{1}{D^2+2D+5} 3^x \,,\ D = a;$

$$\boxed{P.I = \frac{1}{(\ln 3)^3 + 2\ln 3 + 5} 3^x}$$

(37) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 8x^2e^{2x}\sin 2x$

Step 1: A.E $\qquad$ Step 2: $C.F = e^{2x}(c_1 + c_2x)$

$D^2y - 4Dy + 4y = 8x^2e^{2x}\sin 2x$ $\qquad$ Step 3: $P.I = \frac{1}{f(D)} \cdot Q(x) = \frac{1}{D^2-4D+4} \cdot 8x^2e^{2x}\sin x$ EAT

$m^2-4m+4=0$

$m = \frac{4 \pm \sqrt{16-16}}{2} = \boxed{2,2}$ $\qquad P.I = 8e^{2x} \cdot \frac{1}{(D+2)^2-4(D+2)+4} \cdot x^2\sin 2x$

$P.I = 8e^{2x} \cdot \frac{1}{D^2+4D+\cancel{4}-4\cancel{D}-8+\cancel{4}} \cdot x^2\sin 2x = 8e^{2x} \cdot \boxed{\frac{1}{D^2} x^2 \cdot \sin 2x}$

$P.I = 8e^{2x} \cdot \frac{1}{D^2} x^2 \cdot \sin 2x \rightarrow$ Either Integrate 2 times OR Apply case (vi) of P.I

$\rightarrow$ Applying case (vi): $Q(x) = x^n \cdot V(x)$; $V(x)$: $\sin ax$ OR $\cos ax$

(1) $P.I = \frac{1}{f(D)} x^n \sin ax = \frac{1}{f(D)} x^n$ I.P of e^{iax} = IP of $\frac{1}{f(D)} e^{iax} \cdot x^n$

$e^{iax} = \cos\theta + i\sin\theta$ $\rightarrow$ I.P

$\rightarrow$ For Simplicity integrating 2 times

$P.I = 8e^{2x} \cdot \frac{1}{D}\int x^2\sin 2x \, dx$

$$P.I = 8e^{2x}\cdot\frac{1}{D}\int x^2\sin 2x\,dx = 8e^{2x}\cdot\frac{1}{D}\left[\frac{-x^2\cos 2x}{2}+\int\frac{\cos 2x}{2}(2x)\,dx\right]$$

$u=x^2;\ du=2x\,dx;\ dv=\sin 2x,\ v=\frac{-\cos 2x}{2}$

$\rightarrow u=x,\ du=dx;\ dv=\cos 2x;\ v=\sin 2x/2$

$$=8e^{2x}\cdot\frac{1}{D}\left[\frac{-x^2\cos 2x}{2}+\int x\cos 2x\,dx\right] = 8e^{2x}\cdot\frac{1}{D}\left[\frac{-x^2\cos 2x}{2}+\frac{x\sin 2x}{2}-\int\frac{\sin 2x}{2}dx\right]$$

$$=8e^{2x}\cdot\frac{1}{D}\left[\frac{-x^2\cos 2x}{2}+\frac{x\sin 2x}{2}-\int\frac{\sin 2x}{2}dx\right] = 4e^{2x}\cdot\frac{1}{D}\left[-x^2\cos 2x+x\sin 2x+\frac{\cos 2x}{2}\right]$$

$$=4e^{2x}\cdot\frac{1}{D}\left[-x^2\cos 2x+x\sin 2x+\frac{\cos 2x}{2}\right]$$

(1) $u=x^2;\ du=2x\,dx$; $dv=\cos 2x;\ v=\sin 2x/2$

$$=4e^{2x}\left[-\int x^2\cos 2x+\int x\sin 2x+\frac{1}{2}\int\cos 2x\right]dx$$

(2) $u=x;\ du=dx$; $dv=\sin 2x,\ v=\frac{-\cos 2x}{2}$

$$=4e^{2x}\left[-\frac{x^2\sin 2x}{2}+\int\frac{\sin 2x}{2}2x\,dx+\frac{x(-\cos 2x)}{2}+\int\frac{\cos 2x}{2}dx+\frac{1}{2}\frac{\sin 2x}{2}\right]$$

$$=4e^{2x}\left[\frac{-x^2\sin 2x}{2}+\int\sin 2x\cdot x\,dx-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}+\frac{\sin 2x}{4}\right]$$

$$=4e^{2x}\left[\frac{-x^2\sin 2x}{2}-\frac{x\cos 2x}{2}+\int\frac{\cos 2x}{2}dx-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}+\frac{\sin 2x}{4}\right]$$

$$=4e^{2x}\left[\frac{-x^2\sin 2x}{2}-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}-\frac{x\cos 2x}{2}+\frac{\sin 2x}{4}+\frac{\sin 2x}{4}\right]$$

$$=4e^{2x}\left[\frac{-x^2\sin 2x}{2}-\frac{2x\cos 2x}{2}+\frac{2\sin 2x}{4}+\frac{\sin 2x}{4}\right]$$

$$=4e^{2x}\left[\frac{-x^2\sin 2x}{2}-x\cos 2x+\frac{3\sin 2x}{4}\right]$$

$$=e^{2x}\left[-2x^2\sin 2x-4x\cos 2x+3\sin 2x\right]$$

$$=e^{2x}\left[3\sin 2x-2x^2\sin 2x-4x\cos 2x\right]$$

$$e^{2x}\left[\sin 2x(3-2x^2)-4x\cos 2x\right]$$

$$\boxed{G.S = e^{2x}(c_1+c_2x)+e^{2x}\left[\sin 2x(3-2x^2)-4x\cos 2x\right]}$$