Differential Equations

Assignment_03_Laplace_Transformation

Submitted to,

Respected sir:

IFTIKHAR
AHMED
BHUTTO

Submitted by:

Sagar chhabriya CS-2K22

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Signature

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Some useful Formulas:

Euler : $e^{i\theta} = \cos\theta + i\sin\theta$, Gamma $F(n)$: $\int_0^\infty e^{-x} \cdot x^n \, dx = \Gamma(n+1) = n!$

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$ $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}$

$\sin A \cdot \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$ $\cos^3\theta = \frac{3}{4}\cos\theta + \frac{\cos 3\theta}{4}$

$\cos A \cdot \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$\sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ $\sinh = \frac{e^\theta - e^{-\theta}}{2}$

$\cosh = \frac{e^\theta + e^{-\theta}}{2}$

$\sin^2\theta = \frac{1 - \sin 2\theta}{2}$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

Table of Laplace Transforms

$f(t) = L^{-1}\{F(s)\}$	$F(s) = L\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n,\ n=1,2,3,\dots$	$\frac{n!}{s^{n+1}}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{3/2}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$
$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$
$t^n f(t),\ n=1,2,3,\dots$	$(-1)^n F^{(n)}(s)$
$\frac{1}{t} f(t)$	$\int_s^\infty F(u)\,du$
$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
$f'(t)$	$sF(s)-f(0)$
$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$

Laplace Transformation

Suppose that $f(t)$ is a piecewise function. The Laplace transform of $f(t)$ is denoted $L\{f(t)\}$ and defined as

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$$

There is an alternate notation for Laplace transforms. For the sake of Convenience we will often denote Laplace transforms as,

$$L\{f(t)\} = F(s)$$

With this alternate notation, note that the transform is really a function of new variable s, and that all the t's will drop out in the integration process.

We'll start off with probably the simplest laplace transform to compute

① $L\{1\}$, $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt$

$$L\{1\} = \int_0^{\infty} e^{-st} \cdot (1)\, dt = \int_0^{\infty} e^{-st} dt = \left.\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \left[\frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s}\right] = \left[0 + \frac{1}{s}\right] = \boxed{\frac{1}{s}}$$

② Find $L\{e^{at}\}$

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} \cdot e^{at} \cdot dt = \int_0^{\infty} e^{-st+at} dt = \int_0^{\infty} e^{(-s+a)t}$$

$$= \int_0^{\infty} e^{-(s-a)t} dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty} = \left[\frac{e^{-(s-a)\infty}}{-(s-a)} - \frac{e^{-(s-a)(0)}}{-(s-a)}\right]$$

$$= \left[\frac{e^{-\infty}}{-(s-a)} + \frac{e^{0}}{s-a}\right] = 0 + \frac{1}{s-a} = \boxed{\frac{1}{s-a}}$$

③ Find $L\{\sin at\}$ and $L\{\cos at\}$

By Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$

Replace θ by "at" : $e^{iat} = \cos at + i\sin at$

Taking Laplace on b/s

$$L\{e^{iat}\} = L\{\cos at + i\sin at\}$$

$$\frac{1}{s-ia} = L\{\cos at + i\sin at\}$$

$$\frac{1}{s-ia} \times \frac{s+ia}{s+ia} = L\{\cos at + i\sin at\}$$

$$\frac{s+ia}{s^2-i^2a^2} = L\{\cos at + i\sin at\}$$

$$\frac{s+ia}{s^2-(-1)a^2} = L\{\cos at\} + i L\{\sin at\}$$

$$\frac{s+ia}{s^2+a^2} = L\{\cos at\} + i L\{\sin at\}$$

$$\frac{s}{s^2+a^2} + i\cdot\frac{a}{s^2+a^2} = L\{\cos at\} + i L\{\sin at\}$$

$$\frac{s}{s^2+a^2} = L\{\cos at\} \; ; \; \cancel{i}\frac{a}{s^2+a^2} = \cancel{i} L\{\sin at\}$$

④ $L\{t^n\}$

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n \, dt$$

put $\cancel{x} = st$, then $t = \frac{x}{s}$ and $dt = \frac{dx}{s}$

Interchange limits $x = st$ — $t \to 0$ then $x \to 0$; $t \to \infty$ then $x \to \infty$

$$L\{t^n\} = \int_0^\infty e^{-st} \cdot t^n \cdot dt = \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^n \cdot dt$$

$$= \int_0^\infty e^{-x} \cdot \left(\frac{x}{s}\right)^n \frac{dx}{s} = \int_0^\infty e^{-x} \cdot \frac{x^n}{s^n} \cdot \frac{dx}{s}$$

$$= \int_0^\infty e^{-x} \cdot \frac{x^n}{s^n \cdot s} \, dx = \int_0^\infty e^{-x} \cdot \frac{x^n}{s^{n+1}} dx = \frac{1}{s^{n+1}} \int_0^\infty e^{-x} \cdot x^n dx$$

$$= \frac{1}{s^{n+1}} \Gamma_{n+1} \rightarrow \text{Gamma function} = \boxed{\frac{n!}{s^{n+1}}}$$

$\therefore$ Gamma function $= \int_0^\infty e^{-x} \cdot x^n \, dx = \Gamma_{n+1} = n!$

⑤ $L\{\sin^2(3t)\}$ $\quad \sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$L\{\sin^2(3t)\} = L\left\{\frac{1 - \cos 2(3t)}{2}\right\} = L\left\{\frac{1 - \cos 6t}{2}\right\}$$

$$= \frac{1}{2} L\{1 - \cos 6t\} = \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right]$$

⑥ $L\{\cos^2 t\}$ $\therefore \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$L\{\cos^2 t\} = L\left\{\frac{1+\cos 2\theta}{2}\right\} = \frac{1}{2}L\{1+\cos 2\theta\}$$

$$= \frac{1}{2}\left[L\{1\} + L\{\cos 2\theta\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

⑦ $L\{\sin^3(2t)\}$ $\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}$

$$L\{\sin^3(2t)\} = L\left\{\frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}\right\} = \frac{1}{4}L\{3\sin(2t) - \sin 6t\}$$

$$= \frac{1}{4}\left[L\{3\sin(2t)\} - L\{\sin 6t\}\right]$$

$$= \frac{1}{4}\left[3L\{\sin(2t)\} - L\{\sin 6t\}\right]$$

$$= \frac{1}{4}\left[3\left(\frac{2}{s^2+4}\right) - \frac{6}{s^2+36}\right] = \frac{3}{2}\left[\frac{1}{s^2+4} - \frac{1}{s^2+36}\right]$$

⑧ $L\{\cos^3 t\}$ $\therefore \cos^3\theta = \frac{3}{4}\cos\theta + \frac{\cos 3\theta}{4}$

$$L\{\cos^3 t\} = L\left\{\frac{3}{4}\cos t + \frac{\cos 3t}{4}\right\}$$

$$= \frac{3}{4}L\{\cos t\} + \frac{1}{4}L\{\cos 3t\}$$

$$= \frac{3}{4}\left(\frac{s}{s^2+1}\right) + \frac{1}{4}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{4}\left[\frac{3s}{s^2+1} + \frac{s}{s^2+9}\right]$$

→ Laplace Properties

① Property No 1: Multiplication by e^{at}

If $L\{f(t)\} = F(s)$ then proove that, $L\{e^{at} f(t)\} = F(s-a)$

$$L\{e^{at} f(t)\} = F(s-a)$$

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} \cdot e^{at} f(t)\, dt$$

$$= \int_0^\infty e^{-st+at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt \text{ , where}$$

$$\int_0^\infty e^{-st}\, dt = F(s) \quad ; \quad \int_0^\infty e^{-(s-a)t} = F(s-a)$$

② Property No 2: Multiplication by "t"

If $L\{f(t)\} = F(s)$ then proove that, $L\{t f(t)\} = -\frac{d}{ds} F(s)$

$$L\{t f(t)\} = -\frac{d}{ds} F(s)$$

$$= -\frac{d}{ds}\left[\int_0^\infty e^{-st} \cdot f(t)\, dt\right]$$

differentiate integral term using Leibnix rule.

$$L\{t f(t)\} = -\left[\int_0^\infty e^{-st} \cdot (-t) \cdot f(t)\, dt\right]$$

$$= \int_0^\infty t e^{-st} \cdot f(t)\, dt$$

(3) Division by "t" : Property No 3

If $L\{f(t)\} = F(s)$ then prove that $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds$

we'll solve the right side of equation because it includes an integral

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds \text{, where } F(s) = L\{f(t)\}$$

$$= \int_s^{\infty} [L\{f(t)\}]\,ds$$

$$= \int_s^{\infty} \left[\int_0^{\infty} e^{-st} \cdot f(t)\,dt\right] ds \text{, Interchange Integral}$$

$$= \int_0^{\infty} \left[\int_s^{\infty} e^{-st} \underbrace{f(t)}_{\text{Constant}}\,ds\right] dt$$

$$= \int_0^{\infty} \left[f(t) \int_s^{\infty} e^{-st}\,ds\right] dt$$

$$= \int_0^{\infty} f(t) \left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt \text{, put upper } s = \infty \text{, lower } s = s$$

$$= \int_0^{\infty} f(t) \left[\frac{e^{-\infty t}}{-t} - \frac{e^{-st}}{-t}\right] dt$$

$$= \int_0^{\infty} f(t) \left[0 + \frac{e^{-st}}{t}\right] dt$$

$$= \int_0^{\infty} e^{-st} \frac{f(t)}{t}\,dt$$

$\therefore L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\,dt$, so that

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^{\infty} e^{-st} \frac{f(t)}{t}\,dt$$

④ Property No 4: Derivative Property

If $L\{f(t)\} = \bar{F}(s)$ then prove that $L\{f'(t)\} = s\,\bar{F}(s) - f(0)$

Integral | Derivative part of Laplace is solved first.

$L\{f'(t)\} = \bar{F}(s) - f(0)$; $L\{f'(t)\} = \int_0^{\infty} e^{-st}.\, f'(t)\, dt$

$$L\{f'(t)\} = \int_0^{\infty} e^{-st}.\, f'(t)\, dt$$

∴ Integration by parts

$\Rightarrow$ $u = e^{-st}$ $\quad dv = f'(t)\, dt$

$du = e^{-st}.(-s)\, dt$ $\quad v = f(t)$

$$L\{f'(t)\} = \int_0^{\infty} e^{-st}.\, f'(t)\, dt$$

$$= \left[(e^{-st})(f(t))\right]_0^{\infty} - \int_0^{\infty} f(t).\, e^{-st}(-s)\, dt$$

$$= \left[e^{-st} f(t)\right]_0^{\infty} + s\int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$= \left[e^{-s\infty} f(\infty) - e^{-0} f(0)\right] + s\int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$= \left[\, 0 - f(0)\right] + s\int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$= s\int_0^{\infty} e^{-st}.\, f(t)\, dt - f(0)$$

$\Rightarrow$ $L\{f(t)\} = \int_0^{\infty} e^{-st}.\, f(t)\, dt = \bar{F}(s)$

$$= s\,\bar{F}(s) - f(0)$$

⑤ Property No 5: Integral Property of Laplace

If $L\{f(t)\} = \bar{F}(s)$ then prove that, $L\{\int_0^t f(t)\,dt\} = \frac{1}{s}\bar{F}(s)$

$L\{\int_0^t f(t)\,dt\} = \frac{1}{s}\bar{F}(s);$

let $\phi(t) = \int_0^t f(t)\,dt$ — ①

$\phi'(t) = \frac{d}{dt}\left[\int_0^t f(t)\,dt\right]$

$\boxed{\phi'(t) = f(t)}$

equation ① at $t=0$, $\boxed{\phi(0)=0}$

↳ upper lower limit

$L\{\int_0^t f(t)\,dt\} = \frac{1}{s}\bar{F}(s);$

prove $L\{\int_0^t f(t)\,dt\} = \frac{1}{s}\bar{F}(s)$

$\therefore L\{f'(t)\} = s\bar{F}(s) - f(0)$, $f \leftrightarrow \phi$

↳ $L\{\phi'(t)\} = s\bar{\phi}(s) - \phi(0)$; $\phi'(t) = f(t)$, $\phi(0)=0$

$L\{f(t)\} = s\bar{\phi}(s) - 0$

$L\{f(t)\} = s\bar{\phi}(s)$

$\therefore \bar{F}(s) = L\{f(t)\}$, so that $\bar{\phi}(s) = L\{\phi(t)\}$

$L\{f(t)\} = s\,L\{\phi(t)\}$, $f \leftrightarrow \phi$

$L\{f(t)\} = s\,L\{\phi(t)\}$, $\phi(t) = \int_0^t f(t)\,dt$

$L\{f(t)\} = s\,L\{\int_0^t f(t)\,dt\}$

$\bar{F}(s) = s\,L\{\int_0^t f(t)\,dt\}$

$\frac{\bar{F}(s)}{s} = L\{\int_0^t f(t)\,dt\}$

⑥ Property No 6: Laplace of unit step function (or Heaviside)

If $L\{f(t)\} = \bar{F}(s)$, then proove that $L\{u(t-a)f(t-a)\} = e^{-as}\bar{F}(s)$

$$L\{u(t-a)f(t-a)\} = e^{-as}\bar{F}(s)$$

$$\therefore L\{u(t-a)f(t-a)\} = \int_0^\infty e^{-st}.\, u(t-a)f(t-a)\,dt, \text{ then}$$

$$\int_0^\infty e^{-st}.\, u(t-a)f(t-a)\,dt = e^{-as}\bar{F}(s)$$

⇒ Unit step function

$$u(t-a) = \begin{cases} 0, & t<0 \\ 1, & t>a \end{cases}$$

$$\int_0^a e^{-st}.\underbrace{u(t-a)}_{0}f(t-a)\,dt + \int_a^\infty e^{-st}.\underbrace{u(t-a)}_{1}f(t-a)\,dt = e^{-as}\bar{F}(s)$$

$0 < t < a$

$a < t < \infty$

$$\int_0^a e^{-st}.(0).f(t-a)\,dt + \int_a^\infty e^{-st}.(1).f(t-a)\,dt = e^{-as}\bar{F}(s)$$

$$0 + \int_a^\infty e^{-st}.(1).f(t-a)\,dt = e^{-as}\bar{F}(s)$$

$$\int_a^\infty e^{-st}f(t-a)\,dt = e^{-as}\bar{F}(s) \quad , \quad \text{put } t-a = u, \; dt = du$$

$$\int_a^\infty e^{-s(u+a)}f(u)\,du = e^{-as}\bar{F}(s)$$

→ constant term

$$\int_a^\infty e^{-su}.e^{-sa}.f(u)\,du = e^{-as}\bar{F}(s)$$

$$e^{-sa}\int_a^\infty e^{-su}.f(u)\,du = e^{-as}\bar{F}(s)$$

$$\boxed{e^{-sa}\bar{F}(s) = e^{-as}\bar{F}(s)}$$

(1) Property No 7: Periodic Function

Let $f(t)$ be a periodic function with period T, then proove that

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \quad \text{(by definition)}$$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt \cdots$$

Periodic Function Condition ⇒ $f(t) = f(t+T) = f(t+2T) = f(t+3T)$

put $t = u+T$ in Second integral, $dt = du$

$t = u+2T$ in third integral and so on ...

Replace limit points, the first integral's limit points will remain same

$t \to T$ in $t = u+T$, $u \to 0$; $t \to 2T$ in $t = u+2T$, $u \to 0$

$t \to 2T$ in $t = u+T$, $u \to T$; $t \to 3T$ in $t = u+2T$, $u \to T$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_T^{2T} e^{-st} f(t)\,dt + \int_{2T}^{3T} e^{-st} f(t)\,dt \cdots$$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-s(u+T)} f(u+T)\,du + \int_0^T e^{-s(u+2T)} f(u+2T)\,du$$

$\therefore\ f(t) = f(t+T) = f(t+2T) = f(t+3T) \cdots$

$$L\{f(t)\} = \int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-su}\cdot e^{-sT} f(u)\,du + \int_0^T e^{-su}\cdot e^{-2sT} f(u)\,du \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du \cdots$$

$$= (1 + e^{-sT} + e^{-2sT})\left(\int_0^T e^{-st} f(t)\,dt + \int_0^T e^{-su} f(u)\,du + \int_0^T e^{-su} f(u)\,du \cdots\right.$$

replace "u" with "t" in 2nd, 3rd ...

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-su} f(u)\,du + e^{-2sT}\int_0^T e^{-su} f(u)\,du \cdots$$

$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT}\int_0^T e^{-st} f(t) + e^{-2sT}\int_0^T e^{-st} f(t)\,dt \cdots$$

$$= (1 + e^{-sT} + e^{-2sT}) \int_0^T e^{-st} f(t)\,dt$$

$\therefore\ 1 + a + a^2 + a^3 + \cdots = \frac{1}{1-a}$

$$L\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

8) Property no 8: Change of Scale Property

If $L\{f(t)\} = \bar{F}(s)$, then prove that $L\{f(at)\} = \frac{1}{a}\bar{F}\left(\frac{s}{a}\right)$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

put $at = t'$ $\quad t \to 0,\ t' \to 0$

$t = \frac{t'}{a}$, $\quad t \to \infty,\ t' \to \infty$

$dt = \frac{dt'}{a}$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt$$

$$L\{f(t')\} = \int_0^\infty e^{-s\left(\frac{t'}{a}\right)} f(t')\frac{dt'}{a}$$

$$L\{f(t')\} = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)t'} \cdot f(t')\, dt', \quad \text{let } \frac{s}{a} = s'$$

$$L\{f(t')\} = \frac{1}{a}\int_0^\infty e^{-s't'} \cdot f(t')\, dt'$$

$$L\{f(t')\} = \frac{1}{a}\int_0^\infty e^{-st} f(t)\, dt$$
$$= \frac{1}{a}\int_0^\infty e^{-s't'} f(t')\, dt'$$

$$\therefore L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \bar{F}(s)$$
$$\therefore L\{f(t')\} = \int_0^\infty e^{-s't'} f(t')\, dt' = \bar{F}(s)$$

$$L\{f(t')\} = \frac{1}{a}\bar{F}(s'), \text{ when } s' = \frac{s}{a}$$

$$L\{f(t')\} = \frac{1}{a}\bar{F}\left(\frac{s}{a}\right)$$

Exercises

① $L\{\sin 2t \cdot \cos 3t\}$ $\quad \therefore \sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Sol

$L\{\sin 2t \cdot \cos 3t\} = L\{\frac{1}{2}[\sin(2t+3t) + \sin(2t-3t)]\}$

$= \frac{1}{2} L\{\sin 5t + \sin(-t)\}$

$= \frac{1}{2}[L\{\sin 5t\} - L\{\sin(t)\}]$

$= \boxed{\frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]}$ $\quad \therefore L\{\sin at\} = \frac{a}{s^2+a^2}$

② $L\{\cos 2t \cdot \cos 4t\}$ $\quad \therefore \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$L\{\cos 2t \cdot \cos 4t\} = L\{\frac{1}{2}[\cos(2t+4t) + \cos(2t-4t)]\}$

$= \frac{1}{2} L\{\cos(6t) + \cos(2t)\} = \frac{1}{2}[L\{\cos(6t)\} + L\{\cos(2t)\}]$

$= \boxed{\frac{1}{2}\left[\frac{s}{s^2+36} + \frac{s}{s^2+4}\right]}$

③ $L\{\sin 3t \cdot \sin 5t\}$ $\quad \therefore \sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$L\{\sin 3t \cdot \sin 5t\} = L\{\frac{1}{2}[\cos(3t-5t) - \cos(3t+5t)]\}$

$= \frac{1}{2} L\{\cos(2t) - \cos(8t)\} = \frac{1}{2}[L\{\cos 2t\} - L\{\cos(8t)\}]$

$= \boxed{\frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+64}\right]}$

4) $L\{\sinh^2(2t)\}$ $\quad \because \sinh = \frac{e^{+\theta} - e^{-\theta}}{2}$

$L\{\sinh^2(2t)\} = L\{\left(\frac{e^{2t} - e^{-2t}}{2}\right)\}$

$= \frac{1}{2} L\{e^{2t} - e^{-2t}\}$

$= \frac{1}{2}\left[L\{e^{2t}\} - L\{e^{-2t}\}\right]$

$= \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+2}\right]$

(5) $L\{\sinh^2(2t)\}$

$L\{\sinh^2(2t)\} = L\{\left(\frac{e^{2t} - e^{-2t}}{2}\right)^2\}$

$= L\{\frac{(e^{2t} - e^{-2t})^2}{4}\}$

$= \frac{1}{4} L\{e^{4t} - 2e^{2t-2t} + e^{-4t}\}$

$= \frac{1}{4} L\{e^{4t} - 2 + e^{-4t}\}$

$= \frac{1}{4}\left[L\{e^{4t}\} - L\{2\} + L\{e^{-4t}\}\right]$

$= \boxed{\frac{1}{4}\left[\frac{1}{s-4} - \frac{2}{s} + \frac{1}{s+4}\right]}$

(6) $L\{\sinh(2t) \cdot \cosh(3t)\}$ $\quad \because \sinh = \frac{e^{\theta} - e^{-\theta}}{2}, \quad \cosh = \frac{e^{\theta} + e^{-\theta}}{2}$

$L\{\sinh(2t) \cdot \cosh(3t)\} = L\{\frac{e^{2t} - e^{-2t}}{2} \cdot \frac{e^{3t} + e^{-3t}}{2}\}$

$= \frac{1}{4} L\{(e^{2t} - e^{-2t})(e^{3t} + e^{-3t})\} = \frac{1}{4}\{e^{5t} + e^{-t} - e^{t} - e^{-5t}\}$

$= \frac{1}{4}\left[L\{e^{5t}\} + L\{e^{-t}\} - L\{e^{t}\} - L\{e^{-5t}\}\right]$

$= \boxed{\frac{1}{4}\left[\frac{1}{s-5} + \frac{1}{s+1} - \frac{1}{s-1} - \frac{1}{s+5}\right]}$

(7) Find $L\{e^{2t}.\sin 3t\}$

$\Rightarrow L\{\sin 3t\} = \frac{3}{s^2+9}$

multiply e^{2t} on L.H.S and replace s by $(s-a)$ on R.H.S, where $a = 2$.

$$L\{e^{2t}\sin 3t\} = \frac{3}{(s-2)^2+9} = \frac{3}{s^2-4s+4+9}.$$

$$\boxed{L\{e^{2t}\sin 3t\} = \frac{3}{s^2-4s+13}}$$

(8) Find $L\left\{\frac{t}{2a}\sin at\right\}$

$$L\left\{\frac{t}{2a}\sin at\right\} = \frac{1}{2a}L\{t\sin at\}$$

$\therefore L\{\sin at\} = \frac{a}{s^2+a^2}$ — (1) eqⁿ

multiply equation (1) L.H.S with "t" and take $-\frac{d}{ds}$ on R.H.S

$$\frac{1}{2a}L\{t\sin at\} = -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -\left[\frac{(s^2+a^2)(0)-(a)(2s)}{(s^2+a^2)^2}\right] = \frac{2as}{(s^2+a^2)^2}$$

$$\frac{1}{\cancel{2a}}L\{t\sin at\} = \frac{\cancel{2a}\,s}{(s^2+a^2)^2}$$

$$\boxed{L\{\sin at\} = \frac{s}{(s^2+a^2)^2}}$$

9) Find $L\{\sinh(3t)\cos(2t)\}$

$$L\{\sinh(3t)\cos(2t)\} = L\left\{\frac{e^{3t}-e^{-3t}}{2}\cdot\frac{e^{2t}+e^{-2t}}{2}\right\}$$

$$= \frac{1}{4}L\{(e^{3t}-e^{-3t})\cdot(e^{2t}+e^{-2t})\}$$

$$= \frac{1}{4}L\{e^{5t}+e^{t}-e^{-t}-e^{-5t}\}$$

$$= \frac{1}{4}\left[L\{e^{5t}\}+L\{e^{t}\}-L\{e^{-t}\}-L\{e^{-5t}\}\right]$$

$$= \boxed{\frac{1}{4}\left[\frac{1}{s-5}+\frac{1}{s-1}-\frac{1}{s+1}-\frac{1}{s+5}\right]}$$

10) Find $L\{t^2.\cos 3t\}$

$$\Rightarrow L\{\cos 3t\} = \frac{s}{s^2+9} \quad \text{—} ①$$

- multiply "t" on L.H.S and take $-\frac{d}{ds}$ on R.H.S

$$L\{t\cos 3t\} = -\frac{d}{ds}\left[\frac{s}{s^2+9}\right] = -\left[\frac{(s^2+9)(1)-s(2s)}{(s^2+9)^2}\right]$$

$$= -\left[\frac{s^2+9-2s^2}{(s^2+9)^2}\right] = -\left[\frac{9-s^2}{(s^2+9)^2}\right] = \frac{s^2-9}{(s^2+9)^2}$$

- Multiply "t" on L.H.S and take $-\frac{d}{ds}$ on R.H.S again.

$$L\{t^2\cos 3t\} = -\frac{d}{ds}\left[\frac{s^2-9}{(s^2+9)^2}\right] = -\left[\frac{(s^2+9)^2(2s)-(s^2-9)(2(s^2+9)(2s))}{(s^2+9)^4}\right]$$

$$= -\left[\frac{2s(s^2+9)^2-4s(s^2-9)(s^2+9)}{(s^2+9)^4}\right]$$

$$= -\left[\frac{2s(s^2+9)-4s(s^2-9)}{(s^2+9)^3}\right]$$

$$= -\left[\frac{2s^3+18s-4s^3+36s}{(s^2+9)^3}\right] = -\left[\frac{-2s^3+54s}{(s^2+9)^3}\right]$$

$$\boxed{L\{t^2\cos 3t\} = \frac{2s^3-54s}{(s^2+9)^3}}$$

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(ii) $L\{(t-1)^2 \sin t\}$

$$L\{(t^2-2t+1)\sin t\} = L\{t^2\sin t - 2t\sin t + \sin t\}$$

$$= \underset{(1)}{L\{t^2\sin t\}} - 2\underset{(2)}{L\{t\sin t\}} + \underset{(3)}{L\{\sin t\}}$$

(1) $L\{t^2 \sin t\}$

$$\Rightarrow L\{\sin t\} = \frac{1}{s^2+1}$$

- multiply "t" on L.H.S and take $-\frac{d}{ds}$ on R.H.S.

$$L\{t\sin t\} = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\left[\frac{(s^2+1)(0)-(1)(2s)}{(s^2+1)^2}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

- multiply "t" on L.H.S and take $-\frac{d}{ds}$ on R.H.S

$$L\{t^2\sin t\} = -\frac{d}{ds}\left[\frac{2s}{(s^2+1)^2}\right] = -2\left[\frac{(s^2+1)^2(1)-2s(s^2+1)(2s)}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{(s^2+1)^2-4s^2(s^2+1)}{(s^2+1)^4}\right] = -2\left[\frac{(s^2+1)-4s^2}{(s^2+1)^3}\right]$$

$$= -2\left[\frac{1-3s^2}{(s^2+1)^3}\right] = \frac{6s^2-2}{(s^2+1)^3}$$

(2) $L\{t\sin t\} = \frac{2s}{(s^2+1)^2}$; (3) $L\{\sin t\} = \frac{1}{s^2+1}$

$$\therefore L\{(t^2-2t+1)\sin t\} = L\{t^2\sin t\} - 2L\{t\sin t\} + L\{\sin t\}$$

$$= \boxed{\frac{6s^2-2}{(s^2+1)^3} - \frac{4s}{(s^2+1)^2} + \frac{1}{s^2+1}}$$

(13) $L\left\{\frac{\sin^2 t}{t}\right\}$

$$\Rightarrow L\{\sin^2 t\} = L\left\{\frac{1-\cos 2t}{2}\right\} = \frac{1}{2}L\{1-\cos 2t\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\cos 2t\}\right]$$

$$L\{\sin^2 t\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{2}\left[\int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{s}{s^2+4}\,ds\right]$$ let $u = s^2+4$, $\frac{du}{2} = s\,ds$

$$= \frac{1}{2}\left[\log(s)\Big|_s^\infty - \frac{1}{2}\int_s^\infty \frac{du}{u}\right] = \frac{1}{2}\left[\log(s)\Big|_s^\infty - \frac{1}{2}\log(s^2+4)\Big|_s^\infty\right]$$

$$= \frac{1}{2}\left[\log(s) - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{2}{2}\times\frac{1}{2}\left[\log(s) - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[2\log(s) - \log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log(s^2) - \log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty = \frac{1}{4}\left[\log\left(\frac{\cancel{s^2}}{\cancel{s^2}}\,\frac{1}{1+\frac{4}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{1}{1+\frac{4}{s^2}}\right]_s^\infty = \frac{1}{4}\left[\log\frac{1}{1+\frac{4}{\infty}} - \log\frac{1}{1+\frac{4}{s^2}}\right]$$

$$= \frac{1}{4}\left[\log\left(\frac{1}{1+0}\right) - \log\left(\frac{1}{\frac{s^2+4}{s^2}}\right)\right] = \frac{1}{4}\left[0 - \log\left(\frac{s^2}{s^2+4}\right)\right]$$

$$= -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right) = \frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right)^{-1} = \boxed{\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)}$$

(14) Find $L\left\{\dfrac{\cos(at)-\cos(bt)}{t}\right\}$

$$\Rightarrow L\{\cos at\} - L\{\cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

Divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S.

$$L\left\{\frac{\cos at - \cos(bt)}{t}\right\} = \int_s^\infty \frac{s}{s^2+a^2} - \int_s^\infty \frac{s}{s^2+b^2}$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{\left(1+\frac{a^2}{s^2}\right)}{\left(1+\frac{b^2}{s^2}\right)}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+\frac{a^2}{\infty}}{1+\frac{b^2}{\infty}}\right) - \log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1+0}{1+0}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right) = \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)^{-1}$$

$$\boxed{L\left\{\frac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)}$$

(15) Find $L\left\{\frac{e^{at}-e^{bt}}{t}\right\}$

$\Rightarrow L\{e^{at}-e^{bt}\} = \frac{1}{s-a} - \frac{1}{s-b}$

Divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S

$$L\left\{\frac{e^{at}-e^{bt}}{t}\right\} = \int_s^\infty \frac{1}{s-a} - \int_s^\infty \frac{1}{s-b}$$

$$= \left[\log(s-a) - \log(s-b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s-a}{s-b}\right)\right]_s^\infty = \left[\log\left(\frac{s\left(1-\frac{a}{s}\right)}{s\left(1-\frac{b}{s}\right)}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1-\frac{a}{s}}{1-\frac{b}{s}}\right)\right]_s^\infty = \left[\log\left(\frac{1-\frac{a}{\infty}}{1-\frac{b}{\infty}}\right) - \log\left(\frac{1-\frac{a}{s}}{1-\frac{b}{s}}\right)\right]$$

$$= \left[\log\left(\frac{1-0}{1-0}\right) - \log\left(\frac{s-a}{s-b}\right)\right]$$

$$= \left[\log(1) - \log\left(\frac{s-a}{s-b}\right)\right] = 0 - \log\left(\frac{s-a}{s-b}\right)$$

$$= \log\left(\frac{s-a}{s-b}\right)^{-1} = \log\left(\frac{s-b}{s-a}\right)$$

$$\boxed{L\left\{\frac{e^{at}-e^{bt}}{t}\right\} = \log\left(\frac{s-b}{s-a}\right)}$$

(16) Find: $L\{e^{-2t} \cdot t \cdot \cos 3t\}$

$$\Rightarrow L\{\cos 3t\} = \frac{s}{s^2+9}$$

- Multiply L.H.S by "t" and take $-\frac{d}{ds}$ on R.H.S

$$L\{t\cos 3t\} = -\frac{d}{ds}\left[\frac{s}{s^2+9}\right]$$

$$= -\left[\frac{(s^2+9)(1) - (2s)s}{(s^2+9)^2}\right]$$

$$= -\left[\frac{s^2+9-2s^2}{(s^2+9)^2}\right] = -\left[\frac{9-s^2}{(s^2+9)^2}\right]$$

$$L\{t\cos 3t\} = \left[\frac{s^2-9}{(s^2+9)^2}\right]$$

- multiply L.H.S e^{-2t} and Replace s by $(s+2)$ on R.H.S

$$L\{e^{-2t} \cdot t \cdot \cos 3t\} = \left[\frac{(s+2)^2-9}{((s+2)^2+9)^2}\right]$$

$$= \left[\frac{s^2+2(s)(2)+4-9}{((s+2)^2+9)^2}\right]$$

$$= \left[\frac{s^2+4s-5}{((s+2)^2+9)^2}\right]$$

$$\boxed{L\{e^{-2t} \cdot t \cdot \cos 3t\} = \left[\frac{s^2+4s-5}{((s+2)^2+9)^2}\right]}$$

(17) $L\{t\sin^2(3t)\}$ $\quad \therefore \sin^2\theta = \dfrac{1-\sin 2\theta}{2}$

$$L\{\sin^2(3t)\} = \left\{\frac{1-\sin 2(3t)}{2}\right\} = \left\{\frac{1-\sin(6t)}{2}\right\}$$

$$L\{\sin^2(3t)\} = \frac{1}{2} L\{1-\sin(6t)\}$$

$$= \frac{1}{2}\left[L\{1\} - L\{\sin(6t)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{6}{s^2+36}\right]$$

$\Rightarrow$ Multiply L.H.S by "t" and take $-\frac{d}{ds}$ on R.H.S

$$L\{t\sin^2(3t)\} = -\frac{1}{2}\frac{d}{ds}\left[\frac{1}{s} - \frac{6}{s^2+36}\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} - \left(\frac{0-6(2s)}{(s^2+36)^2}\right)\right]$$

$$= -\frac{1}{2}\left[-\frac{1}{s^2} + \frac{12s}{(s^2+36)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s^2} - \frac{12s}{(s^2+36)^2}\right]$$

$$\boxed{L\{t\sin^2(3t)\} = \frac{1}{2}\left[\frac{1}{s^2} - \frac{12s}{(s^2+36)^2}\right]}$$

(18) Find $L\left\{ \frac{e^{3t}\sin(2t)}{t} \right\}$

$\Rightarrow L\{\sin(2t)\} = \frac{2}{s^2+4}$

• divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S

$$L\left\{ \frac{\sin(2t)}{t} \right\} = \int_s^\infty \frac{2}{s^2+4}\, ds = 2\int_s^\infty \frac{ds}{s^2+4}$$

$$= 2\left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty \quad \therefore \text{trigonometric Substitution}$$

$$= 2\left[\tan^{-1}\infty - \tan^{-1}\frac{s}{2}\right] \quad \therefore \tan^{-1}\infty = \frac{\pi}{2}$$

$$= 2\left[\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right]$$

$$\therefore \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$$

$$L\left\{ \frac{\sin(2t)}{t} \right\} = 2\cot^{-1}\left(\frac{s}{2}\right)$$

• Multiply L.H.S by "e^{3t}" and Replace s by (s−3) on R.H.S

$$\boxed{L\left\{ \frac{e^{3t}\sin(2t)}{t} \right\} = 2\cot^{-1}\left(\frac{s-3}{2}\right)}$$

(18) Find $L\left\{ \frac{1-e^t}{t} \right\}$

$L\{1-e^t\} = \frac{1}{s} - \frac{1}{s-1}$

divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S

$$L\left\{ \frac{1-e^t}{t} \right\} = \int_s^\infty \frac{1}{s}\, ds - \int_s^\infty \frac{1}{s-1}\, ds$$

$$= \left[\log(s) - \log(s-1)\right]_s^\infty = \left[\log\left(\frac{s}{s-1}\right)\right]_s^\infty = \left[\log\left(\frac{1}{1-\frac{1}{s}}\right)\right]_s^\infty$$

$$= \log\left(\frac{1}{1-\frac{1}{\infty}}\right) - \log\left(\frac{1}{1-\frac{1}{s}}\right) = \log(1) - \log\left(\frac{s}{s-1}\right)$$

$$= 0 - \log\left(\frac{s}{s-1}\right) = \log\left(\frac{s}{s-1}\right)^{-1} = \boxed{\log\left(\frac{s-1}{s}\right) = L\left\{\frac{1-e^t}{t}\right\}}$$

(20) $\mathcal{L}\left\{\frac{1-\cos t}{t}\right\}$

$$\mathcal{L}\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$$

Divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S

$$\mathcal{L}\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{s}{s^2+1}\,ds$$

$$= \left[\log(s) - \frac{1}{2}\log(s^2+1)\right]_s^\infty \quad \therefore \text{ u- substitution}$$

$$= \frac{1}{2}\left[2\log(s) - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(s^2) - \log(s^2+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2}{s^2\left(1+\frac{1}{s^2}\right)}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{1}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{1}{\infty}}\right) - \log\left(\frac{1}{1+\frac{1}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right) = \frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)^{-1}$$

$$\boxed{\mathcal{L}\left\{\frac{1-\cos t}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)}$$

(21) $\dfrac{\cos 2t - \cos 3t}{t}$

$$L\{\cos 2t - \cos 3t\} = \frac{s}{s^2+4} - \frac{s}{s^2+9}$$

Divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S

$$L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_s^\infty \frac{s}{s^2+4}\,ds - \int_s^\infty \frac{s}{s^2+9}\,ds$$

$\Rightarrow$ $u = s^2+4$ $v = s^2+9$

$\frac{du}{2} = s\,ds$ $\frac{dv}{2} = s\,ds$

$$= \frac{1}{2}\int_s^\infty \frac{du}{u} - \frac{1}{2}\int_s^\infty \frac{dv}{v}$$

$$= \frac{1}{2}\left[\log(u) - \log(v)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(s^2+4) - \log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2+4}{s^2+9}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s^2\left(1+\frac{4}{s^2}\right)}{s^2\left(1+\frac{9}{s^2}\right)}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+\frac{4}{\infty}}{1+\frac{9}{\infty}}\right) - \log\left(\frac{1+\frac{4}{s^2}}{1+\frac{9}{s^2}}\right)\right]$$

$$= \frac{1}{2}\left[\log(1) - \log\left(\frac{s^2+4}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)^{-1}$$

$$\boxed{L\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right)}$$

22) $L\left\{\frac{1-e^{2t}}{t}\right\}$

$\Rightarrow L\{1-e^{2t}\} = \frac{1}{s} - \frac{1}{s-2}$

Divide L.H.S by "t" and take $\int_s^\infty ds$ on R.H.S

$$L\left\{\frac{1-e^{2t}}{t}\right\} = \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{1}{s-2}\,ds$$

$$= \left[\log(s) - \log(s-2)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-2}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s\left(1-\frac{2}{s}\right)}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1}{1-\frac{2}{s}}\right)\right]_s^\infty$$

$$= \left[\log\left(\frac{1}{1-\frac{2}{\infty}}\right) - \log\left(\frac{1}{1-\frac{2}{s}}\right)\right]_s^\infty$$

$$= \left[\log(1) - \log\left(\frac{s}{s-2}\right)\right]_s^\infty$$

$$= \log\left(\frac{s}{s-2}\right)^{-1} = \log\left(\frac{s-2}{s}\right)$$

$$\boxed{L\left\{\frac{1-e^{2t}}{t}\right\} = \log\left(\frac{s-2}{s}\right)}$$

(23) $t^2 \sin 3t$

$$L\{\sin 3t\} = \frac{3}{s^2+9}$$

- multiply L.H.S by "t" and take $-\frac{d}{ds}$ on R.H.S

$$L\{t \sin 3t\} = -3\frac{d}{ds}\left[\frac{1}{s^2+9}\right] = -3\left[\frac{-2s}{(s^2+9)^2}\right]$$

$$L\{t \sin 3t\} = \frac{6s}{(s^2+9)^2}$$

- multiply L.H.S by "t" and take $-\frac{d}{ds}$ on R.H.S

$$L\{t^2 \sin 3t\} = -6\frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{(s^2+9)^2 - 2s(s^2+9)(2s)}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{(s^2+9)^2 - 4s^2(s^2+9)}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{(s^2+9) - 4s^2}{(s^2+9)^3}\right]$$

$$= -6\left[\frac{9-3s^2}{(s^2+9)^3}\right]$$

$$\boxed{L\{t^2 \sin 3t\} = \frac{18s^2-54}{(s^2+9)^3}}$$

(24) $e^{-t}t^2\sin t$

$$L\{\sin t\} = \frac{1}{s^2+1}$$

• multiply L.H.S by "t" and take $-\frac{d}{ds}$ on R.H.S

$$L\{t\sin t\} = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\left[\frac{-2s}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}$$

• multiply L.H.S by "t" and take $-\frac{d}{ds}$ on R.H.S

$$L\{t^2\sin t\} = -\frac{d}{ds}\left[\frac{2s}{(s^2+1)^2}\right]$$

$$= -2\left[\frac{(s^2+1)^2 - 2s(s^2+1)(2s)}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{(s^2+1)^2 - 4s^2(s^2+1)}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{(s^2+1) - 4s^2}{(s^2+1)^3}\right]$$

$$L\{t^2\sin t\} = -2\left[\frac{1-3s^2}{(s^2+1)^3}\right] = \frac{6s^2-2}{(s^2+1)^3}$$

multiply L.H.S by "e^{-t}" and Replace s by $(s+1)$ on R.H.S

$$\boxed{L\{e^{-t}t^2\sin t\} = \frac{6(s+1)^2-2}{((s+1)^2+1)^3}}$$

(25) By using definition of $f(t)$ when

$$f(t) = \begin{cases} t-1, & 1 < t < 2 \\ 3-t, & 2 < t < 3 \\ 0, & \text{otherwise} \end{cases}$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^1 e^{-st} \cdot 0\, dt + \int_1^2 e^{-st}(t-1)\, dt + \int_2^3 e^{-st}(3-t)\, dt + \int_3^\infty e^{-st} \cdot 0\, dt$$

$$= 0 + \int_1^2 e^{-st}(t-1)dt + \int_2^3 e^{-st}(3-t)\, dt + 0$$

$$= \int_1^2 e^{-st}(t-1)\, dt + \int_2^3 e^{-st}(3-t)\, dt$$

$u = t-1$, $dv = e^{-st}dt$; $u = 3-t$, $dv = e^{-st}dt$

$du = dt$, $v = \frac{e^{-st}}{-s}$, $-du = dt$, $v = \frac{e^{-st}}{-s}$

$$= \int_1^2 u\, dv + \int_2^3 u\, dv$$

$$= uv - \int_1^2 v\, du + uv - \int_2^3 v\, du$$

$$= (t-1)\left(\frac{e^{-st}}{-s}\right)\Big]_1^2 - \int_1^2 \frac{e^{-st}}{-s}\, dt + (3-t)\left(\frac{e^{-st}}{-s}\right)\Big]_2^3 + \int_2^3 \frac{e^{-st}}{-s}\, du$$

$$= \frac{e^{-2s}}{-s} - 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_1^2 + \frac{e^{-2s}}{s} + \left[\frac{e^{-st}}{s^2}\right]_2^3$$

$$= \frac{e^{-2s}}{-s} - \frac{1}{s^2}\left[e^{-2s} - e^{-s}\right] + \frac{e^{-2s}}{s} + \frac{e^{-3s} - e^{-2s}}{s^2}$$

$$= -\cancel{\frac{e^{-2s}}{s}} - \frac{1}{s^2}\left[e^{-2s} - e^{-s}\right] + \cancel{\frac{e^{-2s}}{s}} + \frac{e^{-3s} - e^{-2s}}{s^2}$$

$$= -\frac{1}{s^2}\left[e^{-2s} - e^{-s}\right] + \frac{e^{-3s} - e^{-2s}}{s^2}$$

$$= \frac{1}{s^2}\left[e^{-s} - e^{-2s} + e^{-3s} - e^{-2s}\right]$$

$$= \boxed{\frac{1}{s^2}\left[e^{-s} + e^{-3s} - 2e^{-2s}\right]}$$

(26) $e^{3t} + t^4 - 2\sin 4t$

$$L\{e^{3t} + t^4 - 2\sin 4t\} = L\{e^{3t}\} + L\{t^4\} - 2L\{\sin 4t\}$$

$$= \frac{1}{s-3} + \frac{4!}{s^{4+1}} - 2 \times \frac{4}{s^2+16}$$

$$L\{e^{3t} + t^4 - 2\sin 4t\} = \frac{1}{s-3} + \frac{24}{s^5} - \frac{8}{s^2+16}$$

(27) $2 + 2\sqrt{t} + \frac{1}{\sqrt{t}}$

$$2L\{1\} + 2L\{\sqrt{t}\} + L\left\{\frac{1}{\sqrt{t}}\right\}$$

$$\frac{2}{s} + \frac{2\sqrt{\pi}}{2s^{3/2}} + \frac{\sqrt{\pi}}{\sqrt{s}} = \boxed{\frac{2}{s} + \frac{\sqrt{\pi}}{s^{3/2}} + \frac{\sqrt{\pi}}{\sqrt{s}}}$$

(28) $\cos(2t + b)$ $\quad \therefore \cos(A+B) = \cos A \cos B - \sin A \sin B$

$$L\{\cos(2t)\cos(b) - \sin(2t)\sin(b)\} = L\{\cos(2t)\cos(b)\} - L\{\sin(2t)\sin(b)\}$$

$$= \cos(b)\,L\{\cos(2t)\} - \sin(b)\,L\{\sin(2t)\}$$

$$= \cos(b)\left(\frac{s}{s^2+4}\right) - \sin(b)\left(\frac{2}{s^2+4}\right)$$

$$= \boxed{\frac{s\cos(b)}{s^2+4} - \frac{2\sin(b)}{s^2+4}}$$

(29) $(\sin t - \cos t)^2$

$L\{\sin^2 t - 2\sin t\cos t + \cos^2 t\}$

$= L\{\sin^2 t\} - 2L\{\sin t\cos t\} + L\{\cos^2 t\}$

$= L\left\{\frac{1-\cos 2t}{2}\right\} - 2L\{\sin t\cos t\} + L\left\{\frac{1+\cos 2t}{2}\right\}$

$= \frac{1}{2}L\{1-\cos 2t\} - 2L\{\sin t \times \cos t\} + \frac{1}{2}L\{1+\cos 2t\}$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] - 2L\{\sin t\cos t\} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

$\because \sin A \cdot \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] - 2\left[\sin(2t) + \sin(0)\right] + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] - 2L\{\sin(2t)\} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

$= \frac{1}{2}\left[\frac{1}{2} - \frac{s}{s^2+4}\right] - 2\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$

$$\boxed{L\{(\sin t - \cos t)^2\} = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] - 2\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]}$$

(30) $\sin 2t \cos 4t$

$L\{\sin 2t \cos 4t\} = \frac{1}{2}L\{\sin(2t+4t) + \sin(2t-4t)\}$

$= \frac{1}{2}L\{\sin(6t) + \sin(-2t)\} = L\{\sin(6t) - \sin(2t)\}$

$= \frac{1}{2}\left[\frac{6}{s^2+36} - \frac{2}{s^2+4}\right]$

(31) $\sin^2 4t + 3\cosh 4t$

$$L\{\sin^2 4t + 3\cosh 4t\} = L\{\sin^2 4t\} + 3L\{\cosh 4t\}$$

$$= L\left\{\frac{1-\cos 2(4t)}{2}\right\} + 3L\left\{\frac{e^{4t}+e^{-4t}}{2}\right\}$$

$$= \frac{1}{2}L\{1-\cos(8t)\} + \frac{3}{2}L\{e^{4t}+e^{-4t}\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+64}\right] + \frac{3}{2}\left[\frac{1}{s-4} + \frac{1}{s+4}\right]$$

(32) $\sin 2t \cosh 2t$ $\qquad$ $\therefore \sin A\cos B = \frac{1}{2}[\sin(A+B)+\sin(A-B)]$

$$L\{\sin 2t \cos 2t\}$$ $\qquad$ $\therefore \cosh t = \frac{e^{t}+e^{-t}}{2}$

$$= \frac{1}{2}L\{\sin(2t+2t) + \sin(2t-2t)\}$$

$$= \frac{1}{2}L\{\sin(4t) + \sin(0)\}$$

$$= \frac{1}{2}L\{\sin(4t)\} = \frac{1}{2}\left[\frac{4}{s^2+16}\right]$$

(33) $t^2 e^{-4t}$

$$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2!}{s^3} = \frac{2}{s^3}$$

Multiply L.H.S by e^{-4t} and replace R.H.S "s" by (s+4)

$$L\{e^{-4t}t^2\} = \frac{2}{(s+4)^3}$$

(34) If $L\{\cos^2 t\} = \frac{s^2+2}{s(s^2+4)}$, find $L\{\cos^2 4t\}$

$\because \cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$L\{\cos^2 4t\} = L\left\{\frac{1+\cos 2(4t)}{2}\right\}$$

$$= \frac{1}{2} L\{1+\cos(8t)\} = \frac{1}{2}\left[L\{1\} + L\{\cos(8t)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right]$$

(35) $\int_0^t e^{-t}\cos t\, dt$

$\int_0^t e^{-t}\cos t\, dt$ Integration by parts : $uv - \int v\,du = \int u\,dv$

$u = \cos t$, $du = -\sin t\, dt$;

$dv = e^{-t}dt$, $v = -e^{-t}$

$$\int_0^t e^{-t}\cos t\, dt = (\cos t)(-e^{-t})\Big]_0^t + \int_0^t e^{-t}(-\sin t)\,dt$$

$$= \left[-\cos t\, e^{-t} + 1\right] - \int_0^t e^{-t}\sin t\, dt$$

$u = \sin t$, $du = \cos t\, dt$

$dv = e^{-t}dt$, $v = -e^{-t}$

$$= 1 - e^{-t}\cos t - \left[(\sin t)(-e^{-t})\Big|_0^t - \int_0^t (-e^{-t})(\cos t)\,dt\right]$$

$$= 1 - e^{-t}\cos t - \left[-e^{-t}\sin t\Big|_0^t + \int_0^t e^{-t}\cos t\, dt\right]$$

$$= 1 - e^{-t}\cos t - \left[-e^{-t}\sin t - 0 + \int_0^t e^{-t}\cos t\, dt\right]$$

$$\int_0^t e^{-t}\cos t\, dt = 1 - e^{-t}\cos t + e^{-t}\sin t - \int_0^t e^{-t}\cos t\, dt$$

let $I = \int_0^t e^{-t}\cos t\, dt$

$$I = 1 - e^{-t}\cos t + e^{-t}\sin t - I$$

$$I + I = 1 - e^{-t}\cos t + e^{-t}\sin t$$

$$2I = 1 - e^{-t}\cos t + e^{-t}\sin t$$

$$I = \frac{1 - e^{-t}\cos t + e^{-t}\sin t}{2}$$; Replace I

$$\boxed{\int_0^t e^{-t}\cos t\, dt = \frac{1 - e^{-t}\cos t + e^{-t}\sin t}{2}}$$

Inverse Laplace Transform

Definition: If the Laplace transform of function $F(t)$ is $f(s)$ that is $L\{F(t)\} = f(s)$ then $F(t)$ is called an inverse Laplace transform of $f(s)$. Symbolically we may express it as: $L^{-1}\{f(s)\} = F(t)$ where L^{-1} denotes the inverse Laplace transformation operator.

We are going to be given a transform, $F(s)$, and ask what function (or functions) did we have originally. As you will see this can be a more complicated and lengthy process than taking transforms. In these cases we are finding the Inverse Laplace Transform of $F(s)$ and use the following notation.

$$f(t) = L^{-1}\{F(s)\}$$

As with Laplace transform, we have got the following fact to help us take the inverse transform.

Fact

Given two laplace transform $F(s)$ and $G(s)$ then

$$L^{-1}\{aF(s) + bG(s)\} = aL^{-1}\{F(s)\} + bL^{-1}\{G(s)\}$$

for any constant a and b.

① $F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$

$$L^{-1}\{F(s)\} = 6L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{1}{s-8}\right\} + 4L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$= 6(1) - e^{8t} + 4e^{3t}$$

$$= 6 - e^{8t} + 4e^{3t}$$

② $L^{-1}\{1\} = 0$, prove it

$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma n}, \text{ put } n=0$$

$$L^{-1}\left\{\frac{1}{s^0}\right\} = \frac{t^{0-1}}{\Gamma 0}$$

$$L^{-1}\left\{\frac{1}{1}\right\} = \frac{t^{-1}}{\Gamma 0}$$

$$L^{-1}\{1\} = \frac{t^{-1}}{\infty} = 0$$

$$\boxed{L^{-1}\{1\} = 0}$$

③ $L^{-1}\left\{\frac{1}{s-3}\right\}$ and $L^{-1}\left\{\frac{1}{s+3}\right\}$

$\Rightarrow L^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$ $\quad \because L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$

$\Rightarrow L^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$ $\quad \because L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$

④ $L^{-1}\left\{\frac{s}{s^2+25}\right\}$ and $L^{-1}\left\{\frac{s}{s^2-25}\right\}$

$\Rightarrow L^{-1}\left\{\frac{s}{s^2+25}\right\} = \cos(5t)$ $\quad \because L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos(at)$

$\Rightarrow L^{-1}\left\{\frac{s}{s^2-25}\right\} = \cosh(5t)$ $\quad \because L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh(at)$

⑤ $L^{-1}\left\{\frac{1}{s^2+25}\right\}$ and $L^{-1}\left\{\frac{1}{s^2-25}\right\}$

$\Rightarrow L^{-1}\left\{\frac{1}{s^2+25}\right\} = \frac{\sin(5t)}{5}$ $\quad \because L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin(at)}{a}$

$\Rightarrow L^{-1}\left\{\frac{1}{s^2-25}\right\} = \frac{\sinh(5t)}{5}$ $\quad \because L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sinh(at)}{a}$

⑥ $L^{-1}\left\{\frac{1}{s^2+1}\right\}$ and $L^{-1}\left\{\frac{3}{s^2+4}\right\}$

$\Rightarrow L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$ $\quad \because L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin(at)}{a}$

$\Rightarrow L^{-1}\left\{\frac{3}{s^2+4}\right\} = 3L^{-1}\left\{\frac{1}{s^2+4}\right\} = 3 \times \frac{\sin(2t)}{2} = \frac{3}{2}\sin(2t)$

⑦ $L^{-1}\left\{\frac{1}{(s-3)^2}\right\}$ and $L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$

$\Rightarrow L^{-1}\left\{\frac{1}{(s-3)^2}\right\}$ revaluate parameter $\rightarrow e^{3t}$, then evaluate $L^{-1}\left\{\frac{1}{s^n}\right\}$

where $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!}$, $(n-1)! = \Gamma n$ gamma function

$L^{-1}\left\{\frac{1}{(s-3)^2}\right\} = e^{3t} \cdot \frac{t^{2-1}}{(2-1)!}$

$= e^{3t} \cdot \frac{t}{1!} = e^{3t} t$

$$\boxed{L^{-1}\left\{\frac{1}{(s-3)^2}\right\} = e^{3t} \cdot t}$$

$\Rightarrow L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$

$L^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t} \cdot \frac{t^{2-1}}{(2-1)!} = e^{-3t} \cdot \frac{t}{1!} = e^{-3t} \cdot t$

$$\boxed{L^{-1}\left\{\frac{1}{(s+3)^2}\right\} = e^{-3t} \cdot t}$$

⑧ $L^{-1}\left\{\frac{(s-2)}{(s+2)^2+25}\right\}$

$\Rightarrow L^{-1}\left\{\frac{(s-2)}{(s-2)^2+25}\right\}$ first, parameter of exponential e^{2t} then trigonometric $\frac{a}{s^2+a^2} \to \cos(at)$

$L^{-1}\left\{\frac{(s-2)}{(s-2)^2+25}\right\} = e^{2t}\cdot\cos(5t)$

⑨ $L^{-1}\left\{\frac{(s-3)}{(s-3)^2-25}\right\}$, $L^{-1}\left\{\frac{1}{(s-1)^2+25}\right\}$, $L^{-1}\left\{\frac{1}{(s-2)^2-25}\right\}$

$\Rightarrow L^{-1}\left\{\frac{(s-3)}{(s-3)^2-25}\right\} = e^{3t}\cdot\cos(5t)$

$\Rightarrow L^{-1}\left\{\frac{1}{(s-1)^2+25}\right\} = e^{t}\cdot\frac{\sin(5t)}{5}$

$\Rightarrow L^{-1}\left\{\frac{1}{(s-2)^2-25}\right\} = e^{2t}\cdot\frac{\sinh(5t)}{5}$

⑩ $L^{-1}\left\{\frac{2s+3}{3s+7}\right\}$

$\Rightarrow L^{-1}\left\{\frac{2s}{3s+7}\right\} + L^{-1}\left\{\frac{3}{3s+7}\right\}$

$2L^{-1}\left\{\frac{s}{3s+7}\right\} + 3L^{-1}\left\{\frac{1}{3s+7}\right\}$

$2\left[L^{-1}\left\{\frac{s}{s}\frac{1}{(3+\frac{7}{s})}\right\}\right] + 3L^{-1}\left\{\frac{1}{3s+7}\right\}$

$\frac{2s+3}{3s+7} = \frac{2}{3} + \frac{-5/3}{3s+7} = \frac{2}{3} - \frac{5}{3(3s+7)}$ using long division

$L^{-1}\left\{\frac{2s+3}{3s+7}\right\} = \frac{2}{3}L^{-1}\{1\} - \frac{5}{3}L^{-1}\left\{\frac{1}{3s+7}\right\} = 0 - \frac{5}{3}L^{-1}\left\{\frac{1/3}{\frac{3s+7}{3}}\right\}$

$= -\frac{5}{3}L^{-1}\left\{\frac{1/3}{s+\frac{7}{3}}\right\} = -\frac{5}{9}L^{-1}\left\{\frac{1}{s+\frac{7}{3}}\right\}$

$\boxed{L^{-1}\left\{\frac{2s+3}{3s+7}\right\} = -\frac{5}{9}e^{-\frac{7}{3}t}}$

(11) Find $L^{-1}\left\{\dfrac{3s+7}{s^2+6s+9}\right\}$

$\Rightarrow L^{-1}\left\{\dfrac{3s+7}{(s+3)^2}\right\} \Rightarrow$ Solve using partial fraction

$$\frac{3s+7}{(s+3)^2} = \frac{A}{(s+3)} + \frac{B}{(s+3)^2}$$

$$3s+7 = A(s+3) + B$$

$$3s+7 = As + 3A + B$$

$$3s = As \;;\; 7 = 3A + B$$

$$\boxed{A = 3} \qquad \boxed{-2 = B}$$

$$\Rightarrow L^{-1}\left\{\frac{3s+7}{(s+3)^2}\right\} = L^{-1}\left\{\frac{3}{(s+3)} - \frac{2}{(s+3)^2}\right\}$$

$$= 3L^{-1}\left\{\frac{1}{s+3}\right\} - 2L^{-1}\left\{\frac{1}{(s+3)^2}\right\}$$

$$= 3 \times e^{-3t} - 2 \times e^{-3t} \times \frac{t^{2-1}}{(2-1)!}$$

$$= 3e^{-3t} - 2e^{-3t}t$$

$$= \boxed{e^{-3t}(3-2t) = L^{-1}\left\{\frac{3s+7}{(s+3)^2}\right\}}$$

(12) $L^{-1}\left\{\frac{3s+1}{s^2-2s-3}\right\}$

$\Rightarrow L^{-1}\left\{\frac{3s+1}{(s-1)^2-4}\right\}$, Solve using partial Fraction

$$\frac{3s+1}{s^2-2s-3} = \frac{3s+1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1}$$

$$3s+1 = A(s+1) + B(s-3)$$
$$3s+1 = As + A + Bs - 3B$$

$3 = A+B$, $1 = A - 3B$

$\boxed{3-B=A}$ $\quad 1 = 3-B-3B$

$-2 = -4B$

$\boxed{3/2 = A}$ $\quad \boxed{1/2 = B}$

$$L^{-1}\left\{\frac{3s+1}{(s-1)^2-4}\right\} = L^{-1}\left\{\frac{3s+1}{(s-3)(s+1)}\right\} = L^{-1}\left\{\frac{3/2}{s-3} + \frac{1/2}{s+1}\right\}$$

$$\frac{3}{2} L^{-1}\left\{\frac{1}{s-3}\right\} + \frac{1}{2} L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \boxed{\frac{3}{2} e^{3t} + \frac{1}{2} e^{-t}}$$

(13) Find $L^{-1}\left\{\frac{s^2-6}{s^3+4s^2+3s}\right\}$

$$L^{-1}\left\{\frac{s^2-6}{s(s^2+4s+3)}\right\} = L^{-1}\left\{\frac{s^2-6}{s(s+3)(s+1)}\right\}$$

$\Rightarrow$ Solve using partial Fraction

$$\frac{s^2-6}{s(s+3)(s+1)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+1}$$

$$s^2-6 = A(s+3)(s+1) + Bs(s+1) + Cs(s+3)$$
$$s^2-6 = A(s^2+4s+3) + B(s^2+s) + C(s^2+3s)$$
$$s^2-6 = As^2 + Bs^2 + Cs^2 + A4s + Bs + 3Cs + 3A$$

$A+B+C=1$; $\quad 4A+B+3C=0 \quad$ $3A=-6$

$B+C=3 \quad B+3C=8 \quad \boxed{A=-2}$

$\boxed{B=1/2} \quad \boxed{C=5/2}$

$$L^{-1}\left\{\frac{s^2-6}{s(s+3)(s-1)}\right\} = L^{-1}\left\{\frac{-2}{s}+\frac{1/2}{s+3}+\frac{5/2}{s+1}\right\}$$

$$= -2L^{-1}\left\{\frac{1}{s}\right\} + \tfrac{1}{2}L^{-1}\left\{\frac{1}{s+3}\right\} + \frac{5}{2}L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -2(1) + \frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}$$

$$= \boxed{\frac{1}{2}\left[-4 + e^{-3t} + 5e^{-t}\right]}$$

(4) Find $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}$

$$F(s) = \log\left(\frac{s^2+1}{s^2}\right) = \log(s^2+1) - \log(s^2) = \log(s^2+1) - 2\log(s)$$

$$\bar{F}(s) = \log(s^2+1) - 2\log(s)$$

differentiate both sides

$$\frac{d}{ds}\bar{F}(s) = \frac{2s}{s^2+1} - \frac{2}{s}$$

Now take Inverse Laplace on b/s

$$L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = 2L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2L^{-1}\left\{\frac{1}{s}\right\}$$

$$= 2\cos t - 2 = 2(\cos t - 1)$$

$$\therefore L\{t\, f(t)\} = -\frac{d}{ds}\bar{F}(s)$$

$$\therefore L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = -t\, f(t), \text{ So}$$

$$L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = -t\, f(t) = 2(\cos t - 1)$$

$$f(t) = -\frac{2}{t}(\cos t - 1)$$

$$\boxed{L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = \frac{2}{t}(1-\cos t)}$$

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(15) Find $L^{-1}\left\{\frac{16}{s^2+2s+5}\right\}$

$$L^{-1}\left\{\frac{16}{s^2+2s+5}\right\} = 16\,L^{-1}\left\{\frac{1}{s^2+2s+1+4}\right\} = 16L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}$$

$$= 16L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

$$= 16e^{-t}\cdot\frac{\sin(2t)}{2}$$

$$\boxed{L^{-1}\left\{\frac{16}{s^2+2s+5}\right\} = 8e^{-t}\sin(2t)}$$

(16) Find $L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$

$$L^{-1}\{\log(s+1) - \log(s-1)\}$$

$$\bar{F}(s) = \log(s+1) - \log(s-1)$$

$\Rightarrow$ Differentiate both sides

$$\frac{d}{ds}\bar{F}(s) = \frac{1}{s+1} - \frac{1}{s-1}$$

$\Rightarrow$ Now take Inverse laplace on b/s

$$L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = e^{-t} - e^{t}$$

$$L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = -t f(t) = e^{-t} - e^{t}$$

$$= f(t) = \frac{e^{-t}-e^{t}}{-t}$$

$$= f(t) = \frac{2}{2}\times\frac{e^{t}-e^{-t}}{t}$$

$$= f(t) = \frac{2}{t}\times\frac{e^{t}-e^{-t}}{2}$$

$$\therefore \frac{e^{\theta}-e^{-\theta}}{2} = \sinh t$$

$$= \frac{2}{t}\sinh t$$

$$\boxed{L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\} = \frac{2}{t}\sinh t}$$

17) Find $L^{-1}\left\{\frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)\right\}$

$\Rightarrow \frac{1}{2}L^{-1}\left\{\log\left(\frac{s^2+4}{s^2+9}\right)\right\} = \frac{1}{2}L^{-1}\left\{\log(s^2+4)-\log(s^2+9)\right\}$

$\frac{1}{2}\bar{F}(s) = \log(s^2+4)-\log(s^2+9)$

$\Rightarrow$ Differentiate both sides

$$\frac{d}{ds}\bar{F}(s) = \frac{2s}{s^2+4} - \frac{2s}{s^2+9}$$

$\Rightarrow$ take inverse Laplace on b/s

$$\frac{1}{2}L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = 2L^{-1}\left\{\frac{s}{s^2+4}\right\} - 2L^{-1}\left\{\frac{s}{s^2+9}\right\}$$

$$= 2\cos(2t) - 2\cos(3t)$$

$$\therefore L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = -t\,f(t)$$

$$\frac{1}{2}L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = -t\,f(t) = 2(\cos(2t)-\cos(3t))$$

$$= \quad f(t) = \frac{2}{t}(\cos(3t)-\cos(2t))$$

$$\cancel{\tfrac{1}{2}}L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = \frac{\cancel{2}}{t}(\cos(3t)-\cos(2t))$$

$$\boxed{L^{-1}\left\{\frac{d}{ds}\bar{F}(s)\right\} = \frac{4}{t}(\cos 3t - \cos 2t)}$$

(18) Find $L^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$ by Convolution method/Theorem

$$\therefore L\left\{\int_0^t f(u)\,g(t-u)\,du\right\} = \bar{f}(s)\,\bar{g}(s)$$

$$\therefore L^{-1}\left\{\bar{f}(s)\,\bar{g}(s)\right\} = \int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = L^{-1}\left\{\underset{\bar{f}(s)}{\underset{\downarrow}{\frac{1}{s}}} \times \underset{\bar{g}(s)}{\underset{\downarrow}{\frac{1}{s^2+9}}}\right\}$$

$$\therefore f(t) = L^{-1}\left\{\frac{1}{s}\right\}, \quad g(t) = L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$f(t) = 1, \quad g(t) = \frac{\sin 3t}{3}$$

$\therefore$ By Convolution theorem we have

$$L^{-1}\left\{\frac{1}{s} \times \frac{1}{s^2+9}\right\} = \int_0^t \overset{f(u)}{1} \times \overset{g(t-u)}{\frac{\sin 3(t-u)}{3}}\,du$$

$$= \frac{1}{3}\int_0^t \sin 3(t-u)\,du$$

$$= \frac{1}{3}\int_0^t \sin(3t-3u)\,du$$

$$= \frac{1}{3}\left[\frac{-\cos(3t-3u)}{-3}\right]_0^t$$

$$= \frac{1}{9}\left[\cos(3t-3u)\right]_0^t$$

$$= \frac{1}{9}\left[\cos(3t-3t) - \cos(3t-3(0))\right]$$

$$= \frac{1}{9}\left[\cos(0) - \cos(3t)\right]$$

$$\boxed{L^{-1}\left\{\frac{1}{s(s^2+9)}\right\} = \frac{1}{9}\left[1 - \cos(3t)\right]}$$

(13) Find $L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$ by Convolution Theorem

$$L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = L^{-1}\left\{\underset{\bar{f}(s)}{\frac{1}{s^2+1}} \times \underset{\bar{g}(s)}{\frac{1}{s^2+1}}\right\}$$

$$\therefore f(t) = L^{-1}\{\bar{f}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$g(t) = L^{-1}\{\bar{g}(s)\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

by Convolution Theorem, we have:

$$L^{-1}\{\bar{f}(s)\,\bar{g}(s)\} = \int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}\left\{\frac{1}{s^2+1} \times \frac{1}{s^2+1}\right\} = \int_0^t \sin u \cdot \sin(t-u)\,du$$

$$\therefore \sin A \cdot \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$L^{-1}\left\{\frac{1}{s^2+1} \times \frac{1}{s^2+1}\right\} = \frac{1}{2}\int_0^t [\cos(u-(t-u)) - \cos(u+t-u)]\,du$$

$$= \frac{1}{2}\int_0^t [\cos(\cancel{2u+t}\,t) - \cos(t)]\,du$$

$$= \frac{1}{2}\int_0^t [\cos(2u-t) - \cos(t)\cdot 1]\,du$$

$$= \frac{1}{2}\left[\frac{\sin(2u-t)}{2} - u\cos(t)\right]_0^t$$

$$= \frac{1}{2}\left[\left(\frac{\sin(2t-t)}{2} - t\cos(t)\right) - \left(\frac{\sin(2(0)-t)}{2} - 0\cos(t)\right)\right]$$

$$= \frac{1}{2}\left[\frac{\sin(t)}{2} - t\cos(t) - \frac{\sin(-t)}{2} - 0\right]$$

$$= \frac{1}{2}\left[\frac{\sin(t)}{2} - t\cos(t) + \frac{\sin(t)}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\sin(t) + \sin(t)}{2} - t\cos(t)\right]$$

$$\boxed{L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}[\sin(t) - t\cos(t)]}$$