work sheet #01

Q1 - Write the order and degree of the D.E

i- $\frac{d^2y}{dx^2} + a^2x = 0$ ii- $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = \frac{d^2y}{dx^2}$

Order: 2, degree: 1; Order: 2, degree: 1;

iii- $x^2\left(\frac{d^2y}{dx^2}\right)^3 + y\left(\frac{dy}{dx}\right)^4 + y^4 = 0$

Order: 2, degree: 3;

Q2- Obtain the D.E $y^2 = 4a(x+a)$ and $Ax^2 + By^2 = 1$

$\frac{\partial}{\partial x} y^2 = 4a\frac{\partial}{\partial x}(x+a) \Rightarrow 2y\frac{\partial y}{\partial x} = 4a(1+0) \Rightarrow 2y\frac{\partial y}{\partial x} = 4a$

$= y \cdot \frac{\partial y}{\partial x} = 2a$

$\frac{\partial}{\partial x}(Ax^2 + By^2) = \frac{\partial}{\partial x}(1) \Rightarrow 2Ax + 2yB\frac{\partial y}{\partial x} = 0$

Q3: By eliminating constant find D.E $y = e^x(A\cos x + B\sin x)$

$\frac{dy}{dx} = \frac{d}{dx}(Ae^x\cos x + Be^x\sin x) = A[\cos x(e^x) + e^x(-\sin x)] + \left[\frac{d}{dx}Be^x\sin x\right]$

$Ae^x\cos x - Ae^x\sin x + B[\sin x(e^x) + e^x\cos x]$

$= Ae^x\cos x - Ae^x\sin x + B.e^x\sin x + Be^x\cos x$

$= Ae^x\cos x + Be^x\cos x - Ae^x\sin x + Be^x\sin x$

$= e^x\cos x(A+B) - e^x\sin x(A-B)$

Q4- Variable Seperable

i- $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$ Ans: $y\sin y = x^2\log x + c$

$\frac{dy}{dx} = \frac{2x\log x + x}{\sin y + y\cos y}$; Integration by parts

$\int(\sin y + y\cos y)\,dy = \int(2x\log x + x)\,dx$

$\int \sin y\,dy + \int y\cos y\,dy = x^2\log x + c$

$-\cos y + \int y\cos y\,dy = x^2\log x + c$

$-\cancel{\cos y} + y\sin y + \cancel{\cos y} + c_1 = x^2\log x + c_2$

$\boxed{y\sin y = x^2\log x + c}$

$\int(2x\log x + x)\,dx$

$\int u\,dv = uv - \int v\,du$, ILATE

$u = \log x$, $dv = 2x\,dx$

$du = \frac{1}{x}dx$; $v = x^2$

$\int 2x\log x = x^2\log x - \int \frac{x^2}{x}dx$

$= [x^2\log x - \int x\,dx] + \int x\,dx$

$= x^2\log x - \cancel{\frac{x^2}{2}} + \cancel{\frac{x^2}{2}} + c$

$= \boxed{x^2\log x + c}$

$\rightarrow \int y\cos y\,dy$

$u = y$, $dv = \cos y\,dy$

$du = dy$; $v = \sin y$

$\int y\cos y\,dy = y\sin y - \int \sin y\,dy$

$= y\sin y - (-\cos y) + c$

$= \boxed{y\sin y + \cos y + c}$

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ii - $x^4 \frac{dy}{dx} + x^3 y = -\sec(xy)$ Ans: $\sin(xy) = \frac{1}{2x^2} + c$

$x^3 \left(x\frac{dy}{dx} + y \right) = -\sec(xy)$, let $z = xy$

$\frac{dz}{dx} = \frac{d}{dx}(xy)$

$x^3 \left(\frac{dz}{dx} \right) = -\sec(z)$

$\frac{dz}{dx} = y + x\frac{dy}{dx}$.

$\int \frac{dz}{\sec(z)} = \int -\frac{dx}{x^3}$

$\int \cos(z)\,dz = -\int x^{-3} dx \Rightarrow \sin(z) = \frac{-x^{-3+1}}{-3+1} + c$

$= \sin(z) = \frac{+x^{-2}}{+2} + c \Rightarrow \sin(z) = \frac{1}{2x^2} + c \Rightarrow \boxed{\sin(xy) = \frac{1}{2x^2} + c}$

iv — $\frac{dx}{x} = \tan y \, dy$ Ans: $x \cos y = c$

$\int \frac{1}{x} dx = \int \tan y \, dy \Rightarrow \ln(x) = \int \frac{\sin y}{\cos y} dy$; u-substitution

let $u = \cos y$, $du = -\sin y \, dy \Rightarrow -du = \sin y \, dy$

$\ln(x) = \int \frac{\sin y}{\cos y} dy \Rightarrow \ln(x) = \int \frac{-du}{u} \Rightarrow \ln(x) = -\ln(u) + c$

× $\boxed{\cancel{\ln}(x) = -\cancel{\ln}(u) + c} \Rightarrow x = -u + c$; $u = \cos y$

$\ln(x) + \ln(u) = c \Rightarrow \ln(x) + \ln(u) \Rightarrow$ Product Law

$xu = c \Rightarrow \boxed{x \cos y = c}$ ✓

vi — $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1+x^2}}$ Ans: $\sin^{-1} y = \sin^{-1} x + c$

$\frac{dy}{\sqrt{1-y^2}} = \frac{dx}{\sqrt{1+x^2}} \Rightarrow \boxed{\int \frac{1}{\sqrt{1-y^2}} dy = \int \frac{1}{\sqrt{1+x^2}} dx}$

trigonometric Substitution

$\sqrt{a^2 - x^2}$, $x = a \sin\theta$; $\sqrt{a^2 - x^2} = a|\cos\theta|$

$\sqrt{a^2 + x^2}$, $x = a \tan\theta$; $\sqrt{a^2 + x^2} = a|\sec\theta|$

① $\int \frac{1}{\sqrt{1-y^2}} dy$, $y = a \sin\theta = \sqrt{1} \sin\theta = \boxed{\sin\theta}$

$dy = \cos\theta \, d\theta$

$\int \frac{1}{\sqrt{1-(\sin\theta)^2}} \cos\theta \, d\theta = \int \frac{\cos\theta}{\sqrt{1-\sin^2\theta}} d\theta = \int \frac{\cos\theta \, d\theta}{\sqrt{\cos^2\theta}}$

$= \int \frac{\cancel{\cos\theta} \, d\theta}{\cancel{\cos\theta}} = \int d\theta = \theta + c$; $y = a \sin\theta = \sin\theta$

$\sin\theta = y \Rightarrow \theta = \frac{y}{\sin} \Rightarrow \boxed{\theta = \sin^{-1}(y)}$ ✓

$\frac{1}{\cos\theta}$, $du = -\sin\theta\, d\theta$

② $\int \frac{1}{\sqrt{1+x^2}}\, dx$; $x = a\tan\theta = 1\tan\theta$
$dx = \sec^2\theta\, d\theta$

$\int \frac{1}{\sqrt{1+(\tan\theta)^2}} \sec^2\theta\, d\theta = \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{\sqrt{\sec^2\theta}}$

$= \int \frac{\sec^2\theta\, d\theta}{\sec\theta} = \int \sec\theta\, d\theta = \int \sec\theta \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) d\theta$

$= \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$, $u = \sec\theta + \tan\theta$
$du = \sec\theta\tan\theta + \sec^2\theta\, d\theta$

$= \int \frac{du}{u} = \ln(u) + c = \boxed{\ln(\sec\theta + \tan\theta) + c}$

$\int \frac{1}{\sqrt{1-y^2}}\, dy = \int \frac{1}{\sqrt{1+x^2}}\, dx \Rightarrow \boxed{\sin^{-1}(y) = \ln(\sec\theta + \tan\theta) + c}$

vii — $y(1+x^2)^{1/2}\, dy + x\sqrt{1+y^2}\, dx = 0$
Ans: $\sqrt{1+y^2} + \sqrt{1+x^2} = c$

$y\sqrt{1+x^2}\, dy + x\sqrt{1+y^2}\, dx = 0$; Trigonometric substitution, ✓
U- Substitution

$y\sqrt{1+x^2}\, dy = -x\sqrt{1+y^2}\, dx$

$\int \frac{y}{\sqrt{1+y^2}}\, dy = \int \frac{-x}{\sqrt{1+x^2}}\, dx$; $\sqrt{a^2+x^2}$, $x = a\tan\theta$; $a|\sec\theta|$
Using U- Substitution

① $\int \frac{y}{\sqrt{1+y^2}}\, dy$, $y = \tan\theta$, $dy = \sec^2\theta\, d\theta$
$u = 1+y^2$, $du = 2y\, dy \rightarrow du/2 = y\, dy$

$\int \frac{du/2}{\sqrt{u}} \Rightarrow \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du \Rightarrow \frac{1}{2}\ln(\sqrt{u}) + c$

$= \boxed{\frac{1}{2}\ln(\sqrt{1+y^2}) + c}$

(2) $\int \frac{-x}{\sqrt{1+x^2}}\,dx$, $u = 1+x^2$, $du = 2x\,dx \Rightarrow du/2 = x\,dx$

$$\int \frac{-x}{\sqrt{1+x^2}}\,dx \Rightarrow \int \frac{-du/2}{\sqrt{u}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du$$

$$\boxed{-\frac{1}{2}\ln(\sqrt{u}) + c} \Rightarrow -\frac{1}{2}\ln(\sqrt{1+x^2}) + c$$

Answer:

$$\int \frac{y}{\sqrt{1+y^2}}\,dy = -\int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$\frac{\cancel{1}}{\cancel{2}}\ln(\sqrt{1+y^2}) = -\frac{\cancel{1}}{\cancel{2}}\ln(\sqrt{1+x^2}) + c$$

$$\ln(\sqrt{1+y^2}) + \ln(\sqrt{1+x^2}) = \frac{c}{1/2}$$

Using Trigonometric Substitution:

$$y(1+x^2)^{1/2}\,dy + x\sqrt{1+y^2}\,dx = 0$$

$$\int \frac{y\,dy}{\sqrt{1+y^2}} = -\int \frac{x\,dx}{\sqrt{1+x^2}}$$

(1) $\int \frac{y\,dy}{\sqrt{1+y^2}}$ $\sqrt{1+y^2}$, $y = \tan\theta$, $\sqrt{1+y^2} = \sec\theta$

$dy = \sec^2\theta\,d\theta$

$$\int \frac{\tan\theta(\sec^2\theta)\,d\theta}{\sqrt{1+(\tan\theta)^2}} = \int \frac{\tan\theta\sec^2\theta\,d\theta}{\sqrt{1+\tan^2\theta}}$$

$$-\int \frac{\tan\theta\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta}} = \int \frac{\tan\theta\sec^2\theta\,d\theta}{\sec\theta} = \int \tan\theta\sec\theta\,d\theta$$

$$\int \tan\theta\sec\theta\,d\theta = \boxed{\sec\theta + c} \quad \boxed{\sec\theta = \sqrt{1+y^2}}$$

(2) $-\int \frac{x}{\sqrt{1+x^2}}\,dx$, $x = \tan\theta$, $\sqrt{1+x^2} = \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$-\int \frac{\tan\theta\,(\sec^2\theta)\,d\theta}{\sqrt{1+(\tan\theta)^2}} = -\int \frac{\tan\theta\,(\sec^2\theta)\,d\theta}{\sqrt{1+\tan^2\theta}}$$

$$= -\int \frac{\tan\theta\,\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta}} = -\int \frac{\tan\theta\,\sec^{\cancel{2}}\theta\,d\theta}{\cancel{\sec\theta}} = -\int \tan\theta\,\sec\theta\,d\theta$$

$$= -\int \tan\theta\,\sec\theta\,d\theta = \boxed{-\sec\theta + c}$$

$\sec\theta = \sqrt{1+x^2} \rightarrow \quad -\sec\theta + c = \boxed{-\sqrt{1+x^2} + c}$

Answer:

$$y\,(1+x^2)^{1/2}\,dy + x\sqrt{1+y^2}\,dx = 0$$

$$\int \frac{y\,dy}{\sqrt{1+y^2}} = -\int \frac{x\,dx}{\sqrt{1+x^2}}$$

$$\sec\theta = -\sec\theta + c$$

$$\sqrt{1+y^2} = -\sqrt{1+x^2} + c$$

$$\boxed{\sqrt{1+y^2} + \sqrt{1+x^2} = c}$$

viii — $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

Ans: $(e^y+1)\sin x = c$

$$(e^y+1)\cos x\,dx = -e^y \sin x\,dy$$

$$\frac{\cos x\,dx}{\sin x} = \frac{-e^y\,dy}{e^y+1}$$

Let $u = \sin x$, $du = \cos x\,dx$; let $u = e^y+1$, $du = e^y\,dy$

$$\frac{\cos x}{\sin x}\,dx = \frac{-e^y\,dy}{e^y+1}$$

$$\int \frac{du}{u} = \int -\frac{du}{u}$$

$$\ln(u) = -\ln(u) + c$$

$$\ln(\sin x) = -\ln(e^y+1) + c$$

$$\ln(\sin x) + \ln(e^y+1) = c$$

$$\boxed{\sin x\,(e^y+1) = c}$$

ix — $(e^y+2)\sin x\,dx - e^y \cos x\,dy = 0$

Ans: $(e^y+2)\cos x = c$

$$(e^y+2)\sin x\,dx = e^y \cos x\,dy$$

$$\int \frac{\sin x}{\cos x}\,dx = \int \frac{e^y}{e^y+2}\,dy$$

let $u = \cos x$, $du = -\sin x\,dx$; let $u = e^y+2$, $du = e^y\,dy$

$$\int \frac{-du}{u} = \int \frac{du}{u}$$

$$-\ln(u) = \ln(u) + c$$

$$-\ln(\cos x) = \ln(e^y+2) + c$$

$$-\ln(\cos x) - \ln(e^y+2) = c$$

$$-[\ln(\cos x) + \ln(e^y+2)] = c$$

$$\cos x\,(e^y+2) = c/-1$$

$$\boxed{\cos x\,(e^y+2 = c}$$

X — $\frac{dy}{dx} = 1 + \tan(y-x)$ hint [put $y-x=z$]

Ans: $\sin(y-x) = e^{x+c}$

let $y - x = z$, $\frac{d}{dx}(y-x) = \frac{d}{dx}(z)$

$\frac{dy}{dx} = 1 + \tan(y-x)$ $\quad\quad$ $\frac{dy}{dx} - 1 = \frac{dz}{dx}$

$\frac{dz}{dx} + 1 = 1 + \tan(z)$ $\quad\quad$ $\boxed{\frac{dy}{dx} = \frac{dz}{dx} + 1}$

$\frac{dz}{dx} = \tan(z)$

$\int \frac{dz}{\tan(z)} = \int dx \Rightarrow \int \cot(z)\, dz = \int dx$

$\int \frac{\cos(z)}{\sin(z)}\, dz = \int dx \Rightarrow u = \sin(z), \; du = \cos(z)\, dz$

$\int \frac{du}{u} = \int dx \Rightarrow \ln(u) = x + c$

$\ln(\sin(z)) = x + c$, e on both sides

$\ln_e(\sin(z)) = e^{(x+c)}$ OR $\cancel{\ln}_e(\sin(z)) = e^{(x+c)}$

$\sin(z) = e^{x+c}$

$\boxed{\sin(y-x) = e^x + c}$