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KEY To
Applied Calculus

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PREFACE

"KEY to APPLIED CALCULUS" is in your hands. The need of writing this KEY arose because many of our students and other readers felt difficulties in solving the problems given in the worksheets. Keeping in mind the difficulties of the readers, I along with my Ex-colleagues Dr. Asif Ali Shaikh and Ms. Sania Nizamani (Both from Mehran Engineering University, Jamshoro) decided to write guide books on different textbooks that we wrote couple of years ago.

We are sure that readers will find this KEY very much useful. In some problems, direct steps have been taken, for example, the use of Partial Fractions, Integration by Parts; etc is made directly because it is hoped that readers are familiar with such techniques.

There are few problems of worksheets of the textbook whose data were wrongly entered. The problems are corrected and correct solutions are given in the KEY. Such corrections will also be incorporated in the textbook in the next print.

Any suggestion from the readers will highly be appreciated to improve the standard of the KEY.

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Chapter 1 INTRODUCTION

WORKSHEET 01

1. What do you know about calculus and its applications in different fields of sciences?
Solution: Calculus is the mathematics of motion and change. Where there is motion or growth, where variable forces are at work producing acceleration, calculus is the right mathematics to apply. This was true in the beginnings of the subject and it is true today.
Calculus was first invented to meet the mathematical needs of the scientists of the sixteenth and seventeenth centuries, needs that were mainly mechanical in nature.

2. How many branches are there of calculus and what do you know about them?
Solution: There are two main branches of calculus called ***Differential Calculus*** and ***Integral Calculus***.
Differential calculus deals with the problem of calculating rates of change. It enabled people to define slopes of curves, to calculate velocities and accelerations of moving bodies, to find firing angles that would give cannons their greatest range, and to predict the times when planets would be closest together or farthest apart. *Integral calculus deals with the problems of determining a function from information about its rate of change.* It enabled people to calculate the future location of a body from its present position and knowledge of the forces acting on it, to find the areas of irregular regions in the plane, to measure the lengths of curves, and to find the volumes and masses of arbitrary solids.
Today, applications of calculus and its extensions in mathematical analysis are far reaching indeed, and the physicists, mathematicians and astronomers who first invented the subject would surely be amazed and delighted, as we hope you will be, to see what a large number of problems it solves and what a range of fields now use it in the mathematical models that bring understanding about the universe and the world around us.
3. What are the contributions of Sir Isaac Newton and Gottfried Wilhelm Leibniz in the development of calculus?
Solution: The Englishman ***Isaac Newton*** (1642 – 1727) and the German ***Gottfried Wilhelm Leibniz*** (1646–1716) are the mathematicians credited with inventing calculus. They worked independently of each other. Newton invented calculus in 1665 but took more than 20 years to publish his results hence Leibniz's development of calculus was published first. Furthermore, Leibniz's notation was considered superior to Newton's notation, and it is still used today.

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Sir Isaac Newton

Gottfried Wilhelm Leibniz

4. Mention briefly about the uses of calculus.

Solution:

Calculus is widely employed in the physical, biological and social sciences. It is used, for example, in the physical sciences to study the speed of a falling body, the rates of change in a chemical reaction, or the rate of decay of a radioactive material. In the biological sciences, a problem such as rate of growth of a colony of bacteria as a function of time is easily solved using calculus. In the social sciences calculus is widely used in the study of Statistics, Probability and optimization problems.

Calculus can be applied to many problems involving the notion of extreme values, such as the fastest, the most, the slowest or the least. These maximum or minimum amounts may be described as values for which a certain rate of change (increase or decrease) is zero. By using calculus it is possible to determine how high a projectile will go by finding the point at which its change of altitude with respect to time, that is, its velocity, is equal to zero.

The invention of calculus had a great impact on technology as well as on the development of mathematics. Years later, applications of calculus were found in a variety of non – engineering areas, including business and economics, biology, medicine, sociology and psychology. Calculus can be used to:

- Determine the average speed at which blood flows through an artery.
- Select the most economical dimensions for packaging.
- Calculate how high a projectile will travel.
- Find the production level that will maximize a company's profit.

5. How can you learn calculus by taking different steps?

Solution: Learning calculus is not the same as learning arithmetic, algebra and geometry. In these subjects, you learn primarily how to calculate with numbers, how to simplify algebraic expressions and calculate with variables and how to reason about points lines and figures in the plane. Calculus involves those techniques and skills but develops others as well, with greater precision and a deeper level. Calculus introduces so many new concepts and computational operations, in fact, that you will no longer be able to learn everything you need in class. You will have to learn a fair amount on your own by working with other students.

- **Read the text:** You will be able to learn all the meanings and connections you need just by attempting the exercises. You will need to read relevant passages in the book and work through examples step-by-step. Speed-reading will not work here. You are reading and searching for detail in a step-by-step logical fashion. This kind of reading, required by any deep and technical content, takes attention, patience and practice.

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- **Do the given work in time, keeping the following principles in mind:**
  - Sketch diagrams whenever possible.
  - Write your solutions in a connected step-by-step logical fashion, as if you were explaining to someone else.
  - Think about why each exercise is there? Why was it assigned? How it is related to the other assigned exercises?
- **Use your calculator and computer whenever possible:** Graphs provide insight and visual representations of important concepts and relationships.
- **Try to write your own:** Write short descriptions of the key points each time you complete a section of the text. If you succeed, you probably understand the material. If you do not, you will know where there is a gap in your understanding.

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Chapter 2

FUNCTIONS AND THEIR APPLICATIONS

WORKSHEET 02

1. Draw the graphs of the following functions. Also mention the domain and range of these functions.

(a) $y = 2x + 7$ (b) $y = 2x^2 + 1$ (c) $y = (x^2 + 1)/(x - 1)$

Solution: (a) $y = 2x + 7$: Look at the following table.

x:	-2	-1	0	1	2
y:	3	5	7	9	11

The graph is shown below (Fig. i) which, in fact is a straight line with slope 2 and y intercept equal to 7. Since there exists no value of x for which y is either complex or infinite hence, the domain of y is set of all real $x \in R$. Moreover, the range of given function is also a set of real numbers (all possible values of y).

(b) $y = 2x^2 + 1$: Look at the following table.

x:	-2	-1	0	1	2
y:	9	3	1	3	9

The graph is shown below (Fig. ii) which, in fact is a parabola. Since there exists no value of x for which y is either complex or infinite hence, domain of y is the set of all real numbers R. Moreover, range of given function is a set $[1, \infty)$.

(c) $y = (x^2 + 1)/(x - 1)$: Look at the following table.

x:	-2	-1	-0.5	0	0.5	1	2
y:	-5/3	-1	-0.83	-1	-2.5	∞	5

The graph is shown below (Fig. iii). Since y is infinite at x = 1 hence, domain of y is $R - \{1\}$. Moreover, range of given function is a set $(-\infty, 1) \cup (1, \infty)$.

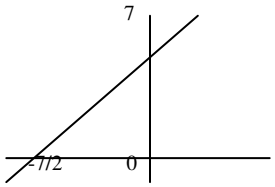

Fig. (i)

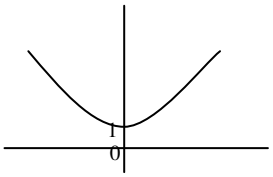

Fig. (ii)

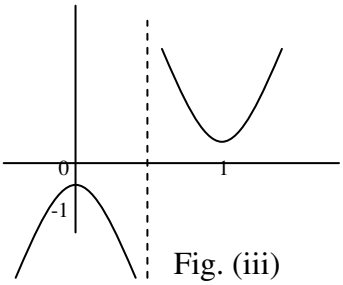

Fig. (iii)

2. Solve each of the following inequalities:

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(a) $|2x + 5| > |2 - 5x|$ (b) $|x| + |x - 1| > 1$ (c) $12x^2 - 25x + 12 > 0$
(d) $|x^2 + x + 1| > 1$ (e) $x^{-2} - 4x^{-1} + 4 > 0$ (f) $2x/(x + 2) \geq x/(x - 2)$

Solution: (a) Given inequality $|2x + 5| > |2 - 5x|$ is equivalent to

$|(2 - 5x)| < |(2x + 5)| \Rightarrow -(2x + 5) < 2 - 5x < (2x + 5)$

Now if, $-(2x + 5) < 2 - 5x \Rightarrow 3x < 7 \Rightarrow x < 7/3$ (i)

And if, $2 - 5x < 2x + 5 \Rightarrow -7x < 3 \Rightarrow -x < 3/7 \Rightarrow x > -3/7$ (ii)

Combining (i) and (ii), the solution of given inequality is: $-3/7 < x < 7/3$.

(b) $|x| + |x - 1| > 1$: Notice that $|x| = \pm x$. Thus we have four possibilities for given inequality.

(i) $x + (x - 1) > 1 \Rightarrow 2x > 2 \Rightarrow x > 1$

(ii) $-x + (x - 1) > 1 \Rightarrow -1 > 1$

(iii) $x - (x - 1) > 1 \Rightarrow 1 > 1$

(iv) $-x - (x - 1) > 1 \Rightarrow -2x > 0 \Rightarrow x < 0$.

Cases (ii) and (iii) won't give any information regarding x. However from (i) and (iv) one concludes that solution of the given inequality is union of two sets, that is; $x \in (-\infty, 0) \cup (1, \infty)$.

(c) $12x^2 - 25x + 12 > 0$: Consider $12x^2 - 25x + 12 = 0 \Rightarrow 12x^2 - 16x - 9x + 12 = 0$
$\Rightarrow (4x - 3)(3x - 4) = 0 \Rightarrow x = 3/4$ and $x = 4/3$. Thus solution of given inequality is an open interval $x \in (-\infty, 3/4) \cup (4/3, \infty)$.

REMARK: If the inequality is $12x^2 - 25x + 12 < 0$ then the solution would be $x \in (3/4, 4/3)$.

(d) $|x^2 + x + 1| > 1 \Rightarrow 1 < |x^2 + x + 1| \Rightarrow -(x^2 + x + 1) < 1 < (x^2 + x + 1)$.

Now if $-(x^2 + x + 1) < 1 \Rightarrow 0 < x^2 + x + 2 \Rightarrow x^2 + x + 2 > 0$. This is true for all real x.

And if $1 < (x^2 + x + 1) \Rightarrow x^2 + x > 0 \Rightarrow x(x + 1) > 0$. This shows that either both x and x + 1 are positive or both are negative.

If $x > 0$ and $x + 1 > 0 \Rightarrow x > 0$ and $x > -1$.

If $x < 0$ and $x + 1 < 0 \Rightarrow x < 0$ and $x < -1$.

Thus solution of given inequality is $x \in (-\infty, -1) \cup (0, \infty)$.

(e) $x^{-2} - 4x^{-1} + 4 > 0$: Multiplying both sides by x^2, we get $1 - 4x + 4x^2 > 0 \Rightarrow (2x - 1)^2 > 0$ which, is true for all real x. Thus solution set for given inequality is set of all real numbers R.

(f) $2x/(x + 2) \geq x/(x - 2)$: Cross multiplying, we get: $2x^2 - 4x \geq x^2 + 2x \Rightarrow x^2 - 6x \geq 0$. We see that for all negative x, the inequality is true. Also for all $x \geq 6$ the inequality is also true. Thus solution set of given inequality is $x \in (-\infty, 0] \cup [6, \infty)$ or $R - (0, 6)$.

3. If $f(x) = \sqrt{x^2 - 1}$ and $g(x) = \dfrac{1}{\sqrt{4 - x^2}}$, show that $fog \neq gof$.

Solution: By definition,

$$fog = f[g(x)] = f\left[\frac{1}{\sqrt{4 - x^2}}\right] = \sqrt{\frac{1}{4 - x^2} - 1} = \sqrt{\frac{1 - 4 + x^2}{4 - x^2}} = \sqrt{\frac{x^2 - 3}{4 - x^2}}$$

$$gof = g[f(x)] = g\left[\sqrt{x^2 - 1}\right] = \frac{1}{\sqrt{4 - (4 - x^2)}} = \frac{1}{\sqrt{x^2}} = \frac{1}{x}.$$ Here $fog \neq gof$

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4. A patient paid \$300 per day for a semiprivate hospital room and \$1500 for an appendectomy operation. Express the total amount for an appendectomy as a function of number of days of hospital confinement.

Solution: If the rent of a room for one day is \$300 then for x days the total room rent will be \$300x. Now dollars 1500 are to be paid for appendectomy operation which is a fixed charge. Hence if A(x) is a function to show the total amount to be paid then

$$A(x) = 1500 + 300x,\ x \geq 0. \quad \text{(x indicates number of day)}$$

5. In some cities you can rent a car for \$18 per day and \$ 0.20 per miles.

a. Find the cost of renting the car for one day and driving 200 miles.
b. If the car is rented for one day then express the total rental expenses as a function of the number x miles driven.

Solution: (a) If the rent of a car for one day is \$18 then for x days the total rent for x days will be \$18x. Similarly, if \$0.20 is the charge for one mile then for y miles the amount to be paid is \$0.20y. Hence, if A(x, y) is a function to show the total amount to be paid then

$$A(x, y) = 18x + 0.20y,\ x \geq 0,\ y \geq 0.$$

(b) Put x = 1, we get $A(y) = 18 + 0.20y,\ y \geq 0.$

6. Suppose the longer side of a rectangle has twice the length of shorter side, and if x is length of shorter side, express the perimeter of the rectangle as a function of x.

Solution: If shorter side is x units then longer side is 2x. Now area of rectangle is

$$A(x) = (\text{Length}).(\text{Breadth}) = (2x).(x) = 2x^2$$

7. The monthly charge (in dollars) for x kilowatt hours (KWH) of electricity used by a customer is given by the following function:

$$C(x) = \begin{cases} 7.52 + 0.1079x, & 0 \leq x \leq 5 \\ 19.22 + 0.1079x, & 5 < x \leq 750 \\ 20.795 + 0.1058x, & 750 < x \leq 1500 \\ 131.345 + 0.0321x, & x > 1500 \end{cases}$$

Find the monthly charge for the following usages.

(a) 5 KWH (b) 6 KWH (c) 3000 KWH.

Solution: (a) C(5) = 7.52 + 0.1079(5) = 8.0595 dollars

(b) C(6) = 19.22 + 0.1097(6) = 19.8674 dollars

(c) C(3000) = 131.345 + 0.0321(3000) = 227.645 dollars

8. The pressure P of a certain gas is related to volume V according to: P = 100/V

(a) Is 0 in the domain of this function?
(b) What are P(100) and P(50)?
(c) As volume decreases, what happens to pressure?

Solution: (a) No, zero can not be in domain of the given function because for V = 0, P is infinite.

(b) P(100) = 100/100 = 1 dollar and P(50) = 100/50 = 2 dollars.

(c) If volume decreases then pressure will increase.

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9. The total cost of producing a product is given by

$$C(x) = 300\,x + 0.1\,x^2 + 1200 \text{ dollars}$$

where x represents the number of units produced.

(a) What is the total cost of producing 10 units?

(b) What is the average cost per unit when 10 units are produced?

Solution: (a) The total cost of producing 10 units of product is

$C(10) = 300(10) + 0.1\,(100) + 1200 = 4210$ dollars.

(b) Average cost of one item = (Total cost)/No. of items = $C(10)/10 = 4210/10 = 421$ dollars per item.

10. A cigar box distributor's revenue is $R(x) = 1.35\,x$ dollars, where x is the number of boxes sold.

(a) How much revenue is obtained from selling 5 boxes?

(b) How much revenue is obtained from the sale of 5th box?

(c) How much revenue is obtained from the sale of 8th box?

Solution: (a) Revenue from the sale of 5 boxes is $R(5) = 1.35\,(5) = 6.75$ dollars.

(b) Revenue obtained on the sale of 5th box is $R(5) – R(4) = 6.75 – 5.4 = 1.35$ dollars.

(c) Revenue obtained on the sale of 8th box is $R(8) – R(7) = 10.8 – 9.45 = 1.35$ dollars.

11. It costs a TV manufacturer $C(x) = 0.1\,x^2 + 150x + 1000$ dollars to produce x TV sets. The revenue from the sale of x TV sets is $R(x) = 280x$ dollars.

(a) Determine the profit function.

(b) What is the profit on the manufacture and sale of 50 TV sets?

Solution: (a) The profit function is

$$P(x) = R(x) – C(x) = 280x – (0.1\,x^2 + 150x + 1000) = -0.1x^2 + 130x – 1000 \text{ dollars}$$

(b) The profit from the sale of 50 TV sets is

$$P(50) = -0.1(50)^2 + 130(50) – 1000 = -250 + 6500 – 1000 = 5250 \text{ dollars}$$

12. The cost on producing x radios is $C(x) = 0.4\,x^2 + 7x + 95$ \$. The revenue received is $R(x) = 40x$ \$. What is the profit function? Find P(24) and P(25). What is the profit on the sale of 25th radio?

Solution: The profit function is

$$P(x) = R(x) – C(x) = 40x – (0.4\,x^2 + 7x + 95) = -0.4x^2 + 33x – 95 \text{ dollars}$$

Now, $P(25) = -0.4(25)^2 + 33(25) -95 = -250 + 825 – 95 = 480$ dollars. Similarly,

$P(24) = -249.6 + 792 – 95 = 447.4$ dollars.

Thus the profit on the sale of 25th radio is: $P(25) – P(24) = 480 – 447.4 = 32.6$ dollars.

13. A psychologist needs volunteers for an experiment. She offers to pay \$ 8 per hour for volunteer who works up to 5 hours. Those who work more than 5 hours are paid \$10 per hour for additional hour.

(a) Write down the function that represents the volunteer's pay V(x), where x represents the hours worked. (b) Also find V(5), V(10) and V(15).

Solution: (a) Let us assume that volunteer works for x hours.

Now, if $x \le 5$ then function V(x) that represents volunteer's pay, will be $V(x) = 8x$.

If $x > 5$ then according to the given condition,

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$$V(x) = 40 + 10(x - 5) = 40 + 10x - 50 = 10x - 10 = 10(x - 1).$$

Here 40 indicates the pay of 5 hours and $(x - 5)$ represents the hours that he/she works more than 5 hours and 10 indicates the rate for extra hours. Thus function representing given phenomenon is:

$$V(x) = \begin{cases} 5x, & 0 \le x \le 5 \\ 10(x-1), & x > 5 \end{cases}$$

(b) Now, $V(5) = 5\times 8 = \$40$

(c) $V(10) = 10(10 - 1) = \$90$

(d) $V(15) = 10(15 - 1) = 10 \times 14 = \$140$

14. If $p = 0.01x + 19$ dollars is the price of one jacket and each jacket is sold for \$80, determine the profit function and then find P(2) and (50). What is BEP?

Solution: Given that $p = 0.01x + 19$ is the price of one jacket hence if x jackets are bought then total cost is $C(x) = (0.01x + 19)x$. Now if each jacket is sold for \$80, the total revenue is $R(x) = 80x$. Thus, profit function is

$$P(x) = R(x) - C(x) = 80x - (0.01x + 19)x = 61x - 0.01x^2$$

Thus profit on the sale of 2 jackets is $P(2) = 122 - 0.04 = 121.96\$$.

Similarly, profit on the sale of 50 jackets is $P(50) = 61(50) - 0.01(50)^2 = \$3025$.

Finally, Break-even Point occurs when $R(x) = C(x)$ or when profit is zero. Thus putting

$$61x - 0.01x^2 = 0 \Rightarrow 61 - 0.01x = 0 \Rightarrow x = 61/0.01 = 6100.$$

So, when 6100 number of jackets are produced and sold, the profit is zero.

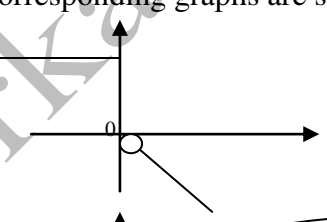
15. Sketch the graph of following piecewise functions.

(a) $f(x) = \begin{cases} 2 & \text{for } x \le 0 \\ -x & \text{for } x > 0 \end{cases}$ (b) $f(x) = \begin{cases} x & \text{for } x < 0 \\ x^2 & \text{for } x \ge 0 \end{cases}$

(c) $f(x) = \begin{cases} x & \text{for } x \le 1 \\ \sqrt{x} & \text{for } x > 1 \end{cases}$ (d) $f(x) = \begin{cases} |x| & \text{for } x \le 0 \\ 2x & \text{for } x > 0 \end{cases}$

Solution: The corresponding graphs are shown as under.

Fig. (a)



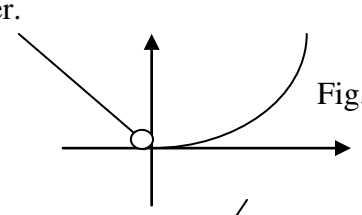
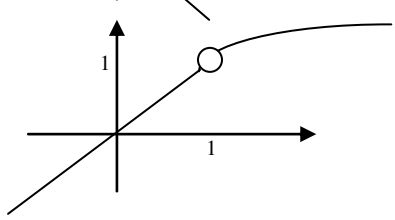


Fig. (b)

Fig. (c)



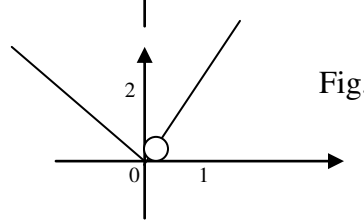


Fig. (d)

16. A motorbike was purchased 2 years ago for Rs. 50, 000 is now worth Rs. 25, 000. Find the linear relationship between its value (y) and the life (x) in years.

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(a) How many years from the time of purchase will it be before motorbike is worth Rs. 35, 000.

(b) What would be its cost after 10 years?

Solution: The following tabular values are taken from the statement of the problem.

x (Time in years):	0	2
y (Value in Rs.) :	50000	25000

Using the equation of line through two points, we get:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-50000}{25000-50000}=\frac{x-0}{2-0} \Rightarrow \frac{y-50000}{-25000}=\frac{x}{2}$$

Therefore, $y = -12500x + 50000$ (1)

This is a linear relationship between x and y.

(a) Putting y = 35000 in equation (1), we obtain: x = 15000/12500 = 6/5. This means it will take 6/5 = 1.2 years when the motorbike will be worth Rs. 35000.

(b) Putting x = 10 in equation (1), we get: y = -125000 – 50000 = -75000. The negative sign shows that after 10 years the motorbike will have no value.

17. A house was purchased 5 years ago for Rs. 5 million is now worth Rs. 8 million. Find the linear relationship between its value (y) and the life (x) in years.

(a) How many years from the time of purchase will it be before house is worth Rs. 6 millions.

(b) What would be its cost after 8 years?

Solution: The following tabular values are taken from the statement of the problem.

x (Time in years):	0	5
y (Value in Million Rs.):	5	8

Using the equation of line through two points, we get:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-5}{8-5}=\frac{x-0}{5-0} \Rightarrow \frac{y-5}{3}=\frac{x}{5}$$

Therefore, $y=\frac{3}{5}x+5$ (1)

This is a linear relationship between x and y.

(a) Putting y = 6 in equation (1) we obtain, x = 5/3 years. This means that it would take 5/3 = 1.67 years when the cost of house will be Rs. 6 million

(b) Putting x = 8 in equation (1), we get y = 49/5 = 9.8. This means that after 8 years the cost of house will be Rs. 9.8 million.

18. State which of the following functions is even or odd.

(a) $f(x) = x^4 + x^2 - 1$

(b) $g(x) = x^3 + \sin x + 1$

(c) $h(x) = x^3 - x$

(d) $k(x) = |x| + \cos x - 2$

Solution: (a) $f(x) = x^4 + x^2 - 1$

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Replacing x by –x we get: $f(-x) = (-x)^4 + (-x)^2 - 1 = x^4 + x^2 - 1 = f(x)$. Thus, f(x) is even function.

(b) $g(x) = x^3 + \sin x + 1$

Replacing x by –x we get: $g(-x) = (-x)^3 + \sin(-x) + 1 = -x^3 - \sin x + 1 \neq g(x)$. Thus, g(x) is not an even function. Also, $g(-x) = -[x^3 + \sin x - 1] \neq -g(x)$. Hence, g(x) is not an odd function. Thus, g(x) is neither even nor odd function.

(c) $h(x) = x^3 - x$

Replacing x by –x we get: $h(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -g(x)$. Thus, g(x) is an odd function.

(d) $k(x) = |x| + \cos x - 2$

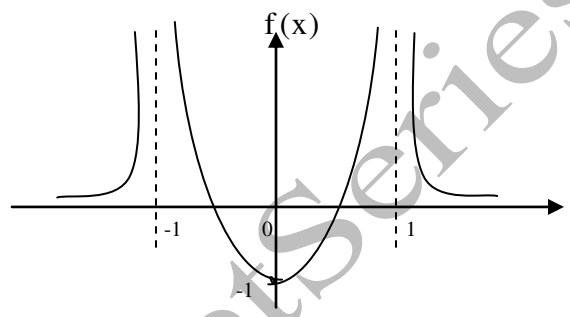
Replacing x by –x we get: $k(-x) = |-x| + \cos(-x) - 2\ 1 = |x| + \cos x - 2 = k(x)$. Thus, k(x) is an even function.

19. What is the domain and range of the following functions?

Solution: (i) $f(x) = 1/(x^2 - 1)$

Here when x = 1 or x = -1, f(x) becomes infinite. Hence $D_f = R - \{-1, 1\}$. Also for x = 0

Fig. (i)



f(x) = -1. Hence, $R_f = [-1, \infty)$. The graph of f(x) depicts this.

(ii) $g(x) = (x + 1)/(2x - 1)$

When $2x - 1 = 0 \Rightarrow x = 1/2$ then g(x) is infinite. Thus, $D_g = R - \{1/2\}$. The graph of this function is depicted below. From the graph of figure (i), we see that the range of g is set of all real R, that is; $R_g = (-\infty,\infty)$.

Fig. (ii)

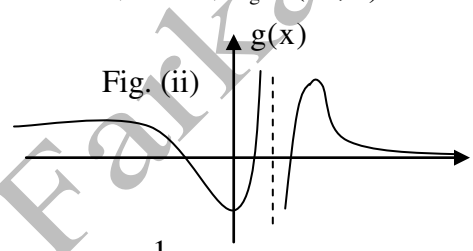
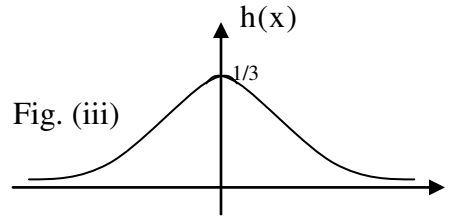


Fig. (iii)



(iii) $h(x) = \dfrac{1}{\sqrt{x^2 + 9}}$

Here D_h is R because for any value of x, h(x) is neither infinite nor complex. Moreover, from the figure (iii) we observe that $R_h = (0, 1/3]$.

(iv) $k(x) = \dfrac{1}{\sqrt{16 - x^2}}$

Here, if $16 - x^2 = 0$ then $x = \pm 4$ and for these two values of x, k(x) is infinite. Moreover,

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if x > 4 or x < -4, k(x) is complex, hence, D_k = (-4, 4). If x = 0 then k(x) =1/4 hence, the range of k(x), that is; R_k = [1/4, ∞). This is depicted in the figure (iv).

Fig. (iv)

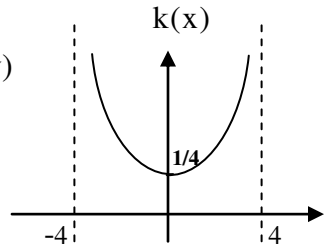
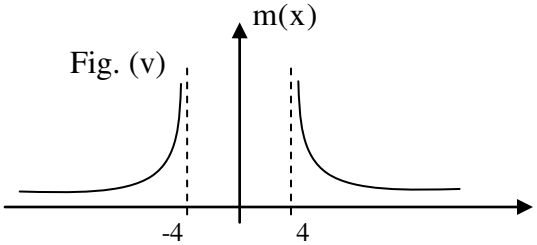


Fig. (v)



(v) $m(x) = \dfrac{1}{\sqrt{x^2 - 16}}$

Here, if $x^2 - 16 = 0$ then x = ± 4 and for these two values of x, m(x) is infinite. Moreover, if x < 4 or x > -4, m(x) is complex. Hence, D_m = R – [-4, 4]. Now, if x tends to infinity, m(x) tends to zero hence, the range of m(x) is (0, ∞). This is shown in the figure (v).

(vi) $n(x) = \sqrt{16 - x^2}$

Here, if $16 - x^2 = 0$ then x = ± 4 and for these two values of x, n(x) is zero. If x = 0 then n(x) = 4. Moreover, if x > 4 or x < -4, n(x) is complex. Hence, D_n = [-4, 4]. The range of n(x) is (0, 4).

Fig. (vi)

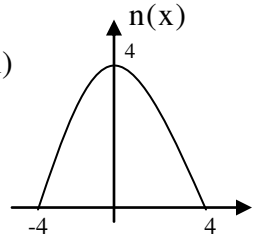
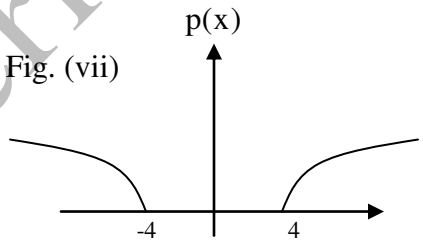


Fig. (vii)



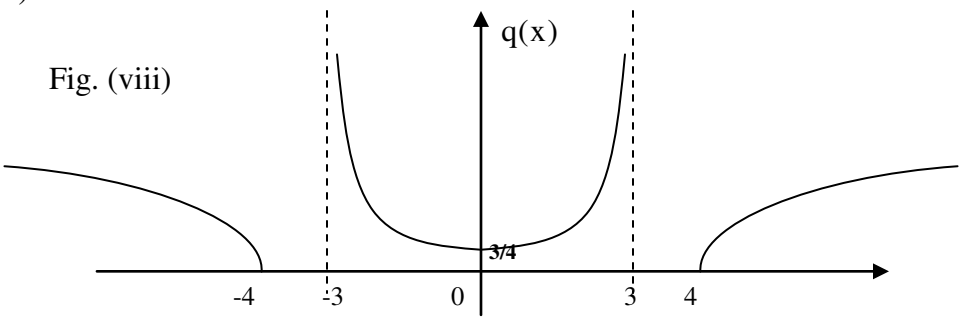
(vii) $p(x) = \sqrt{x^2 - 16}$

Here, if $x^2 - 16 = 0$ then x = ± 4 and for these two values of x, p(x) is zero. Moreover, if x < 4 or x > -4, p(x) is complex. Hence, D_p = R – (-4, 4). The range of p(x) is [0, ∞). This is shown in the figure (vii).

(viii) $q(x) = \sqrt{x^2 - 16} / \sqrt{x^2 - 9}$

Here, if $x^2 - 16 = 0$ then x = ± 4 and for these two values of x, q(x) is zero. If $x^2 - 9 = 0$ then x = ± 3. For these values of x, q(x) is infinite. If -4 < x < -3 and 3 < x < 4, then q(x) is complex. Now if -3 < x < 3 then q(x) is real. For example, if x = 0 then q(x) = 4/3. Hence, D_q = R – {(-4, -3) U (3, 4)}. The range of q(x) is [0, ∞). This is shown in the figure (viii).

Fig. (viii)



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Chapter 3

LIMIT AND CONTINUITY

WORKSHEET 03

1. Evaluate the following limits:

(a) $\lim\limits_{x\to\infty}\dfrac{x^4-3x}{x^5+x^2-9}$ (b) $\lim\limits_{x\to1+0}\dfrac{x-1}{\sqrt{x^2-1}}$ (c) $\lim\limits_{x\to0}\dfrac{|-1+x|-1}{|x|}$ (d) $\lim\limits_{x\to3-0}\dfrac{3-x}{|x-3|}$

Solution:

(a) $\lim\limits_{x\to\infty}\dfrac{x^4-3x}{x^5+x^2-9}=\lim\limits_{x\to\infty}\dfrac{x^4\left(1-3/x^3\right)}{x^5\left(1+1/x^3-9/x^5\right)}=\lim\limits_{x\to\infty}\dfrac{\left(1-3/x^3\right)}{x\left(1+1/x^3-9/x^5\right)}=\dfrac{(1-0)}{\infty(1+0-0)}=0$

(b) $\lim\limits_{x\to1+0}\dfrac{x-1}{\sqrt{x^2-1}}$:Putting $x=1+h\Rightarrow h=1-x$.As $x\to1$, $h\to0$.Thus, we get

$$\lim_{x\to1+0}\frac{x-1}{\sqrt{x^2-1}}=\lim_{h\to0}\frac{1+h-1}{\sqrt{(1+h)^2-1}}=\lim_{h\to0}\frac{h}{\sqrt{1+2h+h^2-1}}=\lim_{h\to0}\sqrt{\frac{h^2}{h(2+h)}}=\lim_{h\to0}\sqrt{\frac{h}{(2+h)}}=0$$

(c) $\lim\limits_{x\to0}\dfrac{|-1+x|-1}{|x|}$.Here we have to consider left and right limits.

Notice that, $\lim\limits_{x\to0-0}\dfrac{|-1-x|-1}{-x}=\lim\limits_{x\to0-0}\dfrac{|-(1+x)|-1}{-x}=\lim\limits_{x\to0-0}\dfrac{1+x-1}{-x}=\lim\limits_{x\to0-0}\dfrac{x}{-x}=\lim\limits_{x\to0-0}\dfrac{1}{-1}=-1$

and, $\lim\limits_{x\to0+0}\dfrac{|-1+x|-1}{+x}=\lim\limits_{x\to0-0}\dfrac{|-(1-x)|-1}{x}=\lim\limits_{x\to0-0}\dfrac{1-x-1}{x}=\lim\limits_{x\to0-0}\dfrac{-x}{x}=\lim\limits_{x\to0-0}\dfrac{-1}{1}=-1$

Since LHL = RHL = -1, thus $\lim\limits_{x\to0}\dfrac{|-1+x|-1}{|x|}=-1$

(d) $\lim\limits_{x\to3-0}\dfrac{3-x}{|x-3|}$: Putting $x=3-h\Rightarrow h=3-x$.If $x\to3$, then $h\to0$. Thus,

$$\lim_{x\to3-0}\frac{3-x}{|x-3|}=\lim_{h\to0}\frac{3-(3-h)}{(3-h)-3}=\lim_{h\to0}\frac{h}{-h}=\lim_{h\to0}(-1)=-1$$

2. Suppose the cost C of removing p percent of the particulate pollution from the smokestacks of an industry plant is given by: $C(p)=\dfrac{7300p}{100-p}$. Find $\lim\limits_{p\to\infty}C(p)$.

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Solution: $\lim_{p\to\infty} C(p) = \lim_{p\to\infty} \frac{7300p}{100-p} = \lim_{p\to\infty} \frac{7300p}{p\left(\frac{100}{p}-1\right)} = \lim_{p\to\infty} \frac{7300}{\left(\frac{100}{p}-1\right)} = \frac{7300}{(0-1)} = -7300$

3. The monthly charge (in dollars) for x kilowatt hours (KWH) of electricity used by a commercial customer is given by the following function:

$$C(x) = \begin{cases} 7.52 + 0.1079x, & 0 \le x \le 5 \\ 19.22 + 0.1079x, & 5 < x \le 750 \\ 20.795 + 0.1058x, & 750 < x \le 1500 \\ 131.345 + 0.0321x, & x > 1500 \end{cases}$$

Find (a) Find $\lim_{x\to 1500-0} C(x)$ and (b) $\lim_{x\to 5} C(x)$.

Solution: (a) $\lim_{x\to 1500-0} C(x) = \lim_{x\to 1500} (20.795 + 0.1058x) = 20.795 + 0.1058(1500) = 179.5$

(b) Consider, $\lim_{x\to 5-0} (7.52 + 0.1079x) = 7.52 + 0.1079(5) = 8.059$

Also, $\lim_{x\to 5+0} (19.22 + 0.1079x) = 19.22 + 0.1079(5) = 19.7595$

Since LHL and RHL are not equal hence, $\lim_{x\to 5} C(x)$ does not exist.

4. The monthly charge for water in a small town is given by:

$$f(x) = \begin{cases} 18, & 0 \le x \le 20 \\ 18 + 0.1(x-20), & x > 20 \end{cases}. \quad \text{Find } \lim_{x\to 20} f(x).$$

Solution: Consider, $\lim_{x\to 20-0} f(x) = \lim_{x\to 20-0} (18) = 18$

Also, $\lim_{x\to 20+0} f(x) = \lim_{x\to 20+0} \left[18 + 0.1(x-20)\right] = 18 + 0.1(20-20) = 18$

Since, LHL = RHL hence, $\lim_{x\to 20} f(x) = 18$.

5. The profit from the sale of x units is: $P(x) = \frac{1400x - 250}{x}$ dollars for $x \ge 1$

What is the limit of profit as the quantity sold increases without bound?

Solution: Consider,

$$\lim_{x\to\infty} P(x) = \lim_{x\to\infty} \frac{1400x - 250}{x} = \lim_{x\to\infty} \frac{x(1400 - 250/x)}{x} = \lim_{p\to\infty} (1400 - 250/x) = 1400$$

Thus the profit will be \$1400 if the sale (x) increases without any bound (means when x tends to infinity).

6. The amount of a drug that remains in a person's bloodstream t hours after being injected is given by: $f(t) = 0.15t/(1 + t^2)$. Find $\lim_{x\to\infty} f(x)$ if it exists. Explain the result you obtain.

Solution: Consider,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{0.15t}{(1+t^2)} = \lim_{t\to\infty} \frac{0.15t}{t^2(1/t^2+1)} = \lim_{t\to\infty} \frac{0.15}{t(1/t^2+1)} = \frac{0.15}{\infty(0+1)} = 0$$

Thus the amount of drug in blood stream is zero as the time t increases indefinitely.

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7. Function f(x) is defined as:

$$f(x)=\begin{cases}(x^3-27)/(x-3), & x\neq 3\\ 9/2, & x=3\end{cases}$$

Is f (x) continuous at x = 3?

Solution: If a function is continuous at x = a, we must have: $\lim\limits_{x\to a} f(x)=f(a)$. Now,

$$\lim_{x\to a} f(x)=\lim_{x\to 3}\frac{x^3-27}{x-3}=\lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{(x-3)}=\lim_{x\to 3}(x^2+3x+9)=27.$$

But $f(3)=9/2$. Since, $\lim\limits_{x\to 3} f(x)\neq f(3)$ hence given function is not continuous at x = 3.

8. Is the following function continuous?

$$f(x)=\begin{cases}x+2 & 0<x<1\\ x & 1\le x<2\\ x+5 & 2\le x<3\end{cases}$$

Solution: We know that a function f(x) is continuous at x = a if f(a - 0), f(a + 0) and f(a) are all equal and finite. Now, the given function is a piecewise function where the possible discontinues occur at points x = 1 and x = 2. We now consider,

Continuity at x = 1: Here, $\lim\limits_{x\to 1-0} f(x)=\lim\limits_{x\to 1}(x+2)=3$ and $\lim\limits_{x\to 1+0} f(x)=\lim\limits_{x\to 1}(x)=1$.

Since, LHL ≠ RHL hence given function is not continuous at x = 1.

Continuity at x = 2: Here, $\lim\limits_{x\to 2-0} f(x)=\lim\limits_{x\to 2}(x)=2$ and $\lim\limits_{x\to 2+0} f(x)=\lim\limits_{x\to 2}(x+5)=7$.

Since, LHL ≠ RHL hence given function is not continuous at x = 2.

9. Examine the continuity of following function at x = 0.

(a) $f(x)=\begin{cases}\dfrac{e^{1/x}}{e^{1/x}+1} & x\neq 0\\ 0 & x=0\end{cases}$ (b) $f(x)=\begin{cases}\dfrac{\sin 7x}{\sin 6x} & x\neq 0\\ 6/7 & x=0\end{cases}$

(c) $f(x)=\begin{cases}(1+4x)^{1/x} & x\neq 0\\ e^4 & x=0\end{cases}$.

Solution: We know that a function f(x) is continuous at x = a if $\lim\limits_{x\to a} f(x)=f(a)=L$, L being finite.

(a) $$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{e^{1/x}}{1+e^{1/x}}=\lim_{x\to 0}\frac{e^{1/x}}{e^{1/x}\left(1/[e^{1/x}]+1\right)}=\lim_{x\to 0}\frac{1}{\left(1/[e^{1/x}]+1\right)}=\frac{1}{(1/\infty+1)}=\frac{1}{0+1}=1$$

But, f(0) = 0. Since, $\lim\limits_{x\to 0} f(x)\neq f(0)$ hence, given function is not continuous at x = 0.

NOTE: $e^{1/0}=e^{\infty}=\infty$. Thus, $1/e^{\infty}=1/\infty=0$.

(b) $$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{\sin 7x}{\sin 6x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\times\frac{7x}{6x}\times\frac{6x}{\sin 6x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\times\lim_{x\to 0}\frac{6}{7}\times\lim_{x\to 0}\frac{6x}{\sin 6x}=1.\frac{6}{7}.1=\frac{6}{7}$$

Also f(0) = 6/7. Since, $\lim\limits_{x\to 0} f(x)=f(0)=6/7$ hence, given function is continuous at x = 0.

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(c) $\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+4x)^{1/x} = \lim_{x\to 0}(1+4x)^{\frac{x}{4}.4} = e^4$. Also, $f(0) = e^4$.

Since, $\lim_{x\to 0} f(x) = f(0) = e^4$ hence given function is continuous at x = 0.

10. Use $\in -\delta$ definition, show that $f(x) = x^3 - 3$ is continuous at x = 4.

Solution: By $\in -\delta$ definition, a function f(x) is continuous at x = a if for given $\in > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \in$ whenever $|x - a| < \delta$.

Now $f(x) = x^3 - 3$, therefore f(4) = 64 – 3 = 61. Now,

$|f(x) - f(a)| = |x^3 - 3 - 61| = |x^3 - 64| = |(x - 4)(x^2 + 4x + 16)| = |x - 4| . |(x^2 + 4x + 16)|$.

Now, let $|x - 4| < \delta$ so $|x - 4| . |(x^2 + 4x + 16)| = \delta |(x^2 + 4x + 16)| = \in$ (say). Thus, $|f(x) - f(a)| < \in$. This proves the result that given function is continuous at x = 4.

11. In 1986, Tax Reform Act created the following tax schedule for a single filers in 1988.

Tax Rate	Tax Income
15%	0 - \$17850
28%	Above \$17850

This means that the tax T is a function of income x as follows:

$$T(x) = \begin{cases} 0.15x, & 0 \le x \le 17850 \\ 2677.50 + 0.28(x - 17850), & x > 17850 \end{cases}.$$

Is T(x) continuous?

Solution: Since the given function is piecewise function hence the possible discontinuity occurs at 17850.

Now consider, LHL at x=17850: $\lim_{x\to 17850-0}(0.15x) = 0.15(17850) = 2677.5$

The RHL at x = 17850:

$$\lim_{x\to 17850+0}[2677.5 + 0.28(x - 17850)] = [2677.5 + 0.28(17850 - 17850)] = 2677.5$$

Since, T(17850 - 0) = T(17850 + 0) = T(17850) = 2677.5 hence, the system of tax deduction is continuous at 17850.

12. Residential customers in a small town have their monthly charge f(x) for x hundred gallons of water given by:

$$f(x) = \begin{cases} 18, & 0 \le x \le 20 \\ 18 + 0.1(x - 20), & x > 20 \end{cases}.$$

Is f(x) continuous?

Solution: Since the given function is piecewise function hence the possible discontinuity occurs at 20.

Now consider, LHL at x= 20;

$$f(20-0) = \lim_{x\to 20-0}(18) = 18$$

And the RHL at x = 20; $f(20+0) = \lim_{x\to 20+0}[18 + 0.1(x - 20)] = [18 + 0.1(20 - 20)] = 18$

Since, f(20 - 0) = f(20 + 0) = f(20) = 18, hence, the system of monthly water charges in a town is continuous.

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13. Suppose that cost C of removing p percent of particulate pollution from the exhaust gases at an industrial site is given by $C(p) = \frac{8100p}{100-p}$. Describe any discontinuity for C (p).

Solution: We observe that C(100) = [8100(100)/(100 – 100)] = ∞. Hence percentage of pollution from the exhaust gases is not continuous when p = 100.

14. The monthly charge (in dollars) for x kilowatt hours (KWH) of electricity used by a commercial customer is given by the following function:

$$C(x) = \begin{cases} 7.52 + 0.1079x, & 0 \le x \le 5 \\ 19.22 + 0.1079x, & 5 < x \le 750 \\ 20.795 + 0.1058x, & 750 < x \le 1500 \\ 131.345 + 0.0321x, & x > 1500 \end{cases}.$$

a. What will be the charges for using the 5 and 750 KWH of electricity?
b. Is the function continuous at x = 5 and x = 15?

Solution: (a) Monthly electricity charges at x = 5 (KWH) are:

C(5) = 7.52 + 0.1079(5) = 8.0595 dollars.

Similarly the monthly electricity electrical charges when x = 15(KWH) are:

C(15) = 19.22 + 0.1079(15) = 20.8385 dollars.

(a) **Continuity at x = 5:** Consider, $C(5-0) = \lim_{x \to 5-0} (7.52 + 0.1079x) = 8.0595$

$$C(5+0) = \lim_{x \to 5+0} (19.22 + 0.1079x) = 19.7595$$

Since LHL and RHL are not equal hence electricity charges at x = 5 (KWH) is not continuous.

Continuity at x = 15:

Since, 15 lies in the interval (5, 750] for which the monthly charges are C(x) = 19.22 + 0.0179x, hence C(x) is continuous at x = 15 KWH.

15. Suppose the size of a population of bacteria in time t is given by:

$$P(t) = \begin{cases} t, & \text{when } 0 \le 1 \le 1/2 \\ 1, & \text{when } t = 1/2 \\ 1 - t, & \text{when } 1/2 < t < 1 \end{cases}.$$

a. How many bacteria were originally present?
b. Is the function continuous at t = 1/2 hours?

Solution: (a) Putting t = 0, we get P = 0. Thus originally zero bacteria were present.

(b) Continuity at t = 1/2: Consider,

$$C\left(\frac{1}{2} - 0\right) = \lim_{x \to \frac{1}{2}-0} (t) = \frac{1}{2} \quad \text{and} \quad C\left(\frac{1}{2} + 0\right) = \lim_{x \to \frac{1}{2}+0} (1-t) = \frac{1}{2}$$

Now, P(1/2) = 1. Since, P(1/2 – 0) = P(1/2 + 0) ≠ P(1/2) hence, growth of bacteria is not continuous at t = 1/2 hours.

16. The amount of a drug that remains in a person's bloodstream t hours after being injected is given by:

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$$f(t)=\begin{cases} t+2, & \text{when } 0 \le t \le 1 \\ t, & \text{when } 1 \le t < 2 \\ t+5, & \text{when } 2 \le t < 3 \end{cases}$$

a. What was the amount of drug at the beginning?
b. Is the function continuous at t = 2 and t = 3 hours?

Solution: (a) The amount of drug at the beginning (t = 0) is f(0) = (0 + 2) = 2 cc.
(b) Continuity of drug amount at t = 2:

$$f(2-0)=\lim_{x\to 2-0}(t)=2 \qquad \text{and} \qquad f(2+0)=\lim_{x\to 2+0}(t+5)=7$$

Since LHL ≠ RHL hence the amount of drug is not continuous at t = 2 hours.

17. Suppose the cost of obtaining water that contains p% impurities is given by:

$$C(p)=\begin{cases} \dfrac{120,000}{p}-1200, & \text{if } 0 < p \le 10 \\ 10800, & \text{if } 10 < p \le 12 \end{cases}.$$

a. What is the cost of obtaining water that contains 12% impurities?
b. What is the cost of obtaining water that contains 5% impurities?
c. Is the function continuous at p = 10%?

Solution: (a) Putting p = 12, we get C(12) = 10800 (This is the cost of purifying water that contains 12% impurities).
(b) Putting p = 5, we get C(5) = (120000/5) = 24000 (This is the cost of purifying water that contains 5% impurities).
(c) Continuity at p = 10: Consider,

$$C(10-0)=\lim_{p\to 10-0}\frac{120000}{p}-1200=10800 \quad \text{and} \quad C(10+0)=\lim_{p\to 10+0}(10800)=10800$$

Also, C(10) = 10800. Since, C(10 – 0) = C(10 + 0) = C(10) = 10800 hence, process of obtaining water that contains 10% impurities is continuous.

18. The velocity V(t) m/sec of a particle is given as:

$$V(t)=\begin{cases} t, & \text{when } t \le 0 \\ t^2, & \text{when } 0 < t \le 4 \\ 2t+4, & \text{when } t > 4 \end{cases}.$$

a. What was the velocity at the beginning?
b. Is the velocity of the particle continuous at t = 0 and t = 4?

Solution: (a) The velocity of a particle at time t = 0 is V(0) = 0.
(b) Continuity of velocity at t = 0: Consider,

$$V(0-0)=\lim_{t\to 0-0}(t)=0 \qquad \text{and} \qquad V(0+0)=\lim_{t\to 0+0}(t^2)=0$$

Since V(0 – 0) = V(0 + 0) = V(0) = 0 hence the velocity of a particle is continuous at t = 0.
Continuity at t = 4: Consider

$$V(4-0)=\lim_{t\to 4-0}(t^2)=16 \qquad \text{and} \qquad V(4+0)=\lim_{t\to 4+0}(2t+4)=12$$

Since, V(4 – 0) ≠ V(4 + 0) hence, the velocity of a particle is not continuous at t = 4.

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19. Suppose that the cost to remove x percent of the pollutants in a lake is given by:

$$C(x) = \begin{cases} \dfrac{900,000}{100 - x}, & \text{if } 0 < x \le 20 \\ 11250, & \text{if } x > 20 \end{cases}.$$

a. What is the cost to remove 10% of the pollutants from the lake?
b. Is the function continuous at x = 20%?

Solution: (a) The cost of removing 10% of the pollutants from the lake is

$$C(10) = \frac{900000}{100 - 10} = 1000$$

(b) Continuity at x = 20%: Consider,

$$C(20-0) = \lim_{x \to 20-0} \frac{900000}{100 - x} = 11250 \quad \text{and} \quad C(20+0) = \lim_{x \to 20+0} (11250) = 11250$$

Since C(20 – 0) = C(20 + 0) = C(20) = 11250, hence cost of removing 20% pollutant from water is continuous.

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Chapter 4

DERIVATIVES

WORKSHEET 04

1. Differentiate the following functions:

$(i)\ f(x)=\frac{1}{2}x^2+\frac{1}{7}x+\frac{1}{4},\ (ii)\ f(x)=\left(x+\frac{1}{x}\right)^2,\ (iii)\ f(x)=\left(x^2+3\right)\left(2x^2-1\right)$

$(iv)\ f(x)=\sqrt{x}\left(2x-1\right)\left(x^2+x+1\right),\ (v)\ f(x)=\frac{\sqrt{x}+1}{\sqrt{x}-1}.$

Solution: (i) Given $f(x)=\frac{1}{2}x^2+\frac{1}{7}x+\frac{1}{4}$. Differentiate both sides w.r.t. x, we get

$$f'(x)=\frac{2x}{2}+\frac{1}{7}.1+0=x+\frac{1}{7}$$

(ii) Given $f(x)=\left(x+\frac{1}{x}\right)^2$. Differentiate w.r.t x, we obtain

$$f'(x)=\frac{d}{dx}\left(x+\frac{1}{x}\right)^2=2\left(x+\frac{1}{x}\right).\frac{d}{dx}\left(x+\frac{1}{x}\right)=2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)$$

(iii) Given $f(x) = (x^2 + 3)(2x^2 - 1) = 2x^4 + 5x^2 -3$. Now differentiating w.r.t x, we get
$f`(x) = 8x^3 + 10x - 0 = 8x^3 + 10x$

(iv) $f(x)=\sqrt{x}\left(2x-1\right)\left(x^2+x+1\right)=2x^{7/2}+x^{5/2}+x^{3/2}-x^{1/2}$

Differentiating both sides w.r.t x, we get:

$$f'(x)=2\frac{7}{2}x^{5/2}+\frac{5}{2}x^{3/2}+\frac{3}{2}x^{1/2}-\frac{1}{2}x^{-1/2}=7x^{5/2}+\frac{5}{2}x^{3/2}+\frac{3}{2}x^{1/2}-\frac{1}{2\sqrt{x}}$$

(v) $f(x)=\frac{\sqrt{x}+1}{\sqrt{x}-1}$ Differentiate w.r.t x, we get

$$f'(x)=\frac{\left(\sqrt{x}-1\right)\frac{d}{dx}\left(\sqrt{x}+1\right)-\left(\sqrt{x}+1\right)\frac{d}{dx}\left(\sqrt{x}-1\right)}{\left(\sqrt{x}-1\right)^2}=\frac{\left(\sqrt{x}-1\right)\left(\frac{1}{2\sqrt{x}}\right)-\left(\sqrt{x}+1\right)\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}-1\right)^2}$$

$$=\frac{\sqrt{x}-1-\sqrt{x}-1}{2\sqrt{x}\left(\sqrt{x}-1\right)^2}=\frac{-2}{2\sqrt{x}\left(\sqrt{x}-1\right)^2}=\frac{-1}{\sqrt{x}\left(\sqrt{x}-1\right)^2}$$

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$$= \frac{\sqrt{x}-1-\sqrt{x}-1}{2\sqrt{x}\left(\sqrt{x}-1\right)^2} = \frac{-2}{2\sqrt{x}\left(\sqrt{x}-1\right)^2} = \frac{-1}{\sqrt{x}\left(\sqrt{x}-1\right)^2}$$

2. If $f(x) = 6x^2 - 5x + 3$, find f `(0). For what value of x is f `(x) = 0

Solution: Given $f(x) = 6x^2 - 5x + 3$. Differentiate w.r.t, we get

f `(x) = 12x -5. Thus f `(0) = 12(0) – 5 = -5. If f `(x) = 0 then 12x – 5 = 0 $\Rightarrow$ x = 5/12.

3. Fnd y` where:

(a) $y = x^{\sin x}$.

Solution: Taking ln on both sides, we get: $\ln y = \ln (x)^{\sin x} \Rightarrow \ln y = \sin x.\ln x$.

Differentiating w.r.t x, we get:

$$\frac{1}{y}\frac{dy}{dx} = \sin x\left(\frac{1}{x}\right) + \ln x.\cos x \Rightarrow \frac{dy}{dx} = y\left[\frac{\sin x + x\ln x\cos x}{x}\right] = x^{\sin x}\left[\frac{\sin x + x\ln x\cos x}{x}\right]$$

(b) $y = \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

Solution: $y = \ln\left(\sqrt{x}+1\right) - \ln\left(\sqrt{x}-1\right)$

Differentiate w.r.t x, we get

$$\frac{dy}{dx} = \frac{1}{\left(\sqrt{x}+1\right)}.\frac{1}{2\sqrt{x}} - \frac{1}{\left(\sqrt{x}-1\right)}.\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}}\left[\frac{1}{\left(\sqrt{x}+1\right)} - \frac{1}{\left(\sqrt{x}-1\right)}\right]$$

$$= \frac{1}{2\sqrt{x}}\left[\frac{\sqrt{x}-1-\sqrt{x}-1}{\left(\sqrt{x}+1\right)\left(\sqrt{x}-1\right)}\right] = \frac{-2}{2\sqrt{x}}\left(\frac{1}{x-1}\right) = -\left[\frac{1}{\sqrt{x}\left(x-1\right)}\right]$$

(c) $y = \tan^{-1}\left(\frac{1+2x}{2-x}\right)$

Solution: Differentiate w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\left[1+\left(\frac{1+2x}{2-x}\right)^2\right]}.\frac{d}{dx}\left(\frac{1+2x}{2-x}\right) = \frac{\left(2-x\right)^2}{\left(2-x\right)^2+\left(1+2x\right)^2}.\frac{\left(2-x\right).2-\left(1+2x\right)(-1)}{\left(2-x\right)^2}$$

$$= \frac{4-2x+1+2x}{4-4x+x^2+1+4x+4x^2} = \frac{5}{5x^2+5} = \frac{5}{5\left(x^2+1\right)} = \frac{1}{x^2+1}$$

(d) $y = \ln\left(\sin^{-1} e^x\right)$

Solution: Differentiate w.r.t x, we have

$$\frac{dy}{dx} = \frac{1}{\left[\sin^{-1}(e^x)\right]}.\frac{d}{dx}\left(\sin^{-1}(e^x)\right) = \frac{1}{\left[\sin^{-1}(e^x)\right]}.\frac{1}{\sqrt{1-e^{2x}}}.\frac{d}{dx}(e^x) = \frac{e^x}{\sqrt{1-e^{2x}}\left[\sin^{-1}(e^x)\right]}$$

(f) $y = \left(\sin^{-1} x^2\right)^4$

Solution: Differentiate w.r.t x, we get

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$$\frac{dy}{dx}=4\left[\sin^{-1}x^2\right]^3.\frac{d}{dx}\left(\sin^{-1}x^2\right)=4\left[\sin^{-1}x^2\right]^3\frac{1}{\sqrt{1-x^4}}.\frac{d}{dx}x^2=\frac{8x\left[\sin^{-1}x^2\right]^3}{\sqrt{1-x^4}}$$

(g) $y=\left(\cos^{-1}x^2\right)^{\pi}$

Solution: Differentiate w.r.t x, we get

$$\frac{dy}{dx}=\pi\left(\cos^{-1}x^2\right)^{\pi-1}\frac{d}{dx}\left(\cos^{-1}x^2\right)=\pi\left(\cos^{-1}x^2\right)^{\pi-1}.\frac{-1}{\sqrt{1-x^4}}\frac{d}{dx}x^2$$

$$=\frac{-2\pi\left(\cos^{-1}x^2\right)^{\pi-1}}{\sqrt{1-x^4}}$$

(h) $y=\dfrac{1-\cos x}{1+\cos x}$

Solution: Differentiate w.r.t x, we get

$$\frac{dy}{dx}=\frac{(1+\cos x)\frac{d}{dx}(1-\cos x)-(1-\cos x)\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x).\sin x-(1-\cos x)(-\sin x)}{(1+\cos x)^2}=\frac{\sin x+\sin x\cos x+\sin x-\sin x\cos x}{(1+\cos x)^2}$$

$$=\frac{2\sin x}{(1+\cos x)^2}$$

(i) $y=\dfrac{1-\cosh x}{1+\cosh x}$

Solution: Differentiate w.r.t x, we have

$$\frac{dy}{dx}=\frac{(1+\cosh x)\frac{d}{dx}(1-\cosh x)-(1-\cosh x)\frac{d}{dx}(1+\cosh x)}{(1+\cosh x)^2}$$

$$=\frac{(1+\cosh x)(-\sinh x)-(1-\cosh x)(\sinh x)}{(1+\cosh x)^2}$$

$$=\frac{-\sinh x-\sinh x\cosh x-\sinh x+\sinh x\cosh x}{(1+\cosh x)^2}=\frac{-2\sinh x}{\left(1+\cosh x\right)^2}$$

(j) $y=\log_{10}\left(\dfrac{x+1}{x}\right)$

Solution: Here y may be expressed as: $y=\log_{10}(x+1)-\log_{10}x$

Now differentiating w.r.t x, we get;

$$\frac{dy}{dx}=\frac{1}{(x+1)\ln 10}-\frac{1}{x\ln 10}=\frac{1}{\ln 10}\left[\frac{1}{x+1}-\frac{1}{x}\right]=\frac{1}{\ln 10}\left[\frac{x-x-1}{x(x+1)}\right]=\frac{-1}{x(x+1)\ln 10}$$

(k) $y=\cos^{-1}\sqrt{1-x^2}$

Solution: Differentiating w.r.t x, we get

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$$\frac{dy}{dx}=\frac{-1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}.\frac{d}{dx}\sqrt{1-x^2}=\frac{-1}{\sqrt{1-1+x^2}}.\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)$$

$$=\frac{2x}{2\sqrt{x^2}}.\frac{1}{\sqrt{1-x^2}}=\frac{2x}{2x}.\frac{1}{\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}$$

(l) $y=\sec^{-1}(\sinh x)$

Solution: Differentiate w.r.t x, we get

$$\frac{dy}{dx}=\frac{1}{\sinh x\sqrt{\sinh^2 x-1}}.\frac{d}{dx}\sinh x=\frac{\cosh x}{\sinh x\sqrt{\sinh^2 x-1}}=\frac{\coth x}{\sqrt{\sinh^2 x-1}}$$

NOTE : $\frac{d}{dx}\sec^{-1}x=\frac{1}{x\sqrt{x^2-1}}$

(m) $y=\cosh^{-1}\left(1+x^2\right)$

Solution: Differentiate w.r.t x, we get

$$\frac{dy}{dx}=\frac{1}{\sqrt{\left(1+x^2\right)^2-1}}\frac{d}{dx}\left(1+x^2\right)=\frac{2x}{\sqrt{1+2x^2+x^4-1}}=\frac{2x}{\sqrt{x^2\left(2+x^2\right)}}=\frac{2x}{x\sqrt{2+x^2}}=\frac{2}{\sqrt{2+x^2}}$$

(n) $\sqrt{x}+\sqrt{y}-\sqrt{2}=0$

Solution: This is an implicit equation hence differentiating both sides w.r.t x, we obtain

$$\frac{d}{dx}\sqrt{x}+\frac{d}{dx}\sqrt{y}-\frac{d}{dx}\sqrt{2}=0\Rightarrow\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\sqrt{\frac{y}{x}}$$

(o) $xy^2-2xy+x=1$

Solution: Differentiating w.r.t x, we get

$$\frac{d}{dx}\left(xy^2\right)-2\frac{d}{dx}(xy)+\frac{d}{dx}(x)=\frac{d}{dx}(1)\Rightarrow x(2yy`)+y^2.1-2\left(xy`+y.1\right)+1=0$$

$$\Rightarrow\left(2xy-2x\right)y`=2y-y^2-1=-\left(y^2-2y+1\right)=-\left(y-1\right)^2\Rightarrow y`=-\frac{\left(y-1\right)^2}{2x\left(y-1\right)}=-\frac{\left(y-1\right)}{2x}$$

(p) $x^3+y^3-3axy=0$

Solution: Differentiating both sides w.r.t x, we get

$$\frac{d}{dx}x^3+\frac{d}{dx}y^3-3a\frac{d}{dx}(xy)=0\Rightarrow 3x^2+3y^2y`-3a(xy`+y.1)=0$$

$$\Rightarrow\left(3y^2-3ax\right)y`=-3x^2+3ay\Rightarrow\frac{dy}{dx}=\frac{3\left(ay-x^2\right)}{3\left(y^2-ax\right)}=\frac{\left(ay-x^2\right)}{\left(y^2-ax\right)}$$

(q) $y=\left(x^2+y^2\right)^3$

Solution: Differentiating w.r.t x on both sides, we have

$$y`=3[(x^2+y^2)]^2\frac{d}{dx}(x^2+y^2)\Rightarrow y`=3[(x^2+y^2)]^2(2x+2yy`)$$

$$\Rightarrow y`=6x[(x^2+y^2)]^2+6y[(x^2+y^2)]^2y`$$

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$$\Rightarrow\ [1-6y[(x^2+y^2)]^2]y` = 6x[(x^2+y^2)]^2 \quad \Rightarrow y` = \frac{6x[(x^2+y^2)]^2}{[1-6y[(x^2+y^2)]^2]}$$

(r) $\tan^{-1}\left(\frac{y}{x}\right)+yx^2=1$

Solution: Differentiating both sides w.r.t x, we get

$$\frac{d}{dx}\tan^{-1}\left(\frac{y}{x}\right)+\frac{d}{dx}(yx^2)=\frac{d}{dx}(1) \quad \Rightarrow \quad \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right)+y(2x)+x^2\frac{d}{dx}(y)=0$$

$$\frac{x^2}{x^2+y^2}.\left(\frac{xy`-y.1}{x^2}\right)+2xy+x^2y`=0 \quad \Rightarrow \quad \frac{xy`}{x^2+y^2}-\frac{y}{x^2+y^2}+2xy+x^2y`=0$$

$$\left[\frac{x}{x^2+y^2}+x^2\right]y` = \frac{y}{x^2+y^2}-2xy \quad \Rightarrow \quad \left[\frac{x+x^4+x^2y^2}{x^2+y^2}\right]y` = \frac{y-2x^3y-2xy^3}{x^2+y^2}$$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{y(1-2x^3-2xy^2)}{x(1+x^3+xy^2)}$$

(s) $y=\tan^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)^{1/2}$

Solution: Simplifying the given equation, we get

$$y=\tan^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}=\tan^{-1}\sqrt{\left(\frac{2\sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)}\right)}=\tan^{-1}\sqrt{\left(\tan\frac{x}{2}\right)^2}=\tan^{-1}\left(\tan\frac{x}{2}\right)$$

Thus, $\quad y=\frac{x}{2} \quad \Rightarrow \quad y`=1/2$

(t) $\tan^{-1}(x+y)=\sin^{-1}(x+y)$ NOTE: See the change in the problem

Solution: Differentiating both sides w.r.t x, we get

$$\frac{d}{dx}\tan^{-1}(x+y)=\frac{d}{dx}\sin^{-1}(x+y)$$

$$\Rightarrow \quad \frac{1}{1+(x+y)^2}.\frac{d}{dx}(x+y)=\frac{1}{\sqrt{1-(x+y)^2}}.\frac{d}{dx}(x+y) \quad \left[\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{1-x^2}}\right]$$

$$\Rightarrow \quad \frac{1}{1+(x+y)^2}.(1+y`)=\frac{1}{\sqrt{1-(x+y)^2}}.(1+y`) \quad \left[\frac{d}{dx}\tan^{-1}x=\frac{1}{1+x^2}\right]$$

$$\Rightarrow \quad \frac{1}{[1+(x+y)^2]}y`-\frac{1}{\sqrt{1-(x+y)^2}}y`=-\frac{1}{[1+(x+y)^2]}-\frac{1}{\sqrt{1-(x+y)^2}}$$

$$\left[\frac{\sqrt{1-(x+y)^2}-\left[1+(x+y)^2\right]}{\left[1+(x+y)^2\right]\sqrt{1-(x+y)^2}}\right]y`=-\left[\frac{\sqrt{1-(x+y)^2}+\left[1+(x+y)^2\right]}{\left[1+(x+y)^2\right]\sqrt{1-(x+y)^2}}\right]$$

$$\Rightarrow y`=-\frac{\sqrt{1-(x+y)^2}+\left[1+(x+y)^2\right]}{\sqrt{1-(x+y)^2}-\left[1+(x+y)^2\right]}$$

(u) $y=\sin^{-1}(\ln x)-\ln\left(\tan^{-1}x\right)$

Solution: Differentiate w.r.t x, we have

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$$y` = \frac{d}{dx}\sin^{-1}(\ln x) - \frac{d}{dx}\ln(\tan^{-1}x)$$

$$\Rightarrow y` = \frac{1}{\sqrt{1-(\ln x)^2}}\frac{d}{dx}(\ln x) - \frac{1}{(\tan^{-1}x)}\frac{d}{dx}((\tan^{-1}x))$$

$$\Rightarrow y` = \frac{1}{x\sqrt{1-(\ln x)^2}} - \frac{1}{(\tan^{-1}x)(1+x^2)}$$

(v) $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+x}{1-x}}\right)$ $[\text{Hint : Put } x = \cos\theta]$

Put $x = \cos\theta$, we get $y = \sin^2\left(\cot^{-1}\sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\right) = \sin^2\left(\cot^{-1}\sqrt{\frac{2\cos^2\theta/2}{2\sin^2\theta/2}}\right)$

$$\Rightarrow y = \sin^2\left(\cot^{-1}\left(\cot\frac{\theta}{2}\right)\right) = \sin^2\left(\frac{\theta}{2}\right)$$

These are now parametric equations, thus:

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.\left(\frac{1}{2}\right) \div \sin\theta = \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) \div \left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{1}{2}$$

(w) x = a(t – sin t) and y = a(1 – cos t)

Solution: Given equations are parametric equations hence,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = a(0-(-\sin t)) \div a(1-\cos t) = \frac{a\sin t}{a(1-\cos t)} = \frac{2\sin\frac{t}{2}.\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2}$$

(x) x = 3 cos t – cos 3t and y = 3 sin t – sin 3t

Solution: Given equations are parametric equations hence,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = (3\cos t - 3\cos 3t) \div (-3\sin t + 3\sin 3t) = \frac{\cos t - \cos 3t}{\sin 3t - \sin t}$$

(y) x = a[cos t + ln tan (t/2)] and y = a sin t

Solution: Given equations are parametric equation, hence

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = a\cos t \div a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}.\frac{d}{dt}\tan\left(\frac{t}{2}\right)\right]$$

$$= \frac{a\cos t}{a\left[-\sin t + \frac{1}{\tan\left(\frac{t}{2}\right)}.\sec^2\left(\frac{t}{2}\right).\frac{1}{2}\right]} = \frac{\cos t}{\left[-\sin t + \frac{\cos\left(\frac{t}{2}\right)}{2\sin\left(\frac{t}{2}\right).\cos^2\left(\frac{t}{2}\right)}\right]}$$

$$= \frac{\cos t}{\left[-\sin t + \frac{1}{2\sin\left(\frac{t}{2}\right).\cos\left(\frac{t}{2}\right)}\right]} = \frac{\cos t}{\left(-\sin t + \frac{1}{\sin t}\right)} = \frac{\cos t.\sin t}{-\sin^2 t + 1} = \frac{\cos t\sin t}{\cos^2 t}$$

$$= \frac{\sin t}{\cos t} = \tan t$$

(z) $x = \tan^{-1}\frac{2\theta}{1-\theta^2}$ and $y = \sin^{-1}\frac{2\theta}{1+\theta^2}$

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Solution: These are parametric equations hence: $\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}$. Now,

$$\frac{dy}{d\theta}=\frac{1}{\sqrt{1-\left(\frac{2\theta}{1+\theta^2}\right)^2}}.\frac{d}{d\theta}\left(\frac{2\theta}{1+\theta^2}\right)=\frac{1+\theta^2}{\sqrt{(1+\theta^2)^2-(2\theta)^2}}.\frac{(1+\theta^2).2-2\theta(2\theta)}{(1+\theta^2)^2}$$

$$=\frac{1}{\sqrt{1+2\,\theta^2+\theta^4-4\theta^2}}.\frac{2+2\theta^2-4\theta^2}{(1+\theta^2)}=\frac{2-2\theta^2}{(1+\theta^2)\sqrt{1-2\,\theta^2+\theta^4}}$$

$$=\frac{2(1-\theta^2)}{(1+\theta^2)(1-\theta^2)}=\frac{2}{(1+\theta^2)}$$

$$\frac{dx}{d\theta}=\frac{1}{1+\left(\frac{2\theta}{1-\theta^2}\right)^2}.\frac{d}{d\theta}\left(\frac{2\theta}{1-\theta^2}\right)=\frac{(1-\theta^2)^2}{(1-\theta^2)^2+(2\theta)^2}.\frac{(1-\theta^2).2-2\theta(-2\theta)}{(1-\theta^2)^2}$$

$$=\frac{2-2\theta^2+4\theta^2}{1-2\theta^2+\theta^4+4\theta^2}=\frac{2\left(1+\theta^2\right)}{\left(1+\theta^2\right)^2}=\frac{2}{\left(1+\theta^2\right)}$$

Thus, $\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=\frac{2}{\left(1+\theta^2\right)}\div\frac{2}{\left(1+\theta^2\right)}=1$

4. Differentiate logarithmically:

(a) $y=\frac{\sqrt{x}\left(1-2x\right)^{2/3}}{\left(2-2x\right)^{3/4}\left(3-4x\right)^{4/3}}$

Solution: Taking ln on both sides, we get

$$\ln y=\ln\sqrt{x}+\ln\left(1-2x\right)^{2/3}-\ln\left(2-2x\right)^{3/4}-\ln\left(3-4x\right)^{4/3}$$

$$\Rightarrow \ln y=\frac{1}{2}\ln x+\frac{2}{3}\ln\left(1-2x\right)-\frac{3}{4}\ln\left(2-2x\right)-\frac{4}{3}\ln\left(3-4x\right)$$

Differentiate both sides w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2x}+\frac{2}{3\left(1-2x\right)}(-2)-\frac{3}{4\left(2-2x\right)}(-2)-\frac{4}{3\left(3-4x\right)}(-4)$$

$\therefore \frac{dy}{dx}=y\left[\frac{1}{2x}-\frac{4}{3\left(1-2x\right)}+\frac{3}{2\left(2-2x\right)}+\frac{16}{3\left(3-4x\right)}\right]$. Putting the value of y, we get

$$\frac{dy}{dx}=\frac{\sqrt{x}\left(1-2x\right)^{2/3}}{\left(2-2x\right)^{3/4}\left(3-4x\right)^{4/3}}\left[\frac{1}{2x}-\frac{4}{3\left(1-2x\right)}+\frac{3}{2\left(2-2x\right)}+\frac{16}{3\left(3-4x\right)}\right]$$

(b) $y=x^x e^x\sin\left(\ln x\right)$

Solution: Taking ln on both sides, we get

$\ln y=\ln x^x+\ln e^x+\ln\left[\sin\left(\ln x\right)\right]=x\ln x+x+\ln\left[\sin\left(\ln x\right)\right]$ NOTE: ln e = 1

Differentiate w.r.t x, we get

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$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \ln x.1 + 1 + \frac{1}{\sin(\ln x)}.\cos(\ln x).\frac{1}{x}$$

$\Rightarrow \frac{dy}{dx} = y\left[1 + \ln x + 1 + \frac{\cot(\ln x)}{x}\right]$. Putting the value of y, we get

$$\frac{dy}{dx} = x^x e^x \sin(\ln x)\left[2 + \ln x + \frac{\cot(\ln x)}{x}\right]$$

(c) $y = e^{\sec^{-1}\left(\frac{1}{x}\right)}$

Solution: Taking ln on both sides, we get

$\ln y = \ln e^{\sec^{-1}\left(\frac{1}{x}\right)} = \sec^{-1}\left(\frac{1}{x}\right).\ln e = \sec^{-1}\left(\frac{1}{x}\right)$. Differentiate w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\frac{1}{x}\sqrt{\left(\frac{1}{x}\right)^2 - 1}}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{x.x}{\sqrt{1-x^2}} \times \left(\frac{-1}{x^2}\right) = -\frac{1}{\sqrt{1-x^2}}.\left[\text{NOTE}: \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}\right]$$

$$\Rightarrow \frac{dy}{dx} = y.\frac{1}{\sqrt{1-x^2}} = e^{\sec^{-1}\left(\frac{1}{x}\right)}.\frac{1}{\sqrt{1-x^2}}$$

(d) $y = \frac{(x+2)^2}{(x+1)(x^2+3)^2}$

Solution: Taking ln on both sides, we get

$$\ln y = \ln(x+2)^2 - \ln(x+1) - \ln(x^2+3)^2 = 2\ln(x+2) - \ln(x+1) - 2\ln(x^2+3)$$

Differentiate both sides w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+2} - \frac{1}{x+1} - \frac{2}{(x^2+3)}(2x) \Rightarrow \frac{dy}{dx} = y\left[\frac{2}{x+2} - \frac{1}{x+1} - \frac{4x}{(x^2+3)}\right]$$

Putting the value of y, we have: $\frac{dy}{dx} = \frac{(x+2)^2}{(x+1)(x^2+3)^2}\left[\frac{2}{x+2} - \frac{1}{x+1} - \frac{4x}{(x^2+3)}\right]$

(e) $y = x^x$

Solution: Taking ln on both sides, we get:

$\ln y = \ln x^x = x.\ln x$. Now differentiating w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + \ln x.1 = 1 + \ln x \qquad \Rightarrow \frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x)$$

(f) $y = (\sin x)^x$

Solution: Taking ln on both sides, we get

$\ln y = \ln(\sin x)^x = x.\ln(\sin x)$. Now differentiating w.r.t x, we get

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$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{\sin x}.\cos x + \ln(\sin x).1 = x\cot x + \ln(\sin x)$$

$$\Rightarrow \frac{dy}{dx} = y\left[x\cot x + \ln(\sin x)\right] = \sin x^{x}\left[x\cot x + \ln(\sin x)\right]$$

(g) $y = x^{\sin 2x}$

Solution: Taking ln on both sides, we get

$\ln y = \ln(x)^{\sin 2x} = \sin 2x.\ln x$. Now differentiating w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = \sin 2x.\frac{1}{x} + \ln x(2\cos 2x) = \frac{\sin 2x + 2x\ln x\cos 2x}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{\sin 2x + 2x\ln x\cos 2x}{x}\right] = x^{\sin 2x}\left[\frac{\sin 2x + 2x\ln x\cos 2x}{x}\right]$$

(h) $y = (\sin x)^{\cos x}$

Solution: Taking ln on both sides, we get

$\ln y = \ln(\sin x)^{\cos x} = \cos x.\ln(\sin x)$. Differentiating w.r.t x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos x.\frac{1}{\sin x}.\cos x + \ln(\sin x).(-\sin x) = \cot x\cos x - \ln(\sin x)\sin x$$

$$\Rightarrow \frac{dy}{dx} = y\left[\cot x\cos x - \ln(\sin x)\sin x\right] = (\sin x)^{\cos x}\left[\cot x\cos x - \ln(\sin x)\sin x\right]$$

(i) $y = (\sin x)^{\cos x} + (\cos x)^{\sin x}$

Solution: Here right side contains sum of two exponential terms hence we cannot take log directly. The procedure to solve this type of problem is as under:

Let $u = (\sin x)^{\cos x}$ and $v = (\cos x)^{\sin x}$. Thus $y = u + v$ and therefore,

$$y` = u` + v` \qquad (1)$$

Now, $u = (\sin x)^{\cos x}$. Taking ln on both sides, we get

$\ln u = \ln(\sin x)^{\cos x} = \cos x.\ln(\sin x)$. Differentiating w.r.t x, we get

$$\frac{1}{u}\frac{du}{dx} = \cos x.\frac{1}{\sin x}.\cos x + \ln(\sin x).(-\sin x) = \cot x\cos x - \ln(\sin x)\sin x$$

$$\Rightarrow \frac{du}{dx} = u\left[\cot x\cos x - \ln(\sin x)\sin x\right] = (\sin x)^{\cos x}\left[\cot x\cos x - \ln(\sin x)\sin x\right]$$

Now, $v = (\cos x)^{\sin x}$. Taking ln on both sides, we get

$\ln v = \ln(\cos x)^{\sin x} = \sin x.\ln(\cos x)$. Now differentiating w.r.t x, we get

$$\frac{1}{v}\frac{dv}{dx} = \sin x.\frac{1}{\cos x}.(-\sin x) + \ln(\cos x).(\cos x) = -\tan x\sin x + \ln(\cos x)\cos x$$

$$\frac{dv}{dx} = v\left[-\tan x\sin x + \ln(\cos x)\cos x\right] = (\cos x)^{\sin x}\left[-\tan x\sin x + \ln(\cos x)\cos x\right]$$

Thus equation (1) becomes:

$$\frac{dy}{dx} = (\sin x)^{\cos x}\left[\cot x\cos x - \ln(\sin x)\sin x\right] + (\cos x)^{\sin x}\left[-\tan x\sin x + \ln(\cos x)\cos x\right]$$

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(k) $y = x^{\sin x} + (\sin x)^x$

Solution: Refer to part (i), let $u = x^{\sin x}$ and $v = (\sin x)^x$ thus $y = u + v$ and therefore,

$$y` = u` + v` \qquad (1)$$

Now, $u = (x)^{\sin x}$. Taking ln on both sides, we get

$\ln u = \ln(x)^{\sin x} = \sin x.\ln(x)$. Differentiating w.r.t x, we get

$$\frac{1}{u}\frac{du}{dx} = \sin x.\frac{1}{x} + \ln(x).(\cos x) = \frac{\sin x + x\ln x.\cos x}{x}$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{\sin x + x\ln x.\cos x}{x}\right] = (x)^{\sin x}\left[\frac{\sin x + x\ln x.\cos x}{x}\right]$$

Also, $v = (\sin x)^x$. Taking ln on both sides, we get

$\ln v = \ln(\sin x)^x = x.\ln(\sin x)$. Now differentiating w.r.t x, we get

$$\frac{1}{v}\frac{dv}{dx} = x.\frac{1}{\sin x}.(\cos x) + \ln(\sin x).(1) = \frac{x\cos x + \sin x.\ln(\sin x)}{\sin x}$$

$$\frac{dv}{dx} = v\left[\frac{x\cos x + \sin x.\ln(\sin x}{\sin x}\right] = (\sin x)^x\left[\frac{x\cos x + \sin x.\ln(\sin x)}{\sin x}\right]$$

Thus equation (1) becomes:

$$\frac{dy}{dx} = (x)^{\sin x}\left[\frac{\sin x + x\ln x.\cos x}{x}\right] + (\sin x)^x\left[\frac{x\cos x + \sin x.\ln(\sin x}{\sin x}\right]$$

5. If $\sin y = x \sin(a - y)$ prove that $y` = \sin^2(a - y)/\sin a$ [The question may be corrected]

Solution: Given $\sin y = x \sin(a - y)$ (1)

Differentiate w.r.t x, we get

$$\cos y.y` = \sin(a-y).1 + x.\cos(a-y).(-y`) \Rightarrow [\cos y + x\cos(a-y)]y` = \sin(a-y)$$

$$\Rightarrow y` = \frac{\sin(a-y)}{[\cos y + x\cos(a-y)]} \qquad (2)$$

Now from (1), $x = \sin y/\sin(a - y)$. Put this in (2), we obtain

$$y` = \frac{\sin(a-y)}{\left[\cos y + \frac{\sin y}{\sin(a-y)}\cos(a-y)\right]} = \frac{\sin^2(a-y)}{[\sin(a-y)\cos y + \cos(a-y)\sin y]}$$

$$= \frac{\sin^2(a-y)}{\sin(a-y+y)} = \frac{\sin^2(a-y)}{\sin a}. \quad \text{NOTE}: \sin a\cos b + \cos a\sin b = \sin(a+b)$$

6. If $x^2 + xy + 3y^2 = 1$, prove that $(x + 6y)^3 y`` + 22 = 0$.

Solution: Since, $x^2 + xy + 3y^2 = 1 \Rightarrow (x^2 + 3y^2) = 1 - xy$. Differentiating

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(xy) + \frac{d}{dx}(3y^2) = \frac{d}{dx}(1) \Rightarrow 2x + xy` + y.1 + 6yy` = 0 \qquad (1)$$

Differentaite again w.r.t x, we get

$$2 + y`.1 + xy`` + y` + 6(yy`` + y`y`) = 0 \quad \Rightarrow (x+6y)y`` + 6(y`)^2 + y` + 2 = 0 \qquad (2)$$

From (1), $y` = -\frac{(y+2x)}{(x+6y)}$. Put this in (2), we get

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$(x+6y)y``+6\left[\frac{y+2x}{x+6y}\right]^2-\left[\frac{y+2x}{x+6y}\right]+2=0.$ Multiplying by $(x+6y)^2$, we get

$(x+6y)^3 y``+6(y+2x)^2-(y+2x)(x+6y)+2(x+6y)^2=0$

or $(x+6y)^3 y''+24x^2-4x^2+2x^2+24xy-2xy-24xy+24xy-12y^2+72y^2+6y^2=0$

or $(x+6y)^3 y''+22x^2+22xy+66y^2=0$

or $(x+6y)^3 y''+22(x^2+xy+3y^2)=0$

or $(x+6y)^3 y''+22(1)=0$ Notice that we have used equation (*).

or $(x+6y)^3 y''+22=0$. Proved

7. Differentiate with respect to x each of the following:

(i) $f(x)=e^{ax}\cos(b\arctan x)$ (ii) $f(x)=\ln(\arcsin e^x)$ (iii) $f(x)=\ln(\tanh 2x)$

(iv) $f(x)=\arccos\left(\sqrt{1-x^2}\right)$ (v) $f(x)=\cosh^{-1}(1+x^2)$.

(i) Solution: Given that $f(x) = e^{ax}\cos(b\tan^{-1}x)$. Differentiate both sides w.r.t x, we get

$$f`(x)=e^{ax}\left[-\sin\left(b\tan^{-1}x\right).\frac{b}{1+x^2}\right]+ae^{ax}\cos\left(b\tan^{-1}x\right)$$

$$=e^{ax}\left[a\cos\left(b\tan^{-1}x\right)-\frac{b\sin\left(b\tan^{-1}x\right)}{1+x^2}\right]$$

(ii) Solution: Given $f(x) = \ln(\sin^{-1}e^x)$. Differentiate w.r.t.x, we get

$$f`(x)=\frac{1}{\sin^{-1}\left(e^x\right)}.\frac{d}{dx}\left(\sin^{-1}(e^x)\right)=\frac{1}{\sin^{-1}\left(e^x\right)}\frac{1}{\sqrt{1-e^{2x}}}.e^x=\frac{e^x}{\sqrt{1-e^{2x}}\sin^{-1}\left(e^x\right)}$$

(iii) Solution: Given that $f(x) = \ln(\tanh 2x)$. Differentiate w.r.t x, we get

$$f`(x)=\frac{1}{\tanh 2x}.\frac{d}{dx}(\tanh 2x)=\frac{1}{\tanh 2x}\sec h^2 2x.(2)=\frac{2\sec h^2 2x}{\tanh 2x}=2\frac{1}{\cosh^2 2x}.\frac{\cosh 2x}{\sinh 2x}$$

$$=\frac{2}{\sinh 2x\cosh 2x}=\frac{4}{2\sinh 2x\cosh 2x}=\frac{4}{\sinh 4x}=4\operatorname{csch} 4x$$

(iv) Solution: Given that $f(x)=\cos^{-1}\sqrt{1-x^2}$. Differentiate w.r.t x, we get

$$f`(x)=\frac{-1}{\sqrt{1-\left(\sqrt{1-x^2}\right)^2}}.\frac{d}{dx}\left(\sqrt{1-x^2}\right)=\frac{-1}{\sqrt{1-\left(1-x^2\right)}}.\frac{1}{2}\left(1-x^2\right)^{-1/2}.(-2x)$$

$$\Rightarrow f`(x)=\frac{-1}{x}.\frac{1}{2\sqrt{1-x^2}}(-2x)=\frac{1}{\sqrt{1-x^2}}$$

(v) Solution: Given that $f(x) = \cosh^{-1}(1 + x^2)$. Differentiate w.r.t x, we get

$$f`(x)=\frac{1}{\sqrt{\left(1+x^2\right)^2-1}}.\frac{d}{dx}\left(1+x^2\right)=\frac{2x}{\sqrt{1+2x^2+x^4-1}}=\frac{2x}{\sqrt{2x^2+x^4}}=\frac{2x}{x\sqrt{2+x^2}}=\frac{2}{\sqrt{2+x^2}}$$

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8. Find $\frac{dy}{dx}$ in each of the following:

(i) $x^3+y^3-3axy=0$ (ii) $y=\arcsin(\ln x)-\ln(\arctan x)$

(iii) $\arctan(x+y)=\arcsin(e^y+x)$ (iv) $y\arcsin x - x\arctan y = 1$

(i) Solution: See Q3. part (p)

(ii) Solution: Given that $y = \sin^{-1}(\ln x) - \ln(\tan^{-1}x)$. Differentiate w.r.t x, we get

$$y`=\frac{1}{\sqrt{1-(\ln x)^2}}.\frac{d}{dx}\ln x-\frac{1}{\tan^{-1}x}.\frac{d}{dx}\left(\tan^{-1}x\right)=\frac{1}{x\sqrt{1-(\ln x)^2}}-\frac{1}{\left(1+x^2\right)\tan^{-1}x}$$

(iii) Solution: Given that $\tan^{-1}(x + y) = \sin^{-1}(e^y)$

Differentiate w.r.t x, we have

$$\frac{1}{1+(x+y)^2}\frac{d}{dx}(x+y)=\frac{1}{\sqrt{1-e^{2y}}}\frac{d}{dx}\left(e^y\right) \quad \Rightarrow \frac{1}{1+(x+y)^2}(1+y`)=\frac{1}{\sqrt{1-e^{2y}}}e^y y`$$

$$\Rightarrow\left[\frac{1}{\left(1+(x+y)^2\right)}-\frac{e^y}{\sqrt{1-e^{2y}}}\right]y`=-\frac{1}{\left(1+(x+y)^2\right)}$$

$$\Rightarrow\left[\frac{\sqrt{1-e^{2y}}-e^y\left(1+(x+y)^2\right)}{\left(1+(x+y)^2\right)\sqrt{1-e^{2y}}}\right]y`=-\frac{1}{\left(1+(x+y)^2\right)}\Rightarrow\left[\frac{\sqrt{1-e^{2y}}-e^y\left(1+(x+y)^2\right)}{\sqrt{1-e^{2y}}}\right]y`=-1$$

$$\therefore y`=-\frac{\sqrt{1-e^{2y}}}{\sqrt{1-e^{2y}}-e^y\left(1+(x+y)^2\right)}$$

(iv) Solution: Given that $y\arcsin x - x\arctan y = 1$. Differentiate w.r.t x, we get

$$\frac{d}{dx}\left(y\sin^{-1}x\right)-\frac{d}{dx}\left(x\tan^{-1}y\right)=\frac{d}{dx}(1) \quad \Rightarrow y`\sin^{-1}x+y\frac{1}{\sqrt{1-x^2}}-x\frac{1}{1+y^2}y`-\tan^{-1}y.1=0$$

$$\left[\sin^{-1}x-\frac{x}{1+y^2}\right]y`=\tan^{-1}y-\frac{y}{\sqrt{1-x^2}} \quad \Rightarrow\left[\frac{\left(1+y^2\right)\sin^{-1}x-x}{\left(1+y^2\right)}\right]y`=\frac{\sqrt{1-x^2}\tan^{-1}y-y}{\sqrt{1-x^2}}$$

$$\Rightarrow y`=\left[\frac{\left(1+y^2\right)}{\left(1+y^2\right)\sin^{-1}x-x}\right]\left[\frac{\sqrt{1-x^2}\tan^{-1}y-y}{\sqrt{1-x^2}}\right]$$

9. Find dy/dx in each of the following:

(i) $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$ (ii) $y=(x)^{\frac{1}{x}}$ (iii) $y=x^{\ln x}$ (iv) $y=x^{\sin y}$.

(i) Solution: Given that $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$. Let $u = (\tan x)^{\cot x}$ and $v = (\cot x)^{\tan x}$.

Thus, $y = u + v$ and therefore, $y` = u` + v`$ (1)

Now consider, $u = (\tan x)^{\cot x}$. Taking ln on both sides, we obtain

$$\ln u=\cot x.\ln(\tan x).\ \text{Differentiating}:\ \frac{1}{u}u`=\cot x.\frac{1}{\tan x}.\sec^2 x+\ln(\tan x)(-\operatorname{cosec}^2 x)$$

$$\Rightarrow u`=u\left[\cot^2 x\sec^2 x-\ln(\tan x).\operatorname{cosec}^2 x\right]=(\tan x)^{\cot x}\left[\cot^2 x\sec^2 x-\ln(\tan x).\operatorname{cosec}^2 x\right]$$

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Now, $v = (\cot x)^{\tan x}$. Taking ln on both sides, we get

$\ln v = \tan x.\ln(\cot x)$. Differentiating w.r.t x, we get

$$\frac{1}{v}v` = \tan x.\frac{1}{\cot x}.\left(-\operatorname{cosec}^2 x\right) + \ln(\cot x)(\sec^2 x)$$

$$\Rightarrow v` = v\left[-\tan^2 x \operatorname{cosec}^2 x + \ln(\cot x).\sec^2 x\right] = (\cot x)^{\tan x}\left[-\tan^2 x \operatorname{cosec}^2 x + \ln(\cot x).\sec^2 x\right]$$

Thus, $y` = (\tan x)^{\cot x}\left[\cot^2 x \sec^2 x - \ln(\tan x).\operatorname{cosec}^2 x\right]$

$$+(\cot x)^{\tan x}\left[-\tan^2 x \operatorname{cosec}^2 x - \ln(\cot x).\sec^2 x\right]$$

(ii) Solution: Given that $y = x^{1/x}$. Taking ln on both sides, we get

$\ln y = \ln(x)^{1/x} = \frac{1}{x}.\ln x = \frac{\ln x}{x}$. Differentiate both sides w.r.t x, we get

$$\frac{1}{y}y` = \frac{x.\frac{1}{x} - \ln x.1}{x^2} = \frac{1-\ln x}{x^2} \Rightarrow y` = y\frac{1-\ln x}{x^2} = (x)^{1/x}\left[\frac{1-\ln x}{x^2}\right]$$

(iii) Solution: Given that $y = x^{\ln x}$. Taking ln on both sides, we have

$\ln y = \ln(x)^{\ln x} = \ln x \ln x = (\ln x)^2$. Differentiating both sides, w.r.t x

$$\frac{1}{y}y` = 2(\ln x).\frac{1}{x} = \frac{2\ln x}{x} \Rightarrow y` = y\frac{2\ln x}{x} = (x)^{\ln x}\left[\frac{2\ln x}{x}\right]$$

10. If $y = e^{ax}\sin bx$, prove that $y`` - 2a\,y` + (a^2 + b^2)\,y = 0$

Solution: Given that $y = e^{ax}\sin bx$

or $y` = a\,e^{ax}\sin bx + b\,e^{ax}\cos bx = e^{ax}(a\sin bx + b\cos bx)$

And $y`` = ae^{ax}(a\sin bx + b\cos bx) + e^{ax}(ab\cos bx - b^2\sin bx)$

$= e^{ax}[(a^2 - b^2)\sin bx + 2ab\cos bx]$. Thus,

$y`` - 2a\,y` + (a^2 + b^2)y$

$= e^{ax}[(a^2 - b^2)\sin bx + 2ab\cos bx - 2a^2\sin bx - 2ab\cos bx + (a^2 + b^2)\sin bx]$

$= e^{ax}[a^2\sin bx - b^2\sin bx + 2ab\cos bx - 2a^2\sin bx - 2ab\cos bx + a^2\sin bx + b^2\sin bx] = 0$

11. The driver of an experimental racing car begins a test run. During the first 6 seconds, the distance s (in feet) of the car from the starting point is

$$s = 14t^2 - t^3/3 \qquad 0 \le t \le 6$$

where t is the number of seconds the car has been moving. What is the velocity of the car after 4 seconds have passed?

Solution: Given $s = 14t^2 - t^3/3$. Differentiating w.r.t t, we get

$v = ds/dt = 28t - t^2$. Putting t = 4, we get: v(4) = 112 – 16 = 96 feet/s.

12. At time t, the position of a body moving along the s – axis is $s = t^3 - 6t^2 + 9t$ m.

(a) Find the body's acceleration each time the velocity is zero. (b) Find the body's speed each time the acceleration is zero. (c) Find the total distance traveled by the body from t = 0 to t = 2.

Solution: Given $s = t^3 - 6t^2 + 9t$. Differentiating w.r.t t, we get

$v = 3t^2 - 12t + 9$. Differentiating again w.r.t t, we get $a = 6t - 12$. .

(a) Now if v = 0 then $3t^2 - 12t + 9 = 0$ ➔ t = 1 and t = 3.

When t = 1 then a = -6 and when t = 3, a = 6. Thus when velocity is 1 ft/sec then retardation is 6 ft/sec^2. Similarly, when velocity is 3 ft/sec then acceleration is 6 ft/sec^2.

(b) Now when a = 0 then 6t – 12 = 0 ➔ t = 2. In this case v = 12 – 24 + 9 = -3 ft/sec.

(c) Putting t = 2 in $s = t^3 - 6t^2 + 9t = 8 - 24 + 18 = 2$ ft.

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13. A bicycle manufacturer estimates that it can price its bicycles at $p = 140 - 0.02x^2$ dollars each, where x is the number sold. The cost of producing x bicycles is $900 – 0.01x^2$ dollars. Determine the marginal profit when 20 bicycles are sold.

Solution: One bicycle is worth \$ $140 – 0.02x^2$ hence revenue from the sale of x bicycle is $R(x) = (140 – 0.02x^2)x = 140x – 0.02x^3$. Thus profit P(x) is

$P(x) = R(x) – C(x) = (140x – 0.02x^3) – (900 – 0.01x^2) = -0.02x^3 + 0.01x^2 + 140x – 900$

Thus marginal profit is $MP = P`(x) = -0.06x^2 + 0.02x + 140$. Putting x = 20, we get

$MP(20) = -0.06(20)^2 + 0.02(20) + 140 = -24 + 0.4 + 140 = 115.6$ \$/bicycle.

14. The pollution in a lake is being reduced over a 5–year period. The amount of pollutants (in pounds) is given by: $A = 110 - \frac{90t}{t+1}$, $0 \le t \le 5$, where t is the time in years.

(a) Determine the amount of pollution in the lake at the beginning, after 2 years, and after 5 years.

(b) Determine the rate at which the pollution in the lake is changing at t = 2 years.

(c) What is the meaning of the given sign obtained in part (b)?

Solution: (a) Putting t = 0, A = 110. Putting t = 2, A = 50. Putting t = 5, A = 35.

(b) Differentiate A w. r. t, we get

$\frac{dA}{dt} = 0 - \left[\frac{(t+1).90 - 90t(1+0)}{(t+1)^2}\right] = \frac{-90}{(t+1)^2}$. Putting t = 2, we get: $\frac{dA}{dt} = \frac{-90}{9} = -10$.

(c) Since dA/dt is negative which means rate of change of pollution is decaying by 10 per year.

15. On Earth, in the absence of air, a rock at a velocity of 24m/sec (about 86km/h) would reach a height of $s = 24t – 4.9t^2$ meters in t seconds.

(a) Find the rock's velocity and acceleration at time t.

(b) How long would it take the rock to reach its highest point?

(c) How high would the rock go?

(d) How long would it take the rock to reach half its maximum height?

(e) How long would the rock be aloft (in the air)?

Solution: (a) Given $s = 24t – 4.9t^2$, differentiating w.r.t t twice, we get

v = 24 - 9.8t and a = -9.8.

(b) When rocket reaches at the highest point, v = 0. Thus, 24 – 9.8t = 0 or t = 2.45 sec.

(c) Putting t = 2.45 we get $s = 24(2.45) – 4.9(2.45)^2 = 58.8 – 29.4 = 29.4$ feet.

(d) Half of maximum height s_{max} is 29.4/2 = 14.7 feet. Thus, $14.7 = 24t – 4.9t^2$

Or $4.9t^2 – 24t + 14.7 = 0$ or t = 0.7 and t = 4.18. The two values of t indicate that the rocket attains half of maximum height twice. Firstly, on the way up which takes t = 0.7 sec and secondly, the time when it returns back to the earth that takes t = 4.18 sec.

(e) The total time that rocket remains in the air is 2(2.45) = 9.9 sec (use part b).

16. A 45–caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t – 2.6t^2$ feet after t seconds. On Earth, in the absence of air, its height would be $s = 832t – 16t^2$ feet after t seconds. How long will the bullet be aloft in each case? How high would the bullet go?

Solution: (a) **On The Moon** $s = 832t - 2.6t^2$ or v = ds/dt = 832 – 5.2t. When it attains maximum height its velocity is zero. Thus 832 – 5.2t = 0 or t = 832/5.2 = 160. Thus the bullet is aloft for 2t = 320 seconds. In this case the maximum height bullet attains is:

$s = 832(160) – 2.6(160)^2 = 133120 – 66560 = 66560$ feet.

On The Earth: $s = 832t - 16t^2$ or v = ds/dt = 832 – 32t. When it attains maximum height its velocity is zero. Thus 832 – 32t = 0 or t = 832/32 = 26. Thus the bullet is aloft for

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2t = 52 seconds. In this case the maximum height bullet attains is:
s = 832(26) – 2.6(26)^2 = 21632 – 1757.6 = 19874.4 feet.

17. Suppose an oil spill has taken the form of a circular region and its area is increasing at the rate of 100 square meters per hour. At what rate is the radius of the region increasing when the radius is 200 meters?

Solution: The area of circle is

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt} = 2\pi r.\frac{dr}{dt} = 2\times 3.14\times 200\times 100 = 12560\ m^2/s \text{ .(r = 200, dr/dt = 100)}$$

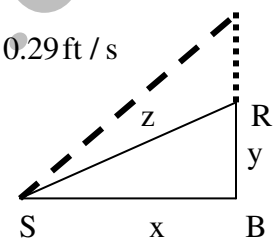
Thus, circular area is increasing at the rate of 12560 m^2/s.

18. Consider a spherical balloon with volume $V = 4\pi r^3/3$ that is being inflated by helium at the rate of 4 cubic feet per minute. At what rate is the radius increasing when the radius is 2 feet?

Solution: The volume of spherical balloon and its rate of change w.r.t time t is:

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{dV}{dr}.\frac{dr}{dt} = \pi r^2.\frac{dr}{dt}$$

$$\text{Now } \frac{dV}{dt} = 4, \text{and } r = 2. \text{Thus, } 4 = 3.14\times 2^2\times\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{4}{4(3.14)} = 0.29\,ft/s$$



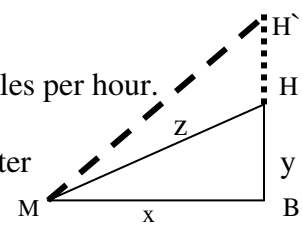
19. A rocket is launched straight up. There is an observation station 7 miles from the launch site. At what rate is the distance between the rocket and the station increasing when the rocket is 24 miles high and traveling at 200 miles per hour?

Solution: From the right triangle z^2 = x^2 + y^2. Here x = 7 (fixed). Thus z^2 = 49 + y^2. Initially, rocket is at the height of y = 24. Thus, z^2 = 49 + 24^2 = 625 or z = 25.

Now, z^2 = 49 + y^2. Differentiating both sides w.r.t.t, we get

$$2z\frac{dz}{dt} = 0 + 2y\frac{dy}{dz} \Rightarrow \frac{dz}{dz} = \frac{y}{z}.\frac{dy}{dt} = \frac{24}{25}(200) = 192\,miles/h$$

Thus rocket is moving away from the station at the rate of 192 miles per hour.



20. A helicopter rises vertically at the rate of 10 feet per second. There is a maker (M in the figure) 80 feet from where the helicopter lifts off. At what rate is the distance between the helicopter and the marker changing when the helicopter is 192 feet high?

Solution: From the right triangle z^2 = x^2 + y^2. Here x = 80 (fixed). Thus,z^2 = 6400 + y^2. Initially, helicopter's height is y = 192 ft.
Thus, z^2 = 6400 + 192^2 or z = 208.
Now, z^2 = 6400 + y^2. Differentiating both sides w.r.t.t, we get

$$2z\frac{dz}{dt} = 0 + 2y\frac{dy}{dz} \Rightarrow \frac{dz}{dz} = \frac{y}{z}.\frac{dy}{dt} = \frac{192}{208}(10) = 9.23\,ft/h$$

Thus, helicopter is moving away from the marker at the rate of 9.23 feet per hour.

21. A ball is thrown straight up from the ground and travels such that its distance from the ground at any time t is $s = -16t^2 + 80t$ feet. Find its acceleration at any time.

Solution: Since, s = -16t^2 + 80t or v = ds/dt = -32t + 80 and a = dv/dt= -32 ft/sec^2.

22. Find the nth derivative of following:

(a) $y = \dfrac{x}{2x^2+3x+1}$ (b) $y = \dfrac{x}{x^2+a^2}$ (c) y = cos^{2}x (d) y = sin 2x cos 3x

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(e) $y = e^x \cos^2 x$ (f) $y = \ln x^2$ (g) $y = e^x \sin 4x \cos 6x$
(h) $y = x^3 \ln x$ (i) $y = x^{n-1} \ln x$ (j) $y = e^x \ln x$
(k) $y = e^{4x} \sin(2x + 3)$ (l) $y = \ln (2x + 3)$

Solution:

$$(a)\ y = \frac{x}{2x^2+3x+1} = \frac{x}{(x+1)(2x+1)} = \frac{A}{x+1} + \frac{B}{2x+1} = \frac{1}{x+1} + \frac{-(1/2)}{2x+1} = (x+1)^{-1} - \frac{1}{2}(2x+1)^{-1}$$

Now differentiating n times w.r.t x, we get

$$y_1 = (-1)(x+1)^{-2} - \frac{1}{2}(-1)(2x+1)^{-2}.2$$

$$y_2 = (-1)(-2)(x+1)^{-3} - \frac{1}{2}(-1)(-2)(2x+1)^{-3}.2^2 = (-1)^2 2!\left[\frac{1}{(x+1)^3} - \frac{1}{2}.\frac{2^2}{(2x+1)^3}\right]$$

Continuing n times, we obtain

$$y_n = (-1)^n n!\left[\frac{1}{(x+1)^{n+1}} - \frac{1}{2}.\frac{2^n}{(2x+1)^{n+1}}\right] = (-1)^n n!\left[\frac{1}{(x+1)^{n+1}} - \frac{2^{n-1}}{(2x+1)^{n+1}}\right]$$

$$(b)\ y = \frac{x}{x^2+a^2} = \frac{x}{(x-ia)(x+ia)} = \frac{A}{x-ia} + \frac{B}{x+ia} = \frac{1/2}{x-ia} + \frac{1/2}{x+ia} = \frac{1}{2}\left[(x-ia)^{-1} + (x+ia)^{-1}\right]$$

Differentiating w.r.t x n times, we get

$$y_1 = \frac{1}{2}\left[(-1)(x-ia)^{-2} + (-1)(x+ia)^{-2}\right]$$

$$y_2 = \frac{1}{2}\left[(-1)(-2)(x-ia)^{-3} + (-1)(-2)(x+ia)^{-3}\right] = (-1)^2 2!\left[\frac{1}{(x-ia)^3} + \frac{1}{(x+ia)^3}\right]$$

Continuing this way n times, we get

$$y_n = (-1)^n n!\left[\frac{1}{(x-ia)^{n+1}} + \frac{1}{(x+ia)^{n+1}}\right]$$

23. If $y = \sin(a \sin^{-1}x)$ prove that $(1 - x^2)y_{n+2} = (2n + 1)x\, y_{n+1} + (n^2 - a^2)\, y_n$.

Solution: Given that, $y = \sin(a \sin^{-1}x)$.

Differentiate w.r.t x, we get

$$y_1 = \cos\left(a\sin^{-1}x\right).\frac{a}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\,y_1 = a\cos\left(a\sin^{-1}x\right).$$ Differentiate again,

$$\sqrt{1-x^2}\,y_2 + \frac{1}{2\sqrt{1-x^2}}(-2x).y_1 = -a\sin\left(a\sin^{-1}x\right).\frac{a}{\sqrt{1-x^2}} = \frac{-a^2 y}{\sqrt{1-x^2}}.$$

Multiplying both sides by $\sqrt{1-x^2}$, we obtain,

$$\left(1-x^2\right)y_2 - xy_1 = -a^2 y \qquad \Rightarrow \left(1-x^2\right)y_2 - xy_1 + a^2 y = 0$$

Now differentiating the lastequation n times using Leibnitz theorem, we obtain

$$\left(1-x^2\right)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n - \left[xy_{n+1} + n.1y_n\right] + a^2 y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n^2 y_n + ny_n - xy_{n+1} - ny_n + a^2 y_n = 0$$

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OR, $\left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2-a^2\right)y_n=0\cdot$

24. If $y=e^{m\arcsin x}$, show that $\left(1-x^2\right)y^{(n+2)}-\left(2n+1\right)xy^{(n+1)}-\left(n^2+m^2\right)y^{(n)}=0\cdot$

Solution: Given that, $y = e^{m \sin\text{-}1x}$

$$y_1=e^{\left(m\sin^{-1}x\right)}.\frac{m}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}\,y_1=me^{\left(m\sin^{-1}x\right)}=my \qquad (1)$$

Differentiate again, $\sqrt{1-x^2}\,y_2+\frac{1}{2\sqrt{1-x^2}}(-2x).y_1=my_1$

Multiplying both sides by $\sqrt{1-x^2}$, we obtain,

$$\left(1-x^2\right)y_2-xy_1=m\sqrt{1-x^2}\,y_1 \qquad \Rightarrow\left(1-x^2\right)y_2-xy_1=m.my \text{ [From (1)]}$$

Now differentiating the last equation n times using Leibnitz theorem, we obtain

$$\left(1-x^2\right)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2!}(-2)y_n-\left[xy_{n+1}+n.1y_n\right]-m^2y_n=0$$

$$\left(1-x^2\right)y_{n+2}-2nxy_{n+1}-n^2y_n+ny_n-xy_{n+1}-ny_n-m^2y_n=0$$

OR, $\left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-\left(n^2+m^2\right)y_n=0\cdot$

25. Show that: $\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right)=\frac{(-1)^n n!}{x^{n+1}}\left[\ln x-1-\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{n}\right]\cdot$

Solution: Let y = ln x/x =u.v where u =1/x and v = ln x. Thus

$$u_1=(-1)x^{-2},\ u_2=(-1)(-2).x^{-3}=\frac{(-1)^2 2!}{x^3} \qquad \Rightarrow\ u_n=\frac{(-1)^n n!}{x^{n+1}}$$

$$v_1=\frac{1}{x}=x^{-1}, v_2=(-1)x^{-2},\ v_3=(-1)(-2)x^{-3}=\frac{(-1)^2 2!}{x^3}\ \Rightarrow\ v_n=\frac{(-1)^{n-1}(n-1)!}{x^n}$$

Now by Leibniz theorem, if y = uv then

$$y_n=vu_n+nv_1u_{n-1}+\frac{n(n-1)}{2!}v_2u_{n-2}+\ldots+v_nu.$$

Putting the above values in this equation and simplifying, we get

$$y_n=\frac{d^ny}{dx^n}=\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right)=\frac{(-1)^n n!}{x^{n+1}}\left[\ln x-1-\frac{1}{2}-\frac{1}{3}-\cdots-\frac{1}{n}\right]\cdot$$

26. Use Leibniz's theorem to find the nth derivative of $y = x^4 e^{2x}$.

Solution: Given $y = x^4 e^{2x}$. Let $v = x^4$ and $u = e^{2x}$. Then,

$u_1 = 2e^{2x}$, $u_2 = 4e^{2x}$, $u_3 = 8e^{2x}$, $u_4 = 16e^{2x}$ and $v_1 = 4x^3$, $v_2 = 12x^2$, $v_3 = 24x$, $v_4 = 24$.

Now by Leibniz theorem,

$y_4 = vu_4 + 4v_1 u_3 + 6v_2 u_2 + 4v_3 u_1 + v_4 u = e^{2x}(16x^4 + 128x^3 + 228x^2 + 192x + 24)$

27. If $y = \sin^{-1}x$ show that $\left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}-n^2y_n=0$. Hence prove that $\left(1-x^2\right)y_5-7xy_4-9y_3=0$.

Given $y=\sin^{-1}x$. Differentiating, we get: $y_1=\frac{1}{\sqrt{1-x^2}}\Rightarrow\sqrt{1-x^2}\,y_1=1$

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Differentiating again, $\sqrt{1-x^2}y_2+\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)y_1=0$. Multiplying both sides by $\sqrt{1-x^2}$

$\left(1-x^2\right)y_2-xy_1=0$. Now differentiating this n times using Leibniz theorem, we get

28. If y = cos(m ln x), show that $x^2y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(m^2+n^2\right)y_n=0$

Solution: (Note: Please correct the problem)

Given that $y=\cos\left(m\ln x\right)\Rightarrow y_1=-\sin(m\ln x).\frac{m}{x}\Rightarrow xy_1=-m\sin(m\ln x)$

Differentiating again, we get

$xy_2+y_1.1=-m\cos(m\ln x).\frac{m}{x}$. Multiplying by x, and putting $\cos(m\ln x)=y$, we get

$x^2y_2+xy_1=-m^2y \quad \Rightarrow x^2y_2+xy_1+m^2y=0$

Differentiating this last equation n times using leibniz theorem, we get

$x^2y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{2!}(2)y_n+xy_{n+1}+n.1.y_n+m^2y_n=0$. Simplifying, we get

$x^2y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(m^2+n^2\right)y_n=0$

29. If $y=\sin^{-1}x/\sqrt{1-x^2}$ then show that $\left(1-x^2\right)y_{n+2}-\left(2n+3\right)xy_{n+1}-\left(n+1\right)^2y_n=0$

Solution: (Notice the correction in the problem) Given that

$y=\left(\frac{\sin^{-1}x}{\sqrt{1-x^2}}\right) \quad \Rightarrow \sqrt{1-x^2}y=\sin^{-1}x$. Differentiate w.r.t x, we obtain

$\sqrt{1-x^2}\,y_1+\frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)y=\frac{1}{\sqrt{1-x^2}}$. Multiplying by $\sqrt{1-x^2}$, we get

$\left(1-x^2\right)y_1-xy=1$. Differentiate w.r.t x again, we obtain

$\left(1-x^2\right)y_2-2xy_1-\left(xy_1+y.1\right)=0 \;\Rightarrow \left(1-x^2\right)y_2-3xy_1-y=0$

Differentiating n times using Leibniz theorem, we get

OR $\left(1-x^2\right)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{2!}(-2)y_n-3\left(xy_{n+1}+n.1y_n\right)-y_n=0$

OR $\left(1-x^2\right)y_{n+2}-2nxy_{n+1}-\left(n^2-n\right)y_n-3xy_{n+1}-3ny_n-y_n=0$

OR $\left(1-x^2\right)y_{n+2}-\left(2n+3\right)xy_{n+1}-\left(n^2+2n+1\right)y_n=0$

Thus, $\left(1-x^2\right)y_{n+2}-\left(2n+3\right)xy_{n+1}-\left(n+1\right)^2y_n=0$

30. If y = sinh(m $\sinh^{-1}$x), prove that $\left(1+x^2\right)y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(n^2-m^2\right)y_n=0\cdot$

Solution: (Notice the correction in the problem)

Given that, y = sinh(m $\sinh^{-1}$x). Differentiating w.r.t x

$y_1=\cosh\left(m\sinh^{-1}x\right).\frac{m}{\sqrt{1+x^2}} \;\Rightarrow \sqrt{1+x^2}y_1=m\cosh\left(m\sinh^{-1}x\right)$. Differentiate again,

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$$\sqrt{1+x^2}\,y_2+\frac{1}{2\sqrt{1+x^2}}(2x).y_1=m\sinh\left(m\sinh^{-1}x\right).\frac{m}{\sqrt{1+x^2}}=\frac{m^2y}{\sqrt{1+x^2}}.$$

Multiplying both sides by $\sqrt{1+x^2}$, we obtain,

$$\left(1+x^2\right)y_2+xy_1=m^2y \qquad \Rightarrow\left(1+x^2\right)y_2+xy_1-m^2y=0$$

Now differentiating the lastequation n times using Leibnitz theorem, we obtain

$$\left(1+x^2\right)y_{n+2}+n(2x)y_{n+1}+\frac{n(n-1)}{2!}(2)y_n+\left[xy_{n+1}+n.1y_n\right]-m^2y_n=0$$

OR, $\left(1+x^2\right)y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(n^2-m^2\right)y_n=0\cdot$

31. If $y^{1/m}+y^{-1/m}=2x$, prove that $\left(x^2-1\right)y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(n^2-m^2\right)y_n=0$

Solution: Given that, $y^{1/m}+y^{-1/m}=2x \Rightarrow y^{1/m}+\frac{1}{y^{1/m}}=2x \Rightarrow y^{2/m}+1=2xy^{1/m}$

$$\Rightarrow y^{2/m}-2xy^{1/m}+x^2=x^2-1 \Rightarrow\left(y^{1/m}-x\right)^2=x^2-1 \Rightarrow\left(y^{1/m}-x\right)=\sqrt{x^2-1}$$

$y^{1/m}=x+\sqrt{x^2-1} \quad \Rightarrow y=\left(x+\sqrt{x^2-1}\right)^m$. Differentiating, we get

$$y_1=m\left(x+\sqrt{1+x^2}\right)^{m-1}\left[1+\frac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}}(2x)\right]=m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(1+\frac{x}{\sqrt{1+x^2}}\right)$$

$$=m\left(x+\sqrt{1+x^2}\right)^{m-1}\left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)\Rightarrow\sqrt{1+x^2}\,y_1=m\left(x+\sqrt{1+x^2}\right)^m$$

$\sqrt{1+x^2}\,y_1=my\Rightarrow\left(1+x^2\right)\left(y_1\right)^2=m^2y^2$. Differentiating again, we get

$\left(1+x^2\right)\left(2y_1\right)\left(y_2\right)+\left(y_1\right)^2(2x)=m^2(2y)\left(y_1\right)$. Dividing by $2y_1$, we get

$$\left(1+x^2\right)y_2+xy_1=m^2y \qquad (1)$$

Differentiating (1) n times using Leibniz's theorem, we get

$$\left[y_{n+2}\left(1+x^2\right)+ny_{n+1}(2x)+\frac{n(n-1)}{2!}y_n(2)\right]+\left[y_{n+1}(x)+ny_n(1)\right]=m^2y_n$$

$$\left(1+x^2\right)y_{n+2}+2nxy_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n-m^2y_n=0$$

$$\left(1+x^2\right)y_{n+2}+\left(2n+1\right)xy_{n+1}+\left(n^2-m^2\right)y_n=0\cdot$$

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Chapter 5

PARTIAL DERIVATIVES

WORKSHEET 05

1. Evaluate the following limits:

(i) $\lim\limits_{\substack{x\to 0\\ y\to 2}} \dfrac{3xy}{x^2+y^2}$ (ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x-y}{x+5y}$ (iii) $\lim\limits_{\substack{x\to 1\\ y\to \infty}} \dfrac{3xy}{x^2+y^2}$ (iv) $\lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{xy-2x}{xy-2y}$

Solution: (i) $\lim\limits_{\substack{x\to 0\\ y\to 2}} \dfrac{3xy}{x^2+y^2} = \dfrac{3.0.2}{0^2+2^2} = \dfrac{0}{4} = 0$

(ii) $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{x-y}{x+5y} = \dfrac{0-0}{0+0} = \dfrac{0}{0}$. Since, 0/0 is an indeterminate quantity hence, limit of given function does not exist.

(iii) $\lim\limits_{\substack{x\to 1\\ y\to \infty}} \dfrac{3xy}{x^2+y^2} = \lim\limits_{y\to\infty} \dfrac{3.1.y}{1^2+y^2} = \lim\limits_{y\to\infty} \dfrac{3y}{1+y^2} = \lim\limits_{y\to\infty} \dfrac{3y}{y\left[\dfrac{1}{y}+y\right]} = \lim\limits_{y\to\infty} \dfrac{3}{\left[\dfrac{1}{y}+y\right]} = \dfrac{3}{0+\infty} = 0$

(iv) $\lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{xy-2x}{xy-2y} = \lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{x(y-2)}{y(x-2)} = \lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{x(1-2)}{y(1-2)} = \lim\limits_{\substack{x\to 1\\ y\to 1}} \dfrac{-x}{-y} = \dfrac{-1}{-1} = 1$

2. If $f(x,y) = \dfrac{2x-y}{2x+y}$ show that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$

Solution: Given that $f(x,y) = \dfrac{2x-y}{2x+y}$. Now consider

$$\lim_{x\to 0}\left[\lim_{y\to 0} f(x,y)\right] = \lim_{x\to 0}\left[\lim_{y\to 0} \frac{2x-y}{2x+y}\right] = \lim_{y\to 0} \frac{2x}{2x} = \lim_{y\to 0}(1) = -1$$

And $\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} \dfrac{2x-y}{2x+y}\right] = \lim\limits_{x\to 0} \dfrac{-y}{y} = \lim\limits_{x\to 0}(-1) = -1$

Thus, we see that $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0} f(x,y)\right] \neq \lim\limits_{y\to 0}\left[\lim\limits_{x\to 0} f(x,y)\right]$

3. Show that the function f(x, y) defined as

$$f(x,y) = \begin{cases} 2x^2+y, & (x,y) \neq (1,2) \\ 0 \quad , & (x,y) = (1,2) \end{cases}$$

is not continuous at (x, y) = (1, 2).

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Solution: Consider,

$\lim\limits_{\substack{x\to 1\\ y\to 2}}\left(2x^2+y\right)=2+2=4.$ But $f(1,2)=0.$

Since, $\lim\limits_{\substack{x\to 1\\ y\to 2}} f(x,y)\neq f(1,2)$ hence given function is not contious at $x=1$ and $y=2.$

4. Show that the function (a) $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$ and (b) $f(x,y)=\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

satisfies the Laplace equation $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$

Solution: (a) Given that $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$. Thus

$$\frac{\partial f}{\partial x}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{dx}\left(\frac{y}{x}\right)=\frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right)=\frac{-y}{x^2+y^2}=-y\left(x^2+y^2\right)^{-1/2}$$

Differentiating again w.r.t x, we get: $\frac{\partial^2 f}{\partial x^2}=-y.\frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2x=\frac{-xy}{x^2+y^2}$ (1)

$$\frac{\partial f}{\partial y}=\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{\partial}{dy}\left(\frac{y}{x}\right)=\frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=x\left(x^2+y^2\right)^{-1/2}$$

Differentiating again w.r.t y, we get: $\frac{\partial^2 f}{\partial y^2}=x.\frac{1}{2}\left(x^2+y^2\right)^{-1/2}.2y=\frac{xy}{x^2+y^2}$ (2)

Adding (1) and (2), we get: $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0.$ Proved

(b) Given that $f(x,y)=\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$

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$$\frac{\partial f}{\partial x}=\frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\frac{\partial}{dx}\left(\frac{2xy}{x^2-y^2}\right)=\frac{\left(x^2-y^2\right)^2}{\left(x^2-y^2\right)^2+4x^2y^2}\left[\frac{\left(x^2-y^2\right).2y-2xy(2x)}{\left(x^2-y^2\right)^2}\right]$$

$$=\frac{2x^2y-2y^3-4x^2y}{x^4-2x^2y^2+y^4+4x^2y^2}=-2y\frac{\left(x^2+y^2\right)}{x^4+2x^2y^2+y^4}=-2y\frac{\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}=\frac{-2y}{\left(x^2+y^2\right)}=-2y\left(x^2+y^2\right)^{-1}$$

Differentiating again w.r.t x, we get : $\frac{\partial^2 f}{\partial x^2}=-2y(-1)\left(x^2+y^2\right)^{-2}.2x=\frac{4xy}{\left(x^2+y^2\right)^2}$ (1)

$$\frac{\partial f}{\partial y}=\frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\frac{\partial}{dy}\left(\frac{2xy}{x^2-y^2}\right)=\frac{\left(x^2-y^2\right)^2}{\left(x^2-y^2\right)^2+4x^2y^2}\left[\frac{\left(x^2-y^2\right).2x-2xy(-2y)}{\left(x^2-y^2\right)^2}\right]$$

$$=\frac{2x^3-2xy^2+4xy^2}{x^4-2x^2y^2+y^4+4x^2y^2}=2x\frac{\left(x^2+y^2\right)}{x^4+2x^2y^2+y^4}=2x\frac{\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}=\frac{2x}{\left(x^2+y^2\right)}=2x\left(x^2+y^2\right)^{-1}$$

Differentiating again w.r.t x, we get : $\frac{\partial^2 f}{\partial y^2}=2x(-1)\left(x^2+y^2\right)^{-2}.2y=\frac{-4xy}{\left(x^2+y^2\right)^2}$ (2)

Adding (1) and (2), we get : $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$ Proved

5. Show that $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$ where:

(a) $f(x, y) = \ln (e^x + e^y)$ (b) $f(x, y) = \ln (x^2 + y^2) - \ln(xy)$

(c) $f(x, y) = x \sin xy + y \cos xy$ (d) $f(x, y) = \ln(e^x + e^y)$

Solution: (a) Given that, $f(x, y) = \ln(e^x + e^y)$. Then

$$\frac{\partial f}{\partial x}=\frac{1}{e^x+e^y}.e^x=e^x\left(e^x+e^y\right)^{-1} \text{ and } \frac{\partial^2 f}{\partial y\partial x}=(-1)e^x\left(e^x+e^y\right)^{-2}(e^y)=-\frac{e^{x+y}}{\left(e^x+e^y\right)^2} \quad (1)$$

$$\frac{\partial f}{\partial y}=\frac{1}{e^x+e^y}.e^y=e^y\left(e^x+e^y\right)^{-1} \text{ and } \frac{\partial^2 f}{\partial x\partial y}=(-1)e^y\left(e^x+e^y\right)^{-2}(e^x)=-\frac{e^{x+y}}{\left(e^x+e^y\right)^2} \quad (2)$$

From (1) and (2) we observe that : $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}$

(b) Given that $f(x, y) = \ln (x^2 + y^2) - \ln(xy) = \ln (x^2 + y^2) - \ln x - \ln y$. Then

$$\frac{\partial f}{\partial x}=\frac{1}{x^2+y^2}.2x-\frac{1}{x}-0=2x\left(x^2+y^2\right)^{-1}-x^{-1} \text{ and } \frac{\partial^2 f}{\partial y\partial x}=-2x\left(x^2+y^2\right)^{-2}2y-0=\frac{-4xy}{\left(x^2+y^2\right)^2} \quad (1)$$

$$\frac{\partial f}{\partial y}=\frac{1}{x^2+y^2}.2y-0-\frac{1}{y}=2y\left(x^2+y^2\right)^{-1}-y^{-1} \text{ and } \frac{\partial^2 f}{\partial x\partial y}=-2y\left(x^2+y^2\right)^{-2}2x-0=\frac{-4xy}{\left(x^2+y^2\right)^2} \quad (2)$$

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From (1) and (2) we observe that : $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

(c) Given that, f(x, y) = x sin xy + y cos xy. Then

$$\frac{\partial f}{\partial x} = x(\cos xy).y + (\sin xy).1 + y(-\sin xy).y = xy(\cos xy) + (\sin xy) - y^2(\sin xy)$$

and $$\frac{\partial^2 f}{\partial y \partial x} = x\left[y(-\sin xy).x + (\cos xy).1\right] + (\cos xy)x - y^2(\cos xy).x$$

$$= -x^2 y(\sin xy) + x(\cos xy) - x(\cos xy) - xy^2(\cos xy)$$

$$= -x^2 y(\sin xy) - xy^2(\cos xy) \qquad (1)$$

$$\frac{\partial f}{\partial y} = x(\cos xy).x + (\cos xy).1 + y(-\sin xy).x = x^2(\cos xy) + (\cos xy) - xy(\sin xy)$$

and $$\frac{\partial^2 f}{\partial x \partial y} = \left[x^2(-\sin xy).y + (\cos xy).2x\right] - (\sin xy).y - y\left[(\sin xy).1 + x(\cos xy).y\right]$$

$$= -x^2 y(\sin xy) + 2x(\cos xy) - y(\sin xy) - y(\sin xy) - xy^2(\cos xy)$$

$$= -x^2 y(\sin xy) - 2xy^2(\cos xy) - 2\left[y(\sin xy) + x(\cos xy)\right]$$

$$= -x^2 y(\sin xy) - 2xy^2(\cos xy) - 2[0] \qquad (\text{By problem statement})$$

$$= -x^2 y(\sin xy) - 2xy^2(\cos xy) \qquad (2)$$

From (1) and (2) we observe that : $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

6. If $Z = \ln(e^x + e^y)$, show that $Z_{xx} \,.\, Z_{yy} = (Z_{xy})^2$

Solution: Given that $Z = \ln(e^x + e^y)$. Then

$$Z_x = \frac{1}{e^x + e^y}.e^x = \frac{e^x}{(e^x + e^y)}, Z_{xx} = \left[\frac{(e^x + e^y)e^x - e^x e^x}{(e^x + e^y)^2}\right] = \frac{e^{2x} + e^{x+y} - e^{2x}}{(e^x + e^y)^2} = \frac{e^{x+y}}{(e^x + e^y)^2}$$

$$Z_y = \frac{1}{e^x + e^y}.e^y = \frac{e^y}{(e^x + e^y)}, Z_{yy} = \left[\frac{(e^x + e^y)e^y - e^y e^y}{(e^x + e^y)^2}\right] = \frac{e^{2y} + e^{x+y} - e^{2y}}{(e^x + e^y)^2} = \frac{e^{x+y}}{(e^x + e^y)^2}$$

$$Z_{xx} Z_{yy} = \left[\frac{e^{x+y}}{(e^x + e^y)^2}\right]\left[\frac{e^{x+y}}{(e^x + e^y)^2}\right] = \frac{e^{2(x+y)}}{(e^x + e^y)^4} \qquad (1)$$

$$\text{Also, } (Z_{xy})^2 = \left[-\frac{e^{x+y}}{(e^x + e^y)^2}\right]^2 = \frac{e^{2(x+y)}}{(e^x + e^y)^4} \qquad (2)$$

From (1) and (2), we see that $Z_{xx}.Z_{yy} = (Z_{xy})^2$.

7. If $u = r^m$ and $r^2 = x^2 + y^2 + z^2$, show that $u_{xx} + u_{yy} + u_{zz} = m(m+1)r^{m-2}$

Solution: Since $r^2 = x^2 + y^2 + z^2$ ➔ $2r\dfrac{\partial r}{\partial x} = 2x \Rightarrow \dfrac{\partial r}{\partial x} = \dfrac{x}{r} \Rightarrow \dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

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Now using chain rule, we have

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = mr^{m-1}\frac{x}{r} = mxr^{m-2}$. Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(mxr^{m-2}\right)$$

$$= m\left[r^{m-2}.1 + x.(m-2)r^{m-3}.\frac{\partial r}{\partial x}\right] = m\left[r^{m-2} + x.(m-2)r^{m-3}.\frac{x}{r}\right]$$

Thus, $\frac{\partial^2 u}{\partial x^2} = m\left[r^{m-2} + (m-2)x^2 r^{m-4}\right]$. Differentiating again and using the chain rule

$\frac{\partial^2 u}{\partial y^2} = m\left[r^{m-2} + (m-2)y^2 r^{m-4}\right]$ and $\frac{\partial^2 u}{\partial z^2} = m\left[r^{m-2} + (m-2)z^2 r^{m-4}\right]$

Adding the last three results, we obtain

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m\left[3r^{m-2} + (m-2)r^{m-4}\left(x^2 + y^2 + z^2\right)\right]$$

$$= m\left[3r^{m-2} + (m-2)r^{m-4}.r^2\right] = m\left[3r^{m-2} + (m-1)r^{m-2}\right]$$

$$= m\left(3 + m - 1\right)r^{m-2} = m\left(m+2\right)r^{m-2}. \text{ Proved}$$

8. If $ux + vy = 0$ and $u/x + v/y = 1$, show that $u_x - v_y = [(x^2 + y^2)/(y^2 - x^2)]$

Solution:

Given that $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1 \Rightarrow yu + xv = xy$. Now, solving these two equations simultaneously, we get: $u = \frac{-xy^2}{x^2 - y^2}$ and $v = \frac{x^2 y}{x^2 - y^2}$. Now,

$$\frac{\partial u}{\partial x} = -\left[\frac{\left(x^2 - y^2\right)\left(y^2\right) - xy^2\left(2x\right)}{\left(x^2 - y^2\right)^2}\right] = \frac{y^2\left(2x^2 - x^2 + y^2\right)}{\left(x^2 - y^2\right)^2} = \frac{y^2\left(x^2 + y^2\right)}{\left(x^2 - y^2\right)^2} \quad (1)$$

And, $$\frac{\partial v}{\partial y} = x^2\left[\frac{\left(x^2 - y^2\right).1 - y\left(-2y\right)}{\left(x^2 - y^2\right)^2}\right] = \frac{x^2\left(x^2 + y^2\right)}{\left(x^2 - y^2\right)^2} \quad (2)$$

Subtracting (2) from (1), we get:

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \frac{y^2\left(x^2 + y^2\right)}{\left(x^2 - y^2\right)^2} - \frac{x^2\left(x^2 + y^2\right)}{\left(x^2 - y^2\right)^2} = \frac{\left(x^2 - y^2\right)\left(x^2 + y^2\right)}{\left(x^2 - y^2\right)^2} = \frac{\left(x^2 + y^2\right)}{\left(x^2 - y^2\right)}. \text{ Proved}$$

9. If $u = x^y$, prove that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial y \partial x \partial x}$

Solution: Given that $u = x^y$. Differentiating w.r.t y, we get

$\partial u / \partial y = x^y \ln y$. Now differentiating twicely w.r.t x,

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$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}.\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}.\frac{\partial}{\partial x}\left[x^y.\ln y\right] = \frac{\partial}{\partial x}\left[yx^{y-1}.\ln y\right] = y(y-1)x^{y-2}\ln y \quad (1)$$

Now, $$\frac{\partial^3 u}{\partial y \partial x \partial x} = \frac{\partial}{\partial y}.\frac{\partial}{\partial x}.\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}.\frac{\partial}{\partial x}\left(yx^{y-1}\right) = \frac{\partial}{\partial y}\left[y(y-1)x^{y-2}\right] = y(y-1)x^{y-2}.\ln y \quad (2)$$

From (1) and (2), the result follows.

NOTE: (i) Correct the problem. (ii) Here we have used the formula that if
(a) $y = x^n$ then $y` = nx^{n-1}$ and if $y = a^x$ then $y` = a^x.\ln a$

10. Find dy/dx in each of the following case. Verify the results by using the formula $dy/dx = -f_x/f_y$.

(a) $y^2 + x^2y + a\,x^4 = 0$ (b) $x^3 + x^2 + xy^2 + \sin y = 0$

Solution: (a) Differentiating w.r.t x, we get: $2yy` + x^2\,y` + y.2x + 4ax^3 = 0$
or $y`(2y + x^2) = -(2xy + 4ax^3)$ or $y` = -(2xy + 4ax^3)/(2y + x^2)$ (1)

Now by definition, $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{0 + 2xy + 4ax^3}{2y + x^2.1 + 0} = -\frac{2xy + 4ax^3}{2y + x^2} \quad (2)$$

The results in (1) and (2) are same.

(b) Differentiating w.r.t x, we get: $3x^2 + 2x + 1.y^2 + 2xyy` + \cos y.y` = 0$
or $(2xy + \cos y)\,y` = -(3x^2 + 2x + y^2)$ or $y` = -(3x^2 + 2x + y^2)/(2xy + \cos y)$ (1)

Now by definition, $$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{3x^2 + 2x + y^2}{2xy + \cos y} \quad (2)$$

The results in (1) and (2) are same.

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Chapter 6

FUNDAMENTAL THEOREMS AND INDETERMINATE FORMS

WORKSHEET 06

1. Verify the Rolle's Theorem and find c (if possible) for the following functions where interval for each function is also given.

Solution:

(i) $f(x)=x^2-3x+2$ on $[1,2]$: Given function $f(x)=x^2-3x+2$ is a polynomial, therefore, it is continuous and derivable on $[1,2]$. Also, $f(1)=1-3+2=0$ and $f(2)=4-6+2=0$ so $f(1)=f(2)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval $(1,2)$ such that $f`(c)=0$. Now, $f`(x)=2x-3 \Rightarrow f`(c)=2c-3=0 \Rightarrow c=3/2\in(1,2)$. This verifies Roll's theorem.

(ii) $f(x)=\sin^2 x$ on $[0,\pi]$: Given function $f(x)=\sin^2 x$ is continuous and derivable on $[0,\pi]$. Also, $f(0)=\sin^2 0=0$ and $f(\pi)=\sin^2\pi=0$ so $f(0)=f(\pi)$.

Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval $(0,\pi)$ such that $f`(c)=0$. Now, $f`(x)=2\sin x\cos x=\sin 2x$ $\Rightarrow f`(c)=\sin 2c=0 \Rightarrow 2c=\sin^{-1}(0)=\pi \Rightarrow c=\pi/2\in(0,\pi)$. This verifies Roll's theorem.

(iii) $f(x)=1-x^{3/4}$ on $[-1,1]$: Given function $f(x)=1-x^{3/4} \Rightarrow f`(x)=\frac{3}{4}x^{-1/4}=\frac{3}{4x^{1/4}}$.

$f`(0)=\infty$ (undefined). Thus one of three conditions of Roll's theorem is not satisfied. Thus Roll's theorem fails for the function $f(x)=1-x^{3/4}$ on $(-1,1)$.

(iv) Given function $f(x)=\frac{1-x^2}{1+x^2}$ on $[-1,1]$ is continuous and derivable on $(-1,1)$ being the division of two polynomials and the denominator is not zero at any point of the interval $(-1,1)$. Also, $f(-1)=0=f(1)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval $(-1,1)$ such that $f`(c)=0$. Now, $f`(x)=\frac{(1+x^2)(-2x)-(1-x^2)(2x)}{(1+x^2)^2}=\frac{2x(-1-x^2-1+x^2)}{(1+x^2)^2}=\frac{-4x}{(1+x^2)^2}$

$\Rightarrow f`(c)=\frac{-4c}{(1+c^2)^2}=0 \Rightarrow c=0\in(-1,1)$. This verifies Roll's theorem.

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(v) $f(x) = x(x+3)e^{-\frac{x}{2}}$ $[-3,0]$. This problem is solved in the textbook.

(vi) $f(x) = 2 + (x-1)^{3/2}$, on $[0,2]$

This function is continuous and derivable on the interval (0, 2).

Now, f(0) = 2 + $(0 - 1)^{3/2}$ = 2 + $(-1)^{3/2}$ = 2 + $(\sqrt{-1})^3$ = 2 + i^3 = 2 - i and f(2) = 3. Since, f(0) ≠ f(2) hence, 3rd condition of Roll's theorem is not satisfied so, the Roll's cannot be verified.

(vii) $f(x) = x^2 - 6x + 8$ on $[2,4]$: Given function $f(x) = x^2 - 6x + 8$ is a polynomial, therefore, it is continuous and derivable on $(2,4)$. Also, $f(2) = 4 - 12 + 8 = 0$ and $f(4) = 16 - 24 + 8 = 0$ so $f(2) = f(4)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval (1, 2) such that $f`(c) = 0$. Now, $f`(x) = 2x - 6 \Rightarrow f`(c) = 2c - 6 = 0 \Rightarrow c = 3 \in (2,4)$. This verifies Roll's theorem.

(viii) $f(x) = x^3 - 4x$ on $[-2,2]$: Given function $f(x) = x^3 - 4x$ is a polynomial, therefore, it is continuous and derivable on $(-2,2)$. Also, $f(-2) = -8 + 8 = 0$ and $f(2) = 8 - 8 = 0$ so $f(2) = f(4)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval (–2, 2) such that $f`(c) = 0$.

Now, $f`(x) = 3x^2 - 4 \Rightarrow f`(c) = 3c^2 - 4 = 0 \Rightarrow c = \pm 2/\sqrt{3} \in (-2,2)$.

This verifies Roll's theorem.

(ix) $f(x) = 8x - x^2$ on $[0,8]$: Given function $f(x) = 8x - x^2$ is a polynomial, therefore, it is continuous and derivable on $(0,8)$. Also, $f(0) = 0 = f(8)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval (0,8) such that $f`(c) = 0$. Now, $f`(x) = 8 - 2x$ $\Rightarrow f`(c) = 8 - 2c = 0 \Rightarrow c = 4 \in (0,8)$. This verifies Roll's theorem.

(x) $f(x) = x^2$ on $[-1,1]$: Given function $f(x) = x^2$ is a polynomial, therefore, it is continuous and derivable on $(-1,1)$. Also, $f(-1) = 1 = f(1)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval (–1,1) such that $f`(c) = 0$. Now, $f`(x) = 2x \Rightarrow f`(c) = 2c = 0$ $\Rightarrow c = 0 \in [0,8]$. This verifies Roll's theorem.

(xi) $f(x) = \sin x$ on $[-\pi,\pi]$: Given function $f(x) = \sin x$ is continuous and derivable on $(0,\pi)$. Also, $f(0) = \sin 0$ and $\sin \pi = 0$. Thus $f(0) = f(\pi)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval $(0,\pi)$ such that $f`(c) = 0$. Now, $f`(x) = \cos x \Rightarrow f`(c) = \cos c = 0$ $\Rightarrow c = \pi/2 \in (0,\pi)$. This verifies Roll's theorem.

(xii) $f(x) = e^x$ on $[0,\pi]$: Given function $f(x) = e^x$ is exponential function hence is continuous and derivable on $(0,\pi)$. Also, $f(0) = e^0 = 1$ and $f(\pi) = e^\pi$. Since, $f(0) \neq f(\pi)$

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hence, one of the three conditions of Rolle's theorem is not satisfied. Thus, Roll's theorem cannot be satisfied.

(xiii) Given function $f(x) = \sin x / e^x$ on $[0, \pi]$ is continuous and derivable as e^x is not zero at any point in $(0, \pi)$. Also, $f(0) = 0 = f(\pi)$. Hence, all three conditions of Rolle's theorem are satisfied. Thus, there must be a point 'c' in the interval $(-1,1)$ such that $f`(c) = 0$. Now, $f`(x) = \dfrac{e^x(\cos x) - e^x \sin x}{e^{2x}} = \dfrac{e^x[\cos x - \sin x]}{e^{2x}} \Rightarrow f`(c) = \dfrac{[\cos c - \sin c]}{e^c} = 0$

$\Rightarrow \cos c - \sin c = 0 \Rightarrow \sin c = \cos c \Rightarrow \tan c = 1 \Rightarrow c = \tan^{-1}(1) = \pi/4 \in (0, \pi)$.

This verifies Roll's theorem.

(xiv) $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$ is continuous and derivable in $(-2, 2)$. Also, $f(-2) = 0 = f(2)$. Thus all three conditions of Roll's theorem are satisfied. Now, $f`(x) = \dfrac{-2x}{2\sqrt{4 - x^2}} \Rightarrow f`(c) = \dfrac{-c}{\sqrt{4 - c^2}} = 0 \Rightarrow c = 0 \in (-2, 2)$. This verifies Roll's theorem.

(xv) $f(x) = \tan x$ on $[0, \pi]$ is not continuous because $\tan(\pi/2) = \infty$ and $\pi/2$ lies in the interval $(0, \pi)$. Thus Roll's theorem cannot be verified.

(xvi) $f(x) = x^2 - 7x + 12$ on $[3, 4]$ is continuous and derivable being a polynomial. Also $f(3) = 0 = f(4)$. Thus all three conditions of Roll's theorem are satisfied. Now, $f`(x) = 2x - 7 \Rightarrow f`(c) = 2c - 7 = 0 \Rightarrow c = 3.5 \in (3, 4)$. This verifies Roll's theorem.

(xvi) $f(x) = x(x-2)^3$ on $[0, 2]$ is continuous and derivable being a polynomial. Also $f(0) = 0 = f(2)$. Thus all three conditions of Roll's theorem are satisfied. Now, $f`(x) = (x-2)^3.1 + x.3(x-2)^2 = (x-2)^2[x - 2 + 3x] = (x-2)^2(4x-2) = 2(x-2)^2(2x-1)$.

$\Rightarrow f`(c) = 2(c-2)^2(2c-1) = 0 \Rightarrow c = 2, 1/2$. Here $x = 1/2 \in (0, 2)$.

This verifies Roll's theorem.

(xvii) $f(x) = \sec x$ on $[0, 2\pi]$ is not continuous and derivable because $\sec \pi/2 = \sec 3\pi/2 = \infty$, whereas, $f(0) = 1 = f(2\pi)$. Thus Roll's theorem cannot be verified.

(xviii) $f(x) = x^3 - 3x^2 + 3x + 2$ on $[1, 2]$ is continuous and derivable $f(x)$ being a polynomial. Also $f(1) = 3$, $f(2) = 4$. Thus, $f(1) \neq f(2)$ so the 3rd condition of Roll's theorem is not satisfied. Hence, Roll's theorem cannot be verified.

2. Discuss the applicability of Roll's Theorem to the function F(x) = | x| on [-1, 1].

Solution: We know that given function is continuous on [-1, 1] but not derivable at x = 0 which lies in the interval [-1, 1]. (For proof, see the textbook). Hence, Roll's theorem cannot be verified.

3. Find c (if possible) of the Mean Value Theorem for the following functions:

(i) The function $f(x) = x^3 - 3x^2 + 3x + 2$ on $[1, 2]$ is continuous and derivable being a polynomial. Thus there exists a point 'c' in $(1, 2)$ such that $f`(c) = [f(b) - f(a)]/(b - a)$. Now,

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$f`(x) = 3x^2 - 6x + 3 \Rightarrow f`(c) = 3c^2 - 6c + 3$. Also $f(2) = 4$ and $f(1) = 3$. Thus, by MVT,

$f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 - 6c + 3 = \frac{4-3}{2-1} = 1 \Rightarrow 3c^2 - 6c + 2 = 0 \Rightarrow c = \frac{6 \pm \sqrt{12}}{6} \Rightarrow c = 1.58$ or 0.423

The only value of c that lies in the interval $(1,2)$ is $c = 1.58$. This verifies MVT.

(ii) The function $f(x) = x^2 - 7x + 12$ on $[3,4]$ is continuous and derivable being a polynomial. Thus there exists a point c in $(3,4)$ such that $f`(c) = \frac{f(b) - f(a)}{b - a}$. Now,

$f`(x) = 2x - 7 \Rightarrow f`(c) = 2c - 7$. Also $f(4) = 0$ and $f(3) = 0$. Thus, by MVT,

$f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 2c - 7 = \frac{0-0}{4-3} = 0 \Rightarrow 2c - 7 = 0 \Rightarrow c = 7/2 = 3.5.$

This value of c lies in the interval $(3,4)$. This verifies MVT.

(iii) The function $f(x) = x(x-2)^3$ on $[0,2]$ is continuous and derivable being a polynomial. Thus there exists a point 'c' in $(0,2)$ such that

$f`(c) = [f(b) - f(a)]/(b - a)$. Now,

$f`(x) = (x-2)^3.1 + x.3(x-2)^2 = (x-2)^2[x - 2 + 3x] = 2(x-2)^2(2x-1)$

$\Rightarrow f`(c) = 2(c-2)^2(2c-1)$. Also $f(2) = 0$ and $f(0) = 0$. Thus, by MVT,

$f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 2(c-2)^2(2c-1) = \frac{0-0}{2-0} = 0 \Rightarrow 2(c-2)^2(2c-1) = 0 \Rightarrow c = 2, 1/2.$

The only value of 'c' that lies in the interval $(0,2)$ is $c = 1/2$. This verifies MVT.

(iv) The function $f(x) = x^3 - 3x - 1$ on $\left[-\frac{11}{7}, \frac{13}{7}\right]$ is continuous and derivable being a polynomial.

Thus there exists a point 'c' in the given interval such that $f`(c) = \frac{f(b) - f(a)}{b - a}$. Now,

$f`(x) = 3x^2 - 3 \Rightarrow f`(c) = 3c^2 - 3$. Also $f\left(\frac{13}{7}\right) = -0.17$ and $f\left(\frac{-11}{7}\right) = -0.17$. Thus, by MVT,

$f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 - 3 = \frac{-0.17 + 0.17}{[(13/7) - (-11/7)]} = 0 \Rightarrow 3c^2 - 3 = 0 \Rightarrow c = \pm 1$. Both values of 'c' lie in the given interval. This verifies MVT.

(v) The function $f(x) = \sqrt{x-2}$ on $[2,4]$ is continuous and derivable.

Thus there exists a point 'c' in the given interval such that $f`(c) = \frac{f(b) - f(a)}{b - a}$. Now,

$f`(x) = \frac{1}{2\sqrt{x-2}} \Rightarrow f`(c) = \frac{1}{2\sqrt{c-2}}$. Also $f(4) = \sqrt{2} = 1.414$ and $f(2) = 0$. Thus, by MVT,

$f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{1}{2\sqrt{c-2}} = \frac{1.414 - 0}{4-2} = 0.707 \Rightarrow \frac{1}{\sqrt{c-2}} = 1.414 \Rightarrow \sqrt{c-2} = \frac{1}{1.414} = 0.707$

$\Rightarrow c - 2 = 0.5$ or $c = 2.5$. This value of 'c' lies in the given interval $(2,4)$. This verifies MVT.

(vi) The function $f(x) = x^3 - 5x^2 + 4x - 2$ on $[1,3]$ is continuous and derivable being a polynomial. Thus there exists a point 'c' in the given interval such that $f`(c) = \frac{f(b) - f(a)}{b - a}$.

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Now, $f`(x) = 3x^2 - 10x + 4 \Rightarrow f`(c) = 3c^2 - 10c + 4$. Also $f(1) = -2$ and $f(3) = -8$. Thus, by MVT, $f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 - 10c + 4 = \frac{-8+2}{3-1} = -3 \Rightarrow 3c^2 - 10c + 7 = 0 \Rightarrow c = 1$ and $c = 2.3$. Now only $c = 2.3$ lies in the given interval $(1,3)$. This verifies MVT.

(vii) Given function $f(x) = x^{2/3}$ on $[-1,1]$ is not derivable because $f`(x) = \frac{2}{3x^{1/3}}$ $\Rightarrow f`(0) = \infty$ as 0 lies in $(-1,1)$. Hence, MVT cannot be verified.

(viii) The function $f(x) = f(x) = x(x-1)(x-2)$ on $[0,1/2]$ is continuous and derivable being a polynomial.Thus there exists a point 'c' in $(0,1/2)$ such that $f`(c) = \frac{f(b) - f(a)}{b - a}$.

Now, $f`(x) = 3x^2 - 6x + 2 \Rightarrow f`(c) = 3c^2 - 6c + 2$. Also $f(0.5) = 0.375$ and $f(0) = 0$.

Thus, by MVT, $f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 - 6c + 2 = \frac{0.375 - 0}{0.5 - 0} = \frac{3}{4} \Rightarrow 12c^2 - 24c + 8 = 3$ $\Rightarrow 12c^2 - 24c + 5 = 0 \Rightarrow c = 1/4$ and $c = 7/4$. Only the value of 'c' that lies in the given interval is $c = 1/4$. This verifies MVT.

(ix) The function $f(x) = (x-1)(x-2)(x-3)$ on $[0,4]$ is continuous and derivable being a polynomial.Thus there exists a point 'c' in $(0,4)$ such that $f`(c) = \frac{f(b) - f(a)}{b - a}$.

Now, $f`(x) = 3x^2 - 12x + 11 \Rightarrow f`(c) = 3c^2 - 12c + 11$. Also $f(4) = 6$ and $f(0) = -6$.

Thus, by MVT, $f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow 3c^2 - 12c + 11 = \frac{6+6}{4-0} = 3 \Rightarrow 3c^2 - 12c + 8 = 0$ $\Rightarrow c = 3.15$ and $c = 0.8$. Both values of 'c' lie in the given interval $(0,4)$.

This verifies MVT.

(x) The function $f(x) = \sqrt{x^2 - 4}$ on $[2,4]$ is continuous and derivable in $(2,4)$.

Now, $f`(x) = \frac{2x}{2\sqrt{x^2-4}} = \frac{x}{\sqrt{x^2-4}} \Rightarrow f`(c) = \frac{c}{\sqrt{c^2-4}}$. Also $f(4) = \sqrt{12} = 2\sqrt{3}$ and $f(2) = 0$.

Thus, by MVT, $f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{c}{\sqrt{c^2-4}} = \frac{2\sqrt{3} - 0}{4 - 2} = \sqrt{3}$. Squaring both sides, we obtain $\frac{c^2}{c^2-4} = 3 \Rightarrow c^2 = 3c^2 - 12 \Rightarrow c^2 = 12 \Rightarrow c = \pm\sqrt{12} = \pm 2\sqrt{3}$. Only $c = 2\sqrt{3}$ lies in the given interval $(2,4)$. This verifies MVT.

(xi) We know that function $f(x) = \log x$ is continuous on $[1,e]$ and derivable in $(1,e)$.

Now, $f`(x) = \frac{1}{x} \Rightarrow f`(c) = \frac{1}{c}$ and $f(1) = \log 1 = 0$, $f(e) = \log e = 1$. Thus, by MVT

$f`(c) = \frac{f(b) - f(a)}{b - a} \Rightarrow \frac{1}{c} = \frac{1-0}{e-1} = \frac{1}{e-1} \Rightarrow c = e - 1 = 1.718$ which lies inside $(1,e)$.

This verifies MVT.

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(xii) We know that function $f(x)=e^x$ on $[0,1]$ is continuous and derivable being an exponential function. Now, $f`(x)=e^x \Rightarrow f`(c)=e^c$ and $f(1)=e,\ f(0)=1$.

Thus, by MVT $f`(c)=\dfrac{f(b)-f(a)}{b-a} \Rightarrow e^c=\dfrac{e-1}{1-0}=e-1 \Rightarrow c=\ln(e-1)=0.54$ which lies inside (0,1). This verifies MVT.

(xiii) The function $f(x)=x^3-5x^2-3x$ on $[1,3]$ is continuous and derivable being a polynomial. Thus there exists a point 'c' in the given interval such that $f`(c)=\dfrac{f(b)-f(a)}{b-a}$.

Now, $f`(x)=3x^2-10x-3 \Rightarrow f`(c)=3c^2-10c-3$. Also $f(1)=-7$ and $f(3)=-27$. Thus, by MVT, $f`(c)=\dfrac{f(b)-f(a)}{b-a} \Rightarrow 3c^2-10c-3=\dfrac{-27+7}{3-1}=-10 \Rightarrow 3c^2-10c+7=0$

$\Rightarrow c=1$ and $c=2.3$. Now only $c=2.3$ lies in the given interval (1,3). This verifies MVT.

(xiv) The function $f(x)=\sqrt{x^2-9}$ on $[3,4]$ is continuous and derivable in $(3,4)$.

Now, $f`(x)=\dfrac{2x}{2\sqrt{x^2-9}}=\dfrac{x}{\sqrt{x^2-9}} \Rightarrow f`(c)=\dfrac{c}{\sqrt{c^2-9}}$. Also $f(4)=\sqrt{7}$ and $f(3)=0$.

Thus, by MVT, $f`(c)=\dfrac{f(b)-f(a)}{b-a} \Rightarrow \dfrac{c}{\sqrt{c^2-9}}=\dfrac{\sqrt{7}-0}{4-3}=\sqrt{7}$. Squaring both sides, we obtain $\dfrac{c^2}{c^2-9}=7 \Rightarrow c^2=7c^2-63 \Rightarrow c^2=63/6=10.5 \Rightarrow c=\pm\sqrt{10.5}=\pm 3.24$. Only $c=3.24$ lies in the given interval (3,4). This verifies MVT.

4. Find the McLaren's series of the following functions:

(i) Given $f(x)=\sin x \Rightarrow f(0)=\sin 0=0,\quad f`(x)=\cos x \quad \Rightarrow f`(0)=\cos 0=1$

$f``(x)=-\sin x \quad \Rightarrow f``(0)=-\sin 0=0,\ f```(x)=-\cos x \Rightarrow f```(0)=-\cos 0=-1$

$f^{iv}(x)=\sin x \quad \Rightarrow f^{iv}(0)=\sin 0=0,\quad f^{v}(x)=\cos x \quad \Rightarrow f^{v}(0)=\cos 0=1$, etc

Now by McLaren's series:

$$f(x)=f(0)+\frac{x}{1!}f`(0)+\frac{x^2}{2!}f``(0)+\frac{x^3}{3!}f```(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\ldots$$

Putting above values: $\Rightarrow \sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\ldots$

(ii) Given $f(x)=\cos x \Rightarrow f(0)=\cos 0=1,\quad f`(x)=-\sin x \quad \Rightarrow f`(0)=-\sin 0=0$

$f``(x)=-\cos x \quad \Rightarrow f``(0)=-\cos 0=-1,\ f```(x)=\sin x \quad \Rightarrow f```(0)=\sin 0=0$

$f^{iv}(x)=\cos x \quad \Rightarrow f^{iv}(0)=\cos 0=1,\quad f^{v}(x)=-\sin x \quad \Rightarrow f^{v}(0)=-\sin 0=0$

$f^{vi}(x)=-\cos x \quad \Rightarrow f^{vi}(0)=-\cos 0=-1$, etc

Now by McLaren's series:

$$f(x)=f(0)+\frac{x}{1!}f`(0)+\frac{x^2}{2!}f``(0)+\frac{x^3}{3!}f```(0)+\frac{x^4}{4!}f^{iv}(0)+\frac{x^5}{5!}f^{v}(0)+\frac{x^6}{6!}f^{vi}(0)\ldots$$

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Putting above values, we get : $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} ...$

(iii) $f(x) = \sec x$ NOTE : $\tan 0 = 0$ and $\sec 0 = 1$

Given $f(x) = \sec x \Rightarrow f(0) = \sec 0 = 1$, $f`(x) = \sec x \tan x$ $\Rightarrow f`(0) = 0$

$f``(x) = \sec x.\sec^2 x + \tan x.\sec x \tan x = \sec^3 x + \sec x \tan^2 x$ $\Rightarrow f``(0) = 1$

$f```(x) = 3\sec^2 x.\sec x \tan x + \tan^2 x.\sec x \tan x + \sec x.2\tan x \sec^2 x$

$= 5\sec^3 x \tan x + \sec x \tan^3 x$ $\Rightarrow f```(0) = 0$

$f^{iv}(x) = 5\sec^3 x.\sec^2 x + 5\tan x.3\sec^2 x \sec x \tan x + \tan^3 x.\sec x \tan x + \sec x.3\tan^2 x \sec^2 x$

$= 5\sec^5 x + 8\tan^2 x \sec^3 x + \tan^4 x \sec x$ $\Rightarrow f^{iv}(0) = 5$

Now by McLaren's series :

$$f(x) = f(0) + \frac{x}{1!}f`(0) + \frac{x^2}{2!}f``(0) + \frac{x^3}{3!}f```(0) + \frac{x^4}{4!}f^{iv}(0) + ...$$

Putting above values, we get : $\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} ...$

(iv) f(x) = tan x NOTE: tan 0 = 0 and sec 0 = 1

Given f(x) = tan x ➔ f(0) = tan 0 = 0, $f`(x) = \sec^2 x$ ➔ $f`(0) = 1$

$f``(x) = 2\sec x.\sec x \tan x = 2\sec^2 x \tan x$ $\Rightarrow f``(0) = 0$

$f```(x) = 2\sec^2 x.\sec^2 x + 2\tan x.2\sec x.\sec x \tan x$

$= 2\sec^4 x + 4\sec^2 x \tan^2 x$ $\Rightarrow f```(0) = 2$

$f^{iv}(x) = 8\sec^3 x.\sec x \tan x + 4\sec^2 x.2\tan x \sec^2 x + 4\tan^2 x.2\sec x.\sec x \tan x$

$= 8\sec^4 x \tan x + 8\tan x \sec^4 x + 8\tan^3 x \sec^2 x$ $\Rightarrow f^{iv}(0) = 0$

$f^{v}(x) = 8\sec^4 x.\sec^2 x + 8\tan x.4\sec^3 x.\sec x \tan x + 8\sec^4 x.\sec^2 x$

$+ 8\tan x.4\sec^3 x.\sec x \tan x + 8\tan^3 x.2\sec x \sec x \tan x$

$+ 8\sec^2 x.3\tan^2 x.\sec^2 x$

$= 16\sec^6 x + 86\sec^4 x \tan^2 x + 16\sec^4 x \tan^2 x$ $\Rightarrow f^{v}(0) = 16$

Now by McLaren's series :

$$f(x) = f(0) + \frac{x}{1!}f`(0) + \frac{x^2}{2!}f``(0) + \frac{x^3}{3!}f```(0) + \frac{x^4}{4!}f^{iv}(0) + ...$$

Putting above values, we get : $\tan x = x + \frac{2x^3}{3!} + \frac{16x^5}{5!} ... = x + \frac{x^3}{3} + \frac{2x^5}{15} + ...$

(v) Given $f(x) = \ln(1-x)$ $\Rightarrow f(0) = \ln 1 = 0$

$f`(x) = -\frac{1}{1-x} = -(1-x)^{-1}$ $\Rightarrow f`(0) = -1$

$f``(x) = -(1-x)^{-2}$ $\Rightarrow f``(0) = -1$

$f```(x) = -2(1-x)^{-3}$ $\Rightarrow f```(0) = -2$

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$f^{iv}(x) = -6(1-x)^{-4} \qquad \Rightarrow f^{iv}(0) = -6$, etc.

Now by McLaren's series:

$$f(x) = f(0) + \frac{x}{1!}f`(0) + \frac{x^2}{2!}f``(0) + \frac{x^3}{3!}f```(0) + \frac{x^4}{4!}f^{iv}(0)...$$

Putting above values, we get: $\ln(1-x) = -x - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{6x^4}{4!}...$

Or, $\ln(1-x) = -\left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}...\right]$

(vi) Given $f(x) = e^{\sin x} \qquad \Rightarrow f(0) = 1 \qquad$ [NOTE: $e^{\sin 0} = e^0 = 1$]

$f`(x) = e^{\sin x}.\cos x \qquad \Rightarrow f`(0) = 1 \qquad [\sin 0 = 0 \ \& \ \cos 0 = 1]$

$f``(x) = e^{\sin x}(-\sin x) + \cos x.e^{\sin x}.\cos x$

$\therefore \ f``(x) = e^{\sin x}\left(-\sin x + \cos^2 x\right) \qquad \Rightarrow f``(0) = 1$

$f```(x) = e^{\sin x}\left(-\cos x - 2\cos x \sin x\right) + \left(-\sin x + \cos^2 x\right).e^{\sin x}.\cos x$

$= e^{\sin x}\left(-\cos x - 3\sin x \cos x + \cos^3 x\right) \qquad \Rightarrow f```(0) = 0$

$f^{iv}(x) = e^{\sin x}\left(\sin x - 3\sin x(-\sin x) - 3\cos x(\cos x) + 3\cos^2 x(-\sin x)\right)$

$+\left(-\cos x - 3\sin x \cos x + \cos^3 x\right).e^{\sin x}\cos x \qquad \Rightarrow f^{iv}(0) = -3$, etc.

Now by McLaren's series:

$$f(x) = f(0) + \frac{x}{1!}f`(0) + \frac{x^2}{2!}f``(0) + \frac{x^3}{3!}f```(0) + \frac{x^4}{4!}f^{iv}(0)...$$

Putting above values, we get: $e^{\sin x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}(0) + \frac{-3x^4}{4!}...$

Or, $e^{\sin x} = \left[1 + x + \frac{x^2}{2} - \frac{x^4}{8}...\right]$

5. Apply Taylor's Theorem to prove that:

(i) $a^{x+h} = a^x\left[1 + h\ln a + \frac{h^2}{2!}(\ln a)^2 + \frac{h^3}{3!}(\ln a)^3 + \cdots\right]$

Solution: By Taylor's series,

$$f(x+a) = f(a) + (x-a)f`(a) + \frac{(x-a)^2}{2!}f``(a) + \frac{(x-a)^3}{3!}f```(a) + \frac{(x-a)4}{4!}f^{iv}(a) + ...$$

Now, $f(x) = a^{x+h} \qquad \Rightarrow f(a) = a^{a+h}, \qquad f`(x) = a^{x+h}\ln a \qquad \Rightarrow f`(a) = a^{a+h}\ln a$

$f``(x) = a^{x+h}(\ln a)^2 \qquad \Rightarrow f``(a) = a^{a+h}(\ln a)^2, \ f^{iv}(x) = a^{x+h}(\ln a)^3 \qquad \Rightarrow f^{iv}(a) = a^{a+h}(\ln a)^4$

Placing these values in the above series, and setting $x - a = h \Rightarrow a + h = x$, we obtain

$$a^{x+h} = a^x\left[1 + h\ln a + \frac{h^2}{2!}(\ln a)^2 + \frac{h^3}{3!}(\ln a)^3 + \cdots\right]$$

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(ii) $e^{x+h} = e^x\left[1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots\right]$

Solution : By Taylor's series,

$$f(x+a) = f(a)+(x-a)f`(a)+\frac{(x-a)^2}{2!}f``(a)+\frac{(x-a)^3}{3!}f```(a)+\frac{(x-a)4}{4!}f^{iv}(a)+...$$

Now, $f(x) = e^{x+h}$ $\Rightarrow f(a) = e^{a+h}$, $f`(x) = e^{x+h}$ $\Rightarrow f`(a) = e^{a+h}$

$f``(x) = e^{x+h}$ $\Rightarrow f``(a) = e^{a+h}$, $f^{iv}(x) = e^{x+h}$ $\Rightarrow f^{iv}(a) = e^{a+h}$

Placing these values in the above series, and setting $x-a=h \Rightarrow a+h=x$, we obtain

$$e^{x+h} = e^x\left[1+h+\frac{h^2}{2!}+\frac{h^3}{3!}+\cdots\right]$$

(iii) $\frac{1}{x+h} = \frac{1}{x}\left[1-\frac{h}{x}+\frac{h^2}{x^2}-\frac{h^3}{x^3}+\cdots\right]$

Solution : By Taylor's series,

$$f(x+a) = f(a)+(x-a)f`(a)+\frac{(x-a)^2}{2!}f``(a)+\frac{(x-a)^3}{3!}f```(a)+\frac{(x-a)4}{4!}f^{iv}(a)+...$$

Now, $f(x) = (x+h)^{-1}$ $\Rightarrow f(a) = (a+h)^{-1}$

$f`(x) = -1(x+h)^{-2}$ $\Rightarrow f`(a) = -1(a+h)^{-2}$

$f``(x) = +2(x+h)^{-3}$ $\Rightarrow f``(a) = 2(a+h)^{-3}$

$f```(x) = -6(x+h)^{-4}$ $\Rightarrow f```(a) = -6(a+h)^{-3}$

Placing these values in the above series, and letting $x-a=h \Rightarrow a+h=x$, we obtain

$$\frac{1}{x+h} = \frac{1}{x}\left[1-\frac{h}{x}+\frac{h^2}{x^2}-\frac{h^3}{x^3}+\cdots\right]$$

(iv) $\ln(x+h) = \ln x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\cdots$

Solution : By Taylor's series,

$$f(x+a) = f(a)+(x-a)f`(a)+\frac{(x-a)^2}{2!}f``(a)+\frac{(x-a)^3}{3!}f```(a)+\frac{(x-a)4}{4!}f^{iv}(a)+...$$

Now, $f(x) = \ln(x+h)$ $\Rightarrow f(a) = \ln(a+h)$

$f`(x) = \frac{1}{x+h} = (x+h)^{-1}$ $\Rightarrow f`(a) = (a+h)^{-1}$

$f``(x) = -(x+h)^{-2}$ $\Rightarrow f``(a) = -(x+h)^{-2}$

$f```(x) = 2(x+h)^{-3}$ $\Rightarrow f```(a) = 2(a+h)^{-3}$

Placing these values in the above series, and setting $x-a=h \Rightarrow a+h=x$, we obtain

$$\ln(x+h) = \left[\ln x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\cdots\right]$$

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(v) $\ln\sin(x+h) = \ln\sin x + h\cot x - \frac{h^2}{2}\csc^2 x + \frac{h^3}{3}\cot x\csc^2 x + \cdots$

Solution : By Taylor's series,

$$f(x+a) = f(a) + (x-a)f`(a) + \frac{(x-a)^2}{2!}f``(a) + \frac{(x-a)^3}{3!}f```(a) + \frac{(x-a)4}{4!}f^{iv}(a) + \ldots$$

Now, $f(x) = \ln\sin(x+h)$ $\Rightarrow f(a) = \ln\sin(a+h)$

$f`(x) = \frac{1}{\sin(x+h)}.\cos(x+h) = \cot(x+h)$ $\Rightarrow f`(a) = \cot(a+h)$

$f``(x) = -\csc^2(x+h)$ $\Rightarrow f``(a) = -\csc^2(a+h)$

$f```(x) = -2\csc(x+h)\left[-\csc(x+h).\cot(x+h)\right]$

$= +2\csc^2(x+h)\cot(x+h)$ $\Rightarrow f```(a) = 2\csc^2(a+h)\cot(a+h)$

Placing these values in the above series, and setting $x-a=h \Rightarrow a+h=x$, we obtain

$$\ln\sin(x+h) = \left[\ln\sin x + h\cot x - \frac{h^2}{2}\csc^2 x + \frac{h^3}{3}\cot x\csc^2 x + \cdots\right]$$

(vi) $\ln\cos(x+h) = \ln\cos x - h\tan x - \frac{h^2}{2}\sec^2 x - \frac{h^3}{3}\sec^2 x\tan x + \cdots$

Solution : By Taylor's series,

$$f(x+a) = f(a) + (x-a)f`(a) + \frac{(x-a)^2}{2!}f``(a) + \frac{(x-a)^3}{3!}f```(a) + \frac{(x-a)4}{4!}f^{iv}(a) + \ldots$$

Now, $f(x) = \ln\cos(x+h)$ $\Rightarrow f(a) = \ln\sin(a+h)$

$f`(x) = \frac{-\sin(x+h)}{\cos(x+h)} = -\tan(x+h)$ $\Rightarrow f`(a) = \tan(a+h)$

$f``(x) = -\sec^2(x+h)$ $\Rightarrow f``(a) = -\sec^2(a+h)$

$f```(x) = -2\sec(x+h)\left[\sec(x+h).\tan(x+h)\right]$

$= -2\sec^2(x+h)\tan(x+h)$ $\Rightarrow f```(a) = -2\sec^2(a+h)\tan(a+h)$

Placing these values in the above series, and setting $x-a=h \Rightarrow a+h=x$, we obtain

$$\ln\cos(x+h) = \left[\ln\cos x - h\tan x - \frac{h^2}{2}\sec^2 x - \frac{h^3}{3}\sec^2 x\tan x + \cdots\right]$$

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$$(\text{vii})\ \sin^{-1}(x+h)=\sin^{-1}x+\frac{h}{\sqrt{1-x^2}}+\frac{x}{\left(1-x^2\right)^{3/2}}\cdot\frac{h^2}{2!}+\cdots$$

Solution : By Taylor's series,

$$f(x+a)=f(a)+(x-a)f`(a)+\frac{(x-a)^2}{2!}f``(a)+\frac{(x-a)^3}{3!}f```(a)+\frac{(x-a)4}{4!}f^{iv}(a)+...$$

Now, $f(x)=\sin^{-1}(x+h)$ $\Rightarrow f(a)=\sin^{-1}(a+h)$

$f`(x)=\dfrac{1}{\sqrt{1-(x+h)^2}}=\left[1-(x+h)\right]^{-1/2}$ $\Rightarrow f`(a)=\dfrac{1}{\sqrt{1-(a+h)^2}}$

$f``(x)=-\dfrac{1}{2}\left[1-(x+h)\right]^{-3/2}=-\dfrac{1}{2\left[1-(x+h)\right]^{3/2}}$ $\Rightarrow f``(a)=-\dfrac{1}{2\left[1-(a+h)\right]^{3/2}}$

Placing these values in the above series, and setting $x-a=h\Rightarrow a+h=x$, we obtain

$$\sin^{-1}(x+h)=\left[\sin^{-1}x+\frac{h}{\sqrt{1-x^2}}+\frac{x}{\left(1-x^2\right)^{3/2}}\cdot\frac{h^2}{2!}+\cdots\right]$$

$$(\text{viii})\ \tan^{-1}(x+h)=\tan^{-1}x+\frac{h}{1+x^2}-\frac{xh^2}{\left(1+x^2\right)^2}+\cdots$$

Solution : By Taylor's series,

$$f(x+a)=f(a)+(x-a)f`(a)+\frac{(x-a)^2}{2!}f``(a)+\frac{(x-a)^3}{3!}f```(a)+\frac{(x-a)4}{4!}f^{iv}(a)+...$$

Now, $f(x)=\tan^{-1}(x+h)$ $\Rightarrow f(a)=\tan^{-1}(a+h)$

$f`(x)=\dfrac{1}{1+(x+h)^2}=\left[1+(x+h)\right]^{-1}$ $\Rightarrow f`(a)=\left[1+(x+h)\right]^{-1}$

$f``(x)=-\left[1+(x+h)\right]^{-2}=\dfrac{1}{\left[1+(x+h)\right]^2}$ $\Rightarrow f``(a)=\dfrac{1}{\left[1+(a+h)\right]^2}$

Placing these values in the above series, and setting $x-a=h\Rightarrow a+h=x$, we obtain

$$\tan^{-1}(x+h)=\left[\tan^{-1}x+\frac{h}{1+x^2}-\frac{xh^2}{\left(1+x^2\right)^2}+\cdots\right]$$

$$(\text{ix})\ \sec^{-1}(x+h)=\sec^{-1}x+\frac{h}{x\sqrt{x^2-1}}-\frac{h^2}{2!}\cdot\frac{2x^2-1}{x^2\left(x^2-1\right)^{3/2}}+\cdots$$

Solution : By Taylor's series,

$$f(x+a)=f(a)+(x-a)f`(a)+\frac{(x-a)^2}{2!}f``(a)+\frac{(x-a)^3}{3!}f```(a)+\frac{(x-a)4}{4!}f^{iv}(a)+...$$

Now, $f(x)=\sec^{-1}(x+h)$ $\Rightarrow f(a)=\sec^{-1}(a+h)$

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$$f`(x)=\frac{1}{(x+h)\sqrt{(x+h)^2-1}}=(x+h)^{-1}\left[(x+h)^2-1\right]^{-1/2} \quad \Rightarrow f`(a)=\frac{1}{(a+h)\sqrt{(a+h)^2-1}}$$

$$f``(x)=\left[(x+h)-1\right]^{-1/2}.\left[-(x+h)^{-2}\right]+(x+h)^{-1}\left(\frac{-1}{2}\right)\left[(x+h)^2-1\right]^{-3/2}.2(x+h)$$

$$=-\frac{1}{(x+h)^2\sqrt{(x+h)^2-1}}-\frac{1}{\left[(x+h)^2-1\right]^{3/2}}=\frac{-\left[(x+h)^2-1\right]-(x+h)^2}{(x+h)^2\left[(x+h)^2-1\right]^{3/2}}$$

$$=-\frac{2(x+h)^2-1}{(x+h)^2\left[(x+h)^2-1\right]^{3/2}} \qquad \Rightarrow f``(a)=-\frac{2(a+h)^2-1}{(a+h)^2\left[(a+h)^2-1\right]^{3/2}}$$

Placing these values in the above series, and setting $x-a=h \Rightarrow a+h=x$, we obtain

$$\sec^{-1}(x+h)=\sec^{-1}x+\frac{h}{x\sqrt{x^2-1}}-\frac{h^2}{2!}\cdot\frac{2x^2-1}{x^2\left(x^2-1\right)^{3/2}}+\cdots$$

(x) $\tan(x+h)=\tan x+h\sec^2 x+h^2\sec^2 x\tan x+\frac{h^3}{3}\sec^2 x\left(1+3\tan^2 x\right)+\cdots$

Solution : By Taylor's series,

$$f(x+a)=f(a)+(x-a)f`(a)+\frac{(x-a)^2}{2!}f``(a)+\frac{(x-a)^3}{3!}f```(a)+\frac{(x-a)4}{4!}f^{iv}(a)+...$$

Now, $f(x)=\tan(x+h) \qquad \Rightarrow f(a)=\tan(a+h)$

$f``(x)=\sec^2(x+h) \qquad \Rightarrow f`(a)=\sec^2(a+h)$

$f```(x)=2\sec(x+h).\sec(x+h)\tan(x+h)=2\sec^2(x+h)\tan(x+h)$

$\Rightarrow f``(a)=2\sec^2(a+h)\tan(a+h)$

Placing these values in the above series, and setting $x-a=h \Rightarrow a+h=x$, we obtain

$$\tan(x+h)=\left[\tan x+h\sec^2 x+h^2\sec^2 x\tan x+\frac{h^3}{3}\sec^2 x\left(1+3\tan^2 x\right)+\cdots\right]$$

6. Prove the following:

(a) $\lim_{x\to 4}\frac{x^4-256}{x-4}=256$

Solution : $\lim_{x\to 4}\frac{x^4-256}{x-4}=\frac{0}{0}$. Thus applying the L'Hospital rule, we get

$$\lim_{x\to 4}\frac{x^4-256}{x-4}=\lim_{x\to 4}\frac{4x^3}{1}=4.(4)^3=256$$

(b) $\lim_{x\to 4}\frac{x^4-256}{x^2-16}=32$

Solution : $\lim_{x\to 4}\frac{x^4-256}{x^2-16}=\frac{0}{0}$. Applying the L'Hospital rule, we get

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$$\lim_{x\to 4}\frac{x^4-256}{x^2-16}=\lim_{x\to 4}\frac{4x^3}{2x}=\lim_{x\to 4}\left(2x^2\right)=2.(16)=32$$

$$(c)\lim_{x\to 3}\frac{x^2-3x}{x^2-9}=\frac{1}{2}$$

Solution : $\lim_{x\to 3}\frac{x^2-3x}{x^2-9}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$$\lim_{x\to 3}\frac{x^2-3x}{x^2-9}=\lim_{x\to 3}\frac{2x-3}{2x}=\frac{6-3}{6}=\frac{3}{6}=\frac{1}{2}$$

$$(d)\lim_{x\to 2}\frac{e^x-e^2}{x-2}=e^2$$

Solution : $\lim_{x\to 2}\frac{e^x-e^2}{x-2}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$$\lim_{x\to 2}\frac{e^x-0}{1-0}=e^2$$

$$(e)\lim_{x\to 0}\frac{xe^x}{1-e^x}=-1$$

Solution : $\lim_{x\to 0}\frac{xe^x}{1-e^x}=\frac{0.e^0}{1-e^0}=\frac{0}{1-1}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$$\lim_{x\to 0}\frac{xe^x}{1-e^x}=\lim_{x\to 0}\frac{xe^x+e^x.1}{0-e^x}=\frac{0e^0+e^0}{-e^0}=\frac{0+1}{-1}=-1$$

$$(f)\lim_{x\to 0}\frac{e^x-1}{\tan 2x}=\frac{1}{2}$$

Solution : $\lim_{x\to 0}\frac{e^x-1}{\tan 2x}=\frac{e^0-1}{\tan 0}=\frac{1-1}{0}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$$\lim_{x\to 0}\frac{e^x-1}{\tan 2x}=\lim_{x\to 0}\frac{e^x}{2\sec^2 2x}=\frac{e^0}{2\sec^2 0}=\frac{1}{2(1)}=\frac{1}{2}$$

$$(g)\lim_{x\to -1}\frac{\ln(2+x)}{x+1}=1$$

Solution : $\lim_{x\to -1}\frac{\ln(2+x)}{x+1}=\frac{\ln 1}{1-1}=\frac{0}{0}$. Thus applying the L'Hospital rule, we get

$$\lim_{x\to -1}\frac{\ln(2+x)}{x+1}=\lim_{x\to -1}\frac{1}{(2+x).(1+0)}=\frac{1}{2-1}=1$$

$$(h)\lim_{x\to 0}\frac{\cos x-1}{\cos 2x-1}=\frac{1}{4}$$

Solution : $\lim_{x\to 0}\frac{\cos x-1}{\cos 2x-1}=\frac{\cos 0-1}{\cos 0-1}=\frac{1-1}{1-1}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$\lim_{x\to 0}\frac{\cos x-1}{\cos 2x-1}=\lim_{x\to 0}\frac{\sin x}{2\sin 2x}=\lim_{x\to 0}\frac{\sin 0}{2\sin 0}=\frac{0}{0}$. Applying the L'Hospital rule once again

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$$\lim_{x\to 0}\frac{\cos x-1}{\cos 2x-1}=\lim_{x\to 0}\frac{\cos x}{4\cos 2x}=\frac{\cos 0}{4\cos 0}=\frac{1}{4(1)}=\frac{1}{4}$$

(i) $\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{\sin x}=4$

Solution : $\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{\sin x}=\frac{e^0-e^0}{\sin 0}=\frac{1-1}{0}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$$\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{\sin x}=\lim_{x\to 0}\frac{2e^{2x}+2e^{-2x}}{\cos x}=\frac{2e^0+2e^0}{\cos 0}=\frac{2+2}{1}=4$$

(j) $\lim_{x\to 0}\frac{8^x-2^x}{4x}=\frac{1}{2}\ln 2$

Solution : $\lim_{x\to 0}\frac{8^x-2^x}{4x}=\frac{8^0-2^0}{4(0)}=\frac{1-1}{0}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$$\lim_{x\to 0}\frac{8^x-2^x}{4x}=\lim_{x\to 0}\frac{8^x.\ln 8-2^x.\ln 2}{4(1)}=\frac{8^0.\ln 8-2^0.\ln 2}{4}=\frac{\ln\left(\frac{8}{2}\right)}{4}=\frac{\ln 4}{4}=\frac{\ln 2^2}{4}=\frac{2.\ln 2}{4}=\frac{\ln 2}{2}$$

(k) $\lim_{x\to 0}\frac{2\tan^{-1}x-x}{2x-\sin^{-1}x}=1$

Solution : $\lim_{x\to 0}\frac{2\tan^{-1}x-x}{2x-\sin^{-1}x}=\frac{2\tan^{-1}0-0}{2(0)-\sin^{-1}0}=\frac{2(0)-0}{0-0}=\frac{0}{0}$.

Thus applying the L'Hospital rule, we get

$$\lim_{x\to 0}\frac{2\tan^{-1}x-x}{2x-\sin^{-1}x}=\lim_{x\to 0}\frac{\frac{2}{1+x^2}-1}{2(1)-\frac{1}{\sqrt{1-x^2}}}=\frac{2-1}{2-1}=\frac{1}{1}=1$$

(l) $\lim_{x\to 0}\frac{\ln\sec 2x}{\ln\sec x}=4$

Solution : $\lim_{x\to 0}\frac{\ln\sec 2x}{\ln\sec x}=\frac{\ln\sec 0}{\ln\sec 0}=\frac{\ln 1}{\ln 1}=\frac{0}{0}$. Applying the L'Hospital rule, we get

$$\lim_{x\to 0}\frac{\ln\sec 2x}{\ln\sec x}=\lim_{x\to 0}\frac{1}{\sec 2x}\left(2\sec 2x\tan 2x\right)\div\frac{1}{\sec x}\left(\sec x\tan x\right)$$

$=\lim_{x\to 0}\frac{2\tan 2x}{\tan x}=\frac{2\tan 0}{\tan 0}=\frac{0}{0}$.Thus applying the L'Hospital rule again

$$=\lim_{x\to 0}\frac{2.2\sec^2 2x}{\sec^2 x}=\frac{4\sec^2 0}{\sec^2 0}=\frac{4}{(1)^2}=4$$

(m) $\lim_{x\to 0}\left[\ln\cos x\right]/x^2=-1/2$

Solution : $\lim_{x\to 0}\frac{\ln\cos x}{x^2}=\frac{\ln\cos 0}{0^2}=\frac{\ln 1}{0}=\frac{0}{0}$. Applying the L'Hospital rule, we get

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$$\lim_{x\to 0}\frac{\ln\cos x}{x^2}=\lim_{x\to 0}\frac{1}{\cos x}\left(-\sin x\right)\div(2x)=\lim_{x\to 0}\frac{-\tan x}{2x}=\frac{0}{0}$$

Thus applying the L'Hospital rule again

$$=\lim_{x\to 0}\frac{-\sec^2 x}{2}=\frac{-\sec^2 0}{2}=\frac{-1}{2}$$

(n) $\lim_{x\to 0}\frac{\cos 2x-\cos x}{\sin^2 x}=-\frac{3}{2}$

Solution: $\lim_{x\to 0}\frac{\cos 2x-\cos x}{\sin^2 x}=\frac{\cos 0-\cos 0}{\sin^2 0}=\frac{1-1}{0}=\frac{0}{0}$.

Thus applying the L'Hospital rule, we get

$$\lim_{x\to 0}\frac{\cos 2x-\cos x}{\sin^2 x}=\lim_{x\to 0}\frac{-2\sin 2x+\sin x}{2\sin x\cos x}=\lim_{x\to 0}\frac{-2\sin 2x+\sin x}{\sin 2x}=\frac{0}{0}$$

Thus applying the L'Hospital rule again

$$=\lim_{x\to 0}\frac{-4\cos 2x+\cos x}{2\cos 2x}=\frac{-4\cos 0+\cos 0}{2\cos 0}=\frac{-4+1}{2(1)}=-\frac{3}{2}$$

(o) $\lim_{x\to +\infty}\frac{\ln x}{\sqrt{x}}=0$

Solution: $\lim_{x\to +\infty}\frac{\ln x}{\sqrt{x}}=\frac{\ln\infty}{\sqrt{\infty}}=\frac{\infty}{\infty}$. Thus applying the L'Hospital rule, we get

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}=\lim_{x\to\infty}\frac{1/x}{(1/2)x^{-1/2}}=\lim_{x\to\infty}\frac{2\sqrt{x}}{x}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=\frac{2}{\infty}=0$$

(p) $\lim_{x\to\frac{\pi}{2}}\frac{\csc 6x}{\csc 2x}=\frac{1}{3}$

Solution: $\lim_{x\to\frac{\pi}{2}}\frac{\csc 6x}{\csc 2x}=\frac{\csc 3\pi}{\csc\pi}=\frac{\infty}{\infty}$. Now, $\lim_{x\to\frac{\pi}{2}}\frac{\csc 6x}{\csc 2x}=\lim_{x\to\frac{\pi}{2}}\frac{\sin 2x}{\sin 6x}=\frac{\sin\pi}{\sin 3\pi}=\frac{0}{0}$

Thus applying the L'Hospital rule, we get

$$\lim_{x\to\frac{\pi}{2}}\frac{\csc 6x}{\csc 2x}=\lim_{x\to\frac{\pi}{2}}\frac{\sin 2x}{\sin 6x}=\lim_{x\to\frac{\pi}{2}}\frac{2\cos 2x}{6\cos 6x}=\frac{2\cos\pi}{6\cos 3\pi}=\frac{2(-1)}{6(-1)}=\frac{1}{3}$$

(q) $\lim_{x\to +\infty}\frac{5x+2\ln x}{x+3\ln x}=5$

Solution: $\lim_{x\to +\infty}\frac{5x+2\ln x}{x+3\ln x}=\frac{\infty}{\infty}$. Thus applying the L'Hospital rule, we get

$$\lim_{x\to +\infty}\frac{5x+2\ln x}{x+3\ln x}=\lim_{x\to +\infty}\frac{5+(2/x)}{1+(3/x)}=\frac{5+0}{1+0}=5$$

(r) $\lim_{x\to +\infty}\frac{x^4+x^2}{e^x+1}=0$

Solution: $\lim_{x\to +\infty}\left(x^4+x^2\right)/\left(e^x+1\right)=\infty/\infty=$ Thus, applying the L'Hospital rule, we get

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$\lim\limits_{x\to+\infty}\dfrac{x^4+x^2}{e^x+1}=\lim\limits_{x\to+\infty}\dfrac{4x^3+2x}{e^x}=\dfrac{\infty}{\infty}$. Applying again the L'Hospital rule, we get

$=\lim\limits_{x\to+\infty}\dfrac{12x^2+2}{e^x}=\dfrac{\infty}{\infty}$. Applying again the L'Hospital rule, we get

$=\lim\limits_{x\to+\infty}\dfrac{24x}{e^x}=\dfrac{\infty}{\infty}$. Applying the L'Hospital rule once again, we get

$=\lim\limits_{x\to+\infty}\dfrac{24}{e^x}=\dfrac{24}{\infty}=0$

(s) $\lim\limits_{x\to0^+}\dfrac{\ln\cot x}{e^{\csc^2 x}}=0$ $\quad$ [NOTE: $\csc 0=\infty=\cot 0$ and $\ln\infty=\infty$]

Solution: $\lim\limits_{x\to0^+}\dfrac{\ln\cot x}{e^{\csc^2 x}}=\dfrac{\ln(\cot 0)}{e^{\csc^2 0}}=\dfrac{\infty}{\infty}$. Thus, applying the L'Hospital rule, we get

$$\lim_{x\to0^+}\frac{\ln\cot x}{e^{\csc^2 x}}=\lim_{x\to0^+}\frac{1}{\cot x}(-\csc^2 x)\div e^{\csc^2 x}\left[2\csc x.(-\csc x.\cot x)\right]$$

$$=\lim_{x\to0^+}\frac{-\csc^2 x}{\cot x.e^{\csc^2 x}\left[2\csc x.(-\csc x.\cot x)\right]}=\lim_{x\to0^+}\frac{\tan^2 x}{2e^{\csc^2 x}}=\frac{0}{2e^{\infty}}=\frac{0}{\infty}=0$$

(t) $\lim\limits_{x\to+\infty}\dfrac{e^x+3x^3}{4e^x+2x^2}=\dfrac{1}{4}$

Solution: $\lim\limits_{x\to+\infty}\dfrac{e^x+3x^3}{4e^x+2x^2}=\dfrac{\infty}{\infty}$. Applying the L'Hospital rule, we get

$\lim\limits_{x\to+\infty}\dfrac{e^x+3x^3}{4e^x+2x^2}=\lim\limits_{x\to+\infty}\dfrac{e^x+9x}{4e^x+4x}=\dfrac{\infty}{\infty}$. Applying again the L'Hospital rule, we get

$=\lim\limits_{x\to+\infty}\dfrac{e^x+9}{4e^x+4}=\dfrac{\infty}{\infty}$. Applying the L'Hospital rule once agian, we get

$=\lim\limits_{x\to+\infty}\dfrac{e^x}{4e^x}=\lim\limits_{x\to+\infty}\dfrac{1}{4}=\dfrac{1}{4}$

(u) $\lim\limits_{x\to0}\left(e^x-1\right)\cot x=1$ $\quad$ [NOTE: See the change in the problem]

Solution: $\lim\limits_{x\to0}\left(e^x-1\right)\cot x=\left(e^0-1\right)\cot 0=(1-1)\cot 0=0\times\infty$.

$\lim\limits_{x\to0}\left(e^x-1\right)\cot x=\lim\limits_{x\to0}\dfrac{\left(e^x-1\right)}{\tan x}=\dfrac{0}{0}$.

Now, applying the L'Hospital rule, we get $\lim\limits_{x\to0}\dfrac{e^x-0}{\sec^2 x}=\dfrac{1-0}{1}=1$

(v) $\lim\limits_{x\to-\infty}x^2e^x=0$

Solution: $\lim\limits_{x\to-\infty}x^2e^x=\infty\times e^{-\infty}=0\times\infty$.

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Thus, $\lim_{x\to-\infty} x^2e^x = \lim_{x\to-\infty}\frac{x^2}{e^{-x}} = \frac{\infty}{\infty}$.

Now, applying the L'Hospital rule, we get

$$= \lim_{x\to-\infty}\frac{2x}{-e^{-x}} = \frac{\infty}{\infty}.$$ Applying the L'Hospital rule once again, we get

$$= \lim_{x\to-\infty}\frac{2}{e^{-x}} = \lim_{x\to-\infty} 2e^x = 2\times e^{-\infty} = 2(0) = 0$$

(w) $\lim_{x\to0} x\csc x = 1$

Solution : $\lim_{x\to0} x\csc x = 0\times\infty$

But, $\lim_{x\to0} x\csc x = \lim_{x\to0}\frac{x}{\sin x} = \frac{0}{0}$. Now, applying the L'Hospital rule, we get

$$= \lim_{x\to0}\frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

(x) $\lim_{x\to1}\csc \pi x\ln x = -\frac{1}{\pi}$

Solution : $\lim_{x\to1}\csc \pi x\ln x = \csc\pi\times\ln 1 = \infty\times 0$

But, $\lim_{x\to1}\csc \pi x\ln x = \lim_{x\to1}\frac{\ln x}{\sin \pi x} = \frac{0}{0}$. Now, applying the L'Hospital rule, we get

$$= \lim_{x\to1}\frac{1}{x\pi\cos\pi x} = \frac{1}{1.\pi.\cos\pi} = \frac{1}{-\pi} = -\frac{1}{\pi}$$

(y) $\lim_{x\to\pi/2} e^{-\tan x}\sec^2 x = 0$

Solution : $\lim_{x\to\pi/2} e^{-\tan x}\sec^2 x = \lim_{x\to\pi/2}\frac{\sec^2 x}{e^{\tan x}} = \frac{\infty}{\infty}$ NOTE : $\tan(\pi/2) = \sec(\pi/2) = \infty$

Now, applying the L'Hospital rule, we get

$$\lim_{x\to\pi/2} e^{-\tan x}\sec^2 x = \lim_{x\to\pi/2}\frac{2\sec x.\sec x\tan x}{e^{\tan x}\sec^2 x} = \lim_{x\to\pi/2}\frac{2\tan x}{e^{\tan x}}$$

As x tends to ∞, $e^{\tan x}$ tends to ∞ faster than $\tan x$. Thus,

$$\lim_{x\to\pi/2} e^{-\tan x}\sec^2 x = \lim_{x\to\pi/2}\frac{2\tan x}{e^{\tan x}} = 0$$

(z) $\lim_{x\to0}\left(x-\sin^{-1}x\right)\csc^3 x = -\frac{1}{6}$

Solution : $\lim_{x\to0}\left(x-\sin^{-1}x\right)\csc^3 x = 0\times\infty$

But, $\lim_{x\to0}\left(x-\sin^{-1}x\right)\csc^3 x = \lim_{x\to0}\frac{\left(x-\sin^{-1}x\right)}{\sin^3 x} = \frac{0}{0}$. Now, applying the L'Hospital rule,

$$= \lim_{x\to1}\frac{1}{x\pi\cos\pi x} = \frac{1}{1.\pi.\cos\pi} = \frac{1}{-\pi} = -\frac{1}{\pi}$$

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$(a`)\ \lim_{x\to 2}\left[\frac{4}{x^2-4}-\frac{1}{x-2}\right]=-\frac{1}{4}$

Solution : $\lim_{x\to 2}\left[\frac{4}{x^2-4}-\frac{1}{x-2}\right]=\frac{4}{0}-\frac{1}{0}=\infty-\infty$

Now, $\lim_{x\to 2}\left[\frac{4}{x^2-4}-\frac{1}{x-2}\right]=\lim_{x\to 2}\left[\frac{4}{(x-2)(x+2)}-\frac{1}{x-2}\right]=\lim_{x\to 2}\frac{1}{(x-2)}\left[\frac{4}{(x+2)}-1\right]$

$$=\lim_{x\to 2}\frac{1}{(x-2)}\left[\frac{4-x-2}{(x+2)}\right]=\lim_{x\to 2}\frac{1}{(x-2)}\left[\frac{2-x}{(x+2)}\right]=\lim_{x\to 2}\frac{1}{(x-2)}\left[\frac{-(x-2)}{(x+2)}\right]$$

$$=\lim_{x\to 2}\frac{-1}{(x+2)}=-\frac{1}{4}$$

$(b')\ \lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=0$

Solution : $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\frac{1}{0}-\frac{1}{0}=\infty-\infty$. Thus,

$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left(\frac{\sin x-x}{x\sin x}\right)=\frac{0}{0}$

Now, applying the L'Hospital rule, we get

$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim_{x\to 0}\left(\frac{\sin x-x}{x\sin x}\right)=\lim_{x\to 0}\left(\frac{\cos x-1}{x\cos x+\sin x}\right)=\frac{0}{0}.$

Applying the L'Hospital rule once again, we get

$$=\lim_{x\to 0}\left(\frac{-\sin x}{-x\sin x+\cos x+\cos x}\right)=\frac{0}{0+1+1}=0$$

$(c')\ \lim_{x\to \pi/2}\left(\sec^3 x-\tan^3 x\right)=\infty$

Solution : $\lim_{x\to \pi/2}\left(\sec^3 x-\tan^3 x\right)=\infty-\infty$. Now,

$\lim_{x\to \pi/2}\left(\sec^3 x-\tan^3 x\right)=\lim_{x\to \pi/2}\left(\frac{1}{\cos^3 x}-\frac{\sin^3 x}{\cos^3 x}\right)=\lim_{x\to \pi/2}\left(\frac{1-\sin^3 x}{\cos^3 x}\right)=\frac{0}{0}$

Thus, applying the L'Hospital rule, we get

$\lim_{x\to \pi/2}\left(\sec^3-\tan^3 x\right)=\lim_{x\to \pi/2}\frac{0-3\sin^2 x.\cos x}{3\cos^2 x.(-\sin x)}=\lim_{x\to \pi/2}\frac{-\sin x}{-\cos x}=\frac{\sin(\pi/2)}{\cos(\pi/2)}=\frac{1}{0}=\infty$

$(d')\ \lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)=-\frac{1}{2}$

Solution : $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)=\frac{1}{\ln 1}-\frac{1}{1-1}=\frac{1}{0}-\frac{1}{0}=\infty-\infty$. Now,

$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)=\lim_{x\to 1}\left(\frac{x-1-x\ln x}{(x-1)\ln x}\right)=\frac{0}{0}$

Thus, applying the L'Hospital rule, we get

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$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right)=\lim_{x\to 1}\left(\frac{1-0-x(1/x)-\ln x.1}{\ln x.1+(x-1)/x}\right)=\lim_{x\to 1}\left(\frac{-\ln x}{\ln x+1-x^{-1}}\right)=\frac{0}{0}$$

$$=\lim_{x\to 1}\left(\frac{-1/x}{1/x+0+x^{-2}}\right)=\frac{-1}{1+1}=-\frac{1}{2}$$

(e`) $\lim_{x\to 0}\left(\frac{4}{x^2}-\frac{2}{1-\cos x}\right)=-\frac{1}{3}$

Solution : $\lim_{x\to 0}\left(\frac{4}{x^2}-\frac{2}{1-\cos x}\right)=\frac{4}{0}-\frac{2}{1-\cos 0}=\frac{4}{0}-\frac{2}{1-1}=\infty-\infty$. Now,

$$\lim_{x\to 0}\left(\frac{4}{x^2}-\frac{2}{1-\cos x}\right)=\lim_{x\to 0}\left(\frac{4-4\cos x-2x^2}{x^2(1-\cos x)}\right)=\frac{0}{0}.$$

Thus, applying the L'Hospital rule, we get

$$=\lim_{x\to 0}\left(\frac{0+4\sin x-4x}{x^2(\sin x)+(1-\cos x)(2x)}\right)=\frac{0}{0}$$

Applying once again the L'Hospital rule, we get

$$=\lim_{x\to 0}\left(\frac{0+4\cos x-4}{x^2\cos x+2x\sin x+2(1-\cos x)+2x\sin x}\right)$$

$$=\lim_{x\to 0}\left(\frac{4\cos x-4}{x^2\cos x+4x\sin x+2(1-\cos x)}\right)=\frac{0}{0}$$

Applying once again the L'Hospital rule, we get

$$=\lim_{x\to 0}\left(\frac{-4\sin x-0}{-x^2\sin x+2x\cos x+4x\cos x+4\sin x+2(0+\sin x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{0-4\sin x}{-x^2\sin x+6x\cos x+6\sin x}\right)=\frac{0}{0}$$

Applying once again the L'Hospital rule, we get

$$=\lim_{x\to 0}\left(\frac{0-4\cos x}{-x^2\cos x-2x\sin x+6\cos x-6x\sin x+6\cos x}\right)=\frac{-4}{12}=-\frac{1}{3}$$

(f`) $\lim_{x\to\infty}\left(\frac{\ln x}{x}-\frac{1}{\sqrt{x}}\right)=0$

Solution : As $x\to\infty$, $\ln x$ also tends to ∞ but not faster than x. Thus, $\lim_{x\to\infty}\left(\frac{\ln x}{x}\right)\to 0$.

Similarly, $\lim_{x\to\infty}\left(\frac{1}{\sqrt{x}}\right)\to 0$. Hence, $\lim_{x\to\infty}\left(\frac{\ln x}{x}-\frac{1}{\sqrt{x}}\right)=0-0=0$

(g`) $\lim_{x\to 0}x^x=1$

Solution : $\lim_{x\to 0^+}x^x=0^0$. Let $y=x^x\Rightarrow \ln y=\ln\left(x^x\right)=x\ln x$. Thus,

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$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} x\ln x = 0\times(-\infty)$. Now,

$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} x\ln x = \lim\limits_{x\to 0} \dfrac{\ln x}{x^{-1}} = \dfrac{\infty}{\infty}$. Thus, applying the L'Hospital rule, we get

$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{1/x}{-x^{-2}} = \lim\limits_{x\to 0} (-x) = 0$. Taking anti log on both sides, we obtain

$\lim\limits_{x\to 0} y = \lim\limits_{x\to 0} x^x = e^0 = 1$

(h`) $\lim\limits_{x\to 0} (\cos x)^{1/x} = 1$

Solution: $\lim\limits_{x\to 0} (\cos x)^{1/x} = 1^\infty$

Let $y = (\cos x)^{1/x} \Rightarrow \ln y = \ln(\cos x)^{1/x} = \dfrac{\ln\cos x}{x}$. Thus,

$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln\cos x}{x} = \dfrac{0}{0}$. Now, apply the L'Hospital rule, we get

$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln\cos x}{x} = \lim\limits_{x\to 0} \dfrac{-\sin x}{\cos x\,(1)} = \lim\limits_{x\to 0} (-\tan x) = 0.$

Taking anti log on both sides, we get

$\lim\limits_{x\to 0} \ln y = e^0 = 1$. Putting the value of y, we obtain: $\lim\limits_{x\to 0} (\cos x)^{1/x} = 1$

(i`) $\lim\limits_{x\to 0} (e^x + 3x)^{1/x} = e^4$

Solution: $\lim\limits_{x\to 0} (e^x + 3x)^{1/x} = 1^\infty$

Let $y = (e^x + 3x)^{1/x} \Rightarrow \ln y = \ln(e^x + 3x)^{1/x} = \dfrac{\ln(e^x + 3x)}{x}$. Thus,

$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln(e^x + 3x)}{x} = \dfrac{0}{0}$. Now, apply the L'Hospital rule, we get

$\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln(e^x + 3x)}{x} = \lim\limits_{x\to 0} \dfrac{e^x + 3}{(e^x + 3x)(1)} = \dfrac{e^0 + 3}{e^0 + 0} = \dfrac{1+3}{1+0} = 4.$

Taking anti log on both sides, we get

$\lim\limits_{x\to 0} \ln y = e^4$. Putting the value of y, we obtain: $\lim\limits_{x\to 0} (e^x + 3x)^{1/x} = e^4$

(j`) $\lim\limits_{x\to \infty} (1 - e^{-x})^{e^x} = e^{-1}$

Solution: $\lim\limits_{x\to \infty} (1 - e^{-x})^{e^x} = 1^\infty$

Let $y = (1 - e^{-x})^{e^x} \Rightarrow \ln y = e^x \ln(1 - e^{-x}) = \dfrac{\ln(1 - e^{-x})}{e^{-x}}$. Thus,

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$\lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln\left(1-e^{-x}\right)}{e^{-x}} = \dfrac{0}{0}$. Now, apply the L'Hospital rule, we get

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\left(0+e^{-x}\right)}{\left(1-e^{-x}\right)(-e^{-x})} = \lim_{x\to\infty} \frac{e^{-x}}{-e^{-x}\left(1-e^{-x}\right)} = \lim_{x\to\infty} \frac{1}{-\left(1-e^{-x}\right)} = -1.$$

Taking anti log on both sides, we get

$\lim\limits_{x\to 0} \ln y = e^{-1}$. Putting the value of y, we obtain : $\lim\limits_{x\to\infty} \left(1-e^{-x}\right)^{e^{x}} = e^{-1}$

$(k`)$ $\lim\limits_{x\to\pi/2} \left(\sin x - \cos x\right)^{\tan x} = e^{-1}$

Solution : $\lim\limits_{x\to\pi/2} \left(\sin x - \cos x\right)^{\tan x} = 1^{\infty}$

Let $y = \left(\sin x - \cos x\right)^{\tan x} \Rightarrow \ln y = \ln\left(\sin x - \cos x\right)^{\tan x} = \tan x \ln\left(\sin x - \cos x\right)$. Thus,

$\lim\limits_{x\to\pi/2} \ln y = \lim\limits_{x\to\pi/2} \tan x \ \ln(\sin x - \cos x) = 0\times\infty$. Now,

$$\lim_{x\to\pi/2} \ln y = \lim_{x\to\pi/2} \tan x \ \ln(\sin x - \cos x) = \lim_{x\to\pi/2} \frac{\ln(\sin x - \cos x)}{\cot x} = \frac{0}{0}$$

Apply the L'Hospital rule now, we get

$$\lim_{x\to\pi/2} \ln y = \lim_{x\to\pi/2} \frac{\left(\cos x + \sin x\right)}{(\sin x - \cos x).(-\csc^2 x)} = -1$$

NOTE : $\sin\left(\dfrac{\pi}{2}\right) = \csc\left(\dfrac{\pi}{2}\right) = 1$

$\cos\left(\dfrac{\pi}{2}\right) = \cot\left(\dfrac{\pi}{2}\right) = 0$

Taking anti log on both sides, we get

$\lim\limits_{x\to 0} \ln y = e^{-1}$. Putting the value of y, we obtain : $\lim\limits_{x\to\pi/2} \left(\sin x - \cos x\right)^{\tan x} = e^{-1}$

$(l`)$ $\lim\limits_{x\to\pi/2} \left(\tan x\right)^{\cos x} = 1$

Solution : $\lim\limits_{x\to\pi/2} \left(\tan x\right)^{\cos x} = \infty^{0}$

Let $y = \left(\tan x\right)^{\cos x} \Rightarrow \ln y = \ln\left(\tan x\right)^{\cos x} = \cos x \ \ln\left(\tan x\right)$. Thus,

$\lim\limits_{x\to\pi/2} \ln y = \lim\limits_{x\to\pi/2} \cos x \ \ln(\tan x) = 0\times\infty$. Now,

$$\lim_{x\to\pi/2} \ln y = \lim_{x\to\pi/2} \cos x \ \ln(\tan x) = \lim_{x\to\pi/2} \frac{\ln(\tan x)}{\sec x} = \frac{\infty}{\infty}.$$

NOTE : $\sin\left(\dfrac{\pi}{2}\right) = 1,\ \cos\left(\dfrac{\pi}{2}\right) = 0$

Apply the L'Hospital rule now, we get

$$\lim_{x\to\pi/2} \ln y = \lim_{x\to\pi/2} \frac{\sec^2 x}{\tan x.(\sec x \ \tan x)} = \lim_{x\to\pi/2} \frac{\sec x}{\tan^2 x} = \lim_{x\to\pi/2} \frac{\cos^2 x}{\cos x \sin^2 x} = \lim_{x\to\pi/2} \frac{\cos x}{\sin^2 x} = 0$$

Taking anti log on both sides, we get

$\lim\limits_{x\to 0} \ln y = e^{0} = 1$. Putting the value of y, we obtain : $\lim\limits_{x\to\pi/2} \left(\tan x\right)^{\cos x} = 1$

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$(m`)\ \lim_{x\to 1}(x)^{\tan\frac{\pi x}{2}} = e^{-2/\pi}$

Solution: $\lim_{x\to 1}(x)^{\tan\frac{\pi x}{2}} = 1^{\infty}$. Let $y = (x)^{\tan\frac{\pi x}{2}} \Rightarrow \ln y = \ln(x)^{\tan\frac{\pi x}{2}} = \tan\frac{\pi x}{2}\ln x$. Thus,

$\lim_{x\to 1}\ln y = \lim_{x\to 1}\left(\tan\frac{\pi x}{2}\ln x\right) = 0\times\infty$. Now,

$\lim_{x\to 1}\ln y = \lim_{x\to 1}\left(\tan\frac{\pi x}{2}\ln x\right) = \lim_{x\to 1}\frac{\ln(x)}{\cot\frac{\pi x}{2}} = \frac{0}{0}$. NOTE: $\cot\left(\frac{\pi}{2}\right) = 0 = \ln 1,\ \csc\left(\frac{\pi}{2}\right) = 1$

Apply the L'Hospital rule now, we get

$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{1}{x.\left(-\csc^2\frac{\pi x}{2}\right).\frac{\pi}{2}} = -\frac{2}{\pi}$

Taking anti log on both sides, we get

$\lim_{x\to 1}\ln y = e^{-2/\pi}$. Putting the value of y, we obtain: $\lim_{x\to 1}(x)^{\tan\frac{\pi x}{2}} = e^{-2/\pi}$

$(n`)\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

Solution: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = 1^{\infty}$

Let $y = \left(1+\frac{1}{x}\right)^{x} \Rightarrow \ln y = \ln\left(1+\frac{1}{x}\right)^{x} = x\ \ln\left(1+\frac{1}{x}\right)$. Thus,

$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ \ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} = \frac{0}{0}$.

Thus, applying the L'Hospital rule, we get: $\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\left(0-x^{-2}\right)}{\left(1+\frac{1}{x}\right)\left(-x^{-2}\right)} = \lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)} = 1$

Taking the anti log and putting the value of y, we get: $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

$(o`)\ \lim_{x\to\infty}\frac{2^x}{3^x} = 0$

Solution: $\lim_{x\to\infty}\left(2^x / 3^x\right) = \infty/\infty$. Apply the L'Hospital rule, we get

$\lim_{x\to\infty}\frac{2^x}{3^x} = \lim_{x\to\infty}\frac{2^x.\ln 2}{3^x.\ln 3} = \frac{\infty}{\infty}$. Thus, process of applying the L'Hospital rule will never end.

Now, see the technique: As $x\to\infty$, 2^x and 3^x both tends to infinity but 3^x moves to

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inf inity faster than 2^x. For example, if $x = 20$ then $2^{20} = 1,048,576$ and $3^{20} = 3,486,784,401$.

Clearly, if x tends to infinity, 3^x tends to inf inity faster than 2^x. Thus, $\lim_{x\to\infty} \frac{2^x}{3^x} = 0$.

(p`) $\lim_{x\to 0} \frac{e^{-3/x}}{x^2} = 0$

Solution : $\lim_{x\to 0} \frac{e^{-3/x}}{x^2} = \frac{0}{0}$. Thus, applying the L'Hospital rule, we get

$\lim_{x\to 0} \frac{e^{-3/x}}{x^2} = \lim_{x\to 0} \frac{e^{-3/x}\left(3/x^2\right)}{2x} = \lim_{x\to 0} \frac{3e^{-3/x}}{2x^3} = \frac{0}{0}$. Thus, this way, the process of applying the L'Hospital rule, will never end. So we have to look at the solution in another way.

We see that as x tends to zero $e^{-3/x} = e^{-\infty}$ and this tends to zero faster than x^2. Thus,

$\lim_{x\to 0} \frac{e^{-3/x}}{x^2} = 0$

(q`) $\lim_{x\to 0} \frac{e^x\left(1-e^x\right)}{(1+x)(1-x)} = \lim_{x\to 0} \frac{e^x}{(1+x)} . \lim_{x\to 0} \frac{\left(1-e^x\right)}{(1-x)} = 0$ NOTE : See the change

Solution : $\lim_{x\to 0} \frac{e^x}{(1+x)} = \frac{e^0}{1+0} = 1$. Now, $\lim_{x\to 0} \frac{\left(1-e^x\right)}{(1-x)} = \frac{1-1}{1-0} = 0$. Thus,

$\lim_{x\to 0} \frac{e^x\left(1-e^x\right)}{(1+x)(1-x)} = \lim_{x\to 0} \frac{e^x}{(1+x)} . \lim_{x\to 0} \frac{\left(1-e^x\right)}{(1-x)} = 1(0) = 0$

7. Determine the horizontal and vertical asymptotes for the following functions:

NOTE: It may be noted that any curve may have at most two asymptotes. They may be (i) Horizontal and vertical asymptotes (ii) Horizontal and oblique asymptotes (iii) Vertical and oblique asymptotes

(i) $y = (x-2)^2 / x^2$

Solution : $y = \frac{\left(x^2 - 4x + 4\right)}{x^2} = 1 - \frac{4}{x} + \frac{4}{x^2}$. Now,

$\lim_{x\to\infty} y = \lim_{x\to\infty}\left(1 - \frac{4}{x} + \frac{4}{x^2}\right) = 1 - 0 + 0 = 1$. Thus, $y = 1$ is Horizontal Asymptote of the curve.

You may observe that if x tends to zero then y tends to inf inity. Thus, $x = 0$ is the vertical asymptote of the curve. The graph is shown in figure (i).

(ii) $x^2y^2 = 12(x-3)$

Solution : $x^2y^2 = 12(x-3) \Rightarrow y^2 = \frac{(x-3)}{x^2} = \frac{x}{x^2} - \frac{3}{x^2} = \frac{1}{x} - \frac{3}{x^2}$

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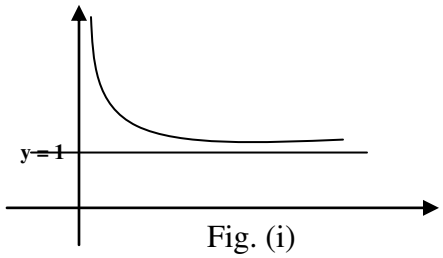

Fig. (i)

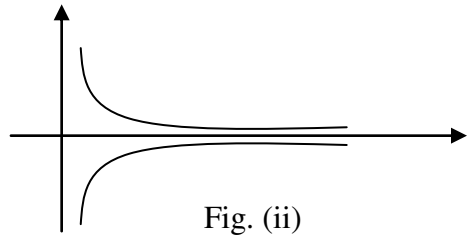

Fig. (ii)

Now, asymptote of the curve. The graph is shown in figure (ii).

(iii) $2xy = x^2 + 3$

Solution : Given $2xy = x^2 + 3 \Rightarrow y = (x/2) + (3/2x)$

Clearly, if $x \to \infty$ then $y \to \infty$. Thus, there is no Horizontal Asymptote.

Similarly, if $y \to \infty$ then $x \to \infty$.Hence, there is no Vertical Asymptote.

(iv) $x^2(x-y)^2 + a^2(x^2 - y^2) = a^2xy$

Solution : Simplifying, $x^4 - 2x^3y - 2x^2y^2 + a^2x^2 - a^2y^2 - a^2xy = 0$

Here the term containing highest power of x is x^4. It's coefficient is 1.

Put this equal to zero, we have $1 = 0$ which is not possible. Hence, there is no horizontal asymptote.

Similarly, the term containing highest power of y is y^2. It's coefficient is $-2x^2 - a^2$.

Put this equal to zero, we have $-2x^2 - a^2 = 0 \Rightarrow x = \pm ia/\sqrt{2}$ which is a complex number.

Hence, there is no vertical asymptote.

(v) $(x-y)^2(x^2+y^2) - 10(x-y)x^2 + 12y^2 + 2x + y = 0$

Solution : Simplifying, $x^4 - 2x^3y - 2xy^3 + x^2y^2 + y^4 - 10x^3 + 10x^2y + 12y^2 + 2x + y = 0$

Here the term containing highest power of x is x^4. It's coefficient is 1.

Put this equal to zero, we have $1 = 0$ which is not possible. Hence, there is no horizontal asymptote.

Similarly, the term containing highest power of y is y^4. It's coefficient is 1.

Put this equal to zero, we have $1 = 0$ which is not possible. Hence, there is no vertical asymptote.

(vi) $x^2y + xy^2 + xy + y^2 + 3x = 0$

Solution : Here the term containing highest power of x is x^2. It's coefficient is y.

Put this equal to zero, we have $y = 0$. This is the horizontal asymptote.

Similarly, the term containing highest power of y is y^2. It's coefficient is $1 + x$.

Put this equal to zero, we have $1 + x = 0$ or $x = -1$ which is vertical asymptote.

8. Find all asymptoteof the curve $2xy + 2y = (x-2)^2$

Solution : Given $2xy + 2y = (x-2)^2 \Rightarrow y = (x-2)^2/2(x+1)$

Clearly, if $x \to \infty$ then $y \to 0$. Thus, $y = 0$ is Horizontal Asymptote.

Similarly, if $y \to \infty$ then $x \to -1$.Hence, the Vertical Asymptote is $x = -1$.

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Chapter 7

FURTHER APPLICATIONS OF DERIVATIVES

WORKSHEET 07

1. Find the equations of tangent and normal to the curves (a) $x^2 - xy + y^2 = 7$ at the point (-1, 2) (b) y(x – 2) (x – 3) – x + 7 = 0 at the point where it cuts the x-axis.

Solution: (a) Given equation is $x^2 - xy + y^2 = 7$. Differentiating w.r.t x, we get

$$2x - xy` - y + 2yy` = 0 \Rightarrow y`(2y - x) = (y - 2x) \Rightarrow y` = (y - 2x)/(2y - x).$$

At (-1, 2), y` = 4/5 = m. Thus equation of tangent line is

$$y - y_1 = m(x - x_1) \Rightarrow y - 2 = 4(x + 1)/5 \Rightarrow 5y - 10 = 4x + 4 \Rightarrow 5y - 4x = 14.$$

The equation of normal is:

$$y - y_1 = -\frac{1}{m}(x - x_1) \Rightarrow y - 2 = -5(x + 1)/4 \Rightarrow 4y - 8 = -5x - 5 \Rightarrow 4y + 5x = 3.$$

(b) Given equation is y(x – 2) (x – 3) – x + 7 = 0. If this curve cuts the x-axis then y = 0. Thus, we have –x + 7 = 0 ⇒ x = 7. Now we have to find the equation of tangent and normal to the given curve y(x – 2) (x – 3) – x + 7 = 0 that passes through the point (7, 0). Now,

$$y(x-2)(x-3) - x + 7 = 0 \Rightarrow y = \frac{(x-7)}{(x^2 - 5x + 6)} \Rightarrow y` = \frac{(x^2 - 5x + 6).1 - (x-7)(2x-5)}{(x^2 - 5x + 6)^2}$$

At (7, 0), y` = 20/400 = 1/20 = m.

Thus equation of tangent line is

$$y - y_1 = m(x - x_1) \Rightarrow y - 0 = (x - 7)/20 \Rightarrow 20y - x + 7 = 0$$

The equation of normal is:

$$y - y_1 = (-1/m)(x - x_1) \Rightarrow y - 0 = -20(x - 7) \Rightarrow y + 20x = 140$$

2. Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x – axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

Solution: If the given curve crosses the x-axis then y = 0 ⇒ $x^2 = 7$ ⇒ $x = \pm\sqrt{7}$. Thus two points are ($\sqrt{7}$, 0) and (-$\sqrt{7}$, 0). Now, $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{2x + y}{x + 2y} = m$

At ($\sqrt{7}$, 0), m_1 = -2 and at (-$\sqrt{7}$, 0), m_2 = -2. Since, $m_1 = m_2$ hence two tangents at ($\sqrt{7}$, 0) and (-$\sqrt{7}$, 0) are parallel. The common slope of these tangents is -2.

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3. At what points of the curve $y = 2x^3 - 3x^2 - 2x + 4 \Rightarrow$ are the tangents parallel to the line $10x - y + 7 = 0$. Find the equation of the normal at each of these points. Also find the angle between the two curves: $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$.

Solution: (a) Given curve is $y = 2x^3 - 3x^2 - 2x + 4 \Rightarrow y` = 6x^2 - 6x - 2 = m_1$. This is the slope of the given curve. Now the slope of the line $10x - y + 7 = 0$ is $m_2 = -(10/-1) = 10$.
Now, $m_1 = m_2 \Rightarrow 6x^2 - 6x - 2 = 10 \Rightarrow 6x^2 - 6x - 12 = 0 \Rightarrow x^2 - x - 2 = 0 \Rightarrow x = -2, 1$.
Put $x = 1 \Rightarrow y = 1$ and when $x = -2$, $y = -20$. Thus, points are (1, 1) and (-2, -20) where the curve $y = 2x^3 - 3x^2 - 2x + 4$ and the line $10x - y + 7 = 0$ are parallel.
The equation of normal at (1, 1) to the given curve is $y - 1 = -(x - 1)/10$
$\Rightarrow 10y - 10 = -x + 1 \Rightarrow x + 10y = 11$.
The equation of normal at (-2, -20) to the given curve is $y + 2 = -(x + 20)/10$
$\Rightarrow 10y + 20 = -x + 20 \Rightarrow x + 10y = 0$.
(b) Adding two equations $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$, we get
$2x^2 = a^2(1 + \sqrt{2}) \Rightarrow x^2 = a^2(\sqrt{2} + 1)/2 \Rightarrow y^2 = x^2 - a^2 = a^2(\sqrt{2} - 1)/2$.
Therefore, $x^2.y^2 = a^4/4 \Rightarrow xy = a^2/2$.
Now, the slope of the curve $x^2 - y^2 = a^2$ is $2x - 2yy` = 0 \Rightarrow y` = x/y = m_1$.
The slope of the curve $x^2 + y^2 = a^2\sqrt{2}$ is $2x + 2yy` = 0 \Rightarrow y` = -x/y = m_2$.
The angle between two curves is:

$$\theta = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = \tan^{-1}\left(\frac{x}{y} - \frac{-x}{y}\right) \div \left(1 + \frac{-x^2}{y^2}\right) = \tan^{-1}\left(\frac{2x}{y}\right) \div \left(\frac{y^2 - x^2}{y^2}\right)$$

$$\theta = \tan^{-1}\left(\frac{2x}{y} \times \frac{-y^2}{x^2 - y^2}\right) = \tan^{-1}\left(\frac{-2xy}{x^2 - y^2}\right) = \tan^{-1}\left(\frac{-a^2/2}{a^2}\right) = \tan^{-1}\left(\frac{-1}{2}\right)$$

4. Find the angle between two curves: $x^2 - y^2 = a^2$ and $x^2 + y^2 = \sqrt{2}\, a^2$

Solution: See part (b) of Q (3).

5. Show that parabolas $y^2 = 4ax$ and $2x^2 = ay$ intersect at an angle $\tan^{-1}(3/5)$.

Solution: From the first equation, $x = y^2/4a$. Putting this value of x in the second equation, we get

$$2\left(\frac{y^2}{4a}\right)^2 = ay \Rightarrow \frac{y^4}{8a^2} = ay \Rightarrow y^4 = 8a^3 y. \text{ Dividing by } y,\ y^3 = 8a^3 \Rightarrow y = 2a$$

Put this value of y in $x = y^2/4a$, we get $x = a$. Thus, point of intersection between two given curves is (a, 2a).
Now, differentiating the first equation, we get: $2yy` = 4a \Rightarrow y` = 2a/y = m_1$.
Differentiating the second equation, we get: $4x = ay` \Rightarrow y` = 4x/a = m_2$.
At (a, 2a): $m_1 = 1$ and $m_2 = 4$. Now, the angle between two curves is

$$\theta = \tan^{-1}\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right) = \tan^{-1}\left(\frac{4 - 1}{1 + 4}\right) = \tan^{-1}\left(\frac{3}{5}\right)$$

6. Prove that curves $x^2/a^2 + y^2/b^2 = 1$ and $x^2/c^2 + y^2/d^2 = 1$ will cut orthogonally if $a - b = c - d$.

Solution: Since, $a - b = c - d \Rightarrow a + c = b + d$ (1) and $a - c = b - d$ (2)
Now solving the two equations simultaneously, we get

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$$x^2 = \frac{a^2c^2\left(d^2-b^2\right)}{\left(a^2d^2-b^2c^2\right)} \text{ and } y^2 = \frac{b^2d^2\left(a^2-c^2\right)}{\left(a^2d^2-b^2c^2\right)}$$

Differentiating both equations and simplifying, we get

$$y` = \frac{-b^2x}{a^2y} = m_1 \text{ and } y` = \frac{-d^2x}{c^2y} = m_2$$

$$m_1m_2 = \left(\frac{-b^2x}{a^2y}\right)\left(\frac{-d^2x}{c^2y}\right) = \frac{b^2d^2x^2}{a^2c^2y^2} = \frac{b^2d^2}{a^2c^2}\cdot\frac{a^2c^2\left(d^2-b^2\right)}{\left(a^2d^2-b^2c^2\right)}\cdot\frac{\left(a^2d^2-b^2c^2\right)}{b^2d^2\left(a^2-c^2\right)} = \frac{\left(d^2-b^2\right)}{\left(a^2-c^2\right)}$$

$$= \frac{(d-b)(d+b)}{(a-c)(a+c)} = -1 \text{ using equations (1) and (2).}$$

Since the product of two slopes is -1 hence, two curves intersect each other orthogonally under the condition that a – b = c – d.

7. Find the lengths of tangent, normal, sub-tangent and sub-normal to the curve:

$$x = a(t + \sin t),\ y = a(1 - \cos t)$$

Solution: We know that: Length of tangent is $y\sqrt{1+(dx/dy)^2}$

Length of normal is $y\sqrt{1+(dy/dx)^2}$

Length of sub-tangent is = y(dx/dy) and length of sub-normal is = y(dy/dx)

From given equations,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{a(0+\sin t)}{a(1+\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\cos^2\frac{t}{2}} = \tan\frac{t}{2} \qquad \Rightarrow \frac{dx}{dy} = \cot\frac{t}{2}$$

Thus length of tangent is:

$$y\sqrt{1+(dx/dy)^2} = a(1-\cos t)\sqrt{1+\cot^2\frac{t}{2}} = a.2\sin^2\frac{t}{2}\csc\frac{t}{2} = 2a\sin\frac{t}{2}$$

The length of normal is: $y\sqrt{1+(dy/dx)^2} = a(1-\cos t)\sqrt{1+\tan^2\frac{t}{2}} = 2a\sin^2\frac{t}{2}\sec\frac{t}{2}$

Length of sub-tangent is:

$$y(dx/dy) = a(1-\cos t).\cot\frac{t}{2} = a.2\sin^2\frac{t}{2}.\cot\frac{t}{2} = 2a\sin\frac{t}{2}\cos\frac{t}{2} = a\sin t$$

Length of sub-normal is: $y(dy/dx) = a(1-\cos t)\tan\frac{t}{2}$

8. For the curve x = a $\cos^3\theta$, y = a $\sin^3\theta$, show that length of tangent is y cosec θ. Also find the length of: sub-tangent, normal and sub-normal.

Solution: $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{a.3\sin^2\theta.\cos\theta}{a.3\cos^2\theta(-\sin\theta)} = -\tan\theta \Rightarrow \frac{dx}{dy} = -\cot\theta$

Thus length of tangent is: $y\sqrt{1+(dx/dy)^2} = y\sqrt{1+\cot^2\theta} = y.\sqrt{\csc^2\theta} = y\csc\theta$

The length of normal is: $y\sqrt{1+(dy/dx)^2} = y\sqrt{1+\tan^2\theta} = y\sec\theta$

Length of sub-tangent is: $y(dx/dy) = |-y\cot\theta|$

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Length of sub-normal is: $y(dy/dx) = |-y\tan\theta|$

9. For the curve x = a(ln cot θ/2 – cos θ), y = a sin θ, find the lengths of sub-tangent and sub-normal at the point θ = π/4.

Solution : $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$. Now, $\frac{dy}{d\theta} = a\cos\theta$ and

$$\frac{dx}{d\theta} = a\left[\frac{1}{\cot\frac{\theta}{2}}\left(-\csc^2\frac{\theta}{2}\right) + \sin\theta\right] = a\left[-\frac{1}{\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} + \sin\theta\right] = a\left[\frac{-2}{\sin\theta} + \sin\theta\right] = a\left[\frac{-2+\sin^2\theta}{\sin\theta}\right]$$

At $\theta = \frac{\pi}{4}$, $\frac{dy}{d\theta} = a\cos\frac{\pi}{4} = \frac{a}{\sqrt{2}}$ and $\frac{dx}{d\theta} = a\left[\frac{-2+\frac{1}{2}}{1/\sqrt{2}}\right] = a\left[\frac{-3}{2} \div \frac{1}{\sqrt{2}}\right] = \frac{-3a}{\sqrt{2}}$ and $y = \frac{a}{\sqrt{2}}$

$$\therefore \frac{dy}{dx} = \frac{a}{\sqrt{2}} \div \left(\frac{-3a}{\sqrt{2}}\right) = -\frac{1}{3}$$

Length of sub-tangent is: $y(dx/dy) = \left|\frac{a}{\sqrt{2}}(-3)\right| = \frac{3a}{\sqrt{2}}$

Length of sub-normal is: $y(dy/dx) = \left|\frac{a}{\sqrt{2}}\left(\frac{-1}{3}\right)\right| = \frac{a}{3\sqrt{2}}$

10. Find the radius of curvature at the given point of each of the following curves:

Solution: Radius of curvature may be found depending on the type of equation of the curve.

(1) Explicit equation: If the equation of the curve is given in the explicit form y = f(x), then

$$\rho = \left|\frac{\left[1+\{f'(x)\}^2\right]^{3/2}}{f''(x)}\right|.$$

(2) Implicit equation: If the equation of the curve is given in the implicit form, f(x, y) = 0, then

$$\rho = \left|\frac{\left[(f_x)^2 + (f_y)^2\right]^{3/2}}{f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2}\right|$$

(3) Parametric equations: If the equation of the curve is given in parametric form, that is; x = f(t), y = g(t), then

$$\rho = \frac{\left[\{f'(t)\}^2 + \{g'(t)\}^2\right]^{3/2}}{|f'(t)\cdot g''(t) - g'(t)\cdot f''(t)|}.$$

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(4) Polar equation of a curve: If the equation of the curve is given in polar form f(r, θ),

then
$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\cdot\frac{d^2r}{d\theta^2}\right|}.$$

(i) $y = x^2 - 5x - 6$ at $(4,-10)$: This is explicit form of the curve. Hence, $y` = 2x - 5$ and $y`` = 2$. At $(4,-10)$, $y` = 3$ and $y`` = 2$. Now,

$$\rho = \left|\frac{\left[1+\{f'(x)\}^2\right]^{3/2}}{f''(x)}\right| = \frac{1+(9)^{3/2}}{2} = \frac{1+27}{2} = 14$$

(ii) $9x^2 + 16y^2 = 180$ at $(2,3)$: This is implicit equation hence we use the formula

$$\rho = \left[(f_x)^2 + (f_y)^2\right]^{3/2} / \left[f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2\right]$$

Here, $f(x,y) = 9x^2 + 16y^2 - 180 \Rightarrow f_x = 18x,\ f_{xx} = 18,\ f_y = 32y,\ f_{yy} = 32$ and $f_{xy} = 0$.
At $(2,3)$, $f_x = 36,\ f_y = 96,\ f_{xx} = 18,\ f_{yy} = 32$, and $f_{xy} = 0$

$$\therefore \rho = \left|\frac{\left[(36)^2 + (96)^2\right]^{3/2}}{18(96)^2 - 2(36)(96)(0) + 32(36)^2}\right| = 5.19$$

(iii) $y = \sec x$ at $\left(\frac{\pi}{3}, 2\right)$: This is explicit form of the curve. Hence, $y` = \sec x \tan x$ and $y`` = \sec x.\sec^2 x + \tan x.\sec x \tan x$. At $x = \pi/3$, $y` = \sec \pi/3 = 2$ and $y`` = 2^3 + 2\left(\sqrt{3}\right)^2 = 14$

NOTE: $\sec \pi/3 = 2$ and $\tan \pi/3 = \sqrt{3}$

$$\rho = \left[1+\{f'(x)\}^2\right]^{3/2} / f''(x) = \frac{\left(1+(2)^2\right)^{3/2}}{14} = 0.8$$

(iv) $x^3 + y^3 = 3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$: This is implicit equation hence we use the formula

$$\rho = \left|\frac{\left[(f_x)^2 + (f_y)^2\right]^{3/2}}{f_{xx}(f_y)^2 - 2f_x f_y f_{xy} + f_{yy}(f_x)^2}\right|$$

Here, $f(x,y) = x^3 + y^3 - 3axy \Rightarrow f_x = 3x^2 - 3ay,\ f_{xx} = 6x,\ f_y = 3y^2 - 3ax,\ f_{yy} = 6y$ and $f_{xy} = -3a$. At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$, $f_x = \frac{9a^2}{4},\ f_y = \frac{9a^2}{4},\ f_{xx} = 9a,\ f_{yy} = 9a$, and $f_{xy} = -3a$

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$$\therefore \rho = \left| \frac{\left[\left(\frac{9a^2}{4}\right)^2 + \left(\frac{9a^2}{4}\right)^2\right]^{3/2}}{(9a)\left(\frac{9a^2}{4}\right)^2 - 2\left(\frac{9a^2}{4}\right)\left(\frac{9a^2}{4}\right)(-3a) + (9a)\left(\frac{9a^2}{4}\right)^2} \right| = \frac{2^3\left(\frac{9a^2}{4}\right)^2}{24\left(\frac{9a^2}{4}\right)^2} = \frac{1}{3}$$

11. Find the radius of curvature of the given curve at the specified point:

(i) $r = 4\sin 2\theta$ at $\left(2, \frac{1}{12}\pi\right)$: Given equation is the polar form, hence

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}\right|}$$

$r = 4\sin 2\theta,\ \frac{dr}{d\theta} = 8\cos 2\theta,\ \frac{d^2r}{d\theta^2} = -16\sin 2\theta.$ At $\theta = \pi/12: r = 4\sin\frac{4\pi}{12} = 4\sin\frac{\pi}{3} = 2\sqrt{3}$

$\frac{dr}{d\theta} = 8\cos 2\theta = 8\cos\frac{\pi}{3} = 8\left(\frac{1}{2}\right) = 4,\ \frac{d^2r}{d\theta^2} = -16\sin\frac{\pi}{3} = -16\left(\frac{\sqrt{3}}{2}\right) = -8\sqrt{3}$

$$\therefore\ \rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}\right|} = \frac{\left[\left(2\sqrt{3}\right)^2 + 4^2\right]^{3/2}}{\left(2\sqrt{3}\right)^2 + 2(4)^2 - \left(2\sqrt{3}\right)\left(-8\sqrt{3}\right)} = \frac{(28)^{3/2}}{12 + 32 + 48} = 1.6$$

(ii) $r = a\sec 2\theta;\ (a, 0),\ a > 0$: Given equation is the polar form, hence

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r \cdot \frac{d^2r}{d\theta^2}\right|}$$

$r = a\sec 2\theta,\ \frac{dr}{d\theta} = 2a\sec 2\theta\tan 2\theta,\ \frac{d^2r}{d\theta^2} = 2a\left[2\sec 2\theta.\sec^2 2\theta + \tan 2\theta.2\sec 2\theta\tan 2\theta\right].$

At $\theta = 0: r = a,\ \frac{dr}{d\theta} = 0,\ \frac{d^2r}{d\theta^2} = 4a$ NOTE : $\tan 0 = 0,\ \sec 0 = 1$

$$\rho = \frac{\left[a^2 + 0^2\right]^{3/2}}{|a^2 + 2(0)^2 - (a)(4a)|} = \frac{a^3}{3a^2} = \frac{a}{3}$$

12. Find the equation of the osculating circle to the curve y = ln x at the point (1, 0).

Solution: Since y = ln x ➔ y ` = 1/x and y `` = -1/x^2.

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At (1, 0), y` = 1 and y`` = -1. Now, the coordinates of the center of curvature are:

$$h = x - y`\frac{1+(y`)^2}{y``} = 1 - 1\frac{1+1}{-1} = +3, \quad k = y + \frac{1+(y`)^2}{y``} = 0 - 1\frac{1+1}{-1} = 2$$

Thus center of the curvature is: (3, 2).

Radius of curvature is $\rho = \left|\frac{\left[1+\{f'(x)\}^2\right]^{3/2}}{f''(x)}\right| = \left|\left[\frac{\left[1+(1)^2\right]^{3/2}}{-1}\right]\right| = 2\sqrt{2}$

The equation of Osculating circle is: $(x-h)^2 + (y-k)^2 = \rho^2$

$\Rightarrow \quad (x-3)^2 + (y-2)^2 = \left(2\sqrt{2}\right)^2 \Rightarrow x^2 + y^2 - 6x - 4y + 5 = 0$

13. Find the equation of the osculating circle to the curve $x^2/4 - y^2/9 = 1$ at the point (-2, 0).

Solution: Since

$\frac{x^2}{4} - \frac{y^2}{9} = 1 \Rightarrow \frac{2x}{4} - \frac{2yy`}{9} = 0 \Rightarrow y` = \frac{9x}{4y} = \frac{9}{4}xy^{-1}$. At $(-2,0)$, $y` = \infty$. Now,

$y`` = \frac{9}{4}\left(y^{-1}.1 + x(-1)y^{-2}y`\right) = \frac{9}{4}\left(\frac{1}{y} - \frac{xy`}{y^2}\right) = \frac{9}{4}\left(\frac{1}{y} - \frac{x}{y^2}.\frac{9x}{4y}\right)$. At $(-2,0)$, $y`` = \infty$.

Now, the coordinates of the center of curvature are:

$$h = x - y`\frac{1+(y`)^2}{y``} = -2 + 0 = -2, \quad k = y + \frac{1+(y`)^2}{y``} = 0 + 0 = 0$$

Thus center of the curvature is: (3, 2).

Radius of curvature is $\rho = \left|\frac{\left[1+\{f'(x)\}^2\right]^{3/2}}{f''(x)}\right| = \left|\left[\frac{\left[1+(1)^2\right]^{3/2}}{-1}\right]\right| = 2\sqrt{2}$

The equation of Osculating circle is: $(x-h)^2 + (y-k)^2 = \rho^2$

$\Rightarrow \quad (x-3)^2 + (y-2)^2 = \left(2\sqrt{2}\right)^2 \Rightarrow x^2 + y^2 - 6x - 4y + 5 = 0$

14. A tennis ball thrown straight up is $-16t^2 + 96t + 7$ feet above the ground after t seconds. How high will the ball go?

Solution: Let $h = -16t^2 + 96t + 7$ be the height at any time t. At the maximum height, the velocity of the ball will be zero. Hence

$$Dh/dt = v = -32t + 96t = 0 \Rightarrow t = 3.$$

Now, at t = 3, h = -144 + 288 + 7 = 151 ft.

15. The sum of two positive numbers is 100. If their product is maximum, what are the two numbers? (Hint: Let one number be x. What expression represents the other number?)

Solution: Let the numbers be x and y. As given, x + y = 100 $\Rightarrow$ y = 100 – x. If P is their product, then $P = xy = x(100 - x) = 100x - x^2$.

$\Rightarrow$ P` = 100 – 2x. For maxima, P` = 0 $\Rightarrow$ 100 – 2x = 0 $\Rightarrow$ x = 50. Now,

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P`` = -2 which is negative. Hence the product is maximum when x = 50 and y = 50. Thus, required numbers are 50 and 50.

16. A farmer has 1600 feet of fencing to make a rectangular pen for his hogs. What should be the dimensions of the pen if he wants the largest area?

Solution: Let the dimensions of pen be x and y. The perimeter

$$P = 2x + 2y = 1600 \Rightarrow y = 800 - x$$

If A is the area then $A = xy = x(800 - x) = 800x - x^2$

For maximum area, A` = 800 – 2x = 0 ⇒ x = 400 and y = 400. Now,

A`` = -2 which is negative. Hence, the area of the pen (enclosure) is maximum if its sides are: 400 ft by 400 ft.

17. What is the smallest amount of fencing that can be used to enclose a rectangular garden having an area of 900 square feet?

Solution: Let the dimensions of garden be x and y. Now A = xy = 900 ⇒ y = 900/x.

Let F be the amount required for fencing.

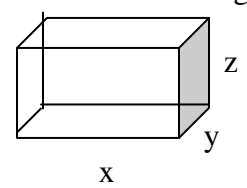
$\Rightarrow \quad F = 2x + 2y = 2x + 2(900/x) = 2x + 1800.x^{-1}$

$\Rightarrow \quad F` = 2 - 1800.\ x^{-2} = 0 \Rightarrow 2x^2 = 1800 \Rightarrow x^2 = 900 \Rightarrow x = 30.$

And therefore, y = 30. Now,

$F`` = 5400\ x^{-3} = 5400/(30)^3 = 6/30 = 1/5$ which is positive. Hence amount of fencing is minimum if the dimensions of the garden are: 30 ft by 30 ft.

18. An open rectangular box (that is, a box with no top) with a capacity of 36,000 cubic inches is needed. If the box must be twice as long as it is wide, what dimensions would require the least material?



Solution:

Let the dimensions of box be x, y and z Then its volume is V = xyz. But V = 36, 000.

Thus, $xyz = 36,000$ (1)

Now, it is given that x = 2y hence,

$2y^2z = 36,000$ (2)

The surface area of the box is S = xy + 2xz + 2yz (Top Open) (3)

From (2) $z = 18,000/y^2$ and x = 2y. Putting these in (3), we get:

$S = 2y^2 + 2(2y)(18,000/y^2) + 2y(18,000/y^2) = 2y^2 + 36,000/y + 36,000$

$z = \dfrac{18000}{y^2}$ and $x = 2y$. Putting these in (3), we get:

$$S = 2y^2 + \frac{72000}{y} + \frac{36000}{y} = 2y^2 + \frac{108000}{y}$$

$$\Rightarrow \frac{dS}{dy} = 4y - \frac{108000}{y^2} = 0 \qquad \text{(For minima)} \quad \Rightarrow y^3 = \frac{108000}{4} = 27000 \Rightarrow y = 30$$

Thus $x = 60,\ y = 30$ and $z = 20.$

Now, $\dfrac{dS^2}{dy^2} = 4 + \dfrac{216000}{y^3} = 4 + \dfrac{216000}{27000} = 4 + 8 = 12$ which is positive.

Hence, the minimum material is required when the dimensions of the box are:

$60 \times 30 \times 20\ \text{in}^3$

19. A builder plans to construct a gutter from a long sheet of metal by making two folds of equal size (see the figure). The folds are made to create perpendicular sides.

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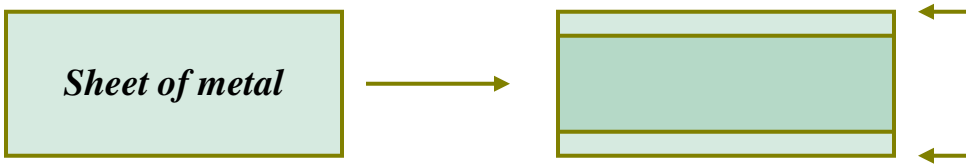


The metal is 28 centimeters wide and 500 centimeters long. How much `x` should be turned up for each side in order for the gutter to hold the most water?

Solution: The volume of gutter after folding the two sides as shown in the figure is

$$V = 500x(28 - 2x) = 14000x - 1000x^2$$

Now, V` = 14000 – 2000x = 0 $\Rightarrow$ x = 7. Also, V`` = -2000 which is negative, hence gutter holds maximum water if folding is made 7 cms on two long sides of the metallic sheet.

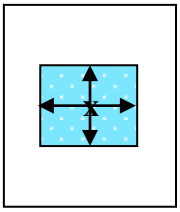
20. Determine the dimensions of the smallest size (that is, smallest area) rectangular piece of paper that satisfies all of the following conditions:

(a) You can print 50 square inches of material on it (the shaded area in the fig).

(b) There will be 2 – inch margins on the top and bottom.

(c) There will be 1 – inch margins on the sides.

Solution: If the dimensions of printing area are x and y then according to 1st condition, xy = 50 $\Rightarrow$ y = 50/x (1)



According to 2nd and 3rd conditions, area of rectangle paper is A = (x + 2)(y + 4) = xy + 4x + 2y + 8. Putting the value of y from (1), we get

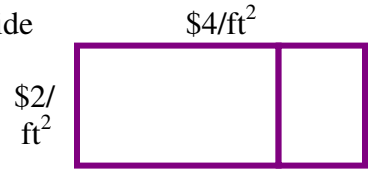
$$A = 50 + 4x + 2\frac{50}{x} + 8 = \frac{100}{x} + 4x + 58$$

$$\therefore A` = \frac{-100}{x^2} + 4 = 0 \quad \Rightarrow x^2 = 25 \text{ or } x = 5 \Rightarrow y = 10. \text{ Now,}$$

$$A`` = +\frac{200}{x^3} \text{ which is positive for } x = 5, \text{ i.e.; } A`` = 8/5$$

Hence minimum dimensions of paper must be 5 by 10 cm^2 to satisfy above conditions.

21. A gardener wishes to fence in a rectangular area of 1728 square feet. He also wants to insert a fence that will divide the area into two rectangular sub areas. The drawing shows that some fencing costs \$4 per $foot^2$ and some costs \$2 per $foot^2$. Find the dimensions that will minimize the cost of the fencing.



Solution: Let the dimensions of the garden be x and y. If A is the area of garden then as given, A = xy = 1728 $\Rightarrow$ y = 1728/x (1)

The perimeter P = 2x + 3y. Thus total cost on fencing is:

C = 2x(4) + 3y(2) = 8x + 6y. Putting the value of y, we get

$$C = 8x + 6\frac{1728}{x} = 8x + \frac{10368}{x} \quad \Rightarrow C` = 8 - \frac{10368}{x^2} = 0 \quad \Rightarrow x^2 = \frac{10638}{8} = 1296 \Rightarrow x = 36$$

$$\text{Now, } C`` = 0 + \frac{20736}{x^3} = \frac{20736}{36^3} = 0.44 \quad \text{which is positive.}$$

Now if x = 36 $\Rightarrow$ y = 48. Thus, for the minimum cost of fencing, the dimensions of the rectangular area must be 48 by 36 $feet^2$. The minimum cost is C = 288 + 144 = \$432.

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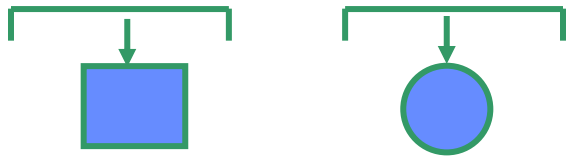
22. A wire 50 centimeters long is cut into two pieces. One piece (call its length x) will be bent to form a square. The other piece (of length 50 - x) will be bent to form a circle. How much wire should be used for the square if the total area (square plus circle) is to be the smallest possible?



Solution: The piece of wire used for square is of length x, hence its each side is x/4. Thus, its area $A_S = (x/4)^2 = x^2/16$. Similarly, the circumference of circle is $C = 2\pi r = (50 - x) \Rightarrow r = (50 - x)/2\pi$. Thus, area of circle is $A_C = \pi r^2 = \pi[(50-x)/2\pi]^2 = (50 - x)^2/4\pi$. Thus total area is:

$$A = A_S + A_C = \frac{x^2}{16} + \frac{(50-x)^2}{4\pi} \Rightarrow A` = \frac{x}{8} + \frac{2(x-50)}{4\pi} = 0 \Rightarrow \frac{x}{8} = -\frac{(x-50)}{2\pi}$$

Or $2\pi x = 8x - 200 \Rightarrow (8 - 2\pi)x = 200 \Rightarrow x = 800/(8-2\pi) = 400/(4-\pi)$. Now,

$$A`` = \frac{1}{8} + \frac{(50-x)}{2\pi}(-1) = \frac{1}{8} - \frac{25}{\pi} + \frac{x}{2\pi} = \frac{1}{8} - \frac{25}{\pi} + \frac{200}{\pi(4-\pi)} = 0.125 - 7.96 + 55.87 \cong 48$$

Since, A`` is positive, hence total area is minimum when $x = 400/(4-\pi)$. This is the amount of wire required to form square.

23. Test the curve $y = x^4$ for point of inflexion.

Solution: $y = x^4 \Rightarrow y` = 4x^3 = 0 \Rightarrow x = 0$. This is a critical point. Now, if $f'(x)$ does not change sign, then x = 0 is a point of inflexion. If x < 0 then y` < 0 and if x > 0 then y` > 0. It means that x = 0 is not a point of inflexion.

24. Find a and b if the function f(x) = a/x + bx such that f(2) = 1 possesses extreme point at x = 2. Is (2, 1) a point of minima or maxima?. Confirm it by drawing the graph.

Solution: Given that

$$f(x) = \frac{a}{x} + bx \Rightarrow f`(x) = \frac{-a}{x^2} + b = 0. \text{ Now, } f(2) = \frac{a}{2} + 2b = 1 \text{ and}$$

$f`(2) = \frac{-a}{4} + b = 0$ because $f(x)$ possesses extreme point at $x = 2$. Solving the above two equations, we get $a = 1, b = 1/4$. Thus, $f(x) = \frac{1}{x} + \frac{x}{4}$

$$\Rightarrow f`(x) = \frac{-1}{x^2} + \frac{1}{4} = 0 \Rightarrow x = \pm 2. \text{ Now, } f``(x) = \frac{2}{x^3}.$$

Putting $x = 2$, we get $f`` = \frac{1}{4} > 0$. Hence $x = 2$ is a point of minima.

The value of function at x = 2 is f = ½ + ½ = 1. Thus, (2, 1) is a point of minima.

The graph is shown as under:

x:	**-2**	**-1**	**0**	**1**	**2**	**4**
y:	**-1**	**-5/4**	∞	**5/4**	**1**	**5/4**

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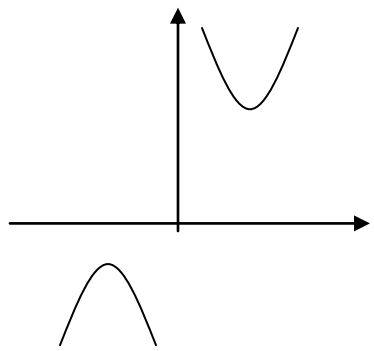

5. Show that function f(x) = $\sin^m x \cos^n x$ attains a maximum when $x = \tan^{-1}\left(\sqrt{m/n}\right)$

Solution:

Given $f(x) = \sin^m x.\cos^n x \quad \Rightarrow f`(x) = m\sin^{m-1} x.\cos x.\cos^n x + n\cos^{n-1} x(-\sin x).\sin^m x$

Or $f`(x) = m\sin^{m-1} x\ \cos^{n+1} x - n\cos^{n-1} x\ \sin^{m+1} x = 0 \Rightarrow m.\cos^2 x = n.\sin^2 x$

$\Rightarrow \tan x = \sqrt{m/n}$ or $x = \tan^{-1}\left(\sqrt{m/n}\right)$. $\left[\text{NOTE}: \tan^2 x = (m/n) \text{ and } \cot^2 x = (n/m)\right]$

Now, $f``(x) = m\left[(m-1)\sin^{m-2} x(\cos x).\cos^{n+1} x + (n+1)\cos^n x(-\sin x).\sin^{m-1} x\right]$

$$-n\left[(n-1)\cos^{n-2} x(-\sin x).\sin^{m+1} x + (m+1)\sin^m x(\cos x).\cos^{n-1} x\right]$$

$$= m(m-1)\sin^{m-2} x.\cos^{n+2} x - m(n+1)\cos^n x\sin^m x + n(n-1)\cos^{n-2} x\sin^{m+2} x$$

$$-n(m+1)\sin^m x\cos^n x$$

$$= \sin^m x\cos^n x\left[m(m-1)\cot^2 x + n(n-1)\tan^2 x - mn - m - mn - n\right]$$

$$= \sin^m x\cos^n x\left[m(m-1)\frac{n}{m} + n(n-1)\frac{m}{n} - mn - m - mn - n\right]$$

$$= \sin^m x\cos^n x\left[mn - n + mn - m - mn - m - mn - n\right]$$

$$= \sin^m x\cos^n x\left[-n - m - m - n\right] = -2(m+n)\sin^m x\cos^n x$$

Since f ``(x) is negative, hence given function has maxima at $x = \tan^{-1}\left(\sqrt{m/n}\right)$

26. The horse power developed by an aircraft traveling horizontally with velocity v feet per second is given by H = $aw^2/v + bv$, where a,b, w are constants. For what value of v is the horse power H is maximum?

Solution:

Given $H = \frac{aw^2}{v} + bv \quad \Rightarrow H` = -\frac{aw^2}{v^2} + b = 0 \Rightarrow v^2 = \frac{aw^2}{b} \quad \Rightarrow v = \pm\sqrt{\frac{a}{b}}\,w.$ Now,

$H`` = +\frac{2aw^2}{v^3}$. Now, if $v = \sqrt{\frac{a}{b}}\,w \quad \Rightarrow H`` = 2aw^2\left(\frac{b}{a}\right)^{3/2}.\frac{1}{w^3} = \frac{2a}{w}\left(\frac{b}{a}\right)^{3/2}$ (+ve)

If $v = -\sqrt{\frac{a}{b}}\,w \quad \Rightarrow H`` = -2aw^2\left(\frac{b}{a}\right)^{3/2}.\frac{1}{w^3} = -\frac{2a}{w}\left(\frac{b}{a}\right)^{3/2}$ (−ve)

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Thus, horse power H is maximum when $v = -\sqrt{\frac{a}{b}}\,w$.

27. The velocity of waves of wave length λ on deep water is proportional to $\sqrt{\lambda/a + a/\lambda}$, where 'a' is some constant. Prove that the velocity is minimum when λ = a.

Solution: Given that velocity $V \;\alpha\; \left(\frac{\lambda}{a}+\frac{a}{\lambda}\right)^{1/2} \Rightarrow V = k\left(\frac{\lambda}{a}+\frac{a}{\lambda}\right)^{1/2}$

$$\Rightarrow V` = \frac{1}{2}k\left(\frac{\lambda}{a}+\frac{a}{\lambda}\right)^{-1/2}.\left(\frac{1}{a}-\frac{a}{\lambda^2}\right) = 0 \Rightarrow \left(\frac{1}{a}-\frac{a}{\lambda^2}\right) = 0 \qquad \Rightarrow \lambda = a$$

$$\text{Also, } V`` = \frac{1}{2}k\left[\left(\frac{\lambda}{a}+\frac{a}{\lambda}\right)^{-1/2}.\left(0+\frac{2a}{\lambda^3}\right)+\left(\frac{1}{a}-\frac{a}{\lambda^2}\right)\left(\frac{-1}{2}\right)\left(\frac{\lambda}{a}+\frac{a}{\lambda}\right)^{-3/2}\left(\frac{1}{a}-\frac{a}{\lambda^2}\right)\right]$$

Putting $\lambda = a$, we get

$$V`` = \frac{1}{2}k\left[\left(\frac{a}{a}+\frac{a}{a}\right)^{-1/2}.\left(0+\frac{2a}{a^3}\right)+\left(\frac{1}{a}-\frac{a}{a^2}\right)\left(\frac{-1}{2}\right)\left(\frac{a}{a}+\frac{a}{a}\right)^{-3/2}\left(\frac{1}{a}-\frac{a}{a^2}\right)\right]$$

$$= \frac{1}{2}k\left[(2)^{-1/2}.\left(0+\frac{2}{a^2}\right)+0\right] = \frac{k}{\sqrt{2}a^2} \text{ which is positive.}$$

Hence, the velocity of waves of wave-length is minimum when λ = a.

28. In a submarine telegraph cable, the speed of signaling varies as $x^2 \ln(1/x)$, where x is the ratio of radius of the core to that of the covering. Show that greatest speed is attained when x is $1/\sqrt{e}$.

Solution : Given that speed $V \;\alpha\; x^2 \ln\left(\frac{1}{x}\right) \Rightarrow V = kx^2 \ln(x)^{-1} = -kx^2 \ln x$

$$\Rightarrow V` = -k\left(x^2.\frac{1}{x}+\ln x.2x\right) = 0 \Rightarrow (x+2x\ln x) = 0 \qquad \Rightarrow \ln x = -1/2 \Rightarrow x = e^{-1/2}$$

Also, $V`` = -k[1+2\ln x+2]$. Putting $\ln x = -1/2$, we get

$$V`` = -k\left[3+2\left(\frac{-1}{2}\right)\right] = -2k, \text{ which is negative.}$$

Hence, the speed of signaling is maximum when $x = e^{-1/2} = 1/\sqrt{e}$.

29. The efficiency E of a screw jack is given by E = tan x/tan(x + a), where a is a constant. Find x if this efficiency is to be maximum. Also find the maximum efficiency.

Solution: Given that the efficiency of screw jack is

$$E = \frac{\tan x}{\tan(x+a)} \Rightarrow E` = \left(\frac{\tan(x+a).\sec^2 x - \tan x.\sec^2(x+a)}{\tan^2(x+a)}\right) = 0$$

$$\Rightarrow \tan(x+a).\sec^2 x - \tan x.\sec^2(x+a) = 0$$

$$\Rightarrow \frac{\sin(x+a)}{\cos(x+a)}.\frac{1}{\cos^2 x} = \frac{\sin x}{\cos x}.\frac{1}{\cos^2(x+a)}$$

$$\Rightarrow \frac{\sin(x+a)}{\cos x} = \frac{\sin x}{\cos(x+a)} \Rightarrow \sin(x+a)\cos(x+a) = \cos x \sin x$$

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$\Rightarrow 2\sin(x+a)\cos(x+a) = 2\sin x\cos x$

$\Rightarrow \sin 2(x+a) = \sin 2x \quad \Rightarrow 2x+2a = 2x \;\Rightarrow 2a = 0 \;\Rightarrow a = 0$

Thus, $y = \dfrac{\tan x}{\tan x} = 1$. Since, y is constant hence we can not determine the efficiency from the given information (equation).

30. Show that all the rectangles of given area, the square has the least parameter.

Solution: Let the dimensions of rectangle be x and y. Then its area and perimeters will be

$A = xy \qquad (1) \qquad\qquad P = 2(x+y) \qquad (2)$

From (1), y = A/x. Putting this in (2), we get:

$$P = 2\left(x+\frac{A}{x}\right) = 2\left(\frac{x^2+A}{x}\right) \Rightarrow P` = 2\left(\frac{x(2x+0)-(x^2+A).1}{x^2}\right) = 2\left(\frac{x^2-A}{x^2}\right) = 0$$

$\Rightarrow x^2 - A = 0$ or $x = \sqrt{A}$. Now,

$$P`` = 2\left(\frac{x^2(2x)-(x^2-A)(2x)}{x^2}\right) = 4x\left(\frac{A}{x^2}\right) = \frac{4A}{x} \text{ which is positive.}$$

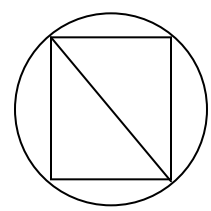
Thus, Perimeter is minimum for all $x = \sqrt{A}$. Now, putting this in $y = \dfrac{A}{x} = \dfrac{A}{\sqrt{A}} = \sqrt{A}$

This shows that $x = y = \sqrt{A}$. Hence, the rectangle is square.

31. Find the rectangle of greatest perimeter that can be inscribed in a circle of radius a.

Solution: Let us assume that equation of circle be $x^2 + y^2 = a^2 \Rightarrow a = \sqrt{x^2+y^2}$.

Let the rectangle have sides u and v. Then from the figure,



$x^2 + y^2 = (2a)^2 \quad \Rightarrow y^2 = 4a^2 - x^2 \quad \Rightarrow y = \sqrt{4a^2 - x^2}$.

Now the perimeter $P = 2(x+y) = 2x + 2\sqrt{4a^2 - x^2}$

$$\Rightarrow \frac{dP}{dx} = 2 + 2.\frac{1}{2}\left(4a^2 - x^2\right)^{-1/2}(-2x) = 2 - \frac{2x}{\sqrt{4a^2 - x^2}} = 0$$

$$\Rightarrow \frac{2x}{\sqrt{4a^2 - x^2}} = 2 \;\Rightarrow x = \sqrt{4a^2 - x^2} \;\Rightarrow x^2 = 4a^2 - x^2 \text{ or } 2x^2 = 4a^2 \quad \Rightarrow x = \sqrt{2}\,a.$$

Now,

$$\frac{d^2P}{dx^2} = 0 - \frac{\sqrt{4a^2 - x^2}(2) - 2x.\frac{1}{2}\left(4a^2 - x^2\right)^{-1/2}(-2x)}{\left(4a^2 - x^2\right)} = -\frac{8a^2 - 2x^2 + 2x^2}{\left(4a^2 - x^2\right)^{3/2}} = -\frac{8a^2}{\left(4a^2 - x^2\right)^{3/2}}$$

Which is negative. Hence rectangle that with largest perimeter inside the circle is having dimensions: $x = \sqrt{2}\,a$ and $y = \sqrt{4a^2 - x^2} = \sqrt{4a^2 - 2a^2} = \sqrt{2}\,a$. Since, $x = y = \sqrt{2}\,a$, thus, the figure that lies inside the circle with largest perimeter is square.

32. After t hours, the number of bacteria in a laboratory culture id given by $n = 6t^2 + 200$. Use differentials to approximate the change in the number of bacteria when t changes from 5 hours to 5 hours and 3 minutes.

Solution: Given $n = 6t^2 + 200$ or $dn = 12t.dt$. Putting t = 5 and dt = 3/60 = 1/20, we get

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dn = 60/20 = 3

This means in three minutes the number of bacteria increases by 3.

33. Use differentials to approximate the change in volume of a sphere – shaped tumor $\left(V = 4/3\pi r^3\right)$ when its radius increases from 1 to 1.1 centimeters.

Solution: Given, $V = 4/3\pi r^3 \quad \Rightarrow dV = \frac{4}{3}3r^2\pi.dr = 4r^2\pi.dr$

Putting r = 1 and dr = 0.1, we get: $dV = 4(1)^2(0.1)(3.14) = 1.256$. Thus, change in the volume of tumor is approximately equal to 1.256 cm^3

34. Leaking sand forms a conical pile in which the height is always thrice the radius (h = 3r). Consider the moment when the radius is 9cm. Use differentials to determine the approximate change in the volume when the radius changes by 1%, and volume of the cone is given by $V = \pi r^2 h/3$.

Solution: Given that the volume of cone is

$$V = \frac{\pi r^2 h}{3} = \frac{\left(\pi r^2\right).3r}{3} = \pi r^3 \quad \Rightarrow dV = 3r^2\pi.dr$$

Now, $r = 9\,\text{cm}$. Then 1% of 9 is : $9/100 = 0.09 = dr$. Thus,

$dV = 3(9)^2(3.14)(0.09) = 68.67.$

Thus approximate change in the volume is 68.67 cm^3

35. A fast food restaurant serves soft drinks in a cylindrical cup that has a radius of 1.5 inches. The volume of soft drink that this cup can hold is given by $V = \pi r^2 h$. Ordinarily, the restaurant fills up to a height of 7.8 inches. Suppose that instead the restaurant decides to fill the cup to a height of 8inches. Use differentials to approximate the number of cubic inches of soft drink the restaurant spend on each serving.

Solution: Given that the volume of cone is

$V = \pi r^2 h.$ Here $r = 1.5``$. Thus, $V = \pi(1.5)^2 h = 2.25\pi h \quad \Rightarrow dV = 2.25\pi.dh$

Now, $h = 7.8``$ and $dh = 8 - 7.8 = 0.2$. Thus, $dV = 2.25(3.14)(0.2) = 4.143.$

Hence, the approximate change in the volume is 4.143 inch cube.

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Chapter 8

CALCULUS OF SEVERAL VARIABLES

WORKSHEET 08

1. Verify the following using Euler's Theorem:

(i) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solution: Given function $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$ is homogeneous of degree zero, hence it is to be proved that: $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$. Now,

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-(x/y)^2}}.\frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{1}{1+(y/x)^2}.\frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{y}{\sqrt{y^2-x^2}}.\frac{1}{y} + \frac{x^2}{(x^2+y^2)}\left(\frac{-y}{x^2}\right)$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{(x^2+y^2)} \qquad (1)$$

Also, $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-(x/y)^2}}.\frac{\partial}{\partial y}\left(\frac{x}{y}\right) + \frac{1}{1+(y/x)^2}.\frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{y}{\sqrt{y^2-x^2}}\left(-\frac{x}{y^2}\right) + \frac{x^2}{(x^2+y^2)}\left(\frac{1}{x}\right)$$

$$\therefore y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{(x^2+y^2)} \qquad (2)$$

Adding (1) and (2), we get : $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

(ii) If $u = \sin^{-1}\frac{x+y}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

Let $z = \frac{x+y}{\sqrt{x+y}}$ (1) Thus, $u = \sin^{-1} z \Rightarrow z = \sin u$ (2)

Now, from (1) we observe that z is a homogeneous function of degree $1/2$. Thus, by Euler's theorem, we must have: $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$ (3)

Now from equation (2), $\partial z/\partial x = \cos u.\partial u/\partial x$ and $\partial z/\partial y = \cos u.\ \partial u/\partial y$

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Putting these in (3), we get: $x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{1}{2}z=\frac{1}{2}\sin u$

Dividing both side by $\cos u$, we obtain: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=\frac{1}{2}\tan u$

(iii) If $u=\sin^{-1}\frac{x^2+y^2}{x+y}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

Solution: Let $z=\frac{x^2+y^2}{x+y}$ (1) Thus, $u=\sin^{-1}z \Rightarrow z=\sin u$ (2)

Now, from (1) we observe that z is a homogeneous function of degree 1. Thus, by Euler's theorem, we must have: $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$ (3)

Now from equation (2), $\frac{\partial z}{\partial x}=\cos u.\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\cos u.\frac{\partial u}{\partial y}$

Putting these in (3), we get: $x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=z=\sin u$

Dividing both side by $\cos u$, we obtain: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=\tan u$

(iv) If $u=\sec^{-1}\left(\frac{x^3-y^3}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\cot u$

Solution: Let $z=\frac{x^3+y^3}{x+y}$ (1) Thus, $u=\sec^{-1}z \Rightarrow z=\sec u$ (2)

Now, from (1) we observe that z is a homogeneous function of degree 2. Thus, by Euler's theorem, we must have: $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$ (3)

Now from equation (2), $\frac{\partial z}{\partial x}=\sec u\tan u\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=\sec u\tan u\frac{\partial u}{\partial y}$

Putting these in (3), we get: $x\sec u\tan u\frac{\partial u}{\partial x}+y\sec u\tan u\frac{\partial u}{\partial y}=2z=2\sec u$

Dividing both side by $\sec u\tan u$, we obtain: $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial x}=2\frac{\sec u}{\sec u\tan u}=\frac{2}{\tan u}=2\cot u$

(v) If $u=\cos^{-1}\frac{x+y}{\sqrt{x+y}}$, prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{2}\cot u$

Let $z=\frac{x+y}{\sqrt{x+y}}$ (1) Thus, $u=\cos^{-1}z \Rightarrow z=\cos u$ (2)

Now, from (1) we observe that z is a homogeneous function of degree $1/2$. Thus, by Euler's theorem, we must have: $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{2}z$ (3)

Now from equation (2), $\frac{\partial z}{\partial x}=-\sin u.\frac{\partial u}{\partial x}$ and $\frac{\partial z}{\partial y}=-\sin u.\frac{\partial u}{\partial y}$

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Putting these in (3), we get: $-x\sin u\dfrac{\partial u}{\partial x}-y\sin u\dfrac{\partial u}{\partial y}=\dfrac{1}{2}z=\dfrac{1}{2}\cos u$

Dividing both side by $-\sin u$, we obtain: $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial x}=-\dfrac{1}{2}\cot u$

(vi) If $u=\ln\left(x^2+xy+y^2\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2$

Solution: Let $z = x^2 + xy + y^2$ (1) Thus, $u = \ln z$ or $z = e^u$ (2)
From (1) we see that z is a homogeneous function of degree 2. Thus, by Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z \qquad (3)$$

Now from equation (2), $\dfrac{\partial z}{\partial x}=e^u\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial z}{\partial y}=e^u\dfrac{\partial u}{\partial y}$

Putting these in (3), we get: $x\,e^u\dfrac{\partial u}{\partial x}+ye^u\dfrac{\partial u}{\partial y}=2z=2e^u$

Dividing both side by e^u, we obtain: $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial x}=2$

2. The area of a triangle is given by $A = 0.5\,ab\sin C$. When a = 20 cm, b = 30 cm and $\angle C = 30^o$, find

(a) The rate of change of A with respect to a, when b and C are constant.
(b) The rate of change of A with respect to b, when a and C are constant.
(c) The rate of change of A with respect to C, when a and b are constant.

Solution: Given $A=0.5\,ab\sin C \Rightarrow \dfrac{\partial A}{\partial a}=0.5b\sin C=0.5(30)\sin 30=.5(30)(0.5)=7.5$

$\Rightarrow \dfrac{\partial A}{\partial b}=0.5\,a\sin C=0.5(20)\sin 30=0.5(20)(0.5)=5$

$\Rightarrow \dfrac{\partial A}{\partial C}=0.5\,ab\cos C=0.5(20)(30)\cos 30=0.5(20)(3)(0.866)=259.8$

3. Determine whether or not the following functions satisfy the Laplace equation $Z_{xx} + Z_{yy} = 0$ (i) $Z = e^x.\cos y$ (ii) $Z = 0.5\,e^{x+y}$ (iii) $Z = x^2 - y^2$.

Solution:

(i) $Z=e^x\cos y \Rightarrow Z_x=e^x\cos y$ and $Z_{xx}=e^x\cos y$. Also,

$Z_y=-e^x\sin y$ and $Z_{yy}=-e^x\cos y$. Thus, $Z_{xx}+Z_{yy}=e^x\cos y-e^x\cos y=0$

Hence, function $Z=e^x\cos y$ satisfies Laplace equation.

(ii) $z=0.5e^{x+y} \Rightarrow Z_x=0.5e^{x+y}$ and $Z_{xx}=0.5e^{x+y}$. Also,

$Z_y=0.5e^{x+y}$ and $Z_{yy}=0.5e^{x+y}$. Thus, $Z_{xx}+Z_{yy}=0.5e^{x+y}+0.5e^{x+y}\neq 0$

Hence, function $Z=0.5e^{x+y}$ does not satisfy Laplace equation.

(iii) $Z=x^2-y^2 \Rightarrow Z_x=2x$ and $Z_{xx}=2$. Also,

$Z_y=-2y$ and $Z_{yy}=-2$. Thus, $Z_{xx}+Z_{yy}=2-2=0$

Hence, function $Z=x^2-y^2$ satisfies Laplace equation.

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4. In a certain electrostatic field, the potential is $u=\left\{(x-1)^2+y^2+(z+2)^2\right\}^{-1/2}$ Find the rate of change of u in the positive x, y, and z directions, respectively, at the point (2, -2, -4).

Solution: Given $u=\left\{(x-1)^2+y^2+(z+2)^2\right\}^{-1/2}$

$\frac{\partial u}{\partial x}=-\frac{1}{2}\left\{(x-1)^2+y^2+(z+2)^2\right\}^{-3/2}.2(x-1)$. At $(2,-2,-4)$: $\frac{\partial u}{\partial x}=-\frac{1}{27}$

$\frac{\partial u}{\partial y}=-\frac{1}{2}\left\{(x-1)^2+y^2+(z+2)^2\right\}^{-3/2}.2y.$ At $(2,-2,-4)$: $\frac{\partial u}{\partial y}=\frac{2}{27}$

$\frac{\partial u}{\partial z}=-\frac{1}{2}\left\{(x-1)^2+y^2+(z+2)^2\right\}^{-3/2}.2(z+2)$. At $(2,-2,-4)$: $\frac{\partial u}{\partial z}=\frac{2}{27}$

5. If resistors of $R_1, R_2,$ and R_3 ohms are connected in parallel to make an R ohm resistor, the value of R can be found from the equation:

$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$. Find the value of $\partial R/\partial R_3$, where $R_1=20, R_2=35,$ and $R_3=50\cdot$

Solution: When $R_1=20, R_2=35,$ and $R_3=50$, 1/R = .05 + .0285 + .02 = 0.0985

➔ R = 10.15.

Now, differentiating given equation partially w.r.t R_3, we get

$$\frac{\partial}{\partial R_3}\left(\frac{1}{R}\right)=\frac{\partial}{\partial R_3}\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right) \quad \Rightarrow -\frac{1}{R^2}\frac{\partial R}{\partial R_3}=0+0-\frac{1}{(R_3)^2}$$

$$\Rightarrow \frac{\partial R}{\partial R_3}=\left(\frac{R}{R_3}\right)^2=\left(\frac{10.15}{50}\right)^2=0.04$$

6. A thin metal plate is being heated in such a way that the temperature T at any point (x, y) on the surface of the plate is given by: $T(x,y)=350-x^2-y^2$ degrees, where x and y are measured in centimeters. What is the temperature at the origin?

(a) On the basis of the function T, why will the temperature be greatest at the origin?
(b) Determine the rate of change of temperature with respect to distance at the point (10, 6) assuming x can vary and y is held constant?
(c) Determine the rate of change of temperature with respect to distance at the point (10, 6) assuming y can vary and x is held constant.
(d) Explain why the rates of change in parts (c) and (d) are negative.

Solution: Given that

$T(x,y)=350-x^2-y^2 \Rightarrow T_x=-2x,\ T_{xx}=-2,\ T_y=-2y,\ T_{yy}=-2$ and $T_{xy}=0$

(a) Now, for stationary point, $T_x=0$ and $T_y=0 \Rightarrow -2x=0$ and $-2y=0 \Rightarrow x=y=0$

$D=T_{xx}T_{yy}-(T_{xy})^2=4$ which is positive and both T_{xx} and T_{yy} are negative. Hence, maximum temperature occurs at (0,0), that is at the origin.

(b) At $(10,6)$: $\frac{\partial T}{\partial x}=-2x=-20$ (c) At $(10,6)$: $\frac{\partial T}{\partial y}=-2y=-12$

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(d) Since the maximum temperature occurs at the origin (0, 0) thus, after (0, 0) the temperature starts decreasing. Thus at the point (10, 6) which lies on the right of origin, the rate of temperature decreases in positive directions of x and y.

7. Concern about body heat loss led to the development of a formula for measuring the surface area S of a person's body on the basis of the individual's weight w in kilograms and height h in centimeters: $S(w,h) = 0.0072w^{0.425}h^{0.725}$

(a) Determine S(80, 178) (b) Find $S_w(w,h)$ and $S_h(w,h)$

(c) Find $S_w(w,h)$ and $S_h(w,h)$ at $(80,178)$ (d) Explain the meaning of S_w and S_h

Solution: (a) $S(80, 178) = 0.0072\ (80)^{0.425}(178)^{0.725} = 0.0072(6.44)(42.8) = 1.98$

(b) $S_w = \dfrac{\partial S}{\partial w} = 0.0072(0.425)w^{-0.575}h^{0.725} = 0.00306\ w^{-0.575}h^{0.725}$

And $S_h = \dfrac{\partial S}{\partial h} = 0.0072\,w^{0.425}(.725)\,h^{-0.275} = 0.00522\ w^{0.425}\,h^{-0.275}$

(c) At $(80, 178)$: $S_w = 0.00306\ (80)^{-0.575}(178)^{0.725} = 0.00306(0.08)(49.24) = 0.012$

$S_h = 0.00522\,(80)^{0.425}\,(178)^{-0.275} = 0.00306(6.44)(0.24) = 0.005$

(d) S_w means rate of change of surface area of human body w.r.t weight while height remains constant while S_h means rate of change of surface area of human body w.r.t height while weight remains constant.

8. Find the approximate change in the hypotenuse of a right triangle of legs 6 and 8 inches when the shorter leg is extended by 1/4 inch and the longer leg is condensed by 1/8 inch.

Solution: Let z be the hypotenuse and x, y be the base and opposite sides of a right triangle. Now x = 6, y = 8, dx = 0.25 and dy = -0.125. By Pythagoras theorem,

$z^2 = x^2 + y^2 \quad \Rightarrow z = \sqrt{x^2 + y^2}$. Then approximate change in z is given by :

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{2x}{2\sqrt{x^2+y^2}}dx + \frac{2y}{2\sqrt{x^2+y^2}}dy = \frac{6}{\sqrt{6^2+8^2}}(.25) + \frac{8}{\sqrt{6^2+y^2}}(-0.125)$$

$= \dfrac{1.5-1}{10} = 0.05$. This means the approximate change in hypotenuse is 0.05 inches.

9. The dimensions of a rectangular block of wood were found to be 10, 12, and 20 inches, with a possible error of 0.05 in each of the measurements. Find (approximately) the greatest error in the surface area of the block and the percentage error in the area caused by errors in the individual measurements.

Solution: Let the dimensions of rectangular block of wood be x, y and z. Then surface area of this block is: S = 2(xy + yz + zx). Here x = 10, y = 12 and z = 20. Also, dx = dy = dz = 0.05. Now the approximate greatest change in the surface area is

$dS = \dfrac{\partial S}{\partial x}dx + \dfrac{\partial S}{\partial y}dy + \dfrac{\partial S}{\partial z}dz = 2(y+z)dx + 2(x+z)dy + 2(x+y)dz$. Putting the values

$= 2\left[(12+20)(0.05) + (10+20)(0.05) + (10+12)(0.05)\right] = 8.4$

10. Two sides of a triangle were measured as 150 ft and 200 ft, and the included angle is of $60°\cdot$ If the possible errors are 0.2ft in measuring the sides and $1^o = \pi/180$ in the angle, what is the greatest possible error in the computed area?

Solution: The area of a triangle with sides a and b and included angle C is given by

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$A = 0.5\,a\,b\sin C$. Thus approximate change in its area is:

$$dA = \frac{\partial A}{\partial a}da + \frac{\partial A}{\partial b}db + \frac{\partial A}{dC}dC = 0.5\left[b\sin C\,da + a\sin C\,db + ab\cos C\,dC\right].$$

Now given that $a = 150,\ b = 200,\ C = 60^o, da = db = 0.2$ and $dC = 1^o = 0.01745$.

Also $\sin 60 = 0.866$ and $\cos 60 = 0.5$. Thus,

$$dA = 0.5\left[200(0.866)(0.2) + 150(0.866)(0.2) + 150(200)(0.5)(0.01745)\right]$$

$$= 0.5\left[34.64 + 25.98 + 261.75\right] = 161.185$$

11. The altitude of a right circular cone is 15 inches and is increasing at 0.2 in/min. The radius of the base is 10 inches and is decreasing at 0.3 in/min. How fast is the volume changing?

Solution: We know that volume of right circular cone is

$V = \pi r^2 h/3$. The rate of change of its volumeis then:

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right].$$

Now given that $h = 15,\ r = 10,\ dh/dt = 0.2$ and $dr/dt = -0.3$. Thus,

$$\frac{dV}{dt} = \frac{3.14}{3}\left[2(10)(15)(-0.3) + (10)^2(0.2)\right] = 1.047\left[-90 + 20\right] = -73.29$$

This shows that volume of cone is decreasing at the rate of 73.29 in^3 /min.

12. At a certain instant the radius of a right circular cylinder is 6 inches and is increasing at the rate 0.2 in/sec while the altitude is 8 inches and is decreasing at the rate 0.4 in/sec. Find the time rate of change (a) of the volume and (b) of the surface, at that instant.

Solution: We know that volume of right circular cylinder is

$V = \pi r^2 h$. The rate of change of its volumeis then:

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \pi\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right].$$

Now given that $h = 8,\ r = 6,\ dh/dt = -0.4$ and $dr/dt = 0.2$. Thus,

$$\frac{dV}{dt} = 3.14\left[2(6)(8)(0.2) + (6)^2(-0.3)\right] = 4.14\left[19.2 - 10.8\right] = 8.4$$

This shows that volume of cylinder is increasing at the rate of 8.4 cm^3/sec.

13. Divide 120 into three non – negative parts such that the sum of their products taken two at a time is a maximum.

Solution: Let the numbers be x, y and x. As given

$$x + y + z = 120 \qquad (1) \qquad\qquad S = xy + yz + zx \quad (2)$$

From (1) $z = 120 - x - y$. Putting this in (2), we get:

$S = xy + y(120 - x - y) + x(120 - x - y) = xy + 120y - xy - y^2 + 120x - x^2 - xy$

$\Rightarrow \qquad S = -x^2 - y^2 - xy + 120x + 120y$

$$\frac{\partial S}{\partial x} = -2x - y + 120 = 0 \quad \text{and} \quad \frac{\partial S}{\partial y} = -x - 2y + 120 = 0$$

Solving, we get: $x = 40,\ y = 40 \Rightarrow z = 40$

Now, $S_{xx} = -2,\ S_{yy} = -2$ and $S_{xy} = -1$. Thus, $D = S_{xx}S_{yy} - \left(S_{xy}\right)^2 = 3$

Now, $D > 0$ and $S_{xx} < 0$, thus, three numbers whose sum is 120 and sum of their

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product taking two at a time is maximum are 40, 40 and 40.

14. Show that a rectangular parallelepiped of maximum volume V with constant surface area S is a cube.

Solution: Let the dimensions of rectangular parallelepiped be x, y and z. Then its volume and surface area respectively, are:

$$V = xyz \quad \text{and} \quad S = 2(xy + yz + zx) = C_1 \quad \text{(given)}$$

$\Rightarrow$ $(xy + yz + zx) = C_1/2 = C$ $\Rightarrow$ $z(y + x) = C - xy$ $\Rightarrow$ $z = (C - xy)/(x + y)$.

Putting this value of z in the first equation, we get

$V = xy\left(\frac{C - xy}{x + y}\right) = \left(\frac{Cxy - x^2y^2}{x + y}\right)$. Now, if $y = x$, then $V = \left(\frac{Cx^2 - x^4}{2x}\right) = \frac{Cx - x^3}{2}$

$\Rightarrow V` = \frac{C}{2} - \frac{3}{2}x^2 = 0 \Rightarrow x = \sqrt{\frac{C}{3}}$. Now, $V`` = -3x = -\sqrt{3C}$ which is negative.

Hence the volume is maximum when $x = \sqrt{C/3} = y$.

And maximum volume is : $V = \frac{1}{2}\left[C\sqrt{\frac{C}{3}} - \left(\frac{C}{3}\right)^{3/2}\right] = \frac{1}{2}\sqrt{\frac{C}{3}}\left[C - \frac{C}{3}\right] = \frac{1}{2}\sqrt{\frac{C}{3}}.\frac{2C}{3} = \frac{C}{3}\sqrt{\frac{C}{3}}$

Now putting the values of V, x and y in the first equation (V = xyz), we get :

$\frac{C}{3}\sqrt{\frac{C}{3}} = \sqrt{\frac{C}{3}}.\sqrt{\frac{C}{3}}.z = \frac{C}{3}z \quad \Rightarrow \quad z = \sqrt{\frac{C}{3}} = x = y.$

Since, x y and z are equal hence, rectangular parallelepiped of maximum volume V with constant surface area S is a cube.

15. A rectangular box, open at the top, is to have a volume of 32 cubic centimeters. Find the dimensions of the box requiring least material for its construction.

Solution: Let the base of the box be x cm by y cm and its height be z cm. Then

$$V = xyz = 32 \Rightarrow z = 32 / xy \qquad (1)$$

The surface area S of the box is given by

$S = xy + 2(xz + yz) = xy + 2\left(x \times \frac{32}{xy} + y \times \frac{32}{xy}\right) = xy + 2\left(\frac{32}{y} + \frac{32}{x}\right)$ **[Top open]**

We need to find the minimum value of S. Therefore,

$S_x = y + 2\left(-\frac{32}{x^2}\right)$, $S_y = x + 2\left(-\frac{32}{y^2}\right)$, $S_{xx} = 2\left(\frac{64}{x^3}\right) = \frac{128}{x^3}$, $S_{yy} = 2\left(\frac{64}{y^3}\right) = \frac{128}{y^3}$, $S_{xy} = 1$.

For critical points, we set

$S_x = 0 \Rightarrow y - \frac{64}{x^2} = 0 \Rightarrow y = \frac{64}{x^2}$ and $S_y = 0 \Rightarrow x - \frac{64}{y^2} = 0 \Rightarrow x = \frac{64}{y^2}$

$\Rightarrow x = \frac{64}{\left(64/x^2\right)^2} = \frac{x^4}{64} \Rightarrow x - \frac{x^4}{64} = 0 \Rightarrow 64x - x^4 = 0$

$x\left(64 - x^3\right) = 0 \Rightarrow 64 - x^3 = 0, \quad x \neq 0$

or $\quad x^3 = 64 \Rightarrow x = 4 \quad$ or $\quad y = 64/(4)^2 = 4$

Now, $\quad D = \left(S_{xx}\right)\left(S_{yy}\right) - \left(S_{xy}\right)^2$

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For (x, y) = (4, 4) $D = \left(\frac{128}{4^3}\right)\left(\frac{128}{4^3}\right) - (1)^2 = 3 > 0 \cdot$

Since D > 0 and $S_{xx} > 0$, therefore S has a minimum value at x = y = 4.

or $z = 32/(4 \times 4) = 2$

Hence, the dimensions of required box should be $4 \times 4 \times 2$ cm^3 to have a minimum cost.

16. A manufacturer of aquarium wants to make a large rectangular box – shaped aquarium that will hold 64 ft^3 of water. If the material for the base costs \$20 per square foot and the material for the sides costs \$10 per square foot, find the dimensions for which the cost of the materials will be the least.

Solution: Let the base of the box be x cm by y cm and its height be z cm. Then

$$V = xyz = 64 \Rightarrow z = 64/xy \qquad (1)$$

The surface area S of the box is given by

$S = xy + 2xz + 2zx$. (In acquirum the top is open]. Thus, the cost of box will be

$$C = 20(xy) + 2(10)(yz + zx) = 20\left[xy + y.\frac{64}{xy} + x\frac{64}{xy}\right] = 20\left(xy + \frac{64}{x} + \frac{64}{y}\right)$$

We need to find the minimum value of C. Therefore,

$$C_x = 20\left(y - \frac{64}{x^2}\right),\ C_y = 20\left(x - \frac{64}{y^2}\right),\ C_{xx} = 20\left(\frac{128}{x^3}\right),\ C_{yy} = 20\left(\frac{128}{y^3}\right),\ C_{xy} = 20 \cdot$$

For critical points, we set

$$C_x = 0 \Rightarrow y - \frac{64}{x^2} = 0 \Rightarrow y = \frac{64}{x^2} \text{ and } C_y = 0 \Rightarrow 2x - \frac{64}{y^2} = 0 \Rightarrow x = \frac{64}{y^2}$$

$$\Rightarrow x = \frac{64}{\left(64/x^2\right)^2} = \frac{x^4}{64} \Rightarrow x - \frac{x^4}{32} = 0 \Rightarrow 64x - x^4 = 0$$

$$x\left(64 - x^3\right) = 0 \Rightarrow 64 - x^3 = 0,\ x \neq 0$$

or $x^3 = 64 \Rightarrow x = 4$ giving $y = 64/(4)^2 = 4$

Now, $D = (C_{xx})(C_{yy}) - (C_{xy})^2$

For (x, y) = (4, 4) $D = 20\left(\frac{128}{4^3}\right).20\left(\frac{128}{4^3}\right) - (20)^2 = 1200 > 0 \cdot$

Since D > 0 and $S_{xx} > 0$, therefore cost C is minimum at x = y = 4.

or $z = 64/(4 \times 4) = 4$

Hence, the dimensions of required box should be $4 \times 4 \times 4$ cm^3 to have a minimum cost.

17. An open rectangular box is being made to contain a volume of 108 cubic feet. Find the dimensions that will minimize the amount of material used to make the box.

Solution: Let the base of the box be x cm by y cm and its height be z cm. Then

$$V = xyz = 108 \Rightarrow z = 108/xy \qquad (1)$$

The surface area S of the box is given by

$$S = xy + 2(xz + yz) = xy + 2\left(x \times \frac{108}{xy} + y \times \frac{108}{xy}\right) = xy + 2\left(\frac{108}{y} + \frac{108}{x}\right) \textbf{[Top open]}$$

We need to find the minimum value of S. Therefore,

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$S_x = y + 2\left(-\frac{108}{x^2}\right)$, $S_y = x + 2\left(-\frac{108}{y^2}\right)$, $S_{xx} = 2\left(\frac{216}{x^3}\right)$, $S_{yy} = 2\left(\frac{216}{y^3}\right)$, $S_{xy} = 1\cdot$

For critical points, we set

$S_x = 0 \Rightarrow y - \frac{216}{x^2} = 0 \Rightarrow y = \frac{216}{x^2}$ and $S_y = 0 \Rightarrow x - \frac{216}{y^2} = 0 \Rightarrow x = \frac{216}{y^2}$

$\Rightarrow x = \frac{216}{\left(216/x^2\right)^2} = \frac{x^4}{216} \Rightarrow x - \frac{x^4}{216} = 0 \Rightarrow 216x - x^4 = 0$

$x\left(216 - x^3\right) = 0 \Rightarrow 216 - x^3 = 0, \quad x \neq 0$

or $\quad x^3 = 216 \Rightarrow x = 6 \quad$ giving $y = 216/(6)^2 = 6$

Now, $\quad D = \left(S_{xx}\right)\left(S_{yy}\right) - \left(S_{xy}\right)^2$

For (x, y) = (6, 6) $\quad D = 2\left(\frac{216}{6^3}\right)2\left(\frac{216}{6^3}\right) - (1)^2 = 3 > 0\cdot$

Since D > 0 and $S_{xx} > 0$, therefore S has a minimum value at x = y = 6.

Thus, $\quad z = 108/(6\times 6) = 3$

Hence, the dimensions of required box should be $6 \times 6 \times 3$ cm^3 to have a minimum cost.

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Chapter 9

INTEGRATION And Its TECHNIQUES

WORKSHEET 09

Evaluate the following integrals:

1. $\int \frac{ax^2+bx+c}{x^2}dx = a\int 1.dx + b\int \frac{1}{x}dx + c\int x^{-2}dx = ax + b\ln x - \frac{c}{x} + C \quad : \int x^n dx = \frac{x^{n+1}}{n+1}$

2. $\int \sqrt{x}\left(ax^2+bx+c\right)dx = a\int x^{5/2}dx + b\int x^{3/2}dx + c\int x^{1/2}dx = 2a\frac{x^{7/2}}{7} + 2b\frac{x^{5/2}}{5} + 2c\frac{x^{3/2}}{3} + C$

3. $\int \left(x+\frac{1}{x}\right)^3 dx = \int x^3dx + 3\int x\ dx + 3\int \frac{1}{x}dx + \int x^{-3}dx = \frac{x^4}{4} + \frac{3x^2}{2} + 3\ln x - \frac{1}{2x^2} + C$

4. $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^3 dx = \int x^{3/2}dx - 3\int x^{1/2}dx + 3\int x^{-1/2}dx - \int x^{-3/2}dx$

$$= \frac{2x^{5/2}}{5} - 2x^{3/2} + 6x^{1/2} + \frac{2}{\sqrt{x}} + C$$

5. $\int \left(a^2-x^2\right)^3 dx = \int \left(a^6 - 3a^4x^2 + 3a^2x^4 - x^6\right)dx = a^6x - 3a^4.\frac{x^3}{3} + 3a^2\frac{x^5}{5} - \frac{x^7}{7} + C$

$$= a^6x - a^4x^3 + 3a^2\frac{x^5}{5} - \frac{x^7}{7} + C$$

6. $\int \left(x+\frac{1}{x}\right)\left(x^2+\frac{1}{x^2}\right)dx = \int \left(x^3 + x + \frac{1}{x} + \frac{1}{x^3}\right)dx = \frac{x^4}{4} + \frac{x^2}{2} + \ln x - \frac{1}{2x^2} + C$

7. $\int \frac{x^3+3x^2+4}{\sqrt{x}}dx = \int \left(x^{5/2} + 3x^{3/2} + 4x^{-1/2}\right)dx = \frac{2x^{7/2}}{7} + \frac{6x^{5/2}}{5} + 8\sqrt{x} + C$

8. $\int \frac{dx}{\sqrt{x+1}-\sqrt{x-2}} = \int \frac{1}{\sqrt{x+1}-\sqrt{x-2}} \times \frac{\sqrt{x+1}+\sqrt{x-2}}{\sqrt{x+1}+\sqrt{x-2}}dx = \int \frac{\sqrt{x+1}+\sqrt{x-2}}{x+1-x+2}dx$

$$= \frac{1}{3}\left[\int \sqrt{x+1}\,dx + \int \sqrt{x-2}\,dx\right] = \frac{1}{3}\left[\frac{2}{3}(x+1)^{3/2} + \frac{2}{3}(x-2)^{3/2}\right] = \frac{2}{9}\left[(x+1)^{3/2} + (x-2)^{3/2}\right] + C$$

9. $\int \frac{(2x+1)^3}{x+1}dx = \int \frac{8x^3+12x^2+6x+1}{x+1}dx = \int \left[\left(8x^2+4x+2\right) - \frac{1}{x+1}\right]dx$

$$= \frac{8x^3}{3} + \frac{4x^2}{2} + 2x - \ln(x+1) + C = \frac{8x^3}{3} + 2x^2 + 2x - \ln(x+1) + C$$

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10. $\int \sin 2x\sqrt{1-\cos 2x}\ dx = \frac{1}{2}\int 2\sin 2x\sqrt{1-\cos 2x}\ dx$: Putting $z = 1-\cos 2x$

$\Rightarrow dz = 2\sin 2x\, dx$. Thus, $I = \frac{1}{2}\int z^{1/2}\, dz = \frac{1}{2}.\frac{2z^{3/2}}{3} + C = \frac{(1-\cos 2x)^{3/2}}{3} + C$

11. $\int\left(e^{a\ln x} + e^{x\ln a}\right)dx = \int\left(e^{\ln x^a} + e^{\ln a^x}\right)dx = \int\left(x^a + a^x\right)dx = \frac{x^{a+1}}{a+1} + \frac{a^x}{\ln a} + C$

12. $\int(1+x)\sqrt{1-x}\ dx$: Putting $\sqrt{1-x} = z \Rightarrow 1-x = z^2 \Rightarrow x = 1-z^2 \Rightarrow dx = -2z\,dz$

Thus, given integral becomes

$\int\left(1+1-z^2\right).z(-2z)\,dz = \int\left(-4z^2 + 2z^4\right)dz = \frac{-4z^3}{3} + \frac{2z^5}{5} = \frac{-4(1-x)^{3/2}}{3} + \frac{2(1-x)^{5/2}}{5} + C$

13. $\int \frac{dx}{\sqrt{x^2-4}}$: Putting $x = 2\sec\theta \Rightarrow dx = 2\sec\theta\tan\theta\, d\theta$. Thus,

$I = \int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec^2\theta - 4}}d\theta = \int \frac{2\sec\theta\tan\theta}{2\sqrt{\sec^2\theta - 1}}d\theta = \int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta = \int \sec\theta\, d\theta = \ln(\sec\theta + \tan\theta)$

$= \ln\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right) = \ln\left(\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 - 1}\right) = \ln\left(\frac{x + \sqrt{x^2-4}}{2}\right) + C$

14. $\int \frac{dx}{\sqrt{4-x^2}}$: Putting $x = 2\sin\theta \Rightarrow dx = 2\cos\theta\, d\theta$. Thus,

$I = \int \frac{2\cos\theta}{\sqrt{4 - 4\sin^2\theta}}d\theta = \int \frac{2\cos\theta}{2\cos\theta}d\theta = \int 1.d\theta = \theta + C = \sin^{-1}\left(\frac{x}{2}\right) + C$

15. $\int \sqrt{\frac{1-x}{1+x}}dx = \int \sqrt{\frac{1-x}{1+x}}\sqrt{\frac{1-x}{1-x}}dx = \int \frac{1-x}{\sqrt{1-x^2}}\ dx = \int \frac{1}{\sqrt{1-x^2}}\ dx + \frac{1}{2}\int\left(1-x^2\right)^{-1/2}(-2x)\,dx$

$= \sin^{-1}x + \frac{1}{2}\frac{\left(1-x^2\right)^{1/2}}{1/2} + C = \sin^{-1}x + \sqrt{1-x^2} + C$

16. $\int \frac{\sin(\ln x)}{x}dx$: Putting $z = \ln x \Rightarrow dz = \frac{1}{x}dx$

$\Rightarrow \int \frac{\sin(\ln x)}{x}dx = \int \sin z\, dz = -\cos z + C = -\cos(\ln x) + C$

17. $\int \frac{2x+7}{\sqrt{x^2+7x+6}}dx$: Putting $z = x^2 + 7x + 6 \Rightarrow dz = (2x+7)\,dx$

$\Rightarrow \int \frac{2x+7}{\sqrt{x^2+7x+6}}dx = \int \frac{1}{\sqrt{z}}dz = \int z^{-1/2}\, dz = \frac{z^{1/2}}{1/2} + C = 2\sqrt{x^2+7x+6} + C$

18. $\int \frac{x+6}{\sqrt{x^2+12x+7}}dx$: Putting $z = x^2 + 12x + 7 \Rightarrow dz = (2x+12)\,dx \Rightarrow \frac{dz}{2} = (x+6)\,dx$

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$$\Rightarrow \int \frac{x+6}{\sqrt{x^2+12x+7}}dx = \frac{1}{2}\int \frac{1}{\sqrt{z}}dz = \frac{1}{2}\int z^{-1/2}\,dz = \frac{1}{2}\left(\frac{z^{1/2}}{1/2}\right)+C = \sqrt{x^2+12x+7}+C$$

19. $\int \sin^5\theta\cos^3\theta\,d\theta = \int \sin^5\theta.\cos^2\theta.\cos\theta\,d\theta = \int \sin^5\theta\left(1-\sin^2\theta\right)\cos\theta.d\theta$

$$= \int\left(\sin^5\theta-\sin^7\theta\right)\cos\theta.d\theta = \int \sin^5\theta.\cos\theta.d\theta - \int \sin^7\theta.\cos\theta.d\theta = \left[\frac{\sin^6\theta}{6}-\frac{\sin^8\theta}{8}\right]+C$$

NOTE: $\int\left(f(x)\right)^n.f`(x)\,dx = \frac{\left(f(x)\right)^{n+1}}{n+1}$. Here derivative of $\sin\theta$ is $\cos\theta$.

20. $\int \sec^5\theta\tan^3\theta\,d\theta = \int \sec^4\theta.\tan^2\theta(\sec\theta\tan\theta)\,d\theta = \int \sec^4\theta.(\sec^2\theta-1)(\sec\theta\tan\theta)\,d\theta$

$$= \int \sec^6\theta\,(\sec\theta\tan\theta)\,d\theta - \int \sec^4\theta.(\sec\theta\tan\theta)\,d\theta = \frac{\sec^7\theta}{7}-\frac{\sec^5\theta}{5}+C$$

21. $\int \csc^5\theta\cot^3\theta\,d\theta = \int csc^4\theta.\cos^2\theta(csc\theta\cot\theta)\,d\theta = \int \csc^4\theta.(\csc^2\theta-1)(\csc\theta\cot\theta)\,d\theta$

$$= \int \csc^6\theta\,cscc\theta\cot\theta)\,d\theta - \int \csc^4\theta.(\csc\theta\cot\theta)\,d\theta = \frac{-\csc^7\theta}{7}+\frac{\csc^5\theta}{5}+C$$

22. $\int\sqrt{a^2+x^2}dx$: Putting $x = a\sinh\theta \Rightarrow dx = a\cosh\theta d\theta$ and $\sinh^{-1}(x/a) = \theta$. Thus,

$$\int\sqrt{a^2+x^2}dx = \int\sqrt{a^2+a^2\sinh^2\theta}.\,a\cosh\theta\,d\theta = \int a\cosh\theta.a\cosh\theta\,d\theta = a^2\int\cosh^2\theta\,d\theta$$

$$= a^2\int\frac{\cosh 2\theta+1}{2}d\theta = \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2}+\theta\right] = \frac{a^2}{2}\left[\frac{2\sinh\theta.\cosh\theta}{2}+\theta\right] = \frac{a^2}{2}\left[\sinh\theta\sqrt{1+\sinh^2\theta}+\theta\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{1+\left(\frac{x}{a}\right)^2}+\sinh^{-1}\frac{x}{a}\right] = \frac{a^2}{2}\left[\frac{x\sqrt{a^2+x^2}}{a^2}+\sinh^{-1}\frac{x}{a}\right] = \frac{1}{2}\left[x\sqrt{a^2+x^2}+a^2\sinh^{-1}\frac{x}{a}\right]+C$$

23. $\int\sqrt{x^2-a^2}dx$: Putting $x = a\cosh\theta \Rightarrow dx = a\sinh\theta\,d\theta$ and $\cosh^{-1}(x/a) = \theta$. Thus,

$$\int\sqrt{x^2-a^2}dx = \int\sqrt{a^2\cosh^2\theta-a^2}.\,a\sinh\theta\,d\theta = \int a\sinh\theta.a\sinh\theta\,d\theta = a^2\int\sinh^2\theta\,d\theta$$

$$= a^2\int\frac{\cosh 2\theta-1}{2}d\theta = \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2}-\theta\right] = \frac{a^2}{2}\left[\frac{2\sinh\theta.\cosh\theta}{2}-\theta\right] = \frac{a^2}{2}\left[\cosh\theta\sqrt{\cosh^2\theta-1}-\theta\right]$$

$$= \frac{a^2}{2}\left[\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1}-\cosh^{-1}\frac{x}{a}\right] = \frac{a^2}{2}\left[\frac{x\sqrt{x^2-1}}{a^2}-\cosh^{-1}\frac{x}{a}\right] = \frac{1}{2}\left[x\sqrt{x^2-1}-a^2\cosh^{-1}\frac{x}{a}\right]+C$$

24. $\int e^{\cos x}\sin x\,dx$: Putting $z = \cos x \Rightarrow -dz = \sin x\,dx$. Thus,

$$\int e^{\cos x}\sin x\,dx = -\int e^z\,dz = -e^z+C = -e^{\cos x}+C$$

25. $\int\frac{\sin 2x}{\sqrt{1+\cos^2 x}}dx$:Putting $z = 1+\cos^2 x \Rightarrow dz = 2\cos x(-\sin x)\,dx \Rightarrow -dz = \sin 2x\,dx$

$$\therefore \int\frac{\sin 2x}{\sqrt{1+\cos^2 x}}dx = \int\frac{-dz}{\sqrt{z}} = -\int z^{-1/2}dz = -\frac{z^{1/2}}{1/2} = -2\sqrt{1+\cos^2 x}+C$$

26. $\int\frac{1}{\sqrt{x}}\operatorname{cosec}\sqrt{x}\cot\sqrt{x}\,dx$: Putting $z = \sqrt{x} \Rightarrow dz = \frac{dx}{2\sqrt{x}} \Rightarrow 2dz = \frac{dx}{\sqrt{x}}$. Thus,

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$\int \frac{1}{\sqrt{x}} \operatorname{cosec}\sqrt{x}\cot\sqrt{x}\ dx = 2\int \csc z\ \cot z\ dz = -2\csc z + C\ = -\csc\sqrt{x} + C$

27. $\int\left(3^{\sin x} + [\sin x]^3\right)\cos x\ dx$:Putting $z = \sin x\ \Rightarrow\ dz = \cos x\ dx$. Thus,

$\int\left(3^{\sin x} + [\sin x]^3\right)\cos x\ dx = \int\left(3^z + z^3\right) dz = \frac{3^z}{\ln 3} + \frac{z^4}{4} + C = \frac{3^{\sin x}}{\ln 3} + \frac{\sin^4 x}{4} + C$

28. $\int \frac{\sin x}{2\cos x + 7\sqrt{\cos x}} dx$: Putting $z = \sqrt{\cos x}\ \Rightarrow z^2 = \cos x\ \ \Rightarrow -2z\ dz = \sin x\ dx$

Thus, $\int \frac{\sin x}{2\cos x + 7\sqrt{\cos x}} dx = -2\int \frac{z\ dz}{2z^2 + 7z} = -2\int \frac{z\ dz}{z(7z+2)} = -2\int \frac{1}{(7z+2)}\ dz$

$= -\frac{2}{7}\ln(7z+2) + C = -\frac{2}{7}\ln\left(7\sqrt{\cos x} + 2\right) + C$

Here we have used partial fractions Case-I.

29. $\int x\ \operatorname{cosec}^2 x\, dx$: Integrating by parts taking $u = x$ and $v = \csc^2 x$

NOTE : $\int u.v\, dx = u\int v\, dx - \int\left[u`\int v\, dx\right]dx$ and $\int \csc^2 x\, dx = -\cot x$, $\int \cot x\, dx = \ln(\sin x)$

$\therefore \int x\ \operatorname{cosec}^2 x\, dx = x(-\cot x) - \int\left[1.(-\cot x)\right]dx = -x\cot x + \ln\sin x + C$

30. $\int x^n \ln x\, dx$: Integrating by part taking $u = \ln x$ & $v = x^n$

$\int x^n \ln x\, dx = \ln x\int x^n\, dx - \int\left[\frac{1}{x}\int x^n dx\right]dx = \frac{x^{n+1}}{n+1}.\ln x - \int \frac{1}{x}.\frac{x^{n+1}}{n+1} dx$

$= \frac{x^{n+1}}{n+1}.\ln x - \frac{1}{n+1}\int x^n dx\ = \frac{x^{n+1}}{n+1}.\ln x - \frac{1}{n+1}\left(\frac{x^{n+1}}{n+1}\right) + C = \frac{x^{n+1}}{n+1}\left[\ln x - \frac{1}{n+1}\right] + C$

31. $\int \csc^3 x\, dx = \int \csc x\ \csc^2 x\, dx = \csc x\int \csc^2 x\, dx - \int\left[-\csc x\cot x\int \csc^2 x\, dx\right]$

$= \csc x(-\cot x) - \int[-\csc x\cot x](-\cot x)dx = -\csc x\cot x - \int \csc x\cot^2 x\, dx$

$= -\csc x\cot x - \int \csc x\left(\csc^2 x - 1\right)dx = -\csc x\cot x - \int \csc^3 x\, dx + \int \csc x\, dx$ (1)

Let $I = \int \csc^3 x\, dx$, then (1) becomes

$I = -\csc x\cot x - I + \ln(\csc x - \cot x)\ \ \Rightarrow 2I = -\csc x\cot x + \ln(\csc x - \cot x)$

$\Rightarrow I = \frac{1}{2}\left[-\csc x\cot x + \ln(\csc x - \cot x)\right] + C$

32. $\int x\sin^{-1} x\ dx = \sin^{-1} x.\int x\, dx - \int\left[\frac{1}{\sqrt{1-x^2}}.\int x\, dx\right] = \frac{x^2\sin^{-1} x}{2} - \int \frac{1}{\sqrt{1-x^2}}.\frac{x^2}{2} dx$

Putting $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$. Now, $\theta = \sin^{-1} x$ and $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$. Thus,

$\int x\sin^{-1} x\ dx = \frac{x^2\sin^{-1} x}{2} - \int \frac{1}{\sqrt{1-\sin^2\theta}}.\frac{\sin^2\theta}{2}.\cos\theta\, d\theta$

$= \frac{x^2\sin^{-1} x}{2} - \int \frac{1}{\cos\theta}.\frac{\sin^2\theta}{2}.\cos\theta\ d\theta = \frac{x^2\sin^{-1} x}{2} - \frac{1}{2}\int \sin^2\theta\ d\theta = \frac{x^2\sin^{-1} x}{2} - \frac{1}{2}\int \frac{1-\cos 2\theta}{2}\ d\theta$

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$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right] = \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4}\left[\theta - \frac{2\sin\theta.\cos\theta}{2}\right]$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{4}\left[\sin^{-1} x - x\sqrt{1-x^2}\right] + C$$

33. $\int e^{ax} \cos(bx+c)\,dx$: Let $I = \int e^{ax} \cos(bx+c)\,dx$ and $u = e^{ax}$, $v = \cos(bx+c)$.Thus,

$$I = e^{ax}\int \cos(bx+c)\,dx - \int\left[ae^{ax}\int \cos(bx+c)\,dx\right] = e^{ax}.\frac{\sin(bx+c)}{b} - a\int e^{ax}\frac{\sin(bx+c)}{b}dx$$

$$= e^{ax}.\frac{\sin(bx+c)}{b} - \frac{a}{b}\left[e^{ax}\frac{-\cos(bx+c)}{b} - \int ae^{ax}\frac{-\cos(bx+c)}{b}\right]dx$$

$$= e^{ax}.\frac{\sin(bx+c)}{b} + \frac{a}{b^2}e^{ax}\cos(bx+c) - \frac{a^2}{b^2}\int e^{ax}\cos(bx+c)\,dx$$

$$\therefore I = e^{ax}.\frac{\sin(bx+c)}{b} + \frac{a}{b^2}e^{ax}\cos(bx+c) - \frac{a^2}{b^2}I$$

$$I\left(1+\frac{a^2}{b^2}\right) = e^{ax}\left[\frac{b\sin(bx+c)+a\cos(bx+c)}{b^2}\right] \Rightarrow I\left(\frac{a^2+b^2}{b^2}\right)$$

$$\Rightarrow I\left(\frac{a^2+b^2}{b^2}\right) = e^{ax}\left[\frac{b\sin(bx+c)+a\cos(bx+c)}{b^2}\right]$$

$$\therefore I = \int e^{ax}\cos(bx+c)\,dx = \frac{e^{ax}}{\left(a^2+b^2\right)}\left[a\cos(bx+c)+b\sin(bx+c)\right]+C$$

34. $\int \frac{\sin x}{4+5\cos x}dx$: Putting $z = 4+5\cos x \Rightarrow dz = -5\sin x\,dx \Rightarrow \frac{-dz}{5} = \sin x\,dx$. Thus,

$$\int \frac{\sin x}{4+5\cos x}dx = -\frac{1}{5}\int\frac{1}{z}dz = -\frac{1}{5}\ln z + C = -\frac{1}{5}\ln\left(4+5\cos x\right)+C$$

35. $\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx$: let $z = e^x + e^{-x} \Rightarrow dz = \left(e^x - e^{-x}\right)dx$.Thus,

$$\int \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int\frac{1}{z}dz = \ln z + C = \ln\left(e^x + e^{-x}\right)+C$$

36. $\int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)}$: Putting $z = 1+\sqrt{x} \Rightarrow dz = \frac{1}{2\sqrt{x}}dx \Rightarrow 2dz = \frac{1}{\sqrt{x}}dx$. Thus,

$$\int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)} = 2\int\frac{1}{z}dz = 2\ln z + C = \ln z^2 + C = \ln\left(1+\sqrt{x}\right)^2 + C$$

37. $\int \frac{dx}{x\sqrt{x^2-a^2}}$: Putting $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\,d\theta$

$\Rightarrow x^2 - a^2 = a^2\left(\sec^2\theta - 1\right) = a^2\tan^2\theta$. Thus, $\sqrt{x^2-a^2} = a\tan\theta$.

$$\therefore \int \frac{dx}{x\sqrt{x^2-a^2}} = \int\frac{a\sec\theta\tan\theta}{a\sec\theta.a\tan\theta}d\theta = \int 1.d\theta = \theta + C = \sec^{-1}\frac{x}{a} + C$$

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38. $\int \frac{dx}{(2x+3)^2+1}$: Putting $z = 2x+3 \Rightarrow dz = 2dx \Rightarrow dz/2 = dx$. Thus,

$$\int \frac{dx}{(2x+3)^2+1} = \frac{1}{2}\int \frac{1}{1+z^2}dz = \frac{1}{2}\tan^{-1} z + C = \frac{1}{2}\tan^{-1}(2x+3) + C$$

39. $\int \frac{x+5}{\sqrt{1-x^2}}dx = \int \frac{x}{\sqrt{1-x^2}}dx + 5\int \frac{1}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx + 5\sin^{-1} x$

$$= -\frac{1}{2}\frac{(1-x^2)^{1/2}}{1/2} + 5\sin^{-1} x = 5\sin^{-1} x - \sqrt{1-x^2} + C$$

40. $\int \frac{e^{\sqrt{x}}\cos\left(e^{\sqrt{x}}\right)}{\sqrt{x}}dx$: Putting $\sqrt{x} = z \Rightarrow \frac{1}{2\sqrt{x}}dx = dz \Rightarrow \frac{1}{\sqrt{x}}dx = 2dz$. Thus,

$I = 2\int e^z \cos z\,dz = 2.\frac{1}{1+1}e^z[\cos z + \sin z]$. Refer Q.33 : Here $a = 1,\ b = 1$

$$= e^{\sqrt{x}}\left[\cos\sqrt{x} + \sin\sqrt{x}\right] + C \qquad \left(\text{put } z = \sqrt{x}\right)$$

41.

$$\int \frac{\ln(1+x^2)}{x^2}dx = \int \ln(1+x^2).x^{-2}dx = \ln(1+x^2)\frac{x^{-1}}{-1} - \int \frac{1}{1+x^2}.2x\left(\frac{x^{-1}}{-1}\right)dx \quad [\text{By Parts}]$$

$$= -\frac{\ln(1+x^2)}{x} + 2\int \frac{1}{1+x^2}dx = -\frac{\ln(1+x^2)}{x} + 2\tan^{-1} x + C$$

42. $\int x\cot^{-1} x\,dx = \cot^{-1} x\int x\,dx - \int\left[\frac{-1}{1+x^2}\int x\,dx\right]$. Integrating by parts.

$$= \frac{x^2}{2}\cot^{-1} x - \int\left[\frac{-1}{1+x^2}.\frac{x^2}{2}\right]dx = \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int \frac{x^2}{1+x^2}dx$$

$$= \frac{x^2}{2}\cot^{-1} x + \frac{1}{2}\int \frac{1+x^2-1}{1+x^2}dx = \frac{x^2}{2}\cot^{-1} x + +\frac{1}{2}\left[\int \frac{1+x^2}{1+x^2}dx - \int \frac{1}{1+x^2}dx\right]$$

$$= \frac{x^2}{2}\cot^{-1} x + +\frac{1}{2}\left[\int 1.dx - \tan^{-1} x\right] = \frac{x^2}{2}\cot^{-1} x + +\frac{1}{2}\left[x - \tan^{-1} x\right] + C$$

43. $\int x^3 e^{x^2}dx = \int x^2 e^{x^2}(x)\,dx$. Putting $z = x^2 \Rightarrow dz = 2x\,dx \Rightarrow dz/2 = x\,dx$. Thus,

$$\int x^3 e^{x^2}dx = \frac{1}{2}\int ze^z dz = \frac{1}{2}\left[ze^z - \int 1.e^z dz\right] = \frac{1}{2}\left[ze^z - e^z\right] = \frac{e^{x^2}}{2}\left[x^2 - 1\right] + C. \quad [\text{By Parts}]$$

44. $\int \sin(\ln x)\,dx$: Putting $z = \ln x \Rightarrow x = e^z \Rightarrow dx = e^z\,dz$. Thus,

$$\int \sin(\ln x)\,dx = \int e^z \sin z\,dz = \frac{e^z}{1+1}[\sin z - \cos z] = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

45. $\int e^x \sec x(1+\tan x)\,dx = \int e^x \sec x\ dx + \int e^x \sec x \tan x\ dx$. Apply by parts on 2nd part,

$$\therefore \int e^x \sec x(1+\tan x)\,dx = \int e^x \sec x\ dx + e^x.\sec x - \int\left[e^x \sec x\ dx\right] = e^x.\sec x + C$$

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46. $\int \frac{\sin(\ln x)}{x^3}dx$: Putting $z = \ln x \Rightarrow x = e^z \Rightarrow dx = e^z dz$. Thus,

$$\int \frac{\sin(\ln x)}{x^3}dx = \int \frac{\sin z}{e^{3z}}.e^z\, dz = \int e^{-2z}\sin z\, dz = \frac{e^{-2z}}{(-2)^2+1^2}\left[-2\sin z - \cos z\right]$$

$$= \frac{x^{-2}}{5}\left[-2\sin(\ln x) - \cos(\ln x)\right] + C == \frac{-1}{5x^2}\left[2\sin(\ln x) + \cos(\ln x)\right] + C$$

47. $\int \frac{dx}{2x^2 - x - 1} = \int \frac{dx}{(2x+1)(x-1)} = -\frac{1}{3}\int \frac{2}{2x+1}dx + \frac{1}{3}\int \frac{1}{x-1}dx$

$= -\frac{1}{3}\left[\ln(2x+1) + \ln(x-1)\right] + C$ (This is by partial fractions)

48. $\int \frac{2x-3}{(x^2-1)(2x+3)}dx = \int \frac{2x-3}{(x-1)(x+1)(2x+3)} = -\frac{1}{10}\int \frac{dx}{x-1} + \frac{5}{2}\int \frac{dx}{x+1} - \frac{24}{5}\int \frac{dx}{2x+3}$

$$= -\frac{1}{10}\ln(x-1) + \frac{5}{2}\ln(x+1) - \frac{12}{5}\ln(2x+3) + C$$

49. $\int \frac{x}{x^2 - 12x + 35}dx = \int \frac{x}{(x-5)(x-7)}dx = -\frac{5}{2}\int \frac{1}{x-5}dx + \frac{7}{2}\int \frac{1}{x-7}dx$

$= \frac{1}{2}\left[7\ln(x-7) - 5\ln(x-5)\right] + C$

50. $\int \frac{dx}{1+3e^x+2e^{2x}} = \int \frac{dx}{(2e^x+1)(e^x+1)}$: Putting $z = e^x \Rightarrow dz = e^x dx = z\,dx \Rightarrow dx = \frac{dz}{z}$

$$\therefore \int \frac{dx}{1+3e^x+2e^{2x}} = \int \frac{dz}{z(z+1)(2z+1)} = \frac{1}{2}\int \frac{dz}{z} + \int \frac{dz}{z+1} - 4\int \frac{dz}{2z+1}$$

$= \frac{1}{2}\ln z + \ln(z+1) - 2\ln(2z+1) + C$. Putting $z = e^x$, we get

$$\int \frac{dx}{1+3e^x+2e^{2x}} = \frac{1}{2}x + \ln(e^x+1) - 2\ln(2e^x+1) + C$$

51. $\int \frac{3x+1}{(x-1)^2(x+3)}dx$: Consider,

$$\frac{3x+1}{(x-1)^2(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)} \Rightarrow 3x+1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2 \qquad (1)$$

Put $x-1=0 \Rightarrow x=1 \Rightarrow 3+1 = A(0) + B(1+3) + C(0) \Rightarrow B = 1$

Put $x+3=0 \Rightarrow x=-3 \Rightarrow -9+1 = A(0) + B(0) + C(-3+1)^2 \Rightarrow C = -2$

To find A, comparing the coefficients of x^2 in (1), we get : $0 = A + C \Rightarrow A = -C = 2$. Thus,

$$\int \frac{3x+1}{(x-1)^2(x+3)}dx = \int \frac{2}{(x-1)}dx + \int \frac{1}{(x-1)^2}dx - 2\int \frac{1}{(x+3)}dx = 2\ln(x-1) - \frac{1}{(x-1)} - 2\ln(x+3) + C$$

52. $\int \frac{x\,dx}{(x-1)(x^2+4)}$: Consider,

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$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+4)} \quad \Rightarrow x = A(x^2+4) + (Bx+C)(x-1) \qquad (1)$$

Put $x-1=0 \Rightarrow x=1 \quad \Rightarrow \quad 1 = A(5)+0 \qquad \Rightarrow A = 1/5$

To find B and C, comparing the coefficients of x^2 in (1), we get : $0 = A + B \;\Rightarrow\; B = -A = -1/5$

Comparing the coefficients of x in (1), we get : $1 = C - B \Rightarrow C = 1 + B = 1 - 1/5 = 4/5$

$$\therefore \int \frac{x}{(x-1)(x^2+4)}dx = \frac{1}{5}\int \frac{1}{(x-1)}dx - \frac{1}{5}\int \frac{x}{(x^2+4)}dx + \frac{4}{5}\int \frac{x}{(x^2+4)}dx$$

$$= \frac{1}{5}\int \frac{1}{(x-1)}dx - \frac{1}{5}.\frac{1}{2}\int \frac{2x}{(x^2+4)}dx + \frac{2}{5}\int \frac{2x}{(x^2+4)}dx$$

$$= \frac{1}{5}\left[\ln(x-1) - \frac{1}{2}\ln(x^2+4) + 2\ln(x^2+4)\right] + C$$

53. $\int \frac{x\,dx}{x^4 - x^2 - 1} = \int \frac{x}{(x^2)^2 - (x^2) - 1}dx$: Putting $z = x^2 \;\Rightarrow dz = 2x\,dx \;\Rightarrow dz/2 = x\,dx$

$$\therefore \int \frac{x\,dx}{x^4 - x^2 - 1} = \frac{1}{2}\int \frac{1}{z^2 - z - 1}dz = \frac{1}{2}\int \frac{1}{(z-a)(z-b)}dz, \text{ where, } a = \frac{1+i\sqrt{3}}{2},\; b = \frac{1-i\sqrt{3}}{2}$$

Now, $\frac{1}{(z-a)(z-b)} = \frac{A}{z-a} + \frac{B}{z-b} = \frac{a-b}{(z-a)} + \frac{b-a}{(z-b)}$, $a - b = i\sqrt{3}$ & $b - a = -i\sqrt{3}$

Thus, $\frac{1}{(z-a)(z-b)} = i\sqrt{3}\left[\frac{1}{z-a} - \frac{1}{z-b}\right]$

$$\therefore \int \frac{x\,dx}{x^4 - x^2 - 1} = i\sqrt{3}\left[\int \frac{1}{(z-a)}dz - \int \frac{1}{(z-b)}dz\right] = i\sqrt{3}\left[\ln(z-a) - \ln(z-b)\right] + C$$

$$= i\sqrt{3}\left[\ln(x^2 - a) - \ln(x^2 - b)\right] + C$$

54. $\int \frac{1-x^2}{(1+x^2)^2}dx = \int \frac{2-(1+x^2)}{(1+x^2)^2}dx = 2\int \frac{1}{(1+x^2)^2}dx - \int \frac{(1+x^2)}{(1+x^2)^2}dx$

$$= 2\int \frac{1}{(1+x^2)^2}dx - \int \frac{1}{(1+x^2)}dx = 2\int \frac{1}{(1+x^2)^2}dx - \tan^{-1}x.$$

Putting $x = \tan\theta \;\Rightarrow dx = \sec^2\theta\, d\theta$. Now, $\tan\theta = P/B = x \;\Rightarrow P = x$ & $B = 1$

$\Rightarrow H = \sqrt{B^2 + P^2} = \sqrt{1+x^2}$. Thus, $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$. Also $\theta = \tan^{-1}x$ and

$\sin\theta = \frac{P}{H} = \frac{x}{\sqrt{1+x^2}}$ and $\cos\theta = \frac{B}{H} = \frac{1}{\sqrt{1+x^2}}$

$$\therefore I = 2\int \frac{1}{(\sec^2\theta)^2}.\sec^2\theta\; d\theta - \tan^{-1}x = 2\int \cos^2\theta\; d\theta - \tan^{-1}x = 2\int \frac{1+\cos 2\theta}{2}d\theta - \tan^{-1}x$$

$$= \left[\theta + \frac{\sin 2\theta}{2}\right] - \tan^{-1}x = \left[\theta + \frac{2\sin\theta\cos\theta}{2}\right] - \tan^{-1}x$$

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$$=\left[\tan^{-1}x+\frac{x}{\sqrt{1+x^2}}.\frac{1}{\sqrt{1+x^2}}\right]-\tan^{-1}x=\frac{x}{1+x^2}+C$$

55.

$$\int\sqrt{5-4x-x^2}dx=\int\sqrt{5-\left(x^2+4x\right)}dx=\int\sqrt{5-\left(x^2+4x+4-4\right)}dx=\int\sqrt{9-\left(x^2+4x+4\right)}dx$$

$=\int\sqrt{9-(x+4)^2}\,dx$: Putting $x+4=z\ \Rightarrow dx=dz$. Thus,

$$I=\int\sqrt{3^2-z^2}\,dz=I=\frac{z}{2}\sqrt{9-z^2}+\frac{9}{2}\sin^{-1}\left(\frac{z}{3}\right)\left[\text{NOTE}:\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

Putting $z=x+4$ and simplifying we get: $I=\frac{(x+4)}{2}\sqrt{5-4x-x^2}+\frac{9}{2}\sin^{-1}\frac{(x+4)}{3}+C$

56. $$\int\frac{dx}{\sqrt{2x^2+3x+4}}=\int\frac{dx}{\sqrt{2\left(x^2+\frac{3}{2}x+2\right)}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{x^2+\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}+2}}$$

$$=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)+\frac{23}{16}}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}}.$$

Now, put $x+\frac{3}{4}=z\ \Rightarrow dx=dz$ and $\frac{\sqrt{23}}{4}=a$, we get

$$I=\frac{1}{\sqrt{2}}\int\frac{dz}{\sqrt{z^2+a^2}}=\frac{1}{\sqrt{2}}\left[\sinh^{-1}\left(\frac{z}{a}\right)\right]=\frac{1}{\sqrt{2}}\left[\sinh^{-1}\frac{x+\frac{3}{4}}{\frac{\sqrt{23}}{4}}\right]=\frac{1}{\sqrt{2}}\left[\sinh^{-1}\frac{4x+3}{\sqrt{23}}\right]+C$$

57. $\int\frac{\cos x dx}{\sqrt{4\sin^2x+4\sin x+5}}$: Putting $z=\sin x\ \Rightarrow dz=\cos x\ dx$. Thus,

$$I=\int\frac{1}{\sqrt{4z^2+4z+5}}dz=\int\frac{1}{\sqrt{4z^2+4z+1+4}}dz=\int\frac{1}{\sqrt{(2z+1)^2+2^2}}dz.$$

Putting $2z+1=u\Rightarrow dz=du/2$

$$\therefore I=\frac{1}{2}\int\frac{1}{\sqrt{u^2+2^2}}du=\frac{1}{2}.\frac{1}{2}\sinh^{-1}\left(\frac{u}{2}\right)+C=\frac{1}{4}\sinh^{-1}\left(\frac{2z+1}{2}\right)+C=\frac{1}{4}\sinh^{-1}\left(\frac{2\sin x+1}{2}\right)+C$$

58. $\int\frac{dx}{\left(x^2+1\right)\sqrt{x}}$: Putting $z=\sqrt{x}\ \Rightarrow z^2=x\ \Rightarrow 2z\,dz=dx$. Thus,

$$\int\frac{dx}{\left(x^2+1\right)\sqrt{x}}=\int\frac{2z\,dz}{\left(z^4+1\right).z}=2\int\frac{1}{\left(z^4+1\right)}dz$$

$$=2.\frac{1}{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{(z^2-1)}{\sqrt{2}z}-\frac{1}{2\sqrt{2}}\ln\frac{(z^2-\sqrt{2}z-1)}{(z^2+\sqrt{2}z-1)}\right]+C\quad\left[\begin{array}{l}\text{Refer Example 03 page \#174}\\ \text{Textbook on Applied Calculus}\end{array}\right]$$

Putting $x=z^2$ and $z=\sqrt{x}$, we get:

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$$\int \frac{dx}{\left(x^2+1\right)\sqrt{x}} = \left[\frac{1}{\sqrt{2}}\tan^{-1}\frac{(x-1)}{\sqrt{2x}z} - \frac{1}{2\sqrt{2}}\ln\frac{(x-\sqrt{2x}-1)}{(x+\sqrt{2x}-1)}\right]+C$$

59. $\int \frac{dx}{\left(2ax+x^2\right)^{3/2}} = \int \frac{dx}{\left(x^2+2ax+a^2-a^2\right)^{3/2}} = \int \frac{dx}{\left[\left(x+a\right)^2-a^2\right]^{3/2}}$

Putting $x+a=z \Rightarrow dx=dz$, we get : $I=\int \frac{dz}{\left(z^2-a^2\right)^{3/2}}$. Put $z=a\sec\theta \Rightarrow dz = a\sec\theta\tan\theta\, d\theta$

$$\therefore I = \int \frac{a\sec\theta\tan\theta}{\sqrt{\left[a^2\left(\sec^2\theta-1\right)\right]^{3/2}}}d\theta = \int \frac{a\sec\theta\tan\theta}{a^3\tan^3\theta}d\theta = \frac{1}{a^2}\int\frac{\sec\theta}{\tan^2\theta}d\theta = \frac{1}{a^2}\int\frac{\cos^2\theta}{\sin^2\theta.\cos\theta}d\theta$$

$= \frac{1}{a^2}\int\frac{\cos\theta}{\sin^2\theta}d\theta$. Putting $u=\sin\theta \Rightarrow du=\cos\theta\, d\theta$

$$\therefore I = \frac{1}{a^2}\int\frac{1}{u^2}du = \frac{1}{a^2}\left(-\frac{1}{u}\right) = -\frac{1}{a^2\sin\theta} = -\frac{z}{a^2\sqrt{z^2-a^2}} = -\frac{x+a}{a^2\sqrt{x^2+2ax}}+C$$

NOTE : $\sec\theta = \frac{z}{a} = \frac{H}{B} \Rightarrow P=\sqrt{H^2-B^2} = \sqrt{z^2-a^2}$. Thus, $\sin\theta = \frac{P}{H} = \frac{\sqrt{z^2-a^2}}{z}$

60. $\int \frac{(x+1)(x+2)}{\sqrt{x^2+x-2}}dx$

$$I = \int \frac{(x+1)(x+2)}{\sqrt{(x+2)(x-1)}}dx = \int \frac{(x+1)\sqrt{x+2}}{\sqrt{x-1}}dx.$$

Putting $\sqrt{x-1}=z \Rightarrow x-1=z^2,\ x=z^2+1, dx=2z\ dz$

$$\therefore I = \int \frac{(z^2+2)(z^2+3)}{z}.\,2z\ dz = 2\int(z^4+5z^2+6)\,dz = 2\left[\frac{z^5}{5}+\frac{5z^3}{3}+6z\right]+C$$

$$= \frac{2z}{15}\left[3z^4+25z^2+90\right]+C = \frac{2}{15}\sqrt{x-1}\left[3(x-1)^2+25(x-1)+90\right]+C$$

61. $\int \frac{\sin x}{(1+\cos x)(2+\cos x)}dx$: Putting $z=\cos x \Rightarrow -dz = \sin x\, dx$. Thus,

$$\int \frac{\sin x}{(1+\cos x)(2+\cos x)}dx = -\int\frac{dz}{(z+1)(z+2)} = -\left[\int\frac{1}{z+1}-\frac{1}{z+2}\right]dz \ [\text{By P.Fractions}]$$

$$= -\left[\ln(z+1)-\ln(z+2)\right] = \ln\left[\frac{2+\cos x}{1+\cos x}\right]+C$$

62. $\int \frac{x^2}{(1+x)\left(x^2-4x+1\right)}dx$: Consider $\frac{x^2}{(1+x)\left(x^2-4x+1\right)} = \frac{A}{(x+1)} + \frac{Bx+C}{\left(x^2-4x+1\right)}$ (1)

$\therefore x^2 = A\left(x^2-4x+1\right)+(Bx+C)(x+1)$ (2)

Putting $x+1=0$ or $x=-1$ in (2), we get : $A=1/6$. To find B and C,

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comparing the coefficients of $x^2 : 1 = A + B \Rightarrow B = 1 - A = 5/6$

$x \ : 0 = -4A + B + C \Rightarrow C = 4A - B = -1/6$. Thus, (1) becomes after integration

$$\int \frac{x^2}{(1+x)(x^2-4x+1)}dx = \frac{1}{6}\left[\int \frac{dx}{x+1} + \int \frac{5x-1}{x^2-4x+1}dx\right] = \frac{1}{6}\left[\ln(x+1) + 5\int \frac{x-(1/5)}{x^2-4x+1}dx\right]$$

$$= \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\int \frac{2x-\frac{2}{5}}{x^2-4x+1}dx\right] = \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\int \frac{2x-4+4-\frac{2}{5}}{x^2-4x+1}dx\right]$$

$$= \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\int \frac{(2x-4)}{x^2-4x+1}dx + 9\int \frac{1}{x^2-4x+1}dx\right]$$

$$= \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\ln\left(x^2-4x+1\right) + 9\int \frac{1}{\left(x^2-4x+4\right)-3}dx\right]$$

$$= \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\ln\left(x^2-4x+1\right) + 9\int \frac{1}{(x-2)^2-\left(\sqrt{3}\right)^2}dx\right]$$

Putting $x - 2 = z$ ➔ $dx = dz$. Also put $\sqrt{3} = a$. Thus

$$I = \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\ln\left(x^2-4x+1\right) + 9\int \frac{1}{z^2-a^2}dz\right]$$

$$= \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\ln\left(x^2-4x+1\right) + 9\frac{1}{a}\sin^{-1}\left(\frac{z}{a}\right)\right]$$

$$= \frac{1}{6}\left[\ln(x+1) + \frac{5}{2}\ln\left(x^2-4x+1\right) + \frac{9}{\sqrt{3}}\sin^{-1}\left(\frac{x-2}{\sqrt{3}}\right)\right] + C$$

63. $$\int \frac{\cos x}{(\cos x + \sin x)}dx = \int \frac{\cos x(\cos x - \sin x)}{(\cos x + \sin x)(\cos x - \sin x)}dx = \int \frac{\cos^2 x - \sin x \cos x}{(\cos^2 x - \sin^2 x)}dx$$

$$= \int \frac{\left(\frac{1-\cos 2x}{2}\right) - \frac{1}{2}(2\sin x \cos x)}{\cos 2x}dx = \frac{1}{2}\left[\int \frac{1}{\cos 2x}dx - \int \frac{\cos 2x}{\cos 2x}dx - \int \frac{\sin 2x}{\cos 2x}dx\right]$$

$$= \frac{1}{2}\left[\int \sec 2x\, dx - \int 1\, dx - \frac{1}{2}\int \tan 2x dx = \frac{1}{2}\ln\left(\sec 2x + \tan 2x\right) - x - \frac{1}{2}.\frac{1}{2}\ln \sec 2x + C\right]$$

$$= \frac{1}{2}\ln\left(\sec 2x + \tan 2x\right) - \frac{1}{4}\ln \sec 2x - x + C$$

64. $$\int \frac{\sin x}{(\sin x - \cos x)}dx = \int -\frac{\sin x}{(\cos x - \sin x)} \times \frac{(\cos x + \sin x)}{(\cos x + \sin x)}dx = -\int \frac{\sin x \cos x + \sin^2 x}{\cos^2 x - \sin^2 x}dx$$

$$= -\int \frac{\frac{1}{2}(2\sin x \cos x) + \left(\frac{1-\cos 2x}{2}\right)}{\cos 2x}dx = -\frac{1}{2}\int \frac{\sin 2x}{\cos 2x} - \frac{1}{2}\int \frac{1}{\cos 2x}dx + \frac{1}{2}\int \frac{\cos 2x}{\cos 2x}dx$$

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$$=\frac{1}{2}\left[-\int \tan 2x\,dx-\int \sec 2x\,dx+\int 1\,dx\right]=\frac{1}{2}\left[-\frac{1}{2}\ln(\sec 2x)-\frac{1}{2}\ln(\sec 2x+\tan 2x)+x\right]+C$$

65. $\int \frac{\tan x}{1+\cos x}dx=\int \frac{\tan x}{1+\cos x}\times\frac{1-\cos x}{1-\cos x}dx=\int \frac{\tan x(1-\cos x)}{(1-\cos^2 x)}dx$

$$=\int \frac{\sin x(1-\cos x)}{\cos x\sin^2 x}dx=\int \frac{1-\cos x}{\cos x.\sin x}dx=\int\left[\frac{1}{\sin x\cos x}-\frac{\cos x}{\sin x\cos x}\right]dx$$

$$=\int \frac{2}{2\sin x\cos x}dx-\int \frac{1}{\sin x}dx=2\int \frac{1}{\sin 2x}dx-\int \csc x\,dx=2\int \csc 2x\,dx-\int \csc x\,dx$$

$$=2\frac{1}{2}\ln\left(\csc 2x-\cot 2x\right)-\ln\left(\csc x-\cot x\right)+C=\ln\left(\csc 2x-\cot 2x\right)-\ln\left(\csc x-\cot x\right)+C$$

66. A tourist accidentally drops his camera from the top of a cliff that is 576 feet above the water below. Assume the acceleration due to gravity to be -32 feet per second per second.

(a) Determine the velocity v(t) of the camera at any time t during its fall.
(b) Determine s(t) the height of the camera above the water at any time t during its fall.
(c) How fast is the camera falling 4 seconds after it is dropped?
(d) How long will it take the camera to hit the water?

Solution:

(a) $a=\frac{dv}{dt}=-32$. Integrating $v=-32\int 1dt=-32t+C_1$. This is the velocity at any time.

(b) $v=ds/dt=-32t+C_1$. Integrating, $s=-32\int t\,dt+C_1\int 1\,dt+C_2=-16t^2+C_1t+C_2$

This is the distance covered by camera at any time.

(c) At time $t=0$, $v=0\Rightarrow C_1=0$. Thus, $v=-32t$. Putting $t=4$, we get $v=-128$ ft / sec.

Thus after 4 seconds camera will be falling at the speed of 128 ft / sec.

(d) At $t=0$, $s=576$ ft $\Rightarrow C_2=576$. Thus, $s=-16t^2+576$.

Now, when the camera hits the ground $s=0\Rightarrow -16t^2+576=0\Rightarrow t=6$.

Thus, it takes camera 6 seconds to hit the ground.

67. To test learning, a psychologist asks people to memorize a long sequence of digits. Assume that the rate at which digits are being memorized is $dy/dt = 5.4e^{-0.3t}$ words per minute, where y is the number of digits memorized and t is the time in minutes.

(a) Find y as a function of t, which gives the number of digits memorized after t minutes.
(b) How many digits will be memorized after 5 minutes?

Solution:

(a) Given $dy/dt=5.4e^{-0.3t}$. Integrating, $y=5.42\int e^{-0.3t}dt=5.4\frac{e^{-0.3t}}{-0.3}+C=-18e^{-0.3t}+C$.

Put $t=0$ and $y=0$ (Thenumber of digits memorized at the begining is zero), we get

$0=-18e^0+C\Rightarrow C=18$. Thus, $y=-18e^{-0.3t}+18$.

(b) Put $t=5$, we get $y=-18e^{-0.3(5)}+18\cong 14$. Thus, after 5 minutes a person can memorize about 14 digits.

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68. The rate of change of the volume of a spherical balloon with respect to its radius is $dV/dr = 4\pi r^2$. Use this fact and the fact that V = 0 when r = 0, determine the volume of the balloon when its radius is 6 centimeters.

Solution: Given $dV/dr = 4\pi r^2$. Integrating, $V = 4\pi\int r^2\,dr = \frac{4\pi r^3}{3} + C$.

Put $r = 0$ and $V = 0$ (Given), thus $0 = 0 + C \Rightarrow C = 0$. Thus, $V = \frac{4\pi r^3}{3}$.

Put $r = 6$, we get $V = \frac{4\pi(6)^3}{3} = 904.78\ cm^3$.

69. If the marginal cost when x units is produced is C(x) = 100 – 0.5x dollars and the overhead cost is $40, what is the cost of producing 10 units?

Solution:

Given $MC = C`(x) = 100 - 0.5x$. Integrating, $C = \int(100 - 0.5x)dx = 100x - 0.25x^2 + c$.

Put $x = 0$ and $C = 40$ (Overhead expensives), thus $40 = 0 + 0 + c \Rightarrow c = 40$. Thus,

$C = 100x - 0.25x^2 + 40$. Now put $x = 10$, we get: $C = 1000 - 25 + 40 = \$1015$.

This is the cost of producing 10 items.

70. Let R(x) be the revenue a company receives from the sale of x units of its product. If its marginal revenue is **R`(x) = 100 – 0.2x** dollars per item, find the:

(a) Revenue function R(x) and (b) the revenue from the sale of 20 units. Assume there is no revenue when zero units are sold.

Solution:

(a) Given $MR = R`(x) = 100 - 0.2x$. Integrating,

$R = \int(100 - 0.2x)dx = 100x - 0.1x^2 + c$. Put $x = 0$ and $R = 0$, thus $0 = 0 + 0 + c \Rightarrow c = 0$.

Thus, $R = 100x - 0.1x^2$.

(b) Now put $x = 20$, we get: $R = 2000 - 4 = \$1996$. 73. A woman gets into her car and

This is the revenue from the sale of 20 items.

71. A man starts his car and then drives it with a constant acceleration of 22 feet per second per second.

(a) Determine the velocity function.
(b) Determine the distance function.
(c) How far does the car go in 6 seconds?

Solution:

(a) $a = dv/dt = 22$. Integrating $v = 22\int 1dt = 22t + C_1$. This is the velocity at any time.

(b) $v = ds/dt = 22t + C_1$. Integrating, $s = 22\int t\,dt + C_1\int 1dt + C_2 = 11t^2 + C_1t + C_2$

(c) When woman gets into the car, certainly at $t = 0$, $s = 0$ and $v = 0$. Thus,

$C_1 = C_2 = 0 \Rightarrow s = 11t^2$. Put $t = 6$, we get : $s = 11(36) = 396$ feet.

72. A colony of 200 bacteria is introduced to a growth-inhibiting environment and grows at the rate of dn/dt = 30 + 2t, where n is the number of bacteria present at any time t (t is measured in hours).

(a) Determine a function that gives the number of bacteria present at any time t.
(b) How many bacteria are present after 3 hours?

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Solution:

(a) $dn/dt = 30 + 2t$. Integrating, $n = 30t + t^2 + C$. Putting $t = 0$ and $n = 200 \Rightarrow C = 200$.

Thus, $n = 30t + t^2 + 200$.

(b) Put $t = 3$, we get : $n = 90 + 9 + 200 = 299$. Thus after 3 hours, number of bacteria is 299

73. The weight of a mold is growing exponentially at the rate of $dw/dt = e^{0.2t}$ milligrams per hour. How much will the mold weigh in 10 hours if it weighs 70 milligrams now?

Solution:

(a) Given $dw/dt = e^{0.23t}$. Integrating, $w = \int e^{0.2t}\,dt = \dfrac{e^{0.2t}}{0.2} + C = 5e^{0.2t} + C$.

Put $t = 0$ and $w = 70$, we get : $70 = 5e^0 + C = 5 + C \Rightarrow C = 65$. Thus, $w = 5e^{0.2t} + 65$

(b) Put $t = 10$, we get $w = 5e^{0.2(10)} + 65 \cong 71$.

Thus, after 10 hours, the weight of mold will be 71 mg

74. The rate of change of the temperature T inside a furnace after x minutes is $dT/dx = 2x + 15$ $(1 < x < 20)$ degree/min. Assume the temperature inside the furnace is 200°F initially.

(a) Find the formula for the temperature at any time x.
(b) What is the temperature inside the furnace after 14 minutes?

Solution:

(a) Given $dT/dx = 2x + 15$. Integrating, $T = \int (2x + 15)\,dx = x^2 + 15x + C$

Put $x = 0$ and $T = 200$, we get : $200 = 0 + 0 + C \Rightarrow C = 200$. Thus, $T = x^2 + 15x + 200$

(b) Put $x = 14$, we get $T = 196 + 210 + 200 = 604^{o}F$.

75. After t hours of production, a coal mine is producing coal at the rate of $dP/dt = 30 + 2t - 0.03t^2$ tons per hour. Find a formula for the total output of the coal mine after t hours of production. (Note: Coal production P = 0 at t = 0).

Solution:

Given $dP/dt = 30 + 2t - 0.03^2$. Integrating, $P = \int (30 + 2t - 0.3t^2)\,dt = 30t + t^2 - 0.1t^3 + C$

Put $tx = 0$ and $P = 0$, we get : $C = 0$. Thus, $P = 30t + t^2 - 0.1t^3$

76. The rate at which atmospheric pressure P changes as the height x above sea level changes is $dP/dx = -3.087\ e^{-0.21x}$ measured in pounds per square inch and x is in miles. Determine P as a function of x when at sea level, P is 14.7 pounds per square inch.

Solution:

(a) Given $dP/dx = -3.078e^{-0.12x}$. Integrating, $P = -3.078\int e^{-0.12x}\,dx = -3.079\dfrac{e^{-0.12x}}{-0.12} + C$

$\Rightarrow P = 25.65e^{-0.12x} + C$. Put $x = 0$ and $P = 14.7$, we get : $14.70 = 25.65 + C \Rightarrow C = -10.95$.

Thus, $P = 25.65e^{-0.12x} - 10.95$.

77. The rate of the change of the area of a circular region with respect to its radius is $dA/dr = 2\pi r$. Use this fact, and the fact that A = 0 when r = 0, to determine the area of a circular region when the radius is 4 centimeters.

Solution: Given $dA/dr = 2\pi r$. Integrating, $A = 2\pi\int r\,dr = \pi r^2 + C$.

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Put $r = 0$ and $A = 0$ (Given), thus $0 = 0 + C \Rightarrow C = 0$. Thus, $A = \pi r^2$.

Put $r = 4$, we get $A = 16\pi \text{ cm}^2$.

78. The rate of change of the volume of a spherical balloon with respect to its radius is $dV/dr = 4\pi r^2$. Use this fact, and the fact that V = 0 when r = 0, to determine volume of the balloon when its radius is 6 centimeters.

Solution: Given $dV/dr = 4\pi r^2$. Integrating, $V = 4\pi\int r^2\,dr = \frac{4\pi r^3}{3} + C$.

Put $r = 0$ and $V = 0$ (Given), thus $0 = 0 + C \Rightarrow C = 0$. Thus, $V = \frac{4\pi r^3}{3}$.

Put $r = 6$, we get $V = \frac{4\pi(6)^3}{3} = 904.78 \text{ cm}^3$.

79. Flu epidemic is spreading at the rate $dn/dt = 180t - 6t^2$, where n is the number of people who are sick with flu on any particular day t after the outbreak started.

(a) Determine an equation for n as a function of t. Assume no one has the flu at the beginning (when t = 0).

(b) How many people have the flu the tenth day after the outbreak begins?

(a) Given $dn/dt = 180t - 6t^2$. Integrating, $n = \int\left(180t - 6t^2\right)dt = 90t^2 - 2t^3 + C$

Put $t = 0$ and $n = 0$, we get : $C = 0$. Thus, $n = 90t^2 - 2t^3$

(b) Put $t = 10$, $n = 9000 - 2000 = 7000$.

Thus, after 10 hours 7000 people have been affected by flu.

80. Plaque builds up on the inside walls of an artery reduces the diameter of the artery (and thus reduces the blood flow). Suppose that an artery has a diameter of 0.4 centimeter and the length of diameter (D) is decreasing at the rate of $dD/dt = -0.03e^{-0.001t}$ cm/year.

(a) Determine D as a function of t.

(b) What will the diameter of the artery be after 10 years?

(a) Given $dD/dt = -0.03e^{-0.001t}$. Integrating, $D = -0.03\int e^{-0.001t}\,dt = \frac{-0.03e^{-0.001t}}{-0.001} + C$

$D = 30e^{-0.001t} + C$. Put $t = 0$ and $D = 0.4$, we get : $0.4 = 30e^0 + C = 30 + C \Rightarrow C = -29.6$.

Thus, $D = 30e^{-0.001t} - 29.6$

(b) Put $t = 10$, we get $D = 30e^{-0.001(10)} - 29.6 = 27.145 - 29.6 = -1.455\text{cm}$.

81. A manufacturer estimates that marginal cost of a certain production process is given by $C'(x) = e^{0.01x} + 3\sqrt{x}$, where x units are produced. What does it cost to produce 10 units if the cost of producing 4 units is \$ 2000?

Solution : $dC/dt = e^{0.01x} + 3\sqrt{x}$. Integrating, $C = \int\left(e^{0.01x} + 3\sqrt{x}\right)dx = \frac{e^{0.01x}}{0.01} + 2x^{3/2} + c$

Put $C = 2000$, $x = 4$, we get : $2000 = \frac{e^{0.04}}{0.01}2(4)^{3/2} + c \Rightarrow c = 335$.

Thus, $C = \frac{e^{0.01x}}{0.01} + 2x^{3/2} + 335$

(b) Put $t = 10$, we get $C = e^{0.01(10)}/(0.01) + 2(10)^{3/2} + 335 = 110.5 + 284 + 335 = 729.5$

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Chapter 10

DEFINITE INTEGRATION

WORKSHEET 10

1. The rate at which petroleum was consumed in Pakistan was approximately $c'(t) = 21t + 281$ million barrels per year from 1983 (t = 0) to 1987 (t = 4). Determine the total amount of petroleum consumed from 1983 to 1987.

Solution : Given $c`(t) = 21t + 281$. Integrating from 0 to 4 we get,

$$c = \int_0^4 (21t + 281)\,dt = \frac{21t^2}{2} + 281t\Big|_0^4 = \frac{21\times 16}{2} + 281(4) = 128 + 1124 = 1252$$

Thus, total consumption of petrol from 1983 to 1987 is about 1252 million barrels.

2. A man starts his car and then drives it with a constant acceleration of 18 feet per second per second. How far does the car go in 5 seconds?

Solution : Given $a = dv/dt = 18$. Integrating we get, $v = \int 18\,dt + C = 18t + C$

Putting $t = 0$ and $v = 0$, we obtain $C = 0$. Thus, $v = s` = 18t$. Now, integrating from 0 to 5,

$$s = \int_0^5 18t\,dt = \frac{18t^2}{2}\Big|_0^5 = \frac{18\times 25}{2} = 225 \text{ ft}$$

Thus car travels for 225 ft during the first 5 seconds.

3. Find the volume of the solid produced by revolving about the x – axis the region whose boundary is given. (Note: y = 0 is the equation of the x – axis.)

Solution: We know that volume of solid of revolution is given by: $V = \int_a^b \pi\{f(x)\}^2\,dx$

(a) Thus volume generated by $y = x^2 + 1$ between $y = 0, x = 0, x = 1$ is

$$V = \int_0^1 \pi\left(x^2+1\right)^2 dx = \pi\int_0^1 \left(x^4 + 2x^2 + 1\right)dx = \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^1 = \pi\left[\frac{1}{5} + \frac{2}{3} + 1\right] = \frac{28\pi}{15}$$

(b) The volume generated by $y = 1/\sqrt{x}$ between $x = e, x = 10$ is

$$V = \int_e^{10} \pi\left(\frac{1}{\sqrt{x}}\right)^2 dx = \pi\int_e^{10} \frac{1}{x}\,dx = \pi[\ln x]_e^{10} = \pi[\ln 10 - \ln e] = \pi[\ln 10 - 1]$$

(c) The volume generated by $y = x^2 - 5x$ between $y = 0, x = 5.1, x = 5.5$ is

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$$V=\int_0^1 \pi\left(x^2-5x\right)^2 dx=\pi\int_{5.1}^{5.5}\left(x^4-10x^3+25x^2\right)dx=\pi\left[\frac{x^5}{5}-\frac{10x^4}{4}+\frac{25x^3}{3}\right]_{5.1}^{5.5}$$

$$=\pi\left[\frac{(5.5-5.1)^5}{5}-\frac{5(5.5-5.1)^4}{2}+\frac{25(5.5-5.1)^3}{3}\right]=0.478\pi$$

4. Determine the area enclosed between the two curves. Graphing is recommended:

(i) $y=x+5,\ y=\sqrt{x}$ from $x=0$ to $x=4$: Area b / w two curves is $\int_a^b\left(f(x)-g(x)\right)dx$

$$\therefore A=\int_0^4\left(x+5-\sqrt{x}\right)dx=\left[\frac{x^2}{2}+5x-\frac{2x^{3/2}}{3}\right]_0^4=8+20-\frac{16}{3}=\frac{68}{3}$$

(ii) $y=x^3,\ y=x^2$: Solving the two equations, we get $x=0$ and $x=1$. Thus,

Area b / w two curves is $\int_a^b\left(f(x)-g(x)\right)dx$

$$\therefore A=\int_0^1\left(x^2-x^3\right)dx=\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}$$

(iii) $y=1-x^2,\ y=x+4$ from $x=-3$ to $x=1$: The area b / w two curves is

$$\int_a^b\left(f(x)-g(x)\right)dx\ \Rightarrow A=\int_{-3}^1\left[\left(x+4\right)-\left(1-x^2\right)\right]dx=\left[\frac{x^2}{2}+4x-x+\frac{x^3}{3}\right]_{-3}^1$$

$$=\left(\frac{1-9}{2}\right)+4\left(1+3\right)-\left(1+3\right)+\left(\frac{1+27}{3}\right)=-4+16-4+\frac{28}{3}=\frac{52}{3}$$

(iv) $y=e^x,\ y=1$ and $x=1$

Since, $y=e^x,\ y=1$; solving the two equations, we get $e^x=1$ gives $x=0$. Thus,

Area b / w two curves is $\int_a^b\left(f(x)-g(x)\right)dx$

$$\therefore A=\int_0^1\left(e^x-1\right)dx=\left[e^x-x\right]_0^1=(e-e^0)-(1-0)=e-1-1=e-2=2.718-2=0.718$$

The graphs and areas are shown below.

5. Determine the area enclosed by the curves. In each case, the region enclosed consists of two regions. A careful study of the graphs is essential.

(i) $y=x^3,\ y=-x$:

Solving the two equations, we get x = 0 and x = ± i. From the graph, we observe that two curves intersect each other at origin only and the area between the curves increases indefinitely on either sides of origin. Thus, area between two curves is infinite.

(ii) $y=x^4,\ \ y=x^2$

Solving the two equations simultaneously, we get: x = 0 and x = ± 1. The area from -1 to 0 and from 0 to 1 is same, as shown in the figure. Hence required area is:

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$$A=\int_a^b\left(f(x)-g(x)\right)dx=2\int_0^1\left(x^2-x^4\right)dx=2\left[\frac{x^3}{3}-\frac{x^5}{5}\right]_0^1=2\left(\frac{1}{3}-\frac{1}{5}\right)=\frac{4}{15}$$

The graphs and areas are shown below.

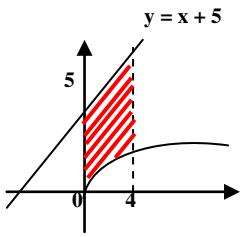

Fig. (i)

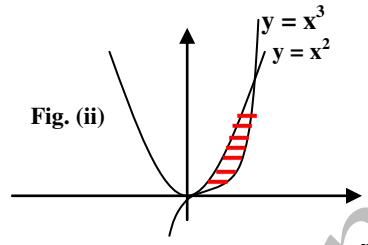

Fig. (ii)

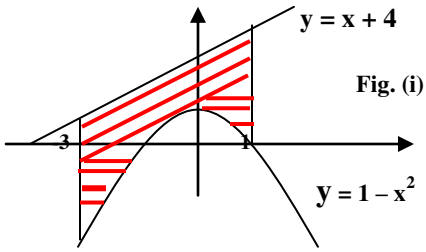

Fig. (i)

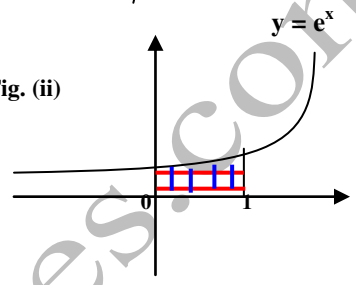

Fig. (ii)

Graphs of Q. 4

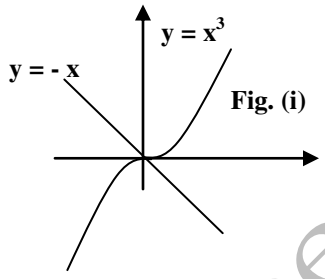

Fig. (i)

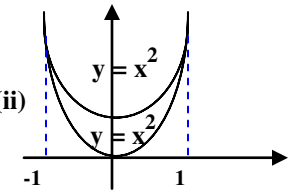

Fig. (ii)

Graphs of Q. 5

6. Let t be the number of years from now. What is the total amount of waste dumped into a lake by a factory that dumps waste at the rate of $5e^{-0.1t}$ pounds per year indefinitely?
Solution: If W is the amount of waste dumped then as given $dW/dt = 5e^{-0.1t}$. Thus, total amount of waste that is dumped from now for indefinite period is:

$$W=\int_0^\infty\frac{dW}{dt}dt=5\int_0^\infty e^{-0.1t}dt=5\left.\frac{e^{-0.1t}}{-0.1}\right|_0^\infty=-50(0-1)=50$$

7. Evaluate (i) $\displaystyle\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx$:

$$\text{Let } I=\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx=\int_0^{\pi/2}\frac{\sqrt{\cot(\pi/2-x)}}{\sqrt{\cot(\pi/2-x)}+\sqrt{\tan(\pi/2-x)}}dx \quad \text{(By P}-4) \qquad (1)$$

$$=\int_0^{\pi/2}\frac{\sqrt{\cot(x)}}{\sqrt{\cot(x)}+\sqrt{\tan(x)}}dx=I \quad [\text{since } \tan(\pi/2-x)=\cot x \text{ and } \cot(\pi/2-x)=\tan x]$$

$$\text{Thus, } I+I=\int_0^{\pi/2}\frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx+\int_0^{\pi/2}\frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}dx=\int_0^{\pi/2}\frac{\sqrt{\tan x}+\sqrt{\cot x}}{\sqrt{\tan x}+\sqrt{\cot x}}dx$$

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$$\therefore \int_0^{\pi/2} 1\,dx = x\Big|_0^{\pi/2} = \frac{\pi}{2} \;\Rightarrow\; 2I = \frac{\pi}{2} \quad \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+\sqrt{\cot x}}\,dx = \frac{\pi}{4}$$

(ii) $\displaystyle\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$$\text{Let } I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2-x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}\,dx \quad (\text{By } P-4) \qquad (1)$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = I \qquad [\text{since, } \sin(\pi/2-x) = \cos x \text{ and } \cos(\pi/2-x) = \sin x]$$

$$\text{Thus, } I + I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$\therefore \int_0^{\pi/2} 1\,dx = x\Big|_0^{\pi/2} = \frac{\pi}{2} \;\Rightarrow\; 2I = \frac{\pi}{2} \quad \Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \frac{\pi}{4}$$

8. Show that:

(i) $\displaystyle\int_0^{\pi/2} \ln(\tan x)\,dx = 0$

$$\text{Solution: Let } I = \int_0^{\pi/2} \ln(\tan x)\,dx = \int_0^{\pi/2} \ln\left[\tan\left(\frac{\pi}{2}-x\right)\right]dx = \int_0^{\pi/2} \ln(\cot x)\,dx$$

$$= \int_0^{\pi/2} \ln\left(\frac{1}{\tan x}\right)dx = \int_0^{\pi/2} \ln(\tan x)^{-1}\,dx = -\int_0^{\pi/2} \ln(\tan x)\,dx = -I$$

$$\text{Thus, } I + I = 0 \;\Rightarrow 2I = 0 \;\Rightarrow I = 0 \quad \text{or} \quad \int_0^{\pi/2} \ln(\tan x)\,dx = 0. \text{ Proved}$$

(ii) $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$

$$\text{Solution: Let } I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}\,dx$$

$$I = \int_0^{\pi} \frac{\pi\sin x}{1+\cos^2 x}\,dx - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx - I.$$

Putting $z = \cos x \;\Rightarrow dz = -\sin x\,dx$ or $-dz = \sin x\,dx$. Also,

when $x = 0$ then $z = \cos 0 = 1$ and when $x = \pi$ then $z = \cos\pi = -1$. Thus,

$$2I = -\pi\int_1^{-1} \frac{1}{1+z^2}\,dz = -\pi\left[\tan^{-1} z\right]_1^{-1} = -\pi\left[\tan^{-1}(-1) - \tan^{-1}(1)\right] = -\pi\left[\frac{-\pi}{4} - \frac{\pi}{4}\right] = \frac{\pi^2}{2}$$

$$\Rightarrow I = \frac{\pi^2}{4} \quad \text{OR} \quad \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \frac{\pi^2}{4}$$

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(iii) $\int_0^1 \ln\left(\frac{1}{x}-1\right)dx = 0$

Solution : $\int_0^1 \ln\left(\frac{1}{x}-1\right)dx = \int_0^1 \ln\left(\frac{1-x}{x}\right)dx = \int_0^1 \ln(1-x)\,dx - \int_0^1 \ln x\ dx$

$= \int_0^1 \ln\left[1-(1-x)\right]\,dx - \int_0^1 \ln x\ dx = \int_0^1 \ln x\ dx - \int_0^1 \ln x\ dx = 0$

(iv) $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$

Solution : Let, $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

Putting $x = \tan\theta \Rightarrow dx = \sec^2\theta\ d\theta$. If $x = 0$ then $\tan\theta = 0 \Rightarrow \theta = 0$.

If $x = 1$ then $\tan\theta = 1 \Rightarrow \theta = \pi/4$. Also, $1+x^2 = 1+\tan^2\theta = \sec^2\theta$

Thus, $I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\ d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)\ d\theta = \int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$

$= \int_0^{\pi/4} \ln\left(1+\left(\frac{1-\tan\theta}{1+\tan\theta}\right)\right)d\theta = \int_0^{\pi/4} \ln\left(\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right)d\theta = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right)d\theta$

$\therefore\ I = \int_0^{\pi/4} \ln 2\ d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\ d\theta = \ln 2\int_0^{\pi/4} 1.\ d\theta - I = \ln 2\left[\theta\right]_0^{\pi/4} - I = \frac{\pi}{4}\ln 2 - I$

$\Rightarrow 2I = \frac{\pi}{4}\ln 2 \quad \Rightarrow I = \frac{\pi}{8}\ln 2$. Thus, $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$

9. Discuss the convergence of

(i) $\int_1^\infty \frac{x}{(1+x)^3}dx = \int_1^\infty \frac{(1+x)-1}{(1+x)^3}dx = \int_1^\infty \frac{1+x}{(1+x)^3}dx - \int_1^\infty \frac{1}{(1+x)^3}dx = \int_1^\infty (1+x)^{-2}dx - \int_1^\infty (1+x)^{-3}dx$

$= -\frac{1}{(1+x)}\Big|_1^\infty + \frac{1}{2(1+x)}\Big|_1^\infty = -\left(0-\frac{1}{2}\right) + \frac{1}{2}\left(0-\frac{1}{2}\right) = \frac{1}{2}$

which is finite. Hence given integral converges.

(ii) $\int_0^\infty xe^{-x^2}dx$: Putting $x^2 = z \Rightarrow x\,dx = dz/2$. If $x = 0 \Rightarrow z = 0$. If $x = \infty$ then $z = \infty$.

Thus, $\int_0^\infty xe^{-x^2}dx = \frac{1}{2}\int_0^\infty e^{-z}dz = -\frac{1}{2e^z}\Big|_0^\infty = -\frac{1}{2}(0-1) = \frac{1}{2}$ which is finite.

Hence given integral converges.

(iii) $\int_1^\infty \frac{dx}{(1+x)\sqrt{x}}$: Putting $z = \sqrt{x} \Rightarrow z^2 = x \Rightarrow 2z\,dz = dx \Rightarrow 2dz = \frac{dx}{z} = \frac{dx}{\sqrt{x}}$. Thus,

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$$\int_1^\infty \frac{dx}{(1+x)\sqrt{x}} = 2\int_1^\infty \frac{dz}{(1+z^2)} = 2\left[\tan^{-1} z\right]_0^\infty = 2\left[\tan^{-1}\infty - \tan^{-1} 0\right] = 2\left[\frac{\pi}{2} - 0\right] = \pi$$

which is finite. Hence given integral converges.

(iv) $\int_1^2 \frac{x\,dx}{\sqrt{x-1}}$: Puting $z = \sqrt{x-1} \Rightarrow z^2 = x-1 \Rightarrow 1+z^2 = x \quad \Rightarrow 2z\,dz = dx.$

Also if $x = 1$ then $z = 0$ and if $x = 2$ then $z = 1$. Thus,

$$\int_1^2 \frac{x\,dx}{\sqrt{x-1}} = \int_0^1 \frac{1+z^2}{z}.2z\,dz = 2\left[z + \frac{z^3}{3}\right]_0^1 = 2\left(1+\frac{1}{3}\right) = \frac{8}{3} \text{ which is finite.}$$

Hence given integral converges.

(v) $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$

Puting $z = \sqrt{1-\sin x} \Rightarrow z^2 = 1-\sin x \Rightarrow 1-z^2 = \sin x \quad \Rightarrow 2z\,dz = -\cos x\,dx.$

Also if $x = 0$ then $z = 1$ and if $x = \pi/2$ then $z = 0$. Thus,

$$\int_0^{\pi/2} \frac{\cos x\,dx}{\sqrt{1-\sin x}} = \int_1^0 \frac{-2z}{z}\,dz = -2\int_1^0 1\,dz = -2\left[z\right]_1^0 = -2(0-1) = 2 \text{ which is finite.}$$

Hence given integral converges.

10 Evaluate (i) $\int_0^\infty x^3 e^{-x^3}\,dx$: Putting $x^3 = z \Rightarrow 3x^2\,dx = dz \Rightarrow dx = dz/3x^2 = dz/3z^{2/3}$

$$\int_0^\infty x^3 e^{-x^3}\,dx = \frac{1}{3}\int_0^\infty z^{-2/3}\,e^{-z}dz = \frac{1}{3}\,\Gamma\left(\frac{1}{3}\right). \qquad \text{NOTE}: \int_0^\infty e^{-z}\,z^n\,dz = \Gamma(n+1)$$

(ii) $\int_0^\infty \sqrt{x}e^{-x^3}\,dx$: Putting $x^3 = z \Rightarrow 3x^2\,dx = dz \Rightarrow dx = dz/3x^2 = dz/3z^{2/3}$. Thus,

$$\int_0^\infty \sqrt{x}e^{-x^3}\,dx = \frac{1}{3}\int_0^\infty z^{-2/3}\,e^{-z}\,dz = \frac{1}{3}\,\Gamma\left(\frac{1}{3}\right).$$

(iii) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\,dx$: We know that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta$. Thus,

$2m-1 = 3 \Rightarrow m = 2$ and $2n-1 = 5/2 \Rightarrow n = 7/4.$

$$\therefore \int_0^{\pi/2} \sin^3 x \cos^{5/2} x\,dx = \frac{\beta(m,n)}{2} = \frac{\beta(2,7/4)}{2}$$

(iv) $\int_0^{\pi/2} \sin^7 x\,dx = \frac{6}{7}\int_0^{\pi/2} \sin^5 x\,dx = \frac{7}{7}.\frac{4}{6}\int_0^{\pi/2} \sin^3 x\,dx = \frac{6}{7}.\frac{4}{6}.\frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \frac{8}{21}\left[-\cos x\right]_0^{\pi/2} = \frac{8}{21}$

(v) $\int_0^{\pi/2} \sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\,d\theta = \int_0^{\pi/2} \sin^{1/2}\theta.\cos^{-1/2}\theta\,d\theta$:

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We know that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$. Thus,

$2m-1 = 1/2 \Rightarrow m = 4/3$ and $2n-1 = -1/2 \Rightarrow n = 1/4$.

$$\therefore \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta.\cos^{-1/2}\theta \, d\theta = \frac{\beta(m,n)}{2} = \frac{\beta\left(\frac{4}{3},\frac{1}{4}\right)}{2}$$

(v) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{\sqrt{\cos\theta}}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta.\cos^{-1/2}\theta \, d\theta$:

We know that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$. Thus,

$2m-1 = 1/2 \Rightarrow m = 4/3$ and $2n-1 = -1/2 \Rightarrow n = 1/4$.

$$\therefore \int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta.\cos^{-1/2}\theta \, d\theta = \frac{\beta(m,n)}{2} = \frac{\beta\left(\frac{4}{3},\frac{1}{4}\right)}{2}$$

11. Use the definition of Gamma Function, evaluate the following:

(i) $\int_0^{\infty} \sqrt{x}e^{-x}dx$: We know that $\Gamma(n) = \int_0^{\infty} e^{-x}x^{n-1}dx$, where $n > 0$.

Comparing we get, $n-1 = 1/2 \Rightarrow n = 3/2$. Thus,

$\int_0^{\infty} \sqrt{x}e^{-x}dx = \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2\sqrt{\pi}}$. NOTE: $\Gamma(n+1) = n\Gamma(n)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

(ii) $\int_0^{\infty} x^4 e^{-x^2}dx$: Putting $x^2 = z \Rightarrow x^4 = z^2$ and $2x\,dx = dz \Rightarrow dx = dz/2x = dz/2\sqrt{z}$

Thus, $\int_0^{\infty} x^4 e^{-x^2}dx = \int_0^{\infty} z^2 e^{-z}\frac{dz}{2\sqrt{z}} = \frac{1}{2}\int_0^{\infty} z^{3/2}e^{-z}\,dz = \frac{1}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{8}\sqrt{\pi}$

(iii) $\int_0^{\infty} e^{-h^2x^2}dx$: Putting $h^2x^2 = z \Rightarrow 2h^2x\,dx = dz$

$\Rightarrow dx = dz/2h^2x = hdz/2h^2\sqrt{z} = dz/2h\sqrt{z}$

Thus, $\int_0^{\infty} e^{-h^2x^2}dx = \int_0^{\infty} e^{-z}\frac{dz}{2h\sqrt{z}} = \frac{1}{2h}\int_0^{\infty} z^{-1/2}e^{-z}\,dz = \frac{1}{2h}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2h}\sqrt{\pi}$

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Chapter 11

VECTOR ANALYSIS

WORKSHEET 11

1. Find the vector whose magnitude is that of the vector 5**i** – 3**j** + 9**k** and is in the direction of the vector 4**i** – 3**j** + **k**.

Solution: Magnitude of the vector 5**i** – 3**j** + 9**k** is $\sqrt{5^2+(-3)^2+9^2}=\sqrt{115}$. Thus, vector of magnitude $\sqrt{115}$ and in the direction of vector 4**i** – 3**j** + **k** is:

$$\frac{4}{\sqrt{115}}i-\frac{3}{\sqrt{115}}j+\frac{1}{\sqrt{115}}k$$

2. Find the value of 't' if the vectors 5**i** – 3**j** + 9**k** and 4**i** – t**j** + **k** have same direction.

Solution: If **a** = 5**i** – 3**j** + 9**k** and **b** = 4**i** – t**j** + **k** have same direction then we must have:

a = m**b**

OR 5 = 4m (i) 3 = mt (ii) 9 = m (iii)

From (i) m = 5/4, and from (iii) m = 9. Since both values are different hence, vectors **a** and **b** are never in the same direction for any value of 't'.

3. For what value of p the vectors **a** = [2, 4, -7] and **b** = [2, 6, p] are perpendicular?

Solution: We know that if two vectors are perpendicular, their dot product is zero, that is;

$$\mathbf{a.b} = a_1b_1 + a_2b_2 + a_3b_3 = 4 + 24 - 7p = 0 \quad \text{giving } p = 4.$$

4. Find the area of the parallelogram determined by 3**i** + 4**j** and **i** + **j** + **k**.

Solution: The area of parallelogram whose adjacent sides are vectors **a, b** is |**a** × **b**|.

Now, $\mathbf{a}\times\mathbf{b}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} i & j & k \\ 3 & 4 & 0 \\ 1 & 1 & 1 \end{vmatrix}=i(4-0)-j(3-0)+k(3-4)=4i-3j-k$.

Thus, $|\mathbf{a}\times\mathbf{b}|=\sqrt{4^2+(-3)^2+(-1)^2}=\sqrt{26}$

5. Find the volume of parallelepiped if **a** = [3, 4, 0], **b** = [2, 3, 4] and **c** = [0, 0, 5] are its edges.

Solution: The volume of parallelepiped whose adjacent sides are vectors **a, b** and **c** is

V = **a.**(b × **c**)

Now, V = **a. b × c**

$$=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\begin{vmatrix} 3 & 4 & 0 \\ 2 & 3 & 4 \\ 0 & 0 & 5 \end{vmatrix}=3(15-0)-4(10-0)+0(0-0)=45-40=5.$$

6. Find the volume of the tetrahedron whose vertices are the points A(2, -1, -3), B(4, 1, 3), C(3, 2, -1) and D(1, 4, 2).

Solution: Let the vectors **a** = AB = [2, 2, 6], **b** = AC = [1, 3, 2] and **c** = AD = [-1, 5, 5]

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The volume of tetrahedron whose adjacent sides are vectors **a, b** and **c** is

$$V = \mathbf{a}.(\mathbf{b} \times \mathbf{c})/6$$

V = **a. (b × c)/6 =**

$$= \frac{1}{6}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 6 \\ 1 & 3 & 2 \\ -1 & 5 & 5 \end{vmatrix} = \frac{1}{6}\left[2(15-10) - 2(5+2) + 6(5+3)\right] = \frac{22}{3}$$

7. The coordinates of a moving particle are given by $x = 4t - t^2/2$, $y = 3 + 6t - t^3$ and $z = 3t^2$. Find the velocity and acceleration of the particle when t = 2 sec.

Solution: Let $\mathbf{s} = xi + yj + zk = (4t - t^2/2)i + (3 + 6t - t^3)j + 3t^2k$. Thus,

$\mathbf{v} = d\mathbf{s}/dt = (4 - t)i + (6 - 3t^2)j + 6t\,k$ and $\mathbf{a} = d\mathbf{v}/dt = -i - 6tj + 6k$.

At t = 2, $\mathbf{v} = -2i - 6j + 12k$ and $\mathbf{a} = -i - 12j + 6k$.

8. A particle moves so that its displacement at time t is given by:

$x(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$. Find the magnitude of the velocity and acceleration of the particle at t = 0.

Solution: Given $\mathbf{x} = 2\cos t\, i + 2\sin t\, j + tk$. Thus,

$\mathbf{v} = d\mathbf{s}/dt = -2\sin t\, i + 2\cos t\, j + k$ and $\mathbf{a} = d\mathbf{v}/dt = -2\cos t\, i - \sin t\, j + 0k$.

At t = 0, $\mathbf{v} = 0i + 2j + k$ and $\mathbf{a} = -2i + 0j + 0k$. Thus, $|\mathbf{v}| = \sqrt{5}$ and $|\mathbf{a}| = 2$.

9. An acceleration of a particle is given by: $\mathbf{f}``(t) = t^2\mathbf{i} + t^2\mathbf{j} - (t-2)\mathbf{k}$

Find the velocity and displacement vector given that $\mathbf{f}`(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{f}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

Solution: Given $\mathbf{f}``(t) = t^2\mathbf{i} + t^2\mathbf{j} - (t-2)\mathbf{k}$, integrating w.r.t t, we get

$\mathbf{f}`(t) = \frac{t^3}{3}\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \left(\frac{t^2}{2} - 2t\right)\mathbf{k} + \mathbf{C}$. Putting t = 0 and $\mathbf{f}` = \mathbf{i} + \mathbf{j} + \mathbf{k}$ we get,

$\mathbf{i} + \mathbf{j} + \mathbf{k} = \mathbf{C}$. Thus, $\mathbf{f}`(t) = \frac{t^3}{3}\mathbf{i} + \frac{t^3}{3}\mathbf{j} - \left(\frac{t^2}{2} - 2t\right)\mathbf{k} + \mathbf{i} + \mathbf{j} + \mathbf{k}$

$$= \left(\frac{t^3}{3} + 1\right)\mathbf{i} + \left(\frac{t^3}{3} + 1\right)\mathbf{j} + \left(\frac{t^2}{2} - 2t + 1\right)\mathbf{k}$$

Integrating again, we get: $\mathbf{f}(t) = \left(\frac{t^4}{4} + t\right)\mathbf{i} + \left(\frac{t^4}{4} + t\right)\mathbf{j} + \left(\frac{t^3}{3} - t^2 + t\right)\mathbf{k} + \mathbf{C}$

Putting t = 0 and $\mathbf{f} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, we get: $\mathbf{C} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$. Put this in the above equation,

$\mathbf{f}(t) = \left(\frac{t^4}{4} + t + 2\right)\mathbf{i} + \left(\frac{t^4}{4} + t + 3\right)\mathbf{j} + \left(\frac{t^3}{3} - t^2 + t - 1\right)\mathbf{k}$

10. Define gradient, divergence and curl. If $\phi = 3x^2yz^2$, find $|\nabla\phi|$ at (2, 1, 2).

Solution: If ϕ is a scalar function of x, y and z, we define

$$\text{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

If $\mathbf{f} = f_1\mathbf{i} + f_2\,\mathbf{j} + f_3\,\mathbf{k}$ is a vector function, we define:

$$\text{Div } \mathbf{f} = \nabla.f = \nabla\phi = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}k \text{ and } \text{curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

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11. Find a unit vector normal to the surface $x^3y^2 - z^2yx^2 = 8$ at (1, 2, 1).

Solution: We know that a vector normal to the surface $\phi(x, y, z) = c$ is :

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

Here $\phi(x, y, z) = x^3y^2 - z^2yx^2 \Rightarrow \frac{\partial\phi}{\partial x} = 3x^2y^2 - 2xyz^2, \frac{\partial\phi}{\partial y} = 2x^3y - x^2z^2, \frac{\partial\phi}{\partial z} = 0 - 2x^2yz$

At $(1,2,1)$, $\frac{\partial\phi}{\partial x} = 8$, $\frac{\partial\phi}{\partial y} = 3$, $\frac{\partial\phi}{\partial z} = -4 \Rightarrow \nabla\phi = 8i + 3j - 4k \Rightarrow |\nabla\phi| = \sqrt{64+9+16} = \sqrt{89}$

$\therefore$ Unit normal vector to given surface is $u = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8}{\sqrt{89}}i + \frac{3}{\sqrt{89}}j - \frac{4}{\sqrt{89}}k$

12. Find a normal vector of magnitude 5 to the surface $x^2y^2 + xy^2z = 10$ at (1, 2, 1).

Solution: We know that a vector normal to the surface $\phi(x, y, z) = c$ is :

$$\nabla\phi = \frac{\partial\phi}{\partial x}\bar{i} + \frac{\partial\phi}{\partial y}\bar{j} + \frac{\partial\phi}{\partial z}\bar{k}$$

Here $\phi(x, y, z) = x^2y^2 + xy^2z \Rightarrow \frac{\partial\phi}{\partial x} = 2xy^2 + zy^2$, $\frac{\partial\phi}{\partial y} = 2x^2y + 2xyz$, $\frac{\partial\phi}{\partial z} = 0 + xy^2$

At $(1,2,1)$, $\frac{\partial\phi}{\partial x} = 12$, $\frac{\partial\phi}{\partial y} = 6$, $\frac{\partial\phi}{\partial z} = 4 \Rightarrow \nabla\phi = 12i + 6j + 4k \Rightarrow |\nabla\phi| = \sqrt{144+36+16} = 14$

$\therefore$ Unit normal vector to given surface is $u = \frac{\nabla\phi}{|\nabla\phi|} = \frac{12}{14}i + \frac{6}{14}j + \frac{4}{14}k$

Thus, a vector of magnitude 5 and perpendicular to given surface is : $5u = \frac{30}{7}i + \frac{15}{7}j + \frac{10}{7}k$

13. Find the directional derivative of $f = 4x^2y^2z^2$ at $(1,2,1)$ in the direction of $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.

Solution: The directional derivative of scalar function f in the direction of vector **a** is:

$D.D = \frac{\nabla f.a}{|a|}$. Here $f(x, y, z) = 4x^2y^2z^2 \Rightarrow \frac{\partial f}{\partial x} = 8xy^2z^2$, $\frac{\partial f}{\partial y} = 8x^2yz^2$, $\frac{\partial\phi}{\partial z} = 8x^2y^2z$

At $(1,2,1)$, $\frac{\partial\phi}{\partial x} = 32$, $\frac{\partial\phi}{\partial y} = 16$, $\frac{\partial\phi}{\partial z} = 32 \Rightarrow \nabla\phi = 32\bar{i} + 16\bar{j} + 32\bar{k}$. Thus,

$$D.D = \frac{(32\bar{i} + 16\bar{j} + 32\bar{k}).(2\bar{i} + \bar{j} + 2\bar{k})}{|2i + j + 2k|} = \frac{64+16+64}{\sqrt{9}} = \frac{144}{3} = 48$$

14. Find the directional derivative of $f = x^2y^2 + y^2z^2$ at $(1,1,2)$ in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Solution: The directional derivative of scalar function f in the direction of vector **a** is:

$D.D = \frac{\nabla f.a}{|a|}$. Here,

$f(x, y, z) = x^2y^2 + y^2z^2 \Rightarrow \frac{\partial f}{\partial x} = 2xy^2$, $\frac{\partial f}{\partial y} = 2x^2y + 2yz^2$, $\frac{\partial\phi}{\partial z} = 2y^2z$

At (1, 1, 2), $\frac{\partial\phi}{\partial x} = 2$, $\frac{\partial\phi}{\partial y} = 10$, $\frac{\partial\phi}{\partial z} = 4 \Rightarrow \nabla\phi = 2\bar{i} + 10\bar{j} + 4\bar{k}$. Thus,

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$$\text{D.D} = \frac{(2\bar{i}+10\bar{j}+4\bar{k}).(\bar{i}+\bar{j}+\bar{k})}{|\bar{i}+\bar{j}+\bar{k}|} = \frac{2+10+4}{\sqrt{3}} = \frac{16}{\sqrt{3}}$$

15. Find the value of the constant "*C*" so that the vector function **f** = x^2y^2z **i** + $(x + Cy^2)$ **j** – xyz^2 **k** is Solenoid at (1, 2, 1)**. (Here we note a vector by f or** $\bar{f}$ **)**

Solution: We know that a vector function **f** is solenoid if div **f** = 0 where

$$\text{div } \bar{f} = \nabla.\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0 \quad \Rightarrow \frac{\partial\left(x^2y^2z\right)}{\partial x} + \frac{\partial\left(x+Cy^2\right)}{\partial y} + \frac{\partial\left(-xyz^2\right)}{\partial z} = 0$$

$$\Rightarrow 2xy^2z + 2Cy - 2xyz = 0 \text{ Put } (x,y,z) = (1,2,1), \text{ we get}: 8 + 4C - 4 = 0 \;\Rightarrow C = -1$$

16. For what value of constant "*C*", the vector function **f** = (3x + y) **i** + (Cy + z) **j** + 2z **k** is Solenoid.

Solution: We know that a vector function **f** is solenoid if div **f** = 0 where

$$\text{div } \bar{f} = \nabla.\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 0 \quad \Rightarrow \frac{\partial(3x+y)}{\partial x} + \frac{\partial(Cy+z)}{\partial y} + \frac{\partial(2z)}{\partial z} = 0$$

$$\Rightarrow 3 + C + 2 = 0 \;\Rightarrow C = -5$$

17. Show that div **r** = 3 and div(**r**/r^3) = 0 where **r** is a position vector.

Solution: We know that **r** = x**i** + y**j** + z**k**. Now,

$$\text{div } \bar{f} = \nabla.\bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \Rightarrow \text{div } \bar{r} = \nabla.\bar{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\text{Now } r = \sqrt{x^2+y^2+z^2} \;\Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2+y^2+z^2\right)^{-1/2}.2x = \frac{x}{r} \quad \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r} \text{ and } \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\text{Now, div}\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) \qquad (1)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^3.1 - x.3r^2\dfrac{\partial r}{\partial x}}{\left(r^3\right)^2} = \frac{r^3.1 - x.3r^2\dfrac{x}{r}}{\left(r^3\right)^2} = \frac{r^3 - 3x^2r}{r^6} = \frac{r^2-3x^2}{r^5}. \text{ Similarly,}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{r^3.1 - x.3r^2\dfrac{\partial r}{\partial x}}{\left(r^3\right)^2} = \frac{r^3.1 - x.3r^2\dfrac{x}{r}}{\left(r^3\right)^2} = \frac{r^3 - 3x^2r}{r^6} = \frac{r^2-3x^2}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{r^2-3y^2}{r^5} \text{ and } \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{r^2-3z^2}{r^5}. \text{ Thus (1) becomes}$$

$$\text{div}\left(\frac{\bar{r}}{r^3}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{3r^2 - 3\left(x^2+y^2+z^2\right)}{r^5} = \frac{3r^2-3r^2}{r^5} = 0$$

18. Show that the curl of the position vector is zero, that is; Curl **r** = 0**.**

Solution : By definition,

$$\text{curl}\,\bar{r} = \nabla\times\bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \bar{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \bar{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \bar{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0\bar{i} + 0\bar{j} + 0\bar{k} = 0$$

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19. Prove that $\nabla r^n = n\, r^{n-2}\, \mathbf{r}$, where **r** is a position vector.

Solution: By definition,

$$\nabla r^n = \text{grad } r^n = \frac{\partial}{\partial x} r^n \bar{i} + \frac{\partial}{\partial y} r^n \bar{j} + \frac{\partial}{\partial z} r^n \bar{k} \qquad (1)$$

$\frac{\partial}{\partial x} r^n = nr^{n-1} \frac{\partial}{\partial x} r = nr^{n-1}\left(\frac{x}{r}\right) = nxr^{n-2}$. Similarly, $\frac{\partial}{\partial y} r^n = nyr^{n-2}$ and $\frac{\partial}{\partial z} r^n = nzr^{n-2}$

Thus (1) becomes : $\nabla r^n = nr^{n-2}\left(x\bar{i} + y\bar{j} + z\bar{k}\right) = nr^{n-2}\bar{r}$

20. Show that Curl r^n **r** $= 0$

Solution: r^n **r** = xr^n **i** + yr^n **j** + zr^n **k**. Thus,

Curl r^n **r**

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xr^n & yr^n & zr^n \end{vmatrix} = i\left[\frac{\partial}{\partial y} zr^n - \frac{\partial}{\partial z} yr^n\right] - j\left[\frac{\partial}{\partial x} zr^n - \frac{\partial}{\partial z} xr^n\right] + k\left[\frac{\partial}{\partial x} yr^n - \frac{\partial}{\partial y} xr^n\right]$$

Now, $r = \sqrt{x^2 + y^2 + z^2}$

$$\Rightarrow \frac{\partial r^n}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}.\frac{1}{2}\left[x^2 + y^2 + z^2\right]^{-1/2}(2x) = nr^{n-2}x$$

Similarly, $\frac{\partial r^n}{\partial y} = nr^{n-2}y$ and $\frac{\partial r^n}{\partial z} = nr^{n-2}z$. Thus,

$$\text{Curl } r^n \mathbf{r} = i\left[\frac{nr^{n-1}yz}{\partial y} \frac{\partial}{\partial y} zr^n - \frac{\partial}{\partial z} yr^n\right] - j\left[\frac{\partial}{\partial x} zr^n - \frac{\partial}{\partial z} xr^n\right] + k\left[\frac{\partial}{\partial x} yr^n - \frac{\partial}{\partial y} xr^n\right]$$

$$= i\left[nr^{n-2}yz - nr^{n-2}yz\right] - j\left[nr^{n-2}xz - nr^{n-2}xz\right] + k\left[nr^{n-2}yx - nr^{n-2}yx\right] = 0$$

21. Evaluate: (a) $\nabla^2 r^2$

$$\text{Solution : } \nabla^2 r^2 = \frac{\partial^2 r^2}{\partial x^2} + \frac{\partial^2 r^2}{\partial y^2} + \frac{\partial^2 r^2}{\partial z^2} \qquad (1)$$

Here $r = \sqrt{x^2 + y^2 + z^2} \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$

Now, $\frac{\partial^2 r^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial r^2}{\partial x}\right) = \frac{\partial}{\partial x}\left(2r.\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(2r.\frac{x}{r}\right) = \frac{\partial}{\partial x}(2x) = 2.$

Similarly, $\frac{\partial^2 r^2}{\partial y^2} = \frac{\partial^2 r^2}{\partial z^2}$. Thus, equation (1) becomes $\nabla^2 r^2 = 2 + 2 + 2 = 6$

(b) $\nabla^2 \ln r$

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Solution : $\nabla^2(\ln r) = \frac{\partial^2(\ln r)}{\partial x^2} + \frac{\partial^2(\ln r)}{\partial y^2} + \frac{\partial^2(\ln r)}{\partial z^2}$ (1)

Here $r = \sqrt{x^2 + y^2 + z^2} \Rightarrow \frac{\partial r}{\partial x} = \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r}$.

Now, $\frac{\partial^2 r^2}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial(\ln r)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}.\frac{\partial r}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}.\frac{x}{r}\right) = \frac{\partial}{\partial x}\left(\frac{x}{r^2}\right) = \frac{r^2.1 - x(2r)\frac{\partial r}{\partial x}}{\left(r^2\right)^2}$

$$= \frac{r^2 - 2xr.\frac{x}{r}}{r^4} = \frac{r^2 - x^2}{r^4}$$

Similarly, $\frac{\partial^2(\ln r)}{\partial y^2} = \frac{r^2 - y^2}{r^4}$ and $\frac{\partial^2(\ln r)}{\partial z^2} = \frac{r^2 - z^2}{r^4}$. Thus, equation (1) becomes

$$\nabla^2(\ln r) = \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^4} = \frac{3r^2 - \left(x^2 + y^2 + z^2\right)}{r^4} = \frac{3r^2 - r^2}{r^4} = \frac{2r^2}{r^4} = \frac{2}{r^2}$$

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