Lecture 17: Triple integrals

If $f(x,y,z)$ is a function and E is a **bounded solid region** in $\mathbb{R}^3$, then $\int\int\int_E f(x,y,z)\ dxdydz$ is defined as the $n \to \infty$ limit of the Riemann sum

$$\frac{1}{n^3} \sum_{(i/n,j/n,k/n)\in E} f(\frac{i}{n},\frac{j}{n},\frac{k}{n}) .$$

As in two dimensions, triple integrals can be evaluated by iterated 1D integral computations. Here is a simple example:

1 If E is the box $\{x \in [1,2], y \in [0,1], z \in [0,1]\}$ and $f(x,y,z) = 24x^2y^3z$.

$$\int_0^1 \int_0^1 \int_0^1 24x^2y^3z\ dz\ dy\ dx .$$

We start from the core $\int_0^1 24x^2y^3z\ dz = 12x^3y^3$, then integrate the middle layer, $\int_0^1 12x^3y^3\ dy = 3x^2$ and finally and finally handle the outer layer: $\int_1^2 3x^2 dx = 7$.
For the most inner integral, $x = x_0$ and $y = y_0$ are fixed. The integral is integrating up the function $z \to f(x_0,y_0,z)$ along the part intersecting the body. After completing the middle integral, we have computed the integral on the plane $z = const$ intersected with R. The most outer integral sums up all these 2-dimensional sections.
In calculus, two important reductions are used to compute triple integrals. In single variable calculus, one reduces the problem directly to a one dimensional integral by slicing the body along an axes. The slices are 2-dimensional. In multi-variable calculus, we usually reduce the problem to an integration problem in two dimensions. This is more flexible.

The **single variable method** slices the solid along a line. If $g(z)$ is the double integral along the two dimensional slice, then $\int_a^b g(z)\ dz$. The **multi-variable method** sees the solid sandwiched between the graphs of two functions $g(x,y)$ and $h(x,y)$ over a common two dimensional region R. The integral reduces to a double integral $\int\int_R [\int_{g(x,y)}^{h(x,y)} f(x,y,z)\ dz] dA$.

2 An important special case is the **volume**

$$\int\int_R \int_0^{f(x,y)} 1\ dzdxdy .$$

below the graph of a function $f(x,y)$ and above a region R, considered part of the xy-plane. It is the integral $\int\int_R f(x,y)\ dA$. We actually have expressed this now as a triple integral. It is more natural to think of volume as a triple integral also when considering physical units as we would measure volume using cubic meters for example.

3 Find the volume of the unit sphere. **Solution:** The sphere is sandwiched between the graphs of two functions obtained by solving for z. Let R be the unit disc in the xy plane. If we use the **sandwich method**, we get

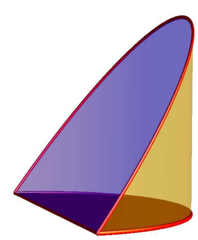
$$V = \int\int_R [\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} 1 dz] dA \ .$$

which gives a double integral $\int\int_R 2\sqrt{1-x^2-y^2}\ dA$ which is of course best solved in polar coordinates. We have $\int_0^{2\pi} \int_0^1 \sqrt{1-r^2} r\ dr d\theta = 4\pi/3$.
With the **washer method** which is in this case also called **disc method**, we slice along the z axes and get a disc of radius $\sqrt{1-z^2}$ with area $\pi(1-z^2)$. This is a method suitable for single variable calculus because we get directly $\int_{-1}^1 \pi(1-z^2)\ dz = 4\pi/3$.

4 The mass of a body with density $\rho(x,y,z)$ is defined as $\int\int\int_R \rho(x,y,z)\ dV$. For bodies with constant density ρ, the mass is ρV, where V is the volume. Compute the mass of a body which is bounded by the parabolic cylinder $z = 4 - x^2$, and the planes $x = 0, y = 0, y = 6, z = 0$ if the density of the body is z. **Solution:**

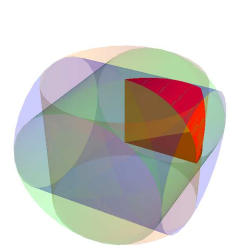
$$\int_0^2 \int_0^6 \int_0^{4-x^2} z\ dz\ dy\ dx = \int_0^2 \int_0^6 (4-x^2)^2/2\ dydx$$
$$= 6\int_0^2 (4-x^2)^2/2\ dx = 6(\frac{x^5}{5} - \frac{8x^3}{3} + 16x)|_0^2 = 2 \cdot 512/5$$

The solid region bound by $x^2 + y^2 = 1$, $x = z$ and $z = 0$ is called the **hoof of Archimedes**. It is historically significant because it is one of the first examples, on which Archimedes probed a Riemann sum integration technique. It appears in every calculus text book.
5 Find the volume of the hoof. **Solution**. Look from the situation from above and picture it in the $x-y$ plane. You see a half disc R. It is the floor of the solid. The roof is the function $z = x$. We have to integrate $\int\int_R x\ dxdy$. We got a double integral problems which is best done in polar coordinates; $\int_{-\pi/2}^{\pi/2} \int_0^1 r^2 \cos(\theta)\ drd\theta = 2/3$.



Finding the volume of the solid region bound by the three cylinders $x^2 + y^2 = 1$, $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ is one of the most famous volume integration problems. **Solution:** look at 1/16'th of the body given in cylindrical coordinates $0 \leq \theta \leq \pi/4, r \leq 1, z > 0$. The roof is $z = \sqrt{1-x^2}$ because above the "one eighth disc" R only the cylinder $x^2 + z^2 = 1$ matters. The polar integration
6 problem

$$16 \int_0^{\pi/4} \int_0^1 \sqrt{1 - r^2\cos^2(\theta)} r\ drd\theta$$

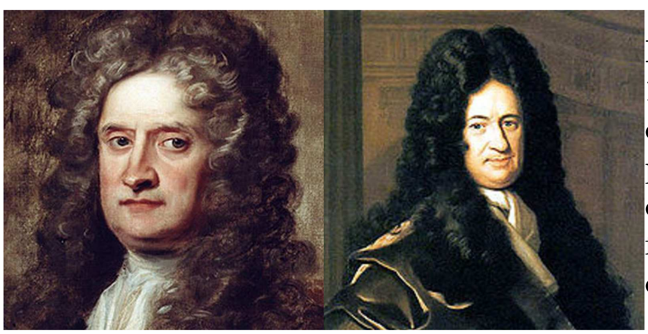


has an inner r-integral of $(16/3)(1 - \sin(\theta)^3)/\cos^2(\theta)$. Integrating this over θ can be done by integrating $(1 + \sin(x)^3)\sec^2(x)$ by parts using $\tan'(x) = \sec^2(x)$ leading to the anti derivative $\cos(x) + \sec(x) + \tan(x)$. The result is $16 - 8\sqrt{2}$.

The problem of computing volumes has been tackled early in mathematics:

Archimedes (287-212 BC) developed an integration method which allowed him to find areas, volumes and surface areas in many cases without calculus. His method of **exhaustion** is close to the numerical method of integration by Riemann sum. In our terminology, Archimedes used the **washer method** to reduce the problem to a single variable problem. The **Archimedes principle** states that any body submerged in a water is acted upon by an upward force which is equal to the weight of the displaced water. This provides a practical way to compute volumes of complicated bodies. A second method, the **displacement method** is a **comparison technique**: the area of a sphere is the area of the cylinder enclosing it. The volume of a sphere is the volume of the complement of a cone in that cylinder. Modern rearrangement techniques use this still today in modern analysis. Heureka!

Cavalieri (1598-1647) would build on Archimedes ideas and determine area and volume using tricks now called the **Cavalieri principle**. An example already due to Archimedes is the computation of the volume the half sphere of radius R, cut away a cone of height and radius R from a cylinder of height R and radius R. At height z, this body has a cross section with area $R^2\pi - r^2\pi$. If we cut the half sphere at height z, we obtain a disc of area $(R^2 - r^2)\pi$. Because these areas are the same, the volume of the half-sphere is the same as the cylinder minus the cone: $\pi R^3 - \pi R^3/3 = 2\pi R^3/3$ and the volume of the sphere is $4\pi R^3/3$.



Newton (1643-1727) and Leibniz(1646-1716) developed calculus independently. It provided a new tool which made it possible to compute integrals through "anti-derivation". Suddenly, it became possible to find integrals using analytic tools. We can do this also in higher dimensions.

Remarks (can be skipped).
1) The **Lebesgue integral** is more powerful than the Riemann integral: suppose we want to calculate the volume of some solid body R which we assumed to be contained inside the unit cube $[0,1] \times [0,1] \times [0,1]$. The **Monte Carlo method** shoots randomly n times onto the unit cube. If we hit the body k times, then k/n approximates the volume. Here is a Mathematica example where an octant of the sphere is computed:

$$\mathrm{R := Random[]; k = 0; Do[x = R; y = R; z = R; If[x^2 + y^2 + z^2 < 1, k++], \{10000\}]; k/10000}$$

Assume, we hit 5277 of n=10000 times. The volume so measured is 0.5277. The actual volume of 1/8'th of the sphere is $\pi/6 = 0.524$. For $n \to \infty$ the Monte Carlo computation gives the actual volume. The Monte-Carlo integral is stronger than the Riemann integral. The law of large

numbers in probability theory proves it to be equivalent to the **Lebesgue integral** and allows to measure much more sets than solids with piecewise smooth boundaries.
2) Is there an "integral' which can measure **every solid** in space and which has the property that the volume of a rotated or translated body remains the same? No! Many sets turn out to be "crazy" in the sense that one can not measure their volume. An example is the **paradox of Banach and Tarski** which tells that one can slice up the unit ball $x^2 + y^2 + z^2 \le 1$ into 5 pieces A, B, C, D, E, rotate and translate them in space so that the pieces A, B, C fit together to be a unit ball and D, E form a second unit ball. Since the volume has obviously doubled and volume should be additive, some of the sets A, B, C, D, E do not have a well defined volume.

Homework

1 Evaluate the triple integral

$$\int_0^5 \int_0^z \int_0^{2y} ze^{-y^2} \, dxdydz \ .$$

2 Find the volume of the solid bounded by the paraboloids $z = x^2 + y^2$ and $z = 36 - (x^2 + y^2)$ and satisfying $x \ge 0$.

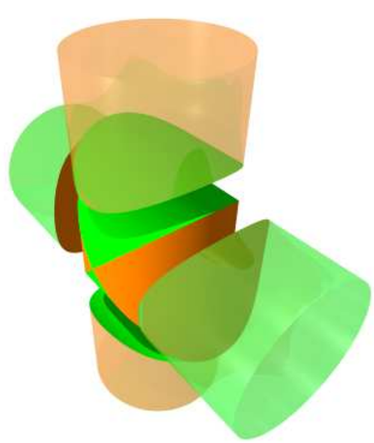
3 Find the **moment of inertia** $\int\int\int_E (x^2 + y^2) \, dV$ of a cone

$$E = \{x^2 + y^2 \le z^2 \ 0 \le z \le 5 \} \ ,$$

which has the z-axis as its center of symmetry.

4 Integrate $f(x, y, z) = x^2 + y^2 - z$ over the tetrahedron with vertices $(0, 0, 0), (4, 4, 0), (0, 4, 0), (0, 0, 12)$.

5 This is a classic by Archimedes: What is the volume of the body obtained by intersecting the solid cylinders $x^2 + z^2 \le 1$ and $y^2 +$

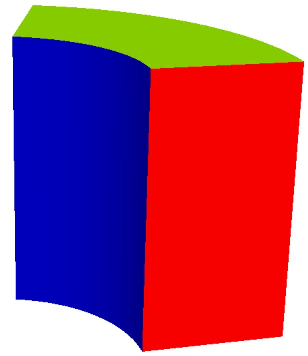


$z^2 \le 1$?

Lecture 18: Spherical Coordinates

Cylindrical coordinates are coordinates in space in polar coordinates are used in the xy-plane and where the z-coordinate is untouched. A surface of revolution $x^2 + y^2 = g(z)^2$ can be described in cylindrical coordinates as $r = g(z)$. The coordinate change transformation $T(r, \theta, z) = (r\cos(\theta), r\sin(\theta), z)$, produces the integration factor $\boxed{r}$. This is easy to remember as it is the factor we know in polar coordinates.

$$\int\int_{T(R)} f(x,y,z)\; dxdydz = \int\int_R g(r,\theta,z)\, \boxed{\mathrm{r}}\, drd\theta dz$$

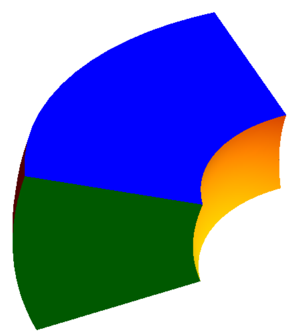


Spherical coordinates use ρ, the distance to the origin as well as two **Euler angles**: θ the polar angle and ϕ, the angle between the vector and the z axis. The coordinate change is

$$T : (x, y, z) = (\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi)) \; .$$

The integration factor measures the volume of a **spherical wedge** which is $d\rho, \rho\sin(\phi)\; d\theta, \rho d\phi = \rho^2\sin(\phi) d\theta d\phi d\rho$.

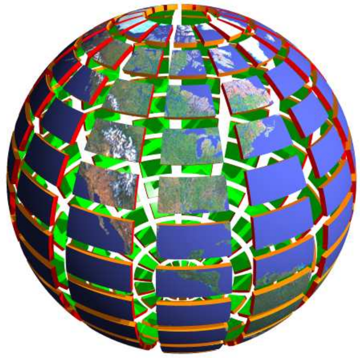
$$\int\int_{T(R)} f(x,y,z)\; dxdydz = \int\int_R g(\rho,\theta,z)\, \boxed{\rho^2\sin(\phi)}\, d\rho d\theta d$$



1 A sphere of radius R has the volume

$$\int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin(\phi)\; d\phi d\theta d\rho \; .$$

The most inner integral $\int_0^\pi \rho^2\sin(\phi) d\phi = -\rho^2\cos(\phi)|_0^\pi = 2\rho^2$. The next layer is, because ϕ does not appear: $\int_0^{2\pi} 2\rho^2\; d\phi = 4\pi\rho^2$. The final integral is $\int_0^R 4\pi\rho^2\; d\rho = 4\pi R^3/3$.



The moment of inertia of a body G with respect to an axis L is defined as the triple integral $\int\int\int_G r(x,y,z)^2 \; dzdydx$, where $r(x,y,z) = \rho\sin(\phi)$ is the distance from the axis L.

2 For a sphere of radius R we obtain with respect to the z-axis:

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 \sin^2(\phi) \rho^2 \sin(\phi) \; d\phi d\theta d\rho$$

$$= (\int_0^{\pi} \sin^3(\phi) \; d\phi)(\int_0^R \rho^4 \; dr)(\int_0^{2\pi} \; d\theta)$$

$$= (\int_0^{\pi} \sin(\phi)(1-\cos^2(\phi)) \; d\phi)(\int_0^R \rho^4 \; dr)(\int_0^{2pi} \; d\theta)$$

$$= (-\cos(\phi) + \cos(\phi)^3/3)|_0^{\pi}(L^5/5)(2\pi) = \frac{4}{3} \cdot \frac{R^5}{5} \cdot 2\pi = \frac{8\pi R^5}{15} \; .$$

If the sphere rotates with angular velocity ω, then $I\omega^2/2$ is the **kinetic energy** of that sphere. **Example:** the moment of inertia of the earth is $8 \cdot 10^{37} kgm^2$. The angular velocity is $\omega = 2\pi/day = 2\pi/(86400s)$. The rotational energy is $8 \cdot 10^{37} kgm^2/(7464960000s^2) \sim 10^{29} J \sim 2.510^{24} kcal$.

3 Find the volume and the center of mass of a diamond, the intersection of the unit sphere with the cone given in cylindrical coordinates as $z = \sqrt{3}r$.
Solution: we use spherical coordinates to find the center of mass

$$\begin{aligned}
\overline{x} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi)\cos(\theta) \; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{y} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \sin^2(\phi)\sin(\theta) \; d\phi d\theta d\rho \frac{1}{V} = 0 \\
\overline{z} &= \int_0^1 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos(\phi)\sin(\phi) \; d\phi d\theta d\rho \frac{1}{V} = \frac{2\pi}{32V}
\end{aligned}$$

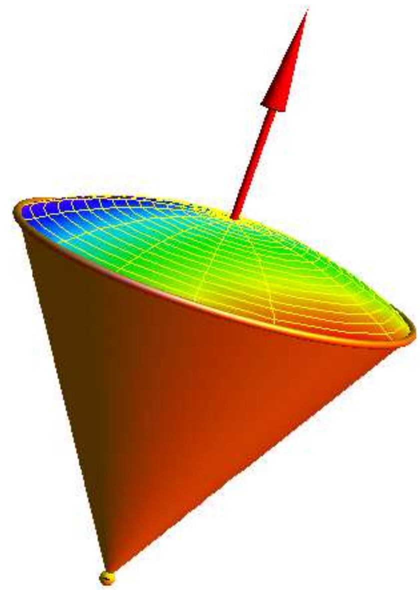
4 Find $\int\int\int_R z^2 \; dV$ for the solid obtained by intersecting $\{1 \leq x^2+y^2+z^2 \leq 4\}$ with the double cone $\{z^2 \geq x^2+y^2\}$.

Solution: since the result for the double cone is twice the result for the single cone, we work with the diamond shaped region R in $\{z > 0\}$ and multiply the result at the end with 2. In spherical coordinates, the solid R is given by $1 \leq \rho \leq 2$ and $0 \leq \phi \leq \pi/4$. With $z = \rho\cos(\phi)$, we have

$$\int_1^2 \int_0^{2\pi} \int_0^{\pi/4} \rho^4 \cos^2(\phi)\sin(\phi) \; d\phi d\theta d\rho$$

$$= (\frac{2^5}{5} - \frac{1^5}{5}) 2\pi (\frac{-\cos^3(\phi))}{3}|_0^{\pi/4} = 2\pi\frac{31}{5}(1-2^{-3/2}) \; .$$

The result for the double cone is $\boxed{4\pi(31/5)(1-1/\sqrt{2}^3)}$.



Remarks: There are other coordinate systems besides Euclidean, cylindrical and spherical. One of them are **torus coordinates**, where $T(r,\phi,\theta) = (1+r\cos(\phi))\cos(\theta), (1+r\cos(\phi)\sin(\theta), r\sin(\phi))$, a coordinate system which works inside the solid torus $r \leq 1$. Are there spherical coordinates in higher dimensions? Yes, there are. They are called **hyperspherical coordinates**. In four dimensions (the space of quaternions) for example we would have a third angle ψ and get

$$(x,y,z,w) = (\rho\sin(\psi)\sin(\phi)\sin(\theta), \rho\sin(\psi)\sin(\phi)\cos(\theta), \rho\sin(\psi)\cos(\phi), \rho\cos(\psi)) \; .$$

The four dimensional case is especially interesting because one can write the sphere S^3 in four dimensions as the set of pairs of complex numbers z,w with $|z|^2+|w|^2 = 1$. The 3 sphere is special because it is the group $SU(2)$ of all unitary 2×2 matrices of determinant 1. It is also the set of all quaternions of length 1. The quaternions are historically interesting for multivariable calculus because they predated vector calculus we teach today and incorporate both the dot and cross product.

Homework

1 Assume the density of a solid $E = x^2 + y^2 - z^2 < 1, -1 < z < 1$ is given by the forth power of the distance to the z-axes: $\sigma(x, y, z) = (x^2 + y^2)^2$. Find its mass

$$M = \int\int\int_E (x^2 + y^2)^2 \, dxdydz \; .$$

2 Find the moment of inertia $\int\int\int_E (x^2 + y^2) \, dV$ of the body E whose volume is given by the integral

$$\mathrm{Vol}(E) = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^3 \rho^2 \sin(\phi) \, d\rho d\theta d\phi \; .$$

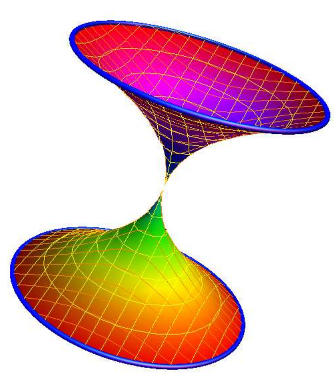
3 A solid is described in spherical coordinates by the inequality $\rho \leq \sin(\phi)$. Find its volume.

4 Integrate the function

$$f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$$

over the solid which lies between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$, which is in the first octant and which is above the cone $x^2 + y^2 = z^2$.

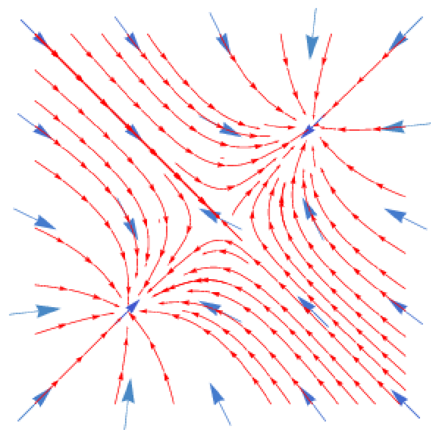
5 Find the volume of the solid $x^2 + y^2 \leq z^4, z^2 \leq 4$.



Lecture 19: Vector fields

A **planar vector field** is a map which assigns to $(x, y) \in \mathbb{R}^2$ a vector $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$. A **vector field in space** is a map, which assigns to each $(x, y, z) \in \mathbb{R}^3$ a vector $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

An example is the **dipol field** $\vec{F}(x, y) = \langle x-1, y\rangle/((x-1)^2+y^2)^{3/2} - \langle x+1, y\rangle/((x+1)^2+y^2)^{3/2}$ generated by a positive and negative point charge. Here is a picture



If $f(x, y)$ is a function of two variables, then $\vec{F}(x, y) = \nabla f(x, y)$ is called a **gradient field**. Gradient fields in space are of the form $\vec{F}(x, y, z) = \nabla f(x, y, z)$.

When is a vector field a gradient field? $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = \nabla f(x, y)$ implies $Q_x(x, y) = P_y(x, y)$. If this does not hold at some point, $\vec{F}$ is no gradient field.

Clairaut test: If $Q_x(x, y) - P_y(x, y)$ is not zero at some point, then $\vec{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is not a gradient field.

We will see that $\text{curl}(\vec{F}) = Q_x - P_y = 0$ is also sufficient for $\vec{F}$ to be a gradient field if $\vec{F}$ is defined everywhere. How do we get f the function with $\vec{F} = \nabla f$? We will look at examples in class.

1 Is the vector field $\vec{F}(x, y) = \langle P, Q\rangle = \langle 3x^2y + y + 2, x^3 + x - 1\rangle$ a gradient field? **Solution**: the Clairaut test shows $Q_x - P_y = 0$. We integrate the equation $f_x = P = 3x^2y + y + 2$ and get $f(x, y) = 2x + xy + x^3y + c(y)$. Now take the derivative of this with respect to y to get $x + x^3 + c'(y)$ and compare with $x^3 + x - 1$. We see $c'(y) = -1$ and so $c(y) = -y + c$. We see the solution $\boxed{x^3y + xy - y + 2x}$.

2 Is the vector field $\vec{F}(x, y) = \langle xy, 2xy^2\rangle$ a gradient field? **Solution:** No: $Q_x - P_y = 2y^2 - x$ is not zero.

Vector fields in weather forecast On weather maps, one can see **isoterms**, curves of constant temperature or **isobars**, curves $p(x, y) = c$ of constant pressure. These are level curves. The wind velocity $\vec{F}(x, y)$ is close but not always exactly perpendicular to the **isobars**, the lines of equal pressure p. In reality, the scalar pressure field p and the velocity field $\vec{F}$ also depend on time. The equations which describe the weather dynamics are called the **Navier Stokes equations**

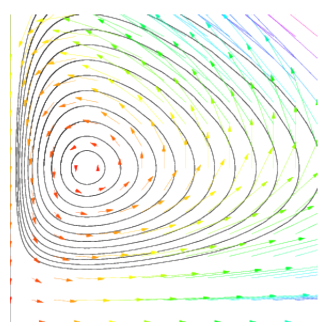
$$\frac{d}{dt}\vec{F} + \vec{F} \cdot \nabla \vec{F} = \nu \Delta \vec{F} - \nabla p + f, \mathrm{div}\vec{F} = 0$$

(where Δ and div are defined later. This is an other example of a **partial differential equation**. It is one of the millenium problems to prove that these equations have smooth solutions in space. Vector fields are important in differential equations. We look at some examples in population dynamics and mechanics. You can skip this motivational part:

3 Let $x(t)$ denote the population of a "prey species" like tuna fish and $y(t)$ is the population size of a "predator" like sharks. We have $x'(t) = ax(t) + bx(t)y(t)$ with positive a, b because both more predators and more prey species will lead to prey consumption. The rate of change of $y(t)$ is $y'(t) = -cy(t) + dxy$, where c, d are positive. This can be written using a vector field $\vec{r}' = \vec{F}(\vec{r}(t))$. We have a negative sign in the first part because predators would die out without food. The second term is explained because both more predators as well as more prey leads to a growth of predators through reproduction. A concrete example is the **Volterra-Lodka system**

$$\begin{aligned} \dot{x} &= 0.4x - 0.4xy \\ \dot{y} &= -0.1y + 0.2xy \ , \end{aligned}$$

where $\vec{F}(x, y) = \langle 0.4x - 0.4xy$
$- 0.1y + 0.2xy\rangle$. Volterra explained with such systems the oscillation of fish populations in the Mediterranean sea. At any specific point $\vec{r}(x, y) = \langle x(t), y(t)\rangle$, there is a curve $= \vec{r}(t) = \langle x(t), y(t)\rangle$ through that point for which the tangent $\vec{r}\ '(t) = (x'(t), y'(t)$ is the vector field.

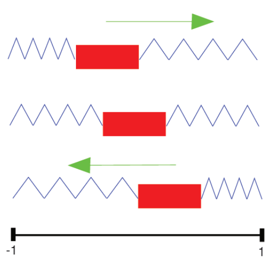


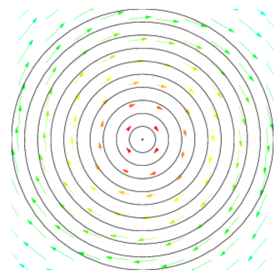
4 A class vector fields important in mechanics are **Hamiltonian fields**: If $H(x, y)$ is a function of two variables, then $\langle H_y(x, y), -H_x(x, y)\rangle$ is called a **Hamiltonian vector field**. An example is the harmonic oscillator $H(x, y) = (x^2+y^2)/2$. Its vector field $(H_y(x, y), -H_x(x, y)) = (y, -x)$. The flow lines of a Hamiltonian vector fields are located on the level curves of H (as you have shown in th homework with the chain rule).

5 Newton's law $m\vec{r}'' = F$ relates the acceleration $\vec{r}''$ of a body with the force F acting at the point. For example, if $x(t)$ is the position of a mass point in $[-1, 1]$ attached at two springs and the mass is $m = 2$, then the point experiences a force $(-x + (-x)) = -2x$ so

that $mx'' = 2x$ or $x''(t) = -x(t)$. If we introduce $y(t) = x'(t)$ of t, then $x'(t) = y(t)$ and $y'(t) = -x(t)$. Of course y is the velocity of the mass point, so a pair (x, y), thought of as an initial condition, describes the system so that nature knows what the future evolution of the system has to be given that data.

6 We don't yet know yet the curve $t \mapsto \vec{r}(t) = \langle x(t), y(t) \rangle$, but we know the tangents $\vec{r}\,'(t) = \langle x'(t), y'(t) \rangle = \langle y(t), -x(t) \rangle$. In other words, we know a direction at each point. The equation $(x' = y, y' = -x)$ is called a system of ordinary differential equations (ODE's) More generally, the problem when studying ODE's is to find solutions $x(t), y(t)$ of equations $x'(t) = f(x(t), y(t)), y'(t) = g(x(t), y(t))$. Here we look for curves $x(t), y(t)$ so that at any given point (x, y), the tangent vector $(x'(t), y'(t))$ is $(y, -x)$. You can check by differentiation that the circles $(x(t), y(t)) = (r\sin(t), r\cos(t))$ are solutions. They form a family of curves.





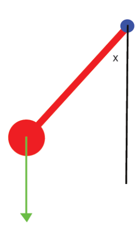
7 If $x(t)$ is the angle of a pendulum, then the gravity acting on it produces a force $G(x) = -gm\sin(x)$, where m is the mass of the pendulum and where g is a constant. For example, if $x = 0$ (pendulum at bottom) or $x = \pi$ (pendulum at the top), then the force is zero. The Newton equation "mass times acceleration = force" gives

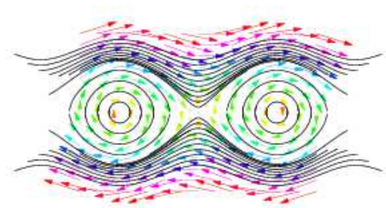
$$\ddot{x}(t) = -g\sin(x(t)) \ .$$

8 The equation of motion for the pendulum $\ddot{x}(t) = -g\sin(x(t))$ can be written with $y = \dot{x}$ also as

$$\frac{d}{dt}(x(t), y(t)) = (y(t), -g\sin(x(t)) \ .$$

Each possible motion of the pendulum $x(t)$ is described by a curve $\vec{r}(t) = (x(t), y(t))$. Writing down explicit formulas for $(x(t), y(t))$ is in this case not possible with known functions like $\sin, \cos, \exp, \log$ etc. However, one still can understand the curves.





Curves on the top of the picture represent situations, where the velocity y is large. They describe the pendulum spinning around fast in the clockwise direction. Curves starting near the point $(0, 0)$, where the pendulum is at a stable rest, describe small oscillations of the pendulum.

Homework

1 a) Draw the gradient vector field of $f(x,y) = \sin(x^2 - y^2)$.
b) Draw the gradient vector field of $f(x,y) = (x-1)^2 + (y-2)^2$.
Hint: In both cases, draw a contour map of f and use gradients to draw the vector field $F(x,y) = \nabla f$.

2 The vector field $\vec{F}(x,y) = \langle x/(x^2+y^2)^{(3/2)}, y/(x^2+y^2)^{(3/2)} \rangle$ appears in electrostatics. Find a function $f(x,y)$ such that $\vec{F} = \nabla f$.

3 a) Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle xy, x^2 \rangle$ a gradient field?
b) Is the vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = \langle \sin(x) + y, \cos(y) + x \rangle$ a gradient field?
In both cases, find $f(x,y)$ satisfying $\nabla f(x,y) = \vec{F}(x,y)$ or give a reason, why it does not exist.

4 Which of the following vector fields $\vec{F} = \langle P, Q \rangle$ can be written as $\vec{F} = \langle P, Q \rangle = \langle f_x, f_y \rangle$? Make use of Clairaut's identity $Q_x = P_y$, to see whether f exists. If yes, find f.
a) $\vec{F}(x,y) = \langle x^{11}, y^9 \rangle$.
b) $\vec{F}(x,y) = \langle y^9, x^7 \rangle$.
c) $\vec{F}(x,y) = \langle 10y + 10x, 10x + 10y \rangle$.
d) $\vec{F}(x,y) = \langle 9 - y^2 + 4x^3y^3, -2xy + 3x^4y^2 \rangle$.

5 Find the potential f to

$$\vec{F}(x,y,z) = \langle 5x^4y + z^4 + y\cos(xy), x^5 + x\cos(xy), 4xz^3 \rangle \ .$$

Lecture 20: Line integral Theorem

If $\vec{F}$ is a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ and $C : t \mapsto \vec{r}(t)$ is a curve, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$$

is called the **line integral** of $\vec{F}$ along the curve C.

We use also the short-hand notation $\int_C \vec{F} \cdot d\vec{r}$. In physics, if $\vec{F}(x, y, z)$ is a **force field**, then $\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$ is called **power** and the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is **work**. In electrodynamics, if $\vec{F}(x, y, z)$ is an electric field, then the line integral $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt$ is the **electric potential**.

1 Let $C : t \mapsto \vec{r}(t) = \langle \cos(t), \sin(t) \rangle$ be a circle parametrized by $t \in [0, 2\pi]$ and let $\vec{F}(x, y) = \langle -y, x \rangle$. Calculate the line integral $I = \int_C \vec{F}(\vec{r}) \cdot d\vec{r}$.
Solution: We have $I = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t))\; dt = \int_0^{2\pi} \sin^2(t) + \cos^2(t)\; dt = 2\pi$

2 Let $\vec{r}(t)$ be a curve given in polar coordinates as $\vec{r}(t) = \cos(t), \phi(t) = t$ defined on $[0, \pi]$. Let $\vec{F}$ be the vector field $\vec{F}(x, y) = (-xy, 0)$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$. **Solution:** In Cartesian coordinates, the curve is $r(t) = (\cos^2(t), \cos(t)\sin(t))$. The velocity vector is then $\vec{r}\,'(t) = \langle -2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t) \rangle = (x(t), y(t)\rangle$. The line integral is

$$\int_0^{\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = \int_0^{\pi} (\cos^3(t)\sin(t), 0) \cdot (-2\sin(t)\cos(t), -\sin^2(t) + \cos^2(t))\; dt$$

$$= \; -2\int_0^{\pi} \sin^2(t)\cos^4(t)\; dt = -2(t/16 + \sin(2t)/64 - \sin(4t)/64 - \sin(6t)/192)|_0^{\pi} = -\pi/8\,.$$

The first generalization of the fundamental theorem of calculus to higher dimensions is the **fundamental theorem of line integrals**.

Fundamental theorem of line integrals: If $\vec{F} = \nabla f$, then

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\; dt = f(\vec{r}(b)) - f(\vec{r}(a))\,.$$

In other words, the line integral is the potential difference between the end points $\vec{r}(b)$ and $\vec{r}(a)$, if $\vec{F}$ is a gradient field.

3 Let $f(x,y,z)$ be the temperature distribution in a room and let $\vec{r}(t)$ the path of a fly in the room, then $f(\vec{r}(t))$ is the temperature, the fly experiences at the point $\vec{r}(t)$ at time t. The change of temperature for the fly is $\frac{d}{dt}f(\vec{r}(t))$. The line-integral of the temperature gradient ∇f along the path of the fly coincides with the temperature difference between the end point and initial point.

4 If $\vec{r}(t)$ is parallel to the level curve of f, then $d/dt f(\vec{r}(t)) = 0$ and $\vec{r}\,'(t)$ orthogonal to $\nabla f(\vec{r}(t))$.

5 If $\vec{r}(t)$ is orthogonal to the level curve, then $|d/dt f(\vec{r}(t))| = |\nabla f||\vec{r}\,'(t)|$ and $\vec{r}\,'(t)$ is parallel to $\nabla f(\vec{r}(t))$.

The proof of the fundamental theorem uses the chain rule in the second equality and the fundamental theorem of calculus in the third equality of the following identities:

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \int_a^b \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \int_a^b \frac{d}{dt} f(\vec{r}(t))\ dt = f(\vec{r}(b)) - f(\vec{r}(a)) .$$

For a gradient field, the line-integral along any closed curve is zero.

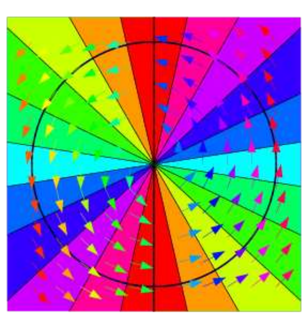
When is a vector field a gradient field? $\vec{F}(x,y) = \nabla f(x,y)$ implies $P_y(x,y) = Q_x(x,y)$. If this does not hold at some point, $\vec{F} = \langle P, Q\rangle$ is no gradient field. This is called the **component test** or Clairot test. We will see later that the condition $\mathrm{curl}(\vec{F}) = Q_x - P_y = 0$ implies that the field is conservative, if the region satisfies a certain property.

6 Let $\vec{F}(x,y) = \langle 2xy^2 + 3x^2, 2yx^2\rangle$. Find a potential f of $\vec{F} = \langle P, Q\rangle$.
Solution: The potential function $f(x,y)$ satisfies $f_x(x,y) = 2xy^2 + 3x^2$ and $f_y(x,y) = 2yx^2$. Integrating the second equation gives $f(x,y) = x^2y^2 + h(x)$. Partial differentiation with respect to x gives $f_x(x,y) = 2xy^2 + h'(x)$ which should be $2xy^2 + 3x^2$ so that we can take $h(x) = x^3$. The potential function is $f(x,y) = x^2y^2 + x^3$. Find g, h from $f(x,y) = \int_0^x P(x,y)\ dx + h(y)$ and $f_y(x,y) = g(x,y)$.

7 Let $\vec{F}(x,y) = \langle P, Q\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$. It is a gradient field because $f(x,y) = \arctan(y/x)$ has the property that $f_x = (-y/x^2)/(1+y^2/x^2) = P$, $f_y = (1/x)/(1+y^2/x^2) = Q$. However, the line integral $\int_\gamma \vec{F}\ d\vec{r}$, where γ is the unit circle is

$$\int_0^{2\pi} \langle \frac{-\sin(t)}{\cos^2(t) + \sin^2(t)}, \frac{\cos(t)}{\cos^2(t) + \sin^2}\rangle \cdot \langle -\sin(t), \cos(t)\rangle\ dt$$

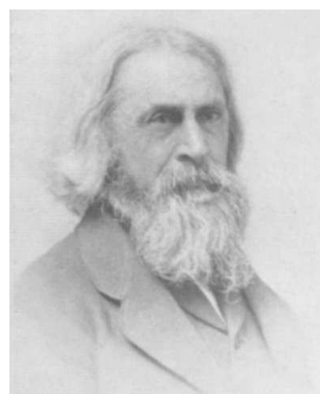
which is $\int_0^{2\pi} 1\ dt = 2\pi$. What is wrong?
Solution: note that the potential f as well as the vector-field F are not differentiable everywhere. The curl of F is zero except at $(0,0)$, where it is not defined.



Remarks: The fundamental theorem of line integrals works in any dimension. You can formulate and check it yourself. The reason is that curves, vector fields, chain rule and integration along curves are easy to generalize to any dimensions. We will see next week that if R is a region "without holes" then $\vec{F}$ is a gradient field if and only if $\text{curl}(\vec{F}) = 0$ everywhere in R. A region R is called **simply connected**, if every curve in R can be contracted to a point in a continuous way and every two points can be connected by a path. A disc is an example of a simply connected region, an annular region is an example which is not. Any region with a hole is not simply connected. For simply connected regions, the existence of a gradient field is equivalent to the field having curl zero everywhere.

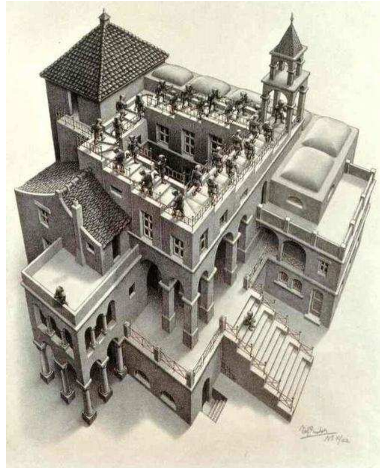
A device which implements a non gradient force field is called a **perpetual motion machine**. It realizes a force field for which the energy gain is positive along some closed loop. The first law of thermodynamics forbids the existence of such a machine. It is informative to contemplate some of the ideas people have come up and to see why they don't work. Here is an example: consider a O-shaped pipe which is filled only on the right side with water. A wooden ball falls on the right hand side in the air and moves up in the water.

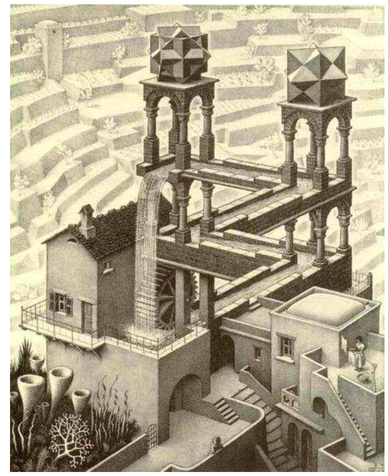
Why are there no "perpetual motion machines". Benjamin Peirce refers in his book "A system of analytic mechanics" of 1855 to the **"antropic principle"**: *"Such a series of motions would receive the technical name of a 'perpetual motion' by which is to be understood, that of a system which would constantly return to the same position, with an increase of power, unless a portion of the power were drawn off in some way and appropriated, if it were desired, to some species of work. A constitution of the fixed forces, such as that here supposed and in which a perpetual motion would possible, may not, perhaps, be incompatible with the unbounded power of the Creator; but, if it had been introduced into nature, it would have proved destructive to human belief, in the spiritual origin of force, and the necessity of a First Cause superior to matter, and would have subjected the grand plans of Divine benevolence to the will and caprice of man".*



Here is a futile attempt featured in Youtube.

Non-conservative fields can also be generated by **optical illusion** as **M.C. Escher** did. The illusion suggests the existence of a force field which is not conservative. Can you figure out how Escher's pictures "work"?





Homework

1 Let C be the space curve $\vec{r}(t) = \langle \cos(t), \sin(t), t \rangle$ for $t \in [0, \pi/4]$ and let $\vec{F}(x, y, z) = \langle y, x, 15 \rangle$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$.

2 What is the work done by moving in the force field $\vec{F}(x, y) = \langle 2x^3 + 1, 2y^4 \rangle$ along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$? a) compute directly b) use the theorem.

3 Let $\vec{F}$ be the vector field $\vec{F}(x, y) = \langle -y, x \rangle/2$. Compute the line integral of F along an ellipse $\vec{r}(t) = \langle a\cos(t), b\sin(t) \rangle$ with width $2a$ and height $2b$. The result should depend on a and b.

4 After summer school, you relax in a Jacuzzi and move along curve C given by part of the curve $x^{20} + y^{20} = 1$ in the first quadrant, oriented counter clockwise. The hot water in the tub has the velocity $\vec{F}(x, y) = \langle x + y, y^4 + x \rangle$. Calculate the line integral $\int_C \vec{F} \cdot d\vec{r}$, the energy you gain from the fluid force when dislocating from $(1, 0)$ to $(0, 1)$. Be lazy.

5 Find a closed curve $C : \vec{r}(t)$ for which the vector field

$$\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle xy, x^2 \rangle$$

satisfies $\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \neq 0$.