Propositional Logic:

⇒ The phrase propositional Logic is composed of two words..

proposition → Logic

Logic:

⇒ Logic is the science of reasoning.

⇒ It helps us to understand and reason about different mathematical statements.

⇒ With rules of Logic, we would be able to think about mathematical statements and finally we would be able to prove or disprove those mathematical statements precisely.

For Example:

Rules of Logic enable us to reason about the mathematical statements like:

"For every positive integer n, the sum of positive integers not exceeding n is $n(n+1)/2$."

we are well familiar with this particular formula which is the sum of first n natural numbers or n positive integers. We know that this mathematical statement is a valid mathematical statement and we will prove this mathematical with the rules of Logic.

Either this is a valid mathematical argument

or it is an invalid mathematical argument.

Purpose of Logic:

⇒ Purpose of logic is to construct valid arguments (or proofs). Once we prove a mathematical statement is True then we call it a theorem and this is the basis of whole mathematics.

Proposition:

⇒ Proposition is a declarative sentence (a sentence that is declaring a fact or stating an argument) which can be either True or False but can not be both.

For Example:

(1) Delhi is the Capital of India
(2) Water froze this morning.
(3) $1 + 1 = 2$

Sentences which are not Propositions

(1) What time is it?
(2) $x + 1 = 2$ (can be both true or False)
(3) Send us your resume before 11 pm.
(4) I request you to please allow me a day off.
(5) Fetch my Umbrella.

Propositional Logic:

→ It is an area of Logic that studies ways of joining and/or modifying propositions to form more complicated propositions and it also studies the logical relationships and properties derived from these combined/altered propositions.

What does this definition really mean?

Ex#

Statement#1: "Adam is good in playing football."

Statement#2: "Adam is good in playing football and this time he is representing his college at national level."

"Adam is good in playing football and this time he is representing his college at national level"

Proposition 1

joining two propositions with logical connective

Proposition 2

We are joining two statements and this is what definition tells.

Ex#

Statement#3: "I enjoy watching television."

Statement#4: "It is not the case that I enjoy watching television."

modifying the statement using negation

Here we are modifying a proposition and that's what we learned in definition.

Fact: "propositional Logic is sometimes called as "Sentential Logic" or "Statement Logic"

Why do we need Compound/Complicated propositions?

because most of the mathematical statements are constructed by combining one or more than one propositions. As simple as that!

Propositional Variables:

Variables that are used to represent propositions are called propositional Variables.

Propositional variables help us to express the statements in a more convenient way.

Ex#

P: Adam is good in playing football.
Q: This time he is representing his college at national level.

here "P" and "Q" are propositional variables.

Now we will discuss about different types of Logical Operators available in Propositional Logic.

⇒ There are 6 Logical operators that we will focus on:

1, Negation
2, Conjunction
3, Disconjunction
4, Exclusive OR
5, Implication
6, Biconditional

Negation:

⇒ let P be a proposition. ¬P is called negation of P which simply states that,

"It is not the case that P"

if P is true then ¬P is false. if P is false then ¬P is true.

"¬" symbol represents Negation.

Ex # let P be a proposition -
"Adam and Eve lived together for many years"

then ¬P will be:
"It is not the case that Adam and Eve lived together for many years."

OR

"Adam and Eve haven't lived together for many years."

Truth Table

p	$\neg p$
T	F
F	T

Conjunction:

⇒ Let p and q be two propositions. Conjunction of p and q is denoted by $p \wedge q$. When both p and q are true then only the compound Proposition ~~is~~ $p \wedge q$ is true.

This "$\wedge$" symbol is used for "and"

Ex# "<u>12 is divisible by 3</u> and <u>3 is a prime number</u>."

Above example contains two propositions which are joined by conjunction operator and both propositions are True so their conjunction is True.

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Note: Sometimes we use "but" instead of "and"

Ex#
"12 is divisible by 3 (but) 3 is a prime number"
and

Disjunction:

→ let p and q be propositions. Disjunction of p and q is denoted by $p \vee q$. When both p and q are false then only the compound proposition $p \vee q$ is false.

This "v" symbol represents the disjunction operator "or".

Ex# "16 − 4 = 10 or 4 is an even number".
Above example contains two propositions one is true and one is false so their disconjunction will be ~~false~~ True.

Truth Table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Exclusive OR

Consider the following statement:

"In order to get a job in this multinational company, experience with C++ (or) java is mandatory."

inclusive OR

here, this statement means

If you have experience in C++ then you are eligible for job

If = -- - = java = =

= = =

If you have experience with both C++ and java then also you are eligible.

Since here if you ~~have~~ fulfill both ~~eq~~ conditions then you are also eligible so here "OR" in this statement is "inclusive "OR" - also called "Disjunction".

Consider another statement:

"When you buy a car from XYZ Company, you get $2500 cashback (or) accessories worth $2500."

Exclusive OR

This statement means,

You will either get

Cashback of $2500 or

Accessories worth $2500.

here,

It is not possible that you get both things so here "OR" operator is an "Exclusive OR" operator. We also write it as "XOR".

Exclusive OR means whole given compound proposition will get satisfied ~~either~~ when either p will get satisfied or q will satisfied but not both.

Exclusive OR excludes "both" possibility.

Definition:

let p and q be two propositions. The exclusive OR of p and q (denoted by $p \oplus q$) is a proposition that simply means exactly one of p and q will be true but both can not be true.

Truth Table

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Implication:

Definition: let p and q be propositions. The proposition "if p then q" denoted by $p \rightarrow q$ is called <u>implication</u> or <u>conditional</u> statement.

The symbol "$\rightarrow$" is used for implication.

Truth Table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

here, p is called <u>hypothesis</u> (or premise) and q is called <u>conclusion</u> (or consequence)

If hypotheses (i.e P) is false then it doesnot matter if conclusion (i.e q) is true or false, the $P \rightarrow q$ is always true.

Ex # 1

"If you try hard for your exam then you will succeed."

here,

P: You try hard for your exam.
q: you will succeed.

Case-I: "You tried hard for your exam" and "You succeed."
P = True
q = True
compound proposition $P \rightarrow q$ is true

Case-II: "You tried hard for your exam." but "you failed."
P = true
q = false
compound proposition $P \rightarrow q$ is false

Case-III: "You haven't tried hard for your exam" and "You succeeded."
P = false
q = True
compound proposition $p \rightarrow q$ is True

because here hypothesis itself is not satisfied then we can not argue about whole compound Proposition to be false so compound proposition is true

Case-IV: "You haven't tried hard for your exam" and "You failed".

p = false $\quad q$ = false

Compound Proposition $p \to q$ is True (same reason as above)

Example # 2

"If you have connection with seniors then you will get promoted."

p	q	$p \to q$
T	T	T
F	T	T
T	F	F
F	F	T

Ex # 3

"If you get 100% marks on the final exam then you will be awarded a trophy."

p	q	$p \to q$
T	T	T
F	T	T
T	F	F
F	F	T

Note: If p is false then it doesn't matter what will be the truth value of q. $p \to q$ is always True.

Implication – Representations

⇒ Different ways to represent conditional statements:

"if P then q"
"P implies q"
"q when P"
"q whenever P"
"q follows from P"

→ all same

Equivalent

"P only if q"
"q is necessary for P"
"P is sufficient for q"
"q unless ¬P"

"P only if q"

How "if P then q" and "P only if q" can be same?

Example: "I will stay at home only if I'm sick."
let
P = "I will stay at home" and q = "I am sick"
Above statement is of the form "P only if q"

According to the above statement, becoming sick is the necessary condition that will make you stay at home.
This means "if you're not sick then you can not stay at home at any cost."
In order to falsify above statement, q must be **false** and p must be True i.e. you are not sick and you will stay at home.

Proof idea: Truth value of P and q must be same in both cases to falsify the statement in both cases.

"if P then q":
"If I'll stay at home then I'm sick"
The only way to falsify the above statement is by making P True and q False.
Therefore, "P only if q" is equivalent to "if P then q" because in both cases (i.e in "P only if q" and "if P then q") we used same truth values of P and q to falsify the statement.

Why "P only if q" is not equivalent to "if q then P"?

Ex# "I'll stay at home if I am sick"

P: I will stay at home, q: I am sick

Proof: P only if q is false when p is True and q is False.

"P only if q"

P	q	P → q
T	T	T
T	F	F
F	T	T
F	F	T

If q then P:
"If I am sick then I will stay at home"

As when P is True and q is False "P only if q" is false but with same truth values of P and q, "If q then P" is True.

If q then P

q	P	q → P
T	T	T
F	T	T
T	F	F
F	F	T

Therefore, "P only if q" is not equivalent to "if q then p".

Hence proved

"q is necessary for p"

Ex # "Good food is necessary to keep us alive."

According to the statement, if we won't have good food then, we'll soon die.

But is it the only factor to keep us alive? NO.

Ex # My friend died at the age of 16 and he has no shortage of good food. (Even though he is having good food but still he died.)

Therefore, when we say A is necessary for B then falsity of A guarantees the falsity of B but we can not guarantee the truth of B from the truth of A.

Similarly, when we say q is necessary for p then we can only guarantee that when q is false then, p is definitely false.

"p is sufficient for q":

Ex # "It is sufficient for you to travel by car in order to reach your destination on time."

Definitely, if you travel by car, you'll reach your destination on time. No doubt. but if you won't travel by car, does it mean you'll never reach your destination on time. May be by flight or other means of transport, you'll reach your destination much earlier.

Therefore, when we say that A is sufficient for B then, truth of A guarantees the truth of B but we can not guarantee the falsity of B from the falsity of A.

Similarly when we say "P is sufficient for q" then we can only guarantee that when p is true the q is definitely true.

Why P is not necessary for q?

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Proof by Contradiction:

Let say P is necessary for q. if P is really necessary for q then if P is false then it is mandatory for q to be false in order to make the whole compound proposition $(P \rightarrow q)$ true. but it is not the case.

According to the truth table, it is not mandatory for q to be false when p is false. Hence "P is not necessary for q."

Why q is not sufficient for P?

P	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Proof By Contradiction:

Let say q is sufficient for P.
if q is really sufficient for P then if q is true then it is mandatory for P to be true in order to make the whole compound proposition true. but it is not the case.
According to the truth table it is not mandatory for P to be true when q is true. Hence "q is not sufficient for P".

"q unless $\neg P$"
→ means "except if"

Evolution steps:

"if P then q" (if P is true then q must be true)

"q is true when p is true"

"q is true except when p is false"

"q is true unless p is false" = "q unless $\neg P$"

Implication, Converse, Contrapositive, Inverse

Implication or Conditional statement: $p \rightarrow q$

Converse $\rightarrow$ $q \rightarrow p$
Contrapositive $\rightarrow$ $\neg q \rightarrow \neg p$

Inverse $\rightarrow$ $\neg p \rightarrow \neg p$

Example:
"If it rains today then I will stay at home."

Converse: If I will stay at home then it rains today.
Contrapositive:
If I will not stay at home then it does not rain today.
Inverse: If it does not rain today then I will not stay at home.

Facts:
- ⇒ Implication and contrapositive both are equivalent.
- ⇒ Converse and inverse both are equivalent
- ⇒ Neither converse nor Inverse is equivalent to implication.

Truth Table (Proof of facts)

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$q \rightarrow p$	$\neg q \rightarrow \neg p$	$\neg p \rightarrow \neg q$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	F	T
F	T	T	F	T	F	T	F
F	F	T	T	T	T	T	T

Biconditional Operator

Definition:

let p and q be two propositions. The biconditional statement of the form $p \leftrightarrow q$ is the proposition "p if and only if q"

$p \leftrightarrow q$ is true whenever the truth values of p and q are same.

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

How "p if and only if q" make sense?
"p if and only if q" composed of two statements.
"p if q" and "p only if q".

"p only if q" = if p then q <u>and</u> "p if q" = if q then p

$$(p \rightarrow q) \wedge (q \rightarrow p) \equiv p \leftrightarrow q$$

Representations:

1) p is necessary and sufficient for q and vice versa.
2) if p then q and conversely.
3) p <u>iff</u> q
↳ if and only if

Ex # let P be a proposition "You get promoted" and let q be proposition "You have connections."
then $P \leftrightarrow q$ is the statement: "You get promoted if and only if you have connections."

Precedence of Logical Operators

→ Precedence of operators helps us to decide which operator will get evaluated first in a complicated looking compound proposition.

Ex # $(P \rightarrow (q \wedge (\sim P)))$

$p = true$, $q = True$

$P \rightarrow q \wedge \neg P$

True True False

$T \rightarrow F = F$

Operators	Names	Precedence
$\neg$	Negation	1
$\wedge$	Conjunction	2
$\vee$	Disjunction	3
$\rightarrow$	Implication	4
$\leftrightarrow$	Biconditional	5

Problem #

Construct the truth table for the compound proposition below:

$$P \rightarrow (\neg q \wedge \neg P) \leftrightarrow \neg q$$

P	q	$\sim P$	$\sim q$	$\sim q \wedge \sim P$	$P \rightarrow (\neg q \wedge \neg P)$	$P \rightarrow (\neg q \wedge \neg P) \leftrightarrow \neg q$
T	T	F	F	F	F	T
T	F	F	T	F	F	F
F	T	T	F	F	T	F
F	F	T	T	T	T	T

Translating English Sentences into Logical Expressions.

Reasons:

1, Removes uncertainty.
2, Easy manipulation
3, Able to solve puzzles

Ex# "You are not allowed to watch adult movies if your age is less than 18 years or you have no age proof."

Step-1: find the connectives which are connecting the propositions together.

Step-2: Rename the Propositions

let q = "You are allowed to watch adult movies"

r = "Your age is less than 18 years"

s = "You have age proof"

$$(r \vee \sim s) \rightarrow \sim q$$

Ans

Problem # Are these system specifications consistent?

1, "The system is in multiuser state if and only if it is operating normally
2, "If the System is operating normally then the kernel is functioning."
3, "The kernel is not functioning or the system is in interrupt mode."
4, "If the system is not in multiuser state then it is in interrupt mode".
5, "The System is not in interrupt mode."

Solution: let

p = "The system is in multi user state"
q = "The system is operating normally."
r = "The kernel is functioning."
s = "The system is in interupt mode."

1, $p \longleftrightarrow q$
2, $q \rightarrow r$
3, $\sim r \vee s$
4, $\sim p \rightarrow s$
5, $\sim s$

Consistent means assigning truth values to propositional variables in such a way that finally we would be able to make all specifications "true".

Tip: Always start from the statement which involves least number of propositions.

1, $\overset{T}{p} \longleftrightarrow \overset{F}{q} = F$

2, $\overset{F}{q} \rightarrow \overset{F}{r}$, $q = F$

3, $\overset{T}{\sim r} \vee \overset{F}{s}$, $r = F$

4, $\overset{F}{\sim p} \rightarrow \overset{F}{s}$, $p = T$

5, $\sim s$, $s =$ False

Since we can not make all specifications true therefore we can say that these system specifications are not consistent.

Tautology, Contradiction, Contigency & Satisfiability

Tautology: Compound proposition which is always **True**

Ex #

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

here, Compound proposition will always be True so this compound proposition ($p \vee \sim p$) is Tautology.

Contradiction: Compound proposition which is always False.

Ex #

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Here, The compound proposition (i.e. $p \wedge \sim p$) will always be false so its Contradiction.

Contigency: Compound proposition which is some times True and sometimes False.

Ex #

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Here, Compound proposition ($p \wedge q$) is sometimes True and sometimes false so it is Contigency.

Satisfiability:

A Compound proposition is satisfiable if there is at least one True result in its Truth Table.

⇒ A tatoulogy is always satisfiable.

Unsatisfiability:
Not even a single True result in its Truth Table

⇒ Contradiction is always Unsatisfiability

Valid: A compound proposition is valid when it is tautology.

Invalid: A compound proposition is invalid when it is either a Contradiction or contigency.

Logical Equivalences

Definition: The Compound Propositions p and q are said to be logically equivalent if $p \leftrightarrow q$ is a Tautology. Logical equivalence is denoted by $\equiv$ or $\Longleftrightarrow$

Ex #

$$p \wedge T \equiv p$$

p	T	$p \wedge T$
T	T	T
F	T	F

Most Common and famous Logical Equivalences.

1, Identity Laws:

(a) $p \wedge T \equiv p$ (b) $p \vee F \equiv p$

2, Domination laws:

(a) $p \vee T \equiv T$ (b) $p \wedge F \equiv F$

3, Idempotent laws:

(a) $p \vee p \equiv p$ (b) $p \wedge p \equiv p$

4, Double Negation Law:

$$\sim(\sim p) \equiv p$$

5, Commutative Laws:

(a) $p \vee q \equiv q \vee p$ (b) $p \wedge q \equiv q \wedge p$

6, Associative Laws:

(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

7, Distributive Laws:

(a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$, (b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

8, De-Morgan's Laws:

(a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$, (b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

9, Absorption Laws:

(a) $p \vee (p \wedge q) \equiv p$ (b) $p \wedge (p \vee q) \equiv p$

10, Negation Laws:

(a) $p \vee \sim p \equiv T$ (b) $p \wedge \sim p \equiv F$

Logical Equivalences involving Conditional statements:

1, $p \rightarrow q \equiv \sim p \vee q$

2, $p \rightarrow q \equiv \sim q \rightarrow \sim p$

3, $p \vee q \equiv \sim p \rightarrow q$

4, $p \wedge q \equiv \sim(q \rightarrow \sim p)$

5, $\sim(p \rightarrow q) \equiv p \wedge \sim q$

6, $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

7, $(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

8, $(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

9, $(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditionals

1, $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

2, $p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

3, $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

4, $\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q$

Rules of Inference In Propositional Logic
(Basic Terminology and Examples)

Premise:
It is a proposition on the basis of which we would be able to draw a conclusion.
You can think of Premise as an evidence or assumption.
Therefore, initially we assume something is True and on the basis of that assumption, we draw some conclusion.

Conclusion:
It is a proposition that is reached from the given set of premises.
You can think of it as the result of the assumptions that we made in an argument.

$\boxed{\text{if } \underline{\text{premise}} \text{ then } \underline{\text{conclusion}}}$

Argument:
It is a sequence of statements that ends with a conclusion.

OR

It is a set of one or more premises and conclusion.

Valid Argument:
An argument is said to be valid if and only if it is not possible to make all premises true and a conclusion false.

Ex #

"If I love cat then I love dog." $(P \to q)$
"I love cat." (P)
Therefore, "I love dog."

Premise - 1 $\leftarrow P \to q$ OR $((P \to q) \wedge P) \to q$
Premise - 2 $\leftarrow P$

$\therefore q$ ← conclusion

here, there is no way that we make all premises true and conclusion false so, it is an valid argument.

Ex #

"If I love cat then I love dog."
"I love dog"
Therefore, "I love cat".

$$\begin{array}{l} F \quad T = T \\ P \to q \\ q \to T \\ \hline \therefore P \to F \end{array}$$

here, there is a case when we make all premises True but the conclusion becomes false. So,

It is an Invalid Argument

Rules of Inference
(Definition & Types)

Rules of Inference:

Inference means deriving conclusions from evidences.
↳ These are premises or assumptions

on the basis of these assumptions we are trying to derive conclusions

Rules of Inference are the templates for constructing valid arguments.

basically Rules of Inference themselves are valid arguments and are used to construct more complicated valid arguments

Types of Inference Rules

1, Modus Ponens:

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array} \quad \text{OR} \quad \underbrace{[(p \rightarrow q) \wedge p] \rightarrow q}_{\text{If Premises then conclusion}}$$

2, Modus Tollens:

$$\begin{array}{l} p \rightarrow q \\ \sim q \\ \hline \therefore \sim p \end{array} \quad \text{OR} \quad [(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$$

3, Hypothetical Syllogism:

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array} \quad \underline{\underline{OR}} \quad [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

4, Disjunctive Syllogism:

$$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore q \end{array} \quad OR \quad [(p \vee q) \wedge \sim p] \rightarrow q$$

5, Addition:

$$\begin{array}{l} p \\ \therefore p \vee q \end{array} \quad OR \quad p \rightarrow (p \vee q)$$

6, Simplification:

$$\begin{array}{l} p \wedge q \\ \therefore p \end{array} \quad OR \quad \begin{array}{l} p \wedge q \\ \therefore q \end{array}$$

OR

$$(p \wedge q) \rightarrow p \quad or \quad (p \wedge q) \rightarrow q$$

7, Conjunction:

$$\begin{array}{l} p \\ q \\ \hline \therefore p \wedge q \end{array} \quad OR \quad [(p) \wedge (q)] \rightarrow (p \wedge q)$$

8, Resolution:

$$\begin{array}{l} p \vee q \\ \sim p \vee r \\ \hline \therefore q \vee r \end{array} \quad OR \quad [(p \vee q) \wedge (\sim p \vee r)] \rightarrow (q \vee r)$$

Logic Gates

- **Basic Gates**
  - And
  - OR
  - Not
- **Universal Gates**
  - NAND
  - NOR
- **Arithmetic Gates**
  - XOR
  - X-NOR

Truth Table

Input		Output						
A	B	$A \cdot B$	$A+B$	$\overline{A \cdot B}$	$\overline{A+B}$	$A \oplus B$	$A \odot B$	
0	0	0	0	1	1	0	1	
0	1	0	1	1	0	1	0	
1	0	0	1	1	0	1	0	
1	1	1	1	0	0	0	1	

Symbols Used:

- (•) "dot" implies "and" operator
- (+) "plus" implies "or" operator
- (—) "bar" implies the Negation
- (⊙) it shows the X-Nor gate

X-NOR is inverse of XOR Gate.

Now we can easily construct circuits from given logical forms and vice versa.

Half Adder

⇒ It is called Two input and Two output Circuit.

⇒ It basically shows the addition of two inputs.

Input

X	Y	Carry	Sum
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

Sum

⇒ Above is simply the addition of two binary inputs showing sum and Carry.

Now we will make an equation for Sum by looking at the above table where sum is 1.

let 1 represents True
0 represents false

So,

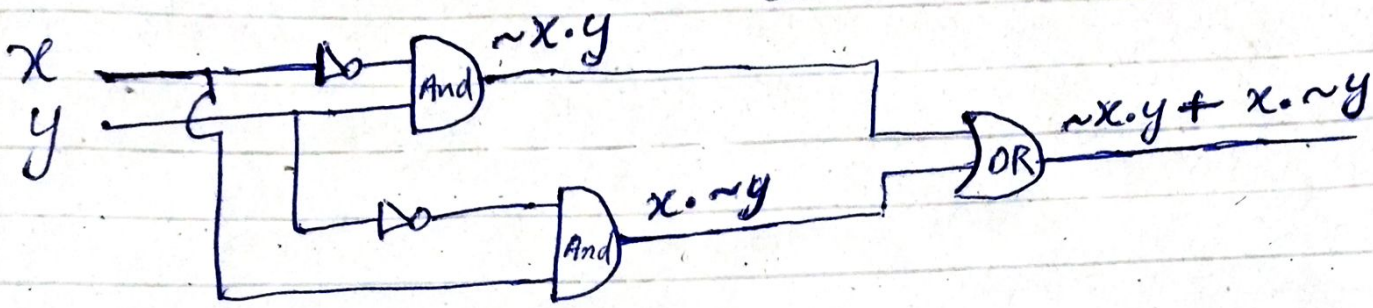
$$Sum = \sim x.y + x.\sim y$$

⇒ for carry, we will make an equation for carry by looking in the table where carry is 1.

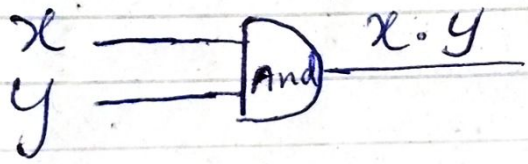
$$Carry = x.y$$

Now we will construct the circuits for sum and carry

for sum $\boxed{Sum = \sim x \cdot y + x \cdot \sim y}$



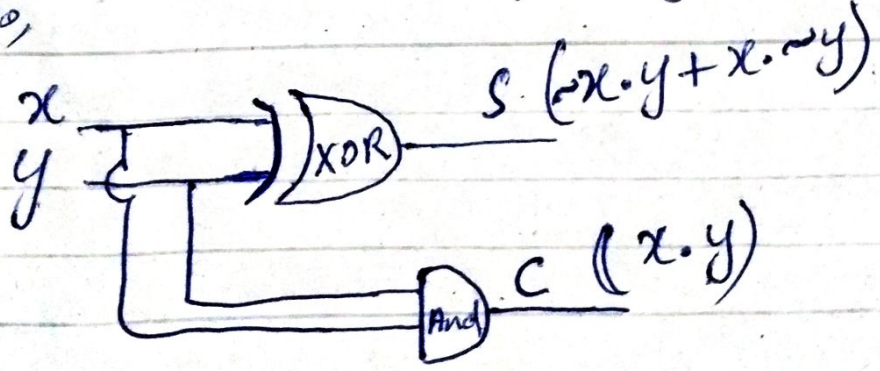
for Carry $c = x \cdot y$



⇒ We can also make the circuit of this whole half adder more precisely like,

$$Sum = \boxed{\sim x \cdot y + x \cdot \sim y}$$

It is the equation of XOR and can be written as,

$$Sum = x \oplus y$$

So,