CHAPTER SIX

APPLICATIONS

6.1 INTRODUCTION

Readers may recall that when higher order non-homogeneous linear differential equations are to be solved the first step is to find out the complementary function y_C and particular integral y_P. Then after finding the general solution $y = y_C + y_P$, the particular solution is found by using the initial and/or boundary conditions. The Laplace transform method of solving differential equations yields particular solution without finding the general solution. It makes use of initial conditions at the beginning of process. This method is shorter than the methods that have been discussed so far in the subject of "Solution of Differential Equations of Higher Orders". This method is also useful in solving differential equations with variable coefficients as well as the system of differential equations.

Solutions of Ordinary Linear Differential Equations with Constant Coefficients

In this section, we shall discuss the solutions of ordinary differential equations with constant coefficients. The procedure is explained by solving some problems.

EXAMPLE 01: Solve the initial value problem

$$Y''(t)+Y(t)=t, \quad Y(0)=1, Y'(0)=-2$$

Solution: Taking Laplace transform on both sides of given differential equation, we get:

$$L\{Y''(t)\}+L\{Y(t)\}=L\{t\} \Rightarrow s^2y(s)-sY(0)-Y'(0)+y(s)=\frac{1}{s^2}$$

Using initial conditions, we get

$$s^2y(s)-s(1)-(-2)+y(s)=\frac{1}{s^2} \qquad \Rightarrow (s^2+1)y(s)=\frac{1}{s^2}+s-2$$

$$\Rightarrow y(s)=\frac{1}{s^2(s^2+1)}+\frac{s}{(s^2+1)}-\frac{2}{(s^2+1)}=\frac{s^2+1-s^2}{s^2(s^2+1)}+\frac{s}{(s^2+1)}-\frac{2}{(s^2+1)}$$

$$=\frac{(s^2+1)}{s^2(s^2+1)}-\frac{s^2}{s^2(s^2+1)}+\frac{s}{(s^2+1)}-\frac{2}{(s^2+1)}=\frac{1}{s^2}-\frac{1}{(s^2+1)}+\frac{s}{(s^2+1)}-\frac{2}{(s^2+1)}$$

Taking inverse Laplace transform on both sides, we obtain

$$L^{-1}\{y(s)\}=L^{-1}\left\{\frac{1}{s^2}\right\}-L^{-1}\left\{\frac{3}{(s^2+1)}\right\}+L^{-1}\left\{\frac{s}{(s^2+1)}\right\} \text{ (By linear property)}$$

$$\Rightarrow Y(t)=t-3\sin t+\cos t$$

EXAMPLE: 02 Solve the initial value problem

$$Y''(t)-3Y'(t)+2Y(t)=4e^{2t}, \quad Y(0)=-3, Y'(0)=5$$

Solution: Given $Y''(t)-3Y'(t)+2Y(t)=4e^{2t}$

Taking the Laplace transform on both sides, we get

$$L\{Y''(t)\}-3L\{Y'(t)\}+2L\{Y(t)\}=4L\{e^{2t}\}$$

$\Rightarrow\left[s^2y(s)-sY(0)-Y'(0)\right]-3\left[sy(s)-Y(0)\right]+2y(s)=\dfrac{4}{s-2}$

Using the given conditions, we get

$\left[s^2y(s)-s(-3)-5\right]-3\left[sy(s)-(-3)\right]+2y(s)=\dfrac{4}{s-2}$

or, $\left(s^2-3s+2\right)y(s)=\dfrac{4}{s-2}-3s+14$

$\Rightarrow y(s)=\dfrac{4}{(s-2)\left(s^2-3s+2\right)}+\dfrac{14-3s}{\left(s^2-3s+2\right)}\rightarrow y(s)=\dfrac{4+14s-28-3s^2+6s}{(s-2)\left(s^2-3s+2\right)}$

$=\dfrac{-3s^2+20s-24}{(s-2)(s-2)(s-1)}\qquad \left(\because s^2-3s+2=(s-2)(s-1)\right)$

$\rightarrow\ y(s)=\dfrac{-3s^2+20s-24}{(s-2)^2(s-1)}$

Now resolving into partial fractions, we get

$$y(s)=\frac{-3s^2+20s-24}{(s-1)(s-2)^2}=\frac{A}{s-1}+\frac{B}{(s-2)^2}+\frac{C}{s-2}\qquad(1)$$

$\rightarrow -3s^2+20s-24=\ A(s-2)^2+B(s-1)+C(s-1)(s-2)\rightarrow(1)$

Now put s = 1 in (1), we get

$-3(1)^2+20(1)-24=\ A(1-2)^2+B(1-1)+C(1-1)(1-2)\quad\rightarrow\quad A=-7$

Now put s = 2 in (1), we obtain

$-3(2)^2+20(2)-24=\ A(2-2)^2+B(2-1)+C(2-1)(2-2)\rightarrow\quad B=4$

To find C, we rewrite equation (1) to get

$-3s^2+20s-24=\ (A+C)s^2-(B-4A-3C)s+4A-B+2C$

Now comparing the coefficients of s^2 on both sided, we get

$A+C=-3\rightarrow -7+C=-3\rightarrow\ C=4$. Thus equation (1) becomes

$y(s)=\dfrac{-3s^2+20s-24}{(s-1)(s-2)^2}=\dfrac{-7}{s-1}+\dfrac{4}{(s-2)^2}+\dfrac{4}{s-2}$

Now taking the inverse Laplace transform on both sides, we get

$L^{-1}\{y(s)\}=L^{-1}\left\{\dfrac{-3s^2+20s-24}{(s-1)(s-2)^2}\right\}=-7L^{-1}\left\{\dfrac{1}{s-1}\right\}+4L^{-1}\left\{\dfrac{1}{(s-2)^2}\right\}+4L^{-1}\left\{\dfrac{1}{s-2}\right\}$

$\Rightarrow Y(t)=-7e^t+4te^{2t}+4e^{2t}$ or $Y(t)=-7e^t+4e^{2t}(t+1)$

EXAMPLE 03: Solve the differential equation

$Y'''(t)-3Y''(t)++3Y'(t)-Y(t)=t^2e^t,\quad Y(0)=1, Y'(0)=0, Y''(0)=-2$

by using Laplace transform method.

Solution: Given differential equation is $Y'''(t)-3Y''(t)++3Y'(t)-Y(t)=t^2e^t$

Taking Laplace transform on both sides, we have

$\{Y'''(t)\}-3L\{Y''(t)\}++3L\{Y'(t)\}-L\{Y(t)\}=L\{t^2e^t\}$

$\Rightarrow\{s^3y(s)-s^2Y(0)-sY'(0)-Y''(0)\}-3\{s^2y(s)-sY(0)-Y'(0)\}+3\{sy(s)-1\}-y(s)=2/(s-$

Now using given conditions, we get

$$\{s^3y(s)-s^2(1)-s(0)-(-2)\}-3\{s^2y(s)-s(1)-0\}+3\{sy(s)-1\}-y(s)=\frac{2}{(s-1)^3}$$

$$\Rightarrow y(s)=\frac{2}{(s-1)^3(s^3-3s^2+3s-1)}+\frac{s^2-3s+1}{(s^3-3s^2+3s-1)}$$

$$\text{or, } y(s)=\frac{2}{(s-1)^3(s-1)^3}+\frac{s^2-3s+1}{(s-1)^3} \qquad \left(\because (a-b^3)=a^3-3a^2b+3ab^2-b^3\right)$$

$$=\frac{2}{(s-1)^6}+\frac{s^2-3s+1}{(s-1)^3}=\frac{s^2-2s+1-s+1-1}{(s-1)^3}+\frac{2}{(s-1)^6}$$

$$=\frac{(s-1)^2}{(s-1)^3}-\frac{s-1}{(s-1)^3}-\frac{1}{(s-1)^3}+\frac{2}{(s-1)^6}=\frac{1}{(s-1)}-\frac{1}{(s-1)^2}-\frac{1}{(s-1)^3}+\frac{2}{(s-1)^6}$$

Now taking the inverse Laplace transform on both sides, we have

$$L^{-1}\{y(s)\}=L^{-1}\left\{\frac{1}{(s-1)}\right\}-L^{-1}\left\{\frac{1}{(s-1)^2}\right\}-L^{-1}\left\{\frac{1}{(s-1)^3}\right\}+2L^{-1}\left\{\frac{1}{(s-1)^6}\right\}$$

$$\Rightarrow \qquad Y(t)=e^t-te^t-\frac{t^2}{2}e^t+2\frac{t^5}{5!}e^t=e^t-te^t-\frac{t^2}{2}e^t+\frac{t^5}{60}e^t$$

EXAMPLE 04: Solve the differential equation

$Y''(t)+2Y'(t)+5Y=e^{-t}\sin t \quad ,Y(0)=0, Y'(0)=1$

Solution: Given $Y''(t)+2Y'(t)+5Y=e^{-t}\sin t$

Taking Laplace transform on both sides, we obtain

$$L\{Y''(t)\}Y''(t)+2L\{Y'(t)\}+5L\{Y\}=L\{e^{-t}\sin t\}$$

$$\Rightarrow \qquad \{s^2y(s)-sY(0)-Y'(0)\}+2\{sy(s)-Y(0)\}+5y(s)=\frac{1}{(s+1)^2+1}$$

Using the given conditions, that is; Y(0)=0, $Y'(0)=1$, we obtain

$$\{s^2y(s)-s(0)-1\}+2\{sy(s)-0\}+5y(s)=\frac{1}{s^2+2s+2}$$

$$s^2y(s)-1+2sy(s)+5y(s)=\frac{1}{s^2+2s+2} \Rightarrow y(s)=\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

Taking the inverse Laplace transform on both sides, we have

$$L^{-1}\{y(s)\}=L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

Now consider, $$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}=\frac{As+B}{(s^2+2s+2)}+\frac{Cs+D}{(s^2+2s+5)} \qquad (1)$$

$$\Rightarrow s^2+2s+3=(As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)$$

$$\Rightarrow s^2+2s+3=(A+C)s^3+(2A+B+2C+D)s^2+(5A+2B+2C+2D)s+5B+2D$$

Now by comparing the coefficients of same powers of:

$s^3: (A+C)=0$ (i)

$s^2: 2A+B+2C+D=1$ (ii)

$s: (5A+2B+2C+2D)=2$ (iii)

$s^0: 5B+2D=3$ (iv)

Now subtracting (ii) from (iii), we have

$3A+B+D=1$ (v)

Now subtract (v) from (ii), we have

$-A+2C=0$ (vi)

Now from (i) A= - C. Put in (vi), we get

$C+2C=0 \Rightarrow C=0$ Put $C=0$ in (vi), we have $A=0$

Put $A=0$ in (v), we get $B+D=1$ or $2B+2D=2$ (vii)

Adding (iv) and (vii), we have: $B=1/3$

Put $A=0$ and $B=1/3$ in (v), we get $D=2/3$

Thus equation (1) becomes after taking inverse Laplace transform

$$L^{-1}(y(s))=L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}=L^{-1}\left\{\frac{1/3}{s^2+2s+2}\right\}+L^{-1}\left\{\frac{2/3}{s^2+2s+5}\right\}$$

$$=\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}+\frac{2}{6}L^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}=\frac{1}{3}e^{-t}\sin t+\frac{2}{6}e^{-t}\sin 2t$$

$$\Rightarrow \qquad Y(t)=\frac{1}{3}e^{-t}(\sin t+\sin 2t)$$

Example 05: Solve the differential equation $Y''+9Y=18t$, $Y(0)=0$, $Y(\pi/2)==2$.

Solution: Taking Laplace transform of given differential equation, we get:

$L[Y'']+9L[Y]=18L[t] \quad \rightarrow s^2y(s)-sY(0)-Y'(0)+9y(s)=18/s^2$ (1)

Here we see that only one condition $Y(0)=0$ is given where as the second condition is not. Thus we assume that $Y'(0)=c$ and so equation (1) becomes:

$(s^2+9)y(s)=c+18/s^2$

$$\rightarrow y(s)=\frac{cs^2+18}{s^2(s^2+9)}=\frac{A}{s^2}+\frac{B}{s}+\frac{Cs+D}{s^2+9} \quad \text{(By PF Case III)} \tag{2}$$

$$\rightarrow \qquad cs^2+18=A(s^2+9)+Bs(s^2+9)+Cs^3+Ds^2 \tag{3}$$

Put $s=0$, we get: $18=9A \quad \rightarrow A=2$.

To find B, C and D, we compare the coefficients of different terms of s in equation (3).

Now comparing the coefficients of:

s^3: $0=B+C$ (i)

s^2: $c=A+D \quad \rightarrow D=c-A=c-2$ (ii)

s: $0=9B \rightarrow B=0 \quad \rightarrow C=0$ (iii)

Thus equation (2) becomes: $y(s)=\dfrac{cs^2+18}{s^2(s^2+9)}=\dfrac{2}{s^2}+\dfrac{c-2}{s^2+9}$

Taking inverse Laplace transform on both sides, we obtain:

$$L^{-1}[y(s)]=Y(t)=2L^{-1}\frac{1}{s^2}+(c-2)L^{-1}\frac{1}{s^2+9}=2t+\frac{(c-2)\sin 3t}{3} \tag{4}$$

To find c we use the remaining condition that $Y(\pi/2)=2$, that put $t=\pi/2$ and $Y=2$.

Scanned with CamScanner

$\Rightarrow 2 = 0 + (c-2)(-1)/3 \quad \Rightarrow c-2=-6 \quad \Rightarrow c=-4.$

Thus equation (4) becomes:

$$Y(t) = 2t + \frac{(-4-2)\sin 3t}{3} = 2t - 2\sin 3t$$

Solutions of Ordinary Differential Equations with Variable Coefficients

This section is devoted towards the solution of ordinary differential equations with variable coefficients. The method is best explained by solving the following few problems.

EXAMPLE 06: Solve the differential equation

$$tY''(t)+Y'(t)+4tY(t)=0, \quad Y(0)=3, Y'(0)=0$$

Solution: Taking the Laplace transform on both sides, we have

$$L\{tY''(t)\}+L\{Y'(t)\}+4L\{tY(t)\}=L\{0\} \qquad (1)$$

We know that $L\{t^n F(t)\}=(-1)^n f^n(s)$ (2)

Now, $L\{Y''(t)\}=\{s^2y(s)-sY(0)-Y'(0)\}$. Thus using (2), we get

$$L\{tY''(t)\}=\frac{-d}{ds}\{s^2y(s)-sY(0)-Y'(0)\}$$

Thus equation (1) becomes

$$\frac{-d}{ds}\{s^2y(s)-sY(0)-Y'(0)\}+\{sy-Y(0)\}-4\frac{dy}{ds}=0$$

Using given conditions, we get

$$\frac{-d}{ds}\{s^2y-s(3)-0\}+\{sy-3\}-4\frac{dy}{ds}=0 \Rightarrow 3\frac{d}{ds}s-\frac{d}{ds}\{s^2y\}+\{sy-3\}-4\frac{dy}{ds}=0$$

$$\Rightarrow 3.1-\left\{2sy+s^2\frac{dy}{ds}\right\}+\{sy-3\}-4\frac{dy}{ds}=0$$

or, $(s-2s)y-(s^2+4)\frac{dy}{ds}=0 \quad \Rightarrow \quad sy+(s^2+4)\frac{dy}{ds}=0$

If we observe this differential equation in y can be solved by separable variable method.

Thus separating the variables, we get $\frac{dy}{y}=\frac{-s}{s^2+4}ds$

Integrating both sides, we obtain

$$\int\frac{dy}{y}=-\int\frac{s}{s^2+4}ds \Rightarrow \ln y=-\frac{1}{2}\ln(s^2+4)+\ln c \Rightarrow y=\frac{c}{\sqrt{(s^2+4)}}$$

Taking inverse Laplace transform on both sides, we get

$$L^{-1}\{y(s)\}=cL^{-1}\left\{\frac{1}{\sqrt{(s^2+4)}}\right\} \Rightarrow Y(t)=cJ_0(2t) \qquad (3)$$

NOTE: $L^{-1}\left(\frac{1}{\sqrt{s^2+a^2}}\right)=J_0(at)$

In order to compute c, we use the condition Y(0) = 3, that is; we put $Y=3, t=0$.

$\therefore \quad Y(0)=3=cJ_0(0) \quad \Rightarrow \quad 3=c \qquad (\because J_0(0)=1)$

Thus equation (3) becomes $Y(t) = 3J_0(2t)$. This is the solution of given differential equation.

EXAMPLE 07: Solve the differential equation

$$tY''(t)+2Y'(t)+tY(t)=0,\quad Y(0)=1, Y(\pi)=0$$

Solution: Taking Laplace transform on both sides of given differential equation, we get

$$L\{tY''(t)\}+2L\{Y'(t)\}+L\{tY(t)\}=L\{0\}$$

$$\Rightarrow \quad \frac{-d}{ds}\left(s^2y-sY(0)-Y'(0)\right)+2\left(sy-Y(0)\right)-\frac{d}{ds}y=0$$

Using the condition Y (0) = 1 and suppose that Y '(0) = c, so that above equation becomes:

$$\frac{-d}{ds}\left(s^2y-s(1)-c\right)+2(sy-1)-\frac{dy}{ds}=0$$

$$\Rightarrow 1-2sy-s^2\frac{dy}{ds}+2sy-2-\frac{dy}{ds}=0 \text{ or } -1-\left(s^2+1\right)\frac{dy}{ds}=0 \text{ or } \left(s^2+1\right)\frac{dy}{ds}=-1$$

Separating the variables and integrating, we get

$$\int dy=\int\frac{-1}{\left(s^2+1\right)}ds \quad \Rightarrow \quad y=-\tan^{-1}s+c \qquad (1)$$

It may be noted that by the definition of inverse Laplace transform:
When $s\to\infty \Rightarrow y\to 0$. Thus from (1) we see that

$$0=-\tan^{-1}\infty+c \Rightarrow 0=-\pi/2+c \Rightarrow c=\pi/2.$$

Hence, (1) becomes: $y=-\tan^{-1}s+\dfrac{\pi}{2}=\tan^{-1}\left(\dfrac{1}{s}\right)$ (2)

Now taking inverse Laplace transform on both sides of (2), we have

$$L^{-1}(y(s))=L^{-1}\left(\tan^{-1}\left(\frac{1}{s}\right)\right) \Rightarrow Y(t)=\frac{\sin t}{t} \qquad \left(\because L^{-1}\left(\frac{\sin t}{t}\right)=\tan^{-1}\left(\frac{1}{s}\right)\right)$$

This is the required solution of given differential equation.

NOTE: $\tan^{-1}a-\tan^{-1}b=\tan^{-1}\left(\dfrac{a-b}{1+ab}\right)=\tan^{-1}\left[\dfrac{a\left(1-\frac{b}{a}\right)}{a\left(\frac{1}{a}+b\right)}\right]=\tan^{-1}\left[\dfrac{\left(1-\frac{b}{a}\right)}{\left(\frac{1}{a}+b\right)}\right]$

Now let $a\to\infty$ then above equation becomes: $\tan^{-1}\infty-\tan^{-1}b=\tan^{-1}\left(\dfrac{1}{b}\right)$.

But $\tan^{-1}\infty=\pi/2$. Thus, $\dfrac{\pi}{2}-\tan^{-1}b=\tan^{-1}\left(\dfrac{1}{b}\right)$. This is same as equation (2).

Solutions of Simultaneous Ordinary Differential Equations

In this section we shall discuss the solution of system of ordinary differential equations. The method is best explained by solving the following problems.

EXAMPLE 08: Solve the system of differential equations

$$\frac{dX}{dt}=2X-3Y \quad ; \quad \frac{dY}{dt}=Y-2X \quad \text{given that } X(0)=8, Y(0)=3$$

Here, dX/dt = X ', dY/dt = Y '

Solution: Taking the Laplace transform of given equations, we get

$$L\{X'\}=2L\{X\}-3L\{Y\} \qquad L\{Y'\}=L\{Y\}-2L\{X\}$$

$$sx-X(0)=2x-3y \qquad sy-Y(0)=y-2x$$

Now using given conditions X(0) = 8 and Y(0) =3, we have

$sx-8=2x-3y$ $\quad$ $sy-3=y-2x$

$(s-2)x+3y=8 \quad$ (i) $\quad$ $(s-1)y+2x=3 \quad$ (ii)

Now solving (i) and (ii) simultaneously using Cramer's rule or otherwise, we get

$$x=\frac{\begin{vmatrix}8 & 3\\ 3 & s-1\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}} \quad \text{and} \quad y=\frac{\begin{vmatrix}s-2 & 8\\ 2 & 3\end{vmatrix}}{\begin{vmatrix}s-2 & 3\\ 2 & s-1\end{vmatrix}}$$

$$=\frac{8s-17}{s^2-3s-4} \quad \text{and} \quad =\frac{3s-22}{s^2-3s-4}$$

or $\quad x=\dfrac{8s-17}{(s+1)(s-4)} \quad$ and $\quad y=\dfrac{3s-22}{(s+1)(s-4)}$

Now by partial fractions,

$$x=\frac{8(-1)-17}{(s+1)(-1-4)}+\frac{8(4)-17}{(4+1)(s-4)} \quad \text{and} \quad y=\frac{3(-1)-22}{(s+1)(-1-4)}+\frac{3(4)-22}{(4+1)(s-4)}$$

$$=\frac{-8-17}{-5(s+1)}+\frac{32-17}{5(s-4)} \quad \text{and} \quad =\frac{-3-22}{-5(s+1)}+\frac{12-22}{5(s-4)}$$

or $x=\dfrac{5}{(s+1)}+\dfrac{3}{(s-4)} \quad$ and $\quad y=\dfrac{5}{(s+1)}+\dfrac{-2}{(s-4)}$

Now taking inverse Laplace transforms on both sides, we obtain

$$L^{-1}\{x(s)\}=5L^{-1}\left\{\frac{1}{(s+1)}\right\}+3L^{-1}\left\{\frac{1}{(s-4)}\right\}, \quad L^{-1}\{y(s)\}=L^{-1}\left\{\frac{5}{(s+1)}\right\}-L^{-1}\left\{\frac{2}{(s-4)}\right\}$$

$\rightarrow$ $X(t)=5e^{-t}+3e^{4t} \quad$ and $\quad Y(t)=5e^{-t}-2e^{4t}$.

This is the solution of given system of linear equations.

EXAMPLE 09: Solv e the system of differential equation

$$\frac{dY}{dt}-\frac{dZ}{dt}-2Y+2Z=\sin t$$

$$\frac{d^2Y}{dt^2}+2\frac{dZ}{dt}+Y=0, \text{ given that } Y(0)=Y'(0)=Z'(0)=0$$

Solution: Taking the Laplace transform of both equations, we have

$$L\{Y'\}-L\{Z'\}-2L\{Y\}+2L\{Z\}=L\{\sin t\}$$

and $L\{Y''\}+2L\{Z'\}+L\{Y\}=L\{0\}$

$$\therefore \{sy-Y(0)\}-\{sz-Z(0)\}-2y+2z=\frac{1}{s^2+1}$$

and $\{s^2y-sY(0)-Y'(0)\}+2\{sz-Z(0)\}+y=0$

Now applying given conditions, we get

$\{sy-0\}-\{sz-\}-2y+2z=1/(s^2+1) \quad$ and $\quad \{s^2y-s(0)-0\}+2\{sz-0\}+y=0$

or $\quad sy-sz-2y+2z=1/(s^2+1) \quad$ and $\quad s^2y+2sz+y=0$

$\therefore \quad (s-2)y+(2-s)z=1/(s^2+1)$ (i)

and $\quad (s^2+1)y+2sz=0$ (ii)

Now solving (i) and (ii) simultaneously, we get

Solving for y:

$$y=\frac{\begin{vmatrix}1/s^2+1 & 2-s\\ 0 & 2s\end{vmatrix}}{\begin{vmatrix}s-2 & 2-s\\ s^2+1 & 2s\end{vmatrix}} \rightarrow y=\frac{2s}{(s^2+1)(s^3-3s-2)} \text{ or } y=\frac{2s}{(s-2)(s+1)^2(s^2+1)}$$

$$\left(\begin{aligned}&\text{NOTE}: s^3-3s-2=s^3-4s+s-2=s(s^2-4)+1(s-2)=s(s-2)(s+2)+(s-2)\\ &=(s-2)(s^2+2s+1)=(s-2)(s+1)^2\end{aligned}\right)$$

$$\therefore \quad y=\frac{2s}{(s-2)(s+1)^2(s^2+1)}$$

By partial fractions,

$$\frac{2s}{(s-2)(s+1)^2(s^2+1)}=\frac{A}{s-2}+\frac{B}{s+1}+\frac{C}{(s+1)^2}+\frac{Ds+E}{(s^2+1)} \quad (1)$$

$$\therefore 2s=A(s+1)^2(s^2+1)+B(s-2)(s+1)(s^2+1)+C(s-2)(s^2+1) \quad (2)$$
$$+(Ds+E)(s-2)(s+1)^2$$

Put s = 2 in (2), we get: $4=A(9)(5) \rightarrow A=4/45$

Now put s = -1 in (2), we get: $-2=A(-3)(2) \rightarrow C=1/3$

To find B, C and D we rewrite equation (2), we get

$$2s=A(s^4+2s^3+2s^2+2s+1)+B(s^4-s^3-s^2-s-2)+C(s^3+s-2s^2-2)$$
$$+(Ds^4-3Ds^2-2Ds+Es^3+-3Es-2E)$$

Now comparing the coefficient of same powers of:

s^4: $A+B+D=0$ $\qquad$ s^3: $2A-B+C+E=0$

s^2: $2A-B-2C-3D=0$ $\qquad$ s: $2A-B+C-2D-3E=2$

s^0: $A-2B-2C-2E=0$

Solving the above five equations simultaneously, we get

B = 1/9, D = -1/5 and E = -2/5

Thus equation (1) becomes

$$y=\frac{4}{45(s-2)}+\frac{1}{9(s+1)}+\frac{1}{3(s+1)^2}-\frac{s+2}{5(s^2+1)}$$

Taking inverse Laplace transform on both sides, we have

$$L^{-1}\{y\}=\frac{4}{5}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{9}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{1}{5}L^{-1}\left\{\frac{s+2}{s^2+1}\right\}$$

$$\therefore \quad Y=\frac{4}{5}e^{2t}+\frac{1}{9}e^{-t}+\frac{1}{3}te^{-t}-\frac{1}{5}\cos t-\frac{2}{5}\sin t$$

Solving for z:

$$z=\frac{\begin{vmatrix} s-2 & 1/s^2+1 \\ s^2+1 & 0 \end{vmatrix}}{\begin{vmatrix} s-2 & 2-s \\ s^2+1 & 2s \end{vmatrix}} \Rightarrow z=\frac{-1}{(s+1)^2(s-2)}$$

By partial fractions, we have

$$\frac{-1}{(s+1)^2(s-2)}=\frac{A}{s-2}+\frac{B}{s+1}+\frac{C}{(s+1)^2} \qquad (3)$$

$$\Rightarrow -1=A(s+1)^2+B(s-2)(s+1)+C(s-2) \qquad (4)$$

Put s = 2, we get

$$-1=A(2+1)^2+B(2-2)(s+1)+C(2-2) \quad \Rightarrow A=-1/9$$

Again put s = -1, we get $\quad -1=-3C \quad \Rightarrow C=1/3$

To find B we rewrite equation (4) to get

$$-1=A(s^2+2s+1)+B(s^2-s-2)+C(s-2)$$

Comparing the coefficients of

s^2: $\quad A+B=0 \;\Rightarrow\; B=-A=-1/9$

Thus equation (3) becomes:

$$z=\frac{-1}{(s+1)^2(s-2)}=\frac{-1/9}{s-2}+\frac{1/9}{s+1}+\frac{1/3}{(s+1)^2}$$

Taking inverse Laplace transform on both sides, we have

$$L^{-1}\{z\}=\frac{-1}{9}L^{-1}\left\{\frac{1}{s-2}\right\}+\frac{1}{9}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$\Rightarrow Z=\frac{-1}{9}e^{2t}+\frac{1}{9}e^{-t}+\frac{1}{3}te^{-t}$$

Thus the solution of given system of differential equation is

$$Y=\frac{4}{5}e^{2t}+\frac{1}{9}e^{-t}+\frac{1}{3}te^{-t}-\frac{1}{5}\cos t-\frac{2}{5}\sin t,\quad Z=\frac{-1}{9}e^{2t}+\frac{1}{9}e^{-t}+\frac{1}{3}te^{-t}$$

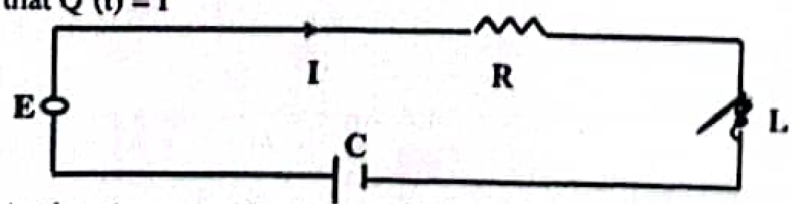
6.2 APPLICATIONS OF LAPLACE TRANSFORMS

Applications to Electrical Circuits

Readers are familiar with RLC circuit. The differential equation that governs this circuit is given by:

$$LQ''+RQ'+\frac{1}{C}Q=E(t)$$

Here we are also given certain conditions on the initial charge and current in the circuit. It may be noted that $Q'(t) = I$



Example 01: At time t = a constant voltage E is applied to RLC-circuit. The current and the charge initially are zero. Find the current in the circuit at ant time t > 0 given that R = 2 ohms, L = 1 H and C 1 F.

Solution: Substituting the values of L, R, C and E = k in the differential equation, we get

$$Q'' + 2Q' + Q = k$$

Taking Laplace transform, we get: $LQ'' + 2LQ' + LQ = kL(1)$

➔ $s^2 q(s) - s\,Q'(0) - Q(0) + 2s\,q(s) - 2Q(0) + q(s) = k/s.$

Now $Q(0) = Q'(0) = 0$. Thus above equation becomes:

$$(s^2 + 2s + 1)\,q(s) = k/s$$

$$\Rightarrow q(s) = \frac{ks}{(s+1)^2} = k\frac{s+1-1}{(s+1)^2} = k\left[\frac{s+1}{(s+1)^2} - \frac{1}{(s+1)^2}\right] = k\left[\frac{1}{(s+1)} - \frac{1}{(s+1)^2}\right]$$

Taking inverse Laplace transform, we obtain

$$L^{-1}q(s) = k\left[L^{-1}\frac{1}{(s+1)} - L^{-1}\frac{1}{(s+1)^2}\right] \Rightarrow Q(t) = k\left(e^{-t} - te^{-t}\right)$$

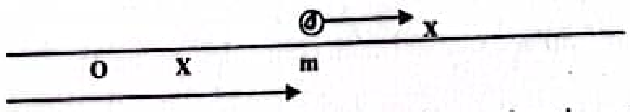
This is the charge in the circuit at any time. Now differentiate both sides w.r.t time t, we get:

$$I = Q'(t) = k\left(-e^{-t} + te^{-t} - e^{-t}\right) = ke^{-t}(t-2)$$

Applications to Mechanics

Example 02: A mass m moves along the x-axis under the influence of a force which is proportional to its instantaneous speed and in a direction opposite to the direction of motion. Assuming that at t = 0 the particle is located at X = a and moving to the right with speed v_0, find the position where the mass comes to rest.

Solution:



According Newton's 2^{nd} law of motion: ma = F. Here a is acceleration and is given by $a = d^2x/dt^2$. The force F is proportional to speed and is acting in the direction of opposite direction. Thus equation governing this problem is:

$$mX'' = -kX' \quad \text{with initial conditions } X(0) = a \text{ and } X'(0) = v_0$$

Taking Laplace transform on each side, we get:

$$mLX'' = -kLX' \Rightarrow m\left(s^2x - sX(0) - X'(0)\right) + k\left(sx - X'(0)\right) = 0$$

$$m\left(s^2x - as - v_0\right) + k\left(sx - v_0\right) = 0 \quad \Rightarrow s(ms+k)x = mas + mv_0 + kv_0$$

$$\Rightarrow x = \frac{mas + v_0(m+k)}{s(ms+k)} = \frac{A}{s} + \frac{B}{ms+k} \quad \Rightarrow mas + v_0(m+k) = A(ms+k) + Bs$$

Now put s = 0, we get: $A = (mv_0 + ak)/k$

Put ms + k = 0, we get: $B = -m^2 v_0/k$

$$x = \frac{(mv_0 + ak)}{ks} - \frac{m^2v_0}{ms+k} = \frac{(mv_0+ak)}{ks} - \frac{mv_0}{(s+k/m)}$$

Thus, taking inverse Laplace transform, we get:

$$L^{-1}(x) = X(t) = \frac{(mv_0+ak)}{k}L^{-1}\left(\frac{1}{s}\right) - mv_0L^{-1}\frac{1}{(s+k/m)}$$

Thus $X(t) = \dfrac{(mv_0+ak)}{k} mv_0e^{-\left(\frac{k}{m}\right)t}$ is the general solution.

WORKSHEET 06

Solve the following differential equations by using Laplace transforms

1. $Y'' + Y = t \quad y(0)=1, y'(0)=-2$
2. $Y'' - 3Y' + 2Y = 4e^{2t}t \quad Y(0)=-3, Y'(0)=5$
3. $Y'' + 2Y' + 5Y = e^{-t}\sin t \quad Y(0)=0, Y'(0)=1$
4. $Y''' - 3Y'' + 3Y' - Y = t^2e^t \quad Y(0)=1, Y'(0)=0, Y''(0)=-2$
5. $Y'' + Y = 8\cos t \quad y(0)=1, y'(0)=-1$
6. $Y'' + 4Y = \cos 2t \quad y(\pi)=0=y'(\pi)$
7. $Y'' + 2Y' + 2Y = 4e^{-t}t\sin t \quad Y(0)=2, Y'(0)=3$
8. $tY'' + Y' + 4tY = 0 \quad Y(0)=3, Y'(0)=0$
9. $Y'' - tY' + Y = 1 \quad Y(0)=1, Y'(0)=2$
10. $X' - Y' - 2X - Y = e^t, 2X' + Y' - 3X - 3Y = 6e^{2t}, \quad X(0)=3, Y(0)=0$
11. $X' + Y' = T, X'' - Y = e^{-t}, \quad X(0)=3, X'(0)=-2, Y(0)=0$
12. Solve the differential equation: $LQ'' + RQ' + \frac{1}{C}Q = E(t)$ where L = 1 H, R = 2 ohms, C = 1 F and E =5V with Q(0) = 0 and I(0) = 0.
13. Find the charge on the capacitor and the current in the given LRC series circuit, where L = 5/3 H, R = 10 ohms, C = 1/30 F and E = 300 V and Q(0) = 0, I(0) = 0.
14. A circuit consists of an inductance of 0.05 Henry, a resistance of 20 ohms, a condenser of capacitance 100 microfarads, and an emf of E = 100V. Find I and Q, given the initial conditions Q = 0, I = 0 at t = 0.
15. A particle of mass 2 grams on the x-axis is attracted towards origin with a force equal to 8X. If it is initially at rest at X = 10, find its position at any time.

Scanned with CamScanner

CHAPTER SEVEN

FOURIER TRANSFORMS

7.1 INTRODUCTION

Joseph Fourier, a French mathematician had invented a method called Fourier transform in 1801, to explain the flow of heat around an anchor ring. Since then, it has become a powerful tool in diverse fields of science and engineering.

Fourier transforms provide a means of solving unwieldy equations that describe dynamic responses to electricity, heat or light. In some cases, it can also identify the regular contributions to a fluctuating signal, thereby helping to make sense observations in Astronomy, Medicine and Chemistry. Fourier transform has become indispensable in the numerical calculations needed to design electrical circuits, to analyze the mechanical vibrations, and to study wave propagation. In order to deal with semi-infinite or totally infinite range of variables or unbounded regions it is necessary to generalize Fourier series to include infinite intervals and to introduce concept of Fourier integral and Fourier transform.

Fourier Transform Pairs

Let f(x) is the function defined on $(-\infty,\infty)$ and is piecewise continuous, differentiable in each finite interval and is absolutely integrable on $(-\infty,\infty)$, then Fourier transform of f(x) is denoted and defined as:

$$F[f(x)]=\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$$

And the inverse Fourier transform is $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\alpha)e^{-i\alpha x}d\alpha$, where α being a Fourier transform parameter.

Fourier Sine transform pair denoted by $\bar{f_s}(\alpha)$ is to be defined as

$$\bar{f_s}(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\alpha x\,dx=F_s(f(x))$$

The inverse Fourier Sine transform pair denoted by $f(x)$ is to be defined as

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\bar{f}_{s(\alpha)}\sin\alpha x\,d\alpha=F_s^{-1}\left[\bar{f_s}(\alpha)\right]$$

Similarly Fourier Cosine transform pair denoted by $\bar{f_c}(\alpha)$ is to be defined as

$$\bar{f_c}(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\alpha x\,dx=F_c(f(x))$$

and the inverse Fourier Cosine transform pair denoted by $f(x)$ is to be defined as

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\bar{f_c}(\alpha)\cos\alpha x\,d\alpha=F_c^{-1}\left[\bar{f_c}(\alpha)\right]$$

Scanned with CamScanner

Example 01: Find the Fourier transform of $f(x)=e^{-x^2/2}$

Solution: By definition, $\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}e^{i\alpha x}dx$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(x-i\alpha)^2}{2}}e^{\frac{-\alpha^2}{2}}dx \text{ (By completing the squares)}$$

$$=\frac{e^{\frac{-\alpha^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-(x-i\alpha)^2}{2}}dx \qquad \text{(i)}$$

Let $\frac{(x-i\alpha)}{\sqrt{2}}=t \Rightarrow \frac{dx}{\sqrt{2}}=dt$. Putting in (i), we get

$$\bar{f}(\alpha)=\frac{e^{\frac{-\alpha^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2}\sqrt{2}\,dt=\frac{e^{\frac{-\alpha^2}{2}}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}dt$$

$$=\frac{e^{\frac{-\alpha^2}{2}}}{\sqrt{\pi}}\sqrt{\pi}=e^{\frac{-\alpha^2}{2}} \qquad \left(\because \int_{-\infty}^{\infty} e^{-t^2}dt=\sqrt{\pi}\right)$$

Example 02: Find the Fourier transform of $f(x)=e^{-ax^2}$, $a>0$

Solution: We know that

$$\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ax^2}e^{i\alpha x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x^2-\frac{i\alpha x}{a}\right)}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x^2-\frac{i\alpha x}{a}+\frac{i^2\alpha^2}{4a^2}-\frac{i^2\alpha^2}{4a^2}\right)}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{i\alpha}{2a}\right)^2}e^{\frac{-\alpha^2}{4a}}dx=\frac{e^{\frac{-\alpha^2}{4a}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{i\alpha}{2a}\right)^2}dx \qquad \text{(i)}$$

Substituting $u=\sqrt{a}\left(x-\frac{i\alpha}{2a}\right)$, then $du=\sqrt{a}\,dx$ and so (i) becomes

$$\bar{f}(\alpha)=\frac{e^{-\alpha^2/4a}}{\sqrt{2\pi a}}\int_{-\infty}^{\infty} e^{-u^2}du=\frac{e^{-\alpha^2/4a}}{\sqrt{2\pi a}}\sqrt{\pi}=\frac{e^{-\alpha^2/4a}}{\sqrt{2a}}$$

Example 03: Find the Fourier transform of $f(x)=e^{-a|x|}$, $-\infty<x<\infty$

Solution: We know that absolute function is of x is defined as:

$$f(x)=\begin{cases} x, & \text{if } x\geq 0 \\ -x, & \text{if } x<0 \end{cases}$$

By definition, $\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0} e^{ax}e^{i\alpha x}dx+\int_{0}^{\infty} e^{-ax}e^{i\alpha x}dx\right]$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a+i\alpha)x}dx+\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{(-a+i\alpha)x}dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{1}{a+i\alpha}-\frac{1}{-a+i\alpha}\right]=\frac{1}{\sqrt{2\pi}}\left[\frac{-a+i\alpha-a-i\alpha}{(a+i\alpha)(a-i\alpha)}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{-2a}{\left(-a^2+i^2\alpha^2\right)}\right]=\frac{1}{\sqrt{2\pi}}\left[\frac{-2a}{\left(-a^2-\alpha^2\right)}\right]=\sqrt{\frac{2}{\pi}}\left[\frac{a}{a^2+x^2}\right]$$

Example 04: Find the Fourier transform of f(x) defined by

$$f(x)=1 \quad ,|x|\le a$$
$$=0 \quad ,|x|>a$$

and hence evaluate $\int_{-\infty}^{\infty}\frac{\sin\alpha a\cos\alpha x}{\alpha}d\alpha$ and $\int_{0}^{\infty}\frac{\sin\alpha a}{\alpha}d\alpha$

Solution: Since by definition

$$\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(1)e^{i\alpha x}dx=\frac{1}{\sqrt{2\pi}}\left|\frac{e^{i\alpha x}}{i\alpha}\right|_{-a}^{a}$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{e^{i\alpha a}}{i\alpha}-\frac{e^{-i\alpha a}}{i\alpha}\right)=\frac{2}{2\sqrt{2\pi}}\left[\frac{e^{i\alpha a}-e^{-i\alpha a}}{i\alpha}\right]=\frac{2\sin\alpha a}{\alpha\sqrt{2\pi}}$$

Therefore $\bar{f}(\alpha)=\frac{2\sin\alpha a}{\alpha\sqrt{2\pi}},\ \alpha>0$ $\qquad \because \left(e^{i\alpha a}-e^{-i\alpha a}\right)/2i=\sin\alpha a$

$$\therefore \lim_{\alpha\to 0}\bar{f}(\alpha)=\frac{2a}{\sqrt{2\pi}}\lim_{\alpha\to 0}\frac{\sin\alpha a}{\alpha a}=\frac{2a}{\sqrt{2\pi}}$$

Now $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}(\alpha)e^{-i\alpha x}d\alpha=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2\sin\alpha a}{\alpha\sqrt{2\pi}}e^{-i\alpha x}d\alpha \begin{matrix}=1 & ,|x|\le a\\ =0 & ,|x|>a\end{matrix}$

So, $\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin\alpha a(\cos\alpha x-i\sin\alpha x)}{\alpha}d\alpha=\begin{cases}1 & ,|x|\le a\\ 0 & ,|x|>a\end{cases}$

Hence $\int_{-\infty}^{\infty}\frac{\sin\alpha a\cos\alpha x}{\alpha}d\alpha=\begin{cases}\pi & ,|x|\le a\\ 0 & ,|x|>a\end{cases}$

Also by setting $x=0$ in above equation, we obtain: $\int_{-\infty}^{\infty}\frac{\sin\alpha a}{\alpha}d\alpha=\pi$

Since the integrand is even, we can have $\int_{0}^{\infty}\frac{\sin\alpha a}{\alpha}d\alpha=\frac{\pi}{2}$

Example 05: Find the Fourier cosine and sine transform of e^{-bx}, hence evaluate the integrals (a) $\int_{0}^{\infty}\frac{\cos\alpha x}{\alpha^2+b^2}d\alpha$ (b) $\int_{0}^{\infty}\frac{\alpha\sin\alpha x}{\alpha^2+b^2}d\alpha$

Solution: Given $f(x)=e^{-bx}$

By definition of Fourier cosine transform, we have

$$\bar{f}_c(\alpha)=\sqrt{\frac{2}{\pi}}\int_{0}^{\infty}f(x)\cos\alpha x\,dx$$. But $f(x)=e^{-bx}$, therefore

$$\overline{f_c}(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\cos\alpha x\,dx \qquad \text{(i)}$$

Now we know that $\int_0^\infty e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$

Taking $a=-b$ and $b=\alpha$ in (i), we have

$$\overline{f_c}(\alpha)=\sqrt{\frac{2}{\pi}}\left|\frac{e^{-bx}}{b^2+\alpha^2}(-b\cos\alpha x+\alpha\sin\alpha x)\right|_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left|0-\frac{e^0}{b^2+\alpha^2}(-b\cos 0+\alpha\sin 0)\right| \quad (\because e^{-\infty}=0)$$

$$=\sqrt{\frac{2}{\pi}}\frac{b}{b^2+\alpha^2}$$

Now Fourier sine transform is given by

$$\overline{f_s}(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\,dx\text{. Put } f(x)=e^{-bx}\text{, we get}$$

$$\overline{f_s}(\alpha)=\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-bx}\sin\alpha x\,dx \qquad \text{(ii)}$$

Now we know that $\int_0^\infty e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)$

Now by taking $a=-b$ and $b=\alpha$ in (ii), we have

$$\overline{f_s}(\alpha)=\sqrt{\frac{2}{\pi}}\left|\frac{e^{-bx}}{b^2+\alpha^2}(-b\sin\alpha x-\alpha\cos\alpha x)\right|_0^\infty$$

$$=\sqrt{\frac{2}{\pi}}\left[0-\frac{e^0}{b^2+\alpha^2}(-b\sin 0-\alpha\cos 0)\right]=\sqrt{\frac{2}{\pi}}\frac{\alpha}{b^2+\alpha^2} \quad (\because e^{-\infty}=0)$$

To prove (a) we have found that $\overline{f_c}(\alpha)=\sqrt{\frac{2}{\pi}}\frac{b}{b^2+\alpha^2}$ and $\overline{f_s}(\alpha)=\sqrt{\frac{2}{\pi}}\frac{\alpha}{b^2+\alpha^2}$

Now the inverse Fourier cosine transform of f(x) is $f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty \overline{f}_{c(\alpha)}\cos\alpha x\,d\alpha$

or $$e^{-bx}=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{b}{b^2+\alpha^2}\cos\alpha x\,d\alpha=\frac{2b}{\pi}\int_0^\infty\frac{\cos\alpha x}{b^2+\alpha^2}d\alpha$$

$$\Rightarrow\int_0^\infty\frac{\cos\alpha x}{b^2+\alpha^2}d\alpha=\frac{\pi}{2b}e^{-bx}$$

Similarly to prove (b), we have an inverse Fourier sine transform of f(x)

$$f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty\overline{f}_{s(\alpha)}\sin\alpha x\,d\alpha \quad\text{or}\quad e^{-bx}=\sqrt{\frac{2}{\pi}}\int_0^\infty\sqrt{\frac{2}{\pi}}\frac{\alpha}{b^2+\alpha^2}\sin\alpha x\,d\alpha=\frac{2}{\pi}\int_0^\infty\frac{\alpha\sin\alpha x}{b^2+\alpha^2}d\alpha$$

$$\Rightarrow\frac{\pi}{2}e^{-bx}=\int_0^\infty\frac{\alpha\sin\alpha x}{b^2+\alpha^2}d\alpha \quad\text{or}\quad\int_0^\infty\frac{\alpha\sin\alpha x}{b^2+\alpha^2}d\alpha=\frac{\pi}{2}e^{-bx}$$

Scanned with CamScanner

Example 06: Find the Fourier sine transform of f(x) defined by

$$f(x)=0 \quad 0<x<a,$$
$$=x \quad a\le x\le b,$$
$$=0 \quad x>b$$

Solution: we know that Fourier sine integral of f(x),is defined as

$$\bar{f}_{s(\alpha)}=\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\alpha x\,dx=\sqrt{\frac{2}{\pi}}\int_a^b x\sin\alpha x\,dx$$

Integrate by parts, we get

$$=\sqrt{\frac{2}{\pi}}\left[\left|\frac{-x\cos\alpha x}{\alpha}\right|_a^b-\int_a^b 1.\frac{-\cos\alpha x}{\alpha}\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\left|\frac{-b\cos\alpha b}{\alpha}+\frac{a\cos\alpha a}{\alpha}\right|+\left|\frac{\sin\alpha b}{\alpha^2}-\frac{\sin\alpha a}{\alpha^2}\right|\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{a\cos\alpha a-b\cos\alpha b}{\alpha}+\frac{\sin\alpha b-\sin\alpha a}{\alpha^2}\right]$$

7.2 PROPERTIES OF FOURIER TRANSFORMS

In many practical situations, determining the Fourier transforms of certain functions are too complex. Once we know the transform of some elementary functions, we can find the transform of many other functions with the help of properties associated with the Fourier transform. We now discuss some of the important properties of Fourier transform.

Linearity Property

If $\bar{f}(\alpha)$ and $\bar{g}(\alpha)$ are the Fourier transforms of f(x) and g(x) respectively, then

$$F[c_1f(x)+c_2g(x)]=c_1F[f(x)]+c_2F[g(x)]=c_1\bar{f}(\alpha)+c_2\bar{g}(\alpha)$$

Where c_1 and c_2 are arbitrary constants?

Proof: By definition.

$$F[c_1f(x)+c_2g(x)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}[c_1f(x)+c_2g(x)]dx$$

$$=\frac{c_1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}f(x)+\frac{c_2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}g(x)dx=c_1\bar{f}(\alpha)+c_2\bar{g}(\alpha)$$

Example 01: Evaluate $F\left[3e^{-2x^2}+4e^{-2|x|}\right]$

Solution: Using linear property $F\left[3e^{-2x^2}+4e^{-2|x|}\right]=3F\left[e^{-2x^2}\right]+4F\left[e^{-2|x|}\right]$

Now, $F\left[e^{-ax^2}\right]=\frac{e^{-\alpha^2/4a}}{\sqrt{2a}}$ and $F\left[e^{-a|x|}\right]=\sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+\alpha^2}\right)$ [See above examples]

$$\therefore F\left[3e^{-2x^2}+4e^{-2|x|}\right]=3F\left[e^{-2x^2}\right]+4F\left[e^{-2|x|}\right]=3\ \frac{e^{-\alpha^2/4(2)}}{\sqrt{2(2)}}+4\ \sqrt{\frac{2}{\pi}}\left(\frac{2}{2^2+\alpha^2}\right)$$

Hence $F\left[3e^{-2x^2}+4e^{-2|x|}\right]=\frac{3}{2}e^{-\alpha^2/8}+8\ \sqrt{\frac{2}{\pi}}\left(\frac{1}{4+\alpha^2}\right)$

Scanned with CamScanner

Change of Scale Property

If $\bar{f}(\alpha)$ is the Fourier transform of f(x), then Fourier transform of f(ax) is $\frac{1}{a}\bar{f}\left(\frac{\alpha}{a}\right)$

Proof: By definition,

$$F[f(x)]=\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)dx$$

$$\therefore F[f(ax)]=\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(ax)dx \qquad \text{(i)}$$

Letting $ax=t \Rightarrow dx=\frac{dt}{a}$, therefore from (i), we have

$$\therefore F[f(ax)]=\frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\left(\frac{\alpha}{a}\right)} f(t)dt$$

Thus, $$F[f(ax)]=\frac{1}{a}\bar{f}\left(\frac{\alpha}{a}\right) \quad ,a>0 \qquad \text{(ii)}$$

Similarly, $$F[f(ax)]=\frac{-1}{a}\bar{f}\left(\frac{\alpha}{a}\right) \quad ,a<0 \qquad \text{(iii)}$$

Combining (i) and (ii), we have

$$F[f(ax)]=\frac{1}{|a|}\bar{f}\left(\frac{\alpha}{a}\right) \quad ,a\neq 0$$

Example 02: Find $F[f(3x)]$ if $f(x)=e^{-x^2/2}$

Solution: We know that $F[f(x)]=F\left[e^{-x^2/2}\right]=e^{-\alpha^2/2}=\bar{f}(\alpha)$

Therefore $F[f(3x)]$ by using scale property i-e $F[f(ax)]=\frac{1}{|a|}\bar{f}\left(\frac{\alpha}{a}\right)$,$a\neq 0$

Put $a=3$, we get $F[f(3x)]=\frac{1}{|3|}=e^{\frac{-\left(\frac{\alpha}{3}\right)^2}{2}}=\frac{1}{3}e^{\frac{-\alpha^2}{18}}$

Shifting Property

If $\bar{f}(\alpha)$ is the Fourier transform of f(x), then Fourier transform of f(x - a) is $F[f(x-a)]=e^{i\alpha a}\bar{f}(\alpha)$.

Proof: By definition, $\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)dx$

$$\therefore F[f(x-a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x-a)dx \qquad \text{(i)}$$

Put $x-a=t \Rightarrow dx=dt$ in (i), we get

$$F[f(x-a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha(a+t)} f(t)dt=e^{i\alpha a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha t} f(t)dt=e^{i\alpha a}\bar{f}(\alpha)$$

Hence $F[f(x-a)]=e^{i\alpha a}\bar{f}(\alpha)$

Scanned with CamScanner

Example 03: Find the Fourier transform of $e^{-3x}f(x)$, where

$$f(x)=1 \quad |x|\le a$$
$$=0 \quad |x|>a$$

Solution: Since we know that $F[f(x-a)]=e^{i\alpha a}\bar{f}(\alpha)$, where $\bar{f}(\alpha)=F[f(x)]$

Therefore $F\left[f(x)e^{-3x}\right]=\bar{f}(\alpha+3i)$ (i)

Now $F[f(x)]=\bar{f}(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}f(x)dx$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\alpha x}f(x)dx=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}e^{i\alpha x}f(x)dx+\int_{-a}^{a}e^{i\alpha x}f(x)dx+\int_{a}^{\infty}e^{i\alpha x}f(x)dx\right]$$

$$=0+\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}e^{i\alpha x}f(x)\,dx+0$$

$$=\frac{1}{\sqrt{2\pi}}\left|\frac{e^{i\alpha x}}{i\alpha}\right|_{-a}^{a}=\frac{1}{\sqrt{2\pi}}\frac{1}{i\alpha}\frac{e^{i\alpha a}-e^{-i\alpha a}}{2i}2i=\sqrt{\frac{2}{\pi}}\frac{\sin\alpha a}{\alpha}$$

Thus, $f(\alpha)=\sqrt{\frac{2}{\pi}}\frac{\sin\alpha a}{\alpha}$

$$\Rightarrow f(\alpha+3i)=\sqrt{\frac{2}{\pi}}\frac{\sin a(\alpha+3i)}{(\alpha+3i)}=\sqrt{\frac{2}{\pi}}\left(\frac{\sin a\alpha\cos 3a+i\cos a\alpha\sinh 3\alpha}{\alpha+3i}\cdot\frac{\alpha-3i}{\alpha-3i}\right)$$

$$=\frac{1}{\alpha^2-9}\sqrt{\frac{2}{\pi}}(\alpha\sin a\alpha\cosh 3a+i\alpha\cos a\alpha\sinh 3-3i\sin a\alpha\cosh 3a+3\cos a\alpha\sinh 3\alpha)$$

$$=\frac{1}{\alpha^2-9}\sqrt{\frac{2}{\pi}}(\alpha\sin a\alpha\cosh 3a+3\cos a\alpha\sinh 3\alpha+i(\alpha\cos a\alpha\sinh 3-3\sin a\alpha\cosh 3a))$$

Substituting these values in (i), we get

$$F\left[f(x)e^{-3x}\right]=\frac{1}{\alpha^2-9}\sqrt{\frac{2}{\pi}}\begin{pmatrix}\alpha\sin a\alpha\cosh 3a+3\cos a\alpha\sinh 3\alpha+\\ i(\alpha\cos a\alpha\sinh 3-3\sin a\alpha\cosh 3a)\end{pmatrix}$$

WORKSHEET 07

1. Find the Fourier transform of the following functions

(i) $f(x)=e^{i2x}$, if $-1<x<1$
$=0$, otherwise

(ii) $f(t)=e^{-t^2/2}$

(iii) $(f(x)=e^{-3x}$, where $f(x)=1, |x|\le 2$
$=0, |x|>2$

(iv) $f(x)=e^{-3x^2}\cos 2x$

2. Find Fourier sine transform of:

(i) $\{e^{-x}\cos x\}$ (ii) $\{xe^{-2x}\}$ (iii) $\{e^{-2x}/x\}$

(iv) $f(x)=\begin{cases}\sin x, & 0<x<a\\ 0, & x>a\end{cases}$

(v) $f(x)=\begin{cases}x, & 0<x<1\\ 2-x, & 1<x<2\\ 0, & x>2\end{cases}$

3. Find Fourier cosine transform of:

(i) $\{e^{-x}\sin x\}$ (ii) $\{te^{-2t}\}$ (iii) $\{1/(1+x^2)\}$

Scanned with CamScanner