LECTURE-8

Diagonalization
&
Orthogonal Diagonalization

Diagonalization

- Diagonalization problem:

  For a square matrix A, does there exist an invertible matrix P such that $P^{-1}AP$ is diagonal?

- Diagonalizable matrix:

  A square matrix A is called diagonalizable if there exists an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

  (P diagonalizes A)

- Notes:

  (1) If there exists an invertible matrix P such that $B = P^{-1}AP$, then two square matrices A and B are called similar.

  (2) The eigenvalue problem is related closely to the diagonalization problem.

- Theorem 8.1: (Similar matrices have the same eigenvalues)

  If A and B are similar $n\times n$ matrices, then they have the same eigenvalues.

Pf:

$$A \text{ and } B \text{ are similar} \Rightarrow B = P^{-1}AP$$

$$\begin{aligned}
\left|\lambda I - B\right| &= \left|\lambda I - P^{-1}AP\right| = \left|P^{-1}\lambda IP - P^{-1}AP\right| = \left|P^{-1}(\lambda I - A)P\right| \\
&= \left|P^{-1}\right|\left|\lambda I - A\right|\left|P\right| = \left|P^{-1}\right|\left|P\right|\left|\lambda I - A\right| = \left|P^{-1}P\right|\left|\lambda I - A\right| \\
&= \left|\lambda I - A\right|
\end{aligned}$$

Thus A and B have the same eigenvalues.

Procedure for Diagonalizing a Matrix

Step 1. Find n linearly independent eigenvectors of A, say $p_1, p_2, \ldots, p_n$.

Step 2. Form the matrix P having $p_1, p_2, \ldots, p_n$ as its column vectors.

Step 3. The matrix $P^{-1}AP$ will then be diagonal with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its successive diagonal entries, where λ_i is the eigenvalue corresponding to p_i for i=1,2,..,n.

- Ex 8.1: (A diagonalizable matrix)

$$A=\begin{bmatrix}1 & 3 & 0\\ 3 & 1 & 0\\ 0 & 0 & -2\end{bmatrix}$$

Sol: Characteristic equation:

$$\left|\lambda\mathrm{I}-A\right|=\begin{vmatrix}\lambda-1 & -3 & 0\\ -3 & \lambda-1 & 0\\ 0 & 0 & \lambda+2\end{vmatrix}=(\lambda-4)(\lambda+2)^2=0$$

$$\text{The eigenvalues: } \lambda_1=4,\ \lambda_2=-2,\ \lambda_3=-2$$

$$(1)\ \lambda=4 \Rightarrow \text{the eigenvector } p_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$$

$$(2)\lambda = -2 \Rightarrow \text{the eigenvector } p_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\text{such that } P^{-1}AP = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Note: If $P = [p_2 \quad p_1 \quad p_3]$

$$= \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Rightarrow P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- Theorem 8.2: (Condition for diagonalization)

  An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

- Ex 8.2: (A matrix that is not diagonalizable)

  Show that the following matrix is not diagonalizable.

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 1 & -2 \\ 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 = 0$$

$$\text{The eigenvalue : } \lambda_1 = 1$$

$$\boldsymbol{\lambda \mathrm{I} - A = I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \textbf{eigenvector } p_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

A does not have two linearly independent eigenvectors, so A is not diagonalizable.

- Ex 8.3: (Diagonalizing a matrix)

$$A=\begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

Find a matrix P such that $P^{-1}AP$ is diagonal.

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix} \lambda-1 & 1 & 1 \\ -1 & \lambda-3 & -1 \\ 3 & -1 & \lambda+1 \end{vmatrix}=(\lambda-2)(\lambda+2)(\lambda-3)=0$$

$$\text{The eigenvalues}:\quad \lambda_1=2,\ \lambda_2=-2,\ \lambda_3=3$$

$\lambda_1 = 2$

$$\Rightarrow \lambda_1 \mathrm{I} - A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} \Rightarrow \text{eigenvector } p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2$

$$\Rightarrow \lambda_2 \mathrm{I} - A = \begin{bmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ -\frac{1}{4}t \\ t \end{bmatrix} \Rightarrow \text{eigenvector } p_2 = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\lambda_3 = 3$$

$$\Rightarrow \lambda_3 \mathrm{I} - A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ t \end{bmatrix} \Rightarrow \text{eigenvector } p_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$P = [p_1 \quad p_2 \quad p_3] = \begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{bmatrix},$$

$$\text{s.t.} P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- Theorem 8.3: (Sufficient conditions for diagonalization)

  If an $n \times n$ matrix A has n distinct eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

- Ex 8.4: (Determining whether a matrix is diagonalizable)

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol: Because A is a triangular matrix, its eigenvalues are

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = -3$$

These three values are distinct, so A is diagonalizable.

Computing Powers of a Matrix

If A is an nxn matrix and P is an invertible matrix, then

$$(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P$$

More generally, for any positive integer *k*, $(P^{-1}AP)^k = P^{-1}A^kP$

It follows from this equation that if *A is diagonalizable, and $P^{-1}AP = D$ is a diagonal matrix, then* $P^{-1}A^kP = (P^{-1}AP)^k = D^k$

Solving this equation for A^k yields $A^k = PD^kP^{-1}$

This last equation expresses the kth power of A in terms of the kth power of the diagonal matrix D. But D^k is easy to compute, for if

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}, \quad \text{then} \quad D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

ORTHOGONAL DIAGONALIZATION

Orthogonal Diagonalization

- Symmetric matrix:

  A square matrix A is symmetric if it is equal to its transpose:

$$A = A^T$$

- Ex 8.5: (Symmetric matrices and nonsymetric matrices)

$$A = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$ (symmetric)

$$B = \begin{bmatrix} 4 & 3 \\ 3 & 1 \end{bmatrix}$$ (symmetric)

$$C = \begin{bmatrix} 3 & 2 & 1 \\ 1 & -4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$ (nonsymmetric)

- Theorem 8.4: (Eigenvalues of symmetric matrices)

  If A is an $n \times n$ symmetric matrix, then the following properties are true.

  (1) A is diagonalizable.

  (2) All eigenvalues of A are real.

  (3) If λ is an eigenvalue of A with multiplicity k, then λ has k linearly independent eigenvectors. That is, the eigenspace of λ has dimension k.

- Ex 8.6:

Prove that a symmetric matrix is diagonalizable.

$$A=\begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

Pf: Characteristic equation:

$$|\lambda I-A|=\begin{vmatrix} \lambda-a & -c \\ -c & \lambda-b \end{vmatrix}=\lambda^2-(a+b)\lambda+ab-c^2=0$$

As a quadratic in λ, this polynomial has a discriminant of

$$\begin{aligned}(a+b)^2-4(ab-c^2) &= a^2+2ab+b^2-4ab+4c^2 \\ &= a^2-2ab+b^2+4c^2 \\ &= (a-b)^2+4c^2 \geq 0\end{aligned}$$

(1) $(a-b)^2+4c^2=0$

$$\Rightarrow\ a=b,\ c=0$$

$A=\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ is a matrix of diagonal.

(2) $(a-b)^2+4c^2>0$

The characteristic polynomial of A has two distinct real roots, which implies that A has two distinct real eigenvalues. Thus, A is diagonalizable.

- Orthogonal matrix:

A square matrix P is called orthogonal if it is invertible and

$$P^{-1} = P^{T}$$

- Theorem 8.5: (Properties of orthogonal matrices)

An $n \times n$ matrix P is orthogonal if and only if its column vectors form an orthogonal set.

- Theorem 8.6: (Properties of symmetric matrices)

Let A be an $n \times n$ symmetric matrix. If λ_1 and λ_2 are distinct eigenvalues of A, then their corresponding eigenvectors x_1 and x_2 are orthogonal.

- Ex 8.7: Show that following matrix is orthogonal.

$$P=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}$$

Sol: If P is a orthogonal matrix, then $P^{-1}=P^T \;\Rightarrow\; PP^T=\mathrm{I}$

$$PP^T=\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt{5}} & \frac{5}{3\sqrt{5}} \end{bmatrix}\begin{bmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=\mathrm{I}$$

$$\text{Let } p_1=\begin{bmatrix} \frac{1}{3} \\ \frac{-2}{\sqrt{5}} \\ \frac{-2}{3\sqrt{5}} \end{bmatrix},\ p_2=\begin{bmatrix} \frac{2}{3} \\ \frac{1}{\sqrt{5}} \\ \frac{-4}{3\sqrt{5}} \end{bmatrix},\ p_3=\begin{bmatrix} \frac{2}{3} \\ 0 \\ \frac{5}{3\sqrt{5}} \end{bmatrix}$$

produces

$$p_1 \cdot p_2 = p_1 \cdot p_3 = p_2 \cdot p_3 = 0$$

$$\|p_1\| = \|p_2\| = \|p_3\| = 1$$

$\{p_1, p_2, p_3\}$ is an orthonormal set.

- Theorem 8.7: (Fundamental theorem of symmetric matrices)

  Let A be an $n \times n$ matrix. Then A is orthogonally diagonalizable and has real eigenvalue if and only if A is symmetric.

- Orthogonal diagonalization of a symmetric matrix:

  Let A be an $n \times n$ symmetric matrix.

  (1) Find all eigenvalues of A and determine the multiplicity of each.

  (2) For each eigenvalue of multiplicity 1, choose a unit eigenvector.

  (3) For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors. If this set is not orthonormal, apply Gram-Schmidt orthonormalization process.

  (4) The composite of steps 2 and 3 produces an orthonormal set of n eigenvectors. Use these eigenvectors to form the columns of P. The matrix $P^{-1}AP = P^{T}AP = D$ will be diagonal.

- Ex 8.8: (Orthogonal diagonalization)
  Find an orthogonal matrix P that diagonalizes A.

$$A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & -1 & 4 \\ -2 & 4 & -1 \end{bmatrix}$$

Sol:

(1) $|\lambda I - A| = (\lambda - 3)^2(\lambda + 6) = 0$

$$\lambda_1 = -6, \lambda_2 = 3 \text{ (has a multiplicity of 2)}$$

(2) $\lambda_1 = -6, \quad v_1 = (1, -2, 2) \Rightarrow u_1 = \dfrac{v_1}{\|v_1\|} = (\tfrac{1}{3}, \tfrac{-2}{3}, \tfrac{2}{3})$

(3) $\lambda_2 = 3, \quad v_2 = (2, 1, 0), \; v_3 = (-2, 0, 1)$

Linear Independent

Gram-Schmidt Process:

$$w_2 = v_2 = (2,1,0), \quad w_3 = v_3 - \frac{v_3 \cdot w_2}{w_2 \cdot w_2} w_2 = (\tfrac{-2}{5}, \tfrac{4}{5}, 1)$$

$$u_2 = \frac{w_2}{\|w_2\|} = (\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}}, 0), \quad u_3 = \frac{w_3}{\|w_3\|} = (\tfrac{-2}{3\sqrt{5}}, \tfrac{4}{3\sqrt{5}}, \tfrac{5}{3\sqrt{5}})$$

$$P = \begin{bmatrix} \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{-2}{3} & \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{bmatrix} \qquad \Rightarrow \quad P^{-1}AP = \begin{bmatrix} -6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$