Lecture #03

Linear D.E

⇒ D.E is said to be linear if satisfies the following conditions

1. Dependent variable and it's derivative(s) occur first degree.
2. No term in the eq: that contains the product of dependent variable and it's derivative(s)
3. No transcedental function with dependent variable.

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

Working Rule:- 1. Integrating factor $= e^{\int P(x)dx}$
2. $y(I.F) = \int Q(x)[I.F]dx + c$

Ex: $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

Soln

$$\frac{dy}{dx} + \frac{2x}{(1+x^2)}.y = \frac{4x^2}{1+x^2} \quad , \quad P(x) = \frac{2x}{1+x^2} \quad Q(x) = \frac{4x^2}{1+x^2}$$

$$I.F = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$$

$$y(1+x^2) = \int \frac{4x^2}{\cancel{1+x^2}}.\cancel{(1+x^2)}\,dx + c$$

$$= 4\int x^2 dx + c$$

$$y(1+x^2) = \frac{4x^3}{3} + c$$

Ans

Ex: $\frac{dy}{dx} + \sec x . y = \tan x$ $\quad P(x) = \sec x \quad Q(x) = \tan x$

$$IF = e^{\int \sec x dx} = e^{\ln|\sec x + \tan x|} = \sec x + \tan x$$

$$y[\sec x + \tan x] = \int \tan x(\sec x + \tan x)dx + c$$

$$= \int \sec x.\tan x\, dx + \int \tan^2 x\, dx + c$$

$$= \sec x + \int(\sec^2 x - 1)dx + c \qquad \tan^2 x = \sec^2 x - 1$$

$$= \sec x + \int \sec^2 x\, dx - \int dx + c$$

$$= \sec x + \tan x - x + c$$

Ans

Q- Find the general solution of $(x^2-9)\frac{dy}{dx}+xy=0$

Solu

$$\frac{dy}{dx}+\frac{x}{x^2-9}\cdot y=0 \quad , \quad P(x)=\frac{x}{x^2-9} \quad Q(x)=0$$

$$IF=e^{\int\frac{x}{x^2-9}dx}=e^{\frac{1}{2}\int\frac{2x}{x^2-9}dx}=e^{\frac{1}{2}\ln(x^2-9)}=\cancel{e}^{\cancel{\ln}(x^2-9)^{1/2}}=\sqrt{x^2-9}$$

$$y\sqrt{x^2-9}=\int 0\cdot\frac{1}{\sqrt{x^2-9}}dx+c$$

$$y\sqrt{x^2-9}=c \quad \Rightarrow \quad y=\frac{c}{\sqrt{x^2-9}} \quad , \quad x>3 \text{ or } x<-3$$

Ans

Initial-Value Problem

Solve: $\frac{dy}{dx}+y=x \quad , \quad y(0)=4$

Solu

$$P(x)=1 \quad Q(x)=x$$

$$IF=e^{\int dx}=e^{x}$$

$$ye^{x}=\int x\cdot e^{x}dx+c \qquad U=x \quad V=e^{x}$$

$$U\int v\,dv-\int\left(\frac{du}{dx}\int v\,dv\right)dv$$

$$ye^{x}=xe^{x}-\int 1\,e^{x}dx+c$$

$$ye^{x}=xe^{x}-e^{x}+c$$

$$\boxed{y=x-1+ce^{-x}} \rightarrow \text{General Solution}$$

$$y(0)=0-1+ce^{-(0)}$$

$$4=-1+c$$

$$\boxed{c=5}$$

$$y=x+5e^{-x}-1 \quad \text{Ans}$$

Bernoulli's Differential Equation

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)y^n}$$

If $n=0$ or $n=1$ then Linear if $n>1$ then Bernoulli

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

let $y^{1-n} = z \Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$$\Rightarrow \frac{1}{(1-n)}\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

$$\frac{1}{(1-n)}\frac{dz}{dx} + P(x)z = Q(x)$$

$$\boxed{\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)} \rightarrow \text{Linear D.E}$$

Ex:- $\frac{dy}{dx} + \frac{y}{x} = y^2$

Sol: $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = 1$

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = 1$$

Let $z = y^{-1}$ $\quad \frac{dz}{dx} = -1y^{-2}\frac{dy}{dx} \Rightarrow -\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$

$$-\frac{dz}{dx} + \frac{1}{x}z = 1$$

$$\frac{dz}{dx} - \frac{1}{x}z = -1, \quad P(x) = -\frac{1}{x} \quad Q(x) = -1$$

$$IF = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = \frac{1}{x}$$

$$z\cdot\frac{1}{x} = \int \frac{1}{x}(-1)dx + c$$

$$\frac{z}{x} = -\ln x + c$$

$$\frac{y^{-1}}{x} + \ln x = c$$

$$\frac{1}{xy} + \ln x = c \quad \text{Ans}$$

ii) $\frac{dy}{dx} - 2y\tan x = y^2\tan^2 x$

Sol: $y^{-2}\frac{dy}{dx} - 2y^{-1}\tan x = \tan^2 x$

Let $z = y^{-1}$

$\frac{dz}{dx} = -1 y^{-2}\frac{dy}{dx} \Rightarrow -\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$

$-\frac{dz}{dx} - 2\tan x \cdot z = \tan^2 x$

$\frac{dz}{dx} + 2\tan x \cdot z = -\tan^2 x$ $\quad P(x) = 2\tan x \quad Q(x) = -\tan^2 x$

$I.F = e^{\int 2\tan x\,dx} = e^{2\int \tan x\,dx} = e^{2\ln\sec x} = \sec^2 x$

$\therefore \int \tan x\,dx$

$\int \frac{\sin x}{\cos x}dx = -\int \frac{-\sin x}{\cos x}$

$-\ln|\cos x| = \ln\cos^{-1} x$

$= \ln\sec x$

$z \cdot \sec^2 x = \int \sec^2 x(-\tan^2 x)dx + c$

$= -\int \tan^2 x \cdot \sec^2 x\,dx + c$

$\therefore \frac{d}{dx}\tan = \sec^2 x$

$z \cdot \sec^2 x = -\frac{\tan^3 x}{3} + c$

$y^{-1} = -\frac{1}{3}\frac{\tan^3 x}{\sec^2 x} + \frac{c}{\sec^2 x}$

$y^{-1} = \frac{3c - \tan^3 x}{3\sec^2 x}$

$y = \frac{3\sec^2 x}{3c - \tan^3 x}$ Ans

iii- Solve the following initial value problem.

$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}, \quad y(1) = 2$

Sol: $\frac{dy}{dx} + \frac{y}{2x} = xy^{-3}$

$\frac{1}{y^{-3}}\frac{dy}{dx} + \frac{y}{2x}\cdot\frac{1}{y^{-3}} = x$

$y^3\frac{dy}{dx} + \frac{1}{2x}y^4 = x$

Let $z = y^4$

$\frac{dz}{dx} = 4y^3\frac{dy}{dx} \Rightarrow \frac{1}{4}\frac{dz}{dx} = y^3\frac{dy}{dx}$

$\frac{1}{4}\frac{dz}{dx} + \frac{1}{2x}z = x$

$\frac{dz}{dx} + \frac{2}{x}z = 4x$

$P(x) = \frac{2}{x} \quad Q(x) = 4x$

$IF = e^{\int \frac{2}{x}dx} = e^{2\int \frac{1}{x}dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

$zx^2 = \int 4x(x^2)dx + c$

$zx^2 = \cancel{4}\frac{x^4}{\cancel{4}} + c$

$z = x^2 + \frac{c}{x^2}$

$y^4 = x^2 + c/x^2 \Rightarrow y = \left(x^2 + \frac{c}{x^2}\right)^{\frac{1}{4}}$

$y(1) = \left(1 + \frac{c}{1}\right)^{\frac{1}{4}}$

$2 = (1+c)^{\frac{1}{4}} \Rightarrow 2^4 = 1 + c$

$16 - 1 = c \Rightarrow \boxed{15 = c}$

$y = \left(x^2 + \frac{15}{x^2}\right)^{\frac{1}{4}}$

OR $x^2y^4 = x^4 + 15$ Ans

Worksheet #03

Linear and Bernoulli's D.E

1. $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$ Ans: $xy = \frac{x^5}{5} - 3\frac{x^2}{2} + c$

2. $(2y - 3x)dx + x\,dy = 0$ Ans: $yx^2 = x^3 + c$

3. $\frac{dy}{dx} + y\cot x = \cos x$ Ans: $y\sin x = \frac{\sin^2 x}{2} + c$

4. $\frac{dy}{dx} + y\sec x = \tan x$ Ans: $y = \frac{c - x}{\sec x + \tan x} + 1$

5. $\cos^2 x \frac{dy}{dx} + y = \tan x$ Ans: $y = \tan x - 1 + ce^{-\tan x}$

6. $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$ Ans: $2y = (x+a)^5 + 2c(x+a)^3$

u

7. $x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1$ Ans: $xy = \sin x + c$

8. $\sec x \frac{dy}{dx} = y + \sin x$ Ans: $y = -\sin x - 1 +$ ...

9. $(1+y^2)dx = (\tan^{-1}y - x)dy$ Ans: $x = -\tan^{-1}y - 1 + ce^{-\tan^{-1}y}$

10. $dr + (2r\cot\theta + \sin 2\theta)d\theta = 0$ Ans: $r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$

11. $x^2 dy + y(x+y)dx = 0$ Ans: $\frac{1}{xy} = -\frac{1}{2x^2} - c$

12. $x\frac{dy}{dx} + y\ln y = xye^x$ Ans: $x\ln y = xe^x - e^x + c$

13. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$. Ans: $\frac{\sin y}{1+x} = e^x + c$

14. $\tan y\frac{dy}{dx} + \tan x = \cos y \cdot \cos^2 x$ Ans: $\sec y = (\sin x + c)\cos x$

dy
dx

15. $x\left[\frac{dx}{dy} + y\right] = 1 - y$ Ans: $y = \frac{1}{x} + \frac{c}{x}e^{-x}$

16. $y\ln y\,dx + (x - \ln y)dy = 0$ Ans: $x\ln y = \frac{(\ln y)^2}{2} + c$

17. $(1+y^2)dx = (\tan^{-1}y - x)dy$ Ans: $x = (\tan^{-1}y - 1) + ce^{-\tan^{-1}y}$

18. $r\sin\theta - \frac{dr}{d\theta}\cos\theta = r^2$ Ans: $r = \frac{1}{\sin\theta + c\cos\theta}$

19. $\cos x\frac{dy}{dx} + 4y\sin x = 4\sqrt{y}\sec x$ Ans: $\sqrt{y}\sec^2 x = 2\left[\tan x + \frac{\tan^3 x}{3}\right] + c$

20. $\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2$ Ans: $\frac{1}{x\ln y} = \frac{1}{2x^2} + c$