

Lecture #04

Non-Exact D.E's

If the equation $M(x,y)dx + N(x,y)dy = 0$ is not exact it may be possible to multiply it by a function $\mu(x,y)$ so that the resulting equation $\mu M(x,y) + \mu N(x,y)dy = 0$ becomes exact.

Rule #01:- If $M(x,y)dx + N(x,y)dy = 0$ is not exact then

$$\boxed{\frac{My - Nx}{N} = P(x)} \rightarrow I.F = e^{\int P(x) dx}$$

Rule #02:- If rule #01 fail then

$$\boxed{\frac{Nx - My}{M} = Q(y)} \rightarrow I.F = e^{\int Q(y) dy}$$

Rule #03:- If $Mdx + Ndy = 0$ is a homogeneous and $xM + yN \neq 0$ then

$$\boxed{I.F = \frac{1}{xM + yN}}$$

Rule #04:- If $M(x,y)dx + N(x,y)dy = 0$ is of the form $y f(x,y)dx + x g(x,y)dy = 0$ and $xM - yN \neq 0$ then

$$\boxed{I.F = \frac{1}{xM - yN}}$$

Ex:- $(x^2 + y^2 + 2x)dx + 2ydy = 0 \rightarrow i$

Solu

$$M = x^2 + y^2 + 2x$$

$$N = 2y$$

$$\frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial N}{\partial x} = 0$$

$$\boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$$

Check rule #1 $\frac{M_y - N_x}{N} = P(x)$

$$\frac{2y - 0}{2y} = \frac{2y}{2y} = 1 \rightarrow P(x) = 1$$

I.F = $e^{\int dx} = e^x \rightarrow$ multiple with eqn

$$e^x(x^2 + y^2 + 2x)dx + e^x(2y)dy = 0$$

$$(x^2e^x + y^2e^x + 2xe^x)dx + 2ye^x dy = 0$$

$$M = x^2e^x + y^2e^x + 2xe^x$$

$$N = 2ye^x$$

$$\frac{\partial M}{\partial y} = 0 + 2ye^x$$

$$\frac{\partial N}{\partial x} = 2ye^x$$

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \text{ satisfied}$$

$$\int_{\text{const}} M dx + \int N dy = c$$

$$\int (x^2e^x + y^2e^x + 2xe^x) dx + \int 0 dy = c$$

$$\int x^2e^x dx - y^2 \int e^x dx + 2 \int xe^x dx = c$$

$$x^2e^x - 2 \int xe^x dx - y^2e^x + 2 \int xe^x dx = c$$

$$e^x(x^2 - y^2) = c \quad \underline{\underline{Ans}}$$

lecture # 04
Exact Differential Equations

Defn D.E $M(x,y)dx + N(x,y)dy = 0$ is exact, that is when $M_y = N_x$.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex: $(2x^3 - 6x^2y + 3xy^2)dx - (2x^3 - 3x^2y + y^3)dy = 0$

Solu $M = 2x^3 - 6x^2y + 3xy^2$ $N = -(2x^3 - 3x^2y + y^3)$

$$\frac{\partial M}{\partial y} = 0 - 6x^2(1) + 3x \cdot (2y) \quad \frac{\partial N}{\partial x} = -(6x^2 - 6xy)$$

$$\frac{\partial M}{\partial y} = -6x^2 + 6xy \quad \frac{\partial N}{\partial x} = -6x^2 + 6xy$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then

$$P(x,y) = \int M dx + \int (\text{term in } N \text{ not containing } x) dy = C$$

$$\Rightarrow \int (2x^3 - 6x^2y + 3xy^2) dx + \int -(y^3) dy = C$$

$$\Rightarrow \cancel{2} \frac{x^4}{\cancel{4}} - \cancel{6} \frac{x^3y}{\cancel{3}} + \frac{3x^2y^2}{2} - \frac{y^4}{4} = C$$

$$\Rightarrow \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 - \frac{y^4}{4} = C \quad \underline{\underline{Ans}}$$

Ex #2 $(e^y + 1) \cos x dx + e^y \sin x dy = 0$

Solu

$$M = (e^y + 1) \cos x$$

$$N = e^y \sin x$$

$$\frac{\partial M}{\partial y} = \cos x (e^y + 0)$$

$$\frac{\partial N}{\partial x} = e^y \cos x$$

$$= e^y \cos x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$P(x,y) = \int M dx + \int (\text{term in } N \text{ not containing } x) dy = C$$

$$\Rightarrow \int (e^y + 1) \cos x dx + \int (0) dy = C$$

$$\Rightarrow \int e^y \cos x dx + \int \cos x dx = C$$

$$\Rightarrow e^y \sin x + \sin x = C \quad \underline{\underline{Ans}}$$

$$\underline{\underline{Ans}} \quad (e^y + 1) \sin x = C \quad \underline{\underline{Ans}}$$

Ex #03 $\sec^2 x \tan y dx + \sec y \tan x dy = 0$

Solve

$$M = \sec^2 x \tan y$$

$$\frac{\partial M}{\partial y} = \sec^2 x \cdot \sec^2 y$$

$$N = \sec y \tan x$$

$$\frac{\partial N}{\partial x} = \sec^2 y \cdot \sec^2 x$$

Worksheet # 04

Exact & Reducible to Exact D.E's

Ans: $\frac{x^3}{3} = xy^2 + C$

1. $(y^2 - x^2)dx + 2xydy = 0$

2. $(1 + 3e^{xy})dx + 3e^{xy}(1 - \frac{x}{y})dy = 0$ Ans: $x + 3ye^{xy} = C$

3. $(2x - y)dx = (x - y)dy$ Ans: $xy = x^2 + \frac{y^2}{2} + C$

4. $(y \sec^2 x + \sec x \cdot \tan x)dx + (\tan x + 2y)dy = 0$ Ans: $y \cdot \tan x + \sec x + y^2 = C$

5. $(ax + hy + g)dx + (hx + by + f)dy = 0$ Ans: $ax^2 + 2hxy + by^2 + 2gx + 2fy + C = 0$

6. $(x^2 + 2ye^{2x})dy + (2xy + 2y^2e^{2x})dx = 0$ Ans: $x^2y + y^2e^{2x} = C$

7. $[y(1 + \frac{1}{x}) + \cos y]dx + (x + \log x - x \cdot \sin y)dy = 0$ Ans: $y(x + \log x) + x \cos y = C$

8. $(x^3 - 3xy^2)dx + (y^3 - 3x^2y)dy = 0, y(0) = 1$ Ans: $x^4 - 6x^2y^2 + y^4 = 1$

9. $(y \cdot \log y)dx + (x - \log y)dy = 0$ Ans: $2x \log y = C + (\log y)^2$

10. $(y + \frac{1}{3}y^3 + \frac{1}{2}x^2)dx + \frac{1}{4}(1 + y^2)x dy = 0$ Ans: $\frac{y^4}{4} + \frac{y^3 x^4}{12} + \frac{x^6}{12} = C$

11. $(x \cdot \sec^2 y - x^2 \cos y)dy = (\tan y - 3x^4)dx$ Ans: $-\frac{1}{x} \tan y - x^3 + \sin y = C$

12. $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$ Ans: $x^2y^2 + \frac{x^2}{y} = C$

13. $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$ Ans: $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C$

14. $(2xy^4e^y + 2xy^3 + y)dx + (xy^4e^y - x^2y^2 - 3x)dy = 0$ Ans: $xe^y + \frac{x^2}{y} + \frac{x}{y^3} = C$

Rule 4

15. $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$ Ans: $-\frac{1}{xy} + 2 \log x - \log y = C$

16. $(y - xy^2)dx - (x + x^2y)dy = 0$ Ans: $\log(\frac{x}{y}) - xy = C$

17. $(xy \cdot \sin(xy) + \cos(xy)) \cdot y dx + (xy \cdot \sin(xy) - \cos(xy)) \cdot x dy = 0$ Ans: $y \cos xy = Cx$

18. $y(1 + xy)dx + x(1 + xy + x^2y^2)dy = 0$ Ans: $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = C$

19. Homogeneous D.E

19. $(y^3 - 2x^2y)dx + (2xy^2 - x^3)dy = 0$ Ans: $\frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = C$

20. $(3y - 2xy^3)dx + (4x - 3x^2y^2)dy = 0$ Ans: $x^3y^4 - \frac{x^4y^6}{2} = C$

21. $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$ Ans: $\frac{7}{5}x^{1/5}y^{-5/2} - \frac{7}{4}x^{-4/5}y^{-12/5} = C$

22. $(y^2 + 2yx^2)dx + (2x^3 - xy)dy = 0$ Ans: $4(xy)^{1/2} - \frac{2}{3}(\frac{y}{x})^{3/2} = C$