Lecture #04 Exact Differential Equations

Definition: D.E $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact, that is when $M_y = N_x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ex: $(2x^3 - 6x^2y + 3xy^2)\,dx - (2x^3 - 3x^2y + y^3)\,dy = 0$

$M = 2x^3 - 6x^2y + 3xy^2$ $\quad$ $N = -(2x^3 - 3x^2y + y^3)$

$\frac{\partial M}{\partial y} = 0 - 6x^2 + 6xy$ $\quad$ $\frac{\partial N}{\partial x} = -6x^2 + 6xy + 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Then,

$$f(x,y) = \int M\,dx + \int (\text{term in } N \text{ not containing } x)\,dy = c$$

$$\Rightarrow \int (2x^3 - 6x^2y + 3xy^2) + \int (-y^3)\,dy = c$$

$$\Rightarrow \frac{2x^4}{4} - \frac{6x^3y}{3} + \frac{3x^2y^2}{2} - \frac{y^4}{4} = c$$

$$\Rightarrow \frac{x^4}{2} - 2x^3y + \frac{3}{2}x^2y^2 - \frac{y^4}{4} = c$$

Ex #2 $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$

$M = (e^y + 1)\cos x$ $\quad$ $N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x$ $\quad$ $\frac{\partial N}{\partial x} = e^y \cos x$; $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$f(x,y) = \int M\,dx + \int (\text{term in } N \text{ not containing } x)\,dy = c$$

$$= \int (e^y + 1)\cos x\,dx + \int 0\,dy = c$$

$$= e^y \int \cos x\,dx + \int \cos x\,dx + 0 = c$$

$$= e^y \sin x + \sin x = 0$$

$$\boxed{(e^y + 1)\sin x = c}$$

Ex #03 $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$M = \sec^2 x \tan y \, dx$, $N = \sec^2 y \tan x$

$$\frac{\partial M}{\partial y} = \sec^2 x \sec^2 y \qquad \frac{\partial N}{\partial x} = \sec^2 y \sec^2 x \, , \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Rightarrow \int M \, dx + \int (\text{term in } N \text{ not containing } x) \, dy = c$$

$$\int \sec^2 x \tan y \, dx + \int (0) \, dy = c$$

$$\tan y \int \sec^2 x \, dx = \boxed{\tan y \tan x = c}$$

→ Non-Exact D.E

When the ~~differential~~ differential equation $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, it might be possible to find an integrating factor $U(x,y)$ that makes the equation exact.

Here are the rules for finding integrating factors:

1. Rule 1: If $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact, and if $\frac{M_y - N_x}{N} = p(x)$ (a function of x only), then the integrating factor is $U(x) = e^{\int p(x)\,dx}$

2. Rule 2: If Rule 1 fails, and if $\frac{N_x - M_y}{M} = Q(y)$ (a function of y only), then the integrating factor is $U(y) = e^{\int Q(y)\,dy}$

3. Rule 3: For homogeneous equations, if $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous and $xM + yN \neq 0$, then the integrating factor is $U = \frac{1}{xM + yN}$

4. Rule 4: For equations of the form $yf(x,y)\,dx + xg(x,y)\,dy = 0$ if $xM - yN \neq 0$, then the integrating factor is $U = \frac{1}{xM - yN}$.

Worksheet #04

Exact & Reducible to Exact D.E's

1. $\underset{M}{(y^2-x^2)}\,dx + \underset{N}{2xy}\,dy = 0$ Ans: $\frac{x^3}{3} = xy^2 + C$

Step 1: $M_y = \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y^2-x^2) = 2y$

$N_x = \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \rightarrow$ Exact D.E

Step 2: $\int M dx + \int$ (term in N not containing x) $dy = C$

$\int (y^2-x^2)\,dx + \int (0)\,dy = C$

$\int y^2 dx - \int x^2 dx + 0 = C$

$y^2 \int dx - \int x^2 dx = C \Rightarrow \boxed{xy^2 - \frac{x^3}{3} = C}$

2. $(1+3e^{x/y})\,dx + 3e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$; Ans: $x + 3ye^{x/y} = C$

Step 1: $M = (1+3e^{x/y})$; $N = 3e^{x/y}(1-x/y)$

$\frac{\partial M}{\partial y} = 3e^{x/y} \times \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = \frac{-3e^{x/y}x}{y^2}$; $\frac{\partial N}{\partial x} = 3e^{x/y} - \frac{3e^{x/y}x}{y} = \frac{3e^{x/y}(-x)}{y} - \frac{3}{y}\frac{\partial}{\partial y}(xe^{x/y})$

$\frac{\partial N}{\partial x} = \frac{-3e^{x/y}x}{y} - \frac{3}{y}\frac{\partial}{\partial y}(xe^{x/y}) = \frac{-3e^{x/y}x}{y} - \frac{3}{y}\left(e^{x/y} + xe^{x/y} \times \frac{\partial}{\partial y}\left(\frac{x}{y}\right)\right)$

$\frac{\partial N}{\partial x} = \frac{-3xe^{x/y}}{y} - \frac{3}{y}\left(e^{x/y} + xe^{x/y} \times \left(-\frac{x}{y^2}\right)\right) = -\frac{3xe^{x/y}}{y} - 3$

$\frac{\partial M}{\partial y} = 3e^{x/y} \times \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = 3e^{x/y} \times \frac{-x}{y^2} = \boxed{\frac{-3e^{x/y}x}{y^2}}$

$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(3e^{x/y} - \frac{3xe^{x/y}}{y}\right) = 3e^{x/y} \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right) - \frac{3}{y}\frac{\partial}{\partial x}(xe^{x/y})$

$= \frac{3e^{x/y}}{y} - \frac{3}{y}\left(e^{x/y} + xe^{x/y} \times \frac{\partial}{\partial x}\left(\frac{x}{y}\right)\right) = \frac{3e^{x/y}}{y} - \frac{3}{y}\left(e^{x/y} + \frac{xe^{x/y}}{y}\right)$

$= 3e^{x/y}\left(\frac{1}{y} - \frac{1}{y} - \frac{x}{y^2}\right) = \boxed{\frac{-3e^{x/y}x}{y^2}}$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step2: $\int M dx + \int (\text{term in } N \text{ not containing } x)\, dy = C$

$$\int (1+3e^{x/y})\, dx + \int (0)\, dy = C$$

$$\int dx + 3\int e^{x/y}\, dx = C \Rightarrow x + \frac{3e^{x/y}}{1/y} = C$$

$$\Rightarrow x + 3e^{x/y} \div \frac{1}{y} = C \Rightarrow \boxed{x + 3y e^{x/y} = C}$$

③ $(2x-y)\,dx = (x-y)\,dy$ Ans: $xy = x^2 + \frac{y^2}{2} + C$

Step1: $M = (2x-y)$; $N = -(x-y)$; $(2x-y)dx - (x-y)dy = 0$

① $\frac{\partial M}{\partial y} = -1$; ② $\frac{\partial N}{\partial x} = -1$ $\rightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Step2: $\int M dx + \int (\text{term in } N \text{ not containing } x)\, dy = C$

$$\int (2x-y)\, dx + \int y\, dy = C$$

$$2\int x\, dx - y\int dx + \int y\, dy = C$$

$$\frac{2x^2}{2} - xy + \frac{y^2}{2} = C$$

$$x^2 + \frac{y^2}{2} - xy = C$$ constant $C = -C$, could be

$$\boxed{x^2 + \frac{y^2}{2} + C = xy}$$

(4) $(y\sec^2 x + \sec x \tan x)\,dx + (\tan x + 2y)\,dy = 0$

Ans: $y\tan x + \sec x + y^2 = C$

Step 1: $M = y\sec^2 x + \sec x \tan x$; $N = \tan x + 2y$

$M_y = \frac{\partial M}{\partial y} = \sec^2 x$; $N_x = \frac{\partial N}{\partial x} = \sec^2 x$; $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Step 2: $\int M\,dx + \int(\text{term in } N \text{ not containing } x)\,dy = C$

$\int(y\sec^2 x + \sec x\tan x)\,dx + 2\int y\,dy = C$

$y\int \sec^2 x\,dx + \int \sec x \tan x\,dx + 2\int y\,dy = C$

$\boxed{y\tan x + \sec x + y^2 = C}$

(5) $(ax + hy + g)\,dx + (hx + by + f)\,dy = 0$; Ans: $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Step 1: $M = ax + hy + g$ $\quad N = hx + by + f$

$M_y = \frac{\partial M}{\partial y} = 0 + h + 0$ $\quad N_x = \frac{\partial N}{\partial x} = h + 0 + 0$; $\boxed{M_y = N_x} \rightarrow$ Exact D.E

Step 2: $\int M\,dx + \int(\text{term in } N \text{ not containing } x)\,dy = C$

$\int(ax + hy + g)\,dx + \int(by + f)\,dy = C$

$\frac{ax^2}{2} + hxy + gx + \frac{by^2}{2} + fy = C$, Multiply 2 on b/s

$ax^2 + 2hxy + 2gx + by^2 + 2fy = 2C$; $\rightarrow$ Constant

$\boxed{ax^2 + 2hxy + 2gx + by^2 + 2fy = C}$

(6) $(x^2 + 2ye^{2x})\,dy + (2xy + 2y^2e^{2x})\,dx = 0$; Ans: $x^2y + y^2e^{2x} = C$

Step 1: $N = x^2 + 2ye^{2x}$ $\quad M = 2xy + 2y^2e^{2x}$

$N_x = 2x + 2e^{2x}(2y)$ $\quad M_y = 2x + 4ye^{2x}$

$N_x = 2x + 4ye^{2x}$ $\quad M_y = 2x + 4ye^{2x}$; $\boxed{M_y = N_x}$

Step 2: $\int M\,dx + \int(\text{term in } N \text{ not containing } x)\,dy = C$

$\int(2xy + 2y^2e^{2x})\,dx + \int(0)\,dy = C$

$2y\int x\,dx + 2y^2\int e^{2x}\,dx = C$

$x^2y + \frac{2y^2e^{2x}}{\frac{d}{dx}(2x)} = C \Rightarrow x^2y + \frac{2y^2e^{2x}}{2} = C$

$\Rightarrow \boxed{x^2y + y^2e^{2x} = C}$

(7) $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx + (x+\log x - x\sin y)\,dy = 0$

Ans: $y(x+\log x) + x\cos y = c$

Step 1: $M = y\left(1+\frac{1}{x}\right)+\cos y$; $N = x + \log x - x\sin y$

$M_y = 1+\frac{1}{x} + (-\sin y)$ $N_x = 1 + \frac{1}{x} - \sin y$; Exact D.E ✓

Step 2: $\int M\,dx + \int (\text{term in } N \text{ not containing } x)\,dy = c$

$\int \left(y\left(1+\frac{1}{x}\right)+\cos y\right)dx + \int (0)\,dy = c$

$y\int\left(1+\frac{1}{x}\right)dx + \cos y\int dx = c$

$$\boxed{y(x+\ln|x|) + x\cos y = c}$$

(8) $(x^3 - 3xy^2)\,dx + (y^3 - 3x^2y)\,dy = 0$; Ans: $x^4 - 6x^2y^2 + y^4 = 1$

Step 1: $M = x^3 - 3xy^2$ $N = y^3 - 3x^2y$

$M_y = 0 - 6xy$ $N_x = 0 - 6xy$; $M_y = N_x \Rightarrow$ Exact D.E ✓

Step 2: $\int M\,dx + \int (\text{term in } N \text{ not containing } x)\,dy = c$

$\int (x^3 - 3xy^2)\,dx + \int y^3\,dy = c$

$$\boxed{\frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} + c = 0}$$

Step 3: Determine the constant C using the initial condition $y(0)=1$;

Substitute $x=0$, and $y=1$

$\frac{0^4}{4} - \frac{3}{2}(0^2)(1)^2 + \frac{(1)^4}{4} + c = 0 \Rightarrow c + \frac{1}{4} = 0$

$\boxed{c = -1/4}$

$\frac{x^4}{4} - \frac{3}{2}x^2y^2 + \frac{y^4}{4} - \frac{1}{4} = 0$; Multiply 4 on b/s

$\frac{4x^4}{4} - \frac{12}{2}x^2y^2 + \frac{4y^4}{4} - \frac{4}{4} = 0$

$$\boxed{x^4 - 6x^2y^2 + y^4 = 1}$$

(8) $(y\log y)\,dx + (x-\log y)\,dy = 0$; Ans: $2x\log y = c + (\log y)^2$

Step 1: $M = y\log y$ $\qquad N = x - \log y$

$M_y = \log y + 1$ $\qquad N_x = 1$; Not Exact D.E

Let's try to write D.E in Standard Linear D.E form.

(1) $y\log y \frac{dx}{dy} + (x-\log y)\frac{dy}{dy} = 0$

$y\log y \frac{dx}{dy} + x - \log y = 0$

$y\log y \frac{dx}{dy} + x = \log y$

$\frac{dx}{dy} + \frac{x}{y\log y} = \frac{\log y}{y\log y}$

$$\boxed{\frac{dx}{dy} + \frac{x}{y\log y} = \frac{1}{y}}$$

$\frac{dx}{dy} + P(y)x = Q(y)$

(2) Integrating factor.

$P(y) = \frac{1}{y\log y}$; $Q(y) = \frac{1}{y}$

$\int P(y)\,dy = \int \frac{1}{y\log y}\,dy$; $u = \log y$, $du = \frac{1}{y}dy$

$\int P(y)\,dy = \int \frac{du}{u} = \ln|u| = \boxed{\ln|\log y|}$

$e^{\int P(y)} = e^{\ln|\log y|} = \boxed{\log y} \to$ I.F

(3) General Solution

$x(\text{I.F}) = \int Q(y)(\text{I.F})\,dy + c$

$x\log y = \int \frac{1}{y} \times \log y\,dy + c$

$x\log y = \int \frac{\log y}{y}\,dy + c$, $u = \log y$

$du = 1/y\,dy$

$x\log y = \int u\,du + c$

$x\log y = \frac{(\log y)^2}{2} + c \Leftarrow x\log y = \frac{u^2}{2} + c$

$\boxed{2x\log y = (\log y)^2 + C}$ i.e., $\boxed{2C = C}$

(10) $\left(y+\frac{1}{3}y^3+\frac{1}{2}x^2\right)dx+\frac{1}{4}(1+y^2)x\,dy=0$

Ans: $\frac{yx^4}{4}+\frac{y^3x^4}{12}+\frac{x^6}{12}=C$

Step 1: Identify D.E and apply technique.

$M=y+\frac{1}{3}y^3+\frac{1}{2}x^2$ $\qquad N=\frac{x}{4}+\frac{xy^2}{4}$

$M_y=1+y^2$ $\qquad N_x=\frac{1}{4}+\frac{y^2}{4}$, Not Exact D.E

Applying Rule 1: $\frac{M_y-N_x}{N}=P(x)$; $I.F=e^{\int P(x)dx}$

$P(x)=\frac{(1+y^2)-\frac{1}{4}(1+y^2)}{\frac{x}{4}(1+y^2)}=\frac{\frac{3}{4}(1+y^2)}{\frac{x}{4}(1+y^2)}=\frac{3}{4}(1+y^2)\div\frac{x(1+y^2)}{4}$

$=\frac{3}{\cancel{4}}\cancel{(1+y^2)}\times\frac{\cancel{4}}{x\cancel{(1+y^2)}}=\boxed{\frac{3}{x}=P(x)}$

$I.F=e^{\int P(x)dx}=e^{3\int\frac{1}{x}dx}=e^{3\ln|x|}=e^{\ln|x^3|}=\boxed{x^3=I.F}$

Step 2: $(I.F)M+(I.F)N=0$

$x^3\left[y+\frac{1}{3}y^3+\frac{1}{2}x^2\right]dx+x^3\left[\frac{x}{4}+\frac{xy^2}{4}\right]dy=0$

$\underbrace{\left[x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right]}_{M_1}dx+\underbrace{\left[\frac{x^4}{4}+\frac{x^4y^2}{4}\right]}_{N_1}dy=0$

$M_1=x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}$ $\qquad N_1=\frac{x^4}{4}+\frac{x^4y^2}{4}$

Step 3: Check $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$ i.e., $M_1=N_1$?

$M_{1y}=\frac{\partial}{\partial y}\left[x^3y+\frac{x^3y^3}{3}+\frac{x^5}{2}\right]=x^3+x^3y^2+0$

$N_{1x}=\frac{\partial}{\partial x}\left[\frac{x^4}{4}+\frac{x^4y^2}{4}\right]=x^3+x^3y^2$ $\quad\boxed{M_{1y}=N_{1x}}$ Exact D.E

Step II: $\int M_1\,dx + \int (\text{Terms of } N, \text{ not containing } x)\,dy = c$

$\int (x^3 y + x^3 y^2 + \frac{x^5}{2})\,dx + \int (0)\,dy = c$

$\frac{x^4 y}{4} + \frac{x^4}{12} y^3 + \frac{x^6}{12} + 0 = c$

$$\boxed{\frac{x^4 y}{4} + \frac{x^4 y^3}{12} + \frac{x^6}{12} = c}$$

(ii) $(x\sec^2 y - x^2\cos y)\,dy = (\tan y - 3x^4)\,dx$

Ans: $-\frac{1}{x}\tan y - x^3 + \sin y = c$

Step 1: Identify D.E and apply suitable method

$(\tan y - 3x^4)\,dx - (x\sec^2 y - x^2\cos y)\,dy = 0$

$M = \tan y - 3x^4$ $\qquad$ $N = -(x\sec^2 y - x^2\cos y)$

$M_y = \sec^2 y - 0$ $\qquad$ $N_x = -\sec^2 y + 2x\cos y$

$\Rightarrow$ Non-Exact D.E

Step 2: Apply Exact D.E Rule(s)

Rule 1. $\frac{M_y - N_x}{N} = \frac{\sec^2 y - (-\sec^2 y + 2x\cos y)}{-x\sec^2 y + x^2\cos y} = \frac{2\sec^2 y - 2x\cos y}{-x\sec^2 y + x^2\cos y}$

$= \frac{2\sec^2 y - \frac{2x}{\sec y}}{-x\sec^2 y + \frac{x^2}{\sec y}} = \frac{\frac{1}{\sec y}\left(2\sec^3 y - 2x\right)}{\frac{1}{\sec y}\left(-x\sec^3 y + x^2\right)} = \frac{2\sec^3 y - 2x}{-x\sec^3 y + x^2}$

$\Rightarrow \frac{2(\sec^3 y - x)}{-x(\sec^3 y - x)} = \boxed{\frac{-2}{x}} \rightarrow f(x) = (I.F)$ according to Rule 1

$I.F = e^{\int f(x)dx} = e^{-2\int \frac{1}{x}dx} = e^{-2\ln|x|} = e^{\ln|x^{-2}|} = \boxed{1/x^2} \Rightarrow I.F$

Step3: $(I.F)\,M\,dx + (I.F)\,N\,dy = 0$

$$\frac{1}{x^2}\left[x\sec^2 y - x^2\cos y\right] dy = \frac{1}{x^2}\left(\tan y - 3x^4\right) dx$$

$$-\frac{1}{x^2}\left(\tan y - 3x^4\right) dx + \frac{1}{x^2}\left[x\sec^2 y - x^2\cos y\right] dy = 0$$

$$\left(-\frac{\tan y}{x^2} + 3x^2\right) dx + \left(\frac{\sec^2 y}{x} - \cos y\right) dy = 0$$

$M_1 = -\frac{\tan y}{x^2} + 3x^2$ $\qquad$ $N_1 = \frac{\sec^2 y}{x} - \cos y$

$M_{1y} = -\frac{\sec^2 y}{x^2}$ $\qquad$ $N_{1x} = -\frac{\sec^2 y}{x^2}$; $\boxed{M_{1y} = N_{1x}}$

Step4: General Solution

$$\int M\,dx + \int (\text{terms in } N_1 \text{ not containing } x)\, dy = C$$

$$\int\left(-\frac{\tan y}{x^2} + 3x^2\right) dx + \int(-\cos y)\, dy = C$$

$$-\tan y\int x^{-2}\,dx + 3\int x^2\,dx - \int \cos y\, dy = C$$

$$-\tan y\,\frac{x^{-2+1}}{-2+1} + 3\frac{x^{2+1}}{2+1} - \sin y = C$$

$$-\tan y\,\frac{x^{-1}}{-1} + \frac{3x^3}{3} - \sin y = C$$

$$\boxed{\frac{\tan y}{x} + x^3 - \sin y = C}$$

(12) $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$; Ans: $x^3y^3 + \frac{x^2}{y} = C$

Step 1: Identify the D.E and Apply suitable method.

The given D.E is in the Form $M(x,y)dx + N(x,y)dy = 0$, that is the general form of Exact D.E.

$M = (3x^2y^4 + 2xy)$ $\quad\quad N = 2x^3y^3 - x^2$

$M_y = 12x^2y^3 + 2x$ $\quad\quad N_x = 6x^2y^3 - 2x$; $\quad M_y \neq N_x$

Non-Exact D.E

Step 2:

Apply Rule # 1 of Exact D.E's Integrating factor

$(I.F) M(x,y)dx + (I.F) N(x,y) dy = 0$; $I.F = e^{\int P(x)}$; $P(x) = \frac{M_y - N_x}{N}$,

where I.F is the only function of x.

$\frac{M_y - N_x}{N} = \frac{(12x^2y^3 + 2x) - (6x^2y^2 - 2x)}{2x^3y^3 - x^2} = \frac{6x^2y^3 + 4x}{2x^3y^3 - x^2} \rightarrow$ Implicit

The $P(x)$ in Rule 1's implementation result in an implicit function, but we need explicit function in terms of x

Apply Rule # 2

$(I.F) M(x,y) dx + (I.F) N(x,y) dy = 0$; $I.F = e^{\int P(y)}$; $P(y) = \frac{N_x - M_y}{M}$

$P(y) = \frac{(6x^2y^3 - 2x) - (12x^2y^3 + 2x)}{3x^2y^4 + 2xy} = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{3x^2y^4 + 2xy}$

$= \frac{6x^2y^3 - 12x^2y^3 - 4x}{3x^2y^4 + 2xy} = \frac{-6x^2y^3 - 4x}{3x^2y^4 + 2xy} = \frac{-2(3x^2y^3 + 2x)}{y(3x^2y^3 + 2x)} = \frac{-2}{y}$

$P(y) = \frac{-2}{y}$; $\quad I.F = e^{\int P(y)dy} = e^{-2\ln|y|} = \boxed{\frac{1}{y^2} = I.F}$

Step 3:

So, the D.E will now become $(I.F) M(x,y)dx + (I.F) N(x,y) dy = 0$

$\frac{1}{y^2}(3x^2y^4 + 2xy)dx + \frac{1}{y^2}(2x^3y^3 - x^2) dy = 0$

$\boxed{\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right) dy = 0}$, so this become

$M_1 dx + N_1 dy = 0$, $\quad M_1 = 3x^2y^2 + \frac{2x}{y}$, $\quad N_1 = 2x^3y - \frac{x^2}{y^2}$

Now Check for $M_{1y} = N_{1x}$

$$M_1(x,y)\,dx + N_1(x,y)\,dy = 0$$

$$\left(3x^2y^2 + \frac{2x}{y}\right)dx + \left(2x^3y - \frac{x^2}{y^2}\right)dy = 0$$

$M_1 = 3x^2y^2 + \frac{2x}{y}$ $\qquad$ $N_1 = 2x^3y - \frac{x^2}{y^2}$

$M_{1y} = 6x^2y - \frac{2x}{y^2}$ $\qquad$ $N_{1x} = 6x^2y - \frac{2x}{y^2}$; $\boxed{M_{1y} = N_{1x}}$ $\rightarrow$ Exact D.E

Step 4: Writing General Solution

$$\int M_1\,dx + \int (\text{term in } N_1 \text{ not containing } x)\,dy = 0$$

$$\int \left(3x^2y - \frac{2x}{y^2}\right)dx + \int (0)\,dy = 0$$

$$\frac{3x^3y}{3} - \frac{x^2}{y^2} = c \implies \boxed{x^3y - \frac{x^2}{y^2} = c}$$

(13) $(xy^3+y)\,dx + 2(x^2y^2+x+y^4)\,dy = 0$; Ans: $\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C$

Step 1: Identify D.E (Assert the Conditions)

$M = xy^3 + y$ $\quad N = 2(x^2y^2 + x + y^4)$

$M_y = 3xy^2 + 1$ $\quad N_x = 4xy^2 + 2$; $\boxed{M_y \neq N_x}$

→ Apply the Exact D.E Rules and reduce the D.E.

Rule 1: $\frac{M_y - N_x}{N} = \frac{(3xy^2+1)-(4xy^2+2)}{2(x^2y^2+x+y^4)} = \frac{-xy^2-1}{2(x^2y^2+x+y^4)}$ → Rule 1 failed ☹

Rule (1) failed because I.F's $= \frac{M_y - N_x}{N} = f(x)$ → the result should be a function of x

Rule (2): $\frac{N_x - M_y}{M} = \frac{(4xy^2+2)-(3xy^2+1)}{xy^3+y} = \frac{xy^2+1}{xy^3+y} = \frac{(xy^2+1)}{y(xy^2+1)} = \boxed{\frac{1}{y}}$

$\frac{N_x - M_y}{M} = f(y) = \frac{1}{y}$, in Rule 2 we need a function of y.

$I.F = e^{\int f(y)\,dy} = e^{\int \frac{1}{y} dy} = e^{\ln|y|} = \boxed{y = I.F}$

Step 2: $(I.F)\,M\,dx + (I.F)\,N\,dy = 0$

$y(xy^3+y)\,dx + 2y(x^2y^2+x+y^4)\,dy = 0$

$(xy^4+y^2)\,dx + 2y(x^2y^2+x+y^4)\,dy = 0$

$M_1 = xy^4 + y^2$ $\quad N_1 = 2y(x^2y^2+x+y^4)$

$M_{1y} = 4xy^3 + 2y$ $\quad N_{1x} = 2y(2xy^2+1+0) = 4xy^3 + 2y$

$\boxed{M_{1y} = N_{1x}}$

Step3: General Solution

$$\int M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = C$$

$$\int (xy^4 + y^2)\,dx + \int (2y^5)\,dy = c$$

$$\frac{x^2y^4}{2} + xy^2 + \frac{2y^6}{6} = c \Rightarrow \boxed{\frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = C}$$

(14) $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$

Ans: $x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = C$

Step1: Identify D.E

$$y(2xy^3e^y + 2xy^2 + 1)\,dx + x(xy^4e^y - xy^2 - 3)\,dy = 0$$

Rule 4 States: For equation of form $yf(x,y)dx + xg(x,y)dy = 0$
if $x.M - y.N \neq 0$, then integrating factor is I.F = $\frac{1}{x.M - y.N}$

but $yf(x,y) = y(2xy^3e^y + 2xy^2 + 1)$; $x.g(x,y) = x(xy^4e^y - xy^2 - 3)\,dy$ → Neither x nor y

Applying Exact D.E Rules to Reduce the D.E

Rule 1: $P(x) = \frac{M_y - N_x}{N} = \frac{(8xy^3e^y + 2xy^4e^y + 6xy^2 + 1) - (2xy^4e^y - 2xy^2 - 3)}{x(xy^4e^y - xy^2 - 3)} \neq P(x)$

Rule 2: $P(y) = \frac{N_x - M_y}{M} = \frac{(2xy^4e^y - 2xy^2 - 3) - (8xy^3e^y + 2xy^4e^y + 6xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)}$

$$P(y) = \frac{(2xy^4e^y - 2xy^2 - 3) - (8xy^3e^y + 2xy^4e^y + 6xy^2 + 1)}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{\cancel{2xy^4e^y} - 8xy^3e^y - 2xy^2 - \cancel{2xy^4e^y} - 6xy^2 - 3 - 1}{2xy^4e^y + 2xy^3 + y}$$

$$= \frac{-8xy^3e^y - 8xy^2 - 4}{2xy^4e^y + 2xy^3 + y} = \frac{-4(2xy^3e^y + 2xy^2 + 1)}{y(2xy^3e^y + 2xy^2 + 1)} = \boxed{\frac{-4}{y}} \to P(y)$$

$I.F = e^{\int p(y)\,dy} = e^{\int -\frac{4}{y}dy} = e^{-4\ln|y|} = e^{\ln|y^{-4}|} = 1/y^4$

Step 3: General Solution $(I.F)M\,dx + (I.F)N\,dy = c$

Step 2: Applying now $\rightarrow$ $(I.F)M\,dx + (I.F)N\,dy = 0$

$$\frac{1}{y^4}\,y\left(2xy^3e^y + 2xy^2 + 1\right)dx + \frac{1}{y^4}\,x\left(xy^4e^y - xy^2 - 3\right)dy = 0$$

$$\left(\frac{2xy^3e^y}{y^3} + \frac{2xy^2}{y^3} + \frac{1}{y^3}\right)dx + \left(\frac{x^2y^4e^y}{y^4} - \frac{x^2y^2}{y^4} - \frac{3x}{y^4}\right)dy = 0$$

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

$M_1 = 2xe^y + \frac{2x}{y} + \frac{1}{y^3}$; $N_1 = x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}$

$M_{1y} = 2xe^y + \frac{2x}{y^2} - \frac{3}{y^4}$ $\boxed{M_{1y} = N_{1x}}$ $N_{1x} = 2xe^y - \frac{2x}{y^2} - \frac{3}{y^4}$

Step 3: General Solution

$$\int M_1\,dx + \int(\text{term in } N_1 \text{ not containing } x)\,dy = C$$

$$\int\left(2xe^y + \frac{2x}{y^2} + \frac{1}{y^3}\right)dx + \int(0)\,dy = C$$

$$\frac{2x^2e^y}{2} + \frac{2x^2}{2y} + \frac{x}{y^3} = C$$

$$\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = C}$$

(18) $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$; Ans: $-\frac{1}{xy} + 2\log x - \log y = b$

Step 1: Identify D.E

The D.E is in the $y f(x,y)\,dx + x g(x,y)\,dy = 0$ form, so we can apply the Rule 4 of Exact D.E's easily but check ~~the~~ the existence of Exact D.E first.

$M = y(xy + 2x^2y^2)\,dx$ $\qquad N = x(xy - x^2y^2)$

$M_y = \frac{\partial}{\partial y}(xy^2 + 2x^2y^3)$ $\qquad N_x = \frac{\partial}{\partial x}(x^2y - x^3y^2)$

$M_y = 2xy + 6x^2y^2$ $\qquad N_x = 2xy - 3x^2y^2$; $M_y \neq N_x \Rightarrow$ Non-Exact

Rule (4) For the equation of form $y f(x,y)\,dx + x g(x,y)\,dy = 0$, if $x\cdot M - y\cdot N \neq 0$, then Integrating Factor is $I.F = \frac{1}{x\cdot M - y\cdot N}$

$x\cdot M = xy(xy + 2x^2y^2)$ $\quad$; $y\cdot N = xy(xy - x^2y^2)$

$x\cdot M - y\cdot N = xy(xy + 2x^2y^2) - xy(xy - x^2y^2) = (xy)^2[(1 + 2xy) - (1 - xy)] \neq 0$

$I.F = \frac{1}{x\cdot M - y\cdot N} = \frac{1}{xy(xy + 2x^2y^2) - xy(xy - x^2y^2)} = \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{+3x^3y^3}$

Step 2: General Solution for reducing the Non Exact D.E

$\frac{1}{x\cdot M - y\cdot N}\, yM\,dx + \frac{1}{x\cdot M - y\cdot N}\, xN\,dy = 0$

$\frac{y(xy + 2x^2y^2)\,dx}{3x^3y^3} + \frac{x(xy - x^2y^2)\,dy}{3x^3y^3} = 0$

$\frac{xy^2(1 + 2xy)}{3x^3y^3} + \frac{x^2y(1 - xy)}{3x^3y^3} = 0$

$\frac{(1 + 2xy)}{3x^2y} + \frac{(1 - xy)}{3xy^2} = 0$

$$M_1 = \frac{1+2xy}{3x^2y} \quad ; \quad N_1 = \frac{1-xy}{3xy^2}$$

$$M_1y = \frac{\partial}{\partial y}\left[\frac{1}{3x^2y}\right] + \frac{\partial}{\partial y}\left[\frac{2xy}{3x^2y}\right] \quad ; \quad N_1x = \frac{\partial}{\partial x}\left[\frac{1}{3xy^2}\right] - \frac{\partial}{\partial x}\left[\frac{xy}{3xy^2}\right]$$

$$M_1y = \frac{-1}{3x^2y^2} + \frac{2}{3} \times 0 \qquad N_1x = \frac{-1}{3x^2y^2} - 0$$

$$M_1y = \frac{-1}{3x^2y^2} \qquad \boxed{M_1y = N_1x} \qquad N_1x = \frac{-1}{3x^2y^2}$$

$\rightarrow$ Exact D.E

Step 3: General Solution

$$\int M_1\,dx + \int (\text{term in } N_1 \text{ not containing } x)\,dy = C$$

$$\int \left(\frac{1+2xy}{3x^2y}\right) dx + \int \left(\frac{-1}{3y}\right) dy = C$$

$$\frac{1}{3y}\int \frac{1}{x^2}\,dx + \frac{2}{3}\int \frac{1}{x}\,dx - \frac{1}{3}\int \frac{1}{y}\,dy = C$$

$$\frac{1}{3y} \times \frac{x^{-2+1}}{-2+1} + \frac{2}{3}\ln|x| - \frac{1}{3}\ln|y| = C$$

$$-\frac{1}{3xy} + \frac{2}{3}\ln|x| - \frac{1}{3}\ln|y| = C$$

$$-\frac{1}{3xy} + \frac{\ln|x^2|}{3} - \frac{\ln|y|}{3} = C$$

$$-\frac{1}{xy} + \ln|x^2| - \ln|y| = 3C$$

$$\boxed{-\frac{1}{xy} + 2\ln|x| - \ln|y| = b} \quad , \; b = 3C$$

(16) $(y-xy^2)\,dx - (x+x^2y)\,dy=0$; Ans: $\log\left(\frac{x}{y}\right) - xy = c$

Step 1: Identify D.E

$$(y-xy^2)\,dx + (x+x^2y)\,dy=0$$

$$y(1-xy)\,dx + x(1+xy)\,dy=0$$

$$\hookrightarrow y\,f(x,y)\,dx + x\,g(x,y)\,dy=0$$

$M = y - xy^2$ $\quad$ $N = -(x+x^2y)$

$M_y = 1 - 2xy$ $\quad$ $N_x = -1 - 2xy$; $\boxed{M_y \neq N_x}$

Applying Rule 4: If $x.M - y.N \neq 0$, then I.F $= \dfrac{1}{x.M - y.N}$

$x.M - y.N = x(y-xy^2) + y(x+x^2y) = xy \cancel{-x^2y^2} + xy \cancel{+x^2y^2} = 2xy$

$$\text{I.F} = \frac{1}{2xy}$$

Step 2: Reduce the Non Exact D.E to Exact using I.F

$$(\text{I.F})\,M\,dx + (\text{I.F})\,N\,dy=0$$

$$\frac{(y-xy^2)}{2xy}\,dx - \frac{(x+x^2y)}{2xy} = 0$$

$$\left(\frac{1-xy}{2x}\right)dx - \left(\frac{1+xy}{2y}\right) = 0$$

$M_1 = \dfrac{1-xy}{2x}$ $\quad$ $N_1 = \left(\dfrac{1+xy}{2y}\right) = -\left(\dfrac{1}{2y} + \dfrac{xy}{2y}\right) = -\dfrac{1}{2y} + \dfrac{x}{2}$

$M_1y = \dfrac{\partial}{\partial y}\left(\dfrac{1}{2x}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{xy}{2x}\right)$ $\quad$ $N_1x = \dfrac{\partial}{\partial x}\left(-\dfrac{1}{2y}\right) - \dfrac{\partial}{\partial x}\left(\dfrac{x}{2}\right)$

$M_1y = 0 - \dfrac{1}{2}$ $\quad$ $N_1x = 0 - \dfrac{1}{2}$

$\boxed{M_1y = N_1x} \rightarrow$ Exact D.E

Step 3: General Solution

$$\int M_1 dx + \int (\text{terms in } N_1 \text{ not containing } x)\, dy = C$$

$$\int \left(\frac{1}{2x} - \frac{y}{2}\right) dx + \int \left(-\frac{1}{2y}\right) dy = c$$

$$\frac{1}{2}\ln|x| - \frac{xy}{2} - \frac{1}{2}\ln|y| = c$$

$$\ln|x| - xy - \ln|y| = 2c$$

$$\ln|x| - \ln|y| - xy = 2c$$

$$\boxed{\ln\left(\frac{x}{y}\right) - xy = b}, \text{ where } b = 2c$$

(17) $(xy \cdot \sin(xy) + \cos(xy))\, y \cdot dx + (xy \sin(xy) - \cos(xy))\, x\, dy = 0$

Ans: $y \cos xy = cx$

Step 1: Identify D.E

$M_1 = y(xy \sin(xy) + \cos(xy))$ $\quad N_1 = x(xy \sin(xy) - \cos(xy))$

$M_{1y} = x[\sin(xy)\, 2y + xy^2 \cos(xy)] + [\cos(xy) + xy(-\sin(xy))]$

$= 2xy \sin(xy) + x^2y^2 \cos(xy) + \cos(xy) - xy \sin(xy)$

$N_{1x} = y[\sin(xy)\, 2x + x^2 \cos(xy)\, x] - [\cos(xy) + x(-\sin(xy))\, x]$

$2xy \sin(xy) + x^3 y \cos(xy) - \cos(xy) + x^2 \sin(xy)$ $\quad \boxed{M_y \neq N_x}$

Applying Rule (4)

$x \cdot M = xy(xy \sin(xy) + \cos(xy))$; $\quad y \cdot N = xy(xy \sin(xy) - \cos(xy))$

$x \cdot M - y \cdot N = \cancel{x^2y^2 \sin(xy)} + xy \cos(xy) - \cancel{x^2y^2 \sin(xy)} + xy \cos(xy)$

$= 2xy \cos(xy)$

$$I.F = \frac{1}{2xy \cos(xy)}$$

Step 2: Reduce the D.E to Exact D.E

$(I.F)\,M\,dx + (I.F)\,N\,dy = 0$

$$y\frac{(xy\sin(xy)+\cos(xy))}{2xy\cos(xy)}\,dx + \frac{(xy\sin(xy)-\cos(xy))x}{2xy\cos(xy)}\,dy = 0$$

$$\left(\frac{xy\sin(xy)}{2x\cos(xy)} + \frac{y\cos(xy)}{2x\cos(xy)}\right)dx + \left(\frac{xy\sin(xy)}{2y\cos(xy)} - \frac{\cos(xy)}{2y\cos(xy)}\right)dy = 0$$

$$\left(\frac{y\sin(xy)}{2\cos(xy)} + \frac{\cos(xy)}{2x\cos(xy)}\right)dx + \left(\frac{x\sin(xy)}{2\cos(xy)} - \frac{\cos(xy)}{2y\cos(xy)}\right)dy = 0$$

$$\left(\frac{y\tan(xy)}{2} + \frac{1}{2x}\right)dx + \left(\frac{x}{2}\tan(xy) - \frac{1}{2y}\right)dy = 0$$

$$M_1 = \frac{y\tan(xy)}{2} + \frac{1}{2x} \qquad N_1 = \frac{x\tan(xy)}{2} - \frac{1}{2y}$$

$$M_{1y} = \frac{\tan(xy) + xy\sec^2(xy)}{2} \qquad N_{1x} = \frac{\tan(xy) + xy\sec^2(xy)}{2}; \quad \boxed{M_{1y} = N_{1x}} \to \text{Exact D.E}$$

Step 3: General Solution

$$\int M\,dx + \int (\text{terms in } N_1 \text{ not containing } x)\,dy = c$$

$$\int \frac{y\tan(xy)}{2}\,dx + \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

step is to be revised on next page

$$\frac{y}{2}\int \frac{\sin(xy)\,dx}{\cos(xy)} + \frac{1}{2}\ln|x| - \frac{1}{2}\ln|y| = c$$

$$-\frac{y}{2}\ln\left|\frac{\cos(xy)}{y}\right| + \frac{1}{2}\ln|x| - \frac{1}{2}\ln|y| = c$$

$$-y\ln|\cos(xy)/y| + \ln|x| - \ln|y| = 2c$$

$$y\ln|\cos(xy)^{-1/2}\tfrac{1}{y}| + \ln|x^{1/2}| - \ln|y^{1/2}| = c$$

$$y\ln|\sec(xy)^{1/2}\tfrac{1}{y}| + \ln|x^{1/2}| - \ln|y^{1/2}| = c$$

$$-\frac{y}{2}\ln|\cos(xy)/y| + \frac{1}{2}\ln|x| - \frac{1}{2}\ln|y| = c$$

$$-y\ln\left|\frac{\cos(xy)}{y}\right| + \ln|x| - \ln|y| = 2c$$

$$y\ln|\cos(xy)^{-1}| + \ln|x| - \ln|y| = 2c$$
$$y\ln|\sec(xy)| + \ln\left|\frac{x}{y}\right| = 2c$$

Step 3: General Solution

$$\int \frac{y\tan(xy)}{2}\,dx + \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{y}\,dy = c$$

$$\int y\frac{\sin(xy)}{2\cos(xy)} + \frac{1}{2}\ln|x| - \frac{1}{2}\ln|y| = c$$

Let $u = \cos(xy)$, then $du = -\sin(xy) \times \frac{d}{dx}(xy) = \quad -du = y\sin(xy)$

$$-\frac{1}{2}\int \frac{du}{u} + \frac{1}{2}(\ln|x| - \ln|y|) = c$$

$\Rightarrow$ log

$$-\frac{1}{2}\ln|u| + \frac{1}{2}\ln\left(\frac{x}{y}\right) = c$$

$$\frac{\ln|\cos(xy)^{-1}|}{2} + \frac{1}{2}\ln\left(\frac{x}{y}\right) = \ln|c|$$

$$\ln|\sec(xy)| + \ln(x/y) = 2\ln|c|$$
$$\ln\left|\frac{x\sec(xy)}{y}\right| = 2\ln|c|$$

$$\frac{x\sec(xy)}{y} = c^2$$

$$\frac{x}{y} = \frac{c^2}{\sec(xy)} \Rightarrow \frac{x}{y} = c^2\cos(xy)$$

$$\frac{x}{c^2} = y\cos(xy) \Rightarrow xc^{-2} = y\cos(xy) = \boxed{xb = y\cos(xy)}$$

(18) $y(1+xy)dx + x(1+xy+x^2y^2)dy = 0$; Ans: $\frac{1}{2x^2y^2} + \frac{1}{xy} - \log y = C$

Step1: Identify D.E

$M = y(1+xy) = y + xy^2$ $\qquad$ $N = x(1+xy+x^2y^2) = x + x^2y + x^3y^2$

$M_y = 1 + 2xy$ $\qquad$ $N_x = 1 + 2xy + 3x^2y^2$

The given D.E is in the form $y f(x,y)dx + x g(x,y)dy = 0$, so we are applying Rule (4)

Rule 4: if $x \cdot M - y \cdot N \neq$, then $I.F = \frac{1}{x \cdot M - y \cdot N}$

$x \cdot M - y \cdot N = (xy + x^2y^2) - (xy + x^2y^2 + x^3y^3) = -x^3y^3$, so

$I.F = -\frac{1}{x^3y^3}$

Step2: Reducing D.E to Exact D.E

$(I.F)Mdx + (I.F)Ndy = 0$

$-\frac{1}{x^3y^3}(y + xy^2)dx - \frac{1}{x^3y^3}(x + x^2y + x^3y^2)dy = 0$

$\left(\frac{-y}{x^3y^3} + \frac{xy^2}{x^3y^3}\right)dx - \left(\frac{x}{x^3y^3} + \frac{x^2y}{x^3y^3} + \frac{x^3y^2}{x^3y^3}\right)dy = 0$

$\left(\frac{-1}{x^3y^2} + \frac{1}{x^2y}\right)dx - \left(\frac{1}{x^2y^3} + \frac{1}{xy^2} + \frac{1}{y}\right)dy = 0$

$M_1 = \frac{-1}{x^3y^2} - \frac{1}{x^2y}$ $\qquad$ $N_1 = \frac{-1}{x^2y^3} - \frac{1}{xy^2} - \frac{1}{y}$

$M_{1y} = \frac{-2y^2}{x^3y^4} + \frac{1}{x^2y^2}$ $\qquad$ $N_{1x} = \frac{-2x}{x^4y^3} + \frac{1}{x^2y^2} - 0$

$M_{1y} = \frac{-2}{x^3y^3} + \frac{1}{x^2y^2}$ $\qquad$ $N_{1x} = \frac{-2}{x^3y^3} + \frac{1}{x^2y^2}$ $\qquad$ $\boxed{M_{1y} = N_{1x}}$ → Exact D.E

Step 3: General Solution

$$\int M\,dx + \int (\text{terms in } N \text{ not containing } x)\,dy = C$$

$$\int\left(\frac{-1}{x^3y^2} - \frac{1}{x^2y}\right)dx + \int\left(-\frac{1}{y}\right)dy = C$$

$$-\frac{1}{y^2}\int\frac{1}{x^3}dx - \frac{1}{y}\int\frac{1}{x^2}dx - \int\frac{1}{y}dy = C$$

$$-\frac{1}{y^2}\times\frac{x^{-3+1}}{-3+1} - \frac{1}{y}\times\frac{x^{-2+1}}{-2+1} - \ln|y| = C$$

$$-\frac{1}{y^2}\times\frac{x^{-2}}{-2} - \frac{1}{y}\times\frac{x^{-1}}{-1} - \ln|y| = C$$

$$\boxed{\frac{1}{2x^2y^2} + \frac{1}{xy} - \ln|y| = C}$$

(19) $(y^3 - 2x^2y)\,dx + (2xy^2 - x^3)\,dy = 0$; Ans: $\frac{x^2y^4}{2} - \frac{2x^4y^2}{4} = C$

Step 1: Identify D.E (A homogeneous D.E it is)

(D.E in the form $y f(x,y)\,dx + x g(x,y)\,dy = 0$)

$M = y^3 - 2x^2y$ $\qquad N = 2xy^2 - x^3$

$M_y = 3y^2 - 2x^2$ $\qquad N_x = 2y^2 - 3x^2$; $M_y \neq N_x \rightarrow$ Non-Exact

Rule (1) $P(x) = \frac{M_y - N_x}{N}$, $I.F = e^{\int P(x)\,dx}$

$$\frac{M_y - N_x}{N} = \frac{(3y^2-2x^2)-(2y^2-3x^2)}{2xy^2-x^3} = \frac{3y^2-2y^2-2x^2+3x^2}{2xy^2-x^3} = \frac{y^2+x^2}{2xy^2-x^3}$$

→ Non-Homo

→ Not a function of x.

↳ Rule Failed

Rule (2) $P(y) = \frac{N_x - M_y}{M}$, $I.F = e^{\int P(y)\,dy}$

$$\frac{N_x - M_y}{M} = \frac{(2y^2-3x^2)-(3y^2-2x^2)}{y^3-2x^2y} = \frac{-y^2-x^2}{y^3-2x^2y} \rightarrow \text{not a function of } y$$

Rule (3) For homogeneous Equations, if $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous and $x\cdot M + y\cdot N \neq 0$, then $I.F = \frac{1}{x\cdot M + y\cdot N}$

Condition 1: The given D.E is Homogeneous ✓

Condition 2: $x\cdot M + y\cdot N = (xy^3 - 2x^3y) + (2xy^3 - x^3y) = -3x^3y + 3xy^3 \neq 0$

$$I.F = \frac{1}{x\cdot M + y\cdot N} = \frac{1}{-3x^3y + 3xy^3} = \frac{1}{-3(x^3y + xy^3)} = \frac{-1}{3x^3y + 3xy^3}$$

$$(I.F)\, M dx + (I.F)\, N dy = 0$$

$$\frac{-1}{3x^3y + 3xy^3}(y^3 - 2x^2y)\, dx - \frac{(2xy^2 - x^3)}{3x^3y + 3xy^3} dy = 0$$

$$\left(\frac{2x^2y - y^2}{3x^3y + 3xy^3}\right) dx - \left(\frac{2xy^2 - x^3}{3x^3y - 3xy^3}\right) dy = 0$$

$$M_1 = \frac{2x^2y - y^2}{3x^3y + 3xy^3} = \frac{2x^2 - y}{3x^3 - 3xy^2} \qquad N_1 = -\left(\frac{2x^2y^2 - x^3}{3x^3y + 3xy^3}\right) = \frac{x^3 - 2xy^2}{3x^3y + 3xy^3} = \frac{x^2 - 2y^2}{3x^2y + 3xy^3}$$

$$M_{1y} = \frac{(3x^3 + 3xy^2)(0 - 1) - (2x^2 - y)(0 + 6xy)}{(3x^3 - 3xy^2)^2} = \frac{-(3x^3 + 3xy^2) - 6xy(2x^2 - y)}{(3x^3 - 3xy^2)^2}$$

$$N_{1x} = \frac{(x^2y + y^3)(2x - 0) - (x^2 - 2y^2)(2x^2 + 0)}{(3x^2y - 3y^3)^2} = \frac{2x(x^2y + y^3) - 2x(x^2 - 2y^2)}{(3x^2y - 3y^3)^2}$$

$\boxed{M_{1y} \neq N_{1x}}$ Rule (3) Failed, will get Complicate to rewrite

Rule (4) For equations of form $y f(x,y) dx + x g(x,y)\, dy = 0$, if $x M - y\cdot N \neq 0$, then $I.F = \frac{1}{x\cdot M - y\cdot N}$

$$x\cdot M - y\cdot N = x(y^3 - 2x^2y) - y(2xy^2 - x^3) = xy^3 - 2x^3y - 2xy^3 + x^3y = -x^3y - xy^3$$

$$I.F = \frac{-1}{x^3y + xy^3}$$

$$(I.F)\, M dx + (I.F)\, N dy = 0$$

$$-\frac{(y^3 - 2x^2y)}{x^3y + xy^3} dx - \frac{(2xy^2 - x^3)}{x^3y + xy^3} dy = 0$$

$$\frac{2x^2y - y^2}{x^3y + xy^3} dx + \frac{x^3 - 2xy^2}{x^3y + xy^3} dy = 0 \Rightarrow \frac{2x^2 - y}{x^3 + xy^2} dx + \frac{x^2 - 2y^2}{x^2y + y^3} dy = 0$$

$$M_2 = \frac{2x^2 - y}{x^3 + xy^2} \qquad N_2 = \frac{x^2 - 2y^2}{x^2y + y^3}$$

$$M_{2y} = \frac{(x^3+xy^2)(0-1) - (2x^2-y)(0+2xy)}{(x^3+xy^2)^2}$$

$$M_{2y} = \frac{-(x^3+xy^2) - 2xy(2x^2-y)}{(x^3+xy^2)^2}$$ $\boxed{M_{2y} \neq N_{2x}}$

$$N_{2x} = \frac{(x^2y+y^3)(2x-0) - (x^2-2y^2)(2xy+0)}{(x^2y+y^3)^2} = \frac{2x(x^2y+y^3) - (x^2-2xy)2xy}{(x^2y+y^3)^2}$$

→ This D.E can be solved by converting the p(x), p(y), or I.Fs from non-homogeneous to homogeneous. So instead of converting these to homogeneous function we already have the given D.E in homo form.

- $(y^3-2x^2y)\,dx + (2xy^2-x^3)\,dy = 0$

Step 1: $(y^3-2x^2y)\,dx = -(2xy^2-x^3)\,dy$

$$-\frac{(y^3-2x^2y)}{(2xy^2-x^3)} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x^2y-y^3}{2xy^2-x^3}, \quad \text{let } y = u\cdot x, \text{ then } \frac{dy}{dx} = x\frac{du}{dx}+u$$

$$x\frac{du}{dx}+u = \frac{2x^2(u\cdot x) - (u\cdot x)^3}{2x(u\cdot x)^2 - (u\cdot x)^3}$$

$$x\frac{du}{dx}+u = \frac{2x^3u - u^3x^3}{2x^3u^2 - \cancel{x^3}} = \frac{x^3(2u-u^3)}{x^3(2u^2-1)} = \frac{2u-u^3}{2u^2-1}$$

$$x\frac{du}{dx} = \frac{2u-u^3}{2u^2-1} - u = \frac{2u-u^3-u(2u^2-1)}{2u^2-1} = \frac{2u-u^3-2u^3+u}{2u^2-1}$$

$$x\frac{du}{dx} = \frac{3u-3u^3}{2u^2-1} \Rightarrow \boxed{\frac{2u^2-1}{3u-3u^3}\,du = \frac{dx}{x}}$$

Step 2: Integrate

$$\int\frac{2u^2-1}{3u-3u^3} = \int\frac{(2u^2-1)}{-3u(u-1^2)} = \int\frac{2u^2-1}{-3u(u-1)(u+1)} = \frac{1}{-3}\int\frac{2u^2-1}{u(u-1)(u+1)}$$

Solve using partial fraction.

$$\frac{-1}{-3}\int\frac{2u^2-1}{u(u-1)(u+1)} \Rightarrow \frac{A}{3}\,\frac{2u^2-1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}$$

$$= \frac{-1}{3}(2u^2-1) = A(u-1)(u+1) + B(u+1)u + C(u-1)u$$

$$= -A + Au^2 + Bu + Bu^2 + Cu + Cu^2$$

$$= u^2(+A+B+C) + u(B+C) + A$$

$-\frac{2}{3}u^2 = u^2(+A+B+C)$

$\frac{-1}{3} = -A$

$0 = u(B+C)$

$$-\frac{1}{3}\int \frac{2u^2-1}{u(u-1)(u+1)}\,du \Rightarrow \frac{2u^2-1}{u(u+1)(u+1)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u-1}$$

$$\frac{2u^2-1}{u(u+1)(u-1)} = \frac{A}{u} + \frac{B}{u+1} + \frac{C}{u-1} \Rightarrow 2u^2-1 = A(u+1)(u-1) + B(u-1)u + C(u+1)u$$

$$2u^2-1 = A(u^2-1) + B(u^2-u) + C(u^2+u)$$

$$2u^2-1 = Au^2 - A + Bu^2 - Bu + Cu^2 + Cu$$

$$2u^2-1 = u^2(A+B+C) + u(C-B) - A$$

$$2u^2 = u^2(A+B+C), \quad -1 = -A \Rightarrow \boxed{A=1} \quad 0 = u(C-B) \Rightarrow C-B=0$$

$$A+B+C=2, \quad A=1, \quad -B+C=0$$

$$1+B+C=2 \qquad \boxed{B=C}, \ \boxed{B=1/2}$$

$$B+C=1, \text{ put } B=C$$

$$C+C=1$$

$$\boxed{C=1/2}$$

$$-\frac{1}{3}\int \frac{2u^2-1}{u(u-1)(u+1)} = -\frac{1}{3}\left(\int \frac{du}{u} + \frac{1}{2}\int \frac{du}{u+1} + \frac{1}{2}\int \frac{du}{u-1}\right)$$

$$\Rightarrow \int \frac{2u^2-1}{3u-3u^3}\,du = \int \frac{dx}{x}$$

$$-\frac{1}{3}\left(\int \frac{du}{u} + \frac{1}{2}\int \frac{du}{u+1} + \frac{1}{2}\int \frac{du}{u-1}\right) = \int \frac{dx}{x}$$

$$\ln|u| + \frac{1}{2}\ln|u+1| + \frac{1}{2}\ln|u-1| = (\ln|x| + \ln|c|)\times -3$$

$$\ln|u| + \frac{1}{2}\ln|(u+1)(u-1)| = -3\ln|x\cdot c|$$

$$\ln|u| + \ln|(u^2-1)^{1/2}| = -3\ln|x\cdot c|$$

$$\ln|u\cdot(u^2-1)^{1/2}| = \ln|(x\cdot c)^{-3}|$$

$$u\cdot\sqrt{u^2-1} = \frac{1}{(xc)^3}, \quad y = u\cdot x$$

$$\frac{y}{x}\sqrt{\frac{y^2}{x^2}-1} = (xc)^{-3} \Rightarrow \frac{y}{x}\sqrt{\frac{y^2-x^2}{x^2}} = \frac{y}{x}\frac{\sqrt{y^2-x^2}}{\sqrt{x^2}} = \frac{y}{x^2}\sqrt{y^2-x^2}$$

$$\frac{y}{\cancel{x^2}}\sqrt{y^2-x^2} = \frac{1}{x^{\cancel{3}}c^3} \Rightarrow xy\sqrt{y^2-x^2} = b \Rightarrow (xy)^2(y^2-x^2) = b^2$$

$$\boxed{x^2y^4 - x^4y^2 = b^2}$$