Change of Variables for Definite Integrals

When using u-substitution with a definite integral, it is often convenient to determine the limits of integration for the variable u rather than to convert the antiderivative back to the variable x and evaluate at the original limits. This change of variables is stated explicitly in the next theorem. The proof follows from Theorem 4.13 combined with the Fundamental Theorem of Calculus.

THEOREM 4.15 CHANGE OF VARIABLES FOR DEFINITE INTEGRALS

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

EXAMPLE 8 Change of Variables

Evaluate $\displaystyle\int_0^1 x(x^2+1)^3\,dx$.

Solution To evaluate this integral, let $u = x^2 + 1$. Then, you obtain

$$u = x^2 + 1 \implies du = 2x\,dx.$$

Before substituting, determine the new upper and lower limits of integration.

Lower Limit	*Upper Limit*
When $x = 0$, $u = 0^2 + 1 = 1$.	When $x = 1$, $u = 1^2 + 1 = 2$.

Now, you can substitute to obtain

$$\begin{aligned}\int_0^1 x(x^2+1)^3\,dx &= \frac{1}{2}\int_0^1 (x^2+1)^3(2x)\,dx && \text{Integration limits for } x\\ &= \frac{1}{2}\int_1^2 u^3\,du && \text{Integration limits for } u\\ &= \frac{1}{2}\left[\frac{u^4}{4}\right]_1^2\\ &= \frac{1}{2}\left(4 - \frac{1}{4}\right)\\ &= \frac{15}{8}.\end{aligned}$$

Try rewriting the antiderivative $\frac{1}{2}(u^4/4)$ in terms of the variable x and evaluate the definite integral at the original limits of integration, as shown.

$$\begin{aligned}\frac{1}{2}\left[\frac{u^4}{4}\right]_1^2 &= \frac{1}{2}\left[\frac{(x^2+1)^4}{4}\right]_0^1\\ &= \frac{1}{2}\left(4 - \frac{1}{4}\right) = \frac{15}{8}\end{aligned}$$

Notice that you obtain the same result. ■