Assignment: Integration of Partial Fractions

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Partial Fractions

Introduction : A fraction is a symbol indicating the divison of integers. For example, $\frac{13}{8}$, $\frac{2}{3}$ are fractions and are called Common fraction. The dividend (upper number) is called the numerator $N(x)$ and the divisor (lower number) is called the denominator, $D(x)$.

From the previous study of elementary algebra we have learnt how the sum of different fractions can be found by taking L.C.M. and then add all the fractions. For example

i) $$\frac{1}{x-2} + \frac{2}{x+2} = \frac{3x}{(x-1)(x+2)}$$

ii) $$\frac{2}{x+2} + \frac{1}{(x+1)^2} + \frac{3}{x-2} = \frac{8x^2+5x-3}{(x+1)^2(x-2)}$$

Here we study the reverse process, i.e., we split up a single fraction into a number of fractions whose denominators are the factors of denominator of that fraction. These fractions are called partial fractions.

Partial Fraction

To express a single rational function into the sum of two or more single rational fractions is called partial fraction resolution | decomposition.

Ex:
$$\frac{2x+x^2-1}{x(x^2-1)} = \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x+1}$$

$\frac{2x+x^2-1}{x(x^2-1)}$ is the resultant fraction and $\frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+1}$ are its partial fractions.

Process of Finding Partial Fraction

A proper fraction $\frac{N(x)}{D(x)}$ can be resolved into partial fractions.

(I) If the denominator $D(x)$ a linear factor $(ax+b)$ occurs and is non-repeating, its partial fraction will be of the form

$\frac{A}{ax+b}$, where A is a constant whose value is to be determined

(II) If in the Denominator $D(x)$ a linear factor $(ax+b)$ occurs n times, i.e., $(ax+b)^n$, then there will be n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \dots + \frac{A_n}{(ax+b)^n},$$ where

$A_1, A_2, A_3, \dots A_n$ are constants whose values are to be deter-

(III) If in the denominator $D(x)$ a quadratic factor ax^2+bx+c occurs and is non-repeating, its partial fraction will be of the form $\frac{Ax+B}{ax^2+bx+c}$, where A and B are constants whose val are to be determined.

(IV) If in the denominator a quadratic factor ax^2+bx+c occurs n times, i.e., $(ax^2+bx+c)^n$, then there will be n partial fractions of the form

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \frac{A_3x+B_3}{(ax^2+bx+c)^3} + \dots + \frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where $A_1, A_2, A_3, \dots A_n$ and $B_1, B_2, B_3, \dots B_n$ are constants whose values are to be determined.

Type I

When the factors of $D(x)$ are all linear and distinct i.e., non-repeating.

Q1) $\int \frac{2x+3}{(x-3)(x+5)} dx$

Step 1: partial fraction resolution/ decomposition

$$\frac{2x+3}{(x-3)(x+5)} = \frac{A}{(x-3)} + \frac{B}{(x+5)}$$

$$2x+3 = \frac{A\cancel{(x-3)}(x+5)}{\cancel{(x-3)}} + \frac{B(x-3)\cancel{(x+5)}}{\cancel{(x+5)}}$$

$$2x+3 = A(x+5) + B(x-3)$$

$$2x+3 = Ax + 5A + Bx - 3B$$

$$2x+3 = Ax + Bx + 5A - 3B$$

$$2x+3 = x(A+B) + (5A-3B)$$

Step 2: Equating method (Identity property)

$2x = x(A+B)$; $3 = (5A+3B)$

$2 = A+B$ $3 = 5A-3B$

Substitute $A = 2-B$ put $A = 2-B$

put $B = 7/8$

$3 = 5(2-B) - 3B$

$3 = 10 - 5B - 3B$

$2 = A + 7/8$

$-7 = -8B$

$\frac{2}{1} - \frac{7}{8} = A$

$\boxed{\frac{7}{8} = B}$

$\boxed{A = 9/8}$

Step 3: Integrate

$$\int \frac{2x+3}{(x-3)(x+5)} dx = \frac{9}{8}\int \frac{dx}{(x-3)} + \frac{7}{8}\int \frac{dx}{(x+5)}$$

$$= \frac{9}{8}\ln(x-3) + \frac{7}{8}\ln(x+5) + C$$

$$= \boxed{\frac{1}{8}\left[(x-3)^9(x+5)^7\right] + C}$$

Q2) $\int \frac{dx}{1-x^2}$

Step1: Partial fraction resolution

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$\frac{1}{(1-x)(1+x)} = \frac{A}{(1-x)} + \frac{B}{(1+x)}$$

Step2:

$$1 = A(1+x) + B(1-x)$$

$$1 = Ax + A + B - Bx$$

$$1 = x(A-B) + (A+B)$$

$$A+B = 1 \quad ; \quad A-B=0$$

$A+B=1$; $A-B=0$

$A = 1-B$ — put $A = 1-B$

$A = 1 - 1/2$ — $1-B-B=0$

$\boxed{A = 1/2}$ — $1-2B=0$

$\boxed{B = 1/2}$

Step3: Integrate

$$\int \frac{dx}{1-x^2} = \int \frac{dx}{(1-x)(1+x)} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x}$$

$$\boxed{\frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+x) + C}$$

Q3) $\int \frac{dx}{x^2+2x}$

Step1: $\frac{1}{x^2+2x} = \frac{1}{x(x+2)} = \frac{A}{x} + \frac{B}{x+2}$

$$1 = A(x+2) + B(x)$$

Step2: $1 = Ax + 2A + Bx$

$1 = x(A+B) + 2A$

$2A = 1$; $x(A+B)=0$

$\boxed{A = 1/2}$ $\quad A+B=0$

$\boxed{B=-1/2}$

Step 3: Integrate

$$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2}$$

$$= \boxed{\frac{1}{2}\ln(x) - \frac{1}{2}\ln(x+2) + C}$$

Q4) $\int_4^8 \frac{y\,dy}{y^2-2y-3}$

y^2-2y-3
$y^2-3y+y-3$
$y(y-3)+1(y-3)$
$(y-3)(y+1)$

Step1: Partial Fraction Resolution.

$$\frac{y}{y^2-2y-3} = \frac{y}{(y-3)(y+1)} = \frac{A}{(y-3)} + \frac{B}{(y+1)}$$

$$\frac{y}{(y-3)(y+1)} = \frac{A}{(y-3)} + \frac{B}{(y+1)} \Rightarrow y = A(y+1) + B(y-3)$$

Step2: Find A, & B.

$y = Ay + A + By - 3B$

$y = y(A+B) + (A-3B)$

$A+B=1$; $A-3B=0$

$A+B=1$; $A-3B=0$.

$A = 1-B$; $1-1B-3B=0$

$A = 1 - 1/4$; $1 - 4B = 0$

$\boxed{A = 3/4}$; $\boxed{B = 1/4}$

Step3: Integrate

$$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1}$$

$$\frac{3}{4}\left[\ln(y-3)\right]_4^8 + \frac{1}{4}\left[\ln(y+1)\right]_4^8$$

$$= \frac{3}{4}\left[\ln(8-3) - \ln(4-3)\right] + \frac{1}{4}\left[\ln(8+1) - \ln(4+1)\right]$$

$$= \frac{3}{4}\left[\ln(5) - \ln(1)\right] + \frac{1}{4}\left[\ln(9) - \ln(5)\right]$$

$$= \frac{3}{4}\left[\ln(5)\right] + \frac{1}{4}\left[\ln(9) - \ln(5)\right]$$

$$= \frac{1}{4}\left[3\ln(5) + \ln(9) - \ln(5)\right]$$

$$= \frac{1}{4}\left[2\ln(5) + \ln(9)\right]$$

$$= \frac{1}{2}\ln(5) + \frac{1}{4}\ln(3)^2 = \frac{\ln(5)}{2} + \frac{\ln(3)}{2}$$

$$= \frac{1}{2}\left[\ln(5) + \ln(3)\right] = \frac{1}{2}\ln(5\times 3) = \boxed{\frac{1}{2}\ln(15)}$$

Q5) $\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy$

Step 1: Partial fraction Resolution

$$\frac{y+4}{y^2+y} = \frac{y+4}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1}$$

$$\frac{y+4}{y(y+1)} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By$$

Step 2:

$y+4 = Ay + A + By$

$y+4 = y(A+B) + A$

$y = y(A+B)$; $\boxed{A=4}$

$1 = A+B$

$\boxed{-3 = B}$

Step 3: Integrate

$$\int_{1/2}^{1} \frac{y+4}{y^2+y}\,dy = \int_{1/2}^{1} \frac{A}{y}\,dy + \int_{1/2}^{1} \frac{B}{y+1}\,dy$$

$$= 4\int_{1/2}^{1} \frac{dy}{y} + (-3)\int_{1/2}^{1} \frac{dy}{y+1}$$

$$= 4\left[\ln(y)\right]_{1/2}^{1} - 3\left[\ln(y+1)\right]_{1/2}^{1}$$

$$= 4\left[\ln(1) - \ln(1/2)\right] - 3\left[\ln(1+1) - \ln(1/2+1)\right]$$

$$= 4\left[0 - \ln(1/2)\right] - 3\left[\ln(2) - \ln(3/2)\right]$$

$$= -4\ln(1/2) - 3\left[\ln\left(\frac{2}{3/2}\right)\right]$$

$$= -4\ln(1/2) - 3\ln(4/3)$$

$$= \ln\left(\frac{1}{2}\right)^{-4} + \ln\left(\frac{4}{3}\right)^{-3}$$

$$= \ln\left(\frac{2}{1}\right)^{4} + \ln\left(\frac{3}{4}\right)^{3}$$

$$= \ln(16) + \ln\left(\frac{27}{64}\right)$$

$$= \ln\left(\cancel{16}^{1} \times \frac{27}{\cancel{64}_{4}}\right) = \boxed{\ln\left(\frac{27}{4}\right)}$$

Type II

When the factors of the denominator are all linear but some are repeated.

Q6) $\int_0^1 \frac{x^3}{x^2+2x+1}\,dx$

Step 1: Partial Fraction Resolution

Long division

$$\frac{x^3}{(x^2+2x+1)} = (x-2) + \boxed{\frac{3x+2}{x^2+2x+1}}$$

By long division, because degree of $N(x) >$ deg $D(x)$

$$\begin{array}{r} x-2 \\ x^2+2x+1 \overline{)\, x^3 \qquad\qquad} \\ -\;x^3+2x^2+x \\ \hline -2x^2-x \\ -(-2x^2-4x-2) \\ \hline 3x+2 \end{array}$$

$\frac{x^3}{x^2} = x$, $\frac{-2x^2}{x^2} = -2$

$$\frac{3x+2}{x^2+2x+1} = \frac{3x+2}{(x+1)(x+1)} = \frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Step 2: A, and B

$$\frac{3x+2}{(x+1)^2} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1)+B$$

$3x+2 = Ax+A+B$

$3x = Ax$; $2 = A+B$

$\boxed{A=3}$ $\boxed{B=-1}$

$$\Rightarrow \int_0^1 \frac{x^3}{x^2+2x+1}\,dx = \int_0^1 (x-2)\,dx + \int_0^1 \frac{3x+2}{x^2+2x+1}\,dx$$

$$= \int_0^1 (x-2)\,dx + \int_0^1 \frac{3}{x+1}\,dx - \int_0^1 \frac{dx}{(x+1)^2}$$

Step 3: Integrate

$$\int_0^1 (x-2)\,dx + \int_0^1 \frac{3}{x+1}\,dx - \int_0^1 \frac{dx}{(x+1)^2}$$, u-Substitution

$u = x+1$

$du = dx$

$$\left[\frac{x^2}{2} - 2x + 3\ln(x+1)\right]_0^1 - \int_0^1 \frac{dx}{(x+1)^2}$$

$$\left[\frac{x^2}{2} - 2x + 3\ln(x+1)\right]_0^1 - \int_0^1 \frac{du}{u^2}$$

$$\left[\left(\frac{1^2}{2} - 2(1) + 3\ln(1+1)\right) - 3\ln(0+1)\right] - \int_0^1 u^{-2}\,du$$

$$\left[\frac{1}{2} - 2 + 3\ln(2) - 0\right] - \left.\frac{u^{-2+1}}{-2+1}\right]_0^1$$

$$3\ln(2) - \frac{3}{2} + \left.\frac{1}{u}\right]_0^1 = 3\ln(2) - \frac{3}{2} + \left[\frac{1}{x+1}\right]_0^1$$

$$3\ln(2) - \frac{3}{2} + \left[\frac{1}{2} - 1\right] = 3\ln(2) - \frac{3}{2} - \frac{1}{2}$$

$$= \boxed{3\ln(2) - 2}$$

Q7) $\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\,dx$, Improper fraction

Step 1: partial Fraction Resolution

$$\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{x^2-2x+1}$$

$$\begin{array}{r} x+2 \\ x^2-2x+1 \overline{)\; x^3 \qquad\qquad} \\ \underline{x^3-2x^2+x} \\ 2x^2-x \\ \underline{-\;2x^2-4x+2} \\ 3x-2 \end{array}$$

$\frac{x^3}{x^2} = x$

$\frac{2x^2}{x^2} = 2$

$$\frac{3x-2}{x^2-2x+1} = \frac{3x-2}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Step 2: A, and B

$$\frac{3x-2}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$$

$3x-2 = Ax - A + B$

$3x = Ax$; $-2 = -A+B$

$\boxed{A=3}$ $\quad -2 = -3+B$

$\boxed{1=B}$

Step 3: Integrate

$$\int_{-1}^{0} \frac{x^3}{x^2-2x+1}\,dx = \int_{-1}^{0} (x+2)\,dx + \int_{-1}^{0} \frac{3}{(x-1)}\,dx + \int_{-1}^{0} \frac{dx}{(x-1)^2}$$

$= \left[\frac{x^2}{2} + 2x + 3\ln(x-1)\right]_{-1}^{0} + \int_{-1}^{0} \frac{dx}{(x-1)^2}$, u-substitution

$u = x-1$,

$= \left[(0^2 + 2(0) + 3\ln(0-1)) - (\frac{1}{2} + 2 + 3\ln[-2]\right] + \int_{-1}^{0} \frac{du}{u^2}$ $\quad du = dx$

$= \left[3\ln(+1) - (-\frac{3}{2} + 3\ln(+2))\right] + \left[\frac{u^{-2+1}}{-2+1}\right]_{-1}^{0}$

$= 3\ln(1) + \frac{3}{2} - 3\ln(2) + \left(-\frac{1}{u}\right)_{-1}^{0}$

$= 3\ln(1) + \frac{3}{2} - 3\ln(2) - \left[\frac{1}{x-1}\right]_{-1}^{0}$

$= +\frac{3}{2} - 3\ln(2) - \left[-1 + \frac{1}{2}\right] = +\frac{3}{2} - 3\ln(2) + \frac{1}{2}$

$= \boxed{2 - 3\ln(2)}$

(8)

Q8) $\int \frac{x^2-3x+1}{(x-1)^2(x-2)}$

Step 1: Partial Fraction Resolution

$$\frac{x^2-3x+1}{(x-1)^2(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$$

$$x^2-3x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$$x^2-3x+1 = A(x^2-2x-x+2) + Bx - 2B + C(x^2-2x+1)$$

$$x^2-3x+1 = A(x^2-3x+2) + Bx - 2B + Cx^2 - 2Cx + C.$$

$$x^2-3x+1 = \underline{Ax^2} - \underline{3Ax} + 2A + \underline{Bx} - 2B + \underline{Cx^2} - \underline{2Cx} + C$$

$$x^2-3x+1 = x^2(A+C) - x(3A-B+2C) + 2A - 2B + C$$

Step 2:

$x^2 = x^2(A+C)$; $-3x = -x(3A-B+2C)$; $1 = 2A-2B+C$

$\boxed{1 = A+C}$ — i $\boxed{3 = 3A-B+2C}$ — ii $\boxed{1 = 2A-2B+C}$ — iii

The comparing method can solve this case but when the number of variables grow there occur problem or It will take alot time.

So, we will find the values of A, B, and C using roots.

$$\boxed{x^2-3x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2} \text{ — (1)}$$

putting $x-1=0 \Rightarrow x=1$ in (1)

$$(1)^2-3(1)+1 = A(1-1)(1-2) + B(1-2) + C(1-1)^2$$

$$1-3+1 = 0 + B(-1) + 0$$

$$-1 = -B \Rightarrow \boxed{B=1}$$

→ Putting $x-2=0 \Rightarrow x=2$

$$(2)^2-3(2)+1 = A(2-1)(2-2) + B(2-2) + C(2-1)^2$$

$$4-6+1 = 0+0+C$$

$$\boxed{-1 = C}$$

→ This why you always need to find the equations put $C=-1$ in any equation (i, ii, iii)

$1 = A+C$ — i

$1 = A-1$

$\boxed{A=2}$

$$\int \frac{x^2-3x+1}{(x-1)^2(x-2)} = \int \frac{2}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx + \int \frac{1}{x-2}\,dx$$

$$= 2\ln(x-1) + \ln(x-2) + \int \frac{1}{(x-1)^2}\,dx \quad , \quad u = x-1, \; du = dx$$

$$= 2\ln(x-1) + \ln(x-2) + \int \frac{du}{u^2}$$

$$= \boxed{2\ln(x-1) + \ln(x-2) - \frac{1}{(x-1)} + C}$$

Q9) $\int \frac{x^2}{(x-1)(x^2+2x+1)}\,dx$

Step 1:

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$$

$$x^2 = A(x+1)^2 + B(x-1)(x+1)^2 + C(x-1)$$

$$x^2 = A(x^2+2x+1) + B(x^2+x-x-1) + Cx - C$$

$$x^2 = Ax^2 + 2Ax + A + Bx^2 - B + Cx - C$$

$$x^2 = x^2(A+B) + x(2A+C) + A - B - C$$

$A+B=1$; $2A+C=\boxed{0}$; $A-B-C=\boxed{0}$

Put $A=-\frac{C}{2}$

$B = 1-A$

$\boxed{B = 1+\frac{C}{2}}$

Put $C=-1/2$

$B = 1-\frac{1}{4} = 3/4$

$\boxed{B=3/4}$

$\boxed{A=-\frac{C}{2}}$

put $C=-1/2$

$A = \frac{1/2}{2} = 1/4$

$\boxed{A=1/4}$

Put $A=-\frac{C}{2}$, $B = 1+\frac{C}{2}$

$-\frac{C}{2} - \left(1+\frac{C}{2}\right) - C = 0$

$-\frac{C}{2} - \frac{C}{2} - C = 1$

$-C - C = 1$

$\boxed{C = -1/2}$

$$\int \frac{x^2}{(x-1)(x+1)^2}\,dx = \frac{1}{4}\int \frac{dx}{(x-1)} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} \quad , \quad u = x+1, \; du = dx$$

$$= \frac{1}{4}\ln(x-1) + \frac{3}{4}\ln(x+1) - \frac{1}{2}\left(-\frac{1}{x+1}\right) + C$$

$$= \boxed{\frac{1}{4}\ln(x-1) + \frac{3}{4}\ln(x+1) + \frac{1}{2}\left(\frac{1}{x+1}\right) + C}$$

Q10) $\int \frac{x+4}{(x-2)^2(x+1)}$

Step 1: Partial Fraction Resolution

$$\frac{x+4}{(x-2)^2(x+1)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2}$$

$$\boxed{x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)} \quad \text{— (1)}$$

$$x+4 = A(x^2-2x+1) + B(x^2-2x+x-2) + Cx + C$$

$$x+4 = Ax^2 - 2Ax + A + Bx^2 - Bx - 2B + Cx + C$$

$$x+4 = x^2(A+B) - x(2A+B-C) + A - 2B + C$$

$0 = x^2(A+B)$; $x = -x(2A+B-C)$; $4 = A-2B+C$

$\boxed{A+B=0}$ $\boxed{2A+B-C=-1}$ $\boxed{A-2B+C=4}$

Step 2:

Put $x+1=0 \Rightarrow x=-1$ in equation (1)

$$x+4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$$

$$-1+4 = A(-1-2)^2 + B(-1+1)(x-2) + 0$$

$$3 = A(9) + 0 + 0$$

$\boxed{A = 1/3}$ $\quad B = -A \Rightarrow \boxed{B = -1/3}$

$A - 2B + C = 4 \Rightarrow \frac{1}{3} + \frac{2}{3} + C = 4 \Rightarrow \boxed{C = 3}$

Step 3: Integrate

$$\int \frac{x+4}{(x-2)^2(x+1)}\,dx = \frac{1}{3}\int \frac{dx}{(x+1)} - \frac{1}{3}\int \frac{dx}{x-2} + \int \frac{3}{(x-2)^2}\,dx$$

$$= \frac{1}{3}\ln(x+1) - \frac{1}{3}\ln(x-2) + 3\left(\frac{-1}{x-2}\right) + C$$

$$= \boxed{\frac{1}{3}\ln(x+1) - \frac{1}{3}\ln(x-2) - \frac{3}{x-2} + C}$$

Type III

1. When the denominator contains ir-reducible quadratic factors which are non-repeated.

Q11) $\int \frac{8x-7}{(x+3)(x^2+1)} dx$

Step 1: Partial Fraction Resolution

$$\frac{8x-7}{(x+3)(x^2+1)} = \frac{A}{(x+3)} + \frac{Bx+C}{(x^2+1)}$$

$$\boxed{8x-7 = A(x^2+1) + (Bx+C)(x+3)} \quad - (1)$$

$8x-7 = Ax^2 + A + Bx^3 + B3 + Cx + 3C$ ×

$8x-7 = Bx^3 + x^2(A+C) + Bx + A + C$

$Bx^3 = 0$; $0 = (A+C)x^2$; $8x = Bx$; $A+C = -7$

Bit Confusing? switch to root method we made a mistake

→ Put $x = -3$ in eq (1)

$8(-3) - 7 = A(9+1) + (-3B+C)(-3+3)$

$-34 = A(10) + 0 \Rightarrow \boxed{A = -\frac{17}{5}}$

$8x-7 = A(x^2+1) + (Bx+C)(x+3)$

$8x-7 = Ax^2 + A + Bx^2 + 3Bx + Cx + 3C$

$8x-7 = x^2(A+B) + Cx + A + 3Bx + 3C$

$A+B=0$; $Cx + 3Bx = 8x$; $A + 3C = -7$

put $A = -\frac{17}{5}$; $C + 3B = 8$; put $A = -\frac{17}{5}$

$\boxed{B = +\frac{17}{5}}$

$3C = -\frac{7}{1} + \frac{17}{5}$

$3C = \frac{-35+17}{5}$

$C = -\frac{18}{15} = \boxed{-\frac{6}{5}}$

Step 3: Integration

$$\int \frac{8x-7}{(x+3)(x^2+1)} dx = -\frac{17}{5}\int \frac{dx}{(x+3)} + \int \frac{17x-6}{5(x^2+1)} dx$$

$u = x^2+1$
$du = 2x$

$$= -\frac{17}{5} \ln(x+3) + \frac{17}{5}\int \frac{x}{x^2+1} dx - \frac{6}{5}\int \frac{dx}{x^2+1}$$

$$= \boxed{-\frac{17}{5} \ln(x+3) + \frac{17}{10} \ln(x^2+1) - \frac{6}{5} \ln(x^2+1) + C}$$

u-Substitution

Q12) $\int \frac{dx}{(x+1)(x^2+1)}$

Step 1: Partial Fraction Resolution

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$1 = \frac{A(x+1)(x^2+1)}{(x+1)} + \frac{(Bx+C)(x+1)(x^2+1)}{(x^2+1)}$$

$$\boxed{1 = A(x^2+1) + (Bx+C)(x+1)}$$

Step 2:

$$1 = Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$1 = x^2(A+B) + x(B+C) + A+C$$

$A+B=0$; $B+C=0$; $A+C=1$

put $B = A-1$; put $C = 1-A$; $\boxed{C = 1-A}$

$A + A - 1 = 0$; $B + 1 - A = 0$; Put $A = 1/2$

$\boxed{A = 1/2}$; $\boxed{B = A-1}$; $\boxed{C = 1/2}$

put $A = 1/2$

$\boxed{B = -1/2}$

Step 3: Integrate

$$\int \frac{dx}{(x+1)(x^2+1)} = \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{-x+1}{x^2+1}dx$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\int \frac{x+1}{x^2+1}dx ,$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\int \frac{x}{x^2+1}dx - \frac{1}{2}\int \frac{1}{x^2+1}dx$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\int \frac{x}{x^2+1}dx ,$$ $u = x^2+1$, $\frac{du}{2} = dx$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2+1) - \frac{1}{4}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln(x+1) - \frac{1}{2}\ln(x^2+1) - \frac{1}{4}\ln(u) + C$$

$$= \boxed{\frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x^2+1| - \frac{1}{4}\ln|x^2+1| + C}$$

Q13) $\int \frac{3t^2+t+4}{t^3+t}\,dt$

Step 1:

$$\frac{3t^2+t+4}{t^3+t} = \frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1}$$

$$\frac{3t^2+t+4}{t(t^2+1)} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1)+(Bt+C)t$$

2:

$3t^2+t+4 = At^2+A+Bt^2+Ct$

$3t^2+t+4 = t^2(A+B)+Ct+A$

$3t^2 = t^2(A+B)$; $t = Ct$; $\boxed{A=4}$

$\boxed{1=C}$

$3 = A+B$

$\boxed{-1=B}$

Step 3: Integrate

$$\int \frac{3t^2+t+4}{t^3+t} = 4\int \frac{dt}{t} + \int \frac{-t+1}{t^2+1}\,dt$$

$$= 4\ln(t) - \int \frac{t}{t^2+1}\,dt + \int \frac{dt}{t^2+1}$$

$$= 4\ln(t) + \ln(t^2+1) - \int \frac{t}{t^2+1}\,dt, \quad u = t^2+1, \quad \frac{du}{2} = t\,dt$$

$$= 4\ln(t) + \ln(t^2+1) - \frac{1}{2}\int \frac{du}{u}$$

$$= 4\ln(t) + \ln(t^2+1) - \frac{1}{2}\ln(u)+C$$

$$= \boxed{4\ln(t) + \ln(t^2+1) - \frac{1}{2}\ln(t^2+1)+C}$$

Q 14) $\int \frac{x^2+3x-1}{(x-2)(x^2+5)} \, dx$

step 1:

$$\frac{x^2+3x-1}{(x-2)(x^2+5)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+5)}$$

$$(x^2+3x-1) = A\cdot(x^2+5) + (Bx+C)(x-2)$$

$$x^2+3x-1 = Ax^2+5A + Bx^2 - 2Bx + Cx - 2C$$

$$x^2+3x-1 = x^2(A+B) - x(2B-C) + 5A - 2C$$

$$x^2 = x^2(A+B) \; ; \; 3x = -x(2B-C) \; ; \; -1 = 5A-2C$$

$1 = A+B$; $-3 = 2B-C$; put $A = 1-B$

$C = 2B+3$

$\boxed{A = 1-B}$ $\boxed{C = 2B+3}$ $-1 = 5(1-B)-2(2B+3)$

$\boxed{A = 1}$ $\boxed{C = 3}$, $B=0$ $-\cancel{1} = \cancel{5} - 5B - 4B - \cancel{6}$

$\boxed{B=0}$

step 3: Integrate

$$\int \frac{x^2+3x-1}{(x-2)(x^2+5)} \, dx = \int \frac{dx}{(x-2)} + 3\int \frac{dx}{(x^2+5)} \quad , \quad u = x^2+5, \; \frac{du}{2} = x\,dx$$

$$= \boxed{\ln(x-2) + 3\ln(x^2+5) + C}$$

Q 15) $\int \frac{x^2-x+2}{(x+1)(x^2+3)} \, dx$

Step 1: $\frac{x^2-x+2}{(x+1)(x^2+3)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+3} \Rightarrow x^2-x+2 = A(x^2+3)+(Bx+C)(x+1)$

$$x^2-x+2 = A(x^2+3) + (Bx+C)(x+1)$$

$$x^2-x+2 = Ax^2 + 3A + Bx^2 + Bx + Cx + C$$

$$x^2-x+2 = x^2(A+B) + x(B+C) + 3A + C$$

$A+B = 1$; $B+C = -1$; $3A+C = 2$

$\boxed{A=1}$ $\boxed{B=0}$ $\boxed{C=-1}$

$$\int \frac{x^2-x+2}{(x+1)(x^2+3)} = \int \frac{dx}{x+1} + \int \frac{-dx}{x^2+3} = \boxed{\ln(x+1) - \ln(x^2+3) + C}$$

Type IV: Quadratic repeated factors

Q 16) $\int \frac{y^2+2y+1}{(y^2+1)^2}\, dy$

Step 1: $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2}$

$y^2+2y+1 = (Ay+B)(y^2+1) + (Cy+D)$

$y^2+2y+1 = Ay^3 + Ay + By^2 + B + Cy + D$

$y^2+2y+1 = Ay^3 + y(A+By+C) + B+D$

$y^2+2y+1 = Ay^3 + By^2 + y(A+C) + B+D$

$Ay^3 = 0$; $y^2 = By^2$; $2y = (A+C)y$; $1 = B+D$

$\boxed{A=0}$ $\boxed{1=B}$ $2 = A+C$ $1 = B+D$

$\boxed{2=C}$ $\boxed{0=D}$

Step 3: Integrate

$\int \frac{y^2+2y+1}{(y^2+1)^2}\, dy = \int \frac{dy}{y^2+1} + \int \frac{2y}{(y^2+1)^2}\, dy$

$= \ln(y^2+1) + \int \frac{2y}{(y^2+1)^2}\, dy$; $u = y^2+1$, $\frac{du}{2} = y\, dy$

$= \ln(y^2+1) + \int \frac{du}{u^2} = \ln(y^2+1) + \frac{u^{-2+1}}{-2+1} + c$

$= \ln(y^2+1) - \frac{1}{u} + c = \boxed{\ln(y^2+1) - \frac{1}{y^2+1} + c}$

Q17) $\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx$

Step 1: $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2}$

$8x^2+8x+2 = (Ax+B)(4x^2+1) + (Cx+D)$

$8x^2+8x+2 = 4Ax^3 + Ax + 4Bx^2 + B + Cx + D$

$8x^2+8x+2 = x^3(4A) + x^2(4B) + x(A+C) + B + D$

$x^3(4A) = 0$; $x^2(4B) = 8x^2$; $8x = x(A+C)$; $2 = B+D$

$\boxed{A=0}$ $\boxed{B=2}$ $\boxed{C=8}$ $\boxed{D=0}$

Step 3: Integrate

$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = \int \frac{2\,dx}{(4x^2+1)} + \int \frac{8x\,dx}{(4x^2+1)^2}$, $u = 4x^2+1$, $\frac{du}{8} = x\,dx$

$= 2\ln(4x^2+1) + \int \frac{du}{u^2} \Rightarrow 2\ln(4x^2+1) - \frac{1}{u} + C$

$= \boxed{2\ln|4x^2+1| - \frac{1}{4x^2+1} + C}$

Q18) $\int \frac{x^2}{(1-x)(1+x^2)^2}\,dx$

Step 1: $\frac{x^2}{(1-x)(1+x^2)^2} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2} + \frac{Dx+E}{(1+x^2)^2}$

$x^2 = A(1+x^2)^2 + (Bx+C)(1-x)(1+x^2) + (Dx+E)(1-x)$

$x^2 = A(1+2x^2+x^4) + (Bx - Bx^2 + C - Cx)\cdot(1+x^2) + (Dx + E - Dx^2 - Ex)$

$x^2 = (Ax^4 + 2Ax^2 + A) + (-Bx^2 + Bx - Cx + C)(1+x^2) + (-Dx^2 + Dx - Ex + E)$

$x^2 = (Ax^4 + 2Ax^2 + A) + (-Bx^2 + Bx - Cx + C - Bx^4 + Bx^3 - Cx^3 + Cx^2) + \text{———}$

$x^2 = x^4(A+B) + x^3(B-C) + x^2(2A+B+C-D) + x(B-C+D-E) + A+C+E$

$A+B=0$; $B-C=0$; $2A+B+C-D=1$; $B-C+D-E=0$; $A+C+E=0$

$\boxed{A=1/4}$ $\boxed{B=3/4}$ $\boxed{C=1/4}$ $\boxed{D=-1/2}$ $\boxed{E=-1/2}$

Step 3: $\int \frac{x^2}{(1-x)(1+x^2)^2}\,dx = \frac{1}{4}\int \frac{dx}{1-x} + \frac{1}{4}\int \frac{x+1}{1+x^2} - \frac{1}{2}\int \frac{x+1}{(1+x^2)^2}$

$= \frac{1}{4}\ln(1-x) + \frac{1}{4}\int \frac{x}{1+x^2}\,dx + \frac{1}{4}\int \frac{1}{1+x^2}\,dx - \frac{1}{2}\int \frac{x\,dx}{(1+x^2)^2} - \frac{1}{2}\int \frac{1\,dx}{(1+x^2)^2}$

$= \boxed{\frac{1}{4}\ln(1-x) + \frac{1}{8}\ln(1+x^2) + \frac{1}{4}\ln(1+x^2) + \frac{1}{4(1+x^2)} - \frac{1}{2}\tan^{-1}(x) + C}$

u-sub

u-sub

trigonometric substitution

Q19) $\int \frac{7}{(x+1)(x^2+2)^2} dx$

step 2: $\frac{7}{(x+1)(x^2+2)^2} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+2)} + \frac{Dx+E}{(x^2+2)^2}$

$\boxed{7 = A(x^2+2)^2 + (Bx+C)(x+1)(x^2+2) + (Dx+E)(x+1)}$ — (1)

$7 = A(x^4+2x^2(2)+4) + (Bx^2+Bx+Cx+C)(x^2+2) + (Dx^2+Dx+Ex+E)$

$7 = A(x^4+4x^2+4) + (Bx^4+Bx^3+Cx^3+Cx^2+2Bx^2+2Bx+2Cx+2C) + (Dx^2+Dx+Ex+E)$

$7 = A(x^4+4x^2+4) + (Bx^4+Bx^3+Cx^3+Cx^2+2Bx^2+2Bx+2Cx+2C) + (Dx^2+Dx+Ex+E)$

$7 = x^4(A+B) + x^2(4A+C+2B+D) + x^3(B+C) + x(2B+2C+D+E) + 4A+2C+E$

$A+B=0$; $4A+C+2B+D=0$; $B+C=0$; $2B+2C+D+E=0$; $4A+2C+E=7$

[Hint put $x=-1$ in eq (1)]

$\boxed{A=7/9}$ $\boxed{B=-7/9}$ $\boxed{C=7/9}$ $\boxed{D=-7/3}$ $\boxed{E=7/3}$

$\int \frac{7}{(x+1)(x^2+2)^2} = \boxed{\frac{7}{9}\int \frac{dx}{x+1} - \frac{7}{9}\int \frac{x-1}{(x^2+2)} - \int \frac{7x-7}{3(x^2+2)^2}}$

Q20) $\int \frac{48}{(x-2)(x^2+3)^2}$

$\frac{48}{(x-2)(x^2+3)^2} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+3)} + \frac{Dx+E}{(x^2+3)^2}$

$48 = A(x^2+3)^2 + (Bx+C)(x-2)(x^2+3) + (Dx+E)(x-2)$

$48 = A(x^4+6x^2+9) + (Bx^2-2Bx+Cx-2C)(x^2+3) + (Dx^2-2Dx+Ex-2E)$

$\boxed{A=1}$ $\boxed{B=-1}$ $\boxed{C=-2}$ $\boxed{D=-7}$ $\boxed{E=-14}$

$\int \frac{48}{(x-2)(x^2+3)^2} = \boxed{\int \frac{dx}{(x-2)} - \int \frac{x+2}{(x^2+3)}dx - \int \frac{7x+14}{(x^2+3)^2} dx}$