Triple Integral Practice

To Set Up A Triple Integral

1. Write down all the conditions (boundary surfaces). Try to visualize the 3D shape if you can.
2. Find the curves of intersections of the boundary surfaces.
3. Make a choice of which innermost variable you want in the integral. Look for a variable that has only two boundary surfaces (the variable only appears in two of the conditions).
4. Then draw the projection region, D, on the plane given by the other two variables.
   (a) Draw all boundaries from the conditions that involve only these two variables.
   (b) Draw all curves of intersection that involve only these two variables (these are only needed if they occur inside the region given by the other boundaries).
5. Then use the techniques of 15.3 and 15.4 to describe D.

Practice Problems (solutions follow)

For each of the following, set up the triple integral: $\iiint\limits_E f(x,y,z)\,dV$.

1. E lies under the plane $z = 1 + x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$ and $x = 1$.
2. E is bounded by the cylinder $y^2 + x^2 = 9$ and the planes $z = 0$, $y = 3z$, and $x = 0$ in the first octant.
3. E is bounded by $x = 3z^2$ and the planes $x = y$, $y = 0$, and $x = 12$.

Solutions

1. (a) Bounding surfaces: $z = 1 + x + y$, $y = \sqrt{x}$, $y = 0$, $x = 1$, and $z = 0$ (this last one because it is 'above' the xy-plane).
   (b) Curves of intersection (these aren't needed in this problem, but I am showing you how you would find all the intersections):
      i. $z = 1 + x + y$ and $z = 0$ intersect when $0 = 1 + x + y$ to give $y = -1 - x$.
      ii. $z = 1 + x + y$ and $x = 1$ intersect when $z = 2 + y$.
      iii. $y = \sqrt{x}$ and $x = 1$ intersect when $y = 1$.
      iv. $z = 1 + x + y$ and $y = 0$ intersect when $z = 1 + x$. $z = 1 + x + y$ and $y = \sqrt{x}$ intersect when $z = 1 + x + \sqrt{x}$ (or if you prefer, when $z = 1 + y^2 + y$).
   (c) There are only two surfaces involving z! Use z as the innermost integral.
   (d) Thus, $0 \le z \le 1 + x + y$.
   (e) Now draw the xy-region bounded by $y = \sqrt{x}$, $y = 0$, and $x = 1$ (the intersection of the z equations is $y = -1 - x$ which occurs outside these other boundaries).
   (f) You can describe this 2D region either as a top/bottom or left/right. Here are the answers each give:

$$\int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} f(x,y,z)\,dzdydx \quad \text{or} \quad \int_0^1 \int_{y^2}^1 \int_0^{1+x+y} f(x,y,z)\,dzdxdy.$$

2. (a) Bounding surfaces: $y^2 + x^2 = 9$, $z = 0$, $y = 3z$, and $x = 0$.

   (b) Curves of intersection (these aren't all needed in this problem, but I am showing you how you would find all the intersections):

      i. $y^2 + x^2 = 9$ and $x = 0$ intersect when $y = 3$.

      ii. $y = 3z$ and $z = 0$ intersect when $y = 0$.

      iii. $y^2 + x^2 = 9$ and $y = 3z$ intersect when $(3z)^2 + x^2 = 9$.

   (c) There are only two conditions on each variable, so you could use any of them. However, looking at intersections, the y equations is complicated, so I wouldn't choose y. Let's try x.

   (d) Thus, $0 \leq x \leq \sqrt{9 - y^2}$

   (e) Now draw the yz-region bounded by $z = 0$ and $y = 3z$ (the intersection of the x equations is $y = 3$ which is needed to determine the region).

   (f) You can describe this 2D region either as a top/bottom or left/right. Here are the answers each give:

$$\int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} f(x, y, z)\, dxdzdy \quad \text{or} \quad \int_0^1 \int_{3z}^3 \int_0^{\sqrt{9-y^2}} f(x, y, z)\, dxdydz.$$

3. (a) Bounding surfaces: $x = 3z^2$, $x = y$, $y = 0$, $x = 12$.

   (b) Curves of intersection:

      i. $x = y$ and $x = 12$ intersect when $y = 12$.

      ii. $x = 3z^2$ and $x = 12$ intersect when $z = \pm 2$.

      iii. $x = y$ and $y = 0$ intersect when $x = 0$.

   (c) There are only three conditions on x, two conditions on y and 'one' condition on z. Let's try y.

   (d) Thus, $0 \leq y \leq x$

   (e) Now draw the xz-region bounded by $x = 3z^2$ and $x = 12$ (the intersection of the y equations is $x = 0$ which is not needed as the region is already determined by the given bounds).

   (f) You can describe this 2D region either as a top/bottom or left/right. Here are the answers each give:

$$\int_{-2}^2 \int_{3z^2}^{12} \int_0^x f(x, y, z)\, dydxdz \quad \text{or} \quad \int_0^{12} \int_{-\sqrt{x/3}}^{\sqrt{x/3}} \int_0^x f(x, y, z)\, dydzdx.$$