Farkaleet Series

KEY TO
Differential Equations and Fourier Series

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PREFACE

"KEY to DIFFERENTIAL EQUATIONS AND FOURIER SERIES" is in your hands. The need of writing this KEY arose because many of our students and other readers felt difficulties in solving the problems given in the worksheets. Keeping in mind the difficulties of the readers, I along with my Ex-colleagues Mr. Asif Ali Shaikh and Ms. Sania Nizamani (Both from Mehran Engineering University, Jamshoro) decided to write guide books on different textbooks that we wrote couple of years ago.

We are sure that readers will find this KEY very much useful. In some problems, direct steps have been taken, for example, the use of Partial Fractions, Integration by Parts; etc is made directly because it is hoped that readers are familiar with such techniques.

There are few problems of worksheets of the textbook whose data were wrongly entered. The problems are corrected and correct solutions are given in the KEY. Such corrections will also be incorporated in the textbook in the next print.

Any suggestion from the readers will highly be appreciated to improve the standard of the KEY.

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Chapter 1

INTRODUCTION TO DIIFERENTIAL EQUATIONS

WORKSHEET 01

1. Find the orders and degrees of following differential equations. Also state whether given differential equations are linear or non-linear and that they are ordinary or partial differential equations.

(a) $e^x\,dx + e^y\,dy = 0$ (b) $y'' + n^2 y = 0$ (c) $y = x y' + x/y'$

(d) $[1 + (y')^2]^{3/2} = c^2 (y'')^2$ (e) $y = x y' + a[1 + (y')^2]$

(f) $y''' + 4y'' - 6y' + y = \cos x$ (g) $U_{xx} + U_{yy} = 0$ (h) $(y')^2 = (y'' + y)^{3/2}$

Solution:

Part	Order	Degree	Type	Kind
(a)	1	1	ODE	Non-Linear
(b)	2	1	ODE	Linear
(c)	1	2	ODE	Non-Linear
(d)	2	4	ODE	Non-Linear
(e)	1	2	ODE	Non-Linear
(f)	3	1	ODE	Linear
(g)	2	1	PDE	Linear
(h)	2	3	ODE	Non-Linear

2. Verify that:

(a) $y = a + be^{-2x} + e^x/3$ is the general solution of equation $y'' + 2y' = e^x$

(b) $y^2 = (x + 5)^3$ is general solution of equation $27y - 8(y')^3 = 0$

Solution: (a) Given $y = a + be^{-2x} + e^x/3 \Rightarrow y' = -2be^{-2x} + e^x/3$ and $y'' = 4be^{-2x} + e^x/3$.

Putting these in given differential equation, we get:

$y'' + 2y' = 4be^{-2x} + e^x/3 + 2(-2be^{-2x} + e^x/3) = 4be^{-2x} + e^x/3 - 4be^{-2x} + 2e^x/3 = e^x$. Proved

(b) Given $y^2 = (x + 5)^3 \Rightarrow 2yy' = 3(x + 5)^2 \Rightarrow 2y' = 3(x + 5)^2/y = 3(x + 5)^2/(x + 5)^{3/2}$

$\Rightarrow 2y' = 3(x + 5)^{1/2}$. Cubing both sides, we get:

$(2y')^3 = 27[(x + 5)^{1/2}]^3 = 27[(x + 5)^3]^{1/2} = 27(y^2)^{1/2} = 27y \Rightarrow 27y - 8(y')^3 = 0$. Proved

3. Obtain the differential equations by eliminating an arbitrary constant(s) from the following equations.

Solution: (a) Given equation contains two arbitrary constants hence we differentiate it twice. Now, $y = ax^3 + bx^2$ (1) $y' = 3ax^2 + 2bx$ (2) $y'' = 6ax + 2b$ (3)

Multiply (1) by 3 and (2) x and subtracting, we obtain

$$xy' - 3y = -bx^2 \qquad (4)$$

Now multiply (1) by 6 and (3) by x^2, we get

$x^2y`` - 6y = -5b\,x^2 = 5(-bx^2) = 5(xy` - 3y)$ using (4)

Simplifying, we get: $x^2y`` - 5xy` + 15y = 0$

(b) Given equation contains two arbitrary constants hence we differentiate it twice. Now,

$xy = A\,e^x + B\,e^{-x}$ (1)

➔ $xy` + y = Ae^x - Be^{-x}$ (2)

and $xy`` + y` + y` = Ae^x + Be^{-x}$ (3)

From (1) and (3), we have: $xy`` + y` + y` = y \Rightarrow xy`` + 2y` + y = 0$

(c) Given equation contains one arbitrary constant hence we differentiate it once. Now,

$y^2 = 4a\,(x + a)$ (1)

➔ $2yy` = 4a \Rightarrow a = yy`/2.$

Put this in (1), we get: $y^2 = 2yy`\,(x + yy`/2) \Rightarrow y^2 = yy`\,(2x + yy`)$

$\Rightarrow$ $y = 2xy` + y(y`)^2 = 0$

(d) Given equation contains three arbitrary constants hence differentiating thrice the given equation, we egt

$y = a\,e^{2x} + b\,e^{-3x} + c\,e^x$ (1) $y` = 2ae^{2x} - 3be^{-3x} + ce^x$ (2)

$y`` = 4ae^{2x} + 9be^{-3x} + ce^x$ (3) $y``` = 8ae^{2x} - 27be^{-3x} + ce^x$ (4)

Subtracting (1) from (2), we get: $y` - y = ae^{2x} - 4be^{-3x}$ (5)

Also subtracting (1) from (3), we get: $y`` - y = 3ae^{2x} + 8be^{-3x}$ (6)

Now, subtracting (1) from (4), we get: $y``` - y = 7ae^{2x} - 28be^{-3x}$ (7)

Multiplying (5) by 7 and subtracting from (7), we get: $y``` - 7y` + 6y = 0$

This is the required differential equation.

(e) Given equation contains two arbitrary constants hence differentiating twice, we get

$A\,x^2 + B\,y^2 = 1$ (1)

$2Ax + 2Byy` = 0 \Rightarrow Ax + Byy` = 0$ (2)

$A + B(yy`` + y`.y`) = 0 \Rightarrow A + B[yy`` + (y`)^2] = 0$ (3)

Multiplying (2) by x and subtracting it from (1), we obtain:

$By(y - xy`) = 1 \Rightarrow B = 1/y(y - xy`)$ (4)

Now, multiplying (3) by x^2 and subtracting it from (1), we get:

$B[y^2 - x^2\{yy`` + (y`)^2\}] = 1 \Rightarrow B = 1/[y^2 - x^2\{yy`` + (y`)^2\}]$ (5)

Equating (4) and (5) and simplifying, we have:

$[y^2 - x^2\{yy`` + (y`)^2\}] = y(y - xy`) \Rightarrow x[yy`` + (y`)^2] - yy` = 0.$

This is the required differential equation.

(f) Given equation $y = a\sin(x + 3)$ contains one arbitrary constant hence differentiating it once, we get: $y` = a\cos(x + 3)$ (1)

From given equation, $a = y/\sin(x + 3)$. Put this in (1), we obtain,

$y` = y\tan(x + 3)$

This is the required differential equation.

(g) Given equation $y = \sqrt{6x + c} \Rightarrow y^2 = 6x + c$ contains one arbitrary constant hence differentiating it once, we get: $yy` = 6$

This is the required differential equation.

(h) Given equation $y = x + c\,e^x$ contains one arbitrary constant hence differentiating once, we get: $y` = 1 + ce^x$ (1)

From given equation, $y = x + ce^x \Rightarrow ce^x = y - x$. Put this in (1), we get:

$y` = 1 + y - x \Rightarrow y` - y = 1 - x$

This is the solution of given differential equation.

(i) Given equation $y = (x^3 + c)\,e^{-3x}$ contains one arbitrary constant hence differentiating once, we get: $y` = -3e^{-3x}(x^3 + c) + 3x^2e^{-3x} = e^{-3x}[3x^2 - 3(x^3 + c)]$ (1)

From given equation, $(x^3 + c) = y/e^{-3x} = ye^{3x}$. Put this in (1), we get:

$$y` = e^{-3x}[3x^2 - 3ye^{3x}] \quad \Rightarrow \quad y` + 3y = 3x^2e^{-3x}$$

This is the solution of given differential equation.

(j) Given equation $a\,x + \ln y = y + b,\ y > 0$ contains two arbitrary constants hence differentiating twice, we get: $a.1 + y`/y = y` + 0 \Rightarrow a + y`/y = y`$. Differentiating again, we get:

$$0 + \frac{yy``-y`y`}{y^2} = y`` \qquad \Rightarrow \qquad y^2y`` = yy`` + \left(y`\right)^2$$

This is the required differential equation.

(k) Given equation $y^2 - 2a\,y + x^2 = a^2$ contains only one arbitrary constant hence differentiating once, we get: $2yy` - 2a\,y` + 2x = 0$. Dividing by 2, we have:

$$yy` - ay` + x = 0 \quad \Rightarrow a = (yy` + x)/y`.$$

Put this value of a in the given equation $y^2 - 2a\,y + x^2 = a^2$, we get:

$$y^2 - 2y\left(\frac{yy`+x}{y`}\right) + x^2 = \left(\frac{yy`+x}{y`}\right)^2$$

This is the required differential equation.

(*l*) Given equation $y = ae^x + b\ln x + cx + d$ contains four arbitrary constants hence,

Differentiating w.r.t x, $y` = ae^x + b/x + c$ (1)

Differentiating again, $y`` = ae^x - b/x^2$ (2)

Differentiating again, $y``` = ae^x + 2b/x^3$ (3)

Differentiating again, $y^{iv} = ae^x - 6b/x^4$ (4)

Subtracting (2) from (3), we obtain :

$$y```-y`` = \frac{2b}{x^3} + \frac{b}{x^2} = b\left(\frac{x+2}{x^3}\right) \Rightarrow b = x^3\left(\frac{y```-y``}{x+2}\right) \qquad (5)$$

Subtracting (2) from (4), we obtain :

$$y^{iv} - y`` = \frac{-6b}{x^4} + \frac{b}{x^3} = b\left(\frac{x^2-62}{x^4}\right) \Rightarrow b = x^4\left(\frac{y^{iv}-y``}{x^2-6}\right) \qquad (6)$$

From (5) and (6), we have:

$$x^3\left(\frac{y```-y``}{x+2}\right) = x^4\left(\frac{y^{iv}-y``}{x^2-6}\right) \Rightarrow \left(x^2-6\right)\left(y```-y``\right) = x\left(x+2\right)\left(y^{iv}-y``\right)$$

(m) Given equation $y = a\sinh 2x + b\cosh 2x$ contains two arbitrary constants hence differentiating twice, we get:

$y` = 2a\cosh 2x + 2b\sinh 2x$ and

$y`` = 4a\sinh 2x + 4b\cosh 2x = 4(a\sinh 2x + b\cosh 2x) = 4y$ **[This is from given equation]**

or $y`` - 4y = 0$. This is the required differential equation.

(n) Given equation $y = a\sin x + b\cos x + x\sin x$ contains two arbitrary constants hence differentiating twice, we get:

$y` = a\cos x - b\sin x + x\cos x + \sin x$ and

$y`` = -a\sin x - b\cos x + x(-\sin x) + \sin x + \cos x$

$\quad = -(a\sin x + b\cos x + x\sin x) + \sin x + \cos x = -y + \sin x + \cos x$

or $y`` + y = \sin x + \cos x$. This is the required differential equation.

4. Solve following initial/boundary problems:

(a) $y` = -x/y,\ y(3) = 4$ given that $x^2 + y^2 = r^2$ is the solution of given differential equation.

Solution: Given that $x^2 + y^2 = r^2$ is the solution of differential equation, $y` = -x/y$, putting $x = 3$ and $y = 4$, we get: $9 + 16 = r^2$ giving $r^2 = 25$. Put this in the general solution, we get the particular solution of given differential equation as: $x^2 + y^2 = 25$.

(b) $y`` - y` - 12y = 0$, $y(0)= -2$, $y`(0) = 6$, given that $y = a\, e^{4x} + b\, e^{-3x}$ is the solution of given differential equations.

Solution: Given that $y = a\, e^{4x} + b\, e^{-3x}$ is the general solution of differential equation $y`` - y` - 12y = 0$. Now,
$y = a\, e^{4x} + b\, e^{-3x}$ or $y` = 4a\, e^{4x} -3b\, e^{-3x}$. Putting $x = 0$ and $y = -2$ and $y` = 6$, we get:
$a + b = -2$ and $4a – 3b = 6$. Solving these equations simultaneously, we get $a = 0$ and $b = -2$. Hence, the particular solution of given differential equation is $y = -2e^{-3x}$.

(c) $y`` + y = 0$, $y(0) = 1$, $y`(\pi) = 1$, given that $y = a \sin x + b \cos x$ is the solution of given differential equations.

Solution: Given that $y = a \sin x + b \cos x$ is general solution of differential equation $y`` + y = 0$. Now,
$y = a \sin x + b \cos x$ or $y` = a \cos x - b \sin x$. Putting $x = 0$, $y = 1$ and $x = \pi$, $y` = 1$, we get: $a \sin0 + b \cos0 = 1$ or $b = 1$ and $a \cos \pi - b \sin \pi = 1$ or $a = -1$. Hence, the particular solution of given differential equation is $y = - \sin x + \cos x$.
NOTE: $\sin0 = \sin \pi = 0$ and $\cos0 = 1$, $\cos \pi = -1$.

(d) $y`` -4y` + 3y = 0$, $y(0) = 1$, $y(1) = 1$, given that $y = a\, e^{x} + b\, e^{3x}$ is the solution of given differential equations.

Solution: Given that $y = a\, e^{x} + b\, e^{3x}$ is the general solution of differential equation $y`` - 4y` + 3y = 0$. Now,
$y = a\, e^{x} + b\, e^{3x}$ or $y` = a\, e^{x} + 3b\, e^{3x}$. Putting $x = 0$ and $y = 1$ and $x = 1$, $y` = 1$, we get:
$a + b = 1$ and $ae + 3be^3 = 1$. Solving these equations simultaneously, we get
$a = 3e^3/(3e^3 – 1)$ and $b = -1/(3e^3 -1)$. Putting these in the general solution, we get:

$$y = \frac{3e^{3+x}}{3e^3 - 1} - \frac{e^{1+x}}{3e^3 - 1} = \left(\frac{3e^{x+3} - e^{x+1}}{3e^3 - 1}\right)$$

This is the particular solution of given differential equation.

Chapter 2

FIRST ORDER DIFFERENTIAL EQUATIONS

WORKSHEET 02

1. Solve the following differential equations by separating the variables method.

(a) $y - x\,y' = (y^2 + ay')$

Solution: Rearranging, we get: $y'(x + a) = y - y^2 = -y(y - 1)$

$\therefore \frac{dy}{dx} = -\frac{y(y-1)}{(x+a)}$. Separating the variables, we get

$$\frac{1}{y(y-1)}dy = -\frac{1}{x+a}dx \Rightarrow \left[\frac{-1}{y} + \frac{1}{y-1}\right]dy = -\frac{1}{x+a}dx.\text{ Integrating, we get}$$

$$-\int \frac{1}{y}dy + \int \frac{1}{y-1}dy = -\int \frac{1}{x+a}dx + \ln c \Rightarrow -\ln y + \ln(y-1) = -\ln(x+a) + \ln c$$

$$\ln\left(\frac{y-1}{y}\right) = \ln\left(\frac{c}{x+a}\right) \Rightarrow \frac{y-1}{y} = \frac{c}{x+a}\ [\text{By taking anti log}] \Rightarrow (y-1)(x+a) = cy$$

This is the general solution of given differential equation.

(b) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

Solution: Rearranging, we get:

$\frac{dy}{dx} = -\frac{\sec^2 x \tan y}{\sec^2 y \tan x}$. Separating the variables, we get

$\frac{\sec^2 y}{\tan y}dy = -\frac{\sec^2 x}{\tan x}dx$. Integrating, we get

$$\int \frac{\sec^2 y}{\tan y}dy = -\int \frac{\sec^2 x}{\tan x}dx + \ln c \Rightarrow \ln \tan y = -\ln \tan x + \ln c = \ln\left(\frac{c}{\tan x}\right)$$

$$\tan y = -\left(\frac{c}{\tan x}\right) \Rightarrow \tan x \tan y + c = 0 \qquad \text{NOTE: } \int \frac{f'(x)}{f(x)}dx = \ln[f(x)]$$

This is the general solution of given differential equation.

(c) $3\,e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$

Solution: Rearranging, we get: $\frac{dy}{dx} = \frac{3e^x \tan y}{(e^x - 1)\sec^2 y}$. Separating the variables, we get

$\frac{\sec^2 y}{\tan y}dy = 3\frac{e^x}{e^x - 1}dx$. Integrating, we get

$$\int \frac{\sec^2 y}{\tan y} dy = 3\int \frac{e^x}{e^x - 1} dx + \ln c \quad \Rightarrow \ln \tan y = 3\ln\left(e^x - 1\right) + \ln c = \ln c\left(e^x - 1\right)^3$$

$$\Rightarrow \tan y = c\left(e^x - 1\right) \qquad \text{NOTE : } \int \frac{f`(x)}{f(x)} dx = \ln\left[f(x)\right]$$

This is the general solution of given differential equation.

(d) $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$

Solution: Rearranging, we get: $\dfrac{dy}{dx} = -\dfrac{y^2(x+1)}{x^2(1-y)} = \dfrac{y^2(x+1)}{x^2(y-1)}$.

Separating the variables, we get : $\dfrac{y-1}{y^2} dy = \dfrac{x+1}{x^2} dx$. Integrating, we get

$$\int \left[\frac{1}{y} - y^{-2}\right] dy = \int \left[\frac{1}{x} + x^{-2}\right] dx + c \quad \Rightarrow \ln y + \frac{1}{y} = \ln x - \frac{1}{x} + c \qquad \Rightarrow \ln\left(\frac{y}{x}\right) + \frac{1}{y} + \frac{1}{x} = c$$

This is the general solution of given differential equation.

(e) $(x - xy^2)\, dx + (y - yx^2)\, dy = 0$

Solution: Rearranging, we get:

$\dfrac{dy}{dx} = \dfrac{-x(y^2-1)}{-y(x^2-1)} = \dfrac{x(y^2-1)}{y(x^2-1)}$. Separating the variables, we get :

$\dfrac{y}{y^2-1} dy = \dfrac{x}{x^2-1} dx$. Dividing 2 and integrating, we get

$\dfrac{1}{2}\displaystyle\int \frac{2y}{y^2-1} dy = \frac{1}{2}\int \frac{2x}{x^2-1} dx + \frac{1}{2}\ln c$. Integrating and multiplying by 2, we have

$$\Rightarrow \ln\left(y^2 - 1\right) = \ln\left(x^2 - 1\right) + \ln c \quad \Rightarrow \ln\left(y^2 - 1\right) = \ln c\left(x^2 - 1\right) \quad \Rightarrow \left(y^2 - 1\right) = c\left(x^2 - 1\right)$$

This is the general solution ofgiven differential equation.

NOTE : $\dfrac{1}{2}\ln c$ is constant of integration.

(f) $x\,(1 + y^2)\, dx + y\,(1 + x^2)\, dy = 0$

Solution: Rearranging, we get:

$\dfrac{dy}{dx} = \dfrac{-x(1+y^2)}{y(1+x^2)}$. Separating the variables, we get :

$\dfrac{y}{1+y^2} dy = -\dfrac{x}{1+x^2} dx$. Dividing by 2 and integrating, we get

$\dfrac{1}{2}\displaystyle\int \frac{2y}{1+y^2} dy = -\frac{1}{2}\int \frac{2x}{1+x^2} dx + \frac{1}{2}\ln c$. Integrating and multiplying by 2, we have

$$\Rightarrow \ln\left(1 + y^2\right) = -\ln\left(1 + x^2\right) + \ln c \quad \Rightarrow \ln\left(1 + y^2\right) = \ln \frac{c}{\left(1 + x^2\right)} \quad \Rightarrow \left(1 + y^2\right) = \frac{c}{\left(1 + x^2\right)}$$

$$\Rightarrow \left(1 + y^2\right)\left(1 + x^2\right) = c \qquad \text{NOTE : } \frac{1}{2}\ln c \text{ is constant of integration.}$$

This is the general solution of given differential equation.

(g) $(e^x + 1) \cos y \, dx + e^x \sin y \, dy = 0$

Solution: Rearranging, we get: $\frac{dy}{dx} = \frac{-(e^x+1)\cos y}{e^x \sin y}$. Separating the variables, we get:

$$\frac{-\sin y}{\cos y} dy = \frac{e^x+1}{e^x} dx = \left(\frac{e^x}{e^x} + \frac{1}{e^x}\right) dx.$$ Integrating, we get

$$\int \frac{-\sin y}{\cos y} dy = \int \left(1 + e^{-x}\right) dx + \ln c \qquad \Rightarrow \ln \cos y = x - e^{-x} + c$$

This is the general solution of given differential equation.

(h) $y` \tan y = \sin(x + y) + \sin(x - y)$

Solution: Using the formula: $\sin a + \sin b = 2 \sin(a + b)/2 \cos(a - b)/2$, we get

$$\tan y \frac{dy}{dx} = 2\left[\sin\left(\frac{x+y+x-y}{2}\right)\cos\left(\frac{x+y-x+y}{2}\right)\right] = 2\sin x \cos y.$$

Separating the variables, we get

$$\frac{\tan y}{\cos y} dy = 2\sin x \, dx \quad \Rightarrow \tan y \sec y \, dy = \sin x \, dx.$$ Integrating, we get:

$$\int \tan y \sec y \, dy = \int \sin x \, dx + c \quad \Rightarrow \sec y = -\cos x + c$$

This is the general solution of given differential equation.

(i) $\ln(y`) = a x + by$

Solution: $\ln(y`) = a x + by$ or $y` = e^{ax + by}$

$$\frac{dy}{dx} = e^{ax} e^{by}.$$ Separating the variables, we get $e^{-by} dy = e^{ax} dx$. Integrating,

$$\int e^{-by} dy = \int e^{ax} dx + c \quad \Rightarrow \ -\frac{e^{by}}{b} = \frac{e^{ax}}{a} + c$$

This is the general solution of given differential equation.

(j) $(x + y)^2 y` = a^2$

Solution: Putting, $x + y = z$ or $1 + y` = z`$ or $y` = z` - 1$.

Thus given equation becomes: $z^2(z`-1) = a^2 \ \Rightarrow z`-1 = \frac{a^2}{z^2} \quad \Rightarrow \frac{dz}{dx} = 1 + \frac{a^2}{z^2} = \frac{\left(z^2+a^2\right)}{z^2}$

Separating the variables, we get: $\frac{z^2}{\left(z^2+a^2\right)} dz = dx \quad \Rightarrow \frac{\left(z^2+a^2\right)-a^2}{\left(z^2+a^2\right)} dz = dx$

$$\left[\frac{\left(z^2+a^2\right)}{\left(z^2+a^2\right)} - \frac{a^2}{\left(z^2+a^2\right)}\right] dz = dx \ \Rightarrow \left[1 - \frac{a^2}{\left(z^2+a^2\right)}\right] dz = dx.$$ Integrating, we get:

$$\int 1 \ dz - a^2 \int \frac{1}{z^2+a^2} dz = \int 1 \ dx + c \quad \Rightarrow z - a^2 . \frac{1}{a} \tan^{-1}\left(\frac{z}{a}\right) = x + c.$$ Putting $z = x + y$, we get

$$x + y - a \tan^{-1}\left(\frac{x+y}{a}\right) = x + c \quad \Rightarrow y - a \tan^{-1}\left(\frac{x+y}{a}\right) = c$$

This is the general solution of given differential equation.

(k) $y' = x^2(1 + y^2)$; when $x = 0$ and $y = \pi/4$

Solution: Separating the variables and integrating, we get

$$\int \frac{1}{(y^2+1)}dy = \int x^3 dx + c \quad \Rightarrow \tan^{-1}y = \frac{x^3}{3} + c.$$

This is the general solution of given differential equation. Now, putting $x = 0$ & $y = \pi/4$, we get: $c = 1$. Thus, particular solution is $\tan^{-1}y = \frac{x^3}{3} + 1$

(l) $\cos u\, du - \sin v\, dv = 0$; $(u, v) = (\pi/2, \pi/2)$

Solution: This equation is already in separable form hence integrating, we get

$\int \cos u\, du + \int -\sin v\, dv = c \qquad \Rightarrow \sin u + \cos v = c$. Putting $u = \pi/2$ and $v = \pi/2$, we get

$$\sin\frac{\pi}{2} + \cos\frac{\pi}{2} = c \quad \Rightarrow 1 + 0 = c \quad \Rightarrow c = 1. \text{ Thus, } \sin u + \cos v = 1$$

This is a aprticular solution of given differential equation.

(m) $y' = 1 + \tan(y - x)$

Solution: Putting, $y - x = z \Rightarrow y' - 1 = z' \Rightarrow y' = 1 + z'$.

Thus given equation becomes: $1 + z' = 1 + \tan z \Rightarrow \frac{dz}{dx} = \tan z \Rightarrow \frac{dz}{\tan z} = dx$.

Integrating, we get $\int \cot z\, dz = \int 1\, dx + c \quad \Rightarrow \ln(\sin z) = x + c \Rightarrow \sin z = e^{x+c} = e^x.e^c$

Or $\sin z = Ce^x$. Putting $z = y - x$, we get $\sin(y - x) = Ce^x$ is general solution.

(n) $(4x + y)\, y' = 1$

Solution: Putting, $4x + y = z \Rightarrow 4 + y' = z' \Rightarrow y' = z' - 4$.

Thus given equation becomes: $z(z' - 4) = 1 \Rightarrow z' - 4 = \frac{1}{z} \quad \Rightarrow \frac{dz}{dx} = 4 + \frac{1}{z} = \frac{4z+1}{z}$.

Separating the variables, we get: $\frac{z}{4z+1}dz = dx \quad \Rightarrow \frac{1}{4}\frac{4z}{(4z+1)}dz = dx$

$\Rightarrow \frac{1}{4}\left[\frac{(4z+1)-1}{4z+1}\right]dz = dx$. Integrating, we get $\frac{1}{4}\int\left[1 - \frac{1}{4z+1}\right]dz = \int 1\, dx + c_1$

$\Rightarrow \frac{1}{4}\left[\int 1 dz - \frac{1}{4}\int \frac{4}{4z+1}dz\right] = \frac{1}{4}\left[z - \frac{1}{4}\ln(4z+1)\right] = x + c_1$. Putting $z = 4x + y$, we get

$$\frac{1}{4}\left[4x + y - \frac{1}{4}\ln(16x + 4y + 1)\right] = x + c \Rightarrow \left[4x + y - \frac{1}{4}\ln(16x + 4y + 1)\right] = 4x + 4c_1$$

$$\Rightarrow y - \frac{1}{4}\ln(16x + 4y + 1) = c \Rightarrow 4y = \ln(16x + 4y + 1) + C \;\; (C = 4c).$$

This is the general solution of given differential equation.

(o) $y' = x(2 \ln x + 1)/(\sin y + y \cos y)$

Solution: Separating the variables, we get

$(\sin y + y\cos y)\, dy = (2x\ln x + x)\, dx$. Integrating, we get

$$\int \sin y\, dy + \int y\cos y\, dy = 2\int x\ln x\, dx + \int x\, dx + c$$

$$-\cos y + \left[y\sin y - \int 1.\sin y\, dy\right] = 2\left[\ln x.\frac{x^2}{2} - \int \frac{1}{x}.\frac{x^2}{2}dx\right] + \frac{x^2}{2} + c$$

$$-\cos y + y\sin y + \cos y = x^2 \ln x - \frac{x^2}{2} + \frac{x^2}{2} + c \;\;\Rightarrow\;\; y\sin y = x^2 \ln x - 2x + c.$$

This the general solution of given differential equation.

(p) y\`= [sin x + lnx/x]/[cos y – sec^{2}y]

Solution: Separating the variables, we get

$(\cos y - \sec^2 y)\, dy = \left(\sin x + \frac{\ln x}{x}\right)dx$. Integrating, we get

$$\int \cos y\, dy - \int \sec^2 y\, dy = \int \sin x\, dx + \int \frac{\ln x}{x} dx + c$$

$\sin y - \tan y = -\cos x + \int \frac{\ln x}{x}\, dx + c.$ Putting $z = \ln x \;\Rightarrow dz = \frac{dx}{x}$

$$\Rightarrow \sin y - \tan y = -\cos x + \int z\, dz + c \;\;\Rightarrow\;\; \sin y - \tan y = -\cos x + \frac{z^2}{2} + c$$

Or, $\sin y - \tan y = -\cos x + \frac{1}{2}(\ln x)^2 + c$

This the general solution of given differential equation.

(q) y\` = e$^{x - y}$ + x^2 e^{-y}

Solution: Rearranging the equation, we get: $y\grave{} = e^x . e^{-y} + x^2 e^{-y} = e^{-y}\left(e^x + x^2\right).$

Sepating the variables, we get

$e^y dy = \left(e^x + x^2\right)dx.$ Integrating, we get : $\int e^y\, dy = \int \left(e^x + x^2\right)dx + c$

$\Rightarrow e^y = e^x + \frac{x^3}{3} + c.$ This the general solution of given differential equation.

(r) y\` = e$^{x - y}$ + 1

Solution: Putting z = x – y giving z\` = 1 – y\` or y\` = 1- z\`. Thus, given equation becomes

1 – z\` = e$^z$ + 1 or z\` = – e^z. Separating the variables and integrating, we get:

$\int e^{-z}\, dz = -\int 1.dx + c \;\;\Rightarrow -e^{-z} = -x + c \;\;\Rightarrow x - e^{(x-y)} = c$

This is the general solution of given differential equation.

(s) y\` = (x + y)^2

Solution: Putting, x + y = z ⇒ 1 + y\` = z\` ⇒ y\` = z\` - 1.

Thus given equation becomes : $(z\grave{}-1) = z^2 \;\;\Rightarrow z\grave{} = 1 + z^2 \quad \Rightarrow \frac{dz}{dx} = 1 + z^2$

Separating the variables, we get : $\frac{1}{(z^2+1)}dz = dx.$ Integrating, we get :

$\int \frac{1}{z^2+1}dz = \int 1\; dx + c \;\;\Rightarrow\;\; \tan^{-1} z = x + c.$ Putting $z = x + y$, we get : $\tan^{-1}(x + y) = x + c$

This is the general solution of given differential equation.

(t) dx + e^x y^2 dy = 0; when x = 0 and y = 1.

Solution: Rearranging, we get:

$y^2\, dy = -\, e^{-x}\, dx$. Integrating, we get

$\int y^2\; dy = -\int e^{-x} dx + c \qquad \Rightarrow y^3/3 = e^{-x} + c.$ Put $x = 0$ and $y = 1 \;\;\Rightarrow c = -2/3$

Thus, $\frac{y^3}{3} = e^{-x} - \frac{2}{3} \Rightarrow y^3 = 3e^{-x} - 2$ is a particular solution.

(u) $y` = \sin(x + y)$

Solution: Putting, $x + y = z \Rightarrow 1 + y` = z` \Rightarrow y` = z` - 1$.

Thus given equation becomes: $(z`-1) = \sin z \Rightarrow \frac{dz}{dx} = 1 + \sin z \Rightarrow \frac{dz}{1+\sin z} = dx.$

Integrating, we get $\int \frac{1}{1+\sin z}\, dz = \int 1\ dx + c$

$\Rightarrow \int \frac{1}{1+\sin z} \times \frac{1-\sin z}{1-\sin z} dz = x + c \Rightarrow \int \frac{1-\sin z}{1-\sin^2 z} dz = x + c \Rightarrow \int \frac{1-\sin z}{\cos^2 z} dz = x + c$

$\Rightarrow \int \frac{1}{\cos^2 z} dz - \int \frac{\sin z}{\cos^2 z} dz = x + c \Rightarrow \int \sec^2 z\, dz - \int \tan z \sec z\ dz = x + c$

$\Rightarrow \tan z - \sec z = x + c.$ Putting $z = x + y$, we get

$\tan(x + y) - \sec(x + y) = x + c$

This is the general solution of given differential equation.

2. Solve the following homogeneous differential equations

(a) $(x^2 - y^2)\, dx + 2xy\, dy = 0$

Solution: Rearranging, we get

$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$. Putting $y = vx \Rightarrow y` = v.1 + x.v`$, we obtain

$$v + xv` = \frac{v^2x^2 - x^2}{2vx^2} = \frac{x^2\left(v^2 - 1\right)}{2vx^2} = \frac{\left(v^2 - 1\right)}{2v} \Rightarrow xv` = \frac{\left(v^2 - 1\right)}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{v^2 + 1}{2v}$$

Separating the variables and integrating, we get: $\int \frac{2v}{v^2 + 1} dv = -\int \frac{1}{x} dx + \ln c$

$\Rightarrow \ln\left(v^2 + 1\right) = -\ln x + \ln c = \ln \frac{c}{x}$. Taking antilog, we get: $\left(v^2 + 1\right) = \frac{c}{x}$

Putting $y = v/x$: $\left(\frac{y^2}{x^2} + 1\right) = \frac{c}{x} \Rightarrow \left(\frac{y^2 + x^2}{x^2}\right) = \frac{c}{x} \Rightarrow \left(y^2 + x^2\right) = cx$

This is the general solution of given differential equation.

(b) $(x^2 + y^2)\, dx - 2xy\, dy = 0$

Solution: Rearranging, we get

$\frac{dy}{dx} = \frac{y^2 + x^2}{2xy}$. Putting $y = vx \Rightarrow y` = v.1 + x.v`$, we obtain

$$v + xv` = \frac{v^2x^2 + x^2}{2vx^2} = \frac{x^2\left(v^2 + 1\right)}{2vx^2} = \frac{\left(v^2 + 1\right)}{2v} \Rightarrow xv` = \frac{\left(v^2 + 1\right)}{2v} - v = \frac{v^2 + 1 - 2v^2}{2v} = -\frac{v^2 - 1}{2v}$$

Separating the variables and integrating, we get: $\int \frac{2v}{v^2 - 1} dv = -\int \frac{1}{x} dx + \ln c$

$\Rightarrow \ln\left(v^2 - 1\right) = -\ln x + \ln c = \ln \frac{c}{x}$. Taking antilog, we get: $\left(v^2 - 1\right) = \frac{c}{x}$

Putting $y = v/x : (y^2/x^2 - 1) = c/x \Rightarrow \left(y^2 - x^2\right) = cx$, the solution of given d.e.

(c) $x^2 y \ dx - (x^3 + y^3)\ dy = 0$

Solution: Rearranging, we get $\dfrac{dy}{dx} = \dfrac{x^2 y}{x^3 + y^3}$.

Putting $y = vx \Rightarrow y` = v.1 + x.v`$, we obtain

$$v + xv` = \frac{vx^3}{x^3 + v^3x^3} = \frac{vx^3}{x^3\left(1+v^3\right)} = \frac{v}{1+v^3} \quad \Rightarrow xv` = \frac{v}{1+v^3} - v = \frac{v - v - v^4}{1+v^3} = -\frac{v^4}{1+v^3}$$

Separating the variables and integrating, we get: $\displaystyle\int \frac{1+v^3}{v^4}dv = -\int \frac{1}{x}dx + \ln c$

$$\Rightarrow \int \left[\frac{1}{v^4} + \frac{1}{v}\right]dv = -\ln x + \ln c = \ln\frac{c}{x} \ \Rightarrow \frac{v^{-3}}{-3} + \ln v = \ln\frac{c}{x} \ \Rightarrow -\frac{1}{3v^3} = \ln\frac{c}{x} - \ln v = \ln\frac{c}{vx}$$

Putting $v = y/x$ and simplifying, we obtain: $-\dfrac{x^3}{3y^3} = \ln\left(\dfrac{c}{y}\right)$

This is the general solution of given differential equation.

(d) $x\,y` - y = (x^2 + y^2)^{1/2}$

Solution: Rearranging, we get

$\dfrac{dy}{dx} = \dfrac{\sqrt{x^2+y^2} + y}{x}$. Putting $y = vx \Rightarrow y` = v.1 + x.v`$, we obtain

$$v + xv` = \frac{\sqrt{x^2 + v^2x^2} + vx}{x} = \frac{x\left(\sqrt{1+v^2} + v\right)}{x} = \left(\sqrt{1+v^2} + v\right)$$

$xv` = \sqrt{v^2+1} + v - v = \sqrt{v^2+1}$. Separating the variables and integrating, we get:

$\displaystyle\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{1}{x}dx + \ln c \Rightarrow \ln\left(v + \sqrt{1+v^2}\right) \Rightarrow \ln x + \ln c = \ln cx$. Taking anti log, we get:

$v + \sqrt{v^2+1} = cx$. Putting $y = v/x$: $\left(\dfrac{y}{x} + \sqrt{\dfrac{y^2}{x^2} + 1}\right) = cx \quad \Rightarrow \left(\dfrac{y + \sqrt{y^2 + x^2}}{x}\right) = cx$

$\Rightarrow y + \sqrt{y^2 + x^2} = cx^2$. This is the general solution of given differential equation.

(e) $y` = (y/x) + \sin(y/x)$

Solution: Putting $y/x = v$ or $y = vx$. Thus, $y` = v + xv`$. Thus given equation becomes $v + xv` = v + \sin v \ \Rightarrow xv` = \sin v$. Separating the variables and integrating, we get:

$\displaystyle\int \frac{1}{\sin v}dv = \int \frac{1}{x}dx + \ln c \ \Rightarrow \int \csc v\, dv = \ln x + \ln c = \ln cx.$

$\Rightarrow \ln\left(\csc v - \cot v\right) = \ln cx$. Taking anti log, we get:

$\csc v - \cot v = cx$. Putting $v = y/x : \csc(y/x) - \cot(y/x) = cx$

This is the general solution of given differential equation.

(f) $y` = (y/x) + \tan(y/x)$

Solution: Putting $y/x = v$ or $y = vx$. Thus, $y` = v + xv`$. Thus given equation becomes

$v + xv` = v + \sin v \ \Rightarrow xv` = \tan v$. Separating the variables and integrating, we get:

$$\int \frac{1}{\tan v} dv = \int \frac{1}{x} dx + \ln c \Rightarrow \int \cot v \, dv = \ln x + \ln c = \ln cx.$$

$\Rightarrow \ln(\sin v) = \ln cx.$ Taking anti log, we get :

$\sin v = cx$. Putting $v = y/x : \sin(y/x) = cx$ which is a general solution.

(g) $(1 + e^{x/y})\,dx + e^{x/y}(1 - x/y)\,dy = 0.$

Solution: Rearranging, we get : $\dfrac{dx}{dy} = \dfrac{e^{x/y}\left(\dfrac{x}{y} - 1\right)}{1 + e^{x/y}}$

Putting $\dfrac{x}{y} = v \Rightarrow x = vy$. Differentiating w.r.t y, we get

$\dfrac{dx}{dy} = v.1 + yv`$. Thus given equation becomes

$$v + yv` = \frac{e^v(v-1)}{1+e^v} \Rightarrow yv` = \frac{e^v(v-1)}{1+e^v} - v = \frac{ve^v - e^v - v - ve^v}{1+e^v} = -\frac{v + e^v}{1+e^v}$$

$$\Rightarrow \frac{1+e^v}{v+e^v} dv = -\frac{1}{y} dy. \text{ Integrating, } \int \frac{1+e^v}{v+e^v} dv = -\int \frac{1}{y} dy + \ln c = -\ln y + \ln c = \ln \frac{c}{y}$$

$$\Rightarrow \ln\left(v + e^v\right) = \ln \frac{c}{y}. \text{ Taking anti log,: } v + e^v = \frac{c}{y} \Rightarrow y\left(v + e^v\right) = c.$$

$$\text{Putting } v = x/y: \quad \frac{x}{y}\left(\frac{x}{y} + e^{x/y}\right) = c \qquad \Rightarrow x\left(x + ye^{x/y}\right) = cy^2$$

This is the general solution of given differential equation.

(h) $[x\cos(y/x) + y\sin(y/x)]y = x\,[y\sin(y/x) - x\cos(y/x)]\,y`$

Solution: Putting $y/x = v$ or $y = vx$ or $y` = v + xv`$. Thus given equation becomes after rearrangement,

$$y` = v + xv` = \frac{vx\left(x\cos v + vx\sin v\right)}{x\left(vx\sin v - x\cos v\right)} = \frac{vx^2\left(\cos v + v\sin v\right)}{x^2\left(v\sin v - \cos v\right)} \Rightarrow xv` = \frac{v\left(\cos v + v\sin v\right)}{\left(v\sin v - \cos v\right)} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{\left(v\sin v - \cos v\right)} = \frac{2v\cos v}{\left(v\sin v - \cos v\right)}.$$

Separating the variables and integrating, we get :

$$\int \frac{v\sin v - \cos v}{2v\cos v} dv = \int \frac{1}{x} dx + \ln c \Rightarrow \frac{1}{2}\int\left[\frac{v\sin v}{v\cos v} - \frac{\cos v}{v\cos v}\right] dv = \ln x + \ln c = \ln cx.$$

$$\Rightarrow \frac{1}{2}\left[\int \tan v\, dv - \int \frac{1}{v} dv\right] = \ln cx \quad \Rightarrow \left[\ln\sec v - \ln v\right] = 2\ln(cx) \quad \Rightarrow \ln\left(\frac{\sec v}{v}\right) = \ln(cx)^2.$$

$$\text{Taking anti log, we get : } \frac{\sec v}{v} = (cx)^2 \Rightarrow \sec v = c^2 v x^2.$$

$$\text{Putting } v = y/x, \text{ we get : } \sec\left(y/x\right) = Cxy \quad \left[C = c^2\right]$$

This is the general solution of given differential equation.

(i) $x\,y` = y[\ln y - \ln x + 1]$ (NOTE: $\ln a - \ln b = \ln(a/b)$)

Solution: Rearranging, we get

$\dfrac{dy}{dx} = \dfrac{y}{x}\left[\ln\left(\dfrac{y}{x}\right)+1\right]$. Putting $\dfrac{y}{x} = v \;\Rightarrow y = vx \quad \Rightarrow y` = v + xv`$. Thus, we get

$v + xv` = v(\ln v + 1) \;\Rightarrow v + xv` = v\ln v + v \quad \Rightarrow x\dfrac{dv}{dx} = v\ln v.$

Separating the variables and integrating, we have

$$\int \frac{1}{v\ln v}dv = \int \frac{1}{x}dx + \ln c \;\Rightarrow \int \frac{1}{v\ln v}dv = \ln x + \ln c = \ln cx \qquad (1)$$

Now put $z = \ln v \;\Rightarrow dz = \dfrac{1}{v}dv$. Thus, (1) becomes :

$\int \dfrac{1}{z}dz = \ln cx \qquad \Rightarrow \ln z = \ln(cx)$. Taking antilog, we get : $z = cx$. Put $z = \ln v = \ln\left(\dfrac{y}{x}\right)$

$\Rightarrow \;\ln\left(\dfrac{y}{x}\right) = cx$. This is the general solution of given differential equation.

3. Solve the following differential equations (Reducible to homogeneous)

(a) y` = [(x + 2y – 3)/(2x + y – 3)]

Solution: Putting x = X + h and y = Y + k or dy/dx = dY/dX. Thus given eq. becomes

$$\frac{dY}{dX} = \frac{X + h + 2Y + 2k - 3}{2X + 2h + Y + k - 3} = \frac{X + 2Y + h + 2k - 3}{2X + Y + 2h + k - 3} \qquad (1)$$

Put : $\quad h + 2k - 3 = 0$ and $2h + k - 3 = 0$ and solving, we get : $h = 1,\ k = 1$

Thus (1) becomes : $\dfrac{dY}{dX} = \dfrac{X + 2Y}{2X + Y} \qquad (2)$

Eq. (2) is homogeneous equation thus putting $Y = VX \;\Rightarrow Y` = V + XV`$, we get

$$V + XV` = \frac{X + VX}{2X + VX} = \frac{X(1+V)}{X(2+V)} = \frac{(1+V)}{(2+V)}$$

$$\Rightarrow XV` = \frac{(1+V)}{(2+V)} - V = \frac{1 + V - 2V - V^2}{(2+V)} = \frac{1 - V - V^2}{2 + V} = -\frac{\left(V^2 + V - 1\right)}{(V+2)}.$$

Separating the variables and integrating, we get : $\displaystyle\int \frac{(V+2)}{\left(V^2 + V - 1\right)}dV = \int \frac{1}{X}\,dX + \ln C$

$$\frac{1}{2}\int \frac{(2V+4)}{\left(V^2 + V - 1\right)}dV = \ln X + \ln C = \ln CX \;\Rightarrow \frac{1}{2}\int \frac{(2V+1)+3}{\left(V^2 + V - 1\right)}dV = \ln CX$$

$$\frac{1}{2}\int \frac{(2V+1)}{\left(V^2 + V - 1\right)}dV + \frac{3}{2}\int \frac{1}{\left(V^2 + V - 1\right)} = \ln CX$$

$$\Rightarrow \frac{1}{2}\ln\left(V^2 + V - 1\right) + \frac{3}{2}\int \frac{1}{\left(V^2 + V + \frac{1}{4}\right) - \frac{5}{4}}dV = \ln CX$$

$$\Rightarrow \frac{1}{2}\ln\left(V^2 + V - 1\right) + \frac{3}{2}\int \frac{1}{\left(V + \frac{1}{2}\right)^2 - \left(\sqrt{\frac{5}{4}}\right)^2}dV = \ln CX$$

$$\left[\text{NOTE}: \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)\right]$$

$$\Rightarrow \frac{1}{2}\ln\left(V^2+V-1\right)+\frac{3}{2}.\frac{1}{2}\frac{2}{\sqrt{5}}\ln\left[\frac{\left(V+\frac{1}{2}\right)-\frac{\sqrt{5}}{2}}{\left(V+\frac{1}{2}\right)+\frac{\sqrt{5}}{2}}\right]=\ln CX$$

$$\Rightarrow \frac{1}{2}\ln\left(V^2+V-1\right)+\frac{3}{2\sqrt{5}}\ln\left[\frac{2V+1-\sqrt{5}}{2V+1+\sqrt{5}}\right]=\ln CX.$$

By putting $V=\frac{Y}{X}=\frac{y-k}{x-h}=\frac{y-1}{x-1}$ a general solution is obtained.

(b) (2x + 3y- 5) dy + (3x + 2y - 5) dx = 0

Solution: Putting x = X + h and y = Y + k or dy/dx = dY/dX. Thus given eq. becomes

$$\frac{dY}{dX}=-\frac{3X+3h+2Y+2k-5}{2X+2h+3Y+3k-5}=-\frac{3X+2Y+3h+2k-5}{2X+3Y+2h+3k-5} \qquad (1)$$

Put : $3h+2k-5=0$ and $2h+3k-5=0$ and solving, we get : $h=1,\ k=1$

Thus (1) becomes : $\frac{dY}{dX}=-\frac{3X+2Y}{2X+3Y}$ (2)

Eq. (2) is homogeneous equation thus putting $Y=VX \Rightarrow Y`=V+XV`$, we get

$$V+XV`=-\frac{3X+2VX}{2X+3VX}=-\frac{X(3+2V)}{X(2+3V)}=-\frac{(3+2V)}{(2+3V)}$$

$$\Rightarrow XV`=-\frac{(3+2V)}{(2+3V)}-V=-\frac{3+2V-2V-3V^2}{(2+3V)}=-\frac{3-3V^2}{2+3V}=3\frac{V^2-1}{3V+2}$$

Separating the variables and integrating, we get : $\int\frac{(3V+2)}{\left(V^2-1\right)}dV=\int\frac{1}{X}dX+\ln C$

$$\frac{3}{2}\int\frac{2V}{\left(V^2-1\right)}dV+2\int\frac{1}{\left(V^2-1\right)}=\ln X+\ln C=\ln CX \Rightarrow \frac{3}{2}\ln\left(V^2-1\right)+2.\frac{1}{2}\ln\left(\frac{V-1}{V+1}\right)=\ln CX$$

$$\Rightarrow \ln\left(V^2-1\right)^{3/2}+\ln\left(\frac{V-1}{V+1}\right)=\ln CX \qquad \Rightarrow \ln\left[\frac{\left(V^2-1\right)^{3/2}(V-1)}{(V+1)}\right]=\ln CX$$

$$\Rightarrow\left[\frac{\left(V^2-1\right)^{3/2}(V-1)}{(V+1)}\right]=CX \qquad \Rightarrow\left[\frac{\left(V^2-1\right)\sqrt{V^2-1}(V-1)}{(V+1)}\right]=CX$$

$$\Rightarrow\left[\frac{(V-1)(V+1)\sqrt{V^2-1}(V-1)}{(V+1)}\right]=CX \qquad \Rightarrow (V-1)^2\sqrt{V^2-1}=CX.$$

Putting $V=\frac{Y}{X}=\frac{y-k}{x-h}=\frac{y-1}{x-1}$ provides the general solution.

(c) (4x + 6y + 3) dx = (6x + 9y + 2) dy

Solution: Rearranging, we get $\frac{dy}{dx}=\frac{4x+6y+3}{6x+9y+2}=\frac{2(2x+3y)+3}{3(2x+3y)+2}$

Here $\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36 - 36 = 0$. Thus we put z = 2x + 3y ➔ z` = 2 + 3y` ➔ y` = (z` - 2)/3.

Thus above equation becomes:

$$\frac{z`-2}{3} = \frac{2z+3}{3z+2} \Rightarrow z`-2 = \frac{6z+9}{3z+2} \Rightarrow \frac{dz}{dx} = \frac{6z+9}{3z+2} + 2 = \frac{6z+9+6z+4}{3z+2} = \frac{12z+13}{3z+2}$$

Separating the variables and integrating, we get

$$\int \frac{3z+2}{12z+3} dz = \int 1.dx + c \Rightarrow \frac{1}{4}\int \frac{12z+8}{12z+3} dz = x + c \Rightarrow \frac{1}{4}\int \frac{12z+3+5}{12z+3} dz = x + c$$

Or $\frac{1}{4}\left[\int \frac{12z+3}{12z+3} + \frac{5}{12}\int \frac{12}{12z+3}\right] dz = x + c \Rightarrow \frac{1}{4}\left[\int 1dz + \frac{5}{12}\ln(12z+3)\right] = x + c$

$\Rightarrow \frac{1}{4}\left[z + \frac{5}{12}\ln(12z+3)\right] = x + c.$ Putting $z = 3x + 2y$, we get

$\frac{1}{4}\left[2x + 3y + \frac{5}{12}\ln\left(24x + 36y + 3\right)\right] = x + c.$ This is a general solution of given d.e.

(d) (2x + 3y + 4) dx - (4x + 6y + 5) dy = 0

Solution: Given equation may be rearranged as

$$\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5} = \frac{(2x+3y)+4}{2(2x+3y)+5}$$

Here $\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$. Thus we put z = 2x + 3y or z` = 2 + 3y` or y` = (z` - 2)/3.

Thus above equation becomes:

$$\frac{z`-2}{3} = \frac{z+4}{2z+5} \Rightarrow z`-2 = \frac{3z+12}{2z+5} \Rightarrow \frac{dz}{dx} = \frac{3z+12}{2z+5} + 2 = \frac{3z+12+4z+10}{2z+5} = \frac{7z+22}{2z+5}$$

Separating the variables and integrating, we get

$$\int \frac{2z+5}{7z+22} dz = \int 1.dx + c \Rightarrow \frac{2}{7}\int \frac{z+5/2}{z+22/7} dz = x + c \text{ OR } \frac{2}{7}\int \frac{(z+22/7) - 22/7 + 5/2}{z+22/7} dz = x + c$$

Or $\frac{2}{7}\left[\int \frac{z+22/7}{z+22/7} - \frac{9}{14}\int \frac{1}{z+22/7}\right] dz = x + c \Rightarrow \frac{2}{7}\left[\int 1dz - \frac{9}{14}\ln\left(z + \frac{22}{7}\right)\right] = x + c$

$\Rightarrow \frac{2}{7}\left[z - \frac{9}{14}\ln\left(z + \frac{22}{7}\right)\right] = x + c.$ Putting $z = 3x + 2y$, we get

$\frac{2}{7}\left[2x + 3y - \frac{9}{14}\ln\left(2x + 3y + \frac{22}{7}\right)\right] = x + c.$ This is a general solution of given d.e.

(e) (x – y) dy = (x + y + 1) dx

Solution: Putting x = X + h and y = Y + k or dy/dx = dY/dX. Thus given eq. becomes

$$\frac{dY}{dX} = \frac{X+h+Y+k+1}{X+h-Y-k} = \frac{X+Y+h+k+1}{X-Y+h-k} \qquad (1)$$

Put : $h + k + 1 = 0$ and $h - k = 0$ and solving, we get : $h = k = -1/2$

Thus (1) becomes : $\frac{dY}{dX} = \frac{X+Y}{X-Y}$ (2)

Eq. (2) is homogeneous equation thus putting $Y = VX \Rightarrow Y` = V + XV`$, we get

$$V + XV` = \frac{X+VX}{X-VX} = \frac{X(1+V)}{X(1-V)} = \frac{(1+V)}{(1-V)}$$

$$\Rightarrow XV` = \frac{(1+V)}{(1-V)} - V = \frac{1+V-V+V^2}{(1-V)} = \frac{V^2+1}{-(V-1)}$$

Separating the variables and integrating, we get : $\int \frac{(V-1)}{(V^2+1)} dV = -\int \frac{1}{X} dX + \ln C$

$$\frac{1}{2}\int \frac{2V}{(V^2+1)} dV - \int \frac{1}{(V^2+1)} = -\ln X + \ln C = \ln\frac{C}{X} \quad \Rightarrow \frac{1}{2}\ln(V^2+1) - \tan^{-1}V = \ln\frac{C}{X}$$

$$\Rightarrow \ln(V^2-1)^{1/2} + \ln\left(\frac{C}{X}\right) = \tan^{-1}V \qquad \Rightarrow \ln\left[\frac{C\sqrt{V^2-1}}{X}\right] = \tan^{-1}V$$

$$\Rightarrow \ln\left[\frac{C\sqrt{Y^2-X^2}}{X^2}\right] = \tan^{-1}\left(\frac{Y}{X}\right).$$

Putting $V = \frac{Y}{X} = \frac{y-k}{x-h} = \frac{y-1}{x-1}$ provides the general solution.

(f) y` = (x + y)/x, y(1) = 1

Solution: Given differential is

$\frac{dy}{dx} = \frac{x+y}{x}$. Putting $y = vx \Rightarrow y` = v.1 + x.v`$, we obtain

$$v + xv` = \frac{x+vx}{x} = \frac{x(1+v)}{x} = 1+v \quad \Rightarrow x\frac{dv}{dx} = 1+v-v = 1$$

Separating the variables and integrating, we get : $\int 1 dv = \int \frac{1}{x} dx + \ln c$

$\Rightarrow v = \ln x + \ln c = \ln cx$. Putting $y = v / x : \frac{y}{x} = cx \quad \Rightarrow y = cx^2.$

This is the general solution of given differential equation.

Now put $x = 1$ and $y = 1$, we obtain : $c = 1 \Rightarrow y = x^2.$

This is the particular solution of given differential equation.

(g) (x + 6y) dx + (4x – y) dy = 0; y(-1) = 6 [Notice the change in the problem]

Solution: Rearranging, we get

$\frac{dy}{dx} = (x+6y)/(4x-y)$. Putting $y = vx \Rightarrow y` = v.1 + x.v`$, we obtain

$$v + xv` = \frac{x+6vx}{4x-vx} = \frac{x(1+6v)}{x(4-v)} = \frac{(1+6v)}{(4-v)} \quad \Rightarrow x\frac{dv}{dx} = \frac{(1+6v)}{(4-v)} - v = \frac{(1+6v-4v+v^2)}{(4-v)}$$

Separating the variables and integrating, we get : $\int \frac{4-v}{1+2v+v^2} dv = \int \frac{1}{x} dx + \ln c$

$$\Rightarrow -\int \frac{v-4}{(v+1)^2} dv = \ln x + \ln c = \ln cx \qquad \Rightarrow -\int \frac{(v+1)-5}{(v+1)^2} dv = \ln cx$$

$$\Rightarrow -\left[\int \frac{(v+1)}{(v+1)^2}dv + 5\int (v+1)^{-2}\,dv\right] = \ln cx \qquad \Rightarrow -\int \frac{1}{(v+1)}dv + 5\frac{(v+1)^{-1}}{-1} = \ln cx$$

$$\Rightarrow -\ln(v+1) - \frac{5}{(v+1)} = \ln cx \qquad \Rightarrow -\frac{5}{(v+1)} = \ln cx + \ln(v+1)$$

Putting $v = y/x$, and simplifying, we get:

$$-\left(\frac{5x}{y+x}\right) = \ln c(y+x).\ \text{ Putting } x=-1 \text{ and } y=6 \Rightarrow 1 = \ln 5c \ \Rightarrow 5c = e \Rightarrow c = e/5$$

$$\Rightarrow \ln\frac{e(y+x)}{5} + \left(\frac{5x}{y+x}\right) = 0 \ \Rightarrow \ \ln e + \ln(y+x) - \ln 5 + \left(\frac{5x}{y+x}\right) = 0$$

$\Rightarrow 1 + \ln(y+x) - \ln 5 + \left(\dfrac{5x}{y+x}\right) = 0$, This is the particular solution of given d.e.

(h) $\left(y + \sqrt{x^2+y^2}\right)dx - x\,dy = 0;\ y(1) = 0$

Solution: Rearranging, we get; $\dfrac{dy}{dx} = \dfrac{\left(y+\sqrt{x^2+y^2}\right)}{x}$. Putting $y = vx \Rightarrow y` = v + xv`$

The given d.e becomes: $v + xv` = \dfrac{\left(vx + \sqrt{x^2+(vx)^2}\right)}{x} = \dfrac{x\left(v+\sqrt{1+v^2}\right)}{x} = \left(v+\sqrt{1+v^2}\right)$

$\Rightarrow xv` = v + \sqrt{1+v^2} - v = \sqrt{1+v^2}$. Separating the variables and integrating, we get 4.

$$\int \frac{1}{\sqrt{1+v^2}}dv = \int \frac{1}{x}dx + \ln c \ \Rightarrow \ln\left(v+\sqrt{1+v^2}\right) = \ln cx \ \Rightarrow v + \sqrt{1+v^2} = cx \ \ (\text{Anti log})$$

Putting $v = y/x$ and simplifying: $y + \sqrt{y^2+x^2} = cx^2$. Putting $x = 1$ and $y = 0$,

$\Rightarrow c = 1$. Thus, particular solution is: $y + \sqrt{y^2+x^2} = x^2$.

4. Solve the following exact differential equations

(a) Solution: Given $(e^y + 1)\cos x\,dx + e^y \sin x\,dy = 0$ (1)

Here $M = (e^y + 1)\cos x$ and $N = e^y \sin x \ \Rightarrow M_y = e^y\cos x$ and $N_x = e^y \cos x$

Since $M_y = N_x$ hence (1) is exact. To solve it, we have:

Step I: Integrating M w.r.t x keepiny y constant. Thus

$$\int M\,dx = \int \left(e^y + 1\right)\cos x\,dx = \left(e^y + 1\right)\int \cos x\,dx = \left(e^y + 1\right)\sin x$$

Step II: Integrating those terms of N w.r.t y which are free from x. As there is no such term so we skip this step. Hence, the general solution is: $(e^y + 1)\sin x = c$

(b) Solution: Given $(a^2 - 2x - y)dx - (x - y)\,dy = 0$ (1)

Here $M = (a^2 - 2x - y)\,dx$ and $N = -(x - y)\,dy \ \Rightarrow M_y = -1$ and $N_x = -1$

Since $M_y = N_x$ hence (1) is exact. To solve it, we have:

Step I: Integrating M w.r.t x keepiny y constant. Thus

$\int M\,dx = \int \left(a^2 - 2x - y\right)dx = a^2x - x^2 - xy$

Step II: Integrating those terms of N w.r.t y which are free from x.

Thus, $\int y\,dy = y^2/2$

Step III: Adding the two results and equate them to c, we get: $a^2x - x^2 - xy + \frac{y^2}{2} = c$

This is the general solution of given differential equation.

(c) Solution: Given $\left(3x^2 + 6xy^2\right)dx + 6\left(1 + x^2y\right)dy = 0$ (1)

Here $M = \left(3x^2 + 6xy^2\right)$ and $N = 6\left(1 + x^2y\right) \Rightarrow M_y = 12xy$ and $N_x = 12xy$

Since $M_y = N_x$ hence (1) is exact. To solve it, we have:

Step I: Integrating M w.r.t x keepiny y constant. Thus

$\int M\,dx = \int \left(3x^2 + 6xy^2\right)dx = x^3 + 3x^2y^2 = x^2\left(x + 3y^2\right)$

Step II: Integrating those terms of N w.r.t y which are free from x.

Thus, $\int 6\,dy = 6y$

Step III: Adding the two results and equate them to c, we get: $x^2\left(x + 3y^2\right) + 6y = c$

This is the general solution.

(d) Solution: Given $(x^2 - ay)\,dx = (ax - y^2)\,dy \Rightarrow (x^2 - ay)\,dx + (y^2 - ax)\,dy$ (1)

Here $M = x^2 - ay$ and $N = y^2 - ax \Rightarrow M_y = -a$ and $N_x = -a$

Since $M_y = N_x$ hence (1) is exact. To solve it, we have:

Step I: Integrating M w.r.t x keepiny y constant. Thus

$\int M\,dx = \int \left(x^2 - ay\right)dx = \frac{x^3}{3} - axy$

Step II: Integrating those terms of N w.r.t y which are free from x.

Thus, $\int y^2 dy = \frac{y^3}{3}$

Step III: Adding the two results and equate them to c, we get: $x^3 - 3axy + y^3 = 3c$

Or, $x^3 - 3axy + y^3 = C$, where $C = 3c$. This is the general solution.

(e) Solution: Given equation is $\sec 2x \tan y\,dx + \sec 2y \tan x\,dy = 0$

Here $M = \sec 2x \tan y$ and $N = \sec 2y \tan x \Rightarrow M_y = \sec 2x \sec^2 y$ and $N_x = \sec 2x \sec^2 x$

Since $M_y = N_x$ hence (1) is exact. To solve it, we have:

Step I: Integrating M w.r.t x keepiny y constant. Thus

$\int M\,dx = \int \sec 2x \tan y\,dx = \tan y \int \sec 2x\,dx = \frac{\tan y \sec 2x \tan 2x}{2}$

Step II: Integrating those terms of N w.r.t y which are free from x.

Since there is no such term hence, we skip this step.

Step III : Adding the two results and equate them to c, we get : $\tan y \dfrac{\sec 2x \tan 2x}{2} = c$

Or $\tan y \sec 2x \tan 2x = 2c = C$, where $C = 2c$. This is the general solution.

(f) Solution: Given equation is $(2y \sin 2x)\,dx - (y^2 + \cos 2x)\,dy = 0$

Here $M = 2y \sin 2x$ and $N = -(y^2 + \cos 2x) \Rightarrow M_y = 2\sin 2x$ and $N_x = 2\sin 2x$

Since $M_y = N_x$ hence given differential equation is exact.

To find its solution we have :

Step I : $\int M\,dx = \int 2y \sin 2x\,dx = 2y\int \sin 2x\,dx = 2y\left(\dfrac{-\cos 2x}{2}\right) = -y\cos 2x$

Step II : Integrating those terms of N w.r.t y which are free from x.

$$\int N\,dy = -\int y^2\,dy = \frac{-y^3}{3}$$

Step III : Adding the two results and equate them to c, we get :

$$-y\cos 2x - \frac{y^3}{3} = c$$

Or $3y\cos 2x + y^3 + C = 0$, where $C = 3c$. This is the general solution.

(g) Solution: Given equation is $(x^4 - 2xy^2 + y^4)\,dx - (2x^2y - 4xy^3 + \sin y)\,dy = 0$

Here $M = (x^4 - 2xy^2 + y^4)$ and $N = -(2x^2y - 4xy^3 + \sin y)$

$\Rightarrow M_y = -4xy + 4y^3$ and $N_x = -4xy + 4y^3$

Since $M_y = N_x$ hence given differential equation is exact.

To find its solution we have :

Step I : $\int M\,dx = \int (x^4 - 2xy^2 + y^4)\,dx = \dfrac{x^5}{5} - 2y^2\dfrac{x^2}{2} + y^4x$

Step II : Integrating those terms of N w.r.t y which are free from x.

$$\int N\,dy = \int -\sin y\,dy = +\cos y$$

Step III : Adding the two results and equate them to c, we get :

$$\frac{x^5}{5} - 2y^2\frac{x^2}{2} + y^4x + \cos y = c$$

Or $x^5 - 5x^2y^2 + 5xy^4 + 5\cos y = C$, where $C = 5c$.

This is the general solution.

(h) Solution: Given equation is:

$\cos x\,(\cos x - \sin a \sin y)\,dx + \cos y\,(\cos y - \sin a \sin x)\,dy = 0$

Here, $M = \cos x\,(\cos x - \sin a \sin y)$ and $N = \cos y\,(\cos y - \sin a \sin x)$

$M = \cos x\,(\cos x - \sin a \sin y)$ and $N = \cos y\,(\cos y - \sin a \sin x)$

$\Rightarrow M_y = \cos x(-\sin a \cos y) = -\sin a \cos x \cos y$ and $N_x = -\sin a \cos x \cos y$

Since $M_y = N_x$ hence given differential equation is exact.

To find its solution we have :

Step I: $\int M\,dx = \int \cos^2 x\,dx - \sin a \sin x \int \cos x\,dx = \int \frac{1+\cos 2x}{2}dx - \sin a \sin x \sin x$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] - \sin a \sin^2 x = \frac{2x + \sin 2x - 4\sin a \sin^2 x}{4}$$

Step II: Integrating those terms of N w.r.t y which are free from x.

$$\int \cos^2 y\,dy = \int \frac{1+\cos 2y}{2}dy = \frac{1}{2}\left[y + \frac{\sin 2y}{2}\right] = \frac{2y + \sin 2y}{4}$$

Step III: Adding the two results and equate them to c, we get:

$$\frac{2x + \sin 2x - 4\sin a \sin^2 x}{4} + \frac{2y + \sin 2y}{4} = c$$

Or $2x + \sin 2x - 4\sin a \sin^2 x + 2y + \sin 2y = C$, where $C = 4c$. This is the general solution.

(i) Solution: Given equation is

$(2y \sin x \cos x + y^2 \sin x)\,dx + (\sin^2 x - 2y \cos x)\,dy = 0;\ y(0) = 3$

Here $M = 2y \sin x \cos x + y^2 \sin x = y\sin 2x + y^2 \sin x$ and $N = \sin^2 x - 2y \cos x$

$\Rightarrow M_y = \sin 2x + 2y \sin x$ and $N_x = 2\sin x \cos x + 2y \sin x = \sin 2x + 2y \sin x$

Since $M_y = N_x$ hence given differential equation is exact.

To find its solution we have:

$$\int M\,dx = 2y\int \sin x \cos x\,dx + y^2 \int \sin x\,dx = 2y\frac{\sin^2 x}{2} + y^2(-\cos x) = y\sin^2 x - y^2 \cos x$$

Step II: Integrating those terms of N w.r.t y which are free from x.

Since there is no such term hence we skip this step

Step III: Adding the two results and equate them to c, we get:

$y\sin^2 x - y^2 \cos x = c$. This is the general solution.

Now put $x = 0$ and $y = 3$, we have $0 - 9 = C$. Thus, above equation becomes

$y\sin^2 x - y^2 \cos x + 9 - 0$. This is a particular solution.

(j) Solution: Here $M = 2xy - 3$ and $N = x^2 + 4y \Rightarrow M_y = 2x$ and $N_x = 2x$

Since $M_y = N_x$ hence given equation is exact. To find its solution we have:

Step: I $\int M\,dx = \int (2xy - 3)dx = x^2 y - 3x$

Step II: Integrating those terms of N w.r.t y which not contain x, we have: $4\int y\,dy = 2y^2$

Step III: Adding the two results and equate them to c, we get:

$x^2 y - 3x + \frac{y^2}{2} = c$. This is the general solution.

Putting $x = 1$ and $y = 2$, we get: $2 - 3 + 2 = c \Rightarrow c = 1$. Thus, above equation becomes

$x^2 y - 3x + \frac{y^2}{2} = 1 \Rightarrow 2x^2 y - 6x + y^2 = 2$. This is a particular solution.

(k) Solution: Here $M = 3x^2y^2 - y^3 + 2x$ and $N = 2x^3y - 3xy^2 + 1$

Thus, $M_y = 6x^2y - 3y^2$ and $N_x = 6x^2y - 3y^2$. We see that $M_y = N_x$.

Hence given equation is exact. To find its solution we have:

Step : I $\int M\,dx = \int\left(3x^2y^2 - y^3 + 2x\right)dx = x^3y^2 - xy^3 + x^2$

Step II: Integrating those terms of N w.r.t y which are free from x : $\int 1\,dy = y$

Step III: Adding the two results and equate them to c, we get:

$x^3y^2 - xy^3 + x^2 + y = c$. This is the general solution.

Putting $x = -2$ and $y = 1$, we get : $-8 + 2 + 4 + 1 = c \Rightarrow c = 1$. Thus, above equation becomes

$x^3y^2 - xy^3 + x^2 + y = 1$. This is a particular solution.

(*l*) Solution: Here $M = \dfrac{3-y}{x^2} = \dfrac{3}{x^2} - \dfrac{y}{x^2}$ and $N = \dfrac{y^2 - 2x}{xy^2} = \dfrac{1}{x} - \dfrac{2}{y^2}$

Thus, $M_y = -\dfrac{1}{x^2}$ and $N_x = -\dfrac{1}{x^2}$. We see that $M_y = N_x$.

Hence given equation is exact. To find its solution, we have:

Step : I $\int M\,dx = \int \dfrac{3-y}{x^2}dx = (3-y)\int x^{-2}\,dx = (3-y)\left(\dfrac{-1}{x}\right) = \dfrac{y-3}{x}$

Step II: Integrating those terms of N w.r.t y which are free from x. : $\int -\dfrac{2}{y^2}\,dy = +\dfrac{2}{y}$.

Step III: Adding the two results and equate them to c, we get:

$\dfrac{y-3}{x} + \dfrac{2}{y} = c$. This is the general solution.

Putting $x = -1$ and $y = 2$, we get : $-1 + 1 = c \Rightarrow c = 2$. Thus, above equation becomes

$\dfrac{y-3}{x} + \dfrac{2}{y} = 2 \quad \Rightarrow y^2 - 3y + 2x = 2xy$. This is a particular solution.

(m) Solution: Here $M = e^{x+y}(4x^3 + x^4) + 2x$ and $N = x^4 e^{x+y} + 2y$

$M_y = e^{x+y}(4x^3 + x^4)$ and $N_x = e^{x+y}(4x^3 + x^4)$

We observe that $M_y = N_x$. Hence, given equation is exact.

To find the solution we have:

Step I: Integrate M w.r.t x keeping y constant

$$\int M\,dx = \int\left(4x^3e^{x+y} + x^4e^{x+y} + 2x\right)dx = 4e^y\int x^3e^x dx + e^y\int x^4e^x dx + 2\int x\,dx$$

$$= 4e^y\int x^3e^x dx + e^y\left[x^4e^x - \int 4x^3e^x\right] + 2\frac{x^2}{2}$$

$$= 4e^y\int x^3e^x dx + x^4e^xe^y - e^y\int 4x^3e^x + x^2 = x^4e^xe^y + x^2$$

Step II: Integrate those terms of N w.r.t y ahich do not contain x : $\int 2y\,dy = y^2$

Step III: Adding the above results and equating them to c.

$x^4e^xe^y + x^2 + y^2 = c$. Putting $x = 0$ and $y = 1$, we get $c = 1$. Thus,

$x^4e^xe^y + x^2 + y^2 = 1$ which is a particular solution.

5. Find an appropriate Integral Factor and hence solve the following differential equations.

(a) Solution: Given equation is $\left(x^2+y^2+2x\right)dx+2y\,dy=0$ (1)

Here $M=x^2+y^2+2x$ and $N=2y \Rightarrow M_y=2y,\ N_x=0$. since, $M_y \neq N_x$ hence given equation is not exact. To find I.F we apply

Rule I: $\dfrac{M_y-N_x}{N}=\dfrac{2y-0}{2y}=1=P(x)$ hence $I.F=e^{\int e^{P\,dx}}=e^{\int 1dx}=e^x.$

Multiplying (1) by I.F, we get: $e^x\left(x^2+y^2+2x\right)dx+2ye^x\,dy=0$ (2)

Equation (2) is now exact. To find its solution, we proceed as:

Step I: Integrating M w.r.t x keeping y constant:

$$\int e^x\left(x^2+y^2+2x\right)dx=\int x^2e^x\,dx+y^2\int e^x\,dx+2\int xe^x\,dx$$

$$=x^2e^x-\int 2xe^x\,dx+y^2e^x+\int 2xe^x\,dx=x^2e^x+y^2e^x=e^x\left(x^2+y^2\right)$$

Step II: Integrating those terms of N w.r.t y which do not contain x.

As there is no such term. hence, we skip this step.

Step III: Adding the two results from step I and II and equating them to c, we get:

$e^x\left(x^2+y^2\right)=c.$ This is the general solution.

(b) Solution: Given equation is $\left[x\,y^2-e^{1/x^3}\right]dx-x^2\,y\,dy=0$ (1)

Here $M=xy^2\text{-}e^{1/x^3}$ and $N=-x^2y \Rightarrow M_y=2xy,\ N_x=-2xy.$

Since, $M_y \neq N_x$ hence given equation is not exact. To find I.F we apply

Rule I: $\dfrac{M_y-N_x}{N}=\dfrac{2xy+2xy}{-x^2y}=\dfrac{4xy}{-x^2y}=\dfrac{-4}{x}=P(x).$

Hence, $I.F=e^{\int e^{P\,dx}}=e^{-4\int \frac{1}{x}dx}=e^{-4\ln x}=e^{\ln x^{-4}}=x^{-4}.$

Multiplying (1) by I.F, we get: $\dfrac{1}{x^4}\left[x\,y^2-e^{1/x^3}\right]dx-\dfrac{y}{x^2}dy=0$ (2)

Equation (2) is now exact. To find its solution, we proceed as:

Step I: Integrating M w.r.t x keeping y constant:

$$y^2\int x^{-3}\,dx-\int \frac{e^{1/x^3}}{x^4}dx=-\frac{y^2}{x^2}-\int \frac{e^{1/x^3}}{x^4}dx.\ \text{Putting } z=1/x^3=x^{-3} \Rightarrow dz=\frac{-3}{x^4}dx$$

$$\Rightarrow \frac{dz}{3}=-\frac{dx}{x^4}.\quad \text{Thus, } \int M\,dx=-\frac{y^2}{x^2}+\frac{1}{3}\int e^z dz=-\frac{y^2}{x^2}+\frac{e^z}{3}=-\frac{y^2}{x^2}+\frac{e^{1/x^3}}{3}$$

Step II: Integrating those terms of N w.r.t y which do not contain x.

As there is no such term hence, we skip this step.

Step III: Adding the two results from step I and II and equating them to c, we get:

$-\dfrac{y^2}{x^2}+\dfrac{e^{1/x^3}}{3}=c$. This is the general solution.

(c) Solution: Given equation is $\left(y-2x^3\right)dx - x\left(1-xy\right)dy = 0$ (1)

Here $M=\left(y-2x^3\right)$ and $N=-x\left(1-xy\right) \Rightarrow M_y=1,\ N_x=-1+2xy.$

Since, $M_y \neq N_x$ hence given equation is not exact. To find I.F we apply

Rule I: $\dfrac{M_y-N_x}{N}=\dfrac{1+1-2xy}{-x\left(1-xy\right)}=\dfrac{2(1-xy)}{-x(1-xy)}=\dfrac{-2}{x}=P(x).$

Hence, $I.F=e^{\int e^{P\,dx}}=e^{-2\int\frac{1}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}.$

Multiplying (1) by I.F, we get: $\dfrac{1}{x^2}\left[y-2x^3\right]dx-\dfrac{(1-xy)}{x}dy=0$ (2)

Equation (2) is now exact. To find its solution, we proceed as:

Step I: Integrating M w.r.t x keeping y constant: $y\int x^{-2}\,dx-\int 2x\,dx=-\dfrac{y}{x}-x^2$

Step II: Integrating those terms of N w.r.t y which do not contain x: $\int y\,dy=y^2/2$

Step III: Adding the two results from step I and II and equating them to c, we get:

$-\dfrac{y}{x}-x^2+\dfrac{y^2}{2}=c$ Or $2y+2x^3-xy^2+2cx=0$. This is the general solution.

(d) Solution: Given equation is $\left(y^2+2xy\right)dx+\left(2x^2-xy\right)dy=0$ (1)

Here $M=y^2+2xy$ and $N=2x^2-xy \Rightarrow M_y=2y+2x$ and $N_x=4x-y$

Since $M_y\neq N_x$ hence (1) is not exact. To find the I.F we observe that eq (1) is

homogeneous hence we apply Rule IV where $I.F=\dfrac{1}{xM+yN}$, provided $xM+yN\neq 0$.

Now, $xM+yN=xy^2+2x^2y+2x^2y-xy^2=4x^2y$. Now, multiplying (1) by I.F, we get

$\dfrac{y(y+2x)}{4x^2y}dx+\dfrac{x(2x-y)}{4x^2y}dy=0$ OR $\dfrac{(y+2x)}{4x^2}dx+\dfrac{(2x-y)}{4xy}dy=0$ (2)

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant.

$\int M\,dx=\dfrac{y}{4}\int x^{-2}dx+\dfrac{1}{2}\int\dfrac{1}{x}dx=-\dfrac{y}{4x}+\dfrac{1}{2}\ln x$

Step II: Integrating those terms of N w.r.t y which do not contain x.

Thus: $\int\dfrac{1}{2y}dy=\dfrac{\ln y}{2}$

Step III: adding the two results and equating them to a constant, we get

$-\dfrac{y}{4x}+\dfrac{1}{2}\ln x+\dfrac{\ln y}{2}=c$. This is the general solution of given equation.

NOTE: See the change in the question.

(e) Solution: Given equation is $\left(y+y^3/3+x^2/2\right)dx+0.25\left(x+xy^2\right)dy=0$ (1)

Here $M=(y+y^3/3+x^2/2)$ and $N=0.25\left(x+xy^2\right)$

$\Rightarrow M_y=1+y^2$ and $N_x=0.25(1+y^2)$. Since, $M_y\neq N_x$ hence (1) is not exact.

To find an I.F we apply:

Rule I: $\dfrac{M_y-N_x}{N}=\dfrac{1+y^2+0.25(1+y^2)}{0.25x(1+y^2)}=\dfrac{1.25(1+y^2)}{0.25x(1+y^2)}=\dfrac{5}{x}=P(x)$. Thus,

I.F $=e^{\int\frac{5}{x}dx}=e^{5\ln x}=e^{\ln x^5}=x^5$. Multiplying (1) by I.F, we get

$$\left(x^5y+\frac{x^5y^3}{3}+\frac{x^7}{2}\right)dx+0.25x^5\left(x+xy^2\right)dy=0 \qquad (2)$$

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant

$$\int\left(x^5y+\frac{x^5y^3}{3}+\frac{x^7}{2}\right)dx=\frac{x^6y}{6}+\frac{x^6y^3}{18}+\frac{x^8}{16}=\frac{x^6}{144}\left(24y+8y^3+9x^2\right)$$

Step II: Integrating those terms of N w.r.t y which do not contains x.

Since there is no such term hence we skip this step.

Step III: The general solution is therefore,

$$\frac{x^6}{144}\left(24y+8y^3+9x^2\right)=c \;\Rightarrow\; x^6\left(24y+8y^3+9x^2\right)=C, \text{ where } C=144c.$$

(f) Solution: Here $M=x-y^2$ and $N=2xy \Rightarrow M_y=-2y$ and $N_x=2y$

Since $M_y\neq N_x$ hence given equation is not exact. To find an I.F we apply:

Rule I: $\dfrac{M_y-N_x}{N}=\dfrac{-2y-2y}{2xy}=\dfrac{-4y}{2xy}=\dfrac{-2}{x}=P(x)$. Thus,

I.F $=e^{\int\frac{-2}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}$. Multiplying given equation by I.F, we get

$$\left(\frac{1}{x}-\frac{y^2}{x^2}\right)dx+\frac{2y}{x}dy=0 \qquad (1)$$

Equation (1) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant

$$\int\left(\frac{1}{x}-\frac{y^2}{x^2}\right)dx=\ln x+\frac{y^2}{x}$$

Step II: Integrating those terms of N w.r.t y which do not contains x.

Since there is no such term hence we skip this step.

Step III: The general solution is therefore, $\ln x+\dfrac{y^2}{x}=c \;\Rightarrow x\ln x+y^2=cx$

(g) Solution: Given equation is $\left(y^4+2y\right)dx+(xy^3+2y^4-4x)dy=0$ (1)

Here $M=y^4+2y$ and $N=xy^3+2y^4-4x \Rightarrow M_y=4y^3+2$ and $N_x=y^3-4$

Since $M_y \neq N_x$ hence given equation is not exact. To find an I.F we apply:

Rule I: $\dfrac{M_y - N_x}{N} = \dfrac{4y^3 + 2 - y^3 + 4}{xy^3 + 2y^4 - 4x} = \dfrac{3y^3 + 6}{xy^3 + 2y^4 - 4x} \neq P(x)$. Thus Rule – I fails.

Rule II: $\dfrac{N_x - M_y}{M} = \dfrac{y^3 - 4 - 4y^3 - 2}{y^4 + 2y} = \dfrac{-3y^3 - 6}{y(y^3+2)} = -3\dfrac{(y^3+2)}{y(y^3+2)} = \dfrac{-3}{y} = P(y).$

Thus I.F $= e^{\int P\,dy} = e^{\int \frac{-3}{y}dy} = e^{-3\ln y} = e^{\ln y^{-3}} = y^{-3}$. Multiplying (1) by I.F, we get

$$\left(y + 2y^{-2}\right) dx + (x + 2y - 4xy^{-3})\, dy = 0 \qquad (2)$$

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant: $\left(y + 2y^{-2}\right)\int 1.dx = \dfrac{x\left(y^3 + 2\right)}{y^2}$

Step II: Integrating those terms of N w.r.t y which do not contains x: $2\int y\,dy = y^2$.

Step III: Adding the results of Steps I and II and equating to a constant, we get

$\dfrac{x\left(y^3 + 2\right)}{y^2} + y^2 = c \Rightarrow x\left(y^3 + 2\right) + y^4 = cy^2$. This is general solution.

(h) Solution: Given equation is $\left(2x^2y^2 + 6y\right)dx + \left(x^3y + 3x\right)dy = 0 \qquad (1)$

Here $M = 2x^2y^2 + 6y$ and $N = x^3y + 3x \Rightarrow M_y = 4x^2y + 6$ and $N_x = 3x^2y + 3$

Since $M_y \neq N_x$ hence given equation is not exact. To find an I.F we apply:

Rule I: $\dfrac{M_y - N_x}{N} = \dfrac{4x^2y + 6 - 3x^2y - 3}{x^3y + 3x} = \dfrac{x^2y + 3}{x(x^2y+3)} = \dfrac{1}{x} = P(x).$

Thus I.F $= e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. Multiplying (1) by I.F, we get

$$\left(2x^3y^2 + 6xy\right) dx + \left(x^4y + 3x^2\right) dy = 0 \qquad (2)$$

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant

$$\int\left(2x^3y^2 + 6xy\right) dx = 2y^2\frac{x^4}{4} + 6y\frac{x^2}{2} = \frac{x^4y^2 + 6x^2y}{2}$$

Step II: Integrating those terms of N w.r.t y which do not contains x.

Since there is no such term hence we skip this step.

Step III: The general solution is therefore $\dfrac{x^4y^2 + 6x^2y}{2} = c \Rightarrow x^4y^2 + 6x^2y = 2c$

(i) Solutio Given equation is $\left(3x^2y^4 + 2xy\right)dx + \left(2x^3y^3 - x^2\right)dy = 0 \qquad (1)$

Here $M = 3x^2y^4 + 2xy$ and $N = 2x^3y^3 - x^2 \Rightarrow M_y = 12x^2y^3 + 2x$ and $N_x = 6x^2y^3 - 2x$

Since $M_y \neq N_x$ hence given equation is not exact. To find an I.F we apply:

Rule I: $\dfrac{M_y - N_x}{N} = \dfrac{12x^2y^3 + 2x - 6x^2y^3 + 2x}{2x^3y^3 - x^2} = \dfrac{6x^2y^3 + 4x}{2x^3y^3 - x^2} = \dfrac{2x\left(3xy^3+2\right)}{x^2\left(2xy^3-1\right)} \neq P(x).$

Thus Rule – I fails.

Rule II: $\dfrac{N_x - M_y}{M} = \dfrac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{3x^2y^4 + 2xy} = \dfrac{-6x^2y^3 - 4x}{xy(3xy^3+2)} = -2\dfrac{x(3xy^3+2)}{xy(3xy^3+2)} = \dfrac{-2}{y} = P(y).$

Thus $I.F = e^{\int P\,dy} = e^{\int \frac{-2}{y}dy} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2}$. Multiplying (1) by I.F, we get

$y^{-2}\left(3x^2y^4 + 2xy\right)dx + y^{-2}\left(2x^3y^3 - x^2\right)dy = 0$

$\left(3x^2y^2 + 2xy^{-1}\right)dx + \left(2x^3y - x^2y^{-2}\right)dy = 0$ (2)

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant

$\int\left(3x^2y^2 + 2xy^{-1}\right)dx = \left(x^3y^2 + x^2y^{-1}\right)$

Step II: Integrating those terms of N w.r.t y which do not contains x.

Since there is no such term hence we neglect this step.

Step III: Thus general solution is: $\left(x^3y^2 + x^2y^{-1}\right) = c \Rightarrow x^3y^3 + x^2 = cy$

(j) Solution: Given equation is $\left(x^2y - 2xy^2\right)dx - \left(x^3 - 3x^2y\right)dy = 0$ (1)

Here $M = x^2y - 2xy^2$ and $N = -x^3 + 3x^2y$

$\Rightarrow M_y = x^2 - 4xy$ and $N_x = -3x^2 + 6xy$

Since $M_y \neq N_x$ hence given equation is not exact. Since given equation is homogeneous hence we apply Rule IV to find an I.F.

$IF = \dfrac{1}{xM + yN} = \dfrac{1}{x^3y - 2x^2y^2 + 3x^2y^2 - x^3y} = \dfrac{1}{x^2y^2}$

Multiplying (1) by I.F, we get: $\dfrac{\left(x^2y - 2xy^2\right)}{x^2y^2}dx - \dfrac{\left(x^3 - 3x^2y\right)}{x^2y^2}dy = 0$ (2)

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant: $\dfrac{1}{y}\int 1\,dx - 2\int\dfrac{1}{x}dx = \dfrac{x}{y} - 2\ln x$

Step II: Integrating those terms of N w.r.t y which do not contains x. $3\int\dfrac{1}{y}dy = 3\ln y$

Step III: Thus general solution is: $(x/y) - 2\ln x + 3\ln y = c$

(k) Solution: Given equation is $\left(3xy^2 - y^3\right)dx - \left(2x^2y - xy^2\right)dy = 0$ (1)

Here $M = 3xy^2 - y^3$ and $N = -2x^2y + xy^2 \Rightarrow M_y = 6xy - 3y^2$ and $N_x = -4xy + y^2$

Since $M_y \neq N_x$ hence given equation is not exact. Since given equation is homogeneous hence; we apply Rule IV to find an I.F.

$$IF = \frac{1}{xM + yN} = \frac{1}{3x^2y^2 - xy^3 - 2x^2y^2 + xy^3} = \frac{1}{x^2y^2}$$

Multiplying (1) by I.F, we get: $\frac{\left(3xy^2 - y^3\right)}{x^2y^2}\,dx - \frac{\left(2x^2y - x\,y^2\right)}{x^2y^2}\,dy = 0$ (2)

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant: $3\int \frac{1}{x}\,dx \text{ - } y\int x^{-2}\,dx = 3\ln x + \frac{y}{x}$

Step II: Integrating those terms of N w.r.t y which do not contains x.: $-2\int \frac{1}{y}dy = -2\ln y$

Step III: Thus general solution is: $3\ln x + (y/x) - 2\ln y = c$

***(l)* Solution:** Given equation is $\left(y^2x + 2\,x^2y^3\right)dx - \left(x^2y - x^3y^2\right)dy = 0$ (1)

Here $M = y^2x + 2\,x^2y^3$ and $N = -x^2\,y + x^3y^2$

$\Rightarrow M_y = 2xy + 6x^2y^2$ and $N_x = -2xy + 3x^2y^2$.

Since $M_y \neq N_x$ hence given equation is not exact.

Given equation may be rewritten as: $y(xy + 2x^2y^2)\,dx + x(x^2y^2 - xy)\,dx = 0$

This equation is of the form: $y\,f(xy)\,dx + x\,g(xy)\,dy = 0$. Hence,

$$I.F = \frac{1}{xM - yN} = \frac{1}{x^2y^2 + 2x^3y^3 + x^2y^2 - x^3y^3} = \frac{1}{x^2y^2\left(2 + xy\right)}$$

Multiplying (1) by I.F, we get: $\frac{xy(y + 2xy^2)}{x^2y^2(2 + xy)}\,dx - \frac{x^2y(1 - x\,y)}{x^2y^2(2 + xy)}\,dy = 0$

$$\Rightarrow \frac{\left(1 + 2xy\right)}{x\left(2 + xy\right)}\,dx - \frac{\left(1 - x\,y\right)}{y\left(2 + xy\right)}\,dy = 0 \qquad (2)$$

Equation (2) is now exact. To find its solution we have:

Step I: Integrating M w.r.t x keeping y constant:

$$\int \frac{\left(1 + 2xy\right)}{x\left(2 + xy\right)}\,dx = \int \frac{1/2}{x}\,dx + \frac{3}{2}\int \frac{y}{2 + xy}\,dx \qquad \left[\text{This is by partial fractions}\right]$$

$$= \frac{1}{2}\ln x + \frac{3}{2}\ln(2 + xy)$$

Step II: Integrating those terms of N w.r.t y which do not contains x: $-\frac{1}{2}\int \frac{1}{y}dy = -\frac{1}{2}\ln y$

Step III: Thus general solution is: $\frac{1}{2}\ln x + \frac{3}{2}\ln(2 + xy) - \frac{1}{2}\ln y = \ln c.$

Mmultiplying by 2, we get:

$$\ln\frac{x(2 + xy)^3}{y} = 2\ln c = \ln c^2 \;\Rightarrow\; \frac{x(2 + xy)^3}{y} = c^2 = C.$$

This is a general solution.

(m) Solution: Given equation is:

$$(xy\sin xy + \cos xy)\,y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0 \qquad (1)$$

Here $M = y(xy\sin xy + \cos xy)$ and $N = x(xy\sin xy - \cos xy)$

$$M_y = (xy\sin xy + \cos xy).1 + y\left(x^2 y\cos xy + x\sin xy.1 - x\sin xy\right)$$

$$= xy\sin xy + \cos xy + x^2y^2\cos xy$$

$$N_x = (xy\sin xy - \cos xy).1 + x\left(xy^2\cos xy + y\sin xy.1 + y\sin xy\right)$$

$$= -\cos xy + x^2y^2\cos xy + 3xy\sin xy$$

We see that $M_y \neq N_x$ hence equation (1) is not exact. To find I.F we apply

Rule I: $\dfrac{M_y - N_x}{N} = \dfrac{xy\sin xy + \cos xy + x^2y^2\cos xy + \cos xy - x^2y^2\cos xy - 3xy\sin xy}{x(xy\sin xy - \cos xy)}$

$$= \frac{2(\cos xy - xy\sin xy)}{-x(\cos xy - xy\sin xy)} = \frac{-2}{x} = P(x)$$

Thus $I.F = e^{\int \frac{-2}{x}dx} = e^{\ln x^{-2}} = x^{-2} = \dfrac{1}{x^2}$. Multiplying (1) by I.F, we get

$$\frac{1}{x^2}(xy\sin xy + \cos xy)\,y\,dx + \frac{1}{x^2}(xy\sin xy - \cos xy)\,x\,dy = 0 \qquad (2)$$

Equation (2) is now exact.To find the solution we have

Step I: Integrating M w.r.t x keeping y constant

$$\int M\,dx = y^2\int\frac{\sin xy}{x}dx + y\int\frac{\cos xy}{x^2}\,dx = y^2\int\frac{\sin xy}{x}dx + y\left[-\frac{\cos xy}{x} - \int -y\sin xy\left(-\frac{1}{x}\right)dx\right]$$

$$= y^2\int\frac{\sin xy}{x}dx - \frac{y\cos xy}{x} - y^2\int\frac{\sin xy}{x}dx = -\frac{y\cos xy}{x}$$

Step – II: Integrating those terms of N w.r.t y which do not contain x

As there is no such term, hence we skip this step.

Step – III: Thus solution is $-\dfrac{y\cos xy}{x} = c \;\Rightarrow y\cos xy + cx = 0$

(n) Solution: Given equation is: $\left(y + 2xy^2\right)dx + \left(x - x^2y\right)dy = 0 \qquad (1)$

Here $M = y + 2xy^2$ and $N = x - x^2y$

$M_y = 1 + 4xy$ and $N_x = 1 - 2xy$. Since, $M_y \neq N_x$ hence equation (1) is not exact.

To find I.F we apply Rule – III because given equation may be expressed as

$y(1+2xy)dx + x(1-xy)dy = 0$ which is of the form: $y f(xy)dx + x\,g(xy)dy = 0$

Rule III: $I.F = \dfrac{1}{xM - yN} = \dfrac{1}{xy + 2x^2y^2 - xy + x^2y^2} = \dfrac{1}{3x^2y^2}$. Multiplying (1) by I.F,

we get, $\dfrac{1}{3x^2y^2}\left(y + 2xy^2\right)dx + \dfrac{1}{3x^2y^2}\left(x - x^2y\right)dy = 0 \qquad (2)$

Equation (2)) is now exact. To find its general solution, we have

Step – I: Integrating M w.r.t x while keping y constant

$$\int M\,dx = \frac{1}{3y}\int \frac{1}{x^2}dx + \frac{2}{3}\int \frac{1}{x}dx = -\frac{1}{3xy} + \frac{2}{3}\ln x$$

Step – II: Integrating those terms of N w.r.t y which do not contain x: $-\frac{1}{3}\int \frac{1}{y}dy = -\frac{1}{3}\ln y$

Step – III: The general solution is, therefore: $-\frac{1}{3xy} + \frac{2}{3}\ln x - \frac{1}{3}\ln y = c$ Or

$3cxy + xy\ln y - 2xy\ln x + 1 = 0.$

(o) Solution: Given equation is $\left(x^2y - y^3\right)dx + \left(x^3 + xy^2\right)dy = 0$ (1)

Here $M = x^2y - y^3$ and $N = x^3 + xy^2$

$M_y = x^2 - 3y^2$ and $N_x = 3x^2 + y^2$. Since, $M_y \neq N_x$ hence equation (1) is not exact.

To find I.F we apply Rule – IV because given equation is homogeneous.

$$\text{Rule IV: I.F} = \frac{1}{xM + yN} = \frac{1}{x^3y - xy^3 + x^3y + xy^3} = \frac{1}{2x^3y}.$$

Multiplying (1) by I.F, we get: $\frac{1}{2x^3y}\left(y + 2xy^2\right)dx + \frac{1}{2x^3y}\left(x - x^2y\right)dy = 0$

$$\text{Or } \frac{1+2xy}{2x^3}dx + \frac{1-xy}{2x^2y}dx = 0 \qquad (2)$$

Equation (2)) is now exact. To find its general solution, we have

Step – I: Integrating M w.r.t x while keping y constant

$$\int M\,dx = \frac{1}{2}\int \frac{1}{x^3}dx + y\int \frac{1}{x^2}dx = -\frac{1}{4x^2} - \frac{y}{x}$$

Step – II: Integrating those terms of N w.r.t y which do not contain x.

We skip this step as there is no such term

Step – III: The general solution is, therefore: $-\frac{1}{4x^2} - \frac{y}{x} = c$ Or $4cx^2 + 4xy + 1 = 0$

(p) Solution: Given equation is $y\left(1+xy\right)dx + x\left(1-xy\right)dy = 0$ (1)

Here $M = y(1 + xy)$ and $N = x(1 - xy)$

$M_y = 1 + 2xy$ and $N_x = 1 - 2xy$. Since, $M_y \neq N_x$ hence equation (1) is not exact. To find I.F we apply Rule – III because given equation is of the form: $y f(xy)dx + x\,g(xy)dy = 0$

Rule III: $\text{I.F} = \frac{1}{xM - yN} = \frac{1}{xy + x^2y^2 - xy + x^2y^2} = \frac{1}{2x^2y^2}$. Multiplying (1) by I.F, we get

$$\frac{1}{2x^2y^2}y\left(1 + xy\right)dx + \frac{1}{2x^2y^2}x\left(1 - xy\right)dy = 0 \text{ Or } \frac{1+xy}{2x^2y}dx + \frac{1-xy}{2xy^2} = 0 \qquad (2)$$

Equation (2) is now exact. To find its general solution, we have

Step – I: Integrating M w.r.t x while keping y constant

$$\int M\,dx = \frac{1}{2y}\int \frac{1}{x^2}dx + \frac{1}{2}\int \frac{1}{x}dx = -\frac{1}{2xy} + \frac{1}{2}\ln x$$

StepII: Integrating those terms of N w.r.t y which do not contain x: $-\frac{1}{2}\int \frac{1}{y}dy = -\frac{1}{2}\ln y$

StepIII : The general solution is, therefore : $-\frac{1}{2xy}+\frac{1}{2}\ln x-\frac{1}{2}\ln y=c$

Or $2cxy-xy\ln x+xy\ln y+1=0.$

(q) Solution: Given equation is $y\ln y\,dx+\left(x-\ln y\right)dy=0$ (1)

Here $M=y\ln y$ and $N=\left(x-\ln y\right) \Rightarrow M_y=1+\ln y$ and $N_x=1.$

Since $M_y\neq N_x$ hence given equation is not exact. To find an I.F we apply :

Rule I : $\frac{M_y-N_x}{N}=\frac{1+\ln y-1}{x-\ln y}=\frac{\ln y}{x-\ln y}\neq P(x)$. Thus Rule – I fails.

Rule II : $\frac{N_x-M_y}{M}=\frac{1-1-\ln y}{y\ln y}=-\frac{\ln y}{y\ln y}=-\frac{1}{y}=P(y).$

Thus $I.F=e^{\int P\,dy}=e^{-\int\frac{1}{y}dy}=e^{-\ln y}=y^{-1}$. Multiplying (1) by I.F, we get

$\ln y\;dx+y^{-1}(x-\ln y)=0$ (2)

Equation (2) is now exact. To find its solution we have :

Step I : Integrating M w.r.t x keeping y constant : $\ln y\int 1.dx=x\ln y$

Step II : Integrating those terms of N w.r.t y which do not contains x.

$-\int\frac{\ln y}{y}dy=-\left(\ln y\right)^2/2$

Step III : Thus general solution is : $x\ln y-\left(\ln y\right)^2/2=c$

(r) Solution: Given equation is $y\left(1+xy\right)dx+x\left(1+xy+x^2y^2\right)dy=0$ (1)

Here $M=y(1+xy)$ and $N=1+xy+x^2y^2 \Rightarrow M_y=1+2xy,\; N_x=y+2xy^2.$

Since $M_y\neq N_x$ hence given equation is not exact. To find an I.F we apply Rule – III since given equation is of the form : $yf(xy)\,dx+xg(xy)\,dy=0$

$I.F=\frac{1}{xM-yN}=\frac{1}{xy+x^2y^2-xy-x^2y^2-x^2y^3}=-\frac{1}{x^2y^3}$. Multiplying (1) by I.F, we obtain

$-\frac{y(1+xy)}{x^2y^3}dx-\frac{x(1+xy+x^2y^2)}{x^2y^3}dy=0 \Rightarrow -\frac{(1+xy)}{x^2y^2}dx-\frac{(1+xy+x^2y^2)}{xy^3}dy=0$ (2)

Equation (2) is now exact. To find its solution, we have

Step – I : Integrating M w.r.t x while keping y constant

$\int M\;dx=-\frac{1}{y^2}\int\frac{1}{x^2}dx-\frac{1}{y}\int\frac{1}{x}dx=\frac{1}{xy^2}-\frac{1}{y}\ln x$

StepII : Integrating those terms of N w.r.t y which do not contain x : $-\int\frac{1}{y^2}dy=+\frac{1}{y}$

Step – III : The general solution is, therefore : $\frac{1}{xy^2}-\frac{1}{y}\ln x+\frac{1}{y}=c$

OR $1-xy\ln y+xy=cxy^2 \quad\Rightarrow\; cxy^2+xy\ln x-xy-1=0.$

(s) Solution: Given equation is $(y^3-3xy^2)dx+\left(2x^2y-xy^2\right)dy=0$ (1)

Here $M=y^3-3xy^2$ and $N=2x^2y-xy^2$.

$M_y=3y^2-6xy$ and $N_x=4xy-y^2$. Since, $M_y\neq N_x$ hence equation (1) is not exact.

To find I.F we apply Rule – III because given equation is homogeneous.

Rule III : $\text{I.F}=\dfrac{1}{xM+yN}=\dfrac{1}{xy^3-3x^2y^2+2x^2y^2-xy^3}=-\dfrac{1}{x^2y^2}.$

Multiplying (1) by I.F, we get

$$-\frac{1}{x^2y^2}\left(y^3-3xy^2\right)dx-\frac{1}{x^2y^2}\left(2x^2y-xy^2\right)dy=0 \quad (2)$$

Equation (2) is now exact. To find its general solution, we have

Step – I : Integrating M w.r.t x while keping y constant.

$$\int M dx=-y\int\frac{1}{x^2}dx+3y^2\int x\,dx=\frac{y}{x}+\frac{3x^2y^2}{2}$$

StepII : Integrating those terms of N w.r.t y which do not contain x : $-2\int\frac{1}{y}dy=-2\ln y$

SrepIII : The general solution is, therefore : $\dfrac{y}{x}+\dfrac{3x^2y^2}{2}-2\ln y=c$

OR $2y+3x^3y^2=2cx$

6. Solve the following Linear/Bernoulli differential equations

(a) Solution: Given equation is $\left(1+x^2\right)y`+2xy=4x^2$

Dividing by $(1+x^2)$, $\dfrac{dy}{dx}+\dfrac{2x}{1+x^2}y=\dfrac{4x^2}{1+x^2}$ (1)

Equation (1) is a linear differential equation(l. d. e). Here,

$\text{I.F}=e^{\int P\,dx}=e^{\int\frac{2x}{1+x^2}dx}e^{\ln\left(1+x^2\right)}=(1+x^2)$. .Multiplying (1) by I. F, we get

$(1+x^2)\dfrac{dy}{dx}+2xy=4x^2 \Rightarrow \dfrac{d}{dx}\left[y\times \text{I. F}\right]=4x^2$. Integrating, we get :

$\left[y\times \text{I. F}\right]=4\int x^2\,dx+c \Rightarrow y\,(1+x^2)=\dfrac{4x^3}{3}+c \Rightarrow y=\dfrac{4x^3+3c}{3(1+x^2)}.$

This is the general solution of equation (1).

(b) Solution: Given equation $y`+y\sec x=\tan x$ (1)

This is a linear differential equation(l. d. e). Here,

$\text{I.F}=e^{\int P\,dx}=e^{\int\sec x\,dx}=e^{\ln(\sec x+\tan x)}=(\sec x+\tan x)$. Multiplying (1) by I. F, we get

$(\sec x+\tan x)(dy/dx)+\sec x(\sec x+\tan x)y=\tan x(\sec x+\tan x)$

$\Rightarrow\dfrac{d}{dx}\left[y\times \text{I. F}\right]=\sec^2 x+\tan x\sec x$. Integrating, we get :

$\left[y\times \text{I. F}\right]=\int\left(\sec^2 x+\sec x\tan x\right)dx+c \Rightarrow y\,(\sec x+\tan x)=\tan x+\sec x+c$

This is the general solution of equation (1).

(c) Solution: Given equiation $y` + y/x = x^2$ (1)

This is a linear differential equation(l. d. e). Here,

$I.F = e^{\int P\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$ Multiplying (1) by I. F, we get

$x\frac{dy}{dx} + y = x^3 \quad \Rightarrow \frac{d}{dx}[y\times I.\ F] = x^3.$ Integrating, we get:

$[y\times I.\ F] = \int x^3\,dx + c \ \Rightarrow y\ x = \frac{x^4}{4} + c \qquad \Rightarrow 4xy = x^4 + C,\ C = 4c.$

This is the general solution of equation (1).

(d) Solution: Given equation $y` - y\tan x = -2\sin x$ (1)

This is a linear differential equation(l. d. e). Here,

$I.F = e^{\int P\,dx} = e^{-\int \tan x\,dx} = e^{\ln\cos x} = \cos x.$ Multiplying (1) by I. F, we get

$\cos x\frac{dy}{dx} - \cos x\tan x\ y = -2\sin x\ \cos x \quad \Rightarrow \frac{d}{dx}[y\times I.\ F] = -\sin 2x.$

Integrating, we get the general solution:

$[y\times I.\ F] = -\int \sin 2x\,dx + c \ \Rightarrow y\cos x = +\frac{\cos 2x}{2} + c \Rightarrow 2\cos x\ y = \cos 2x + C,\ C = 2c.$

(e) Solution: Given equation $\sec x\ y` = y + \sin x \Rightarrow y` - \cos x\ .\ y = \sin x\ \cos x$ (1)

This is a linear differential equation(l. d. e). Here,

$I.F = e^{\int P\,dx} = e^{-\int \cos x\,dx} = e^{-\sin x}.$ Multiplying (1) by I. F, we get

$e^{-\sin x}\frac{dy}{dx} - e^{-\sin x}\cos x\ \ y = \sin x\ \cos x\ e^{-\sin x} \quad \Rightarrow \frac{d}{dx}[y\times I.\ F] = \sin x\ \cos x\ e^{-\sin x}.$

Integrating, we get:

$[y\times I.\ F] = \int \sin x\ \cos x\ e^{-\sin x}\,dx + c \ \Rightarrow ye^{-\sin x} = \int \sin x\ \cos x\ e^{-\sin x}\,dx + c$ (2)

Putting $z = \sin x \ \Rightarrow dz = \cos x\,dx$. Thus (2) becomes

$ye^{-\sin x} = \int ze^{-z}\,dz + c = -ze^{-z} + \int 1.e^{-z}\,dz + c = -ze^{-z} - e^{-z} + c = -e^{-z}(z+1) + c$ [By parts]

Putting $z = \sin x$, we obtain: $ye^{-\sin x} = -e^{-\sin x}(\sin x + 1) + c.$ Multiplying by $e^{\sin x}$, we get

$\Rightarrow\ y = -(\sin x + 1) + ce^{\sin x}.$ This is the general solution of equation (1).

(f) Solution: Given equation is $y` + y\tan x = \sec x$ (1)

This is a l.d.e hence I.F = $e^{\int \tan x\,dx} = e^{\ln(\sec x)} = \sec x.$ Multiplying (1) by I.F, we get

$\sec x\ y`+y\ \tan x\ \sec x = \sec^2 x \ \Rightarrow \frac{d}{dx}[y\times I.F] = \sec^2 x.$ Integrating, we obtain

$[y\times I.F] = \int \sec^2 x\ dx + c = \tan x + c \ \Rightarrow\ y\ \sec x = \tan x + c.$ Dividing by $\sec x$, we get

$y = \sin x + c\cos x.$ This is the general solution of given differential equation.

(g) Solution: Given equation is $\left(1 + x^2\right) y` + 2xy = \cos x$ (1)

$\Rightarrow y` + \frac{2x}{1+x^2}y = \frac{\cos x}{1+x^2}.$ This is a l.d.e hence I.F = $e^{\int \frac{2x}{1+x^2}\,dx} = e^{\ln(1+x^2)} = (1+x^2)$

Multiplying both sides by I.F, we obtain

Thus, LHS is: $\frac{d}{dx}[y\times I.F] = \cos x$. Integrating, we get $[y\times I.F] = \int \cos x\ dx + c = \sin x + c$

$\Rightarrow y\ (1+x^2) = \sin x + c$. This is the general solution of given differential equation.

(h) Solution: Given equation is $x \ln x\ y` + y = 2 \ln x$. Dividing by x lnx, we get:

$$\frac{dy}{dx} + \frac{1}{x\ \ln x}y = \frac{2}{x} \qquad (1)$$

This is a l.d.e hence, I.F = $e^{\int \frac{1}{x\ \ln x}dx}$. Putting $z = \ln x \quad \Rightarrow dz = \frac{1}{x}dx$.

Thus, $I.F = e^{\int \frac{1}{z}dz} = e^{\ln z} = z = \ln x$. Multiplying (1) by I.F, we obtain

$\ln x\frac{dy}{dx} + \frac{y}{x} = \frac{2}{x}\ln x \quad \Rightarrow \frac{d}{dx}[y\times I.F] = \frac{2\ln x}{x}$. Integrating, we obtain:

$[y\times I.F] = 2\int \frac{\ln x}{x}\ dx + c = 2\frac{(\ln x)^2}{2} + c$

$\Rightarrow y\ln x = (\ln x)^2 + c$. This is the general solution of given differential equation.

(i) Solution: Given equation is $\left(x + 2y^3\right)y` = y \Rightarrow \frac{dy}{dx} = \frac{y}{x+2y^3} \quad \Rightarrow \frac{dx}{dy} = \frac{x+2y^3}{y}$

$$\Rightarrow \qquad \frac{dx}{dy} - \frac{1}{y}x = 2y^2 \qquad (1)$$

This is a l.d.e in x hence, I.F = $e^{\int P\ dy} = e^{\int \frac{1}{y}dy} = e^{\ln y} = y$. Multiplying (1) by I.F, we get

$y\frac{dx}{dy} - x = 2y^3 \quad \Rightarrow \frac{d}{dy}[x\times I.F] = 2y^3$. Integrating, we obtain:

$[x\times I.F] = 2\int y^3\ dy + c = 2\frac{y^4}{4} + c = \frac{y^4}{2} + c \Rightarrow xy = \frac{y^4}{2} + c$

Or $2xy = y^4 + C,\ (C = 2c)$. This is the general solution of given differential equation.

(j) Solution: Given equation is $\left(1 + y^2\right)dx = \left(\tan^{-1}y - x\right)dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} \quad \Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2} \qquad (1)$$

Equatiion (1) is l.d.e in x. Hence, I.F = $e^{\int P\ dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$.

Multiplying (1) by I.F, we get

$e^{\tan^{-1}y}\left[\frac{dx}{dy} - \frac{1}{1+y^2}x\right] = \frac{\tan^{-1}y\ e^{\tan^{-1}y}}{1+y^2} \quad \Rightarrow \frac{d}{dy}[x\times I.F] = \frac{\tan^{-1}y\ e^{\tan^{-1}y}}{1+y^2}$.

Integrating, we obtain: $[x\times I.F] = \int \frac{\tan^{-1}y\ e^{\tan^{-1}y}}{1+y^2}dy + c$.

Putting $z = \tan^{-1}y \quad \Rightarrow dz = dy/\left(1+y^2\right)$. Thus, $[x\times I.F] = \int z e^z\ dz + c$

$\Rightarrow \quad x.e^{\tan^{-1}y} = ze^z - \int 1.e^z\, dz + c = ze^z - e^z + c = e^z(z-1) + c \quad [\text{By parts}]$

Putting $z = \tan^{-1} y$, we get : $x\, e^{\tan^{-1}y} = e^{\tan^{-1}y}\left(\tan^{-1} y - 1\right) + c$

This is the general solution of given differential equation.

(k) Solution: Given equation is $\left(2x - 10y^3\right) y` + y = 0$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{\left(2x - 10y^3\right)} \quad \Rightarrow \frac{dx}{dy} = -\frac{2x - 10y^3}{y} \quad \Rightarrow \frac{dx}{dy} + \frac{2}{y}x = 10y^2 \qquad (1)$$

Equation (1) is l.d.e in x. Thus, $\text{I.F} = e^{\int P\, dy} = e^{\int \frac{2}{y} dy} = e^{2\ln y} = e^{\ln y^2} = y^2.$

Multiplying (1) by I.F, we get : $y^2\left[\frac{dx}{dy} + \frac{2}{y}x\right] = 10y^4 \quad \Rightarrow \frac{d}{dy}\left[x \times \text{I.F}\right] = 10y^4$

Integrating, we obtain : $\left[x \times \text{I.F}\right] = 10\int y^4\, dy + c = 10\frac{y^5}{5} + c = 2y^5 + c.$

$\Rightarrow \; xy^2 = 2y^5 + c.$ This is the general solution of given differential equation.

(m) Solution: Given equation is $y` = y/x + y^2 \;\Rightarrow \frac{dy}{dx} - \frac{1}{x}y = y^2 \qquad (1)$

Equation (1) is a Bernoulli differential equation (B.D.E). Dividing by y^2, we get

$y^{-2}\frac{dy}{dx} - \frac{1}{x}y^{-1} = 1.$ Putting $z = y^{-1} \;\Rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx}$ or $y^{-2}\frac{dy}{dx} = -\frac{dz}{dx}$

Thus, equation (1) becomes : $-\frac{dz}{dx} - \frac{1}{x}z = 1 \quad \Rightarrow \frac{dz}{dx} + \frac{1}{x}z = -1 \qquad (2)$

Equation (2) is now a l.d.e. Thus, $\text{I.F} = e^{\int P\, dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x.$

Multiplying (2) by I.F, we get

$x\left[\frac{dz}{dx} + \frac{1}{x}z\right] = -x \quad \Rightarrow \frac{d}{dy}\left[z \times \text{I.F}\right] = -x$

Integrating, we obtain : $\left[z \times \text{I.F}\right] = -\int x\, dx + c = -x + c.$

$\Rightarrow \; xz = -x + c.$ Putting $z = y^{-1}$, we get : $xy^{-1} = -x + c \quad \Rightarrow x = y(c - x).$

This is the general solution of given differential equation.

(n) Solution: Given equation is $\frac{dy}{dx} = \frac{x+y+1}{x+1} = \frac{x+1}{x+1} + \frac{y}{x+1} = 1 + \frac{y}{x+1}$

$$\Rightarrow \qquad \frac{dy}{dx} - \frac{1}{x+1}y = 1 \qquad (1)$$

Equation (1) is a l.d.e with $\text{I.F} = e^{\int p\, dx} = e^{\int \frac{1}{x+1} dx} = e^{\ln(x+1)} = (x+1)$

Multiplying (1) by I.F, we get : $(x+1)\left[\frac{dy}{dx} - \frac{1}{x+1}y\right] = (x+1)$

$\Rightarrow \dfrac{d}{dx}\left[y \times I.F\right] = (x+1).$ Integrating, we get: $\left[y \times I.F\right] = \dfrac{x^2}{2} + x + c$

$\Rightarrow y(1+x) = \dfrac{x^2}{2} + x + c.$ This is a general solution of given differential equation.

(o) Solution: Given equation is $\dfrac{dy}{dx} - \dfrac{1}{2x} y = \dfrac{y^2}{2x^2}$ (1)

Equation (1) is B.D.E. Dividing by y^2, we get: $y^{-2}\dfrac{dy}{dx} - \dfrac{1}{2x} y^{-1} = \dfrac{1}{2x^2}$

Putting $z = y^{-1} \Rightarrow \dfrac{dz}{dx} = -y^{-2}\dfrac{dy}{dx} \quad \Rightarrow -\dfrac{dz}{dx} = y^{-2}\dfrac{dy}{dx}.$

Thus, above equation becomes: $-\dfrac{dz}{dx} - \dfrac{1}{2x} z = \dfrac{1}{2x^2} \qquad \Rightarrow \dfrac{dz}{dx} + \dfrac{1}{2x} z = -\dfrac{1}{2x^2}$ (2)

Equation (2) is a l.d.e. Thus, $I.F = e^{\int p\,dx} = e^{\frac{1}{2}\int \frac{1}{x}dx} = e^{\frac{1}{2}\ln x} = e^{\ln x^{1/2}} = x^{1/2}$

Multiplying (2) by I.F, we get: $x^{1/2}\left[\dfrac{dz}{dx} + \dfrac{1}{2x} z\right] = -\dfrac{x^{1/2}}{2x^2} = -\dfrac{1}{2} x^{-3/2}$

$\Rightarrow \dfrac{d}{dx}\left[z \times I.F\right] = -\dfrac{1}{2} x^{-3/2}.$ Integrating, we get: $\left[z \times I.F\right] = -(-2)\dfrac{x^{-1/2}}{2} + c = x^{-1/2} + c$

Putting $z = y^{-1}$, we get: $y^{-1}x^{1/2} = x^{-1/2} + c \Rightarrow xy^{-1} = 1 + cx^{1/2}.$

This is a general solution of given differential equation.

(p) Solution: Given equation is $y` = x^3y^3 - xy \quad \Rightarrow y` + xy = x^3y^3$ (1)

Equation (1) is B.D.E. Dividing by y^3, we get $y^{-3}\dfrac{dy}{dx} + xy^{-2} = x^3$

Putting $z = y^{-2} \Rightarrow \dfrac{dz}{dx} = -2y^{-3}\dfrac{dy}{dx} \quad \Rightarrow -\dfrac{1}{2}\dfrac{dz}{dx} = y^{-3}\dfrac{dy}{dx}.$ Thus, above equation becomes

$-\dfrac{1}{2}\dfrac{dz}{dx} + xz = x^3 \qquad \Rightarrow \dfrac{dz}{dx} - 2x\,z = -2x^3$ (2)

Equation (2) is a l.d.e. Thus, $I.F = e^{\int p\,dx} = e^{-2\int x\,dx} = e^{-x^2}$

Multiplying (2) by I.F, we get: $e^{-x^2}\left[\dfrac{dz}{dx} - 2x\,z\right] = -2x^3e^{-x^2}$

$\Rightarrow \dfrac{d}{dx}\left[z \times I.F\right] = -2x^3e^{-x^2}.$ Integrating, we get

$\left[z \times I.F\right] = -2\int x^3e^{-x^2}dx + c = -\int x^2e^{-x^2}(2x)\,dx + c$

Putting $u = x^2 \Rightarrow du = 2x\,dx$. Thius, above equation becomes

$\left[z \times I.F\right] = -\int ue^{-u}du + c = -\left[-ue^{-u} + \int 1.e^{-u}\,du\right] + c = ue^{-u} + e^{-u} + c = e^{-u}\left[u+1\right] + c$

Putting $z = y^{-2}$, $u = x^2$ and $I.F = e^{-x^2}$, we get:

$y^{-2}e^{-x^2} = e^{-x^2}\left[x^2 + 1\right] + c \Rightarrow y^{-2} = x^2 + 1 + ce^{x^2}$. This is general solution of given d.e

(q) Solution: Given equation is $\frac{dr}{d\theta}+r\tan\theta=\cos^2\theta;\ r\left(\frac{\pi}{4}\right)=1$ (1)

This is a l.d.e hence, $I.F=e^{\int \tan\theta\, d\theta}=e^{\ln\sec\theta}=\sec\theta$. Multiplying (1) by I.F, we get

$\sec\theta\left[\frac{dr}{d\theta}+r\tan\theta\right]=\sec\theta.\cos^2\theta \Rightarrow \frac{d}{d\theta}[r\times I.F]=\cos\theta$. Integrating, we get

$[r\times I.F]=\int\cos\theta\, d\theta=\sin\theta+c \Rightarrow r\sec\theta=\sin\theta+c$ (2)

Equation (2) is the general solution of (1). Now, put $r=1$ and $\theta=\pi/4$, we get

$\sec(\pi/4)=\sin(\pi/4)+c \Rightarrow \sqrt{2}=\frac{1}{\sqrt{2}}+c \Rightarrow c=\sqrt{2}-\frac{1}{\sqrt{2}}=\frac{2-1}{\sqrt{2}}=\frac{1}{\sqrt{2}}.$

Thus, (2) becomes: $r\sec\theta=\sin\theta+\frac{1}{\sqrt{2}} \Rightarrow \sqrt{2}\ r\sec\theta=\sqrt{2}\sin\theta+1.$

This is a particular solution of equation (1).

(r) Solution: Given equation is $\left(x^2+1\right)y`+4xy=x;\ y(2)=1$. Dividing by $(1+x^2)$

$$\frac{dy}{dx}+\frac{4x}{1+x^2}y=\frac{x}{1+x^2} \qquad (1)$$

Equation (1) is a l.d.e. Thus, $I.F=e^{\int P dx}=e^{2\int\frac{2x}{1+x^2}dx}=e^{2\ln\left(1+x^2\right)}=e^{\ln\left(1+x^2\right)^2}=\left(1+x^2\right)^2.$

Multiplying (1) by I.F, we get: $\left(1+x^2\right)^2\left[\frac{dy}{dx}+\frac{4x}{1+x^2}y\right]=\left(1+x^2\right)^2\frac{x}{1+x^2}=x\left(1+x^2\right)$

$\Rightarrow \frac{d}{dx}[y\times I.F]=x\left(1+x^2\right)$. Integrating, we get

$[y\times I.F]=\int\left(x+x^3\right)dx=\frac{x^2}{2}+\frac{x^4}{4}+c \Rightarrow y\left(1+x^2\right)^2=\frac{x^2}{2}+\frac{x^4}{4}+c$ (2)

Equation (2) is the general solution of (1). Now, put $y=1$ and $x=2$, we get

$25=2+4+c \Rightarrow c=19$. Thus, (2) becomes

$y\left(1+x^2\right)^2=\frac{x^2}{2}+\frac{x^4}{4}+19$. This is a particular solution of equation (1).

(s) Solution: Given equation is $x(2+x)\,y`+2(1+x)\,y=1+3x^2;\ y(-1)=1$

Dividing by x(2 + x), we get: $\frac{dy}{dx}+\frac{2+2x}{x(x+2)}y=\frac{1+3x^2}{x(x+2)}$ (1)

Equation (1) is a l.d.e. Thus, $I.F=e^{\int P dx}=e^{\int\frac{2+2x}{x(x+2)}dx}=e^{\int\left[\frac{1}{x}+\frac{1}{x+2}\right]dx}$

$$=e^{[\ln x+\ln(x+2)]}=e^{\ln x(x+2)}=x(x+2)$$

Multipying (1) by I.F, we get: $x(x+2)\left[\frac{dy}{dx}+\frac{2+2x}{x(x+2)}y\right]=x(x+2).\frac{1+3x^2}{x(x+2)}$

$\Rightarrow \frac{d}{dx}[y\times I.F]=1+3x^2$. Integrating, we get: $[y\times I.F]=\int\left(1+3x^2\right)dx=x+x^3+c$

$\Rightarrow y\left[x(x+2)\right] = x + x^3 + c$ (2)

Equation (2) is a general solution of (1). Putting $x = -1$ and $y = 1$, we obtain

$-1 = -1 - 1 + c \;\; \Rightarrow c = 1$. Put this in (2) we have: $y\left[x(x+2)\right] = x + x^3 + 1$

This is a particular solution of (1).

(t) Solution: Given equation is $y` + 2xy = 2x^3;\; y(0) = 1$ (1)

Equation (1) is a l.d.e. Thus, $I.F = e^{\int P\,dx} = e^{2\int x\,dx} = e^{x^2}$

Multipying (1) by I.F, we get: $e^{x^2}\left[\dfrac{dy}{dx} + 2x\,y\right] = 2x^3 e^{x^2}$

$\Rightarrow \dfrac{d}{dx}\left[y \times I.F\right] = 2x^3 e^{x^2}$. Integrating, we get: $\left[y \times I.F\right] = \int 2x^3 e^{x^2}\,dx = \int x^2 e^{x^2}(2x)\,dx$

Substituting $z = x^2 \Rightarrow dz = 2x\,dx$. Thus, we have

$\left[y \times I.F\right] = \int z\,e^z\,dz = ze^z - \int 1.e^z\,dz = ze^z - e^z = e^z(z-1) + c$. Putting $z = x^2$, we get

$$ye^{x^2} = e^{x^2}(x^2 - 1) + c \qquad (2)$$

Equation (2) is a general solution of (1). Now, substituting $x = 0$ and $y = 1$, we obtain

$1 = (1-1) + c \;\; \Rightarrow c = 1$. Put this in (2) we have: $ye^{x^2} = e^{x^2}(x^2 - 1) + 1$

This is a particular solution of (1).

(u) Solution: Given equation $\dfrac{dy}{dx} + \dfrac{2}{x}y = -x^9 y^5;\;\; y(-1) = 2$ (1)

is a B.D.E. Dividing by y^5, we get: $y^{-5}y` + (2/x)\,y^{-4} = -x^9$.

Substituting $z = y^{-4} \Rightarrow \dfrac{dz}{dx} = -4y^{-5}\dfrac{dy}{dx} \;\; \Rightarrow -\dfrac{1}{4}\dfrac{dy}{dx} = y^{-5}\dfrac{dy}{dx}$.

Thus, above equation becomes

$-\dfrac{1}{4}\dfrac{dz}{dx} + \dfrac{2}{x}z = -x^9 \;\; \Rightarrow \qquad \dfrac{dz}{dx} - \dfrac{8}{x}z = 4x^9$ (2)

Equation (2) is now a l.d.e. Thus, $I.F = e^{\int P\,dx} = e^{-8\int \frac{1}{x}dx} = e^{-8\ln x} = e^{\left[\ln x^{-8}\right]} = x^{-8}$

Multipying (2) by I.F, we get: $x^{-8}\left[\dfrac{dz}{dx} - \dfrac{8}{x}z\right] = 4x^9.x^{-8} = 4x$

$\Rightarrow \dfrac{d}{dx}\left[z \times I.F\right] = 4x$. Integrating, we get: $\left[z \times I.F\right] = 4\int x\,dx = 2x^2 + c$

$\Rightarrow z\,x^{-8} = 2x^2 + c$ or $z = 2x^{10} + cx^8 \qquad \Rightarrow y^{-4} = 2x^{10} + cx^8$ (3)

Equation (3) is a general solution of (1). Putting $x = -1$ and $y = 2$, we obtain

$1/16 = 2 + c \;\; \Rightarrow c = -31/2$. Put this in (3) we have: $y^{-4} = 2x^{10} - \dfrac{31}{2}x^8$

This is a particular solution of (1).

(v) Solution: Given equation is $y` - y/x = y^2 \sin x$ (1)

This is B.D.E. Dividing by y^2, we get: $y^{-2}\,y` - \frac{1}{x}y^{-1} = \sin x.$

Putting $z = y^{-1} \Rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow -\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$. Thus, above equation becomes

$$-\frac{dz}{dx} - \frac{1}{x}z = \sin x \quad \Rightarrow \frac{dz}{dx} + \frac{1}{x}z = -\sin x \qquad (2)$$

Equation (2) is a l.d.e. Thus, $I.F = e^{\int \frac{1}{x}dx} = e^{\ln x} = e^{\ln x} = x$

Multiplying (2) by I.F, we get: $x\frac{dz}{dx} + z = -x\sin x \Rightarrow \frac{d}{dx}[z \times I.F] = -x\sin x$

Integrating, $[z \times I.F] = -\int x\sin x\,dx = -\left[x(-\cos x) - \int 1.(-\cos x)\,dx\right]$

$zx = x\cos x - \sin x + c$. Putting $z = y^{-1}$ we obtain: $xy^{-1} = x\cos x - \sin x + c.$

This is a general solution of equation (1). [NOTE: See the change in the problem]

(w) Solution: Given equation is $xy - y` = y^3 e^{-x^2} \Rightarrow y` - xy = -y^3 e^{-x^2}$ (1)

This is B.D.E. Dividing by y^3, we get: $y^{-3}\,y` - x\,y^{-2} = -e^{x^{-2}}$

Putting $z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}\frac{dz}{dx} = y^{-3}\frac{dy}{dx}$. Thus, above equation becomes

$$-\frac{1}{2}\frac{dz}{dx} - xz = -e^{-x^2} \quad \Rightarrow \frac{dz}{dx} + 2x\,z = 2e^{-x^2} \qquad (2)$$

Equation (2) is a l.d.e. Thus, $I.F = e^{2\int x\,dx} = e^{x^2}$

Multiplying (2) by I.F, we get: $e^{x^2}\left[\frac{dz}{dx} + 2x\,z\right] = 2 \Rightarrow \frac{d}{dx}[z \times I.F] = 2$

Integrating, $[z \times I.F] = 2\int 1\,dx = 2x + c \Rightarrow zx = 2x + c$. Putting $z = y^{-2}$ and $I.F = e^{x^2}$

we obtain: $e^{x^2}y^{-2} = 2x + c$. This is a general solution of equation (1).

7. Find the orthogonal trajectories for the following curves:

(a) Solution: Given that $xy = c$ (1)

Differentiating w.r.t x, we get: $xy` + y = 0 \Rightarrow y` = -y/x$.

This is the slope of curve (1). Thus, slope of O.T is: $\frac{dy}{dx} = +\frac{x}{y}$

Separating the variables and integrating, we get: $\frac{y^2}{2} = \frac{x^2}{2} + c$

Or $y^2 + x^2 = C$, $(C = 2c)$. This is the equation of required O.T

(b) Solution: Given that $y^2 = 4ax \quad \Rightarrow 4a = y^2/x$ (1)

Differentiating w.r.t x, we get: $2y\,y` = 4a \Rightarrow 2y\,y` = y^2/x$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2x}$. This is the slope of curve (1). Thus, slope of O.T is: $\frac{dy}{dx} = -\frac{2x}{y}$

Separating the variables and integrating, we get:

$$\int y\ dy = -2\int x\ dx + c \Rightarrow \frac{y^2}{2} = -x^2 + c$$

$y^2 + 2x^2 = C$, $(C = 2c)$. This is the equation of required O.T.

(c) Solution: Given that $x^{2/3} + y^{2/3} = a^{2/3}$ (1)

Differentiating w.r.t x, we get: $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$

This is the slope of curve (1). Thus, slope of O.T is: $\frac{dy}{dx} = +\left(\frac{x}{y}\right)^{1/3}$

Separating the variables and integrating, we get

$$\int y^{1/3}\ dy = \int x^{1/3}\ dx + c \Rightarrow \frac{3}{4}y^{4/3} = \frac{3}{4}x^{4/3} + c$$

Or $y^{4/3} - x^{4/3} = C$, $\left(C = 4c/3\right)$. This is the equation of required O.T.

(d) Solution: Given that $a\,y^2 = x^3 \quad \Rightarrow a = x^3/y^2$ (1)

Differentiating w.r.t x, we get: $2ay\,y` = 3x^2 \Rightarrow 2y\left(x^3/y^2\right)y` = 3x^2$

$\Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right)$. This is the slope of curve (1). Thus, slope of O.T is: $\frac{dy}{dx} = -\frac{2x}{3y}$

Separating the variables & integrating, we get: $\int y\ dy = -\frac{2}{3}\int x\ dx + c \Rightarrow \frac{y^2}{2} = -\frac{x^2}{3} + c$

$3y^2 + 2x^2 = C$, $\left(C = 6c\right)$. This is the equation of required O.T.

(e) Solution: Given that $r = a\left(1 - \cos\theta\right) \Rightarrow a = r/\left(1 - \cos\theta\right)$ (1)

Differentiating w.r.t r, we get: $1 = a\sin\theta.\frac{d\theta}{dr} \quad \Rightarrow \frac{d\theta}{dr} = \frac{1}{a\sin\theta} = \frac{(1-\cos\theta)}{r\sin\theta}$

$\Rightarrow r\frac{d\theta}{dr} = \frac{(1-\cos\theta)}{\sin\theta}$. This is the slope of curve (1). Thus, slope of O.T is:

$$r\frac{d\theta}{dr} = -\frac{\sin\theta}{1-\cos\theta} = -\frac{2\sin(\theta/2).\cos(\theta/2)}{2\sin^2(\theta/2)} = -\cot(\theta/2).$$

Separating the variables and integrating, we get:

$$\int \tan\frac{\theta}{2}d\theta = -\int\frac{1}{r}dr + \ln c \Rightarrow 2\ln\left(\sec\frac{\theta}{2}\right) = -\ln r + \ln c \Rightarrow \ln\left(\sec\frac{\theta}{2}\right)^2 = \ln\left(\frac{c}{r}\right)$$

$\sec^2\frac{\theta}{2} = \frac{c}{r} \Rightarrow r\sec^2\frac{\theta}{2} = c$. This is the equation of required O.T.

(f) Solution: Given that $r = a\left(1 + \cos\theta\right) \Rightarrow a = r/\left(1 + \cos\theta\right)$ (1)

Differentiating w.r.t r, we get: $1 = -a\sin\theta.\frac{d\theta}{dr} \quad \Rightarrow \frac{d\theta}{dr} = -\frac{1}{a\sin\theta} = -\frac{(1+\cos\theta)}{r\sin\theta}$

$\Rightarrow r\frac{d\theta}{dr} = -\frac{(1+\cos\theta)}{\sin\theta}$. This is the slope of curve (1). Thus, slope of O.T is:

$r\frac{d\theta}{dr} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin(\theta/2).\cos(\theta/2)}{2\cos^2(\theta/2)} = \tan(\theta/2)$. Separating the variables, we get:

$$\int \cot\frac{\theta}{2}d\theta = \int \frac{1}{r}\,dr + \ln c \Rightarrow 2\ln\left(\sin\frac{\theta}{2}\right) = \ln r + \ln c \Rightarrow \ln\left(\sin\frac{\theta}{2}\right)^2 = \ln(cr)$$

$\sin^2(\theta/2) = cr.$ This is the equation of required O.T.

(g) Solution: Given that $r = a^{\theta} \Rightarrow \ln r = \theta.\ln a \Rightarrow \ln a = \ln r/\theta$ (1)

Differentiating w.r.t r, we get: $1/r = \ln a.\dfrac{d\theta}{dr} \Rightarrow r\dfrac{d\theta}{dr} = \dfrac{1}{\ln a} = \dfrac{\theta}{\ln r}$

This is the slope of curve (1). Thus, slope of O.T is:

$r\dfrac{d\theta}{dr} = -\dfrac{\ln r}{\theta}$. Separating the variables, and integrating, we get

$\displaystyle\int \theta\, d\theta = -\int \frac{\ln r}{r}\,dr + c \Rightarrow \frac{\theta^2}{2} = -[\ln r]^2 + c.$ This is the equation of required O.T.

(h) Solution: Given that $r^n \sin n\theta = a^n$ (1)

Differentiating w.r.t r, we get: $r^n(n\cos n\theta)\dfrac{d\theta}{dr} + nr^{n-1}\sin n\theta = 0.$ Dividing by $(n r^n)$

$$\Rightarrow \cos n\theta\frac{d\theta}{dr} + r^{-1}\sin n\theta = 0 \Rightarrow \cos n\theta\frac{d\theta}{dr} = -\frac{\sin n\theta}{r} \Rightarrow r\frac{d\theta}{dr} = -\tan n\theta$$

This is the slope of curve (1). Thus, slope of O.T is: $r\dfrac{d\theta}{dr} = +\cot n\theta$

Separating the variables, we get: $\displaystyle\int \tan n\theta\, d\theta = \int \frac{1}{r}\,dr + \ln c \Rightarrow \frac{\ln\sec n\theta}{2} = \ln r + \ln c = \ln(cr)$

$\Rightarrow \ln\sec n\theta = 2\ln(cr) = \ln(cr)^2 \Rightarrow \sec n\theta = c^2r^2.$ This is the equation of required O.T.

(i) Solution: Given that $r^2 = C\sin 2\theta$ (1)

Differentiating w.r.t r, we get: $2r = 2C\cos\theta\dfrac{d\theta}{dr}.$ From (1) $C = r^2/\sin 2\theta$.

Putting this in (1), we get: $r = r^2\cot 2\theta\dfrac{d\theta}{dr} \Rightarrow \dfrac{1}{\cot 2\theta} = r\dfrac{dr}{d\theta}.$ This is the slope of curve (1).

Thus, slope of O.T is: $r\dfrac{d\theta}{dr} = -\tan 2\theta.$ Separating the variables and integrating, we get:

$$\int \cot 2\theta\, d\theta = -\int \frac{1}{r}\,dr + \ln c \Rightarrow \frac{\ln\sin 2\theta}{2} = -\ln r + \ln c = \ln\left(\frac{c}{r}\right) \Rightarrow \ln\sin 2\theta = 2\ln\left(\frac{c}{r}\right) = \ln\left(\frac{c}{r}\right)^2$$

$\Rightarrow \sin 2\theta = \left(\dfrac{c}{r}\right)^2 \Rightarrow r^2\sin 2\theta = c^2.$ This is the equation of required O.T.

8. Find the member of the orthogonal trajectories for $x + y = c\,e^y$ that passes through (0, 5).

Solution: Given that $x + y = ce^y \Rightarrow c = (x+y)/e^y$ (1)

Differentiating w.r.t x, we get: $1 + y` = ce^y.y` \Rightarrow 1 + y` = (x+y)\,y`$

$\Rightarrow (x+y-1)y` = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x+y-1}.$ This is the slope of curve (1).

Thus, slope of O.T is: $\dfrac{dy}{dx} = -(x+y-1) \Rightarrow \dfrac{dy}{dx} + y = 1 - x$ (2)

Equation (2) is a l.d.e.

Thus, $\text{I.F} = e^{\int P\,dx} = e^{\int 1\,dx} = e^x$. Multiplying (2) by I.F, we get:

$e^x\left[\dfrac{dy}{dx} + y\right] = e^x(1-x) \Rightarrow \dfrac{d}{dx}\left[y \times \text{I.F}\right] = e^x(1-x)$. Integrating, w get

$$\left[y \times \text{I.F}\right] = \int e^x(1-x)\,dx + c = (1-x)e^x - \int (0-1)e^x\,dx + c = (1-x)e^x + e^x + c$$

$$\left[y \times \text{I.F}\right] = e^x(2-x) + c \Rightarrow ye^x = e^x(2-x) + c \Rightarrow y = (2-x) + ce^{-x} \qquad (3)$$

This is the equation of required O.T. Now this O.T is passing through (0, 5).

Thus, putting $x = 0$ and $y = 5$, we get: $5 = 2 + c \Rightarrow c = 3$. Thus (3) becomes $y = (2-x) + 3e^{-x}$

Chapter 3

DIFFERENTIAL EQUATIONS OF FIRST ORDER HIGHER DEGREE

WORKSHEET 03

1. Solve the following non-linear differential equations (Solvable for p).

(i) Solution: Given equation is $p^2 + p(x + y) + xy = 0$

$\Rightarrow p^2 + px + py + xy = 0 \quad \Rightarrow p(p+x) + y(p+x) = 0$

$\Rightarrow (p+x)(p+y) = 0 \quad \Rightarrow p + x = 0$ or $p + y = 0$

Let us consider the fist equation $p + x = 0 \quad \Rightarrow \dfrac{dy}{dx} = -x \quad \Rightarrow dy = -x\,dx$

Integrating, we get: $\int 1\,dy = -\int x\,dx + c \quad \Rightarrow y = -\dfrac{x^2}{2} + c \quad \Rightarrow 2y = -x^2 + 2c$

Or $2y + x^2 - 2c = 0$ (1)

Now consider the second equation $p + y = 0 \quad \Rightarrow \dfrac{dy}{dx} = -y \quad \Rightarrow \dfrac{dy}{y} = -dx$

Integrating, we get: $\int \dfrac{1}{y}\,dy = -\int 1\,dx + c \quad \Rightarrow \ln y = -x + c \quad \Rightarrow y = e^{c-x}$

Or $y - e^{c-x} = 0$ (2)

Combining (1) and (2) we get: $\left(2y + x^2 - 2c\right)\left(y - e^{c-x}\right) = 0$

This is the general solution of given differential equation.

(ii) Solution: Given equation is $p^2 + 2px - 3x^2 = 0 \quad \Rightarrow p^2 + 3px - px - 3x^2 = 0$

$\Rightarrow p(p+3x) - x(p-3x) = 0 \quad \Rightarrow (p+3x)(p-x) = 0 \quad \Rightarrow p + 3x = 0$ or $p - x = 0$

Let us consider the fist equation $p + 3x = 0 \quad \Rightarrow \dfrac{dy}{dx} = -3x \quad \Rightarrow dy = -3x\,dx$

Integrating, we get: $\int 1\,dy = -3\int x\,dx + c \quad \Rightarrow y = -\dfrac{3x^2}{2} + c \quad \Rightarrow 2y = -3x^2 + 2c$

Or $2y + 3x^2 - 2c = 0$ (1)

Now consider the second equation $p - x = 0 \quad \Rightarrow \dfrac{dy}{dx} = -x \quad \Rightarrow dy = -x\,dx$

Integrating, we get: $\int 1\,dy = -\int x\,dx + c \quad \Rightarrow y = -\dfrac{x^2}{2} + c \quad \Rightarrow 2y = -x^2 + 2c$

Or $2y+x^2-2c=0$ (2)

Combining (1) and (2) we get: $\left(2y+3x^2-2c\right)\left(2y+x^2-2c\right)=0$

This is the general solution of given differential equation.

(iii) Solution: Given equation is $x^2p^2+xyp-6y^2=0 \Rightarrow x^2p^2+3xyp-2xyp-6y^2=0$

$xp(xp+3y)-2y(xp+3y)=0 \Rightarrow (xp+3y)(xp-2y)=0 \Rightarrow xp+3y=0$ or $xp-2y=0$

Let us consider the fist equation $xp+3y=0 \Rightarrow x\dfrac{dy}{dx}=-3y \Rightarrow \dfrac{dy}{y}=-3\dfrac{dx}{x}$

Integrating, we get: $\int\frac{1}{y}dy=-3\int\frac{1}{x}dx+\ln c \Rightarrow \ln y=-3\ln x+\ln c=-\ln x^3+\ln c=\ln\frac{c}{x^3}$

$\Rightarrow y=c/x^3 \Rightarrow x^3y=c$ Or $x^3y-c=0$ (1)

Now consider the second equation $xp-2y=0 \Rightarrow x\dfrac{dy}{dx}=2y \Rightarrow \dfrac{dy}{y}=2\dfrac{dx}{x}$

Integrating, we get: $\int\frac{1}{y}dy=2\int\frac{1}{x}dx+\ln c \Rightarrow \ln y=2\ln x+\ln c=\ln x^2+\ln c=\ln cx^2$

$\Rightarrow y=cx^2 \Rightarrow y-cx^2=0$ (2)

Combining (1) and (2) we get: $\left(x^3y-c\right)\left(y-cx^2\right)=0$

This is the general solution of given differential equation.

(iv) Solution: Given equation is $x^2p^2-2xyp+y^2=0 \Rightarrow (xp-y)^2=0$

$(xp-y)(xp-y)=0$ $(xp-y)=0$ or $(xp-y)=0$

Now consider the first equation $xp-y=0 \Rightarrow x\dfrac{dy}{dx}=y \Rightarrow \dfrac{dy}{y}=\dfrac{dx}{x}$

Integrating, we get: $\int\frac{1}{y}dy=\int\frac{1}{x}dx+\ln c \Rightarrow \ln y=\ln x+\ln c \Rightarrow \ln y=\ln cx$

$\Rightarrow y=cx \Rightarrow (y-cx)=0$

Since the second equation is same hence its solution is $(y-cx)=0$. Thus general solution is $(y-cx)^2=0$ Or $y-cx=0$. [Note the change in the problem]

(v) Solution: Given equation is $p^3-\left(x^2-xy+y^2\right)p+xy(x+y)=0$

$\Rightarrow p^3-\left(x^2+y^2\right)p+xyp+xy(x+y)=0 \Rightarrow p\left[p^2-\left(x^2+y^2\right)\right]+xy\left[p+(x+y)\right]=0$

$\Rightarrow p\left[(p-x-y)(p+x+y)\right]+xy(p+x+y)=0$

$\Rightarrow (p+x+y)\left[p^2-px-py+xy\right]=0 \Rightarrow (p+x+y)\left[p(p-x)-y(p-x)\right]=0$

$\Rightarrow (p+x+y)(p-x)(p-y)=0 \Rightarrow p+x+y=0,\ p-x=0,\ p-y=0$

Now, consider $p+x+y=0 \Rightarrow \dfrac{dy}{dx}+y=-x$. This is a l.d.e. Hence I.F $=e^{\int P dx}=e^{\int 1 dx}=e^x$

Multiplying above equation by I.F, we get: $e^x\left[\dfrac{dy}{dx}+y\right]=-xe^x \Rightarrow \dfrac{d}{dx}\left[y\times \text{I.F}\right]=-xe^x$

Integrating, $y\times \text{I.F}=-\int xe^x dx=-\left[xe^x-\int e^x dx\right]=-\left[xe^x-e^x\right]+c=e^x(1-x)+c$

$\Rightarrow ye^x = e^x(1-x)+c \quad \Rightarrow e^x(y+x-1)-c=0$ (1)

Now consider $p-x=0 \quad \Rightarrow \dfrac{dy}{dx}=x \quad \Rightarrow \int 1.dy = \int x\,dx + c \Rightarrow y=\dfrac{x^2}{2}+c$

Or $\quad 2y-x^2-2c=0$ (2)

Finally consider $p-y=0 \quad \Rightarrow \dfrac{dy}{dx}=y \quad \Rightarrow \int \dfrac{1}{y}.dy = \int 1dx + c \Rightarrow \ln y = x+c$

Or $\quad y=e^{x+c} \quad \Rightarrow y-e^{x+c}=0$ (3)

Combining three equations, the general solution is

$$\left[e^x(y+x-1)-c\right]\left[2y-x^2-2c\right]\left[y-e^{x+c}\right]=0$$

NOTE: Please see the change in the problem

(vi) Solution: Given equation is $\quad xy\,p^2+\left(x^2+xy+y^2\right)p+x^2+y^2=0$

By quadratic formula, we get : $p=\dfrac{-\left(x^2+xy+y^2\right)\pm\sqrt{\left(x^2+xy+y^2\right)^2-4(xy)\left(x^2+y^2\right)}}{2xy}$

$$\Rightarrow p=\frac{-\left(x^2+xy+y^2\right)\pm\sqrt{x^4+x^2y^2+y^4+2x^3y+2xy^3+2x^2y^2-4x^3y-4xy^3}}{2xy}$$

$$=\frac{-\left(x^2+xy+y^2\right)\pm\sqrt{x^4+x^2y^2+y^4-2x^3y-2xy^3+2x^2y^2}}{2xy}$$

$$=\frac{-\left(x^2+xy+y^2\right)\pm\sqrt{\left(x^2-xy+y^2\right)^2}}{2xy}=\frac{-\left(x^2+xy+y^2\right)\pm\left(x^2-xy+y^2\right)}{2xy}$$

Thus, $p=\dfrac{-x^2-xy-y^2+x^2-xy+y^2}{2xy}$ or $p=\dfrac{-x^2-xy-y^2-x^2+xy-y^2}{2xy}$

$\Rightarrow p=\dfrac{-2xy}{2xy}=-1$ or $p=-\dfrac{x^2+y^2}{xy}=\dfrac{dy}{dx}$. From first equation

$\dfrac{dy}{dx}=-1 \quad \Rightarrow dy=-dx$. Integrating, $\quad y=-x+c \quad \Rightarrow y+x-c=0$ (1)

The second equation is homogeneous equation so putting $y=vx \ \Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$.

Thus equation becomes : $v+x\dfrac{dv}{dx}=-\dfrac{x^2+v^2x^2}{vx^2}=\dfrac{x^2\left(1+v^2\right)}{vx^2}=\dfrac{1+v^2}{v}$

$\Rightarrow x\dfrac{dv}{dx}=\dfrac{1+v^2}{v}-v=\dfrac{1+v^2-v^2}{v}=\dfrac{1}{v}$. Separating the variables and integrating, we get

$\int \dfrac{1}{v}dv=\int\dfrac{1}{x}dx+\ln c \ \Rightarrow \ln v=\ln x+\ln c=\ln cx \quad \Rightarrow v=cx \ \Rightarrow \dfrac{y}{x}=cx$

$\Rightarrow y=cx^2 \qquad \Rightarrow y-cx^2=0$ (2)

Combining equations (1) and (2), we get the general solution $\quad (y+x-c)\left(y-cx^2\right)=0$

NOTE: See the change in the problem.

(vii) Solution: Given equation is $\left(x^2 + x\right)p^2 + \left(x^2 + x - xy - y\right)p - y^2 - xy = 0$

$\Rightarrow \left(x^2+x\right)p^2+\left(x^2+x\right)p-y\left(x+y\right)p-y\left(y+x\right)=0$

$\Rightarrow p\left(x^2+x\right)\left(p+1\right)-y\left(x+y\right)\left(p+1\right)=0$

$\Rightarrow (p+1)\left[p\left(x^2+x\right)-y\left(x+y\right)\right]$

$\Rightarrow p+1=0 \quad$ (i) or $p\left(x^2+x\right)-y\left(x+y\right)=0 \quad$ (ii)

From (i), $y`=-1$ or $dy=-dx$. Integrating, we get: $y=c-x \qquad (1)$

From (ii), $y`\left(x^2+x\right)-xy=y^2 \Rightarrow y`-\dfrac{1}{(x+1)}y=\dfrac{y^2}{(x^2+x)}$

This is B.D.E. Dividing by y^2, we get

$y^{-2}y`-\dfrac{1}{x+1}y^{-1}=\dfrac{1}{x(x+1)} \qquad$ (iii)

Putting $z=y^{-1}$ then $z`=-y^{-2}y`$ or $y^{-2}y`=-z`$. Thus, equation (iii) becomes

$-z`-\dfrac{1}{1+x}z=\dfrac{1}{x(x+1)}$ or $z`+\dfrac{1}{1+x}z=-\dfrac{1}{x(x+1)} \quad$ (iv)

This is l.d.e with $I.F=e^{\int\frac{1}{1+x}dx}=e^{\ln(x+1)}=(x+1)$. Multiplying (iv) by I.F, we get

$LHS=\dfrac{d}{dx}\left[z\times I.F\right]=-\dfrac{1}{x}$. Integrating, we get: $z(x+1)=-\ln x+\ln c=\ln\left(c/x\right)$

Thus, $z=y^{-1}=\dfrac{\ln\left(c/x\right)}{\left(x+1\right)}$ or $y=\dfrac{\left(x+1\right)}{\ln\left(c/x\right)} \qquad (2)$

Equations (1) and (2) combinely form a solution of given differential equation.

(vii) $(x^2 + x)\, p^2 + (x^2 + x - xy - y)\, p - y^2 - xy = 0$

Solution: $(x^2 + x)\, p^2 + (x^2 + x)\, p - y(x + y)\, p - y(y + x) = 0$

Or $p(x^2 + x)(p + 1) - y(x + y)(p + 1) = 0$

Or $(p + 1)[p(x^2 + x) - y(x + y)] = 0$ or $(p + 1) = 0$ or $p(x^2 + x) - y(x + y) = 0$

$\therefore\ p=1 \Rightarrow dy/dx=1 \Rightarrow dy=dx$. Integrating, we get: $y=x+c \Rightarrow y-x-c=0 \qquad (1)$

Also, $p\left(x^2 + x\right) - y\left(x + y\right) = 0 \Rightarrow \dfrac{dy}{dx}-\dfrac{x}{x(x+1)}y=\dfrac{y^2}{x(x+1)}$

$$\frac{dy}{dx}-\frac{1}{(x+1)}y=\frac{y^2}{x(x+1)} \qquad (2)$$

Equation (2) is B.D.E. Thus dividing by y^2, we get: $y^{-2}\dfrac{dy}{dx}-\dfrac{1}{(x+1)}y^{-1}=\dfrac{1}{x(x+1)} \quad (3)$

Putting $z=y^{-1} \Rightarrow -z`=y^{-2}y`$. Thus, (3) becomes: $-\dfrac{dz}{dx}-\dfrac{1}{x+1}z=\dfrac{1}{x(x+1)}$

Or $\dfrac{dz}{dx}+\dfrac{1}{x+1}z=\dfrac{-1}{x(x+1)}$. This is a l.d.e with $I.F=e^{\int Pdx}=e^{\int\frac{1}{x+1}dx}=e^{\ln(x+1)}=(x+1)$

Multiplying above equation by I.F, we get : $(x+1)\dfrac{dz}{dx}+z=-\dfrac{(x+1)}{x}$

$\Rightarrow \dfrac{d}{dx}\left[z.(x+1)\right]=-\dfrac{x+1}{x}$

Integrating : $z(x+1)=-\int\left(1+\dfrac{1}{x}\right)dx+c=-\left(x+\ln x\right)+c$. Putting $z=y^{-1}$ we obtain

$y^{-1}(x+1)+x+\ln x-c=0 \quad \Rightarrow (x+1)+xy+y\ln x-cy=0$ (4)

Combining (1) and (4) we get : $\left(y-x-c\right)\left[(x+1)+xy+y\ln x-cy\right]=0$

This is the general solution. NOTE : Please see the change in the problem.

(viii) Solution: Given equation is $p^2 + 2py\cot x - y^2 = 0$

Apply quadratic formula, we get :

$$p=\frac{-2y\cot x\pm\sqrt{4y^2\cot^2 x+4y^2}}{2}=\frac{-2y\cot x\pm 2y\sqrt{\cot^2 x+1}}{2}=\frac{-2y\cot x\pm 2y\csc x}{2}$$

$\therefore\ p=y\left(\cot x+\csc x\right)$ OR $p=y\left(\cot x-\csc x\right)$, where $p=dy/dx$

Now consider $\dfrac{dy}{dx}=y\left(\cot x+\csc x\right)$. Separating the variables and integrating

$\int\dfrac{1}{y}dy=\int\left(\cot x+\csc x\right)dx+\ln c \ \Rightarrow \ln y=\ln(\sin x)+\ln(\cot x+\csc x)+\ln c$

$\ln y=\ln(\sin x+\cot x+\csc x+c) \ \Rightarrow y=\sin x+\cot x+\csc x+c$

$\Rightarrow\ \sin x+\cot x+\csc x+c-y=0$ (1)

Now consider $\dfrac{dy}{dx}=y\left(\cot x-\csc x\right)$. Separating the variables and integrating

$\int\dfrac{1}{y}dy=\int\left(\cot x-\csc x\right)dx+\ln c \ \Rightarrow \ln y=\ln(\sin x)-\ln(\cot x+\csc x)+\ln c$

$\ln y=\ln\left(\dfrac{c\sin x}{\cot x+\csc x}\right)\Rightarrow y=\left(\dfrac{c\sin x}{\cot x+\csc x}\right) \ \Rightarrow y(\cot x+\csc x)-c\sin x=0$ (2)

Combining (1) and (2) $\left(\sin x+\cot x+\csc x+c-y\right)\left[y(\cot x+\csc x)-c\sin x\right]=0$

This is the general solution.

2. Solve the following differential equations (Solvable for y)

(i) Solution: Given equation is $y = x + p^3$. (1)

Differentiating w.r.t x : $y` = p = 1+3p^2.\dfrac{dp}{dx} \quad \Rightarrow \dfrac{dp}{dx}=\dfrac{(p-1)}{3p^2}$.

Separating the variables and integrating, we have

$3\int\dfrac{p^2}{p-1}\,dp=\int 1.dx+c \Rightarrow 3\int\left[\left(p+1\right)+\dfrac{1}{p-1}\right]dp=x+c$

$3\left[\dfrac{p^2}{2}+p-\ln(p-1)\right]=x+c$ (2)

From (1) $p=(y-x)^{1/3}$. Placing this value of p in (2) we get a general solution

$$3\left[\frac{(y-x)^{2/3}}{2}+(y-x)^{1/3}-\ln\left[\left((y-x)^{1/3}\right)-1\right]\right]=x+c.$$

(ii) Solution: Given equation is $p-y=\ln(p^2-1)$ (1)

Differentiating w.r.t x, $\frac{dp}{dx}-\frac{dy}{dx}=\frac{1}{p^2-1}.2p\frac{dp}{dx}$ $\left[\text{NOTE}:\frac{dy}{dx}=p\right]$

$$\Rightarrow\frac{dp}{dx}\left[1-\frac{2p}{p^2-1}\right]-p=0\Rightarrow\frac{dp}{dx}\left[\frac{p^2-1-2p}{p^2-1}\right]=p\Rightarrow\frac{dp}{dx}=\frac{p\left(p^2-1\right)}{p^2-2p-1}$$

Separating the variables and integrating

$$\int\frac{p^2-2p-1}{p(p-1)(p+1)}dp=\int 1.dx+c\Rightarrow\int\left(\frac{1}{p}-\frac{1}{p-1}+\frac{1}{p+1}\right)dp=x+c.\quad\text{[By P. fractions]}$$

$$\Rightarrow\ln p-\ln(p-1)+\ln(p+1)=x+c\ \Rightarrow\ln\left(\frac{p(p+1}{(p-1)}\right)=x+c \qquad (2)$$

Now equations (1) and (2) together form a general solution.

(iii) Solution: Given equation is $y=2px-xp^2$ (1)

Differentiating w.r.t x : $\frac{dy}{dx}=p=2\left(x\frac{dp}{dx}+p.1\right)-\left(p^2.1+x.2p\frac{dp}{dx}\right)$

$$\Rightarrow p-2p+p^2=\left(2x-2xp\right)\frac{dp}{dx}\quad\Rightarrow p(p-1)=-2x(p-1)\frac{dp}{dx}$$

$$p(p-1)+2x(p-1)\frac{dp}{dx}=0\ \Rightarrow(p-1)\left[2x\frac{dp}{dx}+p\right]=0\ \Rightarrow p-1=0\ \text{ or }\ 2x\frac{dp}{dx}+p=0$$

If $p-1=0\ \Rightarrow\frac{dy}{dx}=1\ \Rightarrow\int 1.dy=x+c\ \Rightarrow y=x+c\ \Rightarrow y-x-c=0$ (i)

If $2x\frac{dp}{dx}+p=0\Rightarrow 2x\frac{dp}{dx}=-p$. Separating the variables and integrating,

$$2\int\frac{1}{p}dp=-\int\frac{1}{x}dx+\ln c\ \Rightarrow 2\ln p=-\ln x+\ln c\ \Rightarrow\ln p^2=\ln\frac{c}{x}\quad\Rightarrow p^2=\frac{c}{x}\ \Rightarrow p=\sqrt{\frac{c}{x}}$$

Put this value of p and p^2 in (1), we get:

$$y=2x\sqrt{\frac{c}{x}}-x\frac{c}{x}=2\sqrt{cx}-c\ \Rightarrow y+c-2\sqrt{cx}=0\quad\text{(ii)}$$

Combining (i) and (ii), the general solution is $\left(y-x-c\right)\left(y+c-2\sqrt{cx}\right)=0$

(iv) Solution: Given equation is: $xp^2-2py+ax=0$. By quadratic formulat

$$p=\frac{2y\pm\sqrt{4y^2-4ax^2}}{2x}=\frac{y\pm\sqrt{y^2-ax^2}}{x}\ \Rightarrow\frac{dy}{dx}=\frac{y\pm\sqrt{y^2-ax^2}}{x}.\text{ Thus,}$$

$$\frac{dy}{dx}=\frac{y+\sqrt{y^2-ax^2}}{x}\quad(1)\qquad\qquad\frac{dy}{dx}=\frac{y-\sqrt{y^2-ax^2}}{x}\quad(2)$$

Consider equation (1) which is homogeneous equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{v^2x^2 - ax^2}}{x} = v + \sqrt{v^2 - a} \Rightarrow x\frac{dv}{dx} = \sqrt{v^2 - a} = \sqrt{v^2 - b^2} \quad \left(a = b^2\right).$$

Sparating the variables and integrating: $\int \frac{1}{\sqrt{v^2 - b^2}} dv = \int \frac{1}{x} dx + \ln c = \ln x + \ln c$

$\ln\left(v - \sqrt{v^2 - b^2}\right) = \ln cx \quad \Rightarrow v - \sqrt{v^2 - b^2} = cx$. Put $v = y/x$, we get

$$\frac{y}{x} - \sqrt{\frac{y^2}{x^2} - b^2} = cx \quad \Rightarrow y - \sqrt{y^2 - b^2x^2} = cx^2 \quad \text{OR} \quad y - \sqrt{y^2 - b^2x^2} - cx^2 = 0 \qquad (3)$$

Now consider equation (2) which is also homogeneous equation.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx - \sqrt{v^2x^2 - ax^2}}{x} = v - \sqrt{v^2 - a} \Rightarrow x\frac{dv}{dx} = -\sqrt{v^2 - a} = -\sqrt{v^2 - b^2} \quad \left(a = b^2\right)$$

Sparating the variables and integrating: $\int \frac{1}{\sqrt{v^2 - b^2}} dv = -\int \frac{1}{x} dx + \ln c = -\ln x + \ln c$

$\ln\left(v - \sqrt{v^2 - b^2}\right) = \ln\left(\frac{c}{x}\right) \quad \Rightarrow v - \sqrt{v^2 - b^2} = \frac{c}{x}$. Put $v = y/x$, we get

$$\frac{y}{x} - \sqrt{\frac{y^2}{x^2} - b^2} = \frac{c}{x} \quad \Rightarrow y - \sqrt{y^2 - b^2x^2} = c \quad \text{OR} \quad y - \sqrt{y^2 - b^2x^2} - c = 0 \qquad (4)$$

Combining equations (3) and (4) to get general solution:

$$\left(y - \sqrt{y^2 - b^2x^2} - cx^2\right)\left(y - \sqrt{y^2 - b^2x^2} - c\right) = 0$$

(v) Solution: Given equation is $y = 2px + p^4x^2$. Differentiating w.r.t x, we get

$$\frac{dy}{dx} = p = 2\left(p.1 + x\frac{dp}{dx}\right) + \left(x^2.4p^3\frac{dp}{dx} + p^4.2x\right) = 0 \Rightarrow 0 = p + 2x\frac{dp}{dx} + 4x^2p^3\frac{dp}{dx} + 2xp^4$$

$$\frac{dp}{dx}\left(2x + 4x^2p^3\right) + \left(p + 2xp^4\right) = 0 \quad \text{OR} \quad \left(2p + 2xp^4\right)dx + \left(2x + 4x^2p^3\right)dp = 0 \qquad (1)$$

Here $M = 2p + 2xp^4$ and $N = 2x + 4x^2p^3 \Rightarrow M_p = 2 + 8xp^3$ and $N_x = 2 + 8xp^3$

Since, $M_p = N_x$ thus (1) is exact. To find its solution, we have

Step – I: Integrating M w.r.t x keeping p constant.

$\int\left(2p + 2xp^4\right)dx = 2px + x^2p^4.$

Step II: Integrating those terms of n w.r.t p which are free from x. Since there is no such term hence we skip this step. The solution is therefore: $\left(2px + x^2p^4\right) = c \qquad (2)$

Combining given equation and equation (2) we get: $y = c$.

(vi) Solution: Given equation is $y = (2+p)x + p^2$ (1)

Differentiating w.r.t x

$$\frac{dy}{dx} = p = 2 + p + x.\frac{dp}{dx} + 2p\frac{dp}{dx} \Rightarrow (2p+x)\frac{dp}{dx} + 2 = 0 \Rightarrow \frac{dp}{dx} = \frac{-2}{2p+x} \Rightarrow \frac{dx}{dp} = -\frac{2p+x}{2}$$

$$\Rightarrow \frac{dx}{dp} + \frac{1}{2}x = -p \qquad (2)$$

Equation (2) is a l.d.e in x. Thus, I.F $= e^{\int P\,dp} = e^{\int \frac{1}{2}dp} = e^{p/2}$. Multiplying (2) by I.F, we get

$e^{p/2}\left(\frac{dx}{dp} + \frac{1}{2}x\right) = -pe^{p/2} \Rightarrow \frac{d}{dp}[x \times \text{I.F}] = -pe^{p/2}$. Integrating, we get

$$[x \times \text{I.F}] = -\int p.e^{p/2}dp + c = -\left[2pe^{p/2} - 2\int e^{p/2}dp\right] + c = -e^{p/2}[2p-4] + c$$

$$\Rightarrow xe^{p/2} = -e^{p/2}[2p-4] + c \Rightarrow x = 2(2-p) + ce^{-p/2} \qquad (3)$$

Now (1) and (3) together form a solution of given differential equation.

(vii) Solution: Given equation is $xp^2 - yp - y = 0$. Using quadraticd equation, we get

$$p = \frac{y \pm \sqrt{y^2+4xy}}{2x} \Rightarrow \frac{dy}{dx} = \frac{y+\sqrt{y^2+4xy}}{2x} \quad (1) \qquad \frac{dy}{dx} = \frac{y-\sqrt{y^2+4xy}}{2x} \quad (2)$$

Both equations are homogeneous. Consider (1). Putting $y = vx \Rightarrow y` = v + xv`$.

Thus (1) becomes: $v + x\frac{dv}{dx} = \frac{vx+\sqrt{v^2x^2+4vx^2}}{x} = v + \sqrt{v^2+4v} \Rightarrow x\frac{dv}{dx} = \sqrt{v^2+4v}$

Separating the variables and integrating: $\int \frac{1}{\sqrt{v^2+4v}}dv = \int \frac{1}{x}dx + \ln c$

$$\Rightarrow \int \frac{1}{\sqrt{(v^2+4v+4)-4}}dv = \ln x + \ln c \Rightarrow \int \frac{1}{\sqrt{(v+2)^2-2^2}}dv = \ln cx \Rightarrow \frac{1}{2}\sin^{-1}\frac{v+2}{2} = \ln cx$$

$$\Rightarrow \sin^{-1}\frac{v+2}{2} = 2\ln cx \text{ or } \sin^{-1}\frac{v+2}{2} - 2\ln cx = 0 \qquad (i)$$

Now consider (2). Putting $y = vx \Rightarrow y` = v + xv`$.

Thus (1) becomes: $v + x\frac{dv}{dx} = \frac{vx-\sqrt{v^2x^2+4vx^2}}{x} = v - \sqrt{v^2+4v} \Rightarrow x\frac{dv}{dx} = -\sqrt{v^2+4v}$

Separating the variables and integrating: $\int \frac{1}{\sqrt{v^2+4v}}dv = -\int \frac{1}{x}dx + \ln c$

$$\Rightarrow \int \frac{1}{\sqrt{(v^2+4v+4)-4}}dv = -\ln x + \ln c \Rightarrow \int \frac{1}{\sqrt{(v+2)^2-2^2}}dv = \ln\left(\frac{c}{x}\right) \Rightarrow \frac{1}{2}\sin^{-1}\frac{v+2}{2} = \ln\left(\frac{c}{x}\right)$$

$$\Rightarrow \sin^{-1}\frac{v+2}{2} = 2\ln\left(\frac{c}{x}\right) \text{ or } \sin^{-1}\frac{v+2}{2} - 2\ln\left(\frac{c}{x}\right) = 0 \qquad (ii)$$

Combining (i) and (ii), we get: $\left(\sin^{-1}\frac{v+2}{2} - 2\ln cx\right)\left(\sin^{-1}\frac{v+2}{2} - 2\ln\left(\frac{c}{x}\right)\right) = 0$

This is the general solution of given differential equation.

(viii) Solution: Given equation is $p^3 + p = e^y \Rightarrow y = \ln(p^3+p)$ (1)

$$\Rightarrow \frac{dy}{dx} = \frac{1}{p^3+p}\left(3p^2\frac{dp}{dx} + \frac{dp}{dx}\right)$$

$\Rightarrow \left(p^3+p\right).p=\left(3p^2+1\right)\frac{dp}{dx} \quad \Rightarrow \frac{dp}{dx}=\frac{p^2\left(p^2+1\right)}{3p^2+1}$ $\left[\text{NOTE}: \frac{dy}{dx}=p\right]$

$\Rightarrow \frac{dx}{dp}=\frac{3p^2+1}{p^2\left(p^2+1\right)}=\frac{A}{p}+\frac{B}{p^2}+\frac{Cp+D}{p^2+1}$, giving $A=0, B=1, C=0, D=2$ [By P. Fraction]

$\therefore \frac{dx}{dp}=\frac{1}{p^2}+\frac{2}{p^2+1} \Rightarrow dx=\left[\frac{1}{p^2}+\frac{2}{p^2+1}\right]dp$. Integrating, we have:

$\int 1.dx=\int p^{-2}\,dx+\int\frac{1}{p^2+1}.dx+c \quad \Rightarrow x=-\frac{1}{p}+\tan^{-1}p+c \quad \Rightarrow xp=-1+p\tan^{-1}p+cp$

Or $\quad xp+1-p\tan^{-1}p-cp=0$ (2)

Equations (1) and (2) together form a solution.

3. Solve the following Clairaut's equations:

(i) Given equation is $y = x y` + (y`)^2$. But $y`=p$. Thus, given equation is: $y=xp+p^2$ (1)

This is Clairaut's equation. To find its solution differentiating w.r.t x, we get:

$y`=p=xp`+p.1+2pp`$

$\Rightarrow 2pp`+xp`=0 \quad \Rightarrow (2p+x).p`=0 \Rightarrow p`=\frac{0}{2p+x}=0 \quad \Rightarrow p=c$. Putting this in (1), we get

$y=cx+c^2$. This is the general solution of (1).

NOTE: You may observe solution of Clairaut's equation is just to replace y` = p by c.

(ii) Solution: Given equations is $y = px + [a^2 p^2 + b^2]^{1/2}$. This is Clairaut's equation and its solution is obtained upon replacing p by c, that is; $y = cx + [a^2 c^2 + b^2]^{1/2}$ is a solution.

(iii) Solution: Given equations is $y = px - pb + a/p$. This is Clairaut's equation and its solution is obtained upon replacing p by c, that is; $y = cx - cb + a/c$ is a solution.

(iv) Solution: Given equations is $(y - xp)(p - 1) = p$ or $y - xp = p/(p - 1)$ or $y = xp + p/(p - 1)$. This is Clairaut's equation and its solution is obtained upon replacing p by c, that is; $y = xc + p/(c - 1)$ is a solution.

(v) Solution: Given equations is $p = \ln(px - y)$ or $e^p = px - y$ or $y = px - e^p$. This is Clairaut's equation and its solution is obtained upon replacing p by c, that is; $y = xc - e^c$ is a solution.

(vi) Solution: Given equations is $x^2 (y - px) = yp^2$ or $y - px = yp^2/x^2$ or $y = px + yp^2/x^2$. This is Clairaut's equation and its solution is obtained upon replacing p by c, that is; $y = cx + yc^2/x^2$ is a solution.

4. Solve the following Lagrange differential equations:

(i) Solution: Given equation is $y = p + x/p$. (1)

Differentiating, $y`=p=p`+\frac{1}{p}-x\frac{p`}{p^2} \Rightarrow p^3=p^2p`+p-xp` \Rightarrow \left(p^2-x\right)p`=p^3-p$

$$\Rightarrow \frac{dp}{dx}=\frac{p\left(p^2-1\right)}{\left(p^2-x\right)} \quad \Rightarrow \frac{dx}{dp}=\frac{\left(p^2-x\right)}{p\left(p^2-1\right)}=\frac{p^2}{p\left(p^2-1\right)}-\frac{x}{p\left(p^2-1\right)}$$

$$\Rightarrow \frac{dx}{dp}+\frac{1}{p\left(p^2-1\right)}x=\frac{p}{\left(p^2-1\right)}$$

This is a l.d.e with $\text{I.F} = e^{\int P\,dp} = e^{\int \frac{1}{p(p^2-1)}dp} = e^{\int \frac{1}{p(p-1)(p+1)}dp} = e^{\int\left(\frac{-1}{p}+\frac{1}{2(p-1)}+\frac{1}{2(p+1)}\right)dp}$

$$= e^{\left(-\ln p+\frac{1}{2}\ln(p-1)+\frac{1}{2}\ln(p+1)\right)} = e^{\ln\frac{\sqrt{(p^2-1)}}{p}} = \frac{\sqrt{p^2-1}}{p}$$

Multiplying by I.F: $\frac{\sqrt{p^2-1}}{p}\left[\frac{dx}{dp}+\frac{1}{p(p^2-1)}x\right] = \frac{\sqrt{p^2-1}}{p}.\frac{p}{(p^2-1)} = \frac{1}{\sqrt{p^2-1}}$

$\frac{d}{dp}\left[x.\frac{\sqrt{p^2-1}}{p}\right] = \frac{1}{\sqrt{p^2-1}}$. Integrating, $\left[x.\frac{\sqrt{p^2-1}}{p}\right] = \int \frac{1}{\sqrt{p^2-1}}dp + c = \cosh^{-1}p + c.$ (2)

Equations (1) and (2) together form a solution.

(ii) Solution: Give equation is $y = 2px - xp^2.$ (1)

Differentiating, $y` = p = 2(p + xp`) - (p^2 + 2pxp`) \Rightarrow p + 2x\,p` - p^2 - 2xp\,p` = 0$

$\Rightarrow 2x(1-p)\frac{dp}{dx} = p^2 - p \quad \Rightarrow \frac{dp}{dx} = \frac{p(p-1)}{-2x(p-1)} = -\frac{p}{2x}.$

Separating the variables and integrating, we get : $\int \frac{1}{p}dp = -\frac{1}{2}\int \frac{1}{x}dx \Rightarrow \ln p = -\frac{1}{2}\ln x + c_1$

$\Rightarrow 2\ln p = -\ln x + 2c_1 \quad \Rightarrow \ln p^2 + \ln x = \ln c \quad \Rightarrow \ln xp^2 = \ln c \quad [2c_1 = \ln c]$ (2)

$\Rightarrow xp^2 = c \Rightarrow p^2 = c/x \Rightarrow p = \sqrt{c/x}.$ Putting these values in (1), we get :

$y = 2\sqrt{cx} - c.$ This is a general solution.

5. Solve for x the following differential equations:

(i) Solution: Given equation is $y = 2px + y^2p^3 \quad \Rightarrow x = \frac{y - y^2p^3}{2p}$

Differentiating w.r.t y, $\frac{dx}{dy} = \frac{1}{p} = \frac{2p(1-2yp^3-3y^2p^2p`) - (y-y^2p^3)(2p`)}{4p^2}$ $\quad \left[p` = \frac{dp}{dy}\right]$

$\Rightarrow 4p = 2p - 4yp^4 - 6y^2p^3p` - 2yp` + 2y^2p^3p` \Rightarrow 2p = -4yp^4 - 4y^2p^3p` - 2yp`$

$p` = -\frac{2p(1+2yp^3)}{2y(1+2yp^3)} = -\frac{p}{y}$. Sparating the variables and integrating, we get :

$\int \frac{1}{p}dp = -\int \frac{1}{y}dy + \ln c \Rightarrow \ln p = -\ln y + \ln c \Rightarrow \ln p + \ln y = \ln c \quad \Rightarrow \ln(yp) = \ln c$

(ii) Solution: Given equation is $2xp = yp^2 + y \Rightarrow x = \frac{yp^2+y}{2p}.$

Differentiating w.r.t y

$\frac{dx}{dy} = \frac{1}{p} = \frac{2p(p^2+2ypp`+1)) - (yp^2+y)(2p`)}{4p^4} \Rightarrow 4p^3 = 2p^3 + 4yp^2p` + 2p - 2yp^2p` - 2yp`$

or $2p^3 = 2yp^2p` + 2p - 2yp` \Rightarrow 2\left(p^3 - p\right) = 2y\left(p^2 - 1\right)p`$

or $p` = \frac{dp}{dy} = \frac{2p\left(p^2-1\right)}{2y\left(p^2-1\right)} = \frac{p}{y}$. Separating the variables and integrating, we get

$$\int \frac{dp}{p} = \int \frac{dy}{y} + \ln c \quad \Rightarrow \ln p = \ln y + \ln c = \ln cy \quad \Rightarrow p = cy$$

Put this value of p in given differential equation $2xp = yp^2 + y$, we get

$2x(cy) = y(cy)^2 + y \quad \Rightarrow 2cxy = c^2y^3 + y.$

This is the general solution of given differential equation.

(iii) Solution: Given equation is $x = y + p^2$. Differentiating w.r.t y

$$\frac{dx}{dy} = \frac{1}{p} = 1 + 2pp` \Rightarrow 2pp` = \frac{1}{p} - 1 = \frac{1-p}{p} \quad \Rightarrow p` = \frac{1-p}{2p^2} = \frac{dp}{dy}$$

Separating the variables and integrating: $-2\int \frac{p^2}{p-1}dp + c = \int 1.dy$

$$\Rightarrow -2\int\left[(p+1) + \frac{1}{p-1}\right]dp + c = y \quad \Rightarrow y = -p^2 - 2p + \ln(p-1) + c \qquad (1)$$

From given differential equation $p^2 = x - y \Rightarrow p = \sqrt{x-y}$. Put this in (1), we get

$y = -(x-y) - 2\sqrt{x-y} + \ln\left(\sqrt{x-y} - 1\right) + c$. This is the required solution.

(iv) $y = 3px + 6p^2y^2$

Solving for x, we get: $x = \frac{y}{3p} - 2py^2$. Differentiating with respect to y, we get

$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{3p} - \frac{y}{3p^2}p' - 4py - 2y^2p'$. Multiplying by p^2,

$p = \frac{1}{3}p - \frac{1}{3}yp' - 2p^2y^2p' - 4p^3y \quad \Rightarrow \frac{2}{3}p = -yp'\left(\frac{1}{3} + 2p^2y\right) - 4p^3y$. Multiplying by 3,

$$\Rightarrow 2p = -yp'\left(1 + 6p^2y\right) - 12p^3y \quad \Rightarrow yp'\left(1+6p^2y\right) = -2p\left(1+6p^2y\right) \Rightarrow yp' = -2p$$

$$\Rightarrow y\frac{dp}{dy} = -2p \Rightarrow \frac{1}{p}dp = \frac{-2}{y}dy \quad \Rightarrow \ln p = -2\ln y + \ln c \Rightarrow \ln p + \ln y^2 = \ln c$$

$\Rightarrow py^2 = c$ OR $p = \frac{c}{y^2}$. Substituting this value of p in (1) and simplifying, we have

$y^3 = 3cx + 6c^2$ · This is the general solution.

(v) $y = 2px + p^2y \qquad (1)$

Solution: Solving for x, we get $x = \frac{1}{2p}\left(y - p^2y\right) = \frac{1}{2}\left(\frac{y}{p} - py\right)$

Differentiating with respect to y, we get: $\frac{dx}{dy}=\frac{1}{p}=\frac{1}{2}\left(\frac{p.1-yp`}{p^2}-p.1-y.p`\right)$

$\Rightarrow \frac{1}{p}=\frac{1}{2p^2}\left(p-yp`-p^3-yp^2p`\right)$ $\Rightarrow \frac{1}{p}=\frac{1}{2p}-\frac{1}{2}p-\frac{1}{2}yp'\left(\frac{p^2+1}{p^2}\right)$

OR $\frac{1}{2p}+\frac{1}{2}p=-\frac{1}{2}yp'\left(\frac{p^2+1}{p^2}\right)$ $\Rightarrow \frac{1+p^2}{2p}=-\frac{1}{2}yp'\left(\frac{p^2+1}{p^2}\right)$

OR $\frac{1}{p}=-\frac{yp'}{p^2}\Rightarrow yp'=-p \Rightarrow y\frac{dp}{dy}=-p\Rightarrow \frac{1}{p}dp=-\frac{1}{y}dy$. Integrating both sides,

we get: $\ln p=-\ln y+\ln c\Rightarrow \ln yp=\ln c\Rightarrow yp=c \Rightarrow p=c/y$.

Substituting this value of p in (1), we get: $y^2=2cx+c^2$ is a general solution.

(vi) $y^2\ln y=xyp+p^2$: Solving for x, we get

$$xyp=y^2\ln y-p^2 \Rightarrow x=\frac{y\ln y}{p}-\frac{p}{y} \qquad (1)$$

Differentiating with respect to y, we get

$$\frac{dx}{dy}=\frac{1}{p}=\frac{p\left(y.\frac{1}{y}+\ln y.1\right)}{p^2}-\frac{yp`-p.1}{y^2}=\frac{p\left(1+\ln y\right)y^2-yp^2p`+p^3}{p^2y^2}$$

$\frac{1}{p}=\frac{y^2p\left(1+\ln y\right)-y^3\ln yp'-yp^2p'+p^3}{p^2y^2}$ OR $py^2=py^2+py^2\ln y-y^3\ln yp'-yp^2p'+p^3$

$0=-yp'\left(y^2\ln y+p^2\right)+y^2p\ln y+p^3$ OR $yp'\left(y^2\ln y+p^2\right)=p\left(y^2\ln y+p^2\right)\Rightarrow yp'=p$

Or $y\frac{dp}{dy}=p \Rightarrow \frac{1}{p}dp=\frac{1}{y}dy$. Integrating both sides, we get

$\ln p=\ln y+\ln c\Rightarrow \ln\left(\frac{p}{y}\right)=\ln c\Rightarrow \frac{p}{y}=c\Rightarrow p=cy$

Substituting the value of p from last equation into thegiven equation, we get

$y^2\ln y=cxy^2+c^2y^2 \Rightarrow \ln y=cx+c^2\cdot$

Chapter 4

APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

WORKSHEET 04

1. The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after 2 years population has been doubled and after 3 years the population is 20,000, find the number of people initially living in the country.

Solution: Let the population at any time be P. As given:

$$\frac{dP}{dt}\alpha P \Rightarrow \frac{dP}{dt} = kP. \text{ Separating the variables and integrating: } \int \frac{1}{P}dP = k\int 1.dt + c$$

$$\Rightarrow \ln P = kt + c \quad \Rightarrow P = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow \quad P = Ce^{kt} \tag{1}$$

Let at time $t = 0$ the population be $P = P_0 \Rightarrow P_0 = Ce^0 = C$. Thus (1) becomes

$$P = P_0e^{kt} \tag{2}$$

Now, at $t = 2$, $P = 2P_0 \Rightarrow 2P_0 = P_0e^{2k} \Rightarrow e^{2k} = 2 \Rightarrow e^{k} = \sqrt{2}$

And at $t = 3$, $P = 20,000 \Rightarrow 20,000 = P_0\, e^{3k} = P_0.e^{2k+k} = P_0.e^{2k}.e^{k} = 2\sqrt{2}P_0$

$\Rightarrow P_0 = \dfrac{20,000}{2\sqrt{2}} = 7072.$ Thus initially the population was 7072.

2. A certain culture of bacteria grows at a rate that is proportional to the number present. If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?

Solution: Let the population of bacteria at any time be P. As given:

$$\frac{dP}{dt}\alpha P \Rightarrow \frac{dP}{dt} = kP. \text{ Separating the variables and integrating: } \int \frac{1}{P}dP = k\int dt + c$$

$$\Rightarrow \ln P = kt + c \quad \Rightarrow P = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow \quad P = Ce^{kt} \tag{1}$$

Let at time $t = 0$ the population be $P = P_0 \Rightarrow P_0 = Ce^0 = C$. Thus (1) becomes

$$P = P_0e^{kt} \tag{2}$$

Then at $t = 4$, $P = 2P_0 \Rightarrow 2P_0 = P_0e^{4k} \Rightarrow e^{4k} = 2$

Now at $t = 12$, $P = P_0\, e^{12k} = P_0.\left(e^{12k}\right) = P_0.\left(e^{4k}\right)^3 = P_0(2)^3 = 8P_0.$

Thus after 12 hours it is expected that population will be 8 times the initial population.

3. The radioactive isotope thorium 234 disintegrates at a rate proportional to the amount present. It is found that in one week 17.96% of this material has disintegrated. Determine how long will it take for one half of this material to disintegrate?

Solution: Let the amount of radioactive material at any time be A. As given:

$$\frac{dA}{dt} \alpha A \Rightarrow \frac{dA}{dt} = kA. \text{ Separating the variables and integrating : } \int \frac{1}{A} dA = k\int dt + c$$

$$\Rightarrow \ln A = kt + c \quad \Rightarrow A = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow \quad A = Ce^{kt} \qquad (1)$$

$$\text{Let at time } t = 0 \text{ the amount be } A = A_0 \quad \Rightarrow \quad A_0 = Ce^0 = C. \text{ Thus (1) becomes :}$$

$$A = A_0 e^{kt} \qquad (2)$$

$$\text{Then at } t = 1, A = A_0 - 0.1796A_0 = 0.8204A_0 \Rightarrow 0.8204A_0 = A_0 e^k \quad \Rightarrow \quad e^k = 0.8204$$

$$\text{Now if } A = 0.5A_0, \text{ then } 0.5A_0 = A_0.\left(e^{kt}\right) \Rightarrow 0.5 = \left(0.8204\right)^t \Rightarrow \ln 0.5 = t\ln .8204$$

$$\Rightarrow \quad t = \ln 0.5 / \ln 0.8204 = (-0.693)/(-0.198) = 3.5$$

Thus, it takes 3.5 weeks for radioactive material to be disintegrated.

4. A certain radioactive material is known to decay at a rate proportional to the amount present. If initially, there are 100 milligrams of the material present and if after two years it is observed that 5 percent of the original mass has decayed, find (a) an expression for the mass at any time t and (b) the time necessary for 10 percent of the original mass to have decayed.

Solution: Let the amount of radioactive material at any time be A. As given:

$$\frac{dA}{dt} \alpha A \Rightarrow \frac{dA}{dt} = kA. \text{ Separating the variables and integrating : } \int \frac{1}{A} dA = k\int dt + c$$

$$\Rightarrow \ln A = kt + c \quad \Rightarrow A = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow \quad A = Ce^{kt} \qquad (1)$$

$$\text{At time } t = 0 \text{ the amount be } A = 100 \text{ mg} \quad \Rightarrow \quad 100 = Ce^0 = C. \text{ Thus (1) becomes :}$$

$$A = 100e^{kt} \qquad (2)$$

$$\text{Now, when } t = 2, A = 100 - 5 = 95 \Rightarrow 95 = 100e^{2k} \quad \Rightarrow \quad e^{2k} = 0.95$$

$$\text{Now if } A = 100 - 10 = 90, \text{ then } 90 = 100.\left(e^{kt}\right) \Rightarrow 0.9 = \left(e^{2k}\right)^{t/2} \Rightarrow 0.9 = (0.95)^{t/2}$$

$$\Rightarrow \frac{t}{2} = \frac{\ln 0.9}{\ln 0.95} \cong 2 \quad \Rightarrow t = 4. \text{ Thus, it takes 4 years for radioactive material to be}$$

disintegrated 10% of the original amount.

5. A man currently has Rs. 12,000 and plans to invest it in an account that accrues interest continuously. What interest rate must he receive, if his goal is to have Rs. 16,000 in 3 years?

Solution: Let the amount at any time be A. As given:

$$\frac{dA}{dt} \alpha A \Rightarrow \frac{dA}{dt} = kA. \text{ Separating the variables and integrating : } \int \frac{1}{A} dA = k\int dt + c$$

$$\Rightarrow \ln A = kt + c \quad \Rightarrow A = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow \quad A = Ce^{kt} \qquad (1)$$

$$\text{At time } t = 0 \text{ the amount } A = 12,000 \quad \Rightarrow \quad 12,000 = Ce^0 = C. \text{ Thus (1) becomes :}$$

$$A = 12,000\, e^{kt} \qquad (2)$$

$$\text{Now, when } t = 3, A = 16000 \quad \Rightarrow 16000 = 12000e^{3k} \quad \Rightarrow \quad e^{3k} = 1.34$$

$$\Rightarrow 3k = \ln 1.34 = 0.3 \quad \Rightarrow k = 0.1 = 10\%. \text{ Thus, required rate of interest is 10\%}$$

6. How long will it take a bank deposit to double if interest is compounded continuously at a constant rate of 8 percent pr annum?

Solution: Let the amount at any time be A. As given:

$\frac{dA}{dt} = 0.08\,A$. Separating the variables and integrating : $\int \frac{1}{A} dA = 0.08\int dt + c$

$\Rightarrow \ln A = 0.08\,t + c \quad \Rightarrow A = e^{0.08t+c} = e^{0.08t}.e^{c} \quad \Rightarrow \quad A = Ce^{0.08t}$ (1)

At time $t = 0$ let the amount $A = A_0$ $\Rightarrow \quad A_0 = Ce^0 = C.$

Thus (1) becomes: $A = A_0\, e^{0.08t}$ (2)

Now, when $A = 2A_0$ then $2A_0 = A_0\, e^{0.08t} \quad \Rightarrow \quad 2 = e^{0.08t}$

$\Rightarrow 0.08t = \ln 2 \quad \Rightarrow t = \ln 2 / 0.08 \cong 8.5.$

Thus it takes about 8.5 years the amount becomes doubled.

7. A depositor places Rs. 100,000 in a certificate of deposit account which pays 7 percent interest per annum, compounded continuously. How much will be in the account after 2 years?

Solution: Let the amount at any time be A. As given:

$\frac{dA}{dt} = 0.07\,A$. Separating the variables and integrating : $\int \frac{1}{A} dA = 0.07\int dt + c$

$\Rightarrow \ln A = 0.07\,t + c \quad \Rightarrow A = e^{0.07t+c} = e^{0.07t}.e^{c} \quad \Rightarrow \quad A = Ce^{0.07t}$ (1)

At time $t = 0$ the amount $A = 100,000$ $\Rightarrow \quad 100,000 = Ce^0 = C.$

Thus (1) becomes: $A = 100,000\, e^{0.07t}$ (2)

Now, when $t = 2$ then $A = 100,000\; e^{2(0.07)} \quad \Rightarrow \quad A = 115,027$

Thus, after 2 years the initial amount of Rs.100,000 becomes 115,027.

8. A man places \$700 in an account that accrues interest continuously. Assuming no additional deposits and no withdraws, how much will be in the account after 10 years if the interest rate is a constant 7.5 percent for first 6 years and a constant 8.25 percent for the last 4 years?

Solution: Let the amount at any time be A. As given:

$\frac{dA}{dt} = 0.075\,A$. Separating the variables and integrating : $\int \frac{1}{A} dA = 0.075\int 1.dt + c$

$\Rightarrow \ln A = 0.075\,t + c \quad \Rightarrow A = e^{0.075t+c} = e^{0.075t}.e^{c} \quad \Rightarrow \quad A = Ce^{0.075t}$ (1)

At time $t = 0$ the amount $A = \$700$ $\Rightarrow \quad 700 = Ce^0 = C.$

Thus (1) becomes: $A = 700\, e^{0.075t}$ (2)

Now, when $t = 6$ then $A = 700\; e^{6(0.075)} \quad \Rightarrow \quad A \cong \$1098$

Thus, after 6 years the initial amount of \$700 becomes \$1098.

Now this will be the amount at the begining of first year of next four years.

By the above procedure, $dA/dt = 0.0825\,A.$

Separating the variables and integrating: $\int \frac{1}{A} dA = 0.0825\int 1.dt + c$

$\Rightarrow \ln A = 0.0825\,t + c \quad \Rightarrow A = e^{0.0825t+c} = e^{0.0825t}.e^{c} \quad \Rightarrow \quad A = Ce^{0.0825t}$ (3)

At time $t = 0$ (Begining of 7^{th} year, the amount $A = \$1098$ $\Rightarrow \quad 1098 = Ce^0 = C.$

Thus (3) becomes: $A = 1098\, e^{0.0825t}$ (4)

Now, when $t = 4$ (the 10^{th} year) then $A = 1098\; e^{4(0.0825)} \Rightarrow A \cong \$1527$

Thus, after anothe 4 years the initial amount of \$700 becomes \$1527.

9. A body at a temperature of 50 F^o is placed outdoors where the temperature is 100 F^o. If after 5 minutes the temperature of the body is 60 F^o, determine the temperature of the body after 20 minutes.

Solution: According to Newton's law of cooling,

$\frac{dT}{dt} = k(T - T_0) = k(T - 100)$ where $T_0 = 100$ is the temperature of the surroundings.

Separating the variables and integrating: $\int \frac{1}{T-100} dT = k\int 1.dt + c$

$\Rightarrow \ln(T-100) = kt + c \quad \Rightarrow T - 100 = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow T - 100 = Ce^{kt}$ (1)

At time $t = 0$ the temperature $T = 50$ F^o $\Rightarrow 50 - 100 = -50 = Ce^0 = C.$

Thus (1) becomes: $T - 100 = -50e^{kt}$ (2)

Now, when $t = 5$, $T = 50$. Thus, $50 - 100 = -50\; e^{5k} \Rightarrow e^{5k} = 1.$

Now when $t = 20$, $T - 100 = -50e^{20k} = -50\left(e^{5k}\right)^4 = -50(1)^4 = -50 \Rightarrow T = 50.$

Thus, after 20 minutes the temperature of the body becomes $T = 50$ F^o.

10. A substance cools in air from 100 C^o to 70 C^o in 15 minutes. If the temperature of air is 30 C^o, find when the temperature will be 40 C^o.

Solution: According to Newton's law of cooling,

$\frac{dT}{dt} = k(T - T_0) = k(T - 30)$ where $T_0 = 30$ is the temperature of the surroundings (air).

Separating the variables and integrating: $\int \frac{1}{T-30} dT = k\int 1.dt + c$

$\Rightarrow \ln(T-30) = kt + c \quad \Rightarrow T - 30 = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow T - 30 = Ce^{kt}$ (1)

At time $t = 0$ the temperature $T = 100$ C^o $\Rightarrow 100 - 30 = 70 = Ce^0 = C.$

Thus (1) becomes: $T - 30 = 70e^{kt}$ (2)

Now, when $t = 15$, $T = 70$. Thus, $70 - 30 = 70\; e^{15k} \Rightarrow e^{15k} = 4/7$

Now when $T = 40$, $40 - 30 = 10 = 70e^{kt} = 70\left(e^{15k}\right)^{t/15} = 70(4/7)^{t/15} = 70\left(0.4714\right)^{t/15}$

Thus, $\left(0.4714\right)^{t/15} = 10/70 = 0.143 \Rightarrow \frac{t}{15} = \frac{\ln 0.143}{\ln 0.4714} = \frac{-1.946}{-0.75} = 2.595 \Rightarrow t \cong 39$ minutes

Thus, when $T = 40$ C^o the time $t = 39$ minutes.

11. One liter of ice cream at a temperature of -15 C^o is removed from the deep freezer and placed in a room where the temperature is 20 C^o. If after 15 minutes the temperature of the ice cream is -10 C^o how long will it take the ice cream to reach a temperature of 0 C^o.

Solution: By Newton's law of cooling,

$\frac{dT}{dt} = k(T - T_0) = k(T - 20)$ where $T_0 = 20$ is the temperature of the surroundings (room).

Separating the variables and integrating: $\int \frac{1}{T-20} dT = k\int 1.dt + c$

$\Rightarrow \ln(T-20) = kt + c \quad \Rightarrow T - 20 = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow T - 20 = Ce^{kt}$ (1)

At time $t = 0$ the temperature $T = -15\ C^{o} \quad \Rightarrow -15 - 30 = -70 = Ce^{0} = C.$

Thus (1) becomes: $T - 20 = -70e^{kt}$ (2)

Now when $t = 15,\ T = -10 \Rightarrow -10 - 20 = -70e^{15k} \Rightarrow e^{15k} = 3/7.$

Now when $T = 0,\quad 0 - 20 = -70e^{kt} = -70\left(e^{15k}\right)^{t/15} = -70(3/7)^{t/15}$

$$\text{Thus, } \left(\frac{3}{7}\right)^{t/15} = \frac{2}{7} \Rightarrow \frac{t}{15}\ln\left(\frac{3}{7}\right) = \ln\left(\frac{2}{7}\right) \Rightarrow \frac{t}{15} = \frac{\ln(2/7)}{\ln(3/7)} = \frac{-1.2527}{-0.8473} = 1.48 \quad \Rightarrow t = 22.$$

Thus, after 22 minutes the temperature of the body becomes $T = 0\ C^{o}$.

12. The temperature of a machine, when it is first shut down after operating, is 220^{o}C and temperature of the surrounding air is 30^{o}C. After 20 minutes, the temperature of the machine is 160 C^{o}. Find the temperature of the machine 30 minutes after it is shut down?

Solution: According to Newton's law of cooling,

$\frac{dT}{dt} = k(T - T_0) = k(T - 30)$ where $T_0 = 30$ is the temperature of the surroundings (air).

Separating the variables and integrating: $\int \frac{1}{T-30}dT = k\int 1.dt + c$

$\Rightarrow \ln(T-30) = kt + c \quad \Rightarrow T - 30 = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow T - 30 = Ce^{kt}$ (1)

At time $t = 0$ the temperature $T = 220\ C^{o} \quad \Rightarrow 220 - 30 = 190 = Ce^{0} = C.$

Thus (1) becomes: $T - 30 = 190e^{kt}$ (2)

Now, when $t = 20,\ T = 160.$ Thus, $160 - 30 = 190\ e^{20k} \quad \Rightarrow \quad e^{20k} = 13/19 = 0.684.$

Now when $t = 30,\quad T - 30 = 190e^{30k} = 190\left(e^{20k}\right)^{30/20} = 190(0.684)^{1.5} = 107.5$

Thus, after 30 minutes the temperature of the machine becomes $T = 107.5\ C^{o}$.

13. A copper ball is heated to a temperature of 100 C^{o}. Then at time t = 0 it is placed in water which is maintained at a temperature of 30 C^{o}. At the end of 3 minutes the temperature of the ball is reduced to 70 C^{o}. Find the time at which the temperature of the ball is reduced to 31 C^{o}.

Solution: By Newton's law of cooling,

$\frac{dT}{dt} = k(T - T_0) = k(T - 100)$ where $T_0 = 30$ is the temperature of the surroundings.

Separating the variables and integrating: $\int \frac{1}{T-30}dT = k\int 1.dt + c$

$\Rightarrow \ln(T-30) = kt + c \quad \Rightarrow T - 30 = e^{kt+c} = e^{kt}.e^{c} \quad \Rightarrow T - 30 = Ce^{kt}$ (1)

At time $t = 0$ the temperature $T = 100\ C^{o} \quad \Rightarrow 100 - 30 = 70 = Ce^{0} = C.$

Thus (1) becomes: $T - 30 = 70e^{kt}$ (2)

Now, when $t = 3,\ T = 70.$ Thus, $70 - 30 = 70\ e^{3k} \quad \Rightarrow \quad e^{3k} = 40/70 = 0.57$

Now when $T = 31$, then $31 - 30 = 1 = 70e^{kt} \Rightarrow e^{kt} = 1/70 = 0.0143 \Rightarrow \left(e^{3k}\right)^{t/3} = 0.0143$

$$\left(0.57\right)^{t/3} = 0.0143 \Rightarrow t/3 = \frac{\ln(0.0143)}{\ln(0.57)} = \frac{-4.25}{-0.56} = 7.6 \Rightarrow t \cong 22.$$

Thus it takes 22 minutes when the temperature of copper ball reduces to 50 F^{o}.

14. A body is dropped from a height of 300 feet with an initial velocity of 30 ft/sec. Assuming, no air resistance, find (a) an expression for the velocity of the body at any time t and (b) the time required for the body to hit the ground?

Solution: We know that

(a) $a = \frac{dv}{dt} = -32.$ Integrating w.r.t t, w get : $v = -32t + c.$ At $t = 0,\ v = 30,$ thus

$30 = 0 + c$ or $c = 30.$ Hence, $v = -32t + 30.$

(b) Now, $v = \frac{ds}{dt},$ therefore, $\frac{ds}{dt} = -32t + 30.$

Integrating w.r.t t, we get : $s = -16t^2 + 30t + c.$ Now at $t = 0,\ s = 300,$ thus, $c = 300.$

$\therefore s = -16t^2 + 30t + 300.$ Now when the body hits the ground, $s = 0$

$\Rightarrow -16t^2 + 30t + 300 = 0$ or $-8t^2 + 15t + 150 = 0.$ By quadratic formula, we have :

$t = \frac{-15 \pm \sqrt{225 + 4800}}{-16} = \frac{-15 \pm 70.88}{-16}.$ This gives, $t = -3.5$ or $t = 5.36.$

Since time is never negative hence, $t = 5.36$ is the valid value of t. This means that it takes body 5.36 seconds to hit the ground.

15. The magnitude of the velocity (in meters per second) of a particle moving along the t-axis is given by v = t/4,. When t = 0, the particle is 2 meters to the right of the origin. Determine the position of the particle when t = 3 seconds.

Solution: Given that

$v = \frac{ds}{dt} = \frac{t}{4}.$ Integrating w.r.t t, we get : $s = (t^2/8) + c.$ Now at $t = 0,\ s = 2,$ thus, $c = 2.$

$\therefore s = (t^2/8) + 2.$ Now when $t = 3$ then $s = 3.125.$

This means that after 3 seconds the body is 3.125 metrs on the right of origin.

16. A particle initially at rest, moves from a fixed point in a straight line so that its acceleration is given by sin t + $[1/(t+1)]^2$. What is its distance at the end of π seconds from the start?

Solution: We know that

$a = \frac{dv}{dt} = \sin t + \frac{1}{(t+1)^2}.$ Integrating w.r.t t, w get : $v = -\cos t - \frac{1}{t+1} + c.$ At $t = 0,\ v = 0,$ thus

$0 = -\cos 0 - 1 + c$ or $c = 2$ $(\cos 0 = 1).$ Hence, $v = -\cos t - \frac{1}{t+1} + 2$

(b) Now, $v = \frac{ds}{dt},$ therefore, $\frac{ds}{dt} = -32t + 30.$

Integrating w.r.t t, we get : $s = -16t^2 + 30t + c.$ Now at $t = 0,\ s = 300,$ thus, $c = 300.$

$\therefore s = -16t^2 + 30t + 300.$ Now when the body hits the ground, $s = 0$

$\Rightarrow -16t^2 + 30t + 300 = 0$ or $-8t^2 + 15t + 150 = 0.$ By quadratic formula, we have :

$t = \frac{-15 \pm \sqrt{225 + 4800}}{-16} = \frac{-15 \pm 70.88}{-16}.$ This gives, $t = -3.5$ or $t = 5.36.$

Since time is never negative hence, $t = 5.36$ is the valid value of t. This means that it takes body 5.36 seconds to hit the ground.

17. A particle free to move along a straight line, becomes subject to an acceleration a = cos pt. If initially, the particle is at rest at the origin, what is its distance at any instant?

Solution: We are given

$a = \frac{dv}{dt} = \cos pt$. Integrating w.r.t t, we get : $v = \frac{\sin pt}{p} + c$. At $t = 0$, $v = 0$, thus $c = 0$

Hence, $v = \frac{\sin pt}{p}$. Now, $v = \frac{ds}{dt} = \frac{\sin pt}{p}$. Integrating again, we get : $s = \frac{-\cos pt}{p^2} + c$.

Now at $t = 0$, $s = 0$, thus, $c = 1/p^2$.

Therefore, $s = -\frac{\cos pt}{p^2} + \frac{1}{p^2} = \frac{(1-\cos pt)}{p^2}$. This is the expression for distance.

18. A train starting from rest is accelerated that is given by 10/(v + 1) ft/sec^2, where v is the velocity in ft/sec. Find the distance in which the train attains a velocity of 44 ft/sec.

Solution: We are given

$a = \frac{dv}{dt} = \frac{10}{v+1}$. Separating the variables and integrating w.r.t t, we get :

$\int (v+1)\,dv = \int 10\,dt + c \quad \Rightarrow \frac{v^2}{2} + v = 10t + c$. At $t = 0$, $v = 0$, thus $c = 0$.

Hence, $\frac{v^2}{2} + v = 10t$. Now, $v = 44 \quad \Rightarrow t = \frac{1}{10}\left(\frac{v^2}{2} + v\right) = 101.2$

But s = vt = 44(101.2) = 4452.8 ft.

19. An RC circuit has an emf of 5 volts, a resistance of 10 ohms, a capacitance of 10^{-2} Farad, and initially a charge of 5 coulombs on the capacitor. Find (a) the transient current and (b) the steady – state current.

Solution: We know that a differential equation governing RC circuit is given by

$R\frac{dQ}{dt} + \frac{1}{C}Q = E$. Here $E = 5$, $R = 10$, and $C = 10^{-2}$. Putting these values, we get

$10\frac{dQ}{dt} + 100Q = 5 \Rightarrow 2\frac{dQ}{dt} + 20Q = 1 \Rightarrow \frac{dQ}{dt} = -\frac{1}{2}(20Q-1)$.

Separating the variables and integrating w.r.t t, we get :

$\int \frac{1}{20Q-1}dQ = -\frac{1}{2}\int 1.dt + c \quad \Rightarrow \frac{1}{20}\ln(20Q-1) = -\frac{t}{2} + c.$

$\ln(20Q-1) = \frac{20(c-2t)}{2} \quad \Rightarrow \ln(20Q-1) = 10(2c-t) \Rightarrow 20Q-1 = e^{10(2c-t)}$

$$20Q - 1 = e^{20c}.e^{-10t} = Ce^{-10t} \qquad (1)$$

Now at $t = 0$, $Q = 5$ giving $C = 99$.

Diffrentiating (1) w.r.t t, w get : $20\frac{dQ}{dt} = C.\left(-10e^{-10t}\right)$. But $\frac{dQ}{dt} = I$

$\therefore I = -\frac{1}{2}Ce^{-10t}$. This is the transient current (current at any time).

Putting $C = 99$, we get : $I = -\frac{99}{2}e^{-10t}$. This is the steady – state current.

20. An RC circuit has an emf of 300 cos 2t volts, a resistance of 150 ohms, a capacitance of $\frac{1}{6}\times 10^{-2}$ Farad, and an initial charge on the capacitor of 5 coulombs. Find (a) the charge on the capacitor at any time t and (b) the steady – state current.

Solution: We know that a differential equation governing RC circuit is given by

$R\frac{dQ}{dt}+\frac{1}{C}Q=E.$ Here $E=300\cos 2t,\ R=150,$ and $C=\frac{1}{6}10^{-2}.$ Putting these values, we get

$150\frac{dQ}{dt}+600Q=300\cos 2t \;\Rightarrow \frac{dQ}{dt}+4Q=2\cos 2t.$ (Dividing by 150)

This is a l.d.e with $I.F=e^{\int 4\,dt}=e^{4t}.$ Multiplying by I.F, we get :

$$e^{4t}\left[\frac{dQ}{dt}+4Q\right]=2e^{4t}\cos 2t \;\Rightarrow \frac{d}{dt}\left[Q\times I.F\right]=2e^{4t}\cos 2t.\ \text{Integrating, we get :}$$

$$\left[Q\times I.F\right]=2\int e^{4t}\cos 2t\ dt+c=2\frac{e^{4t}}{\left(4^2+2^2\right)}\left[4\cos 2t-2\sin 2t\right]+c$$

$$\Rightarrow Q.\ e^{4t}=\frac{e^{4t}}{10}\left[4\cos 2t-2\sin 2t\right]+c \;\Rightarrow Q=\frac{1}{10}\left[4\cos 2t-2\sin 2t\right]+c\ e^{-4t}.$$

This is the cahrge at any time. Putting $t=0$ and $Q=5,$ we get : $5=\frac{4}{10}+c\Rightarrow c=4.6$

Thus, $Q=\frac{1}{10}\left[4\cos 2t-2\sin 2t\right]+4.6\ e^{-4t}.$ Differentiating w.r.t t, we get

$$\frac{dQ}{dt}=I=\frac{1}{10}\left[-8\sin 2t-4\cos 2t\right]+4.6(-4)e^{-4t}=-\frac{2}{5}\left[2\sin 2t+\cos 2t\right]-18.4e^{-4t}$$

This is the steady state current.

21. An RL circuit has a resistance of 10 ohms, an inductance of 1.5 henries, an applied emf of 9 volts, and an initial current of 6 amperes. Find (a) the current in the circuit at any time t and (b) its transient component.

Solution: We know that a differential equation governing RL circuit is given by

$L\frac{dI}{dt}+RI=E.$ Here $E=9,\ R=10,$ and $L=1.5.$ Putting these values, we get

$$1.5\frac{dI}{dt}+10I=9 \;\Rightarrow \frac{dI}{dt}+6.7\ I=0.7 \qquad (1)$$

This is a l.d.e with $I.F=e^{\int 6.7\,dt}=e^{6.7t}.$ Multiplying (1) by I.F, we get :

$$e^{6.7t}\left[\frac{dI}{dt}+6.7I\right]=0.7e^{6.7t} \;\Rightarrow \frac{d}{dt}\left[I\times I.F\right]=0.7e^{6.7t}.\ \text{Integrating, we get :}$$

$$\left[I\times I.F\right]=0.7\int e^{6.7t}\ dt+c=\frac{0.7}{6.7}e^{6.7t}+c=0.105e^{6.7t}+c$$

$$\Rightarrow Ie^{6.7t}=0.105e^{6.7t}+c \qquad \Rightarrow\ I=0.105+ce^{-6.7t}$$

This is the current at any time. Putting $t=0$ and $I=6,$ we get : $6=0.015+c\Rightarrow c=5.985$

Thus, $I=0.105+5.985e^{-6.7t}.$ This is the steady state current.

Chapter 5

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS

WORKSHEET 05

1. Solve the following homogeneous linear differential equations:

(a) $(D^2 - 3D - 4)\,y = 0$

Solution: The auxiliary equation is $m^2 - 3m - 4 = 0$ or $m = 4$ or $m = -1$ (Using Quadratic Formula). The roots of AE are real and distinct, thus general solution is

$$y = c_1 e^{4x} + c_2 e^{-x}.$$

(b) $(D^3 - 7D - 6)\,y = 0$

Solution: The auxiliary equation is $m^3 - 7m - 6 = 0.$

Putting $m = -1$ we get: $-1 + 7 - 6 = 0$. Thus, $m = -1$ is a root of given auxiliary equation. To find other two roots, we use synthetic division:

-1	1	0	-7	-6
		-1	1	6
	1	-1	-6	0

This gives $m^2 - m - 6 = 0$ or $m = 3$ or $m = -2$ (Using Quadratic Formula). The roots of AE are real and distinct, thus general solution is: $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x}$.

(c) $(D^3 - 9D^2 + 23D - 15)\,y = 0$

Solution: The auxiliary equation is $m^3 - 9m^2 + 23m - 15 = 0.$

Putting $m = 1$ we get: $1 - 9 + 23 - 15 = 0$. Thus, $m = 1$ is a root of given auxiliary equation. To find other two roots, we use synthetic division:

1	1	-9	23	-15
		1	-8	15
	1	-8	15	0

This gives $m^2 - 8m + 15 = 0$ or $m = 3$ or $m = 5$ (Using Quadratic Formula). The roots of AE are real and distinct, thus general solution is: $y = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}$.

(d) $(D^2 + (a + b)D + ab)\,y = 0$

Solution: The auxiliary equation is $m^2 + am + bm + ab = 0$ or $m(m + a) + b(m + a) = 0$ or $(m + a)(m + b) = 0$ or $m + a = 0$ or $m + b = 0$ or $m = -a$, $m = -b$. The roots of AE are real and distinct, thus general solution is: $y = c_1 e^{ax} + c_2 e^{bx}$.

(e) $(D^3 - 2D^2 + 4D - 8)\,y = 0$

Solution: The auxiliary equation is $m^3 - 2m^2 + 4m - 8 = 0.$

Putting $m = 2$ we get: $8 - 8 + 8 - 8 = 0$. Thus, $m = 2$ is a root of given auxiliary equation.

To find other two roots, we use synthetic division:

$$\begin{array}{c|cccc} 2 & 1 & -2 & 4 & -8 \\ & & 2 & 0 & 8 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

This gives $m^2 + 4 = 0$ or $m^2 = -4$ giving $m = \pm$ i. One root of AE is real and two are complex, thus general solution is:

$$y = c_1e^{2x} + c_2 \cos 2x + c_3 \sin 2x.$$

(f) $(D^4 - 5D^2 + 4)\, y = 0$

Solution: The auxiliary equation is $m^4 - 5m^2 + 4 = 0$ or $(m^2 - 4)(m^2 - 1)$ or $m = \pm 2$, $m = \pm 1$: Thus general solution is: $y = c_1e^{2x} + c_2e^{-2x} + c_3e^{x} + c_4e^{-x}$

(g) $(D^2 - 4D + 1)\, y = 0$

Solution: The auxiliary equation is $m^2 - 4m + 1 = 0$ or $m = 2, 2$. Thus general solution is: $y = (c_1 + c_2x)\, e^{2x}$.

(h) $(D^4 + k^4)\, y = 0$

Solution: The auxiliary equation is $m^4 + k^4 = 0$ or $m^4 + 2m^2k^2 + k^4 = 2m^2k^2$ or $(m^2 + k^2)^2 = 2m^2 k^2$. Taking the square root on both sides, we get

$$m^2 + k^2 = \pm\sqrt{2}\,mk \quad \Rightarrow m^2 - \sqrt{2}\,mk + k^2 = 0 \text{ or } m^2 + \sqrt{2}\,mk + k^2 = 0$$

$$\text{If } \; m^2 - \sqrt{2}\,mk + k^2 = 0 \Rightarrow m = \frac{\sqrt{2}\,k \pm \sqrt{2k^2 - 4k^2}}{2} = \frac{\sqrt{2}\,k \pm \sqrt{-2k^2}}{2}$$

$$\Rightarrow m = \frac{\sqrt{2}\,k \pm i\sqrt{2}k}{2} = \frac{k}{\sqrt{2}} \pm i\frac{k}{\sqrt{2}}$$

$$\text{If } \; m^2 + \sqrt{2}\,mk + k^2 = 0 \Rightarrow m = \frac{-\sqrt{2}\,k \pm \sqrt{2k^2 - 4k^2}}{2} = \frac{-\sqrt{2}\,k \pm \sqrt{-2k^2}}{2}$$

$$\Rightarrow m = \frac{-\sqrt{2}\,k \pm i\sqrt{2}k}{2} = \frac{-k}{\sqrt{2}} \pm i\frac{k}{\sqrt{2}}$$

$$\text{Thus general solution is}: y = e^{k/\sqrt{2}}\left[c_1\cos\frac{kx}{\sqrt{2}} + c_2\sin\frac{kx}{\sqrt{2}}\right] + e^{-k/\sqrt{2}}\left[c_3\cos\frac{kx}{\sqrt{2}} + c_4\sin\frac{kx}{\sqrt{2}}\right]$$

(i) $(D^3 - D^2 - D - 2)\, y = 0$

Solution: The auxiliary equation is $m^3 - m^2 - m - 2 = 0$.

Putting $m = 2$ we get: $8 - 4 - 2 - 2 = 0$. Thus, $m = 2$ is a root of given auxiliary equation.

To find other two roots, we use synthetic division:

$$\begin{array}{c|cccc} 2 & 1 & -1 & -1 & -2 \\ & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

This gives $m^2 + m + 1 = 0$ or $m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$.

Thus general solution is: $y = c_1e^{2x} + e^{-x/2}\left[c_2\cos\frac{\sqrt{3}x}{2} + c_3\sin\frac{\sqrt{3}x}{2}\right]$

(j) $(D^4 + 2D^3 + 3D^2 + 2D + 1)\, y = 0$

Solution: The auxiliary equation is $m^4 + 2m^3 + 3m^2 + 2m + 1 = 0$ or $(m^2 + m + 1)^2 = 0$

This gives $m^2 + m + 1 = 0$ m $m = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2} = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2}$.

Since $(m^2 + m + 1)$ is repeated twice hence general solution is:

$$y = e^{-x/2}\left[(c_1 + c_2 x)\cos\frac{\sqrt{3}x}{2} + (c_3 + c_4 x)\sin\frac{\sqrt{3}x}{2}\right]$$

(k) $(D^3 + 3D^2 + 3D + 1)\, y = 0$

Solution: The auxiliary equation is $m^3 + 3m^2 + 3m + 1 = 0$ or $(m + 1)^3 = 0$

or $m = -1, -1, -1$. The roots are real and repeated thrice. Thus general solution is:

$$y = \left(c_1 + c_2 x + c_3 x^2\right)e^{-x}$$

(*l*) $(D^4 - 7D^3 + 18D^2 - 20D + 8)\, y = 0$

Solution: The auxiliary equation is $m^4 - 7m^3 + 18m^2 - 20m + 8 = 0$.

Putting $m = 1$ we get: $1 - 7 + 18 - 20 + 8 = 0$. Thus, $m = 1$ is a root of given auxiliary equation. To find other three roots, we use synthetic division:

1	1	-7	18	-20	8
		1	-6	12	-8
	1	-6	12	-8	0

This gives $m^3 - 6m^2 + 12m - 8 = 0$. Putting $m = 2$ we get: $8 - 24 + 24 - 8 = 0$. Thus, $m = 2$ is a root of given auxiliary equation. To find remaining two roots, we use synthetic division:

2	1	-6	12	-8
		2	-8	8
	1	-4	4	0

This gives $m^2 - 4m + 4 = 0$ or $(m - 2)^2 = 0$ or $m = 2, 2$. Thus, roots of auxiliary equation are: $m = 1, 2, 2, 2$. The general solution is therefore: $y = c_1 e^x + (c_2 + c_3 x + c_4 x^2)\, e^{2x}$.

(m) $(D^4 - 8D^2 + 16)\, y = 0$

Solution: The auxiliary equation is $m^4 - 8m^2 + 16 = 0$ or $(m^2 - 4)^2 = 0$

or $(m^2 - 4)(m^2 - 4) = 0$ or $m = -2, -2, 2, 2$. Thus, general solution is:

$$y = (c_1 + c_2 x)\, e^{-2x} + (c_1 + c_2 x)\, e^{2x}$$

(n) $(D^4 - k^4)\, y = 0$

Solution: The auxiliary equation is $m^4 - k^4 = 0$ or $(m^2 - k^2)(m^2 + k^2) = 0$

or $m^2 - k^2 = 0$ or $m^2 + k^2 = 0$ or $m = -k, -k, k, k$. Thus, general solution is:

$$y = (c_1 + c_2 x)e^{-kx} + (c_3 + c_4 x)e^{kx}$$

(o) $(D^6 - k^6)\, y = 0$

Solution: The auxiliary equation is $m^6 - k^6 = 0$ or $(m^3)^2 - (k^3)^2 = 0$

or $(m^3 - k^3)(m^3 + k^3) = 0$ or $(m^3 - k^3) = 0$ OR $(m^3 + k^3) = 0$

or $(m - k)(m^2 + km + k^2) = 0$ OR $(m + k)(m^2 - km + k^2) = 0$

If $m - k = 0 \Rightarrow m = k$ and if $m + k = 0 \Rightarrow m = -k$

If $m^2 + km + k^2 = 0 \Rightarrow m = \dfrac{-k \pm \sqrt{k^2 - 4k^2}}{2} = \dfrac{-k}{2} \pm i\dfrac{\sqrt{3}k}{2}$

and if $m^2 - km + k^2 = 0 \Rightarrow m = \dfrac{k \pm \sqrt{k^2 - 4k^2}}{2} = \dfrac{k}{2} \pm i\dfrac{\sqrt{3}k}{2}$

Thus, general solution is:

$$y = c_1 e^{kx} + c_2 e^{-kx} + e^{-kx/2}\left[c_3 \cos\frac{\sqrt{3}k}{2}x + c_4 \sin\frac{\sqrt{3}k}{2}x\right] + e^{kx/2}\left[c_5 \cos\frac{\sqrt{3}k}{2}x + c_6 \sin\frac{\sqrt{3}k}{2}x\right]$$

(p) $(D^4 + 2D^3 - 3D^2 - 4D + 4)\, y = 0$

Solution: The auxiliary equation is $m^4 + 2m^3 - 3m^2 - 4m + 4 = 0$.

Putting m = 1 we get: 1 + 2 - 3 – 4 + 4 = 0. Thus, m = 1 is a root of given auxiliary equation. To find other three roots, we use synthetic division:

1	1	2	-3	-4	4
		1	3	0	-4
	1	3	0	-4	0

This gives $m^3 + 3m^2 + 0m - 4 = 0$. Putting m = 1 we get: 1 + 3 - 4 = 0. Thus, m = 1 is a root of given auxiliary equation. To find remaining two roots, we use synthetic division:

1	1	3	0	-4
		1	4	4
	1	4	4	0

This gives $m^2 + 4m + 4 = 0$ or $(m + 2)^2 = 0$ or m = -2, -2. Thus, roots of auxiliary equation are: m = 1, 1, -2, -2. The general solution is therefore:

$$y = (c_1 + c_2x)\, e^x + (c_3 + c_4x)\, e^{-2x}$$

$= \dfrac{1}{0^3+1}e^{0x} + \dfrac{1}{D^2.D+1}\cos 2x = 1 + \dfrac{1}{(-2^2)D+1}\cos 2x = 1 + \dfrac{1}{1-4D} \times \dfrac{1+4D}{1+4D}\cos 2x$ $(D^2 + 4D + 3)\, y = 0$, $y(0) = 0$, $y`(0) = 12$.

Solution: The auxiliary equation is $m^2 + 4m + 3 = 0$ or m = -3 or m = -1 (Using Quadratic Formula). Thus general solution is: $y = c_1 e^{-x} + c_2 e^{-3x}$.

Now, $y = c_1 e^{-x} + c_2 e^{-3x}$ or $y` = -c_1 e^{-x} - 3c_2 e^{-3x}$. Putting x = 0, y = 0 and y` = 12, we get: $c_1 + c_2 = 0$ and $-c_1 - 3c_2 = 12$

Solving the above equations, we get $c_1 = 6$ and $c_2 = -6$. Thus particular solution is

$$y = 6(e^{-x} - e^{-3x})$$

(r) $(D^4 + D^2)\, y = 0$; $y(0) = y`(0) = y``(0) = 0$, $y```(0) = 1$

Solution: The auxiliary equation is $m^4 + m^2 = 0$ or $m^2(m^2 + 4) = 0$ or $m = 0, 0, \pm 2i$ Thus general solution is:

$y = (c_1 + c_2\, x)e^{0x} + (c_3 \cos x + c_4 \sin x)$ (1)

$y` = c_2 - c_3 \sin x + c_4 \cos x$ (2)

$y`` = -c_3 \cos x - c_4 \sin x$ (3)

$y``` = c_3 \sin x - c_4 \cos x$

Putting x = 0, y = 0, y` = 0, y`` = 0, y``` = 1, we get:

$0 = c_1 + c_3$, $0 = c_2 + c_4$, $0 = -c_3$ and $1 = -c_4$

Solving the above equations, we get $c_1 = c_3 = 0$, $c_4 = -1$ and $c_2 = 1$. Thus particular solution is $y = x - \sin x$

$= \dfrac{1}{0^3+1}e^{0x} + \dfrac{1}{D^2.D+1}\cos 2x = 1 + \dfrac{1}{(-2^2)D+1}\cos 2x = 1 + \dfrac{1}{1-4D} \times \dfrac{1+4D}{1+4D}\cos 2x$ $(D^3 +$

(s) $6D^{2} + 12D + 8)\,y = 0;\ y(0) = y`(0) = 0,\ y``(0) = 2$
Solution: The auxiliary equation is $m^3 + 6m^2 + 12m + 8 = 0$.
Putting m = 2 we get: 8 – 4 – 2 - 2 = 0. Thus, m = 2 is a root of given auxiliary equation. To find other two roots, we use synthetic division:

$$\begin{array}{c|cccc} 2 & 1 & -1 & -1 & -2 \\ & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

This gives $m^2 + m + 1 = 0$ or $m = \dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1 \pm i\sqrt{3}}{2} = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2}$.
Thus general solution is:

$$y = c_1 e^{2x} + e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right] \qquad (1)$$

$$\Rightarrow y` = 2c_1 e^{2x} + e^{-x/2}\left[-\frac{\sqrt{3}}{2}c_2 \sin\frac{\sqrt{3}x}{2} + \frac{\sqrt{3}}{2}c_3 \cos\frac{\sqrt{3}x}{2}\right] - \frac{1}{2}e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right]$$

$$y`` = 4c_1 e^{2x} + e^{-x/2}\left[-\frac{3}{4}c_2 \cos\frac{\sqrt{3}x}{2} - \frac{3}{4}c_3 \sin\frac{\sqrt{3}x}{2}\right] - \frac{1}{2}e^{-x/2}\left[-\frac{\sqrt{3}}{2}c_2 \sin\frac{\sqrt{3}x}{2} + \frac{\sqrt{3}}{2}c_3 \cos\frac{\sqrt{3}x}{2}\right]$$

$$-\frac{1}{2}e^{-x/2}\left[-\frac{\sqrt{3}}{2}c_2 \sin\frac{\sqrt{3}x}{2} + c_3\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}x}{2}\right] + \frac{1}{4}e^{-x/2}\left[c_2 \cos\frac{\sqrt{3}x}{2} + c_3 \sin\frac{\sqrt{3}x}{2}\right]$$

Putting $x = 0, y = 0,\ y` = 0,\ y`` = 2$, we get :

$$c_1 + c_2 = 0, \quad 2c_1 - \frac{1}{2}c_2 + \frac{\sqrt{3}}{2}c_3 = 0 \text{ and } 4c_1 - \frac{3}{4}c_2 - \frac{\sqrt{3}}{4}c_3 - \frac{\sqrt{3}}{4}c_3 + \frac{1}{4}c_2 = 2 \qquad (t)$$

$\Rightarrow c_1 + c_2 = 0,\ 4c_1 - c_2 + \sqrt{3}c_3 = 0$ and $8c_1 - c_2 - \sqrt{3}c_3 = 4$. Solving, we get

$c_1 = 2/7,\ c_2 = -2/7$ and $-10/7\sqrt{3} = c_3$. Thus, equation (1) becomes :

$$y = \frac{3}{7}e^{2x} - \frac{2}{7}e^{-x/2}\left[\cos\frac{\sqrt{3}x}{2} + \frac{5}{\sqrt{3}}\sin\frac{\sqrt{3}x}{2}\right].$$ This is a particular solution

(t) $(D^{3} + D^{2} + 4D + 4)\,y = 0;\ y(0) = y`(0) = 0,\ y``(0) = -5$
Solution: The auxiliary equation is $m^3 + m^2 + 4m + 4 = 0$.
Putting m = -1 we get: -1 + 1 -4 + 4 = 0. Thus, m = -1 is a root of given auxiliary equation. To find other two roots, we use synthetic division:

$$\begin{array}{c|cccc} -1 & 1 & 1 & 4 & 4 \\ & & -1 & 0 & -4 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

This gives $m^2 + 4 = 0 \Rightarrow m^2 = -4 \Rightarrow m = \pm i2$. Thus general solution is:

$$y = c_1 e^{-x} + \left[c_2 \cos 2x + c_3 \sin 2x\right] \qquad (1)$$

$$\Rightarrow y` = -c_1 e^{-x} + \left[-2c_2 \sin 2x + 2c_3 \cos 2x\right] \text{ and } y`` = c_1 e^{-x} + \left[-4c_2 \cos 2x - 4c_3 \sin 2x\right]$$

Putting $x = 0, y = 0, y` = 0$ and $y`` = -5$, we get:

$c_1 + c_2 = 0, \quad -c_1 + 2c_3 = 0$ and $c_1 - 4c_2 = -5$. Solving, we get

$c_1 = -1,\ c_2 = 1$ and $c_3 = -1/2$. Thus, equation (1) becomes

$y = -e^{-x} + \left[\cos 2x - \frac{1}{2}\sin 2x\right]$. This is a particular solution.

(u) $(D^3 - 6D^2 - 12D + 8)\,y = 0;\ y(0) = y`(0) = 0,\ y``(0) = 2$

Solution: The auxiliary equation is $m^3 - 6m^2 - 12m + 8 = 0$.

Putting m = -2 we get: -8 - 24 + 24 + 8 = 0. Thus, m = -2 is a root of given auxiliary equation. To find other two roots, we use synthetic division:

-2	1	-6	-12	8
		-2	16	-8
	1	-8	4	0

This gives $m^2 - 8m + 4 = 0 \Rightarrow m = \frac{8 \pm \sqrt{64-16}}{2} = \frac{8 \pm 4\sqrt{3}}{2} \Rightarrow m = 4 \pm 2\sqrt{3}$.

Thus general solution is:

$$y = c_1 e^{-2x} + \left[c_2 e^{(4+2\sqrt{3})x} + c_3 e^{(4-2\sqrt{3})x}\right] \qquad (1)$$

$$\Rightarrow y` = -2c_1 e^{-2x} + \left[(4+2\sqrt{3})c_2 e^{(4+2\sqrt{3})x} + c_3(4-2\sqrt{3})e^{(4-2\sqrt{3})x}\right] \text{ and}$$

$$y`` = 4c_1 e^{-2x} + \left[(4+2\sqrt{3})^2 c_2 e^{(4+2\sqrt{3})x} + c_3 (4-2\sqrt{3})^2 e^{(4-2\sqrt{3})x}\right]$$

Putting $x = 0, y = 0, y` = 0$ and $y`` = 2$, we get:

$c_1 + c_2 + c_3 = 0, \quad -2c_1 + (4+2\sqrt{3})c_2 + (4-2\sqrt{3})c_3 = 0$ and

$4c_1 + (4+2\sqrt{3})^2 c_2 + (4-2\sqrt{3})^2 c_3 = 2$. Solving, we get

$c_1 =, c_2 =$ and $c_3 =$. Thus, equation (1) becomes:

This is a particular solution.

2. Solve the following non-homogeneous linear differential equations:

(a) $(D^2 + D + 1)\,y = e^{-x}$

Solution: The auxiliary equation is $m^2 + m + 1 = 0$. Thus using quadratic formula, we get

$\Rightarrow m = \frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$. Thus, $y_c = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$

$$y_p = \frac{1}{(D^2 + D + 1)}e^{-x} = \frac{1}{(-1)^2 + (-1) + 1}e^{-x} = e^{-x}$$

Thus, general solution is: $y = y_c + y_p = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + e^{-x}$

(b) $(D^2 - 3D + 2)\,y = e^{5x}$

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0$. Thus using quadratic formula, we get

$$\Rightarrow m=\frac{3\pm\sqrt{9-8}}{2}=\frac{3\pm1}{2}=1,\ 2.\text{ Thus, } y_c=\left[c_1e^{x}+c_2e^{2x}\right]$$

$$y_p=\frac{1}{\left(D^2-3D+2\right)}e^{5x}=\frac{1}{(5)^2-3(5)+2}e^{5x}=\frac{e^{5x}}{12}$$

$$\text{Thus, general solution is: } y=y_c+y_p=\left[c_1e^{x}+c_2e^{2x}\right]+\frac{e^{5x}}{12}$$

(c) $(D^2 - 5D + 6)\, y = \sinh 2x$

Solution: The auxiliary equation is $m^2 - 5m + 6 = 0$. Thus using quadratic formula, we get

$$\Rightarrow m=\frac{5\pm\sqrt{25-24}}{2}=\frac{5\pm1}{2}=2,\ 3.\text{ Thus, } y_c=\left[c_1e^{2x}+c_2e^{3x}\right]$$

$$\text{Now, } y_p=\frac{1}{\left(D^2-5D+6\right)}\sinh 2x=\frac{1}{D^2-5D+6}\left(\frac{e^{2x}-e^{-2x}}{2}\right).\left[\text{NOTE: } e^{2x}\text{ is in } y_c\right]$$

$$=\frac{1}{2}\left[\frac{1}{D^2-5D+6}e^{2x}-\frac{1}{D^2-5D+6}e^{-2x}\right]=\frac{1}{2}\left[\frac{x}{2D-5}e^{2x}-\frac{1}{(-2)^2-5(-2)+6}e^{-2x}\right]$$

$$=\frac{1}{2}\left[\frac{x}{2(2)-5}e^{2x}-\frac{1}{20}e^{-2x}\right]=\frac{1}{2}\left[-xe^{2x}-\frac{1}{20}e^{-2x}\right]=-\frac{1}{40}\left[-20xe^{2x}+e^{-2x}\right]$$

$$\text{Thus, general solution is: } y=y_c+y_p=\left[c_1e^{2x}+c_2e^{3x}\right]-\frac{1}{40}\left[-20xe^{2x}+e^{-2x}\right]$$

(d) $(D^3 + 1)\, y = 5e^x - \cosh x$

Solution: The auxiliary equation is $m^3 + 1 = 0$ or $(m + 1)(m^2 - m + 1) = 0$

$$\Rightarrow m=-1\ \text{ OR }\ m=\frac{1\pm i\sqrt{3}}{2}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}.\text{ Thus, } y_c=c_1e^{-x}+e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$y_p=\frac{1}{\left(D^3+1\right)}\left[5e^{x}-\cosh x\right]=5\frac{1}{D^3+1}e^{x}+\frac{1}{D^3+1}\left[\frac{e^{x}+e^{-x}}{2}\right]\qquad\left[\text{NOTE: } e^{-x}\text{ is in } y_c\right]$$

$$=5\frac{1}{1^3+1}e^{x}+\frac{1}{2}\left[\frac{1}{1^3+1}e^{x}\right]+\frac{1}{2}\left[\frac{x}{3D^2}e^{-x}\right]=\frac{5e^{x}}{2}+\frac{e^{x}}{4}+\frac{1}{2}\left[\frac{xe^{-x}}{3(-1)^2}\right]=\frac{11e^{x}}{4}+\frac{xe^{-x}}{6}$$

Thus, general solution is:

$$y=y_c+y_p=c_1e^{-x}+e^{x/2}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right]+\frac{11e^{x}}{4}+\frac{xe^{-x}}{6}$$

(e) $(4D^2 + 4D - 3)\, y = e^{2x}$

Solution: The auxiliary equation is $4m^2 + 4m - 3 = 0$. Thus using quadratic formula, we get

$$\Rightarrow m=\frac{-4\pm\sqrt{16+48}}{8}=\frac{-4\pm8}{8}=\frac{1}{2},\ -\frac{3}{2}.\text{ Thus, } y_c=\left[c_1e^{x/2}+c_2e^{-3x/2}\right]$$

$$y_p=\frac{1}{\left(4D^2+4D-3\right)}e^{2x}=\frac{1}{4(2)^2+4(2)-3}e^{2x}=\frac{e^{2x}}{21}$$

$$\text{Thus, general solution is: } y=y_c+y_p=\left[c_1e^{x/2}+c_2e^{-3x/2}\right]+\frac{e^{2x}}{21}$$

(f) $(D^2 + D + 1)\,y = \sin 2x$

Solution: The auxiliary equation is $m^2 + m + 1 = 0$. Thus using quadratic formula, we get

$$\Rightarrow m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \text{ Thus, } y_c = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}\,x}{2} + c_2 \sin\frac{\sqrt{3}\,x}{2}\right]$$

$$y_p = \frac{1}{\left(D^2 + D + 1\right)}\sin 2x = \frac{1}{(-2^2) + D + 1}\sin 2x = \frac{1}{D-3} \times \frac{D+3}{D+3}\sin 2x = \frac{D+3}{D^2 - 9}\sin 2x$$

$$= \frac{D+3}{(-2^2) - 9}\sin 2x = -\frac{1}{13}(D+3)\sin 2x = -\frac{1}{13}(D\sin 2x + 3\sin 2x) = -\frac{1}{13}(2\cos 2x + 3\sin 2x)$$

Thus, general solution is :

$$y = y_c + y_p = e^{-x/2}\left[c_1 \cos\frac{\sqrt{3}\,x}{2} + c_2 \sin\frac{\sqrt{3}\,x}{2}\right] - \frac{1}{13}\left(2\cos 2x + 3\sin 2x\right)$$

(g) $(D^4 - 1)\,y = \cos x$

Solution: The auxiliary equation is $m^4 - 1 = 0$ or $(m^2 - 1)(m^2 + 1) = 0$ or $m = -1, 1, \pm i$.

Thus, $y_c = c_1 e^x + c_1 e^{-x} + c_3 \cos x + c_4 \sin x$.

$$y_p = \frac{1}{D^4 - 1}\cos x = \frac{x}{4D^3}\cos x \quad [\text{NOTE} : \cos x \text{ is in } y_c]$$

$$\Rightarrow y_p = \frac{x}{4}\int\!\!\int\!\!\int (\cos x\ dx).dx.dx = \frac{x}{4}\int\!\!\int (\sin dx).dx = \frac{x}{4}\int -\cos x\ dx = -\frac{x\sin x}{4}$$

Thus, general solution is : $y = y_c + y_p = c_1 e^x + c_1 e^{-x} + c_3 \cos x + c_4 \sin x - \dfrac{x\sin x}{4}$

(h) $(D^2 - 5D + 6)\,y = \sin 3x$

Solution: The auxiliary equation is $m^2 - 5m + 6 = 0$ or $m = 2, 3$. Thus, $y_c = c_1 e^{2x} + c_1 e^{3x}$

$$y_p = \frac{1}{\left(D^2 - 5D + 6\right)}\sin 3x = \frac{1}{(-3^2) + D + 1}\sin 3x = \frac{1}{D-8} \times \frac{D+8}{D+8}\sin 3x = \frac{D+8}{D^2 - 64}\sin 3x$$

$$= \frac{D+8}{(-3^2) - 64}\sin 3x = -\frac{1}{73}(D+8)\sin 3x = -\frac{1}{73}(D\sin 3x + 8\sin 3x) = -\frac{1}{73}(3\cos 3x + 8\sin 3x)$$

Thus, general solution is :

$$y = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} - \frac{1}{73}\left(3\cos 3x + 8\sin 3x\right)$$

(i) $(D^4 - 2D^2 + 1)\,y = \cos x$

Solution: The auxiliary equation is $m^4 - 2m^2 + 1 = 0$ or $(m^2 - 1)^2 = 0$

or $(m^2 - 1)(m^2 - 1) = 0$ or $m = -1, 1, -1, 1$. Thus, $y_c = (c_1 + c_2 x)\,e^x + (c_3 + c_4 x)\,e^{-x}$.

$$y_p = \frac{1}{\left(D^4 - 2D^2 + 1\right)}\cos x = \frac{1}{(D^2)^2 - 2D^2 + 1}\cos x = \frac{1}{(-1^2)^2 - 2(-1^2) + 1}\cos x$$

$$= \frac{1}{1 + 2 + 1}\cos x = \frac{\cos x}{4}$$

Thus, general solution is : $y = y_c + y_p = (c_1 + c_2 x)e^x + (c_3 + c_4 x)e^{-x} + \cos x / 4$

(j) $(D^2 + D - 6)\,y = x\,e^{2x}$ [Note the change in the problem]

Solution: The auxiliary equation is $m^2 + m - 6 = 0$ or $m = -3, 2$.

Thus, $y_c = (c_1 e^{-3x} + c_2 e^{2x})$.

$$y_p = \frac{1}{\left(D^2+D-6\right)} x\ e^{2x} = \left[e^{2x}\frac{1}{(D+2)^2+(D+2)-6}x\right] \qquad [\text{By shift property}]$$

$$= \left[e^{2x}\frac{1}{D^2+4D+4+D+2-6}x\right] = \left[e^{2x}\frac{1}{5D}x\right]$$

$$\left[\text{Neglecting } D^2 \text{ since power of x in the numerator is 1}\right]$$

$$= \frac{e^{2x}}{5}\left[\int x\ dx\right] = \frac{x^2e^{2x}}{10}. \text{ Thus, general solution is:}$$

$$y = y_c + y_p = \left[c_1e^{2x} + c_2e^{-3x}\right] + \frac{x^2e^{2x}}{10}$$

(k) $(D^3 - 3D - 2)\ y = x^2$

Solution: The auxiliary equation is $m^2 - 3m - 2 = 0$

$$\Rightarrow m = \frac{3 \pm \sqrt{9+8}}{2} = \frac{3 \pm \sqrt{17}}{2}. \text{ Thus, } m = \frac{3+\sqrt{17}}{2} \text{ OR } m = \frac{3-\sqrt{17}}{2}$$

$$\text{Thus, } y_c = c_1e^{\left(\frac{3+\sqrt{17}}{2}\right)x} + c_1e^{\left(\frac{3-\sqrt{17}}{2}\right)x}$$

$$y_p = \frac{1}{D^2-3D-2}x^2 = -\frac{1}{2}\frac{1}{1-\frac{D^2-3D}{2}}x^2 = -\frac{1}{2}\left(1-\frac{D^2-3D}{2}\right)^{-1}x^2$$

$$= -\frac{1}{2}\left[1+\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^2-3D}{2}\right)^2+\dots\right]x^2$$

$$= -\frac{1}{2}\left[1+\frac{D^2-3D}{2}+\frac{9D^2}{4}\right]x^2 \quad \left[\text{Neglecting } D^3 \text{and higher powers as x has exponent 2}\right]$$

$$= -\frac{1}{2}\left[\frac{4-6D+11D^2}{4}\right]x^2 = -\frac{1}{8}\left[4x^2-6Dx^2+11D^2x^2\right] = -\frac{1}{8}\left[4x^2-12x+22\right]$$

$$= -\frac{1}{4}\left[2x^2-6x+11\right]$$

$$\text{Thus, general solution is } y = y_c + y_p = c_1e^{\left(\frac{3+\sqrt{17}}{2}\right)x} + c_1e^{\left(\frac{3-\sqrt{17}}{2}\right)x} - \frac{1}{4}\left[2x^2-6x+11\right]$$

(*l*) $(D^3 - 13D + 12)\ y = x$

Solution: The auxiliary equation is $m^3 - 13m + 12 = 0$

Putting m = 1 we get: 1 - 13 + 12 = 0. Thus, m = 1 is a root of given auxiliary equation. To find other two roots, we use synthetic division:

1	1	0	-13	12
		1	1	-12
	1	1	-12	0

This gives $m^2 + m - 12 = 0$ or m = 3, -4. Thus, $y_c = c_1e^x + c_2e^{3x} + c_3e^{-4x}$

$$y_p = \frac{1}{\left(D^3 - 13D + 12\right)} x = \left[\frac{1}{12 - 13D} x\right] \left[\text{Neglecting } D^2 \text{ since power of x in is } 1\right]$$

$$= \frac{1}{12}\frac{1}{1-(13D/12)} x = \frac{1}{12}\left(1 - \frac{13D}{12}\right)^{-1} x = \frac{1}{12}\left[1 + \frac{13D}{12}\right] x = \frac{1}{12}\left(x + \frac{13}{12}Dx\right) = \frac{1}{12}\left(x + \frac{13}{12}\right)$$

$$= \frac{1}{12}\left(\frac{12x + 13}{12}\right) = \frac{1}{144}(12x + 13)$$

Thus, general solution is: $y = y_c + y_p = \left[c_1 e^x + c_2 e^{3x} + c_3 e^{-4x}\right] + \frac{1}{144}(12x + 13)$

(m) $(D^2 - 4)\, y = x^2$

Solution: The auxiliary equation is $m^2 - 4 = 0$ or $m = -2, 2$. Thus: $y_c = c_1 e^{-2x} + c_2 e^{2x}$

$$y_p = \frac{1}{\left(D^2 - 4\right)} x^2 = \frac{1}{-4\left(1 - \frac{D^2}{4}\right)} x^2 = -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} x^2 = -\frac{1}{4}\left[1 + \frac{D^2}{4}\right] x^2$$

(Neglecting D^4 and higher terms)

$$y_p = -\frac{1}{4}\left(x^2 + \frac{1}{4}D^2 x^2\right) = -\frac{1}{4}\left(x^2 + \frac{2}{4}\right) = -\frac{1}{4}\left(\frac{4x^2 + 2}{4}\right) = -\frac{1}{8}\left(2x^2 + 1\right)$$

Thus, general solution is: $y = y_c + y_p = \left[c_1 e^{-2x} + c_2 e^{2x}\right] - \frac{1}{8}\left(2x^2 + 1\right)$

(n) $(D^2 - 2D + 4)\, y = e^x \cos x$

Solution: The auxiliary equation is $m^2 - 2m + 4 = 0$. Using quadratic formula, we get

$m = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm i2\sqrt{3}}{2} = 1 \pm i\sqrt{3}$. Thus, $y_c = e^x\left(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x\right)$

$$y_p = \frac{1}{\left(D^2 - 2D + 4\right)} e^x \cos x = \left[e^{2x} \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x\right] \left[\text{By shift property}\right]$$

$$= e^x\left[\frac{1}{D^2 + 2D + 1 - 2D - 2 + 4} x\right] = e^x\left[\frac{1}{D^2 + 5} \cos x\right] = e^x\left[\frac{1}{(-1^2) + 5} \cos x\right]$$

$= \frac{e^x \cos x}{4}$. Thus, general solution is:

$$y = y_c + y_p = e^x\left(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x\right) + \frac{e^x \cos x}{4}$$

(o) $(D^2 - 5D + 6)\, y = xe^{4x}$

Solution: The auxiliary equation is $m^2 - 5m + 6 = 0$ or $m = 2, 3$.

Thus, $y_c = (c_1 e^{2x} + c_2 e^{3x})$.

$$y_p = \frac{1}{\left(D^2 - 5D + 6\right)} x\, e^{4x} = \left[e^{4x} \frac{1}{(D+4)^2 - 5(D+4) + 6} x\right] \qquad \left[\text{By shift property}\right]$$

$$= \left[e^{4x} \frac{1}{D^2 + 8D + 16 - 5D - 20 + 6} x\right] = \left[e^{4x} \frac{1}{3D + 2} x\right]$$

$\left[\text{Neglecting } D^2 \text{ since power of x in the numerator is } 1\right]$

$$=\frac{e^{4x}}{2}\left(\frac{1}{1+3D/2}\right)=\frac{e^{4x}}{2}\left(1+\frac{3D}{2}\right)^{-1}x=\frac{e^{4x}}{2}\left(1-\frac{3D}{2}\right)x$$

[Using binomial expansion and neglecting D^2 since power of x in the numerator is 1]

$$=\frac{e^{4x}}{2}\left(x-\frac{3Dx}{2}\right)=\frac{e^{4x}}{2}\left(\frac{2x-3}{2}\right)=\frac{e^{4x}}{4}(2x-3).$$ Thus, general solution is:

$$y=y_c+y_p=\left[c_1e^{2x}+c_2e^{3x}\right]+\frac{e^{4x}}{4}(2x-3)$$

(p) $(D^2+1)\,y = x\sin 2x$

Solution: The auxiliary equation is $m^2+1=0$ or $m=\pm i$. Thus, $y_c = c_1\cos x + c_2 \sin x$.

$$y_p=\frac{1}{(D^2+1)}\,x\sin 2x=\mathrm{Im}\left(\frac{1}{D^2+1}x\,e^{ix}\right)=\mathrm{Im}\left(e^{ix}\frac{1}{(D+i)^2+1}x\right)\ \text{[By shift property]}$$

$$=\mathrm{Im}\left(e^{ix}\frac{1}{D^2+2iD+i^2+1}x\right)=\mathrm{Im}\left(e^{ix}\frac{1}{D^2+2iD-1+1}x\right)$$

$$=\mathrm{Im}\left(e^{ix}\frac{1}{D^2+2iD}x\right)=\mathrm{Im}\left(e^{ix}.\frac{1}{2iD[1+(D/2i)]}x\right)=\mathrm{Im}\left(e^{ix}.\frac{1}{2iD}[1+(D/2i)]^{-1}x\right)$$

$$=\mathrm{Im}\left(e^{ix}.\frac{1}{2iD}\left[1-\frac{D}{2i}\right]x\right)=\mathrm{Im}\left(e^{ix}.\frac{1}{2iD}\left[x-\frac{Dx}{2i}\right]x\right)=\mathrm{Im}\left(e^{ix}.\frac{1}{2iD}\left[\frac{2ix-1}{2i}\right]\right)$$

[Applying the binomial expansion and keeping the terms upto power 1 of D due to x]

$$=\mathrm{Im}\left(-\frac{e^{ix}}{4}\int(2ix-1)\,dx\right)=\mathrm{Im}\left(-\frac{e^{ix}}{4}\left[ix^2-x\right]\right)=\frac{x}{4}\,\mathrm{Im}(1-ix)(\cos x+i\sin x)$$

$$=\frac{x}{4}\,\mathrm{Im}\left[(\cos x+x\sin x)+i(\sin x-x\cos x)\right]=\frac{x}{4}(\sin x-x\cos x)$$

NOTE: "Im" means the imaginary part of : and $e^{ix}=\cos x+i\sin x$ [By Euler formula]

Thus, general solution is: $y=y_c+y_p=(c_1\cos x+c_2\sin x)+\frac{x}{4}(\sin x-x\cos x)$.

(q) $(D-1)^3\,y = 16\,e^x$ (Note the change in the problem)

Solution: The auxiliary equation is $(m-1)^3=0$ or $m=1, 1, 1$. Thus,
$y_c = (c_1 + c_2 x + c_3 x^2)\,e^x$.

$$y_p=16\frac{1}{(D-1)^3}e^x=16\frac{x}{3(D-1)^2}e^x=16\frac{x^2}{3.2(D-1)}e^x=\frac{x^3}{3.2.1}e^x=16\frac{x^3e^x}{6}$$

NOTE: e^x, xe^x, x^2e^x are all in y_c.

Thus, general solution is: $y=y_c+y_p=\left(c_1+c_2x+c_3x^2\right)e^x+\frac{8x^3e^x}{3}$

(r) $(D^3 + 2D^2 - D - 2)\,y = e^x$

Solution: The auxiliary equation is $(m^3 + 2m^2 - m - 2) = 0$. Putting m= 1, we get
$1 + 2 - 1 - 2 = 0$. Thus, m = 1 is a root. To find other two roots, we apply synthetic division.

$$\begin{array}{c|cccc} 1 & 1 & 2 & -1 & -2 \\ & & 1 & 3 & 2 \\ \hline & 1 & 3 & 2 & \boxed{0} \end{array}$$

This gives $m^2 + 3m - 2 = 0$. Applying quadratic formula, we get

$$m = \frac{-3 \pm \sqrt{9+8}}{2} = \frac{-3 \pm \sqrt{17}}{2}. \text{ Thus, } y_c = c_1 e^x + c_2 e^{\left(\frac{-3+\sqrt{17}}{2}\right)x} + c_3 e^{\left(\frac{-3-\sqrt{17}}{2}\right)x}$$

$$y_p = \frac{1}{\left(D^3 + 2D^2 - D - 2\right)} e^x = \frac{x}{3D^2 + 4D - 1} e^x = \frac{xe^x}{6} \left(\text{Note}: e^x \text{ is in } y_c\right)$$

$$\text{Thus, general solution is}: y = y_c + y_p = c_1 e^x + c_2 e^{\left(\frac{-3+\sqrt{17}}{2}\right)x} + c_3 e^{\left(\frac{-3-\sqrt{17}}{2}\right)x} + \frac{xe^x}{6}$$

(s) $(D^3 - 2D^2 - 5D + 6)y = e^{3x}$

Solution: The auxiliary equation is $(m^3 - 2m^2 - 5m + 6) = 0$. Putting $m = 1$, we get $1 - 2 - 5 + 6 = 0$. Thus, $m = 1$ is a root. To find other two roots, we apply synthetic division.

$$\begin{array}{c|cccc} 1 & 1 & -2 & -5 & 6 \\ & & 1 & -1 & -6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

This gives $m^2 - m - 6 = 0$. Applying quadratic formula, we get: $m = 3, -2$

Thus, $y_c = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x}$

$$y_p = \frac{1}{\left(D^3 - 2D^2 - 5D + 6\right)} e^{3x} = \frac{x}{3D^2 - 4D - 5} e^{3x} = \frac{xe^{3x}}{3.3^2 - 4.3 + 6} = \frac{xe^{3x}}{21}$$

$$\left(\text{Note}: e^{3x} \text{ is in } y_c\right).\text{Thus, general solution is}: y = y_c + y_p = c_1 e^x + c_2 e^{3x} + c_3 e^{-2x} + \frac{xe^{3x}}{21}$$

(t) $(D^2 - D - 6)y = e^x \sinh 3x$

Solution: The auxiliary equation is $m^2 - m - 6 = 0$ or $m = 3, -2$.

Thus, $y_c = (c_1 e^{3x} + c_2 e^{-2x})$.

$$y_p = \frac{1}{\left(D^2 - D - 6\right)} e^x \sinh 3x = \left[\frac{1}{\left(D^2 - D - 6\right)} e^x \left(\frac{e^{3x} - e^{-3x}}{2}\right)\right] = \frac{1}{2}\frac{1}{\left(D^2 - D - 6\right)}\left(e^{4x} - e^{-2x}\right)$$

$$= \frac{1}{2}\left[\frac{1}{\left(D^2 - D - 6\right)} e^{4x} - \frac{1}{\left(D^2 - D - 6\right)} e^{-2x}\right] = \frac{1}{2}\left[\frac{e^{4x}}{4^2 - 4 - 6} - \frac{x}{2D - 1} e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{e^{4x}}{6} - \frac{x}{2(-2) - 1} e^{-2x}\right] = \frac{1}{2}\left[\frac{e^{4x}}{6} + \frac{xe^{-2x}}{5}\right] = \frac{5e^{4x} + 6xe^{-2x}}{60} \quad \text{NOTE}: e^{-2x} \text{ is in } y_c$$

$$\text{Thus, general solution is}: y = y_c + y_p = \left[c_1 e^{3x} + c_2 e^{-2x}\right] + \frac{5e^{4x} + 6xe^{-2x}}{60}$$

(u) $(D^4 + 1)y = x^4$

Solution: The auxiliary equation is $m^4 + 1 = 0$ or $m^4 + 2m^2 + 1 = 2m^2$
or $(m^2 + 1)^2 = (\sqrt{2}\, m)^2$ or $m^2 + 1 = \pm (\sqrt{2}\, m)$. Thus,

$m^2 - \sqrt{2}\,m + 1 = 0$ OR $m^2 + \sqrt{2}\,m + 1 = 0$. Applying quadratic formula, we have:

$$\text{If } m^2 - \sqrt{2}\,m + 1 = 0 \Rightarrow m = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm i\sqrt{2}}{2} = \frac{1 \pm i}{\sqrt{2}} = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$$

$$\text{And, if } m^2 + \sqrt{2}\,m + 1 = 0 \Rightarrow m = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{-\sqrt{2} \pm i\sqrt{2}}{2} = \frac{-1 \pm i}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$$

$$\text{Thus, } y_c = e^{x/\sqrt{2}}\left[c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[c_3 \cos\frac{x}{\sqrt{2}} + c_4 \sin\frac{x}{\sqrt{2}}\right]$$

$$y_p = \frac{1}{\left(D^4+1\right)}x^4 = \left(1+D^4\right)^{-1}x^4 = \left(1 - D^4 + D^8 \ldots\right)x^4 = x^4 - D^4x^4 = x^4 - 24$$

$\left[\text{Neglecting } D^8 \text{ and higher terms as x has power 4}\right]$. Thus, general solution is:

$$y = y_c + y_p = e^{x/\sqrt{2}}\left[c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[c_3 \cos\frac{x}{\sqrt{2}} + c_4 \sin\frac{x}{\sqrt{2}}\right] + \left(x^4 - 24\right)$$

(v) $(5D^3 - D^2 - 6D)y = x^2$

Solution: The auxiliary equation is $m(5m^2 - m - 6) = 0$ or $m = 0, 3, -2$. Thus,

$$y_c = c_1 + c_2e^{3x} + c_3e^{-2x}$$

$$\text{Thus, } y_p = \frac{1}{\left(5D^3 - D^2 - 6D\right)}x^2 = \frac{1}{D\left(5D^2 - D - 6\right)}x^2 = \frac{1}{D}\cdot\frac{1}{-6\left(1 + \frac{D - 5D^2}{6}\right)}x^2$$

$$= \frac{1}{-6D}\left(1 + \frac{D-5D^2}{6}\right)^{-1}x^2 = \frac{1}{-6D}\left(1 - \frac{D-5D^2}{6} + \left(\frac{D-5D^2}{6}\right)^2 + \ldots\right)x^2 \left[\text{Binomial Exp:}\right]$$

$$= \frac{1}{-6D}\left(1 - \frac{D-5D^2}{6} + \frac{D^2}{36}\right)x^2 \quad \left[\text{Neglecting } D^3 \text{and the higher powers as x has power 2}\right]$$

$$= -\frac{1}{6D}\left[\frac{36 - 6D - 30D^2 + D^2}{36}\right]x^2 = -\frac{1}{196D}\left[36x^2 - 6Dx^2 - 29D^2x^2\right]$$

$$= -\frac{1}{196}\int\left(36x^2 - 12x - 58\right)dx = -\frac{1}{196}\left(\frac{36x^3}{3} - \frac{12x^2}{2} - 58x\right) = -\frac{1}{196}\left(12x^3 - 6x^2 - 58x\right)$$

$$\text{Now, general solution is: } y = y_c + y_p = y_c = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{196}\left(12x^3 - 6x^2 - 58x\right)$$

$$\left[\text{NOTE: } \frac{1}{D}f(x) = \int f(x)\,dx\right]$$

(w) $(D^3 + 1)y = 2\cos^2 x$

Solution: The auxiliary equation is $(m^3 + 1) = 0$ or $(m + 1)(m^2 - m + 1) = 0$

$$m = -1 \text{ or } m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}. \text{ Thus, } y_c = c_1e^{-x} + e^{x/2}\left(c_2 \cos\frac{\sqrt{3}\,x}{2} + c_3 \sin\frac{\sqrt{3}\,x}{2}\right)$$

$$y_p = 2\frac{1}{D^3+1}\cos^2 x = 2\frac{1}{D^3+1}\left(\frac{1+\cos 2x}{2}\right) = \frac{1}{D^3+1}e^{0x} + \frac{1}{D^3+1}\cos 2x$$

$$= \frac{1}{0^3+1}e^{0x} + \frac{1}{D^2.D+1}\cos 2x = 1 + \frac{1}{(-2^2)D+1}\cos 2x = 1 + \frac{1}{1-4D}\times\frac{1+4D}{1+4D}\cos 2x$$

$$= 1 + \frac{1+4D}{1-16D^2}\cos 2x = 1 + \frac{1+4D}{1-16(-2^2)}\cos 2x = 1 + \frac{1}{65}(\cos 2x + 4D\cos 2x)$$

$$= 1 + \frac{1}{65}(\cos 2x - 8\sin 2x) = \frac{65+\cos 2x - 8\sin 2x}{65}. \text{ Thus}$$

$$y = y_c + y_p = c_1 e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}\,x}{2} + c_3\sin\frac{\sqrt{3}\,x}{2}\right) + \frac{65+\cos 2x - 8\sin 2x}{65}$$

(x) $(D^2 - 3D + 2)y = x^2e^{4x}$

Solution: The auxiliary equation is $(m^2 - 3m + 2) = 0$ or $m = 1, 2$ or $y_c = c_1e^x + c_2 e^{2x}$.

$$y_p = \frac{1}{D^2-3D+2}x^2e^{4x} = e^{4x}\frac{1}{(D+4)^2-3(D+4)+2}x^2 \quad [\text{Using shift theorem}]$$

$$= e^{4x}\frac{1}{D^2+8D+16-3D-12+2}x^2 = e^{4x}\frac{1}{D^2+5D+6}x^2 = \frac{e^{4x}}{6}\left(1+\frac{D^2+5D}{6}\right)^{-1}x^2$$

$$= \frac{e^{4x}}{6}\left(1+\frac{D^2+5D}{6}\right)^{-1}x^2 = \frac{e^{4x}}{6}\left[1-\left(\frac{D^2+5D}{6}\right)+\left(\frac{D^2+5D}{6}\right)^2-\ldots\right]x^2$$

$$= \frac{e^{4x}}{6}\left[1-\left(\frac{D^2+5D}{6}\right)+\frac{25D^2}{36}\right]x^2 \quad [\text{Neglecting } D^3 \text{ and higher terms}]$$

$$= \frac{e^{4x}}{6}\left[\frac{36-6D^2-30D+25D^2}{36}\right]x^2 = \frac{e^{4x}}{196}\left[36x^2-30Dx^2+19D^2x^2\right]$$

$$= \frac{e^{4x}}{196}\left[36x^2-60x+38\right]. \text{ Now,}$$

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + \frac{e^{4x}}{196}\left[36x^2-60x+38\right]$$

(y) $(D^3 - 4D^2 + 3D)y = \cos 2x$

Solution: The auxiliary equation is $m(m^2 - 4m + 3) = 0$ or $m = 0, 1, 3$
or $y_c = c_1e^{0x} + c_2 e^x + c_3 e^{3x} = (c_1 + c_2 e^x + c_3 e^{3x})$

$$y_p = \frac{1}{D^2.D-4D^2+3D}\cos 2x = \frac{1}{(-2)^2D-4(-2^2)+3D}\cos 2x = \frac{1}{16-D}\times\frac{16+D}{16+D}\cos 2x$$

$$= \frac{16+D}{256-D^2}\cos 2x = \frac{16+D}{256-(-2^2)}\cos 2x = \frac{16\cos 2x + D\cos 2x}{260} = \frac{16\cos 2x - 2\sin 2x}{260}$$

$$= \frac{8\cos 2x - \sin 2x}{130}. \quad \text{Now, } y = y_c + y_p = \left(c_1 + c_2\,e^x + c_3\,e^{3x}\right) + \frac{8\cos 2x - \sin 2x}{130}$$

(z) $(D^2 - 7D + 10)y = e^{2x}\sin x + xe^x$

Solution: The auxiliary equation is $(m^2 - 7m + 10) = 0$. or $m = 2, 5$
or $y_c = c_1e^{2x} + c_2 e^{5x}$ [NOTE: See the change in the problem]

$$y_p = \frac{1}{D^2 - 7D + 10}\left(e^{2x}\sin x + xe^x\right) = e^{2x}\frac{1}{(D+2)^2 - 7(D+2) + 10}\sin x$$

$$+ e^x \frac{1}{(D+1)^2 - 7(D+1) + 10} x \quad \left[\text{Using shift theorem separately}\right]$$

$$= e^{2x}\frac{1}{D^2 + 4D + 4 - 7D - 14 + 10}\sin x + e^x \frac{1}{D^2 + 2D + 1 - 7D - 7 + 10} x$$

$$= e^{2x}\frac{1}{-1^2 - 3D}\sin x + e^x\frac{1}{4 - 5D} x \quad \left[\text{Neglecting } D^2 \text{ in the 2nd expression for x has power 1}\right]$$

$$= -e^{2x}\frac{1}{1+3D}\times\frac{1-3D}{1-3D}\sin x + e^x\frac{1}{4(1-5D/4)} x = -e^{2x}\frac{1-3D}{1-9D^2}\sin x + \frac{e^x}{4}\left(1 - \frac{5D}{4}\right)^{-1} x$$

$$= -e^{2x}\frac{1-3D}{1-9(-1^2)}\sin x + \frac{e^x}{4}\left(1 + \frac{5D}{4}\right) x \quad \left[\text{Using binomial expansion and neglecting } D^2\right]$$

$$= -\frac{e^{2x}}{10}\left(\sin x - 3D\sin x\right) + \frac{e^x}{4}\left(x + \frac{5Dx}{4}\right) = -\frac{e^{2x}}{10}\left(\sin x - 3\cos x\right) + \frac{e^x}{4}\left(x + \frac{5}{4}\right)$$

$$= -\frac{e^{2x}}{10}\left(\sin x - 3\cos x\right) + \frac{e^x}{16}\left(4x + 5\right)$$

$$\text{Now, } y = y_c + y_p = \left(c_1 e^{2x} + c_2 e^{5x}\right) + -\frac{e^{2x}}{10}\left(\sin x - 3\cos x\right) + \frac{e^x}{16}\left(4x + 5\right)$$

NOTE: The following table will be used to find the "Particular Integrals" while using Indeterminate Coefficients Method.

f(x)	Form of y_p
1. 1 (any constant)	A
2. 5x + 7	Ax + B
3. $3x^2$ - 2	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + D$
5. sin 4x	Acos 4x + Bsin 4x
6. cos 4x	Acos 4x + Bsin 4x
7. e^{5x}	(Ax + B) e^{5x} or A e^{5x}
8. (9x -2) e^{5x}	$(Ax + B)e^{5x}$
9. $x^2 e^{5x}$	$(Ax^2 + Bx + C)e^{5x}$
10. e^{3x} sin 4x	(Acos 4x + B sin 4x) e^{3x}
11. $5x^2$ sin 4x	$(Ax^2 + Bx + C)$ cos4x + $(Dx^2 + Ex + F)$ sin4x
12. xe^{3x} cos 4x	$(Ax + B)e^{3x}$ cos 4x + $(Cx + D)e^{3x}$ sin 4x

3. Solve the following differential equations by method of undetermined coefficient

(a) $(D^2 + D + 1)\, y = e^{-x}$ (1)

Solution: The auxiliary equation is $m^2 + m + 1 = 0$ or $m = \left(-1 \pm i\sqrt{3}\right)/2$. Thus,

$$y_c = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}\,x}{2} + c_2\sin\frac{\sqrt{3}\,x}{2}\right)$$

To find y_p we proceed as under: Since right side contains e^{-x} and it is not in y_c hence we

let: $y = Ae^{-x}$ or $y` = -Ae^{-x}$ and $y`` = Ae^{-x}$. Thus, equation (1) becomes $(A - A + A)e^{-x} = e^{-x}$. Comparing both sides, we get A = 1. Thus, $y_p = e^{-x}$. Now,

$$y = y_c + y_p = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}\,x}{2} + c_2 \sin\frac{\sqrt{3}\,x}{2}\right) + e^{-x}$$

(b) $(D^2 - 3D + 2)\, y = e^{5x}$ (1)

Solution: The auxiliary equation $m^2 - 3m + 2 = 0$ or $m = 1, 2$. Thus, $y_c = \left(c_1 e^x + c_2 e^{2x}\right)$

To find y_p we proceed as under: Since right side contains $e^{5x}$and iy is not in y_c hence, we let: $y = Ae^{5x}$ or $y` = 5Ae^{5x}$ and $y`` = 25Ae^{5x}$. Thus, equation (1) becomes $(25A - 15A + 2A)e^{5x} = e^{5x}$. Comparing both sides, we get: $12A = 1$ giving $A = 1/12$.

Thus, $y_p = e^{5x}/12$. Now, $y = y_c + y_p = \left(c_1 e^x + c_2 e^{2x}\right) + e^{5x}/12$

(c) $(D^2 - 5D + 6)\, y = e^{4x}$ (1)

Solution: The auxiliary equation $m^2 - 5m + 6 = 0$ or $m = 2, 3$. Thus, $y_c = \left(c_1 e^{2x} + c_2 e^{3x}\right)$

To find y_p we proceed as under: Since right side contains $e^{4x}$and iy is not in y_c hence, we let: $y = Ae^{4x}$ or $y` = 4Ae^{4x}$ and $y`` = 16Ae^{4x}$. Thus, equation (1) becomes $(16A - 12A + 2A)e^{5x} = e^{5x}$. Comparing both sides, we get: $6A = 1$ giving $A = 1/6$.

Thus, $y_p = e^{4x}/6$. Now, $y = y_c + y_p = \left(c_1 e^{2x} + c_2 e^{3x}\right) + e^{4x}/6$

(d) $(D^3 + 1)\, y = 3 + 5e^x$ (1)

Solution: The auxiliary equation $m^3 + 1 = 0 \Rightarrow (m+1)(m^2 - m + 1) = 0$.

$\Rightarrow m = -1$ OR $m = \dfrac{1 \pm i\sqrt{3}}{2}$. Thus, $y_c = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\dfrac{\sqrt{3}\,x}{2} + c_3 \sin\dfrac{\sqrt{3}\,x}{2}\right)$

To find y_p we proceed as under: Since right side contains constant and e^x and both terms are not in y_c hence, we let $y = B + Ae^x$ or $y` = Ae^x$, $y`` = Ae^x$and $y``` = Ae^x$. Thus, equation (1) becomes: $Ae^x + B + Ae^x = 3 + e^x$. Comparing both sides, we get: $2A = 1$ and $B = 0$ or $A = 1/2$. Thus, $y_p = 3 + e^x/2$.

Now, $y = y_c + y_p = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\dfrac{\sqrt{3}\,x}{2} + c_3 \sin\dfrac{\sqrt{3}\,x}{2}\right) + 3 + \dfrac{e^x}{2}$

(e) $(4D^2 + 4D - 3)\, y = e^{2x}$ (1)

Solution: The auxiliary equation $4m^2 + 4m - 3 = 0 \Rightarrow m = 1/2, -3/2$. (By Q. Formula)

Thus, $y_c = c_1 e^{x/2} + c_2 e^{-3x/2}$.

To find y_p we proceed as under: Since right side contains constant and e^{2x} and this term is not in y_c hence, we let $y = Ae^{2x}$ or $y` = 2Ae^x$, $y`` = 4Ae^x$. Thus, equation (1) becomes: $16Ae^{2x} + 8Ae^{2x} - 3Ae^{2x} = e^{2x}$. Comparing both sides, we get: $21A = 1$ or $A = 1/21$. Thus, $y_p = e^{2x}/21$. Now, $y = y_c + y_p = c_1 e^{x/2} + c_2 e^{-3x/2} + e^{2x}/21$

(f) $(D^2 + D + 1)\, y = \sin 2x$ (1)

Solution: The auxiliary equation is $m^2 + m + 1 = 0$ or $m = \left(-1 \pm i\sqrt{3}\right)/2$. Thus,

$$y_c = e^{-x/2}\left(c_1 \cos\frac{\sqrt{3}\,x}{2} + c_2 \sin\frac{\sqrt{3}\,x}{2}\right)$$

To find y_p we proceed as under: Since right side contains constant and $\sin 2x$ and this term is not in y_c hence, we let $y = A\cos 2x + B\sin 2x$ or $y` = -2A \sin 2x + 2B \cos 2x$ and, $y`` = -4A \cos 2x - 4B \sin 2x$. Thus, equation (1) becomes

$-4A \cos 2x - 4B \sin 2x - 2A \sin 2x + 2B \cos 2x + A \cos 2x + B \sin 2x = \sin 2x$. or $(-3A + 2B) \cos 2x + (-3B - 2A) \sin 2x = \sin 2x$. Comparing both sides, we get:

-3A + 2B = 0 and -3B – 2A = 1 or A = 10/13 and B = -3/13. Thus,

$y_p = (10\cos 2x - 3\sin 2x)/13$. Now,

$$y = y_c + y_p = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}\,x}{2} + c_2\sin\frac{\sqrt{3}\,x}{2}\right) + \frac{(10\cos 2x - 3\sin 2x)}{10}$$

(g) $(D^4 + 1)\,y = \cos x$ (1)

Solution: The auxiliary equation is $m^4 + 1 = 0$ or $m^4 + 2m^2 + 1 = 2m^2$

Or $(m^2 + 1)^2 = (\sqrt{2}\,m)^2$ or $m^2 + 1 = \pm(\sqrt{2}\,m)$. Thus,

$m^2 - \sqrt{2}\,m + 1 = 0$ OR $m^2 + \sqrt{2}\,m + 1 = 0$. Applying quadratic formula, we get

$$\text{Now, if } m^2 - \sqrt{2}\,m + 1 = 0 \Rightarrow m = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{\sqrt{2} \pm i\sqrt{2}}{2} = \frac{1 \pm i}{\sqrt{2}} = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$$

$$\text{And, if } m^2 + \sqrt{2}\,m + 1 = 0 \Rightarrow m = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{-\sqrt{2} \pm i\sqrt{2}}{2} = \frac{-1 \pm i}{\sqrt{2}} = \frac{-1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$$

$$\text{Thus, } y_c = e^{x/\sqrt{2}}\left[c_1\cos\frac{x}{\sqrt{2}} + c_2\sin\frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[c_3\cos\frac{x}{\sqrt{2}} + c_4\sin\frac{x}{\sqrt{2}}\right]$$

To find y_p we proceed as under: Since right side contains constant and cos x and this term is not in y_c hence, we let y = Acos x + Bsin x or y` = -A sin x + B cos x, y`` = -Acos x – B sin x, y``` = Asin x – Bcos x and y^{iv} = Acos x + Bsin x. Thus, (1) becomes: 2Acos x + 2Bsin x = cos x. Comparing both sides, we get: 2A = 0 and 2B = 1. This implies that A = 0 and B = 1/2. Thus, y_p = (sin x)/2.

$$\text{Now, } y = y_c + y_p = e^{x/\sqrt{2}}\left[c_1\cos\frac{x}{\sqrt{2}} + c_2\sin\frac{x}{\sqrt{2}}\right] + e^{-x/\sqrt{2}}\left[c_3\cos\frac{x}{\sqrt{2}} + c_4\sin\frac{x}{\sqrt{2}}\right] + \frac{\sin x}{2}$$

(h) $(D^2 - 5D + 6)\,y = \sin 3x$ (1)

Solution: The auxiliary equation is $m^2 - 5m + 6 = 0$ or m = 2, 3. Thus, $y_c = c_1e^{2x} + c_2e^{3x}$.

To find y_p we proceed as under: Since right side contains constant and sin 3x and this term is not in y_c hence, we let y = Acos 3x + Bsin 3x or y` = -3A sin 3x + 3B cos 3x and, y`` = -9A cos 3x – 9B sin 3x. Thus, equation (1) becomes:

-9Acos 3x – 9Bsin 3x + 15Asin 3x - 15Bcos 3x + 6Acos 3x + 6Bsin 3x = sin 3x. or (-3A - 15B) cos 3x + (-3B + 15A) sin 3x = sin 3x. Comparing both sides, we get:

-3A - 15B = 0 and -3B + 15A = 1 or A = 5/78 and B = -1/78. Thus,

$y_p = (5\cos 3x - \sin 3x)/78$. Now,

$$y = y_c + y_p = \left(c_1e^{2x} + c_2e^{3x}\right) + (5\cos 3x - \sin 3x)/78$$

(i) $(D^4 - 2D^2 + 1)\,y = \cos x$ (1)

Solution: The auxiliary equation is $m^4 - 2m^2 + 1 = 0$ or $(m^2 - 1)^2 = 0$

or $[(m - 1)(m + 1)]^2 = 0$ giving m = -1, -1, 1, and 1. Thus, $y_c = (c_1 + c_2x)e^x + (c_2 + c_4x)e^{-x}$.

To find y_p we proceed as under: Since right side contains cos x and this term is not in y_c hence, we let y = Acos x + Bsin x or y` = -A sin x + B cos x, y`` = -A cos x – B sin x, y``` = Asin x – Bcos x and y^{iv} = Acos x + Bsin x. Thus, equation (1) becomes:

(Acos x + Bsin x) -2(-A cos x – B sin x) + (A cos x + B sin x) = cos x

or 4A cos x + 4B sin x = cos x or 4A = 1 and 4B = 0 giving A = 1/4 and B = 0. Thus,

y_p = (cos x)/4. Now,

$$y = y_c + y_p = (c_1 + c_2x)e^x + (c_2 + c_4x)e^{-x} + \cos x/4$$

(k) $(D^2 + D - 6)\,y = x$ (1)

Solution: The auxiliary equation is $m^2 + m - 6 = 0$ or m = 2, -3. Thus, $y_c = c_1e^{2x} + c_2e^{-3x}$.

To find y_p we proceed as under: Since right side contains 'x' and this term is not in y_c hence, we let y = A + Bx or y\` = B and, y\`\` = 0. Thus, equation (1) becomes:
0 + B – 6A – 6Bx = x or (-6A + B) – 6Bx = x or -6B = 1 and -6A + B = 0 or B = -1/6 and B – 6A = 0 or A = B/6 = -1/36. Thus,

$$y_p = -\frac{1}{36} - \frac{x}{6} = -\left(\frac{1+6x}{36}\right). \text{ Now, } y = y_c + y_p = \left(c_1e^{2x} + c_2e^{-3x}\right) - \left(\frac{1+6x}{36}\right)$$

(*l*) $(D^3 - 3D - 2)y = x^2$ (1)

Solution: The auxiliary equation is $m^3 - 3m - 2 = 0$ or m = -1, -1, 2. Thus, $y_c = (c_1 + c_2x)e^{-x} + c_3e^{2x}$.

To find y_p we proceed as under: Since right side contains 'x^2' and this term is not in y_c hence, we let $y = A + Bx + Cx^2$ or y\` = B + 2Cx, y\`\` = 2C and y\`\`\` = 0. Thus, equation (1) becomes:

$0 - 3(B + 2Cx) - 2(A + Bx + Cx^2) = x^2$ or -3B – 2A = 0, -6C – 2B = 0 and -2C = 1

Or C = -1/2, B = 3/2 and A = -1. Thus, $y_p = -\frac{x^2}{2} + \frac{3x}{2} - 1$

$$\text{Now, } y = y_c + y_p = \left(c_1 + c_2x\right)e^{-x} + c_3e^{2x} - \frac{x^2}{2} + \frac{3x}{2} - 1$$

(m) $(D^3 - 13D + 12)y = x$ (1)

Solution: The auxiliary equation is $m^3 - 13m + 12 = 0$ giving, m = 1, 3, -4. Thus, $y_c = c_1e^x + c_2e^{3x} + c_3e^{-4x}$.

To find y_p we proceed as under: Since right side contains 'x' and this term is not in y_c hence, we let y = A + Bx or y\` = B, y\`\` = 0 and y\`\`\` = 0. Thus, equation (1) becomes:
0 -13B + 12(A + Bx) = x or -13B + 12A = 0, 12B = 1 giving B = 1/12 and A = 13/144.

$$\text{Thus, } y_p = \frac{13}{144} + \frac{x}{12} = \left(\frac{13+12x}{144}\right)$$

$$\text{Now, } y = y_c + y_p = c_1e^x + c_2e^{3x} + c_3e^{-4x} + \left(\frac{13+12x}{144}\right)$$

(n) $(D^2 - 4)y = x^2$ (1)

Solution: The auxiliary equation is $m^2 - 4 = 0$ giving, m = -2, 2.
Thus, $y_c = c_1e^{-2x} + c_2e^{2x}$.

To find y_p we proceed as under: Since right side contains 'x^2' and this term is not in y_c hence, we let $y = A + Bx + Cx^2$ or y\` = B + 2Cx, y\`\` = 2C. Thus, equation (1) becomes:
$2C - 4A - 4Bx - 4Cx^2 = x^2$ or -4C = 1, -4B = 0 and 2C – 4A = 0 or C = -1/4, B = 0 and A = -1/8. Thus, $y_p = -\frac{x^2}{4} - \frac{1}{8} = -\left(\frac{2x^2+1}{8}\right)$

$$\text{Now, } y = y_c + y_p = c_1e^{-2x} + c_2e^{2x} - \left(\frac{2x^2+1}{8}\right)$$

(o) $(D^2 - 5D + 6)y = xe^{4x}$ (1)

Solution: The auxiliary equation is $m^2 - 5m + 6 = 0$ or m = 2, 3. Thus, $y_c = c_1e^{2x} + c_2e^{3x}$.

To find y_p we proceed as under: Since right side contains 'xe^{4x}' and this term is not in y_c hence, we let $y = (A + Bx)e^{4x}$ or $y\grave{} = 4(A + Bx)e^{4x} + Be^{4x} = (4A + 4Bx + B)e^{4x}$, $y\grave{}\grave{} = 4(4A + 4Bx + B)e^{4x} + e^{4x}(4B) = (16A + 16Bx + 8B)e^{4x}$. Thus, equation (1) becomes: $[(16A + 16Bx + 8B) - 5(4A + 4Bx + B) + 6(A + Bx)]e^{4x} = xe^{4x}$
Or $[(2A + 3B)e^{4x} + 2B\,xe^{4x} = xe^{4x}$. Comparing the two sides, we get
2A + 3B = 0 and 2B = 1 or B = 1/2 and A = -3/4. Thus,

$$y_p = \left(-\frac{3}{4} + \frac{1}{2}x\right)e^{4x} = \left(\frac{2x-3}{4}\right)e^{4x}$$

Now, $y = y_c + y_p = c_1\ e^{2x}\ +\ c_2 e^{3x} + \left(\frac{2x-3}{4}\right)e^{4x}$

(p) $(D^2 + 1)\ y = x\ e^{2x}$ (1)

Solution: The auxiliary equation is $m^2 + 1 = 0$ or $m = \pm\, i$. Thus, $y_c = c_1 \cos x + c_2 \sin x$. To find y_p we proceed as under: Since right side contains 'xe^{2x}' and this term is not in y_c hence, we let $y = (A + Bx)\ e^{2x}$ or $y` = 2(A + Bx)\ e^{2x} + Be^{2x} = (2A + 2Bx + B)\ e^{2x}$, $y`` = 2(2A + 2Bx + B)\ e^{2x} + e^{2x}(2B) = (4A + 4Bx + 4B)e^{4x}$. Thus, equation (1) becomes: $[(4A + 4Bx + 4B) + (A + Bx)]e^{4x} = xe^{4x}$ or $[(5A + 4B)e^{4x} + 5B\ xe^{4x} = xe^{4x}$.
Comparing the two sides, we get: $5A + 4B = 0$ and $5B = 1$ or $B = 1/5$ and $A = -4/25$.
Thus,

$$y_p = \left(-\frac{4}{25} + \frac{1}{5}x\right)e^{2x} = \left(\frac{5x-4}{25}\right)e^{2x}$$

Now, $y = y_c + y_p = c_1\ \cos x +\ c_2 \sin x + \left(\frac{5x-4}{25}\right)e^{2x}$

(q) $(D^2 - 2D + 4)\ y = e^x \cos x$ (1)

Solution: The auxiliary equation is $m^2 - 2m + 4 = 0$ giving $m = 1 \pm i\sqrt{3}$. Thus,

$$y_c = e^x\left(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x\right)$$

To find y_p we proceed as under: Since right side contains '$e^x\cos x$' and this term is not in y_c hence, we let $y = (A\cos x + B \sin x)e^x$
or $y` = (A\cos x + B \sin x)e^x + (-A\sin x + B \cos x)e^x = [(A + B)\cos x + ((B - A)\sin x]e^x$.
$y`` = [(A + B)\cos x + ((B - A)\sin x]e^x + [-(A + B)\sin x + ((B - A)\cos x]e^x$
or $y`` = [(A + B + B - A)\cos x + (B - A - A - B)\sin x]e^x = (2B \cos x - 2A \sin x)\ e^x$ Thus, equation (1) becomes:
$[(2B \cos x - 2A \sin x) - 2(A + B)\cos x + 2(B - A)\sin x) + 4(A\cos x+B\sin x)]e^x = e^x \cos x$
Or $[2B - 2A - 2B + 4A]e^x\cos x + [-2A + 2B - 2A + B]e^x \sin x = e^x\cos x$. Comparing the two sides, we get
$2A = 1$ and $-4A + 3B = 0$ giving $A = 1/2$ and $B = 2/3$. Thus,

$$y_p = \left(\frac{1}{2}\cos x + \frac{2}{3}\sin x\right)e^x = \left(\frac{3\cos x + 4\cos x}{6}\right)e^x$$

Now, $y = y_c + y_p = y_c = e^x\left(c_1 \cos\sqrt{3}\,x + c_2 \sin\sqrt{3}\,x\right) + \left(\frac{3\cos x + 4\cos x}{6}\right)e^x$

NOTE: In the following Cauchy-Euler differential equations we shall be substituting

$$x = e^t \quad\text{or}\quad t = \ln x$$

Then by Chain Rule, we have

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{d}{dx}(\ln x) = \frac{dy}{dt}\cdot\left(\frac{1}{x}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt} \qquad (2)$$

Differentiating (2), we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)\frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right) - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt} \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right) \qquad (3)$$

Similarly, $$\frac{d^3y}{dx^3} = \frac{1}{x^3}\left(\frac{d^3y}{dt^3} - 3\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right) \qquad (4)$$

Now if we write $D = d/dx$ and $\Delta = d/dt$, then

from (2), $xDy = \Delta y$,

from (3), $x^2D^2y = (\Delta^2 - \Delta)y = \Delta(\Delta - 1)y$

from (4), $x^3D^3y = (\Delta^3 - 3\Delta^2 + 2\Delta)y = \Delta(\Delta-1)(\Delta-2)y$

In general, $x^nD^ny = \Delta(\Delta-1)(\Delta-2)\{\Delta-(n-1)\}y = \Delta(\Delta-1)(\Delta-2)(\Delta-n+1)y\cdot$

4. Solve the following Cauchy-Euler differential equations

(a) $x^2 y`` - x y` + y = \ln x$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$ defined as above, the given differential equation becomes: $[\Delta(\Delta - 1) - \Delta + 1]y = t$ or $(\Delta^2 - 2\Delta + 1)y = t$.

The auxiliary equation is $m^2 - 2m + 1 = 0$ giving $m = 1, 1$.

Thus, $y_c = (c_1 + c_2t)e^t = [(c_1 + c_2\ln x)]x$.

$$y_p = \frac{1}{\Delta^2 - 2\Delta + 1}(t) = \frac{1}{1-2\Delta}(t) = (1-2\Delta)^{-1}t = (1+2\Delta)t = t + 2\Delta t = t + 2 = (\ln x + 2)$$

NOTE: Here we have applied binomial expansion and neglected Δ^2 and higher powers as t appears with exponent 1. Now, $y = y_c + y_p = [(c_1 + c_2\ln x)]x + (\ln x + 2)$

(b) $x^2 y`` - 4 x y` + 6 y = x^2$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$ defined as above, the given differential equation becomes: $[\Delta(\Delta - 1) - 4\Delta + 6]y = e^{2t}$ or $(\Delta^2 - 5\Delta + 6)y = e^{2t}$.

The auxiliary equation is $m^2 - 5m + 6 = 0$ or $m = 2, 3$.

Thus, $y_c = (c_1e^{2t} + c_2e^{3t}) = [(c_1x^2 + c_2x^3]$.

$$y_p = \frac{1}{\Delta^2 - 5\Delta + 6}e^{2t} = \frac{t}{2\Delta - 5}e^{2t} = \frac{te^{2t}}{4-5} = -te^{2t} = -x^2\ln x. \qquad \text{NOTE}: e^{2t} \text{ is in } y_c.$$

Thus, $y = y_c + y_p = (c_1 + c_2\ln x)x - x^2\ln x$

(c) $x^2 y`` - 2 x y` - 4 y = x^4$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$ defined as above, the given differential equation becomes: $[\Delta(\Delta - 1) - 2\Delta - 4]y = e^{4t}$ or $(\Delta^2 - 3\Delta - 4)y = e^{4t}$.

The auxiliary equation is $m^2 - 3m - 4 = 0$ giving $m = -1, 4$.

Thus, $y_c = (c_1e^{-t} + c_2e^{4t}) = [(c_1x^{-1} + c_2x^4]$.

$$y_p = \frac{1}{\Delta^2 - 3\Delta - 4}e^{4t} = \frac{t}{2\Delta - 3}e^{4t} = \frac{te^{4t}}{8-3} = \frac{te^{4t}}{5} = \frac{x^4\ln x}{5}. \qquad \text{NOTE}: e^{2t} \text{ is in } y_c.$$

Thus, $y = y_c + y_p = (c_1x^{-1} + c_2x^4) + x^4\ln x/5$

(d) $x^2 y`` - 3 x y` + 4 y = (x + 1)^2 = x^2 + 2x + 1$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$ defined as above, the given differential equation becomes: $[\Delta(\Delta - 1) - 3\Delta + 4]y = e^{2t} + 2e^t + 1$

or $(\Delta^2 - 4\Delta + 4)y = e^{2t} + 2e^t + e^{0t}$.

The auxiliary equation is $m^2 - 4m + 4 = 0$ or $m = 2, 2$.

Thus, $y_c = (c_1 + c_2t)e^{2t} = [(c_1 + c_2\ln x)]x^2$.

$$y_p = \frac{1}{\Delta^2 - 4\Delta + 4}\left(e^{2t} + 2e^t + e^{0t}\right) = \frac{1}{\Delta^2 - 4\Delta + 4}e^{2t} + 2\frac{1}{\Delta^2 - 4\Delta + 4}e^t + \frac{1}{\Delta^2 - 4\Delta + 4}e^{0t}$$

$$= \frac{t}{2\Delta - 4}e^{2t} + 2\frac{1}{1-4+4}e^t + \frac{1}{0-0+4} = \frac{t^2}{2}e^{2t} + 2e^t + \frac{1}{4} = \frac{x^2(\ln x)^2}{2} + 2x + \frac{1}{4}$$

Thus, $y = y_c + y_p = \left(c_1 + c_2 \ln x\right)x^2 + \dfrac{x^2(\ln x)^2}{2} + 2x + \dfrac{1}{4}$

NOTE: e^{2t} and te^{2t} both terms are in y_c.

(e) $x^2 y'' - 2y = x^2 + 1/x$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$ defined as above, the given differential equation becomes: $[\Delta(\Delta - 1) - 4]y = e^{2t} + e^{-t}$ or $(\Delta^2 - \Delta - 4)y = e^{2t} + e^{-t}$.

The auxiliary equation is $m^2 - m - 4 = 0 \quad \Rightarrow m = \dfrac{1 \pm \sqrt{1+16}}{2} = \dfrac{1 \pm \sqrt{17}}{2}$.

$$\text{Thus, } y_c = c_1 e^{\left(\frac{1+\sqrt{17}}{2}\right)t} + c_2 e^{\left(\frac{1-\sqrt{17}}{2}\right)t} = c_1 x^{\left(\frac{1+\sqrt{17}}{2}\right)} + c_2 x^{\left(\frac{1-\sqrt{17}}{2}\right)}$$

$$y_p = \frac{1}{\Delta^2 - \Delta - 4}\left(e^{2t} + e^{-t}\right) = \frac{1}{\Delta^2 - \Delta - 4}e^{2t} + \frac{1}{\Delta^2 - \Delta - 4}e^{-t} = \frac{e^{2t}}{4 - 2 - 4} + \frac{e^{-t}}{1 + 1 - 4}$$

$$= -\frac{e^{2t}}{2} - \frac{e^{-t}}{2} = -\frac{1}{2}\left(e^{2t} + e^{-t}\right) = -\frac{1}{2}\left(x^2 + x^{-1}\right)$$

$$\text{Now, } y = y_c + y_p = c_1 x^{\left(\frac{1+\sqrt{17}}{2}\right)} + c_2 x^{\left(\frac{1-\sqrt{17}}{2}\right)} - \frac{1}{2}\left(x^2 + x^{-1}\right)$$

(f) $x^3 y''' + 2x^2 y'' + 2y = 10(x + 1/x)$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$ defined as above, the given differential equation becomes: $[\Delta(\Delta - 1)(\Delta - 2) + 2\Delta(\Delta - 1) + 2]y = 10(e^t + e^{-t})$
or $(\Delta^3 - \Delta^2 + 2)y = 10(e^t + e^{-t})$.

The auxiliary equation is $m^3 - m^2 + 2 = 0 \ \Rightarrow m = -1, 1 \pm i$.

Thus, $y_c = c_1 e^{-t} + e^t(\cos t + c_3 \sin t) = c_1 x^{-1} + x[c_1 \cos(\ln x) + c_2 \sin(\ln x)]$.

$$y_p = 10\frac{1}{\Delta^3 - \Delta^2 + 2}\left(e^t + e^{-t}\right) = 10\left[\frac{1}{\Delta^3 - \Delta^2 + 2}e^t + \frac{1}{\Delta^3 - \Delta^2 + 2}e^{-t}\right]$$

$$= 10\left[\frac{e^t}{1 - 1 + 2} + \frac{t}{3\Delta^2 - 2\Delta}e^{-t}\right] = 10\left[\frac{e^t}{2} + \frac{te^{-t}}{3 + 2}\right] = 10\frac{5e^t + 2te^{-t}}{10} = \left(5e^t + 2te^{-t}\right)$$

$= \left(5x + 2x^{-1}\ln x\right)$ NOTE: e^{-t} is in y_c. Now,

$$y = y_c + y_p = c_1 x^{-1} + x\left[c_2 \cos(\ln x) + c_2 \sin(\ln x)\right] + \left(5x + 2x^{-1}\ln x\right).$$

(g) $x^2 y'' + x y' + y = \ln x \sin(\ln x)$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$ defined as above, the given differential equation becomes: $[\Delta(\Delta - 1) + \Delta + 1]y = t \sin t$ or $(\Delta^2 + 1)y = t \sin t$.

The auxiliary equation is $m^2 + 1 = 0 \ \Rightarrow m = \pm i$.

Thus, $y_c = (c_1 \cos t + c_2 \sin t) = c_1 \cos(\ln x) + c_2 \sin(\ln x)$.

$$y_p = \text{Im}\left[\frac{1}{\Delta^2 + 1}te^{it}\right] = \text{Im}\left[e^{it}\frac{1}{(\Delta + i)^2 + 1}t\right] = \text{Im}\left[e^{it}\frac{1}{\Delta^2 + 2i\Delta - 1 + 1}t\right] = \text{Im}\left[e^{it}\frac{1}{2i\Delta}t\right]$$

$\left[\text{Neglecting } \Delta^2 \text{ and higher powers because power of } t \text{ is } 1\right]$

$$y_p = \text{Im}\left[\frac{1}{2i}e^{it}\int t\,dt\right] = \text{Im}\left[\frac{1}{2i}e^{it}\frac{t^2}{2}\right] = \frac{t^2}{4}\text{Im}\left[\frac{\cos t + i\sin t}{i}\right] = \frac{t^2}{4}\text{Im}\left[-i\cos t + \sin t\right]$$

$$= -\frac{t^2 \cos t}{4} = -\frac{\left[(\ln x)^2 \cos(\ln x)\right]}{4}$$

Now, $y = y_c + y_p = c_1 \cos(\ln x) + c_2 \sin(\ln x) - \frac{\left[(\ln x)^2 \cos(\ln x)\right]}{4}$

(h) $(2x + 3)^2$ y`` - (2x + 3) y` -12 y = 6x

Solution: Using the substitutions $2x + 3 = e^t$ and t = ln (2x + 3), the given differential equation becomes: $[2\Delta.2(\Delta - 1) -2\Delta + 3]y = 6(e^t - 3)/2$ or $(4\Delta^2 - 6\Delta + 3)y = 3(e^t - 3)$.
This change is explained in the solved example of the textbook.
The auxiliary equation is $4m^2 - 6m + 3 = 0$

$$\Rightarrow m = \frac{6 \pm \sqrt{36-48}}{8} = \frac{6 \pm \sqrt{-12}}{8} = \frac{6 \pm i2\sqrt{3}}{8} = \frac{3}{4} \pm i\frac{\sqrt{3}}{4}. \text{ Thus,}$$

$$y_c = e^{3t/4}\left[a\cos\frac{\sqrt{3}t}{4} + b\sin\frac{\sqrt{3}}{4}\right] = (2x+3)^{3/4}\left[a\cos\left(\frac{\sqrt{3}\ln(2x+3)}{4}\right) + b\sin\left(\frac{\sqrt{3}\ln(2x+3)}{4}\right)\right]$$

$$y_p = 3\frac{1}{\left(4\Delta^2 - 6\Delta + 3\right)}\left(e^t - 3\right) = 3\left[\frac{1}{\left(4\Delta^2 - 6\Delta + 3\right)}e^t - 3\frac{1}{\left(4\Delta^2 - 6\Delta + 3\right)}e^{0t}\right]$$

$$= 3\left[\frac{e^t}{4-6+3} - 3\frac{e^{0t}}{0-0+3}\right] = 3\left[e^t - 1\right] = 3\left[\ln(2x+3) - 1\right]. \text{ Now,}$$

$$y = y_c + y_p = (2x+3)^{3/4}\left[a\cos\left(\frac{\sqrt{3}\ln(2x+3)}{4}\right) + b\sin\left(\frac{\sqrt{3}\ln(2x+3)}{4}\right)\right] + 3\left[\ln(2x+3) - 1\right]$$

(i) $(x + 1)^2$ y`` + (x + 1) y` + y = 4 cos [ln (x + 1)]

Solution: Using the substitutions $x + 1 = e^t$ and t = ln(x + 1), the given differential equation becomes: $[\Delta(\Delta - 1) + \Delta + 1]y = 4\cos t$ ➔ $(\Delta^2 + 1)y = 4 \cos t$.
The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$.
Thus, $y_c = (c_1 \cos t + c_2 \sin t) = c_1 \cos[\ln(x+1) + c_2 \sin[(\ln(x+1)$.

$$y_p = 4\frac{1}{\Delta^2+1}\cos t = 4\frac{t}{2\Delta}\cos t = 2t\int \cos t\, dt = 2t\sin t = 2\ln(x+1)[\sin(\ln(x+1))].$$

$[\text{NOTE}: \cos t \text{ is in } y_c]$.

Now, $y = y_c + y_p = c_1 \cos[\ln(x+1)] + c_2 \sin[\ln(x+1)] + 2\ln(x+1)[\sin(\ln(x+1))]$

(j) $(x + 1)^2$ y`` + (x + 1) y` + y = sin 2[ln (x + 1)]

Solution: Using the substitutions $x + 1 = e^t$ and t = ln(x + 1), the given differential equation becomes: $[\Delta(\Delta - 1) + \Delta + 1]y = \sin 2t$ or $(\Delta^2 + 1)y = \sin 2t$.
The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$.
Thus, $y_c = (c_1 \cos t + c_2 \sin t) = c_1 \cos[\ln(x+1) + c_2 \sin[(\ln(x+1)$.

$$y_p = \frac{1}{\Delta^2+1}\sin 2t = \frac{1}{(-2^2)+1}\sin 2t = -\frac{\sin 2t}{3} = -\frac{\sin 2\ln(x+1)}{3}.$$

Now, $y = y_c + y_p = c_1 \cos[\ln(x+1)] + c_2 \sin[\ln(x+1)] - \frac{\sin 2\ln(x+1)}{3}$

(k) x^2 y``` + 3xy`` + y` = $x^2 \ln x$ [Hint: Multiply both sides by x]

Solution: Multiplying by x, we get: x^3 y``` + $3x^2$ y`` + xy` = $x^3 \ln x$. Using the substitutions $x = e^t$ and t = ln x, the given differential equation becomes:

$[\Delta(\Delta - 1)(\Delta - 2) + 3\Delta(\Delta - 1) + \Delta]y = te^{3t}$ or $\Delta^3 y = t e^{3t}$.

The auxiliary equation is $m^3 = 0$ giving $m = 0, 0, 0$.

Thus, $y_c = (c_1 + c_2 t + c_3 t^2) = c_1 + c_2 \ln(x + 1) + c_3 \ln(x + 1)^2$.

$$y_p = \frac{1}{\Delta^3} te^{3t} = e^{3t} \frac{1}{(\Delta+3)^3} t = e^{3t} \frac{1}{\Delta^3 + 9\Delta^2 + 27\Delta + 27} t = e^{3t} \frac{1}{27(1+\Delta)} t$$

$$\left[\text{Neglecting } \Delta^2 \text{and higher terms since t appears with power 1}\right]$$

$$y_p = e^{3t}(1+\Delta)^{-1} t = e^{3t}\left(1-\Delta+\Delta^2 ...\right)t = e^{3t}\left(t - \Delta t + \Delta^2 t...\right) = e^{3t}(t-1) = x^3(\ln x - 1)$$

$$\text{Now, } y = y_c + y_p = c_1 + c_2 \ln(x+1) + c_3 \ln(x+1)^2 + x^3(\ln x - 1)$$

(*l*) $x^3 y``` + 3x^2 y`` + xy` + y = x + \ln x$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$, the given differential equation becomes: $[\Delta(\Delta - 1)(\Delta - 2) + 3\Delta(\Delta - 1) + \Delta + 1]y = e^t + t \Rightarrow (\Delta^3 + 1)y = e^t + t$.

The auxiliary equation is $m^3 + 1 = 0 \Rightarrow (m + 1)(m^2 - m + 1) = 0 \Rightarrow m = -1$ or

$$m = \frac{1 \pm i\sqrt{3}}{2}. \text{ Thus, } y_c = c_1 e^{-t} + e^{t/2}\left(c_2 \cos\frac{\sqrt{3}\,t}{2} + c_3 \sin\frac{\sqrt{3}\,t}{2}\right)$$

$$y_c = c_1 x^{-1} + x^{1/2}\left(c_2 \cos\frac{\sqrt{3}\ln x}{2} + c_3 \sin\frac{\sqrt{3}\ln x}{2}\right)$$

$$y_p = \frac{1}{\Delta^3+1} e^t + \frac{1}{\Delta^3+1} t = \frac{e^t}{1^3+1} + t = \frac{e^t}{2} + t = \frac{x}{2} + \ln x$$

$$\left[\text{Neglecting } \Delta^3 \text{and higher terms since t appears with power 1}\right]$$

$$\text{Now, } y = y_c + y_p = c_1 x^{-1} + x^{1/2}\left(c_2 \cos\frac{\sqrt{3}\ln x}{2} + c_3 \sin\frac{\sqrt{3}\ln x}{2}\right) + \frac{x}{2} + \ln x$$

(m) $y`` + y`/x = 12 \ln x/x^2$ [Hint: Multiply by x^2]

Solution: Multiplying by x^2, we get: $x^2 y`` + xy` = 12\ln x$. Using the substitutions $x = e^t$ and $t = \ln x$, the given differential equation becomes:

$[\Delta(\Delta - 1) + \Delta]y = t \Rightarrow \Delta^2 y = t$. The auxiliary equation is $m^2 = 0 \Rightarrow m = 0, 0$.

Thus, $y_c = (c_1 + c_2 t) = c_1 + c_2 \ln x$.

$$y_p = 12\frac{1}{\Delta^2} t = 12\int\int t\ dt.\ dt = 12\int \frac{t^2}{2}\ dt = 12\frac{t^3}{6} = 2(\ln x)^3.$$

$$\text{Now, } y = y_c + y_p = c_1 + c_2 \ln x + 2(\ln x)^2$$

(n) $x^2 y`` - 2x y` + 2y = x^2 + \sin(5\ln x)$

Solution: Using the substitutions $x = e^t$ and $t = \ln x$, the given differential equation becomes: $[\Delta(\Delta - 1) - 2\Delta + 2]y = e^{2t} + \sin 5t \Rightarrow (\Delta^2 - 3\Delta + 2)y = e^{2t} + \sin 5t$.

The auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$. Thus,

$y_c = c_1 e^{2t} + c_2 e^{3t} = c_1 x^2 + c_2 x^3$.

$$y_p = \frac{1}{\Delta^2 - 3\Delta + 2}\left(e^{2t} + \sin 5t\right) = \frac{1}{\Delta^2 - 3\Delta + 2} e^{2t} + \frac{1}{\Delta^2 - 3\Delta + 2} \sin 5t$$

$$= \frac{t}{2\Delta - 3} e^{2t} + \frac{1}{-5^2 - 3\Delta + 2} \sin 5t = \frac{te^{2t}}{2.2 - 3} + \frac{1}{-(3\Delta + 23)} \times \frac{(3\Delta - 23)}{(3\Delta - 23)} \sin 5t$$

$$= te^{2t} - \frac{(3\Delta - 23)}{9\Delta^2 - 529}\sin 5t = te^{2t} - \frac{(3\Delta - 23)}{9(-5^2) - 529}\sin 5t = te^{2t} + \frac{(3\Delta \sin 5t - 23\sin 5t)}{754}$$

$$= te^{2t} + \frac{(15\cos 5t - 23\sin 5t)}{754} = x^2 \ln x + \frac{15\cos 5(\ln x) - 23\sin 5(\ln x)}{754}$$

$$\text{Now, } y = y_c + y_p = c_1x^2 + c_2 x^3 + \frac{15\cos 5(\ln x) - 23\sin 5(\ln x)}{754}$$

(o) $x^3 y`` + 3x^2 y` + xy = \sin(\ln x)$ [Hint: Divide by x]

Solution: Dividing by x , we get: $x^2 y`` + 3xy` + y = x^{-1} \sin(\ln x)$. Using the substitutions $x = e^t$ and $t = \ln x$, the given differential equation becomes:

$[\Delta(\Delta - 1) + 3\Delta + 1]y = e^{-t} \sin t \Rightarrow (\Delta + 1)^2 y = e^{-t} \sin t$. The auxiliary equation is $(m + 1)^2 = 0 \Rightarrow m = -1, -1$. Thus, $y_c = (c_1 + c_2 t)e^{-t} = (c_1 + c_2 \ln x)x^{-1}$.

$$y_p = \frac{1}{(\Delta+1)^2} e^{-t} \sin t = e^{-t} \frac{1}{(\Delta + 1 - 1)^2} \sin t = e^{-t} \frac{1}{\Delta^2} \sin t = e^{-t} \int\int \sin t \, dt. \, dt$$

$$= e^{-t} \int -\cos t \, dt = -e^{-t} \sin t = -x^{-1} \sin(\ln x).$$

$$\text{Now, } y = y_c + y_p = x^{-1}(c_1 + c_2 \ln x) - x^{-1} \sin(\ln x) = \frac{(c_1 + c_2 \ln x) - \sin(\ln x)}{x}$$

5. Solve the following differential equations by the method of variation of parameters:

(a) $y`` + y = \operatorname{cosec} x$

Solution: The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$. Thus, $y_c = c_1 \cos x + c_2 \sin x$. Let, $y_1 = \cos x$ and $y_2 = \sin x \Rightarrow y`_1 = -\sin x$ and $y`_2 = \cos x$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We know that $y_p = u_1y_1 + u_2y_2$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{\sin x \csc x}{1} dx = -\int 1 dx = -x$$

$$\text{and } u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos x \ \csc x}{1} dx = \int \frac{\cos x}{\sin x} dx = \ln(\sin x)$$

$$\therefore y_p = u_1y_2 + u_2y_1 = -x \sin x + \cos x \ln(\sin x)$$

$$\text{Now, } y = y_c + y_p = c_1 \cos x + c_2 \sin x - x \sin x + \cos x \ln(\sin x)$$

(b) $y`` + a^2 = \sec ax$

Solution: The auxiliary equation is $m^2 + a^2 = 0 \Rightarrow m = \pm ia$. Thus, $y_c = c_1 \cos ax + c_2 \sin ax$.

Let, $y_1 = \cos ax$ and $y_2 = \sin ax \Rightarrow y`_1 = -a \sin ax$ and $y`_2 = a \cos ax$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a\left(\cos^2 ax + \sin^2 ax\right) = a.1 = a$$

We know that $y_p = u_1y_1 + u_2y_2$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{\sin ax \sec ax}{a} dx = -\frac{1}{a}\int \tan ax \, dx = -\frac{\ln(\sec ax)}{a} = \frac{\ln(\cos ax)}{a}$$

$$\text{and } u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos ax \ \sec ax}{a} dx = \frac{1}{a}\int 1.dx = \frac{x}{a}$$

$$\therefore y_p = u_1y_2 + u_2y_1 = \left[-\ln(\sec ax) + x\right]/a$$

Now, $y = y_c + y_p = c_1 \cos ax + c_2 \sin ax + \left[\ln(\cos ax) + x\right]/a$

(c) y`` + y = tan x

Solution: The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$. Thus, $y_c = c_1 \cos x + c_2 \sin x$.
Let, $y_1 = \cos x$ and $y_2 = \sin x \Rightarrow y`_1 = -\sin x$ and $y`_2 = \cos x$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We know that $y_p = u_1 y_1 + u_2 y_2$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{\sin x \tan x}{1} dx = -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1-\cos^2 x}{\cos x} dx$$

$$= \left[-\int \frac{1}{\cos x} dx + \int \cos x \; dx\right] = -\int \sec x \, dx + \sin x = -\ln(\sec x + \tan x) + \sin x$$

and $u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos x \;\; \tan x}{1} dx = \int \sin x \, dx = -\cos x$

$$\therefore y_p = u_1 y_2 + u_2 y_1 = \left[-\ln(\sec x + \tan x) + \sin x\right] \sin x - \cos x(\sin x)$$

Now, $y = y_c + y_p = c_1 \cos x + c_2 \sin x + \left[-\ln(\sec x + \tan x) + \sin x\right] \sin x - \cos x \;\sin x$

(d) y`` + y = x sin x

Solution: The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$. Thus, $y_c = c_1 \cos x + c_2 \sin x$.
Let, $y_1 = \cos x$ and $y_2 = \sin x \Rightarrow y`_1 = -\sin x$ and $y`_2 = \cos x$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We know that $y_p = u_1 y_1 + u_2 y_2$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{x \sin^2}{1} dx = -\int x \frac{(1-\cos 2x)}{2} dx = \frac{-1}{2}\left[\int x\, dx - \int x \cos 2x \, dx\right]$$

$$= \frac{-1}{2}\left[\frac{x^2}{2} - \left(x \frac{\sin 2x}{2}\right) + \int 1 . \frac{\sin 2x}{2} \; dx\right] = \frac{-1}{4}\left[x^2 - x \sin 2x - \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{8}\left[2x \sin 2x + \cos 2x - 2x^2\right]$$

and $u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos x \;\; x \sin x}{1} dx = \frac{1}{2}\int x(2 \sin x \cos x) dx = \frac{1}{2}\int x \sin 2x \, dx$

$$= \frac{1}{4}\left[-x \cos 2x + \int \cos 2x \; dx\right] = \frac{1}{2}\left[-x \cos 2x + \frac{\sin 2x}{2}\right] = \frac{1}{4}\left[\sin 2x - 2x \cos 2x\right]$$

$$\therefore y_p = u_1 y_2 + u_2 y_1 = \frac{1}{8} \sin x\left[2x \sin 2x + \cos 2x - 2x^2\right] + \frac{1}{4} \cos x\left[\sin 2x - 2x \cos 2x\right]$$

Now, $y = y_c + y_p = c_1 \cos x + c_2 \sin x - x \sin x$

$$+\frac{1}{8} \sin x\left[2x \sin 2x + \cos 2x - 2x^2\right] + \frac{1}{4} \cos x\left[\sin 2x - 2x \cos 2x\right]$$

(e) y`` - 6 y` + 9 y = e^{3x}/x^2

Solution: The auxiliary equation is $m^2 - 6m + 9 = 0 \Rightarrow m = 3, 3$. Thus,
$y_c = (c_1 + c_2 x)\, e^{3x}$.
Let, $y_1 = e^{3x}$ and $y_2 = xe^{3x} \Rightarrow y`_1 = 3e^{3x}$ and $y`_2 = e^{3x} + 3xe^{3x} = (1 + 3x)e^{3x}$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & (1+3x)e^{3x} \end{vmatrix} = e^{6x}\left(1+3x-3x\right) = e^{6x}$$

We know that $y_p = u_1y_1 + u_2y_2$. $u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{xe^{3x}e^{3x}}{x^2.e^{6x}} dx = -\int \frac{1}{x} dx = -\ln x$

and $u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{e^{3x}.e^{3x}}{x^2.e^{6x}} dx = \int x^{-2}\,dx = -\frac{1}{x}$

$\therefore y_p = u_1y_2 + u_2y_1 = -e^{3x}(1+3x)\ln x - e^{3x}\frac{1}{x} = -e^{3x}\left((1+3x)\ln x + \frac{1}{x}\right)$

Now, $y = y_c + y_p = (c_1 + c_2x)e^{3x} - e^{3x}\left((1+3x)\ln x + \frac{1}{x}\right)$

(f) $y`` - 2y` + 2y = e^x \tan x$

Solution: The auxiliary equation is $m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Thus,

$y_c = e^x(c_1\cos x + c_2 \sin x)$.

Let, $y_1 = e^x \cos x$ and $y_2 = e^x \sin x \Rightarrow y`_1 = e^x(-\sin x + \cos x)$, $y`_2 = e^x(\cos x + \sin x)$

Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(-\sin x + \cos x) & e^x(\cos x + \sin x) \end{vmatrix}$$

$= e^{2x}\left(\cos^2 x + \cos x \sin x + \sin^2 x - \sin x \cos x\right) = e^{2x}$

We know that $y_p = u_1y_1 + u_2y_2$ where,

$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{e^x \sin x.e^x \tan x}{e^{2x}} dx = -\int \frac{\sin x.\sin x}{\cos x} dx = -\int \frac{(1-\cos^2 x)}{\cos x} dx$

$= -\int \frac{1}{\cos x} dx + \int \frac{\cos^2 x}{\cos x} dx = -\int \sec x\,dx + \int \cos x\,dx = -\ln(\sec x + \tan x) + \sin x$

and $u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{e^x \cos x.e^x \tan x}{.e^{2x}} dx = \int \frac{\cos x.\sin x}{\cos x} dx = \int \sin x\,dx = -\cos x$

$\therefore y_p = u_1y_2 + u_2y_1 = e^x \sin x\left(-\ln(\sec x + \tan x) + \sin x\right) + e^x \cos x(-\cos x)$

Now, $y = y_c + y_p$

$= e^x(c_1\cos x + c_2 \sin x) + e^x \sin x\left(-\ln(\sec x + \tan x) + \sin x\right) + e^x \cos x(-\cos x)$

$= e^x\left[(c_1\cos x + c_2 \sin x) - \sin x\ \ln(\sec x + \tan x) - \left(\cos^2 x - \sin^2 x\right)\right]$

$\therefore y = e^x\left[(c_1\cos x + c_2\sin x) - \sin x\ \ln(\sec x + \tan x) - \cos 2x\right]$

(g) $y`` - y = e^{-2x}\sin(e^{-x})$

Solution: The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = -1, 1$. Thus, $y_c = c_1 e^x + c_2 e^{-x}$.

Let, $y_1 = e^x$ and $y_2 = e^{-x} \Rightarrow y`_1 = e^x$ , $y`_2 = -e^{-x}$. Now,

$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 = -2.$ We know that $y_p = u_1y_1 + u_2y_2$ where,

$$=\frac{1}{2}\left[\frac{z^3\cos z}{3}-\left(z\sin z-\int 1.\sin z\,dz\right)\right]=\frac{1}{2}\left[\frac{z^3\cos z}{3}-z\sin z-\cos z\right]$$

$$=\frac{1}{6}\left[z^3\cos z-3z\sin z-3\cos z\right]=\frac{e^{-3x}\cos(e^{-x})-3e^{-x}\sin(e^{-x})-3\cos(e^{-x})}{6}$$

$$u_2=\int\frac{y_1 f(x)}{W}dx=\int\frac{e^x.e^{-2x}\sin\left(e^{-x}\right)}{-2}dx=-\frac{1}{2}\int e^{-x}\sin(e^{-x})\,dx$$

Putting $z=e^{-x}\ \Rightarrow -dz=e^{-x}\,dx$. Thus,

$$u_2=-\frac{1}{2}\int\sin z(-dz)=\frac{1}{2}(-\cos z)=-\frac{\cos(e^{-x})}{2}$$

$$\therefore\ y_p=u_1y_2+u_2y_1=e^{-x}\ \frac{e^{-3x}\cos(e^{-x})-3e^{-x}\sin(e^{-x})-3\cos(e^{-x})}{6}+e^x\left(-\frac{\cos(e^{-x})}{2}\right)$$

$$\therefore y=y_c+y_p=(c_1e^x+c_2e^{-x})+\frac{e^{-3x}\cos(e^{-x})-3e^{-x}\sin(e^{-x})-3\cos(e^{-x})}{6}-\frac{e^x\cos(e^{-x})}{2}$$

(h) y`` + y = sin x

Solution: The auxiliary equation is $m^2 + 1 = 0\ \Rightarrow\ m = \pm i$. Thus, $y_c = c_1\cos x + c_2\sin x$. Let, $y_1 = \cos x$ and $y_2 = \sin x\ \Rightarrow\ y`_1 = -\sin x$ and $y`_2 = \cos x$. Now,

Now, $y=y_c+y_p$

$$=e^x(c_1\cos x+c_2\sin x)+e^x\sin x\left(-\ln(\sec x+\tan x)+\sin x\right)+e^x\cos x(-\cos x)$$

$$=e^x\left[(c_1\cos x+c_2\sin x)-\sin x\ \ln(\sec x+\tan x)-\left(\cos^2 x-\sin^2 x\right)\right]$$

$$\therefore\ y=e^x\left[(c_1\cos x+c_2\sin x)-\sin x\ \ln(\sec x+\tan x)-\cos 2x\right]$$

$$W=\begin{vmatrix}y_1 & y_2\\ y`_1 & y`_2\end{vmatrix}=\begin{vmatrix}\cos x & \sin x\\ -\sin x & \cos x\end{vmatrix}=\cos^2 x+\sin^2 x=1$$

We know that $y_p=u_1y_1+u_2y_2$ where,

$$u_1=-\int\frac{y_2 f(x)}{W}dx=-\int\frac{\sin^2 x}{1}dx=-\int\frac{(1-\cos 2x)}{2}dx=\frac{-1}{2}\left[\int 1dx-\int\cos 2x\,dx\right]$$

$$=\frac{-1}{2}\left[x-\frac{\sin 2x}{2}\right]=\frac{\sin 2x-2x}{4}$$

and $u_2=\int\frac{y_1 f(x)}{W}dx=\int\frac{\cos x.\sin x}{1}dx=\frac{1}{2}\int(2\sin x\cos x)\,dx=\frac{1}{2}\int\sin 2x\,dx$

$$=\frac{1}{2}\left[-\frac{\cos 2x}{2}\right]=-\frac{\cos 2x}{4}$$

$$\therefore\ y_p=u_1y_2+u_2y_1=\frac{\sin 2x-2x}{4}.\sin x-\frac{\cos x.\cos 2x}{4}$$

Now, $y=y_c+y_p=c_1\cos x+c_2\sin x+\frac{\sin 2x-2x}{4}.\sin x-\frac{\cos x.\cos 2x}{4}$

$$=\frac{4(c_1\cos x+c_2\sin x)-2x\sin 2x+\sin 2x.\sin x-\cos 2x.\cos x}{4}$$

$$= \frac{4(c_1 \cos x + c_2 \sin x) - 2x \sin 2x + \sin(2x - x)}{4} \left[\text{NOTE}: \sin(a+b) = \sin a \cos b - \cos a \sin b\right]$$

$$\therefore\ y = \frac{1}{4}\left[4(c_1 \cos x + c_2 \sin x) - 2x \sin 2x + \sin x\right]$$

(i) y`` - 3y` + 2y = sin x

Solution: The auxiliary equation is $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2.$

Thus, $y_c = c_1e^x + c_2\, e^{2x}$

Let, $y_1 = e^x$ and $y_2 = e^{2x} \Rightarrow y`_1 = e^x$ and $y`_2 = 2e^{2x}$

$$\text{Now, } W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

We know that $y_p = u_1y_1 + u_2y_2$, where

$$u_1 = -\int \frac{y_2\, f(x)}{W} dx = -\int \frac{e^{2x} \sin x}{e^{3x}} dx = -\int e^{-x} \sin x\, dx$$

$$= -\frac{e^{-x}}{2}\left[-\sin x - \cos x\right] = \frac{e^{-x}}{2}\left(\sin x + \cos x\right)$$

$$\text{Now, } u_2 = \int \frac{y_1\, f(x)}{W} dx = \int \frac{e^x \sin x}{.e^{3x}} dx = \int e^{-2x} \sin x\, dx$$

$$= \frac{e^{-2x}}{5}\left[-2\sin x - \cos x\right] = -\frac{e^{-2x}}{5}\left[2\sin x + \cos x\right]$$

$$\therefore\ y_p = u_1y_2 + u_2y_1 = e^{2x}\, \frac{e^{-x}}{2}\left(\sin x + \cos x\right) - e^x\, \frac{e^{-2x}}{5}\left[2\sin x + \cos x\right]$$

$$= \frac{e^x\left(\sin x + \cos x\right)}{2} - \frac{e^{-x}\left(2\sin x + \cos x\right)}{5}$$

$$\text{Now, } y = y_c + y_p = c_1e^x + c_2e^{2x} + \frac{e^x\left(\sin x + \cos x\right)}{2} - \frac{e^{-x}\left(2\sin x + \cos x\right)}{5}$$

$$\Rightarrow y = \frac{1}{10}\left[10\left(c_1e^x + c_2e^{2x}\right) + 5e^x\left(\sin x + \cos x\right) - 2e^{-x}\left(2\sin x + \cos x\right)\right]$$

$$\text{NOTE}: \int e^{ax} \sin bx\ dx = \frac{e^{ax}}{a^2 + b^2}\left[a \sin bx - b \cos bx\right]$$

(j) y`` + y = sec x tan x

Solution: The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm\, i$. Thus, $y_c = c_1 \cos x + c_2 \sin x$.

Let, $y_1 = \cos x$ and $y_2 = \sin x \Rightarrow y`_1 = -\sin x$ and $y`_2 = \cos x$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We know that $y_p = u_1y_1 + u_2y_2$ where,

$$u_1 = -\int \frac{y_2\, f(x)}{W} dx = -\int \frac{\sin x. \tan x \sec x}{1} dx = -\int \frac{\sin x. \tan x}{\cos x} dx = -\int \tan^2 x\, dx$$

$$= -\int\left(\sec^2 x - 1\right) dx = -(\tan x - 1) = 1 - \tan x$$

and $u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos x . \tan x \sec x}{1} dx = \int \tan x \, dx = \ln(\sec x)$

$\therefore y_p = u_1 y_2 + u_2 y_1 = \sin x(1 - \tan x) + \cos x . \ln(\sec x)$

Now, $y = y_c + y_p = c_1 \cos x + c_2 \sin x + \sin x(1 - \tan x) + \cos x . \ln(\sec x)$

(k) $y`` + 2y` + 2y = e^{-x} \sec^3 x$

Solution: The auxiliary equation is $m^2 + 2m + 2 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$. Thus,

$$y_c = e^{-x}(c_1 \cos x + c_2 \sin x)$$

Let, $y_1 = e^{-x} \cos x$ and $y_2 = e^{-x} \sin x \Rightarrow y`_1 = -e^{-x} (\sin x + \cos x)$, $y`_2 = e^{-x} (\cos x - \sin x)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} e^{-x} \cos x & e^{-x} \sin x \\ -e^{-x}(\sin x + \cos x) & e^{-x}(\cos x - \sin x) \end{vmatrix}$$

$$= e^{-2x}\left(\cos^2 x - \cos x \sin x + \sin^2 x + \sin x \cos x\right) = e^{-2x}$$

We know that $y_p = u_1 y_1 + u_2 y_2$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{e^{-x} \sin x . e^{-x} \sec^3 x}{e^{-2x}} dx = \int (\cos x)^{-3} (-\sin x) dx = \frac{(\cos x)^{-2}}{-2} = -\frac{\sec^2 x}{2}$$

$$u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{e^{-x} \cos x . e^{-x} \sec^3 x}{e^{-2x}} dx = \int \sec^2 x \, dx = \tan x$$

$$\therefore y_p = u_1 y_2 + u_2 y_1 = e^{-x} \sin x \left(-\frac{\sec^2 x}{2}\right) + e^{-x} \cos x(\tan x) = \frac{e^{-x}}{2}\left(\sin x - 2\tan x \sec x\right)$$

NOTE: $\cos x \tan x = \frac{\cos x . \sin x}{\cos x} = \sin x$ and $\sin x . \sec^2 x = \frac{\sin x . \sec x}{\cos x} = \tan x \cos x$

Now, $y = y_c + y_p = e^{-x}(c_1 \cos x + c_2 \sin x) + \frac{e^{-x}}{2}\left(\sin x - 2\tan x \sec x\right)$

(*l*) $y`` + 4y = \tan 2x$

Solution: The auxiliary equation is $m^2 + 4 = 0 \Rightarrow m = \pm 2i$. Thus,

$$y_c = c_1 \cos 2x + c_2 \sin 2x.$$

Let, $y_1 = \cos 2x$ and $y_2 = \sin 2x \Rightarrow y`_1 = -2\sin x$ and $y`_2 = 2\cos x$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y`_1 & y`_2 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2\left(\cos^2 2x + \sin^2 2x\right) = 2$$

We know that $y_p = u_1 y_1 + u_2 y_2$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W} dx = -\int \frac{\sin 2x . \tan 2x}{2} dx = -\frac{1}{2}\int \frac{\sin 2x . \sin 2x}{\cos 2x} dx = -\frac{1}{2}\int \frac{1 - \cos^2 2x}{\cos 2x} dx$$

$$= -\frac{1}{2}\left[\int \frac{1}{\cos 2x} dx - \int \frac{\cos^2 2x}{\cos 2x} dx\right] = -\frac{1}{2}\left[\int \sec 2x \, dx - \int \cos 2x \, dx\right]$$

$$= -\frac{1}{2}\left[\frac{\ln(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2}\right] = \frac{1}{4}\left[\sin 2x - \ln(\sec 2x + \tan 2x)\right]$$

and $u_2 = \int \frac{y_1 f(x)}{W} dx = \int \frac{\cos 2x . \tan 2x}{2} dx = \frac{1}{2}\int \sin 2x \, dx = -\frac{\cos 2x}{4}$

$$\therefore\ y_p = u_1y_2 + u_2y_1 = \frac{1}{4}\left[\sin 2x - \ln(\sec 2x + \tan 2x)\right]\sin 2x - \frac{\cos 2x}{4}\cos 2x$$

$$\text{Now, } y = y_c + y_p = c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\left[\sin 2x - \ln(\sec 2x + \tan 2x)\right]\sin 2x - \frac{\cos^2 2x}{4}$$

(m) $y'' - 2y' + 2y = e^x \tan x$

Solution: The auxiliary equation is $m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$. Thus,

$$y_c = e^x(c_1\cos x + c_2 \sin x)$$

Let, $y_1 = e^x \cos x$ and $y_2 = e^x \sin x \Rightarrow y'_1 = e^x(-\sin x + \cos x)$, $y'_2 = e^x(\cos x + \sin x)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} e^x\cos x & e^x \sin x \\ e^x(-\sin x + \cos x) & e^x(\cos x + \sin x) \end{vmatrix}$$

$$= e^{2x}\left(\cos^2 x + \cos x \sin x + \sin^2 x - \sin x \cos x\right) = e^{2x}$$

We know that $y_p = u_1y_2 + u_2y_1$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W}dx = -\int \frac{e^x \sin x . e^x\ \sec^3 x}{e^{2x}}dx = \int (\cos x)^{-3}(-\sin x)\,dx = \frac{(\cos x)^{-2}}{-2} = \frac{-\sec^2 x}{2}$$

$$u_2 = \int \frac{y_1 f(x)}{W}dx = -\int \frac{e^x \cos x . e^x\ \sec^3 x}{e^{2x}}dx = \int \sec^2 x\,dx = \tan x$$

$$\text{Thys, } y_p = y_1u_2 + y_2u_1 = e^x \cos x\left(\frac{-\sec^2 x}{2}\right) + e^x \sin x\left(\tan x\right)$$

$$\text{Now, } y = y_c + y_p = e^x\left(c_1\cos x + c_2 \sin x\right) + \frac{e^x}{2}\left[2\sin x\ \tan x\ - \cos x\right]$$

(n) $y'' + y = x - \cot x$

Solution: The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$. Thus, $y_c = c_1 \cos x + c_2 \sin x$.
Let, $y_1 = \cos x$ and $y_2 = \sin x \Rightarrow y'_1 = -\sin x$ and $y'_2 = \cos x$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We know that $y_p = u_1y_1 + u_2y_2$ where,

$$u_1 = -\int \frac{y_2 f(x)}{W}dx = -\int \frac{\sin x.(x - \cot x)}{1}dx = -\int x \sin x\,dx + \int \cos x\,dx$$

$$= -\left[x(-\cos x) - \int 1.(-\cos x)dx\right] + \int \cos x\,dx = x\cos x - \int \cos x\,dx + \int \cos x\,dx = x\cos x$$

$$\text{and } u_2 = \int \frac{y_1 f(x)}{W}dx = \int \frac{\cos x.(x - \cot x)}{1}dx = \int x\ \cos x\,dx - \int \frac{\cos^2 x}{\sin x}dx$$

$$= x\sin x - \int 1.\sin x\ dx - \int \frac{1 - \sin^2 x}{\sin x}dx = x\sin x - \int \sin x\ dx - \int \frac{1}{\sin x}\ dx + \int \sin x\ dx$$

$$= x\sin x - \int \csc x\,dx = x \sin x - \ln(\csc x + \cot x)$$

$$\therefore\ y_p = u_1y_2 + u_2y_1 = \sin x(x\cos x) + \cos x.\left[x\sin x - \ln(\csc x + \cot x)\right]$$

$$\text{Now, } y = y_c + y_p = c_1\cos x + c_2 \sin x + 2x\sin x\cos x - \ln(\csc x + \cot x$$

6. Solve the following differential equation [Form: y`` = f(x)]

(a) $y^{v} = x$

Solution: Integrating, we get

$$y^{iv} = \frac{x^2}{2} + c_1.\ \text{Integrating again : } y``` = \frac{x^3}{6} + c_1 x + c_2$$

$$\text{Integrating again : } y`` = \frac{x^4}{12} + c_1\frac{x^2}{2} + c_2 x.\ \text{Integrating again : } y`` = \frac{x^4}{12} + c_1\frac{x^2}{2} + c_2 x$$

$$\text{Integrating again : } y` = \frac{x^5}{60} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3$$

$$\text{Integrating again : } y = \frac{x^6}{360} + c_1\frac{x^4}{24} + c_2\frac{x^3}{6} + c_3 x + c_4.\ \text{This is general solution.}$$

(b) $y`` = x\, e^x$

Solution: Integrating, we get

$$y` = \int xe^x\,dx + c_1 = xe^x - \int 1.e^x\,dx + c_1 = e^x(x-1) + c_1$$

$$\text{Integrating again : } y = \int\left[e^x(x-1) + c_1\right]dx + c_2 = (x-1)e^x - \int 1.e^x dx + c_1 x + c_2$$

$$y = (x-1)e^x - e^x + c_1 x + c_2 = e^x(x-2) + c_1 x + c_2$$

This is general solution.

(c) $y^{iv} = x + e^{-x} - \cos x$

Solution: Integrating, we get

$$y``` = \frac{x^2}{2} - e^{-x} - \sin x + c_1.\ \text{Integrating again : } y`` = \frac{x^3}{6} + e^{-x} + \cos x + c_1 x + c_2$$

$$\text{Integrating again : } y` = \frac{x^4}{12} - e^{-x} + \sin x + c_1\frac{x^2}{2} + c_2 x + c_3.\ \text{Integrating again,}$$

$$y = \frac{x^5}{60} + e^{-x} - \cos x + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3 x + c_4.\ \text{This is general solution.}$$

(d) $x^2\, y`` = \ln x$

Solution: $y`` = \ln x/x^2$. Integrating, we get

$$y` = \int \ln x . x^{-2} dx + c_1 = \ln x\left(-x^{-1}\right) - \int \frac{1}{x}\left(-x^{-1}\right)dx + c_1$$

$$= -\frac{\ln x}{x} + \int x^{-2} dx + c_1 = -\frac{\ln x}{x} - \frac{1}{x} + c_1$$

$$\text{Integrating again : } y = \int\left(-\frac{\ln x}{x} - \frac{1}{x} + c_1\right)dx + c_2 = -\left(\ln x\right)^2 - \ln x + c_1 x + c_2$$

$$\therefore\ y = -\left(\ln x\right)^2 - \ln x + c_1 x + c_2\ \text{ is a general solution.}$$

(e) $y``` = \ln x$

Solution: $y``` = \ln x$. Integrating, we get

$$y`` = \int \ln x . 1\, dx + c_1 = \ln x\left(x\right) - \int \frac{1}{x}\left(x\right)dx + c_1 = x\ln x - x + c_1$$

$$\text{Integrating again : } y` = \int\left(x\ln x - x + c_1\right)dx + c_2 = \ln x.\frac{x^2}{2} - \int\frac{1}{x}\left(\frac{x^2}{2}\right)dx - \frac{x^2}{2} + c_1 x + c_2$$

$$=\frac{x^2\ln x}{2}-\frac{x^2}{4}-\frac{x^2}{2}+c_1x+c_2=\frac{x^2\ln x}{2}-\frac{3x^2}{4}+c_1x+c_2$$

$$\text{Integrating again : } y=\int\left(\frac{x^2\ln x}{2}-\frac{3x^2}{4}+c_1x+c_2\right)dx+c_3$$

$$=\frac{1}{2}\left(\frac{\ln x.x^3}{3}-\int\frac{1}{x}.\frac{x^3}{3}dx\right)-\frac{3x^3}{3.4}+c_1\frac{x^2}{2}+c_2x+c_3$$

$$\text{Thus, } y=\frac{x^3\ln x}{6}-\frac{x^3}{18}-\frac{x^3}{4}+c_1\frac{x^2}{2}+c_2x+c_3=\frac{x^3\ln x}{6}-\frac{11x^3}{36}+c_1\frac{x^2}{2}+c_2x+c_3$$

is a general solution.

(f) $y``` = \sin^2 x$

Solution: $y``` = \sin^2 x = (1 - \cos 2x)/2 \ln x$. Integrating, we get

$$y``=\frac{1}{2}\int(1-\cos 2x)dx+c_1=\frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+c_1$$

$$\text{Integrating again : } y`=\frac{1}{2}\int\left(x-\frac{\sin 2x}{2}\right)dx+\int c_1\,dx+c_2=\frac{1}{2}\left(\frac{x^2}{2}+\frac{\cos 2x}{4}\right)+c_1x+c_2$$

$$=\frac{1}{8}\left(2x^2+\cos 2x\right)+c_1x+c_2$$

$$\text{Integrating again : } y=\frac{1}{8}\int\left(2x^2+\cos 2x\right)dx+\int\left(c_1x+c_2\right)dx+c_3$$

$$=\frac{1}{8}\left(\frac{2x^3}{3}+\frac{\sin 2x}{2}\right)+c_1\frac{x^2}{2}+c_2x+c_3$$

$$\text{Thus, } y=\frac{\left(4x^3+3\sin 2x\right)}{48}+c_1\frac{x^2}{2}+c_2x+c_3 \text{ is a general solution.}$$

7. Solve the following differential equation [Form: $y`` = f(y)$]

(a) $y`` = 2(y^3 + y)$, subject to conditions: $x = 0$, $y = 0$ and $y` = 1$

Solution: Multiplying both sides by $2y`$, we get:

$$2y`y`` = 4y`(y^3+y) \Rightarrow \frac{d}{dx}(y`)^2=4\left(y^3+y\right)y` \Rightarrow \int\frac{d}{dx}(y`)^2dx=4\int\left(y^3+y\right)y`dx+c_1$$

$$\Rightarrow (y`)^2=4\left(\frac{y^4}{4}+\frac{y^2}{2}\right)+c_1 \quad \text{NOTE: } \int\left[f(x)\right]^n f`(x)\,dx=\left[f(x)\right]^{n+1}/(n+1)] \qquad (1)$$

Put $y = 0$ and $y` = 1$ in (1), we get: $1 = 0 + c_1 \Rightarrow c_1 = 1$.

Thus (1) becomes: $(y`)^2=y^4+2y^2+1=\left(y^2+1\right)^2 \Rightarrow y`=y^2+1$

Separating the variables and integrating, we obtain

$$\int\frac{1}{1+y^2}dy=\int 1dx+c_2 \Rightarrow \tan^{-1}y=x+c_2 \qquad (2)$$

Substituting $x = 0$ and $y = 1$ in (2), we get: $c_2 = \pi/4$. Thus (2) becomes

$\tan^{-1}y=x+\frac{\pi}{4} \Rightarrow y=\tan\left(\frac{4x+\pi}{4}\right)$. This is a particular solution of given equation

(b) $y^2 y`` = 36$, subject to conditions: x = 0, y = 8 and y` = 0

Solution: $y`` = 36/y^2$. Multiplying both sides by 2y`, we get:

$$2y`y`` = 72y`.y^{-2} \Rightarrow \frac{d}{dx}(y`)^2 = 72y^{-2}y` \Rightarrow \int \frac{d}{dx}(y`)^2\,dx = 72\int y^{-2}y`dx + c_1$$

$$\Rightarrow (y`)^2 = -\frac{72}{y} + c_1 \quad \text{NOTE: } \int [f(x)]^n f`(x)\,dx = [f(x)]^{n+1}/(n+1)] \qquad (1)$$

Put y = 8 and y` = 0 in (1), we get: $0 = -9 + c_1 \Rightarrow c_1 = 9$.

Thus (1) becomes: $(y`)^2 = -\frac{72}{y} + 9 = \frac{9y-72}{y} = 9\left(\frac{y-8}{y}\right) \Rightarrow y` = 3\sqrt{\frac{y-8}{y}}$

Separating the variables and integrating, we obtain

$$\int \sqrt{\frac{y}{y-8}}\,dy = 3\int 1\,dx + c_2 = 3x + c_2 \qquad (2)$$

Putting $y = 8\cosh^2 u \Rightarrow dy = 16\cosh u \sinh u\, du$. Thus

$$\int \sqrt{\frac{y}{y-8}}\,dy = \int \sqrt{\frac{8\cosh^2 u}{8(\cosh^2 u - 1)}}.16\cosh u \sinh u\,du = \int \frac{\cosh u}{\sinh u}.16\cosh u \sinh u\,du$$

$$= 16\int \cosh^2 u\,du = 16\int \frac{\cosh 2u + 1}{2}\,du = 8\left(\frac{\sinh 2u}{2} + 1\right) = 8\left(\frac{2\sinh u \cosh u}{2} + 1\right)$$

$$= 8\left(\cosh u\sqrt{\cosh^2 u - 1} + 1\right) = 8\left(\sqrt{\frac{y}{8}}\sqrt{\frac{y}{8} - 1} + 1\right) = 8\left(\frac{\sqrt{y}\sqrt{y-8} + 8}{8}\right) = \sqrt{y^2 - 8y} + 8$$

Thus equation (2) becomes: $\sqrt{y^2 - 8y} + 8 = 3x + c_2$ (3)

Substituting x = 0 and y = 8 in (3), we get: $c_2 = 8$. Thus (3) becomes

$\sqrt{y^2 - 8y} + 8 = 3x + 8 \Rightarrow \sqrt{y^2 - 8y} = 3x \Rightarrow y^2 - 8y = 9x^2$.

This is a particular solution of given equation.

(c) $y^3 y`` = a$, y = 2a, y` = 0 when x = 0.

Solution: $y`` = a/y^3$. Multiplying both sides by 2y`, we get:

$$2y`y`` = 2ay^{-3} \Rightarrow \frac{d}{dx}(y`)^2 = 2ay^{-2}y` \Rightarrow \int \frac{d}{dx}(y`)^2\,dx = 2a\int y^{-2}y`dx + c_1$$

$$\Rightarrow (y`)^2 = -\frac{2a}{y} + c_1 \quad \text{NOTE: } \int [f(x)]^n f`(x)\,dx = [f(x)]^{n+1}/(n+1)] \qquad (1)$$

Put y = 2a and y` = 0 in (1), we get: $0 = -1 + c_1 \Rightarrow c_1 = 1$.

Thus (1) becomes: $(y`)^2 = -\frac{2a}{y} + 1 = \frac{y-2a}{y} \Rightarrow y` = \sqrt{\frac{y-2a}{y}}$

Separating the variables and integrating, we obtain

$$\int \sqrt{\frac{y}{y-2a}}\,dy = \int 1\,dx + c_2 = x + c_2 \qquad (2)$$

Putting $y = 2a\cosh^2 u \Rightarrow dy = 4a\cosh u \sinh u\, du$. Thus,

$$\int \sqrt{\frac{y}{y-2a}}\,dy = \int \sqrt{\frac{2a\cosh^2 u}{2a(\cosh^2 u - 1)}}4a\cosh u \sinh u\,du = 4a\int \frac{\cosh u}{\sinh u}.\cosh u \sinh u\,du$$

$$= 4a\int \cosh^2 u\,du = 4a\int \frac{\cosh 2u + 1}{2}\,du = 2a\left(\frac{\sinh 2u}{2} + 1\right) = 2a\left(\frac{2\sinh u \cosh u}{2} + 1\right)$$

$$= 2a\left(\cosh u\sqrt{\cosh^2 u - 1} + 1\right) = 2a\left(\sqrt{\frac{y}{2a}}\sqrt{\frac{y}{2a} - 1} + 1\right) = 2a\left(\frac{\sqrt{y}\sqrt{y-2a} + 2a}{2a}\right) = \sqrt{y^2 - 2ay} + 2a$$

Thus equation (2) becomes: $\sqrt{y^2 - 2ay} + 2a = x + c_2$ (3)

Substituting x = 0 and y = 2a in (3), we get: $c_2 = 2a$. Thus (3) becomes

$\sqrt{y^2 - 2ay} + 2a = x + 2a \Rightarrow \sqrt{y^2 - 2ay} = x \Rightarrow y^2 - 2ay = x^2$.

This is a particular solution of given equation.

(d) e^{2y} y`` = 1, x = 0, y = 0 and y` = 0.

Solution: y`` = e^{-2y}. Multiplying both sides by 2y`, we get:

$$2y\text{`}y\text{``} = 2y\text{`}e^{-2y} \Rightarrow \frac{d}{dx}(y\text{`})^2 = 2e^{-2x}y\text{`} \Rightarrow \int \frac{d}{dx}(y\text{`})^2 dx = 2\int e^{-2y}y\text{`}dx + c_1$$

$$\Rightarrow (y\text{`})^2 = \frac{2e^{-2y}}{-2} + c_1 \Rightarrow (y\text{`})^2 = -e^{-2y} + c_1 \quad (1)$$

Put y = 0 and y` = 0 in (1), we get: 0 = -1 + c_1 $\Rightarrow$ c_1 = 1.

Thus (1) becomes: $(y\text{`})^2 = -e^{-2y} + 1 = \dfrac{e^{2y} - 1}{e^{2y}} \Rightarrow y\text{`} = \sqrt{\dfrac{e^{2y} - 1}{e^{2y}}}$

Separating the variables and integrating, we obtain

$\int \dfrac{e^y}{\sqrt{e^{2y} - 1}} dy = \int 1 dx + c_2 = x + c_2$. Putting $z = e^y \Rightarrow dz = e^y\, dy$

Thus, $\int \dfrac{e^y}{\sqrt{e^{2y} - 1}} dy = \int \dfrac{1}{\sqrt{z^2 - 1}} dz = \cosh^{-1} z = \cosh^{-1}\left(e^y\right)$ (2)

Thus equation (2) becomes: $\cosh^{-1}(e^y) = x + c_2$. (3)

Substituting x = 0 and y = 0 in (3), we get: c_2 = 1. Thus (3) becomes

$\cosh^{-1}(e^y) = x + 1$. This is a particular solution of given equation.

(e) $\sin^3$ y y`` = - cos y subject to conditions; x = 0, y = π/4 and y` = 2

Solution: y`` = - cos y / $\sin^3$ y = - cot y $\csc^2$ y. Multiplying both sides by 2y`, we get:

2y`y`` = - cot y $\csc^2$y (2y`)

$$\Rightarrow \frac{d}{dx}(y\text{`})^2 = -2\cot y \csc^2 y\ y\text{`} \Rightarrow \int \frac{d}{dx}(y\text{`})^2 dx = 2\int \cot y(-\csc^2 y)\ (y\text{`}dx) + c_1$$

$$\Rightarrow \quad (y\text{`})^2 = 2\cot^2 y + c_1 \quad (1)$$

Put y = π/4 and y` = 2 in (1), we get: 4 = 2 + c_1 $\Rightarrow$ c_1 = 2. NOTE: cot (π/4) = 1

Thus (1) becomes: $(y\text{`})^2 = 2\cot^2 y + 2 = 2\left(\cot^2 y + 1\right) = 2\csc^2 \Rightarrow y\text{`} = \sqrt{2}\csc y$

Separating the variables and integrating, we obtain

$\int \dfrac{1}{\csc y} dy = \sqrt{2}\int 1 dx + c_2 \Rightarrow \int \sin y\, dy = \sqrt{x} + c_2$ OR $-\cos y = \sqrt{x} + c_2$ (2)

Substituting x = 0 and y = π/4 in (2), we get: c_2 = 1/√2. Thus (2) becomes

$\cos y + \sqrt{x} + \dfrac{1}{\sqrt{2}}$. This is a particular solution of given equation.

8. Solve the following differential equations [Form: F(y``, y`, x) = k]

(a) xy`` + xy` - y` = 0

Solution: Substituting z = y` $\Rightarrow$ z` = y``. Thus given equation becomes; xz` + xz – z = 0

$\Rightarrow$ xz` +(x – 1)z = 0 $\Rightarrow$ xz` = -(x – 1) z.

Separating the variables and integrating, we get:

$$\int \frac{1}{z}dz = -\int \frac{x-1}{x}dx + c` \Rightarrow \ln z = -\int\left(\frac{x}{x}-\frac{1}{x}\right)dx + c` \Rightarrow \ln z = -x + \ln x + c`$$

$$\ln\left(\frac{z}{x}\right) = (c`-x) \Rightarrow z/x = e^{c`-x} = e^{c`}.e^{-x} = c_1 e^{x} \Rightarrow z =_1 xe^{-x}.$$

Now, $z = y`$. Thus, $\frac{dy}{dx} = c_1 e^{-x}$.Separating the variables and integrating, we get

$$\int 1.dy = c_1\int xe^{-x}\,dx + c_2 = c_1\left(-xe^{-x} + \int 1.e^{-x}dx\right) + c_2$$

$\Rightarrow y = -c_1 e^{-x}\left(x+1\right) + c_2$. this is the general solution of given equation.

(b) $(1 + x^2)\, y`` + xy` + ax = 0$

Solution: Substituting $z = y` \Rightarrow z` = y``$. Thus given equation becomes;

$$(1+x^2)\, z` + xz + ax = 0 \Rightarrow (1+x^2)\, z` + x(z + a) = 0.$$

Separating the variables and integrating, we get:

$$\int \frac{1}{z+a}dz = -\int \frac{x}{1+x^2}dx + c_1 \Rightarrow \ln(z+a) = -\frac{1}{2}\int \frac{2x}{1+x^2}dx + c_1 = -\frac{1}{2}\ln(1+x^2) + \ln c_1$$

$$\ln(z+a) = -\ln\left(1+x^2\right)^{1/2} + \ln c_1 \Rightarrow \ln(z+a) = \ln\left(\frac{c_1}{\sqrt{1+x^2}}\right) \Rightarrow z + a = \frac{c_1}{\sqrt{1+x^2}}$$

Thus, $\frac{dy}{dx} = -a + \frac{c_1}{\sqrt{1+x^2}}$.Separating the variables and integrating, we get

$$\int 1.dy = \int\left(-a + \frac{c_1}{\sqrt{1+x^2}}\right)dx + c_2 \Rightarrow y = -ax + c_1 \sinh^{-1} x + c_2.$$

This is the general solution of given equation.

(c) $(1 + x^2)y`` + 1 + (y`)^2 = 0$

Solution: Substituting $z = y` \Rightarrow z` = y``$. Thus given equation becomes;

$$(1+x^2)\, z` + 1 + z^2 = 0 \Rightarrow (1+x^2)\, z` = -(1+z^2).$$

Separating the variables and integrating, we get:

$$\int \frac{1}{\left(1+z^2\right)}dz = -\int \frac{1}{1+x^2}dx + \tan^{-1} c_1 \Rightarrow \tan^{-1} z = -\tan^{-1} x + \tan^{-1} c_1$$

[NOTE: $\tan^{-1} c_1$ is a constant of integration]

$$\Rightarrow \tan^{-1} z = \tan^{-1}\left(\frac{c_1 - x}{1 + c_1 x}\right) \Rightarrow z = \left(\frac{c_1 - x}{1 + c_1 x}\right). \text{ Thus, } \frac{dy}{dx} = \left(\frac{c_1 - x}{1 + c_1 x}\right)$$

Separating the variables and integrating :

$$\int 1.dy = \int\left(\frac{c_1 - x}{1 + c_1 x}\right)dx + c_2 = \int \frac{c_1}{1 + c_1 x}dx - \int \frac{x}{1 + c_1 x}dx + c_2$$

$$\Rightarrow y = \ln\left(1 + c_1 x\right) - \frac{1}{c_1}\int \frac{c_1 x}{1 + c_1 x}dx + c_2 = \ln\left(1 + c_1 x\right) - \frac{1}{c_1}\int \frac{1 + c_1 x - 1}{1 + c_1 x}dx + c_2$$

$$\Rightarrow y = \ln\left(1 + c_1 x\right) - \frac{1}{c_1}\int\left(1 - \frac{1}{1 + c_1 x}\right)dx + c_2 = \ln\left(1 + c_1 x\right) - \frac{1}{c_1}\left[x - \frac{1}{c_1}\int \frac{c_1}{1 + c_1 x}dx\right] + c_2$$

$$\Rightarrow \quad y = \ln\left(1+c_1x\right) - \frac{x}{c_1} + \frac{\ln\left(1+c_1x\right)}{c_1^{\,2}} + c_2$$

This is the general solution of given equation.

(d) $y^{iv} - \cot x\, y``` = 0$

Solution: Substituting z = y` ⇒ z` = y``. Thus given equation becomes; z``` - z`` cot x = 0. Substituting u = z` ⇒ u` = z``, the given equation becomes: u`` - u` cot x = 0. Again substituting v = u` ⇒ v` = u``, the given equation becomes: v` - v cot x = 0. Separating the variables and integrating, we get:

$$\int \frac{1}{v} dv = \int \cot x\, dx + \ln c_1 \quad \Rightarrow \ln v = \ln(\sin x) + \ln c_1 = \ln\left(c_1 \sin x\right)$$

$$\Rightarrow v = c_1 \sin x. \text{ But, } v = u`, \text{ thus, } u` = c_1 \sin x. \text{ Integrating, } u = -$$

$$\ln(z+a) = -\ln\left(1+x^2\right)^{1/2} + \ln c_1 \;\Rightarrow \ln(z+a) = \ln\left(\frac{c_1}{\sqrt{1+x^2}}\right) \Rightarrow z+a = \frac{c_1}{\sqrt{1+x^2}}$$

Thus, $\dfrac{dy}{dx} = -a + \dfrac{c_1}{\sqrt{1+x^2}}$. Separating the variables and integrating, we get

$$\int 1.dy = \int\left(-a + \frac{c_1}{\sqrt{1+x^2}}\right)dx + c_2 \Rightarrow y = -ax + c_1 \sinh^{-1} x + c_2.$$

This is the general solution of given equation.

9. Solve the following differential equations [Form: F(y``, y`, y) = k]

(a) y`` + (y`)^2 = 1

Solution: Given equation is y y`` + (y`)^2 = 1.

Putting z = y`= (dy/dx) ⇒ y`` = (dz/dy).(dy/dx) = z`z. Thus given equation becomes:
zz` + z$^2$ = 1 ⇒ zz` = (1 – z^2) ⇒ z` = (1 – z$^2$)/z. But z` = dz/dy.
Thus separating the variables and integrating, we get

$$\int \frac{z}{1-z^2} dz = \int 1.dy + \ln c \;\Rightarrow -\frac{1}{2}\ln(1-z)^2 = y + c \;\Rightarrow \ln(1-z^2)^{-1/2} = y + c$$

$$\frac{1}{\sqrt{1-z^2}} = e^{y+c} \;\Rightarrow \frac{1}{1-z^2} = e^{2y+2c} \;\Rightarrow 1-z^2 = e^{-2(y+c)} \;\Rightarrow z^2 = 1 - e^{-2(y+c)}$$

$\Rightarrow z = \sqrt{1-e^{-2(y+c)}}$. But $z = \dfrac{dy}{dx}$, thus separating the variables and integrating.

$$\int \frac{1}{\sqrt{1-e^{-2(y+c)}}} dy = \int 1.dx + b = x + b.$$

$$\int \frac{e^{(y+c)}}{\sqrt{e^{(2y+2c)}-1}} dy = x + b. \text{ Putting } u = e^{y+c} \Rightarrow du = e^{y+c}dy$$

Thus, $\displaystyle\int \frac{e^{(y+c)}}{\sqrt{e^{(2y+2c)}-1}} dy = \int \frac{1}{\sqrt{u^2-1}} du = \cosh^{-1} u = x + b \Rightarrow \cosh^{-1}\left(e^{y+c}\right) = cx + b$

This is the solution of given differential equation.

(b) yy``- (y`)$^2$ + 2y`= 0

Solution: Given equation is y y`` + (y`)^2 = 1.

Putting z = y`= (dy/dx) ⇒ y`` = (dz/dy).(dy/dx) = z`z. Thus given equation becomes:
yzz` - z$^2$ + 2z = 0 ⇒ zz` = z(z -2) ⇒ yz` = (z - 1). But z` = dz/dy.

Thus separating the variables and integrating, we get

$$\int\frac{1}{z-1}dz=\int\frac{1}{y}dy+\ln c \;\Rightarrow \ln(z-1)=\ln cy \Rightarrow z-1=cy \;\Rightarrow z=\frac{dy}{dx}=1+cy$$

$$\int\frac{1}{1+cy}dy=\int 1.dx+b=x+b\Rightarrow\frac{\ln(1+cy)}{c}=(x+b)\;\Rightarrow \ln(1+cy)=c(x+b)$$

This is the solution of given differential equation.

(c) $y`` + y` + (y`)^3 = 0$

Solution: Given equation is $y`` + y` + (y`)^3 = 0$.

Putting $z = y` = (dy/dx)$ $\Rightarrow$ $y`` = (dz/dy).(dy/dx) = z`z$. Thus given equation becomes:

$zz` + z + z^3 = 0$ $\Rightarrow zz` = -z(1 + z^2)$ $\Rightarrow z` = -(1 + z^2)$. But $z` = dz/dy$.

Thus separating the variables and integrating, we get

$$\int\frac{1}{1+z^2}dz=-\int 1.dy+\ln c \;\Rightarrow \tan^{-1}z=c-x\Rightarrow z=\tan(c-x)\Rightarrow\frac{dy}{dx}=\tan(c-x)$$

$$\int 1.dy=\int\tan(c-x)\,dx+b\Rightarrow y=-\ln\left[\sec\left(c-x\right)\right]+b$$

This is the solution of given differential equation. [Note the change in the problem]

(d) $2yy`` - 3(y`)^2 - 4y^2 = 0$

Solution: Given equation is $2yy`` -3(y`)^2 - 4y^2 = 0$.

Putting $z = y` = (dy/dx)$ $\Rightarrow$ $y`` = (dz/dy).(dy/dx) = z`z$. Thus given equation becomes:

$$2yzz`-3z^2-4y^2=0\;\Rightarrow 2yzz`=3z^2+4y^2\;\Rightarrow z`=\frac{3z}{2y}+\frac{2y}{z}\Rightarrow\frac{dz}{dy}-\frac{3}{2y}z=2yz^{-1}$$

This is B.D.E hence multiplying boh sides by z, we get

$$z\frac{dz}{dy}-\frac{3}{2y}z^2=2y.\text{ Putting }u=z^2\;\Rightarrow\frac{1}{2}\frac{du}{dy}=z\frac{dz}{dy}.\text{ Thus above equation becomes}$$

$$\frac{1}{2}\frac{du}{dy}-\frac{3}{2y}u=2y\;\Rightarrow\frac{du}{dy}-\frac{3}{y}u=4y.\text{ This is a l.d.e in u, where}$$

$$\text{I.F}=e^{\int p\,dy}=e^{-3\int\frac{1}{y}dy}=e^{-3\ln y}=y^{-3}.\text{ Multiplyiing above equation by I.f, we get}$$

$$y^{-3}\left(\frac{du}{dy}-\frac{3}{y}u\right)=4y^{-2}\;\Rightarrow\frac{d}{dy}\left(u\times\text{I.F}\right)=4y^{-2}.\text{ Integrating w.r.t y, we get:}$$

$$\left(u\times\text{I.F}\right)=-4y^{-1}+c.\text{ Putting the values of u and I.F, we get: }z^2y^{-3}=-4y^{-1}+c$$

$$\text{Multiplying by }y^3\text{, we get: }z^2=y^2\left(cy-4\right)$$

$$\Rightarrow z=\frac{dy}{dx}=y\sqrt{cy-4}.\text{ Separatingthe variables and integrating, we get}$$

$$\int\frac{1}{y\sqrt{c(y-4/c)}}\,dy=\int 1.dx+b\;\Rightarrow\frac{1}{\sqrt{c}}\int\frac{1}{y\sqrt{y-k}}dy=x+b\quad\left[k=4/c\right]$$

$$\text{Putting }y=k\sec^2 v,\;\; dy=2k\sec v\,.\sec v\,\tan v\,dv.\text{ Thus above equation becomes}$$

$$\frac{1}{\sqrt{c}}\int\frac{1}{\sec^2 v\sqrt{k\left(\sec^2 v-1\right)}}2k\sec^2 v\tan v\,dv=x+b$$

$$\text{Or }\frac{2k}{\sqrt{ck}}\int\frac{\tan v}{\tan v}dv=x+b\quad\Rightarrow 2\sqrt{\frac{k}{c}}\int 1.dv=x+b\;\Rightarrow 2\sqrt{\frac{k}{c}}\;v=x+b$$

Now, $\sec^2 v = y/k \Rightarrow \sec v = \sqrt{y/k} \Rightarrow v = \sec^{-1}\sqrt{(y/k)}.$

Thus, $2\sqrt{\frac{k}{c}}\sec^{-1}\sqrt{\frac{y}{k}} = x + b.$ Putting $k = \frac{4}{c}$, we get :

$\frac{4}{c}\sec^{-1}\frac{\sqrt{cy}}{2} = x + b.$ This is the general solutin.

(e) $yy'' - (y')^2 = y^2 \ln y$

Solution: Given equation is $yy'' - (y')^2 = y^2 \ln y$.

Putting $z = y' = (dy/dx) \Rightarrow y'' = (dz/dy).(dy/dx) = z'z$. Thus given equation becomes:

$yzz' - z^2 = y^2 \ln y \Rightarrow zz' - \frac{1}{y}z^2 = y\ln y$. This is the B.D.E hence, putting $u = z^2$

$\Rightarrow \frac{1}{2}\frac{du}{dy} = z\frac{dz}{dy}$. Thus above equation becomes:

$\frac{1}{2}\frac{du}{dy} - \frac{1}{y}u = y\ln y \Rightarrow \frac{du}{dy} - \frac{2}{y}u = 2y\ln y.$ This is a l.d.e with $\text{I.F} = e^{\int P dy} = e^{\int -\frac{2}{y}dy} = y^{-2}$

Multiplying by I.F, we get: $y^{-2}\frac{du}{dy} - 2y^{-3}u = 2\frac{\ln y}{y} \Rightarrow \frac{d}{dy}[u \times \text{I.F}] = 2\frac{\ln y}{y}.$

Integrating : $u \times \text{I.F} = 2\int \frac{\ln y}{y}dy + c \Rightarrow uy^{-2} = 2(\ln y)^2 + c \Rightarrow u = y^2\left[2(\ln y)^2 + c\right]$

$\Rightarrow z^2 = y^2\left[2(\ln y)^2 + c\right] \Rightarrow z = \frac{dy}{dx} = y\sqrt{\left[2(\ln y)^2 + c\right]}.$

Separating the variables and integrating : $\int \frac{1}{y\sqrt{\left[2(\ln y)^2 + c\right]}}dy = \int 1.dx + b.$

Putting $v = 2\ln y \Rightarrow \frac{dv}{2} = \frac{dy}{y}$. Thus above equation becomes : $\frac{1}{2}\int \frac{1}{\sqrt{v^2 + c}}dv = x + b$

$\Rightarrow \frac{1}{2\sqrt{c}}\sinh^{-1}\left(\frac{v}{\sqrt{c}}\right) = x + b.$ Putting $v = 2\ln y$, we get : $\frac{1}{2\sqrt{c}}\sinh^{-1}\left(\frac{2\ln y}{\sqrt{c}}\right) = x + b.$

This is the solution of given differential equation.

10. Solve the following system of differential equations

(a) $x' + 2x - 3y = 0;\ y' - 3x + 2y = 0$

Solution: Using D-operator where $D = d/dt$, we get

$(D + 2)x - 3y = 0$ (1) $-3x + (D + 2)y = 0$ (2)

Multiplying (1) by 3 and (2) by (D + 2), we get

$3(D + 2)x - 9y = 0$

$-3(D + 2)x + (D + 2)^2 y = 0$

Adding, $[(D + 2)^2 - 9]\,y = 0$

The auxiliary equation is $(m + 2)^2 - 9 = 0 \Rightarrow m + 2 = \pm 3 \Rightarrow m = 1$ or $m = -5$

Thus, $y = c_1e^t + c_2 e^{-5t}$. Putting this in (2), we get

$-3x + D(c_1e^t + c_2 e^{-5t})y + 2(c_1e^t + c_2 e^{-5t}) = 0$

$\Rightarrow -3x + (c_1e^t - 5c_2 e^{-5t}) + 2(c_1e^t + c_2 e^{-5t}) = 0$

$\Rightarrow 3x = 3c_1e^t - 3c_2 e^{-5t} \Rightarrow x = c_1e^t - c_2 e^{-5t}$. Thus,

$x = c_1e^t - c_2 e^{-5t}$, and $y = c_1e^t + c_2 e^{-5t}$. This is a general solution of given system.

(b) x`` - y = 0; y`` - x = 0

Solution: Using D-operator where D = d/dt, we get

$D^2 x - y = 0$ (1) $-x + D^2 y = 0$ (2)

Multiplying (2) by D^2, we get

$$D^2 x - y = 0$$
$$-D^2 x + D^2 y = 0$$

Adding, $(D^2 - 1)\, y = 0$

The auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = -1, .$

Thus, $y = c_1e^t + c_2 e^{-t}$. Putting this in (2), we get

$x = D^2 (c_1e^t + c_2 e^{-t}) = c_1e^t + c_2 e^{-t}$

Thus, $x = c_1e^t + c_2 e^{-t}$ and $y = c_1e^t + c_2 e^{-t}$ form a general solution of given system.

(c) x` - y = t; y` + x = t^2

Solution: Using D-operator where D = d/dt, we get

$D x - y = t$ (1) $x + D y = t^2$ (2)

Multiplying (2) by D, we get

$$D x - y = t$$
$$D x + D^2 y = Dt^2 = 2t$$

Subtracting, $(D^2 + 1)\, y = t$

The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$. Thus, $y_c = c_1 \cos t + c_2 \sin t$.

$$y_p = \frac{1}{D^2+1}(t) = \left(1+D^2\right)^{-1} t = \left(1 - D^2 + D^4 - ...\right)t = t - 0 + 0... = t \text{ . Thus,}$$

$y = y_c + y_p = c_1 \cos t + c_2 \sin t + t$. Put this in (2), we get:

$x + D(c_1 \cos t + c_2 \sin t + t) = t^2 \Rightarrow x = c_1 \sin t - c_2 \sin t -1 + t^2$.

Thus, $x = c_1 \sin t - c_2 \sin t -1 + t^2$ and $y = c_1 \cos t + c_2 \sin t + t$.

This is a general solution of given system.

(d) x` + 4x + 3y = t; y` + 2x + 5y = e^t

Solution: Using D-operator where D = d/dt, we get

$(D + 4)x + 3y = t$ (1) $2x + (D + 5)y = e^t$ (2)

Multiplying (1) by 2 and (2) by (D + 4), we get

$$2(D + 4)\, x + 6y = 2t$$
$$2(D + 4)\, x + (D + 4)(D+ 5)\, y = (D + 4)e^t = e^t + 4e^t = 5e^t.$$

Subtracting, $[(D + 4)(D + 5) - 6]\, y = 5e^t - 2t.$

The auxiliary equation is $(m + 4)(m + 5) - 6 = 0 \Rightarrow m^2 + 9m + 20 - 6 = 0.$

$\Rightarrow m^2 + 9m + 14 = 0 \Rightarrow m = -2, -7$. Thus, $y_c = c_1e^{-2t} + c_2 e^{-7t}$.

$$y_p = \frac{1}{D^2+9D+14}\left(5e^t - 2t\right) = 5\frac{1}{D^2+9D+14}e^t - 2\frac{1}{D^2+9D+14}t$$

$$= \frac{e^t}{1+9+14} - 2\frac{1}{9D+14}t \left[\text{Neglecting } D^2 \text{ and higher terms}\right]$$

$$= \frac{e^t}{24} - \frac{2}{14}\left(1+9D/14\right)^{-1} t = \frac{e^t}{24} - \frac{1}{7}\left(1 - \frac{9}{14}D\right)t = \frac{e^t}{24} - \frac{1}{7}\left(t - \frac{9}{14}\right).$$

$$\text{Now, } y = y_c + y_p = c_1e^{-2t} + c_2e^{-7t} + \frac{e^t}{24} - \frac{1}{7}\left(t - \frac{9}{14}\right) \qquad (3)$$

Put this in (2), we get :

$$2x + D\left[c_1e^{-2t} + c_2e^{-7t} + \frac{e^t}{24} - \frac{1}{7}\left(t - \frac{9}{14}\right)\right]y + 5\left[c_1e^{-2t} + c_2e^{-7t} + \frac{e^t}{24} - \frac{1}{7}\left(t - \frac{9}{14}\right)\right] = e^t$$

$$2x - 2c_1e^{-2t} - 7c_2e^{-7t} + \frac{e^t}{24} - \frac{1}{7} + 5c_1e^{-2t} + 5c_2e^{-7t} + \frac{5e^t}{24} - \frac{5}{7}\left(t - \frac{9}{14}\right) = e^t.$$

$$2x + 3c_1e^{-2t} - 2c_2e^{-7t} + \frac{e^t}{4} - \frac{1}{7} - \frac{5t}{7} + \frac{45}{91} = e^t \Rightarrow 2x = e^t - \frac{e^t}{4} + \frac{5t}{7} - \frac{31}{91} - 3c_1e^{-2t} + 2c_2e^{-7t}$$

$$\Rightarrow x = \frac{1}{2}\left[\frac{3e^t}{4} + \frac{5t}{7} - \frac{31}{91} - 3c_1e^{-2t} + 2c_2e^{-7t}\right] \qquad (4)$$

Equations (3) and (4) form a solution of given system.

(e) 2x` + 6x – y = 2sin 2t; y` - 2x + 5y = 0

Solution: Using D-operator where D = d/dt, we get

$(2D + 6)x - y = 2\sin 2t$ (1) $\qquad -2x + (D + 5)y = 0$ (2)

Multiplying (1) by 2 and (2) by (2D + 6), we get

$$2(2D + 6)x - 2y = 4\sin 2t$$
$$-2(2D + 6)x + (2D + 6)(D + 5)y = 0$$

Adding, $\quad [(2D + 6)(D + 5) - 2]\, y = 4 \sin 2t$

$\Rightarrow \quad [(D + 3)(D + 5) - 1]\, y = 2 \sin 2t$

The auxiliary equation is $m^2 + 8m + 14 = 0 \Rightarrow m = \dfrac{-8 \pm \sqrt{64-56}}{2} = \dfrac{-8 \pm \sqrt{8}}{2} = -4 \pm \sqrt{2}$

$$y_c = c_1e^{\left(-4+\sqrt{2}\right)t} + c_2e^{\left(-4-\sqrt{2}\right)t}$$

$$y_p = 2\frac{1}{D^2 + 8D + 14}\sin 2t = \frac{2}{-2^2 + 8D + 14}\sin 2t = \frac{1}{4D + 5} \times \frac{4D - 5}{4D - 5}\sin 2t = \frac{4D - 5}{16D^2 - 25}\sin 2t$$

$$= \frac{4D - 5}{16(-2^2) - 25}\sin 2t = \frac{4D\sin 2t - 5\sin 2t}{-64 - 25} = -\frac{1}{89}\left(8\cos 2t - 5\sin 2t\right)$$

Thus, $y = y_c + y_p = c_1e^{\left(-4+\sqrt{2}\right)t} + c_2e^{\left(-4-\sqrt{2}\right)t} - \dfrac{1}{89}\left(8\cos 2t - 5\sin 2t\right).$ (3)

Put this in (2), we get : $-2x + \left(D + 5\right)\left[c_1e^{\left(-4+\sqrt{2}\right)t} + c_2e^{\left(-4-\sqrt{2}\right)t} - \dfrac{1}{89}\left(8\cos 2t - 5\sin 2t\right) = 0\right]$

$$2x = \left(D + 5\right)\left[c_1e^{\left(-4+\sqrt{2}\right)t} + c_2e^{\left(-4-\sqrt{2}\right)t} - \frac{1}{89}\left(8\cos 2t - 5\sin 2t\right)\right]$$

$$\Rightarrow x = \frac{1}{2}\left[\begin{array}{l}\left(-4 + \sqrt{2}\right)c_1e^{\left(-4+\sqrt{2}\right)t} + \left(-4 - \sqrt{2}\right)c_2e^{\left(-4-\sqrt{2}\right)t} - \frac{1}{89}\left(-16\sin 2t - 10\cos 2t\right) \\ +5\left(c_1e^{\left(-4+\sqrt{2}\right)t} + c_2e^{\left(-4-\sqrt{2}\right)t} - \frac{1}{89}\left(8\cos 2t - 5\sin 2t\right)\right)\end{array}\right] \qquad (4)$$

Equations (3) and (4) together form a solution.

(f) x` + 5x – 2y = 0; y` + 2x + y = 0, x = y = 0 at t = 0

Solution: Using D-operator where D = d/dt, we get

$(D + 5)\, x - y = 0$ (1) $\qquad 2x + (D + 1)y = 0$ (2)

Multiplying (2) by (D + 5) and (1) by 2, we get

$$2(D + 5)x - 2y = 0$$
$$2(D + 5)x + (D + 5)(D + 1) y = 0$$

Subtracting, $[(D + 5)(D + 1) - 2]y = 0$

The auxiliary equation is $m^2 + 6m + 3 = 0 \Rightarrow m = \dfrac{-6 \pm \sqrt{36-12}}{2} = -3 \pm \sqrt{6}$. Thus,

$$y = c_1e^{\left(-3+\sqrt{6}\right)t} + c_2e^{\left(-3-\sqrt{6}\right)t} \quad (3)$$

Put this in (2), we get : $2x + (D+1)\left[c_1e^{\left(-3+\sqrt{6}\right)t} + c_2e^{\left(-3-\sqrt{6}\right)t}\right] = 0$

$$\Rightarrow x = -\frac{1}{2}\left\{\left[\left(-3+\sqrt{6}\right)c_1e^{\left(-3+\sqrt{6}\right)t} + \left(-3-\sqrt{6}\right)c_2e^{\left(-3-\sqrt{6}\right)t}\right] + \left[c_1e^{\left(-3+\sqrt{6}\right)t} + c_2e^{\left(-3-\sqrt{6}\right)t}\right]\right\} (4)$$

Putting $t = 0$, $x = 0$ and $y = 0$ in (3) & (4), we get : $c_1 + c_2 = 0$;

$\left(-3+\sqrt{6}+1\right)c_1 + \left(-3+\sqrt{6}+1\right)c_2 = 0$. This gives $c_1 = 0 = c_2$.

Thus, $y = 0$ and $x = 0$ is a particular solution of given system.

(g) $x` + 2y = - \sin t$; $y` - 2x = \cos t$

Solution: Using D-operator where D = d/dt, we get

$Dx + 2y = -\sin t$ (1) $-2x + Dy = \cos t$ (2)

Multiplying (2) by D, and (1) by -2, we get

$$2Dx - 2y = -2 \sin t$$
$$-2Dx + D^2 y = D\cos t = - \sin t$$

Adding, $(D^2 - 2) y = -3 \sin t$

The auxiliary equation is $m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$. Thus, $y_c = c_1e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t}$.

$y_p = -3\dfrac{1}{D^2-2}\sin t = -3\dfrac{1}{-1^2-2}\sin t = \sin t$. Thus,

$y = y_c + y_p = c_1e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} + \sin t$. Put this in (2), we get:

$2x = D(c_1e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} + \sin t) - \cos t$ or $x = (\sqrt{2}c_1e^{\sqrt{2}\,t} + \sqrt{2}c_2 e^{-\sqrt{2}\,t} + \sin t - \cos t)/2$.

Thus, $x = (\sqrt{2}c_1e^{\sqrt{2}\,t} + \sqrt{2}c_2 e^{-\sqrt{2}\,t} + \sin t - \cos t)/2$ and $y = y_c + y_p = c_1e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} + \sin t$ gives a general solution of given system.

11. Solve RLC circuit under given conditions

(a) Given: L = 10 mili H, R = 200 ohms, C = 0.1micro F, E = 0 and at t = 0, Q = 0.01 and I = 0. Find Q

Solution: The governing RLC circuit is: $L\dfrac{d^2Q}{dt^2} + R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E$

Putting the given values, we get: $10\dfrac{d^2Q}{dt^2} + 200\dfrac{dQ}{dt} + 10Q = 0$

Dividing by 10, we have: $\dfrac{d^2Q}{dt^2} + 20\dfrac{dQ}{dt} + Q = 0$. Using D-operator, (D = d/dt) the equation becomes: $\left(D^2 + 2D + 1)\right)Q = 0$. The auxiliary equation is $m^2 + 2m + 1 = 0$. This gives m = -1, -1. Thus,

$$Q = (c_1 + c_2 t) e^{-t} \quad (i) \quad \text{and} \quad Q` = I = e^{-t} (- c_1 - c_2 t + c_2) \quad (ii)$$

Putting t = 0, Q = 0.01 and Q` = I = 1, we get from (i) and (ii): $c_1 = 0$ and $c_2 = 1$. Putting these in (i), we get: $Q = te^{-t}$. This is the required charge in the circuit.

(b) Given: L = 0.5 H, R = 300 ohms, C = 2 $(10)^{-6}$ F, E = 0 and at t = 0, Q = 0.01 and I = 0. Find Q and I.

Solution: The governing RLC circuit is: $L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E$

Putting the given values, we get: $0.5\frac{d^2Q}{dt^2}+300\frac{dQ}{dt}+\frac{10^6}{2}Q=0$

Multiplying by 2, we have: $\frac{d^2Q}{dt^2}+600\frac{dQ}{dt}+10^6Q=0$. Using D-operator, (D = d/dt) the equation becomes: $\left(D^2+600D+10^6\right)Q=0$. The auxiliary equation is

$m^2 + 600m + 10^6 = 0$. This gives, m = - 600 ± 954i

Thus, $Q = e^{-600t}(c_1 \cos 954t + c_2 \sin 954t)$ (i)

$Q` = I = e^{-600t}(-954c_1 \sin 954t + 954c_2 \cos 954t - 600c_1 \cos 954t - 600c_2 \sin 954t)$ (ii)

Putting t = 0, Q = 0.01 and Q` = I = 1, we get from (i) and (ii):

$c_1 = 0.01$ and $c_2 = 0.0063$. Putting these in (i), we get:

$Q = e^{-600t}(0.01 \cos 954t + 0.0063 \sin 954t)$. This is the required charge in the circuit.

(c) Given: L = 0.05 H, R = 10 ohms, C = 10^{-3} F, E = 50 sin 200 t and at t = 0, Q = 0 and I = 1. Find Q.

Solution: The governing RLC circuit is: $L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E$

Putting the given values, we get: $0.5\frac{d^2Q}{dt^2}+10\frac{dQ}{dt}+10^3Q=50\sin 200t$

Multiplying by 2, we have: $\frac{d^2Q}{dt^2}+20\frac{dQ}{dt}+2000Q=100\sin 200t$.

Using D-operator, D = d/dt above equation becomes: $\left(D^2+20D+2000\right)Q=100\sin 200t$

The auxiliary equation is $m^2 + 20m + 2000 = 0$. This gives, m = - 10 ± 43.6i

Thus, $Q = e^{-10t}(c_1 \cos 43.6t + c_2 \sin 43.6t)$ (i)

$Q` = I = e^{-10t}(-43.6c_1 \sin 43.6t + 43.6c_2 \cos 43.6t - 10c_1 \cos 43.6t - 10c_2 \sin 43.6t)$ (ii)

Putting t = 0, Q = 0 and Q` = I = 1, we get from (i) and (ii):

$c_1 = 0$ and $c_2 = 0.023$. Putting these in (i), we get:

$Q = 0.023e^{-10t} \sin 43.6t$. This is the required charge in a circuit.

(d) Given: L = 1 H, R = 100 ohms, C = 10^{-4} F, E = 100 and at t = 0, Q = 0 and I = 0. Find Q and I.

Solution: The governing RLC circuit is: $LQ``+RQ`+Q/C=E$

Putting the given values, we get: $\frac{d^2Q}{dt^2}+100\frac{dQ}{dt}+10^4Q=100$

Using D-operator, (D = d/dt) the equation becomes: $\left(D^2+100D+10000\right)Q=100$.

The auxiliary equation is $m^2 + 100m + 10000 = 0$ giving, m = -50 ± 86.6 i.

Thus, $Q = e^{-50t}(c_1 \cos 86.6t + c_2 \sin 86.6t)$ (i)

$Q` = I = e^{-50t}(-86.6c_1 \sin 86.6t + 86.6c_2 \cos 43.6t - 50c_1 \cos 86.6t - 50c_2 \sin 86.6t)$ (ii)

Putting t = 0, Q = 0 and Q` = I = 1, we get from (i) and (ii):

$c_1 = 0$ and $c_2 = 0.011$. Putting these in (i), we get:

$Q = 0.011e^{-50t} \sin 86.6t$. This is the required charge in a circuit.

Chapter 6

INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS

WORKSHEET 06

1. Form the partial differential equations by eliminating arbitrary constants.
NOTE: a, b and c are arbitrary constants.

(a) $z = ax + by + a^2 + b^2$ (1)

Solution: Given equation contains two arbitrary constants hence we differentiate it partially w.r.t x and y respectively. Now,

$\frac{\partial z}{\partial x} = a$ and $\frac{\partial z}{\partial y} = b$. Subsitituting these in (1), we get :

$z = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$. This is required partial differential equation.

(b) $z = (a + x)(b + y)$ (1)

Solution: Given equation contains two arbitrary constants hence we differentiate it partially w.r.t x and y respectively. Now,

$\frac{\partial z}{\partial x} = (b + y)$ and $\frac{\partial z}{\partial y} = (a + x)$. Subsitituting these in (1), we get :

$z = \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}$. This is required partial differential equation.

(c) $(x - a)^2 + (y - b)^2 + z^2 = c^2$ (1)

Solution: Differentiating (1) partially w.r.t x, we get,

$$2(x - a) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow (x - a) + z\frac{\partial z}{\partial x} = 0 \qquad (2)$$

Differentiating again w.r.t x,

$$1 + z\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial x} = 0 \Rightarrow z\frac{\partial^2 z}{\partial x^2} + \left(\frac{\partial z}{\partial x}\right)^2 = -1 \text{ is required p.d eq.}$$

Differentiating w.r.t y, we get : $2(y - b) + 2z\frac{\partial z}{\partial x} = 0 \Rightarrow (y - b) + z\frac{\partial z}{\partial y} = 0$ (3)

Differentiating again w.r.t y,

$$1 + z\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\frac{\partial z}{\partial y} = 0 \Rightarrow z\frac{\partial^2 z}{\partial y^2} + \left(\frac{\partial z}{\partial y}\right)^2 = -1. \text{ This is the required p.d e.}$$

If we differentiate (2) w.r.t y, we get : $z\dfrac{\partial^2 z.}{\partial y\partial x}+\dfrac{\partial z}{\partial x}.\dfrac{\partial z}{\partial y}=0$ (4)

If we differentiate (3) w.r.t x, we get : $z\dfrac{\partial^2 z}{\partial x\partial y}+\dfrac{\partial z}{\partial y}.\dfrac{\partial z}{\partial x}=0$ (5)

Equations (4) and (5) are the required p.d e's.

It may be noted that (1) is the solution of p.d eqs (2), (3), (4) and (5).

(d) $2z = (ax + y)^2 + b$ (1)

Solution: Differentiating (1) partially w.r.t x, we get

$2\dfrac{\partial z}{\partial x}=2a(ax+y) \Rightarrow z_x = a(ax+y)$ (2)

Similarly, differentiating partially w.r.t y, we get:

$z_y = (ax+y)$. Substituting this in (2), we get : $z_x = az_y \Rightarrow a = z_x \div z_y$

Now substituting this value of 'a' in (2), we get:

$z_x = \dfrac{z_x}{z_y}\left(\dfrac{z_x}{z_y}x+y\right) \Rightarrow z_x\left(z_y\right)^2 = xz_x + yz_y$ OR $xz_x + yz_y - z_x\left(z_y\right)^2 = 0$

This is the required partial differential equation.

(e) $ax^2 + by^2 + z^2 = 1$ (1)

Solution: Differentiating (1) partially w.r.t x, we get: $2a + 2zz_x = 0 \Rightarrow a = zz_x$.

Now differentiating (1) partially w.r.t y, we get: $2b + 2zz_y = 0 \Rightarrow b = zz_y$

Substituting these values of 'a' and 'b' in (1), we obtain: $x^2zz_x + y^2zz_y + z^2 - 1 = 0$.

This is the required partial differential equation.

(f) $x^2 + y^2 = (z - c)^2 \tan^2 a$ (1)

Solution: Differentiating (1) partially w.r.t x, we get:

$2x = 2(z-c)\tan^2 a.z_x \Rightarrow (z-c)\tan^2 a = x/z_x$ (2)

Now differentiating (1) partially w.r.t y and simplifying, we get:

$(z-c)\tan^2 a = y/z_y$ (3)

Equating (2) and (3), we get: $x/z_x = y/z_y \Rightarrow x.z_y = y.z_x \Rightarrow x.z_y - y.z_x = 0$.

This is the required partial differential equation.

(g) $z = ax + by + ab$

Solution: Differentiating (1) partially w.r.t x, we get: $z_x = a$ (2)

Now differentiating (1) partially w.r.t y and simplifying, we get: $z_y = b$ (3)

Substituting the values of 'a' and 'b' in (1), we get: $xz_x + yz_y + z_x.z_y - z = 0$.

This is the required partial differential equation.

(h) $z = a(x + y) + b$ (1)

Solution: Differentiating (1) partially w.r.t x, we get: $z_x = a$ (2)

Now differentiating (1) partially w.r.t y, we get: $z_y = a$ (3)

Equating (2) and (3), we get: $z_x = z_y \Rightarrow z_x - z_y = 0$.

This is the required partial differential equation.

(i) $z = ax + a^2 y^2 + b$ (1)

Solution: Differentiating (1) partially w.r.t x, we get: $z_x = a$ (2)

Now differentiating (1) partially w.r.t y and simplifying, we get: $z_y = 2a^2yz_y$ (3)

Substituting the value of 'a' from (2) in (3), we get:

$$z_y = 2y(z_x)^2 z_y \quad \Rightarrow \quad 2y(z_x)^2 z_y - z_y = 0$$

This is the required partial differential equation.

(j) z = ax e^y + 1/2 a^2 e^{2y} + b (1)

Solution: Differentiating (1) partially w.r.t x, we get:

$$z_x = ae^y \Rightarrow a = e^{-y}z_x \tag{2}$$

Now differentiating (1) partially w.r.t y and simplifying, we get:

$$z_y = axe^y + a^2e^{2y} \tag{3}$$

Substituting the value of 'a' and 'ae^y' from (2) in (3), we get:

$$z_y = xz_x + e^{-2y}.e^{2y}(z_x)^2 \quad \Rightarrow \quad xz_x + (z_x)^2 - z_y = 0$$

This is the required partial differential equation.

(k) z = a e^{bx} sin by (1)

Solution: Differentiating (1) partially w.r.t x, twice we get:

$$z_{xx} = ab^2e^{bx}\sin by = b^2\left(ae^{bx}\sin by\right) \Rightarrow z_{xx} = b^2z \tag{2}$$

Now differentiating (1) partially w.r.t y twice, we get:

$$z_{yy} = -ab^2e^{bx}\sin by = -b^2\left(ab^2e^{bx}\sin by\right) \Rightarrow z_{yy} = -b^2z \tag{3}$$

Adding (2) and (3), we get: $z_{xx} + z_{yy} = 0$

This is the required partial differential equation.

(*l*) az + b = a^2 x + y (1)

Solution: Differentiating (1) partially w.r.t x, we get:

$$az_x = a^2 \Rightarrow z_x = a \tag{2}$$

Now differentiating (1) partially w.r.t y, we get:

$$az_y = 1 \Rightarrow z_y = 1/a \tag{3}$$

Substituting the value of 'a' from (2) in (3), we get:

$$z_y = 1/z_x \quad \Rightarrow z_x z_y = 1$$

This is the required partial differential equation.

2. Form the partial differential equation by eliminating arbitrary functions.

NOTE: f and g are arbitrary functions.

(a) z = f(x^2 – y^2) (1)

Solution: Differentiating (1) partially w.r.t x, we get:

$$z_x = 2x\,f`\left(x^2 - y^2\right) \Rightarrow f`\left(x^2 - y^2\right) = z_x / 2x \tag{2}$$

Now differentiating (1) partially w.r.t y, we get:

$$z_y = -2y\,f`\left(x^2 - y^2\right) \Rightarrow f`\left(x^2 - y^2\right) = -z_y / 2y \tag{3}$$

From (2) and (3), we get:

$$z_x / 2x = -z_y / 2y \quad \Rightarrow yz_x = -xz_y \Rightarrow yz_x + xz_y = 0$$

This is the required partial differential equation.

(b) z = x f(y) + y g(x) (1)

Solution: Differentiating (1) partially twice w.r.t x, we get:

$z_x = f(y) + yg`(x)$ (2) $z_{xx} = yg``(x)$ (2a)

Now differentiating (1) partially twice w.r.t y, we get:

$z_y = xf`(y) + g(x)$ (3) $z_{yy} = xf``(y)$ (3a)

Multiplying (2) by x and (3) by y and adding, we get:

$$xz_x + yz_y = (xf + yg) + xy(f` + g`) = z + xy(f` + g`) \qquad (4)$$

Now, differentiating (4) partially w.r.t x, we obtain:

$$xz_{xx} + z_x + yz_{xy} = z_x + y(f` + g`) + xyg`` \Rightarrow xz_{xx} + yz_{xy} = y(f` + g`) + xyg`` \qquad (5)$$

Now, differentiating (4) partially w.r.t y, we obtain:

$$xz_{xy} + yz_{yy} + z_y = z_y + x(f` + g`) + xyf`` \Rightarrow xz_{xy} + yz_{yy} = x(f` + g`) + xyf`` \qquad (6)$$

Adding (5) and (6), we have:

$$xz_{xx} + yz_{yy} + (x+y)z_{xy} = (x+y)(f` + g`) + xy(f`` + g``) \qquad (7)$$

Now, from (4) $\dfrac{xz_x + yz_y - z}{xy} = (f` + g`)$ and from (2a) and (3a), $\dfrac{z_{xx}}{y} = g``$ and $\dfrac{z_{yy}}{x} = f``$

Substituting these in (7), we obtain:

$$\Rightarrow\ xz_{xx} + yz_{yy} + (x+y)z_{xy} = (x+y)\frac{\left(xz_x + yz_y - z\right)}{xy} + xy\left(\frac{z_{xx}}{y} + \frac{z_{yy}}{x}\right)$$

$$\Rightarrow\ xz_{xx} + yz_{yy} + (x+y)z_{xy} = (x+y)\frac{\left(xz_x + yz_y - z\right)}{xy} + \left(xz_{xx} + yz_{yy}\right)$$

Multiplying by xy, we get

$$x^2yz_{xx} + xy^2z_{yy} + xy(x+y)z_{xy} = (x+y)\left(xz_x + yz_y - z\right) + xy\left(xz_{xx} + yz_{yy}\right)$$

$$x^2yz_{xx} + xy^2z_{yy} + x^2yz_{xy} + xy^2z_{xy} = x^2z_x + xyz_x + xyz_y + y^2z_y - xz - yz + x^2yz_{xx} + xy^2z_{yy}$$

$$\Rightarrow \left(x^2 + y^2\right)z_{xy} - \left(x^2z_x + y^2z_y\right) - xy\left(z_x + z_y\right) + \left(x+y\right)z = 0$$

This is the required partial differential equation.

(c) z = f(x/y) (1)

Solution: Differentiating (1) partially w.r.t x, we get:

$$z_x = \frac{f`\left(x/y\right)}{y} \Rightarrow f`(x/y) = yz_x \qquad (2)$$

Now differentiating (1) partially w.r.t y, we get:

$$z_y = -\frac{x}{y^2}f`\left(x/y\right) \quad \Rightarrow f`\left(x/y\right) = -\frac{y^2}{x}z_y \qquad (3)$$

Equating (2) and (3), we get:

$$yz_x = -\frac{y^2}{x}z_y \quad \Rightarrow xyz_x = -y^2z_y \quad \Rightarrow y\left(xz_x + yz_y\right) = 0 \quad \Rightarrow \left(xz_x + yz_y\right) = 0$$

This is the required partial differential equation.

(d) z = f(x cos a + y sin a – a) Note the change in the problem (1)

Solution: Differentiating (1) partially w.r.t x, we get:

$$z_x = \cos a\, f` \quad \Rightarrow f` = z_x/(\cos a) \qquad (2)$$

Now differentiating (1) partially w.r.t y, we get:

$$z_y = \sin a\, f` \quad \Rightarrow f` = z_y/\sin a \qquad (3)$$

Equating (2) and (3), we get: $\dfrac{z_x}{\cos a} = \dfrac{z_y}{\sin a} \ \Rightarrow \left(\sin a\ z_x - \cos a\ z_y\right) = 0$

This is the required partial differential equation.

3. Classify the following partial differential equations

(a) $u_{xx} + u_{xy} + u_{yy} = 0$

Solution: We know that the most general form of second order partial differential equation is: $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = 0$ (1)

Here A, B and C are linear functions of independent variable x and y or constants. This equation is:

Elliptic if $B^2 - AC < 0$: Parabolic if $B^2 - AC = 0$: Hyperbolic if $B^2 - AC > 0$.

Comparing the given equation $u_{xx} + u_{xy} + u_{yy} = 0$ with (1), we see that: A = 1, B = 1 and C = 1. Now, $B^2 - AC = 1 - 1 = 0$. Thus, equation (1) is parabolic.

(b) $xu_{xx} + y\, u_{xy} + u_{yy} = 0$

Solution: Comparing the given equation with (1), we see that: A = x, B = y and C = 1. Now, $B^2 - AC = y^2 - x$. If $y^2 - x = 0$, then (1) is parabolic, if $y^2 - x < 0$ then (1) is elliptic and if $y^2 - x > 0$ then (1) is hyperbolic differential equation.

(c) $-u_{xx} + x^2 u_{yy} + u = 0$

Solution: Comparing the given equation with (1), we see that: A = -1, B = 0 and C = x^2. Now, $B^2 - AC = 0 - (-1)\, x^2 = x^2$ which is positive for all x. Hence, equation (1) is hyperbolic differential equation.

(d) $xu_{xx} + 2u_{xy} + yu_{yy} + u_x = 0$

Solution: Comparing the given equation with (1), we see that: A = x, B = 2 and C = y. Now, $B^2 - AC = 4 - xy$. If $4 - xy = 0$, then (1) is parabolic, if $4 - xy < 0$ then (1) is elliptic and if $4 - xy > 0$ then (1) is hyperbolic differential equation.

(e) $z_{xx} + z_y = 0$

Solution: Comparing the given equation with (1), we see that: A = 1, B = 0 and C = 0. Now, $B^2 - AC = 0 - 0 = 0$. Thus, (1) is parabolic differential equation.

(f) $a^2 z_{xx} = z_y$

Solution: Comparing the given equation with (1), we see that: A = a^2, B = 0 and C = 0. Now, $B^2 - AC = 0 - 0$. Thus, (1) is parabolic differential equation.

4. Solve the following partial differential equations by direct method

(a) $z_{xx} = xy$

Solution: Integrating w.r.t x keeping y constant, we get

$z_x = y\int x\,dx + f(y) = \frac{x^2y}{2} + f(y)$ where, f(y) is constant of integration. Integrating once again w.r.t x keeping y constant, we obtain:

$z = \frac{y}{2}\int x^2\,dx + f(y)\int 1.dx + g(y) = \frac{x^3y}{4} + xf(y) + g(y)$. Here, g(y) is also a constant of integration.

(b) $z_{xxy} = -18xy^2 - \sin(2x - y)$

Solution: Integrating w.r.t x keeping y constant, we get

$$z_{xy} = -18y^2\int x\,dx - \int \sin(2x-y)\,dx + f(y) = -9x^2y^2 + \frac{\cos(2x-y)}{2} + f(y)$$

Here, f(y) is constant of integration. Integrating once again w.r.t x keeping y constant, we obtain:

$z_y = -3x^3y^2 + \frac{\sin(2x-y)}{4} + xf(y) + g(y)$. Here, g(y) is also a constant of integration.

Finally, integrating w.r.t y keeping x constant, we get

$$z = -x^3y^3 + \frac{\cos(2x-y)}{4} + x\int f(y)\,dy + \int g(y)\,dy$$

$$\Rightarrow z = -x^3y^3 + \frac{\cos(2x-y)}{4} + xF(y) + G(y) + h(x),\ h(x) \text{ being a constant of integration.}$$

NOTE: If we first integrate w.r.t y and then integrate two times w.r.t x, we would get the same result.

(c) $z_{xy} = e^y \cos x$

Solution: Integrating w.r.t y keeping x constant, we get

$z_x = \cos x \int e^y\, dy + f(x) = \cos x e^y + f(x)$. Here, f(x) is constant of integration.

Integrating now w.r.t x keeping y constant, we obtain:

$z = \sin x\, e^y + \int f(x)\, dx + g(y) \Rightarrow z = e^y \sin x + F(x) + g(y)$ Here, g(y) is also a constant of integration and $\int f(x)\, dx = F(x)$.

(d) $z_{xy} = y/x + 2$

Solution: Integrating w.r.t y keeping x constant, we get

$z_x = \frac{1}{x}\int y\,dy + 2\int 1 dy + f(x) = \frac{y^2}{2x} + 2y + f(x)$. Here, f(x) is constant of integration.

Integrating now w.r.t x keeping y constant, we obtain:

$z = \frac{y^2}{2}\int \frac{1}{x}\,dx + 2y\int 1.dx + \int f(x)dx + g(y) \Rightarrow z = \frac{y^2 \ln x}{2} + 2xy + F(x) + g(y)$.

Here, g(y) is also a constant of integration.

(e) $z_{xx} = a^2 z$ given that $z_x = a \sin y$ and $z_y = 0$ when $x = 0$

Solution: Using D-operator $D = \partial/\partial x$, given equation becomes

$(D^2 - a^2)\, z = 0$. Auxiliary equation is $m^2 - a^2 = 0 \Rightarrow m = \pm a$. Thus,

$$z = e^{ax} f(y) + e^{-ax} g(y) \qquad (1)$$

Now, $z_x = a e^{ax} f(y) - a e^{-ax} g(y)$. Putting $x = 0$ and $z_x = a \sin y$, we get

$a f(y) - a g(y) = a \sin y \Rightarrow f(y) - g(y) = \sin y \quad \Rightarrow f`(y) - g`(y) = \cos y \qquad$ (i)

Also $z_y = e^{ax} f`(y) + e^{-ax} g`(y)$. Putting $x = 0$ and $z_y = 0$, we get

$$f`(y) + g`(y) = 0 \qquad \text{(ii)}$$

Adding (i) and (ii), we get: $2f`(y) = \cos y \Rightarrow f`(y) = \cos y / 2$. Integrating,

$f(y) = \frac{\sin y}{2} \Rightarrow g(y) = f(y) - \sin y = \frac{\sin y}{2} - \sin y = -\frac{\sin y}{2}$. Thus, equation (1) becomes

$z = \frac{\sin y}{2}\left(e^{ax} - e^{-ax}\right) = \sin y\left(\frac{e^{ax} - e^{-ax}}{2}\right) = \sin y \sinh ax.$

This is the particular solution of given equation.

(f) $z_{yy} = z$ subject to conditions $z = e^x$ and $z_y = e^{-x}$ when $y = 0$

Solution: Using D-operator $D = \partial/\partial y$, given equation becomes

$(D^2 - 1)\, z = 0$. Auxiliary equation is $m^2 - 1 = 0 \Rightarrow m = \pm 1$. Thus,

$$z = e^y f(x) + e^{-y} g(x) \qquad (1)$$

Putting $y = 0$ and $z = e^x$, we get

$$f(x) + g(x) = e^x \qquad \text{(i)}$$

Also $z_y = e^y f(x) - e^{-y} g(x)$. Putting $y = 0$ and $z_y = e^{-x}$, we get

$$f(x) - g(x) = e^{-x} \qquad \text{(ii)}$$

Adding (i) and (ii), we get: $2f(x) = e^x + e^{-x} \Rightarrow f(x) = \left(e^x + e^{-x}\right)/2 = \cosh x$

Subtracting (ii) from (i), we get: $2g(x) = e^x - e^{-x} \Rightarrow g(x) = \left(e^x - e^{-x}\right)/2 = \sinh x$

Thus, (1) becomes: $z = e^{y}\cosh x + e^{-y}\sinh x$. This is a particular solution.

(g) $z_{xy} = \sin x \sin y$ subject to $z_y = -2\sin y$ and $z = \cos y$ when $x = 0$.

Solution: Integrating w.r.t x keeping y constant, we get

$$z_y = \sin y\int \sin x\,dx + f(y) = -\sin y\cos x + f(y)$$

Putting $z_y = -\sin y$ and $x = 0: -\sin y = -\sin y + f(y) \Rightarrow f(y) = 0.$

Thus, $z_y = -\sin y\cos x$. Now integrating w.r.t y keeping x constant .

$z = \cos x\,\cos y + g(x)$. Putting $z = \cos y,\ x = 0$, we get, $g(0) = 0 \Rightarrow g(x) = x\,\varphi(x)$

Thus, $z = \cos x\,\cos y + x\,\varphi(x)$. NOTE the change in the problem.

5. Solve the following partial differential equations by the method of separable variables.

(a) $z_x y^3 + z_y x^2 = 0$

Solution: Putting $z = X(x)\,Y(y) \Rightarrow z_x = yX`$ and $z_y = xY`$.

Thus, given equation becomes: $YX`y^3 + XY`x^2 = 0 \Rightarrow YX`y^3 = -XY`x^2$

$$\Rightarrow \frac{X`}{x^2X} = -\frac{Y`}{y^3Y} = k. \text{ Thus, } \frac{X`}{x^2X} = k \text{ and } \frac{Y`}{y^3Y} = -k$$

$$\Rightarrow \frac{X`}{X} = kx^2 \text{ and } \frac{Y`}{Y} = -ky^3. \text{ Integrating :}$$

$$\ln X = k\frac{x^3}{3} + c_1 = \frac{kx^3 + 3c_1}{3} \quad \text{and} \quad \ln Y = -k\frac{y^4}{4} + c_2 = \frac{-ky^4 + 4c_2}{4}$$

$$\Rightarrow X = e^{\frac{kx^3}{3}+c_1} = C_1e^{kx^3/3} \text{ and } Y = e^{\frac{-ky^4}{4}+c_2} = C_2e^{-ky^4/4}, \left[C_1 = e^{c_1} \text{ and } C_2 = e^{c_2}\right]$$

$$\text{Thus, } z = XY = C_1C_2\,e^{kx^3/3}e^{ky^3/4} = Ce^{\frac{kx^3}{3}+\frac{ky^4}{4}} = Ce^{k\left(\frac{4x^3+3y^4}{12}\right)}, \text{ where } C = C_1C_2$$

(b) $4z_x + z_y = 0,\ z(0, y) = 3e^{-y}$

Solution: Putting $z = X(x)\,Y(y) \Rightarrow z_x = yX`$ and $z_y = xY`$.

Thus, given equation becomes: $4YX` + XY` = 0 \Rightarrow 4YX` = -XY`$

$$\Rightarrow \frac{4X`}{X} = -\frac{Y`}{Y} = k. \text{ Thus, } \frac{4X`}{X} = k \text{ and } \frac{Y`}{Y} = -k. \text{ Integrating :}$$

$4\ln X = kx + c_1$ and $\ln Y = -ky + c_2$

$$\Rightarrow X = e^{\frac{kx+c_1}{4}} = C_1e^{kx/4} \text{ and } Y = e^{-ky+c_2} = C_2e^{-ky}, \left[C_1 = e^{c_1/4} \text{ and } C_2 = e^{c_2}\right]$$

$$\text{Thus, } z = XY = C_1C_2\,e^{kx/4}e^{-ky} = Ce^{k\left(\frac{x-4y}{4}\right)}, \text{ where } C = C_1C_2.$$

Putting $x = 0$ and $z = 3e^{-y}$, we get: $3e^{-y} = Ce^{-ky} \Rightarrow C = 3$ and $k = 1$.

Thus, $z = 3e^{\frac{x-4y}{4}}$ is a particular solution.

Note the change in the problem.

(c) $3z_x + 2z_y = 0$ given that $z(x, 0) = 3e^{-x}$.

Solution: Putting $z = X(x)\,Y(y) \Rightarrow z_x = YX`$ and $z_y = XY`$.

Thus, given equation becomes: $3YX` + 2XY` = 0$ or $3YX` = -2XY`$

$\Rightarrow \dfrac{3X'}{X} = -\dfrac{2Y'}{Y} = k.$ Thus, $\dfrac{3X'}{X} = k$ and $\dfrac{2Y'}{Y} = -k.$ Integrating :

$3\ln X = kx + c_1$ and $2\ln Y = -ky + c_2$

$\Rightarrow X = e^{\frac{kx+c_1}{3}} = C_1 e^{kx/3}$ and $Y = e^{\frac{-ky+c_2}{2}} = C_2 e^{-ky/2}, \left[C_1 = e^{c_1/3} \text{ and } C_2 = e^{c_2/2}\right]$

Thus, $z = XY = C_1C_2\, e^{kx/3}e^{-ky/2} = C e^{k\left(\frac{2x-3y}{6}\right)}$, where $C = C_1C_2$.

Putting $y = 0$ and $z = 3e^{-x}$, we get : $3e^{-x} = Ce^{kx/3} \Rightarrow C = 3$ and $k = -3$.

Thus, $z = 3e^{-(2x-3y)}$ is a particular solution.

(d) $u_x + u = u_t$ if $u = 4e^{-3x}$ when $t = 0$.

Solution: Putting $u = X(x)\,Y(y) \Rightarrow u_x = YX'$ and $u_y = XY'$.

Thus, given equation becomes : $3YX' + 2XY' = 0 \Rightarrow 3\,YX' = -2\,XY'$

$\Rightarrow \dfrac{3X'}{X} = -\dfrac{2Y'}{Y} = k.$ Thus, $\dfrac{3X'}{X} = k$ and $\dfrac{2Y'}{Y} = -k.$ Integrating :

$3\ln X = kx + c_1$ and $2\ln Y = -ky + c_2$

$\Rightarrow X = e^{\frac{kx+c_1}{3}} = C_1 e^{kx/3}$ and $Y = e^{\frac{-ky+c_2}{2}} = C_2 e^{-ky/2}, \left[C_1 = e^{c_1/3} \text{ and } C_2 = e^{c_2/2}\right]$

Thus, $z = XY = C_1C_2\, e^{kx/3}e^{-ky/2} = C e^{k\left(\frac{2x-3y}{6}\right)}$, where $C = C_1C_2$.

Putting $y = 0$ and $z = 3e^{-x}$, we get : $3e^{-x} = Ce^{kx/3} \Rightarrow C = 3$ and $k = -3$.

Thus, $z = 3e^{-(2x-3y)}$ is a particular solution.

(e) $u_x + u_y = 2(x + y)u$

Solution: Putting $u = X(x)\,Y(y) \Rightarrow u_x = YX'$ and $u_y = XY'$.

Thus, given equation becomes : $YX' + XY' = 2xXY + 2yXY$

$XY' - 2yXY = 2xXY - YX' \Rightarrow X(Y' - 2yY) = Y(2xX - X')$. Separating the variables,

$\dfrac{(Y'-2yY)}{Y} = -\dfrac{(X'-2xX)}{X} = k \Rightarrow \dfrac{(Y'-2yY)}{Y} = k$ and $\dfrac{(X'-2xX)}{X} = -k.$

Let us consider $\dfrac{(Y'-2yY)}{Y} = k \Rightarrow Y' - 2yY = kY \Rightarrow Y' = Y(2y + k)$

$\Rightarrow \dfrac{dY}{Y} = (2y + k)\,dy.$ Integrating : $\ln Y = \left(y^2 + ky\right) + c_1 \Rightarrow Y = e^{\left(y^2+ky\right)+c_1} = C_1 e^{y(y+k)}$

Let us consider $\dfrac{(X'-2xX)}{X} = -k \Rightarrow X' - 2xX = -kX \Rightarrow X' = X(2x - k)$

$\Rightarrow \dfrac{dX}{X} = (2x - k)\,dx.$ Integrating : $\ln X = \left(x^2 - kx\right) + c_2 \Rightarrow X = e^{\left(x^2-kx\right)+c_1} = C_2 e^{x(x-k)}$

Thus, $u = XY = C_1C_2\, e^{y(y+k)}e^{x(x-k)}$, where $C = C_1C_2$.

This is the general solution.

(f) $u_t = 4u_{xx}$

Solution: Putting $u = X(x)\,T(t) \Rightarrow u_{xx} = TX''$ and $u_t = XT'$.

Thus, given equation becomes: $XT` = 4TX``$. Separating the variables,

$\frac{T`}{T} = 4\frac{X``}{X} = k \Rightarrow \frac{T`}{T} = k$ and $\frac{4X``}{X} = k$. Let us consider,

$\frac{T`}{T} = k \Rightarrow T` = kT \Rightarrow \frac{dT}{T} = k\,dt$. Integrating: $\ln T = kt + c_1 \Rightarrow T = e^{kt+c_1} = C_1e^{kt}$.

Now, consider $\frac{4X``}{X} = k \Rightarrow 4X`` = kX \Rightarrow 4X`` - kX = 0 \Rightarrow \left(4D^2 - k\right)X = 0$

The auxiliary equation is $4m^2 - k = 0 \Rightarrow m = \pm\sqrt{k}/2$. Thus, $X = C_2e^{\left(\sqrt{k}/2\right)x} + C_3e^{\left(-\sqrt{k}/2\right)x}$

Hence, $u = XT = C_1e^{kt}\left(C_2e^{\left(\sqrt{k}/2\right)x} + C_3e^{\left(-\sqrt{k}/2\right)x}\right)$ is the general solution.

(g) $u_{xx} = 2\,u_t$ [Note the change in the problem]

Solution: Putting $u = X(x)\,T(t) \Rightarrow u_{xx} = TX``$ and $u_t = XT`$.

Thus, given equation becomes: $X`` = 2XT`$. Separating the variables,

$\frac{X``}{X} = 2\frac{T`}{T} = k \Rightarrow \frac{T`}{T} = \frac{k}{2}$ and $\frac{X``}{X} = k$. Let us consider,

$\frac{T`}{T} = \frac{k}{2} \Rightarrow \Rightarrow \frac{dT}{T} = \frac{k}{2}\,dt$. Integrating: $\ln T = (kt)/2 + c_1 \Rightarrow T = e^{(kt)/2+c_1} = C_1e^{kt/2}$.

Now, consider $\frac{X``}{X} = k \Rightarrow X`` = kX \Rightarrow X`` - kX = 0 \Rightarrow \left(D^2 - k\right)X = 0$

The auxiliary equation is $m^2 - k = 0 \Rightarrow m = \pm\sqrt{k}$. Thus, $X = C_2e^{\sqrt{k}\,x} + C_3e^{-\sqrt{k}\,x}$

Hence, $u = XT = C_1e^{(k/2)t}\left(C_2e^{\sqrt{k}\,x} + C_3e^{\sqrt{k}\,x}\right)$ is the general solution.

(h) $u_{xx} = u_t$

Solution: Putting $u = X(x)\,T(t) \Rightarrow u_{xx} = TX``$ and $u_t = XT`$.

Thus, given equation becomes: $X`` = XT`$. Separating the variables,

$\frac{X``}{X} = \frac{T`}{T} = k \Rightarrow \frac{T`}{T} = k$ and $\frac{X``}{X} = k$. Let us consider,

$\frac{T`}{T} = k \Rightarrow \frac{dT}{T} = k\,dt$. Integrating: $\ln T = kt + c_1 \Rightarrow T = e^{kt+c_1} = C_1e^{kt}$.

Now, consider $\frac{X``}{X} = k \Rightarrow X`` = kX \Rightarrow X`` - kX = 0 \Rightarrow \left(D^2 - k\right)X = 0$

The auxiliary equation is $m^2 - k = 0 \Rightarrow m = \pm\sqrt{k}$. Thus, $X = C_2e^{\sqrt{k}\,x} + C_3e^{-\sqrt{k}\,x}$

Hence, $u = XT = C_1e^{kt}\left(C_2e^{\sqrt{k}\,x} + C_3e^{\sqrt{k}\,x}\right)$ is the general solution.

(i) $u_t = u_{xx}$ if $u(x, 0) = \sin \pi x$

Solution: Putting $u = X(x)\,T(t) \Rightarrow u_{xx} = TX``$ and $u_t = XT`$.

Thus, given equation becomes: $X`` = XT`$. Separating the variables,

$\frac{X``}{X} = \frac{T`}{T} = k \Rightarrow \frac{T`}{T} = k$ and $\frac{X``}{X} = k$. Let us consider,

Now, conbsider $\frac{T`}{T} = k \Rightarrow \frac{dT}{T} = k\,dt$. Integrating: $\ln T = kt + c_1 \Rightarrow T = e^{kt+c_1} = C_1e^{kt}$

Now, consider $X``/X = k \Rightarrow X`` = kX \Rightarrow X`` - kX = 0 \Rightarrow (D^2 - k)X = 0$

The auxiliary equation is $m^2 - k = 0 \Rightarrow m^2 + p^2 = 0 \Rightarrow m = \pm\, ip.$ $\left(\text{where } k = -p^2\right)$

Thus, $X = C\cos px + C_3 \sin px$

NOTE: The initial condition $u(x,\ 0) = \sin \pi x$ suggests that k must be negative so we have assumed $k = -p^2$. Therefore, $u = XT = C_1 e^{-p^2 t}\left(C_2 \cos px + C_3 \sin px\right)$ is the general solution. Now substituting $t = 0$ and $u = \sin \pi x$, we get:

$\sin \pi x = C_2 \cos px + C_3 \sin px$. Comparing two sides, we observe that $C_2 = 0$ and $p = \pi$.

Thus, particular solution is: $u = XT = C_1 e^{-\pi^2 t}\left(C_3 \sin \pi x\right) = Ce^{-p^2 t}\sin \pi x.$

(j) $z_{xx} - 2z_x - z_y = 0$

Solution: Putting $u = X(x)\,Y(y) \Rightarrow u_x = YX`$, $u_{xx} = YX``$ and $u_y = XY`$.

Thus, given equation becomes: $YX`` - 2YX` - XY` = 0 \Rightarrow Y(X`` - 2X`) = XY`$

Separating the variables, we get: $\dfrac{Y`}{Y} = -\dfrac{(X``-2X`)}{X} = k \Rightarrow \dfrac{Y`}{Y} = k$ and $\dfrac{(X``-2X`)}{X} = -k.$

Let us consider $\dfrac{Y`}{Y} = k \Rightarrow \dfrac{dY}{Y} = k\,dy.$ Integrating: $\ln Y = ky + c_1 \Rightarrow Y = e^{ky+c_1} = C_1 e^{ky}$

Let us consider $\dfrac{(X``-2X`)}{X} = -k \Rightarrow X`` - 2X` = -kX \Rightarrow X`` - 2X` + kX = 0$

Auxiliary equation is $m^2 - 2m + k = 0 \Rightarrow m = \dfrac{2 \pm \sqrt{4-4k}}{2} = 1 \pm \sqrt{1-k}$

$\therefore\ X = C_2 e^{\left(1+\sqrt{1-k}\right)x} + C_3 e^{\left(1-\sqrt{1-k}\right)x}$. Thus,

$u = XY = C_1\, e^{ky}.\left(C_2 e^{\left(1+\sqrt{1-k}\right)x} + C_3 e^{\left(1-\sqrt{1-k}\right)x}\right)$ is the general solution.

(k) $x^2 u_{xx} + 3y^2 u_y = 0$

Solution: Putting $u = X(x)\,Y(y) \Rightarrow u_x = YX`$, $u_{xx} = YX``$ and $u_y = XY`$

Thus, given equation becomes: $x^2 YX`` + 3y^2 XY` = 0 \Rightarrow x^2 X`` = -3y^2 XY`.$

Separating the variables, we get:

$\dfrac{x^2 X``}{X} = -3y^2 \dfrac{Y`}{Y} = k \Rightarrow -3y^2 \dfrac{Y`}{Y} = k$ and $\dfrac{x^2 X``}{X} = k.$

Let us consider $x^2 X`` - kX = 0 \Rightarrow \left(x^2 D^2 - k\right)X = 0.$ This is Cauchy – Euler differential equation. thus, putting $X = e^t$ and $t = \ln X$, we get:

$x^2 D^2 = \Delta(\Delta - 1)$. Hence, above equation becomes:

$\left(\Delta^2 - \Delta - k\right)X = 0.$ The auxiliary equation is $m^2 - m - k = 0 \Rightarrow m = \dfrac{1 \pm \sqrt{1+4k}}{2}$

$\therefore\ X = C_1 e^{\left(\frac{\left(1+\sqrt{1+4k}\right)}{2}\right)t} + C_2 e^{\left(\frac{\left(1-\sqrt{1+4k}\right)}{2}\right)t} = C_1 x^{\left(\frac{\left(1+\sqrt{1+4k}\right)}{2}\right)} + C_2 x^{\left(\frac{\left(1-\sqrt{1+4k}\right)}{2}\right)}$ where, $e^t = x.$

Now consider, $-3y^2 \dfrac{Y`}{Y} = k \Rightarrow y^2 Y`` + \dfrac{k}{3} Y = 0.$ This is Cauchy – Euler differential equation.

Thus, putting $Y = e^t$ and $t = \ln Y$, we get: $y^2 D^2 = \Delta(\Delta - 1)$

Hence, above equation becomes:

$\left(\Delta^2-\Delta+\frac{k}{3}\right)Y=0$. The auxiliary equation is $m^2-m-\frac{k}{3}=0 \Rightarrow 3m^2-3m-k=0$

$$\Rightarrow m=\frac{3\pm\sqrt{9+12k}}{6}$$

$$\therefore\ Y=C_3e^{\left(\frac{\left(3+\sqrt{9+12k}\right)}{6}\right)t}+C_4e^{\left(\frac{\left(3-\sqrt{9+12k}\right)}{6}\right)t}=C_3y^{\left(\frac{\left(3+\sqrt{9+12k}\right)}{6}\right)}+C_4y^{\left(\frac{\left(3-\sqrt{9+12k}\right)}{6}\right)}$$

where, $e^t=y$. Thus,

$$u=XY=\left(C_1x^{\left(\frac{\left(1+\sqrt{1+4k}\right)}{2}\right)}+C_2x^{\left(\frac{\left(1-\sqrt{1+4k}\right)}{2}\right)}\right)\left(C_3y^{\left(\frac{\left(3+\sqrt{9+12k}\right)}{6}\right)}+C_4y^{\left(\frac{\left(3-\sqrt{9+12k}\right)}{6}\right)}\right)$$

is the general solution.

(*l*) $u_{tt}=u_{xx}$ Note the change in the problem

Solution: Putting u = X(x) Y(y) $\Rightarrow$ $u_{xx}=YX``$ and $u_{yy}=XY``$

Thus, given equation becomes: $x^2X``=XY``=0$. Separating the variables, we get:

$$\frac{X``}{X}=\frac{Y``}{Y}=k \Rightarrow \frac{Y``}{Y}=k \text{ and } \frac{X``}{X}=k.$$

Let us consider $X``-kX=0\Rightarrow\left(D^2-k\right)X=0$. The auxiliary equation is $m^2-k=0 \Rightarrow m=\pm\sqrt{k}$

$\therefore\ X=C_1e^{\sqrt{k}\,x}+C_2e^{-\sqrt{k}\,x}$. Similarly, $Y=C_3e^{\sqrt{k}\,y}+C_4e^{-\sqrt{k}\,y}$.

Thus, $u=XY=\left(C_1e^{\sqrt{k}\,x}+C_2e^{-\sqrt{k}\,x}\right)\left(C_3e^{\sqrt{k}\,y}+C_4e^{-\sqrt{k}\,y}\right)$ is the general solution.

Chapter 7

SOLUTIONS IN SERIES

WORKSHEET 07

1. Solve the following differential equations using solution in series technique.

(a) $y`` + x^2 y = 0$

Solution:

Method-I Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots + a_r x^r + \ldots$ (1)

$\Rightarrow y` = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots + r a_r x^{r-1} + \ldots$ and

$y`` = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + 5.4a_5 x^4 + \ldots + r(r-1)\, a_r x^{r-2} + \ldots .$ Thus, (1) becomes

$$2a_2 + 3.2a_3 x + 4.3a_4 x^2 + 5.4a_5 x^3 + \ldots + r(r-1)a_r x^{r-2} + \ldots$$
$$+ \quad a_0 x^2 + \quad a_1 x^3 + \ldots\ldots + a_{r-4} x^{r-2} + \ldots = 0$$

$$\Rightarrow 2a_2 + 6a_3 x + (4.3a_4 + a_0)x^2 + (5.4a_5 + a_1)x^3 + \ldots + [r(r-1)a_r + a_{r-4}]x^{r-2} + \ldots = 0$$

Equating the coeffcients to zeros, we get :

$2a_2 = 0 \qquad \Rightarrow a_2 = 0 \qquad\qquad 6a_3 = 0 \qquad \Rightarrow a_3 = 0$

$4.3a_4 + a_0 = 0 \quad \Rightarrow a_4 = -\dfrac{1}{4.3}a_0 \qquad\qquad 5.4a_5 + a_1 = 0 \quad \Rightarrow a_5 = -\dfrac{1}{5.4}a_1$

$$r(r-1)a_r + a_{r-4} = 0 \qquad \Rightarrow a_r = -\frac{1}{r(r-1)}a_{r-4} \qquad (2)$$

Equation (2) is "Recurrence Relation".

Putting $\quad r = 6,\ a_6 = -\dfrac{1}{6.5}a_2 = 0\ \ (\because a_2 = 0)$

Putting $\quad r = 7,\ a_7 = -\dfrac{1}{7.6}a_3 = 0\ \ (\because a_2 = 0)$

Putting $\quad r = 8,\ a_8 = -\dfrac{1}{8.7}a_4 = +\dfrac{1}{8.7.4.3}a_0.$

Putting $\quad r = 9,\ a_9 = -\dfrac{1}{9.8}a_5 = +\dfrac{1}{9.8.5.4}a_1$

Thus (1) becomes : $y = a_0\left(-\dfrac{1}{4.3}x^4 + \dfrac{1}{8.7.4.3}x^8 - \ldots\right) + a_1\left(-\dfrac{1}{5.4}x^5 + \dfrac{1}{9.8.5.4}x^8 - \ldots\right)$

This is the solution in the form of series of given differential equation.

Method-II: Let the solution be

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1},\ \ y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting these in given equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

The next step is to combine every term into a single series. To do this we require that both series should start at the same point and that the exponent on the x be the same in both series. Now, in the first series the exponent of x is (n-2) hence reduce n by (n – 4) in the second series, we get

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_{n-4} x^{n-2} = 0 \quad \Rightarrow \sum_{n=0}^{\infty}\left[n(n-1)a_n + a_{n-4}\right]x^{n-2} = 0$$

Equating the coefficient to zero, we get:

$$n(n-1)a_n + a_{n-4} = 0 \quad \Rightarrow a_n = -\frac{1}{n(n-1)}a_{n-4},\ n = 4, 5, 6, \ldots$$

This is the recurrence relation which is same as shown in Method-I.
Rest of the process is same. It may be noted that coefficients a_4, a_5, a_6, … in terms of a0 and a_1 can be computed from the above recurrence relation. The coefficients a_2 and a_3 will therefore be considered as zero. This is shown in Method-I.

(b) y`` + x y` + y = 0

Solution: Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots + a_r x^r + \ldots$ (1)

$\Rightarrow y` = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots + r a_r x^{r-1} + \ldots$ and

$y`` = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + 65.4a_5 x^4 + \ldots + r(r-1)\, a_r x^{r-2} + \ldots.$ Thus, (1) becomes

$$\begin{aligned} &2a_2 + 3.2a_3 x + 4.3a_4 x^2 + 5.4a_5 x^3 + \ldots + r(r-1)a_r x^{r-2} + \ldots \\ &\quad + a_1 x + 2a_2 x^2 + 3a_3 x^3 + \ldots + (r-2)a_{r-2} x^{r-2} + \ldots \\ &+ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_{r-2} x^{r-2} + \ldots = 0. \text{ This gives:} \end{aligned}$$

$$(2a_2 + a_0) + (3.2a_3 + 2a_1)x + (4.3a_4 + 3a_2)x^2 + \ldots + \left[r(r-1)a_r + (r-1)a_{r-2}\right]x^{r-2} + \ldots = 0$$

Equating the coefficients to zero:

$$2a_2 + a_0 \ \Rightarrow\ a_2 = -\frac{1}{2}a_0 = -\frac{1}{2}a_0; \qquad 3.2a_3 + 2a_1 = 0 \Rightarrow a_3 = -\frac{2}{3.2}a_1 = -\frac{1}{3}a_1$$

$$r(r-1)a_r + (r-1)a_{r-2} = 0 \quad \Rightarrow a_r - \frac{r-1}{r(r-1)}a_{r-2} = -\frac{1}{r}a_{r-2} \qquad (2)$$

Equation (2) is RR and is used to compute other coefficients.

Putting $r = 4,\ a_4 = -\frac{1}{4}a_2 = \left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)a_0 = \frac{1}{2.4}a_0$

Putting $r = 5,\ a_5 = -\frac{1}{5}a_3 = \left(-\frac{1}{5}\right)\left(-\frac{1}{3}\right)a_1 = \frac{1}{3.5}a_1$

Thus, $y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{2.4}x^2 - \frac{1}{2.4.6}x^4 + \ldots\right) + a_1\left(x - \frac{1}{3}x^3 + \frac{1}{3.5}x^5 - \frac{1}{3.5.7}x^7 + \ldots\right)$

This is the solution in series form of given differential equation.

(c) y`` + xy = 0 Notice the change in the problem.

Solution: Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots + a_r x^r + \ldots$ (1)

$\Rightarrow y` = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots + r a_r x^{r-1} + \ldots$ and

$y`` = 2a_2 + 3.2a_3x + 4.3a_4x^2 + 65.4a_5x^4 + ... + r(r-1)\,a_r\,x^{r-2} + ...$ Thus, (1) becomes

$$2a_2 + 3.2a_3x + 4.3a_4x^2 + 5.4a_5x^3 + ... + r(r-1)a_rx^{r-2} + ...$$
$$+ a_0x \quad + a_1x^2 + a_2x^3 + ... \qquad + a_{r-3}x^{r-2} + ... = 0$$

$$\Rightarrow 2a_2 + (3.2a_3 + a_0)x + (4.3a_4 + a_1)x^2 + ... + [r(r-1)a_r + a_{r-3}]x^{r-2} + ... = 0$$

Equating the coeffcients to zeros, we get:

$$2a_2 = 0 \qquad \Rightarrow a_2 = 0; \qquad 3.2a_3 + a_0 = 0 \qquad \Rightarrow a_3 = -\frac{1}{2.3}a_0$$

$$4.3a_4 + a_1 = 0 \quad \Rightarrow a_4 = -\frac{1}{4.3}a_1 = -\frac{2}{4!}a_1; \quad r(r-1)a_r + a_{r-3} = 0 \quad \Rightarrow a_r - \frac{1}{r(r-1)}a_{r-3} \quad (2)$$

Equation (2) is "Recurrence Relation".

Putting $r = 5,\ a_6 = -\frac{1}{5.4}a_2 = 0\ (\because a_2 = 0)$

Putting $r = 6,\ a_7 = -\frac{1}{6.5}a_3 = -\frac{1}{6.5}\left(-\frac{1}{2.3}a_0\right) = \frac{1}{2.3.5.6}a_0 = \frac{4}{6!}a_0$

Putting $r = 7,\ a_8 = -\frac{1}{7.6}a_4 = -\frac{1}{7.6}\left(-\frac{1}{3.4}a_1\right) = \frac{2.5}{7.6.5.4.3.2}a_1 = \frac{10}{7!}a_1$

Putting $r = 8,\ a_9 = -\frac{1}{8.7}a_6 = -\frac{1}{8.7}(0) = 0$

Thus (1) becomes: $y = a_0\left(1 - \frac{1}{3!}x^3 + \frac{4}{6!}x^7 - ...\right) + a_1\left(x - \frac{2}{4!}x^4 + \frac{10}{7!}x^8 - ...\right)$

This is the solution in the form of series of given differential equation.

(d) $y`` + xy` + x^2y = 0$

Solution: Let $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + ... + a_rx^r + ...$ (1)

$\Rightarrow y` = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + ... + r\,a_r\,x^{r-1} + ...$ and

$y`` = 2a_2 + 3.2a_3x + 4.3a_4x^2 + 65.4a_5x^4 + ... + r(r-1)\,a_r\,x^{r-2} + ...$ Thus, (1) becomes

$$2a_2 + 3.2a_3x + 4.3a_4x^2 + 5.4a_5x^3 + ... + r(r-1)a_rx^{r-2} + ...$$
$$+ a_1x \quad + 2a_2x^2 \quad + 3a_3x^3 + ... \quad + (r-2)a_{r-2}x^{r-2} + ...$$
$$+ a_0x^2 \qquad + a_1x^3 \quad + ... \quad + a_{r-4}x^{r-2} + ... = 0.$$ This gives:

$$2a_2 + (2.3a_3 + a_1)x + (4.3a_4 + 2a_2 + a_0)x^2 + ... + [r(r-1)a_r + (r-2)a_{r-2} + a_{r-4}]x^{r-2} + ... = 0$$

Equating the coefficients to zero:

$$2a_2 = 0 \Rightarrow a_2 = 0; \qquad 2.3a_3 + a_1 = 0 \Rightarrow a_3 = -\frac{1}{3.2}a_1 = -\frac{1}{6}a_1;$$

$$4.3a_4 + 2a_2 + a_0 = 0 \Rightarrow a_4 = -\frac{1}{4.3}a_0 = -\frac{1}{12}a_0\ (\because a_2 = 0);$$

$$r(r-1)a_r + (r-2)a_{r-2} + a_{r-4} = 0 \Rightarrow a_r = -\frac{(r-2)a_{r-2} + a_{r-4}}{r(r-1)} \qquad (2)$$

Putting $r = 5,\ a_5 = -\frac{3a_2 + a_1}{5.4} = -\frac{1}{5.4}[0 + a_1] = \frac{1}{20}a_1;$

Putting $r = 6,\ a_6 = -\frac{4a_4 + a_2}{5.4} = -\frac{1}{5.4}\left[4\left(-\frac{1}{12}\right)a_0 + 0\right] = \frac{1}{60}a_0;$

Putting $r = 7,\ a_7 = -\frac{5a_5 + a_3}{7.6} = -\frac{1}{7.6}\left[5\left(\frac{1}{20}\right)a_1 - \frac{1}{6}a_1\right] = -\frac{1}{42}\left[\frac{1}{12}a_1\right] = -\frac{1}{504}a_1;$

Thus, equation becomes : $y = a_0\left(1 - \frac{1}{12}x^4 + \frac{1}{60}x^6 - \ldots\right) + a_1\left(x - \frac{1}{6}x^3 + \frac{1}{20}x^5 - \frac{1}{504}x^7 + \ldots\right)$

This is the series solution of given differential equation.

(e) $(x^2 + 1)\, y`` + xy` - xy = 0$

Solution: Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \ldots + a_r x^r + \ldots$ (1)

$\Rightarrow y` = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \ldots + r a_r x^{r-1} + \ldots$ and

$y`` = 2a_2 + 3.2a_3 x + 4.3a_4 x^2 + 65.4a_5 x^4 + \ldots + r(r-1)\, a_r x^{r-2} + \ldots.$ Thus, (1) becomes

$$\begin{array}{llllll}
 & & 2a_2x^2 & +3.2a_3x^3 & +\ldots+r(r-1)a_rx^r+\ldots & \\
2a_2 & +3.2a_3x & +4.3a_4x^2 & +5.4a_5x^3 & +\ldots+(r+2)(r+1)a_{r+2}x^r+\ldots & \\
 & +a_1x & +2a_2x^2 & +3a_3x^3+\ldots & +ra_rx^r+\ldots & \\
 & -a_0x & -a_1x^2 & -a_2x^3-\ldots. & -a_{r-1}x^r-\ldots=0. &
\end{array}$$

This gives :

$$2a_2 + (2.3a_3 + a_1 - a_0)x + (4.3a_4 + 2a_2 - a_1)x^2 + \ldots$$
$$+\left[r(r-1)a_r + (r+2)(r+1)a_{r-2} + ra_r - a_{r-1}\right]x^r + \ldots = 0$$

Equating the coefficients to zero :

$2a_2 = 0 \Rightarrow a_2 = 0;\qquad 2.3a_3 + a_1 - a_0 = 0 \Rightarrow a_3 = \frac{a_0 - a_1}{6};$

$4.3a_4 + 2a_2 - a_1 = 0 \Rightarrow a_4 = a_0/12\ (\because a_2 = 0);$

$r(r-1)a_r + (r+2)(r+1)a_{r-2} + ra_r - a_{r-1} \Rightarrow \left(r(r-1) + r\right)a_r = a_{r-1} - (r+1)(r+2)a_{r-2}$

$$\Rightarrow a_r = \frac{a_{r-1} - (r+1)(r+2)a_{r-2}}{r^2} \qquad (2)$$

Equation (2) is RR. Thus,

Putting $r = 5,\ a_5 = \frac{a_4 - 6.7a_3}{5^2} = \frac{1}{5^2}\left[\frac{1}{12}a_0 - 42\left(\frac{a_0 - a_1}{6}\right)\right] = \left[\frac{84a_1 - 83a_0}{12(5)^2}\right]$

Putting $r = 6,\ a_6 = \frac{a_5 - 7.8a_4}{6^2} = \frac{1}{6^2}\left[\frac{84a_1 - 83a_0}{300} - 56\left(\frac{1}{12}\right)a_0\right]$

$$= \frac{1}{6^2}\left[\frac{84a_1 - 83a_0 - 1400a_0}{300}\right] = \frac{84a_1 - 1483a_0}{12(5^2)(6^2)}$$

Thus, series solution of given differential equation is :

$$y = a_0 + a_1x + \left(\frac{a_0 - a_1}{6}\right)x^3 + \left(\frac{a_0}{12}\right)x^4 + \left(\frac{84a_1 - 83a_0}{12(5)^2}\right)x^5 + \left(\frac{84a_1 - 1483a_0}{12(5^2)(6^2)}\right)x^6 + \ldots$$

(f) $y`` - y = 0$

Solution: Let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ be the solution in series form.

$$\Rightarrow \qquad y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}\ , \qquad y''(x)=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Substituting these in given equation, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} a_n x^n=0 \Rightarrow \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty} a_n x^n=0$$

Equating the coefficient to zero, we get:

$$(n+2)(n+1)a_{n+2}-a_n=0 \Rightarrow a_{n+2}=\frac{1}{(n+1)(n+2)}a_n,\ n=0,1,2,\dots\ n=0,1,2,\dots$$

This is the recurrence relation. To find a_2, a_3, etc; we:

Put n = 0, we get: $a_2=\frac{a_0}{2.1}$. Put n = 1, we get: $a_3=\frac{a_1}{3.2}$,

Put n = 2, we get: $a_4=\frac{a_2}{4.3}=\frac{a_0}{4.3.2.1}$. Put n = 3, we get: $a_5=\frac{a_3}{5.4}=\frac{a_1}{5.4.3.2}$.

Put n = 4, we get: $a_6=\frac{a_4}{6.5}=\frac{a_0}{6.5.4.3.2.1}$,

Put n = 5, we get: $a_7=\frac{a_5}{7.6}=\frac{a_1}{7.6.5.4.3.2},\dots$

Continuing the process, we get

For $n=2k,\ a_{2k}=\frac{a_0}{(2k)!},\ k=1,2,\dots$ and $n=2k+1,\ a_{2k+1}=\frac{a_1}{(2k+1)!},\ k=1,2,\dots$

$$\text{Thus, } y(x)=\sum_{n=0}^{\infty} a_n x^n=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\dots+a_{2k}x^{2k}+a_{2k+1}x^{2k+1}+\dots$$

$$=a_0+a_1x-\frac{a_0}{2!}x^2-\frac{a_1}{3!}x^3+\frac{a_0}{4!}x^4+\dots+\frac{(-1)^k a_0}{(2k)!}x^{2k}+\frac{(-1)^k a_1}{(2k+1)!}x^{2k+1}+\dots$$

$$\therefore\ y(x)=a_0\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots+\frac{(-1)^k}{(2k)!}x^{2k}+\dots\right\}+a_1\left\{x-\frac{x^3}{3!}+\dots+\frac{(-1)^k}{(2k+1)!}x^{2k+1}+\dots\right\}$$

$$\text{OR}\quad y(x)=a_0\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}+a_1\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

This is the series solution in compact form.

2. Solve the following differential equations in power series by Frobenius method.

(a) $2x^2y'' - xy' + (1-x^2)y = 0$

Solution: Let the solution be $y=\sum_{k=0}^{\infty} a_k x^{m+k}$

$$\Rightarrow \quad \frac{dy}{dx}=\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} \text{ and } \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$2x^2\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}-x\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1}+\left(1-x^2\right)\sum_{k=0}^{\infty} a_k x^{m+k}=0$$

$$\sum_{k=0}^{\infty} 2a_k(m+k)(m+k-1)x^{m+k} - \sum_{k=0}^{\infty} a_k(m+k)x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k} - \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k\{2(m+k)(m+k-1)-(m+k)+1\}x^{m+k} - \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^m in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$$a_0\{2m(m-1)-m+1=0 \Rightarrow a_0 2m^2-2m-m+1=0 \Rightarrow a_0(2m-1)(m-1)=0.$$

Since $a_0 \neq 0$, therefore, (2m -1) (m – 1) = 0 or m = 1 and m = 1/2 are the roots of indicial equation. We see that roots of indicial equation are distinct and do not differ by an integer.

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$a_k\{2(m+k)(m+k-1)-(m+k)+1\}-a_{k-2}=0$$

$$\Rightarrow \quad a_k = \frac{1}{\{[(m+k)(2m+2k-3)]+1\}}a_{k-2}$$

For k = 2: $\quad a_2 = \dfrac{a_0}{(m+2)(2m+1)+1},$

For k = 3: $\quad a_3 = \dfrac{a_1}{(m+3)(2m+3)+1},$

For k = 4: $\quad a_4 = \dfrac{a_2}{(m+4)(2m+5)+1} = \dfrac{a_0}{[(m+2)(2m+1)+1][(m+4)(2m+5)+1]}$

For k = 5: $\quad a_5 = \dfrac{a_3}{(m+5)(2m+7)+1} = \dfrac{a_0}{[(m+3)(2m+3)+1][(m+5)(2m+7)+1]}$

and so on.

Now put m = 1/2, the first value of root from indicial equation:

$a_2 = \dfrac{a_0}{6}$, $a_3 = \dfrac{2a_1}{15}$, $a_4 = \dfrac{a_0}{168}$, $a_5 = \dfrac{a_1}{675}$ and so on.

Hence, $\quad y_1 = x^{1/2}\left[a_0\left(1+\frac{1}{6}x^2+\frac{1}{168}x^4+\ldots\right)+a_1\left(x+\frac{1}{15}x^3+\frac{1}{675}x^5+\ldots\right)\right]$

For m = 1: $a_2 = \dfrac{a_0}{7}, a_3 = \dfrac{a_1}{17}, a_4 = \dfrac{a_0}{252}$, $a_5 = \dfrac{a_1}{935}$ and so on.

Hence, $\quad y_2 = x\left[a_0\left(1+\frac{1}{7}x^2+\frac{1}{252}x^4+\ldots\right)+a_1\left(x+\frac{1}{17}x^3+\frac{1}{935}x^5+\ldots\right)\right]$

Thus the complete solution is: $y = c_1(y)_{m_1=1/2} + c_2(y)_{m_2=1}$.

(b) 4 x y`` +2 y` + y = 0 (1)

Solution: Let the solution be $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

$$\Rightarrow \quad y` = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} \text{ and } y`` = \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$4x\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2} + 2\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} 4a_k(m+k)(m+k-1)x^{m+k-1} + \sum_{k=0}^{\infty} 2a_k(m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k\{4(m+k)(m+k-1)+2(m+k)\}x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$$a_0\{4m(m-1)+2m\} = 0 \Rightarrow a_0(4m^2-2m) = 0 \Rightarrow a_0 2m(2m-1) = 0.$$

Since $a_0 \neq 0$, therefore, 2m(2m -1) = 0 or m = 0 and m = 1/2 are the roots of indicial equation. We see that roots of indicial equation are distinct and do not differ by an integer.

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$a_k\{4(m+k)(m+k-1)+(m+k)\} + a_{k-1} = 0 \quad \Rightarrow \quad a_k = -\frac{1}{(m+k)(4m+4k-3)}a_{k-1}$$

For k = 1: $\quad a_1 = -\dfrac{a_0}{(m+1)(4m+1)}$

For k = 2: $\quad a_2 = -\dfrac{a_1}{(m+2)(2m+5)} = \dfrac{a_0}{(m+1)(m+2)(4m+1)(4m+5)}$

For k = 3:

$$a_3 = -\frac{a_2}{(m+3)(4m+9)} = -\frac{a_0}{(m+1)(m+2)(m+3)(4m+1)(4m+5)(4m+9)}$$

and so on.

Now put m = 0, the first value of root from indicial equation:

$a_1 = -\dfrac{a_0}{1.1}$, $a_2 = \dfrac{a_0}{(1.2)(1.5)}$, $a_3 = -\dfrac{a_0}{(1.2.3)(1.5.9)}$ and so on.

Hence, $y_1 = x^0 a_0\left(1 - \dfrac{1}{1.1}x + \dfrac{1}{(1.2)(1.5)}x^2 - \dfrac{1}{(1.2.3)(1.5.9)}x^3 + \ldots\right)$

$$= a_0\left(1 - \frac{1}{1.1}x + \frac{1}{(1.2)(1.5)}x^2 - \frac{1}{(1.2.3)(1.5.9)}x^3 + \ldots\right)$$

For m = 1/2 : $a_1 = -\dfrac{2}{(3)(3)}a_0$, $a_2 = \dfrac{2.2}{(3.5)(3.7)}a_0$, $a_3 = -\dfrac{2.2.2}{(3.5.7)(3.7.11)}a_0$ and so on.

Hence, $y_2 = a_0\sqrt{x}\left(1 - \dfrac{2}{(3)(3)}x + \dfrac{2^2}{(3.5)(3.7)}x^2 - \dfrac{2^3}{(3.5.7)(3.7.11)}x^3 + \ldots\right)$

Thus the complete solution is: $y = c_1y_1 + c_2y_2$ where y_1 and y_2 are given as above.

(c) x y`` + y` - y = 0 (1)

Solution: Let the solution be $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

$$\Rightarrow \quad y` = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} \text{ and } y`` = \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-1} + \sum_{k=0}^{\infty} a_k (m+k)x^{m+k-1} - \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k \{(m+k)(m+k-1)+(m+k)\} x^{m+k-1} - \sum_{k=0}^{\infty} a_k x^{m+k} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$$a_0\{m(m-1)+m\} = 0 \Rightarrow a_0\left(m^2-m+m\right) = 0 \Rightarrow a_0 m^2 = 0.$$

Since $a_0 \neq 0$, therefore, $m^2 = 0$ or m = 0, 0 are the roots of indicial equation. We see that roots of indicial equation are equal.

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$a_k\{(m+k)(m+k-1)+(m+k)\} - a_{k-1} = 0 \quad \Rightarrow \quad a_k = \frac{1}{(m+k)^2} a_{k-1}$$

For k = 1: $\quad a_1 = \dfrac{a_0}{(m+1)^2}$

For k = 2: $\quad a_2 = \dfrac{a_1}{(m+2)^2} = \dfrac{a_0}{(m+1)^2(m+2)^2}$

For k = 3: $\quad a_3 = \dfrac{a_2}{(m+3)^2} = \dfrac{a_0}{(m+1)^2(m+2)^2(m+3)^2}$

and so on.

Now put m = 0, the first value of root from indicial equation:

$a_1 = \dfrac{a_0}{1^2},\ a_2 = \dfrac{a_0}{1^2.2^2},\ a_3 = \dfrac{a_0}{1^1.2^2.3^2}$ and so on.

Hence, $y_1 = x^0 a_0\left(1+\dfrac{1}{1^2}x+\dfrac{1}{1^2.2^2}x^2+\dfrac{1}{1^2.2^2.3^2}x^3+\ldots\right)$

$$\Rightarrow \quad y_1 = a_0\left(1+\frac{1}{1^2}x+\frac{1}{1^2.2^2}x^2+\frac{1}{1^2.2^2.3^2}x^3+\ldots\right)$$

Since the roots of indicial equation are repeated, the second solution is obtained as follows:

$$y = x^m\left(a_0+a_1x+a_2x^2+a_3x^3+\ldots\right) = a_0x^m\left(1+\frac{1}{1^2}x+\frac{1}{1^2.2^2}x^2+\frac{1}{1^2.2^2.3^2}x^3+\ldots\right)$$

$$\therefore \left(\frac{\partial y}{\partial m}\right)_{m_1} = \left[a_0x^m \ln x\left(1+\frac{1}{1^2}x+\frac{1}{1^2.2^2}x^2+\frac{1}{1^2.2^2.3^2}x^3+\ldots\right)\right]_{m=0}$$

$$= a_0 \ln x\left(1+\frac{1}{1^2}x+\frac{1}{1^2.2^2}x^2+\frac{1}{1^2.2^2.3^2}x^3+\ldots\right)$$

NOTE : If $y = a^x$ then $y` = a^x \ln a$. Similary if $y = x^m f(x)$ then $\left(\dfrac{\partial y}{\partial m}\right) = x^m.\ln x.f(x)$

Here m is variable and x is constant.

Hence the general solution is: $y = c_1(y)_{m_1} + c_2\left(\dfrac{\partial y}{\partial m}\right)_{m_1}$

$$y = A\left(1+\frac{1}{1^2}x+\frac{1}{1^2.2^2}x^2+\frac{1}{1^2.2^2.3^2}x^3+...\right)+B\ln x\left(1+\frac{1}{1^2}x+\frac{1}{1^2.2^2}x^2+\frac{1}{1^2.2^2.3^2}x^3+...\right)$$

Here $A = a_0c_1$ and $B = a_0c_2$.

(d) $x\,y`` + y` + x^2\,y = 0$

Solution: Let the solution be $y=\sum_{k=0}^{\infty}a_k x^{m+k}$

$$\Rightarrow \quad y`=\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k-1} \text{ and } y``=\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}$$

Substituting these values in (1), we get

$$\sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-1}+\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k-1}+\sum_{k=0}^{\infty}a_k x^{m+k+2}=0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty}a_k\left\{\left(m+k\right)\left(m+k-1\right)+\left(m+k\right)\right\}x^{m+k-1}+\sum_{k=0}^{\infty}a_k x^{m+k+2}=0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty}a_k\left\{\left(m+k\right)^2\right\}x^{m+k-1}+\sum_{k=0}^{\infty}a_k x^{m+k+2}=0 \qquad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$$a_0\left\{m\left(m-1\right)+m\right\}=0\Rightarrow a_0\left(m^2-m+m\right)=0\Rightarrow a_0\,m^2=0.$$

Since $a_0\neq 0,$ therefore, $m^2 = 0$ or m = 0, 0 are the roots of indicial equation. We see that roots of indicial equation are equal.

Equating to zero the coefficient of x^{m+k-1}, the recurrence relation is given by

$$a_k\left(m+k\right)^2+a_{k-3}=0 \quad\Rightarrow\quad a_k=-\frac{1}{(m+k)^2}a_{k-3} \qquad (3)$$

We observe that least value of k that we can take is 3 because if we take k = 1 or 2 the right side coefficients will be a_{-2} and a_{-1} respectively which are not defined. To find $a_1$and a_2 we proceed as under: Expanding (2), we get

$$a_0m^2x^{m-1}+a_1(m+1)^2x^m+a_2(m+2)^2x^{m+1}+...+a_0x^{m+2}+a_1x^{m+3}+a_2x^{m+2}+$$

The coefficient of x^{m-1} is $a_0\,m^2 = 0$ gives m = 0, 0 because $a_0 \neq 0$. These are the roots of indicial equation.

Now, the coefficient of x^m is $a_1\ (m + 1)^2 = 0$ gives $a_1 = 0$. The coefficient of x^m is $a_2\ (m + 2)^2 = 0$ gives $a_2 = 0$. Now reconsider the recurrence relation (3)

For k = 3: $\quad a_3=-\frac{a_0}{(m+3)^2}$: For k = 4: $a_4=-\frac{a_1}{(m+4)^2}=0 \quad \left(\because a_1=0\right)$

For k = 5: $\quad a_5=-\frac{a_2}{(m+5)^2}=0 \quad \left(\because\ a_2=0\right)$

For k = 6: $\quad a_6=-\frac{a_3}{(m+6)^2}=\frac{a_0}{(m+3)^2(m+6)^2}$

Similarly, a_7, a_8 are zero and $a_9=-\frac{a_6}{(m+9)^2}=-\frac{a_0}{(m+3)^2(m+6)^2(m+9)^2}$

Now put m = 0 we get, $a_3 = -\dfrac{a_0}{3^2},\ a_6 = \dfrac{a_0}{3^2.6^2},\ a_3 = -\dfrac{a_0}{3^2.6^2.9^2}$, etc. and so on.

Hence, $y_1 = a_0\left(1 - \dfrac{1}{3^2}x^3 + \dfrac{1}{3^2.6^2}x^6 - \dfrac{1}{3^2.6^2.9^2}x^9 + \ldots\right)$

Since the roots of indicial equation are repeated, the second solution is obtained as follows:

$$y = x^m\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\right) = a_0x^m\left(1 - \frac{1}{3^2}x + \frac{1}{3^2.6^2}x^2 - \frac{1}{3^2.6^2.9^2}x^3 + \ldots\right)$$

$$\therefore \left(\frac{\partial y}{\partial m}\right)_{m_1} = \left[a_0x^m \ln x\left(1 - \frac{1}{3^2}x + \frac{1}{3^2.6^2}x^2 - \frac{1}{3^2.6^2.9^2}x^3 + \ldots\right)\right]_{m=0}$$

$$= a_0 \ln x\left(1 - \frac{1}{3^2}x + \frac{1}{3^2.6^2}x^2 - \frac{1}{3^2.6^2.9^2}x^3 + \ldots\right)$$

NOTE : If $y = a^x$ then $y` = a^x \ln a$. Similary if $y = x^m f(x)$ then $\left(\dfrac{\partial y}{\partial m}\right) = x^m . \ln x . f(x)$

Here m is variable and x is constant.

Hence the general solution is: $y = c_1(y)_{m_1} + c_2\left(\dfrac{\partial y}{\partial m}\right)_{m_1}$

$$y = A\left(1 - \frac{1}{3^2}x + \frac{1}{3^2.6^2}x^2 - \frac{1}{3^2.6^2.9^2}x^3 + \ldots\right) + B\ln x\left(1 - \frac{1}{3^2}x + \frac{1}{3^2.6^2}x^2 - \frac{1}{3^2.6^2.9^2}x^3 + \ldots\right)$$

(e) $x^2 y`` + 6\,xy` + (6 - x^2)\,y = 0$ (1)

Solution: Let the solution be $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

$$\Rightarrow \quad \frac{dy}{dx} = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k} + 6\sum_{k=0}^{\infty} a_k(m+k)x^{m+k} + 6\sum_{k=0}^{\infty} a_k x^{m+k} - \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k\left\{(m+k)(m+k-1+6) + 6\right\}x^{m+k} - \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k\left\{(m+k)(m+k+5) + 6\right\}x^{m+k} - \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^m in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$a_0[m(m+5) + 6] = 0 \Rightarrow m^2 + 5m + 6 = 0 \Rightarrow m = -2,\ m = -3,$ since $a_0 \neq 0$.

We see that roots of indicial equation are distinct and differ by an integer.

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$a_k\left\{(m+k)(m+k+5) + 6\right\} - a_{k-2} = 0 \qquad \Rightarrow a_k = \frac{1}{\left\{(m+k)(m+k+5) + 6\right\}}a_{k-2}$$

From this recurrence relation, we see that a_1 can not be determined hence its value considered as zero as discussed in the part (d).

For k = 2: $a_2 = \dfrac{a_0}{(m+2)(m+7)+6} = \dfrac{a_0}{m^2+9m+20} = \dfrac{a_0}{(m+4)(m+5)}$

For k = 3: $a_3 = \dfrac{a_1}{(m+3)(m+8)+6} = 0 \quad (\because a_1 = 0)$

For k = 4:

$$a_4 = \frac{a_2}{(m+4)(m+9)+6} = \frac{a_2}{m^2+13m+42} = \frac{a_2}{(m+6)(m+7)} = \frac{a_0}{(m+4)(m+5)(m+6)(m+7)}$$

Similarly, $a_5 = 0$ and $a_6 = \dfrac{a_0}{(m+4)(m+5)(m+6)(m+7)(m+8)(m+9)}$ and so on.

Now put m = -2, the first value of root from indicial equation:

$$a_2 = \frac{a_0}{2.3} = \frac{a_0}{3!},\ a_4 = \frac{a_0}{2.3.4.5} = \frac{a_0}{5!},\ a_6 = \frac{a_0}{2.3.4.5.6.7} = \frac{a_0}{7!},\ \ldots \text{ and so on.}$$

Hence, $y_1 = x^{-2}a_0\left(1+\dfrac{1}{3!}x^2+\dfrac{1}{5!}x^4+\dfrac{1}{7!}x^6+\ldots\right)$

Since the second root of indicial equation is m = -3 and differ by an integer from the first root hence the second solution is given by

For m = -3 :

$$y_2 = \left(\frac{\partial y_1}{\partial m}\right)_{m2}.\ \text{Now, } \left(\frac{\partial y_1}{\partial m}\right)_{m2} = \frac{\partial}{\partial m}y_1 = x^m a_0\left(1+\frac{1}{3!}x^2+\frac{1}{5!}x^4+\frac{1}{7!}x^6+\ldots\right)_{m=-3}$$

$$y_2 = a_0x^m \ln x a_0\left(1+\frac{1}{3!}x^2+\frac{1}{5!}x^4+\frac{1}{7!}x^6+\ldots\right)_{m=-3}$$

$$y_2 = a_0x^{-3}\ln x\left(1+\frac{1}{3!}x^2+\frac{1}{5!}x^4+\frac{1}{7!}x^6+\ldots\right)$$

Thus the complete solution is: $y = c_1y_1 + c_2y_2$ where y_1 and y_2 are given as above.

(f) $2x^2 y'' - x y' + (1+x^2)y = 0$ (1)

Solution: Let the solution be $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

$$\Rightarrow \quad \frac{dy}{dx} = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$2x^2\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2} - x\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} + \left(1+x^2\right)\sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} 2a_k(m+k)(m+k-1)x^{m+k} - \sum_{k=0}^{\infty} a_k(m+k)x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k\left\{2(m+k)(m+k-1)-(m+k)+1\right\}x^{m+k} + \sum_{k=0}^{\infty} a_k x^{m+k+2} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^m in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$a_0\{2m(m-1)-m+1 = 0 \Rightarrow a_0 2m^2 - 2m - m + 1 = 0 \Rightarrow a_0(2m-1)(m-1) = 0$.

Since $a_0 \neq 0$, therefore, (2m -1) (m – 1) = 0 or m = 1 and m = 1/2 are the roots of indicial equation. We see that roots of indicial equation are distinct and do not differ by an integer.

Equating to zero the coefficient of x^{m+k}, the recurrence relation is given by

$$a_k\left\{2(m+k)(m+k-1)-(m+k)+1\right\}+a_{k-2}=0$$

$$\Rightarrow \qquad a_k=-\frac{a_{k-2}}{\left[(m+k)(2m+2k-3)\right]+1}$$

For k = 2: $\quad a_2=-\dfrac{a_0}{(m+2)(2m+1)+1}$,

For k = 3: $\quad a_3=-\dfrac{a_1}{(m+3)(2m+3)+1}$,

For k = 4: $\quad a_4=-\dfrac{a_2}{(m+4)(2m+5)+1}=\dfrac{a_0}{\left[(m+2)(2m+1)+1\right]\left[(m+4)(2m+5)+1\right]}$

For k = 5: $\quad a_5=-\dfrac{a_3}{(m+5)(2m+7)+1}=\dfrac{a_1}{\left[(m+3)(2m+3)+1\right]\left[(m+5)(2m+7)+1\right]}$

and so on.

Now put m = 1/2, the first value of root from indicial equation:

$a_2=-\dfrac{a_0}{6},\ a_3=-\dfrac{2a_1}{15},\ a_4=\dfrac{a_0}{168},\ a_5=-\dfrac{a_1}{675}$ and so on.

Hence, $\quad y_1=x^{1/2}\left[a_0\left(1-\dfrac{1}{6}x^2+\dfrac{1}{168}x^4-...\right)+a_1\left(x-\dfrac{1}{15}x^3+\dfrac{1}{675}x^5-...\right)\right]$

For m = 1: $a_2=-\dfrac{a_0}{7}, a_3=-\dfrac{a_1}{17}, a_4=\dfrac{a_0}{252}, a_5=\dfrac{a_1}{935}$ and so on.

Hence, $\quad y_2=x\left[a_0\left(1-\dfrac{1}{7}x^2+\dfrac{1}{252}x^4-...\right)+a_1\left(x-\dfrac{1}{17}x^3+\dfrac{1}{935}x^5-...\right)\right]$

Thus the complete solution is: $y=c_1(y)_{m_1=1/2}+c_2(y)_{m_2=1}$.

(g) $2x(1-x)y``-7x\,y`-3y=0$ Note: the change in the problem

Solution: The given equation may be re-written as

$2xy``-2x^2y``-7x\,y`-3y=0$ (1)

Let the solution be $y=\sum_{k=0}^{\infty}a_kx^{m+k}$

$$\Rightarrow \quad y`=\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1} \text{ and } y``=\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^{m+k-2}$$

Substituting these values in (1), we get

$$\sum_{k=0}^{\infty}2a_k(m+k)(m+k-1)x^{m+k-1}-\sum_{k=0}^{\infty}2a_k(m+k)(m+k-1)x^{m+k}-\sum_{k=0}^{\infty}7a_kx^{m+k}-\sum_{k=0}^{\infty}3a_kx^{m+k}=0$$

$$\Rightarrow \sum_{k=0}^{\infty}a_k2(m+k)(m+k-1)x^{m+k-1}-\sum_{k=0}^{\infty}a_k\left\{2(m+k)(m+k-1)+7+3\right\}x^{m+k}=0 \quad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$a_0 2m(m-1)=0 \Rightarrow m=0$ or $m=1$, are the roots of indicial equation. We see that roots of indicial equation are distinct and differ by an integer.

Equating to zero the coefficient of x^{m+k-1}, the recurrence relation is given by

$$a_k\{2(m+k)(m+k-1)\}-a_{k-2}\{2(m+k-1)(m+k-2)+10\}=0$$

$$\Rightarrow \qquad a_k=\frac{\{2(m+k-1)(m+k-2)+10\}}{[2(m+k)(m+k-2)]}a_{k-2}$$

For k = 2: $a_2=\dfrac{2m(m+1)+10}{2m(m+2)}a_0=\dfrac{m(m+1)+5}{m(m+2)}$

NOTE: Since a_1 cannot be determined from RR hence it is considered to be zero.

For k = 3: $a_3=\dfrac{2(m+2)(m+1)+10}{2(m+3)(m+1)}a_1=0,$

For k = 4: $a_4=\dfrac{2(m+3)(m+2)+10}{2(m+4)(m+2)}a_2=\dfrac{[m(m+1)+5][(m+3)(m+2)+5]}{[m(m+2)][(m+2)(m+4)]}a_0$

and so on.

Now put m = 1, the first value of root from indicial equation:

$a_2=\dfrac{1.2+5}{1.3}a_0,\ a_4=\dfrac{(1.2+5)(3.4+5)}{(1.3)(3.4)}a_2$, and so on.

Hence, $y_1=a_0x\left(1+\dfrac{(1.2+5)}{1.3}x^2+\dfrac{(1.2+5)(3.4+5)}{(1.3)(3.4)}x^4\ldots\right)$

Since the second root of indicial equation is m = 0 which is also an integer so the second solution may be found by using the formula:

$$y_2=\left(\frac{\partial y}{\partial m}\right)_{m=0}=\frac{\partial}{\partial m}a_0x^m\left(1+\frac{(1.2+5)}{1.3}x^2+\frac{(1.2+5)(3.4+5)}{(1.3)(3.4)}x^4\ldots\right)_{m=0}$$

$$=a_0x^m\ln x\left(1+\frac{(1.2+5)}{1.3}x^2+\frac{(1.2+5)(3.4+5)}{(1.3)(3.4)}x^4\ldots\right)_{m=0}$$

$$=a_0\ln x\left(1+\frac{(1.2+5)}{1.3}x^2+\frac{(1.2+5)(3.4+5)}{(1.3)(3.4)}x^4\ldots\right)$$

Thus the complete solution is: $y=c_1(y)_{m_1=1}+c_2(y)_{m_2=0}$

$$y=Ax\left(1+\frac{(1.2+5)}{1.3}x^2+\frac{(1.2+5)(3.4+5)}{(1.3)(3.4)}x^4\ldots\right)+B\ln x\left(1+\frac{(1.2+5)}{1.3}x^2+\frac{(1.2+5)(3.4+5)}{(1.3)(3.4)}x^4\ldots\right)$$

where $A=a_0c_1$ and $B=a_0c_2$.

(h) 2x(1 – x)y``+ (1- x) y`+ 3y = 0

Solution: The given equation may be re-written as

2xy`` -2x²y`` + y` - xy` + 3y = 0 (1)

Let the solution be $y=\sum_{k=0}^{\infty}a_kx^{m+k}$

$$\Rightarrow \quad \frac{dy}{dx}=\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1} \text{ and } \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^{m+k-2}$$

Thus, (1) becoms $\sum_{k=0}^{\infty}2a_k(m+k)(m+k-1)x^{m+k-1}-\sum_{k=0}^{\infty}2a_k(m+k)(m+k-1)x^{m+k}$

$$+\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1}-\sum_{k=0}^{\infty}a_k(m+k)x^{m+k}+\sum_{k=0}^{\infty}3a_kx^{m+k}=0$$

Or $\sum_{k=0}^{\infty} a_k (m+k)(2m+2k-1)x^{m+k-1} - \sum_{k=0}^{\infty} a_k \left[(m+k)(2m+2k-1)-3\right]x^{m+k} = 0$ (2)

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$a_0[2m(2m-1)] = 0 \Rightarrow m = 0,\ 1/2$. We see that roots of indicial equation are distinct and do not differ by an integer.

Or $\sum_{k=0}^{\infty} a_k (m+k)(2m+2k-1)x^{m+k-1} - \sum_{k=0}^{\infty} a_k \left[(m+k)(2m+2k-1)-3\right]x^{m+k} = 0$

Equating to zero the coefficient of x^{m+k-1}, the recurrence relation is given by

$a_k (m+k)(2m+2k-1) - a_{k-1}\left[(m+k-1)(2m+2k-2-1)-3\right] = 0$

$$\Rightarrow a_k = \frac{\left[(m+k-1)(2m+2k-3)-3\right]}{(m+k)(2m+2k-1)} a_{k-1}$$

For k = 1: $\quad a_1 = \frac{m(2m-1)-3}{(m+1)(2m+1)} a_0$

For k = 2: $\quad a_2 = \frac{(m+1)(2m+1)-3}{(m+2)(2m+3)} a_1 = \frac{[m(2m-1)-3][(m+1)(2m+1)-3]}{(m+1)(2m+1)(m+2)(2m+3)} a_0$

and so on.

Now put m = 0, the first value of root from indicial equation:

$a_1 = \frac{0-3}{1.1} a_0 = \frac{-3}{1} a_0,\ a_2 = \frac{(0-3)(1-3)}{1.1.2.3} a_0 = a_1$ and so on.

Hence, $y_1 = x^0 a_0 \left(1 - \frac{1}{3}x + x^2 - ...\right) = a_0\left(1 - \frac{1}{3}x + x^2 - ...\right)$

For m = 1/2: $a_1 = -a_0,\ a_2 = 0, \Rightarrow a_3 = 0,\ a_4 = 0,$ etc.

Hence, $y_2 = a_0 x^{1/2}(1-x)$

Thus the complete solution is: $y = c_1 y_1 + c_2 y_2$ where y_1 and y_2 are given as above.

(i) $xy'' + 3y' + 4x^3 y = 0$ (1)

Solution: Let the solution be $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

$\Rightarrow \quad y' = \sum_{k=0}^{\infty} a_k (m+k)x^{m+k-1}$ and $y'' = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-2}$

Substituting these values in (1), we get

$$\sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-1} + \sum_{k=0}^{\infty} 3a_k (m+k)x^{m+k-1} + \sum_{k=0}^{\infty} 4a_k x^{m+k+3} = 0$$

$\Rightarrow \quad \sum_{k=0}^{\infty} a_k (m+k)(m+k+2)x^{m+k-1} + \sum_{k=0}^{\infty} 4a_k x^{m+k+3} = 0$ (2)

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$a_0 m(m+2) = 0 \Rightarrow m = 0, m = -2.$ The roots of indicial equation are unequal and differ by an integer.

Equating to zero the coefficient of x^{m+k-1}, the recurrence relation is given by

$$a_k\left(m+k\right)(m+k+2)+4a_{k-4}=0 \quad \Rightarrow \quad a_k=-\frac{1}{(m+k)(m+k+2)}a_{k-4} \qquad (3)$$

We observe that least value of k that we can take is 4 because if we take k = 1, 2 or 3 the right side coefficients will be a_{-3}, a_{-2} and a_{-1} respectively which are not defined. Thus, a_1, a_2 and a_3 are considered to be zero as is explained in part (d). Now reconsider the recurrence relation (3).

For k = 4: $\quad a_4=-\dfrac{a_0}{(m+2)(m+4)}$

For k = 8: $\quad a_8=-\dfrac{a_4}{(m+6)(m+8)}=\dfrac{a_0}{(m+2)(m+4)(m+6)(m+8)}$

NOTE: $a_5 = a_6 = a_7 = 0$ because a_{-3}, a_{-2} and $a_{-1}$are zeros.

Now put m = 0 we get, $a_3=-\dfrac{a_0}{2.4}$, $a_8=\dfrac{a_0}{2.4.6.8}$, $a_{12}=-\dfrac{a_0}{2.4.6.8.10.12}$, etc. and so on.

Hence, $y_1=a_0\left(1-\dfrac{1}{2.4}x^4+\dfrac{1}{2.4.6.8}x^8-\dfrac{1}{2.4.6.8.10.12}x^{12}+...\right)$

Since the second root of indicial equation is m = -2 and it differs by an integer from the first root m = 0 hence, the second solution is obtained as follows:

$$y=x^m\left(a_0+a_1x+a_2x^2+a_3x^3+...\right)$$

$$y_2=\left(\frac{\partial y}{\partial m}\right)_{m_1}=\left[a_0x^m\ln x\left(a_0+a_1x+a_2x^2+a_3x^3+...\right)\right]_{m=0}$$

$$=a_0\ln x\left(1-\frac{1}{2.4}x^4+\frac{1}{2.4.6.8}x^8-\frac{1}{2.4.6.8.10.12}x^{12}+...\right)$$

NOTE : If $y=a^x$ then $y`=a^x\ln a$. Similary if $y=x^m f(x)$ then $\left(\dfrac{\partial y}{\partial m}\right)=x^m.\ln x.f(x)$

Here m is variable and x is constant.

Hence the general solution is: $y=c_1\left(y\right)_{m_1}+c_2\left(\dfrac{\partial y}{\partial m}\right)_{m_2}$

$$y=A\left(1-\frac{1}{2.4}x^4+\frac{1}{2.4.6.8}x^8-\frac{1}{2.4.6.8.10.12}x^{12}+...\right)$$
$$+Bx^{-2}\ln x\left(1-\frac{1}{2.4}x^4+\frac{1}{2.4.6.8}x^8-\frac{1}{2.4.6.8.10.12}x^{12}+...\right)$$

(j) xy``+(x – 1)y`- y = 0 (1)

Solution: Let the solution be $y=\sum\limits_{k=0}^{\infty}a_kx^{m+k}$

$\Rightarrow$ $y`=\sum\limits_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k-1}$ and $y``=\sum\limits_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-2}$. Thus, (1) becomes

$$\therefore \sum_{k=0}^{\infty}a_k\left(m+k\right)\left(m+k-1\right)x^{m+k-1}+\sum_{k=0}^{\infty}a_k\left(m+k\right)x^{m+k}-\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1}-\sum_{k=0}^{\infty}a_kx^{m+k}=0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k (m+k)(m+k-2)x^{m+k-1} + \sum_{k=0}^{\infty} a_k (m+k-1)x^{m+k} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$a_0 m(m-2) = 0 \Rightarrow m = 0, m = 2.$ The roots of indicial equation are unequal and differ by an integer.

Equating to zero the coefficient of x^{m+k-1}, the recurrence relation is given by

$$a_k (m+k)(m+k-2) + (m+k-2)a_{k-1} = 0 \quad \Rightarrow \quad a_k = -\frac{1}{(m+k)} a_{k-1} \qquad (3)$$

For k = 1: $a_1 = -\dfrac{a_0}{(m+1)}$ For k = 2: $a_2 = -\dfrac{a_1}{(m+2)} = \dfrac{a_0}{(m+1)(m+2)}$

For k = 3: $a_3 = -\dfrac{a_0}{(m+1)(m+2)(m+3)}$, etc.

Now put m = 0 we get, $a_1 = -\dfrac{a_0}{1}$, $a_2 = \dfrac{a_0}{1.2}$, $a_3 = -\dfrac{a_0}{1.2.3}$, etc. and so on.

Hence, $y_1 = a_0\left(1 - \dfrac{1}{1!}x + \dfrac{1}{2!}x^2 - \dfrac{1}{3!}x^3 + ...\right)$

Since the second root of indicial equation is m = 2 and it differs by an integer from the first root m = 0 hence, the second solution is obtained as follows:

$$y = x^m\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right)$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m_2} = \left[a_0 x^m \ln x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ...\right)\right]_{m=2}$$

$$= a_0 x^2 \ln x\left(1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + ...\right)$$

NOTE : If $y = a^x$ then $y` = a^x \ln a$. Similary if $y = x^m f(x)$ then $\left(\dfrac{\partial y}{\partial m}\right) = x^m . \ln x . f(x)$

Here m is variable and x is constant.

Hence the general solution is: $y = c_1 (y)_{m_1} + c_2\left(\dfrac{\partial y}{\partial m}\right)_{m_2}$

$$y = A\left(1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + ...\right) + B\ln x\left(1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + ...\right)$$

(k) $xy`` + y` + xy = 0$ (1)

Solution: Let the solution be $y = \sum_{k=0}^{\infty} a_k x^{m+k}$

$\Rightarrow y` = \sum_{k=0}^{\infty} a_k (m+k)x^{m+k-1}$ and $y`` = \sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-2}$. Thus, (1) becomes

$$\therefore \sum_{k=0}^{\infty} a_k (m+k)(m+k-1)x^{m+k-1} + \sum_{k=0}^{\infty} a_k (m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k x^{m+k+1} = 0$$

$$\Rightarrow \quad \sum_{k=0}^{\infty} a_k (m+k)(m+k)x^{m+k-1} + \sum_{k=0}^{\infty} a_k{}^{m+k+1} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:
$a_0 m.m = 0 \Rightarrow m = 0, m = 0.$ The roots of indicial equation are equal.

Equating to zero the coefficient of x^{m+k-1}, the recurrence relation is given by

$$a_k(m+k)^2 + a_{k-2} = 0 \quad \Rightarrow \quad a_k = -\frac{1}{(m+k)^2} a_{k-2} \qquad (3)$$

Since the coefficient a_1 cannot be determined from the RR hence its value will be considered as zero. Hence, $a_3 = a_5 = \ldots = 0$.

For k = 2: $a_1 = -\frac{a_0}{(m+2)^2}$ For k = 4: $a_4 = -\frac{a_2}{(m+4)^2} = \frac{a_0}{(m+2)^2(m+4)^2}$

For k = 6: $a_6 = -\frac{a_0}{(m+2)^2(m+4)^2(m+6)^2}$, etc.

Now put m = 0 we get, $a_2 = -\frac{a_0}{2^2}$, $a_4 = \frac{a_0}{2^2.4^2}$, $a_6 = -\frac{a_0}{2^2.4^2.6^2}$, etc. and so on.

Hence, $y_1 = a_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} + \frac{x^6}{2^2.4^2.6^2} - \ldots\right)$

Since the second root of indicial equation is also m = 0 hence, the second solution is obtained as follows:

$$y = x^m\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\right)$$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m_2} = \left[a_0 x^m \ln x\left(a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\right)\right]_{m=2}$$

$$= a_0 x^2 \ln x\left(1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots\right)$$

NOTE: If $y = a^x$ then $y` = a^x \ln a$. Similary if $y = x^m f(x)$ then $\left(\frac{\partial y}{\partial m}\right) = x^m.\ln x.f(x)$

Here m is variable and x is constant.

Hence the general solution is: $y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}$

$$y = A\left(1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots\right) + B\ln x\left(1 - \frac{1}{1!}x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots\right)$$

(*l*) x(1 – x) y`` + 4y` + 2y = 0 or xy`` - x^2 y`` + 4y` + 2y = 0 (1)

Solution: Let the solution be $y = \sum_{k=0}^{\infty} a_k x^{m+k} \Rightarrow y` = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1}$ and

$y`` = \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2}$. Thus, (1) becomes

$$\therefore \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-1} - \sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k}$$

$$+ \sum_{k=0}^{\infty} 4a_k(m+k)x^{m+k-1} + \sum_{k=0}^{\infty} 2a_k x^{m+k} = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} a_k\,(m+k)(m+k+3)x^{m+k-1} - \sum_{k=0}^{\infty} a_k[(m+k)(m+k-1)-2]x^{m+k} = 0 \qquad (2)$$

The coefficient of the lowest degree term x^{m-1} in (2) is obtained by putting k = 0 in the first summation only then equating it to zero, the indicial equation will be:

$a_0 m.(m+3) = 0 \Rightarrow m = 0, m = -3.$ The roots of indicial equation are different but differ by an integer.

Equating to zero the coefficient of x^{m+k-1}, the recurrence relation is given by

$a_k(m+k)(m+k+3) - a_{k-2}[(m+k-2)(m+k-1)-2] = 0$

$$\Rightarrow \quad a_k = \frac{[(m+k-2)(m+k-1)-2]}{(m+k)(m+k+3)} a_{k-2} \qquad (3)$$

Since the coefficient a_1 cannot be determined from the RR hence its value will be considered as zero. Hence, $a_3 = a_5 = \ldots = 0$.

For k = 2: $\quad a_2 = \dfrac{m(m+1)-2}{(m+2)(m+5)} a_0$

For k = 4: $\quad a_4 = \dfrac{(m+2)(m+3)}{(m+4)(m+7)} a_2 = \dfrac{[m(m+1)-2][(m+2)(m+3)-2]}{(m+2)(m+5)(m+4)(m+7)} a_0$

For k = 4: $\quad a_6 = \dfrac{[m(m+1)-2][(m+2)(m+3)-2][(m+4)(m+5)-2]}{(m+2)(m+5)(m+4)(m+7)(m+6)(m+9)} a_0$, etc.

Now put m = 0 we get, $a_2 = \dfrac{-2}{2.5} a_0$, $a_4 = \dfrac{-2(2.3)}{(2.5)(4.7)} a_0$, $a_6 = \dfrac{-2(2.3-2)(4.5-2)}{(2.5)(4.7)(6.9)} a_0$, etc.

and so on.

Hence, $y_1 = 2a_0\left(1 - \dfrac{1}{2.5}x^2 - \dfrac{(2.3-2)}{(2.5)(4.7)}x^4 - \dfrac{(2.3-2)(4.5-2)}{(2.5)(4.7)(6.9)}x^6 - \ldots\right)$

Since the second root of indicial equation is m = -3 which differ by an integer hence, the second solution is obtained as follows:

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m_2} = \left[a_0 x^m \ln x\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right)\right]_{m=-3}$$

$= 2a_0 x^{-3} \ln x\left(1 - \dfrac{1}{2.5}x^2 - \dfrac{(2.3-2)}{(2.5)(4.7)}x^4 - \dfrac{(2.3-2)(4.5-2)}{(2.5)(4.7)(6.9)}x^6 - \ldots\right)$. Thus,

$$y = c_1(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2} = A\left(1 - \frac{1}{2.5}x^2 - \frac{(2.3-2)}{(2.5)(4.7)}x^4 - \frac{(2.3-2)(4.5-2)}{(2.5)(4.7)(6.9)}x^6 - \ldots\right) +$$

$$+Bx^{-3}\ln x\left(1 - \frac{1}{2.5}x^2 - \frac{(2.3-2)}{(2.5)(4.7)}x^4 - \frac{(2.3-2)(4.5-2)}{(2.5)(4.7)(6.9)}x^6 - \ldots\right)$$

3. Show that $J_4(x) = \left(\dfrac{48}{x^3} - \dfrac{8}{x}\right)J_1(x) + \left(1 - \dfrac{24}{x^2}\right)J_0(x)$

Solution: We know that $J_{n+1}(x) = \dfrac{2n}{x}J_n(x) - J_{n-1}(x)$

Putting $n = 1;\ J_2(x) = \dfrac{2}{x}J_1(x) - J_0(x)$

Putting $n = 2;\ J_3(x) = \dfrac{4}{x}J_2(x) - J_1(x) = \dfrac{4}{x}\left[\dfrac{2}{x}J_1(x) - J_0(x)\right] - J_1(x) = \dfrac{8}{x^2}J_1(x) - \dfrac{4}{x}J_0(x)$ 4

Putting $n=3;\ J_4(x)=\frac{6}{x}J_3(x)-J_2(x)=\frac{6}{x}\left[\frac{8}{x^2}J_1(x)-\frac{4}{x}J_0(x)\right]-\left[\frac{2}{x}J_1(x)-J_0(x)\right]$

$$=\frac{48}{x^3}J_1(x)-\frac{24}{x^2}J_0(x)-\frac{2}{x}J_1(x)+J_0(x)$$

$$=\left[\frac{48}{x^3}-\frac{2}{x}\right]J_1(x)+\left[1-\frac{24}{x^2}\right]J_0(x)$$

4. Show that (a) $J_{1/2}(x)=J_{-1/2}(x)\cot x$

Solution: We know that $J_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sin x$ and $J_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cos x$

$$\frac{J_{-1/2}(x)}{J_{1/2}(x)}=\sqrt{\frac{2}{\pi x}}\sin x\div\sqrt{\frac{2}{\pi x}}\cos x=\frac{\sin x}{\cos x}=\tan x=\frac{1}{\cot x}$$

$\Rightarrow\ \cot x\ J_{-1/2}(x)=J_{1/2}(x)$

(b) $J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{3}{2}\sin x+\frac{3-x^2}{x^2}\cos x\right)$

We know that $J_{n+1}(x)=\frac{2n}{x}J_n(x)-J_{n-1}(x)\ \Rightarrow J_{n-1}(x)=\frac{2n}{x}J_n(x)-J_{n+1}(x)$ (1)

Putting $n=-1/2$ in (1); $J_{-3/2}(x)=\frac{-1}{x}J_{-1/2}(x)-J_{1/2}(x)$

Putting $n=-3/2$ in (1);

$$J_{-5/2}(x)=\frac{-5}{x}J_{-3/2}(x)-J_{-1/2}(x)=\frac{-5}{x}\left[\frac{-1}{x}J_{-1/2}(x)-J_{1/2}(x)\right]-J_{-1/2}(x)$$

$$=\left[\frac{5}{x^2}-1\right]J_{-1/2}(x)+\frac{5}{x}J_{1/2}(x)=\left[\frac{5-x^2}{x^2}\right]J_{-1/2}(x)+\frac{5}{x}J_{1/2}(x)$$

$\therefore\ J_{-5/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\frac{5}{2}\sin x+\frac{3-x^2}{x^2}\cos x\right)$. Using the values of $J_{1/2}$ and $J_{-1/2}$.

5. Show that

(a) $J_2 = J_0'' - x^{-1} J_0'$ **(b)** $J_1''(x)=\left(\frac{2}{x^2}-1\right)J_1(x)-\frac{1}{x}J_0(x)$

Solution: (a) We know that $J_n'(x)=\frac{1}{2}\left(J_{n-1}-J_{n+1}\right)$. Putting $n=0$, we get

$J_0'(x)=\frac{1}{2}\left(J_{-1}-J_1\right)$. But $J_{-n}=(-1)^nJ_n\ \Rightarrow J_{-1}=-J_1$

Thus, $J_0'=\frac{1}{2}\left(-J_1-J_1\right)=-J_1\ \Rightarrow J_0''=-J_1'$ (1)

We also know that $J_n'(x)=\frac{n}{x}J_n-J_{n+1}$. Putting $n=1$, we get $J_1'(x)=\frac{1}{x}J_1-J_2$.

Thus (1) becomes: $J_0''=-\left[x^{-1}J_1-J_2\right]=-x^{-1}J_1+J_2\ \Rightarrow J_2(x)=J_0''(x)+x^{-1}J_1(x)$

$\Rightarrow J_2(x)=J_0''(x)-x^{-1}J_0'(x)$ $\left[\text{From (1) } J_1=-J_0'\right]$

(b) We know that $J_n'(x)=\frac{1}{2}\left(J_{n-1}-J_{n+1}\right)$. Putting $n=1$, we get

$$J_1'(x)=\frac{1}{2}\left(J_0-J_2\right) \Rightarrow J_1''(x)=\frac{1}{2}\left(J_0'-J_2'\right). \text{ But, } J_0'=-J_1$$

$$\Rightarrow J_1''=\frac{1}{2}\left(-J_1-J_2'\right)=\frac{1}{2}\left(-J_1-J_1+\frac{2}{x}J_2\right)=-J_1+\frac{1}{x}J_2 \qquad (1) \quad \text{NOTE}: J_2'=J_1-\frac{2}{x}J_2$$

We also know that $J_{n+1}=\frac{2n}{x}J_n-J_{n-1}$. Putting $n=1$, we get $J_2=\frac{2}{x}J_1-J_0$.

Thus (1) becomes: $J_1''=-J_1+\frac{1}{x}\left[\frac{2}{x}J_1-J_0\right]=\left(\frac{2}{x^2}-1\right)J_1-\frac{1}{x}J_0$

Thus, $J_1''(x)=\left(\frac{2}{x^2}-1\right)J_1(x)-\frac{1}{x}J_0(x)$

6. Evaluate $\int_0^{\pi} x^{n+1}J_n(x)\,dx$

Solution: We know that $\frac{d}{dx}\left[x^nJ_n(x)\right]=x^nJ_{n-1}(x)$

Replacing n by (n + 1), we get:

$\frac{d}{dx}\left[x^{n+1}J_{n+1}(x)\right]=x^{n+1}J_n(x)$. Integrating from 0 to π, we get:

$$\int_0^{\pi} x^{n+1}J_n(x)\,dx=\left[x^{n+1}J_{n+1}(x)\right]_0^{\pi}=\pi^{n+1}J_{n+1}(\pi).$$

7. If $J_0(2)=a$, $J_1(2)=b$ find $J_1`(2)$ and $J_2`(2)$ in terms of a and b.

Solution: We know that $J_n'(x)=\frac{1}{2}\left(J_{n-1}-J_{n+1}\right)$. Putting $n=1$, we get

$$J_1'(x)=\frac{1}{2}\left(J_0-J_2\right)=\frac{1}{2}\left(J_0-\left(\frac{2}{x}J_1-J_0\right)\right)=\frac{3}{2}J_0-\frac{2}{x}J_1.$$

Thus, $J_1'(x)=\frac{3}{2}J_0(x)-\frac{2}{x}J_1(x) \qquad (1)$

$$\Rightarrow \quad J_2'(x)=\frac{3}{2}J_1(x)-\frac{2}{x}J_2(x)=\frac{3}{2}J_1(x)-\frac{2}{x}\left(\frac{2}{x}J_1-J_0\right)$$

$$\Rightarrow \quad J_2'(x)=\left(\frac{3}{2}-\frac{4}{x^2}\right)J_1(x)+\frac{2}{x}J_0(x) \qquad (2)$$

Putting $x=2$, $J_0(2)=a$ and $J_1(2)=b$ in (1) and (2), we get:

$$J_1'(2)=\frac{3}{2}J_0(2)-\frac{2}{2}J_1(2)=\frac{3}{2}a-b$$

$$J_2'(2)=\left(\frac{3}{2}-\frac{4}{2^2}\right)J_1(2)+\frac{2}{2}J_0(2)=\frac{1}{2}b+a$$

8. Show that $P_n(-x)=(-1)^n P_n(x)$

Solution: One form of Legendre's polynomial is derived from Rodriquez's formula

$$P_n(x)=\frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$$

Another formula to compute $P_n(x)$ is called "Explicit Formula" and is given by

$$P_n(x)=\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r},\ \text{where } N=\frac{n}{2} \text{ if } n \text{ is even and}$$

$N=\dfrac{(n-1)}{2}$ if N is odd. This formula is derived by series solution method. For det ail, the reader is advised to read "Advanced Engineering Mathematics" by Grewall.

For example, let $n=1$, then $N=\dfrac{1-1}{2}=0$ and therefore,

$$P_1(x)=\sum_{r=0}^{0}\frac{(-1)^r(2-2r)!}{2^1 r!(1-r)!(1-2r)!}x^{1-2r}=\frac{(-1)^0(2-0)!}{2.(0!)(0!)(1!)}x^{1-0}=x$$

Let $n=2$, then $N=\dfrac{2}{2}=1$ and therefore,

$$P_2(x)=\sum_{r=0}^{1}\frac{(-1)^r(4-2r)!}{2^2 r!(2-r)!(2-2r)!}x^{2-2r}=\frac{(-1)^0(4-0)!}{4.(0!)(2!)(2!)}x^{2-0}+\frac{(-1)^1(4-2)!}{4.1!.0!}x^0$$

$$\Rightarrow P_2(x)=\frac{3}{2}x^2-\frac{1}{2}=\frac{3x^2-1}{2}$$ (These results are same as is shown in the textbook)

$$\text{Now, } P_n(-x)=\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}(-x)^{n-2r}=\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}(-1)^{n-2r}x^{n-2r}$$

$$=(-1)^n\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}(-1)^{2r}x^{n-2r}.\ \text{But } (-1)^{-2r}=\frac{1}{(-1)^{2r}}=\frac{1}{1}=1$$

$$\text{Thus, } P_n(-x)=(-1)^n\sum_{r=0}^{N}\frac{(-1)^r(2n-2r)!}{2^n r!(n-r)!(n-2r)!}x^{n-2r}=(-1)^n P_n(x)$$

9. Express x^3+2x^2-x-3 in terms of Legendre polynomials.

Solution: Let $x^3+2x^2-x-3=aP_3(x)+bP_2(x)+cP_1(x)+dP_0(x)$ (1)

$$x^3+2x^2-x-3=a\left(\frac{5x^3}{2}-\frac{3x}{2}\right)+b\left(\frac{3x^2}{2}-\frac{1}{2}\right)+c(x)+d(1)$$

$$x^3+2x^2-x-3=\frac{5a}{2}x^3-\frac{3a}{2}x+\frac{3b}{2}x^2-\frac{b}{2}+cx+d$$

$$x^3+2x^2-x-3=\frac{5a}{2}x^3+\frac{3b}{2}x^2+\left(c-\frac{3a}{2}\right)x+\left(d-\frac{b}{2}\right)$$

Equating the coefficients of like powers of x from both sides, we obtain

$$\frac{5a}{2}=1\Rightarrow a=\frac{2}{5},\qquad \frac{3b}{2}=2\Rightarrow b=\frac{4}{3},$$

$$c-\frac{3a}{2}=-1\Rightarrow c-\frac{3}{2}\left(\frac{2}{5}\right)=-1\Rightarrow c=\frac{3}{5}-1\Rightarrow c=-\frac{2}{5},$$

$$d-\frac{b}{2}=-3\Rightarrow d-\frac{1}{2}\left(\frac{4}{3}\right)=-3\Rightarrow d=-3+\frac{2}{3}\Rightarrow d=\frac{-7}{3}\cdot$$

Substituting these in (1), we get

$$x^3+2x^2-5x+8=\frac{2}{5}P_3(x)+\frac{4}{3}P_2(x)-\frac{2}{5}P_1(x)-\frac{7}{3}P_0(x)\cdot$$

10. Find $\int_{-1}^{1}\left[P_9(x)\,P_5(x)\right]\,dx$ and $\int_{-1}^{1}\left[P_9(x)\right]^2\,dx$

Solution: We know that $\int_{-1}^{1}P_n(x)P_m(x)\,dx=0$, provided $m\neq n$.

Thus, $\int_{-1}^{1}\left[P_9(x)\,P_5(x)\right]\,dx=0$

We also know that $\int_{-1}^{1}\left[P_n(x)\right]^2 dx=\dfrac{2}{2n+1}$. Thus, $\int_{-1}^{1}\left[P_9(x)\right]^2\,dx=\dfrac{2}{2(9)+1}=\dfrac{2}{19}$

11. Show that $x^5=\dfrac{8}{63}\left[P_5(x)+\dfrac{7}{2}P_3(x)+\dfrac{27}{8}P_1(x)\right]$ & $x^3=\left[\dfrac{2}{5}P_3(x)+\dfrac{3}{5}P_1(x)\right]$

Solution: We know that,

(i) $P_0(x)=1$ (ii) $P_1(x)=x$ (iii) $P_2(x)=\dfrac{1}{2}\left(3x^2-1\right)$

(iv) $P_3(x)=\dfrac{1}{2}\left(5x^3-3x\right)$ (iv) $P_4(x)=\dfrac{1}{8}\left(35x^4-30x^2+3\right)$

(v) $P_5(x)=\dfrac{1}{8}\left(63x^5-70x^3+15x\right)$

From (v), $8P_5(x)=\left(63x^5-70x^3+15x\right)\Rightarrow x^5=\dfrac{1}{63}\left(8P_5(x)+70x^3-15x\right)$ (vi)

From (iv), $2P_3(x)=\left(5x^3-3x\right)\Rightarrow x^3=\dfrac{1}{5}\left(2P_3(x)+3x\right)$

Putting this in (vii), we get

$$x^5=\frac{1}{63}\left[8P_5(x)+\frac{70}{5}\left(2P_3(x)+3x\right)-15x\right]=\frac{1}{63}\left[8P_5(x)+28P_3(x)+42x-15x\right]$$

$$=\frac{1}{63}\left[8P_5(x)+28P_3(x)+27x\right]=\frac{8}{63}\left[P_5(x)+\frac{7}{2}P_3(x)+\frac{27}{8}P_1(x)\right]\ \left(\because\ P_1(x)=x\right)$$

Now, from (iv), $2P_3(x)=\left(5x^3-3x\right)\Rightarrow x^3=\dfrac{1}{5}\left(2P_3(x)+3x\right)=\left[\dfrac{2}{5}P_3(x)+\dfrac{3}{5}P_1(x)\right]$

12. Evaluate the following integrals:

(a) $\int_{-1}^{1}x^3P_4(x)\,dx$ **(b)** $\int_{-1}^{1}x^3P_3(x)\,dx$

Solution: Refer the above problem, $\int_{-1}^{1}f(x)P_n(x)dx=\dfrac{(-1)^n}{2^n\cdot n!}\int_{-1}^{1}\left(x^2-1\right)^n f^n(x)\,dx$. Thus,

(a) $\int_{-1}^{1}x^3P_4(x)\,dx=\dfrac{(-1)^4}{2^4.4!}\int_{-1}^{1}\left(x^2-1\right)^4.(0)\,dx=0$ NOTE: Here $f(x)=x^3\Rightarrow f````(x)=0$

(b) $\int_{-1}^{1}x^3P_3(x)\,dx=\dfrac{(-1)^3}{2^3.3!}\int_{-1}^{1}\left(x^2-1\right)^3.6\,dx$ NOTE: Here $f(x)=x^3\Rightarrow f```(x)=6$

Chapter 8

INFINITE SERIES

WORKSHEET 08

1. Write the general term of each of the following sequences and show if each one of them is convergent or divergent.

(i) $\frac{1}{3}, \frac{-2}{3^2}, \frac{3}{3^3}, \frac{-4}{3^4}, \ldots$

Solutin : $a_1 = \frac{1}{3} \qquad a_2 = -\frac{2}{3^2} \qquad a_3 = \frac{3}{3^3} \quad \Rightarrow \quad a_n = (-1)^n \frac{n}{3^n}$

Now consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{3^n} = \lim\limits_{n\to\infty} \frac{1}{3^n . \ln 3} = \frac{1}{\infty} = 0 \quad (\text{Using L'Hopital Rule})$

Thus given sequence is convergent.

(ii) $\left(\frac{1}{2}\right)^2, \left(\frac{2}{3}\right)^2, \left(\frac{3}{4}\right)^2, \left(\frac{4}{5}\right)^2, \ldots$

Solutin : $a_1 = \left(\frac{1}{2}\right)^2 \qquad a_2 = \left(\frac{2}{3}\right)^2 \qquad a_3 = \left(\frac{3}{4}\right)^2 \quad \Rightarrow \quad a_n = \left(\frac{n}{n+1}\right)^2$

Now consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = \lim\limits_{n\to\infty} \frac{2n}{2(n+1)} = \lim\limits_{n\to\infty} \frac{1}{1} = 1 \quad (\text{By L'Hopital Rule})$

Thus given sequence is convergent.

(iii) $\frac{2^2}{2!}, \frac{3^2}{3!}, \frac{4^2}{4!}, \frac{5^2}{5!}, \ldots$

Solutin : $a_1 = \frac{2^2}{2!} \qquad a_2 = \frac{3^2}{3!} \qquad a_3 = \frac{4^2}{4!} \quad \Rightarrow \quad a_n = \frac{(n+1)^2}{(n+1)!}$

Now consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{(n+1)^2}{(n+1)!} = \lim\limits_{n\to\infty} \frac{(n+1)(n+1)}{(n+1).n!} = \lim\limits_{n\to\infty} \frac{n+1}{n!} = 0$

Here $n! > (n+1)$ as $n \to \infty$. Thus given sequence is convergent.

2. Examine the convergence of each of the following sequences.

(i) $a_n = 2n$ (ii) $a_n = 1 + 1/n$ (iii) $a_n = [n + (-1)^n]^{-1}$

(i) Solutin :Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} (2n) = \infty$. Thus given sequence is divergent.

(ii) Solutin : Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1$. Thus given sequence is convergent.

(iii) Solutin : Consider $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n+(-1)^n} = \frac{1}{\infty} = 0$. Thus given sequence is convergent.

3. Examine the following series for convergence:

(i) 1 + 3 + 5 + 7 + …

(ii) 1 + 1/3 + 1/9 + 1/27 + …

(iii) $1 - 1/3 + 1/3^2 - 1/3^3 + 1/3^4 \dots$

(iv) 6 – 10 + 4 + 6 – 10 + 4 + 6 – 10 + …

(v) $1^2 + 3^2 + 5^2 + 7^2 + \dots$

(vi) 1/(1 . 2) + 1/(2 . 3) + 1/(3 . 4) + …

Solution : (i) The given series is arithmetic series hence its sum to n terms is

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right] = \frac{n}{2}\left(2.1+(n-1).1\right) = \frac{n(n+1)}{2}.$$

Now consider $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\left(\frac{n(n+1)}{2}\right) = \infty$. Hence given series is divergent series.

Solution : (ii) The given series is infinite geometric series hence, its sum to n terms is

$$S_n = a\,\frac{1-r^n}{1-r} = 1.\frac{1-(1/3)^n}{1-(1/3)} = \frac{3}{2}\left[1-\left(\frac{1}{3}\right)^n\right] \quad (\text{Here } a=1,\ r=1/3)$$

Now consider $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\frac{3}{2}\left[1-\left(\frac{1}{3}\right)^n\right] = \frac{3}{2}(1-0) = \frac{3}{2}$ which is finite.

Hence given series is convergent series.

Solution : (iii) The given series is infinite geometric series hence, its sum to n terms is

$$S_n = a\,\frac{1-r^n}{1-r} = 1.\frac{1-(-1/3)^n}{1-(-1/3)} = \frac{3}{4}\left[1-\left(-\frac{1}{3}\right)^n\right] \quad (\text{Here } a=1,\ r=-1/3)$$

Now consider $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\frac{3}{4}\left[1-\left(-\frac{1}{3}\right)^n\right] = \frac{3}{4}(1-0) = \frac{3}{4}$ which is finite.

Hence given series is convergent series.

Solution : (iv) The given series may be expressed as

$$(6-10+4)+(6-10+4)+(6-10+4)+\dots = 0+0+0+\dots = 0$$

Hence given series is convergent.

Solution : (v) Here $S_1 = 1^2 = 1,\ S_2 = 1^2+3^2 = 10,\ S_3 = 1^2+3^2+5^2 = 35, \dots$

Since $S_3 > S_2,\ S_2 > S_1 \Rightarrow S_{n+1} > S_n \Rightarrow \frac{S_{n+1}}{S_n} > 1$. Hence, given series is divergent.

4. Test the convergence of the following series:

(i) $\frac{1}{1.2}+\frac{2}{3.4}+\frac{3}{5.6}+\dots$

Solution : Note the theorem : Let $\sum a_n$ and $\sum b_n$ be two series and $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = L$.

If L is non – zero real then either both series converge or both diverge. Now,

$$a_1 = \frac{1}{1.2},\ a_2 = \frac{2}{3.4},\ a_3 = \frac{3}{5.6} \Rightarrow a_n = \frac{n}{(2n-1)(2n)} = \frac{1}{2(2n-1)}.\ \text{Let } b_n = \frac{1}{n}$$

Now consider $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lim\limits_{n\to\infty}\frac{1}{2(2n-1)}\div\frac{1}{n}=\lim\limits_{n\to\infty}\frac{n}{2(2n-1)}=\lim\limits_{n\to\infty}\frac{n}{2n\left(2-\frac{1}{n}\right)}$

$=\lim\limits_{n\to\infty}\frac{1}{2\left(2-\frac{1}{n}\right)}=\frac{1}{2}$ which is a non-zero real.

Now, the series $\sum\frac{1}{n}$ diverges hence the series $\sum\frac{1}{2(2n-1)}$ also diverges.

(ii) $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\ldots$

Solution: Refer to Example 06 Chapter 8 of Textbook the "Harmonic series" of order `p` $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots$ converges if $p>1$ and diverges if $p\le 1$.

Thus, the series $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\ldots$converges converges if $p>1$ and diverges if $p\le 1$.

(iii) $\sum\frac{\sqrt{n}}{n^2+1}$

Solution: Since $n^2+1>n^2$ therefore, $\frac{1}{n^2+1}<\frac{1}{n^2}\Rightarrow\frac{\sqrt{n}}{n^2+1}<\frac{\sqrt{n}}{n^2}$

Or $\frac{\sqrt{n}}{n^2+1}<\frac{1}{n^{3/2}}$. Now by p-series test $\sum\frac{1}{n^{3/2}}$ is convergent series hence, by comparision test $\sum\frac{\sqrt{n}}{n^2+1}$ is also convergent series.

(iv) $\sum\frac{2n^3+5}{4n^3+1}$

Solution: Let $a_n=\frac{2n^3+5}{4n^3+1}$ and $b_n=\frac{1}{n}$. Now all $n\in N$, $a_n>b_n$. Since, $\sum\frac{1}{n}$ is divergent series hece by comparision test $\sum\frac{2n^3+5}{4n^3+1}$ is also divergent series.

(v) $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\ \ldots$

Solution: $a_1=\frac{1}{1.2}$, $a_2=\frac{1}{2.3}$, $a_3=\frac{1}{3.4}$, $\Rightarrow\ a_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$ (By P.F)

$\therefore\ a_1=1-\frac{1}{2}$, $a_2=\frac{1}{2}-\frac{1}{3}$, $a_3=\frac{1}{3}-\frac{1}{4}$, etc. Thus,

$a_1+a_2+a_3+\ldots=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\frac{1}{4}+\ldots=1.$

Since, sum of infinite series is finite (here 1) hence given series is convergent and its sum is 1.

(vi) $\sum \frac{1}{n \ln n}$

Slolution : Consider $\int_{2}^{\infty} \frac{1}{n \ln n} dn$. Putting $z = \ln n \Rightarrow dz = \frac{1}{n} dn$

If $z = 2$ then $\ln n = 2 \Rightarrow n = e^2$ and if $n = \infty$ then $z = \infty$.

Thus, $\int_{2}^{\infty} \frac{1}{n \ln n} dn = \int_{e^2}^{\infty} \frac{1}{z} dz = [\ln z]_{e^2}^{\infty} = \ln \infty - \ln(e^2) = \infty - 2 = \infty$

Since integral diverges hence, series diverges.

5. Apply the "LIMIT COMPARISON TEST" to determine whether the given series converges or diverges:

(i) $\sum \frac{1}{n+10}$

Solution : Solution : Let $a_n = \frac{1}{n+10}$ and $b_n = \frac{1}{n}$. Now

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+10} = \lim_{n\to\infty} \frac{n}{n\left(1+\frac{10}{n}\right)} = \frac{1}{1+0} = 1 \text{ (non – zero real number)}$$

Since $\sum \frac{1}{n}$ is divergent series hence, $\sum \frac{1}{n+10}$ is also divergent.

(ii) $\sum \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7n}}$

Solution : Let $a_n = \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7n}}$ and $b_n = \frac{n^{2/3}}{n^{3/4}} = n^{1/12}$. Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7n}} \div n^{1/12} = \lim_{n\to\infty} \frac{n^{2/3}\sqrt{3+\frac{1}{n^2}}}{n^{3/4}.n^{1/12}\sqrt{4+\frac{2}{n^2}+\frac{7}{n^3}}}$$

$$= \lim_{n\to\infty} \frac{n^{2/3}\sqrt{3+\frac{1}{n^2}}}{n^{3/4}.n^{1/12}\sqrt{4+\frac{2}{n^2}+\frac{7}{n^3}}} = \lim_{n\to\infty} \frac{\sqrt{3+\frac{1}{n^2}}}{\sqrt{4+\frac{2}{n^2}+\frac{7}{n^3}}} = \frac{\sqrt{3}}{\sqrt{4}} \text{ (non – zero real number)}$$

Since $\sum n^{1/12}$ is divergent series (By p – test series) hence, $\sum \frac{\sqrt[3]{3n^2+1}}{\sqrt[4]{4n^3+2n+7n}}$ is also divergent.

(iii) $\sum n^{\ln n}$

Solution : Let $a_n = n^{\ln n}$ and $b_n = 1/n$. Now by Limit comparision test

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^{\ln n}}{(1/n)} = \lim_{n\to\infty} n.n^{\ln n} = \infty$. Now, $\sum \frac{1}{n}$ is divergent series hence, $\sum n^{\ln n}$

is also divergent series.

(iv) $\sum \frac{n^n}{n!} = \frac{1^1}{1!} + \frac{2^2}{2!} + \frac{3^3}{3!} + \ldots$

Solution : Let $a_n = \sum \frac{n^n}{n!}$ and $b_n = \sum \frac{1}{n}$. Now, for all $n \in N$, $\frac{n^n}{n!} \ge \frac{1}{n}$; that is $a_n > b_n$

Since $\sum \frac{1}{n}$ is divergent series hence, $\sum \frac{n^n}{n!}$ is also divergent.

(v) $\sum \frac{2^n + 1}{3^n + 1}$

Solution : Let $a_n = \frac{2^n + 1}{3^n + 1}$ and $b_n = \left(\frac{2}{3}\right)^n$. Now, consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{2^n + 1}{3^n + 1}\right) \div \left(\frac{2^n}{3^n}\right) = \lim_{n\to\infty} \frac{2^n\left(1 + \frac{1}{2^n}\right)}{3^n\left(1 + \frac{1}{3^n}\right)} \times \frac{3^n}{2^n} = 1 \text{ (a finite real number)}$$

Since $\sum \left(\frac{2}{3}\right)^n$ is infinte geometric series with common ratio $r = 2/3 < 1$ hence

it is convergent series. Thus, $\sum \frac{2^n + 1}{3^n + 1}$ is also convergent series.

(vi) $\sum \frac{1}{x^n + x^{-n}}$

Solution : Here $a_n = \frac{1}{x^n + x^{-n}} = \frac{x^n}{x^{2n} + 1}$. Let $b_n = \frac{1}{x^n}$. Now, consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{x^n}{x^{2n} + 1}\right) \div \left(\frac{1}{x^n}\right) = \lim_{n\to\infty} \frac{x^{2n}}{x^{2n}\left(1 + \frac{1}{x^{2n}}\right)} = 1 \text{ (a finite real number)}$$

Since $\sum b_n = \sum \frac{1}{x^n}$ is infinte geometric series with common ratio $r = 1/x^n$.

Now, if $x > 1$ then $r < 1$ and hence the series $\sum \frac{1}{x^n}$ is convergent and

so does the series $\sum \frac{1}{x^n + x^{-n}}$. If $x \le 1$ then $|r| > 1$, hence the series $\sum \frac{1}{x^n}$

is divergent and so does the series $\sum \frac{1}{x^n + x^{-n}}$.

(vii) $\sum \frac{\sqrt{n}}{n^2 + 1}$

Solution : Here $a_n = \frac{\sqrt{n}}{n^2 + 1}$. Let $b_n = \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$. Now, consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{n^2 + 1}\right) \div \left(\frac{1}{n^{3/2}}\right) = \lim_{n\to\infty} \frac{n^2}{n^2\left(1 + \frac{1}{n^2}\right)} = 1 \text{ (a finite real number)}$$

Since $\sum b_n = \sum \frac{1}{n^{3/2}}$ is convergent (By p – test) hence, so does the series $\sum \frac{\sqrt{n}}{n^2+1}$.

(viii) $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$

Solution : Here $a_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$. Let $b_n = \frac{1}{\sqrt{n}}$. Now, consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} \right) = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\left(1+\sqrt{1+(1/n)}\right)} = 1 \text{ (a finite real number)}$$

Since $\sum b_n = \sum \frac{1}{n^{1/2}}$ is divergent series (By p – test) hence, so does the series $\sum \frac{1}{\sqrt{n}+\sqrt{n+1}}$.

(ix) $\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + ...$

Solution : Given series in summation form maybe expressed as $\sum \frac{1}{n(n+1)(n+2)}$.

Thus, $a_n \frac{1}{n(n+1)(n+2)}$ and let $b_n = \frac{1}{n^3}$. Now, consider

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n^3}{n(n+1)(n+2)} \right) = \lim_{n\to\infty} \frac{n^3}{n^3\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = 1 \text{ (a finite real number)}$$

Since $\sum b_n = \sum \frac{1}{n^3}$ is convergent series (By p – test) hence, so does the series $\sum \frac{1}{n(n+1)(n+2)}$.

6. Apply "D'ALEMBERT'S RATIO TEST" for the convergence/divergence of the series:

(i) $\sum \frac{2^n}{3^n}$

Solution : Let $a_n = \frac{2^n}{3^n} \Rightarrow a_{n+1} = \frac{2^{n+1}}{3^{n+1}}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}}{3^{n+1}} \div \frac{2^n}{3^n} = \lim_{n\to\infty} \frac{2.2^n}{3.3^n} \times \frac{3^n}{2^n} = \frac{2}{3} < 1$$

Hence, by ratio test given series is convergent series.

(ii) $\sum \frac{n^2}{2^n}$

Solution : Let $a_n = \frac{2^n}{3^n} \Rightarrow a_{n+1} = \frac{2^{n+1}}{3^{n+1}}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \div \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)^2}{2.2^n} \times \frac{2^n}{n^2} = \frac{1}{2} < 1$$

Hence, by ratio test given series is convergent series.

(iii) $1 + \frac{2^2}{3^2} + \frac{2^2.4^2}{3^2.5^2} + \frac{2^2.4^2.6^2}{3^2.5^2.7^2} + \ldots$

Solution : The n^{th} term of this sequence is $a_n = \frac{2^2.4^2.6^2\ldots(2n)^2}{3^2.5^2.7^2\ldots(2n+1)^2}$

$\Rightarrow a_{n+1} = \frac{2^2.4^2.6^2\ldots(2n)^2(2n+2)^2}{3^2.5^2.7^2\ldots(2n+1)^2(2n+3)^2}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^2.4^2.6^2\ldots(2n)^2(2n+2)^2}{3^2.5^2.7^2\ldots(2n+1)^2(2n+3)^2} \div \frac{2^2.4^2.6^2\ldots(2n)^2}{3^2.5^2.7^2\ldots(2n+1)^2}$$

$$= \lim_{n\to\infty} \frac{(2n+2)^2}{(2n+3)^2} = \lim_{n\to\infty} \frac{(2n)^2\left(1+\frac{1}{n}\right)^2}{(2n)^2\left(1+\frac{3}{n}\right)^2} = 1.$$ Hence, ratio test fails.

7. Test for convergence the following series:

(i) $\sum \frac{n!2^n}{n^2}$

Solution : Let $a_n = \frac{n!\ 2^n}{n^2} \Rightarrow a_{n+1} = \frac{(n+1)!\ 2^{n+1}}{(n+1)^2}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\ 2^{n+1}}{(n+1)^2} \div \frac{n!\ 2^n}{n^2} = \lim_{n\to\infty} \frac{2.2^n\ (n+1)\ n!}{(n+1)^2} \times \frac{n^2}{n!\ 2^n}$$

$$= \lim_{n\to\infty} \frac{2n^2}{(n+1)} = \lim_{n\to\infty} \frac{2.n^2}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{2.n^2}{n\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{2.n}{\left(1+\frac{1}{n}\right)} = \frac{2.\infty}{1} = \infty > 1.$$

Hence, by ratio test given series is divergent series.

(ii) $\sum \frac{x^{2n}}{2n}$

Solution : Let $a_n = \frac{x^{2n}}{2n} \Rightarrow a_{n+1} = \frac{x^{2n+2}}{2(n+1)}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{2n+2}}{2(n+1)} \div \frac{x^{2n}}{2n} = \lim_{n\to\infty} \frac{x^{2n}.x^2}{2n\left(1+\frac{1}{n}\right)} \times \frac{2n}{x^{2n}} = \lim_{n\to\infty} \frac{x^2}{\left(1+\frac{1}{n}\right)} = x^2.$$

Hence, by ratio test given series is convergent if $|x| < 1$, divergent if $|x| > 1$. The test if $|x| = 1$.

(iii) $\frac{2}{1} + \frac{2.5.8}{1.5.9} + \frac{2.5.8.11}{1.5.9.13} + \ldots$

Solution : The n^{th} term of this sequence is $a_n = \dfrac{2.5.8...(3n-1)}{1.5.9....(4n-3)}$

$\Rightarrow a_{n+1} = \dfrac{2.5.8...(3n-1)(3n+2)}{1.5.9....(4n-3)(4n+1)}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2.5.8...(3n-1)(3n+2)}{1.5.9....(4n-3)(4n+1)} \div \frac{2.5.8...(3n-1)(3n-1)}{1.5.9....(4n-3)(4n-3)}$$

$$= \lim_{n\to\infty} \frac{(3n+2)}{(4n+1)} = \lim_{n\to\infty} \frac{n\left(3+\frac{2}{n}\right)^2}{n\left(4+\frac{1}{n}\right)^2} = \lim_{n\to\infty} \frac{\left(3+\frac{2}{n}\right)^2}{\left(4+\frac{1}{n}\right)^2} = \frac{3}{4} < 1.$$

Hence, given series is convergent series.

(iv) $\left(\dfrac{1}{3}\right)^2 + \left(\dfrac{1.2}{3.5}\right)^2 + \left(\dfrac{1.2.3}{3.5.7}\right)^2 + ...$

Solution : The n^{th} term of this sequence is $a_n = \left(\dfrac{1.2.3....n}{3.5.7....(2n+1)}\right)$

$\Rightarrow a_{n+1} = \left(\dfrac{1.2.3....n(n+1)}{3.5.7....(2n+1)(2n+3)}\right)$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{1.2.3....n(n+1)}{3.5.7....(2n+1)(2n+3)}\right) \div \left(\frac{1.2.3....n}{3.5.7....(2n+1)}\right)$$

$$= \lim_{n\to\infty} \frac{(n+1)}{(2n+3)} = \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n\left(2+\frac{3}{n}\right)} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{\left(2+\frac{3}{n}\right)} = \frac{1}{2} < 1.$$

Hence, given series is convergent series.

(v) $\sum \dfrac{n^2}{2^n}$

Solution : Let $a_n = \dfrac{n^2}{2^n} \Rightarrow a_{n+1} = \dfrac{(n+1)^2}{2^{n+1}}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \div \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)}{2.2^n} \times \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)}{2} = \frac{1}{2} < 1$$

Hence, by ratio test given series is convergent series.

8. Apply "CAUCHY'S ROOT TEST" for the convergence/divergence of the following series:

(i) $\sum \left(1+\dfrac{1}{n^2}\right)^{n^2}$ NOTE : $\lim_{n\to\infty} \left(1+\dfrac{1}{x}\right)^x = e$ and $\lim_{n\to\infty} n^{1/n} = 1$

Solution : Let $a_n = \left(1+\dfrac{1}{n^2}\right)^{n^2}$. Then by root test

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{1/n}=\lim_{n\to\infty}(e)^{1/n}=e^0=1.\ \text{Hence, the test fails.}$$

(ii) $\sum\frac{1}{n^n}$

Solution : Let $a_n=\frac{1}{n^n}$. Then by root test

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left(\frac{1}{n^n}\right)^{1/n}=\lim_{n\to\infty}\left(\frac{1}{n}\right)=0<1.\ \text{Hence, given series is convergent.}$$

(iii) $\sum n\left(\frac{\pi}{n}\right)^n$

Solution : Let $a_n=n\left(\frac{\pi}{n}\right)^n$. Then by root test

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left(n\left(\frac{\pi}{n}\right)^n\right)^{1/n}=\lim_{n\to\infty}n^{1/n}.\left(\frac{\pi}{n}\right)=1.0=0<1.$$

Hence, given series is convergent.

(iv) $\sum\left(\frac{n}{1+n^3}\right)^n$

Solution : Let $a_n=\left(\frac{n}{1+n^3}\right)^n$. Then by root test

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{n}{1+n^3}\right)^n\right]^{1/n}=\lim_{n\to\infty}\left(\frac{n}{1+n^3}\right)=0<1.$$

Hence, given series is convergent.

(v) $\sum\frac{1}{(\ln n)^n}$

Solution : Let $a_n=\left(\frac{1}{\ln n}\right)^n$. Then by root test

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{1}{\ln n}\right)^n\right]^{1/n}=\lim_{n\to\infty}\left(\frac{1}{\ln n}\right)=0<1.$$

Hence, given series is convergent.

$$\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{3n+2}{2n-1}\right)^n\right]^{1/n}=\lim_{n\to\infty}\left(\frac{3n+2}{2n-1}\right)=\lim_{n\to\infty}\frac{n\left(3+\frac{2}{n}\right)}{n\left(2-\frac{1}{n}\right)}=\lim_{n\to\infty}\frac{\left(3+\frac{2}{n}\right)}{\left(2-\frac{1}{n}\right)}$$

$$=\frac{3+0}{2-1}=\frac{3}{2}>1.\ \text{Hence, given series is divergent.}$$

9. Apply "THE INTEGRAL TEST" for the convergence/divergence of the following series:

(i) $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$

Solution : Consider $\int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} dx$. Let $z = \tan^{-1} x \quad \Rightarrow \quad dz = \frac{1}{1+x^2} dx.$

If $x = 1$ then $z = \tan^{-1}(1) = \pi/4$. If $x = \infty$ then $x = \tan^{-1}(\infty) = \pi/2$. Thus,

$I = \int_{\pi/4}^{\pi/2} z\, dz = \left[\frac{z^2}{2}\right]_{\pi/4}^{\pi/2} = \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{16}\right)$, which is finite hence, given integral is convergent so does the series.

(ii) $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

Solution : Consider $\int_2^{\infty} \frac{1}{x.\ln x} dx$. Let $z = \ln x \quad \Rightarrow \quad dz = \frac{1}{x} dx.$

If $x = 2$ then $z = \ln 2$. If $x = \infty$ then $z = \ln(\infty) = \infty$. Thus,

$I = \int_{\ln 2}^{\infty} \frac{1}{z} dz = \left[\ln z\right]_{\ln 2}^{\infty} = \left(\ln(\ln 2) - \ln(\infty)\right)$, which is infinite hence, given integral is divergent so does the series.

(iii) $\sum_{n=1}^{\infty} ne^{-n^2}$

Solution : Consider $\int_1^{\infty} x.e^{-x^2} dx$. Let $z = x^2 \quad \Rightarrow \quad dz = 2x\, dx \Rightarrow \frac{dz}{2} = x\, dx.$

If $x = 1$ then $z = 1$. If $x = \infty$ then $z = \infty$. Thus,

$I = \int_1^{\infty} e^{-z} \frac{dz}{2} = -\frac{1}{2}\left[e^{-z}\right]_1^{\infty} = -\frac{1}{2}\left(e^{-\infty} - e^{-1}\right) = \frac{1}{2}\left(0 - e^{-1}\right) = \frac{e^{-1}}{2}$, which is finite hence, given integral is convergent so does the series.

(iv) $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$

Solution : Consider $\int_1^{\infty} \frac{1}{1+x^2} dx = \left[\tan^{-1} x\right]_1^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(1) = \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{4}$, which is finite hence, given integral is convergent so does the series.

10. Test whether the following series are "ABSOLUTELY AND CONDITIONALLY CONVERGENT".

(i) $\sum \frac{(-1)^n (n+2)}{n(n+1)}$

Solution : Consider $a_n = \frac{(-1)^n (n+2)}{n(n+1)} \Rightarrow a_{n+1} = \frac{(-1)^{n+1}(n+3)}{(n+1)(n+2)}$

Thus, $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{(n+3)}{(n+1)(n+2)} \div \frac{(n+2)}{n(n+1)}$

$= \lim\limits_{n\to\infty}\frac{(n+3)}{(n+1)(n+2)} \times \frac{n(n+1)}{(n+1)(n+2)} = \lim\limits_{n\to\infty}\frac{n(n+3)}{(n+1)(n+2)^2}$

$= \lim\limits_{n\to\infty}\frac{n.n\left(1+\frac{3}{n}\right)}{n\left(1+\frac{1}{n}\right)n^2\left(1+\frac{2}{n}\right)^2} = \lim\limits_{n\to\infty}\frac{\left(1+\frac{3}{n}\right)}{n\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^2} = \frac{(1+0)}{\infty(1+0)^2} = 0 < 1$

Hence by ratio test the given series is absolutely convergent.

(ii) $\sum\frac{(-1)^n n!}{1.3.5...(2n-1)}$

Solution : Consider $a_n = \frac{(-1)^n n!}{1.3.5...(2n-1)} \Rightarrow a_{n+1} = \frac{(-1)^{n+1}(n+1)!}{1.3.5...(2n-1)(2n+1)}$

Thus, $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\frac{(n+1)!}{1.3.5...(2n-1)(2n+1)} \div \frac{n!}{1.3.5...(2n-1)}$

$= \lim\limits_{n\to\infty}\frac{(n+1)!}{1.3.5...(2n-1)(2n+1)} \times \frac{1.3.5...(2n-1)}{n!} = \lim\limits_{n\to\infty}\frac{(n+1)n!}{n!(2n+1)} = \lim\limits_{n\to\infty}\frac{(n+1)}{(2n+1)}$

$= \lim\limits_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{n\left(2+\frac{1}{n}\right)} = \lim\limits_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)}{\left(2+\frac{1}{n}\right)} = \frac{(1+0)}{(2+0)} = \frac{1}{2} < 1$

Hence by ratio test, the given series is absolutely convergent.

(iii) $\sum\frac{(-1)^n \sin\sqrt{n}}{\sqrt{n^3+1}}$

Solution : Consider $a_n = \frac{(-1)^n \sin\sqrt{n}}{\sqrt{n^3+1}} \Rightarrow |a_n| = \frac{\sin\sqrt{n}}{\sqrt{n^3+1}}$. Consider $|b_n| = \frac{1}{\sqrt{n^3}}$

Now, $|a_n| < |b_n|$ and $\sum|b_n| = \sum\frac{1}{\sqrt{n^3}}$ is convergent hence $\sum|a_n|$ also is convergent.

(iv) $\sum(-1)^n n\tan^{-1}\left(\frac{1}{n}\right)$

Solution : Let $a_n = (-1)^n n\tan^{-1}\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Now for all $n \in N$, $|a_n| > |b_n|$.

Since $\sum b_n = \sum\frac{1}{n}$ is divergent series hence, by comparision test series $\sum|a_n|$ is divergent hence, given series is absolutely divergent.

(v) $\sum(-1)^{n-1}\left(\frac{n+2}{3n-1}\right)^n$

Solution : Let $a_n = (-1)^{n-1}\left(\frac{n+2}{3n-1}\right)^n$. Then by root test,

$$\lim_{n\to\infty}\left(|a_n|\right)^{1/n} = \lim_{n\to\infty}\left(\left(\frac{n+2}{3n-1}\right)^n\right)^{1/n} = \lim_{n\to\infty}\left(\frac{n+2}{3n-1}\right) = \lim_{n\to\infty}\left(\frac{n\left(1+\frac{2}{n}\right)}{n\left(3-\frac{1}{n}\right)}\right)$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{2}{n}\right)}{\left(3-\frac{1}{n}\right)} = \frac{1+0}{3-0} = \frac{1}{3} < 1.$$ Hence, given series is absolutely convergent.

(vi) $\sum(-1)^{n-1}\frac{1}{\ln(\ln n)}$

Solution : Let $a_n = (-1)^{n-1}\frac{1}{\ln(\ln n)}$. Then by integral test, consider

$$\int_2^\infty |a_n|dn = \int_2^\infty \frac{1}{\ln(\ln x)}dx.$$ Putting $z = \ln x \Rightarrow dz = \frac{dx}{x}$. Thus,

$$\int_2^\infty |a_n|dn = \int_{\ln 2}^\infty \frac{1}{z}dz = \ln z\Big|_{\ln 2}^\infty = \ln(\infty) - \ln(\ln 2) = \infty.$$

Since the integral is divergent hence the series is absolutely divergent.

(vii) $\sum(-1)^{n+1}\tan^{-1} n$

Solution : Let $a_n = (-1)^{n+1}\tan^{-1}(n)$ and $b_n = \frac{1}{n}$. Now for all $n \in N, |a_n| > |b_n|$.

Since $\sum b_n = \sum\frac{1}{n}$ is divergent series hence, by comparision test series $\sum|a_n|$ is divergent hence, given series is absolutely divergent.

(viii) $\sum(-1)^{n-1}\frac{n^2}{(n+2)!}$

Solution : Let $|a_n| = \frac{n^2}{(n+2)!}$ and $|a_{n+1}| = \frac{(n+1)^2}{(n+3)!}$. Then by ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{(n+3)!} \div \frac{n^2}{(n+2)!} = \lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)^2}{(n+3)(n+2)!}\times\frac{(n+2)!}{n^2}$$

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^2}{(n+3)} = \frac{1+0}{\infty} = 0 < 1.$$ Hence the given series is absolutely convergent.

11. Determine the values of x for which the given series (i) converges absolutely (ii) converges conditionally (iii) diverges:

(i) $\sum \frac{x^n}{\sqrt{n}}$

Solution : Let $a_n = \frac{x^n}{\sqrt{n}}$ and $a_{n+1} = \frac{x^{n+1}}{\sqrt{n+1}}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{\sqrt{n+1}} \div \frac{x^n}{\sqrt{n}} = \lim_{n\to\infty} \frac{x^{n+1}}{\sqrt{n+1}} \times \frac{\sqrt{n}}{x^n} = \lim_{n\to\infty} \frac{x^n . x\sqrt{n}}{x^n \sqrt{n}\left(1+\frac{1}{\sqrt{n}}\right)} = \frac{x}{1+0} = x$$

If $x = 1$ then series diverges. If $x < 1$ then series converges. If $x > 1$ then series diverges.

(ii) $\sum \frac{nx^n}{n^3}$

Solution : Let $a_n = \frac{nx^n}{n^3}$ and $a_{n+1} = \frac{(n+1)x^{n+1}}{(n+1)^3}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)x^{n+1}}{(n+1)^3} \div \frac{nx^n}{n^3} = \lim_{n\to\infty} \frac{(n+1)x^{n+1}}{(n+1)^3} \times \frac{n^3}{n\,x^n}$$

$$= \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)x^n . n^3}{n\,x^n n^3\left(1+\frac{1}{n^3}\right)} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)x}{\left(1+\frac{1}{n^3}\right)} = \frac{(1+0)\,x}{(1+0)} = x$$

If $x = 1$ then series diverges. If $x < 1$ then series converges. If $x > 1$ then series diverges.

(iii) $\sum \frac{\left(-1^n\right)x^n}{3^n\left(n+1\right)!}$

Solution : Let $|a_n| = \frac{x^n}{3^n(n+1)!}$ and $|a_{n+1}| = \frac{x^{n+1}}{3^{n+1}(n+2)!}$. Then by ratio test

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{x^{n+1}}{3^{n+1}(n+2)!} \div \frac{x^n}{3^n(n+1)!} = \lim_{n\to\infty} \frac{x . x^n . 3^n (n+1)!}{3^n . (n+2)(n+1)! . x^n}$$

$$= \lim_{n\to\infty} \frac{x}{(n+2)} = \frac{1+0}{\infty} = 0 < 1.$$ Hence the given series is absolutely convergent.

(iv) $\sum \frac{4^n}{x^n}$

Solution : Let $a_n = \frac{4^n}{x^n}$ and $a_{n+1} = \frac{4^{n+1}}{x^{n+1}}$. Then by ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{4^{n+1}}{x^{n+1}} \div \frac{4^n}{x^n} = \lim_{n\to\infty} \frac{4.4^n . x^n}{x . x^n 4^n} = \lim_{n\to\infty} \frac{4}{x} = \frac{4}{x}$$

If $\frac{4}{x} = 1$ or $x = 4$ then series diverges. If $\frac{4}{x} < 1$ or $x > 4$ then series converges.

If $\frac{4}{x} > 1$ or $x < 4$ then series diverges.

$(v)\sum\frac{(-1)^{n-1}x^n}{n(n+1)}$

Solution : Let $|a_n| = \dfrac{x^n}{n(n+1)!}$ and $|a_{n+1}|=\dfrac{x^{n+1}}{(n+1)(n+2)}$. Then by ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)(n+2)} \div \frac{x^n}{n(n+1)} = \lim_{n\to\infty} \frac{x.x^n.n(n+1)}{(n+1)(n+2)\,x^n}$$

$= \lim\limits_{n\to\infty} \dfrac{x.n}{n\left(1+\dfrac{1}{n}\right)} = \dfrac{x}{1+0} = x$. Hence the given series is absolutely convergent if $x<1$.

It absolutely diverges if $x>1$.

Chapter 9

FOURIER SERIES

WORKSHEET 09

NOTE: The Fourier series of a function f(x) of period 2L defined on the interval (-L, L) is given by

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)\ \text{where}$$

$$a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx,\ a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}\,dx,\ b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\frac{n\pi x}{L}\,dx.$$

Here a_0, a_n and b_n are called Fourier or Euler's coefficients. If the function f(x) is defined on (0, 2L) then the Fourier coefficients are given by

$$a_0=\frac{1}{L}\int_{0}^{2L}f(x)\,dx,\ a_n=\frac{1}{L}\int_{0}^{2L}f(x)\cos\frac{n\pi x}{L}\,dx,\ b_n=\frac{1}{L}\int_{0}^{2L}f(x)\sin\frac{n\pi x}{L}\,dx.$$

1. Find a Fourier series to represent $f(x)=\pi-x$ for $0<x<\pi$.

Solution : Here $2L=\pi \Rightarrow L=\pi/2.$ and $f(x)=\pi-x.$ Thus,

$$a_0=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\,dx=\frac{2}{\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}=\frac{2}{\pi}\left[\pi^2-\frac{\pi^2}{2}\right]=\frac{2}{\pi}.\frac{\pi^2}{2}=\pi$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\cos 2nx\,dx=\frac{2}{\pi}\left[\left[(\pi-x).\frac{\sin 2nx}{2n}\right]_0^{\pi}-\int_0^{\pi}(-1)\frac{\sin 2nx}{2n}dx\right]$$

$$=\frac{2}{2n\pi}\left[(0-0)+\left[-\frac{\cos 2nx}{2n}\right]_0^{\pi}\right]=-\frac{1}{2n^2\pi}\left[\cos 2n\pi-\cos(0)\right]=-\frac{1}{2n^2\pi}\left[1-1\right]=0$$

NOTE : $\sin 2n\pi=\sin(0)=0$ and $\cos 2n\pi=\cos 0=1.$

$$b_n=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\sin 2nx\,dx=\frac{2}{\pi}\left[\left[(\pi-x).\left(\frac{-\cos 2nx}{2n}\right)\right]_0^{\pi}-\int_0^{\pi}(-1)\left(\frac{-\cos 2nx}{2n}\right)dx\right]$$

$$=\frac{2}{2n\pi}\left[-\left(0-\pi.\cos 2n\pi\right)-\left[\frac{\sin 2nx}{2n}\right]_0^{\pi}\right]=\frac{1}{n\pi}\left(\pi-\frac{1}{2n}\left(0-0\right)\right)=\frac{1}{n}$$

Now, F.S of given function is : $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$

$$f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{\sin 2nx}{n}=\frac{\pi}{2}+\left(\frac{\sin 2x}{1}+\frac{\sin 4x}{2}+\frac{\sin 6x}{3}+...\right)$$

2. If $f(x) = [(\pi - x)/2]^2$ in the range 0 to 2π, show that $f(x)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$

Solution : Here $2L=2\pi \Rightarrow L=\pi$, and $f(x)=\left(\frac{\pi-x}{2}\right)^2$. Thus,

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2 dx=\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}=-\frac{1}{12\pi}\left[-\pi^3-\pi^3\right]=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2\cos nx\,dx=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$=\frac{1}{4\pi}\left\{\left[(\pi-x)^2\frac{\sin nx}{n}\right]_0^{2\pi}-\int_0^{2\pi}2(\pi-x)(-1)\frac{\sin nx}{n}dx\right\}$$

$$=\frac{1}{4n\pi}\left\{(0-0)+2\left[(\pi-x).\left(-\frac{\cos nx}{n}\right)\right]_0^{2\pi}-2\int_0^{2\pi}(-1)\left(-\frac{\cos nx}{n}\right)dx\right\}$$

$$=\frac{1}{4n^2\pi}\left\{0-2\left[(\pi-2\pi)\cos 2n\pi-(\pi-0)\cos 0\right]-2\left[\frac{\sin nx}{n}\right]_0^{2\pi}\right\}$$

$$=\frac{1}{4n^2\pi}\left[-2(-\pi-\pi)-2(0-0)\right]=\frac{1}{4n^2\pi}[4\pi]=\frac{1}{n^2}$$

NOTE : $\sin 2n\pi=\sin(0)=0$ and $\cos 2n\pi=\cos 0=1$.

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi-x}{2}\right)^2\sin nx\,dx=\frac{1}{4\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$$=\frac{1}{4\pi}\left\{\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right)\right]_0^{2\pi}-\int_0^{2\pi}2(\pi-x)(-1)\left(-\frac{\cos nx}{n}\right)dx\right\}$$

$$=\frac{1}{4n\pi}\left\{(0-0)-2\left[(\pi-x).\left(\frac{\sin nx}{n}\right)\right]_0^{2\pi}+2\int_0^{2\pi}(-1)\left(\frac{\sin nx}{n}\right)dx\right\}$$

$$=\frac{1}{4n^2\pi}\left\{0-2[0-0]-2\left[-\frac{\cos nx}{n}\right]_0^{2\pi}\right\}=\frac{1}{4n^3\pi}\{0-2[0-0]-2[0-0]\}=0$$

Now, F.S of given function is : $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$

3. Given $f(x) = x + x^2$ for $-\pi < x < \pi$ and $f(x) = \pi^2$ for $x=\pm\pi\cdot$ Expand f(x) in Fourier series and show that $f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right)$.

(a) Here $f(x) = (x + x^2)$ and $2L = 2\pi$ hence, $L = \pi$. Now, it may be noted that:

$g(x)=x$ is odd function and $h(x)=x^2$ is even function. Also notice that for odd function $g(x)$, $\int_{-\pi}^{\pi} g(x)\,dx=0$ and for even function $h(x)$, $\int_{-\pi}^{\pi} h(x)\,dx=2\int_{0}^{\pi} h(x)\,dx$.

Now, $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)dx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx+\int_{-\pi}^{\pi} x^2\,dx\right]=\frac{1}{\pi}\left[0+2\int_{-\pi}^{\pi} x^2\,dx\right]=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2\pi^2}{3}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx+\int_{-\pi}^{\pi} x^2\cos nx\,dx\right]=\frac{1}{\pi}\left[0+2\int_{0}^{\pi} x^2\cos nx\,dx\right]$$

NOTE: $x\cos nx$ is odd function and $x^2\cos nx$ is even function.

$$=\frac{2}{\pi}\left\{\left[x^2\frac{\sin nx}{n}\right]_0^{\pi}-\int_0^{\pi}2x.\frac{\sin nx}{n}dx\right\}=\frac{2}{n\pi}\left\{(0-0)-2\int_0^{\pi}x.\sin nx\;dx\right\}$$

NOTE: $\sin 0=\sin n\pi=0$

$$=\frac{-4}{n\pi}\left\{\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi}-\int_0^{\pi}1.\left(-\frac{\cos nx}{n}\right)dx\right\}=\frac{-4}{n^2\pi}\left\{-\left(\pi\cos nx-0\right)+\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

Thus, $a_n=\frac{4\pi}{n^2\pi}\left\{(-1)^n+(0-0)\right\}=\frac{4}{n^2}(-1)^n$. NOTE: $\cos 0=1$ and $\cos n\pi=(-1)^n$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\,dx+\int_{-\pi}^{\pi} x^2\sin nx\,dx\right]=\frac{1}{\pi}\left[2\int_{0}^{\pi} x\sin nx\,dx+0\right]$$

NOTE: $x^2\sin nx$ is odd function and $x\sin nx$ is even function.

$$=\frac{2}{\pi}\left\{\left[x\frac{-\cos nx}{n}\right]_0^{\pi}-\int_0^{\pi}1.\frac{-\cos nx}{n}dx\right\}=\frac{2}{n\pi}\left\{-\left(\pi\cos n\pi-0\right)+\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

$$=\frac{2}{n\pi}\left\{-\pi(-1)^n+\left(\frac{0-0}{n}\right)\right\}=\frac{2}{n}(-1)^{n+1}$$ NOTE: $\sin 0=\sin\pi=0$

Now, F.S of given function is: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$

$$f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}+2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left(\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right)$$

(b) Here $f(x)=\pi^2$. Thus,

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{\pi^2}{\pi}\int_{-\pi}^{\pi}1.dx=\pi\left[x\right]_{-\pi}^{\pi}=\pi\left[\pi-(-\pi)\right]=2\pi^2$$

$$a_n=\frac{\pi^2}{\pi}\int_{-\pi}^{\pi}\cos nx\,dx=\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}=(0-0)=0$$

$$b_n=\frac{\pi^2}{\pi}\int_{-\pi}^{\pi}\sin nx\,dx=\pi\left[\frac{-\cos nx}{n}\right]_{-\pi}^{\pi}=\frac{-\pi}{n}\left[\cos\,n\pi-\cos(-n\pi)\right]=-\frac{\pi}{n}\left[(-1)^n-(-1)^n\right]$$

$\therefore b_n = 0$ NOTE: $\sin 0 = \sin n\pi = 0$ and $\cos 0 = 1$, $\cos n\pi = \cos(-n\pi) = (-1)^n$

Now, F.S of given function is: $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) = \dfrac{2\pi^2}{2} + 0 = \pi^2$

4. Expand $f(x) = x\sin x,\ 0 < x < 2\pi,$ in a Fourier series.

Solution: Here $2L = 2\pi \Rightarrow L = \pi$. Thus,

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x.\sin x\,dx = \frac{1}{\pi}\left\{\left[x(-\cos x)\right]_0^{2\pi} - \int_0^{2\pi} 1.(-\cos x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\left(2\pi\cos 2\pi - 0\right) + \left[\sin x\right]_0^{2\pi}\right\} = \frac{1}{\pi}\left\{-\left(2\pi.1\right) + \sin 2\pi - \sin 0\right\} = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x.\sin x\cos nx\ dx = \frac{1}{2\pi}\int_0^{2\pi} x.(2\sin x\cos nx)\ dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x.\left[\sin(1+n)x + \sin(1-n)x\right] dx \quad \text{NOTE: } 2\sin u\cos v = \sin(u+v) + \sin(u-v)$$

If $n = 1$ then $a_1 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} x.\left[\sin 2x + \sin 0x\right] dx = \dfrac{1}{2\pi}\left[x\left(\dfrac{-\cos 2x}{2}\right)\Bigg|_0^{2\pi} - \displaystyle\int_0^{2\pi} 1.\left(\dfrac{-\cos 2x}{2}\right)dx\right]$

$$a_1 = \frac{1}{2\pi}\left[-\frac{2\pi\cos 4\pi - 0}{2} + \frac{\sin 2x}{4}\Bigg|_0^{2\pi}\right] = -\frac{1}{2}. \quad \text{NOTE: } \sin 0 = \sin 2\pi = 0 \text{ and } \cos 4\pi = 1$$

If $n \neq 1$ then $a_n = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right] dx$ NOTE: $\sin(-u) = -\sin u$

$$a_n = \frac{1}{2\pi}\left\{\left[x\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right)\right]_0^{2\pi} - \int_0^{2\pi} 1.\left(\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right) dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[2\pi\left(\frac{-1}{n+1} + \frac{1}{n-1}\right) - 0\right] + \left[\frac{\sin(n+1)x}{(n+1)^2} - \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left[2\pi\frac{-(n-1)+(n+1)}{n^2-1} + (0-0)\right] = \frac{2}{n^2-1}. \quad \text{NOTE: } \sin 0 = \sin 2(n\pm1)\pi = 0$$

Thus, $a_1 = -\dfrac{1}{2}$ and $a_n = \dfrac{2}{n^2-1}$ when $n \neq 1$.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x.\sin x\sin nx\ dx = \frac{1}{2\pi}\int_0^{2\pi} x.(2\sin x\sin nx)\ dx = \frac{1}{2\pi}\int_0^{2\pi} x.\left[\cos(1-n)x - \cos(1+n)x\right] dx$$

If $n = 1$ then $b_1 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} x.\left[\cos 0 - \cos 2x\right] dx = \dfrac{1}{2\pi}\left\{\displaystyle\int_0^{2\pi} x\ dx - \int_0^{2\pi} x.\cos 2x\ dx\right\}$

$$= \frac{1}{2\pi}\left\{\left(\frac{x^2}{2}\right)\Bigg|_0^{2\pi} - \left[x.\frac{\sin 2x}{2}\Bigg|_0^{2\pi} - \int_0^{2\pi} 1.\left(\frac{\sin 2x}{2}\right)dx\right]\right\} = \frac{1}{2\pi}\left\{2\pi^2 - \left[(0-0) + \frac{\cos 2x}{4}\right]_0^{2\pi}\right\}$$

$$b_1 = \frac{1}{2\pi}\left[2\pi^2 - \frac{\cos 4\pi - \cos 0}{4}\right] = \frac{1}{2\pi}\left[2\pi^2 - \frac{1-1}{4}\right] = \pi.$$

If $n \neq 1$ then $a_n = \frac{1}{2\pi}\int_0^{2\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx \qquad [\cos(-\theta) = \cos\theta]$

$$b_n = \frac{1}{2\pi}\left\{\left[x\left(\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right)\right]_0^{2\pi} - \int_0^{2\pi} 1.\left(\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right) dx\right\}$$

$$= \frac{1}{2\pi}\left\{\left[2\pi(0-0) - 0\right] - \left[\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi}\right\}$$

$= \frac{1}{2\pi}[0-0] = 0.$ NOTE: $\sin 0 = \sin 2(n\pm1)\pi = 0, \ \cos 2(n\pm1)\pi = \cos 0 = 1$

Thus, $b_1 = \pi$ and $b_n = 0$, when $n \neq 1$.

Now, F.S of given function is: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$

$$f(x) = -\frac{2}{2} - \frac{\cos x}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{n^2-1}\cos nx\right) = -\left(\frac{2+\cos x}{2}\right) + 2\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2-1}\right)$$

5. If f(x) = |cos x|, expand f(x) as a Fourier series in the interval $(-\pi, \pi)$.

Solution: Since $f(x) = |\cos x|$ is an even function hence,

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$, $a_n = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx$ and $b_n = 0$. Now,

$a_0 = \frac{2}{\pi}\int_0^{\pi}\cos x\, dx = \frac{2}{\pi}[\sin x]_0^{\pi} = \frac{2}{\pi}[\sin\pi - \sin 0] = 0.$ NOTE: $\sin 0 = \sin\pi = 0.$

Now consider, $a_n = \frac{2}{\pi}\int_0^{\pi}\cos x \cos nx\, dx$

If $n = 1$ then $a_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \cos x\, dx = \frac{2}{\pi}\int_0^{\pi}\cos^2 x\, dx = \frac{2}{\pi}\int_0^{\pi}\frac{1+\cos 2x}{2} dx = \frac{1}{\pi}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi}$

$= \frac{1}{\pi}\left[\pi + \frac{0}{2}\right] = 1.$ If $n \neq 1$ then

$$a_n = \frac{2}{\pi}\int_0^{\pi}\cos nx \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} 2\cos nx \cos x\, dx = \frac{1}{\pi}\int_0^{\pi}\left[\cos(n+1)x + \cos(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right]_0^{\pi} = 0$$

Thus, F.S of given function is: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$

$|\cos x| = 0 + \cos x + 0 + 0 = \cos x$

6. Show that for $-\pi < x < \pi$,

$$x\cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{1\cdot 3}\sin 2x - \frac{3}{2\cdot 4}\sin 3x + \frac{4}{3\cdot 5}\sin 4x - ...\right)$$

Solution : Here $2L = 2\pi \Rightarrow L = \pi$. Since, $f(x) = x\cos x$ is odd function hence,

$a_0 = a_n = 0$, and $b_n = \frac{2}{\pi}\int_0^{\pi} x.\cos x \sin nx\, dx$. Let us see this :

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x.\cos x\, dx = \frac{1}{\pi}\left\{\left[x(\sin x)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 1.(\sin x)\, dx\right\}$$

$$= \frac{1}{\pi}\left\{(0-0) - \left[-\cos x\right]_{-\pi}^{\pi}\right\} = \frac{1}{\pi}\left(\cos\pi - \cos(-\pi)\right) = \frac{1}{\pi}\left[\cos\pi - \cos\pi\right] = 0.$$

Similarly, $a_n = 0$. NOTE : $2\sin u \cos v = \sin(u+v) + \sin(u-v)$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} x.(2\cos x\sin nx)\, dx = \frac{1}{\pi}\int_0^{\pi} x.\left[\sin(n+1)x + \sin(n-1)x\right] dx$$

If $n = 1$ then $b_1 = \frac{1}{\pi}\int_0^{\pi} x.\sin 2x\, dx = \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos 2x}{2}\right)\right]_0^{\pi} - \int_0^{\pi} 1.\left(-\frac{\cos 2x}{2}\right) dx\right\}$

$$= \frac{1}{2\pi}\left\{-\left(\pi\cos 2\pi - 0\cos 0\right) + \frac{\sin 2x}{2}\bigg|_0^{\pi}\right\} = \frac{1}{2\pi}\left\{-\pi + (0-0)\right\} = -\frac{1}{2}.$$ Thus, $b_1 = -\frac{1}{2}$.

If $n \neq 1$, $b_n = \frac{1}{\pi}\left\{\begin{array}{l}\left[x\left(-\frac{\cos(n+1)x}{n+1}\right) + x\left(-\frac{\cos(n-1)x}{n-1}\right)\right]_0^{\pi} \\ -\int_0^{\pi} 1.\left[\left(-\frac{\cos(n+1)x}{n+1}\right) + \left(-\frac{\cos(n-1)x}{n-1}\right)\right]\end{array}\right\}$

$$= \frac{1}{\pi}\left\{\left[-\pi\left(\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right) - (0-0)\right] + \left[\left(\frac{\sin(n+1)x}{(n+1)^2}\right) + \left(\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\pi(-1)^n\left(\frac{-1}{n+1} + \frac{-1}{n-1}\right)\right] + (0-0)\right\} = \frac{1}{\pi}\left\{\left[-\pi(-1)^n\left(\frac{-n+1-n-1}{(n+1)(n-1)}\right)\right]\right\}$$

$\therefore\ b_n = \frac{1}{\pi}\left[\frac{2n\pi(-1)^n}{(n^2-1)}\right] = \frac{2n(-1)^n}{(n^2-1)}$. NOTE : $a_0 = a_n = 0$ and $b_1 = -\frac{1}{2}$

Now, F.S of given function is : $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$

$$f(x) = -\frac{\sin x}{2} + \sum_{n=2}^{\infty}\frac{2n(-1)^n\sin nx}{(n^2-1)}$$

$$= -\frac{\sin x}{2} + 2\left(\frac{2}{2^2-1}\sin 2x - \frac{3}{3^2-1}\sin 3x + \frac{4}{4^2-1}\sin 4x - ...\right)$$

$$\therefore\ x\cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{3}\sin 2x - \frac{3}{8}\sin 3x + \frac{4}{15}\sin 4x - \ldots\right)$$

$$\text{Or, } x\cos x = -\frac{1}{2}\sin x + 2\left(\frac{2}{1.3}\sin 2x - \frac{3}{2.4}\sin 3x + \frac{4}{3.5}\sin 4x - \ldots\right)$$

7. Find the Fourier series of the function

$$f(x) = \begin{cases} -1 & \text{for} \quad -\pi < x < -\frac{\pi}{2} \\ 0 & \text{for} \quad -\frac{\pi}{2} < x < \frac{\pi}{2} \\ +1 & \text{for} \quad \frac{\pi}{2} < x < \pi \end{cases} \cdot \ \text{NOTE: } \sin(-u) = -\sin u \quad \text{and} \quad \cos(-u) = \cos u$$

$$\text{Solution: } a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -1.dx + \int_{-\pi/2}^{\pi/2} 0.dx + \int_{\pi/2}^{\pi} 1.dx\right] = \frac{1}{\pi}\left[-x\Big|_{-\pi}^{-\pi/2} + 0 + x\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\left(-\frac{\pi}{2} - (-\pi)\right) + 0 + \left(\pi - \frac{\pi}{2}\right)\right] = \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \pi - \frac{\pi}{2}\right] = \frac{1}{\pi}(0) = 0 \quad \Rightarrow a_0 = 0$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -1.\cos nx\ dx + \int_{-\pi/2}^{\pi/2} 0.\cos nx\, dx + \int_{\pi/2}^{\pi} 1.\cos nx\, dx\right] = \frac{1}{\pi}\left[\frac{-\sin nx}{n}\Bigg|_{-\pi}^{-\pi/2} + \frac{\sin nx}{n}\Bigg|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[-\left(\sin\frac{-n\pi}{2} - \sin(-n\pi)\right) + \sin n\pi - \sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{n\pi}\left[\sin\frac{n\pi}{2} - \sin n\pi + \sin n\pi - \sin\frac{n\pi}{2}\right] = 0$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -1.\sin nx\ dx + \int_{-\pi/2}^{\pi/2} 0.\sin nx\, dx + \int_{\pi/2}^{\pi} 1.\sin nx\, dx\right] = \frac{1}{\pi}\left[\frac{\cos nx}{n}\Bigg|_{-\pi}^{-\pi/2} - \frac{\cos nx}{n}\Bigg|_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\left(\cos\frac{-n\pi}{2} - \cos(-n\pi)\right) - \left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right] = \frac{1}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi - \cos n\pi + \cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

Now, if $n = 1, 3, 5, \ldots$ (odd) then $b_n = \frac{2}{n\pi}[0 - (-1)] = \frac{2}{n\pi}$

If $n = 2, 6, 10, \ldots$ then $b_n = \frac{2}{n\pi}[0 - (1)] = -\frac{2}{n\pi}$

If $n = 4, 8, 12, \ldots$ then $b_n = \frac{2}{n\pi}[1 - 1] = 0$

NOTE: $\cos\pi = \cos 3\pi = \cos 5\pi = \ldots = -1$ and $\cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = \cos\frac{5\pi}{2} = \ldots = 0$

$\cos 2\pi = \cos 4\pi = \cos 6\pi = \ldots = 1$

Now, F.S of given function is: $f(x) = \frac{a_0}{2} + \sum\left(a_n\cos nx + b_n\sin nx\right)$

$$f(x)=\frac{2}{\pi}\left[\sum_{n=1,\,3,\,5,\,\ldots}^{\infty}\frac{\sin nx}{n}-\sum_{n=2,\,6,\,10,\,\ldots}^{\infty}\frac{\sin nx}{n}\right] \qquad \text{NOTE}: a_0=a_n=0$$

$$=\frac{2}{\pi}\left[\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right)-\left(\frac{\sin 2x}{2}+\frac{\sin 6x}{6}+\frac{\sin 10x}{10}+\ldots\right)\right]$$

8. Find the Fourier series of the function

$$f(x)=\begin{cases}-1 & \text{for} \quad -\pi<x<0\\ +1 & \text{for} \quad 0<x<\pi\end{cases} \quad \text{where } f(x+2\pi)=f(x).$$

$$\text{Solution}: \ a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}-1.dx+\int_{0}^{\pi}1.dx\right]=\frac{1}{\pi}\left[-x\Big|_{-\pi}^{0}+x\Big|_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}\left[-\left(0-(-\pi)\right)+\left(\pi-0\right)\right]=\frac{1}{\pi}\left[-\pi+\pi\right]=\frac{1}{\pi}(0)=0 \qquad \Rightarrow a_0=0$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-1.\cos nx\;dx+\int_{0}^{\pi}1.\cos nx\,dx\right]=\frac{1}{\pi}\left[\frac{-\sin nx}{n}\Big|_{-\pi}^{0}+\frac{\sin nx}{n}\Big|_{0}^{\pi}\right]$$

$$=\frac{1}{n\pi}\left[-\left(\sin 0-\sin(-n\pi)\right)+\sin n\pi-\sin 0\right]=\frac{1}{n\pi}[0]=0$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-1.\sin nx\;dx+\int_{0}^{\pi}1.\sin nx\,dx\right]=\frac{1}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{0}-\frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$

$$=\frac{1}{n\pi}\left[\left(\cos 0-\cos(-n\pi)\right)-\left(\cos n\pi-\cos 0\right)\right]=\frac{1}{n\pi}\left[1-(-1)^n-(-1)^n+1\right]$$

$$=\frac{1}{n\pi}\left[2-2(-1)^n\right]=\frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore\ b_n=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{4}{n\pi} & \text{if } n \text{ is odd}\end{cases}$$

$$\text{Now, F.S of given function is}: \ f(x)=\frac{a_0}{2}+\sum\left(a_n\cos nx+b_n\sin nx\right)$$

$$\text{NOTE}: a_0=a_n=0$$

$$f(x)=\frac{4}{\pi}\left[\sum_{n=1,\,3,\,5,\,\ldots}^{\infty}\frac{\sin nx}{n}\right]=\frac{4}{\pi}\left(\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\ldots\right)$$

9. Find the Fourier series of the function defined as

$$f(x)=\begin{cases}x+\pi & \text{for} \quad 0<x<\pi\\ -x-\pi & \text{for} \quad -\pi<x<0\end{cases}$$

$$\text{Solution}: \ a_0=\frac{1}{\pi}\left[\int_{-\pi}^{0}-(x+\pi).dx+\int_{0}^{\pi}(x+\pi).dx\right]=\frac{1}{\pi}\left[-\left(\frac{x^2}{2}+\pi x\right)\Bigg|_{-\pi}^{0}+\left(\frac{x^2}{2}+\pi x\right)\Bigg|_{0}^{\pi}\right]$$

$$=\frac{1}{\pi}\left\{-\left[(0-0)-\left(\frac{\pi^2}{2}-\pi^2\right)\right]+\left(\frac{\pi^2}{2}+\pi^2\right)-(0-0)\right\}=\frac{1}{\pi}\left[2\pi^2\right]=0 \quad \Rightarrow a_0=2\pi$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi).\cos nx\ dx + \int_{0}^{\pi}(x+\pi).\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-(x+\pi)\frac{\sin nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0} -1.\frac{\sin nx}{n}dx + (x+\pi)\frac{\sin nx}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi}1.\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{n\pi}\left[-(0-0) - \frac{\cos nx}{n}\Big|_{-\pi}^{0} + (0-0) + \frac{\cos nx}{n}\Big|_{0}^{\pi}\right] \qquad \text{NOTE: } \sin 0 = \sin n\pi = 0$$

$$= \frac{1}{n^2\pi}\left[-\cos 0 + \cos(-n\pi) + \cos n\pi - \cos 0\right] = \frac{1}{n^2\pi}\left[-1 + (-1)^n + (-1)^n - 1\right] = \frac{2}{n^2\pi}\left[-1+(-1)^n\right]$$

$$\therefore\ a_n = \begin{cases} 0 & \text{if n is even} \\ \dfrac{-4}{n^2\pi} & \text{if n is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -(x+\pi).\sin nx\ dx + \int_{0}^{\pi}(x+\pi).\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-(x+\pi)\frac{-\cos nx}{n}\Big|_{-\pi}^{0} - \int_{-\pi}^{0} -1.\frac{-\cos nx}{n}dx + (x+\pi)\frac{-\cos nx}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi}1.\frac{-\cos nx}{n}dx\right]$$

$$= \frac{1}{n\pi}\left[(0+\pi)\cos 0 - 0 - \frac{\sin nx}{n}\Big|_{-\pi}^{0} - (\pi+\pi)\cos n\pi + (0+\pi)\cos 0 + \frac{\sin nx}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\pi - (0-0) - 2\pi(-1)^n + \pi + (0-0)\right] = \frac{2\pi}{n\pi}\left[1-(-1)^n\right] = \frac{2}{n}\left[1-(-1)^n\right]$$

$$\therefore\ b_n = \begin{cases} 0 & \text{if n is even} \\ \dfrac{4}{n} & \text{if n is odd} \end{cases}$$

Now, F.S of given function is: $f(x) = \frac{a_0}{2} + \sum\left(a_n \cos nx + b_n \sin nx\right)$

$$f(x) = \pi - \frac{4}{\pi}\sum_{n=1,2,3\ldots}^{\infty}\frac{\cos nx}{n^2} + 4\sum_{n=1,2,3\ldots}^{\infty}\frac{\sin nx}{n}$$

10. Find the Fourier series for the function

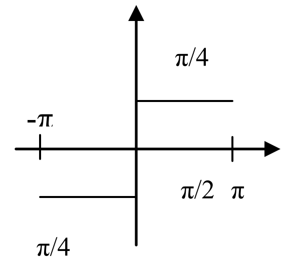
$$f(x) = \begin{cases} -\dfrac{\pi}{4} & \text{for} \quad -\pi < x < 0 \\ +\dfrac{\pi}{4} & \text{for} \quad 0 < x < \pi \end{cases},$$

and $f(-\pi) = f(0) = f(\pi) = 0$, $f(x) = f(x+2\pi)$ for all x. Deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Solution: $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4}.dx + \int_{0}^{\pi}\frac{\pi}{4}.dx\right] = \frac{1}{4}\left[-(x)\big|_{-\pi}^{0} + x\big|_{0}^{\pi}\right]$

$$=\frac{1}{4}\left\{-\left[0-(-\pi)\right]+(\pi-0)\right\}=\frac{1}{4}(-\pi+\pi)=0$$



$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\frac{\pi}{4}.\cos nx\ dx+\int_{0}^{\pi}\frac{\pi}{4}.\cos nx\,dx\right]$$

$$=\frac{1}{4}\left[-\frac{\sin nx}{n}\bigg|_{-\pi}^{0}+\frac{\sin nx}{n}\bigg|_{0}^{\pi}\right]=0$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-\frac{\pi}{4}.\sin nx\ dx+\int_{0}^{\pi}\frac{\pi}{4}.\sin nx\,dx\right]\qquad \text{NOTE: } \sin 0=\sin n\pi=\sin(-n\pi)=0$$

$$=\frac{1}{4}\left[\frac{\cos nx}{n}\bigg|_{-\pi}^{0}+\frac{-\cos nx}{n}\bigg|_{0}^{\pi}\right]=\frac{1}{4n}\left[\left(\cos 0-\cos(-n\pi)\right)-\left(\cos n\pi-\cos 0\right)\right]$$

$$=\frac{1}{4n}\left[1-(-1)^n-(-1)^n+1\right]=\frac{1}{4n}\left[2-2(-1)^n\right]=\frac{1}{2n}\left[1-(-1)^n\right]$$

$$\therefore\ b_n=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{1}{n} & \text{if } n \text{ is odd}\end{cases}$$

Now, F.S of given function is: $f(x)=\dfrac{a_0}{2}+\sum\left(a_n\cos nx+b_n\sin nx\right)$.

NOTE: $a_0=a_n=0$

$$f(x)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{\sin nx}{n}=\frac{\sin x}{1}+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+\frac{\sin 7x}{7}+\frac{\sin 9x}{9}+\frac{\sin 11x}{11}+\ldots$$

Putting $x=\pi/2$ and notice that $\sin\dfrac{\pi}{2}=\sin\dfrac{5\pi}{2}=\sin\dfrac{9\pi}{2}=\ldots=1$, and

$\sin\dfrac{3\pi}{2}=\sin\dfrac{7\pi}{2}=\sin\dfrac{11\pi}{2}=\ldots=-1$, we have

$f\left(\dfrac{\pi}{2}\right)=1-\dfrac{1}{3}+\dfrac{1}{5}-\dfrac{1}{7}+\dfrac{1}{9}-\dfrac{1}{11}+\ldots$ But $f\left(\dfrac{\pi}{2}\right)=\dfrac{\pi}{4}$ (see the above figure). Thus,

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\ldots$$

11. Find the Fourier series for f(x) if

$$f(x)=\begin{cases}-\pi & \text{for} \quad -\pi<x\le 0\\ x & \text{for} \quad 0<x<\pi.\\ -\pi/2 & \text{for} \quad x=0\end{cases}\quad \text{Deduce that } \frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots$$

Solution: $a_0=\dfrac{1}{\pi}\left[\displaystyle\int_{-\pi}^{0}-\pi.dx+\int_{0}^{\pi}x.dx\right]=\dfrac{1}{\pi}\left[-\pi x\Big|_{-\pi}^{0}+\dfrac{x^2}{2}\bigg|_{0}^{\pi}\right]=\dfrac{1}{\pi}\left[-\pi\left(0-(-\pi)\right)+\dfrac{\pi^2}{2}\right]$

$$=\frac{1}{\pi}\left(-\pi^2+\frac{\pi^2}{2}\right)=\frac{1}{\pi}\left(\frac{-\pi^2}{2}\right)=-\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi.\cos nx\ dx + \int_{0}^{\pi} x.\cos nx\, dx\right] = \frac{1}{\pi}\left[-\pi\frac{\sin nx}{n}\Big|_{-\pi}^{0} + \frac{x\sin nx}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi} 1.\frac{\sin nx}{n}dx\right]$$

$$= \frac{1}{n\pi}\left[-(0-0)+(0-0)+\frac{\cos nx}{n}\Big|_{0}^{\pi}\right] \qquad \text{NOTE: } \sin 0 = \sin n\pi = 0$$

$$= \frac{1}{n^2\pi}\left[\cos(n\pi)-\cos 0\right] = \frac{1}{n^2\pi}\left[(-1)^n - 1\right]$$

$$\therefore\ a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{-2}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi.\sin nx\ dx + \int_{0}^{\pi} x.\sin nx\, dx\right] = \frac{1}{\pi}\left[-\pi\frac{-\cos nx}{n}\Big|_{-\pi}^{0} + \frac{-x\cos nx}{n}\Big|_{0}^{\pi} - \int_{0}^{\pi} 1.\frac{-\cos nx}{n}dx\right]$$

$$= \frac{1}{n\pi}\left[\pi\left[\cos 0 - \cos(-n\pi)\right] - \left(\pi\cos n\pi - 0\right) + \frac{\sin nx}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{n\pi}\left[\pi\left[1-2(-1)^n\right]+(0-0)\right] = \frac{1}{n}\left(1-2(-1)^n\right) \qquad \text{NOTE: } \cos 0 = 1 \text{ and } \cos n\pi = (-1)^n$$

$$\therefore\ b_n = \begin{cases} -\dfrac{1}{n} & \text{if } n \text{ is even} \\ \dfrac{3}{n} & \text{if } n \text{ is odd} \end{cases}$$

Now, F.S of given function is: $f(x) = \frac{a_0}{2} + \sum\left(a_n \cos nx + b_n \sin nx\right)$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,5...}^{\infty}\frac{\cos nx}{n^2} + 3\sum_{n=1,3,5...}^{\infty}\frac{\sin nx}{n} - \sum_{n=2,4,6...}^{\infty}\frac{\sin nx}{n}$$

Putting $x = 0$ and notice that $f(0) = -\frac{\pi}{2}$ (given), $\sin 0 = 0$, $\cos 0 = 0$

$$\therefore\ -\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2} \Rightarrow -\frac{\pi}{4} = -\frac{2}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right) \Rightarrow \frac{\pi^2}{8} = \left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right)$$

12. Expand f(x) as a Fourier series, the function defined as

$$f(x) = \begin{cases} \pi + x & \text{for} \quad -\pi < x < -\dfrac{\pi}{2} \\ \pi/2 & \text{for} \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \pi - x & \text{for} \quad \dfrac{\pi}{2} < x < \pi \end{cases}$$

Solution: $a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(\pi + x)\,dx + \int_{-\pi/2}^{\pi/2}\frac{\pi}{2}dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$

$$=\frac{1}{\pi}\left\{\left[\pi x+\frac{x^2}{2}\right]_{-\pi}^{-\pi/2}+\frac{\pi x}{2}\Bigg|_{-\pi/2}^{\pi/2}+\left[\pi x-\frac{x^2}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$=\frac{1}{\pi}\left[\left(-\frac{\pi^2}{2}+\frac{\pi^2}{8}\right)-\left(-\pi^2+\frac{\pi^2}{2}\right)+\left(\frac{\pi^2}{4}+\frac{\pi^2}{4}\right)+\left(\pi^2-\frac{\pi^2}{2}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^2}{8}\right)\right]=\frac{3\pi}{4}$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(\pi+x)\cos nx dx+\int_{-\pi/2}^{\pi/2}\frac{\pi}{2}\cos nx\,dx+\int_{\pi/2}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left\{\begin{array}{l}\left[\dfrac{(\pi+x)\sin nx}{n}\Bigg|_{-\pi}^{-\pi/2}-\displaystyle\int_{-\pi}^{-\pi/2}1.\frac{\sin nx}{n}dx\right]+\dfrac{\pi}{2}\dfrac{\sin nx}{n}\Bigg|_{-\pi/2}^{\pi/2}\\ +\left[\dfrac{(\pi-x)\sin nx}{n}\Bigg|_{\pi/2}^{\pi}-\displaystyle\int_{\pi/2}^{\pi}-1.\frac{\sin nx}{n}dx\right]\end{array}\right\}$$

$$=\frac{1}{n\pi}\left\{\begin{array}{l}\left(\dfrac{\pi}{2}\sin\dfrac{-n\pi}{2}-0\sin(-n\pi)\right)+\dfrac{\cos n\pi}{n}\Bigg|_{-\pi}^{-\pi/2}+\dfrac{\pi}{2}\left(\sin\dfrac{n\pi}{2}-\sin\dfrac{-n\pi}{2}\right)\\ +\left(0-\dfrac{\pi}{2}\sin\dfrac{n\pi}{2}\right)-\dfrac{\cos nx}{n}\Bigg|_{\pi/2}^{\pi}\end{array}\right\}$$

$$=\frac{1}{n\pi}\left[-\frac{\pi}{2}\sin\frac{n\pi}{2}+\frac{1}{n}\left(\cos\frac{n\pi}{2}-\cos n\pi\right)+\frac{\pi}{2}\left(2\sin\frac{n\pi}{2}\right)-\frac{\pi}{2}\sin\frac{n\pi}{2}-\frac{1}{n}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\right]$$

$$=\frac{1}{n\pi}\left[-\frac{\pi}{2}\sin\frac{n\pi}{2}+\frac{1}{n}\cos\frac{n\pi}{2}-\frac{\cos n\pi}{n}+\pi\sin\frac{n\pi}{2}-\frac{\pi}{2}\sin\frac{n\pi}{2}-\frac{\cos n\pi}{n}+\frac{1}{n}\cos\frac{n\pi}{2}\right]$$

$$=\frac{1}{n\pi}\left[\frac{2}{n}\cos\frac{n\pi}{2}-\frac{2}{n}\cos n\pi\right]=\frac{2}{n^2\pi}\left(\cos\frac{n\pi}{2}-\cos n\pi\right)=\frac{2}{n^2\pi}\left(\cos\frac{n\pi}{2}-(-1)^n\right)$$

$$\therefore\ a_n=\begin{cases}-4/n^2\pi & \text{if } n=2,6,10,\ldots\\ 0 & \text{if } n=4,8,12,\ldots\\ 2/n^2\pi & \text{if } n \text{ is odd}\end{cases}$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(\pi+x)\sin nx dx+\int_{-\pi/2}^{\pi/2}\frac{\pi}{2}\sin nx\,dx+\int_{\pi/2}^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$=\frac{1}{\pi}\left\{\begin{array}{l}\left[\dfrac{-(\pi+x)\cos nx}{n}\Bigg|_{-\pi}^{-\pi/2}-\displaystyle\int_{-\pi}^{-\pi/2}1.\frac{-\cos nx}{n}dx\right]+\dfrac{\pi}{2}.\dfrac{-\cos nx}{n}\Bigg|_{-\pi/2}^{\pi/2}\\ +\left[\dfrac{-(\pi-x)\cos nx}{n}\Bigg|_{\pi/2}^{\pi}-\displaystyle\int_{\pi/2}^{\pi}-1.\frac{-\cos nx}{n}dx\right]\end{array}\right\}$$

$$=\frac{1}{n\pi}\left\{\begin{array}{l}-\left(\dfrac{\pi}{2}\cos\dfrac{-n\pi}{2}-0\cos(-n\pi)\right)+\dfrac{\sin n\pi}{n}\Bigg|_{-\pi}^{-\pi/2}-\dfrac{\pi}{2}\left(\cos\dfrac{n\pi}{2}-\cos\dfrac{-n\pi}{2}\right)\\ -\left(0-\dfrac{\pi}{2}\cos\dfrac{n\pi}{2}\right)-\dfrac{\sin nx}{n}\Bigg|_{\pi/2}^{\pi}\end{array}\right\}$$

$$=\frac{1}{n\pi}\left[-\frac{\pi}{2}\cos\frac{n\pi}{2}+\frac{1}{n}\left(\sin\frac{-n\pi}{2}-\sin(-n\pi)\right)-\frac{\pi}{2}(0)+\frac{\pi}{2}\cos\frac{n\pi}{2}-\frac{1}{n}\left(\sin n\pi-\sin\frac{n\pi}{2}\right)\right]$$

$$=\frac{1}{n\pi}\left[-\frac{\pi}{2}\cos\frac{n\pi}{2}-\frac{1}{n}\sin\frac{n\pi}{2}-0+\frac{\pi}{2}\cos\frac{n\pi}{2}-\frac{0}{n}+\frac{1}{n}\sin\frac{n\pi}{2}\right]=0$$

Now, F.S of given function is: $f(x)=\frac{a_0}{2}+\sum\left(a_n\cos nx+b_n\sin nx\right)$

$$f(x)=\frac{3\pi}{8}-\frac{4}{\pi}\sum_{n=2,\,6,10,\ldots}\frac{\cos nx}{n^2}+\frac{2}{\pi}\sum_{n=4,\,8,12,\ldots}\frac{\cos nx}{n^2}$$

13. Obtain a Fourier series to represent the function

$$f(x)=|\sin x| \quad \text{for } -\pi<x<\pi, \quad \begin{cases}\text{Hint } f(x)=-\sin x & \text{for } -\pi<x<0\\ \qquad\quad\ \ =\sin x & \text{for } 0<x<\pi\end{cases}$$

Solution: Since $f(x)=|\sin x|$ is an even function hence,

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx,\ a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx \text{ and } b_n=0. \text{ Now,}$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}\sin x\,dx=\frac{2}{\pi}\left[-\cos x\right]_0^{\pi}=\frac{-2}{\pi}\left[\cos\pi-\cos 0\right]=\frac{-2}{\pi}\left[-1-1\right]=\frac{4}{\pi}$$

Now consider, $a_n=\frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx$

If $n=1$ then $a_1=\frac{1}{\pi}\int_0^{\pi}2\sin x\cos x\,dx=\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx=\frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi}$

$$=\frac{-1}{2\pi}\left[\cos 2\pi-\cos 0\right]=\frac{-1}{2\pi}(1-1)=0$$

If $n\neq 1$ then $a_n=\frac{1}{\pi}\int_0^{\pi}2\sin x\cos nx\,dx$. NOTE: $\sin(u+v)+\sin(u-v)=2\sin u\cos v$

$$=\frac{1}{\pi}\int_0^{\pi}\left[\sin(n+1)x-\sin(n-1)x\right]dx=\frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[-\frac{\left[\cos(n+1)\pi-\cos 0\right]}{n+1}+\frac{\left[\cos(n-1)\pi-\cos 0\right]}{n-1}\right]=\frac{1}{\pi}\left[\frac{-(-1)^{n+1}+1}{n+1}+\frac{(-1)^{n-1}-1}{n-1}\right]$$

$$=\frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1}-\frac{1}{n-1}\right)+\left(\frac{1}{n+1}-\frac{1}{n-1}\right)\right]=\frac{1}{\pi}\left[(-1)^n\left(\frac{n-1-n-1}{(n+1)(n-1)}\right)+\left(\frac{n-1-n-1}{(n+1)(n-1)}\right)\right]$$

$$=\frac{1}{\pi}\left[(-1)^n\left(\frac{-2}{n^2-1}\right)+\left(\frac{-2}{n^2-1}\right)\right]=\frac{-2}{(n^2-1)\pi}\left[(-1)^n+1\right]$$

$$\therefore\ a_n=\begin{cases}\dfrac{-4}{(n^2-1)\pi} & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases}$$

Now, F.S of given function is : $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$

$$|\sin x|=\frac{2}{\pi}-\frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1}+\frac{\cos 4x}{4^2-1}+\frac{\cos 6x}{6^2-1}+\dots\right)$$

14. A function is defined as follows: $f(x)=\begin{cases}-x & \text{when} \quad -\pi<x<0\\ x & \text{when} \quad 0<x<\pi\end{cases}$.

Show that $f(x)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\dots\right)$. Deduce that $\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}=\dfrac{\pi^2}{8}$

Solution : $a_0=\dfrac{1}{\pi}\left[\int_{-\pi}^{0}-x.dx+\int_{0}^{\pi}x.dx\right]=\dfrac{1}{\pi}\left[-\left(\dfrac{x^2}{2}\right)\Bigg|_{-\pi}^{0}+\left(\dfrac{x^2}{2}\right)\Bigg|_{0}^{\pi}\right]$

$$=\frac{1}{\pi}\left\{-\left(0-\frac{\pi^2}{2}\right)+\left(\frac{\pi^2}{2}-0\right)\right\}=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\frac{\pi^2}{2}\right]=\frac{\pi^2}{\pi}=\pi$$

$$a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-x.\cos nx\ dx+\int_{0}^{\pi}x.\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[-x\frac{\sin nx}{n}\Bigg|_{-\pi}^{0}-\int_{-\pi}^{0}-1.\frac{\sin nx}{n}dx+x\frac{\sin nx}{n}\Bigg|_{0}^{\pi}-\int_{0}^{\pi}1.\frac{\sin nx}{n}dx\right]$$

$$=\frac{1}{n\pi}\left[-(0-0)-\frac{\cos nx}{n}\Bigg|_{-\pi}^{0}+(0-0)+\frac{\cos nx}{n}\Bigg|_{0}^{\pi}\right] \qquad \text{NOTE: } \sin 0=\sin n\pi=0$$

$$=\frac{1}{n^2\pi}\left[-\cos 0+\cos(-n\pi)+\cos n\pi-\cos 0\right]=\frac{1}{n^2\pi}\left[-1+(-1)^n+(-1)^n-1\right]=\frac{2}{n^2\pi}\left[-1+(-1)^n\right]$$

$$\therefore\ a_n=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{-4}{n^2\pi} & \text{if } n \text{ is odd}\end{cases}$$

$$b_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}-x.\sin nx\ dx+\int_{0}^{\pi}x.\sin nx\,dx\right]$$

$$=\frac{1}{\pi}\left[(-x)\frac{-\cos nx}{n}\Bigg|_{-\pi}^{0}-\int_{-\pi}^{0}-1.\frac{-\cos nx}{n}dx+x.\frac{-\cos nx}{n}\Bigg|_{0}^{\pi}-\int_{0}^{\pi}1.\frac{-\cos nx}{n}dx\right]$$

$$=\frac{1}{n\pi}\left[\left(0+\pi\cos(-n\pi)-\frac{\sin nx}{n}\Bigg|_{-\pi}^{0}-\pi\cos n\pi+0+\frac{\sin nx}{n}\Bigg|_{0}^{\pi}\right]$$

$$=\frac{1}{n\pi}\left[\pi\cos n\pi-(0-0)-\pi\cos n\pi+(0-0)\right]=\frac{\pi}{n\pi}\left[(-1)^n-(-1)^n\right]=0$$

Now, F.S of given function is : $f(x)=\dfrac{a_0}{2}+\sum\left(a_n\cos nx+b_n\sin nx\right)$

$$f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5...}^{\infty}\frac{\cos nx}{n^2}=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+...\right)$$

Putting $x=0$ and notice that $\cos 0=1$ and $f(0)=0$. Thus,

$$0=\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right)\Rightarrow\frac{\pi}{2}=\frac{4}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right)\Rightarrow\frac{\pi^2}{8}=\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+...\right)$$

$$\Rightarrow\quad\frac{\pi^2}{8}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

NOTE: The n^{th} term of sequence $1, 3, 5, ...$ is $(2n-1)$

15. Find the Fourier cosine series for the function

$$f(x)=\begin{cases}1 & \text{for}\quad 0<x<\dfrac{\pi}{2}\\[2ex] 0 & \text{for}\quad \dfrac{\pi}{2}<x<\pi\end{cases}$$

Solution: Fourier cosine series defined on the interval $(0,L)$ is $f(x)=\frac{a_0}{2}+\sum a_n\cos\frac{n\pi x}{L}$

where, $a_0=\frac{2}{L}\int_0^L f(x)\,dx$ and $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$. Here $L=\pi$ hence,

$$a_0=\frac{2}{\pi}\left[\int_0^{\pi/2}1\,dx+\int_{\pi/2}^{\pi}(0)\,dx\right]=\frac{2}{\pi}[x]_0^{\pi/2}=\frac{2}{\pi}.\frac{\pi}{2}=1$$

$$a_n=\frac{2}{\pi}\left[\int_0^{\pi/2}1.\cos nx\,dx+\int_{\pi/2}^{\pi}0.\cos nx\,dx\right]=\frac{2}{\pi}\left[\frac{\sin nx}{n}\Bigg|_0^{\pi/2}+0\right]=\frac{2}{n\pi}\left[\sin\frac{n\pi}{2}-0\right].$$

$$\therefore\ a_n=\begin{cases}0 & \text{if } n \text{ is even} & \text{since, } \sin\pi=\sin 2\pi=\sin 3\pi=...=0\\ \dfrac{2}{n\pi} & \text{if } n=1, 5, 9, ... & \text{since, } \sin(\pi/2)=\sin(5\pi/2)=...=1\\ \dfrac{-2}{n\pi} & \text{if } n=3, 7, 11, ... & \text{since, } \sin(3\pi/2)=\sin(7\pi/2)=...=-1\end{cases}$$

Now, F.C.S of given function is: $f(x)=\frac{a_0}{2}+\sum\left(a_n\cos nx\right)$

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\left[\left(\frac{\cos x}{1}+\frac{\cos 5x}{5}+\frac{\cos 9x}{9}+...\right)-\left(\frac{\cos 3x}{3}+\frac{\cos 7x}{7}+\frac{\cos 11x}{11}+...\right)\right]$$

16. Find the Fourier sine series for the function $f(x)=e^{ax}$ for $0<x<\pi$, a is constant.

Solution: Fourier sine series defined on the interval $(0,L)$ is $f(x)=\sum b_n\sin\frac{n\pi x}{L}$

where, $b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$. Here $L=\pi$ hence,

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{ax}\sin nx\,dx = \frac{2}{\pi}\left[\frac{e^{ax}}{n^2+a^2}\left(a\sin nx - n\cos nx\right)\right]_0^{\pi}$$

$$= \frac{2}{\left(n^2+a^2\right)\pi}\left[e^{a\pi}\left(a\sin n\pi - n\cos n\pi\right) - e^0\left(a\sin 0 - n\cos 0\right)\right]$$

$$= \frac{2}{\left(n^2+a^2\right)\pi}\left[e^{a\pi}\left(0 - n(-1)^n\right) - e^0\left(0 - n.1\right)\right] = \frac{2}{\left(n^2+a^2\right)\pi}\left[(-1)^{n+1} n e^{a\pi} + n\right]$$

Now, F.S.S of given function is:

$$f(x) = \sum b_n \sin nx = \frac{2}{\pi}\sum \frac{1}{\left(n^2+a^2\right)}\left[(-1)^{n+1} n e^{a\pi} + n\right]\sin nx$$

17. Find a series of cosine of multiples of x which will represent $f(x)$ in $(0,\pi)$ where

$$f(x) = \begin{cases} 0 & \text{for} \quad 0 < x < \dfrac{\pi}{2} \\ \dfrac{\pi}{2} & \text{for} \quad \dfrac{\pi}{2} < x < \pi \end{cases}$$ Also deduce that $\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots\right) = \frac{4}{\pi}$

Solution: Fourier cosine series defined on the interval $(0,L)$ is $f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L}$

where, $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$. Here $L = \pi$ hence,

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} 0\,dx + \int_{\pi/2}^{\pi}\frac{\pi}{2}.dx\right] = \frac{2}{\pi}.\frac{\pi}{2}\left[x\right]_{\pi/2}^{\pi} = \left[\pi - \frac{\pi}{2}\right] = \frac{\pi}{2}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2} 0.\cos nx dx + \int_{\pi/2}^{\pi}\frac{\pi}{2}.\cos nx\,dx\right] = \left[\frac{\sin nx}{n}\Bigg|_{\pi/2}^{\pi}\right] = \left[\sin n\pi - \sin\frac{n\pi}{2}\right] = -\sin\frac{n\pi}{2}.$$ Thus,

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \quad \text{since, } \sin\pi = \sin 2\pi = \sin 3\pi = \ldots = 0 \\ -1 & \text{if } n = 1, 5, 9, \ldots \text{ since, } \sin(\pi/2) = \sin(5\pi/2) = \ldots = 1 \\ 1 & \text{if } n = 3, 7, 11, \ldots \text{ since, } \sin(3\pi/2) = \sin(7\pi/2) = \ldots = -1 \end{cases}$$

Now, F.C.S of given function is: $f(x) = \frac{a_0}{2} + \sum\left(a_n \cos nx\right)$

$$f(x) = \frac{4}{\pi} + \left[-\left(\frac{\cos x}{1} + \frac{\cos 5x}{5} + \frac{\cos 9x}{9} + \ldots\right) + \left(\frac{\cos 3x}{3} + \frac{\cos 7x}{7} + \frac{\cos 11x}{11} + \ldots\right)\right]$$

Putting $x = 0$ and notice that $f(0) = 0$ and $\cos 0 = 1$, we get

$$0 = \frac{\pi}{4} + \left[-\left(\frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \ldots\right) + \left(\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots\right)\right] \Rightarrow -\frac{4}{\pi} = -\left(\frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \ldots\right) + \left(\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots\right)$$

$$\Rightarrow \frac{4}{\pi} = \left(\frac{1}{1} + \frac{1}{5} + \frac{1}{9} + \ldots\right) - \left(\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots\right) \Rightarrow \frac{4}{\pi} = \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \ldots\right)$$

18. Express $f(x) = x$ as a sine series in $0 < x < \pi$.

Solution: Fourier sine series defined on the interval $(0,L)$ is $f(x) = \sum b_n \sin\frac{n\pi x}{L}$

where, $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$. Here $L = \pi$ hence,

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\ dx = \frac{2}{\pi}\left[x.\frac{-\cos nx}{n}\Big|_0^\pi - \int_0^\pi 1.\frac{-\cos nx}{n}dx\right] = \frac{2}{n\pi}\left[-\left(\pi\cos n\pi - 0\right) + \frac{\sin nx}{n}\Big|_0^\pi\right]$$

$= \frac{2}{n\pi}\left[\pi(-1)^{n+1} + 0 - 0\right] = \frac{2(-1)^{n+1}}{n}$. Now, F.S.S of given function is :

$$f(x) = \sum b_n \sin nx = 2\sum\frac{(-1)^{n+1}}{n}\sin nx \qquad \Rightarrow\ x = 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...\right)$$

19. Find the cosine series for f(x) = π - x in the interval 0 < x < π.

Solution : Fourier cosine series defined on the interval $(0, L)$ is $f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L}$

where, $a_0 = \frac{2}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$. Here $L = \pi$ hence,

$$a_0 = \frac{2}{\pi}\left[\int_0^\pi(\pi - x).dx\right] = \frac{2}{\pi}.\left[\pi x - \frac{x^2}{2}\right]_0^\pi = \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - (0-0)\right] = \frac{2}{\pi}.\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\left[\int_0^\pi(\pi - x).\cos nx\ dx\right] = \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\Big|_0^\pi - \int_0^\pi -1.\frac{\sin nx}{n}dx\right] = \frac{2}{n\pi}\left[(0-0) - \frac{\cos nx}{n}\Big|_0^\pi\right]$$

NOTE : $\sin 0 = \sin n\pi = 0$. Thus, $a_n = -\frac{2}{n^2\pi}\left[\cos n\pi - \cos 0\right] = -\frac{2}{n^2\pi}\left[(-1)^n - 1\right]$

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}.$$ Now, F.C.S of given function is : $f(x) = \frac{a_0}{2} + \sum a_n \cos nx$

$$\Rightarrow (\pi - x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right).$$

20. If $f(x) = \begin{cases} x, & \text{for } 0 < x < \frac{\pi}{2} \\ \pi - x, & \text{for } \frac{\pi}{2} < x < \pi \end{cases}.$

Show that (i) $f(x) = \frac{4}{\pi} - \frac{4}{\pi^2}\left(\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + ...\right)$

(ii) $f(x) = \frac{4}{\pi}\left(\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - ...\right)$ NOTE : See the change

Solution : Fourier cosine series defined on the interval $(0, L)$ is $f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{n\pi x}{L}$

where, $a_0 = \frac{2}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$. Here $L = \pi$ hence,

$$a_0=\frac{2}{\pi}\left[\int_0^{\pi/2}x\,dx+\int_{\pi/2}^{\pi}(\pi-x).dx\right]=\frac{2}{\pi}\left[\frac{x^2}{2}\Bigg|_0^{\pi/2}+\left(\pi x-\frac{x^2}{2}\right)\Bigg|_{\pi/2}^{\pi}\right]$$

$$=\frac{2}{\pi}\left[\frac{\pi^2}{8}+\pi^2-\frac{\pi^2}{2}-\frac{\pi^2}{2}+\frac{\pi^2}{8}\right]=\frac{2}{\pi}\left(\frac{\pi^2}{4}\right)=\frac{\pi}{2}$$

$$a_n=\frac{2}{\pi}\left[\int_0^{\pi/2}x.\cos nx dx+\int_{\pi/2}^{\pi}(\pi-x).\cos nx\,dx\right]$$

$$=\frac{2}{\pi}\left[x\frac{\sin nx}{n}\Bigg|_0^{\pi/2}-\int_0^{\pi/2}1.\frac{\sin nx}{n}dx+(\pi-x)\frac{\sin nx}{n}\Bigg|_{\pi/2}^{\pi}-\int_{\pi/2}^{\pi}-1.\frac{\sin nx}{n}dx\right]$$

$$=\frac{2}{n\pi}\left[\left(\frac{\pi}{2}\sin\frac{n\pi}{2}-0\right)+\frac{\cos nx}{n}\Bigg|_0^{\pi/2}+\left(0-\frac{\pi}{2}\sin\frac{n\pi}{2}\right)-\frac{\cos nx}{n}\Bigg|_{\pi/2}^{\pi}\right]$$

$$=\frac{2}{n\pi}\left[\frac{\pi}{2}\sin\frac{n\pi}{2}+\frac{1}{n}\left(\cos\frac{n\pi}{2}-\cos 0\right)-\frac{\pi}{2}\sin\frac{n\pi}{2}-\frac{1}{n}\left(\cos n\pi-\cos\frac{n\pi}{2}\right)\right]$$

$$=\frac{2}{n\pi}\left[\frac{\pi}{2}\sin\frac{n\pi}{2}+\frac{1}{n}\cos\frac{n\pi}{2}-\frac{1}{n}-\frac{\pi}{2}\sin\frac{n\pi}{2}-\frac{1}{n}\cos n\pi-\frac{1}{n}\cos\frac{n\pi}{2}\right]=\frac{2}{n\pi}\left[-\frac{1}{n}-\frac{1}{n}\cos n\pi\right]$$

$$=\frac{-2}{n^2\pi}\left[1+(-1)^n\right].\text{ Thus, }\quad a_n=\begin{cases}0 & \text{if } n \text{ is odd}\\ \dfrac{-4}{n^2\pi} & \text{if } n \text{ is even}\end{cases}$$

Now, F.C.S of given function is: $f(x)=\frac{a_0}{2}+\sum\left(a_n\cos nx\right)$

$$\Rightarrow\quad f(x)=\frac{4}{\pi}-\frac{4}{\pi^2}\left(\frac{\cos 2x}{2^2}+\frac{\cos 4x}{4^2}+\frac{\cos 6x}{6^2}+\ldots\right)$$

(ii) Solution: Fourier sine series defined on the interval $(0,L)$ is $f(x)=\sum a_n\sin\frac{n\pi x}{L}$

where, $b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$. Here $L=\pi$ hence,

$$b_n=\frac{2}{\pi}\left[\int_0^{\pi/2}x.\sin nx dx+\int_{\pi/2}^{\pi}(\pi-x).\sin nx\,dx\right]$$

$$=\frac{2}{\pi}\left[x\frac{-\cos nx}{n}\Bigg|_0^{\pi/2}-\int_0^{\pi/2}1.\frac{-\cos nx}{n}dx+(\pi-x)\frac{-\cos nx}{n}\Bigg|_{\pi/2}^{\pi}-\int_{\pi/2}^{\pi}-1.\frac{-\cos nx}{n}dx\right]$$

$$=\frac{2}{n\pi}\left[-\left(\frac{\pi}{2}\cos\frac{n\pi}{2}-0\right)+\frac{\sin nx}{n}\Bigg|_0^{\pi/2}-\left(0-\frac{\pi}{2}\cos\frac{n\pi}{2}\right)-\frac{\sin nx}{n}\Bigg|_{\pi/2}^{\pi}\right]$$

$$=\frac{2}{n\pi}\left[-\frac{\pi}{2}\cos\frac{n\pi}{2}+\frac{1}{n}\left(\sin\frac{n\pi}{2}-\sin 0\right)+\frac{\pi}{2}\cos\frac{n\pi}{2}-\frac{1}{n}\left(\sin n\pi-\sin\frac{n\pi}{2}\right)\right]\quad \text{NOTE}: \sin n\pi=0$$

$$=\frac{2}{n\pi}\left[-\frac{\pi}{2}\cos\frac{n\pi}{2}+\frac{1}{n}\sin\frac{n\pi}{2}-0+\frac{\pi}{2}\cos\frac{n\pi}{2}-\frac{1}{n}\sin n\pi+\frac{1}{n}\sin\frac{n\pi}{2}\right]=\frac{2}{n\pi}\left[\frac{2}{n}\sin\frac{n\pi}{2}\right]$$

$$=\frac{4}{n^2\pi}\sin\frac{n\pi}{2}. \text{ Thus,}$$

$$b_n=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{4}{n^2\pi}(-1)^n & \text{if } n \text{ is odd.}\end{cases} \quad \text{Now, F.S.S of given function is : } f(x)=\sum b_n\sin nx$$

$$\Rightarrow\ f(x)=\frac{4}{\pi}\left(\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\frac{\sin 7x}{7^2}+\dots\right)$$

21. A periodic function of period 4 is defined as: f(x) = |x|, -2 < x < 2
Find its Fourier series expansion.

Solution : Here $2L=4\Rightarrow L=2$ and $f(x)=|x|$ which is an even function hence,

$$b_n=0 \text{ and } a_0=\frac{2}{L}\int_0^L f(x)\,dx=\frac{2}{2}\int_0^2 x\,dx=\left[\frac{x^2}{2}\right]_0^2=\left[\frac{4-0}{2}\right]=2$$

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx=\frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx=\left[\left[x.\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2-\int_0^2 1.\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}dx\right]$$

$$=\frac{2}{n\pi}\left[(2\sin n\pi-0\sin 0)+\left[\frac{\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right]_0^2\right]=\frac{4}{n^2\pi^2}\left[(0-0)+\cos n\pi-\cos(0)\right]$$

$$=\frac{4}{n^2\pi^2}\left[(-1)^n-1\right]. \text{ Thus, } a_n=\begin{cases}0 & \text{if } n \text{ is even}\\ -\dfrac{8}{n^2\pi^2} & \text{if is odd}\end{cases}$$

$$\text{Now, F.S of given function is : } f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

$$f(x)=1-\frac{8}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{1^2}\sin\frac{\pi x}{2}-\frac{1}{3^2}\sin\frac{3\pi x}{2}+\frac{1}{5^2}\sin\frac{5\pi x}{2}-\dots\right)$$

22. Find the Fourier series expansion of the periodic function of period 1:

$$f(x)=\begin{cases}\dfrac{1}{2}+x=\dfrac{1+2x}{2}, & \text{for } -\dfrac{1}{2}<x\le 0\\ \dfrac{1}{2}-x=\dfrac{1-2x}{2}, & \text{for } 0<x<\dfrac{1}{2}\end{cases}.$$

Solution : Here $2L=1\Rightarrow L=1/2.$

$$a_0=\frac{2}{L}\int_{-L}^{L}f(x)\,dx=4\left[\int_{-1/2}^{0}\frac{1+2x}{2}dx+\int_0^{1/2}\frac{1-2x}{2}dx\right]=\frac{4}{2}\left[\left(x+x^2\right)\Big|_{-1/2}^{0}+\left(x-x^2\right)\Big|_0^{1/2}\right]$$

$$=2\left[(0-0)-\left(-\frac{1}{2}+\frac{1}{4}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)-(0-0)\right]=2\times\frac{1}{2}=1$$

$$a_n = \frac{2}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = 4\left[\int_{-1/2}^{0}\frac{1+2x}{2}.\cos 2n\pi x\ dx + \int_{0}^{1/2}\frac{1-2x}{2}.\cos 2n\pi x\ dx\right]$$

$$= \frac{4}{2}\left[(1+2x).\frac{\sin 2n\pi x}{2n\pi}\Big|_{-1/2}^{0} - \int_{-1/2}^{0}(0+2)\frac{\sin 2n\pi x}{2n\pi}dx + (1-2x).\frac{\sin 2n\pi x}{2n\pi}\Big|_{0}^{1/2} - \int_{0}^{1/2}(0-2)\frac{\sin 2n\pi x}{2n\pi}dx\right]$$

$$= \frac{2}{2n\pi}\left[(0-0) - 2.\frac{-\cos 2n\pi x}{2n\pi}\Big|_{-1/2}^{0} + (0-0) + 2.\frac{-\cos 2n\pi x}{2n\pi}\Big|_{0}^{1/2}\right] \qquad \text{NOTE}: \sin 0 = \sin n\pi = 0$$

$$= \frac{2(2)}{(2n\pi)(2n\pi)}\left[(\cos 0 - \cos n\pi) - (\cos n\pi - \cos 0)\right] = \frac{1}{n^2\pi^2}\left[2 - 2(-1)^n\right]$$

$$\text{Thus,}\quad a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{4}{n^2\pi^2} & \text{if is odd} \end{cases} \qquad \text{NOTE}: \cos 0 = 1,\ \cos n\pi = (-1)^n$$

$$b_n = \frac{2}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = 4\left[\int_{-1/2}^{0}\frac{1+2x}{2}.\sin 2n\pi x\ dx + \int_{0}^{1/2}\frac{1-2x}{2}.\sin 2n\pi x\ dx\right]$$

$$= \frac{4}{2}\left[(1+2x).\frac{-\cos 2n\pi x}{2n\pi}\Big|_{-1/2}^{0} - \int_{-1/2}^{0}(0+2)\frac{-\cos 2n\pi x}{2n\pi}dx + (1-2x).\frac{-\cos 2n\pi x}{2n\pi}\Big|_{0}^{1/2} - \int_{0}^{1/2}(0-2)\frac{-\cos 2n\pi x}{2n\pi}dx\right]$$

$$= \frac{2}{2n\pi}\left[-(1.\cos\ 0 - 0) + 2.\frac{\sin 2n\pi x}{2n\pi}\Big|_{-1/2}^{0} - (0-\cos 0) - 2.\frac{\sin 2n\pi x}{2n\pi}\Big|_{0}^{1/2}\right] \qquad \text{NOTE}: \sin 0 = \sin n\pi = 0,\ \cos 0 = 1$$

$$\therefore\ b_n = \frac{1}{n\pi}\left[-1 + 2(0-0) + 1 - 2(0-0)\right] = 0$$

$$\text{Now, F.S of given function is}: f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{1^2}\cos 2\pi x + \frac{1}{3^2}\cos 6\pi x + \frac{1}{5^2}\cos 10\pi x + ...\right)$$

23. Obtain the half – range cosine series for $f(x) = x^2$ in $0 < x < \pi$

Solution : Fourier cosine series defined on the interval $(0, L)$ is $f(x) = \frac{a_0}{2} + \sum a_n\cos\frac{n\pi x}{L}$

where, $a_0 = \frac{2}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$. Here $L = \pi$ hence,

$$a_0 = \frac{2}{\pi}\left[\int_0^{\pi} x^2.dx\right] = \frac{2}{\pi}.\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi} x^2.\cos nx\ dx\right] = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 2x.\frac{\sin nx}{n}dx\right]$$

$$= \frac{2}{n\pi}\left\{(0-0) - 2\left[x\frac{-\cos nx}{n}\Big|_0^{\pi} - \int_0^{\pi} 1.\frac{-\cos nx}{n}dx\right]\right\}$$

NOTE: $\sin 0 = \sin n\pi = 0$. Thus, $a_n = \dfrac{2}{n^2\pi}\left[2\left(\pi\cos n\pi - 0\cos 0\right) + \dfrac{\sin nx}{n}\Big|_0^\pi\right]$

$\therefore\ a_n = \dfrac{2}{n^2\pi}\left[2\pi(-1)^n - 0 + (0-0)\right] = \dfrac{4}{n^2}(-1)^n$

Now, half range F.C.S of given function is: $f(x) = \dfrac{a_0}{2} + \sum a_n \cos nx$

$$\Rightarrow\ x^2 = \frac{2\pi^2}{2(3)} + 4\left(-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2} + \ldots\right)$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right)$$

24. Express f(x) = x as a cosine, half range series in 0 < x < 2.

Solution: Fourier cosine series defined on the interval $(0, L)$ is $f(x) = \dfrac{a_0}{2} + \sum a_n \cos\dfrac{n\pi x}{L}$

where, $a_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)\,dx$ and $a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\dfrac{n\pi x}{L}dx$. Here $L = 2$ hence,

$$a_0 = \frac{2}{2}\left[\int_0^2 x.dx\right] = \left[\frac{x^2}{2}\right]_0^2 = \frac{4}{2} = 2.$$

$$a_n = \frac{2}{2}\left[\int_0^\pi x.\cos\frac{n\pi x}{2}\,dx\right] = \left(\left[x\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2 - \int_0^\pi 1.\frac{2}{n\pi}\cos\frac{n\pi x}{2}dx\right)\left\{\int\cos\frac{x}{m}dx = m\sin\frac{x}{m}\right\}$$

$$= \frac{2}{n\pi}\left\{(2\cos n\pi - 0) - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_0^\pi\right\} = \frac{2}{n\pi}\left\{(2\cos n\pi - 0) - \frac{2}{n\pi}\left(0-0\right)\right\}$$

NOTE: $\sin 0 = \sin n\pi = 0$. Thus, $a_n = \dfrac{4}{n\pi}(-1)^n$

Now, half range F.C.S of given function is: $f(x) = \dfrac{a_0}{2} + \sum a_n \cos\dfrac{n\pi x}{2}$

$$\Rightarrow\ x = 1 + \frac{4}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{2}\cos\frac{2\pi x}{2} + \frac{1}{3}\cos\frac{3\pi x}{2} - \frac{1}{4}\cos\frac{4\pi x}{2} + \ldots\right)$$

25. Find the Fourier series of the function:

$$f\left(x\right) = \begin{cases} -2 & \text{for} \quad -4 < x < -2 \\ x & \text{for} \quad -2 < x < 2 \\ 2 & \text{for} \quad 2 < x < 4 \end{cases} \cdot \text{ Here } 2L = 8 \Rightarrow L = 4.$$

Solution: $a_0 = \dfrac{1}{4}\left[\displaystyle\int_{-4}^{-2} -2.dx + \int_{-2}^{2} x.dx + \int_{2}^{4} 2.dx\right] = \dfrac{1}{4}\left[-2x\Big|_{-4}^{-2} + \dfrac{x^2}{2}\Big|_{-2}^{2} + 2x\Big|_2^4\right]$

$$= \frac{1}{4}\left[-2\left(-2+4\right) + \frac{4-4}{2} + 2(4-2)\right] = \frac{1}{4}\left[-4+0+4\right] = 0$$

$$a_n = \frac{1}{4}\left[\int_{-4}^{-2} -2.\cos\frac{n\pi x}{4}\,dx + \int_{-2}^{2} x.\cos\frac{n\pi x}{4}\,dx + \int_{2}^{4} 2.\cos\frac{n\pi x}{4}\,dx\right]$$

$$= \frac{1}{4}\left[-2\frac{4}{n\pi}\sin\frac{n\pi x}{4}\Big|_{-4}^{-2} + x\frac{4}{n\pi}\sin\frac{n\pi x}{4}\Big|_{-2}^{2} - \int_{2}^{4} 1.\frac{4}{n\pi}\sin\frac{n\pi x}{4}\,dx + 2\frac{4}{n\pi}\sin\frac{n\pi x}{4}\Big|_{2}^{4}\right]$$

$$= \frac{1}{4}.\frac{4}{n\pi}\left[-2\left(-\sin\frac{n\pi}{2} + \sin n\pi\right) + \left(2\sin\frac{n\pi}{2} - 2\sin\frac{n\pi}{2}\right) + \frac{4}{n\pi}\cos\frac{n\pi x}{4}\Big|_{2}^{4} + 2\left(\sin n\pi - \sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\left[2\sin\frac{n\pi}{2} - 0 + 0 + \frac{4}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right) + 2\left(0 - \sin\frac{n\pi}{2}\right)\right]$$

NOTE : $\sin 0 = \sin n\pi = 0$

$$= \frac{1}{n\pi}\left[2\sin\frac{n\pi}{2} - 2\sin\frac{n\pi}{2} + \frac{4}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right] = \frac{4}{n^2\pi^2}\left[0 + (-1)^n - \cos\frac{n\pi}{2}\right]$$

$$\therefore\ a_n = \begin{cases} \dfrac{8}{n^2\pi^2} & \text{if } n \text{ is } 2, 6, 10, \ldots \\ 0 & \text{if } n \text{ is } 4, 8, 12, \ldots \\ (-1)^n & \text{if } n \text{ is odd} \end{cases}$$

NOTE : $\sin 0 = \sin n\pi = 0$

$$b_n = \frac{1}{4}\left[\int_{-4}^{-2} -2.\sin\frac{n\pi x}{4}\,dx + \int_{-2}^{2} x.\sin\frac{n\pi x}{4}\,dx + \int_{2}^{4} 2.\sin\frac{n\pi x}{4}\,dx\right]$$

$$= \frac{1}{4}\left[2\frac{4}{n\pi}\cos\frac{n\pi x}{4}\Big|_{-4}^{-2} - x\frac{4}{n\pi}\cos\frac{n\pi x}{4}\Big|_{-2}^{2} + \int_{-2}^{2} 1.\frac{4}{n\pi}\cos\frac{n\pi x}{4}\,dx - 2\frac{4}{n\pi}\cos\frac{n\pi x}{4}\Big|_{2}^{4}\right]$$

$$= \frac{1}{4}.\frac{4}{n\pi}\left[2\left(\cos\frac{n\pi}{2} - \cos n\pi\right) - \left(2\cos\frac{n\pi}{2} + 2\cos\frac{n\pi}{2}\right) + \frac{4}{n\pi}\sin\frac{n\pi x}{4}\Big|_{-2}^{2} - 2\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{n\pi}\left[2\cos\frac{n\pi}{2} - 2\cos n\pi - 4\cos\frac{n\pi}{2} + \frac{4}{n\pi}\left(\sin\frac{n\pi}{2} + \sin\frac{n\pi}{2}\right) - 2\cos n\pi + 2\cos\frac{n\pi}{2}\right]$$

$$= \frac{1}{n\pi}\left[-4\cos n\pi + \frac{4}{n\pi}\left(2\sin\frac{n\pi}{2}\right)\right] = \frac{4}{n\pi}\left[-\cos n\pi + \frac{2}{n\pi}\sin\frac{n\pi}{2}\right]$$

NOTE : $\sin(-u) = -\sin u$ and $\cos(-u) = \cos u$

$$\therefore\ b_n = \begin{cases} -\dfrac{4}{n\pi} & \text{if } n \text{ is even} \\ \dfrac{4}{n\pi}\left(1 + \dfrac{2}{n\pi}\right) & \text{if } n = 1, 5, 9, \ldots \\ \dfrac{4}{n\pi}\left(1 - \dfrac{2}{n\pi}\right) & \text{if } n = 3, 7, 11, \ldots \end{cases}$$

Now, FS of given function is $f(x) = \frac{a_0}{2} + \sum\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$. Here $L = 4$:

$$\therefore f(x)=\frac{8}{\pi^2}\sum_{n=2,\,6,\,10,...}\frac{1}{n^2}\cos\frac{n\pi x}{4}+\sum_{n=1,36,5,...}(-1)^n\cos\frac{n\pi x}{4}-\frac{4}{\pi}\sum_{n=2,\,4,\,6,...}\frac{1}{n}\sin\frac{n\pi x}{4}$$
$$+\frac{4}{\pi}\sum_{n=1,5,\,9...}\frac{1}{n}\left(1+\frac{2}{n\pi}\right)\sin\frac{n\pi x}{4}+\frac{4}{\pi}\sum_{n=3,7,\,11...}\frac{1}{n}\left(1-\frac{2}{n\pi}\right)\sin\frac{n\pi x}{4}$$

26. Expand f(x) = 1 - |x|, -1 < x < 1 into Fourier series.

Here $2L=2 \Rightarrow L=1$.

$$\text{Solution: NOTE: } 1-|x|=\begin{cases}1-(-x)=1+x & \text{if } x \text{ is negative (Here b/w } -1 \text{ to } 0)\\ 1-(+x)=1-x & \text{if } x \text{ is positive (Here b/w } 0 \text{ to } +1)\end{cases}$$

$$\therefore a_0=\frac{1}{L}\int_{-L}^{L}f(x)\,dx=\frac{1}{1}\left[\int_{-1}^{0}(1+x)\,dx+\int_{0}^{1}(1-x)\,dx\right]=\left[\left(x+\frac{x^2}{2}\right)\Big|_{-1}^{0}+\left(x-\frac{x^2}{2}\right)\Big|_{0}^{1}\right]$$

$$=\left[\left((0+0)-\left(-1+\frac{1}{2}\right)\right)+\left(\left(1-\frac{1}{2}\right)-(0+0)\right)\right]=1-\frac{1}{2}+1-\frac{1}{2}=1$$

$$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx=\frac{1}{1}\left[\int_{-1}^{0}(1+x).\cos n\pi x\,dx+\int_{0}^{1}(1-x).\cos n\pi x\,dx\right]$$

$$=\left[\left((1+x)\frac{\sin n\pi x}{n\pi}\right)\Big|_{-1}^{0}-\int_{-1}^{0}(0+1)\frac{\sin n\pi x}{n\pi}dx\right]+\left[\left((1-x)\frac{\sin n\pi x}{n\pi}\right)\Big|_{0}^{1}-\int_{0}^{1}(0-1)\frac{\sin n\pi x}{n\pi}dx\right]$$

$$=\frac{1}{n\pi}\left[\left(1.\sin 0-0\sin(-n\pi)\right)+\frac{\cos n\pi}{n\pi}\Big|_{-1}^{0}+\left(0.\sin n\pi-1.\sin 0\right)-\frac{\cos n\pi}{n\pi}\Big|_{0}^{1}\right]$$

$$=\frac{1}{n\pi}\left[(0-0)+\left(\frac{\cos 0-\cos n\pi}{n\pi}\right)+(0-0)-\left(\frac{\cos n\pi-\cos 0}{n\pi}\right)\right]\quad \text{NOTE: } \sin 0=\sin n\pi=0$$

$$=\frac{1}{n^2\pi^2}\left(2\cos 0-2\cos n\pi\right)=\frac{2}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\therefore\ a_n=\begin{cases}0 & \text{if } n \text{ is even}\\ \dfrac{4}{n^2\pi^2} & \text{if } n \text{ is odd}\end{cases}$$

$$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\frac{n\pi x}{L}dx=\frac{1}{1}\left[\int_{-1}^{0}(1+x).\sin n\pi x\,dx+\int_{0}^{1}(1-x).\sin n\pi x\,dx\right]$$

$$=\left[\left((1+x)\frac{-\cos n\pi x}{n\pi}\right)\Big|_{-1}^{0}-\int_{-1}^{0}(0+1)\frac{-\cos n\pi x}{n\pi}dx\right]+\left[\left((1-x)\frac{-\cos n\pi x}{n\pi}\right)\Big|_{0}^{1}-\int_{0}^{1}(0-1)\frac{-\cos n\pi x}{n\pi}dx\right]$$

$$=\frac{1}{n\pi}\left[-\left(1.\cos 0-0\cos(-n\pi)\right)+\frac{\sin n\pi}{n\pi}\Big|_{-1}^{0}-\left(0.\cos n\pi-1.\cos 0\right)-\frac{\sin n\pi}{n\pi}\Big|_{0}^{1}\right]$$

$$=\frac{1}{n\pi}\left[-1+0+\left(\frac{\sin 0+\sin n\pi}{n\pi}\right)-0+1-\left(\frac{\sin n\pi-\sin 0}{n\pi}\right)\right]\quad \text{NOTE: } \cos 0=1,\ \sin 0=\sin n\pi=0$$

$\therefore\ b_n = \dfrac{1}{n^2\pi^2}(-1+1) = 0.$

Now, FS of given function is $f(x) = \dfrac{1}{2}a_0 + \sum\left(a_n \cos\dfrac{n\pi x}{L} + b_n \sin\dfrac{n\pi x}{L}\right)$. With $L = 1$,

$$f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1,3,5...}\frac{\cos n\pi x}{n^2} = \frac{1}{2} + \frac{4}{\pi^2}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + ...\right)$$

27. Represent f(x) = sin x, $0 < x < \pi$ into Fourier sine and cosine series.

Solution : Consider,

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi \sin x\,dx = -\frac{2}{\pi}\cos x\Big|_0^\pi = -\frac{2}{\pi}(\cos\pi - \cos 0) = -\frac{2}{\pi}(-1-1) = \frac{4}{\pi}$$

Now consider, $a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin x\cos nx\,dx$

If $n = 1$ then $a_1 = \dfrac{1}{\pi}\displaystyle\int_0^\pi 2\sin x\cos x\,dx = \dfrac{1}{\pi}\int_0^\pi \sin 2x\,dx = \dfrac{1}{\pi}\left[\dfrac{-\cos 2x}{2}\right]_0^\pi$

$$= \frac{-1}{2\pi}[\cos 2\pi - \cos 0] = \frac{-1}{2\pi}(1-1) = 0$$

If $n \neq 1$ then $a_n = \dfrac{1}{\pi}\displaystyle\int_0^\pi 2\sin x\cos nx\,dx$

$$= \frac{1}{\pi}\int_0^\pi[\sin(n+1)x - \sin(n-1)x]dx = \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[-\frac{[\cos(n+1)\pi - \cos 0]}{n+1} + \frac{[\cos(n-1)\pi - \cos 0]}{n-1}\right] = \frac{1}{\pi}\left[\frac{-(-1)^{n+1}+1}{n+1} + \frac{(-1)^{n-1}-1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right] = \frac{1}{\pi}\left[(-1)^n\left(\frac{n-1-n-1}{(n+1)(n-1)}\right) + \left(\frac{n-1-n-1}{(n+1)(n-1)}\right)\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(\frac{-2}{n^2-1}\right) + \left(\frac{-2}{n^2-1}\right)\right] = \frac{-2}{(n^2-1)\pi}\left[(-1)^n + 1\right]$$

$$\therefore\ a_n = \begin{cases}\dfrac{-4}{(n^2-1)\pi} & \text{if } n \text{ is even}\\ 0 & \text{if } n \text{ is odd}\end{cases}$$

Now, F.C.S of a function is : $f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty}(a_n\cos nx)$ (Here $L = \pi$)

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + ...\right)$$

Now consider, $b_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi \sin x\sin nx\,dx$

If $n=1$ then $b_1=\dfrac{2}{\pi}\int_0^{\pi}\sin x\,\sin x\,dx=\dfrac{2}{\pi}\int_0^{\pi}\sin^2 x\,dx=\dfrac{2}{\pi}\int_0^{\pi}\dfrac{1-\cos 2x}{2}\,dx=\dfrac{1}{\pi}\left[x-\dfrac{\sin 2x}{2}\right]_0^{\pi}$

$$=\frac{1}{\pi}\left[(\pi-0)-\frac{\sin 2\pi-\sin 0}{2}\right]=\frac{1}{\pi}\left[\pi-\frac{0-0}{2}\right]=1$$

If $n\neq 1$ then $b_n=\dfrac{2}{\pi}\int_0^{\pi}\sin x\sin nx\,dx=\dfrac{1}{\pi}\int_0^{\pi}2\sin x\sin nx\,dx$

$$=\frac{1}{\pi}\int_0^{\pi}\left[\cos(n-1)x-\cos(n+1)x\right]dx=\frac{1}{\pi}\left[\frac{\sin(n-1)x}{n-1}-\frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$=\frac{1}{\pi}\left[\frac{\left[\sin(n-1)\pi-\sin 0\right]}{n-1}-\frac{\left[\sin(n+1)\pi-\sin 0\right]}{n+1}\right]=\frac{1}{\pi}\left[(0-0)-(0-0)\right]=0$$

$$\therefore\ b_n=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n\neq 1\end{cases}$$

Now, F.S.S of a function $f(x)$ is : $f(x)=\sum_{n=1}^{\infty}\left(b_n\sin nx\right)$ (Here $L=\pi$)

Thus, Fourier sine series of $\sin x$: $\sin x=\sin x$

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