

5.2 The Natural Logarithmic Function: Integration

- Use the Log Rule for Integration to integrate a rational function.
- Integrate trigonometric functions.

Log Rule for Integration

The differentiation rules

$$\frac{d}{dx}[\ln|x|] = \frac{1}{x} \quad \text{and} \quad \frac{d}{dx}[\ln|u|] = \frac{u'}{u}$$

that you studied in the preceding section produce the following integration rule.

THEOREM 5.5 LOG RULE FOR INTEGRATION

Let u be a differentiable function of x .

$$1. \int \frac{1}{x} dx = \ln|x| + C \quad 2. \int \frac{1}{u} du = \ln|u| + C$$

Because $du = u' dx$, the second formula can also be written as

$$\int \frac{u'}{u} dx = \ln|u| + C. \quad \text{Alternative form of Log Rule}$$

EXAMPLE 1 Using the Log Rule for Integration

$$\begin{aligned} \int \frac{2}{x} dx &= 2 \int \frac{1}{x} dx && \text{Constant Multiple Rule} \\ &= 2 \ln|x| + C && \text{Log Rule for Integration} \\ &= \ln(x^2) + C && \text{Property of logarithms} \end{aligned}$$

Because x^2 cannot be negative, the absolute value notation is unnecessary in the final form of the antiderivative.

EXAMPLE 2 Using the Log Rule with a Change of Variables

Find $\int \frac{1}{4x-1} dx$.

Solution If you let $u = 4x - 1$, then $du = 4 dx$.

$$\begin{aligned} \int \frac{1}{4x-1} dx &= \frac{1}{4} \int \left(\frac{1}{4x-1} \right) 4 dx && \text{Multiply and divide by 4.} \\ &= \frac{1}{4} \int \frac{1}{u} du && \text{Substitute: } u = 4x - 1. \\ &= \frac{1}{4} \ln|u| + C && \text{Apply Log Rule.} \\ &= \frac{1}{4} \ln|4x-1| + C && \text{Back-substitute.} \end{aligned}$$

EXPLORATION

Integrating Rational Functions

Early in Chapter 4, you learned rules that allowed you to integrate any polynomial function. The Log Rule presented in this section goes a long way toward enabling you to integrate rational functions. For instance, each of the following functions can be integrated with the Log Rule.

$$\frac{2}{x} \quad \text{Example 1}$$

$$\frac{1}{4x-1} \quad \text{Example 2}$$

$$\frac{x}{x^2+1} \quad \text{Example 3}$$

$$\frac{3x^2+1}{x^3+x} \quad \text{Example 4(a)}$$

$$\frac{x+1}{x^2+2x} \quad \text{Example 4(c)}$$

$$\frac{1}{3x+2} \quad \text{Example 4(d)}$$

$$\frac{x^2+x+1}{x^2+1} \quad \text{Example 5}$$

$$\frac{2x}{(x+1)^2} \quad \text{Example 6}$$

There are still some rational functions that cannot be integrated using the Log Rule. Give examples of these functions, and explain your reasoning.

Example 3 uses the alternative form of the Log Rule. To apply this rule, look for quotients in which the numerator is the derivative of the denominator.

EXAMPLE 3 Finding Area with the Log Rule

Find the area of the region bounded by the graph of

$$y = \frac{x}{x^2 + 1}$$

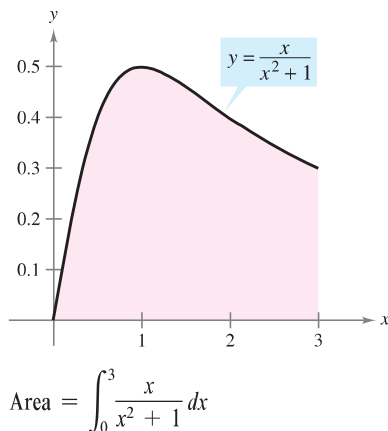
the x -axis, and the line $x = 3$.

Solution In Figure 5.8, you can see that the area of the region is given by the definite integral

$$\int_0^3 \frac{x}{x^2 + 1} dx.$$

If you let $u = x^2 + 1$, then $u' = 2x$. To apply the Log Rule, multiply and divide by 2 as shown.

$$\begin{aligned} \int_0^3 \frac{x}{x^2 + 1} dx &= \frac{1}{2} \int_0^3 \frac{2x}{x^2 + 1} dx && \text{Multiply and divide by 2.} \\ &= \frac{1}{2} \left[\ln(x^2 + 1) \right]_0^3 && \int \frac{u'}{u} dx = \ln|u| + C \\ &= \frac{1}{2} (\ln 10 - \ln 1) \\ &= \frac{1}{2} \ln 10 && \ln 1 = 0 \\ &\approx 1.151 \end{aligned}$$



The area of the region bounded by the graph of y , the x -axis, and $x = 3$ is $\frac{1}{2} \ln 10$.

Figure 5.8

EXAMPLE 4 Recognizing Quotient Forms of the Log Rule

- $\int \frac{3x^2 + 1}{x^3 + x} dx = \ln|x^3 + x| + C$ $u = x^3 + x$
- $\int \frac{\sec^2 x}{\tan x} dx = \ln|\tan x| + C$ $u = \tan x$
- $\int \frac{x + 1}{x^2 + 2x} dx = \frac{1}{2} \int \frac{2x + 2}{x^2 + 2x} dx$ $u = x^2 + 2x$
 $= \frac{1}{2} \ln|x^2 + 2x| + C$
- $\int \frac{1}{3x + 2} dx = \frac{1}{3} \int \frac{3}{3x + 2} dx$ $u = 3x + 2$
 $= \frac{1}{3} \ln|3x + 2| + C$

With antiderivatives involving logarithms, it is easy to obtain forms that look quite different but are still equivalent. For instance, both of the following are equivalent to the antiderivative listed in Example 4(d).

$$\ln|(3x + 2)^{1/3}| + C \quad \text{and} \quad \ln|3x + 2|^{1/3} + C$$

Integrals to which the Log Rule can be applied often appear in disguised form. For instance, if a rational function has a *numerator of degree greater than or equal to that of the denominator*, division may reveal a form to which you can apply the Log Rule. This is shown in Example 5.

EXAMPLE 5 Using Long Division Before Integrating

Find $\int \frac{x^2 + x + 1}{x^2 + 1} dx$.

Solution Begin by using long division to rewrite the integrand.

$$\frac{x^2 + x + 1}{x^2 + 1} \Rightarrow \begin{array}{r} 1 \\ x^2 + 1 \overline{) x^2 + x + 1} \\ \underline{x^2 } \\ x \\ \underline{x} \\ 1 \end{array} \Rightarrow 1 + \frac{x}{x^2 + 1}$$

Now, you can integrate to obtain

$$\begin{aligned} \int \frac{x^2 + x + 1}{x^2 + 1} dx &= \int \left(1 + \frac{x}{x^2 + 1} \right) dx && \text{Rewrite using long division.} \\ &= \int dx + \frac{1}{2} \int \frac{2x}{x^2 + 1} dx && \text{Rewrite as two integrals.} \\ &= x + \frac{1}{2} \ln(x^2 + 1) + C. && \text{Integrate.} \end{aligned}$$

Check this result by differentiating to obtain the original integrand. ■

The next example presents another instance in which the use of the Log Rule is disguised. In this case, a change of variables helps you recognize the Log Rule.

EXAMPLE 6 Change of Variables with the Log Rule

Find $\int \frac{2x}{(x+1)^2} dx$.

Solution If you let $u = x + 1$, then $du = dx$ and $x = u - 1$.

$$\begin{aligned} \int \frac{2x}{(x+1)^2} dx &= \int \frac{2(u-1)}{u^2} du && \text{Substitute.} \\ &= 2 \int \left(\frac{u}{u^2} - \frac{1}{u^2} \right) du && \text{Rewrite as two fractions.} \\ &= 2 \int \frac{du}{u} - 2 \int u^{-2} du && \text{Rewrite as two integrals.} \\ &= 2 \ln|u| - 2 \left(\frac{u^{-1}}{-1} \right) + C && \text{Integrate.} \\ &= 2 \ln|u| + \frac{2}{u} + C && \text{Simplify.} \\ &= 2 \ln|x+1| + \frac{2}{x+1} + C && \text{Back-substitute.} \end{aligned}$$

Check this result by differentiating to obtain the original integrand. ■

TECHNOLOGY If you have access to a computer algebra system, use it to find the indefinite integrals in Examples 5 and 6. How does the form of the antiderivative that it gives you compare with that given in Examples 5 and 6?

As you study the methods shown in Examples 5 and 6, be aware that both methods involve rewriting a disguised integrand so that it fits one or more of the basic integration formulas. Throughout the remaining sections of Chapter 5 and in Chapter 8, much time will be devoted to integration techniques. To master these techniques, you must recognize the “form-fitting” nature of integration. In this sense, integration is not nearly as straightforward as differentiation. Differentiation takes the form

“Here is the question; what is the answer?”

Integration is more like

“Here is the answer; what is the question?”

The following are guidelines you can use for integration.

STUDY TIP Keep in mind that you can check your answer to an integration problem by differentiating the answer. For instance, in Example 7, the derivative of $y = \ln|\ln x| + C$ is $y' = 1/(x \ln x)$.

GUIDELINES FOR INTEGRATION

1. Learn a basic list of integration formulas. (Including those given in this section, you now have 12 formulas: the Power Rule, the Log Rule, and ten trigonometric rules. By the end of Section 5.7, this list will have expanded to 20 basic rules.)
2. Find an integration formula that resembles all or part of the integrand, and, by trial and error, find a choice of u that will make the integrand conform to the formula.
3. If you cannot find a u -substitution that works, try altering the integrand. You might try a trigonometric identity, multiplication and division by the same quantity, addition and subtraction of the same quantity, or long division. Be creative.
4. If you have access to computer software that will find antiderivatives symbolically, use it.

EXAMPLE 7 u -Substitution and the Log Rule

Solve the differential equation $\frac{dy}{dx} = \frac{1}{x \ln x}$.

Solution The solution can be written as an indefinite integral.

$$y = \int \frac{1}{x \ln x} dx$$

Because the integrand is a quotient whose denominator is raised to the first power, you should try the Log Rule. There are three basic choices for u . The choices $u = x$ and $u = x \ln x$ fail to fit the u'/u form of the Log Rule. However, the third choice does fit. Letting $u = \ln x$ produces $u' = 1/x$, and you obtain the following.

$$\begin{aligned} \int \frac{1}{x \ln x} dx &= \int \frac{1/x}{\ln x} dx && \text{Divide numerator and denominator by } x. \\ &= \int \frac{u'}{u} dx && \text{Substitute: } u = \ln x. \\ &= \ln|u| + C && \text{Apply Log Rule.} \\ &= \ln|\ln x| + C && \text{Back-substitute.} \end{aligned}$$

So, the solution is $y = \ln|\ln x| + C$.

Integrals of Trigonometric Functions

In Section 4.1, you looked at six trigonometric integration rules—the six that correspond directly to differentiation rules. With the Log Rule, you can now complete the set of basic trigonometric integration formulas.

EXAMPLE 8 Using a Trigonometric Identity

Find $\int \tan x \, dx$.

Solution This integral does not seem to fit any formulas on our basic list. However, by using a trigonometric identity, you obtain

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx.$$

Knowing that $D_x[\cos x] = -\sin x$, you can let $u = \cos x$ and write

$$\begin{aligned} \int \tan x \, dx &= -\int \frac{-\sin x}{\cos x} \, dx && \text{Trigonometric identity} \\ &= -\int \frac{u'}{u} \, dx && \text{Substitute: } u = \cos x. \\ &= -\ln|u| + C && \text{Apply Log Rule.} \\ &= -\ln|\cos x| + C. && \text{Back-substitute.} \end{aligned}$$

Example 8 uses a trigonometric identity to derive an integration rule for the tangent function. The next example takes a rather unusual step (multiplying and dividing by the same quantity) to derive an integration rule for the secant function.

EXAMPLE 9 Derivation of the Secant Formula

Find $\int \sec x \, dx$.

Solution Consider the following procedure.

$$\begin{aligned} \int \sec x \, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \end{aligned}$$

Letting u be the denominator of this quotient produces

$$u = \sec x + \tan x \quad \Rightarrow \quad u' = \sec x \tan x + \sec^2 x.$$

So, you can conclude that

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx && \text{Rewrite integrand.} \\ &= \int \frac{u'}{u} \, dx && \text{Substitute: } u = \sec x + \tan x. \\ &= \ln|u| + C && \text{Apply Log Rule.} \\ &= \ln|\sec x + \tan x| + C. && \text{Back-substitute.} \end{aligned}$$

With the results of Examples 8 and 9, you now have integration formulas for $\sin x$, $\cos x$, $\tan x$, and $\sec x$. All six trigonometric rules are summarized below. (For proofs of $\cot u$ and $\csc u$, see Exercises 91 and 92.)

NOTE Using trigonometric identities and properties of logarithms, you could rewrite these six integration rules in other forms. For instance, you could write

$$\int \csc u \, du = \ln|\csc u - \cot u| + C.$$

(See Exercises 93–96.)

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$$\int \sin u \, du = -\cos u + C$$

$$\int \cos u \, du = \sin u + C$$

$$\int \tan u \, du = -\ln|\cos u| + C$$

$$\int \cot u \, du = \ln|\sin u| + C$$

$$\int \sec u \, du = \ln|\sec u + \tan u| + C$$

$$\int \csc u \, du = -\ln|\csc u + \cot u| + C$$

EXAMPLE 10 Integrating Trigonometric Functions

Evaluate $\int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx$.

Solution Using $1 + \tan^2 x = \sec^2 x$, you can write

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \tan^2 x} \, dx &= \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx \\ &= \int_0^{\pi/4} \sec x \, dx && \sec x \geq 0 \text{ for } 0 \leq x \leq \frac{\pi}{4}. \\ &= \ln|\sec x + \tan x| \Big|_0^{\pi/4} \\ &= \ln(\sqrt{2} + 1) - \ln 1 \\ &\approx 0.881. \end{aligned}$$

EXAMPLE 11 Finding an Average Value

Find the average value of $f(x) = \tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

Solution

$$\begin{aligned} \text{Average value} &= \frac{1}{(\pi/4) - 0} \int_0^{\pi/4} \tan x \, dx && \text{Average value} = \frac{1}{b-a} \int_a^b f(x) \, dx \\ &= \frac{4}{\pi} \int_0^{\pi/4} \tan x \, dx && \text{Simplify.} \\ &= \frac{4}{\pi} \left[-\ln|\cos x| \right]_0^{\pi/4} && \text{Integrate.} \\ &= -\frac{4}{\pi} \left[\ln\left(\frac{\sqrt{2}}{2}\right) - \ln(1) \right] \\ &= -\frac{4}{\pi} \ln\left(\frac{\sqrt{2}}{2}\right) \\ &\approx 0.441 \end{aligned}$$

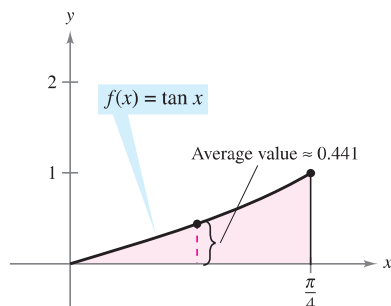


Figure 5.9

The average value is about 0.441, as shown in Figure 5.9. ■