(3) Linear differential equations

A differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, P and Q are constants or functions of x only, is known as a first order linear differential equation. Some examples of the first order linear differential equation are

$$\frac{dy}{dx} + y = \sin x \quad , \quad \frac{dy}{dx} + \left(\frac{1}{x}\right)y = e^{x}$$

$$\frac{dy}{dx} + \left(\frac{y}{x\log x}\right) = \frac{1}{x}$$

Another form of first order linear differential equation is

$$\frac{dx}{dy} + P_1 x = Q_1$$

where P_1 and Q_1 are constants or functions of y only. Some examples of this type of differential equation are

$$\frac{dx}{dy} + x = \cos y$$

$$\frac{dx}{dy} + \frac{-2x}{y} = y^2 e^{-y}$$

To solve the first order linear differential equation of the type

$$\frac{dy}{dx} + Py = Q \quad -(1)$$

Multiply both sides of the equation by a function of x say $g(x)$ to get

$$g(x)\frac{dy}{dx} + P(g(x))y = Q \cdot g(x) \quad -(2)$$

Choose $g(x)$ in such a way that R.H.S becomes a derivative of $y \cdot g(x)$

i.e. $$g(x)\frac{dy}{dx} + P \cdot g(x)\, y = \frac{d}{dx}[y \cdot g(x)]$$

or $$g(x)\frac{dy}{dx} + P \cdot g(x)\, y = g(x)\frac{dy}{dx} + y\, g'(x) \Rightarrow P = \frac{g'(x)}{g(x)}$$

Integrating both sides with respect to x, we get

$$\int P\,dx = \int \frac{g'(x)}{g(x)}\,dx$$

$$\int P.\,dx = \log(g(x))$$

$$g(x) = e^{\int P dx}$$

On multiplying the equation (1) by $g(x) = e^{\int P dx}$, the L.H.S becomes the derivative of some function of x and y. This function $g(x) = e^{\int P dx}$ is called Integrating Factor. Substituting the value of $g(x)$ in equation (2), we get

$$e^{\int P dx} \frac{dy}{dx} + P e^{\int P dx} y = Q \cdot e^{\int P dx}$$

or
$$\frac{d}{dx}\left(y\, e^{\int P dx}\right) = Q e^{\int P dx}$$

Integrating both sides with respect to x, we get

$$y \cdot e^{\int P dx} = \int \left(Q \cdot e^{\int P dx} \right) dx$$

or
$$y = e^{-\int P dx} \cdot \int \left(Q \cdot e^{\int P dx} \right) dx + C$$

which is general solution of the differential equation.

→ Steps involved to solve first order linear differential equation:

(i) Write the given differential equation in the form $\frac{dy}{dx} + Py = Q$ where P, Q are constants or functions of x only.

(ii) Find the Integrating Factor $(I.F) = e^{\int P dx}$.

(iii) Write the solution of the given differential equation as

$$y\,(I.F) = \int (Q \times I.F)\,dx + c$$

In case, the first order differential equation is in the form $\frac{dx}{dy} + P_1 x = Q_1$, where P_1 and Q_1 are constants or functions of y only. Then $I.F = e^{\int P_1 dy}$ and the solution of the differential equation is given by

$$x \cdot (I.F) = \int (Q_1 \times I.F)\,dy + c$$

Ex: Find the general solution of D.E $x\frac{dy}{dx} + 2y = x^2$ $(x \neq 0)$.

Solution

$$x\frac{dy}{dx} + 2y = x^2, \quad -(1)$$

Step 1:

Dividing both sides of equation (1) by x, we get

$$\frac{dy}{dx} + \frac{2}{x}y = x$$

which is linear diff eq of the type $\frac{dy}{dx} + Py = Q$, where $P = \frac{2}{x}$, & $Q = x$

Step 2: So,

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2 \quad [\text{as } e^{\log f(x)} = f(x)]$$

Therefore, solution of the given equation is given by $y\,(I.F) = \int Q\,(I.F)\,dx + c$

Step 3: Solution

$$y \cdot x^2 = \int (x)(x^2)\,dx + c = \int x^3 dx + c$$

$$y = \frac{x^2}{4} + c x^{-2}$$

which is the general solution of the given differential equation

Lecture #03 Linear D.E

⇒ D.E is said to be linear if it satisfies the following conditions

1. Dependent variable and it's derivative(s) occur first degree
2. No term in the equation that contains the product of dependent variable and its derivative(s).
3. No transcendental function with dependent variable.

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Ex: $(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solution ① $\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{4x^2}{1+x^2}$, $P(x) = \frac{2x}{1+x^2}$, $Q(x) = \frac{4x^2}{1+x^2}$

② I.F $e^{\int P(x)dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = \boxed{1+x^2}$

③ Solution $y.(I.F) = \int Q(I.F)dx + c$

$$y(1+x^2) = \int \frac{4x^2}{1+x^2}(1+x^2)\,dx + c$$

$$y(1+x^2) = 4\int x^2\,dx + c$$

$$y(1+x^2) = \frac{4}{3}x^3 + c \quad \text{Ans.}$$

- Bernaulli's Differential Equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

If $n=0,1$ then D.E is linear, If $n>1$ then Bernaulli

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x)$$

let $y^{1-n} = z \Rightarrow \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}$

$$\Rightarrow \frac{1}{1-n}\frac{dz}{dx} = y^{-n}\frac{dy}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$$

$$\boxed{\frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)} \Rightarrow \text{Linear D.E}$$

Ex: $\frac{dy}{dx} + \frac{y}{x} = y^2$

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\cdot\frac{1}{y} = 1 \Rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = 1$

let $z = y^{-1}$, $\frac{dz}{dx} = -1\,y^{-2}\frac{dy}{dx} \Rightarrow -\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$

$-\frac{dz}{dx} + \frac{1}{x}z = 1 \Rightarrow \frac{dz}{dx} - \frac{1}{x}z = -1$, $P(x) = -\frac{1}{x}$, $Q = -1$

I.F $\Rightarrow e^{-\int \frac{1}{x}dx} = e^{-\ln(x)} = e^{\ln(x^{-1})} = 1/x$

Solution $\Rightarrow \frac{z}{x} = \int(-1)\frac{1}{x}\,dx + C \Rightarrow \frac{z}{x} = -\ln(x) + C$

$\Rightarrow$ Replace (z) : $\frac{y^{-1}}{x} + \ln(x) = C \Rightarrow \boxed{\frac{1}{xy} + \ln(x) = C}$

Worksheet #03 Linear & Bernoulli

1. $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$ Ans: $xy = \frac{x^5}{5} - 3\frac{x^2}{2} + c$

Solution

Check | assert Conditions : Degree, Product, Transcedental (DPT)

Step1: Write the D.E in the form $\frac{dy}{dx} + P(x)y = Q(x)$

$$\frac{dy}{dx} + \left(\frac{1}{x}\right)y = (x^3 - 3), \quad P(x) = \frac{1}{x}, \quad Q(x) = x^3 - 3$$

Step2: Integrating Factor $\Rightarrow$ $I.F = e^{\int P(x)\,dx}$

$$I.F = e^{\int \frac{1}{x}dx} = e^{\ln(x)} = x$$

Step3: Write Solution $\Rightarrow$ $y(I.F) = \int Q\,(I.F)\,dx + c$

$$y(x) = \int (x^3 - 3)(x)\,dx + c$$

$$xy = \int x^4\,dx - 3\int x^2\,dx + c$$

$$\boxed{xy = \frac{x^5}{5} - 3\frac{x^2}{2} + c}$$

2. $(2y-3x)\,dx + x\,dy = 0$. Ans: $yx^2 = x^3 + c$

Solution (DPT)

Step 1: Write down D.E in $\frac{dy}{dx} + p(x)y = Q(x)$ form

let $y = u\cdot x$, then $\frac{dy}{dx} = x\frac{du}{dx} + u$

$$(2y-3x)\,dx = -x\,dy$$

$$\frac{3x-2y}{x} = \frac{dy}{dx}$$, Put $y = u\cdot x$, and its derivative

$$\frac{3x - 2(u\cdot x)}{x} = x\frac{du}{dx} + u$$

$$3 - 2u = x\frac{du}{dx} + u$$, multiply $\frac{1}{x}$ on both sides

$$\frac{du}{dx} + (3u)\frac{1}{x} = \frac{3}{x} \Rightarrow \boxed{\frac{du}{dx} + \overbrace{\frac{3}{x}}^{p(x)}\cdot u = \overbrace{\frac{3}{x}}^{Q(x)}}$$

Step 2: Integrating Factor $\Rightarrow$ $I.F = e^{\int p(x)\,dx}$

$$e^{\int p(x)\,dx} = e^{3\int \frac{1}{x}dx} = e^{3\ln(x)} = \cancel{e}^{\cancel{\ln}(x)^3} = x^3$$

Step 3: Write down Solution $y\,(I.F) = \int Q\,(I.F)\,dx + c$

$$u\,(x^3) = \int\left(\frac{3}{x}\right)(x^3)\,dx + C$$, where $u = \frac{y}{x}$

$$\frac{x^3y}{x} = 3\int x^2\,dx + c$$

$$x^2y = \frac{3x^3}{3} + c \Rightarrow \boxed{x^2y = x^3 + c}$$, or

$$\boxed{y = x + cx^{-2}}$$

(3) $\frac{dy}{dx} + y\cot x = \cos x$ Ans: $y\sin x = \frac{\sin^2 x}{2} + C$

Solution DPT: Degree, Product, Transcedental

Step1: Write D.E in the standard form $\frac{dy}{dx} + P(x)y = Q(x)$

$$\frac{dy}{dx} + y\cot x = \cos(x) \quad , P(x) = \cot(x), \ Q(x) = \cos(x)$$

Step2: Integrating Factor $\Rightarrow$ I.F $= e^{\int P(x)\,dx}$

$$e^{\int P(x)\,dx} = e^{\int \cot x\,dx} = e^{\ln|\sin(x)|} = \sin(x)$$

Step3: Solution $y(\text{I.F}) = \int Q(\text{I.F})\,dx + C$

$$y\sin(x) = \int \cos(x)\sin(x)\,dx + C \quad , \text{ let } u = \sin(x) \text{ then } du = \cos(x)\,dx$$

$$y\sin(x) = \int u\,du + C$$

$$y\sin(x) = \frac{u^2}{2} + C \Rightarrow \boxed{y\sin(x) = \frac{\sin^2(x)}{2} + C}$$

(4) $\frac{dy}{dx} + y\sec(x) = \tan(x)$ $y = \frac{C - x}{\sec(x) + \tan(x)} + 1$

Step 1: $\frac{dy}{dx} + y\sec(x) = \tan(x)$, $P(x) = \sec(x)$, $Q(x) = \tan(x)$

Step2: Integrating factor : $e^{\int \sec(x)\,dx}$, let $u = \sec x + \tan(x)$, $du = \sec x \tan x + \sec^2 x\,dx$

$$e^{\int \sec(x)\left(\frac{\sec(x) + \tan(x)}{\sec(x) + \tan(x)}\right)dx} = e^{\int \frac{\sec^2(x) + \sec(x)\tan(x)}{\sec(x) + \tan(x)}dx} = e^{\int \frac{du}{u}} = e^{\ln|u|} = u$$

$u = \sec(x) + \tan(x) = $ I.F

Step3: Solution : $y(\sec x + \tan x) = \int \tan(x)(\sec x + \tan x)\,dx + C$

$$y(\sec(x)+\tan(x)) = \int \tan(x)(\sec(x)+\tan(x))\,dx + c$$

$$y(\sec(x)+\tan(x)) = \int (\sec(x)\tan(x)+\tan^2(x))\,dx + c$$

$$y(\sec(x)+\tan(x)) = \int \sec(x)\tan(x)\,dx + \int \tan^2(x)\,dx + c$$

$$y(\sec x+\tan x) = \sec x + \int (\sec^2 x - 1)\,dx + c$$

$$y(\sec x+\tan x) = \sec x + \tan x \cancel{+x} + c$$

$$y = \frac{\sec x+\tan(x)}{\sec(x)+\tan(x)} + \frac{c-x}{\tan(x)+\sec(x)} = \boxed{\frac{c-x}{\sec(x)+\tan x}+1 = y}$$

⑤ $\cos^2 x \frac{dy}{dx} + y = \tan(x)$, Ans: $y = \tan x - 1 + ce^{-\tan x}$

Solution DPT

Step 1: Write the D.E in standard form $\frac{dy}{dx} + P(x)y = Q(x)$

$$\frac{dy}{dx} + \overbrace{\left(\frac{1}{\cos^2 x}\right)}^{\to P(x)} y = \overbrace{\left(\frac{\tan(x)}{\cos^2(x)}\right)}^{\to Q(x)}$$

Step 2: Integrating factor $\Rightarrow$ I.F $= e^{\int P(x)\,dx}$

$$e^{\int P(x)\,dx} = e^{\int \frac{1}{\cos^2 x}\,dx} = e^{\int \sec^2 x\,dx} = e^{\tan(x)} = I.F$$

Step 3: Solution $y\,(I.F) = \int Q\,(I.F)\,dx + c$

$$y e^{\tan(x)} = \int \left(\frac{\tan(x)}{\cos^2(x)}\right) e^{\tan(x)}\,dx + c$$

$$\boxed{y e^{\tan(x)} = \int (\tan(x)\sec^2(x))\, e^{\tan(x)}\,dx + C}$$

$$ye^{\tan(x)} = \int (\tan(x)\sec^2(x))e^{\tan(x)}\,dx + c$$

let $u = \tan x$, $du = \sec^2 x\,dx$
only for right side eq.
So,

$$\int \tan x \sec^2 x\, e^{\tan(x)}\,dx + c$$

$$\int (u)\,du\,(e^u) + c$$

$$\int u e^u\,du + c$$

Tabular Integration $\rightarrow \int u e^u\,du$

derivatives of u		Integrations of e^u
u	$+$	e^u
$\sec^2 x$	$-$	e^u

We cannot apply the Tabular Integration to this problem because the u is a trigonometric term. So, we need to apply the Integration by parts

Tabular Integration ✗
Integration by parts

$$\int u\,dv = uv - \int v\,du$$

Integrate

$$\int \tan(x)\sec^2(x)\,e^{\tan(x)}\,dx$$

$u = \tan(x)$ $\quad \boxed{dv = \sec^2 x\, e^{\tan(x)}\,dx}$

$du = \sec^2 x\,dx$ $\quad \boxed{v = e^{\tan x}}$

After integration

$$= (\tan x)(e^{\tan x}) - \int e^{\tan x}\sec^2 x\,dx$$

$$= \tan(x)e^{\tan(x)} - e^{\tan x}$$

$$ye^{\tan(x)} = \int \tan x \sec^2 x\, e^{\tan(x)}\,dx + c$$

$$ye^{\tan x} = \tan(x)e^{\tan x} - e^{\tan x} + c$$

$$y = \frac{\tan(x)e^{\tan x}}{e^{\tan x}} - \frac{e^{\tan x}}{e^{\tan x}} + \frac{c}{e^{\tan x}}$$

$$\boxed{y = \tan(x) - 1 + ce^{-\tan x}}$$

(6) $(x+a)\frac{dy}{dx} - 3y = (x+a)^5$ Ans: $2y = (x+a)^5 + 2c = (x+a)^3$

Step 1: Write the D.E in standard form (Assert, check conditions DPT)

$(x+a)\frac{dy}{dx} - 3y = (x+a)^5$, $\frac{1}{(x+a)}$ on b/s

$\frac{dy}{dx} - \left(\frac{3}{x+a}\right)y = (x+a)^4$

$\frac{dy}{dx} + P(x)y = Q(x)$, $P(x) = -\frac{3}{x+a}$, $Q(x) = (x+a)^4$

Step 2: $I.F = e^{\int P(x)\,dx} = e^{-3\int \frac{1}{x+a}dx} = e^{-3\ln|x+a|} = e^{\ln|x+a|^{-3}} = (x+a)^{-3}$

$I.F = (x+a)^{-3}$

Step 3: Solution

$y \cdot (I.F) = \int (I.F)\,Q(x)\,dx + c$

$y(x+a)^{-3} = \int (x+a)^{-3}(x+a)^4\,dx + c$

$y(x+a)^{-3} = \int (x+a)\,dx + c$

$y(x+a)^{-3} = \frac{(x+a)^2}{2} + c$

$y = \frac{(x+a)^2}{2(x+a)^{-3}} + \frac{c}{(x+a)^{-3}}$

$y = \frac{(x+a)^5}{2} + c(x+a)^{-3}$

$\boxed{2y = (x+a)^5 + 2c(x+a)^{-3}}$

(7) $x\cos x \frac{dy}{dx} + y(x\sin x + \cos x) = 1$; Ans: $xy = \sin x + C$

Step 1: Write D.E in standard form + check condition DPT

$$\frac{dy}{dx} + y\left(\frac{x\sin x}{x\cos x} + \frac{\cos x}{x\cos x}\right) = \frac{1}{x\cos x}$$

$$\frac{dy}{dx} + y\left(\frac{\sin x}{\cos x} + \frac{1}{x}\right) = \frac{1}{x\cos x}$$

$$\frac{dy}{dx} + y\left(\tan x + \frac{1}{x}\right) = \frac{\sec x}{x}$$

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

Step 2: I.F $= e^{\int P(x)}$

$$e^{\int P(x)dx} = e^{\int \tan x + \frac{1}{x}\,dx} = e^{\cancel{-\ln|\cos x| + \ln|x|}}$$

$$e^{\ln\left(\frac{x}{\cos x}\right) \cancel{+\ln|c|}} = \frac{cx}{\cancel{c}\cos x} = \frac{x}{\cancel{c}\cos x}$$

Step 3: Write general Solution $\rightarrow$ $y(\text{I.F}) = \int (\text{I.F})\,Q(x)\,dx + c$

$$\frac{xy}{\cos x} = \int \left(\frac{\cancel{x}}{\cos x} \times \frac{\sec x}{\cancel{x}}\right) dx + c \Rightarrow \frac{xy}{\cos x} = \int \sec^2 x\,dx + C$$

$$\frac{xy}{\cos x} = \tan x + c \Rightarrow xy = \cos x \times \frac{\sin x}{\cos x} + (C)\cos x$$

$$xy = \sin x + (C)\cos x$$

Step 2: I.F $\Rightarrow \int P(x)\,dx = \int \left(\tan x + \frac{1}{x}\right)dx = -\ln|\cos x| + \ln|x| = \ln|\sec x| + \ln|x|$ -

$= \ln|\sec x| + |\ln|x|| = \ln|x\sec x|$

$e^{\int P(x)dx} = e^{\ln|x\sec x|} = \boxed{x\sec x} \rightarrow$ I.F

Step 3: $y(\text{I.F}) = \int Q(x)(\text{I.F})\,dx + c$

$$xy\sec x = \int (x\sec x)\frac{\sec x}{x}\,dx$$

$$xy\sec x = \int \sec^2 x\,dx + c$$

$$xy\sec x = \tan x + c$$

$$xy\sec x = \tan x + c$$

(8) $\sec x \frac{dy}{dx} = y + \sin x$ Ans: $y = -\sin x - 1 +$

Step 1: $\frac{dy}{dx} - \frac{y}{\sec x} = \frac{\sin x}{\sec x} \Rightarrow \frac{dy}{dx} - \left(\frac{1}{\sec x}\right) y = \frac{\sin x}{\sec x}$

Step 2: $e^{\int P(x)dx} = e^{-\int \cos x \, dx} = e^{-\sin x}$

Step 3: $y\,(I.F) = \int (I.F)\, Q(x)\, dx + c$

$y e^{-\sin x} = \int e^{-\sin x} \frac{\sin x}{\sec x} dx + c \Rightarrow y e^{-\sin x} = \int e^{-\sin x} \sin x \cos x \, dx + c$

$\Rightarrow$ let $u = \sin x$, $du = \cos x \, dx \Rightarrow y e^{-\sin x} = \int e^{-u} u \, du + c$.

$dv = e^{-\sin x}$, $v = -e^{-\sin x}/\cos x$

$\Rightarrow y e^{-\sin x} = \int u e^{-u} du + c \Rightarrow$ Integration by parts $\int u\,dv = uv - \int v\,du$

$\Rightarrow y e^{-\sin x} = \int e^{-\sin x} \sin x \cos x \, dx + c$ $\int e^{-\sin x} \cos x \, dx$, $u = \sin x$, $du = \cos x\,dx$

let $u = \sin x$, $dv = e^{-\sin x} \cos x \, dx \rightarrow$ $\int e^{-u} du = -e^{-u}$

$du = \cos x$, $v = -e^{-\sin x}$

$y e^{-\sin x} = \int e^{-\sin x} \cos x \sin x \, dx + c$

$y e^{-\sin x} = (\sin x)(-e^{-\sin x}) + \int e^{-\sin x} \cos x \, dx$

$y \cancel{e^{-\sin x}} = -\cancel{e^{-\sin x}} \sin x + (-\cancel{e^{-\sin x}}) + c$

$$\boxed{y = -\sin x - 1 + c}$$

(9) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ Ans: $x = -\tan^{-1}y - 1 + ce^{\tan^{-1}y}$

Step 1: Write the D.E in standard form and check conditions

$$\frac{dy}{dx}(\tan^{-1}y - x) - y^2 = 1 \quad \text{Nope}$$

let us change the independent variable

$$(1+y^2)\,dx = (\tan^{-1}y - x)\,dy \;, \quad \frac{1}{dy} \text{ on b/s}$$

$$(1+y^2)\frac{dx}{dy} = (\tan^{-1}y - x)\cancel{\frac{dy}{dy}} \;, \quad (1+y^2) \text{ divide on b/s}$$

$$\frac{(1+y^2)}{(1+y^2)}\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2} \qquad \frac{dx}{dy} + P(y)\,x = Q(y)$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2} \Rightarrow \boxed{\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}}$$

Step 2: Integrating factor

$$\int P(y)\,dy = \int \frac{1}{1+y^2}\,dy = \tan^{-1}y \;; \quad e^{\int P(y)dy} = \boxed{e^{\tan^{-1}y}}$$

Step 3: General Solution

$$x(I.F) = \int Q(y)(I.F)\,dy + c$$

$$xe^{\tan^{-1}y} = \int \left(\frac{\tan^{-1}y}{1+y^2}\right)\left(e^{\tan^{-1}y}\right)dy + c$$

$$xe^{\tan^{-1}y} = \int \tan^{-1}y \times \frac{e^{\tan^{-1}y}}{1+y^2} + c$$

let $u = \tan^{-1}y$, $dv = \frac{e^{\tan^{-1}y}}{1+y^2}$

$du = \frac{1}{1+y^2}$

$$xe^{\tan^{-1}y} = \int u\,dv + c$$

$$xe^{\tan^{-1}y} = uv - \int v\,du + c$$

$$xe^{\tan^{-1}y} = (\tan^{-1}y)v - \int v\,du + c$$

$\int dv = \int \frac{e^{\tan^{-1}y}}{1+y^2}$, let $a = \tan^{-1}y$, $da = \frac{1}{1+y^2}$

$v = \int e^a\,da \Rightarrow v = e^a \Rightarrow \boxed{v = e^{\tan^{-1}y}}$

$$xe^{\tan^{-1}y} = uv - \int v\,du + c$$

$$xe^{\tan^{-1}y} = (\tan^{-1}y)e^{\tan^{-1}y} - \int \frac{e^{\tan^{-1}y}}{1+y^2} + c$$

$$xe^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

$$xe^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

$$x = \tan^{-1}y - 1 + \frac{c}{e^{\tan^{-1}y}}$$

$$\boxed{x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}}$$

(10) $\frac{dr}{d\theta} + (2r\cot\theta + \sin 2\theta)\, d\theta = 0$; Ans: $r\sin^2\theta = -\frac{\sin^4\theta}{2} + c$

Step 1: Write the D.E in Linear D.E standard form and check the conditions (DPT)

$$\frac{dr}{d\theta} = -(2r\cot\theta + \sin 2\theta)$$

$$\frac{dr}{d\theta} + 2r\cot\theta = -\sin 2\theta$$

$$\boxed{\frac{dr}{d\theta} + p(\theta)r = Q(\theta)}$$

$p(\theta) = 2\cot\theta$, $Q(\theta) = -\sin 2\theta$

Step 2: I.F

$$\int p(\theta)\, d\theta = \int 2\cot\theta\, d\theta = 2\int \frac{\sin\theta}{\cos\theta}\, d\theta = 2(-\ln|\cos\theta|)$$

$$e^{-2\ln|\cos\theta|} = e^{\ln|\cos\theta^{-2}|} = \cos\theta^{-2} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

$$\boxed{I.F = \sec^2\theta}$$

Step 3: General Solution $r(I.F) = \int Q(\theta)\, I.F\, d\theta + c$

$$r\sec^2\theta = -\int \sin 2\theta \sec^2\theta\, d\theta + c$$

$$r\sec^2\theta = -2\int \sin\theta\cos\theta\sec^2\theta\, d\theta + c$$

where $\sin 2\theta = 2\sin\theta\cos\theta$

$$r\sec^2\theta = -2\int \frac{\sin\theta\cos\theta}{\cos^2\theta}\, d\theta + c$$

$$r\sec^2\theta = -2\int \frac{\sin\theta}{\cos\theta}\, d\theta + c$$

$$r\sec^2\theta = (-2)(-\ln|\cos\theta|) + c$$

Step 2: I.F

$$\int p(\theta)\, d\theta = 2\int \cot\theta\, d\theta = 2\int \frac{\cos\theta}{\sin\theta}\, d\theta$$

$$= (2)\ln|\sin\theta| = \ln|\sin^2\theta|$$

$$e^{\int p(\theta)} = e^{\ln|\sin^2\theta|} = \boxed{\sin^2\theta = I.F}$$

Step 3: General Solution

$$r(I.F) = \int Q(\theta)\,(I.F)\, d\theta + c$$

$$r(\sin^2\theta) = \int -\sin 2\theta \sin^2\theta\, d\theta + c$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$r\sin^2\theta = -2\int \sin\theta\cos\theta\sin^2\theta\, d\theta + c$$

$$r\sin^2\theta = -2\int \sin^3\theta\cos\theta\, d\theta + c$$

let $u = \sin\theta$, $du = \cos\theta$

$$r\sin^2\theta = -2\int u^3\, du + c$$

$$r\sin^2\theta = \frac{-2u^4}{4} + c$$

$$r\sin^2\theta = -\frac{u^4}{2} + c$$

$$\boxed{r\sin^2\theta = -\frac{\sin^4\theta}{2} + c}$$

(Q) $x^2 dy + y(y+x)\,dx = 0$ Ans: $\frac{1}{xy} = -\frac{1}{2x^2} - c$

Step 1: Write the D.E in Linear D.E standard form and assert the conditions.

$x^2\frac{dy}{dx} = -y(y+x)$

$\frac{dy}{dx} = -\frac{y}{x^2}(y+x)$

$\frac{dy}{dx} = \frac{-y^2}{x^2} - \frac{y}{x}$

$\boxed{\frac{dy}{dx} + \frac{y}{x} = -\frac{y^2}{x^2}}$

Bernoulli

$\frac{dy}{dx} + \frac{y}{x} = \frac{-y^2}{x^2}$, $\frac{1}{y^2}$ on b/s

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = -\frac{1}{x^2}$, (let $z = \frac{1}{y}$, $\frac{dz}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$)

$-\frac{dz}{dx} + \frac{z}{x} = -\frac{1}{x^2}$

$\boxed{\frac{dz}{dx} - \frac{z}{x} = \frac{1}{x^2}}$ $\Rightarrow \frac{dz}{dx} + p(x)z = Q(x)$

Step 2: I.F, $p(x) = -\frac{1}{x}$, $Q(x) = \frac{1}{x^2}$.

$\int p(x)\,dx = \int -\frac{1}{x}\,dx = -\ln|x| = \ln|x^{-1}| \rightarrow e^{\int p} = e^{\ln|x^{-1}|} = \boxed{1/x = \text{I.F}}$

Step 3: General Solution $z(\text{I.F}) = \int Q(x)\,\text{I.F}\,dx + c$

$\frac{z}{x} = \int \frac{1}{x^2} \times \frac{1}{x}\,dx + c$

$\frac{z}{x} = \int \frac{1}{x^3}\,dx + c$

$\frac{z}{x} = \frac{x^{-3+1}}{-3+1} + c$

$\frac{z}{x} = \frac{x^{-2}}{-2} + c$

$\frac{z}{x} = \frac{1}{-2x^2} + c$

$\frac{z}{x} = \frac{1}{-2x^2} + c$, $z = \frac{1}{y}$.

$\boxed{\frac{1}{xy} = \frac{1}{-2x^2} + c}$

(12) $x\frac{dy}{dx} + y\ln y = xye^x$ Ans: $x\ln y = xe^x - e^x + c$

Step 1: Write the D.E in the Standard Linear D.E & check Conditions

$$x\frac{dy}{dx} + y\ln y = xye^x$$

$$\frac{dy}{dx} + \frac{y\ln y}{x} = ye^x$$

$$\frac{1}{y}\frac{dy}{dx} + \frac{\cancel{y}\ln y}{x\cancel{y}} = e^x$$

let $z = \ln y$, $\frac{dz}{dx} = \frac{1}{y}\frac{dy}{dx}$

$$\frac{dz}{dx} + \frac{z}{x} = e^x$$

Step 2: I.F $P(x) = 1/x$, $Q(x) = e^x$

$$\int P(x)\,dx = \int \frac{1}{x}\,dx = \ln|x|$$

$$e^{\int P(x)} = e^{\ln|x|} = \boxed{x = I.F}$$

Step 3: General Solution $z(I.F) = \int Q(x)\, I.F\, dx + c$

$$z(x) = \int xe^x\, dx + c$$

$$zx = \int u\, dv + c$$

Tabular Integration

$\frac{d}{dx}(u)$		$\int dv$
x	+	e^x
1	−	e^x
0	Stop	e^x

$$xe^x - e^x = \int xe^x\,dx$$

$$zx = \int xe^x\,dx + c$$

$$\boxed{zx = xe^x - e^x + c}$$

$$zx = xe^x - e^x + c, \quad z = \ln y$$

$$\boxed{x\ln y = xe^x - e^x + c}$$

(13) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ Ans: $\frac{\sin y}{1+x} = e^x + c$

Step 1: Write D.E in Standard Linear D.E and check Conditions

$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y \cos y}{1+x} = (1+x)e^x$

$\cos y \frac{dy}{dx} - \frac{\frac{\sin y}{\cos y} \times \cos y}{1+x} = (1+x)e^x$

$\cos y \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$

Let $z = \sin y$, $\frac{dz}{dx} = \cos y \frac{dy}{dx}$

$\frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x$

Step 2: I.F

$\int p(x) = -\int \frac{1}{1+x} = -\ln|1+x|$

$e^{-\ln|1+x|} = e^{\ln|1+x|^{-1}} = 1/1+x$

Step 3: General Solution $z(I.F) = \int Q(x)\, I.F\, dx + c$

$\frac{z}{1+x} = \int \frac{(1+x)e^x}{1+x} dx + c$

$\frac{z}{1+x} = \int e^x dx + c$

$\frac{z}{1+x} = e^x + c$

$$\boxed{\frac{\sin y}{1+x} = e^x + c}$$

(14) $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$ Ans: $\sec y = (\sin x + c)\cos x$.

Step 1: $\frac{dy}{dx} + \frac{\tan x}{\tan y} = \frac{\cos y \cos^2 x}{\tan y}$

$\frac{dy}{dx} + \frac{\tan x}{\tan y} = \cos y \cos^2 x \times \frac{\cos y}{\sin y}$

$\frac{dy}{dx} + \frac{\tan x}{\tan y} = \frac{\cos^2 y \cos^2 x}{\sin y}$

Step 1: $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$\frac{\sin y}{\cos y} \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

let $z = \frac{1}{\cos y}$, $\frac{dz}{dx} = \sec y \tan y \frac{dy}{dx}$

$\boxed{\frac{dz}{dx} = \frac{\tan y}{\cos y} \frac{dy}{dx}}$

$\frac{\sin y}{\cos y} \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$\frac{\sin y}{\cos^2 y} \frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x$

$\boxed{\frac{\tan y}{\cos y} \frac{dy}{dx} + \frac{\tan x}{\cos y} = \cos^2 x}$

$\frac{dz}{dx} + z \tan x = \cos^2 x$

$P(x) = \tan x$; $Q(x) = \cos^2 x$

$\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

$\frac{\sin y}{\cos y} \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

let $z = \cos y$, $\frac{dz}{dx} = -\sin y \frac{dy}{dx}$

$-\frac{1}{z} \frac{dz}{dx} + \tan x = z \cos^2 x$

Step 2: I.F $\Rightarrow e^{\int P(x)dx}$

$\int P(x)dx = \int \tan x \, dx = \ln|\sec x|$

$e^{\int P(x)dx} = e^{\ln|\sec x|} = \boxed{\sec x = I.F}$

Step 3: General Solution

$y(I.F) = \int Q(x)(I.F)dx + c$

$y \sec x = \int \sec x \cos^2 x \, dx + c$

$y \sec x = \int \frac{\cos^2 x}{\cos x} dx + c$

$y \sec x = \int \cos x \, dx + c$

$\boxed{y \sec x = \sin x + c}$

$z \sec x = \sin x + c$

where $z = \frac{1}{\cos y}$

$\frac{\sec x}{\cos y} = \sin x + c$

$\sec x \sec y = (\sin x + c) \cancel{\cos y}$

$\frac{\sec y}{\cos x} = (\sin x + c)$

$\boxed{\sec y = (\sin x + c)\cos x}$

(18) $x\left[\frac{dy}{dx}+y\right]=1-y$ Ans: $y=\frac{1}{x}+\frac{c}{x}e^{-x}$

Step 1: $x\frac{dy}{dx}+xy=1-y$

$\frac{dy}{dx}+y=\frac{1-y}{x}$, $\div x$ on b/s

$\frac{dy}{dx}+y=\frac{1}{x}-\frac{y}{x}$

$\frac{dy}{dx}+y+\frac{y}{x}=\frac{1}{x}$

$\frac{dy}{dx}+y\left(1+\frac{1}{x}\right)=\frac{1}{x}$

$P(x)=1+\frac{1}{x}$; $Q(x)=\frac{1}{x}$

Step 2: I.F

$\int P(x)dx=\int\left(1+\frac{1}{x}\right)dx=x+\ln|x|$

$e^{\int P(x)dx}=e^{x+\ln|x|}=e^{x}\cdot e^{\ln|x|}=\boxed{xe^{x}}$

Step 3: Solution

$yxe^{x}=\int\frac{xe^{x}}{x}dx+c$

$xye^{x}=\int e^{x}+c$

$xye^{x}=e^{x}+c$

$y=\frac{e^{x}}{xe^{x}}+\frac{c}{xe^{x}}$

$$\boxed{y=\frac{1}{x}+\frac{c}{x}e^{-x}}$$

(16) $y\ln y\cdot dx+(x-\ln y)dy=0$ Ans: $x\ln y=\frac{(\ln y)^2}{2}+c$

Step 1: Writing D.E in standard form

$\frac{1}{dy}$ on b/s

$y\ln y\frac{dx}{dy}+(x-\ln y)\frac{dy}{dx}=\frac{0}{dy}$

$y\ln y\frac{dx}{dy}+(x-\ln y)=0$

x is dependent var here

So, D.E will look like

$\frac{dx}{dy}+P(y)x=Q(y)$

$y\ln y\frac{dx}{dy}+x=\ln y$

$\frac{dx}{dy}+\frac{x}{y\ln y}=\frac{\ln y}{y\ln y}$

$\frac{dx}{dy}+\frac{x}{y\ln y}=\frac{\cancel{\ln y}}{y\cancel{\ln y}}$

$\frac{dx}{dy}+\frac{x}{y\ln y}=\frac{1}{y}\rightarrow Q(y)$

Step 2: Integrating Factor $\rightarrow P(y)$

$\int P(y)\,dy=\int\frac{1}{y\ln y}dy$, let $u=\ln y$, $du=\frac{1}{y}dy$

$\int\frac{1}{y\ln y}dy=\int\frac{du}{u}=\ln|u|=\ln|\ln y|$

$e^{\int P(y)dy}=e^{\ln|\ln y|}=\boxed{\ln y}\rightarrow$ I.F

Step 3: General Solution $x(I.F) = \int Q(y)(I.F)\,dy + c$

$x \ln y = \int \frac{1}{y} \times \ln y \, dy + c \Rightarrow x \ln y = \int \frac{\ln y}{y} dy + c, \quad u = \ln y, \; du = \frac{1}{y} dy$

$x \ln y = \int u \, du \Rightarrow x \ln y = \frac{u^2}{2} + c \Rightarrow \boxed{x \ln y = \frac{(\ln y)^2}{2} + c}$

(17) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$ Ans: $x = (\tan^{-1}y - 1) + ce^{-\tan^{-1}y}$

Step 1: $(1+y^2)\frac{dx}{dy} = (\tan^{-1}y - x)$

$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{(1+y^2)}$

$\frac{dx}{dy} = \frac{\tan y^{-1}}{1+y^2} - \frac{x}{1+y^2}$

$$\boxed{\begin{aligned}\frac{dx}{dy} + \frac{x}{1+y^2} &= \frac{\tan^{-1}y}{1+y^2}\\ \frac{dx}{dy} + P(y)x &= Q(y)\end{aligned}}$$

Step 2: I.F

$\int p(y)dy = \int \frac{1}{1+y^2} dy = \tan^{-1}y$ → trigonometric substitution

$e^{\int p(y)dy} = \boxed{e^{\tan^{-1}y} = I.F}$

Step 3: General Solution

$x(I.F) = \int Q(y)\, I.F\, dy + c$

$xe^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} \times e^{\tan^{-1}y} dy + c$

$xe^{\tan^{-1}y} = \int \frac{e^{\tan^{-1}y}}{1+y^2} \times \tan^{-1}y \, dy + c$

Integration By Parts

$\int u\,dv = uv - \int v\,du$ (ILATE)

$u = \tan^{-1}y, \quad dv = e^{\tan^{-1}y}/1+y^2 \, dy$

$du = \frac{1}{1+y^2} dy, \quad v = e^{\tan^{-1}y}$

$\int dv = \int \frac{e^{\tan^{-1}y}}{1+y^2} dy, \quad u = \tan^{-1}y, \quad du = 1/1+y^2 \, dy$

$\int dv = \int e^u du$

$v = e^u = e^{\tan^{-1}y}$

$xe^{\tan^{-1}y} = uv - \int v\,du + c$

$xe^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - \int \frac{e^{\tan^{-1}y}}{1+y^2} dy + c$

$xe^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$

$x = \frac{\tan^{-1}y\, \cancel{e^{\tan^{-1}y}}}{\cancel{e^{\tan^{-1}y}}} - \frac{\cancel{e^{\tan^{-1}y}}}{\cancel{e^{\tan^{-1}y}}} + \frac{c}{e^{\tan^{-1}y}}$

$x = \tan^{-1}y - 1 + \frac{c}{e^{\tan^{-1}y}}$

$$\boxed{x = \tan^{-1}y - 1 + ce^{-\tan^{-1}y}}$$

(18) $r\sin\theta - \frac{dr}{d\theta}\cos\theta = r^2$ Ans: $r = \frac{1}{\sin\theta + (c)\cos\theta}$

Step 1: Write D.E in Standard Linear D.E

$-\frac{dr}{d\theta}\cos\theta + r\sin\theta = r^2$, $-1/\cos\theta$ on b/s

$\frac{dr}{d\theta} - \frac{r\sin\theta}{\cos\theta} = \frac{-r^2}{\cos\theta}$; $\frac{1}{r^2}$ on b/s

$\frac{1}{r^2}\frac{dr}{d\theta} - \frac{\sin\theta}{r\cos\theta} = \frac{-1}{\cos\theta}$

$z = \frac{1}{r}$, $\frac{dz}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$$\boxed{\frac{dz}{d\theta} + \frac{z\sin\theta}{\cos\theta} = +\frac{1}{\cos\theta}}$$

Step 2: Integrating Factor

$\int p(\theta)\,d\theta = +\int \frac{\sin\theta}{\cos\theta}d\theta = -\ln|\cos\theta|$

$e^{\int p(\theta)d\theta} = e^{-\ln|\cos\theta|} = \boxed{\sec\theta}$

Step 3: General Solution

$z(I.F) = \int Q(\theta)\, I.F\, d\theta + c$

$z\sec\theta = +\int \frac{1}{\cos\theta} \times \sec\theta\, d\theta + c$

$z\sec\theta = +\int \sec^2\theta + c$

$z\sec\theta = +\tan\theta + c$

$z\sec\theta = \tan\theta + c$

$z\sec\theta = \tan\theta + c$, $z = 1/r$

$\frac{\sec\theta}{r} = \tan\theta + c$

$\frac{\sec\theta}{\tan\theta + c} = r \Rightarrow \frac{1}{\cos\theta} \times \frac{1}{\frac{\sin\theta}{\cos\theta} + c} = r$

$\frac{1}{\cos\theta} \times \frac{1}{\frac{1}{\cos\theta}(\sin\theta + c\cos\theta)} = r \Rightarrow \boxed{\frac{1}{\sin\theta + c\cos\theta} = r}$

(18) $\cos x \frac{dy}{dx} + 4y \sin x = 4\sqrt{y} \sec x$ Ans: $\sqrt{y}\sec^2 x = 2\left[\tan x + \frac{\tan^3 x}{3}\right] + c$

Step 1: Write the D.E in Standard Linear Form

$\cos x \frac{dy}{dx} + 4y \sin x = 4\sqrt{y} \sec x$; $\frac{1}{\cos x}$ on b/s

$\frac{dy}{dx} + 4y \frac{\sin x}{\cos x} = 4\sqrt{y} \frac{\sec x}{\cos x}$; $\frac{1}{\sqrt{y}}$ on b/S

$\frac{1}{\sqrt{y}} \frac{dy}{dx} + \frac{4y}{\sqrt{y}} \tan x = \frac{4\cancel{\sqrt{y}}}{\cancel{\sqrt{y}}} \frac{\sec x}{\cos x}$; $\frac{1}{\cos x} = \sec x$

$\frac{1}{\sqrt{y}} \frac{dy}{dx} + 4\sqrt{y} \tan x = 4 \sec^2 x$; let $z = \sqrt{y}$; $\frac{dz}{dx} = \frac{1}{2\sqrt{y}} \frac{dy}{dx}$

$\boxed{2\frac{dz}{dx} + 4z \tan x = 4 \sec^2 x}$ $\rightarrow$ $\frac{dz}{dx} + 2z\tan x = 2\sec^2 x$

$P(x) = 2\tan x$; $Q(x) = 2\sec^2 x$

Step 2: Integrating Factor

$\int P(x)\, dx = 2\int \tan x\, dx = -2\ln|\cos x|$

$e^{-2\ln|\cos x|} = e^{\ln|\cos^{-2} x|} = e^{\ln|\sec^2 x|} = \boxed{\sec^2 x} \rightarrow$ I.F

Step 3: General Solution

$z\,(I.F) = \int Q(x)\, I.F\, dx + C$

$z \sec^2 x = \int 2\sec^2 x \sec^2 x\, dx + c$

$z \sec^2 x = 2\int \sec^2 x \sec^2 x\, dx + c$

$z \sec^2 x = 2\int \sec^2 x (1 + \tan^2 x)\, dx + c$

$z \sec^2 x = 2\int \sec^2 x\, dx + 2\int \sec^2 x \tan^2 x\, dx + c$

$\sec^2 x\, z = 2\tan x + 2\int \sec^2 x \tan^2 x\, dx + c$

$u = \tan x$, $du = \sec^2 x\, dx$

$z \sec^2 x = 2\tan x + 2\int u^2\, du + c$

$z \sec^2 x = 2\tan x + \frac{2}{3} u^3 + c$

$z \sec^2 x = 2\tan x + \frac{2}{3} u^3 + c$

$z \sec^2 x = 2\tan x + \frac{2}{3} \tan^3 x + c$

$\sqrt{y} \sec^2 x = 2\tan x + \frac{2}{3} \tan^3 x + c$

$\boxed{\sqrt{y} \sec^2 x = 2\left[\tan x + \frac{\tan^3 x}{3}\right] + c}$

(20) $\frac{dy}{dx} + \frac{y}{x}\ln y = \frac{y}{x^2}(\ln y)^2$ Ans: $\frac{1}{x\ln y} = \frac{1}{2x^2} + c$

Step 1: Write the D.E in Standard Linear D.E form.

$\frac{1}{y(\ln y)^2}\frac{dy}{dx} + \frac{y\ln y}{xy(\ln y)^2} = \frac{y(\ln y)^2}{x^2 y(\ln y)^2}$, $\frac{1}{y(\ln y)^2}$ on b/s

$\frac{1}{y(\ln y)^2}\frac{dy}{dx} + \frac{1}{x\ln y} = \frac{1}{x^2}$, let $\boxed{z = \ln y}$ $\frac{dz}{dx} = \frac{1}{y}\frac{dy}{dx}$

$\frac{1}{z^2}\frac{dz}{dx} + \frac{1}{xz} = \frac{1}{x^2}$, let $\boxed{u = \frac{1}{z}}$; $\frac{du}{dx} = -\frac{1}{z^2}\frac{dz}{dx}$

$-\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^2} \rightarrow \frac{du}{dx} - \frac{u}{x} = -\frac{1}{x^2} \rightarrow \frac{dy}{dx} + P(x)y = Q(x)$

Step 2: Integrating Factor

So, $P(x) = -\frac{1}{x}$, $\int P(x) = -\ln|x| = \ln|1/x| \Rightarrow e^{\ln|1/x|} = \boxed{\frac{1}{x}}$ → I.F

Step 3: General Solution $u(I.F) = \int Q(x)(I.F)\,dx + c$

$\frac{u}{x} = -\int \frac{1}{x^2}\times\frac{1}{x}\,dx + c$

$\frac{u}{x} = -\int x^{-3}\,dx + c$

$\frac{u}{x} = -\frac{x^{-3+1}}{-3+1} + c$

$\boxed{\frac{u}{x} = \frac{1}{2x^2} + c}$

$u = \frac{1}{z}$

$\frac{1}{zx} = \frac{1}{2x^2} + c$

$\frac{1}{zx} = \frac{1}{2x^2} + c$, $z = \ln y$

$\boxed{\frac{1}{x\ln y} = \frac{1}{2x^2} + c}$