# LECTURE-7

## Eigen Value and Eigen Vector

# Eigenvalues and Eigenvectors

- Linear equations Ax = b come from steady state problems. Eigen values have their greatest importance in *dynamic problems. The solution of du/dt= Au is changing with time*—growing or decaying or oscillating.

- Almost all vectors change direction, when they are multiplied by square matrix A.

- *Certain exceptional vectors "x" are in the same direction as Ax. Those are the "Eigen vectors".*

- The basic equation is $Ax = \lambda x$. The number $\lambda$ is an "Eigen value" of A.

- The eigen value tells whether the special vector "x" is stretched or shrunk or reversed or left unchanged—when it is multiplied by A.

Note: The prefix *eigen-* is adopted from the German word "eigen" for "own" in the sense of a characteristic description (that is why the eigenvectors are sometimes also called characteristic vectors, and, similarly, the eigenvalues are also known as characteristic values).

# Eigenvalues and Eigenvectors

- Eigenvalue problem:

If $A$ is an $n \times n$ matrix, do there exist nonzero vectors $\boldsymbol{x}$ in $R^n$ such that $A\boldsymbol{x}$ is a scalar multiple of $\boldsymbol{x}$？

- Eigenvalue and eigenvector:

$A$： an $n \times n$ matrix

$\lambda$： a scalar

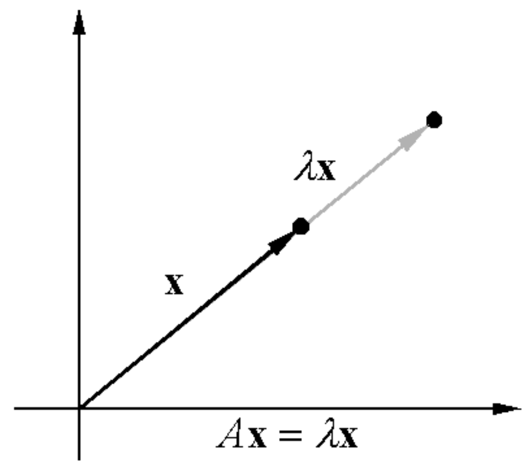
$\boldsymbol{x}$： a nonzero vector in $R^n$

Eigenvalue

$$Ax = \lambda x$$

Eigenvector

- Geometrical Interpretation



- Ex 7.1: (Verifying eigenvalues and eigenvectors)

$$A=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix} \quad x_1=\begin{bmatrix}1\\ 0\end{bmatrix} \quad x_2=\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$Ax_1=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}2\\ 0\end{bmatrix}=\overset{\substack{\text{Eigenvalue}\\ \downarrow}}{2}\underset{\substack{\uparrow\\ \text{Eigenvector}}}{\begin{bmatrix}1\\ 0\end{bmatrix}}=2x_1$$

$$Ax_2=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}=\overset{\substack{\text{Eigenvalue}\\ \downarrow}}{-1}\underset{\substack{\uparrow\\ \text{Eigenvector}}}{\begin{bmatrix}0\\ 1\end{bmatrix}}=(-1)x_2$$

- Theorem 7.1: (The eigenspace of A corresponding to $\lambda$)

  If $A$ is an $n\times n$ matrix with an eigenvalue $\lambda$, then the set of all eigenvectors of $\lambda$ together with the zero vector is a subspace of $R^n$. This subspace is called the eigenspace of $\lambda$ .

Pf:

$x_1$ and $x_2$ are eigenvectors corresponding to $\lambda$

$$(i.e.\ \ Ax_1 = \lambda x_1,\ \ Ax_2 = \lambda x_2)$$

$$(1)\ A(x_1 + x_2) = Ax_1 + Ax_2 = \lambda x_1 + \lambda x_2 = \lambda(x_1 + x_2)$$

$$(i.e.\ \ x_1 + x_2 \text{ is an eigenvector corresponding to } \lambda)$$

$$(2)\ A(cx_1) = c(Ax_1) = c(\lambda x_1) = \lambda(cx_1)$$

$$(i.e.\ \ cx_1 \text{ is an eigenvector corresponding to } \lambda)$$

- Ex 7.2: (An example of eigenspaces in the plane)

Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Sol:

If $\quad \mathrm{v} = (x, y)$

$$A\mathrm{v} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$$

For a vector on the *x*-axis

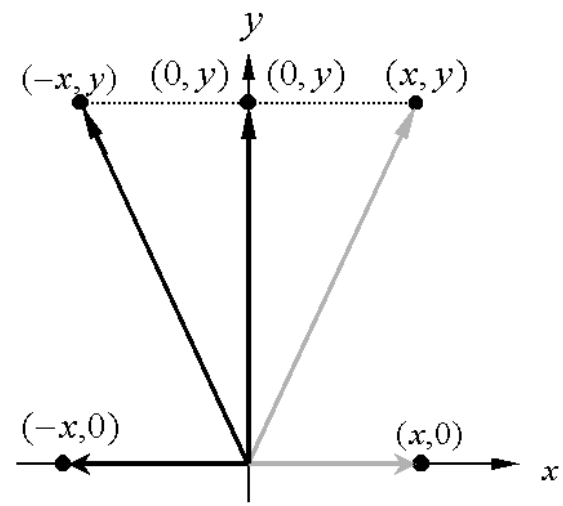
Eigenvalue $\lambda = -1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1 \begin{bmatrix} x \\ 0 \end{bmatrix}$$

For a vector on the *y*-axis

Eigenvalue $\lambda = 1$

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1 \begin{bmatrix} 0 \\ y \end{bmatrix}$$




Geometrically, multiplying a vector (x, y) in $R^2$ by the matrix *A* corresponds to a reflection in the *y*-axis.

The eigenspace corresponding to $\lambda_1 = -1$ is the *x*-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the *y*-axis.

- Theorem 7.2: (Finding eigenvalues and eigenvectors of a matrix $A \in M_{n\times n}$)

Let $A$ is an $n\times n$ matrix.

(1) An eigenvalue of $A$ is a scalar $\lambda$ such that $\det(\lambda \mathrm{I} - A) = 0$

(2) The eigenvectors of $A$ corresponding to λ are the nonzero solutions of $(\lambda \mathrm{I} - A)x = 0$

- Note:

$$Ax = \lambda x \;\Rightarrow\; (\lambda \mathrm{I} - A)x = 0$$ (homogeneous system)

$(\lambda \mathrm{I} - A)x = 0$ has nonzero solutions iff $\det(\lambda \mathrm{I} - A) = 0$

- Characteristic equation of $A$:

$$\det(\lambda \mathrm{I} - A) = 0$$

- Ex 7.3: (Finding eigenvalues and eigenvectors)

Find the eigenvalues and eigenvectors of matrix A.

$$A=\begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Sol: Characteristic equation:

$$(\lambda \mathrm{I}-A)=\begin{vmatrix} \lambda-2 & 12 \\ -1 & \lambda+5 \end{vmatrix}$$

$$=\lambda^2+3\lambda+2=(\lambda+1)(\lambda+2)=0$$

$$\Rightarrow \lambda=-1,-2$$

Eigenvalue: $\lambda_1=-1,\ \lambda_2=-2$

$$(1)\lambda_1 = -1 \qquad \Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} -3 & 12 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4t \\ t \end{bmatrix} = t \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ t \neq 0$$

$$(2)\lambda_2 = -2 \qquad \Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} -4 & 12 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3t \\ t \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ t \neq 0$$

- Ex 7.4: (Finding eigenvalues and eigenvectors)

  Find the eigenvalues and corresponding eigenvectors for the matrix $A$.

  What is the dimension of the eigenspace of each eigenvalue?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^3 = 0$$

Eigenvalue: $\lambda = 2$

The eigenspace of A corresponding to : $\lambda = 2$

$$(\lambda \mathrm{I} - A)x = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; s, t \neq 0$$

$$\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \middle| s, t \in R \right\} : \text{the eigenspace of A corresponding to } \lambda = 2$$

Thus, the dimension of its eigenspace is 2.

- Ex 7.5：Find the eigenvalues of the matrix $A$ and find a basis for each of the corresponding eigenspaces.

$$A=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 5 & -10\\ 1 & 0 & 2 & 0\\ 1 & 0 & 0 & 3\end{bmatrix}$$

Sol: Characteristic equation:

$$|\lambda \mathrm{I}-A|=\begin{vmatrix}\lambda-1 & 0 & 0 & 0\\ 0 & \lambda-1 & -5 & 10\\ -1 & 0 & \lambda-2 & 0\\ -1 & 0 & 0 & \lambda-3\end{vmatrix}$$

$$=(\lambda-1)^2(\lambda-2)(\lambda-3)=0$$

Eigenvalue: $\lambda_1=1, \lambda_2=2, \lambda_3=3$

$$(1)\lambda_1 = 1$$

$$\Rightarrow (\lambda_1 \mathrm{I} - A)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \ \ s,t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

is a basis for the eigenspace of A corresponding to $\lambda = 1$

$$(2)\lambda_2 = 2$$

$$\Rightarrow (\lambda_2 \mathrm{I} - A)x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 10 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \ t \neq 0$$

$$\Rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for the eigenspace of A corresponding to $\lambda = 2$

$$(3)\lambda_3 = 3$$

$$\Rightarrow (\lambda_3 \mathrm{I} - A)x = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -5 & 10 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -5t \\ 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}, \quad t \neq 0$$

$\Rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda = 3$

- Theorem 7.3: (Eigenvalues for triangular matrices)

  If $A$ is an $n \times n$ triangular matrix, then its eigenvalues are the entries on its main diagonal.

- Ex 7.6: (Finding eigenvalues for diagonal and triangular matrices)

$$(a)A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 5 & 3 & -3 \end{bmatrix} \quad (b)A = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Sol:

$$(a) \ |\lambda \mathrm{I} - A| = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 1 & \lambda - 1 & 0 \\ -5 & -3 & \lambda + 3 \end{vmatrix} = (\lambda - 2)(\lambda - 1)(\lambda + 3)$$

$$\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = -3$$

$$(b) \ \lambda_1 = -1, \lambda_2 = 2, \lambda_3 = 0, \lambda_4 = -4, \lambda_5 = 3$$

- Eigenvalues and eigenvectors of linear transformations:

A number $\lambda$ is called an eigenvalue of a linear transformation $T: V \rightarrow V$ if there is a nonzero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \lambda \mathbf{x}$. The vector $\mathbf{x}$ is called an eigenvector of $T$ corresponding to $\lambda$, and the setof all eigenvectors of $\lambda$ (with the zero vector) is called the eigenspace of $\lambda$.

- Ex 7.7: (Finding eigenvalues and eigenspaces)

Find the eigenvalues and corresponding eigenspaces

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Sol:

$$|\lambda I - A| = \begin{bmatrix} \lambda-1 & -3 & 0 \\ -3 & \lambda-1 & 0 \\ 0 & 0 & \lambda+2 \end{bmatrix} = (\lambda+2)^2(\lambda-4)$$

eigenvalues $\lambda_1 = 4, \lambda_2 = -2$

The eigenspaces for these two eigenvalues are as follows.

$B_1 = \{(1, 1, 0)\}$ Basis for $\lambda_1 = 4$

$B_2 = \{(1, -1, 0), (0, 0, 1)\}$ Basis for $\lambda_2 = -2$

- Notes:

(1) Let $T: R^3 \to R^3$ be the linear transformation whose standard matrix is A in 7.7, and let $B'$ be the basis of $R^3$ made up of three linear independent eigenvectors found in 7.7. Then $A'$, the matrix of $T$ relative to the basis $B'$, is diagonal.

$$B' = \{(1,1,0),(1,-1,0),(0,0,1)\}$$

Eigenvectors

$$A' = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Eigenvalues

(2) The main diagonal entries of the matrix $A'$ are the eigenvalues of $A$.

# Complex Eigenvalues

It is possible for the characteristic equation of a matrix with real entries to have complex solutions. In fact, because the eigenvalues of a matrix are the roots of a polynomial of precise degree *n*, *every matrix has exactly n eigenvalues if* we count them as we count the roots of a polynomial (meaning that they may be repeated, and may occur in complex conjugate pairs). For example, the characteristic polynomial of the matrix

$$A = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda + 2 & 1 \\ -5 & \lambda - 2 \end{bmatrix} = \lambda^2 + 1$$

so the characteristic equation is $\lambda^2+1=0$ , the solutions of which are the imaginary numbers $\lambda=i$ and $\lambda=-i$.

# Eigenvalues and Invertibility

## Theorem 7.4:

A square matrix A is invertible if and only if $\lambda=0$ is not an eigenvalue of A.

# Eigenvalues of the Power of a Matrix

## Theorem 7.5:

If k is a positive integer, $\lambda$ is an eigenvalue of a matrix A, and x is a corresponding eigenvector, then $\lambda^k$ is an eigenvalue of $A^k$ and x is a corresponding eigenvector.