LECTURE-2

VECTOR SPACES

2.1 Vectors in R^n

- An ordered n-tuple :

  a sequence of n real numbers $(x_1, x_2, \ldots, x_n)$

- R^n-space :

  the set of all ordered n-tuples

$n = 1$ $\quad$ R^1-space = set of all real numbers
(R^1-space can be represented geometrically by the x-axis)

$n = 2$ $\quad$ R^2-space = set of all ordered pair of real numbers (x_1, x_2)
(R^2-space can be represented geometrically by the xy-plane)

$n = 3$ $\quad$ R^3-space = set of all ordered triple of real numbers (x_1, x_2, x_3)
(R^3-space can be represented geometrically by the xyz-space)

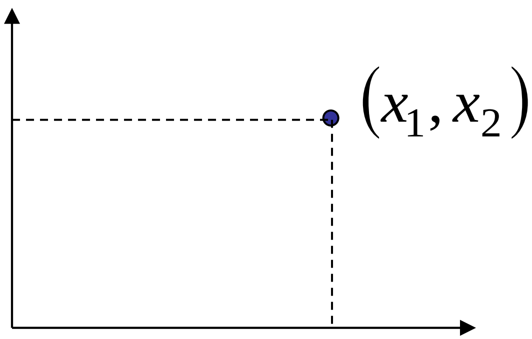
$n = 4$ $\quad$ R^4-space = set of all ordered quadruple of real numbers (x_1, x_2, x_3, x_4)

- Notes:

  (1) An n-tuple $(x_1, x_2, \cdots, x_n)$ can be viewed as a point in R^n with the x_i's as its coordinates

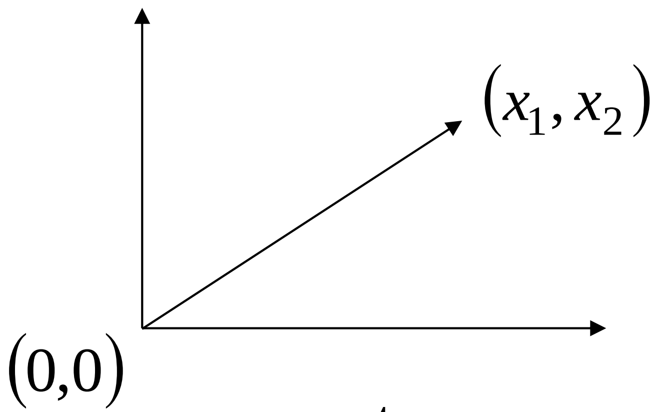
  (2) An n-tuple $(x_1, x_2, \cdots, x_n)$ also can be viewed as a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in R^n with the x_i's as its components

- Ex:1



a point

or



a vector

※ A vector on the plane is expressed geometrically by a directed line segment whose initial point is the origin and whose terminal point is the point (x_1, x_2)

$$\mathbf{u}=(u_1,u_2,\cdots,u_n),\quad \mathbf{v}=(v_1,v_2,\cdots,v_n) \qquad \text{(two vectors in } R^n\text{)}$$

- Equality:

  $\mathbf{u}=\mathbf{v}$ if and only if $u_1= v_1$, $u_2= v_2$,……, $u_n= v_n$

- Vector addition (the sum of **u** and **v**):

  $$\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

- Scalar multiplication (the scalar multiple of **u** by *c*):

  $$c\mathbf{u}=(cu_1,cu_2,\cdots,cu_n)$$

- Notes:

  The sum of two vectors and the scalar multiple of a vector in R^n are called the standard operations in R^n

- Difference between **u** and **v**:

$$\mathbf{u}-\mathbf{v} \equiv \mathbf{u}+(-1)\mathbf{v} = (u_1 - v_1,\ u_2 - v_2,\ u_3 - v_3, \dots,\ u_n - v_n)$$

- Zero vector :

$$\mathbf{0} = (0, 0, \dots, 0)$$

- Notes:

A vector $\mathbf{u}=(u_1,u_2,\ldots,u_n)$ in R^n can be viewed as:

Use comma to separate components

a $1\times n$ row matrix (row vector): $\mathbf{u}=[u_1\ u_2\ \cdots\ u_n]$

Use blank space to separate entries

or

a $n\times 1$ column matrix (column vector): $\mathbf{u}=\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$

※ Therefore, the operations of matrix addition and scalar multiplication generate the same results as the corresponding vector operations (see the next slide)

Vector addition

Scalar multiplication

$$\mathbf{u}+\mathbf{v}=(u_1,u_2,\cdots,u_n)+(v_1,v_2,\cdots,v_n)$$
$$=(u_1+v_1,u_2+v_2,\cdots,u_n+v_n)$$

$$c\mathbf{u}=c(u_1,\ u_2,\cdots,\ u_n)$$
$$=(cu_1,cu_2,\cdots,\ cu_n)$$

Regarded as $1\times n$ row matrix

$$\mathbf{u}+\mathbf{v}=[u_1\ u_2\ \cdots\ u_n]+[v_1\ v_2\ \cdots\ v_n]$$
$$=[u_1+v_1\ u_2+v_2\ \cdots\ u_n+v_n]$$

$$c\mathbf{u}=c[u_1\ u_2\cdots u_n]$$
$$=[cu_1\ cu_2\ \cdots\ cu_n]$$

Regarded as $n\times 1$ column matrix

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}+\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}=\begin{bmatrix}u_1+v_1\\u_2+v_2\\\vdots\\u_n+v_n\end{bmatrix}$$

$$c\mathbf{u}=c\begin{bmatrix}u_1\\u_2\\\vdots\\u_n\end{bmatrix}=\begin{bmatrix}cu_1\\cu_2\\\vdots\\cu_n\end{bmatrix}$$

- Theorem 2.1: Properties of vector addition and scalar multiplication

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in R^n, and let c and d be scalars

(1) $\mathbf{u}+\mathbf{v}$ is a vector in R^n (closure under vector addition)

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$ (commutative property of vector addition)

(3) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ (associative property of vector addition)

(4) $\mathbf{u}+\mathbf{0} = \mathbf{u}$ (additive identity property)

(5) $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$ (additive inverse property)

(6) $c\mathbf{u}$ is a vector in R^n (closure under scalar multiplication)

(7) $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$ (distributive property of scalar multiplication over vector addition)

(8) $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$ (distributive property of scalar multiplication over real-number addition)

(9) $c(d\mathbf{u}) = (cd)\mathbf{u}$ (associative property of multiplication)

(10) $1(\mathbf{u}) = \mathbf{u}$ (multiplicative identity property)

- Notes:

  (1) The zero vector **0** in R^n is called the additive identity in R^n (see Property 4)

  (2) The vector –**u** is called the additive inverse of **u** (see Property 5)

- Theorem 2.2: (Properties of additive identity and additive inverse)

  Let **v** be a vector in R^n and c be a scalar. Then the following properties are true

  (1) The additive identity is unique, i.e., if **v**+**u** = **v**, **u** must be **0**

  (2) The additive inverse of **v** is unique, i.e., if **v**+**u** = **0**, **u** must be –**v**

  (3) 0**v** = **0**

  (4) c**0** = **0**

  (5) If c**v** = **0**, either c = 0 or **v** = **0**

  (6) –(–**v**) = **v** (Since –**v** + **v** = **0**, the additive inverse of –**v** is **v**, i.e., **v** can be expressed as –(–**v**). Note that **v** and –**v** are the additive inverses for each other)

2.2 Vector Spaces

- Vector spaces:

Let V be a set on which two operations (addition and scalar multiplication) are defined. **If the following ten axioms are satisfied** for every element $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and every scalar (real number) c and d, then ***V* is called a vector space**, and the **elements** in ***V*** are called **vectors**

Addition:

(1) $\mathbf{u}+\mathbf{v}$ is in V

(2) $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$

(3) $\mathbf{u}+(\mathbf{v}+\mathbf{w}) = (\mathbf{u}+\mathbf{v})+\mathbf{w}$

(4) V has a zero vector $\mathbf{0}$ such that for every $\mathbf{u}$ in V, $\mathbf{u}+\mathbf{0} = \mathbf{u}$

(5) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u}) = \mathbf{0}$

Scalar multiplication:

(6) $c\mathbf{u}$ is in V

(7) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$

(8) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$

(9) $c(d\mathbf{u})=(cd)\mathbf{u}$

(10) $1(\mathbf{u})=\mathbf{u}$

※ This type of definition is called an **abstraction** because you abstract a collection of properties from R^n to form the axioms for defining a more general space V

※ Thus, we can conclude that R^n is of course a vector space

- Notes:

A vector space consists of four entities:

a set of vectors, a set of real-number scalars, and two operations

V: nonempty set

c: scalar

$+(\mathbf{u}, \mathbf{v}) = \mathbf{u} + \mathbf{v}$: vector addition

$\cdot(c, \mathbf{u}) = c\mathbf{u}$: scalar multiplication

$(V, \ +, \ \cdot)$ is called a vector space

※ The set V together with the definitions of vector addition and scalar multiplication satisfying the above ten axioms is called a vector space

- Four examples of vector spaces are introduced as follows. (It is straightforward to show that these vector spaces satisfy the above ten axioms)

(1) n-tuple space: R^n

$$(u_1, u_2, \cdots u_n) + (v_1, v_2, \cdots v_2) = (u_1 + v_1, u_2 + v_2, \cdots u_n + v_n)$$ (standard vector addition)

$$k(u_1, u_2, \cdots u_n) = (ku_1, ku_2, \cdots ku_n)$$ (standard scalar multiplication for vectors)

(2) Matrix space : $V = M_{m \times n}$

(the set of all $m\times n$ matrices with real-number entries)

Ex: ($m = n = 2$)

$$\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$ (standard matrix addition)

$$k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$ (standard scalar multiplication for matrices)

(3) n-th degree or less polynomial space : $V = P_n$

(the set of all real-valued polynomials of degree n or less)

$$p(x)+q(x)=(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$$ (standard polynomial addition)

$$kp(x)=ka_0+ka_1x+\cdots+ka_nx^n$$ (standard scalar multiplication for polynomials)

※ By the fact that the set of real numbers is closed under addition and multiplication, it is straightforward to show that P_n satisfies the ten axioms and thus is a vector space

(4) Continuous function space : $V = C(-\infty,\infty)$

(the set of all real-valued continuous functions defined on the entire real line)

$$(f+g)(x)=f(x)+g(x)$$ (standard addition for functions)

$$(kf)(x)=kf(x)$$ (standard scalar multiplication for functions)

※ By the fact that the sum of two continuous function is continuous and the product of a scalar and a continuous function is still a continuous function, $C(-\infty,\infty)$ is a vector space

- Summary of important vector spaces

$$R = \text{set of all real numbers}$$

$$R^2 = \text{set of all ordered pairs}$$

$$R^3 = \text{set of all ordered triples}$$

$$R^n = \text{set of all } n\text{-tuples}$$

$$C(-\infty,\infty) = \text{set of all continuous functions defined on the real number line}$$

$$C[a,b] = \text{set of all continuous functions defined on a closed interval } [a,b]$$

$$P = \text{set of all polynomials}$$

$$P_n = \text{set of all polynomials of degree } \le n$$

$$M_{m,n} = \text{set of } m \times n \text{ matrices}$$

$$M_{n,n} = \text{set of } n \times n \text{ square matrices}$$

※ The standard addition and scalar multiplication operations are considered if there is no other specifications

※ Each element in a vector space is called a vector, so a vector can be a real number, an n-tuple, a matrix, a polynomial, a continuous function, etc.

- Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied

- Ex 2: The set of all integers is not a vector space

Pf: $1 \in V$, and $\frac{1}{2}$ is a real-number scalar

$$(\underset{\substack{\uparrow \\ \text{scalar}}}{\tfrac{1}{2}})(\underset{\substack{\uparrow \\ \text{integer}}}{1}) = \underset{\substack{\uparrow \\ \text{noninteger}}}{\tfrac{1}{2}} \notin V$$ (it is not closed under scalar multiplication)

- Ex 3: The set of all (exact) second-degree polynomial functions is not a vector space

Pf: Let $p(x) = x^2$ and $q(x) = -x^2 + x + 1$

$$\Rightarrow p(x) + q(x) = x + 1 \notin V$$

(it is not closed under vector addition)

- Ex 4:

$V=R^2$=the set of all ordered pairs of real numbers

vector addition: $(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$

scalar multiplication: $c(u_1,u_2)=(cu_1,0)$ (nonstandard definition)

Verify V is not a vector space

Sol:

This kind of setting can satisfy the first nine axioms of the definition of a vector space (you can try to show that), but it violates the tenth axiom

$\because 1(1,1)=(1,0)\neq(1,1)$

$\therefore$ the set (together with the two given operations) is not a vector space

- Theorem 2.3: Properties of scalar multiplication

  Let $\mathbf{v}$ be any element of a vector space V, and let c be any scalar. Then the following properties are true

  (1) $0\mathbf{v} = \mathbf{0}$

  (2) $c\mathbf{0} = \mathbf{0}$

  (3) If $c\mathbf{v} = \mathbf{0}$, either $c = 0$ or $\mathbf{v} = \mathbf{0}$

  (4) $(-1)\mathbf{v} = -\mathbf{v}$ (the additive inverse of $\mathbf{v}$ equals $((-1)\mathbf{v})$

※ The first three properties are extension of Theorem 2.2, which simply considers the space of R^n. In fact, these four properties are not only valid for R^n but also for any vector space, e.g.., for all vector spaces mentioned on the previous slide.

Pf:

$$(1)\ 0\mathbf{v} = (c + (-c))\mathbf{v} \overset{(8)}{=} c\mathbf{v} + (-c)\mathbf{v} \overset{(9)}{=} c\mathbf{v} + (-(c\mathbf{v})) \overset{(5)}{=} \mathbf{0}$$

(2) $c\mathbf{0} \overset{(4)}{=} c(\mathbf{0}+\mathbf{0}) \overset{(7)}{=} c\mathbf{0}+c\mathbf{0}$

$\Rightarrow c\mathbf{0}+(-c\mathbf{0}) = (c\mathbf{0}+c\mathbf{0})+(-c\mathbf{0})$ (add ($-c\mathbf{0}$) to both sides)

$\overset{(3)}{\Rightarrow} c\mathbf{0}+(-c\mathbf{0}) = c\mathbf{0}+[c\mathbf{0}+(-c\mathbf{0})]$

$\overset{(5)}{\Rightarrow} \mathbf{0} = c\mathbf{0}+\mathbf{0} \quad \overset{(4)}{\Rightarrow} \mathbf{0} = c\mathbf{0}$

(3) prove by contradiction: suppose that $c\mathbf{v}=0$, but $c \neq 0$ and $\mathbf{v} \neq \mathbf{0}$

$\mathbf{v} \overset{(10)}{=} 1\mathbf{v} = \left(\frac{1}{c}c\right)\mathbf{v} \overset{(9)}{=} \frac{1}{c}(c\mathbf{v}) = \frac{1}{c}(\mathbf{0}) = \mathbf{0}$ (By the second property, $c\mathbf{0} = \mathbf{0}$)

$\Rightarrow \rightarrow\leftarrow \quad \Rightarrow$ if $c\mathbf{v}=\mathbf{0}$, either $c=0$ or $\mathbf{v}=\mathbf{0}$

(4) $0\mathbf{v} = (1+(-1))\mathbf{v} \overset{(8)}{=} 1\mathbf{v}+(-1)\mathbf{v}$

$\Rightarrow \mathbf{0} = \mathbf{v}+(-1)\mathbf{v}$ (By the first property, $0\mathbf{v} = \mathbf{0}$)

$\overset{(5)}{\Rightarrow} (-1)\mathbf{v} = -\mathbf{v}$ (By comparing with Axiom (5), $(-1)\mathbf{v}$ is the additive inverse of $\mathbf{v}$)

※ The proofs are valid as long as they are logical. It is not necessary to follow the same proofs in the text book.

2.3 Subspaces of Vector Spaces

- Subspace:

$(V, +, \cdot):$ a vector space

$\left.\begin{array}{l} W \neq \Phi \\ W \subseteq V \end{array}\right\}:$ a nonempty subset of V

$(W, +, \cdot):$ The nonempty subset W is called a subspace **if** W **is a vector space** under the operations of addition and scalar multiplication defined on V

- Trivial subspace:

Every vector space V has at least two subspaces

(1) Zero vector space $\{\mathbf{0}\}$ is a subspace of V (It satisfies the ten axioms

(2) V is a subspace of V

※ Any subspaces other than these two are called proper (or nontrivial) subspaces

- Examination of whether W being a subspace
  - Since the operations defined on W are the same as those defined on V, and most of the ten axioms are inherited from the properties for operations, it is not needed to verify these axioms
  - Therefore, the following theorem tells us it is sufficient to test for the closure conditions under vector addition and scalar multiplication to identify that a nonempty subset of a vector space is a subspace
- Theorem 2.4: Test whether a nonempty subset being a subspace

If W is a nonempty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold

(1) If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ is in W

(2) If $\mathbf{u}$ is in W and c is any scalar, then $c\mathbf{u}$ is in W

Pf:

1. Note that if **u**, **v**, and **w** are in W, then they are also in V. Furthermore, W and V share the same operations. Consequently, vector space axioms 2, 3, 7, 8, 9, and 10 are satisfied automatically

2. Suppose that the closure conditions hold in Theorem 2.2, i.e., the axioms 1 and 6 for vector spaces are satisfied

3. Since the axiom 6 is satisfied (i.e., $c\mathbf{u}$ is in W if $\mathbf{u}$ is in W), we can obtain

3.1. for scalar $c=0,\ c\mathbf{u}=\mathbf{0}\in W \Rightarrow \exists$ zero vector in W

$\Rightarrow$ axiom 4 is satisfied

3.2. for scalar $c=-1,\ (-1)\mathbf{u}\in W \Rightarrow \exists\ -\mathbf{u}\equiv(-1)\mathbf{u}$

st. $\mathbf{u}+(-\mathbf{u})=\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$

$\Rightarrow$ axiom 5 is satisfied

- Ex 5: A subspace of $M_{2\times2}$

Let W be the set of all 2×2 symmetric matrices. Show that W is a subspace of the vector space $M_{2\times2}$, with the standard operations of matrix addition and scalar multiplication

Sol:

First, we knon that W, the set of all 2×2 symmetric matrices, is an nonempty subset of the vector space $M_{2\times2}$

Second,

$$A_1 \in W, A_2 \in W \Rightarrow (A_1 + A_2)^T = A_1^T + A_2^T = A_1 + A_2 \quad (A_1 + A_2 \in W)$$

$$c \in R,\ A \in W \Rightarrow (cA)^T = cA^T = cA \quad (cA \in W)$$

The definition of a symmetric matrix A is that $A^T = A$

Thus, Th. 2.4 is applied to obtain that W is a subspace of M_{2x2}

- Ex 6: The set of singular matrices is not a subspace of $M_{2\times2}$

  Let W be the set of singular (noninvertible) matrices of order 2. Show that W is not a subspace of $M_{2\times2}$ with the standard matrix operations

Sol:

$$A=\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\in W,\ B=\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\in W$$

$$\because A+B=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I\notin W$$ (W is not closed under vector addition)

$$\therefore W \text{ is not a subspace of } M_{2\times2}$$

- Ex 7: The set of first-quadrant vectors is not a subspace of R^2

  Show that $W=\{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, with the standard operations, is not a subspace of R^2

Sol:

Let $\mathbf{u}=(1,\ 1) \in W$

$\because (-1)\mathbf{u}=(-1)(1,1)=(-1,-1) \notin W$

(*W* is not closed under scalar multiplication)

$\therefore W$ is not a subspace of R^2

- Ex 8: Identify subspaces of R^2

Which of the following two subsets is a subspace of R^2?

(a) The set of points on the line given by $x+2y=0$

(b) The set of points on the line given by $x+2y=1$

Sol:

(a) $W=\{(x,y) \mid x+2y=0\}=\{(-2t,t) \mid t\in R\}$ (Note: the zero vector (0,0) is on this line)

Let $\mathbf{v}_1=(-2t_1,t_1)\in W$ and $\mathbf{v}_2=(-2t_2,t_2)\in W$

$\because \mathbf{v}_1+\mathbf{v}_2=\left(-2(t_1+t_2),t_1+t_2\right)\in W$ (closed under vector addition)

$c\mathbf{v}_1=\left(-2(ct_1),ct_1\right)\in W$ (closed under scalar multiplication)

$\therefore W$ is a subspace of R^2

(b) $W = \{(x, y) \mid x + 2y = 1\}$ (Note: the zero vector (0, 0) is not on this line)

Consider $\mathbf{v} = (1, 0) \in W$

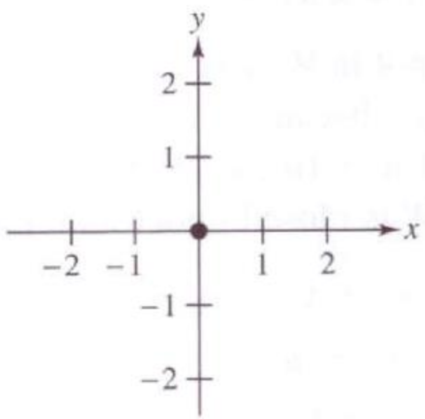
$\because (-1)\mathbf{v} = (-1, 0) \notin W \quad \therefore W$ is not a subspace of R^2

- Note: Subspaces of R^2

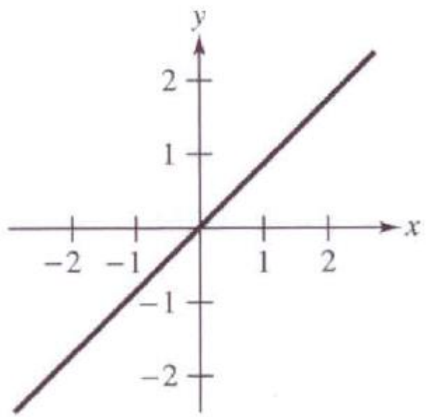
(1) W consists of the *single point* $\mathbf{0} = (0, 0)$

(2) W consists of all points on a *line* passing through the origin

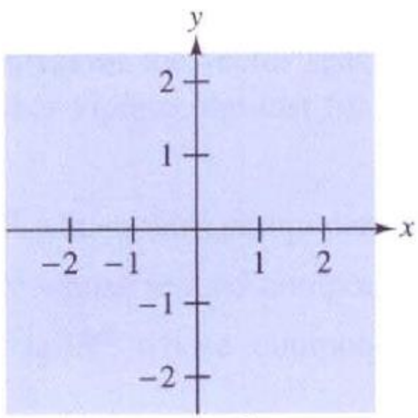
(3) R^2



$W = \{(0, 0)\}$



W = all points on a line passing through the origin



$W = R^2$

- Ex 9: Identify subspaces of R^3

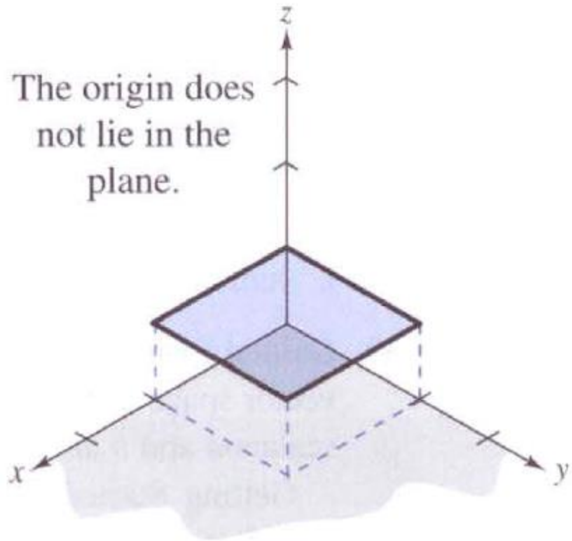
Which of the following subsets is a subspace of R^3?

(a) $W = \{(x_1, x_2, 1) \mid x_1, x_2 \in R\}$ (Note: the zero vector is not in W)

(b) $W = \{(x_1, x_1 + x_3, x_3) \mid x_1, x_3 \in R\}$ (Note: the zero vector is in W)

Sol:

(a)

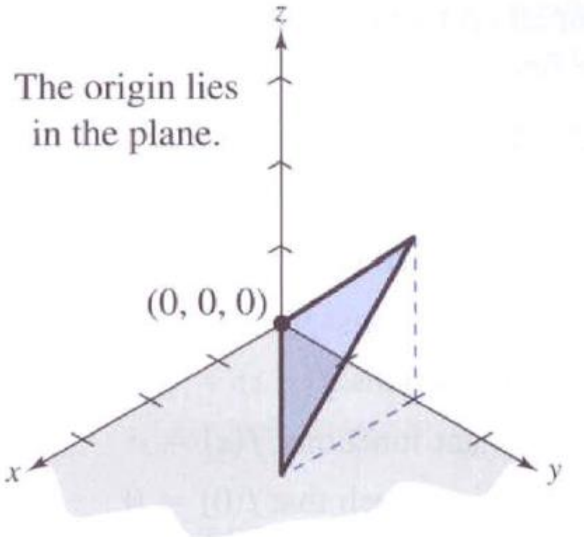


Consider $\mathbf{v} = (0,0,1) \in W$

$\because (-1)\mathbf{v} = (0,0,-1) \notin W$

$\therefore W$ is not a subspace of R^3

(b)



Consider $\mathbf{v} = (v_1, v_1 + v_3, v_3) \in W$ and $\mathbf{u} = (u_1, u_1 + u_3, u_3) \in W$

$\because \mathbf{v} + \mathbf{u} = \left(v_1 + u_1, (v_1 + u_1) + (v_3 + u_3), v_3 + u_3\right) \in W$

$c\mathbf{v} = \left(cv_1, (cv_1) + (cv_3), cv_3\right) \in W$

$\therefore W$ is closed under vector addition and scalar multiplication,

so W is a subspace of R^3

- Note: Subspaces of R^3

(1) W consists of the *single point* $\mathbf{0} = (0,0,0)$

(2) W consists of all points on a *line* passing through the origin

(3) W consists of all points on a *plane* passing through the origin

(The W in problem (b) is a plane passing through the origin)

(4) R^3

※ According to Ex. 8 and Ex. 9, we can infer that if W is a subspace of a vector space V, then both W and V must contain the same zero vector $\mathbf{0}$

Linear Combination in a Vector Space

- Linear combination:

A vector $\mathbf{u}$ in a vector space V is called a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ in V if $\mathbf{u}$ can be written in the form

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are real-number scalars

- Ex 10: Finding a linear combination

$$\mathbf{v}_1=(1,2,3) \quad \mathbf{v}_2=(0,1,2) \quad \mathbf{v}_3=(-1,0,1)$$

Prove (a) $\mathbf{w}=(1,1,1)$ is a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

(b) $\mathbf{w}=(1,-2,2)$ is not a linear combination of $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$

Sol:

(a) $\mathbf{w}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3$

$$\begin{aligned}(1,1,1)&=c_1(1,2,3)+c_2(0,1,2)+c_3(-1,0,1)\\&=(c_1-c_3,\ 2c_1+c_2,\ 3c_1+2c_2+c_3)\end{aligned}$$

$$\Rightarrow \begin{array}{rcl} c_1 \qquad\quad -c_3 &=& 1\\ 2c_1+\ \ c_2 \qquad\quad &=& 1\\ 3c_1+2c_2+c_3 &=& 1\end{array}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = 1 + t\,,\ c_2 = -1 - 2t\,,\ c_3 = t$$

(this system has infinitely many solutions)

$$\overset{t=1}{\Rightarrow} \mathbf{w} = 2\mathbf{v}_1 - 3\mathbf{v}_2 + \mathbf{v}_3$$

$$\overset{t=2}{\Rightarrow} \mathbf{w} = 3\mathbf{v}_1 - 5\mathbf{v}_2 + 2\mathbf{v}_3$$

$$\vdots$$

(b)

$$\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 2 & 1 & 2 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 0 & 7 \end{array}\right]$$

$\Rightarrow$ This system has no solution since the third row means

$$0 \cdot c_1 + 0 \cdot c_2 + 0 \cdot c_3 = 7$$

$\Rightarrow$ $\mathbf{w}$ can not be expressed as $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$