Linear Combinations, Basis, Span, and Independence

We're interested is pinning down what it means for a vector space to have a basis, and that's described in terms of the concept of linear combination. Span and independence are two more related concepts.

Generally, in mathematics, you say that a linear combination of things is a sum of multiples of those things. So, for example, one linear combination of the functions $f(x)$, $g(x)$, and $h(x)$ is

$$2f(x) + 3g(x) - 4h(x).$$

Definition 1 (Linear combination)**.** A *linear combination* of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in a vector space V is an expression of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

where the c_i's are scalars, that is, it's a sum of scalar multiples of them. More generally, if S is a set of vectors in V, not necessarily finite, then a linear combination of S refers to a linear combination of some finite subset of S.

Of course, differences are allowed, too, since negations of scalars are scalars.

We can use linear combinations to characterize subspaces as mentioned previously when we talked about subspaces.

Theorem 2. A nonempty subset W of a vector space V is a subspace of V if and only if W is closed under linear combinations, that is, whenever $\mathbf{w}_1$, $\mathbf{w}_2, \ldots, \mathbf{w}_k$ all belong to W, then so does each linear combination $c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_k$ of them belong to W.

A basis for a vector space. You know some bases for vector spaces already even if you haven't know them by that name.

For instance, in $\mathbf{R}^3$ the three vectors $\mathbf{i} = (1, 0, 0)$ which points along the x-axis, $\mathbf{j} = (0, 1, 0)$ which points along the y-axis, and $\mathbf{k} = (0, 0, 1)$ which points along the z-axis together form the *standard basis* for $\mathbf{R}^3$. Every vector (x, y, z) in $\mathbf{R}^3$ is a unique linear combination of the standard basis vectors

$$(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

That's the one and only linear combination of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ that gives (x, y, z). (Why?)

We'll generally use Greek letters like β and γ to distinguish bases ('bases' is the plural of 'basis') from other subsets of a set. Thus $\epsilon = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is the standard basis for $\mathbf{R}^3$. We'll want our bases to have an ordering to correspond to a coordinate system. So, for this basis ϵ of $\mathbf{R}^3$, $\mathbf{i}$ comes before $\mathbf{j}$, and $\mathbf{j}$ comes before $\mathbf{k}$.

The plane $\mathbf{R}^2$ has a standard basis of two vectors, namely, $\epsilon = \{\mathbf{i}, \mathbf{j}\}$ where $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$. (Although we're using $\mathbf{i}$ and $\mathbf{j}$ for different things, you can tell what's meant by context.)

There is an analogue for $\mathbf{R}^n$. Its standard basis is

$$\epsilon = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$$

where

$$\begin{aligned} \mathbf{e}_1 &= (1, 0, \ldots, 0), \\ \mathbf{e}_2 &= (0, 1, \ldots, 0), \\ &\cdots \\ \mathbf{e}_n &= (0, 0, \ldots, 1). \end{aligned}$$

Sometimes it's nice to have a notation without the ellipsis $(\ldots)$, and the Kronecker delta symbol helps here. Let δ_{ij} be defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the j^{th} coordinate e_{ij} of the i^{th} standard unit vector $\mathbf{e}_i$ is δ_{ij}.

Coordinates are related to bases. Let $\mathbf{v}$ be a vector in $\mathbf{R}^n$. It can be uniquely written as a linear combinations of the standard basis vectors

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

and the coefficients that appear in this unique linear combination are the coordinates of $\mathbf{v}$

$$\mathbf{v} = (v_1, v_2, \ldots, v_n).$$

That leads us to the definition of for the concept of *basis* of a vector space. Whenever we used a basis in conjunction with coordinates, we'll need an ordering on it, but for other purposes the ordering won't matter.

Definition 3. An (ordered) subset

$$\beta = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$$

of a vector space V is an *(ordered) basis* of V if each vector $\mathbf{v}$ in V may be uniquely represented as a linear combination of vectors from β

$$\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2 + \cdots + v_n\mathbf{b}_n.$$

For an ordered basis, the coefficients in that linear combination are called the *coordinates* of the vector with respect to β.

Later on, when we study coordinates in more detail, we'll write the coordinates of a vector $\mathbf{v}$ as a column vector and give it a special notation

$$[\mathbf{v}]_\beta = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

Although we have a standard basis for $\mathbf{R}^n$, there are other bases.

Example 4. For example, the two vectors $\mathbf{b}_1 = (1, 1)$ and $\mathbf{b}_2 = (1, -1)$ form a basis $\beta = (\mathbf{b}_1, \mathbf{b}_2)$ for $\mathbf{R}^2$. Each vector $\mathbf{v} = (v_1, v_2)$ can be written as a unique linear combination of them, namely

$$\mathbf{v} = (v_1, v_2) = \tfrac{1}{2}(v_1 + v_2)\mathbf{b}_1 + \tfrac{1}{2}(v_1 - v_2)\mathbf{b}_2.$$

So the β-coordinates of $\mathbf{v}$ are

$$[\mathbf{v}]_\beta = \begin{bmatrix} v_1 + v_2 \\ v_1 - v_2 \end{bmatrix}$$

So, for instance, the vector which has standard coordinates $(2, 4)$ has the β-coordinates $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ because $(2, 4) = 3\mathbf{b}_1 - \mathbf{b}_2$.

There are lots of other bases for $\mathbf{R}^2$. In fact, if you take any two vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ that don't lie on a line, they'll form a basis.