Triple integrals in Cartesian coordinates (Sect. 15.4)

- Review: Triple integrals in arbitrary domains.
- Examples: Changing the order of integration.
- The average value of a function in a region in space.
- Triple integrals in arbitrary domains.

Review: Triple integrals in arbitrary domains.

Theorem

If $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the domain

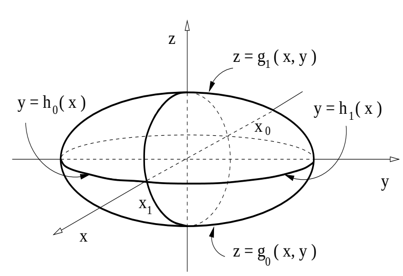
$$D = \big\{x \in [x_0, x_1],\ y \in [h_0(x), h_1(x)],\ z \in [g_0(x,y), g_1(x,y)]\big\},$$

where $g_0, g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ are continuous, then the triple integral of the function f in the region D is given by

$$\iiint_D f\, dv = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

Example

In the case that D is an ellipsoid, the figure represents the graph of functions g_1, g_0 and h_1, h_0.



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- Review: Triple integrals in arbitrary domains.
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Changing the order of integration.

Example

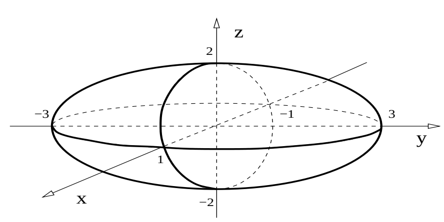
Change the order of integration in the triple integral

$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx.$$

Solution: First: Sketch the integration region.

Start from the outer integration limits to the inner limits.

- Limits in x: $x \in [-1, 1]$.
- Limits in y: $|y| \leqslant 3\sqrt{1-x^2}$, so, $x^2 + \dfrac{y^2}{3^2} \leqslant 1$.
- The limits in z: $|z| \leqslant 2\sqrt{1 - x^2 - \dfrac{y^2}{3^2}}$, so, $x^2 + \dfrac{y^2}{3^2} + \dfrac{z^2}{2^2} \leqslant 1$.



Changing the order of integration.

Example

Change the order of integration in the triple integral

$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx.$$

Solution: Region: $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} \leqslant 1$. We conclude:

$$V = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy\, dz\, dx.$$

$$V = \int_{-2}^{2} \int_{-\sqrt{1-(z/2)^2}}^{\sqrt{1-(z/2)^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy\, dx\, dz.$$

$$V = \int_{-2}^{2} \int_{-3\sqrt{1-(z/2)^2}}^{3\sqrt{1-(z/2)^2}} \int_{-\sqrt{1-(y/3)^2-(z/2)^2}}^{\sqrt{1-(y/3)^2-(z/2)^2}} dx\, dy\, dz.$$

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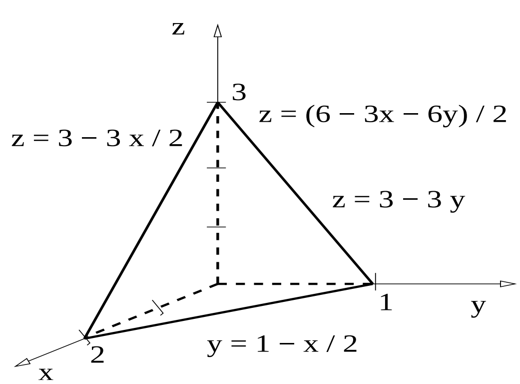
Changing the order of integration.

Example

Interchange the limits in $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx.$

Solution: Sketch the integration region starting from the outer integration limits to the inner integration limits.

- $x \in [0, 2]$,
- $y \in \left[0, 1 - \frac{x}{2}\right]$ so the upper limit is the line $y = 1 - \frac{x}{2}$.
- $z \in \left[0, 3 - \frac{3x}{2} - 3y\right]$ so the upper limit is the plane $z = 3 - \frac{3x}{2} - 3y$. This plane contains the points $(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$.

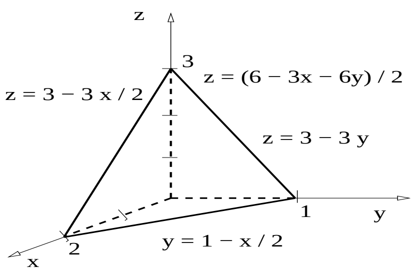


Changing the order of integration.

Example

Interchange the limits in $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx$.

Solution: The region: $x \geqslant 0$, $y \geqslant 0$, $z \geqslant 0$ and $6 \geqslant 3x + 6y + 2z$.
We conclude:



$$V = \int_0^3 \int_0^{1-z/3} \int_0^{2-2y-2z/3} dx\, dy\, dz.$$

$$V = \int_0^1 \int_0^{3-3y} \int_0^{2-2y-2z/3} dx\, dz\, dy.$$

$$V = \int_0^2 \int_0^{3-3x/2} \int_0^{1-x/2-z/3} dy\, dz\, dx.$$

$$V = \int_0^3 \int_0^{2-2z/3} \int_0^{1-x/2-z/3} dy\, dx\, dz.$$

$\triangleleft$

Triple integrals in Cartesian coordinates (Sect. 15.4)

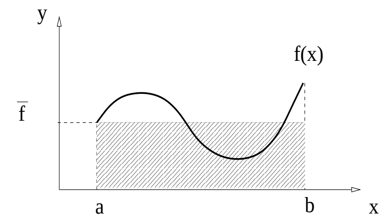
- Review: Triple integrals in arbitrary domains.
- Examples: Changing the order of integration.
- **The average value of a function in a region in space.**
- Triple integrals in arbitrary domains.

Average value of a function in a region in space.

Review: The average of a single variable function.

Definition
The *average* of a function $f : [a, b] \to \mathbb{R}$ on the interval $[a, b]$, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{(b-a)} \int_a^b f(x)\, dx.$$



Definition
The *average* of a function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ on the region R with volume V, denoted by $\overline{f}$, is given by

$$\overline{f} = \frac{1}{V} \iiint_R f\, dv.$$

Average value of a function in a region in space.

Example
Find the average of $f(x, y, z) = xyz$ in the first octant bounded by the planes $x = 1$, $y = 2$, $z = 3$.

Solution: The volume of the rectangular integration region is

$$V = \int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx \quad \Rightarrow \quad V = 6.$$

The average of function f is:

$$\overline{f} = \frac{1}{6} \int_0^1 \int_0^2 \int_0^3 xyz\, dz\, dy\, dx = \frac{1}{6} \Big[\int_0^1 x\, dx\Big] \Big[\int_0^2 y\, dy\Big] \Big[\int_0^3 z\, dz\Big]$$

$$\overline{f} = \frac{1}{6} \Big(\frac{x^2}{2}\Big|_0^1\Big) \Big(\frac{y^2}{2}\Big|_0^2\Big) \Big(\frac{z^2}{2}\Big|_0^3\Big) = \frac{1}{6} \Big(\frac{1}{2}\Big) \Big(\frac{4}{2}\Big) \Big(\frac{9}{2}\Big).$$

We conclude: $\overline{f} = 1/4$. $\triangleleft$

Triple integrals in Cartesian coordinates (Sect. 15.4)

- Review: Triple integrals in arbitrary domains.
- Examples: Changing the order of integration.
- The average value of a function in a region in space.
- **Triple integrals in arbitrary domains.**

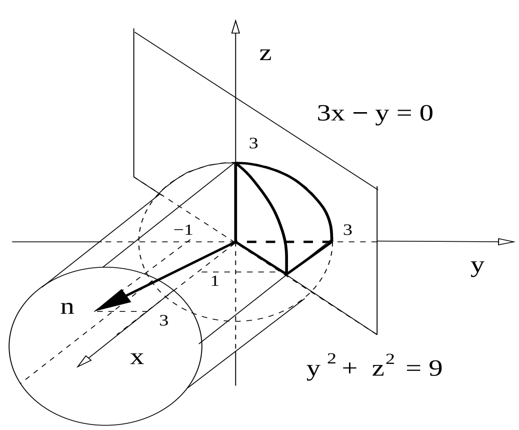
Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Sketch the integration region.

- The integration region is in the first octant.
- It is inside the cylinder $y^2 + z^2 = 9$.
- It is on one side of the plane $3x - y = 0$. The plane has normal vector $\mathbf{n} = \langle 3, -1, 0 \rangle$ and contains $(0, 0, 0)$.

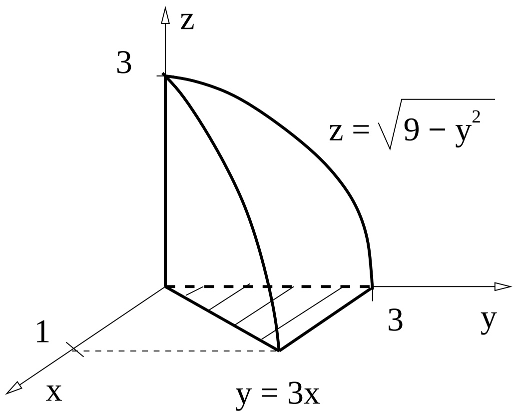


Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: We have found the region:



The integration limits are:

- Limits in z: $0 \leqslant z \leqslant \sqrt{9 - y^2}$.
- Limits in x: $0 \leqslant x \leqslant y/3$.
- Limits in y: $0 \leqslant y \leqslant 3$.

We obtain $I = \int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z \, dz \, dx \, dy$.

Triple integrals in arbitrary domains.

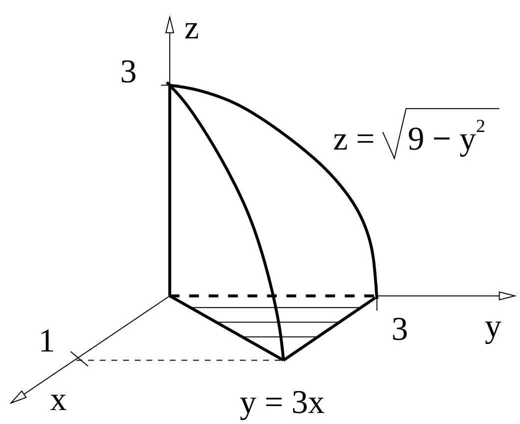
Example

Compute the triple integral of $f(x, y, z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall:

$$\int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z \, dz \, dx \, dy.$$

For practice purpose only, let us change the integration order to $dz \, dy \, dx$:



The result is: $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx$.

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x,y,z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z\, dz\, dy\, dx.$

We now compute the integral:

$$\iiint_D f\, dv = \int_0^1 \int_{3x}^3 \Big(\frac{z^2}{2}\Big|_0^{\sqrt{9-y^2}} \Big)\, dy\, dx,$$

$$= \frac{1}{2}\int_0^1 \int_{3x}^3 (9 - y^2) dy\, dx,$$

$$= \frac{1}{2}\int_0^1 \Big[9\Big(y\Big|_{3x}^3 \Big) - \Big(\frac{y^3}{3}\Big|_{3x}^3 \Big) \Big]\, dx.$$

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x,y,z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution: Recall: $\iiint_D f\, dv = \frac{1}{2}\int_0^1 \Big[9\Big(y\Big|_{3x}^3 \Big) - \Big(\frac{y^3}{3}\Big|_{3x}^3 \Big) \Big]\, dx.$

Therefore,

$$\iiint_D f\, dv = \frac{1}{2}\int_0^1 \Big[27(1-x) - 9(1-x)^3 \Big]\, dx,$$

$$= \frac{9}{2}\int_0^1 \Big[3(1-x) - (1-x)^3 \Big]\, dx.$$

Substitute $u = 1 - x$, then $du = -dx$, so,

$$\iiint_D f\, dv = \frac{9}{2}\int_0^1 (3u - u^3) du.$$

Triple integrals in arbitrary domains.

Example

Compute the triple integral of $f(x,y,z) = z$ in the region bounded by $x \geqslant 0$, $z \geqslant 0$, $y \geqslant 3x$, and $9 \geqslant y^2 + z^2$.

Solution:

$$\iiint_D f\,dv = \frac{9}{2}\int_0^1 (3u - u^3)\,du,$$

$$= \frac{9}{2}\Big[3\Big(\frac{u^2}{2}\Big|_0^1\Big) - \Big(\frac{u^4}{4}\Big|_0^1\Big)\Big],$$

$$= \frac{9}{2}\Big(\frac{3}{2} - \frac{1}{4}\Big).$$

We conclude $\displaystyle\iiint_D f\,dv = \frac{45}{8}$. $\triangleleft$

Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- Integration in cylindrical coordinates.
  - Review: Polar coordinates in a plane.
  - Cylindrical coordinates in space.
  - Triple integral in cylindrical coordinates.

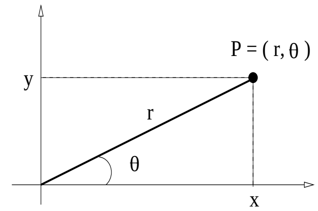
Next class:

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - Spherical coordinates in space.
  - Triple integral in spherical coordinates.

Review: Polar coordinates in plane.

Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ) defined by the picture.



Theorem (Cartesian-polar transformations)

The Cartesian coordinates of a point $P = (r, \theta)$ in the first quadrant are given by

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

The polar coordinates of a point $P = (x, y)$ in the first quadrant are given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Recall: Polar coordinates in a plane.

Example

Express in polar coordinates the integral $I = \int_0^2 \int_0^y x \, dx \, dy$.

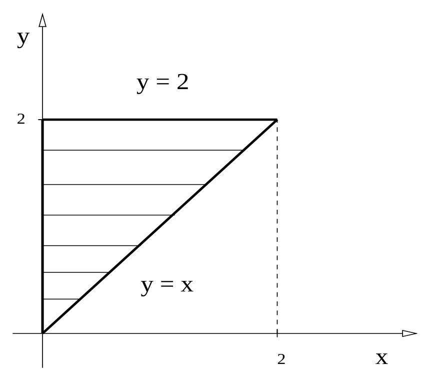
Solution: Recall: $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

More often than not helps to sketch the integration region.

The outer integration limit: $y \in [0, 2]$.

Then, for every $y \in [0, 2]$ the x coordinate satisfies $x \in [0, y]$.

The upper limit for x is the curve $y = x$.



Now is simple to describe this domain in polar coordinates:
The line $y = x$ is $\theta_0 = \pi/4$; the line $x = 0$ is $\theta_1 = \pi/2$.

Recall: Polar coordinates in a plane.

Example

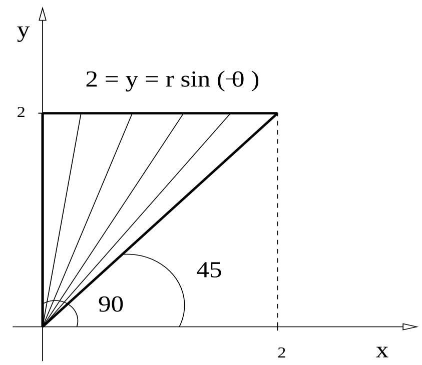
Express in polar coordinates the integral $I = \int_0^2 \int_0^y x\,dx\,dy$.

Solution: Recall: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $\theta_0 = \pi/4$, $\theta_1 = \pi/2$.

The lower integration limit in r is $r = 0$.

The upper integration limit is $y = 2$, that is, $2 = y = r\sin(\theta)$.

Hence $r = 2/\sin(\theta)$.



We conclude: $\int_0^2 \int_0^y x\,dx\,dy = \int_{\pi/4}^{\pi/2} \int_0^{2/\sin(\theta)} r\cos(\theta)(r\,dr)\,d\theta. \triangleleft$

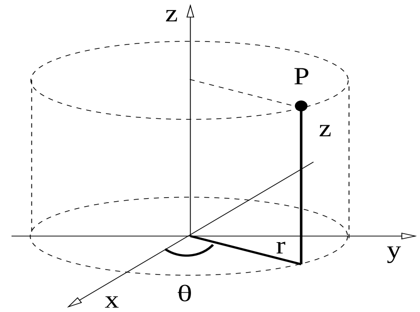
Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- Integration in cylindrical coordinates.
  - Review: Polar coordinates in a plane.
  - **Cylindrical coordinates in space.**
  - Triple integral in cylindrical coordinates.

Cylindrical coordinates in space.

Definition

The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (r, θ, z) defined by the picture.



Remark: Cylindrical coordinates are just polar coordinates on the plane $z = 0$ together with the vertical coordinate z.

Theorem (Cartesian-cylindrical transformations)

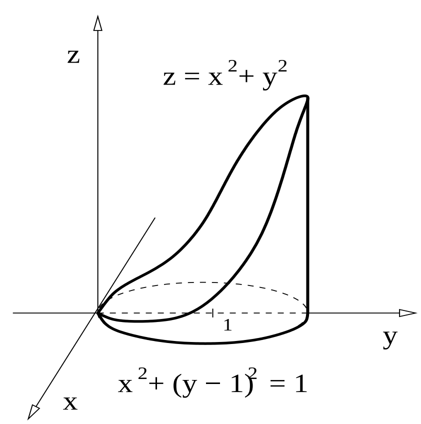
The Cartesian coordinates of a point $P = (r, \theta, z)$ in the first quadrant are given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.

The cylindrical coordinates of a point $P = (x, y, z)$ in the first quadrant are given by $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and $z = z$.

Cylindrical coordinates in space.

Example

Use cylindrical coordinates to describe the region $R = \{(x, y, z) \;:\; x^2 + (y-1)^2 \leqslant 1,\; 0 \leqslant z \leqslant x^2 + y^2\}$.



Solution: We first sketch the region.

The base of the region is at $z = 0$, given by the disk $x^2 + (y-1)^2 \leqslant 1$.

The top of the region is the paraboloid $z = x^2 + y^2$.

In cylindrical coordinates: $z = x^2 + y^2 \Leftrightarrow z = r^2$, and

$$x^2 + y^2 - 2y + 1 \leqslant 1 \;\Leftrightarrow\; r^2 - 2r\sin(\theta) \leqslant 0 \;\Leftrightarrow\; r \leqslant 2\sin(\theta)$$

Hence: $R = \{(r, \theta, z) \;:\; \theta \in [0, \pi],\; r \in [0, 2\sin(\theta)],\; z \in [0, r^2]\}.\triangleleft$

Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- Integration in cylindrical coordinates.
  - Review: Polar coordinates in a plane.
  - Cylindrical coordinates in space.
  - **Triple integral in cylindrical coordinates.**

Triple integrals using cylindrical coordinates.

Theorem

If the function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in cylindrical coordinates as follows,

$$\iiint_R f\, dv = \iiint_R f(r,\theta,z)\, r\, dr\, d\theta\, dz.$$

Remark:

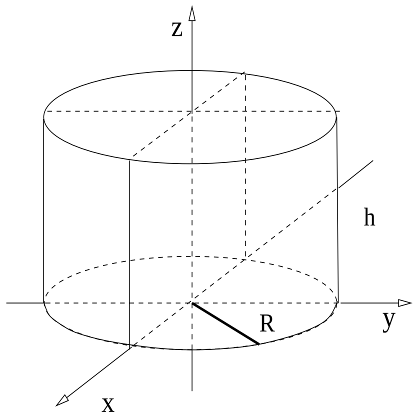
- Cylindrical coordinates are useful when the integration region R is described in a simple way using cylindrical coordinates.
- Notice the extra factor r on the right-hand side.

Triple integrals using cylindrical coordinates.

Example

Find the volume of a cylinder of radius R and height h.

Solution: $R = \{(r, \theta, z) \ : \ \theta \in [0, 2\pi], \ r \in [0, R], \ z \in [0, h]\}$.



$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^R \int_0^h dz \, (r \, dr) \, d\theta, \\
&= h \int_0^{2\pi} \int_0^R r \, dr \, d\theta, \\
&= h \frac{R^2}{2} \int_0^{2\pi} d\theta, \\
&= h \frac{R^2}{2} 2\pi,
\end{aligned}
$$

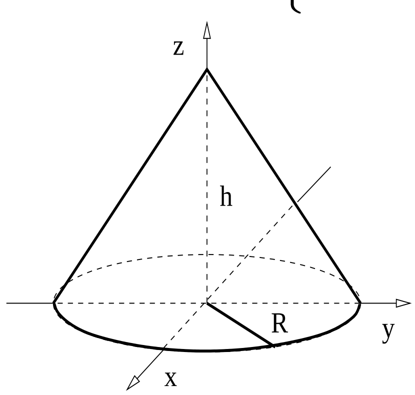
We conclude: $V = \pi R^2 h$. $\triangleleft$

Triple integrals using cylindrical coordinates.

Example

Find the volume of a cone of base radius R and height h.

Solution: $R = \Big\{\theta \in [0, 2\pi], \ r \in [0, R], \ z \in \Big[0, -\frac{h}{R} r + h\Big]\Big\}$.



$$
\begin{aligned}
V &= \int_0^{2\pi} \int_0^R \int_0^{h(1-r/R)} dz \, (r \, dr) \, d\theta, \\
&= h \int_0^{2\pi} \int_0^R \Big(1 - \frac{r}{R}\Big) r \, dr \, d\theta, \\
&= h \int_0^{2\pi} \int_0^R \Big(r - \frac{r^2}{R}\Big) \, dr \, d\theta, \\
&= h \Big(\frac{R^2}{2} - \frac{R^3}{3R}\Big) \int_0^{2\pi} d\theta = 2\pi h R^2 \frac{1}{6}.
\end{aligned}
$$

We conclude: $V = \frac{1}{3} \pi R^2 h$. $\triangleleft$

Triple integrals using cylindrical coordinates.

Example

Sketch the region with volume $V = \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{9-r^2}} r dz\, dr\, d\theta.$

Solution: The integration region is

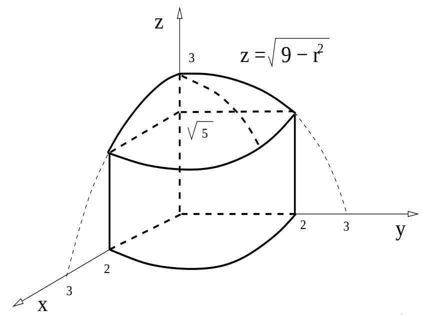
$$R = \{(r, \theta, z) \ : \ \theta \in [0, \pi/2],\ r \in [0, 2],\ z \in [0, \sqrt{9 - r^2}]\}.$$

We upper boundary is a sphere, since

$$z^2 = 9 - r^2 \quad \Leftrightarrow \quad x^2 + y^2 + z^2 = 3^2.$$

The upper limit for r is $r = 2$, so

$$z = \sqrt{9 - 2^2} \quad \Rightarrow \quad z = \sqrt{5}.$$



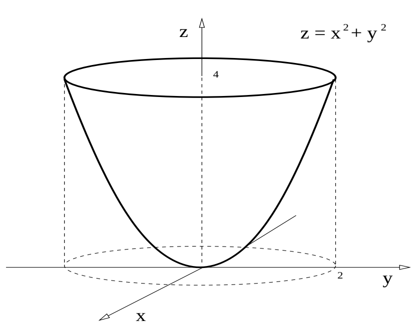
$\triangleleft$

Triple integrals using cylindrical coordinates.

Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \overline{x}, \overline{y}, \overline{z} \rangle$ of the region in space
$R = \{(x, y, z) \ : \ x^2 + y^2 \leqslant 2^2,\ x^2 + y^2 \leqslant z \leqslant 4\}.$

Solution:



The symmetry of the region implies $\overline{x} = 0$ and $\overline{y} = 0$. (We verify this result later on.) We only need to compute $\overline{z}$.

Since $\overline{z} = \dfrac{1}{V} \iiint_R z\, dv$, we start computing the total volume V.

We use cylindrical coordinates.

$$V = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 dz\, r dr\, d\theta = 2\pi \int_0^2 \Big(z \Big|_{r^2}^4 \Big) r dr = 2\pi \int_0^2 (4r - r^3) dr.$$

Triple integrals using cylindrical coordinates.

Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \overline{x}, \overline{y}, \overline{z} \rangle$ of the region in space $R = \{(x, y, z) \ : \ x^2 + y^2 \leqslant 2^2, \ x^2 + y^2 \leqslant z \leqslant 4\}$.

Solution: $V = 2\pi \displaystyle\int_0^2 (4r - r^3)\, dr = 2\pi\Big[4\Big(\frac{r^2}{2}\Big|_0^2\Big) - \Big(\frac{r^4}{4}\Big|_0^2\Big)\Big]$.

Hence $V = 2\pi(8 - 4)$, so $V = 8\pi$. Then, $\overline{z}$ is given by,

$$\overline{z} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^2\int_{r^2}^4 z dz\, r dr\, d\theta = \frac{2\pi}{8\pi}\int_0^2 \Big(\frac{z^2}{2}\Big|_{r^2}^4\Big)\, r dr;$$

$$\overline{z} = \frac{1}{8}\int_0^2 (16r - r^5)\, dr = \frac{1}{8}\Big[16\Big(\frac{r^2}{2}\Big|_0^2\Big) - \Big(\frac{r^6}{6}\Big|_0^2\Big)\Big];$$

$$\overline{z} = \frac{1}{8}\Big(32 - \frac{64}{6}\Big) = 4 - \frac{4}{3} \quad \Rightarrow \quad \overline{z} = \frac{8}{3}.$$

Triple integrals using cylindrical coordinates.

Example

Find the centroid vector $\bar{\mathbf{r}} = \langle \overline{x}, \overline{y}, \overline{z} \rangle$ of the region in space $R = \{(x, y, z) \ : \ x^2 + y^2 \leqslant 2^2, \ x^2 + y^2 \leqslant z \leqslant 4\}$.

Solution: We obtained $\overline{z} = \dfrac{8}{3}$.

It is simple to see that $\overline{x} = 0$ and $\overline{y} = 0$. For example,

$$\overline{x} = \frac{1}{8\pi}\int_0^{2\pi}\int_0^2\int_{r^2}^4 \big[r\cos(\theta)\big]\, dz\, r dr\, d\theta$$
$$= \frac{1}{8\pi}\Big[\int_0^{2\pi}\cos(\theta) d\theta\Big]\Big[\int_0^2\int_{r^2}^4 dz\, r^2 dr\Big].$$

But $\displaystyle\int_0^{2\pi}\cos(\theta) d\theta = \sin(2\pi) - \sin(0) = 0$, so $\overline{x} = 0$.

A similar calculation shows $\overline{y} = 0$. Hence $\bar{\mathbf{r}} = \langle 0, 0, 8/3 \rangle$. $\triangleleft$

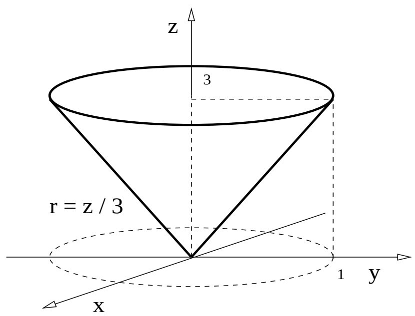
Triple integrals using cylindrical coordinates.

Example

Change the integration order and compute the integral

$$I = \int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3 dr\, dz\, d\theta.$$

Solution:



$$I = \int_0^{2\pi} \int_0^1 \int_{3r}^3 dz\, r^3 dr\, d\theta$$

$$= 2\pi \int_0^1 \Big(z\Big|_{3r}^3 \Big)\, r^3 dr$$

$$= 2\pi \int_0^1 3(r^3 - r^4)\, dr$$

$$= 6\pi \Big(\frac{r^4}{4} - \frac{r^5}{5} \Big)\Big|_0^1.$$

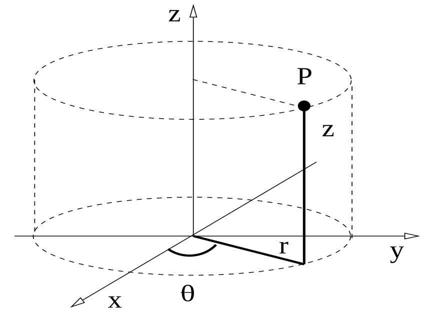
So, $I = 6\pi \dfrac{1}{20}$, that is, $I = \dfrac{3\pi}{10}$. $\triangleleft$

Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - Spherical coordinates in space.
  - Triple integral in spherical coordinates.

Cylindrical coordinates in space.

Definition
The *cylindrical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (r, θ, z) defined by the picture.



Remark: Cylindrical coordinates are just polar coordinates on the plane $z = 0$ together with the vertical coordinate z.

Theorem (Cartesian-cylindrical transformations)
The Cartesian coordinates of a point $P = (r, \theta, z)$ in the first quadrant are given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z = z$.

The cylindrical coordinates of a point $P = (x, y, z)$ in the first quadrant are given by $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$, and $z = z$.

Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
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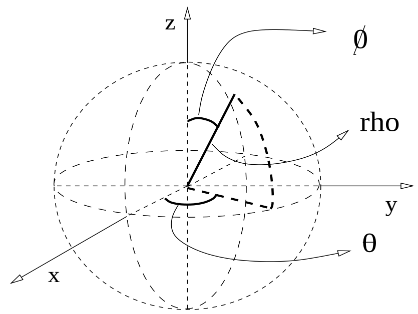
  - Triple integral in spherical coordinates.

Spherical coordinates in $\mathbb{R}^3$

Definition

The *spherical coordinates* of a point $P \in \mathbb{R}^3$ is the ordered triple (ρ, ϕ, θ) defined by the picture.



Theorem (Cartesian-spherical transformations)

The Cartesian coordinates of $P = (\rho, \phi, \theta)$ *in the first quadrant are given by* $x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, *and* $z = \rho\cos(\phi)$.

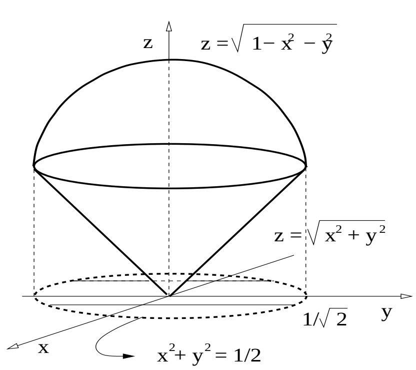
The spherical coordinates of $P = (x, y, z)$ *in the first quadrant are* $\rho = \sqrt{x^2 + y^2 + z^2}$, $\theta = \arctan\left(\dfrac{y}{x}\right)$, *and* $\phi = \arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)$.

Spherical coordinates in $\mathbb{R}^3$

Example

Use spherical coordinates to express region between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution: ($x = \rho \sin(\phi)\cos(\theta)$, $y = \rho \sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.)



The top surface is the sphere $\rho = 1$. The bottom surface is the cone:

$$\rho\cos(\phi) = \sqrt{\rho^2 \sin^2(\phi)}$$
$$\cos(\phi) = \sin(\phi),$$

so the cone is $\phi = \dfrac{\pi}{4}$.

Hence: $R = \left\{(\rho, \phi, \theta) \;:\; \theta \in [0, 2\pi],\ \phi \in \left[0, \frac{\pi}{4}\right],\ \rho \in [0, 1]\right\}$.

Integrals in cylindrical, spherical coordinates (Sect. 15.6).

- Integration in spherical coordinates.
  - Review: Cylindrical coordinates.
  - Spherical coordinates in space.
  - **Triple integral in spherical coordinates.**

Triple integral in spherical coordinates.

Theorem

If the function $f : R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in spherical coordinates as follows,

$$\iiint_R f\, dv = \iiint_R f(\rho, \phi, \theta)\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$$

Remark:

- Spherical coordinates are useful when the integration region R is described in a simple way using spherical coordinates.
- Notice the extra factor $\rho^2 \sin(\phi)$ on the right-hand side.

Triple integral in spherical coordinates.

Example

Find the volume of a sphere of radius R.

Solution: Sphere: $S = \{\theta \in [0, 2\pi],\ \phi \in [0, \pi],\ \rho \in [0, R]\}$.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi} \int_0^{R} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta, \\ &= \Big[\int_0^{2\pi} d\theta\Big] \Big[\int_0^{\pi} \sin(\phi)\, d\phi\Big] \Big[\int_0^{R} \rho^2\, d\rho\Big], \\ &= 2\pi \Big[-\cos(\phi)\Big|_0^{\pi}\Big] \frac{R^3}{3}, \\ &= 2\pi \big[-\cos(\pi) + \cos(0)\big] \frac{R^3}{3}; \end{aligned}$$

hence: $V = \frac{4}{3}\pi R^3$. $\triangleleft$

Triple integral in spherical coordinates.

Example

Use spherical coordinates to find the volume below the sphere $x^2 + y^2 + z^2 = 1$ and above the cone $z = \sqrt{x^2 + y^2}$.

Solution: $R = \Big\{(\rho, \phi, \theta)\ :\ \theta \in [0, 2\pi],\ \phi \in \Big[0, \frac{\pi}{4}\Big],\ \rho \in [0, 1]\Big\}$.
The calculation is simple, the region is a simple section of a sphere.

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^{1} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta, \\ &= \Big[\int_0^{2\pi} d\theta\Big] \Big[\int_0^{\pi/4} \sin(\phi)\, d\phi\Big] \Big[\int_0^{1} \rho^2\, d\rho\Big], \\ &= 2\pi \Big[-\cos(\phi)\Big|_0^{\pi/4}\Big] \Big(\frac{\rho^3}{3}\Big|_0^1\Big), \\ &= 2\pi \Big[-\frac{\sqrt{2}}{2} + 1\Big] \frac{1}{3} \quad \Rightarrow \quad V = \frac{\pi}{3}(2 - \sqrt{2}). \end{aligned}$$

$\triangleleft$

Triple integral in spherical coordinates.

Example

Find the integral of $f(x, y, z) = e^{(x^2+y^2+z^2)^{3/2}}$ in the region $R = \{x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0,\ x^2 + y^2 + z^2 \leqslant 1\}$ using spherical coordinates.

Solution: $R = \Big\{\theta \in \Big[0, \frac{\pi}{2}\Big],\ \phi \in \Big[0, \frac{\pi}{2}\Big],\ \rho \in [0, 1]\Big\}$. Hence,

$$\iiint_R f\,dv = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 e^{\rho^3}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta,$$

$$= \Big[\int_0^{\pi/2} d\theta\Big]\Big[\int_0^{\pi/2}\sin(\phi)\,d\phi\Big]\Big[\int_0^1 e^{\rho^3}\rho^2\,d\rho\Big].$$

Use substitution: $u = \rho^3$, hence $du = 3\rho^2\,d\rho$, so

$$\iiint_R f\,dv = \frac{\pi}{2}\Big[-\cos(\phi)\Big|_0^{\frac{\pi}{2}}\Big]\int_0^1 \frac{e^u}{3}\,du \Rightarrow \iiint_R f\,dv = \frac{\pi}{6}(e-1).$$

Triple integral in spherical coordinates.

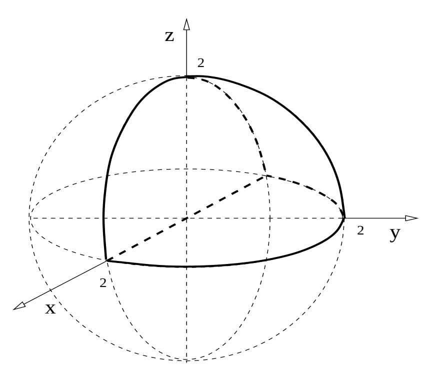
Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\,dz\,dy\,dx.$$

Solution: ($x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.)

- Limits in x: $|x| \leqslant 2$;
- Limits in y: $0 \leqslant y \leqslant \sqrt{4-x^2}$, so the positive side of the disk $x^2 + y^2 \leqslant 4$.
- Limits in z: $0 \leqslant z \leqslant \sqrt{4-x^2-y^2}$, so a positive quarter of the ball $x^2 + y^2 + z^2 \leqslant 4$.



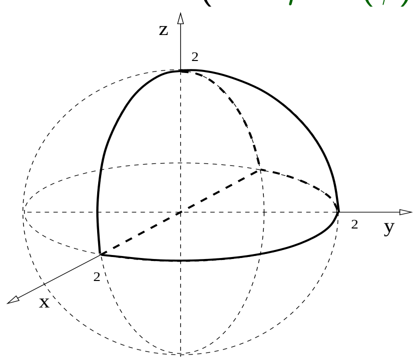
Triple integral in spherical coordinates.

Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

Solution: ($x = \rho \sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$, $z = \rho\cos(\phi)$.)



- Limits in θ: $\theta \in [0.\pi]$;
- Limits in ϕ: $\phi \in [0, \pi/2]$;
- Limits in ρ: $\rho \in [0, 2]$.
- The function to integrate is: $f = \rho^2 \sin(\phi)\sin(\theta)$.

$$I = \int_0^{\pi} \int_0^{\pi/2} \int_0^{2} \rho^2 \sin(\phi)\sin(\theta)\big(\rho^2\sin(\phi)\big)\, d\rho\, d\phi\, d\theta.$$

Triple integral in spherical coordinates.

Example

Change to spherical coordinates and compute the integral

$$I = \int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \int_{0}^{\sqrt{4-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dy\, dx.$$

Solution: $I = \int_0^{\pi} \int_0^{\pi/2} \int_0^{2} \rho^2 \sin(\phi)\sin(\theta)\big(\rho^2\sin(\phi)\big)\, d\rho\, d\phi\, d\theta.$

$$I = \Big[\int_0^{\pi} \sin(\theta)\, d\theta\Big]\Big[\int_0^{\pi/2} \sin^2(\phi)\, d\phi\Big]\Big[\int_0^{2} \rho^4\, d\rho\Big],$$

$$= \Big(-\cos(\theta)\Big|_0^{\pi}\Big)\Big[\int_0^{\pi/2} \frac{1}{2}\big(1-\cos(2\phi)\big)\, d\phi\Big]\Big(\frac{\rho^5}{5}\Big|_0^2\Big),$$

$$= 2\,\frac{1}{2}\Big[\Big(\frac{\pi}{2}-0\Big) - \frac{1}{2}\Big(\sin(2\phi)\Big|_0^{\pi/2}\Big)\Big]\frac{2^5}{5} \quad \Rightarrow \quad I = \frac{2^4\pi}{5}.$$

Triple integral in spherical coordinates.

Example

Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^2 3\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$

Solution: Recall: $\sec(\phi) = 1/\cos(\phi)$.

$$I = 2\pi \int_0^{\pi/3} \Big(\rho^3 \Big|_{\sec(\phi)}^2 \Big) \sin(\phi)\, d\phi,$$

$$= 2\pi \int_0^{\pi/3} \Big(2^3 - \frac{1}{\cos^3(\phi)} \Big) \sin(\phi)\, d\phi$$

In the second term substitute: $u = \cos(\phi)$, $du = -\sin(\phi)\, d\phi$.

$$I = 2\pi \Big[2^3 \Big(-\cos(\phi) \Big|_0^{\pi/3} \Big) + \int_1^{1/2} \frac{du}{u^3} \Big].$$

Triple integral in spherical coordinates.

Example

Compute the integral $I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\sec(\phi)}^2 3\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta.$

Solution: $I = 2\pi \Big[2^3 \Big(-\cos(\phi) \Big|_0^{\pi/3} \Big) + \int_1^{1/2} \frac{du}{u^3} \Big].$

$$I = 2\pi \Big[2^3 \Big(-\frac{1}{2} + 1 \Big) - \int_{1/2}^1 u^{-3} du \Big] = 2\pi \Big[4 - \Big(\frac{u^{-2}}{-2} \Big|_{1/2}^1 \Big) \Big],$$

$$I = 2\pi \Big[4 + \frac{1}{2} \big(u^{-2} \big|_{1/2}^1 \big) \Big] = 2\pi \Big[4 + \frac{1}{2} \big(1 - 2^2 \big) \Big] = 2\pi \Big[\frac{8}{2} - \frac{3}{2} \Big]$$

We conclude: $I = 5\pi$. $\triangleleft$