LECTURE-3

Spanning and Linearly Independent Sets

Spanning Sets and Linear Independence

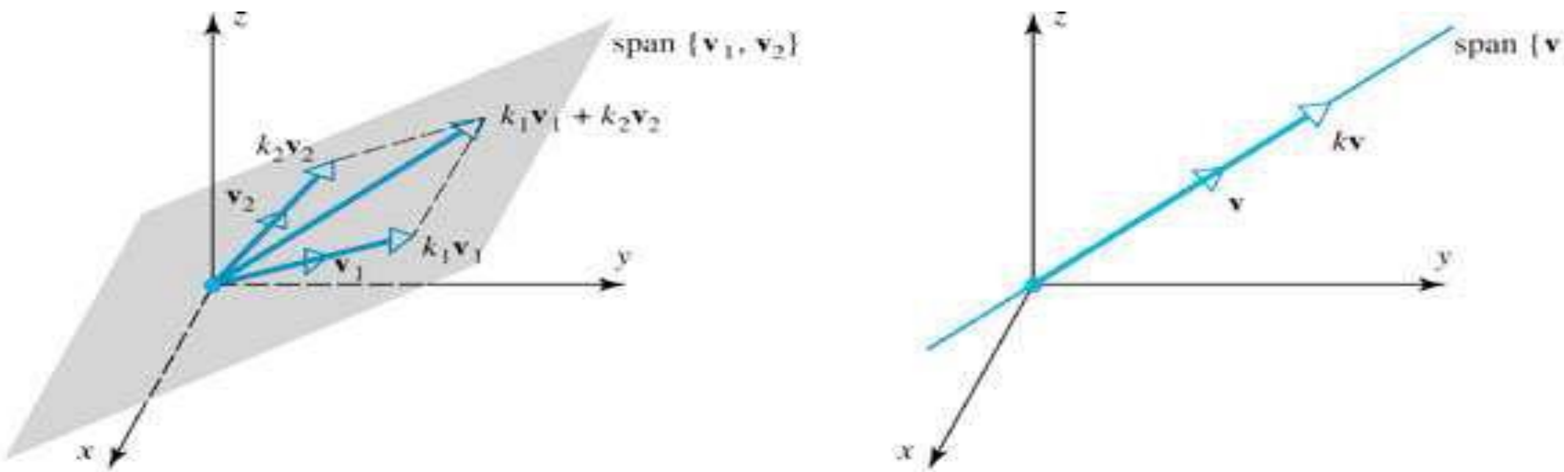
- This section introduces the spanning set, linear independence, and linear dependence
- The above three notions are associated with the representation of any vector in a vector space as a **linear combination** of a selected set of vectors in that vector space.
- Spanning Set:

  If S={$\mathbf{v}_1,\mathbf{v}_2,\ldots, \mathbf{v}_k$} is a set of vectors in a vector space W of V consisting of all linear combinations of the vector in S is called space spanned by $\mathbf{v}_1, \mathbf{v}_2,\ldots,\mathbf{v}_k$ and we say that the vectors $\mathbf{v}_1, \mathbf{v}_2,\ldots,\mathbf{v}_k$ span W. It is denoted by

$$W = \text{Span}(S) \text{ or } W = \text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$$

- The span of a set: span(S)

  If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then the span of S is the set of all linear combinations of the vectors in S.



(a) Span $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(b) Span $\{\mathbf{v}\}$ is the line through the origin determined by $\mathbf{v}$.

- Definition of a spanning set of a vector space:

  If every vector in a given vector space V can be written as a linear combination of vectors in a set S, then S is called a **spanning set** of the vector space V.

- Note: The above statement can be expressed as follows

$$\begin{aligned} &\text{span}(S) = V \\ \Leftrightarrow\ & S \text{ spans (generated) } V \\ \Leftrightarrow\ & V \text{ is spanned (generated) by } S \\ \Leftrightarrow\ & S \text{ is a spanning set of } V \end{aligned}$$

- Ex 3.1:

(a) The set $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ spans R^3 because any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be written as

$$\mathbf{u} = u_1(1,0,0) + u_2(0,1,0) + u_3(0,0,1)$$

(b) The set $S = \{1, x, x^2\}$ spans P_2 because any polynomial function $p(x) = a + bx + cx^2$ in P_2 can be written as

$$p(x) = a(1) + b(x) + c(x^2)$$

- Ex3.2: A spanning set for R^3

Show that the set $S = \{(1,2,3),(0,1,2),(-2,0,1)\}$ spans R^3

Sol:

We must examine whether any vector $\mathbf{u} = (u_1, u_2, u_3)$ in R^3 can be expressed as a linear combination of $\mathbf{v}_1 = (1,2,3)$, $\mathbf{v}_2 = (0,1,2)$, and $\mathbf{v}_3 = (-2,0,1)$

If $\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$
$$\Rightarrow \begin{aligned} c_1 \qquad\quad - 2c_3 &= u_1 \\ 2c_1 + c_2 \qquad\quad &= u_2 \\ 3c_1 + 2c_2 + c_3 &= u_3 \end{aligned}$$

The above problem thus reduces to determine whether this system is consistent for all values of u_1, u_2, and u_3

$$\because |A| = \begin{vmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} \neq 0$$

※ From , if A is an invertible matrix, then the system of linear equations $A\mathbf{x} = \mathbf{b}$ has a unique solution ($\mathbf{x} = A^{-1}\mathbf{b}$) given any $\mathbf{b}$

※ From , a square matrix A is invertible (nonsingular) if and only if det (A) $\neq 0$

$$\therefore A\mathbf{x} = \mathbf{u} \text{ has exactly one solution for every } \mathbf{u}$$

$$\Rightarrow \text{span}(S) = R^3$$

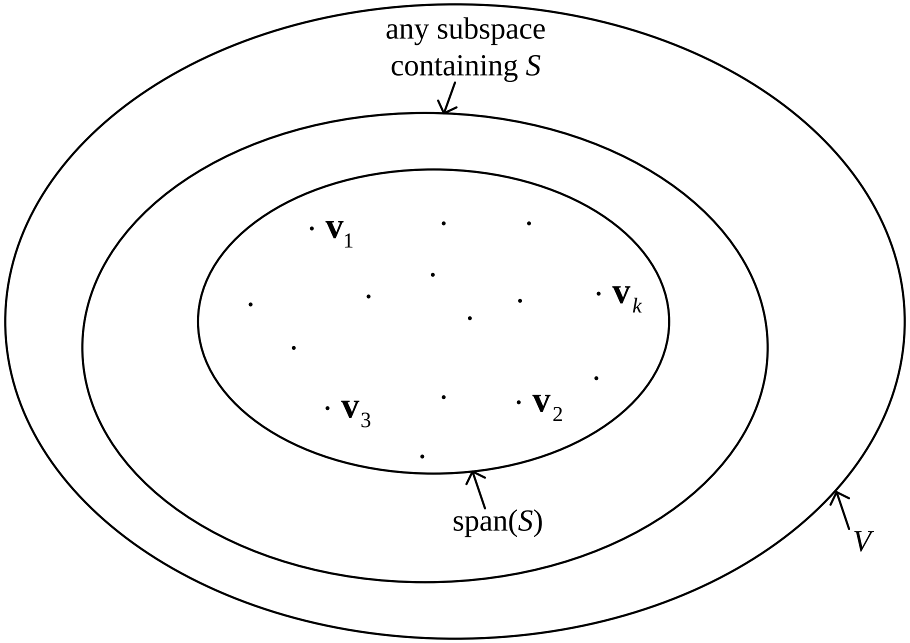
- Note:
  - ※ For any set S_1 containing the set S, if S can span R^3, S_1 can span R^3 as well (e.g., $S_1 = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \{(1,2,3), (0,1,2), (-2,0,1), (1,0,0)\}$).
  - ※ Actually, in this case, what S_1 can span is only R^3. Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span R^3, $\mathbf{v}_4$ must be a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. So, adding $\mathbf{v}_4$ will not generate more combinations. As a consequence, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ can only span R^3

- Theorem 3.1: Span(S) is a subspace of V

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is a set of vectors in a vector space V, then

(a) span(S) is a subspace of V

(b) span(S) is the smallest subspace of V that contains S, i.e., every other subspace of V containing S must contain span(S)



Proof:

(a) Let $S=\{v_1, v_2, \ldots\ldots, v_k\}$ and consider any two vectors **u** and **v** in span(S), that is,

$$\mathbf{u}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots+c_k\mathbf{v}_k$$

and $$\mathbf{v}=d_1\mathbf{v}_1+d_2\mathbf{v}_2+\ldots+d_k\mathbf{v}_k$$

Then

$\mathbf{u}+\mathbf{v}=(c_1+d_1)\mathbf{v}_1 + (c_2+d_2)\mathbf{v}_2+\ldots+(c_k+d_k)\mathbf{v}_k$ belongs to span(S)

and

$c\mathbf{u}=(cc_1)\mathbf{v}_1 +(cc_2)\mathbf{v}_2+\ldots+(cc_k)\mathbf{v}_k$ belongs to span(S), because they can be written as linear combinations of vectors in S.

So, according to the definition of subspace, span(S) is a subspace of V.

(b)

Let U be another subspace of V that contains S, and we want to show $\text{span}(S) \subset U$

Consider any $\mathbf{u} \in \text{span}(S)$, i.e., $\mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{v}_i$, where $\mathbf{v}_i \in S$

$$U \text{ contains } S \Rightarrow \mathbf{v}_i \in U \overset{U \text{ is a subspace}}{\Rightarrow} \mathbf{u} = \sum_{i=1}^{k} c_i \mathbf{v}_i \in U$$

(because U is closed under vector addition and scalar multiplication, and any linear combination can be evaluated with finite vector additions and scalar multiplications)

Since for any vector $\mathbf{u} \in \text{span}(S)$, $\mathbf{u}$ also belongs to U, we can conclude that $\text{span}(S) \subset U$, and therefore $\text{span}(S)$ is the smallest subspace of V that contains $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$

※ For example, $V = R^5$, $S = \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$ and thus $\text{span}(S) = R^3$, and $U = R^4$, U contains S and $\text{span}(S)$ as well

A criteria for spanning

Let A be an $n \times k$ *matrix with columns* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. *Then the following are equivalent:*

1. For every n−vector b the system of linear equations with augmented matrix $[A|b]$ *is consistent;*

2. Every n−vector b is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$.

3. A has n leading - 1 position (equivalently, every row contains a leading - 1).

Solve Problems

Linear Independence and Linear Dependence

- Definitions :

If $S=\{\mathbf{v}_1,\mathbf{v}_2,\ldots,\mathbf{v}_k\}$ is a nonempty set of vectors, then the vector equation

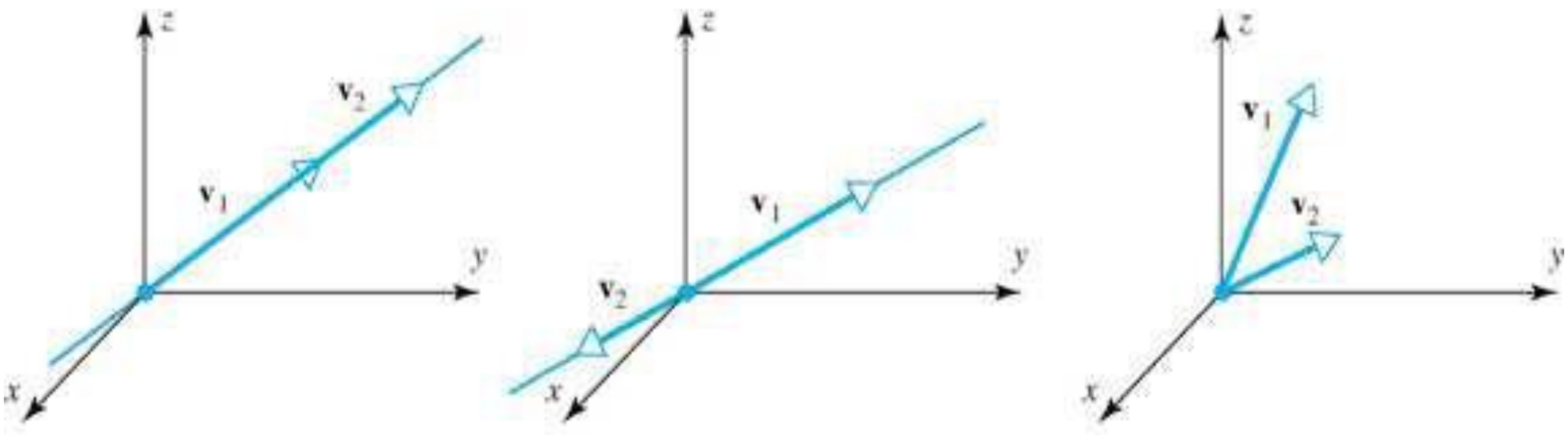
$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+\ldots.+c_k\mathbf{v}_k=\mathbf{0}$$

has at least one solution, namely $c_1=0$, $c_2=0,\ldots,c_k=0$.

If this is the only solution, then *S is called a* linearly independent set. If there are other solutions, then S is called a linearly dependent set.

Geometric Interpretation of Linear Independence

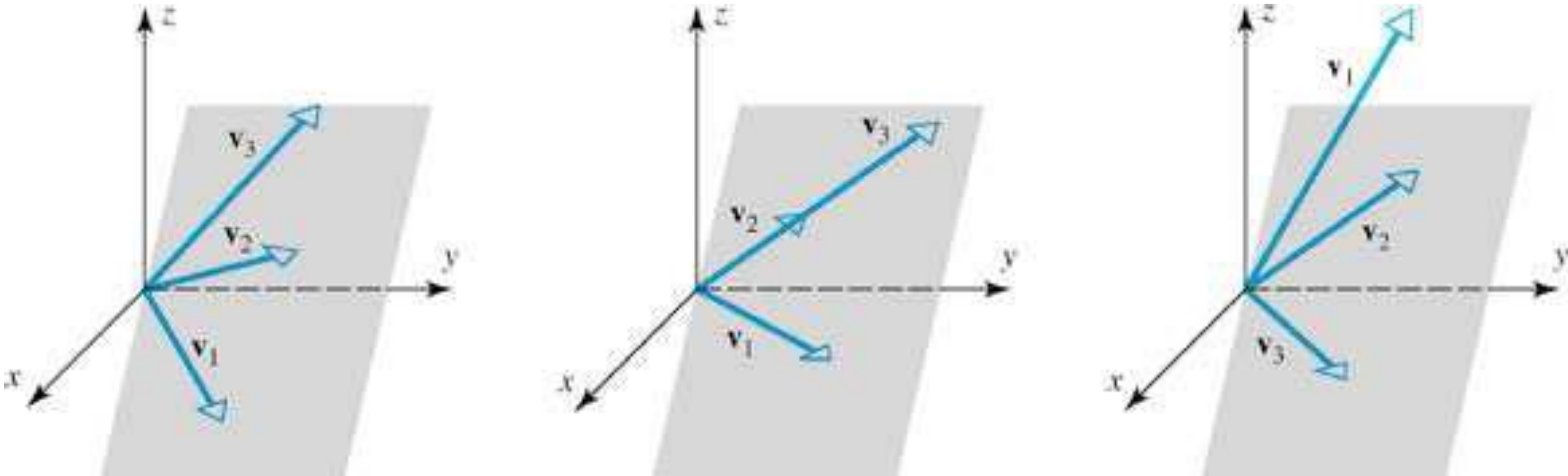
In R^2 or R^3, a set of two vectors is linearly independent if and only if the vectors do not lie on the same line when they are placed with their initial points at the origin.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Geometric Interpretation of Linear Independence

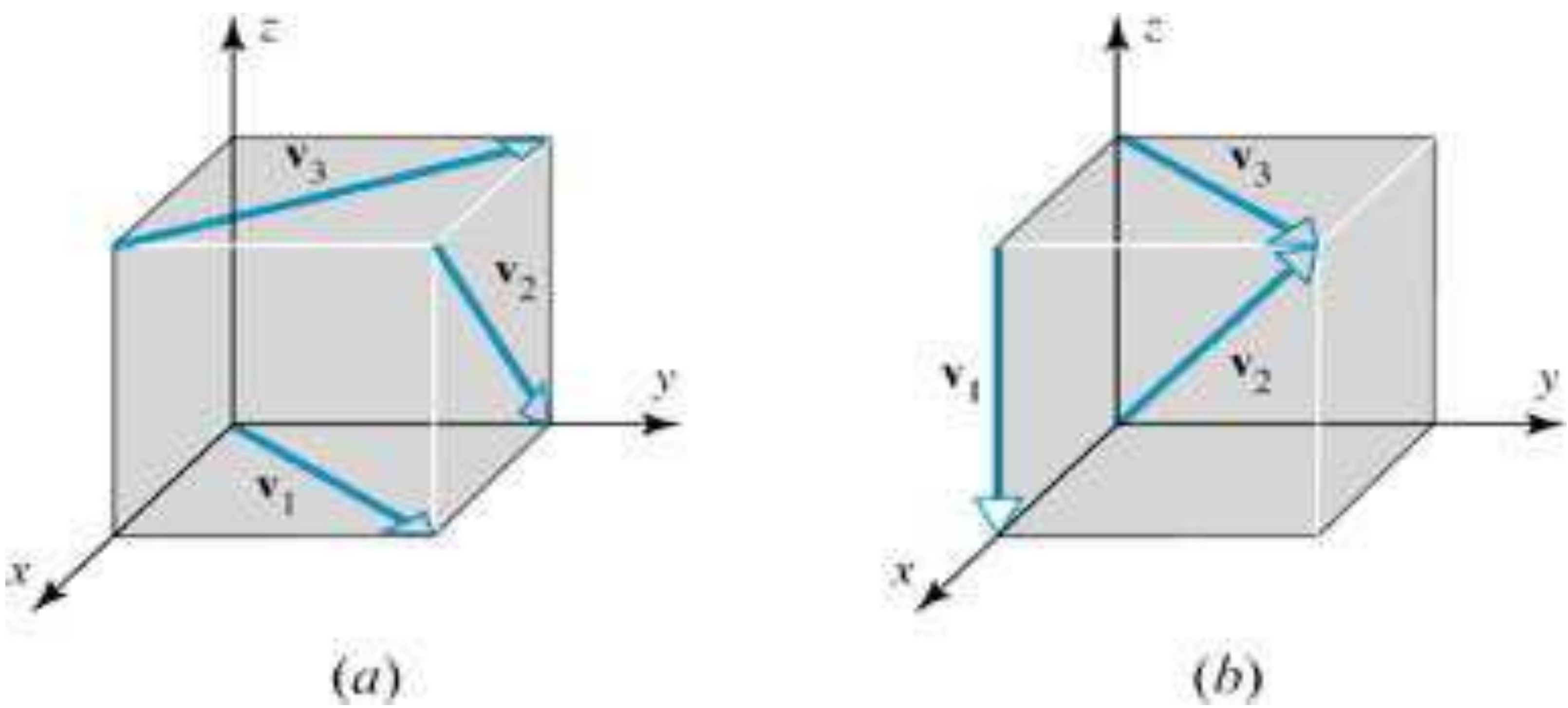
In R^3, a set of three vectors is linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

Geometric Interpretation of Linear Independence

Are the vectors ,$\mathbf{v}_1$,$\mathbf{v}_2$,and $\mathbf{v}_3$ in part (*a) of the accompanying figure linearly independent? What about those in part (b)? Explain.*



- Ex 3.3: Testing for linear independence

Determine whether the following set of vectors in R^3 is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{(1,\ 2,\ 3), (0,\ 1,\ 2), (-2,\ 0,\ 1)\}$$

Sol:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \ \Rightarrow \ \begin{aligned} c_1 \qquad\quad - 2c_3 &= 0 \\ 2c_1 + \ c_2 + \qquad &= 0 \\ 3c_1 + 2c_2 + \ c_3 &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow c_1 = c_2 = c_3 = 0$ (only the trivial solution)

(or $\det(A) = -1 \neq 0$, so there is only the trivial solution)

$\Rightarrow S$ is (or $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are) linearly independent

- Ex 3.4: Testing for linear independence

  Determine whether the following set of vectors in P_2 is L.I. or L.D.

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\}$$

Sol:

$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$

i.e., $c_1(1+x - 2x^2) + c_2(2+5x - x^2) + c_3(x+x^2) = 0+0x+0x^2$

$$\Rightarrow \begin{aligned} c_1+2c_2 \quad\;\; &= 0 \\ c_1+5c_2+c_3 &= 0 \\ -2c_1 - c_2+c_3 &= 0 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 1 & 5 & 1 & 0 \\ -2 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{G. E.}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ This system has infinitely many solutions

(i.e., this system has nontrivial solutions, e.g., $c_1=2$, $c_2= -1$, $c_3=3$)

$\Rightarrow$ S is (or $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are) linearly dependent

- Ex 3.5: Testing for linear independence

  Determine whether the following set of vectors in the 2×2 matrix space is L.I. or L.D.

$$S=\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}2&1\\0&1\end{bmatrix},\begin{bmatrix}3&0\\2&1\end{bmatrix},\begin{bmatrix}1&0\\2&0\end{bmatrix}\right\}$$

Sol:

$$c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3=\mathbf{0}$$

$$c_1\begin{bmatrix}2&1\\0&1\end{bmatrix}+c_2\begin{bmatrix}3&0\\2&1\end{bmatrix}+c_3\begin{bmatrix}1&0\\2&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned} 2c_1 + 3c_2 + c_3 &= 0 \\ c_1 \qquad\qquad &= 0 \\ 2c_2 + 2c_3 &= 0 \\ c_1 + c_2 \qquad &= 0 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{\text{G.-J. E.}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = c_2 = c_3 = 0$ (This system has only the trivial solution)

$\Rightarrow$ S is linearly independent

- Theorem 3.2: A property of linearly dependent sets

  A set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$, $k \geq 2$, is linearly dependent if and only if at least one of the vectors $\mathbf{v}_i$ in S can be written as a linear combination of the other vectors in S

Pf:

If S is linearly dependent set then this can be written as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$

$$\because S \text{ is linearly dependent (there exist nontrivial solutions)}$$

$$\Rightarrow c_i \neq 0 \text{ for some } i$$

$$\Rightarrow \mathbf{v}_i = \left(-\frac{c_1}{c_i}\right)\mathbf{v}_1 + \cdots + \left(-\frac{c_{i-1}}{c_i}\right)\mathbf{v}_{i-1} + \left(-\frac{c_{i+1}}{c_i}\right)\mathbf{v}_{i+1} + \cdots + \left(-\frac{c_k}{c_i}\right)\mathbf{v}_k$$

Let $\mathbf{v}_i = d_1\mathbf{v}_1 + \ldots + d_{i-1}\mathbf{v}_{i-1} + d_{i+1}\mathbf{v}_{i+1} + \ldots + d_k\mathbf{v}_k$

$\Rightarrow d_1\mathbf{v}_1 + \ldots + d_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + d_{i+1}\mathbf{v}_{i+1} + \ldots + d_k\mathbf{v}_k = \mathbf{0}$

$\Rightarrow c_1 = d_1,\ c_2 = d_2, \ldots, c_i = -1, \ldots, c_k = d_k$ (there exits at least this nontrivial solution)

$\Rightarrow S$ is linearly dependent

- Corollary to Theorem 3.2: (A corollary is a must-be-true result based on the already proved theorem)

  Two vectors **u** and **v** in a vector space *V* are linearly dependent (for $k = 2$ in Theorem 3.2) if and only if one is a scalar multiple of the other.

Theorem 3.3

Let $S=\{v_1,v_2,\ldots,v_k\}$ be a set of vectors in R^n. If $k>n$, then S is linearly dependent.

Solve Problems