CHAPTER FOUR

INTRODUCTION TO DIFFERENTIAL EQUATIONS

4.1 INTRODUCTION

Differential equations are of fundamental importance not only in engineering disciplines but in almost every field of applied sciences as many physical laws and relations appear mathematically in the form of differential equations. Various laws of nature can be translated into differential equations. When a physical problem consisting of rate of change of dependent variable with respect to one or more independent variables is transformed into a mathematical model a differential equation is formed. More often we obtain a system of differential equations along with certain conditions from such situations. The solution of such differential equation/system of differential equations provides solution to the original problem. In the coming sections, we shall consider various physical and geometrical problems that lead to differential equations and we shall explain the most important standard methods for solving such problems. Before explaining the physical problems, we will learn some important terms given below:

Definition: An equation involving derivatives or differentials of one dependent variable with respect to one or more independent variables is called a **differential equation**.

For example, equations listed below are differential equations:

$$\frac{dy}{dx} = x \log x \tag{1}$$

$$\frac{d^4y}{dt^4} + \frac{d^2y}{dt^2} + \left(\frac{dy}{dt}\right)^5 + \sin y = e^t \tag{2}$$

$$y = \sqrt{x}\frac{dy}{dx} + \frac{k}{dy/dx} \tag{3}$$

$$k\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \tag{4}$$

$$\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2 \tag{5}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{6}$$

Ordinary Differential Equation

A differential equation involving derivatives with respect to a single independent variable is called an **ordinary differential equation**. Equations (1), (2), (3) and (4) as shown above are ordinary differential equations.

Partial Differential Equation

Differential equations that involve partial derivatives with respect to two or more independent variables is called a **partial differential equation**. Equation (5) and (6) as shown above are partial differential equations.

Order and Degree of a Differential Equation

Order of a differential equation is the order of highest derivative involved in the differential equation.

For equations shown above, equation (2) is of order four, equations (1) and (3) are of order one, equations (4) and (6) are of second order and equation (5) is of order three.

Degree of a differential equation is the power/exponent of highest derivative that occurs in it. In the above given equations, (1), (2) and (6) are of first degree. Equation (3) may be re-written as:

$$y\frac{dy}{dx} = \sqrt{x}\left(\frac{dy}{dx}\right)^2 + k$$

It shows that this equation is of degree two. Again if we square both sides of (4) to make it free from radicals, then by definition equations (4) and (5) are of degree two.

Linear and Non–linear Differential Equations

A differential equation is called **linear** if it satisfies the following three conditions:

- **(i)** The dependent variable and its derivative(s) in the equation occur in first degree only.
- **(ii)** There is no term in the equation that contains the product of dependent variable and/or its derivative.
- **(iii)** No transcendental function with dependent variable as its argument occurs in the equation.

A differential equation that is not linear is called a **non–linear differential equation.** Equations (1) and (6) shown above are linear and all other equations are non-linear.

Solution of a Differential Equation

Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a solution of the differential equation. To understand this, let us consider few examples.

Example 01: Show that $y = c\,e^{2x}$ is solution of differential equation

$$y'' - 5y' + 6y = 0.$$

Solution: Since $y = c\,e^{2x} \rightarrow y' = 2c\,e^{2x}$ and $y'' = 4c\,e^{2x}$. Substituting these in given differential equation, we get

$$4c\,e^{2x} - 10c\,e^{2x} + 6c\,e^{2x} = 10c\,e^{2x} - 10c\,e^{2x} = 0$$

Since, given differential equation is satisfied hence, we say that $y = ce^{2x}$ is solution of differential equation $y'' - 5y' + 6y = 0$. Observe that $y = ce^{2x}$ is solution of given differential equation for any real 'c'. This constant 'c' is known as **arbitrary constant.**

Example 02: Show that $y = c/x + d$ is solution of $y'' + 2y'/x = 0$ and $x^2 + 4y = 0$ is a solution of $(y')^2 + xy' - y = 0$

Solution: (i) The given differential equation is

$$\frac{d^2y}{dx^2} + \frac{2}{x}\cdot\frac{dy}{dx} = 0 \qquad (1)$$

The given function is: $y = \frac{c}{x} + d \qquad (2)$

Differentiating (2), with respect to x, we get

$$\frac{dy}{dx} = -\frac{c}{x^2} \qquad (3)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{2c}{x^3} \qquad (4)$$

Substituting for dy/dx and d^2y/dx^2 in (1), we get

$$\frac{2c}{x^3}-\frac{2c}{x^3}=0 \Rightarrow 0=0$$

This is true, therefore, (2) is the solution of equation (1).

(ii) Given differential equation is

$$\left(\frac{dy}{dx}\right)^2+x\frac{dy}{dx}-y=0 \qquad (1)$$

And given function is

$$x^2+4y=0 \qquad (2)$$

Differentiating (2) with respect to x, we get

$$2x+4\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{x}{2} \qquad (3)$$

Substituting for y and dy/dx into (1), we get

$$\left(\frac{-x}{2}\right)^2+x\left(\frac{-x}{2}\right)-\left(\frac{-x^2}{2}\right)=0 \Rightarrow \frac{x^2}{4}-\frac{x^2}{2}+\frac{x^2}{4}=0 \Rightarrow 0=0$$

This is true, therefore, (2) is a solution of (1).

Types of Solutions

There are five types of solutions of any differential equation. They are:

(i) **Explicit Solution:** When the solution of differential equation is expressed in the form $y = f(x)$, we call such solution as explicit solution of given differential equation. For instance, $y = ce^{2x}$ is explicit solution of differential equation $y'' - 5y' + 6y = 0$ as shown in Example 1.

(ii) **Implicit Solution:** An implicit solution of differential equation is of form $f(x, y) = 0$. For example, the solution of differential equation $x + 3yy' = 0$ is $x^2 + 3y^2 = 4$. [Students may verify this]

(iii) **General Solution:** Solution of a differential equation in which the number of arbitrary constants is equal to order of the differential equation is called **general solution** or **complete solution** or **primitive** or **integral**. For instance, in Example 1, we see that given differential equation is of order two and we proved that $y = ce^{2x}$ satisfies the equation. As y involves only one arbitrary constant thus according to definition, $y = ce^{2x}$ does not form a general solution of this differential equation. However, $y = c_1e^{2x} + c_2e^{3x}$ forms a general solution of this differential equation because y contains two arbitrary constants and given differential equation is also of order two. Students may verify this by finding y' and y'' and then substituting the values of y, y' and y'' in the equation $y'' - 5y' + 6y = 0$. Similarly, Differential equation in Example 2 (i) is of second order and the solution contains two arbitrary constants, that is; c and d. Hence this is the general solution.

(iv) **Particular Solution:** A solution of differential equation that does not contain an arbitrary constant is called particular solution. In Example 2 (ii) the solution contains no arbitrary constant, hence it is a particular solution of given differential equation.

(v) **Singular Solution:** A solution of a differential equation that cannot be obtained from its general solution by assigning any particular values to the arbitrary constants is called a singular solution. For example, consider the differential equation $y\,y' - x(y')^2 = 1$, its general solution is $y = cx + 1/c$. If we put c = 1 we get $y = x + 1$. This is a particular solution. Equation $y^2 = 4x$ also satisfies given differential equation but cannot be obtained from the general solution by assigning any value to c. Hence, this is singular solution.

Let us see whether or not the equation $y^2 = 4x$ satisfies differential equation $y y' - x(y')^2 = 1$. Since $y^2 = 4x$ ➔ $2yy' = 4$ ➔ $yy' = 2$ ➔ $y' = 2/y$.
Now substituting $yy' = 2$ and $y' = 2/y$ in the equation $yy' - x(y')^2 = 1$, we obtain
$2 - x(2/y)^2 = 1$ ➔ $2y^2 - 4x = y^2$ ➔ $y^2 = 4x$

This shows that solution of differential equation $y y' - x(y')^2 = 1$ is $y^2 = 4x$.

REMARK: (i) Some authors are of the opinion that it is not necessary that a general solution must contain arbitrary constants but this is not true. For instance, the equation $(y')^2 + y^2 = 0$ has only one solution $y = 0$ which contains no arbitrary constant.
(ii) We normally expect that a differential equation will have a solution. But this not true! For instance, the equation $(y')^2 + 4 = 0$ has no real-valued solution.

4.2 FORMATION OF DIFFERENTIAL EQUATIONS

In this section, we shall discuss how differential equations arise in different situations and how they are modeled. It may be noted that a differential can be obtained in three different situations:

Differential Equations from Physical Phenomenon

The problems that involve rate of change of one variable with respect to time always give rise to differential equation. We shall provide couple of examples that will help you to understand how differential equations arise in such situations.

Example 01: The rate of growth of a population in a small town is proportional to population present. Find the differential equation.

Solution: Let P be the population at any time. As given:

$$\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP \Rightarrow \frac{dP}{dt} - kP = 0$$

This is required differential equation. Here, k is the constant of proportionality.

Example 02: The rate of decrease in a radio active material (say Sodium) is proportional to the amount present. Find the differential equation that governs this problem.

Solution: Let A be the amount of radioactive material at any time. Then as given:

$$\frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = -kA \Rightarrow \frac{dA}{dt} + kA = 0$$

This is the required differential equation. Here 'k' is the constant of proportionality and the negative sign indicates that the amount of radioactive material decreases.

Example 03: A man jump from an airplane using parachute. If m is the mass of a man with parachutes, find the equation of his motion in the form of differential equation.

Solution: By Newton's second law of motion: $m a = F$ (1)

Here, F is composed of two forces, F_U and F_D, that is $F = F_U + F_D$.

F_D is the downward force that is equal to weight of the body. Thus

$$F_D = w = m g.$$

F_U is the upward force that is proportional to the velocity of body when it falls down. Thus

$$F_U = \alpha v \text{ or } F_U = -k v,$$

where k is the constant of proportionality and negative sign shows that body faces the air resistance. Also, 'a' the acceleration of the body is given by $a = dv/dt$. Thus equation (1) becomes:

$$m\frac{dv}{dt} = mg - kv \Rightarrow m\frac{dv}{dt} + kv = mg.$$

This is the differential equation that governs the motion of falling object. If we put v = ds/dt, above differential equation becomes:

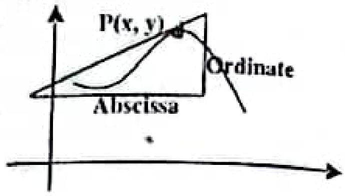
$$m\frac{d^2s}{dt^2}+k\frac{ds}{dt}=mg$$

This is another form of differential equation that governs the motion of a falling body.

Differential Equations from Geometrical Phenomenon

Problems that involve slope of a function of one variable with respect to another variable, always give rise to the differential equations. The following example shows formation of differential equation using the concept of slope.

Example 04: The slope of a curve at any point P (x, y) is given by ratio of an ordinate to twice its abscissa. Find the differential equation that governs this problem.



Solution: We know that slope of a curve at any point P (x, y) is given by dy/dt. Now as per condition

$$\frac{dy}{dt}=\frac{\text{Ordinate}}{\text{Twice the abscissa}}=\frac{y}{2x}$$

$$\Rightarrow \quad 2x\frac{dy}{dt}=y \;\Rightarrow\; 2x\frac{dy}{dt}-y=0$$

This is differential equation that arises from the situation explained above.

Differential Equations by Eliminating an Arbitrary Constant(s)

If an equation involves a dependent variable, an independent variable and some arbitrary constants, we can obtain a differential equation by eliminating the arbitrary constants. Following steps will help to find required equation.

- Write down given equation and differentiate with respect to x successively as many times as the number of arbitrary constants.
- Eliminate arbitrary constants from the equations obtained.
- The resulting equation is required differential equation.

Example 05: Form the differential equation by eliminating arbitrary constant(s).

(i) $y = (x^3 + c)\,e^{-3x}$ (ii) $y = A \sin x + B \cos x$

Solution: (i) Given equation is

$$y=\left(x^3+c\right)e^{-3x} \qquad (1)$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx}=\left(x^3+c\right)\left(-3e^{-3x}\right)+e^{-3x}\left(3x^2\right)$$

$$=3x^2e^{-3x}-3\left(x^3+c\right)e^{-3x}\Rightarrow\frac{dy}{dx}=3x^2e^{-3x}-3y \qquad \text{[Using (1)]}$$

or

$$\frac{dy}{dx}+3y=3x^2e^{-3x}.$$

(ii) Given equation is

$$y=A\sin x+B\cos x \qquad (1)$$

Differentiating (1) with respect to x, we get

$$\frac{dy}{dx}=A\cos x-B\sin x \qquad (2)$$

Again differentiating, we get

$$\frac{d^2y}{dx^2}=-A\sin x-B\cos x\Rightarrow\frac{d^2y}{dx^2}=-(A\sin x+B\cos x)$$

$$\Rightarrow\frac{d^2y}{dx^2}=-y\Rightarrow\frac{d^2y}{dx^2}+y=0. \qquad [\text{using } (1)]$$

Example 06: Form the differential equation of simple harmonic motion given by $x = A\cos(nt+a)$ where A and a are arbitrary constants.

Solution: The given equation is: $x=A\cos(n\,t+\alpha)$ (1)

To eliminate constants A and α we differentiate (1) twice w.r.t t and get

$$\frac{dx}{dt}=-A\sin(nt+\alpha)(n)\Rightarrow\frac{dx}{dt}=-nA\sin(nt+\alpha)$$

Differentiating again w.r.t t, we get

$$\frac{d^2x}{dt^2}=-nA\cos(nt+\alpha)(n)\Rightarrow\frac{d^2x}{dt^2}=-n^2A\cos(nt+\alpha)$$

$$\Rightarrow\frac{d^2x}{dt^2}=-n^2x\Rightarrow\frac{d^2x}{dt^2}+n^2x=0 \qquad [\text{using } (1)]$$

Example 07: Obtain differential equation of all circles of radius r and centered at (h, k).

Solution: Equation of all circles centered at (h, k) and of radius r is:

$$(x-h)^2+(y-k)^2=a^2, \qquad (1)$$

where h and k are the coordinates of center and are arbitrary constants.

Differentiating equation (1) with respect to x, we get

$$2(x-h)+2(y-k)\frac{dy}{dx}=0\Rightarrow(x-h)+(y-k)\frac{dy}{dx}=0 \qquad (2)$$

Again differentiating, we get

$$1+(y-k)\frac{d^2y}{dx^2}+\frac{dy}{dx}\left(\frac{dy}{dx}\right)=0\Rightarrow1+(y-k)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \qquad (3)$$

From (3), we have $y-k=-\dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$ and from (2), we have

$$x-h=-(y-k)\frac{dy}{dx}=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx}}{\frac{d^2y}{dx^2}}.$$

Substituting the values of x- h and y – k in (1), we get

$$\frac{\left[1+(y')^2\right]^2(y')^2}{(y'')^2}+\frac{\left[1+(y')^2\right]^2}{(y'')^2}=r^2\Rightarrow\left[1+(y')^2\right]^2(y')^2+\left[1+(y')^2\right]^2=r^2(y'')^2$$

$$\left[1+(y')^2\right]^2\left[1+(y')^2\right]=r^2(y'')^2 \qquad \Rightarrow\frac{\left[1+(y')^2\right]^3}{(y'')^2}=r^2\ \Rightarrow\frac{\left[1+(y')^2\right]^{3/2}}{(y'')}=r$$

Example 08: Obtain differential equations by eliminating an arbitrary constant(s) From the following equations.

(i) $y = x + c\,e^{-x}$

Solution: Given equation contains only one arbitrary constant, hence we differentiate it once to get: $y' = 1 - c\,e^{-x}$.

Now, $y = x + c\,e^{-x}$

Adding, we get: $y' + y = x + 1$.

(ii) $a x + \ln y = y + b$

Solution: Given equation is: $a x + \ln y = y + b$

It contains two arbitrary constants, hence differentiate it twice, we get:

$$1 + y'/y = y'$$

Differentiate once again, we get: $0 + (y\,y'' - y'\,y')/y^2 = y''$

$\rightarrow \quad y\,y'' - (y')^2 = y^2\,y'' \quad \rightarrow \quad (y - y^2)\,y'' - (y')^2 = 0$

(iii) $y = a\,e^x + b \ln x + cx + d$

Solution: Given equation contains four arbitrary constant, hence we differentiate it four times.

$y' = a\,e^x + b/x + c$ (1)

$y'' = a\,e^x - b/x^2$ (2)

$y''' = a\,e^x + 2b/x^3$ (3)

$y^{iv} = a\,e^x - 6b/x^4$ (4)

Subtracting (2) from (3), we get

$y''' - y'' = 2b/x^3 + b/x^2 = b(2 + x)/x^3 \rightarrow b = x^3(y''' - y'')/(2 + x)$ (5)

Subtracting (3) from (4), we get

$y^{iv} - y''' = -6b/x^4 - 2b/x^3 = b(-6 - 3x)/x^4 \rightarrow b = x^4(y^{iv} - y''')/(-6 - 3x)$ (6)

Equating (5) and (6), we get

$$\frac{x^4(y^{iv} - y''')}{(-6-3x)} = \frac{x^3(y'''-y'')}{(2+x)} \Rightarrow \frac{x(y^{iv} - y''')}{-3(2+x)} = \frac{(y'''-y'')}{(2+x)}$$

or, $x\,(2 + x)\,(y^{iv} - y''') = -3(2 + x)\,(y''' - y'')$

$\rightarrow x\,(y^{iv} - y''') = -3\,(y''' - y'') \rightarrow x\,(y^{iv} - y''') + 3\,(y''' - y'') = 0$

(iv) $y = a\,x^2 + b\,x$

Solution: Given equation contains two arbitrary constants hence we differentiate it twice.

Now,

$y = a\,x^2 + b\,x$ (1) $\quad y' = 2\,ax + b$ (2) $\quad y'' = 2a$ (3)

From equation (3), $a = y''/2$. Put this is (2), we get

$y' \times y'' + b \rightarrow b = y' - x\,y''$.

Now substituting the values of a and b in (1), we obtain

$y = x^2\,(y''/2) + x\,(y' - x\,y'') \rightarrow 2y = x^2\,y'' + 2\,xy' - 2\,x^2\,y''$

or $\quad x^2\,y'' - 2\,xy' + 2\,y = 0$.

(v) $y = A\,e^{2x} + B\,e^{-3x}$

Solution: This equation contains two arbitrary constants hence we differentiate it twice.

Now,

$y = A\,e^{2x} + B\,e^{-3x}$ (1) $\quad y' = 2\,A\,e^{2x} - 3\,B\,e^{-3x}$ (2) $\quad y'' = 4\,A\,e^{2x} + 9\,B\,e^{-3x}$ (3)

Multiply (1) by 2 and subtracting it from (2), we obtain:

$y' - 2\,y = -5\,B\,e^{-3x}$

Now multiply (1) by 4 and subtracting it from (3), we get (4)

$y'' - 4\,y = 5\,B\,e^{-3x}$

Adding (4) and (5), we get: $y'' + y' - 6\,y = 0$. (5)

(vi) $y = e^x (A \cos 2x + B \sin 2x)$

Solution: Given differential equation contains two arbitrary constants, hence we differentiate it twice. Now,

$$y = e^x (A \cos 2x + B \sin 2x)$$

$\Rightarrow$ $y' = e^x (-2A \sin 2x + 2B \cos 2x) + e^x (A \cos 2x + B \sin 2x)$ (1)

$\Rightarrow$ $y' = e^x (-2A \sin 2x + 2B \cos 2x) + y$

Or $y' - y = e^x (-2A \sin 2x + 2B \cos 2x)$ (2)

Differentiate (2) again, we get

$$y'' - y' = e^x (-4A \sin 2x - 4B \cos 2x) + (-2A \sin 2x + 2B \cos 2x) e^x$$
$$= -4 e^x (A \sin 2x + B \cos 2x) + y' - y \quad \text{[from 2]}$$

$\Rightarrow$ $y'' - y' = -4y + y' - y$ $\Rightarrow$ $y'' - 2y' + 5y = 0$

(vii) $y = ax + a^2$

Solution: Given: $y = ax + a^2$. (1)

On differentiating, we obtain: $y' = a$.

Substituting this value of a in equation (1), we get: $y = xy' + (y')^2$

(viii) $y^2 = ax^2 + bx + c$

Solution: Given equation is: $y^2 = ax^2 + bx + c$.

On differentiating, we get: $2yy' = 2ax + b$

Again differentiating, we obtain: $2yy'' + 2(y')^2 = 2a$ $\Rightarrow$ $yy'' + (y')^2 = a$

On differentiating again, we get: $yy''' + y'.y'' + 2y'.y'' = 0$

$\Rightarrow$ $yy''' + 3y'y'' = 0$

(ix) $y = a e^x + bx e^x + cx^2 e^x$

Solution: Given $y = a e^x + bx e^x + cx^2 e^x$

Differentiating both sides, we get

$$y' = a e^x + bx e^x + b e^x + cx^2 e^x + 2cx e^x$$
$$= (a e^x + bx e^x + cx^2 e^x) + (b e^x + 2cx e^x)$$
$$= y + (b e^x + 2cx e^x)$$

$\Rightarrow$ $y' - y = (b e^x + 2cx e^x)$ (1)

Again differentiate both sides, we get

$$y'' - y' = b e^x + 2c e^x + 2cx e^x = (b e^x + 2cx e^x) + 2c e^x$$
$$= y' - y + 2c e^x \quad \text{[From (1)]}$$

$\Rightarrow$ $y'' - 2y' + y = 2c e^x$ (2)

Finally, on differentiating again, we obtain

$$y''' - 2y'' + y' = 2c e^x = y'' - 2y' + y \quad \text{[From (2)]}$$

$\Rightarrow$ $y''' - 3y'' + 3y' - y = 0$

4.3 INITIAL AND BOUNDARY CONDITIONS

The arbitrary constants in general solution of a differential equation can often be determined by giving additional conditions on the unknown function and/or its derivatives. If these conditions are specified at the same value of an independent variable, we call them **initial conditions**. If the conditions are given at different values of independent variable, we call them **boundary conditions**. A differential equation together with its initial conditions is called an **initial value problem** and a differential equation together with its boundary conditions is called a **boundary value problem**.

The general solution of a first order differential equation contains one arbitrary constant and so requires one additional condition to determine this arbitrary constant. Therefore, first order differential equations always present initial value problems. Boundary value problems involve differential equations that are at least of second order. For example, the differential equation

$y''' + 3y'' - y' + y = \sin x$ with initial conditions $y(0) = 0$, $y'(0) = 3$, $y''(0) = 5$ is an initial value problem because all conditions are given at $x = 0$. A general solution of given differential equation will contain three arbitrary constants, which can be determined by three conditions.

Again consider $y'' + 5\,xy = \cos x$, $y(0) = 0$, $y'(\pi) = 3$. This is a boundary value problem where two conditions are given at $x = 0$ and $x = \pi$ which will determine two arbitrary constants in the general solution of given differential equation.

Example 01: Solve the following initial value problem $y'' - y' - 12y = 0$, $y(0) = -2$, $y'(0) = 6$ where it is given that $y = A\,e^{4x} + B\,e^{-3x}$ is the general solution of given equation.

Solution: We have: $y = Ae^{4x} + Be^{-3x}$ (1)

Differentiating (1) with respect to x, we get: $y' = 4Ae^{4x} - 3Be^{-3x}$ (2)

Since $y = -2$ when $x = 0$ then (1) gives

$$-2 = A + B \quad (3)$$

Also $y = 6$ when $x = 0$ then (2) gives

$$6 = 4A - 3B \quad (4)$$

Multiplying equation (3) by 3 and adding to (4), we get $0 = 7A \rightarrow A = 0$

Substituting it into (3), we get $B = -2$.

Hence from (1), we have $y = -2e^{-3x}$ as particular solution of given differential equation.

WORKSHEET 04

1. Find the orders and degrees of following differential equations. Also state that given differential equations are linear or non-linear and that they are ordinary or partial differential equations.

(a) $e^x\,dx + e^y\,dy = 0$ (b) $y'' + n^2\,y = 0$ (c) $y = x\,y' + x/y'$

(d) $[1 + (y')^2]^{3/2} = c^2\,(y'')^2$ (e) $y = x\,y' + a\,[1 + (y')^2]$

(f) $y''' + 4y'' - 6y' + y = \cos x$ (g) $U_{xx} + U_{yy} = 0$ (h) $(y')^2 = (y'' + y)^{3/2}$

2. Verify that:

(a) $y = a + be^{-2x} + e^x/3$ is the general solution of equation $y'' + 2y' = e^x$

(b) $y^2 = (x + 5)^3$ is general solution of equation $27y - 8(y')^3 = 0$

3. Obtain the differential equations by eliminating an arbitrary constant(s) from the following equations.

(a) $y = a\,x^3 + b\,x^2$ (b) $xy = A\,e^x + B\,e^{-x}$ (c) $y^2 = 4a\,(x + a)$

(d) $y = a\,e^{2x} + b\,e^{-3x} + c\,e^x$ (e) $A\,x^2 + B\,y^2 = 1$ (f) $y = a\sin(x + 3)$

(g) $y = \sqrt{6x + c}$ (h) $y = x + c\,e^x$ (i) $y = (x^3 + c)\,e^{-3x}$

(j) $a\,x + \ln y = y + b$, $y > 0$ (k) $y^2 - 2\,a\,y + x^2 = a^2$ (l) $y = ae^x + b\ln x + c\,x + d$

(m) $y = a\sinh 2x + b\cosh 2x$ (n) $y = a\sin x + b\cos x + x\sin x$

4. Solve following initial/boundary problems:

(a). $y' = -x/y$, $y(3) = 4$ given that $x^2 + y^2 = r^2$ is the solution of given differential equations.

(b) $y'' - y' - 12\,y = 0$, $y(0) = -2$, $y'(0) = 6$, given that $y = a\,e^{4x} + b\,e^{-3x}$ is the solution of given differential equations.

(c) $y'' + y = 0$, $y(0) = 1$, $y'(\pi) = 1$, given that $y = a\sin x + b\cos x$ is the solution of given differential equations.

(d) $y'' - y' + 2\,y = 0$, $y(0) = 1$, $y(1) = 1$, given that $y = a\,e^x + b\,e^{3x}$ is the solution of given differential equations.

CHAPTER FIVE

FIRST ORDER DIFFERENTIAL EQUATIONS

$\theta = (a + \lambda x)$

5.1 SOLUTION OF FIRST ORDER DIFFERENTAI EQUATIONS

In chapter one, we discussed the solution of differential equation. By solution of a differential equation, we mean the value of a dependent variable (in explicit form $y = f(x)$ or any relation between x and y in implicit form $f(x, y) = 0$) that satisfies given differential equation. In this chapter, we shall study/learn various methods to solve the differential equations of first and first degree because not every first order differential equation can be solved exactly in the same manner or same method. They can be solved if they belong to standard forms discussed in the following sections.

Before we discuss these methods it may be noted that most general form of differential equation of first order and first degree is:

$$\frac{dy}{dx} = f(x, y) \text{ or } M(x, y)\,dx + N(x, y)\,dy = 0. \qquad \text{(I)}$$

To find a solution of this differential equation, various methods have been developed depending upon the type of the functions M and N as mentioned earlier. We shall discuss these methods now.

Separable Variable Method

If equation (I) as shown above is expressed as:

$$\frac{dy}{dx} = \frac{M(x)}{N(y)},$$

where M(x) is a function of x only and N(y) is a function of y only, then we may rewrite given differential equation as:

$$N(y)dy = M(x)dx.$$

This equation is in separable variable form. Integrating both sides, we obtain

$$\int N(y)dy = \int M(x)dx + c$$

where c is an arbitrary constant known as constant of integration. Solving the integrals will produce the solution of given differential equation.

REMARKS:

i. Never forget to add an arbitrary constant on any side of solution but it is preferred to be placed on right side.
ii. The nature of arbitrary constant depends upon the nature of the solution.
iii. The solution of a differential equation must be put in a form as simple as possible.

Example 01: Find the solution of differential equation $y\,y' = x$

Solution: Given that $\quad y\,dy/dx = x \Rightarrow ydy = xdx$

Integrating both sides, we get: $\quad \int ydy = \int xdx + c \Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + c \qquad (1)$

Multiplying equation (1) by 2, we get: $y^2 = x^2 + r^2$ $\qquad$ [Assuming $2c = r^2$]

This is general solution of given differential equation.

Example 02: Solve the following differential equations:

(i) $y' = e^{x-y} + x^2 e^{-y}$ (ii) $y' + y^2 \sin x = 0$ (iii) $x e^{x^2+y} dx = y dy$

Solution: (i) Given that $y' = e^{x-y} + x^2 e^{-y} = e^{-y}(e^x + x^2)$ (1)

Separating the variables, (1) becomes: $e^y dy = \left(e^x + x^2\right)dx$

Integrating, $\int e^y dy = \int \left(e^x + x^2\right)dx \Rightarrow e^y = e^x + x^3/3 + c$

This is the solution of given differential equation.

(ii) We have $y' + y^2 \sin x = 0$ → $y' = -y^2 \sin x$ (2)

Separating the variables, (2) becomes: $dy/y^2 = -\sin x\, dx$

Integrating, $\int y^{-2} dy = \int -\sin x\, dx + c \Rightarrow -1/y = +\cos x + c \Rightarrow y\cos x + cy + 1 = 0$

(iii) We have $x e^{x^2+y} dx = y dy \Rightarrow x e^{x^2} e^y dx = y dy$ (1)

Separating the variables, (1) becomes: $x e^{x^2} dx = y e^{-y} dy$

Integrating both sides, we get

$$\int x e^{x^2} dx = \int y e^{-y} dy + c' \tag{2}$$

Left side of (2) is solved using integration by substitution as follows:

Let $z = x^2$ → $dz = 2x\, dx$ → $dz/2 = x\, dx$. Thus

$$\int x e^{x^2} dx = \frac{1}{2}\int e^z dz = e^z/2 = e^{x^2}/2$$

Right side of (2) will be solved using integration by parts as follows:

$$\int y e^{-y} dy = y\left(-e^{-y}\right) + \int e^{-y} dy = -y e^{-y} - e^{-y} = -e^{-y}(y+1)$$

Thus, (2) becomes

$$\frac{e^{x^2}}{2} = -e^{-y}(y+1) + c' \Rightarrow e^{x^2} = -2e^{-y}(y+1) + 2c'$$

$$\Rightarrow e^{x^2} = -2e^{-y}(y+1) + c \qquad [\text{assuming } 2c' = c]$$

[**NOTE: Integration by Parts formula is** $\int u\, v\, dx = u\int v\, dx - \int [u' \int v\, dx]dx$]

Example 03: Solve the following initial value problem

$$(3x + 8)(y^2 + 4)\, dx - 4y\,(x^2 + 5x + 6) = 0,\ y(1) = 2$$

Solution: Given $(3x + 8)(y^2 + 4)\, dx - 4y\,(x^2 + 5x + 6) = 0$ (1)

→ $(3x + 8)(y^2 + 4)\, dx = 4y\,(x + 5)(x + 3)$

Separating the variables, we get

$$\frac{(3x+8)}{(x+2)(x+3)}dx = \frac{4y}{y^2+4}dy$$

Integrating both sides, we get

$$\int \frac{(3x+8)}{(x+2)(x+3)}dx = \int \frac{4y}{y^2+4}dy + c_1 \tag{2}$$

Left side of (2) will be solved using integration by partial fractions.

$$\frac{3x+8}{(x+2)(x+3)} = \frac{A}{x+2} + \frac{B}{x+3} \tag{3}$$

Multiplying both sides by $(x + 2)(x + 3)$, we get

$3x + 8 = A(x + 3) + B(x + 2)$ (4)

Put $x + 3 = 0 \rightarrow x = -3$ then (4) gives $B = 1$.

Put $x + 2 = 0 \rightarrow x = -2$ then (4) gives $A = 2$.

Thus equation (3) becomes: $\frac{3x+8}{(x+2)(x+3)} = \frac{2}{x+2} + \frac{1}{x+3}$

Integrating, we get:

$$\int \frac{(3x+8)}{(x+2)(x+3)} dx = 2\int \frac{1}{x+2} dx + \int \frac{1}{x+3} dx = 2\ln(x+2) + \ln(x+3) = \ln(x+2)^2(x+3)$$

Now the right side of (2) is: $2\int \frac{2y}{y^2+4} dy = 2\ln\left(y^2+4\right) = \ln\left(y^2+4\right)^2$

Hence, equation (2) becomes: $\ln(x + 2)^2 (x + 3) = \ln(y^2 + 4)^2 + \ln c = \ln c(y^2 + 4)^2$

$\rightarrow \quad (x + 2)^2 (x + 3) = c (y^2 + 4)^2$ (5)

Applying initial conditions, that is, put $y = 2$ and $x = 1$, we obtain

$(1 + 2)^2 (1 + 3) = c (4 + 4)^2 \rightarrow c = 9/16$

Substituting in equation (5), we get

$(x + 2)^2 (x + 3) = 9(y^2 + 4)/16 \rightarrow 16 (x + 2)^2 (x + 3) = 9(y^2 + 4)$

This is a particular solution of given differential equation.

Example 04: Solve the following differential equations

(i) $dy/dx = x^2/[y(1 + x^3)]$

Solution: Given differential equation is $dy/dx = x^2/[y(1 + x^3)]$.

Separating the variables and integrating, we get

$$\int y\, dy = \int \frac{x^2}{(1+x^3)} dx \Rightarrow \frac{y^2}{2} = \frac{1}{3}\int \frac{3x^2}{(1+x^3)} dx + C \Rightarrow \frac{y^2}{2} = \frac{1}{3}\ln(1+x^3) + C$$

(ii) $dy/dx = 1 + x + y^2 + x y^2$

Solution: Given differential equation is

$$\frac{dy}{dx} = (1+x) + y^2(1+x) \quad \Rightarrow \frac{dy}{dx} = (1+x)(1+y^2)$$

Separating the variables and integrating, we get

$$\int \frac{1}{1+y^2} dy = \int (1+x)dx + C \quad \Rightarrow \tan^{-1} y = x + \frac{x^2}{2} + C$$

(iii) $(xy + 2x + y + 2)\, dx + (x^2 + 2x)\, dy = 0$

Solution: Given differential equation is:

$(x^2 + 2x)\, dy = -[x(y + 2) + 1\,(y + 2)]\, dx \quad \rightarrow \quad (x^2 + 2x)\, dy = -(y + 2)(x + 1)$

Separating the variables and integrating, we get

$$\int \frac{1}{y+2} dy = \int \frac{(1+x)}{(x^2+2x)} dx + C_1 \Rightarrow \ln(y+2) = \frac{1}{2}\int \frac{2x+2}{x^2+2x} dx + \ln C$$

$$\Rightarrow \ln(y+2) = \frac{1}{2}\ln(x^2+2x) + \ln C \quad \Rightarrow \ln(y+2) = \ln C\sqrt{x^2+2x}$$

$$\Rightarrow y+2 = C\sqrt{x^2+2x} \quad \text{or} \quad y = C\sqrt{x^2+2x} - 2$$

(iv) $y' = 2x^2 + y - x^2y + xy - 2x - 2$

Solution: Given that: $y' = 2x^2 + y - x^2y + xy - 2x - 2$. Rearranging the terms on right side, we get: $y' = -x^2 (y - 2) - x (y - 2) + (y - 2) = (y - 2)(-x^2 - x + 1)$

Separating the variables and integrating, we get

$$\int \frac{1}{y-2}dy = \int (-x^2 - x + 1)dx \ +C \qquad \Rightarrow \ln(y-2) = -\frac{x^3}{3} - \frac{x^2}{2} + x + C$$

(v) cosec y dx + sec x dy = 0

Solution: Rewriting the given equation as: sec x dy = - cosec y dx

Now separating the variables and integrating, we get

$$\int \frac{1}{-\operatorname{cosec} y} dy = \int \frac{1}{\sec x} dx + C \quad \Rightarrow -\int \sin y \, dy = \int \cos x \, dx + C \quad \Rightarrow \cos y = \sin x + C$$

(vi) y (1 + x) dx + x (1 + y) dy = 0

Solution: Rearranging the terms, we get: x (1 + y) dy = - y (1 + x) dx

Separating the variables and integrating, we get

$$\int \frac{1+y}{y} dy = -\int \frac{1+x}{x} dx + C \quad \Rightarrow \int (1/y+1)dy + \int (1/x+1)dx = C$$

$$\Rightarrow \ln y + y + \ln x + x = C \quad \rightarrow \ln xy = C - x - y$$

(vii) $y\sqrt{1+x^2}dx + x\sqrt{1+y^2}dy = 0$

Solution: Re-arranging the terms, separating the variables and integrating, we get

$$\int \frac{\sqrt{1+y^2}}{y} dy = -\int \frac{\sqrt{1+x^2}}{x} dx \quad \Rightarrow \int \frac{\sqrt{1+y^2}}{y} dy + \int \frac{\sqrt{1+x^2}}{x} dx \ = C \qquad (1)$$

Let us consider, $\int \frac{\sqrt{1+x^2}}{x} dx$. Put $z = \sqrt{1+x^2} \Rightarrow z^2 = 1 + x^2 \Rightarrow 2z dz = 2x\,dx$

$\rightarrow z dz / x = dx$. Thus

$$\int \frac{\sqrt{1+x^2}}{x} dx = \int \frac{z}{x} \cdot \frac{z}{x} dz = \int \frac{z^2}{x^2} dz = \int \frac{z^2}{z^2-1} dz = \int \frac{(z^2-1)+1}{z^2-1} dz = \int \left(1 + \frac{1}{z^2-1}\right) dz$$

$$= z + \frac{1}{2}\ln\left(\frac{z-1}{z+1}\right) = \sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right)$$

Similarly, $\int \frac{\sqrt{1+y^2}}{y} dy = \sqrt{1+y^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right)$. Thus equation (1) becomes:

$$\sqrt{1+y^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+y^2}-1}{\sqrt{1+y^2}+1}\right) + \sqrt{1+x^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}\right) = C.$$

This is the general solution of given differential equation.

(viii) $(e^x + 1)\, y\, dy = (y + 1)\, e^x\, dx$

Solution: Separating the variables and integrating, we get

$$\int \frac{y}{y+1} dy = \int \frac{e^x}{e^x+1} dx + C \quad \Rightarrow \int \frac{(y+1)-1}{(y+1)} dy = \ln(e^x+1) + C$$

$$\Rightarrow \int \left(1 - \frac{1}{y+1}\right) dy = \ln(e^x+1) + C \qquad \Rightarrow y + \ln(y+1) = \ln(e^x+1) + C$$

(ix) $\frac{dy}{dx} = \frac{y^3+2y}{x^2+3x}$

Solution: Separating the variables and integrating, we get

$\int \frac{1}{y(y^2+2)} dy = \int \frac{1}{x(x+3)} dx + \ln C$ (1)

Consider,

$\frac{1}{y(y^2+2)} = \frac{A}{y} + \frac{By+C}{y^2+2} = \frac{1/2}{y} + \frac{-1/2y+0}{y^2+2}$ [Note: This is by partial fractions]

Thus, $\int \frac{1}{y(y^2+2)} dy = \frac{1}{2}\left(\int \frac{1}{y} dy - \frac{1}{2}\int \frac{2y}{y^2+2} dy\right) = \frac{1}{2}\left(\ln y - \frac{1}{2}\ln(y^2+2)\right) = \frac{1}{2}\ln \frac{y}{\sqrt{y^2+2}}$

Also $\frac{1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3} = \frac{1/3}{x} + \frac{-1/3}{x+3}$ [Note: This is by partial fractions]

Thus, $\int \frac{1}{x(x+3)} dx = \frac{1}{3}\left(\int \frac{1}{x} dx - \int \frac{1}{x+3} dx\right) = \frac{1}{3}(\ln x - \ln(x+3)) = \frac{1}{3}\ln\left(\frac{x}{x+3}\right)$

Hence equation (1) becomes: $\frac{1}{2}\ln \frac{y}{\sqrt{y^2+2}} = \frac{1}{3}\ln\left(\frac{x}{x+3}\right) + \ln C$

(x) (sinx + cos x) dy + (cos x – sin x) dx = 0

Solution: Separating the variables and integrating, we get:

$\int 1.dy = -\int \frac{\cos x - \sin x}{\sin x + \cos x} dx + \ln C \Rightarrow y = -\ln(\sin x + \cos x) + \ln C = \ln\left(\frac{C}{\sin x + \cos x}\right)$

(xi) $e^x (1 + y') = x e^{-y}$

Solution: Rearranging the terms, we get:

$1 + \frac{dy}{dx} = \frac{x}{e^x e^y} = \frac{x}{e^{x+y}}$ (1)

Put $x + y = z$ ➔ $1 + y' = z'$. Thus equation (1) becomes:

$\frac{dz}{dx} = \frac{x}{e^z} \Rightarrow e^z dz = x\, dx$ Integrating, we get: $\int e^z dz = \int x\, dx + C_1 \Rightarrow e^z = \frac{x^2}{2} + C_1$

Substituting the value of z, we get:

$e^{x+y} = \frac{x^2}{2} + C \Rightarrow 2e^{x+y} = x^2 + 2C_1 \Rightarrow 2e^{x+y} = x^2 + C$, where $2C_1 = C$

(xii) $x e^{x^2+y} dx = y\, dy$

Solution: Rearranging, we get: $x e^{x^2} e^y dx = y\, dy$. Separating the variables and integrating, we get:

$$\int y e^{-y} dy = \int x e^{x^2} dx + C \quad (1)$$

Consider, $\int y e^{-y} dy = -y e^{-y} + \int e^{-y} dy = -e^{-y}(y+1)$

Now consider, $\int x e^{x^2} dx$. Put $z = x^2 \Rightarrow dz = 2x\, dx$

Thus $\int x e^{x^2} dx = \frac{1}{2}\int e^z dz = \frac{1}{2}e^z = \frac{1}{2}e^{x^2}$

Thus equation (1) becomes: $-e^{-y}(y+1) = \frac{1}{2}e^{x^2} + C$

(xiii) $2x \cos y\, dx + x^2 (\sec y - \sin y)\, dy = 0$

Solution: Rearranging the terms and integrating, we get

x^2 (sec y – sin y) dy = - 2x cos y dx ➔ $\int \frac{\sec y - \sin y}{\cos y} dy = -2\int \frac{1}{x} dx + C$

➔ $\int (\sec^2 y - \tan y) dy = -2\ln x + C \quad \Rightarrow \tan y - \ln \sec y = -2\ln x + C$

(xiv) $(x+y)^2 (xy'+y) = xy(1+y')$

Solution: Given $(x+y)^2 (xy'+y) = xy(1+y')$

➔ $\frac{(xy'+y)}{xy} = \frac{(1+y')}{(x+y)^2}$. Integrating, we obtain : $\int \frac{(xy'+y)}{xy} dx = \int \frac{(1+y')}{(x+y)^2} dx$

➔ ln xy = -1/(x + y) + C

REMARK: In the first integral, we have used the formula: $\int \frac{f'(x)}{f(x)} dx = \ln f(x)$

And in the second integral, we have used the formula: $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1}$ ($n \neq -1$).

Equations Reducible To Separable Variable Form

Equations of the form

$$\frac{dy}{dx} = f(ax+by) \quad \text{or} \quad \frac{dy}{dx} = f(ax+by+c)$$

can be reduced to separable variables form on substituting ax + by = z or ax + by + c = z. On differentiating with respect to x, we get

$$a + b\frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right).$$

Then given equations becomes:

$$\frac{1}{b}\left(\frac{dz}{dx} - a\right) = f(z) \quad \text{or} \quad \frac{dz}{dx} - a = bf(z) \quad \text{or} \quad \frac{dz}{dx} = a + bf(z).$$

Separating the variables, we get: $\frac{dz}{a + bf(z)} = dx$

which can be integrated easily to get the required solution.

Example 05: Solve the differential equation y` = (4x + y + 1)²

Solution: Given that

$$y' = (4x + y + 1)^2 \tag{1}$$

Put 4x + y + 1 = z. Differentiating with respect to x, we get

$$4 + \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dz}{dx} - 4.$$

Therefore equation (1) becomes: $\frac{dz}{dx} - 4 = z^2$ or $\frac{dz}{dx} = 4 + z^2 \Rightarrow \frac{dz}{z^2+4} = dx$

Integrating, we get

$\int \frac{dz}{z^2+4} = \int 1.dx + c \Rightarrow \frac{1}{2}\tan^{-1}\frac{z}{2} = x + c.$ $\left(\text{Note}: \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c\right)$

Replacing z by 4x + y + 1, we get: $\frac{1}{2}\tan^{-1}\frac{4x+y+1}{2} = x + c$

This is the solution of given equation.

Example 06: Solve the differential equation $\frac{dy}{dx} = \cos(x+y)$

Solution: Put $z = x + y$ → $dz/dx = dy/dx + 1$ → $dy/dx = dz/dx - 1$.

Then given differential equation becomes:

$$\frac{dz}{dx} - 1 = \cos z \Rightarrow \frac{dz}{dx} = 1 + \cos z.$$

Separating the variables and integrating, we get:

$$\int \frac{dz}{1+\cos z} = \int 1.dx + c \Rightarrow \int \frac{dz}{2\cos^2 z/2} = x + c \Rightarrow \frac{1}{2}\int \sec^2(z/2)\, dz = x + c$$

$$\Rightarrow \frac{1}{2}\frac{\tan z/2}{1/2} = x + c \Rightarrow \tan\left(\frac{x+y}{2}\right) = x + c$$

Homogeneous Equations

Definition: A function f(x, y) is said to be **homogeneous function of degree n** if it can be expressed in the form:

$$f(x,y) = x^n f\left(\frac{y}{x}\right) \text{ or } f(tx,ty) = t^n f(x,y), \text{ where } t \neq 0.$$

For instance, let $f(x,y) = \frac{x^3+y^3}{x-y}$. Replacing x by tx and y by ty, we get

$$f(tx,ty) = \frac{t^3x^3 + t^3y^3}{tx - ty} = \frac{t^3(x^3+y^3)}{t(x-y)} = t^2\frac{x^3+y^3}{x-y} = t^2 f(x,y).$$

Thus, f(x, y) is a homogeneous function of degree 2.

Definition: A differential equation $\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)}$ where M(x, y) and N(x, y) are homogeneous functions of the same degree in x, y is called a homogeneous equation.

Since M(x, y) and N(x, y) are homogeneous functions of same degree in x, y.

Hence, $M(x,y) = x^n g\left(\frac{y}{x}\right)$ and $N(x,y) = x^n h\left(\frac{y}{x}\right)$

Then given differential equation becomes:

$$\frac{dy}{dx} = \frac{x^n g\left(\frac{y}{x}\right)}{x^n h\left(\frac{y}{x}\right)} = \frac{g\left(\frac{y}{x}\right)}{h\left(\frac{y}{x}\right)} = f\left(\frac{y}{x}\right) \text{ (say)} \qquad (1)$$

Put $y = vx$ → $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Then (1) becomes,

$$v + x\frac{dv}{dx} = f(v) \text{ or } x\frac{dv}{dx} = f(v) - v.$$

Separating the variables, we have $\frac{dv}{f(v)-v} = \frac{1}{x}dx.$

This can now easily be integrated. Finally, put v = y/x we get the required solution.

Here we produce the tips that will help to solve the homogeneous equation as given above.

- Put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

- Put the above values of y and dy/dx in given equation.
- Separate the variables and integrate.
- Replace v by y/x to get the required solution.

Example 01: Solve the differential equation $(x^2 + y^2)\,dx + 2xy\,dy = 0$

Solution: Give that $(x^2 + y^2)\,dx + 2xy\,dy = 0$

This can be re – written as: $\dfrac{dy}{dx} = -\dfrac{x^2+y^2}{2xy}$ (1)

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Then (1) becomes:

$$x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v \Rightarrow x\frac{dv}{dx} = \frac{-1-v^2-2v^2}{2v} \Rightarrow x\frac{dv}{dx} = -\frac{1+3v^2}{2v}.$$

Separating the variables, we get

$$\frac{2v}{1+3v^2}dv = -\frac{1}{x}dx \quad \text{or} \quad \frac{1}{3}\left(\frac{6v}{1+3v^2}\right)dv = -\frac{1}{x}dx\cdot$$

Integrating both the sides, we get: $\dfrac{1}{3}\displaystyle\int\left(\frac{6v}{1+3v^2}\right)dv = -\int\frac{1}{x}dx + \ln C'$

$$\frac{1}{3}\ln\left(1+3v^2\right) = -\ln x + \ln C' \Rightarrow \ln\left(1+3v^2\right) = -3\ln x + 3\ln C' = \ln\frac{C}{x^3}$$

Substituting v = y/x and notice that we have put 3ln C` = ln C.

$$\ln\left(\frac{x^2+3y^2}{x^2}\right) = \ln\frac{C}{x^3} \Rightarrow \left(\frac{x^2+3y^2}{x^2}\right) = \frac{C}{x^3} \Rightarrow x\left(\frac{x^2+3y^2}{x^2}\right) = C \Rightarrow x\left(x^2+3y^2\right) = Cx$$

$x = C$

This is the solution of given differential equation.

Example 02: Solve the differential equation $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0\cdot$

Solution: Given that $\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\dfrac{x}{y}\right)dy = 0\cdot$

This can be rewritten as:

$$\left(1+e^{x/y}\right)dx = -e^{x/y}\left(1-\frac{x}{y}\right)dy = 0 \Rightarrow \frac{dx}{dy} = \frac{-e^{x/y}\left(1-\frac{x}{y}\right)}{\left(1+e^{x/y}\right)} \quad (1)$$

Put $\dfrac{x}{y} = v \Rightarrow x = vy$, then $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$. [See the change in the problem]

Therefore from (1), we have

$$v + y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v} \Rightarrow y\frac{dv}{dy} = \frac{-e^v(1-v)}{1+e^v} - v$$

$$y\frac{dv}{dy} = \frac{-\left(e^v - e^v v + v + e^v v\right)}{1+e^v} \Rightarrow y\frac{dv}{dy} = -\frac{v+e^v}{1+e^v}.$$

Separating the variables, we have: $\frac{1+e^v}{v+e^v}dv=-\frac{dy}{y}$

Integrating both sides, we have

$$\int\frac{1+e^v}{v+e^v}dv=-\int\frac{1}{y}dy+\ln c \Rightarrow \ln\left(v+e^v\right)=-\ln y+\ln c=\ln\frac{c}{y}$$

$\rightarrow v+e^v=c/y$. Substituting $v=x/y$, we get: $\frac{x}{y}+e^{x/y}=\frac{c}{y}\Rightarrow x+ye^{x/y}=c$

Example 03: Solve the differential equation $x\sin\left(\frac{y}{x}\right)dy=\left[y\sin\left(\frac{y}{x}\right)-x\right]dx$

Solution: The given equation is: $x\sin\left(\frac{y}{x}\right)dy=\left[y\sin\left(\frac{y}{x}\right)-x\right]dx$.

Rewrite this as

$$\frac{dy}{dx}=\frac{\left[y\sin\left(\frac{y}{x}\right)-x\right]}{x\sin\left(\frac{y}{x}\right)}=\frac{x\left[\frac{y}{x}\sin\left(\frac{y}{x}\right)-1\right]}{x\sin\left(\frac{y}{x}\right)}\Rightarrow\frac{dy}{dx}=\frac{\frac{y}{x}\sin\left(\frac{y}{x}\right)-1}{\sin\left(\frac{y}{x}\right)} \qquad (1)$$

Put $y=vx$ then $\frac{dy}{dx}=v+x\frac{dv}{dx}$. Therefore from (1), we have

$$v+x\frac{dv}{dx}=\frac{v\sin v-1}{\sin v}\Rightarrow x\frac{dv}{dx}=\frac{v\sin v-1}{\sin v}-v=\frac{v\sin v-1-v\sin v}{\sin v}\Rightarrow x\frac{dv}{dx}=-\frac{1}{\sin v}.$$

Separating the variables, we get

$$-\sin v\,dv=\frac{1}{x}dx\Rightarrow-\int\sin v\,dv=\int\frac{1}{x}dx+c\Rightarrow-(-\cos v)=\ln x+c$$

$$\cos v=\ln x+c\Rightarrow\cos\left(\frac{y}{x}\right)=\ln x+c.$$ Hence the solution.

Example 4: Solve the following homogeneous equations:

(i) $(x-y)\,dx+(x+y)\,dy=0$

Solution: Rearranging the terms, we get: $\frac{dy}{dx}=\frac{y-x}{y+x}$ (1)

Put $y=vx$ $\rightarrow$ $y'=v+x\,dv/dx$. Thus equation (1) becomes:

$$v+x\frac{dv}{dx}=\frac{vx-x}{vx+x}=\frac{x(v-1)}{x(v+1)}=\frac{v-1}{v+1}\Rightarrow x\frac{dv}{dx}=\frac{v-1}{v+1}-v=\frac{v-1-v^2-v}{v+1}=-\frac{v^2+1}{v+1}.$$

Separating the variables and integrating, we get $\int\frac{v+1}{v^2+1}dv=-\int\frac{1}{x}dx+\ln C$

$$\Rightarrow\frac{1}{2}\int\frac{2v}{v^2+1}dv+\int\frac{1}{v^2+1}dv=-\int\frac{1}{x}dx+\ln C\Rightarrow\ln\sqrt{v^2+1}+\tan^{-1}v=-\ln x+\ln C$$

$\tan^{-1}v=\ln\frac{C}{x\sqrt{v^2+1}}$. Substituting $v=y/x$, we get: $\tan^{-1}\left(\frac{y}{x}\right)=\ln\frac{C}{\sqrt{y^2+x^2}}$

(ii) $(y^2+2xy)\,dx+x^2\,dy=0$

Solution: Rearranging the terms, we get: $\frac{dy}{dx}=-\frac{y^2+2xy}{x^2}$ (1)

Put $y = vx \rightarrow y' = v \cdot 1 + x\, dv/dx$. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = -\frac{v^2x^2 + 2x.vx}{x^2} = -\frac{x^2(v^2+2v)}{x^2} = -(v^2+2v)$$

$\Rightarrow x\frac{dv}{dx} = -v^2 - 2v - v = -(v^2+3v)$. Separating the variables and integrating:

$$\int \frac{1}{v(v+3)}\,dv = -\int \frac{1}{x}dx + \ln C \qquad (2)$$

Using partial fractions, we have: $\frac{1}{v(v+3)} = \frac{1}{3}\left(\frac{1}{v} - \frac{1}{v+3}\right)$.

Thus equation (2) becomes:

$$\Rightarrow \frac{1}{3}\left(\int \frac{1}{v}\,dv - \int \frac{1}{v+3}\,dv\right)dv = -\int \frac{1}{x}dx + \ln C \Rightarrow \ln v - \ln(v+3) = 3[-\ln x + \ln C]$$

$\Rightarrow \ln\frac{v}{v+3} = 3\ln\frac{C}{x} = \ln\left(\frac{C}{x}\right)^3 \Rightarrow \frac{v}{v+3} = \left(\frac{C}{x}\right)^3$. Substituting $v = y/x$ and simplifying, we get

$$\left(\frac{y}{y+3x}\right) = \left(\frac{C}{x}\right)^3 \quad \Rightarrow yx^3 = C^3(y+3x)$$

(iii) $(x^2 - 3y^2)\,dx + 2xy\,dy = 0$

Solution: Rearranging the terms, we get: $\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$ (1)

Put $y = vx \rightarrow y' = v \cdot 1 + x\, dv/dx$. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = \frac{3v^2x^2 - x^2}{2vx^2} = \frac{x^2(3v^2-1)}{2vx^2} = \frac{(3v^2-1)}{2v} \Rightarrow x\frac{dv}{dx} = \frac{(3v^2-1)}{2v} - v = \frac{3v^2-1-2v^2}{2v} = \frac{(v^2-1)}{2v}$$

Separating the variables and integrating, we get:

$$\int \frac{2v}{v^2-1}dv = \int \frac{1}{x}dx + \ln C \quad \Rightarrow \ln(v^2-1) = \ln Cx \quad \Rightarrow v^2 - 1 = Cx.$$

Substituting $v = y/x$, and simplifying, we get: $y^2 - x^2 = C\,x^3$

(iv) $(x^2 + xy + y^2)\,dx - x^2\,dy = 0$

Solution: Rearranging th terms, we get: $\frac{dy}{dx} = \frac{x^2 + xy + y^2}{x^2}$ (1)

Put $y = vx \rightarrow y' = v \cdot 1 + x\, dv/dx$. Thus equation (1) becomes:

$$v + x\frac{dv}{dx} = \frac{x^2 + x^2v + x^2v^2}{x^2} = \frac{x^2(1+v+v^2)}{x^2} = 1 + v + v^2 \qquad \Rightarrow x\frac{dv}{dx} = 1 + v + v^2 - v = 1 + v^2$$

Separating the variables and integrating, we get:

$$\int \frac{1}{1+v^2}dv = \int \frac{1}{x}dx + \ln C \quad \Rightarrow \tan^{-1} v = \ln Cx.$$

Substituting $v = y/x$, and simplifying, we get: $\tan^{-1}\left(\frac{y}{x}\right) = \ln Cx$

(v) $(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$

Solution: Rearranging the terms, we get: $\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$ (1)

Put $y = vx \rightarrow y' = v + x\, dv/dx$. Thus equation (1) becomes:

$$v+x\frac{dv}{dx}=\frac{x^2+3x^2v+x^2v^2}{x^2}=\frac{x^2(1+3v+v^2)}{x^2}=1+3v+v^2$$

$$\Rightarrow x\frac{dv}{dx}=1+3v+v^2-v=1+2v+v^2$$

Separating the variables and integrating, we get:

$$\int\frac{1}{(1+v)^2}dv=\int\frac{1}{x}dx+\ln C \quad\Rightarrow -\frac{1}{(1+v)}=\ln Cx.$$

Substituting v = y/x, and simplifying, we get $-\left(\frac{x}{x+y}\right)=\ln Cx$

(vi) dy/dx = (4y – 3x)/(2x – y) (1)

Solution: Put y = vx ➔ y' = v + x dv/dx. Thus equation (1) becomes:

$$v+x\frac{dv}{dx}=\frac{4vx-3x}{2x-vx}=\frac{x(4v-3)}{x(2-v)}=\frac{(4v-3)}{(2-v)} \qquad \Rightarrow x\frac{dv}{dx}=\frac{(4v-3)}{(2-v)}-v=\frac{4v-3-2v+v^2}{(2-v)}$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2+2v-3}{-(v-2)}=\frac{(v+3)(v-1)}{-(v-2)}$$

Separating the variables and integrating, we get:

$$\int\frac{v-2}{(v+3)(v-1)}dv=-\int\frac{1}{x}dx+\ln C \quad\Rightarrow\int\frac{v-2}{(v+3)(v-1)}dv=-\int\frac{1}{x}dx+\ln C$$

$$\int\frac{5/4}{(v+3)}dv+\int\frac{-1/4}{(v-1)}dv=-\int\frac{1}{x}dx+\ln C \quad\Rightarrow\frac{1}{4}(5\ln(v+3)-\ln(v-1))=\ln\frac{C}{x}$$

Substituting v = y/x, and simplifying, we get:

$$\frac{1}{4}\left(5\ln\left[\frac{y+3x}{x}\right]-\ln\left[\frac{y-x}{x}\right]\right)=\ln\frac{C}{x}$$

(vii) $\left(x^3+y^2\sqrt{x^2+y^2}\right)dx-xy\sqrt{x^2+y^2}dy=0$

Solution: Rewriting the equation, we get: $\frac{dy}{dx}=\frac{\left(x^3+y^2\sqrt{x^2+y^2}\right)}{xy\sqrt{x^2+y^2}}$ (1)

Substituting y = vx so that y' = v + x v'. Thus equation (1) becomes

$$v+x\frac{dv}{dx}=\frac{x^3+v^2x^2\sqrt{x^2+v^2x^2}}{x(vx)\sqrt{x^2+v^2x^2}}=\frac{x^3(1+v^2\sqrt{1+v^2})}{x^3v\sqrt{1+v^2}}=\frac{(1+v^2\sqrt{1+v^2})}{v\sqrt{1+v^2}}$$

$$x\frac{dv}{dx}=\frac{(1+v^2\sqrt{1+v^2})}{v\sqrt{1+v^2}}-v \quad\Rightarrow x\frac{dv}{dx}=\frac{(1+v^2\sqrt{1+v^2}-v^2\sqrt{1+v^2})}{v\sqrt{1+v^2}}=\frac{1}{v\sqrt{1+v^2}}$$

Separating the variables and integrating, we get:

$$\int v\sqrt{1+v^2}\,dv=\int\frac{1}{x}dx+\ln C\Rightarrow\frac{1}{2}\int\sqrt{1+v^2}\,(2v)dv=\int\frac{1}{x}dx+\ln C$$

$$\frac{2}{3}(1+v^2)^{3/2}=\ln Cx\Rightarrow 2(1+v^2)^{3/2}=3\ln Cx.$$

Substituting $v=y/x$, we get: $2\left(x^2+y^2\right)^{3/2}=3x^{3/2}\ln Cx$

(viii) $\left(\sqrt{x+y}+\sqrt{x-y}\right)dx-\left(\sqrt{x+y}-\sqrt{x-y}\right)dy=0$

Solution: Rearranging the equation, we get: $\frac{dy}{dx}=\frac{\sqrt{x+y}+\sqrt{x-y}}{\sqrt{x+y}-\sqrt{x-y}}$ (1)

Substituting $y=vx$ so that $y'=v+xv'$. Thus equation (1) becomes:

$$v+x\frac{dv}{dx}=\frac{\sqrt{x+vx}+\sqrt{x-vx}}{\sqrt{x+vx}-\sqrt{x-vx}}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}\Rightarrow x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}-v$$

$$x\frac{dv}{dx}=\frac{\sqrt{1+v}+\sqrt{1-v}-v\sqrt{1+v}+v\sqrt{1-v}}{\sqrt{1+v}-\sqrt{1-v}}=\frac{\sqrt{1+v}(1-v)+\sqrt{1-v}(1+v)}{\sqrt{1+v}-\sqrt{1-v}}$$

$$=\frac{\sqrt{1+v}\sqrt{1-v}\sqrt{1-v}+\sqrt{1-v}\sqrt{1+v}\sqrt{1+v}}{\sqrt{1+v}-\sqrt{1-v}}=\frac{\sqrt{1+v}\sqrt{1-v}\left(\sqrt{1-v}+\sqrt{1+v}\right)}{\sqrt{1+v}-\sqrt{1-v}}$$

Rationalizing, we get

$$x\frac{dv}{dx}=\frac{\sqrt{1+v}\sqrt{1-v}\left(\sqrt{1-v}+\sqrt{1+v}\right)}{\sqrt{1+v}-\sqrt{1-v}}\times\frac{\sqrt{1+v}+\sqrt{1-v}}{\sqrt{1+v}+\sqrt{1-v}}=\frac{\sqrt{1-v^2}\left(\sqrt{1-v}+\sqrt{1+v}\right)^2}{(1+v)-(1-v)}$$

$$=\frac{\sqrt{1-v^2}\left(1-v+1+v-2\sqrt{1-v^2}\right)}{1+v-1+v}=\frac{2\sqrt{1-v^2}\left(1-\sqrt{1-v^2}\right)}{2v}=\frac{\sqrt{1-v^2}\left(1-\sqrt{1-v^2}\right)}{v}$$

Separating the variables and integrating we get:

$$\int\frac{v}{\sqrt{1-v^2}\left(1-\sqrt{1-v^2}\right)}dv=\int\frac{1}{x}dx+\ln C=\ln Cx \qquad (2)$$

Substituting $v=\sin\theta$ ➔ $dv=\cos\theta\, d\theta$. Thus left side of equation (2) becomes:

$$\int\frac{\sin\theta}{\cos\theta\,(1-\cos\theta)}\cos\theta\, d\theta=\int\frac{\sin\theta}{(1-\cos\theta)}d\theta=\int\frac{2\sin\theta/2.\cos\theta/2}{2\sin^2\theta/2}d\theta=\int\cot\frac{\theta}{2}\,d\theta$$

Thus equation (2) becomes:

$$\int\cot\frac{\theta}{2}\,d\theta=\ln Cx\ \Rightarrow 2\ln(\sin\theta/2)=\ln Cx\ \Rightarrow \ln(\sin\theta/2)^2=\ln Cx\ \Rightarrow \sin^2\frac{\theta}{2}=Cx$$

$$\frac{1-\cos\theta}{2}=Cx\ \Rightarrow\ 1-\sqrt{1-\sin^2\theta}=2Cx\ \Rightarrow 1-\sqrt{1-v^2}=2Cx$$

Substituting $y=vx$, we get:

$$x-\sqrt{x^2-y^2}=2Cx^2\ \Rightarrow x-\sqrt{x^2-y^2}=C_1x^2. \qquad [\text{NOTE: } C_1=2C]$$

(ix) $\frac{dy}{dx}=\frac{x+y}{x},\ y(1)=1$

Solution: The given equation is homogeneous hence we put $y=vx$ so that $y'=v+xv'$. Thus given differential equation becomes:

$$v+x\frac{dv}{dx}=\frac{x+vx}{x}=1+v\ \Rightarrow x\frac{dv}{dx}=1+v-v=1.$$

Separating the variables and integrating, we get:

$$\int 1.dv=\int\frac{1}{x}dx+\ln C\Rightarrow v=\ln Cx\ \Rightarrow y/x=\ln Cx\qquad \Rightarrow y=x\ln Cx \qquad (1)$$

Now using the initial condition that is put $x=1$ and $y=1$, we obtain:

$1=\ln C$ ➔ $C=e^1=e$.

Thus equation (1) becomes: $y=x\ln ex$ ➔ $y=x\,[\ln x+\ln e]$

$\Rightarrow$ $y = x[\ln x + 1]$. This is the particular solution. [NOTE: $\ln e = 1$]

Equations Reducible to Homogeneous

The differential equation of the form: $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$

is not homogeneous. Method of solving such differential equation depends on the coefficients a_1, b_1, a_2 and b_2. We shall consider two cases:

Case I: Given:
$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \qquad (1)$$

If $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 \neq 0$, then it can be reduced to homogeneous form. This is explained as follows: Substituting $x = X + h$ and $y = Y + k$ (2)
where X and Y are new variables and h, k are constants to be chosen so that resulting equation in terms of X and Y becomes homogeneous.

From (2), we have $dx = dX,\ dy = dY \Rightarrow dy/dx = dY/dX$ (3)

Using (2) and (3), equation (1) becomes
$$\frac{dY}{dX} = \frac{a_1(X+h) + b_1(Y+k) + c_1}{a_2(X+h) + b_2(Y+k) + c_2} = \frac{a_1X + b_1Y + (a_1h + b_1k + c_1)}{a_2X + b_2Y + (a_2h + b_2k + c_2)} \qquad (4)$$

Now (4) becomes homogeneous if we put:
$$a_1h + b_1k + c_1 = 0 \ \text{ and } \ a_2h + b_2k + c_2 = 0 \qquad (5)$$

Solving (5), we get
$$h = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \ \text{ and } \ k = \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1} \qquad (6)$$

Given that $a_1b_2 - a_2b_1 \neq 0$, hence, h and k given by (6) are meaningful, that is; h and k exist. Now h and k are known, so in view of (5), equation (4) reduces to
$$\frac{dY}{dX} = \frac{a_1X + b_1Y}{a_2X + b_2Y}$$

which is homogeneous equation in X and Y and can be solved by putting $Y/X = v$ as usual. After getting solution in terms of X and Y, substituting $X = x - h$ and $Y = y - k$ and obtain solution in terms of original variables x and y.

Example 05: Solve the differential equation $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

Solution: The given equation is: $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$ (1)

[Here $a_1 = 1, a_2 = 2, b_1 = 2, b_2 = 1 \Rightarrow a_1b_2 - a_2b_1 = -3 \neq 0$]

Put $x = X + h$ and $y = Y + k$, therefore $dx = dX$, $dy = dY$. Now (1) becomes
$$\frac{dY}{dX} = \frac{X + h + 2Y + 2k - 3}{2X + 2h + Y + k - 3} = \frac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)} \qquad (2)$$

Choose h and k such that $h + 2k - 3 = 0$ and $2h + k - 3 = 0$. Multiplying first equation with 2 and subtracting second equation from the new equation, we get
$$3k - 3 = 0 \Rightarrow k = 1.$$

Substituting this value of k in the first equation, we get $h = 1$.

Thus,
$$k = h = 1 \qquad (3)$$

Now, from equation (2) we have

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y} \tag{4}$$

This is homogeneous equation.

Put $Y=vX$, then $\frac{dY}{dX}=v+X\frac{dv}{dX}$. Therefore, equation (4) becomes

$$v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}=\frac{1+2v}{2+v} \Rightarrow X\frac{dv}{dX}=\frac{1+2v}{2+v}-v=\frac{1+2v-2v-v^2}{2+v}=\frac{1-v^2}{2+v}$$

Separating the variables, we get

$$\frac{2+v}{(1-v)(1+v)}dv=\frac{1}{X}dX.$$

Integrating,
$$\int\frac{2+v}{(1-v)(1+v)}dv=\int\frac{1}{X}dX+c_1 \tag{5}$$

Left side of (5) will be solved by resolving into partial fractions.

$$\frac{2+v}{(1-v)(1+v)}=\frac{A}{1-v}+\frac{B}{1+v} \tag{6}$$

$$2+v=A(1+v)+B(1-v) \tag{7}$$

Put $1+v=0 \Rightarrow v=-1$ into (7), we get $B=1/2$

Put $1-v=0 \Rightarrow v=1$ into (7), we get $A = 3/2$. Thus, equation (6) becomes

$$\frac{2+v}{(1-v)(1+v)}=\frac{3}{2(1-v)}+\frac{1}{2(1+v)}$$

Integrating, $\int\frac{2+v}{(1-v)(1+v)}dv=-\frac{3}{2}\int\frac{-1}{1-v}dv+\frac{1}{2}\int\frac{1}{1+v}dv=-\frac{3}{2}\ln(1-v)+\frac{1}{2}\ln(1+v)$

Equation (5) becomes

$$\ln\left[\frac{1+v}{(1-v)^3}\right]=\ln(c_1X)^2 \Rightarrow \left[\frac{1+v}{(1-v)^3}\right]=cX^2 \qquad [(c_1)^2=c]$$

Replacing v by Y/X we get

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3}=cX^2 \Rightarrow \frac{\frac{X+Y}{X}}{\frac{(X-Y)^3}{X^3}}=cX^2 \Rightarrow \frac{X+Y}{(X-Y)^3}=c \Rightarrow X+Y=c(X-Y)^3.$$

Substituting $X=x-1$ and $Y=y-1$, we get

$$x-1+y-1=c(x-1-y+1)^3 \Rightarrow x+y-2=c(x-y)^3.$$

The solution of given equation.

Case II. Given $\frac{dy}{dx}=\frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$, where $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=a_1b_2-a_2b_1=0$.

This means either $a_1 = a_2$ and $b_1 = b_2$ or they are multiple of each other. In this case we substitute $z = a_1 x + b_1 y$. This will change above equation directly in separable variable form. The procedure is shown in the following example.

Example 06: Solve the differential equation $\frac{dy}{dx}=\frac{x+2y+1}{2x+4y+3}$

Solution: The given differential equation is

$$\frac{dy}{dx}=\frac{x+2y+1}{2x+4y+3} \text{ or } \frac{dy}{dx}=\frac{x+2y+1}{2(x+2y)+3} \qquad (1)$$

Put $x+2y=z$, then $1+2\frac{dy}{dx}=\frac{dz}{dx}$ or $\frac{dy}{dx}=\frac{1}{2}\left(\frac{dz}{dx}-1\right)$.

From equation (1), we get

$$\frac{1}{2}\left(\frac{dz}{dx}-1\right)=\frac{z+1}{2z+3} \Rightarrow \frac{dz}{dx}=\frac{2z+2}{2z+3}+1 \Rightarrow \frac{dz}{dx}=\frac{2z+2+2z+3}{2z+3}=\frac{4z+5}{2z+3}.$$

Separating the variables, we get

$$\frac{2z+3}{4z+5}dz=dx \text{ or } \left[\frac{1}{2}+\frac{1}{2(4z+5)}\right]dz=dx$$

Integrating, $\int\left[\frac{1}{2}+\frac{1}{2(4z+5)}\right]dz=\int 1dx+c_1 \Rightarrow \frac{1}{2}z+\frac{1}{8}\ln(4z+5)=x+c_1$

$$4z+\ln(4z+5)=8x+8c_1 \Rightarrow 4(x+2y)+\ln(4x+8y+5)=8x+c \quad (8c_1=c)$$

$$4x+8y-8x+\ln(4x+8y+5)=c \Rightarrow 4(2y-x)+\ln(4x+8y+5)=c$$

The solution of given equation.

Example 07: Solve the differential equation $\frac{dy}{dx}=\frac{x-2y+3}{2x-4y+5}$

Solution: The given differential equation is

$$\frac{dy}{dx}=\frac{x-2y+3}{2x-4y+5} \text{ or } \frac{dy}{dx}=\frac{(x-2y)+3}{2(x-2y)+5} \qquad (1)$$

Put $x-2y=z$, then $1-2\frac{dy}{dx}=\frac{dz}{dx}$ or $\frac{dy}{dx}=-\frac{1}{2}\left(\frac{dz}{dx}-1\right)$.

From equation (1), we have

$$\frac{1}{2}\left(1-\frac{dz}{dx}\right)=\frac{z+3}{2z+5} \Rightarrow \frac{dz}{dx}=1-\left(\frac{2z+6}{2z+5}\right) \Rightarrow \frac{dz}{dx}=\frac{2z+5-2z-6}{2z+5} \Rightarrow \frac{dz}{dx}=-\frac{1}{2z+5}.$$

Separating the variables, we have

$$(2z+5)dz=-dx \Rightarrow \int(2z+5)dz=-\int 1dx \Rightarrow z^2+5z=-x+c \Rightarrow z^2+5z+x=c \text{ Re}$$

placing z by $x-2y$, we have

$$(x-2y)^2+5(x-2y)+x=c \Rightarrow x^2-4xy+4y^2+5x-10y+x=c$$

$$\Rightarrow \quad x^2-4xy+4y^2+6x-10y=c.$$

Exact Differential Equations

We know that general form of first order differential equation is

$$\frac{dy}{dx}=f(x,y)=\frac{M(x,y)}{N(x,y)}$$

This may also be expressed as: $M(x, y)\,dx + N(x, y)\,dy = 0$ (1)

Readers are aware of the fact, that if f(x, y) be a function of two variables, x and y, and f(x, y) = c, then **total or exact differential** 'df' of f is defined by

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy=0 \qquad (2)$$

Comparing (1) and (2), we see that, $\frac{\partial f}{\partial x}=M(x,y)$ and $\frac{\partial f}{\partial y}=N(x,y)$.

This implies that: $\frac{\partial M}{\partial y}=\frac{\partial^2 f}{\partial y\partial x}, \frac{\partial N}{\partial x}=\frac{\partial^2 f}{\partial x\partial y}$.

Now we know that if first partial derivatives of f exist then: $\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial^2 f}{\partial x\partial y}$

$$\Rightarrow \quad \frac{\partial M}{\partial x}=\frac{\partial N}{\partial y}$$

Therefore, equation (1) is total or exact differential equation, if and only if,

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \quad \Rightarrow M_y=N_x \tag{3}$$

Equation (3) is the test for differential equation (1) to be exact.

Method of Solving Exact Differential Equation

When differential equation M(x, y) dx + N(x, y) dy = 0 is exact, that is when $M_y = N_x$ then it can be solved as depicted in the following example.

Example 01: Solve the differential equation

$$\left(2x^3-6x^2y+3xy^2\right)dx-\left(2x^3-3x^2y+y^3\right)dy=0$$

Solution: We have

$$\left(2x^3-6x^2y+3xy^2\right)dx-\left(2x^3-3x^2y+y^3\right)dy=0 \tag{1}$$

Here $\quad M=2x^3-6x^2y+3xy^2$ and $N=-\left(2x^3-3x^2y+y^3\right)$.

Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y}=-6x^2+6xy \text{ and } \frac{\partial N}{\partial x}=-6x^2+6xy.$$

Thus, $\quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, which implies that (1) is exact.

Now, $$\frac{\partial f}{\partial x}=M=2x^3-6x^2y+3xy^2 \tag{2}$$

And $$\frac{\partial f}{\partial y}=N=-2x^3+3x^2y-y^3 \tag{3}$$

We try to find f(x, y) from equations (2) and (3).

Integrating (2) with respect to x, holding y constant, we obtain

$$f(x,y)=\int Mdx=\int\left(2x^3-6x^2y+3xy^2\right)dx+g(y)$$

$$f(x,y)=\frac{x^4}{2}-2x^3y+\frac{3}{2}x^2y^2+g(y)$$

Here g(y) is constant of integration to be determined. To find g(y), we use the fact that the function f must also satisfy (3). Hence

$$\frac{\partial f}{\partial y}=-2x^3+3x^2y+\frac{dg}{dy}=N=-2x^3+3x^2y-y^3$$

$$\rightarrow \quad \frac{dg}{dy}=-y^3. \text{ Integrating, we get } g(y)=-\frac{1}{4}y^4$$

Thus, $$f(x,y)=\frac{x^4}{2}-2x^3y+\frac{3}{2}x^2y^2-\frac{y^4}{4}$$

Therefore, general solution of the given differential equation is

$$\frac{x^4}{2}-2x^3y+\frac{3}{2}x^2y^2-\frac{y^4}{4}=c.$$

An Alternative Method of Solving Exact Differential Equation

An alternative method which is more quick and easy to apply is presented here. For this the following three steps will help to find the solution of exact differential equation $Mdx+Ndy=0$.

- **I.** First integrate M with respect to x holding y constant.
- **II.** Integrate with respect to y those terms in N which do not contain x.
- **III.** The sum of the expressions so obtained equated to an arbitrary constant will be the solution.

Let us solve above differential equation by using this technique.

I. Integrate M w. r. t x keeping y constant

$$\int Mdx=\int(2x^3-6x^2y+3xy^2)dx=2\frac{x^4}{4}-6y\frac{x^3}{3}+3y^2\frac{x^2}{2}=\frac{x^4}{2}-2x^3y+\frac{3}{2}x^2y^2$$

II. Integrate those terms of N w.r.t y which, do not contain x:

$$-\int y^3dy=-y^4/4$$

III. Adding the above results and equate to a constant, we get:

$$\frac{x^4}{2}-2x^3y+\frac{3}{2}x^2y^2-\frac{y^4}{4}=c$$

This is the same solution as given above.

REMAK: The above three steps can be summarized as follows to get the solution of Exact Equation.

$$\int Mdx+\int(\text{term in N not containing } x)dy=\text{constant}$$

OR $$\int Ndy+\int(\text{term in M not containing } y)dx=\text{constant}$$

Example 02: Solve $(xy\sin xy-\cos xy-e^{2x})dx+(y^2-x^2\sin xy)dy=0$

Solution: Here $M=xy\sin xy-\cos xy-e^{2x}$ and $N=y^2-x^2\sin xy$.

Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y}=x(y\cos xy\times x+\sin xy)+\sin xy(x)-0\Rightarrow\frac{\partial M}{\partial y}=x^2y\cos xy+2x\sin xy.$$

$$\frac{\partial N}{\partial x}=x^2\cos xy(y)+\sin xy(2x)\Rightarrow\frac{\partial N}{\partial x}=x^2y\cos xy+2x\sin xy.$$

We see that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. This shows that given differential equation is exact.

Using the alternative method the solution of given differential equation is

$$\int Mdx+\int(\text{term in N not containing } x)\,dy=\text{constant}$$

$$\int(xy\sin xy-\cos xy-e^{2x})\,dx+\int(-y^2)\,dy=c$$

$$y\int x\sin xy\,dx-\int\cos xy\,dx-\int e^{2x}dx-\int y^2dy=c$$

$$y\left[\left(-x\frac{\cos xy}{y}\right)+\frac{1}{y}\int\cos xy\,dx\right]-\int\cos xy\,dx-\frac{1}{2}e^{2x}-\frac{1}{3}y^3=c \quad \text{[using by parts formula]}$$

$$-x\cos xy+\int\cos xy\,dx-\int\cos xy\,dx-\frac{1}{2}e^{2x}-\frac{1}{3}y^3=c$$

$-x\cos xy-\frac{1}{2}e^{2x}-\frac{1}{2}y^3=c \Rightarrow x\cos xy+\frac{1}{2}e^{2x}+\frac{1}{3}y^3=c_1$, $(c_1=-c)$.

REMARK: We can also start with

$\int N\,dy+\int(\text{term in M not containing y})\,dx=\text{constant}$

$\int\left(-y^2+x^2\sin xy\right)dy+\int\left(-e^{2x}\right)dx=\text{constant} \Rightarrow -\frac{1}{3}y^3+x^2\left(-\frac{\cos xy}{x}\right)-\frac{e^{2x}}{2}=c$

$-\frac{1}{3}y^3-x\cos xy-\frac{1}{2}e^{2x}=c$ or $x\cos xy+\frac{1}{2}e^{2x}+\frac{1}{3}y^3=c_1$, as before.

Example 03: Solve $\left(y\sec^2 x+\sec x\tan x\right)dx+(\tan x+2y)dy=0$

Solution: Here $M=y\sec^2 x+\sec x\tan x$ and $N=\tan x+2y$.
Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y}=\sec^2 x,\ \frac{\partial N}{\partial x}=\sec^2 x$$

Thus, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ the given differential equation is exact.

Now its solution is $\int M\,dx+\int(\text{term in N not containing x})\,dy=\text{constant}$,

$$\int\left(y\sec^2 x+\sec x\tan x\right)dx+\int(2y)\,dy=c \Rightarrow y\tan x+\sec x+y^2=c.$$

Example 04: Solve the following initial value problem

$$\left(2x\cos y+3x^2y\right)dx+\left(x^3-x^2\sin y-y\right)dy,\ y(0)=2$$

Solution: Here $M=2x\cos y+3x^2y$ and $N=x^3-x^2\sin y-y$.
Differentiating M with respect to y and N with respect to x partially, we get

$$\frac{\partial M}{\partial y}=-2x\sin y+3x^2 \text{ and } \frac{\partial N}{\partial x}=3x^2-2x\sin y.$$

Thus, $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x} \Rightarrow$ the given differential equation is exact.

Now the solution is: $\int M dx+\int(\text{term in N not containing x})dy=\text{constant}$,

$$\Rightarrow \int\left(2x\cos y+3x^2y\right)dx+\int(-y)dy=c \Rightarrow x^2\cos y+x^3y-\frac{1}{2}y^2=c.$$

Using the initial conditions, we have: $(0)\cos(2)+(0)(2)-2=c \Rightarrow c=-2$.

Thus particular solution of given equation is $x^2\cos y+x^3y-y^2/2=-2$

Example 05: Solve the following exact differential equations:

(i) $(3x^2+4xy)\,dx+(2x^2+2y)\,dy=0$

Solution: Here $M=(3x^2+4xy)$ and $N=(2x^2+2y)$. Now

$\frac{\partial M}{\partial y}=4x$ and $\frac{\partial N}{\partial x}=4x \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, hence given equation is exact. Its solution is:

$\int M dx+\int(\text{terms of N not containing x})\,dy=C$

→ $\int(3x^2+4xy)dx+\int 2y\,dy=C \Rightarrow x^3+2x^2y+y^2=C$

(ii) $(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$

Solution: Here $M = (2xy + y - \tan y)$ and $N = (x^2 - x\tan^2 y + \sec^2 y)$. Now

$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y = 2x - \tan^2 y$ and $\frac{\partial N}{\partial x} = 2x - \sec^2 y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$,

hence given equation is exact. Its solution is:

$\int M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

➔ $\int (2xy + y - \tan y)dx + \int \sec^2 y\, dy = C \Rightarrow x^2 y + xy - x\tan y + \tan y = C$

(iii) $\frac{x+y}{y-1}dx - \frac{1}{2}\left(\frac{x+1}{y-1}\right)^2 dy = 0$

Solution: Here $M = \frac{x+y}{y-1}$ and $N = -\frac{1}{2}\left(\frac{x+1}{y-1}\right)^2$. Now

$\frac{\partial M}{\partial y} = -\frac{x+1}{(y-1)^2}$ and $\frac{\partial N}{\partial x} = -\frac{x+1}{(y-1)^2} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

➔ $\int \frac{x+y}{y-1}dx - \frac{1}{2}\int \frac{1}{(y-1)^2}dy = C \Rightarrow \frac{1}{y-1}\left(\frac{x^2}{2} + xy\right) + \frac{1}{2(y-1)} = C$

(iv) $\frac{dy}{dx} = -\frac{ax+hy}{hx+by}$

Solution: Re-writing the given differential equation as:

$(ax + hy)\,dx + (hx + by)\,dy = 0$, we see that $M = ax + hy$ and $N = hx + by$. Now

$\frac{\partial M}{\partial y} = h$ and $\frac{\partial N}{\partial x} = h \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact. Its solution is:

$\int M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

➔ $\int (ax + hy)dx + \int by\, dy = C \Rightarrow \left(\frac{ax^2}{2} + hxy\right) + \frac{by}{2} = C$

(v) $(1 + \ln xy)dx + (1 + x/y)\,dy = 0$

Solution: Here $M = (1 + \ln xy) = 1 + \ln x + \ln y$ and $N = 1 + x/y$. Now

$\frac{\partial M}{\partial y} = 1/y$ and $\frac{\partial N}{\partial x} = 1/y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, hence given equation is exact. Its solution is:

$\int M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = C$

➔ $\int (1 + \ln x + \ln y)dx + \int 1\, dy = C \Rightarrow x + x\ln x - x + y = C \Rightarrow x\ln x - y = C$

(vi) $\frac{ydx + xdy}{1 - x^2y^2} + xdx = 0$

Solution: Re-writing given differential equation as:

$\left(\frac{y}{1-x^2y^2}+x\right)dx+\frac{x}{1-x^2y^2}dy=0$. Here $M=\left(\frac{y}{1-x^2y^2}+x\right)$ and $N=\frac{x}{1-x^2y^2}$. Now

$\frac{\partial M}{\partial y}=\frac{1+x^2y^2}{(1-x^2y^2)^2}, \frac{\partial N}{\partial x}=\frac{1+x^2y^2}{(1-x^2y^2)^2} \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M\,dx+\int(\text{terms of } N \text{ not containing } x)\,dy=C$.

REMARK: Since N has no term that is free from x, hence we skip this step.

$$\rightarrow \int\frac{y}{(1-x^2y^2)}dx+\int x\,dx=C \Rightarrow \frac{y}{y^2}\int\frac{1}{((1/y)^2-x^2)}dx+\frac{x^2}{2}=C$$

$$\frac{1}{y}\frac{1}{(2/y^2)}\ln\left(\frac{(1/y)+x}{(1/y)-x}\right)+\frac{x^2}{2}=C \Rightarrow y\ln\left(\frac{1+xy}{1-xy}\right)+x^2y=2C.$$

NOTE: $\int\frac{1}{(a^2-x^2)}dx=\frac{1}{2a}\ln[(a+x)/(a-x)]$

(vii) $(6xy+2y^2-5)\,dx+(3x^2+4xy-6)\,dy=0$

Solution: Here $M=(6xy+2y^2-5)$ and $N=(3x^2+4xy-6)$. Thus

$\frac{\partial M}{\partial y}=6x+4y$ and $\frac{\partial N}{\partial x}=6x+4y \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M\,dx+\int(\text{terms of } N \text{ not containing } x)\,dy=C$

$$\rightarrow \int(6xy+2y^2-5)\,dx+\int -6\,dy=C \quad \Rightarrow 3x^2y+2xy^2-5x-6y=C$$

(viii) $(y\cos x+2x\,e^y)\,dx+(\sin x+x^2e^y-1)\,dy=0$

Solution: Here $M=(y\cos x+2x\,e^y)$ and $N=(\sin x+x^2e^y-1)$. Thus

$\frac{\partial M}{\partial y}=\cos x+2xe^y$ and $\frac{\partial N}{\partial x}=\cos x+2xe^y \Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, hence given equation is exact.

Its solution is: $\int M\,dx+\int(\text{terms of } N \text{ not containing } x)\,dy=C$

$$\rightarrow \int(6xy+2y^2-5)\,dx+\int -6\,dy=C \quad \Rightarrow 3x^2y+2xy^2-5x-6y=C$$

(ix) $(y\,e^{xy}\cos 2x-2\,e^{xy}\sin 2x+2x)\,dx+(x\,e^{xy}\cos 2x-3)\,dy=0$

Solution: $M=(y\,e^{xy}\cos 2x-2\,e^{xy}\sin 2x+2x)$ and $N=(x\,e^{xy}\cos 2x-3)\,dy$. Thus

$\frac{\partial M}{\partial y}=e^{xy}[\cos 2x+xy\cos 2x-2x\sin 2x]$ and $\frac{\partial N}{\partial x}=e^{xy}[\cos 2x+xy\cos 2x-2x\sin 2x]$

$\Rightarrow \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, hence given equation is exact. Its solution is:

$\int M\,dx+\int(\text{terms of } N \text{ not containing } x)\,dy=C$

$$\rightarrow \int(ye^{xy}\cos 2x-2e^{xy}\sin 2x+2x)\,dx+\int -3\,dy=C$$

$$\rightarrow y\int ye^{xy}\cos 2x\,dx-2\int e^{xy}\sin 2x\,dx+2\int x\,dx-3\int 1\,dy=C$$

$$y\frac{e^{xy}}{y^2+4}[y\cos 2x+2\sin 2x]-2\frac{e^{xy}}{y^2+4}[y\sin 2x-2\cos 2x]+x^2-3y=C$$

NOTE: The following TWO formulae are used here.

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{(a^2+b^2)}[a\sin bx - b\cos bx]$ and

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{(a^2+b^2)}[a\cos bx + b\sin bx]$

Non-Exact Differential Equations

If equation:

$$M(x,y)dx + N(x,y)dy = 0,$$

is not exact, it may be possible to multiply it by a function $\mu(x,y)$ so that the resulting equation $\mu\, M(x,y)dx + \mu\, N(x,y)dy = 0$ is exact. Such a function $\mu(x,y)$ is called an integrating factor and we find the solution of the original equation by solving the new (exact) equation.

For instance, the equation

$$(3x+2y)\,dx + \left(x^2+x+xy\right)dy = 0 \tag{1}$$

is not exact because, with $M = 3x+2y, N = x^2+x+xy$, we have

$$\frac{\partial M}{\partial y} = 2,\ \frac{\partial N}{\partial x} = 2x+1+y \quad \Rightarrow \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

If we multiply given equation by $\mu(x,y) = xe^y$, we get

$$\left(3x^2+2xy\right)e^y dx + \left(x^3+x^2+x^2y\right)e^y dy = 0 \tag{2}$$

For this new equation, we have

$$M = \left(3x^2+2xy\right)e^y, N = \left(x^3+x^2+x^2y\right)e^y.$$

$$\frac{\partial M}{\partial y} = \left(3x^2+2xy\right)e^y + e^y(2x) = \left(3x^2+2x+2xy\right)e^y, \frac{\partial N}{\partial x} = \left(3x^2+2x+2xy\right)e^y,$$

$$\Rightarrow \qquad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Thus, equation (2) is exact and $\mu(x,y) = xe^y$ is an integrating factor.

We find solution of equation (2) by considering:

$$\int M dx + \int (\text{term in } N \text{ not containing } x)dy = \text{constant}$$

$$\int \left(3x^2+2xy\right)e^y + \int 0\,dy = c \Rightarrow x^3e^y + x^2ye^y = c$$

This is general solution.

Rules for Finding Integrating Factor

The problem of finding an integrating factor $\mu(x,y)$ for the equation $Mdx + Ndy = 0$ is generally difficult. However, in some special cases, it is easy to determine $\mu(x,y)$.

RULE 1: If $\frac{M_y - N_x}{N}$ is a function of x alone, say f(x) then $e^{\int f(x)dx}$ is an integrating factor of equation $M(x,y)\,dx + N(x,y)\,dy = 0$.

Example 01: Solve $\left(x^2+y^2\right)dx - 2xy\,dy = 0$

Solution: The given equation is: $\left(x^2+y^2\right)dx-2xy\,dy=0$ (1)

Here $M=x^2+y^2, N=-2xy$.

$\frac{\partial M}{\partial y}=2y, \frac{\partial N}{\partial x}=-2y \Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. This implies that (1) is not exact.

Now by **RULE 1,** $\frac{M_y-N_x}{N}=\frac{2y+2y}{-2xy}=\frac{4y}{-2xy}=-\frac{2}{x}=f(x)$.

Therefore, $\text{I.F.}=e^{\int f(x)dx}=e^{-2\int\frac{1}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=x^{-2}=\frac{1}{x^2}$.

Multiplying (1) by this I.F., we get

$$\left(1+\frac{y^2}{x^2}\right)dx-\frac{2y}{x}dy=0 \tag{2}$$

For the new equation, we have: $M=1+\frac{y^2}{x^2}, N=-\frac{2y}{x}$.

$\frac{\partial M}{\partial y}=\frac{2y}{x^2}, \frac{\partial N}{\partial x}=\frac{2y}{x^2}\Rightarrow\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. This shows that equation (2) is exact.

Now the solution is found by using the formula:

$$\int M dx+\int(\text{term in N not containing } x)dy=\text{constant}$$

$$\int\left(1+\frac{y^2}{x^2}\right)dx+\int 0\,dy=c\Rightarrow x+y^2\left(-\frac{1}{x}\right)=c\Rightarrow x^2-y^2=cx$$

RULE 2: If $\frac{N_x-M_y}{M}$ is a function of y alone, say g(y) then $e^{\int g(y)dy}$ is an integrating factor of the equation $M(x,y)dx+N(x,y)dy=0$.

Example 02: Solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$

Solution: The given equation is

$$\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0 \tag{1}$$

Here $M=y^4+2y, N=xy^3+2y^4-4x$.

$\frac{\partial M}{\partial y}=4y^3+2, \frac{\partial N}{\partial x}=y^3-4\Rightarrow\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\Rightarrow$ (1) is not exact.

Now by **RULE 2,** $\frac{N_x-M_y}{M}=\frac{y^3-4-4y^3-2}{y^4+2y}=\frac{-3\left(y^3+2\right)}{y\left(y^3+2\right)}=-\frac{3}{y}$.

Therefore, $\text{I.F.}=e^{-3\int\frac{1}{y}dy}=e^{-3\ln y}=e^{\ln y^{-3}}=y^{-3}=\frac{1}{y^3}$.

Multiplying (1) by I.F., we get

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0 \tag{2}$$

For the new equation, we have

$$M = y + \frac{2}{y^2}, N = x + 2y - \frac{4x}{y^3}.$$

$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}, \frac{\partial N}{\partial x} = 1 - \frac{4}{y^3} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow (2)$ is exact. Now the solution is:

$$\int M dx + \int (\text{term in N not containing } x) dy = \text{constant}$$

$$\int \left(y + \frac{2}{y^2} \right) dx + \int 2y dy = c \Rightarrow xy + \frac{2x}{y^2} + y^2 = c.$$

This is solution of given differential equation.

RULE 3: If $xM - yN \neq 0$ and the equation $Mdx + Ndy = 0$ has the form $yf(xy)dx + xg(xy)dy = 0$, then $\frac{1}{xM - yN}$ is an integrating factor.

Example 03: Solve $(3y + 4xy^2) dx + (2x + 3x^2y) dy = 0$

Solution: The given differential equation is

$$(3y + 4xy^2) dx + (2x + 3x^2y) dy = 0 \quad (1)$$

Here $M = 3y + 4xy^2$, $N = 2x + 3x^2y$.

$$\frac{\partial M}{\partial y} = 3 + 8xy, \frac{\partial N}{\partial x} = 2 + 6xy \Rightarrow \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow (1) \text{ is not exact.}$$

But equation (1) can be re – written as: $y(3 + 4xy)dx + x(2 + 3xy)dy = 0$

This if of the form: y f(x y) dx + x g(x y) dy = 0.

Now, $xM - yN = 3xy + 4x^2y^2 - 2xy - 3x^2y^2 = xy + x^2y^2 \neq 0$

Thus by Rule 3, $\frac{1}{xy + x^2y^2}$ is an I.F.

Multiplying (1) by I.F., we get

$$\frac{(3+4xy)y}{(1+xy)xy} dx + \frac{(2+3xy)x}{(1+xy)xy} dy = 0 \Rightarrow \frac{(3+4xy)}{(1+xy)x} dx + \frac{(2+3xy)}{(1+xy)y} dy = 0 \quad (2)$$

Equation (2) is an exact differential equation. It may now be written as

$$\frac{2+2xy+1+2xy}{(1+xy)x} dx + \frac{2+2xy+xy}{(1+xy)y} dy = 0$$

$$\Rightarrow \left[\frac{2(1+xy)}{(1+xy)x} + \frac{1+2xy}{(1+xy)x} \right] dx + \left[\frac{2(1+xy)}{(1+xy)y} + \frac{xy}{(1+xy)y} \right] dy = 0$$

$$\text{or} \left[\frac{2}{x} + \frac{1+xy}{(1+xy)x} + \frac{xy}{(1+xy)x} \right] dx + \left[\frac{2}{y} + \frac{x}{(1+xy)} \right] dy = 0$$

$$\rightarrow \left[\frac{2}{x} + \frac{1}{x} + \frac{y}{(1+xy)} \right] dx + \left[\frac{2}{y} + \frac{x}{(1+xy)} \right] dy = 0 \quad (3)$$

Now the solution is: $\int M dx + \int (\text{term in N not containing } x) dy = c$

$$\int\left[\frac{2}{x}+\frac{1}{x}+\frac{y}{(1+xy)}\right]dx+\int\frac{2}{y}dy=c\Rightarrow 3\int\frac{1}{x}dx+\int\frac{y}{(1+xy)}dx+2\int\frac{1}{y}dy=c$$

$3\ln x+\ln(1+xy)+2\ln y=c\Rightarrow \ln x^3+\ln y^2+\ln(1+xy)=\ln k \quad (c=\ln k)$

Or $\ln x^3 y^2(1+xy)=\ln k$ ➔ $x^3y^2(1+xy)=k$ is the solution.

RULE 4: If equation $Mdx+Ndy=0$ is homogeneous in x and y, that is; if M, N are homogeneous functions of same degree in x and y, then $1/(xM+yN)$ is an integrating factor.

Example 04: Solve $(3xy+y^2)dx+(x^2+xy)dy=0$

Solution: The given equation is $(3xy+y^2)dx+(x^2+xy)dy=0$ (1)

Here $M=3xy+y^2, N=x^2+xy$.

$\frac{\partial M}{\partial y}=3x+2y, \frac{\partial N}{\partial x}=2x+y\Rightarrow \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}\Rightarrow (1)$ is not exact.

But (1) is homogeneous of degree 2. Therefore

$$I.F.=\frac{1}{xM+yN}=\frac{1}{3x^2y+xy^2+x^2y+xy^2}=\frac{1}{4x^2y+2xy^2}=\frac{1}{2xy(2x+y)}.$$

Multiplying (1) by I.F, we get

$$\frac{(3xy+y^2)}{2xy(2x+y)}dx+\frac{(x^2+xy)}{2xy(2x+y)}dy=0\Rightarrow\frac{3x+y}{2x(2x+y)}dx+\frac{x+y}{2y(2x+y)}dy=0 \quad (2)$$

This is an exact equation. Now (2) may be written as:

$$\frac{2x+y+x}{2x(2x+y)}dx+\frac{2x+y-x}{2y(2x+y)}dy=0$$

$$\Rightarrow\left[\frac{2x+y}{2x(2x+y)}+\frac{x}{2x(2x+y)}\right]dx+\left[\frac{2x+y}{2y(2x+y)}-\frac{x}{2y(2x+y)}\right]dy=0$$

$$\Rightarrow\left[\frac{1}{2x}+\frac{1}{2(2x+y)}\right]dx+\left[\frac{1}{2y}-\frac{x}{2y(2x+y)}\right]dy=0 \quad (3)$$

Now the solution is $\int Mdx+\int(\text{term in N not containing } x)dy=c$

$$\Rightarrow\int\left[\frac{1}{2x}+\frac{1}{2(2x+y)}\right]dx+\int\frac{1}{2y}dy=c\Rightarrow\frac{1}{2}\int\frac{1}{x}dx+\frac{1}{2}\cdot\frac{1}{2}\int\frac{2}{2x+y}dx+\frac{1}{2}\int\frac{1}{y}dy=c$$

or $\frac{1}{2}\ln x+\frac{1}{4}\ln(2x+y)+\frac{1}{2}\ln y=c\Rightarrow 2\ln x+\ln(2x+y)+2\ln y=4c$

➔ $\ln x^2+\ln(2x+y)+\ln y^2=\ln k$

Or $\ln x^2(2x+y)y^2=\ln k$ ➔ $x^2+y2(2x+y)=k$.

This is the solution of given equation.

RULE 5: For appropriate values of m and n, $x^m y^n$ may be an integrating factor of equation $Mdx+Ndy=0$. The procedure of finding m and n is illustrated in the following examples.

Example 05: Solve $\left(y^3-2yx^2\right)dx+\left(2xy^2-x^3\right)dy=0$

Solution: The given equation is

$$\left(y^3-2yx^2\right)dx+\left(2xy^2-x^3\right)dy=0 \tag{1}$$

Here $M=y^3-2yx^2,\ N=2xy^2-x^3$.

$$\frac{\partial M}{\partial y}=3y^2-2x^2,\ \frac{\partial N}{\partial x}=2y^2-3x^2 \Rightarrow (1) \text{ is not exact.}$$

Let $x^m y^n$ be an integrating factor of (1). Then equation (1) becomes

$$x^m y^n\left(y^3-2yx^2\right)dx+x^m y^n\left(2xy^2-x^3\right)dy=0 \text{ is exact.}$$

Hence, $$\frac{\partial}{\partial y}\left[x^m y^n\left(y^3-2yx^2\right)\right]=\frac{\partial}{\partial x}\left[x^m y^n\left(2xy^2-x^3\right)\right]$$

$$\rightarrow x^m\left\{y^n\left(3y^2-2x^2\right)+\left(y^3-2yx^2\right)ny^{n-1}\right\}=y^n\left\{x^m\left(2y^2-3x^2\right)+\left(2xy^2-x^3\right)mx^{m-1}\right\}$$

$$\rightarrow x^m\left(3y^{n+2}-2x^2y^n+2y^{n+2}-2ny^nx^2\right)=y^n\left(2x^my^2-3x^{m+2}+2mx^my^2-mx^{m+2}\right)$$

$$\rightarrow (n+3)x^my^{n+2}-2(n+1)x^{m+2}y^n=2(m+1)x^my^{n+2}-(m+3)x^{m+2}y^n$$

Equating the coefficients of like powers of x and y, we obtain

$$n+3=2(m+1) \quad \text{and} \quad 2(n+1)=m+3.$$

Solving these equations simultaneously, we get $m=n=1$.

Thus, the integrating factor is xy. Therefore equation (1) when multiplied by x y gives:

$$\left(xy^4-2y^2x^3\right)dx+\left(2x^2y^3-x^4y\right)dy=0 \tag{2}$$

which is exact, because $\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}=4xy^3-4x^3y$

The solution of (2) is:

$$\int Mdx+\int(\text{term in N not containing x})dy=\text{constant}$$

$$\int\left(xy^4-2y^2x^3\right)dx+\int 0dy=c\Rightarrow\frac{1}{2}x^2y^4-\frac{2}{4}x^4y^2+c_1=c$$

$$x^2y^4-x^4y^2=k\Rightarrow x^2y^2\left(y^2-x^2\right)=k\cdot\ \left[2(c-c_1)=k\right]$$

Example 06: Solve the following differential equations by finding appropriate Integrating Factor.

(i) $(x\,y^2+y)\,dx-x\,dy=0$ (1)

Solution: Here $M=(x^2\,y+y)$ and $N=-x$. Thus

$\dfrac{\partial M}{\partial y}=2xy+1$ and $\dfrac{\partial N}{\partial x}=-1$. We see that $\dfrac{\partial M}{\partial y}\neq\dfrac{\partial N}{\partial x}$ so given equatuion is not exact.

To find an I.F we apply:

RULE I: $\dfrac{M_y-N_x}{N}=\dfrac{2xy+1+1}{-x}\neq P(x)$. Hence Rule I fails.

RULE II: $\dfrac{N_x - M_y}{M} = \dfrac{-1-2xy-1}{y(xy+1)} = \dfrac{-2(xy+1)}{y(xy+1)} = \dfrac{-2}{y} = P(y)$. Thus

$$I.F = e^{\int P(y)dy} = e^{-2\int dy/y} = e^{-2\ln y} = e^{\ln y^{-2}} = y^{-2} = 1/y^2$$

Multiply (1) by I.F, we get: $\dfrac{1}{y^2}\left(xy^2 + y\right)dx - \dfrac{x}{y^2}dy = 0$ (2)

Equation (2) is now exact. Its solution is:

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$\int (x + 1/y)\,dx = C \Rightarrow \frac{x^2}{2} + \frac{x}{y} = C \Rightarrow x^2y + 2x = 2Cy$$

NOTE: N contains no term free of x.

(ii) $x\,dy - y\,dx = (x^2 + y^2)\,dx$

Solution: Given differential equation may be rewritten as: $\dfrac{xdy - ydx}{x^2 + y^2} = dx$

This is equivalent to: $d\left(\tan^{-1}\dfrac{x}{y}\right) = dx$

Now integrating both sides, we obtain: $\tan^{-1}\dfrac{x}{y} = x + c$.

This is the solution of given differential equation.

(iii) $(x^2 + x - y)\,dx + x\,dy = 0$ (1)

Solution: Here $M = (x^2 + x - y)$ and $N = x$. Thus

$\dfrac{\partial M}{\partial y} = -1$ and $\dfrac{\partial N}{\partial x} = 1$. We see that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Hence given equatuion is not exact.

To find an I.F we apply:

RULE I: $\dfrac{M_y - N_x}{N} = \dfrac{-1-1}{x} = \dfrac{-2}{x} = P(x)$. Thus

$$I.F = e^{\int P(x)dx} = e^{-2\int dx/x} = e^{-2\ln x} = e^{\ln x^{-2}} = x^{-2} = 1/x^2$$

Multiply (1) by I.F, we get: $\dfrac{1}{x^2}\left(x^2 + x - y\right)dx + \dfrac{1}{x}dy = 0$ (2)

Equation (2) is now exact. Its solution is:

$$\int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C$$

$$\int (1 + 1/x - yx^{-2})\,dx = C \Rightarrow x + \ln x + \frac{y}{x} = C \Rightarrow x^2 + x\ln x + y = Cx$$

NOTE: N contains no term free of x.

(iv) $dy + \dfrac{y - \sin x}{x}dx = 0$ (1)

Solution: Here $M = y/x - \sin x/x$ and $N = 1$. Thus

$\dfrac{\partial M}{\partial y} = 1/x$ and $\dfrac{\partial N}{\partial x} = 0$. We see that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$. Hence given equat ion is not exact.

To find an I.F we apply:

RULE I: $\frac{M_y - N_x}{N} = \frac{1/x - 0}{1} = \frac{1}{x} = P(x)$. Thus, $I.F = e^{\int P(x)dx} = e^{\int dx/x} = e^{\ln x} = e^{\ln x} = x$

Multiply (1) by I.F, we get : $(y - \sin x)\,dx + x\,dy = 0$ (2)

Equation (2) is now exact. Its solution is :

$\int M dx + \int(\text{terms of } N \text{ not containing } x)\,dy = C$

$y\int 1.dx - \int \sin x\,dx = C \quad \Rightarrow yx + \cos x = C$ [NOTE : N contains no term free of x] (

(v) $y(2xy + e^x)\,dx - e^x\,dy = 0$ (1)

Solution: Here $M = 2xy^2 + y e^x$ and $N = -e^x$. Thus

$\frac{\partial M}{\partial y} = 4xy + e^x$ and $\frac{\partial N}{\partial x} = -e^x$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. Hence given equatuion is not exact.

To find an I.F we apply :

RULE I: $\frac{M_y - N_x}{N} = \frac{4xy + e^x + e^x}{-e^x} \neq P(x)$. Hence RULE I fails.

RULE II: $\frac{N_x - M_y}{M} = \frac{-e^x - 4xy - e^x}{y(2xy + e^x)} = -2\frac{(2xy + e^x)}{y(2xy + e^x)} = -2/y = P(y)$.

$2\int x.dx + y^{-2}\int e^x\,dx = C \quad \Rightarrow x^2 + e^x y^{-2} = C \Rightarrow x^2y^2 + e^x = Cy^2$

NOTE : N contains no term free from x.

(vi) $(x^2 + y^2 + 2x)\,dx + 2y\,dy = 0$ (1)

Solution: Here $M = x^2 + y^2 + 2x$ and $N = 2y$. Thus

$\frac{\partial M}{\partial y} = 2y$ and $\frac{\partial N}{\partial x} = 0$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply :

RULE I: $\frac{M_y - N_x}{N} = \frac{2y - 0}{2y} = 1 = P(x)$.

Thus, $I.F = e^{\int 1\,dx} = e^x$. Multiply eq(1) by I.F, we get :

$e^x(x^2 + y^2 + 2x)\,dx + 2e^x y\,dy = 0$ (2)

Equation (2) is now exact. Its solution is :

$\int M dx + \int(\text{terms of } N \text{ not containing } x)\,dy = C$

$\int(x^2 + 2x)e^x\,dx + y^2\int e^x\,dx = C \quad \Rightarrow (x^2 + 2x)e^x - \int(2x+2)e^x dx + y^2e^x = C$

$\Rightarrow (x^2 + 2x)e^x - [(2x+2)e^x - \int 2e^x dx + y^2e^x = C$

$\Rightarrow (x^2 + 2x)e^x - (2x+2)e^x + 2e^x + y^2e^x = C$

NOTE : N contains no term free from x.

Thus solution of given differential equation is: $(x^2 + y^2)e^x = C$

(vii) $(4x + 3y^2)\,dx + 2xy\,dy = 0$ (1)

Solution: Here $M = 4x + 3y^2$ and $N = 2xy$. Thus

$\frac{\partial M}{\partial y}=6y$ and $\frac{\partial N}{\partial x}=2y$. We see that $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

RULE I: $\frac{M_y-N_x}{N}=\frac{6y-2y}{2xy}=2/x=P(x)$.

Thus, $I.F=e^{2\int 1/x\,dx}=e^{2\ln x}=e^{\ln x^2}=x^2$. Multiply (1) by I.F, we get:

$(4x^3+3x^2y^2)dx+2x^3y\,dy=0$ (2)

Equation (2) is now exact. Its solution is:

$\int M\,dx+\int(\text{terms of } N \text{ not containing } x)\,dy=C$

$4\int x^3dx+3y^2\int x^2\,dx=C \Rightarrow x^4+x^3y^2=C$ [NOTE: N contains no term free of x]

(viii) $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$

Solution: Here $M=(3x^2y^4+2xy)$ and $N=(2x^3y^3-x^2)$. Thus (1)

$\frac{\partial M}{\partial y}=12x^2y^3+2x$ and $\frac{\partial N}{\partial x}=6x^2y^3-2x$. We see that $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

RULE I: $\frac{M_y-N_x}{N}=\frac{12x^2y^3+2x-6x^2y^3+2x}{x^2(2xy^3-1)}=\frac{6x^2y^3+4x}{x^2(2xy^3-1)}\neq P(x)$.RULE I fails.

RULE II: $\frac{N_x-M_y}{M}=\frac{6x^2y^3-2x-12x^2y^3-2x}{xy(3xy^3+2)}=\frac{-6x^2y^3-4x}{xy(3xy^3+2)}$

$$=\frac{-2x(3x^2y^3+2)}{xy(3xy^3-1)}=\frac{-2}{y}=P(y)$$

Thus, $I.F=e^{-2\int 1/y\,dy}=e^{-2\ln y}=e^{\ln y^{-2}}=y^{-2}$. Multiply (1) by I.F, we get:

$(3x^2y^2+2xy^{-1})dx+(2x^3y-x^2y^{-2})dy=0$ (2)

Equation (2) is now exact. Its solution is:

$\int M\,dx+\int(\text{terms of } N \text{ not containing } x)\,dy=C$

$\int(3x^2y^2+2xy^{-1})dx=C \Rightarrow x^3y^2+x^2/y=C$ [NOTE: N contains no term free of x]

(ix) $y-x\,y`=x+y\,y`$

Solution: Re-writing the given differential equation, we obtain:

$(x-y)\,dx+(x+y)\,dy=0$ (1)

This is a homogeneous equation where $M=x-y$ and $N=x+y$. Now

$\frac{\partial M}{\partial y}=-1$ and $\frac{\partial N}{\partial x}=1$. We see that $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

Hence given equation is not exact. To find an I.F we have

RULE III: $I.F=\frac{1}{xM+yN}=\frac{1}{x^2-xy+xy+y^2}=\frac{1}{x^2+y^2}$

Now multiply equation (1) by an I.F, we obtain:

$\frac{x-y}{x^2+y^2}dx+\frac{x+y}{x^2+y^2}dy=0$ (2)

Equation (2) is now exact. Its solution is obtained as follows:

$\int M dx+\int(\text{terms of } N \text{ not containing } x)\, dy = C$

$\Rightarrow \int\frac{x}{x^2+y^2}dx-\int\frac{y}{x^2+y^2}dx=C \Rightarrow \frac{1}{2}\int\frac{2x}{x^2+y^2}dx-y.\frac{1}{y}\tan^{-1}\left(\frac{x}{y}\right)=C$

$\frac{1}{2}\ln(x^2+y^2)-\tan^{-1}\left(\frac{x}{y}\right)=C.$

NOTE: (i) Here y is treated as constant (ii) N contains no term free of x.

(iii) $\int\frac{1}{x^2+a^2}dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

(x) $y' = e^{2x} + y - 1 = 0$

Solution: Given equation may be re-written as: $dy = (e^{2x} + y - 1)\, dx$

Or $(1- y - e^{2x})\, dx - dy = 0$ (1)

Here $M = (e^{2x} + y - 1)$ and $N = -1$. Thus

$\frac{\partial M}{\partial y}=1$ and $\frac{\partial N}{\partial x}=-1$. We see that $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

RULE I: $\frac{M_y-N_x}{N}=\frac{1+1}{(-1)}=-2=P(x).$

Thus, $I.F=e^{\int P(x)dx}=e^{-2\int 1.dx}=e^{-2x}$. Multiply eq(1) by I.F, we get:

$e^{-2x}(1-y-e^{2x})dx-e^{-2x}dy=0$ (2)

Equation (2) is now exact. Its solution is:

$\int M dx+\int(\text{terms of } N \text{ not containing } x)\, dy = C$

$\int e^{-2x}dx-y\int e^{-2x}dx-\int 1dx=C \Rightarrow -\frac{e^{-2x}}{2}(1-y)-x=C$

NOTE: N contains no term free from x.

(xi) $(y^2 + xy)\, dx - x^2\, dy = 0$

Solution: Given equation is a homogeneous equation where $M = (y^2 + xy)$ and $N = -x^2$. Now,

$\frac{\partial M}{\partial y}=2y+x$ and $\frac{\partial N}{\partial x}=2x$. We see that $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

RULE III: $I.F=\frac{1}{xM+yN}=\frac{1}{xy^2+x^2y-yx^2}=\frac{1}{xy^2}$

Now multiply equation (1) by an I.F, we obtain:

$\left(\frac{1}{x}+\frac{1}{y}\right)dx-\frac{x}{y}dy=0$ (2)

Equation (2) is now exact. Its solution is obtained as follows:

$\int M dx+\int(\text{terms of } N \text{ not containing } x)\, dy = C$

$\Rightarrow \int \frac{1}{x}dx + \frac{1}{y}\int dx = C \Rightarrow \ln x + \frac{x}{y} = C$ [NOTE: N contains no term free of x]

(xii) $(3x^2y + 2xy + y^3)\,dx + (x^2 + y^2)\,dy = 0$

Solution: Here $M = (3x^2y + 2xy + y^3)$ and $N = (x^2 + y^2)$. Thus (1)

$\frac{\partial M}{\partial y} = 3x^2 + 2x + 3y^2$ and $\frac{\partial N}{\partial x} = 2x$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

RULE I: $\frac{M_y - N_x}{N} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2 + y^2} = \frac{3(x^2 + y^2)}{(x^2 + y^2)} = 3 = P(x)$.

Thus, $I.F = e^{\int P(x)dx} = e^{3x}$. Multiply eq(1) by I.F, we get:

$e^{3x}(3x^2y + 2xy + y^2)dx + e^{3x}(x^2 + y^2)dy = 0$

Equation (2) is now exact. Its solution is: (2)

$\int M\,dx + \int(\text{terms of } N \text{ not containing } x)\,dy = C$

$3y\int e^{3x}x^2dx + 2y\int xe^{3x}dx + y^2\int e^{3x}dx = C$ [NOTE: N contains no term free of x]

Integrating by parts and simplifying, we obtain the solution:

$$e^{3x}(x^2y + y^3/3) = C$$

(xiii) $y\,dx + (2xy - e^{-2y})\,dy = 0$

Solution: Here $M = y$ and $N = (2xy - e^{-2y})$. Thus

$\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 2y$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply:

RULE I: $\frac{M_y - N_x}{N} = \frac{1 - 2y}{2xy - e^{-2y}} \neq P(x)$.RULE I fails.

RULE II: $\frac{N_x - M_y}{M} = \frac{2y - 1}{y} = (2 - 1/y) = P(y)$

Thus, $I.F = e^{\int P(y)dy} = e^{2\int dy - \int 1/y\,dy} = e^{2y - \ln y} = e^{2y}.e^{-\ln y} = y^{-1}e^{2y}$.

Multiply eq(1) by I.F, we get: $e^{2y}dx + (2xe^{2y} - y^{-1})dy = 0$

Equation (2) is now exact. Its solution is: (2)

$\int M\,dx + \int(\text{terms of } N \text{ not containing } x)\,dy = C$

$e^{2y}\int 1.dx - \int 1/y\;dy = C \Rightarrow xe^{2y} - \ln y = C$

(xiv) $e^x\,dx + (e^x\cot y + 2y\,\mathrm{cosec}\,y)\,dy = 0$

Solution: Here $M = e^x$ and $N = (e^x\cot y + 2y\,\mathrm{cosec}\,y)$. Thus

$\frac{\partial M}{\partial y} = 0$ and $\frac{\partial N}{\partial x} = e^x\cot y$. We see that $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Hence given equation is not exact. To find an I.F we apply

RULE I: $\frac{M_y - N_x}{N} = \frac{0 - e^x\cot y}{e^x\cot y + 2y\,\mathrm{cosec} y} \neq P(x)$.RULE I fails.

RULE II: $\dfrac{N_x - M_y}{M} = \dfrac{e^x \cot y - 0}{e^x} = \cot y = P(y)$

Thus, $I.F = e^{\int P(y)dy} = e^{\int \cot y dy} = e^{\ln \sin y} = \sin y.$

Multiply (1) by I.F, we get : $e^x \sin y dx + (e^x \cos y + 2y)dy = 0$ (2)

Equation (2) is now exact. Its solution is :

$\int M dx + \int (\text{terms of } N \text{ not containing } x)\ dy = C$

$\sin y \int e^x dx + 2\int y\ dy = C \Rightarrow \sin y e^x + y^2 = C$

(xv) $(x + 2) \sin y\ dx + x \cos y\ dy = 0$

Solution: Here $M = (x + 2) \sin y$ and $N = x \cos y$. Thus

$\dfrac{\partial M}{\partial y} = (x+2)\cos y$ and $\dfrac{\partial N}{\partial x} = \cos y$. We see that $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$.

Hence given equatuion is not exact.To find an I.F we apply :

RULE I: $\dfrac{M_y - N_x}{N} = \dfrac{(x+2)\cos y - \cos y}{x \cos y} = \dfrac{\cos y(x+1)}{x \cos y} = \dfrac{x+1}{x} = P(x).$

Thus, $I.F = e^{\int P(x)dx} = e^{\int (1+1/x)\,dx} = e^{x+\ln x} = e^x e^{\ln x} = xe^x$

Multiply (1) by I.F, we get : $xe^x (x+2)\sin y dx + x^2 e^x \cos y dy = 0$ (2)

Equation (2) is now exact. Its solution is :

$\int M dx + \int (\text{terms of } N \text{ not containing } x)\ dy = C$

$\sin y \int e^x dx + 2\int y\ dy = C \Rightarrow \sin y e^x + y^2 = C$

Linear Differential Equations

A first order differential equation is called linear, if it is of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = f(x) \qquad (1)$$

Equation (1) is known as linear because, y and its derivative dy/dx appear in first degree. By dividing each member of (1) by $a_1(x)$, we obtain

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad (2)$$

where $P(x) = \dfrac{a_0(x)}{a_1(x)}$ and $Q(x) = \dfrac{f(x)}{a_1(x)}$.

We call (2) the standard form of the first order linear differential equation. To solve equation (2), we write it in the form

$$dy + P(x)y\ dx = Q(x)\ dx \text{ or } \{P(x)y - Q(x)\}dx + dy = 0$$

Apply the exactness test to find that it is an exact differential or not, comparing it with $Mdx + Ndy = 0$, we have $M = P(x)y - Q(x)$ and $N = 1$. Now $M_y = P(x)$ and $N_x = 0$. Since $M_y \neq N_x$ hence equation (2) is not exact. To find an integrating factor, we apply RULE I, we see that $\dfrac{M_y - N_x}{N} = \dfrac{P(x) - 0}{1} = P(x)$

Hence, $e^{\int P(x)dx}$ is an integrating factor of (2). Let us multiply (2) by integrating factor to obtain: $$e^{\int P(x)\,dx}\frac{dy}{dx}+e^{\int P(x)\,dx}P(x)\,y=Q(x)e^{\int P(x)\,dx} \quad (3)$$

The left member of (3) is the derivative of the product $ye^{\int P(x)\,dx}$. Thus (3) gives

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right)=Q(x)e^{\int P(x)\,dx}$$

Integrating, we get : $\left(ye^{\int P(x)\,dx}\right)=\int Q(x)e^{\int P(x)\,dx}\,dx$

Solving this equation we get the solution of equation (1) in explicit form:

$$y=f(x)+c.$$

Here we summarize the steps involved in finding the solution of linear differential equation of first order.

Step 1: Put the given equation into standard form:

$$\frac{dy}{dx}+P(x)y=Q(x).$$

Step 2: Obtain the integrating factor $e^{\int P(x)\,dx}$

Step 3: Multiply the equation in step 1 by the integrating factor.

$$e^{\int P(x)\,dx}\frac{dy}{dx}+e^{\int P(x)\,dx}P(x)\,y=Q(x)e^{\int P(x)\,dx}$$

Step 4: The left side of equation in step 3 is the derivative of the product of the dependent variable and integrating factor, that is:

$$\frac{d}{dx}\left(ye^{\int P(x)\,dx}\right)=Q(x)e^{\int P(x)\,dx}$$

Step 5: Integrate both sides of equation in the step 4 to get the solution.

REMARK: Sometimes an equation that is not linear in dependent variable y but it can be made linear in x by interchanging the roles of dependent and independent variables

Example 01: Solve the linear differential equation $\frac{dy}{dx}+\frac{1}{x\ln x}y=\frac{3x^2}{\ln x}$.

Solution: The equation in standard form is:

$$\frac{dy}{dx}+\frac{1}{x\ln x}y=\frac{3x^2}{\ln x} \quad (1)$$

Here the integrating factor is:

$$e^{\int\frac{1}{x\ln x}dx}=e^{\int\frac{1}{\ln x}\left(\frac{1}{x}\right)dx}=e^{\ln(\ln x)}=e^{\ln x} \quad \left[\text{Formula}: \int\frac{f'(x)}{f(x)}dx=\ln\left(f(x)\right)\right]$$

Multiplying (1) by integrating factor, we have

$$\ln x\frac{dy}{dx}+\frac{1}{x}y=3x^2 \Rightarrow \frac{d}{dx}(y\ln x)=3x^2.$$

Integrating, $$y\ln x=x^3+c \quad \Rightarrow y=\left(x^3+c\right)/\ln x.$$

Example 02: Solve the linear differential equation $x\frac{dy}{dx}+(1+x\cot x)y=x$

Solution: Dividing given equation by x, we get: $\frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1 \quad (1)$

Here the integrating factor is:

$$e^{\int \frac{1}{x\ln x}dx} = e^{\int\left(\frac{1}{x}+\cot x\right)dx} = e^{\int \frac{1}{x}dx+\int \cot x\,dx} = e^{\ln x+\ln\sin x} = e^{\ln(x\sin x)} x\sin x$$

Multiplying (1) by the integrating factor, we have

$$x\sin x\frac{dy}{dx}+x\sin x\left(\frac{1}{x}+\cot x\right)y = x\sin x \Rightarrow x\sin x\frac{dy}{dx}+y\sin x+xy\cos x = x\sin x$$

$$\frac{d}{dx}(xy\sin x) = x\sin x \Rightarrow \int\frac{d}{dx}(xy\sin x)\,dx = \int x\sin x\,dx \Rightarrow xy\sin x = \int x\sin x\,dx.$$

Now, $\int x\sin x\,dx = -x\cos x+\int\cos x\,dx = -x\cos x+\sin x$.

Thus solution of given differential equation is:

$$xy\sin x = -x\cos x+\sin x+c \Rightarrow y = \frac{-x\cos x+\sin x+c}{x\sin x} \Rightarrow y = -\cot x+\frac{1}{x}+\frac{c}{x}\csc x.$$

Example 03: Solve $(x + y)\,dy + dx = 0$

Solution: Given equation may be put in the form: $\frac{dy}{dx}+\frac{1}{x+y}=0$

This is not linear in y. However, we may write it as

$$\frac{dy}{dx} = -\frac{1}{x+y} \Rightarrow \frac{dx}{dy} = -x-y \Rightarrow \frac{dx}{dy}+x = -y \qquad (1)$$

This is linear differential equation in x. Thus, reversing the roles of x and y, $P(y) = 1$. So integrating factor is $e^{\int 1 dy} = e^{y}$. Equation (1) after multiplication by e^{y} becomes

$$e^{y}\frac{dy}{dx}+e^{y}x = -ye^{y} \Rightarrow \frac{d}{dy}\left(xe^{y}\right) = -ye^{y} \Rightarrow xe^{y} = -\int ye^{y}\,dy+c \quad \text{[Integrating w.r.t y]}$$

$$\Rightarrow xe^{y} = (-y+1)e^{y}+c \Rightarrow x = (1-y)+ce^{-y} \quad \text{[Integrating by parts]}$$

This is solution of given differential equation.

Example 04: Solve the following initial value problem

$$e^{x}\left\{y-3\left(e^{x}+1\right)^{2}\right\}dx+\left(e^{x}+1\right)dy = 0,\ y(0) = 4$$

Solution: Given differential equation is:

$$e^{x}\left\{y-3\left(e^{x}+1\right)^{2}\right\}dx+\left(e^{x}+1\right)dy = 0$$

$$\frac{dy}{dx} = -\frac{e^{x}\left\{y-3\left(e^{x}+1\right)^{2}\right\}}{\left(e^{x}+1\right)} \Rightarrow \frac{dy}{dx}+\frac{e^{x}}{\left(e^{x}+1\right)}y = 3e^{x}\left(e^{x}+1\right) \qquad (1)$$

which is linear differential equation.

Here the integrating factor is: $e^{\int \frac{e^{x}}{e^{x}+1}dx} = e^{\ln(e^{x}+1)} = \left(e^{x}+1\right)$

Multiplying (1) by integrating factor, we have

$$\left(e^{x}+1\right)\frac{dy}{dx}+e^{x}y = 3e^{x}\left(e^{x}+1\right)^{2} \Rightarrow \frac{d}{dx}\left\{y\left(e^{x}+1\right)\right\} = 3e^{x}\left(e^{x}+1\right)^{2}.$$

$$\left(e^{x}+1\right)\frac{dy}{dx}+e^{x}\ y = 3e^{x}\left(e^{x}+1\right)^{2} \Rightarrow \frac{d}{dx}\left\{y\left(e^{x}+1\right)\right\} = 3e^{x}\left(e^{x}+1\right)^{2}$$

Integrating, $\{y(e^x+1)\}=(e^x+1)^3+c \Rightarrow y=(e^x+1)^2+c(e^x+1)^{-1}$ (2)

Applying the initial conditions, that is, put x = 0 and y = 4, we get

$$4=(e^0+1)^2+c(e^0+1)^{-1} \Rightarrow 4=4+c/2 \Rightarrow c=0.$$

Hence the equation (2) becomes: $y = (e^x + 1)^2$. This is particular solution of given differential equation.

Example 05: Solve the following differential equations:

(i) $\dfrac{dy}{dx}+\left(\dfrac{2x+1}{x}\right)y=e^{-2x}$ (1)

Solution: Here P(x) = (2x + 1)/x

$$\Rightarrow I.F=e^{\int\left(\frac{2x+1}{x}\right)dx}=e^{2\int 1.dx+\int 1/x\,dx}=e^{2x+\ln x}=e^{2x}e^{\ln x}=x\,e^{2x}.$$

Multiplying equation (1) by an I. F, we get

$$x\,e^{2x}\left[\frac{dy}{dx}+\left(\frac{2x+1}{x}\right)y\right]=x\,e^{2x}e^{-2x} \Rightarrow \frac{d}{dx}[y\times I.F]=x.$$

Integrating both sides and placing the value of an I.F we obtain: $y\,x\,e^{2x} = x^2/2 + C$.
This is the solution of given differential equation.

(ii) $\dfrac{dy}{dx}+\dfrac{3}{x}y=6x^2$ (1)

Solution: Here $P(x) = 3/x \Rightarrow I.F=e^{3\int 1/x\,dx}=e^{3\ln x}=e^{\ln x^3}=x^3$.
Multiplying equation (1) by an I. F, we get

$$x^3\left[\frac{dy}{dx}+\frac{3}{x}y\right]=6x^3x^2 \Rightarrow \frac{d}{dx}[y\times I.F]=6x^5$$

Integrating both sides and placing the value of an I.F we obtain: $y\,x^3 = x^6 + C$.
This is the solution of given differential equation.

(iii) $\dfrac{dy}{dx}+3y=3x^2e^{-3x}$ (1)

Solution: Here $P(x) = 3 \Rightarrow I.F=e^{3\int 1\,dx}=e^{3x}$.
Multiplying equation (1) by an I. F, we get

$$e^{3x}\left[\frac{dy}{dx}+3y\right]=3x^2e^{-3x}.e^{3x} \Rightarrow \frac{d}{dx}[y\times I.F]=3x^2$$

Integrating both sides and placing the value of an I.F we obtain: $y\,e^{3x} = x^3 + C$.
This is the solution of given differential equation.

(iv) $\cos^3 x\dfrac{dy}{dx}+y\cos x=\sin x$ (1)

Solution: Dividing both sides of equation (1) by $\cos^3 x$, we obtain:

$$y' + \sec^2 x\,.\,y = \sin x/\cos^3 x = \tan x\sec^2 x \quad (2)$$

Here $P(x)=\sec^2 x \Rightarrow I.F=e^{\int\sec^2 x\,dx}=e^{\tan x}$. Multiplying equation (1) by an I. F, we get: $e^{\tan x}\left[\dfrac{dy}{dx}+\sec^2 x.y\right]=e^{\tan x}\tan x\sec^2 x \Rightarrow \dfrac{d}{dx}[y\times I.F]=e^{\tan x}\tan x\sec^2 x$

Integrating both sides and placing the value of an I.F we obtain:

$$y.e^{\tan x} = \int e^{\tan x} \tan x \sec^2 x\, dx + C$$

Substituting $z = \tan x$ ➔ $dz = \sec^2 x\, dx$, we have:

$$y.e^{\tan x} = \int z.e^z dz + C = z.e^z - \int e^z dz + C = ze^z - e^z + C = e^z(z-1) + C$$

Substituting $z = \tan x$, get: $y\, e^{\tan x} = e^{\tan x}(\tan x - 1) + C$. Dividing both sides by $e^{\tan x}$, we get: $y = (\tan x - 1) + C.\, e^{-\tan x}$.

This is the solution of given differential equation.

(v) $(x+1)\dfrac{dy}{dx} - ny = e^x (x+1)^{n+1}$ (1)

Solution: Dividing both sides of equation (1) by $(x + 1)$, we obtain:

$$y' - n/(x+1)\ .\ y = e^x (x+1)^n \qquad (2)$$

Here $P(x) = -n/(x+1) \Rightarrow I.F = e^{-n\int 1/(x+1)\, dx} = e^{-n\ln(x+1)} = e^{\ln(x+1)^{-n}} = (x+1)^{-n}$.

Multiplying equation (2) by an I. F, we get

$$(x+1)^{-n}\left[\frac{dy}{dx} - \frac{n}{x+1}y\right] = e^x \Rightarrow \frac{d}{dx}[y \times I.F] = e^x$$

Integrating both sides and placing the value of an I.F we obtain:

$$y.(x+1)^{-n} = \int e^x\, dx + C = e^x + C \Rightarrow y = (x+1)^n[e^x + C]$$

This is the solution of given differential equation.

(vi) $(x^2+1)\dfrac{dy}{dx} + 2xy = 4x^2$ (1)

Solution: Dividing both sides of equation (1) by $(x^2 + 1)$, we obtain:

$$y' + 2xy/(x^2+1) = 4x^2/(x^2+1) \qquad (2)$$

Here $P(x) = 2x/(x^2+1) \Rightarrow I.F = e^{\int 2x/(x^2+1)\, dx} = e^{\ln(x^2+1)} = (x^2+1)$.

Multiplying equation (2) by an I. F, we get

$$(x^2+1)\left[\frac{dy}{dx} + \frac{2x}{x^2+1}y\right] = 4x^2 \Rightarrow \frac{d}{dx}[y \times I.F] = 4x^2$$

Integrating both sides and placing the value of an I.F, we obtain

$$y.(x^2+1) = 4\int x^2\, dx + C = \frac{4x^3}{3} + C \quad \Rightarrow \quad (x^2+1)\,y = \frac{4x^3}{3} + C.$$

This is the solution of given differential equation.

(vii) $x\dfrac{dy}{dx} + 2y = \sin x$ (1)

Solution: Dividing both sides of equation (1) by x, we obtain:

$$y' + 2y/x = \sin x/x \qquad (2)$$

Here $P(x) = 2/x \Rightarrow I.F = e^{2\int 1/x\, dx} = e^{2\ln x} = e^{\ln x^2} = x^2$.

Multiplying equation (2) by an I. F, we get

$$x^2\left[\frac{dy}{dx} + 2y\right] = x^2 \sin x \Rightarrow \frac{d}{dx}[y \times I.F] = x^2 \sin x$$

Integrating both sides and placing the value of an I.F, we obtain

$$y.x^2 = \int x^2 \sin x\, dx + C = x^2(-\cos x) + 2\int x\cos x\, dx + C$$

$$= -x^2\cos x + 2[x\sin x - \int \sin x\, dx] + C \quad \Rightarrow x^2 y = -x^2\cos x + 2x\sin x + 2\cos x + C$$

This is the solution of given differential equation.

(viii) $(1+x^2)\dfrac{dy}{dx}+4xy=1/(1+x^2)^2$ (1)

Solution: Dividing both sides of equation (1) by $(1+x^2)$, we obtain:

$$\frac{dy}{dx}+\frac{4x}{(1+x^2)}y=\frac{1}{(1+x^2)^3} \quad (2)$$

Here $P(x)=4x/(1+x^2) \Rightarrow I.F=e^{2\int 2x/(1+x^2)\,dx}=e^{2\ln(1+x^2)}=e^{\ln(1+x^2)^2}=(1+x^2)^2$.

Multiplying equation (2) by an I. F, we get

$$(1+x^2)^2\left[\frac{dy}{dx}+\frac{4x}{(1+x^2)}y\right]=\frac{1}{(1+x^2)} \Rightarrow \frac{d}{dx}[y\times I.F]=\frac{1}{(1+x^2)}$$

Integrating both sides and placing the value of an I.F we obtain:

$$y.(1+x^2)^2=\int\frac{1}{(1+x^2)}dx+C=\tan^{-1}x+C$$

→ $y.(1+x^2)^2=\tan^{-1}x+C$. This is the solution of given differential equation.

(ix) $\dfrac{dy}{dx}=\dfrac{1}{(e^y-x)}$

Solution: This equation is not linear in y. If we write it in the form

$$\frac{dx}{dy}=(e^y-x) \Rightarrow \frac{dx}{dy}+x=e^y \quad (1)$$

Equation (1) is linear in x. Here $P(y)=1 \Rightarrow I.F=e^{\int 1\,dy}=e^y$.

Multiplying equation (1) by an I. F, we get

$$e^y\left[\frac{dx}{dy}+x\right]=ye^y \Rightarrow \frac{d}{dy}[x\times I.F]=ye^y$$

Integrating both sides and placing the value of an I.F, we obtain

$$xe^y=\int ye^y dy+C=ye^y-e^y+C=e^y(y-1)+C \Rightarrow x=(y-1)+Ce^{-y}$$

This is the solution of given differential equation.

(x) $(x+2y^3)\dfrac{dy}{dx}=y$

Solution: This equation is not linear in y. If we write it in the form

$$\frac{dy}{dx}=\frac{y}{(x+2y^3)} \Rightarrow \frac{dx}{dy}=\frac{(x+2y^3)}{y} \Rightarrow \frac{dx}{dy}-\frac{1}{y}x=2y^2 \quad (1)$$

Equation (1) is linear in x.

Here $P(y)=1 \Rightarrow I.F=e^{-\int 1/y\,dy}=e^{-\ln y}=y^{-1}$.

Multiplying equation (1) by an I. F, we get

$$y^{-1}\left[\frac{dx}{dy}-\frac{1}{y}x\right]=2y \Rightarrow \frac{d}{dy}[x\times I.F]=2y$$

Integrating both sides and placing the value of an I.F we obtain:

$$xy^{-1}=2\int y\,dy+C=y^2+C= \Rightarrow x=y(y^2+C)$$

This is the solution of given differential equation.

(xi) $x\dfrac{dy}{dx}-2x^2y=y\ln y$

Solution: Dividing both sides by x, we get:

$$\frac{dy}{dx} - 2xy = y\ln y / x \qquad (1)$$

Substituting $z = \ln y$ → $y = e^z$. Differentiating w.r.t x, we get, $dy/dx = e^z\, dz/dx$. Thus equation (1) becomes:

$e^z \frac{dz}{dx} - 2x\, e^z = (z.e^z)/x$. Dividing by e^z, we obtain : $\frac{dz}{dx} - 2x = z./x$

→ $$\frac{dz}{dx} - \frac{1}{x}z = 2x \qquad (1)$$

Equation (1) is Here $P(x) = -1/x \Rightarrow I.F = e^{-\int 1/x\, dx} = e^{-\ln x} = x^{-1}$.
Multiplying equation (1) by an I. F, we get

$$x^{-1}\left[\frac{dz}{dx} - \frac{1}{x}z\right] = 2 \quad \Rightarrow \frac{d}{dx}[z \times I.F] = 2$$

Integrating both sides and placing the value of an I.F we obtain:

$$zx^{-1} = 2\int 1 dx + C = 2x + C = \quad \Rightarrow z = 2x^2 + Cx \quad \Rightarrow \ln y = 2x^2 + Cx$$

A solution of given differential equation.

⇒ Bernoulli's Differential Equation

A well-known equation that is not linear but can be transformed to a linear equation is

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad (1)$$

This is known as **Bernoulli's differential equation.** If n = 0 or n = 1 then (1) is linear. So we consider the cases where $n \neq 0$ and $n \neq 1$. Dividing both sides of (1) by y^n we get

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x) \qquad (2)$$

Substituting $y^{1-n} = z \Rightarrow (1-n)y' = z' \Rightarrow y^{-n}y' = z'/(1-n)$

Therefore (2) becomes

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x) \quad \text{or} \quad \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x) \qquad (3)$$

Equation (3) is a linear differential equation in standard form and can be solved for z and substitution $y^{1-n} = z$ gives the solution of (1).

Example 06: Solve Bernoulli differential equation $x\frac{dy}{dx} + y = y^2 \ln x$

Solution: Dividing the given equation by x, we have

$$\frac{dy}{dx} + \frac{1}{x}y = \frac{\ln x}{x}y^2 \qquad (1)$$

This is Bernoulli's differential equation where n = 2.

Dividing by y^2, we get

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}y^{-1} = \frac{\ln x}{x} \qquad (2)$$

Let $z = y^{-1} \Rightarrow \frac{dz}{dx} = -y^{-2}\frac{dy}{dx} \Rightarrow -\frac{dz}{dx} = y^{-2}\frac{dy}{dx}$. Then equation (2) becomes:

$$-\frac{dz}{dx} + \frac{1}{x}z = \frac{\ln x}{x} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x}z = -\frac{\ln x}{x}. \qquad (3)$$

which is linear in z. Therefore,

$\Rightarrow$ I.F $= e^{-\int 1/x\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = 1/x$

Multiplying (3) by I.F, we get

$$\frac{1}{x}\frac{dz}{dx} - \frac{1}{x^2}z = -\frac{\ln x}{x^2} \Rightarrow \frac{d}{dx}\left(z \times \frac{1}{x}\right) = -\frac{\ln x}{x^2}.$$

Integrating both sides, we get

$$\frac{z}{x} = -\int \frac{\ln x}{x^2}dx + c \qquad (4)$$

Now using integration by parts, we get

$$\int \frac{\ln x}{x^2}dx = -\frac{\ln x}{x} + \int \frac{1}{x^2}dx = -\frac{\ln x}{x} - \frac{1}{x}.$$

Thus (4) becomes

$$\frac{z}{x} = -\left(-\frac{\ln x}{x} - \frac{1}{x}\right) + c \Rightarrow \frac{z}{x} = \frac{\ln x}{x} + \frac{1}{x} \Rightarrow z = \ln x + 1 + cx \Rightarrow \frac{1}{y} = \ln x + cx + 1.$$

This is the required solution.

Example 07: Solve $y' + y = xy^3$

Solution: We have $y' + y = xy^3$ (1)

This is a Bernoulli's differential equation with $n = 3$.

Dividing both sides of (1) by y^3, we get

$$y^{-3}\frac{dy}{dx} + y^{-2} = x \qquad (2)$$

Let $z = y^{-2} \Rightarrow \frac{dz}{dx} = -2y^{-3}\frac{dy}{dx} \Rightarrow -\frac{1}{2}\frac{dz}{dx} = y^{-3}\frac{dy}{dx}$

Thus equation (2) becomes: $-\frac{1}{2}\frac{dz}{dx} + z = x \qquad \Rightarrow \frac{dz}{dx} - 2z = -2x$ (3)

which is linear in z. This implies that I.F $= e^{-\int 2dx} = e^{-2x}$

Multiplying (3) by I. F, we get:

$$e^{-2x}\frac{dz}{dx} - 2e^{-2x}z = -2xe^{-2x} \qquad \Rightarrow \frac{d}{dx}\left(ze^{-2x}\right) = -2xe^{-2x}$$

Integrating both sides, we get: $ze^{-2x} = -2\int xe^{-2x}dx + c$ (4)

The right side of (4) will be solved using integration by parts.

$$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}$$

Now (4) becomes

$$ze^{-2x} = -2\left(-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x}\right) + c \Rightarrow z = x + ce^{2x} + \frac{1}{2}.$$

Put $z = y^{-2}$, we get : $y^{-2} = x + ce^{2x} + \frac{1}{2}$.

This is solution of given differential equation.

Example 07: Solve the following initial value problem

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3},\ y(1) = 2$$

Solution: We have

$$\frac{dy}{dx}+\frac{y}{2x}=\frac{x}{y^3} \Rightarrow \frac{dy}{dx}+\frac{1}{2x}y=xy^{-3}$$

This is a Bernoulli's differential equation with $n=-3$. (1)

Dividing both sides of (1) by y^{-3}, we get

$$y^3\frac{dy}{dx}+\frac{1}{2x}y^4=x$$

Let $z=y^4 \Rightarrow \frac{dz}{dx}=4y^3\frac{dy}{dx} \Rightarrow \frac{1}{4}\frac{dz}{dx}=y^3\frac{dy}{dx}$. Thus equation (2) becomes: (2)

$$\frac{1}{4}\frac{dz}{dx}+\frac{1}{2x}z=x \Rightarrow \frac{dz}{dx}+\frac{2}{x}z=4x$$

$$\text{I.F.}=e^{\int\frac{2}{x}dx}=e^{2\ln x}=e^{\ln x^2}=x^2. \quad (3)$$

Multiplying (3) by I. F, we get

$$x^2\frac{dz}{dx}+2xz=4x^3 \Rightarrow \frac{d}{dx}\left(zx^2\right)=4x^3.$$

Integrating both sides, we get

$$x^2z=x^4+c \Rightarrow x^2y^4=x^4+c$$

Applying the initial condition, we get (4)

$$(1)^2(2)^4=(1)^4+c \Rightarrow c=15.$$

Equation (4) becomes $x^2y^4=x^4+15$.

This is the particular solution of given differential equation.

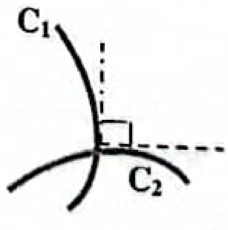
5.2 ORTHOGONAL TRAJECTORIES

Definition: An n–parameter family of curves is a set of relations of the form:

$$\{(x,y): f(x,y,c_1,c_2,...,c_n)=0\},$$

where 'f' is a real valued function of x,y and each c_i $(i=1,2,...,n)$ ranges over an interval of real values. For example, the set of concentric circles defined by:

$$x^2+y^2=c$$



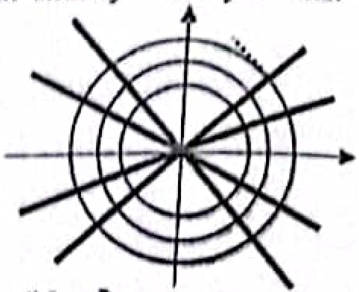
has one non-negative parameter c. Again the set of circles, defined by: $(x-c_1)^2+(y-c_2)^2=c_3$

has three–parameter family where c_1, c_2 are real and c_3 is non–negative real.

Definition: An angle between two curves is defined as angle between their tangents at the point of their intersection. If these tangents are perpendicular to each other we say that curves are normal/perpendicular/orthogonal to each other. This is shown in the following figure.

Definition: If a family of curves cuts every member of other family of curves at right angle, the two families of curves are known as **Orthogonal/ Perpendicular/Normal Trajectories** of each. If the angle is not right angle the trajectories are known as **oblique trajectories.**

As an example, consider two family of curves $y=mx$ and $x^2+y^2=r^2$, where m and r are parameters. Then we find that every line given by y = mx through the origin is orthogonal to every circle given by $x^2 + y^2 = r^2$. We shall shortly show you this. The following figure depicts this fact.



We shall discuss the "Orthogonal Trajectories" in two systems of coordinates.

(a) Cartesian coordinate system

(b) Polar coordinate system.

Before we present the working rule of finding an O.T the following may be noted:

(i) If two line are parallel their slopes are equal, that is; if $L_1 \parallel L_2 \Rightarrow m_1 = m_2$.

(ii) If two lines are perpendicular then the product of their slopes is equal to -1, that is; if $L_1 \perp L_2$ then $m_1 \cdot m_2 = -1 \Rightarrow m_2 = -1/m_1$.

Orthogonal Trajectories in Cartesian Coordinates

Working Rule

Step1: Given an equation

$$f(x, y, c) \tag{1}$$

Differentiate (1) and eliminate an arbitrary constant and get

$$dy/dx = F(x, y). \tag{2}$$

This is the slope of equation (1).

Step 2: Replace dy/dx by $-dx/dy$ to get

$$-dx/dy = F(x, y) \tag{3}$$

This is the slope of O.T.

Step 3: Solving (3) to get required O.T of family of curves (1).

REMARK: Orthogonal trajectories are important in various fields of applied sciences, for example, in hydrodynamics and heat conduction.

Example 01: Find orthogonal trajectories of the family of circles $x^2 + y^2 = r^2$

Solution: Differentiate w. r. t x, we get:

$$2x + 2yy' = \Rightarrow dy/dx = -x/y$$

This is the slope of given family of circles. Hence, the slope of the family of O.T is

$-dx/dy = -x/y$ or equivalently, $dy/dx = + y/x$

Separating the variables and integrating, we have:

$$\int \frac{1}{y} dy = \int \frac{1}{x} dx + \ln c \Rightarrow \ln y = \ln x + \ln c \quad \Rightarrow \ln y = \ln cx \quad \Rightarrow y = mx$$

This is the "Orthogonal Trajectories" of family of circles. The figure of these "Orthogonal Trajectories" is depicted in the above.

Example 02: Find the orthogonal trajectories of family of curves $y = c x^2$.

Solution: We are given $y = c x^2$ (1)

The family (1) consists of parabolas symmetric about the y – axis with vertices at the origin. Differentiating equation (1) with respect to x, we get

$$dy/dx = 2c x \tag{2}$$

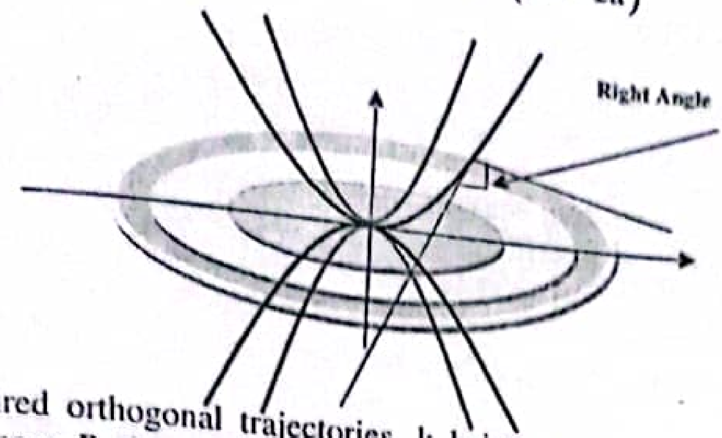
To eliminate c, we observe from equation (1) that $c = y/x^2$. Put this in (2)

$$\frac{dy}{dx} = 2\left(\frac{y}{x^2}\right)x \Rightarrow \frac{dy}{dx} = \frac{2y}{x}, \tag{3}$$

which is the slope of (1). Hence slope of O.T is

$$-\frac{dx}{dy} = \frac{2y}{x} \Rightarrow -xdx = 2ydy. \text{ Integrating,}$$

$$-\frac{1}{2}x^2 = y^2 + a \Rightarrow -\left(\frac{1}{2}x^2 + y^2\right) = a \Rightarrow x^2 + 2y^2 = k, \quad (k = -2a)$$



which is the required orthogonal trajectories, k being a parameter. These orthogonal trajectories are ellipses. Both parabolas and ellipses are shown in the above figure. Note that each ellipse intersects each parabola at right angle.

Example 03: Find the orthogonal trajectories of family of curves $x^2 + y^2 = cx$.

Solution: Equation of curve is $x^2 + y^2 = cx$. (1)

Differentiating with respect to x, we get: $2x + 2y\frac{dy}{dx} = c$ (2)

From (1), we have: $c = (x^2 + y^2)/x$

Substituting this into (2), we get

$$2x + 2y\frac{dy}{dx} = \frac{x^2+y^2}{x} \Rightarrow x + y\frac{dy}{dx} = \frac{x^2+y^2}{2x} \Rightarrow y\frac{dy}{dx} = \frac{x^2+y^2}{2x} - x$$

$$\Rightarrow y\frac{dy}{dx} = \frac{x^2+y^2-2x^2}{2x} \Rightarrow y\frac{dy}{dx} = \frac{y^2-x^2}{2x} \Rightarrow \frac{dy}{dx} = \frac{y^2-x^2}{2xy}.$$

Replacing dy/dx by –dx/dy, we obtain

$$-\frac{dx}{dy} = \frac{y^2-x^2}{2xy} \Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2} \Rightarrow \frac{dy}{dx} = \frac{2(y/x)}{1-(y/x)^2} \quad (3)$$

Equation (3) is a homogeneous differential equation. Put

$\frac{y}{x} = v$ or $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Then (3) becomes

$$v + x\frac{dv}{dx} = \frac{2v}{1-v^2} \Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} - v \Rightarrow x\frac{dv}{dx} = \frac{2v-v+v^3}{1-v^2} \Rightarrow x\frac{dv}{dx} = \frac{v+v^3}{1-v^2}.$$

Separating the variables, we have

$$\frac{1-v^2}{v+v^3}dv = \frac{1}{x}dx \quad \text{or} \quad \frac{1-v^2}{v(1+v^2)}dv = \frac{1}{x}dx.$$

Integrating, $\int \frac{1-v^2}{v(1+v^2)}dv = \int \frac{1}{x}dx + d$ (4)

Left side of (4) will be solved using integration by partial fractions.

$$\frac{1-v^2}{v(1+v^2)} = \frac{A}{v} + \frac{Bv+C}{1+v^2} \quad (5)$$

$$1-v^2 = A\left(1+v^2\right)+(Bv+C)\,v$$

Put $v = 0$, we get $1 = A$

Now, $\quad 1 - v^2 = A(1 + v^2) + B\,v^2 + C\,v.$

Comparing the coefficients of v^2 and v on both sides, we have

$$A+B=-1 \Rightarrow B=-2 \text{ and } C=0\cdot \text{ [Note} : A = 1]$$

Thus, (5) becomes:

$$\frac{1-v^2}{v\left(1+v^2\right)}=\frac{1}{v}-\frac{2v}{1+v^2} \Rightarrow \int\frac{1-v^2}{v\left(1+v^2\right)}dv=\int\frac{1}{v}dv-\int\frac{2v}{1+v^2}dv$$

$$=\ln v-\ln\left(1+v^2\right)=\ln\frac{v}{1+v^2}.$$

Hence equation (4) becomes after taking antilog:

$$\frac{v}{1+v^2}=xd_1 \Rightarrow \frac{y\ x}{\left(x^2+y^2\right)/x^2}=xd_1=\frac{y}{x}\times\frac{x^2}{x^2+y^2}=xd_1 \Rightarrow \frac{y}{x^2+y^2}=d_1$$

$$\Rightarrow x^2+y^2=ky \quad (k=1/d_1), \text{ where } d_1=\ln d$$

This solution gives the orthogonal trajectories of the equation (1).

Example 04: Find an equation of orthogonal trajectories of the curves of the family $y^2 = x^2 + cx$.

Solution: Given curve is: $y^2 = x^2 + cx$ (1)

Differentiating (1) with respect to x, we get: $\quad 2y\ dy/dx=2x+c$ (2)

From (1), we have: $\quad c=\left(y^2-x^2\right)/x$

Put this in (2), we get

$$2y\frac{dy}{dx}=2x+\frac{y^2-x^2}{x} \Rightarrow 2y\frac{dy}{dx}=\frac{2x^2+y^2-x^2}{x} \Rightarrow 2xy\frac{dy}{dx}=x^2+y^2 \Rightarrow \frac{dy}{dx}=\frac{x^2+y^2}{2xy}.$$

Replacing dy/dx by –dx/dy, we get

$$-\frac{dx}{dy}=\frac{x^2+y^2}{2xy} \quad \text{or} \quad \frac{dy}{dx}=\frac{-2xy}{x^2+y^2}.$$

This is homogeneous equation and can be written as

$$\frac{dy}{dx}=\frac{-2(y/x)}{1+(y/x)^2} \tag{3}$$

Putting: $y/x = v$ or $y=vx \Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$. Thus (3) becomes

$$v+x\frac{dv}{dx}=\frac{-2v}{1+v^2} \Rightarrow x\frac{dv}{dx}=-\left(\frac{2v}{1+v^2}+v\right)$$

$$\Rightarrow x\frac{dv}{dx}=-\left(\frac{2v+v+v^3}{1+v^2}\right) \Rightarrow x\frac{dv}{dx}=-\left(\frac{3v+v^3}{1+v^2}\right) \tag{4}$$

Separating the variables, we get

$$-\frac{1+v^2}{3v+v^3}dv=\frac{1}{x}dx \Rightarrow -\int\frac{1+v^2}{3v+v^3}dv=\int\frac{1}{x}dx+d \Rightarrow -\frac{1}{3}\int\frac{3+3v^2}{3v+v^3}dv=\int\frac{1}{x}dx+d$$

$$-\frac{1}{3}\ln\left(3v+v^3\right)=\ln x+\ln d_1 \Rightarrow \ln\left(3v+v^3\right)=-3\ln\left(xd_1\right)\Rightarrow \ln\left(3v+v^3\right)=\ln\left(xd_1\right)^{-3}$$

$$3v+v^3=\left(xd_1\right)^{-3} \Rightarrow 3\frac{y}{x}+\left(\frac{y}{x}\right)^3=\frac{1}{\left(xd_1\right)^3} \Rightarrow 3x^2y+y^3=k$$

which is the required equation of the orthogonal trajectories of the given family.

Definition: A family of curves which is orthogonal to itself is called self – orthogonal.

REMARK: From the above definition, it follows that if the differential equation of the family of curves is identical with the differential equation of its orthogonal trajectories, then such family of curves is self–orthogonal.

Example 05: Show that the orthogonal trajectories of the system of parabolas $y^2 = 4c\,(x + c)$ belong to the system itself, c being parameter.

Solution: We have

$$y^2 = 4c\,(x + c) \qquad (1)$$

Differentiating (1) with respect to x, we get

$$2y\frac{dy}{dx}=4c(1)\Rightarrow y\frac{dy}{dx}=2c\Rightarrow c=\frac{1}{2}y\frac{dy}{dx}.$$

Substituting this value of c into (1), we get

$$y^2=4\left(\frac{1}{2}y\frac{dy}{dx}\right)\left(x+\frac{1}{2}y\frac{dy}{dx}\right)\Rightarrow y^2=2xy\frac{dy}{dx}+y^2\left(\frac{dy}{dx}\right)^2\Rightarrow y=2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2 \qquad (2)$$

Replacing dy/dx by –dx/dy, we get

$$y=2x\left(-\frac{dx}{dy}\right)+y\left(\frac{dx}{dy}\right)^2\Rightarrow y=\frac{-2x}{dy/dx}+\frac{y}{(dy/dx)^2}$$

$$\Rightarrow y\left(\frac{dy}{dx}\right)^2=-2x\frac{dy}{dx}+y\Rightarrow y=2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2 \qquad (3)$$

Equation (2) and (3) are identical. Hence the system of parabolas (1) is itself orthogonal, that is; each member of given family of parabolas intersects its own member orthogonally.

Example 06: Find an Orthogonal Trajectories for the following curves

(i) $x^2 - y^2 = c$

Solution: Differentiate w.r.t x, we get: $2x - 2y\,y` = 0 \rightarrow y` = x/y$

This is the slope of equation (1). Hence slope of an O.T is: $dy/dx = -y/x$

Separating the variables and integrating, we get:

$$\int\frac{1}{y}dy=-\int\frac{1}{x}dx+\ln C \quad \Rightarrow \ln y=-\ln x+\ln C=\ln C/x$$

$$\Rightarrow y=C/x \qquad \Rightarrow xy=C.$$ This is the O.T of given family.

(ii) $x = c\ y^2$

Solution: Differentiate w.r.t x, we get: $1 = 2\,c\,y\,y` \rightarrow y` = 1/2cy$

Substituting the value of c from equation (1), we get: $dy/dx = y/2x$.

This is the slope of equation (1). Hence slope of an O.T is:

$$dy/dx = -2x/y$$

Separating the variables and integrating, we get:

$$\int y\,dy=-2\int x\,dx+C \quad \Rightarrow y^2/2=-2x^2/2+C \quad \Rightarrow y^2+2x^2=K,\ \text{where } 2C=K$$

This is O.T of a given family.

(iii) $y = e^{c\,x}$ (1)

Solution: Differentiate w.r.t x, we get: $y`=c\,e^{c\,x}=c\,y$ (from 1) (2)

Now from equation (1), $\ln y = cx$ → $c = \ln y/x$.
Substituting this in equation (2), we obtain:

$$dy/dx = y \ln y /x.$$

This is slope of equation (1). Hence slope of an O.T is: $dy/dx = -x/y. \ln y$
Separating the variables and integrating, we get:

$$\int y \ln y \, dy = -\int x \, dx + C \Rightarrow \frac{y^2 \ln y}{2} - \int \frac{1}{y} \cdot \frac{y^2}{2} dy = -x^2/2 + C$$

$$\Rightarrow \frac{y^2 \ln y}{2} - \frac{1}{2}\int y \, dy = -\frac{x^2}{2} + C$$

$$\frac{y^2 \ln y}{2} - \frac{y^2}{4} = -\frac{x^2}{2} + C \Rightarrow 2y^2 \ln y - y^2 + 2x^2 = K, \quad \text{where } K = 4C$$

This is an O.T of a family.

(iv) $y = x - 1 + c e^{-x}$ (1)

Solution: Differentiate w.r.t x, we get: $y' = 1 - c e^{-x}$. (2)

Adding (1) and (2), we obtain: $y' + y = x$ → $y' = x - y$. This is the slope of equation (1). Hence slope of an O.T is: $dy/dx = -1/(x - y)$ (3)

Equation (3) may be written as: $dx/dy = -(x - y)$ → $dx/dy + x = y$ (4)

We see that equation (4) is a linear differential equation in x with $P(y) = 1$.

The integrating factor is: $e^{\int P(y)dy} = e^{\int 1.dy} = e^y$.

Multiplying (4) by an I.F, we get: $e^y\left(\frac{dx}{dy} + x\right) = ye^y \Rightarrow \frac{d}{dy}(x \times \text{I.F}) = ye^y$

Integrating both sides w.r.t y to get: $x \times \text{I.F} = \int ye^y dy + C \Rightarrow xe^y = e^y(y-1) + c$

Dividing both sides by e^y, we get: $x = (y - 1) + C e^{-y}$.

This is an O.T of given family.

(v) $x = \frac{y^2}{4} + -\frac{c}{y^2}$ (1)

Solution: Differentiate both sides w.r.t x, we get:

$$1 = \frac{2y}{4}y' - \frac{2c}{y^3}y' = \left(\frac{y}{2} - \frac{2c}{y^3}\right)y' = \left(\frac{y^2 - 4c}{2y^3}\right)y' \Rightarrow \frac{dy}{dx} = \left(\frac{2y^3}{y^2 - 4c}\right) \quad (2)$$

From (1), $c = (4xy^2 - y^4)/4$. Substituting this value of c in (2) and simplifying, we get:

$$\frac{dy}{dx} = \left(\frac{y}{y^2 - 2x}\right)$$

This is a slope of equation (1). Hence the slope of an O. T is:

$$\frac{dy}{dx} = -\frac{y^2 - 2x}{y} = \Rightarrow y\frac{dy}{dx} = -y^2 + 2x \Rightarrow y\frac{dy}{dx} + y^2 = 2x \quad (3)$$

Equation (3) is a Bernoulli differential equation. Substituting $z = y^2$ and $dz/dx = 2y \, dy/dx$, we get:

$$\frac{dz}{dx} + 2z = 4x \quad (4)$$

This is a linear differential equation in z. Here $P = 2$, hence an I.F $= e^{\int 2dx} = e^{2x}$.

Multiplying (4) by an I. F, we get: $e^{2x}\left(\frac{dz}{dx} + 2z\right) = 4xe^{2x} \Rightarrow \frac{d}{dx}(z \times \text{I.F}) = 4xe^{2x}$

Integrating both sides, we get: $ze^{2x} = 4\int xe^{2x}dx + C = 2xe^{2x} - e^{2x} + C$.

Dividing both sides by e^{2x} and substituting the value of $z = y^2$, we get:

$$y^2 = (2x-1) + Ce^{-2x}$$

This is the orthogonal trajectory of given family.

(vi) $y = (x-c)^2$ (1)

Solution: From equation (1), we have $\sqrt{y} = x - c$.

Differentiate both sides w.r.t x, we get: $\frac{1}{2\sqrt{y}}\frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = 2\sqrt{y}$.

This is a slope of equation (1). Hence the slope of an O.T is: $\frac{dy}{dx} = \frac{1}{2\sqrt{y}}$

Separating the variables we get: $2\int\sqrt{y}dy = \int 1\,dx + C \quad \Rightarrow 4\frac{y^{3/2}}{3} = x + C$.

This is the required O.T.

(vii) $x^2 + y^2 = 1 + 2cy$ (1)

Solution: Differentiate w.r.t x, we get: $2x + 2yy' = 0 + 2cy'$

$\rightarrow \quad x + yy' = cy'$ (2)

From equation (1), $c = (x^2 + y^2 - 1)/2y$. Substituting in (2) and simplifying, we get:

$x + yy' = (x^2 + y^2 - 1)y'/2y \quad \rightarrow yy' = (x^2 + y^2 - 1)y'/2y - x$

$\rightarrow \quad yy' = [(x^2 + y^2 - 1)\,y' - 2xy]/2y \rightarrow 2y^2y' = (x^2 + y^2 - 1)y' - 2xy$

$\rightarrow \quad (2y^2 - x^2 - y^2 + 1)y' = -2xy \quad \rightarrow (y^2 - x^2 + 1)y' = -2xy$

$\rightarrow \quad y' = -2xy/(y^2 - x^2 + 1)$. This is the slope of equation (1).

Hence slope of O.T is: $y' = +(y^2 - x^2 + 1)/2xy$

$\rightarrow \frac{dy}{dx} = \frac{y^2}{2xy} + \frac{1-x^2}{2xy} \Rightarrow y\frac{dy}{dx} - \frac{1}{2x}y^2 = \frac{1-x^2}{2x}$ (3)

Equation (3) is a Bernoulli differential equation.

Substituting $y^2 = z \rightarrow yy' = z'/2$, equation (3) after simplifying becomes:

$$\frac{dz}{dx} - \frac{1}{x}z = \frac{1-x^2}{x} \qquad (4)$$

Equation (4) is a linear differential equation with $P(x) = -1/x$. Thus its I.F is:

$$e^{-\int 1/x\,dx} = e^{-\ln x} = x^{-1} = 1/x$$

Multiplying equation (4) by an I.F, we get:

$$\frac{1}{x}\left(\frac{dz}{dx} - \frac{1}{x}z\right) = \frac{1-x^2}{x^2} \Rightarrow \frac{d}{dx}(z \times \text{I.F}) = \frac{1-x^2}{x^2}$$

Integrating both sides and then substituting $z = y^2$, we get

$z \times \text{I.F} = \int (x^{-2} - 1)\,dx + C \quad \Rightarrow y^2.\frac{1}{x} = -x^{-1} - 1 + C = -\left(\frac{1+x}{x}\right) + C$

$\Rightarrow y^2 = -(1+x) + Cx$. This is an O.T of given family (1).

Orthogonal Trajectories in Polar Coordinates

If equation of a curve is given in polar form $f(r, \theta, c) = 0$, the following working rule will help to find the orthogonal trajectories.

Step 1. Differentiate given equation $f(r, \theta, c) = 0$ w.r.t r, eliminate an arbitrary constant and find $r\frac{d\theta}{dr} = F(r, \theta)$. This is slope of given family $f(r, \theta, c) = 0$.

Step 2. The slope of O.T is obtained by replacing the right side by $-1/F(r, \theta)$, that is, $r\frac{d\theta}{dr} = -1/F(r, \theta)$

Step 3. Solving this equation to obtain required O.T.

Example 06: Prove that the orthogonal trajectories of $r^n \cos n\theta = a^n$ is $r^n \sin n\theta = c^n$.

Solution: Given equation of family of curves is: $r^n \cos n\theta = a^n$ (1)

Taking logarithm of both sides, we get

$$\ln r^n \cos n\theta = \ln a^n \Rightarrow \ln r^n + \ln \cos n\theta = \ln a^n \Rightarrow n\ln r + \ln \cos n\theta = \ln a^n \quad (2)$$

Differentiating (2) with respect to r,we get

$$n\left(\frac{1}{r}\right) + \frac{1}{\cos n\theta}(-\sin n\theta)(n)\frac{d\theta}{dr} = 0 \Rightarrow -\tan n\theta\frac{d\theta}{dr} = -\frac{1}{r} \Rightarrow r\frac{d\theta}{dr} = \cot n\theta \quad (3)$$

This is slope given family of curves (1). Hence slope of O.T is

$$r\frac{d\theta}{dr} = -\tan n\theta$$

Separating the variables, we get

$$-\frac{1}{\tan n\theta}d\theta = \frac{1}{r}dr \Rightarrow -\cot\ n\theta d\theta = \frac{1}{r}dr.$$

Integrating,

$$-\int \cot n\theta d\theta = \int \frac{1}{r}dr + \ln c_1 \Rightarrow -\frac{\ln \sin n\theta}{n} = \ln r + \ln c_1$$

$$\Rightarrow \frac{1}{n}\ln \sin n\theta + \ln r = \ln c \quad (-\ln c_1 = \ln c)$$

$$\ln \sin n\theta + n\ln r = n\ln c \Rightarrow \ln \sin n\theta + \ln r^n = \ln c^n$$

$$\Rightarrow \ln r^n \sin n\theta = \ln c^n \Rightarrow r^n \sin n\theta = c^n$$

This is an O.T of given family (1).

Example 07: Find an orthogonal trajectory of

(i) $r = a(1 + \sin\theta)$ (1)

Solution: From (1) we have $a = r/(1 + \sin\theta)$. Now differentiating equation (1) w.r.t r to get: $1 = a(0 + \cos\theta)\, d\theta/dr$.

Substituting the value of a, we get: $1 = \frac{r\cos\theta}{(1+\sin\theta)}.\frac{d\theta}{dr} \Rightarrow r\frac{d\theta}{dr} = \frac{1+\sin\theta}{\cos\theta}$

This is the slope of equation (1). Hence the slope of O.T is:

$$r\frac{d\theta}{dr} = -\frac{\cos\theta}{1+\sin\theta}$$

Separating the variables and integrating, we get:

$$\int \frac{1+\sin\theta}{\cos\theta}\,d\theta = -\int \frac{1}{r}dr + \ln C \Rightarrow \int \sec\theta\, d\theta + \int \tan\theta\, d\theta = -\ln r + \ln C$$

$$\ln(\sec\theta + \tan\theta) + \ln \sec\theta = \ln C/r \quad \Rightarrow \ln \sec\theta(\sec\theta + \tan\theta) = \ln C/r$$

$$\Rightarrow \sec\theta(\sec\theta + \tan\theta) = \frac{C}{r} \quad \Rightarrow r\sec\theta(\sec\theta + \tan\theta) = C$$. This is the required O.T.

(ii) $r^2 = a \sin 2\theta$ (1)

Solution: From (1) we have $a = r^2/\sin 2\theta$. Now differentiating equation (1) w.r.t r to get:

$$2r = a(2\cos 2\theta)\, d\theta/dr \quad \rightarrow r = a(\cos 2\theta)\, d\theta/dr$$

Substituting the value of a, we get:

$$r = \frac{r^2\cos 2\theta}{\sin 2\theta}.\frac{d\theta}{dr} \Rightarrow r\frac{d\theta}{dr} = \frac{\sin 2\theta}{\cos 2\theta}$$

This is the slope of equation (1). Hence the slope of O.T is:

$$r\frac{d\theta}{dr} = -\frac{\cos 2\theta}{\sin 2\theta}$$

Separating the variables and integrating, we get:

$$\int \frac{\sin 2\theta}{\cos 2\theta}\, d\theta = -\int \frac{1}{r} dr + \ln C' \Rightarrow \int \tan 2\theta\, d\theta = -\ln r + \ln C' \quad \Rightarrow \frac{1}{2}\ln \sec 2\theta = \ln \frac{C'}{r}$$

$$\ln \sec 2\theta = 2\ln(C'/r) \quad \Rightarrow \ln \sec 2\theta = \ln(C'/r)^2 \quad \Rightarrow \quad \sec 2\theta = (C'/r)^2$$

$$\Rightarrow \quad r^2 \sec 2\theta = C \quad [C'^2 = C]$$

This is the required O.T.

(iii) $r^n = a^n \cos n\theta$

Solution: From (1) we have $a^n = r^n/\cos n\theta$. Now differentiating equation (1) w.r.t r to get: (1)

$nr^{n-1} = a^n(-n \sin n\theta)\, d\theta/dr \quad \rightarrow r^n . r^{-1} = a^n(-\sin n\theta)\, d\theta/dr$

Substituting the value of a^n, we get:

$$r^n . r^{-1} = \frac{r^n(-\sin\theta)}{\cos n\theta}.\frac{d\theta}{dr} \quad \Rightarrow r\frac{d\theta}{dr} = -\frac{\cos n\theta}{\sin n\theta}$$

This is the slope of equation (1). Hence the slope of O.T is:

$$r\frac{d\theta}{dr} = +\frac{\sin n\theta}{\cos n\theta}$$

Separating the variables and integrating, we get:

$$\int \frac{\cos n\theta}{\sin n\theta}\, d\theta = \int \frac{1}{r} dr + \ln C' \Rightarrow \int \cot n\theta\, d\theta = \ln r + \ln C \quad \Rightarrow \frac{1}{2}\ln \sin n\theta = \ln Cr$$

$$\ln \sin n\theta = 2\ln(Cr) \quad \Rightarrow \ln \sin n\theta = \ln(Cr)^2$$

$$\Rightarrow \quad \sin n\theta = (Cr)^2 \quad \Rightarrow \sin n\theta = Kr^2 \quad [\ K = C^2]$$

This is the required O.T.

(iv) $r = a/(2 + \cos\theta)$

Solution: From (1) we have: $r(2 + \cos\theta) = a$. (1)

Now differentiating equation (1) w.r.t r to get:

$$r(-\sin\theta)\, d\theta/dr + (2 + \cos\theta) = 0 \quad \Rightarrow r\frac{d\theta}{dr} = \frac{2+\cos\theta}{\sin\theta}$$

This is the slope of equation (1). Hence the slope of O.T is: $r\frac{d\theta}{dr} = -\frac{\sin\theta}{2+\cos\theta}$

Separating the variables and integrating, we get:

$$\int \frac{2+\cos\theta}{\sin\theta}\, d\theta = -\int \frac{1}{r} dr + \ln C' \Rightarrow 2\int \operatorname{cosec}\theta\, d\theta + \int \cot\theta\, d\theta = -\ln r + \ln C$$

$$2\ln(\operatorname{cosec}\theta - \cot\theta) + \ln \sin\theta = \ln(C/r) \quad \Rightarrow \ln \sin\theta(\operatorname{cosec}\theta - \cot\theta)^2 = \ln(C/r)$$

$$\Rightarrow \quad \sin\theta(\operatorname{cosec}\theta - \cot\theta)^2 = C/r \quad \Rightarrow r\sin\theta(\operatorname{cosec}\theta - \cot\theta)^2 = C$$

This is the required O.T.

(v) $r^n = a \sin n\theta$

Solution: From (1) we have: $a = r^n/\sin n\theta$. (1)

Now differentiating equation (1) w.r.t r to get:

$$n\, r^{n-1} = a\,(n \cos n\theta)\, d\theta/dr$$

Substituting the value of a, we get:

$$r^n r^{-1} = \frac{r^n(\cos n\theta)}{\sin n\theta}.\frac{d\theta}{dr} \quad \Rightarrow r\frac{d\theta}{dr} = \frac{\sin n\theta}{\cos n\theta} = \tan n\theta$$

This is the slope of equation (1). Hence the slope of O.T is:

$$r\frac{d\theta}{dr} = -\cot n\theta$$

Separating the variables and integrating, we get:

$$\int \frac{1}{\cot n\theta}\,d\theta = -\int \frac{1}{r}\,dr + \ln C \quad \Rightarrow \int \tan n\theta\, d\theta = -\ln r + \ln C \quad \Rightarrow \frac{1}{n}\ln \sec n\theta = \ln\frac{C}{r}$$

$$\ln \sec n\theta = n\ln(C/r) \quad \Rightarrow \ln \sec n\theta = \ln(C/r)^n \quad \Rightarrow \quad \sec n\theta = (C/r)^n$$

$$\Rightarrow \quad r^n \sec n\theta = K \quad [C^n = K]$$

This is the required O.T.

WORKSHEET 05

1. Solve the following differential equations by separating the variables method.

a. $y - x y' = a(y^2 + y')$

b. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

c. $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0$

d. $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0$

e. $(x - xy^2)\, dx + (y - yx^2)\, dy = 0$

f. $x(1 + y^2)\, dx + y(1 + x^2)\, dy = 0$

g. $(e^x + 1)\cos x\, dx + e^x \sin x\, dy = 0$

h. $y' \tan y = \sin(x + y) + \sin(x - y)$

[Hint: For part (h) use the formula: sin a + sin b = 2 sin(a + b)/2 cos(a – b)/2]

i. $\ln(y') = ax + by \rightarrow y' = e^{ax + by}$

j. $(x + y)^2 y' = a^2$

k. $y' = (4x + y + 1)^2$

l. $y' = (x + y)^2$

m. $y' = x^2(1 + y^2)$; when $x = 0$

n. $\cos u\, du - \sin v\, dv$; $(u, v) = (\pi/2, \pi/2)$

o. $dx + e^x y^2\, dy = 0$; when $x = 0$

p. $y' = \sin(x + y)$

q. $y' = 1 + \tan(y - x)$

r. $(4x + y)\, y' = 1$

s. $y' = x(2\ln x + 1)/(\sin y + y\cos y)$

t. $y' = [\sin x + \ln x/x]/[\cos y - \sec^2 y]$

u. $y' = e^{x - y} + x^2 e^{-y}$

v. $y' = e^{x - y} - 1$

2. Solve the following homogeneous differential equations

a. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

b. $(x^2 + y^2)\, dx - 2xy\, dy = 0$

c. $x^2 y\, dx - (x^3 + y^3)\, dy = 0$

d. $x y' - y = (x^2 + y^2)^{1/2}$

e. $y' = (y/x) + \sin(y/x)$

f. $y' = (y/x) + \tan(y/x)$

g. $(1 + e^{x/y})\, dx + e^{x/y}(1 - x/y)\, dy = 0.$

h. $y' - (y/x) = \tan(y/x)$

i. $[x\cos(y/x) + y\sin(y/x)]y = x[y\sin(y/x) - x\cos(y/x)]\, y'$

j. $x y' = y[\ln y - \ln x + 1]$ (NOTE: $\ln a - \ln b = \ln(a/b)$)

3. Solve the following differential equations (Reducible to homogeneous)

a. $y' = [(x + 2y - 3)/(2x + y - 3)]$

b. $(2x + 3y - 5)\, dy + (3x + 2y - 5)\, dx = 0$

c. $(4x + 6y + 3)\, dx = (6x + 9y + 2)\, dy$

d. $(2x + 3y + 4)\, dx - (4x + 6y + 5)\, dy = 0$

e. $(x - y)\, dy = (x + y + 1)\, dx$

f. $y' = (x + y)/x$, $y(1) = 1$

g. $(2x - 5y)\, dx + (4x - y)\, dy = 0$; $y(1) = 4$

h. $\left(y + \sqrt{x^2 + y^2}\right)dx - x\,dy = 0$; $y(1) = 0$

4. Solve the following exact differential equations

a. $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$

b. $(a^2 - 2xy - y^2)\, dx - (x - y)\, dy = 0$

c. $(3x^2 + 6xy^2)\, dx + (x + y^2)\, dy = 0$

d. $(x^2 - ay)\, dx = (ax - y^2)\, dy$

e. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

f. $y\sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

g. $(x^4 - 2xy^2 + y^4)\, dx - (2x^2 y - 4xy^3 + \sin y)\, dy = 0$

h. $\cos x(\cos x - \sin a \sin y)\, dx + \cos y(\cos y - \sin a \sin x)\, dy = 0$

i. $(2y\sin x\cos x + y^2 \sin x)\, dx + (\sin^2 x + 2y\cos x)\, dy = 0$; $y(0) = 3$

j. $(2xy - 3)\, dx + (x^2 + 4y)\, dy = 0$; $y(1) = 2$

k. $(3x^2y^2 - y^3 + 2x)\, dx + (2x^3 y - 3xy^2 + 1)\, dy = 0$; $y(-2) = 1$

l. $\frac{3-y}{x^2}dx + \frac{y^2-2x}{xy^2}dy = 0;\ y(-1) = 2$

m. $(4x^3 e^{x+y} + x^4 e^{x+y} + 2x)\,dx + (x^4 e^{x+y} + 2y)\,dy = 0;\ y(0) = 1$

5. Find an appropriate Integral Factor and hence solve the following differential equations

a.. $(x^2 + y^2 + 2x)\,dx + 2\,y\,dy = 0$

b. $[x\,y^2 - \exp(1/x^3)]\,dx - x^2\,y\,dy = 0$

c. $(y - 2x^3)\,dx - x(1 - xy)\,dy = 0$

d. $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$

e. $(y + y^3/3 + x^2/2)\,dx + 0.25\,(x + x\,y^2)\,dy$

f. $(x - y^2)\,dx + 2xy\,dy = 0$

g. $(y^4 + 2y)\,dx + x\,y^3 + 2\,y^4 - 4\,x)\,dy = 0$

h. $(2\,x^2\,y^2 + y)\,dx - (x^3\,y - 3\,x)\,dy = 0$

i. $(3\,x^2\,y^4 + 2\,x\,y)\,dx + (2\,x^3\,y^3 - x^2)\,dy = 0$

j. $(x^2y - 2\,x\,y^2)\,dx - (x^3 - 3\,x^2\,y)\,dy = 0$

k. $(3xy^2 - y^3)\,dx - (2x^2\,y - x\,y^2)\,dy = 0$

l. $(y^2x + 2\,x^2\,y^3)\,dx - (x^2\,y - x^3\,y^2)\,dy = 0$

m. $(xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0$

n. $y\,(xy \sin xy + \text{cox}\,xy)\,dx + x\,(xy \sin xy - \cos xy)\,dy = 0$

o. $(y^2 + 2\,x^2\,y)\,dx + (2\,x^3 - x\,y)\,dy = 0$

p. $(2\,x^2\,y - 3\,y^4)\,dx + (3x^3 + 2\,x\,y^3)\,dy = 0$

q. $y(1 + xy)\,dx + x(1 - xy)\,dy = 0$

r. $y \ln y\,dx + (x - \ln y)\,dy = 0$

s. $y\,(1 + xy)\,dx + x\,(1 + xy + x^2\,y^2)\,dy = 0$

t. $(y^3 - 3xy^2\,dx + (2x^2y - xy^2)\,dy = 0$

6. Solve the following linear/Bernoulli differential equations

a. $(1 + x^2)\,y` + 2\,x\,y = 4\,x^2$

b. $y` + y \sec x = \tan x$

c. $y` + y/x = x^2$

d. $y` = y \tan x - 2 \sin x$

e. $\sec x\,y` = y + \sin x$

f. $y` + y \tan x = \sec x$

g. $(1 + x^2)\,y` + 2\,x\,y = \cos x$

h. $x \ln x\,y` + y = 2 \ln x$

i. $(x + 2y^3)\,y` = y$

j. $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$

k. $(2x - 10y^3)\,y` + y = 0$

m. $y` = y/x + y^2$

n. $y` = (x + y + 1)/(x + 1)$

o. $2\,y` = y/x + y^2/x^2$

p. $y` = x^3\,y^3 - xy$

q. $\frac{dr}{d\theta} + r \tan\theta = \cos^2\theta;\ r\left(\frac{\pi}{4}\right) = 1$

r. $(x^2 + 1)\,y` + 4xy = x;\ y(2) = 1$

s. $x(2 + x)\,y` + 2(1 + x)\,y = 1 + 3x^2;\ y(-1) = 1$

t. $y` + 2xy = 2x^3;\ y(0) = 1$

u. $y` + 2y/x = -\,x^9y^5;\ y(-1) = 2$

v. $y` + y/x = y^2 \sin x$

w. $xy - y` = y^3 \exp(-x^2)$

7. Find the orthogonal trajectories for the following curves:

a. $xy = c$

b. $y^2 = 4\,a\,x$

c. $x^{2/3} + y^{2/3} = a^{2/3}$

d. $a\,y^2 = x^3$

e. $r = a\,(1 - \cos\theta)$

f. $r = a\,(1 + \cos\theta)$

g. $r = a^{\theta}$

h. $r^n \sin n\theta = a^n$

i. $r^2 = C \sin 2\theta$

8. Find the member of the orthogonal trajectories for $x + y = c\,e^y$ that passes through (0, 5).

CHAPTER SIX

APPLICATIONS OF DIFFERENTIAL EQUATIONS OF FIRST ORDER

6.1 INTRODUCTION

Linear first order differential equations have various applications in Physical sciences and Engineering. To solve physical problems one needs the techniques of differential equations. If a physical problem is translated into a mathematical model then quite often we obtain a set of differential equations along with conditions and the solution of differential equations provides the solution of the physical problem. In this section we shall discuss applications of differential equations to physical and engineering problems.

Growth and Decay Problems

Through experiments scientists have learnt that population of bacterial culture, population of people etc; grows or decays at a rate proportional to the population present or otherwise. That is, if A(t) denotes the amount of substance that is either growing or decaying, then we define dA/dt as the rate of change of this amount of substance with respect to time and according to the given law:

$$\frac{dA}{dt} \propto A \text{ or } \frac{dA}{dt} = kA \text{ or } \frac{dA}{dt} - kA = 0 \qquad (1)$$

where k is constant of proportionality. We shall now discus few problems regarding such applications.

Example 01: The population of a town was 1000 a year ago and the present population is 2000. What will be the population after the end of 4^{th} year if the rate of increase in population is proportional to present population?

Solution: Let P be the population at time t. The initial population was 1000 so we can say that P = 1000 when t = 0. If unit of time is year then at t = 1, P = 2000. Since rate of growth is proportional to P, therefore,

$$\frac{dP}{dt} \propto P \text{ or } \frac{dP}{dt} = kP \qquad (1)$$

where k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore

$$\frac{dP}{P} = k\,dt \Rightarrow \int \frac{1}{P} dP = k\int 1dt + c \Rightarrow \ln P = kt + c$$

$$\Rightarrow P = e^{kt+c} \Rightarrow P = e^c e^{kt} \Rightarrow P = me^{kt}, \quad \left(e^c = m\right) \qquad (2)$$

This is general solution of differential equation (1).

Now N = 1000 when t = 0, therefore, $1000 = m\,e^0 \rightarrow m = 1000$

Thus (2) becomes $P = 1000\,e^{kt}$ (3)

Also, N = 2000 when t = 1 and so $2000 = 1000\,e^k \rightarrow e^k = 2$

Thus, from (3) we have: $P = 1000(e^k)^t \Rightarrow P = 1000(2)^t$

Thus at the end of 4^{th} year the population of the town will be:

$$P = 1000\,(2)^4 = 16,000$$

Example 02: A mold grows at a rate proportional to itself. At the beginning, the amount was 2 grams. In 2 days the amount increased to 3 grams. Find the amount after 8 days?

Solution: Let A be the amount of mold in grams at any time t, and unit of time be a day at time t. Then according to the law of growth, we have

$$\frac{dA}{dt} \propto A \quad \text{or} \quad \frac{dA}{dt} = kA \qquad (1)$$

where k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore

$$\frac{dA}{A} = kdt \Rightarrow \int \frac{1}{A} dA = k\int 1dt + c \Rightarrow \ln A = kt + c$$

$$\Rightarrow A = e^{kt+c} \qquad \Rightarrow A = e^c e^{kt} \qquad \Rightarrow A = me^{kt}, \qquad (e^c = m) \qquad (2)$$

This is general solution of differential equation (1).

Now, at $t = 0$, $A = 2$. Then from (2), we have $\quad 2 = m\, e^{k(0)} \rightarrow m = 2.$

Thus, (2) becomes: $\quad A = 2\, e^{kt} \qquad (3)$

Now at $t = 2$, $A = 2$ hence from (3), we have:

$$3 = 2e^{2k} \Rightarrow e^{2k} = 3/2 \Rightarrow e^k = (3/2)^{1/2}.$$

Thus, (3) becomes $\quad A = 2(3/2)^{t/2}.$

When $t = 8$, $\quad A = 2(3/2)^4 = \frac{1}{2^3}(3)^4 = \frac{81}{8} \Rightarrow A \approx 10$ gms.

Hence the amount of mold will be 10 grams (approximately) after 8 days.

Example 03: Bacteria grow in a nutrient solution at a rate proportional to the amount present. Initially, there are 250 strands of bacteria in the solution that grows to 800 strands after seven hours. Find (a) an expression for the approximate number of strands in the culture at any time t (b) the time needed for the bacteria to grow to 1600 strands.

Solution: (a) Let N(t) denote the number of bacteria strands in the culture at any time t. Then according to law of growth, we have

$$\frac{dN}{dt} \propto N \quad \text{or} \quad \frac{dN}{dt} = kN, \qquad (1)$$

where k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and

$$\frac{dN}{N} = kdt \Rightarrow \int \frac{1}{N} dN = k\int 1dt + c \Rightarrow \ln N = kt + c$$

$$\Rightarrow N = e^{kt+c} \quad \Rightarrow N = e^c e^{kt} \quad \Rightarrow N = me^{kt} \quad (e^c = m) \qquad (2)$$

is general solution of differential equation (1).

(b) At $t = 0$, $N = 250$, hence $250 = m\, e^{k(0)} \rightarrow m = 250.$

Thus, (2) becomes: $N = 250\, e^{kt} \qquad (3)$

At $t = 7$, $N = 800$, hence

$$800 = 250\, e^{7k} \rightarrow e^{7k} = 3.2 \rightarrow 7k = \ln 3.2 \rightarrow k = 0.\,166$$

Thus, (3) becomes: $\quad N = 250\, e^{0.166\,t} \qquad (4)$

which is the required expression for the approximate number of strands in the culture at any time t.

We require t when $N = 1600$. Substituting the N into (4), we get

$$1600 = 250e^{0.166t} \Rightarrow e^{0.166t} = 6.4 \Rightarrow 0.166t = \ln(6.4) \Rightarrow t \approx 11.2 \text{ hours.}$$

Hence 11.2 hours are needed for bacteria to grow to 1600 strands.

Example 04: The population of a certain country has grown at a rate proportional to number of people in the country. At present, the country has 80 million inhabitants. Ten years ago it had 70 million. Assuming that this trend continues, find (a) an expression for the approximate number of people living in the country at time t (taking t = 0 to be the present time) and (b) the approximate number of people at the end of the next ten year period.

Solution: (a) Let P(t) denote the number of people in the country at any time t. Then according to the law of growth, we have

$$\frac{dP}{dt} \propto P \text{ or } \frac{dP}{dt} = kP, \qquad (1)$$

where k is called constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore

$$\frac{dP}{P} = kdt \Rightarrow \int \frac{1}{P} dP = k\int 1dt + c \Rightarrow \ln P = kt + c$$

$$\Rightarrow P = e^{kt+c} \Rightarrow P = e^c e^{kt} \Rightarrow P = me^{kt}, \quad \left(e^c = m\right) \qquad (2)$$

is general solution of the differential equation (1).

(b) At t = 0; P = 80 million ➔ $80 = m\,e^{k(0)}$ ➔ 80 = m.

Thus, (2) becomes: $P = 80\,e^{kt}$ (3)

At t = -10 , P = 70 million:

➔ $70 = 80\,e^{-10k}$ ➔ $e^{-10k} = 7/8$ ➔ $-10\,k = \ln(7/8)$ ➔ $k = 0.01335$

Thus, (3) becomes $P = 80\,e^{0.01335\,t}$ million.

At t = 10, $P = 80\,e^{0.01335(10)} = 91.43$ million.

Example 05: The population of a certain state is known to grow at a rate proportional to the number of people presently living in the state. If after 10 years the population has trebled and if after 20 years the population is 150,000, find the number of people initially living in the state.

Solution: Let P(t) denote the population in the state at any time t. Then according to the law of growth, we have

$$\frac{dP}{dt} \propto P \text{ or } \frac{dP}{dt} = kP \qquad (1)$$

where k is constant of proportionality. Equation (1) is clearly a separable first order differential equation and

$$\frac{dP}{P} = kdt \Rightarrow \int \frac{1}{P} dP = k\int 1dt + c \Rightarrow \ln P = kt + c$$

$$\Rightarrow \quad P = e^{kt+c} \Rightarrow P = e^c e^{kt} \quad \Rightarrow \quad P = me^{kt}, \quad \left(e^c = m\right) \qquad (2)$$

is general solution of differential equation (1).

At t = 10, P = 3 P_0 ➔ $3P_0 = m\,e^{10k}$ (3)

At t = 20, N = 150, 000 ➔ $150,000 = m\,e^{20k}$ (4)

At t = 0, let P = P_0. Thus, (2) becomes $P_0 = m\,e^0$ ➔ $m = P_0$

Substituting in (2), we have

$3P_0 = P_0\,e^{10k}$ ➔ $3 = e^{10k}$ ➔ $10k = \ln 3$ ➔ $k = 0.11$

Thus equation (4) becomes $150,000 = me^{20(0.11)} \Rightarrow m = \dfrac{150,000}{e^{2.2}} \Rightarrow m = 16,620$.

Thus, the number of people initially living in the state is m = P0 =16,620.

Example 06: A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there is 50 milligrams of the material present and after two hours it is observed that the material has lost 10 percent of its original mass, find (a) an expression for the mass of the material remaining at any time t, (b) the mass of the material after four hours, and (c) the time at which the material has decayed to one half of its initial mass.

Solution: (a) Let A denote the amount of material present at any time t. Then accordingly, we have

$$\frac{dA}{dt} \propto A \quad \text{or} \quad \frac{dA}{dt} = kA \qquad (1)$$

where k is constant of proportionality. Equation (1) is clearly a separable first order differential equation and, therefore

$$\frac{dA}{A} = kdt \Rightarrow \int \frac{1}{A}dA = k\int 1dt + c \Rightarrow \ln A = kt + c$$

$$\Rightarrow A = e^{kt+c} \Rightarrow A = e^c e^{kt} \Rightarrow A = me^{kt}, \qquad \left(e^c = m\right) \qquad (2)$$

is general solution of the differential equation (1).

(b) At $t = 0$, we are given that $A = 50$. Therefore, from (2),we have

$$50 = m\,e^{k(0)} \rightarrow m = 50$$

Thus, (2), becomes: $\quad A = 50\,e^{kt} \qquad (3)$

At $t = 2$, 10 percent of the original mass of 50 mg, that is, 5mg, is lost/decayed. Hence, at $t = 2$, $A = 50 - 5 = 45$ mg. Substituting these values into (3), we have

$$45 = 50e^{2k} \Rightarrow e^{2k} = 0.9 \Rightarrow 2k = \ln(0.9) \Rightarrow k \approx -0.053.$$

Hence, equation (3), becomes: $\quad A = 50\,e^{-0.053t} \qquad (4)$

We require A at $t = 4$. Substituting it into (4), we obtain

$$A = 50e^{-0.053(4)} \Rightarrow A = 50e^{-0.212} \Rightarrow A \approx 40.5\text{mg}.$$

(c) We require t when $A = 50/2 = 25$ Substituting it into (4), we obtain

$$25 = 50e^{-0.053t} \Rightarrow -0.053t = \ln(1/2) \Rightarrow t = \frac{-0.693}{-0.053} \Rightarrow t \approx 13\,\text{hours}.$$

Note. The time required reducing a decaying material to one half its original mass is called the **half – life** of the material. For this problem, the half – life is 13 hours.

Investments

The growth of capital compounded continuously is similar to the growth of population, and first order differential equations are helpful in finding the compounded capital.

Example 07: If Rs.10, 000 are invested with annual interest of 10% compounded continuously. What will be the total amount after 5 years?

Solution: Let A denote the amount after t years. Then the rate of growth

$$\frac{dA}{dt} = 10\% \text{ of } A \qquad \text{or} \qquad \frac{dA}{dt} = 0.1A$$

After separating the variables and integrating, we get $A = me^{0.1t} \qquad (1)$

Initially, A(0) = 10, 000, therefore, $10{,}000 = me^{0.1(0)} \Rightarrow m = 10{,}000$.

Equation (1) becomes $A = 10{,}000e^{0.1t} \qquad (2)$

Substituting $t = 5$ in (2), we find the balance after 5 years to be

$$A = 10{,}000e^{0.1(5)} \Rightarrow A \approx 16{,}487.$$

Thus, after 5 years the capital grows to approximately Rs.16, 487.

Example 08: A depositor places \$5000 in an account established for a child at birth. Assuming no additional deposits or withdrawals, how much will the child have upon reaching the age of 21 if the bank pays 5 percent interest per annum compounded continuously for the entire time period?

Solution: Let A denote the amount at any time t.

Then the rate of growth $\frac{dA}{dt} = 5\%A$ or $\frac{dA}{dt} = 0.05N$ (1)

Separating the variables and integrating, we get: $A = me^{0.05t}$

At $t = 0$; $A = 5000$ therefore, $5000 = me^{0.05(0)} \Rightarrow m = 5000$.

Therefore (1) becomes: $A = 5000e^{0.05t}$ (2)

Substituting $t = 21$ in (2), we find the balance after twenty-one years, that is,

$$A(21) = 5000e^{0.05(21)} \Rightarrow A \approx 14,288.$$

Thus, the child will have approximately \$14, 288 reaching at the age of 21 years.

Example 09: How long will it take a bank deposit to triple in value if interest is compounded continuously at a rate of 21/4 percent per annum?

Solution: Let A denote the amount at any time t. Then the rate of growth, that is,

$$\frac{dA}{dt} = 21/4\%A \text{ or } \frac{dA}{dt} = 0.0525\ A \quad (1)$$

Separating the variables and integrating, we get: $N = e^{0.0525\,t}$

At $t = 0$; $A = A_0$, therefore, $A_0 = me^{0.0525(0)} \Rightarrow m = A_0$.

Therefore (1) becomes $A = A_0e^{0.0525t}$. (2)

Now, $A = 3A_0$, then

$$3A_0 = A_0e^{0.0525t} \Rightarrow 3 = e^{0.0525t} \Rightarrow t = \frac{\ln(3)}{0.0525} \Rightarrow t \approx 21 \text{ years.}$$

Thus, bank will take approximately 21 years to make the deposit tripled.

Example 10: A depositor currently has \$6000 and plans to invest it in an account that accrues interest continuously. What interest rate must the bank pay if the depositor needs to have \$10,000 in four years?

Solution: Let A denote the amount invested in the account at any time t. Then the rate of growth

$\frac{dA}{dt} = kA$. Separating variables and Integrating, we get $A = me^{kt}$ (1)

At $t = 0$, $A = 6000$, therefore, $6000 = me^{k(0)} \Rightarrow m = 6000$.

Therefore (1) becomes $N = 6000e^{kt}$ (2)

At $t = 4$, $A = 10,000$, then $1000 = 6000e^{4k} \Rightarrow k = \frac{\ln(5/3)}{4} \Rightarrow k \approx 12.5$.

Thus, bank should pay approximately 12.5% interest.

Example 11: A depositor currently has \$8000 and plans to invest it in an account that accrues interest continuously at the rate of 25/4 percent. How long will it take for the account to grow to \$13,500?

Solution: Let A denote the amount invested in the account at any time t. Then the rate of growth $\frac{dA}{dt} = 25/4\%A$ or $\frac{dA}{dt} = 0.0625A \Rightarrow A = me^{0.0625t}$ (1)

At $t = 0$, $A = 8000$, therefore, $8000 = Ae^{0.0625(0)} \Rightarrow A = 8000$.

Therefore (1) becomes: $A = 8000\,e^{0.0625t}$

When, $A = 13,500$ then we have

$$13500 = 8000e^{0.0625t} \Rightarrow e^{0.0625t} = 1.6875 \Rightarrow 0.0625t = \ln(1.6875) \Rightarrow t \approx 8.4 \quad (2)$$

Thus, approximately 8.4years will take for the amount to grow to \$13500.

Newton's Law of Cooling

A hot body cools at a rate proportional to the difference of temperature of the body and temperature of its surroundings. This law also translates to a first order differential equation.

Example 12: A cup of coffee was initially boiling at 120^{o} C It was placed in air at $20°C$. After 10 minutes the temperature was 80^{o} C. What will be the temperature of coffee after another 10 minutes?

Solution: Let T be the temperature of the cup of coffee at any time t and T_m be the temperature of the surroundings (air). Then according to the **"Newton's Law of Cooling"**, we have

$$\frac{dT}{dt} \propto (T - T_m) \text{ or } \frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{dt} = k(T - 20) \quad (1)$$

From (1), we have $\frac{1}{T-20}dT = kdt \Rightarrow \int \frac{1}{T-20}dT = \int kdt + c$

$\Rightarrow \ln(T-20) = kt + c \Rightarrow T - 20 = e^{kt+c} \Rightarrow T - 20 = e^{kt}e^{c} \Rightarrow T - 20 = me^{kt}$ (2)

Substituting $t = 0$, $T = 120$, we have

$$120 - 20 = me^{k(0)} \Rightarrow m = 100.$$

Now (2) becomes $T - 20 = 100e^{kt}$ (3)

At $t = 0$, $T = 80$, we have: $60 = 100e^{10k} \Rightarrow e^{10k} = \frac{3}{5} \Rightarrow k = \frac{\ln(3/5)}{10} \Rightarrow k \approx -0.0511$.

Thus, (3) becomes $T - 20 = 100e^{-0.0511t}$ (4)

At $t = 20$, we have, $T - 20 = 100e^{-0.0511(20)} \Rightarrow T = 100e^{-1.022} + 20 \Rightarrow T \approx 56°C$

Hence, the temperature of the coffee after another 10 minutes will be approximately 56 °C.

Example 13: A body was heated to 100^{o} C and then placed in a freezer at 0^{o} C. After 30 minutes its temperature was 80^{o} C. How much additional time is required for it to cool to 50^{o} C,

Solution: Let T be the temperature of the body at any time t. Then according to the Newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T - T_m) \text{ or } \frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{dt} = k(T - 0) \Rightarrow \frac{dT}{dt} = kT \quad (1)$$

which has the solution $T = me^{kt}$ (2)

Since at $t = 0$, $T = 100$, we have $100 = me^{k(0)} \Rightarrow m = 100$.

Equation (2) becomes: $T = 100e^{kt}$ (3)

Now at $t = 30$, $T = 80$. Thus we have

$$80 = 100e^{30k} \Rightarrow e^{30k} = \frac{4}{5} \Rightarrow 30k = \ln(4/5) \Rightarrow k = -0.0074.$$

Thus, (3) becomes $\quad T = 100e^{-0.0074t}$ (4)

Now when T = 50, we have

$$50 = 100e^{-0.0074t} \Rightarrow e^{-0.0074t} = 0.5 \Rightarrow -0.0074t = \ln(0.5) \Rightarrow t = 94.$$

Thus, the additional time is $(94-30) = 64$ minutes approximately for the body to cool to 50° C.

Example 14: A body at a temperature of 0° F is placed in a room whose temperature is kept at 100° F. If after 10 minutes the temperature of the body is 25° F find (a) the time required for the body to reach a temperature of 50° F and (b) the temperature of body after 20 minutes.

Solution: Let T be the temperature of the body at any time t. Then according to the Newton's law of cooling, we have

$$\frac{dT}{dt} \propto (T - T_m) \text{ or } \frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{dt} = k(T-100) \Rightarrow \frac{1}{T-100}dT = kdt$$

Integrating, $\quad \int \frac{1}{T-100}dT = k\int 1dt \Rightarrow \ln(T-100) = kt + c \Rightarrow T - 100 = e^{kt+c}$

$$\Rightarrow \quad T - 100 = me^{kt} \quad \Rightarrow \quad T = me^{kt} + 100 \qquad (1)$$

At t = 0, T = 0° F, therefore,

$$0 = me^{k(0)} + 100 \Rightarrow m = -100.$$

Therefore (1) becomes $\quad T = -100e^{kt} + 100$ (2)

At t = 10, T = 25° F, therefore

$$25 = -100e^{10k} + 100 \Rightarrow e^{10k} = \frac{3}{4} \Rightarrow k = \frac{\ln(3/4)}{10} \Rightarrow k = -0.029.$$

Equation (2) becomes $\quad T = -100\,e^{-0.029\,t}$ (3)

(a) When $T = 50°F$:

$$50 = -100e^{-0.029t} + 100 \Rightarrow e^{-0.029t} = 0.5 \Rightarrow t = \frac{\ln(0.5)}{-0.029} \approx 24 \text{ minutes.}$$

(b) When t = 20 minute:

$$T = -100e^{-0.029(20)} + 100 \Rightarrow T \approx 44°F$$

Thus, the temperature of the body after 20 minutes will be approximately $44°F$.

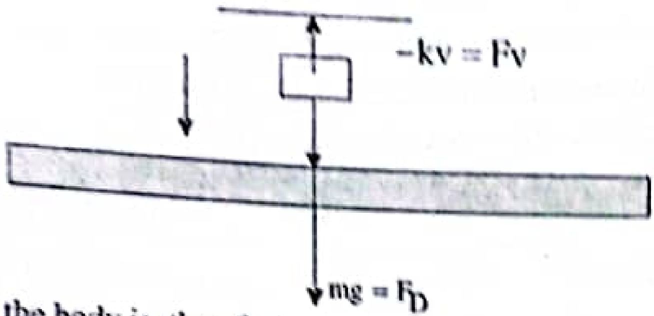
Falling Body Problems

Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction.

Newton's Second Law of Motion: The net force acting on a body is equal to the time rate of change of the momentum of the body; or, for constant mass,

$$F = ma = m\frac{dv}{dt}. \qquad (1)$$

where F is the net force on the body and v is the velocity of the body and a the acceleration at any time t. For the problem at hand, there are two forces acting on the body: (i) the force due to gravity given by the weight w of the body equals mg, and (ii) the force due to air resistance given by $-kv$ where $k \geq 0$ is a constant of proportionality. The minus sign is required because this force opposes the velocity; that is, it acts in the upward, or negative direction. (See the figure):



The net force F on the body is, therefore,

$$F = mg + (-kv) \Rightarrow F = mg - kv.$$

Substituting this in (1) we obtain

$$mg - kv = m\frac{dv}{dt} \Rightarrow g - \frac{kv}{m} = \frac{dv}{dt} \Rightarrow \frac{dv}{dt} + \frac{kv}{m} = g \quad (2)$$

as the equation of motion for falling body.

If air resistance is negligible or non existing, then $k = 0$ and (2) simplifies to

$$dv/dt = g \quad (3)$$

When $k > 0$, the limiting velocity v_l is defined by

$$v_l = mg/k \quad (4)$$

Caution: Equations (2), (3) and (4) are valid only if the given conditions are satisfied. These equations are not valid if, for example, air resistance is not proportional to velocity but to the velocity squared, or if the upward direction is taken to be as positive direction.

Example 15: A body of mass 3 slugs is dropped from a height of 500 feet with zero velocity. Assuming no air resistance, find (a) an expression for the velocity of the body at any time t (b) an expression for the position of body at any time t and (c) the time required for the body to hit the ground?

Solution: (a) Since there is no resistance, therefore $dv/dt = g$, which is the governing equation.

Integrating gives $\int 1dv = g\int 1dt + c \Rightarrow v = gt + c \quad (1)$

When $t = 0$, $v = 0$, hence $0 = g(0) + c \Rightarrow c = 0$.

Thus, (1) becomes: $v = gt$. Assuming $g = 32$ ft/sec^2, we have $v = 32t \quad (2)$

(b) Recall that velocity is the time rate of change of displacement, designated here by x. Hence, $v = dx/dt$ and (2) becomes:

$$\frac{dx}{dt} = 32t \Rightarrow dx = 32tdt \Rightarrow \int 1dx = 32\int tdt + c \Rightarrow x = 16t^2 + c \quad (3)$$

But at $t = 0$, $x = 0$, therefore, $0 = 16(0)^2 + c \Rightarrow c = 0$.

Substituting this value in (3), we obtain: $x = 16t^2 \quad (4)$

(c) We require t when $x = 500$, from (4), we have

$$500 = 16t^2 \Rightarrow t = \sqrt{500/16} \Rightarrow t = 5.6\text{sec}.$$

Example 16: A body of mass 2 slugs is dropped from a height of 450 feet with an initial velocity of 10 ft/sec. Assuming no air resistance, find (a) an expression for the velocity of the body at any time t and (b) the time required for the body to hit the ground.

Solution: (a) Since there is no air resistance, therefore $dv/dt = g$, which is the governing equation.

$$dv = gdt \Rightarrow \int 1dv = g\int 1dt + c \Rightarrow v = gt + c \quad (1)$$

We are given that at $t = 0$, $v = 0$, then from (1), $10 = g(0) + c \rightarrow c = 10$.

Put this in (1),we obtain: $v = 32t + 10$, assuming $g = 32ft/sec^2$.

(b) Since $v = dx/dt$, therefore above equation becomes

$$\frac{dx}{dt} = 32t + 10 \Rightarrow dx = (32t + 10)\,dt \Rightarrow \int 1dx = \int (32t + 10)\,dt + c \Rightarrow x = 16t^2 + 10t + c$$

But at $t = 0, x = 0$, therefore, $0 = 16(0)^2 + 10(0) + c \Rightarrow c = 0$. (2)

Therefore equation (2) becomes $\quad x = 16t^2 + 10\,t$ (3)

We require t when $x = 450$, therefore, from (3), we have

$$450 = 16t^2 + 10t \Rightarrow 16t^2 + 10t - 450 = 0 \Rightarrow 8t^2 + 5t - 225 = 0.$$

Using quadratic formula, we have

$$t = \frac{-5 \pm \sqrt{25 + 7200}}{16} = \frac{-5 \pm 85}{16} = \frac{-5 - 85}{16}, \frac{-5 + 85}{16} \Rightarrow t = -5.625, 5.$$

Neglecting negative sign, we get $t = 5.6$ sec.

Example 17: A ball is propelled straight up with an initial velocity of 250ft/sec in a vacuum with no air resistance. How high will it go?

Solution: Since the direction is upward, therefore v = -25 ft/sec. Because of no air resistance, we have: $\quad dv/dt = g$, which is the governing equation.

$$\rightarrow \quad dv = gdt \Rightarrow \int 1dv = g\int 1dt + c \Rightarrow v = gt + c \quad (1)$$

At $t = 0$, $v = -250$, therefore, $-250 = 32(0) + c \Rightarrow c = -250$. $\left(g = 32ft/sec^2\right)$

Equation (1) becomes: $\quad v = 32\,t - 250$ (2)

Since $v = dx/dt$, therefore

$$\frac{dx}{dt} = 32t - 250 \Rightarrow dx = (32t - 250)\,dt \Rightarrow \int 1\,dx = \int (32t - 250)\,dt + c$$

$$\Rightarrow x = 16t^2 - 250t + c$$

At $t = 0$, $x = 0$, therefore $0 = 16(0)^2 - 250(0) + c \Rightarrow c = 0$

$$\rightarrow \quad x = 16\,t^2 - 250\,t \quad (3)$$

Thus at the maximum height, $v = 0$.Therefore from (2), we have

$$0 = 32t - 250 \Rightarrow t = 7.8125.$$

Substituting this value in (3), we get: $x = 16(7.8125)^2 - 250(7.8125)$

$$\Rightarrow x = -976.5625 \text{ or } x = 976.5625 \quad (\text{ignoring the negativ sign}).$$

Example 18: A body of mass 10 slugs is dropped from a height of 100 feet with no initial velocity. The body encounters an air resistance proportional to its velocity. If the limiting velocity is known to be 320ft/sec, find (a) an expression for the velocity of the body at any time t, (b) an expression for the position of the body at any time t and (c) the time required for the body to attain a velocity of 160ft/sec.

Solution: (a) The limiting velocity is defined to be

$$v_l = mg/k \Rightarrow 320 \text{ or } 10 \times 32 = 320k \quad \Rightarrow k = 1.$$

Equation of motion of the body is

$$\frac{dv}{dt}+\frac{k}{m}v=g \Rightarrow \frac{dv}{dt}+\frac{1}{10}v=32 \Rightarrow \frac{dv}{dt}+0.1v=32 \qquad (1)$$

which is a linear differential equation in v. Therefore, $I.F=e^{\int 0.1dt}=e^{0.1t}$

Thus, (1) becomes:

$$e^{0.1t}\frac{dv}{dt}+0.1ve^{0.1t}=32e^{0.1t} \Rightarrow \frac{d}{dt}\left(e^{0.1t}v\right)=32e^{0.1t} \Rightarrow e^{0.1t}v=32\int e^{0.1t}dt+c$$

or $\quad e^{0.1t}v=32\frac{e^{0.1t}}{0.1}+c \Rightarrow e^{0.1t}v=32e^{0.1t}+c \Rightarrow v=ce^{-0.1t}+320 \qquad (2)$

At t = 0, we are given that v = 0. Substituting these values into (2) we get c = 320. The velocity at any time is therefore given by:

$$v=320e^{-0.1t}+320$$

(b) Since $v=dx/dt$, therefore (3)

$$\frac{dx}{dt}=-320e^{-0.1t}+320 \Rightarrow dx=\left(-320e^{-0.1t}+320\right)dt \Rightarrow \int 1dx=\int\left(-320e^{-0.1t}+320\right)dt$$

or $x=-\frac{320}{-0.1}e^{-0.1t}+320t+c \Rightarrow x=3200e^{-01.t}+320t+c$

When t = 0, x = 0, therefore $0=3200e^{0}+320(0)+c \Rightarrow c=-3200$.

Thus, $x=3200e^{-0.1t}+320t-3200$

(c) Since $v=160\text{ft/sec}^2$. Therefore, from (3), we have

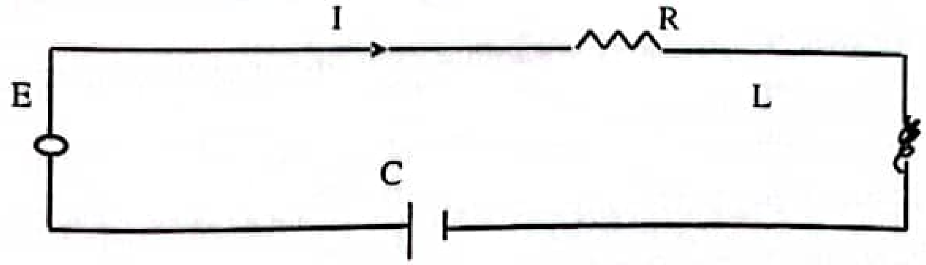
$160=-320e^{-0.1t}+320 \Rightarrow -160=-320e^{-0.1t} \Rightarrow e^{-0.1t}=0.5$

$\Rightarrow -0.1t=\ln(0.5) \Rightarrow t\approx 6.93$ sec.

Electrical Circuits

In a simple electrical circuit like the one shown in following figure:

E is the source of power (known as electromotive force), measured in volts, I is the current produced, measured in amperes, R is resistance of a resistor, measured in ohms, L is inductance of an inductor, measured in henries, and C is capacitance of a capacitor which stores the charge Q, measured in farads.



According to Ohm's Law, $V_R=IR$, where V_R is voltage drop in emf at the resistor. Similarly V_L is the voltage drop in emf at inductor and is given by, $V_L=L\cdot dI/dt$. Voltage drop in emf at the capacitor is $V_C=Q/C$.

Now Kirchoff's law states that the algebraic sum of voltage drops around a closed circuit is equal to total voltage in the circuit. Therefore

$$E=V_R+V_L+V_C \Rightarrow E=IR+L\cdot\frac{dI}{dt}+\frac{1}{C}Q \qquad (1)$$

At present we consider two types of electric circuits, first, in which there is no capacitor. In this case equation (1) becomes

$$E = IR + L\cdot\frac{dI}{dt} \Rightarrow L\cdot\frac{dI}{dt} + IR = E, \qquad (2)$$

The second case when there is no inductor. In this case equation (1) becomes:

$$E = IR + \frac{1}{C}Q \Rightarrow R\cdot\frac{dQ}{dt} + \frac{1}{C}Q = E \qquad (3)$$

Equations (2) and (3) are linear first order differential equations. It may be noted that $I = dQ/dt$.

Example 19: An RL circuit has an e.m.f of 5 volts, a resistance of 50 ohms, an inductance of 1henry, and no initial current. Find the current in the circuit at any time t.

Solution: Given that $E = 5$ volts, $R = 50$ ohms, $L = 1$henry$\cdot$ The governing equation is:

$$L\cdot\frac{dI}{dt} + IR = E.$$

Substituting the given values, we get

$$\frac{dI}{dt} + 50I = 5 \Rightarrow \frac{dI}{dt} + 50I = 5 \qquad (1)$$

This is linear equation; therefore its integrating factor is

$$I.F = e^{\int 50dt} = e^{50t}$$

Multiplying equation (1) by I.F, we get

$$e^{50t}\frac{dI}{dt} + 50e^{50t}I = 5e^{50t} \Rightarrow \frac{d}{dt}\left(e^{50t}I\right) = 5e^{50t}.$$

Integrating, we obtain

$$e^{50t}I = 5\int e^{50t}\,dt + c \Rightarrow e^{50t}I = \frac{1}{10}e^{50t} + c \Rightarrow I = \frac{1}{10} + ce^{-50t} \qquad (2)$$

At $t = 0, I = 0$, then (2) becomes, $0 = \frac{1}{10} + ce^{-50(0)} \Rightarrow c = -\frac{1}{10}$.

Thus equation (2) becomes, $I = \frac{1}{10} - \frac{1}{10}e^{-50t} \Rightarrow I = \frac{1}{10}\left(1 - e^{-50t}\right)$

Example 20: An RL circuit has an e.m.f of 9 volts, a resistance of 10 ohms, an inductance of 1.5 Henry, and an initial current of 6amperes. Find the current in the circuit at any time t.

Solution: Given that $E = 9$ volts, $R = 10$ ohms, $L = 9$henries$\cdot$ The governing equation is:

1.5

$$L\cdot\frac{dI}{dt} + IR = E.$$

Substituting the given values, we get: $(1.5)\frac{dI}{dt} + 10I = 9 \Rightarrow \frac{dI}{dt} + \frac{20}{3}I = 6 \qquad (1)$

This is linear equation; therefore, its integrating factor is

$$I.F = e^{\int \frac{20}{3}t\,dt} = e^{\frac{20}{3}t}$$

Multiplying equation (1) by I.F: $e^{\frac{20}{3}t}\frac{dI}{dt} + \frac{20}{3}e^{\frac{20}{3}t}I = 6e^{\frac{20}{3}t} \Rightarrow \frac{d}{dt}\left(e^{\frac{20}{3}t}I\right) = 6e^{\frac{20}{3}t}$.

Integrating, we obtain: $e^{\frac{20}{3}t}I = 6\int e^{\frac{20}{3}t}dt + c \Rightarrow e^{\frac{20}{3}t}I = 0.9e^{\frac{20}{3}t} + c \Rightarrow I = 0.9 + ce^{-\frac{20}{3}t}$ (2)

At $t=0$, $I=6$ then (2) becomes, $6=0.9+ce^{-\frac{20}{3}(0)}$ $\Rightarrow c=5.1$

Put this in equation (2), we get $I=0.9+5.1e^{-\frac{20}{3}t}$

Example 21: An RL circuit has an e.m.f given by $4\sin t$ (in volts) a resistance of 100 ohms, an inductance of 4henries and no initial current. Find the current at any time t.

Solution: Here we have $E=4\sin t, R=100, L=4$. The governing equation is thus

$$L\cdot\frac{dI}{dt}+IR=E.$$

Substituting the given values, we get

$$4\frac{dI}{dt}+100I=4\sin t\Rightarrow\frac{dI}{dt}+25I=\sin t \quad (1)$$

This equation is linear; therefore we find its integrating factor: $\quad I.F.=e^{\int 25dt}=e^{25t}$.

Multiply equation (1) by integrating factor, we obtain

$$e^{25t}\frac{dI}{dt}+25e^{25t}I=e^{25t}\sin t\Rightarrow\frac{d}{dt}\left(e^{25t}I\right)=e^{25t}\sin t\Rightarrow e^{25t}I=\int e^{25t}\sin t\,dt+c \quad (2)$$

Use integration by parts on right side of (2), we get

$$e^{25t}I=\frac{25}{226}e^{25t}\left(\sin t-\frac{1}{25}\cos t\right)+c\Rightarrow I=\frac{25}{226}\sin t-\frac{1}{226}\cos t+ce^{-25t} \quad (4)$$

At $t=0, I=0$, then (4) becomes, $0=-\frac{1}{226}+c\Rightarrow c=\frac{1}{226}$.

Hence (4) becomes, $I=\frac{25}{226}\sin t-\frac{1}{226}\cos t+\frac{1}{226}e^{-25t}\Rightarrow I=\frac{1}{226}\left(25\sin t-\cos t+e^{-25t}\right)$

Example 22: An RC circuit has an emf given (in volts) by 400 cos 2t a resistance of 100 ohms and a capacitance of 10^{-2} Farad. Initially there is no charge on the capacitor. Find the current in the circuit at any time t.

Solution: Here we have $E=400\cos 2t, R=100, C=10^{-2}$. The governing equation is therefore

$$R\cdot\frac{dQ}{dt}+\frac{1}{C}Q=E.$$

Substituting the given values, we get

$$(100)\frac{dQ}{dt}+\frac{1}{10^{-2}}Q=400\cos 2t\Rightarrow\frac{dQ}{dt}+Q=4\cos 2t \quad (1)$$

This is linear equation and its solution is given by:

$$Q=\frac{8}{5}\sin 2t+\frac{4}{5}\cos 2t+ce^{-t} \quad (2)$$

[Readers are advised to solve this differential equation (1) for Q.

At $t=0, Q=0$, then (2) becomes, $0=(0)+\frac{4}{5}+c\Rightarrow c=-\frac{4}{5}$

Thus, from (2): $\quad Q=\frac{8}{5}\sin 2t+\frac{4}{5}\cos 2t-\frac{4}{5}e^{-t}$

Differentiate w.r.t t: $\quad I=\frac{dQ}{dt}=\frac{d}{dt}\left(\frac{8}{5}\sin 2t+\frac{4}{5}\cos 2t-\frac{4}{5}e^{-t}\right)$

$$I=\frac{8}{5}\cos 2t(2)+\frac{4}{5}(-\sin 2t)(2)-\frac{4}{5}e^{-t}(-1)\Rightarrow I=\frac{2}{5}\left(8\cos 2t-4\sin 2t+2e^{-t}\right)$$

Example 23: An RC circuit has an emf of 100 volts, a resistance of 5 ohms, a capacitance of 0.02 farad, and an initial charge on the capacitor of 5 coulombs. Find (a) the expression for the charge on the capacitor at any time t and (b) the current in the circuit at any time t.

Solution: (a) Here we have $E=100, R=5, C=0.02$. The governing equation is therefore:

$$R\cdot\frac{dQ}{dt}+\frac{1}{C}Q=E.$$

Substituting the given values, we get

$$5.\frac{dQ}{dt}+\frac{1}{0.02}Q=100\Rightarrow\frac{dQ}{dt}+10Q=20 \qquad (1)$$

This is linear equation; therefore its integrating factor: $I.F=e^{\int 10dt}$

Therefore equation (1) becomes

$$e^{10t}\frac{dQ}{dt}+10e^{10t}Q=20e^{10t}\Rightarrow\frac{d}{dt}\left(e^{10t}Q\right)=20e^{10t}\Rightarrow e^{10t}Q=20\int e^{10t}dt+c$$

$$\Rightarrow e^{10t}Q=20e^{10t}\left(\frac{1}{10}\right)+c\Rightarrow e^{10t}Q=2e^{10t}+c\Rightarrow Q=2+ce^{-10t} \qquad (2)$$

At $t=0, Q=5$, then (2) becomes $5=2+ce^{0}\Rightarrow c=3$.

Thus, from (2) we have $Q=2+3e^{-10t}$.

(b) Differentiate w.r.t. t, we get

$$I=\frac{dQ}{dt}=\frac{d}{dt}\left(2+3e^{-10t}\right)=-30e^{-10t}$$

WORKSHEET 06

1. The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after 2 years population has been doubled and after 3 years the population is 20,000, find the number of people initially living in the country.
2. A certain culture of bacteria grows at a rate that is proportional to the number present. If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?
3. The radioactive isotope thorium 234 disintegrates at a rate proportional to the amount present. It is found that in one week 17.96% of this material has disintegrated. Determine how long will it take for one half of this material to disintegrate?
4. A mold grows at a rate proportional to its present size. If the original amount doubles in one day, what proportion of the original amount will be present in five days?
5. A certain radioactive material is known to decay at a rate proportional to the amount present. If initially there are 100 milligrams of the material present and if after two years it is observed that 5 percent of the original mass has decayed, find (a) an expression for the mass at any time t and (b) the time necessary for 10 percent of the original mass to have decayed.
6. A man currently has Rs. 12,000 and plans to invest it in an account that accrues interest continuously. What interest rate must he receive, if his goal is to have Rs. 16,000 in 3 years?

7. How long will it take a bank deposit to double if interest is compounded continuously at a constant rate of 8 percent pr annum?

8. A depositor places Rs. 100,000 in a certificate of deposit account which pays 7 percent interest per annum, compounded continuously. How much will be in the account after 2 years?

9. A man places $700 in an account that accrues interest continuously. Assuming no additional deposits and no withdraws, how much will be in the account after 10 years if the interest rate is a constant $7\frac{1}{2}$ percent for first 6 years and a constant $8\frac{1}{4}$ percent for the last 4 years?

10. A body at a temperature of 50 F° is placed outdoors where the temperature is 100 F°. If after 5 minutes the temperature of the body is 60 F°. Determine the temperature of the body after 20 minutes.

11. A substance cools in air from 100°C to 70 C° in 15 minutes. If the temperature of air is 30 C°, find when the temperature will be 40 C°.

12. One liter of ice cream at a temperature of -15 C° is removed from the deep freezer and placed in a room where the temperature is 20 C°. If after 15 minutes the temperature of the ice cream is -10 C° how long will it take the ice cream to reach a temperature of 0 C°.

13. The temperature of a machine, when it is first shut down after operating, is 220°C and temperature of the surrounding air is 30°C. After 20 minutes, the temperature of the machine is 160 C°. Find the temperature of the machine 30 minutes after it is shut down?

14. A copper ball is heated to a temperature of 100 C° Then at time $t = 0$ it is placed in water which is maintained at a temperature of 30 C°. At the end of 3 minutes the temperature of the ball is reduced to 70 C°. Find the time at which the temperature of the ball is reduced to 31 C°.

15. A body is dropped from a height of 300 feet with an initial velocity of 30 ft/sec. Assuming no air resistance, find (a) an expression for the velocity of the body at any time t and (b) the time required for the body to hit the ground?

16. The magnitude of the velocity (in meters per second) of a particle moving along the x-axis is given by
$v = x/4$,. When $t = 0$, the particle is 2 meters to the right of the origin. Determine the position of the particle when $t = 3$ seconds.

17. A particle initially at rest, moves from a fixed point in a straight line so that its acceleration is given by $\sin t + [1/(t+1)]^2$. What is its distance at the end of π seconds from the start?

18. A particle free to move along a straight line, becomes subject to an acceleration $F_0 \cos pt$. If initially, the particle is at rest at the origin, what is its distance at ant instant?

19. A train starting from rest is accelerated that is given by $10/(v + 1)$ ft/sec, where v is the velocity in ft/sec. Find the distance in which the train attains a velocity of 44 ft/sec.

20. An RC circuit has an emf of 5 volts, a resistance of 10 ohms, a capacitance of 10^{-2} Farad, and initially a charge of 5 coulombs on the capacitor. Find (a) the transient current and (b) the steady – state current.

21. An RC circuit has an emf of $300\cos 2t$ volts, a resistance of 150 ohms, a resistance of $1/6\times10^{-2}$ Farad, and an initial charge on the capacitor of 5 coulombs. Find (a) the charge on the capacitor at any time t and (b) the steady – state current.

22. An RL circuit has a resistance of 10 ohms, an inductance of 1.5 henries, an applied emf of 9 volts, and an initial current of 6 amperes. Find (a) the current in the circuit at any time t and (b) its transient component.

PO;

CHAPTER SEVEN

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

7.1 INTRODUCTION

Higher order differential equations are extensively applied in finding the solution of physical problems. It is not easy, in general to solve a higher order differential equation but particular types of differential equations known as, linear differential equations, are easier to deal with. Although quite a few differential equations that appear in physical problems are linear, nevertheless, learning the solution of linear differential equations is very much important as they provide a foundation for the solution of non-linear differential equations. In this chapter only linear differential equations are studied and discussed.

Definition: A linear differential equation of order n is of the form

$$a_0(x)\frac{d^n y}{dx^n}+a_1(x)\frac{d^{n-1}y}{dx^{n-1}}+\dots+a_{n-1}(x)\frac{dy}{dx}+a_n(x)y=f(x) \qquad (1)$$

where $a_0(x), a_1(x), \dots, a_{n-1}(x), a_n(x)$ and $f(x)$ are functions of independent variable x only and $a_0(x)$ is not zero. If $a_0, a_1, \dots, a_{n-1}, a_n$ are real constants then we have

$$a_0 y^{(n)}+a_1 y^{(n-1)}+\dots+a_{n-1}y'+a_n y=f(x) \qquad (2)$$

Equation (1) is known as differential equation with variable coefficients while (2) is known as differential equation with constant coefficients. We shall first discuss the solution of equation (2).

If f(x) = 0, we get

$$a_0 y^{(n)}+a_1 y^{(n-1)}+\dots+a_{n-1}y'+a_n y=0 \qquad (3)$$

Differential equation (3) is called **homogeneous linear differential equation** of order n. If f(x) is not identically zero then (2) is called **non-homogeneous linear homogeneous differential equation.**

D-Operator

Differential coefficient d/dx sometimes is expressed as an operator D = d/dx. In this case, we write: $\frac{dy}{dx}=Dy, \frac{d^2y}{dx^2}=D^2y, \dots, \frac{d^ny}{dx^n}=D^ny$.

Then equation (2) may be expressed as:

$$a_0D^ny+a_1D^{n-1}y+\dots+a_{n-1}Dy+a_ny=f(x)$$

Or, $$\left(a_0D^n+a_1D^{n-1}+\dots+a_{n-1}D+a_n\right)y=f(x)$$

We may write $F(D) = a_0D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n$.

In this case above equation may be expressed as F(D) y = f(x) and F(D) is regarded as single operator that operates on y.

Definition: If $y_1, y_2, \dots, y_n$ are n functions of an independent variable x and $c_1, c_2, \dots, c_n$

are constants, then $c_1y_1+c_2y_2+...+c_ny_n$ is called a **linear combination** of $y_1,y_2,...,y_n$.

Functions $y_1,y_2,...,y_n$ are called **linearly dependent** if and only if, there exist constants $c_1,c_2,...,c_n$ where at least one of which is nonzero, such that

$$c_1y_1+c_2y_2+...+c_ny_n=0$$

Functions $y_1,y_2,...,y_n$ are called **linearly independent** if and only if, they are not linearly dependent, that is;

$$c_1y_1+c_2y_2+...+c_ny_n=0 \rightarrow c_1=c_2=...=c_n=0.$$

Complementary Function and Particular Integral

Before we talk about the solution of (2) we state the following facts:

- If $y_1,y_2,...,y_n$ are solutions of (3) then any linear combination $c_1y_1+c_2y_2+...+c_ny_n$ is also a solution of (3) where $c_1,c_2,...,c_n$ are arbitrary constants.
- Every homogeneous linear n^{th} - order differential equation
$$a_0y^{(n)}+a_1y^{(n-1)}+...+a_{n-1}y'+a_ny=0,$$
has n linearly independent solutions $y_1,y_2,...,y_n$.
- If $y_1,y_2,...,y_n$ are n linearly independent solutions of (3) then any linear combination $y_c=c_1y_1+c_2y_2+...+c_ny_n$ is also its solution and this solution is called **complementary function** of (2). $c_1,c_2,...,c_n$ being arbitrary constants.
- Equation (2) has two solutions y_c called **complementary function** and y_p the **particular integral**. The complete solution of (2) is therefore $y = y_c + y_p$.
- The particular integral y_p is obtained by solving the equation $y_p=\frac{1}{F(D)}f(x)$.

These statements are also true for equation (1) with variable coefficients.

7.2 HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

Consider the equation

$$a_0\frac{d^ny}{dx^n}+a_1\frac{d^{n-1}y}{dx^{n-1}}+...+a_{n-1}\frac{dy}{dx}+a_ny=0 \quad (1)$$

Using D-operator, equation (1) may be expressed as:

$$(a_0D^n+a_1D^{n-1}+...+a_{n-1}D+a_n)y=0 \quad (1a)$$

where $a_0,a_1,...,a_{n-1},a_n$ are real constants. To find a solution of (1), let us consider a simple case of (1) by taking n = 1. This gives: $a_0y' + a_1y = 0$ (1b)

$a_0\frac{dy}{dx}+a_1y=0 \Rightarrow \frac{dy}{dx}=my,$ where $m=-a_1/a_0$. Separating the variables,

$$\frac{dy}{y}=m\,dx \Rightarrow \int\frac{1}{y}dy=m\int 1.dx+c' \Rightarrow \ln y=mx+c' \Rightarrow y=e^{mx+c'}=ce^{mx}$$

Here $c=e^{c'}$. This shows that $y=ce^{mx}$ is solution of (1b). Now equation (1b) is the special case of (1). This suggests that solution of (1) is also of the form $y=ce^{mx}$.

$$\rightarrow \frac{dy}{dx}=cme^{mx}, \frac{d^2y}{dx^2}=cm^2e^{mx},...,\frac{d^{n-1}y}{dx^{n-1}}=cm^{n-1}e^{mx}, \frac{d^ny}{dx^n}=cm^ne^{mx}.$$

Substituting into (1), we have

$$ce^{mx}\left(a_0m^n+a_1m^{n-1}+...+a_{n-1}m+a_n\right)=0$$

Since $ce^{mx}\neq 0$, we have

$$a_0m^n+a_1m^{n-1}+...+a_{n-1}m+a_n=0 \qquad (2)$$

Equation (2) is called **characteristic or auxiliary equation** of differential equation (1). Observe that (2) can be obtained directly from (1) by merely replacing the k^{th} derivative in (1) by m^k $(k=0,1,2,...,n)$.

Three cases arise according to as the roots of (2) are:

- **Real and Distinct**
- **Real and Repeated**
- **Complex**

CASE I: When the auxiliary equation has real and distinct roots

Let $m_1,m_2,...,m_n$ be n distinct real roots of (2), then $e^{m_1x},e^{m_2x},...,e^{m_nx}$ are n distinct solutions of (1). These n solutions are linearly independent. Therefore, the general solution of (2) is

$$y=c_1e^{m_1x}+c_2e^{m_2x}+...+c_ne^{m_nx},$$

where $c_1,c_2,...,c_n$ are arbitrary constants.

Example 01: Solve the following differential equations:

(i) $(D^3-4D^2+D+6)y=0$

(ii) $(D^3-6D^2+11D-6)y=0;\ y(0)=0=y`(0),\ y``(0)=2$

Solution: (i) Auxiliary equation is

$$m^3-4m^2+m+6=0 \qquad (1)$$

Using hit-and-trial method, if we put $m=-1$ in (1), we get

$$(-1)^3-4(-1)^2+(-1)+6=0\Rightarrow -1-4-1+6=0\Rightarrow 0=0.$$

This implies that m = -1 is a root of the auxiliary equation (1). This means (m + 1) is one of the factors of (1). Now, to find the other roots, we use the synthetic division method* or otherwise,

$$m^3-4m^2+m+6=(m+1)\left(m^2-5m+6\right)=(m+1)(m-2)(m-3)=0 \Rightarrow m=-1,2,3.$$

We observe that roots of auxiliary equation are real and distinct, therefore, general solution of given differential equation is: $y=c_1e^{-x}+c_2e^{2x}+c_3e^{3x}$.

(ii) Auxiliary equation is

$$m^3-6m^2+11m-6=0 \qquad (1)$$

Put $m=1$ into (1) to get

$$(1)^3-6(1)^2+11(1)-6=0\Rightarrow 1-6+11-6=0\Rightarrow 0=0.$$

It implies that (m – 1) is one of the factors of (1) or m = 1 is a root of auxiliary equation (1). Now

$$m^3-6m^2+11m-6=(m-1)\left(m^2-5m+6\right)=(m-1)(m-2)(m-3)=0 \Rightarrow m=1,2,3$$

Other method to find the remaining roots of auxiliary equation (1) is by using the method of **"Synthetic Division"** which is shown below. **[Students are advised to ask their tutors to help them to understand the method].**

$$\begin{array}{c|cccc}
1 & 1 & -6 & 11 & -6 \\
 & & 1 & -5 & 6 \\
\hline
 & 1 & -5 & 6 & \boxed{0}
\end{array}$$

Thus ,

$m^3-6m^2+11m-6=(m-1)\left(m^2-5m+6\right)=(m-1)(m-2)(m-3)=0 \Rightarrow m=1,2,3.$

The roots are real and distinct roots, therefore, general solution is given by

$$y=c_1e^{x}+c_2e^{2x}+c_3e^{3x} \qquad (2)$$

Now,
$$y'=c_1e^{x}+2c_2e^{2x}+3c_3e^{3x} \qquad (3)$$

$$y''=c_1e^{x}+4c_2e^{2x}+9c_3e^{3x} \qquad (4)$$

Now applying the given initial conditions, we get

$$0=c_1+c_2+c_3 \qquad (5)$$

$$0=c_1+2c_2+3c_3 \qquad (6)$$

$$2=c_1+4c_2+9c_3 \qquad (7)$$

Solving equations (5), (6) and (7), we get

$$c_1=1,\ c_2=-2, \text{ and } c_3=1\cdot$$

Now substituting these values in equation (2), we get a particular solution of given differential equation as: $y=e^{x}-2e^{2x}+e^{3x}$

CASE II: When auxiliary equation has real and repeated roots

Given $(a_0D^n+a_1D^{n-1}+\ldots+a_{n-1}D+a_n)\,y=0$ (1)

and the corresponding auxiliary equation is

$$F(m)=(a_0m^n+a_1m^{n-1}+\ldots+a_{n-1}m+a_n)=0 \qquad (2)$$

If $m_1,m_2,\ldots,m_n$ are the roots of $F(m)=0$ then (1) may be written as

$$(D-m_1)(D-m_2)\ldots(D-m_n)y=0\cdot$$

If the root m_1 is repeated twice say $m_1=m_2$, then general solution of (1) is

$$y=c_1e^{m_1x}+c_2e^{m_1x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx} \Rightarrow y=(c_1+c_2)e^{m_1x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx}$$

$$\Rightarrow y=c_0e^{m_1x}+c_3e^{m_3x}+\ldots+c_ne^{m_nx},\ c_1+c_2=c_0$$

This solution contains only (n + 1) arbitrary constants and therefore, it is not the general solution of (1). To obtain general solution we proceed as follows:

The part of general solution of (1) corresponding to twice-repeated root m_1 of equation (2) is the solution of

$$(D-m_1)^2y=0 \text{ or } (D-m_1)(D-m_1)y=0 \qquad (3)$$

Let $(D-m_1)y=v\cdot$ Then (3) becomes

$$(D-m_1)\,v=0\Rightarrow\left(\frac{d}{dx}-m_1\right)v=0\Rightarrow\frac{dv}{dx}-m_1v=0\cdot$$

Separating the variables and integrating, we get

$$\int\frac{1}{v}dv=m_1\int 1dx+k\Rightarrow \ln v=m_1x+k$$

→ $$v=e^{(m_1x+k)}\Rightarrow v=e^{k}e^{m_1x}\Rightarrow v=c_2e^{m_1x}.$$

Replacing v by $(D - m_1)$ y, we obtain

$$(D-m_1)y = c_2e^{m_1x} \Rightarrow \frac{dy}{dx} - m_1y = c_2e^{m_1x} \qquad (4)$$

which is a linear first order differential equation.

$$\text{I.F.} = e^{\int(-m_1)dx} = e^{-m_1x}.$$

Multiplying (4) by the integrating factor, we get

$$e^{m_1x}y' - m_1e^{m_1x}y = c_2 \quad \Rightarrow \frac{d}{dx}\left(e^{m_1x}y\right) = c_2$$

Integrating, we get: $\quad e^{-m_1x}y = c_2x + c_1 \quad \Rightarrow y = (c_2x + c_1)e^{m_1x}$

This is the part of general solution corresponding to repeated root m_1. The general solution of (1) is, therefore

$$y = (c_1 + c_2x)e^{m_1x} + c_3e^{m_3x} + \ldots + c_ne^{m_nx}.$$

Similarly, if auxiliary equation (2) has the root which repeat three times, the corresponding part of the general solution of (1) is the solution of $(D-m_1)^3 y = 0$. Proceeding as before, we can easily find

$$y = (c_1 + c_2x + c_3 x^2)\ e^{m_1x}$$

as the part of general solution corresponding to this triple root m_1. If the auxiliary equation (2) has the real root m_1 occurring k times, then part of the general solution of (1) corresponding to the k times repeated root m_1 is

$$y = \left(c_1 + c_2x + c_3x^2 + \ldots + c_kx^{k-1}\right)e^{m_1x}$$

Example 02: Solve the following differential equations
(i) $(4D^4 - 4D^3 - 3D^2 + 4D - 1)y = 0$ (ii) $(D^2 + 6D + 9)y = 0$, $y(0) = 2$, $y`(0) = -3$.
Solution: (i) Auxiliary equation is

$$4m^4 - 4m^3 - 3m^2 + 4m - 1 = 0 \qquad (1)$$

Put m = 1 in (1), we get: $4(1)^4 - 4(1)^3 - 3(1)^2 + 4(1) - 1 = 0 \Rightarrow 0 = 0$.
It implies that m = 1 is one of the roots of equation (1). To find the other roots, we use the synthetic division method, that is,

1	4	-4	-3	4	-1
		4	0	-3	1
	4	0	-3	1	0

Thus, $4m^4 - 4m^3 - 3m^2 + 4m - 1 = (m-1)\left(4m^3 - 3m + 1\right) = 0.$

Again, by hit and trial method, if we put m = -1, we see that:
$4m^3 - 3m + 1 = -4 + 3 + 1 = 0$. This shows that m = -1 is also the root of auxiliary equation. Thus, again by synthetic division method,

-1	4	0	-3	1
		-4	4	-1
	4	-4	1	0

→ $4m^3 - 3m + 1 = (m+1)\left(4m^2 - 4m + 1\right) = (m+1)(2m-1)(2m-1)$

→ $4m^4 - 4m^3 - 3m^2 + 4m - 1 = (m-1)(m+1)(2m-1)(2m-1) = 0$

→
$$m = 1, -1, 1/2, 1/2.$$
Two roots of auxiliary equation are real, distinct and two are real and repeated, therefore, the general solution of given differential equation is given by
$$y = c_1e^{x} + c_2e^{-x} + (c_3 + c_4x)e^{x/2}.$$
(ii) The auxiliary equation is
$$m^2 + 6m + 9 = 0 \qquad (1)$$
$m^2 + 3m + 3m + 9 = 0 \Rightarrow m(m+3) + 3(m+3) = 0$
$\Rightarrow (m+3)(m+3) = 0 \Rightarrow m = -3, -3.$
The roots are real and repeated, therefore the general solution of given equation is
$$y = (c_1 + c_2x)e^{-3x} \qquad (2)$$
Now
$$y' = (c_1 + c_2x)e^{-3x}(-3) + e^{-3x}(c_2) \qquad (3)$$
Applying the given initial conditions, y(0) = 2 anf y'(0) -3, we get
$$2 = c_1 \text{ and } -3 = -3c_1 + c_2 \Rightarrow c_2 = 3.$$
Substituting these values in (2), we obtain particular solution of given differential equation as: $y = (2+3x)e^{-3x}$

CASE III: When the auxiliary equation has complex roots

Suppose that auxiliary equation has complex number $a+ib$ as a non-repeated root. Since coefficients of auxiliary equation are real and is a polynomial, then the conjugate complex number $a-ib$ is also its non-repeated root. The corresponding part of general solution is: $y = k_1e^{(a+ib)x} + k_2e^{(a-ib)x}$, where k_1 and k_2 are arbitrary constants.

→ $y = k_1e^{ax}e^{ibx} + k_2e^{ax}e^{-ibx} = e^{ax}\left(k_1e^{ibx} + k_2e^{-ibx}\right)$

$= e^{ax}\{k_1(\cos bx + i\sin bx) + k_2(\cos bx - i\sin bx)\}$

$= e^{ax}\{i(k_1 + k_2)\sin bx + (k_1 + k_2)\cos bx\}$ **[Note: $e^{ia} = \cos a + i \sin a$]**

$\Rightarrow \quad y = e^{ax}(c_1\sin bx + c_2\cos bx) \qquad [c_1 = i(k_1 + k_2), c_2 = (k_1 + k_2)]$

It may be noted that if $a+ib$ and $a-ib$ are conjugate complex roots and each one is repeated k times, then the corresponding part of general solution of (1) may be written as
$$y = e^{ax}\left\{\left(c_1 + c_2x + c_3x^2 + \ldots + c_kx^{k-1}\right)\sin bx + \left(c_{k+1} + c_{k+2}x + \ldots + c_{2k}x^{k-1}\right)\cos bx\right\}.$$

Example 03: Solve the following differential equations

(i) $(75D^2 + 50D + 12)y = 0$
(ii) $(D^4 + 2D^3 - 2D^2 - 6D + 5)y = 0$
(iii) $(D^2 + 6D + 13)y = 0$; $y(0) = 3$, $y'(0) = -1$

Solution: (i) Auxiliary equation is: $75m^2 + 50m + 12 = 0 \qquad (1)$
Using quadratic formula, we get
$$m = \frac{-50 \pm \sqrt{(50)^2 - (4)(75)(12)}}{2(75)} = \frac{-50 \pm \sqrt{2500 - 3600}}{150} = \frac{-50 \pm \sqrt{-1100}}{150}$$
$$m = \frac{-50 \pm 10i\sqrt{11}}{150} = \frac{-5 \pm i\sqrt{11}}{15} = \frac{-1}{3} \pm \frac{\sqrt{11}}{15}i$$

The roots of auxiliary equation are complex, therefore general solution is:

$$y = e^{-x/3}\left(c_1 \sin\frac{\sqrt{11}}{15}x + c_2 \cos\frac{\sqrt{11}}{15}x\right).$$

(ii) Auxiliary equation is

$$m^4 + 2m^3 - 2m^2 - 6m + 5 = 0 \qquad (1)$$

Put m = 1in (1), we get

$$(1)^4 + 2(1)^3 - 2(1)^2 - 6(1) + 5 = 0 \Rightarrow 1 + 2 - 2 - 6 + 5 = 0 \Rightarrow 0 = 0.$$

This implies that $(m-1)$ is one of the factors of (1).

Now, $m^4 + 2m^3 - 2m^2 - 6m + 5 = 0$

➔ $$(m-1)(m^3 + 3m^2 + m - 5) = (m-1)(m-1)(m^2 + 4m + 5) = 0$$

Using quadratic formula for $(m^2 + 4m + 5) = 0$, we get

$$m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i.$$

Hence the roots of (1) are $m = 1, 1, -2 \pm i$. Thus general solution of given differential equation is: $y = (c_1 + c_2 x)e^x + e^{-2x}(c_3 \sin x + c_4 \cos x)$.

(iii) Auxiliary equation is: $m^2 + 6m + 13 = 0$ (1)

Using quadratic formula, we get

$$m = \frac{-6 \pm \sqrt{36-52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i.$$

The general solution of given equation is: $y = e^{-3x}(c_1 \sin 2x + c_2 \cos 2x)$ (2)

$$y' = e^{-3x}(2c_1 \cos 2x - 2c_2 \sin 2x) - 3e^{-3x}(c_1 \sin 2x + c_2 \cos 2x) \qquad (3)$$

Applying the given initial conditions, y(0) = 3 and y'(0) = -1, we get

$$3 = c_2 \text{ and } -1 = (2c_1) - 3(c_2) \Rightarrow c_1 = 4.$$

Thus, particular solution of given differential equation is

$$y = e^{-3x}(4\sin 2x + 3\cos 2x)$$

Example 04: Solve the following differential equations

(i) $(D^3 - 2D^2 + 4D - 8)\, y = 0$ **(ii) $(D^4 - 4D^3 + 8D^2 - 8D + 4)\, y = 0$**

Solution: (i) Given $(D^3 - 2D^2 + 4D - 8)\, y = 0$. Auxiliary equation is $m^3 - 2m^2 + 4m - 8 = 0$ ➔ $m = 2, \pm i2$ [Students may verify this]

Thus general solution is: $y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$.

(ii) Given differential equation is $(D^4 - 4D^3 + 8D^2 - 8D + 4)\, y = 0$

Auxiliary equation is: $m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$

➔ $(m^2 - 2m + 2)^2 = 0$ ➔ $(m^2 - 2m + 2)(m^2 - 2m + 2) = 0$

➔ $(m^2 - 2m + 2) = 0$ and $(m^2 - 2m + 2) = 0$

➔ $m = 1 \pm i$ and $m = 1 \pm i$. Thus roots are complx and repeated as well. Hence the general solution is: $y = e^x[(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x]$

Example 05: Solve the following differential equations:

i. $(9D^2 - 12D + 4)\, y = 0$

Solution: Auxiliary equation (A. E.) is:

$$9m^2 - 12m + 4 = 0 \rightarrow m = 2/3, 2/3.$$

Since the roots are real and equal, hence the general solution is:

$$y = (C_1 + C_2\,x)\,e^{2/3\,x}$$

ii. $(D^3 + D^2 + D + 1)\,y = 0$

Solution: Auxiliary equation is:

$$m^3 + m^2 + m + 1 = 0 \rightarrow m^2(m+1) + 1\,(m+1) = 0$$

$\rightarrow \quad (m+1)(m^2+1) = 0 \rightarrow \quad m + 1 = 0$ or $m^2 + 1 = 0 \rightarrow m = -1$ or $m = \pm\, i$.

Since one root of auxiliary equation is real and other two are complex, hence general solution of given differential equation is: $y = C_1\,e^{-x} + (C_2 \cos x + C_3 \sin x)$

iii. $(D^3 - 6\,D^2 + 3\,D + 10)\,y = 0$

Solution: Auxiliary equation is: $m^3 - 6m^2 + 3m + 10 = 0$.

It may be noted that in problem 2, it was easy to find the factors whereas in this problem, we can not do so. Thus we employ "Hit & Trial" method.

Put $m = 1$, we get: $\quad 1 - 6 + 3 + 10 \neq 0$

Put $m = -1$, we get: $\;-1 - 6 - 3 + 10 = 0$

This shows that $m = -1$ is a root of A. Eq. Now to find other two roots, we use the method of **"Synthetic Division"**.

-1	1	-6	3	10
		-1	7	-10
	1	-7	10	0

$\rightarrow m^2 - 7\,m + 10 = 0 \rightarrow (m-2)(m-5) = 0 \quad \rightarrow \quad m = 2, m = 5.$

Thus roots of A. E are $m = -1$, 2 and 5 which are all real and distinct. Hence the general solution of given differential equation is:

$$y = C_1\,e^{-x} + C_2\,e^{\,2x} + C_3\,e^{\,5x}$$

iv. $(D^3 - 27)\,y = 0$

Solution: Auxiliary equation is: $m^3 - 27 = 0$

$\rightarrow (m^3 - 3^3) = 0 \rightarrow (m-3)(m^2 + 3m + 9) = 0 \rightarrow (m-3) = 0$ or $(m^2 + 3m + 9) = 0$

$\rightarrow \quad m = 3$, or $m = \dfrac{3}{2} \pm i\dfrac{3\sqrt{3}}{2}$. Thus one roots of A. E is real and other two are complex.

Hence the general solution of given differential equation is:

$$y = C_1 e^{3x} + e^{\frac{3}{2}x}\left[C_2 \cos\frac{3\sqrt{3}}{2}x + C_3 \sin\frac{3\sqrt{3}}{2}x\right]$$

v. $(D^4 - 5\,D^3 + 6\,D^2 + 4\,D - 8)\,y = 0$

Solution: Auxiliary equation is: $m^4 - 5m^3 + 6m^2 + 4m - 8 = 0$.

Using "Hit & Trial" method, put:

$m = 1$, we get: $\quad 1 - 5 + 6 + 4 - 8 \neq 0$

$m = -1$, we get: $\quad 1 + 5 + 6 - 4 - 8 = 0$

This shows that $m = -1$ is a root of A. E. Now to find other three roots, we use the method of **"Synthetic Division"** once again

-1	1	-5	6	4	-8
		-1	6	-12	8
	1	-6	12	-8	0

$\rightarrow (m^4 - 5m^3 + 6m^2 + 4m - 8) = (m+1)(4m^3 - 6m^2 + 12m - 8) = 0$. Now consider cubic equation: $4m^3 - 6m^2 + 12m - 8 = 0$. Putting

$m = -1$, we get: $\quad -1 - 6 + 12 - 8 \neq 0$

$m = 2$, we get: $8 - 24 + 24 - 8 = 0$

This shows that m = 2 is also a root of A. E. To find the remaining two roots, we once again use **"Synthetic Division"**.

$$\begin{array}{c|cccc} 2 & 1 & -6 & 12 & -8 \\ & & 2 & -8 & 8 \\ \hline & 1 & -4 & 4 & \multicolumn{1}{|c}{0} \end{array}$$

→ $m^2 - 4m + 4 = 0$ → $m = 2, 2$. Thus roots of A. Eq. are $m = -1, 2, 2, 2$. All roots are real where one root is distinct and other three roots are repeated. Thus, general solution of given differential equation is:

$$y = C_1 e^{-x} + (C_2 + C_3 x + C_4 x^2) e^{2x}$$

vi. $(D^4 - 4D^3 - 7D^2 + 22D + 24)\, y = 0$

Solution: Auxiliary equation is: $m^4 - 4m^3 - 7m^2 + 22m + 24 = 0$.

By "Hit & Trial" method, put:

m = 1, we get: $1 - 4 - 7 + 22 + 24 \neq 0$

m = -1, we get: $1 + 4 - 7 - 22 + 24 = 0$

This shows that m = -1 is a root of A. E. Now to find other three roots, we use **"Synthetic Division"** again.

$$\begin{array}{c|ccccc} -1 & 1 & -4 & -7 & 22 & 24 \\ & & -1 & 5 & 2 & -24 \\ \hline & 1 & -5 & -2 & 24 & \multicolumn{1}{|c}{0} \end{array}$$

→ $m^3 - 5m^2 - 2m + 24 = 0$. This is a cubic equation. To find its roots, we put:

m = -1 , we get: $-1 - 5 + 2 + 24 \neq 0$,

m = 2, we get: $8 - 20 - 4 + 24 \neq 0$

m = -2, we get: $-8 - 20 + 4 + 24 = 0$

This shows that m = -2 is also a root of A. E. To find the remaining two roots, we use **"Synthetic Division"**.

$$\begin{array}{c|cccc} -2 & 1 & -5 & -2 & 24 \\ & & -2 & 14 & -24 \\ \hline & 1 & -7 & 12 & \multicolumn{1}{|c}{0} \end{array}$$

→ $m^2 - 7m + 12 = 0$ → $m = 3, 4$. Thus roots of A. Eq. are $m = -1, -2, 3, 4$. All roots are real and distinct. Thus general solution of given differential equation is:

$y = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x} + C_4 e^{4x}$

vii. $(D^4 + 4)\, y = 0$

Solution: A. Eq. is: $m^4 + 4 = 0$

→ $m^4 + 2^2 = 0$ → $(m^2)^2 + 2.m^2.2 + 2^2 = 2.m^2.2$ → $(m^2 + 2)^2 = 4m^2$

Taking square root on both sides, we get: $m^2 + 2 = \pm 2m$

→ $m^2 + 2 = 2m$ or $m^2 + 2 = -2m$

→ $m^2 - 2m + 2 = 0$ or $m^2 + 2m + 2 = 0$

→ $m = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$ or $m = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

The roots of A. Eq. are complex hence the general solution of given differential equation is: $y = e^x (C_1 \cos x + C_2 \sin x) + e^{-x} (C_3 \cos x + C_4 \sin x)$

viii. $(D^4 - D^3 - 3D^2 + D + 2)\, y = 0$

Solution: Auxiliary equation is: $m^4 - m^3 - 3m^2 + m + 2 = 0$.

By "Hit & Trial" method, put:

m = 1, we get: $1 - 1 - 3 + 1 + 2 = 0$

This shows that m = 1 is a root of A. E. Now to find other three roots, we use **"Synthetic Division"**.

1	1	-1	-3	1	2
		1	0	-3	-2
	1	0	-3	-2	0

→ $m^3 + 0\,m^2 - 3m - 2 = 0$. This is a cubic equation. By hit and trial method put:
$m = 1$, we get: $1 + 0 - 3 - 2 \neq 0$. Put $m = 2$, we get: $-8 + 6 + 2 = 0$
This shows that $m = -2$ is also a root of A. E. To find the remaining two roots, we use once again the **"Synthetic Division"**.

2	1	0	-3	-2
		2	4	2
	1	2	1	0

→ $m^2 + 2m + 1 = 0$ → $m = -1, -1$. Thus roots of A. E. are $m = 1, 2, -1, -1$. All roots are real but two of them are repeated. Thus general solution of given differential equation is:

$$y = C_1 e^{x} + C_2 e^{2x} + (C_3 + C_4 x)\,e^{-x}$$

ix. $(16D^6 + 8D^4 + D^2)\,y = 0$

Solution: Auxiliary equation is: $16m^6 + 8m^4 + m^2 = 0$
→ $m^2(16m^4 + 8m^2 + 1) = 0$ → $m^2 = 0$ or $16m^4 + 8m^2 + 1 = 0$.
Now $m^2 = 0$ gives $m = 0, 0$. To solve $16m^4 + 8m^2 + 1 = 0$, we let $m^2 = k$, hence given equation becomes: $16k^2 + 8k + 1 = 0$
→ $k = 1/4, 1/4$ → $m^2 = 1/4, 1/4$ → $m = \pm 1/2, \pm 1/2$.
The roots of A. E. are: 0, 0, 1/2, 1/2, -1/2, -1/2. All six roots are real but pair wise repeated. Thus general solution of given differential equation is:

$$y = (C_1 + C_2 x)e^{0x} + (C_3 + C_4 x)\,e^{x/2} + (C_5 + C_6 x)\,e^{-x/2}$$

x. $(D^4 + 6D^3 + 15D^2 + 20D + 12)\,y = 0$

Solution: Auxiliary equation is: $m^4 + 6m^3 + 15m^2 + 20m + 12 = 0$
By "Hit & Trial" method, put:
$m = -1$, we get: $1 - 6 + 15 - 20 + 12 \neq 0$
$m = -2$, we get: $16 - 48 + 60 - 40 + 12 = 0$
This shows that $m = -2$ is a root of A. Eq. Now to find other three roots, we use **"Synthetic Division"**.

-2	1	6	15	20	12
		-2	-8	-14	-12
	1	4	7	6	0

→ $m^3 + 4m^2 + 7m + 6 = 0$. This is a cubic equation. To find its roots, we put:
$m = -2$, we get: $-8 + 16 - 14 + 6 = 0$.
This shows that $m = -2$ is also a root of A. Eq. To find the remaining two roots, we use once again the **"Synthetic Division"**.

-2	1	4	7	6
		-2	-4	-6
	1	2	3	0

→ $m^2 + 2m + 3 = 0$ → $m = -1 \pm i\sqrt{2}$. Thus roots of A. E. are $m = -2, -2, -1 \pm i\sqrt{2}$. Two of the roots are real and repeated and two are complex. Thus general solution of given differential equation is: $y = (C_1 + C_2 x)\,e^{-2x} + e^{-x}\left(C_3 \cos\sqrt{2}\,x + C_4 \sin\sqrt{2}\,x\right)$

7.3 NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

In this section we shall discuss the solution of non-homogeneous linear differential equation:

$$\left(a_0D^n + a_1D^{n-1} + \ldots + a_{n-1}D + a_n\right)y = f(x) \qquad (1)$$

This differential equation has two solutions: $y = y_c$ and $y = y_p$. y_c is called complementary function and is obtained by solving $F(D)\,y = 0$ as shown in the previous section and y_p is found by solving $y_p = \frac{1}{F(D)}f(x)$. Then complete solution of (1) is:

$$y = y_c + y_p.$$

We shall now discuss how to find y_p depending upon the nature of function $f(x)$.

Working Rule For Finding Particular Integral (P.I)

When $f(x) = e^{ax}$

CASE-I: $y_p = \frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}$, provided $F(a) \neq 0$

Proof: Consider,

$D\,e^{ax} = a\,e^{ax}$, $\quad D^2 e^{ax} = a^2 e^{ax}$, $\quad D^3 e^{ax} = a^3 e^{ax}$,

$\rightarrow D^n e^{ax} = a^n e^{ax}$ $\quad \rightarrow \frac{1}{D^n}e^{ax} = \frac{1}{a^n}e^{ax}$ $\Rightarrow$, $\frac{1}{F(D)}e^{ax} = \frac{1}{F(a)}e^{ax}$, provided $F(a) \neq 0$.

In case $F(a) = 0$ then an other approach is employed. But before we show you this, we prove the following.

> **Prove that** $\frac{1}{(D-a)}f(x) = e^{ax}\int e^{-ax}f(x)\,dx$
>
> **Proof:** Let $\frac{1}{(D-a)}f(x) = y \Rightarrow f(x) = (D-a)y = Dy - ay$
>
> $\therefore y' - ay = f(x)$ which is linear differential equation.
>
> I.F $= e^{-ax}$. Multiplying by I. F and solving, we get:
>
> $ye^{-ax} = \int e^{-ax}f(x)\,dx \quad \Rightarrow y = e^{ax}\int e^{-ax}f(x)\,dx$
>
> Thus, $y = \frac{1}{(D-a)}f(x) = e^{ax}\int e^{-ax}f(x)\,dx$

CASE-II: Above formula for computing y_p is valid provided that $F(a) \neq 0$.
If $F(a) = 0$ then it implies that 'a' is a root of equation:

$$a_0m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n = 0.$$

This means that $(m - a)$ is a factor of $a_0m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n = 0$. In this case $F(D) = (D - a)\,\varphi(D)$. Thus,

$$\frac{1}{F(D)}e^{ax} = \frac{1}{(D-a)\phi(D)}e^{ax} = \frac{1}{\phi(D)}\cdot\frac{1}{(D-a)}e^{ax} = \frac{1}{\phi(D)}e^{ax}\int e^{ax}.e^{-ax}dx. \text{ [See above box]}$$

$$= \frac{1}{\phi(D)}e^{ax}\int 1\,dx = \frac{1}{\phi(D)}xe^{ax}$$

In general, if 'a' is a repeated root of $a_0m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n = 0$, we have

$$\frac{1}{(D-a)^k \varphi(D)} e^{ax} = \frac{x^k e^{ax}}{k!\varphi(a)}, F(a)=0, \text{ where } F(D)=(D-a)^k \phi(D)$$

Another simple method to compute y_p in case $F(a) = 0$ is as follows:

$$\frac{1}{F(D)} e^{ax} = \frac{x}{F'(a)} e^{ax} \text{ provided } F'(a) \neq 0$$

$$= \frac{x^2}{F''(a)} e^{ax} \text{ provided } F''(a) \neq 0 \text{ ,etc.}$$

AN IMPORTANT REMARK: After computation of complementary function y_c you must see whether or not the right side function f(x) contains any function that is present in y_c. If there exists such function in f(x) then F(D) will become zero and therefore we use appropriate case.

Example 01: Find the general solution of

(i) $(D^2 + 3D - 4)y = 15 e^{2x}$ (ii) $(D^2 - 3D + 2)y = e^x + e^{2x}$

Solution: (i) The auxiliary equation is $m^2 + 3m - 4 = 0 \Rightarrow m = 1, -4$.

The Complementary Function is: $y_c = c_1e^x + c_2e^{-4x}$.

Now, $y_p = \frac{1}{(D^2+3D-4)}(15e^{2x}) = 15\frac{1}{2^2+3(2)-4}e^{2x} = \frac{5}{2}e^{2x}$ **[Use Case I]**

Thus general solution of given differential equation is:

$$y_c = c_1e^x + c_2e^{-4x} + 5e^{2x}/2.$$

(ii) The auxiliary equation is: $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, 2$.

Thus, complementary function is: $y_c = c_1e^x + c_2e^{2x}$.

$$\text{Now } y_p = \frac{1}{(D^2-3D+2)}(e^x + e^{2x}) = \frac{1}{(D^2-3D+2)}e^x + \frac{1}{(D^2-3D+2)}e^{2x}$$

$$= \frac{x}{2D-3}e^x + \frac{x}{2D-3}e^{2x} = \frac{xe^x}{(2-3)} + \frac{xe^{2x}}{(4-3)}. \qquad \text{[Use Case II]}$$

Thus, $\quad y_p = -xe^x + xe^{2x}$.

Hence the general solution of the given equation is

$$y = y_c + y_p = c_1e^x + c_2e^{2x} - xe^x + xe^{2x}.$$

REMARK: Observe that function f(x) contains both e^x and e^{2x} which are already in y_c. Thus we use Case II. Students are advised to check this for every problem before finding y_p.

Example 02: Solve the following differential equations

(i) $(D^3 - 2D^2 - 5D + 6) y = e^{3x}$ (ii) $(D^3 - D^2 + 4D - 6) y = e^x$

(iii) $(D^3 + 3D^2 + 3D + 1) y = e^{-x}$ (iv) $(D^3 - 3D - 2) y = e^{2x} + e^x$

(v) $(D^2 - D - 6) y = e^x \cosh 2x$

Solution: (i) Given differential equation is: $(D^3 - 2D^2 - 5D + 6) y = e^{3x}$.

A.E is: $m^3 - 2m^2 - 5m + 6 = 0 \rightarrow m = 1, -2, 3$. Thus, $y_c = A e^x + B e^{-2x} + C e^{3x}$.

$$y_p = \frac{1}{(D^3-2D^2-5D+2)}e^{3x} = \frac{x}{(3D^2-4D-5)}e^{3x} = \frac{x}{(27-12-5)}e^{3x} = \frac{xe^{3x}}{10}$$

[Use Case II]. Thus general solution is:

$$y = y_c + y_p = A e^x + B e^{-2x} + C e^{3x} + xe^{3x}/10$$

(ii) Given equation is $(D^3 - D^2 + 4D - 6)\, y = e^x$
A.E is: $m^3 - m^2 + 4m - 6 = 0 \Rightarrow m = 1, \pm 2i$. Thus, $y_c = A\, e^x + B \cos 2x + C \sin 2x$

$$y_p = \frac{1}{(D^3 - D^2 + 4D - 6)} e^x = \frac{x}{(3D^2 - 2D + 4)} e^x = \frac{x}{(3 - 2 + 4)} e^x = \frac{xe^x}{5} \quad \textbf{[Use Case II]}.$$

Thus general solution is: $y = y_c + y_p = A\, e^x + B \cos 2x + C \sin 2x + xe^x/5$
(iii) Given equation is $(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$
A.E is: $m^3 + 3m^2 + 3m + 1 = 0 \quad \Rightarrow (m+1)^3 = 0 \quad \Rightarrow m = -1, -1, -1$
Thus, $y_c = (A + Bx + C\, x^2)\, e^{-x}$

$$y_p = \frac{1}{(D^3 + 3D^2 + 3D + 1)} e^{-x} = \frac{x}{(3D^2 + 6D + 3)} e^{-x} = \frac{x.x}{(6D + 6)} e^{-x} = \frac{x.x.x}{6} e^{-x} = \frac{x^3 e^{-x}}{6}$$ **[U**

se Case II]. Thus general solution is: $y = y_c + y_p = (A + Bx + C\, x^2)\, e^{-x} + x^3\, e^{-x}/6$
(iv) Given equation is $(D^3 - 3D - 2)\, y = e^{2x} + e^x$
A.E is: $m^3 - 3m - 2 = 0 \quad \Rightarrow m = 2, -1, -1$
Thus, $y_c = A\, e^{2x} + (B + C\, x)\, e^{-x}$

$$y_p = \frac{1}{(D^3 - 3D - 2)}(e^{2x} + e^x) = \frac{1}{(D^3 - 3D - 2)} e^{2x} + \frac{1}{(D^3 - 3D - 2)} e^x$$

$$= \frac{x}{(3D^2 - 3)} e^{2x} + \frac{1}{(1 - 3 - 2)} e^x = \frac{x}{(3.4 - 3)} e^{2x} + \frac{1}{(-4)} e^x = \frac{xe^{2x}}{9} - \frac{e^x}{4} \textbf{[By Cases I \& II]}$$

Thus general solution is: $y = y_c + y_p = A\, e^{2x} + (B + C\, x)\, e^{-x} + x\, e^{2x}/9 - e^x/4$
(v) Given equation is $(D^2 - D - 6)\, y = e^x \cosh 2x$
A.E is: $m^2 - m - 6 = 0 \quad \Rightarrow m = 3, -2$
Thus, $y_c = A\, e^{3x} + B\, e^{-2x}$
Now $e^x \cosh 2x = e^x\, (e^{2x} + e^{-2x})/2 = (e^{3x} + e^{-x})/2$. Thus

$$y_p = \frac{1}{(D^2 - D - 6)} e^x \cosh 2x = \frac{1}{(D^2 - D - 6)}\left(\frac{e^{3x} + e^{-x}}{2}\right)$$

$$= \frac{1}{2(D^2 - D - 6)} e^{3x} + \frac{1}{2(D^2 - D - 6)} e^{-x}$$

$$= \frac{x}{2(2D - 1)} e^{3x} + \frac{1}{2(1 + 1 - 6)} e^{-x} = \frac{xe^{3x}}{2(2.3 - 1)} + \frac{1}{2(-4)} e^{-x} = \frac{xe^{3x}}{10} - \frac{e^{-x}}{8}$$

[Use Cases I & II]. Thus general solution is:

$$y = y_c + y_p = A\, e^{3x} + B\, e^{-2x} + x\, e^{3x}/10 - e^{-x}/8$$

When f(x) = sin ax or cos ax

CASE-I: $\frac{1}{F(D^2)} \sin ax = \frac{1}{F(-a^2)} \sin ax$. Similarly, $\frac{1}{F(D^2)} \cos ax = \frac{1}{F(-a^2)} \cos ax$, provided, $F(-a^2) \neq 0$.

Proof: To find y_p when $f(x)$ is either sin ax or cos ax we proceed as under:

$D\,(\sin ax) = a \cos ax,$
$D^2\,(\sin ax) = D(D(\sin ax)) = D(a \cos ax) = a\,(-a \sin ax) = -\, a^2 \sin ax$

$$\Rightarrow \quad \frac{1}{D^2} \sin ax = \frac{1}{-a^2} \sin ax, \quad \Rightarrow \quad \frac{1}{F(D^2)} \sin ax = \frac{1}{F(-a^2)} \sin ax$$

Similarly, $\frac{1}{F(D^2)} \cos ax = \frac{1}{F(-a^2)} \cos ax$, provided, $F(-a^2) \neq 0$.

CASE II: If $F(-a^2) = 0$ then $\frac{1}{F(D)}\sin ax = \frac{x}{F'(-a^2)}\sin ax$

REMARK: If F(D) contains D^3 and higher terms then following may be noted:
Write $D^3 = D^2 \cdot D$ and, $D^4 = D^2 \cdot D^2$ and so on. Then replace D^2 by $-(a^2)$.
In case, if F(D) reduces to linear factor such as $(D + \alpha)$ after using above identities, we proceed as under:

$$\frac{1}{D+\alpha}\sin ax = \frac{D-\alpha}{(D+\alpha)(D-\alpha)}\sin ax = \frac{D-\alpha}{D^2-\alpha^2}\sin ax$$

$$= \frac{D-\alpha}{-a^2-\alpha^2}\sin ax, \ \left(\text{putting } -a^2 \text{ for } D^2 \text{ in the denominator}\right).$$

$$= -\frac{1}{a^2+\alpha^2}(D-\alpha)\sin ax = -\frac{1}{a^2+\alpha^2}\left\{\frac{d}{dx}(\sin ax)-\alpha\sin ax\right\}$$

$$\therefore \frac{1}{D+\alpha}\sin ax = -\frac{1}{a^2+\alpha^2}(a\cos ax - \alpha\sin ax).$$

Similar steps are taken when f(x) is cos ax.
This process of finding y_p is valid if $F(-a^2) \neq 0$.

In case if $F(-a^2) = 0$ then: $\frac{1}{F(D)}\sin ax = \frac{x}{F'(-a^2)}\sin ax$.

Example 03: Find the general solution of (i) $(D^2 - 5D + 6)y = \sin 2x$ (ii) $(D^3 - D^2 + D - 1)\,y = 4\sin x$ (iii) $(D^3 - 3D^2 + 4D - 2)\,y = e^x + \cos x$

Solution: (i) Auxiliary equation is $m^2 - 5m + 6 = 0 \rightarrow m = 2, 3$. Thus $y_c = A\,e^{2x} + B\,e^{3x}$. Now,

$$y_p = \frac{1}{D^2-5D+6}\sin 2x = \frac{1}{-2^2-5D+6}\sin 2x = \frac{1}{2-5D}\times\frac{2+5D}{2+5D}\sin 2x = \frac{2+5D}{4-25D^2}\sin 2x$$

$$= \frac{2+5D}{4-25(-2^2)}\sin 2x = \frac{2\sin 2x + 5D\sin 2x}{104} = \frac{2\sin 2x + 10\cos 2x}{104} = \frac{\sin 2x + 5\cos 2x}{52}$$ Thus

general solution is: $y = y_c + y_p = Ae^{2x} + Be^{3x} + \frac{\sin 2x + 5\cos 2x}{52}$

(ii) Auxiliary equation is: $m^3 - m^2 + m - 1 = 0 \Rightarrow (m-1)(m^2+1) = 0 \Rightarrow m = 1, \pm i$.

Thus Complementary Function is

$$y_c = c_1e^x + e^{0x}(c_2\sin x + c_3\cos x) = c_1e^x + c_2\sin x + c_3\cos x.$$

$$y_p = \frac{1}{(D^3-D^2+D-1)}(4\sin x) = 4\frac{x}{3D^2-2D+1}\sin x = 4\frac{x}{3(-1^2)-2D+1}\sin x$$

$$= 4\frac{x}{-2(1+D)}\times\frac{(1-D)}{(1-D)}\sin x = -2\frac{x(1-D)}{(1-D^2)}\sin x = -2\frac{x(1-D)}{1-(-1^2)}\sin x$$

$$= -2\frac{x(\sin x - D\sin x)}{2} = -x(\sin x - \cos x) \quad [\text{Note: } \sin x \text{ is present in } y_c]$$

Thus general solution is: $y = y_c + y_p = c_1e^x + c_2\sin x + c_3\cos x - x(\sin x - \cos x)$

(iii) Auxiliary equation is $m^3 - 3m^2 + 4m - 2 = 0 \Rightarrow m = 1,\ 1 \pm i$.

Thus Complementary Function is: $y_c = c_1e^x + e^x(c_2\sin x + c_3\cos x)$

$$y_p = \frac{1}{D^3-3D^2+4D-2}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x$$

$$= \frac{x}{3D^2-6D+4}e^x + \frac{1}{D^2.D-3D^2+4D-2}\cos x$$

$$= \frac{xe^x}{3-6+4} + \frac{1}{(-1^2)D-3(-1^2)+4D-2}\cos x$$

$$= xe^x + \frac{1}{3D+1}\cos x = xe^x + \frac{1}{3D+1}\times\frac{3D-1}{3D-1}\cos x$$

$$= xe^x + \frac{3D-1}{9D^2-1}\cos x = xe^x + \frac{3D-1}{9(-1^2)-1}\cos x = xe^x - \frac{1}{10}(3D\cos x - \cos x)$$

$$= xe^x - (-3\sin x - \cos x)/10 = xe^x + (3\sin x + \cos x)/10$$

Thus general solution is:

$$y = y_c + y_p = c_1e^x + e^x(c_2\sin x + c_3\cos x) + (3\sin x + \cos x)/10$$

Example 04: Solve the initial value problem: y'' - 4y' + 13y = 8 sin 3x; y(0) = 1, y'(0) = 2.

Solution: Given equation can be written as:

$$D^2y - 4Dy + 13y = 8\sin 3x \Rightarrow \left(D^2-4D+13\right)y = 8\sin 3x.$$

Auxiliary equation is: $m^2 - 4m + 13 = 0$.

Using quadratic formula, we get: $m = \frac{4\pm\sqrt{16-52}}{2} = \frac{4\pm 6i}{2} = 2\pm 3i$.

Thus Complementary Function is: $y_c = e^{2x}(c_1\sin 3x + c_2\cos 3x)$. Now

$$y_p = \frac{1}{D^2-4D+13}(8\sin 3x) = 8\frac{1}{-9-4D+13}\sin 3x, \text{ [Replacing } D^2 \text{ by } -3^2]$$

$$= 8\frac{1}{4-4D}\sin 3x = -2\frac{1}{D-1}\sin 3x = \frac{-2(D+1)}{(D-1)(D+1)}\sin 3x = -2\frac{1}{D^2-1}(D+1)\sin 3x$$

$$\Rightarrow \quad y_p = \frac{-2}{-9-1}\{D\sin 3x + \sin 3x\} = \frac{1}{5}(3\cos 3x + \sin 3x)$$

Hence, the general solution is

$$y = y_c + y_p = e^{2x}(c_1\sin 3x + c_2\cos 3x) + \frac{1}{5}(3\cos 3x + \sin 3x).$$

Now $y' = e^{2x}(3c_1\cos 3x - 3c_2\sin 3x) + 2e^{2x}(c_1\sin 3x + c_2\cos 3x) + \frac{1}{5}(-9\sin 3x + 3\cos 3x).$

Applying the given initial conditions, we get

$$1 = (c_1\times 0 + c_2) + \frac{1}{5}(3+0) \Rightarrow c_2 = \frac{2}{5}.$$

$$2 = (3c_1 - 0) + 2(0 + c_2) + \frac{1}{5}(0+3) \Rightarrow 2 - \frac{4}{5} - \frac{3}{5} = 3c_1 \Rightarrow c_1 = \frac{1}{5}.$$

Hence, $y = e^{2x}\left(\frac{1}{5}\sin 3x + \frac{2}{5}\cos 3x\right) + \frac{1}{5}(3\cos 3x + \sin 3x)$

or $\quad y=\frac{1}{5}\left\{e^{2x}(\sin 3x+2\cos 3x)+3\cos 3x+\sin 3x\right\}$ is a particular solution.

Example 05: Solve $(D^2+4)\,y=\sin^2 x$

Solution: Auxiliary equation is $m^2+4=0 \rightarrow m=\pm 2i$. Thus,

$$y_c = c_1 \sin 2x + c_2 \cos 2x.$$

$$y_p=\frac{1}{D^2+4}\sin^2 x=\frac{1}{D^2+4}\left(\frac{1-\cos 2x}{2}\right)=\frac{1}{2}\left\{\frac{1}{D^2+4}e^{0x}-\frac{1}{D^2+4}\cos 2x\right\}$$

$$=\frac{1}{2}\left\{\frac{1}{0^2+4}e^{0x}-\frac{x}{2D}\cos 2x\right\}$$

$$=\frac{1}{8}-\frac{x}{4}\int\cos 2x\,dx=\frac{1}{8}-\frac{x}{4}\cdot\frac{\sin 2x}{2}=\frac{1}{8}[1-x\sin 2x].$$

$$\text{NOTE}: Df(x)=\frac{df}{dx}\Rightarrow\frac{1}{D}f(x)=D^{-1}f(x)=\int f(x)\ dx$$

Thus, $\quad y=y_c+y_p=c_1\sin 2x+c_2\cos 2x+\frac{1}{4}\left[\frac{1}{2}-x\sin 2x\right]$

When $f(x)=e^{ax}\,g(x)$

If $f(x)=e^{ax}\,g(x)$, then $\frac{1}{F(D)}e^{ax}g(x)=e^{ax}\frac{1}{F(D+a)}g(x)$

This is known as **Exponential shift.**

Proof: Consider, $D(e^{ax}u)=e^{ax}Du+a\,u\,e^{ax}=e^{ax}(D+a)\,u$.

Here u is a function of x. Similarly,

$$D^2(e^{ax}u)=D[\,e^{ax}Du+au\,e^{ax}\,]=e^{ax}D^2u+Du\,a\,e^{ax}+a.\,u\,a.\,e^{ax}+a\,e^{ax}Du$$

$$=e^{ax}(D^2+2aD+a^2)=e^{ax}(D+a)^2$$

$\rightarrow \quad D^n(e^{ax}u)=e^{ax}(D+a)^n \quad \rightarrow \quad F(D)(e^{ax}u)=e^{ax}F(D+a)$

$\rightarrow \quad \frac{1}{F(D)}e^{ax}u=e^{ax}\frac{1}{F(D+a)}u$. Put $u=g(x)$, we get:

$$\frac{1}{F(D)}e^{ax}g(x)=e^{ax}\frac{1}{F(D+a)}g(x)$$

Example 06: Find the general solution of (i) $(D^2-2D+4)\,y=e^x\cos x$ (ii) $(D^3-D^2+3D+5)\,y=e^x\cos 2x$

Solution: Auxiliary equation is: $m^2-2m+4=0$.

Using quadratic equation, we get: $m=\frac{2\pm\sqrt{4-16}}{2}=\frac{2\pm 2i\sqrt{3}}{2}=1\pm i\sqrt{3}$.

The complementary function is: $y_c=e^x\left(c_1\sin\sqrt{3}x+c_2\cos\sqrt{3}x\right)$

Now using shift property,

$$y_p=\frac{1}{D^2-2D+4}\left(e^x\cos x\right)=e^x\frac{1}{(D+1)^2-2(D+1)+4}\cos x=e^x\frac{1}{D^2+3}\cos x$$

$$=e^x\frac{1}{-1^2+3}\cos x=\frac{e^x\cos x}{2}$$

Hence, the general solution of the given differential equation is

$$y=y_c+y_p=e^x(c_1\sin\sqrt{3}x+c_2\cos\sqrt{3}x)+(e^x\cos x)/2$$

(ii) A. E is $m^3 - m^2 + 3m + 5 = 0$ → $m = -1, 1 \pm 2i$ (verify by synthetic division)
Thus $y_c = Ae^{-x} + e^{x}(B\cos 2x + C\sin 2x)$
You may observe that $e^x \cos 2x$ is a function on right side of differential equation.

$y_p = \frac{1}{D^3 - D^2 + 3D + 5} e^x \cos 2x = \frac{x}{3D^2 - 2D + 3} e^x \cos 2x$. Apply shift property, we get:

$$= e^x \frac{x}{3(D+1)^2 - 2(D+1) + 3} \cos 2x = e^x \frac{x}{3D^2 + 4D + 4} \cos 2x$$

$$= e^x \frac{x}{3(-2^2) + 4D + 4} \cos 2x = e^x \frac{x}{4D - 8} \cos 2x = \frac{e^x}{4} \frac{x}{D-2} \cos 2x$$

$$= \frac{e^x}{4} \frac{x}{D-2} \times \frac{D+2}{D+2} \cos 2x = \frac{xe^x(D+2)}{4(D^2-4)} \cos 2x = \frac{xe^x(D+2)}{4(-2^2-4)} \cos 2x$$

$$= \frac{xe^x(D\cos 2x + 2\cos 2x)}{-32} = \frac{2xe^x(-\sin 2x + \cos 2x)}{-32} = -\frac{xe^x(\cos 2x - \sin 2x)}{16}$$

Thus general solution is:

$y = y_c + y_p = Ae^{-x} + e^x(B\cos 2x + C\sin 2x) - xe^x(\cos 2x - \sin 2x)/16$

When $f(x) = x^m$

To evaluate $\frac{1}{F(D)} f(x)$, where f(x) is a polynomial of degree m.

- Take out lowest degree term common from F(D) to make the first term unity. The remaining term will contain $\{1 + \varphi(D)\}$ or $\{1 - \varphi(D)\}$.
- Take this factor in the numerator where it takes the form $\{1 + \varphi(D)\}^{-1}$ or $\{1 - \varphi(D)\}^{-1}$.
- Expand by binomial theorem, using either of the results:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \text{ or } (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

The expansion to be carried up to the term D^m, since f(x) is a polynomial of degree m and hence, $D^{m+1}f(x) = 0$, $D^{m+2}f(x) = 0$, all higher derivatives of f(x) vanish.

EXAMPLE 07: Solve the initial value problem $y'' - 4y = (2 - 8x)$; $y(0) = 0$, $y'(0) = 5$.

Solution: A.E is $m^2 - 4 = 0$ → $m = -2, 2$. Thus $y_c = c_1e^{2x} + c_2e^{-2x}$.

$$y_p = \frac{1}{D^2 - 4}(2 - 8x) = -\frac{1}{4(1 - D^2/4)}(2 - 8x) = \frac{-1}{4}\left(D^2 - 1/4\right)^{-1}(2 - 8x)$$

Expanding by Binomial Theorem, and keeping only the term contains D as $f(x) = (2 - 8x)$ is a polynomial of degree 1, hence $D^2(2 - 8x) = 0$. Thus,

$$y_p = -\frac{1}{4}\left(1 + \frac{D^2}{4} + \dots\right)(2 - 8x) = -\frac{1}{4}(2 - 8x) = 2x - \frac{1}{2}$$

Hence, the general solution of given equation is

$$y = y_c + y_p = c_1e^{2x} + c_2e^{-2x} + 2x - 1/2$$

Now $y' = 2c_1e^{2x} - 2c_2e^{-2x} + 2$

Applying given initial conditions, we get

$$0=c_1+c_2-\frac{1}{2}\Rightarrow 1=2c_1+2c_2 \qquad (1)$$

$$5=2c_1-2c_2+2\Rightarrow 3=2c_1-2c_2 \qquad (2)$$

Adding (1) and (2), we get: $c_1=1$. Put this in (1), we get $c_2=-1/2$.

Hence $y=e^{2x}-e^{-2x}/2+2x-1/2$ is particular solution of given differential equation.

When $f(x)=x^n\sin ax$ or $f(x)=x^n\cos ax$

$\frac{1}{F(D)}x^n(\cos ax+i\sin ax)=\frac{1}{F(D)}x^ne^{iax}=e^{iax}\frac{1}{F(D+ia)}x^n$. Thus

$\frac{1}{F(D)}x^n\cos ax=\text{Real part of}\left\{e^{iax}\frac{1}{F(D+ia)}x^n\right\}$ and

$\frac{1}{F(D)}x^n\sin ax=\text{Imaginary part of}\left\{e^{iax}\frac{1}{F(D+ia)}x^n\right\}$

Example 08: Solve $(D^2-2D+1)\,y=x\sin x$

Solution: A. E is $m^2-2m+1=0\rightarrow m=1,1$. Thus $y_c=(A+Bx)e^x$

$$y_p=\text{Im}\left\{\frac{1}{D^2-2D+1}xe^{ix}\right\}=\text{Im}\left\{e^{ix}\frac{1}{(D+i)^2-2(D+i)+1}x\right\}\text{[By shift property]}$$

$$=\text{Im}\left\{e^{ix}\frac{1}{D^2+2iD-1-2D-2i+1}x\right\}$$

$$=\text{Im}\left\{e^{ix}\frac{1}{D^2-2(1-i)D-2i}x\right\}=\text{Im}\left\{e^{ix}\frac{1}{-2(1-i)D-2i}\right\}x. \qquad \left[\text{Neglecting } D^2\right]$$

$$=\text{Im}\frac{1}{-2i}\left\{e^{ix}\frac{1}{1-(1+i)D}x\right\}=\text{Im}\frac{1}{-2i}e^{ix}\left[1-(1+i)D\right]^{-1}x=\text{Im}\frac{1}{-2i}e^{ix}\left[1+(1+i)D\right]x$$

NOTE: $-2(1-i)D-2i=2i^2(1-i)D-2i=-2i[-i(1-i)D+1]=-2i\,[1-i(1-i)D]$
$=-2i\,[1-(i-i^2)D]=-2i\,[1-(i+1)D]=-2i\,[1-(1+i)D]$

$$=\text{Im}\frac{1}{-2i}(\cos x+i\sin x)(x+x+iDx)=\text{Im}\frac{1}{2}(i\cos x-\sin x)[2x+i]$$

$$=\text{Im}\frac{1}{2}\{[-2x\sin x-\cos x]+i[2x\cos x-\sin x]\}=\frac{1}{2}[2x\cos x-\sin x]$$

Thus complete solution is $y=(A+Bx)e^x+(2x\cos x-\sin x)/2$ **(Note: $-1/i=i$)**

Example 09: Solve the following differential equations

i. $(D^2-2D-3)y=2e^x-10\sin x$

Solution: Auxiliary equation is: $m^2-2m-3=0\rightarrow m=-1,3$. Thus

$$y_c=C_1e^{-x}+C_2e^{3x}$$

Now,

$$y_p=\frac{1}{(D^2-2D-3)}[2e^x-10\sin x]=2\frac{1}{(D^2-2D-3)}e^x-10\frac{1}{(D^2-2D-3)}\sin x$$

$$=2\frac{1}{1-2-3}e^x-10\frac{1}{-1^2-2D-3}\sin x=-\frac{e^x}{2}+\frac{10}{2(D+2)}\times\frac{(D-2)}{(D-2)}\sin x$$

$$=-\frac{e^x}{2}+5\frac{(D-2)}{(D^2-4)}\sin x=-\frac{e^x}{2}+5\frac{(D-2)}{(-1^2-4)}\sin x=-\frac{e^x}{2}+5\frac{(D\sin x-2\sin x)}{-5}$$

$$= -\frac{e^x}{2}-(\cos x - 2\sin x)$$

Thus, general solution of given differential equation is $y = y_c + y_p$

→ $$y = C_1 e^{x} + C_2 e^{3x} - \frac{e^x}{2}-(\cos x - 2\sin x)$$

ii. $(D^3 - 2D^2 - 3D + 10)\, y = 40 \cos x$

Solution: Auxiliary equation is $m^3 - 2m^2 - 3m + 10 = 0$. Its roots are: $m = -2,\ 2 \pm i$. Thus,

$$y_c = C_1 e^{-2x} + e^{2x}(C_2 \cos x + C_3 \sin x)$$

$$y_p = 40\frac{1}{(D^3-2D^2-3D+10)}\cos x = 40\frac{1}{(D^2.D-2D^2-3D+10)}\cos x$$

$$= 40\frac{1}{(D^2.D-2D^2-3D+10)}\cos x = 40\frac{1}{[(-1^2).D-2(-1^2)-3D+10]}\cos x$$

$$= 40\frac{1}{(-.D+2-3D+10)}\cos x = 40\frac{1}{(12-4D)}\cos x = \frac{40}{4}\frac{(3+D)}{(3-D)(3+D)}\cos x$$

$$= 10\frac{(3+D)}{(9-D^2)}\cos x = 10\frac{(3+D)}{(9-(-1^2))}\cos x = 10\frac{(3\cos x + D\cos x)}{10} = 3\cos x - \sin x$$

Thus, general solution of given differential equation is $y = y_c + y_p$

→ $$y = C_1 e^{-2x} + e^{2x}(C_2 \cos x + C_3 \sin x) + 3 \cos x - \sin x$$

iii. $(D^3 + D)\, y = 2x^2 + 4 \sin x$

Solution: Auxiliary equation is $m^3 + m = 0$ → $m(m^2 + 1) = 0$ → $m = 0, \pm i$.

Thus: $$y_c = C_1 e^{0x} + C_2 \cos x + C_3 \sin x = C_1 + C_2 \cos x + C_3 \sin x$$

Now,

$$y_p = \frac{1}{(D^3+D)}(2x^2+4\sin x) = 2\frac{1}{D(D^2+1)}x^2 + 4\frac{1}{(D^3+D)}\sin x$$

$$= 2\frac{(1+D^2)^{-1}}{D}x^2 + 4\frac{x}{(3D^2+D)}\sin x = \frac{2}{D}\left(1-D^2+D^4-...\right)x^2 + 4\frac{x}{[3(-1^2)+D]}\sin x$$

$$= \frac{2}{D}\left(x^2 - D^2x^2\right) + 4\frac{x(D+3)}{(D-3)(D+3)}\sin x = \frac{2}{D}\left(x^2-2x\right) + 4\frac{x(D+3)}{(D^2-9)}\sin x$$

$$= 2\int(x^2-2x)\,dx + 4\frac{x(D+3)}{[(-1^2)-9]}\sin x = 2\left(\frac{x^3}{3} - 2\frac{x^2}{2}\right) + 4\frac{x(D\sin x + 3\sin x)}{-10}$$

$$= 2\left(\frac{x^3}{3} - x^2\right) - 2\frac{x(\cos x + 3\sin x)}{5}$$

Thus, the general solution of given differential equation is $y = y_c + y_p$

→ $$y = C_1 + C_2 \cos x + C_3 \sin x + 2\left(\frac{x^3}{3} - x^2\right) - \frac{2}{5}x(\cos x + 3\sin x)$$

iv. $(D^4 + D^2)\, y = 3x^2 + 4 \sin x - 2 \cos x$

Solution: Auxiliary equation is $m^4 + m^2 = 0$

→ $m^2(m^2 + 1) = 0$ → $m = 0, 0, \pm i$.

Thus, $y_c = (C_1 + C_2 x)\, e^{0x} + C_3 \cos x + C_4 \sin x$

$= y_c = (C_1 + C_2 x) + C_3 \cos x + C_4 \sin x$

$$y_p = \frac{1}{(D^4 + D^2)}(3x^2 + 4\sin x - 2\cos x)$$

$$= 3\frac{1}{D^2(1+D^2)}x^2 + 4\frac{1}{(D^4+D^2)}\sin x - 2\frac{1}{(D^4+D^2)}\cos x$$

Note: Both sin x and cos x are in y_c.

$$= 3\frac{(1+D^2)^{-1}}{D^2}x^2 + 4\frac{x}{(4D^3+2D)}\sin x - 2\frac{x}{(4D^3+2D)}\cos x$$

$$= 3\frac{1}{D^2}\left(1 - D^2 + D^4 - \ldots\right)x^2 + 4\frac{x}{2D(2D^2+1)}\sin x - 2\frac{x}{2D(2D^2+1)}\cos x$$

$$= 3\int\int(x^2 - 2)dx\, dx + 2\frac{x}{D[2(-1^2)+1]}\sin x - \frac{x}{D[2(-1^2)+1]}\cos x$$

$$= 3\int\left(\frac{x^3}{3} - 2x)\right)dx - 2x\frac{1}{D}\sin x + x\frac{1}{D}\cos x = \frac{x^4}{4} - 3x^2 - 2x\int \sin x\, dx + x\int \cos x\, dx$$

$$= \left(x^4/4\right) - 3x^2 + 2x\cos x + x\sin x$$

Now the general solution of given differential equation is $y = y_c + y_p$

$\rightarrow y = (C_1 + C_2 x) + C_3 \cos x + C_4 \sin x + \left(x^4/4\right) - 3x^2 + 2x\cos x + x\sin x$

v. $(D^3 - 2D + 4)\, y = e^x \cos x$

Solution: Auxiliary equation is $m^3 - 2m + 4 = 0 \rightarrow m = -2,\ 1 \pm i$. Thus

$$y_c = C_1 e^{-2x} + e^x (C_2 \cos x + C_3 \sin x)$$

Now,

$$y_p = \frac{1}{(D^3 - 2D + 4)}e^x \cos x = \frac{x}{3D^2 - 2}e^x \cos x = xe^x \frac{1}{3(D+1)^2 - 2}\cos x \text{ [by shift theorem]}$$

$$= xe^x \frac{1}{3D^2 + 6D + 3 - 2}\cos x = xe^x \frac{1}{3(-1^2) + 6D + 1}\cos x = xe^x \frac{1}{6D - 2}\cos x$$

$$= xe^x \frac{1}{2(3D-1)}\frac{(3D+1)}{(3D+1)}\cos x = \frac{xe^x}{2}\frac{(3D+1)}{(9D^2-1)}\cos x = \frac{xe^x}{2}\frac{(3D+1)}{[9(-1^2)-1]}\cos x$$

$$= -\frac{xe^x}{20}(3D\cos x + \cos x) = -\frac{xe^x}{20}(-3\sin x + \cos x) = \frac{xe^x}{20}(3\sin x - \cos x)$$

Hence the general solution is:

$y = y_c + y_p = C_1 e^{-2x} + e^x (C_2 \cos x + C_3 \sin x) + xe^x(3\sin x - \cos x)/20$

vi. $(D^3 - D^2 + 3D + 5)\, y = e^x \sin 2x$

Solution: The auxiliary equation is $m^3 - m^2 + 3m + 5 = 0 \rightarrow m = -1,\ 1 \pm 2i$.

Thus, $\quad y_c = C_1 e^{-x} + e^x (C_2 \cos 2x + C_3 \sin 2x)$

Now, $$y_p = \frac{1}{(D^3 - D^2 + 3D + 5)}e^x \sin 2x = \frac{x}{3D^2 - 2D + 3}e^x \sin 2x$$

$$= xe^x \frac{1}{3(D+1)^2 - 2(D+1) + 3}\cos x \text{ [by shift theorem]}$$

$$= xe^x \frac{1}{3D^2+6D-2D+3}\sin 2x = xe^x \frac{1}{3(-2^2)+4D+3}\sin 2x = xe^x \frac{1}{4D-9}\sin 2x$$

$$= xe^x \frac{1}{(4D-9)}\frac{(4D+9)}{(4D+9)}\sin 2x = xe^x \frac{(4D+9)}{(16D^2-81)}\sin 2x = xe^x \frac{(4D+9)}{[16(-2^2)-81]}\sin 2x$$

$$= -\frac{xe^x}{145}(4D\sin 2x+9\sin 2x) = -\frac{xe^x}{145}(8\cos 2x+9\sin 2x) = -\frac{xe^x}{145}(8\cos 2x+9\sin 2x)$$

Hence general solution is:

$y = y_c + y_p = C_1 e^{-x} + e^x (C_2 \cos 2x + C_3 \sin 2x) - xe^x(8\cos 2x + 9\sin 2x)/145$

vii. $(D^3 - 7D - 6)\, y = e^{2x}(1 + x)$

Solution: Auxiliary equation is $m^3 - 7m - 6 = 0 \rightarrow m = -1, -2, 3$. Thus

$$y_c = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x}$$

$$y_p = \frac{1}{(D^3-7D-6)}e^{2x}(1+x) = e^{2x}\frac{1}{(D+2)^3-7(D+2)-6}(x+1) \text{ [by shift theorem]}$$

$$= e^{2x}\frac{1}{D^3+6D^2+12D+8-7D-14-6}(x+1)$$

$$= e^{2x}\frac{1}{5D-12}(x+1)\text{[neglecting } D^2 \text{ \& high power terms]}$$

$$= e^{2x}\frac{1}{-12(1-5D/12)}(x+1) = -\frac{e^{2x}}{12}\left[1-\frac{5}{12}D\right]^{-1}(x+1) = -\frac{e^{2x}}{12}\left[1+\frac{5}{12}D\right](x+1)$$

$$= -\frac{e^{2x}}{12}\left[(x+1)-\frac{5}{12}D(x+1)\right] = -\frac{e^{2x}}{12}\left[(x+1)-\frac{5}{12}\right] = -\frac{e^{2x}}{144}[12x+7]$$

Hence general solution is:

$y = y_c + y_p = C_1 e^{-x} + C_2 e^{-2x} + C_3 e^{3x} - e^{2x}[12x+7]/144$

viii. $(D^4 + 3D^2 - 4)\, y = \sinh x - \cos^2 x$

Solution: Auxiliary equation is $m^4 + 3m^2 - 4 = 0 \rightarrow m = \pm 1, \pm 2i$. Thus

$$y_c = C_1 e^{-x} + C_2 e^x + C_3 \cos 2x + C4 \sin 2x$$

NOTE: $\sinh x = (e^x - e^{-x})/2$ and $\cos^2 x = (1 + \cos 2x)/2$. Hence,

$$y_p = \frac{1}{(D^4+3D^2-4)}\left(\frac{e^x-e^{-x}-1-\cos 2x}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{(D^4+3D^2-4)}e^x - \frac{1}{(D^4+3D^2-4)}e^{-x} - \frac{1}{(D^4+3D^2-4)}e^{0x} - \frac{1}{(D^4+3D^2-4)}\cos 2x\right]$$

$$= \frac{1}{2}\left(\frac{x}{(4D^3+6D)}e^x - \frac{x}{(4D^3+6D)}e^{-x} - \frac{1}{(0^4+3.0^2-4)}e^{0x} - \frac{x}{(4D^3+6D)}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{x}{(4.1^3+6.1)}e^x - \frac{x}{(4.(-1^3)+6(-1))}e^{-x} - \frac{1}{(-4)} - \frac{1}{(D^2)(D)+6D}\cos 2x\right)$$

$$= \frac{1}{2}\left(\frac{x}{10}e^x + \frac{x}{10}e^{-x} + \frac{1}{4} - \frac{1}{(-2^2.D+6D)}\cos 2x\right) = \frac{1}{2}\left(\frac{x}{10}(e^x+e^{-x}) + \frac{1}{4} - \frac{1}{2D}\cos 2x\right)$$

$$=\frac{x}{10}\frac{(e^x+e^{-x})}{2}+\frac{1}{8}-\frac{1}{4}\int\cos 2x\,dx=\frac{x}{10}\cosh x-\frac{1}{8}\sin 2x+\frac{1}{8}$$

$$=\frac{x}{10}\frac{(e^x+e^{-x})}{2}+\frac{1}{8}-\frac{1}{4}\int\cos 2x\,dx=\frac{x}{10}\cosh x-\frac{1}{8}\sin 2x+\frac{1}{8}$$

Hence the general solution is:

$$y=y_c+y_p=C_1e^{-x}+C_2e^{x}+C_3\cos 2x+C4\sin 2x+\frac{x}{10}\cosh x-\frac{1}{8}\sin 2x+\frac{1}{8}$$

7.4 CAUCHY-EULER DIFFERENTIAL EQUATION

A differential equation of the form

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}}+\dots+a_{n-1}x\frac{dy}{dx}+a_ny=F(x) \qquad (1)$$

where $a_0,a_1,a_2,\dots,a_{n-1},a_n$ are constants, is known as **Cauchy – Euler Equation, or equi-dimensional equation.** The unique characteristic of this type of differential equation is that **degree** of each monomial coefficient matches the **order** of differential equation. The equation can be reduced to a linear differential equation with constants coefficients by transformation:

$$x=e^t \text{ or } t=\ln x$$

Using Chain Rule, we can write

$$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}\cdot\frac{d}{dx}(\ln x)=\frac{dy}{dt}\cdot\left(\frac{1}{x}\right)\Rightarrow\frac{dy}{dx}=\frac{1}{x}\frac{dy}{dt} \qquad (2)$$

Differentiating (2), we have

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)=\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{dy}{dt}\right)+\frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)\frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)+\frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$\frac{d^2y}{dx^2}=\frac{1}{x}\frac{d^2y}{dt^2}\left(\frac{1}{x}\right)-\frac{1}{x^2}\frac{dy}{dt}=\frac{1}{x^2}\frac{d^2y}{dt^2}-\frac{1}{x^2}\frac{dy}{dt}\Rightarrow\frac{d^2y}{dx^2}=\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right) \qquad (3)$$

Differentiating (3), we obtain

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\left\{\frac{1}{x^2}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\right\}=\frac{1}{x^2}\frac{d}{dx}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)+\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$\frac{d^3y}{dx^3}=\frac{1}{x^2}\left\{\frac{d}{dt}\left(\frac{d^2y}{dt^2}\right)\left(\frac{dt}{dx}\right)-\frac{d}{dt}\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)\right\}-\frac{2}{x^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3}=\frac{1}{x^2}\left(\frac{d^3y}{dt^3}\frac{1}{x}-\frac{d^2y}{dt^2}\frac{1}{x}\right)-\frac{2}{x^3}\left(\frac{d^2y}{dt^2}-\frac{dy}{dt}\right)=\frac{1}{x^3}\left(\frac{d^3y}{dt^3}-\frac{d^2y}{dt^2}-2\frac{d^2y}{dt^2}+2\frac{dy}{dt}\right)$$

$$\frac{d^3y}{dx^3}=\frac{1}{x^3}\left(\frac{d^3y}{dt^3}-3\frac{d^2y}{dt^2}+2\frac{dy}{dt}\right) \qquad (4)$$

Now if we write $D=d/dx$ and $\Delta=d/dt$, then

from (2), $xDy=\Delta y$,

from (3), $x^2D^2y=(\Delta^2-\Delta)y=\Delta(\Delta-1)y$

from (4), $x^3D^3y=(\Delta^3-3\Delta^2+2\Delta)y=\Delta(\Delta-1)(\Delta-2)y$

In general,

$$x^n D^n y = \Delta(\Delta-1)(\Delta-2)\{\Delta-(n-1)\}y = \Delta(\Delta-1)(\Delta-2)(\Delta-n+1)y.$$

Substituting these values of $xD, x^2D^2, x^3D^3, ..., x^nD^n$ in (1), we obtain an equation of n^{th} order with constant coefficients having t as independent variable. This new equation may be solved by methods discussed in previous sections.

Example 01: Solve: $x^2 y'' - 3xy' + 5y = x^2 + \sin(\ln x)$

Solution: Given that $x^2y''-3xy'+5y = x^2\sin(\ln x)$ (1)

Let $x = e^t$ and $t = \ln x$. Also substitute $xD = \Delta$ and $x^2D^2 = \Delta(\Delta-1) = \Delta^2-\Delta$.

Equation (1) becomes

$$\left(\Delta^2-\Delta\right)y - 3\Delta y + 5y = e^{2t}\sin t \Rightarrow \left(\Delta^2-4\Delta+5\right)y = e^{2t}\sin t \qquad (2)$$

Auxiliary equation is: $m^2-4m+5=0 \Rightarrow m = \dfrac{-(-4)\pm\sqrt{16-20}}{2} = \dfrac{4\pm 2i}{2} = 2\pm i.$

Thus Complementary Function is: $y_c = e^{2t}(c_1\sin t + c_2\cos t).$

Particular Integral of (2) is

$$y_p = \frac{1}{\Delta^2-4\Delta+5}\left(e^{2t}\sin t\right) \Rightarrow y_p = e^{2t}\frac{1}{[(\Delta+2)^2-4(\Delta+2)+5]}\sin t, \text{[By shift theorem]}$$

$$y_p = e^{2t}\frac{1}{\Delta^2+1}\sin t = e^{2t}\frac{t}{2\Delta}\sin t = \frac{te^{2t}}{2}\int \sin t\,dt = -\frac{te^{2t}}{2}\cos t \text{ [Observe that } e^{2t}\sin t \text{ is in } y_c\text{]}.$$

Thus, general solution of the given equation is

$$y = y_c + y_p = e^{2t}(c_1\sin t + c_2\cos t) - te^{2t}\cos t/2$$

Now replacing t by ln x and e^t by x we get:

$$y = x^2\{c_1\sin(\ln x) + c_2\cos(\ln x)\} - \frac{x^2\ln x\cos(\ln x)}{2}.$$

Example 02: Solve: $x^3 y''' + 4x^2 y'' - 5xy' - 15y = x^4$

Solution: We have $x^3y'''+4x^2y''-5xy'-15y = x^4$ (1)

Let $x = e^t$ and $t = \ln x$. Then, $xD = \Delta$, $x^2D^2 = \Delta(\Delta-1) = \Delta^2-\Delta$, and $x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3-3\Delta^2+2\Delta$. Thus equation (1) becomes

$$\left(\Delta^3-3\Delta^2+2\Delta\right)y + 4\left(\Delta^2-\Delta\right)y - 5\Delta y - 15y = e^{4t}$$

$$\Rightarrow \left(\Delta^3-3\Delta^2+2\Delta+4\Delta^2-4\Delta-5\Delta-15\right)y = e^{4t}$$

$$\rightarrow \quad \left(\Delta^3+\Delta^2-7\Delta-15\right)y = e^{4t} \qquad (2)$$

Auxiliary equation is: $m^3 + m^2 - 7m - 15 = 0 \rightarrow (m-3)(m^2+4m+5) = 0$

$\rightarrow m = 3, \ -2\pm i.$

The roots of the auxiliary equation are: $m = 3, -2\pm i.$

Thus Complementary Function is: $y_c = c_1e^{3t} + e^{-2t}(c_2\sin t + c_3\cos t).$

Particular Integral of (2) is

$$y_p = \frac{1}{\Delta^3+\Delta^2-7\Delta-15}e^{4t} = \frac{e^t}{4^3+4^2-7(4)-15} = \frac{e^t}{47}$$

Thus, general solution of given equation is

$$y = y_c + y_p = c_1e^{3t} + e^{-2t}(c_2\sin t + c_3\cos t) + \frac{1}{47}e^{4t}.$$

Replacing t by lnx and e^t by x, we get:

$$y = y_c + y_p = c_1x^3 + x^{-2}\left[c_2\sin(\ln x) + c_3\cos(\ln x)\right] + \frac{1}{47}x^4.$$

Example 03: Solve: $(2x+1)\,y'' - 6(2x+1)\,y' + 16y = 8(2x+1)^2$

Solution: Given that $(2x+1)\,y'' - 6(2x+1)\,y' + 16y = 8(2x+1)^2$ (1)

Let $2x+1 = e^t \Rightarrow t = \ln(2x+1)$. Now

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\left\{\frac{d}{dx}\ln(2x+1)\right\} = \frac{dy}{dt}\cdot\left(\frac{1}{2x+1}(2)\right) = \frac{2}{2x+1}\frac{dy}{dt}$$

$$(2x+1)\frac{dy}{dx} = 2\frac{dy}{dt} \Rightarrow (2x+1)D = 2\Delta.$$

Again differentiating, we get

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = 2\frac{d}{dx}\left\{(2x+1)^{-1}\frac{dy}{dt}\right\} = 2\left\{(2x+1)^{-1}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}(2x+1)^{-1}\right\}$$

$$\frac{d^2y}{dx^2} = 2\left\{\frac{1}{2x+1}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} + \frac{dy}{dt}\left(\frac{-2}{(2x+1)^2}\right)\right\} = 2\left\{\frac{1}{2x+1}\frac{d^2y}{dt^2}\left(\frac{2}{2x+1}\right) - \frac{2}{(2x+1)^2}\frac{dy}{dt}\right\}$$

$$\frac{d^2y}{dx^2} = \frac{4}{(2x+1)^2}\frac{d^2y}{dt^2} - \frac{4}{(2x+1)^2}\frac{dy}{dt} = \frac{4\frac{d^2y}{dt^2} - 4\frac{dy}{dt}}{(2x+1)^2}$$

$\rightarrow (2x+1)^2D^2y = 4\Delta^2y - 4\Delta y \Rightarrow (2x+1)^2D^2 = 4\Delta^2 - 4\Delta$. Thus, equation (1) becomes

$$\left\{4\Delta^2 - 4\Delta - 6(2\Delta) + 16\right\}y = 8e^{2t} \Rightarrow \left(4\Delta^2 - 16\Delta + 16\right)y = 8e^{2t}$$

$$\Rightarrow \left(\Delta^2 - 4\Delta + 4\right)y = 2e^{2t} \qquad (2)$$

Auxiliary equation is: $m^2 - 4m + 4 = 0 \rightarrow m = 2, 2$

Hence, complementary function is: $y_c = (c_1 + c_2t)e^{2t}$.

The particular Integral

$$y_p = \frac{1}{\Delta^2 - 4\Delta + 4}\left(2e^{2t}\right) = \frac{2t}{2\Delta - 4}e^{2t} = \frac{2t^2}{2}e^{2t} = t^2e^{2t}$$

Thus, general solution is: $y = y_c + y_p = (c_1 + c_2t)e^{2t} + t^2e^{2t}$.

Now replacing t by $\ln(2x+1)$ and e^t by $(2x+1)$, we get

$$y = \{c_1 + c_2\ln(2x+1)\}(2x+1)^2 + \{\ln(2x+1)\}^2(2x+1)^2$$

Example 04: Solve: $x^2y'' - 2xy' + 2y = x\ln x$; $y(1) = 1$, $y'(1) = 0$

Solution: Given that $x^2y'' - 2xy' + 2y = x\ln x$ or $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x\ln x$ (1)

Let $x = e^t$ and $t = \ln x$. Then, $xD = \Delta$ and $x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$

Equation (1) becomes: $\left(\Delta^2 - 3\Delta + 2\right)y = te^t$ (2)

Auxiliary equation is: $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$ → $y_c = c_1e^t + c_2e^{2t}$.

$$y_p = \frac{1}{\Delta^2 - 3\Delta + 2}\left(te^t\right) = \frac{1}{(\Delta-1)(\Delta-2)}\left(te^t\right) = \frac{1}{(\Delta-1)(\Delta-1-1)}\left(te^t\right) \text{[By shift Prop]}$$

$$y_p = e^t\frac{1}{\Delta(\Delta-1)}t = -e^t\frac{1}{\Delta(1-\Delta)}t = -e^t\frac{1}{\Delta}(1-\Delta)^{-1}(t) = -e^t\frac{1}{\Delta}\left(1+\Delta+\Delta^2+\ldots\right)(t)$$

$$y_p = -e^t\frac{1}{\Delta}(1+\Delta)(t) = -e^t\frac{1}{\Delta}(t+1) = -e^t\int(t+1)\Delta t = -e^t\left(\frac{t^2}{2}+t\right).$$

Thus, general solution is: $y = y_c + y_p = c_1e^t + c_2e^{2t} - e^t\left(\frac{t^2}{2}+t\right)$.

Now replacing t by lnx and e^t by x, we get

$$y = c_1x + c_2x^2 - x\left\{\frac{(\ln x)^2}{2} + \ln x\right\} = c_1x + c_2x^2 - \frac{1}{2}x(\ln x)^2 - x\ln x \qquad (3)$$

$$\rightarrow \quad y' = c_1 + 2c_2x - \frac{1}{2}\left\{x(2\ln x)\frac{1}{x} + (\ln x)^2(1)\right\} - \left(x\frac{1}{x} + \ln x\right)$$

$$= c_1 + 2c_2x - \ln x - \ln x - 1 - \ln x = c_1 + 2c_2x - 3\ln x - 1.$$

Applying the given initial conditions, we get: $1 = c_1 + c_2$ and $0 = c_1 + 2c_2 - 1$.

Subtracting first equation from second, we get $c_2 = 0$.

Putting it into first, we get $c_1 = 1$.

Thus, (3) becomes: $y = x - x\ln x - \frac{x(\ln x)^2}{2}$.

This is particular solution of given differential equation.

Example 05: Solve the following Cauchy-Euler differential Equations

i. $(x^2D^2 + 7xD + 5)y = x^5$

Solution: Substituting $x = e^t$, we get: $xD = \Delta$ and $x^2D^2 = \Delta(\Delta - 1)$. Thus given differential equation becomes:

$$[\Delta(\Delta-1) + 7\Delta + 5]y = e^{5t} \rightarrow (\Delta^2 + 6\Delta + 5)y = e^{5t}.$$

The auxiliary equation is

$$m^2 + 6m + 5 = 0 \rightarrow m = -1, -5.$$

Thus, $y_c = C_1e^{-t} + C_2e^{-5t}$

Now, $y_p = \frac{1}{(\Delta^2 + 5\Delta + 6)}e^{5t} = \frac{1}{5^2 + 5.5 + 6}e^{5t} = \frac{1}{56}e^{5t}$.

Thus general solution of given differential equation is

$y = y_c + y_p = C_1e^{-t} + C_2e^{-5t} + \frac{1}{56}e^{5t} = C_1x^{-1} + C_2x^{-5} + x^5/56$ [Substituting $e^t = x$]

ii. $[x^2D^2 - (2m-1)xD + (m^2+n^2)]y = n^2x^m\ln x$

Solution: Substituting $x = e^t$ and $\ln x = t$ we get: $xD = \Delta$ and $x^2 D^2 = \Delta(\Delta - 1)$. Thus given differential equation becomes:

$[\Delta(\Delta-1)-(2m-1)\Delta+(m^2+n^2)]\,y=n^2e^{mt}\,t$

$\rightarrow \quad [\Delta^2-2m\Delta+(m^2+n^2)]\,y=n^2e^{mt}\,t$

Auxiliary equation is: $k^2-2mk+(m^2+n^2)=0 \rightarrow k=m\pm n\,i.$

Thus, $y_c=e^{mt}(C_1\cos nt+C_2\sin nt)$

Now,

$$y_p=n^2\frac{1}{[\Delta^2-2m\Delta+(m^2+n^2)]}e^{mt}t=n^2e^{mt}\frac{1}{[(\Delta+m)^2-2m(\Delta+m)+(m^2+n^2)]}t$$

$$=n^2e^{mt}\frac{1}{\Delta^2+n^2}t=n^2e^{mt}\frac{1}{n^2(1+\Delta^2/n^2)}t=e^{mt}\left(1+\frac{\Delta^2}{n^2}\right)^{-1}t=e^{mt}\left(1-\frac{\Delta^2}{n^2}+\dots\right)t.$$

$$=e^{mt}\left(t-\frac{\Delta^2}{n^2}t\right)=e^{mt}t$$

Thus general solution of given differential equation is

$y=y_c+y_p=e^{mt}(C_1\cos nt+C_2\sin nt)+e^{mt}t$

$\quad =x^m[C_1\cos(n\ln x)+C_2\sin(n\ln x)]+x^m\ln x$ [Substituting $e^t=x$ & $t=\ln x$]

iii. $(4x^2D^2-4xD+3)\,y=\sin\ln(-x),\ x<0$

Solution: Since x is negative hence, we let $x=-u$ where $u>0$. Then $x^2=u^2$ and $du/dx=-1$. Thus,

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=-\frac{dy}{du}=-Dy.\ \text{Also}\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(-\frac{dy}{du}\right)=-\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx}$$

$$=-\frac{d^2y}{du^2}(-1)=+\frac{d^2y}{du^2}=D^2y.\qquad [\text{NOTE: We define } Dy=dy/du]$$

Thus given differential equation becomes:

$[4u^2D^2-4(-u)(-D)+3]y=\sin(\ln u) \rightarrow (4u^2D^2-4uD-3)\,y=\sin(\ln u).$

Now substituting $u=e^t$ and $\ln u=t$ and $uD=\Delta$, $u^2D^2=\Delta(\Delta-1)$, above differential equation becomes:

$$(4\Delta(\Delta-1)-4\Delta-3)\,y=\sin t \rightarrow (4\Delta^2-8\Delta-3)\,y=\sin t$$

The auxiliary equation is

$4m^2-8m-3=0 \rightarrow m=3/2,\ 1/2.$ Thus, $y_c=C_1e^{t/2}+C_2e^{3t/2}$

Now

$$y_p=\frac{1}{4\Delta^2-8\Delta-3}\sin t=\frac{1}{4(-1^2)-8\Delta-3}\sin t=\frac{1}{-(8\Delta+7)}\sin t=-\frac{1}{(8\Delta+7)}\frac{(8\Delta-7)}{(8\Delta-7)}\sin t$$

$$=-\frac{(8\Delta-7)}{(64\Delta^2-49)}\sin t=-\frac{(8\Delta-7)}{64(-1^2)-49}\sin t=+\frac{1}{113}(8\Delta\sin t-7\sin t)=\frac{1}{113}(8\cos t-7\sin t)$$

Thus general solution is

$y=y_c+y_p=C_1e^{t/2}+C_2e^{3t/2}+(8\cos t-7\sin t)/113$

$\quad =C_1u^{1/2}+C_2u^{3/2}+[8\cos(\ln u)-7\sin(\ln u)]/113$, where $u=-x$.

iv. $(x^4D^3+2x^3D^2-x^2D+x)\,y=1$

Solution: Dividing both sides by x, we obtain: $(x^3D^3+2x^2D^2-xD+1)\,y=1/x$

Now put $x=e^t$ and $xD=\Delta$, $x^2D^2=\Delta(\Delta-1)$, $x^3D^3=\Delta(\Delta-1)(\Delta-2)$, we obtain

$(\Delta(\Delta-1)(\Delta-2)+2\Delta(\Delta-1)-\Delta+1)\,y=e^{-t} \quad \rightarrow \quad (\Delta^3-\Delta^2-\Delta+1)\,y=e^{-t}$

Auxiliary equation is: $m^3-m^2-m+1=0 \rightarrow m=-1,1,1.$

Thus, $y_c=C_1e^{-t}+(C_2+C_3t)e^t$

$$y_p = \frac{1}{\Delta^3 - \Delta^2 - \Delta + 1} e^{-t} = \frac{t}{3\Delta^2 - 2\Delta - 1} e^{-t} = \frac{t^2}{6\Delta - 2} e^{-t} = \frac{t^2}{6(-1) - 2} e^{-t} = -\frac{t^2 e^{-t}}{8}$$

Thus general solution of given differential equation is:

$$y = y_c + y_p = C_1 e^{-t} + (C_2 + C_3 t) e^t - t^2 e^{-t}/8$$

$$= C_1 x^{-1} + (C_2 + C_3 \ln x) - \frac{x^{-1}(\ln x)^2}{8} \qquad [\text{Substituting } x = e^t \text{ and } \ln x = t]$$

7.5 METHOD OF VARIATION OF PARAMETERS

The most general for of linear non-homogeneous differential equation of order two is:

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = F(x) \qquad (1)$$

The solution of such equation may be determined by a procedure known as **method of variation of parameters.** This method can be applied even to equations of higher orders, but we shall restrict to second order differential equations. However, we shall present one example of order three to make the readers acquainted with such problems.

Suppose that linearly independent solutions of

$$\frac{d^2y}{d^2x} + P(x)\frac{dy}{dx} + Q(x)y = 0, \qquad (2)$$

are given by $y = y_1$ and $y = y_2$. Hence

$$\frac{d^2y_1}{d^2x} + P(x)\frac{dy_1}{dx} + Q(x)y_1 = 0 \text{ and } \frac{d^2y_2}{d^2x} + P(x)\frac{dy_2}{dx} + Q(x)y_2 = 0.$$

Then complementary function is: $y_c = c_1y_1 + c_2y_2$,

where c_1 and c_2 are arbitrary constants. We replace arbitrary constants c_1 and c_2 by unknown functions $u_1(x)$ and $u_2(x)$ and require that

$$y_p = u_1y_1 + u_2y_2, \qquad (3)$$

be a particular solution of (1).

Note. The arbitrary constants that occur in the former case are changed into functions of the independent variables. For this reason, the method is known as **variation of parameters.**

In order to determine two functions u_1 and u_2, we need two conditions. One condition is that (3) must satisfy (1). A second condition can be imposed arbitrarily.

Differentiating (3) with respect to x, we get

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2' \Rightarrow y_p' = (u_1'y_1 + u_2'y_2) + (u_1y_1' + u_2y_2').$$

If we differentiate above equation, y_p'' will contain u_1'' and u_2''. To avoid second derivatives of u_1 and u_2, we set

$$u_1'y_1 + u_2'y_2 = 0$$

With this condition, we have: $y_p' = u_1y_1' + u_2y_2'$ (4)

so that, $y_p'' = u_1y_1'' + y_1'u_1' + u_2y_2'' + y_2'u_2' \Rightarrow y_p'' = u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2''$.

Substituting y_p, y_p', y_p'' into equation (1), we obtain

$$(u_1'y_1' + u_2'y_2' + u_1y_1'' + u_2y_2'') + P(u_1y_1' + u_2y_2') + Q(u_1y_1 + u_2y_2) = F(x)$$

or $$u_1(y_1'' + Py_1' + Qy_1) + u_2(y_2'' + Py_2' + Qy_2) + u_1'y_1' + u_2'y_2' = F(x),$$

Expressions in the parenthesis are zero since y_1 and y_2 are solutions of (2). Hence

$$u_1'y_1' + u_2'y_2' = F(x) \tag{5}$$

Taking (4) and (5) together, we have two equations in the two unknowns u_1' and u_2'. This is: $u'_1 y_1 + u'_2 y_2 = 0$ and $u'_1 y'_1 + u'_2 y'_2 = F(x)$

Multiplying (4) by y_2' and (5) by y_2, we get

$$u_1'y_1y_2' + u_2'y_2y_2' = 0 \tag{6}$$

$$u_1'y_1'y_2 + u_2'y_2y_2' = y_2F(x) \tag{7}$$

Subtracting (7) from (6), we have

$$u_1'y_1y_2' - u_1'y_1'y_2 = -y_2F(x) \Rightarrow u_1'(y_1y_2' - y_1'y_2) = -y_2F(x) \Rightarrow u_1' = \frac{-y_2F(x)}{(y_1y_2' - y_1'y_2)}.$$

Similarly, $u_2' = \dfrac{y_1F(x)}{(y_1y_2' - y_1'y_2)}$.

Thus,

$$\left.\begin{aligned} u_1' &= \frac{-y_2F(x)}{(y_1y_2' - y_1'y_2)} \\ u_2' &= \frac{y_1F(x)}{(y_1y_2' - y_1'y_2)} \end{aligned}\right\} \tag{8}$$

In (8) $y_1y_2' - y_1'y_2 \neq 0$, since y_1, y_2 are linearly independent solutions of (2)

Integrating (8), we find u_1 and u_2 as

$$\left.\begin{aligned} u_1 &= \int \frac{-y_2F(x)}{(y_1y_2' - y_1'y_2)}dx \\ u_2 &= \int \frac{y_1F(x)}{(y_1y_2' - y_1'y_2)}dx \end{aligned}\right\} \tag{9}$$

Thus, y_p is completely determined.

In numerical problems, instead of performing the complete process, formulas (9) will directly be applied to evaluate u_1 and u_2.

It may be noted that expression in the denominator of (9) is known as WRONSKIN and is some times denoted by $W = \begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix} = y_1y'_2 - y'_1y_2$.

Example 01: Find a general solution of $y'' + y = \tan x \sec x$

Solution: Given that $y'' + y = \tan x \sec x$ (1)

$$\Rightarrow (D^2 + 1)y = \tan x \sec x.$$

Auxiliary equation is: $m^2 + 1 = 0 \Rightarrow m = \pm i \rightarrow y_c = c_1 \sin x + c_2 \cos x$.

Let

$$y_p = u_1 \sin x + u_2 \cos x \tag{2}$$

Here $y_1 = \sin x, y_2 = \cos x, F(x) = \tan x \sec x, y_1' = \cos x, y_2' = -\sin x$.

Also $W = y_1y_2' - y_1'y_2 = \sin x(-\sin x) - \cos x(\cos x) = -\sin^2 x - \cos^2 x = -1$.

By formulas (9), we have

$$\left.\begin{aligned} u_1 &= \int \frac{-y_2F(x)}{(y_1y_2' - y_1'y_2)}dx \\ u_2 &= \int \frac{y_1F(x)}{(y_1y_2' - y_1'y_2)}dx \end{aligned}\right\}$$

➔ $$u_1 = \int \frac{-\cos x \tan x \sec x}{-1}dx = \int \tan x dx = \ln(\sec x).$$

$$u_2 = \int \frac{\sin x \tan x \sec x}{-1}dx = -\int \frac{\sin^2 x}{\cos^2 x}dx = -\int \tan^2 x dx$$

$$= -\int (\sec^2 x - 1)dx = \int 1dx - \int \sec^2 x dx = x - \tan x.$$

Thus, (2) becomes

$$y_p = \ln(\sec x)\sin x + (x - \tan x)\cos x = \sin x \ln(\sec x) + x\cos x - \sin x.$$

Thus general solution of given differential equation is: $y = y_c + y_p$

$\Rightarrow$ $$y = c_1 \sin x + c_2 \cos x + \sin x \ln(\sec x) + x\cos x - \sin x.$$

Example 02: Apply the method of variation of parameters to find the solution of $y''' - 3y'' + 3y' - y = 2e^x/x^2$

Solution: We have $\quad y''' - 3y'' + 3y' - y = 2e^x/x^2 \qquad (1)$

The auxiliary equation is: $m^3 - 3m^2 + 3m - 1 = 0 \Rightarrow (m-1)^3 = 0 \Rightarrow m = 1, 1, 1$

Hence, Complementary function is:

$$y_c = (c_1 + c_2x + c_3x^2)e^x = c_1e^x + c_2xe^x + c_3x^2e^x.$$

Let $\quad y_p = u_1e^x + u_2xe^x + u_3x^2e^x.$

Here $\quad y_1 = e^x, y_2 = xe^x, y_3 = x^2e^x, F(x) = 2e^x/x^2$

Also $\quad y_1' = e^x, y_2' = xe^x + e^x, y_3' = x^2e^x + 2xe^x, y_1'' = e^x, y_2'' = xe^x + 2e^x,$

$$y_3'' = x^2e^x + 2xe^x + 2(xe^x + e^x) = x^2e^x + 4xe^x + 2e^x.$$

By the following formulas:

$$\left.\begin{aligned} u_1'y_1 + u_2'y_2 + u_3'y_3 &= 0 \\ u_1'y_1' + u_2'y_2' + u_3'y_3' &= 0 \\ u_1'y_1'' + u_2'y_2'' + u_3'y_3'' &= F(x) \end{aligned}\right\}$$

Substituting the values, we get

$$\left.\begin{aligned} &u_1'e^x + u_2'xe^x + u_3'x^2e^x = 0,\ u_1'e^x + u_2'(xe^x + e^x) + u_3'(x^2e^x + 2xe^x) = 0 \\ &u_1'e^x + u_2'(xe^x + 2e^x) + u_3'(x^2e^x + 4xe^x + 2e^x) = \frac{2e^x}{x^2} \end{aligned}\right\}$$

Solving these equations for u_1', u_2', u' by Cramer's Rule, we get

$$u_1' = \frac{\begin{vmatrix} 0 & xe^x & x^2e^x \\ 0 & xe^x+e^x & x^2e^x+2xe^x \\ \frac{2e^x}{x^2} & xe^x+2e^x & x^2e^x+4xe^x+2e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x+e^x & x^2e^x+2xe^x \\ e^x & xe^x+2e^x & x^2e^x+4xe^x+2e^x \end{vmatrix}} = \frac{\frac{2e^x}{x^2}\begin{vmatrix} xe^x & x^2e^x \\ xe^x+e^x & x^2e^x+2xe^x \end{vmatrix}}{e^{3x}\begin{vmatrix} 1 & x & x^2 \\ 1 & x+1 & x^2+2x \\ 1 & x+2 & x^2+4x+2 \end{vmatrix}}$$

$$u_1' = \frac{\frac{2}{x}\begin{vmatrix} 1 & x \\ x+1 & x^2+2x \end{vmatrix}}{\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 2 & 4x+2 \end{vmatrix}} = \frac{2/x\left(x^2+2x-x^2-x\right)}{(4x+2-4x)} = \frac{2}{x}(x)\left(\frac{1}{2}\right) = 1.$$

$$u_2' = \frac{\begin{vmatrix} e^x & 0 & x^2e^x \\ e^x & 0 & x^2e^x+2xe^x \\ e^x & 2e^x/x^2 & x^2e^x+4xe^x+2e^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x+e^x & x^2e^x+2xe^x \\ e^x & xe^x+2e^x & x^2e^x+4xe^x+2e^x \end{vmatrix}} = \frac{-\frac{2e^x}{x^2}\begin{vmatrix} e^x & x^2e^x \\ e^x & 2xe^x+x^2e^x \end{vmatrix}}{2e^{3x}}$$

$$u_2' = \frac{-\frac{2e^{3x}}{x^2}\begin{vmatrix} 1 & x^2 \\ 1 & 2x+x^2 \end{vmatrix}}{2e^{3x}} = -\frac{1}{x^2}\left(2x+x^2-x^2\right) = -\frac{2}{x}.$$

$$u_3' = \frac{\begin{vmatrix} e^x & xe^x & 0 \\ e^x & xe^x+e^x & 0 \\ e^x & xe^x+2e^x & 2e^x/x^2 \end{vmatrix}}{\begin{vmatrix} e^x & xe^x & x^2e^x \\ e^x & xe^x+e^x & x^2e^x+2xe^x \\ e^x & xe^x+2e^x & x^2e^x+4xe^x+2e^x \end{vmatrix}} = \frac{\frac{2e^x}{x^2}\begin{vmatrix} e^x & xe^x \\ e^x & xe^x+2e^x \end{vmatrix}}{2e^{3x}}$$

$$u_3' = \frac{\frac{2e^{3x}}{x^2}\begin{vmatrix} 1 & x \\ 1 & x+1 \end{vmatrix}}{2e^{3x}} = \frac{1}{x^2}(x+1-x) = \frac{1}{x^2}.$$

Therefore,

$$u_1' = \int 1dx = x,\ u_2' = \int \frac{-2}{x}dx = -2\ln x,\ u_3' = \int \frac{1}{x^2}dx = -\frac{1}{x}.$$

Thus, $y_p = xe^x - 2\ln x\left(xe^x\right) + \left(-\frac{1}{x}\right)\left(x^2e^x\right) = xe^x - 2xe^x \ln x - xe^x = -2xe^x \ln x$.

Thus general solution is: $y = c_1 e^x + c_2 xe^x + c_3 x^2 e^x - 2xe^x \ln x$

Example 03: Solve the following differential equations by method of variation of parameters

i. $y'' + 4y = \sec 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0 \rightarrow m = \pm 2i$. Thus

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = \cos 2x$, $y_2 = \sin 2x$ and

$$u_1 = -\int \frac{y_2 f(x)}{W}dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W}dx$$

Here,

$W = y_1 y_2' - y_2 y_1' = \cos 2x\,(2\cos 2x) - \sin 2x\,(-2\sin 2x) = 2(\cos^2 2x + \sin^2 2x) = 2.$

Thus, $u_1 = -\int \frac{\sin 2x \sec 2x}{2}dx = -\frac{1}{2}\int \tan 2x\, dx = -\frac{1}{2}\left(\frac{\ln \sec 2x}{2}\right) = -\frac{\ln \sec 2x}{4}$ and

$$u_2 = \int \frac{\cos 2x \sec 2x}{2}dx = \frac{1}{2}\int 1.dx = \frac{x}{2}$$

Therefore, $y_p = -\frac{\ln \sec 2x}{4}.\cos 2x + \frac{x}{2}.\sin 2x$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x - \frac{\ln \sec 2x}{4} + \frac{x}{2}.\sin 2x$$

ii. $y'' - 3y' + 2y = (1 + e^{-x})^{-1}$

Solution: The auxiliary equation is: $m^2 - 3m + 2 = 0 \rightarrow m = 1, 2$. Thus

$$y_c = C_1 e^x + C_2 e^{2x}$$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^x$, $y_2 = e^{2x}$ and

$$u_1 = -\int \frac{y_2 f(x)}{W}dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W}dx.$$

Here, $W = y_1 y_2' - y_2 y_1' = e^x(2e^{2x}) - e^{2x}(e^x) = e^{3x}.$

Thus, $u_1 = -\int \frac{e^{2x}(1+e^{-x})^{-1}}{e^{3x}}dx = -\int \frac{1}{(e^x+1)}dx.$

Now put $e^x = z \Rightarrow e^x dx = dz \Rightarrow zdx = dz \Rightarrow dx = dz/z$

$$\therefore u_1 = -\int \frac{1}{z(z+1)}dz = -\int\left(\frac{1}{z} - \frac{1}{z+1}\right)dz = -(\ln z - \ln(z+1)) = \ln\frac{z+1}{z} = \ln\left(\frac{e^x+1}{e^x}\right)$$

Also $u_2 = \int \frac{e^x(1+e^{-x})^{-1}}{e^{3x}}dx = \int \frac{1}{e^x(e^x+1)}dx.$

Now put $e^x = z \Rightarrow e^x dx = dz \Rightarrow z dx = dz \Rightarrow dx = dz/z$

$$\therefore u_2 = \int \frac{1}{z^2(z+1)} du = \int\left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{z+1}\right) du = \left(-\frac{1}{z} - \ln z + \ln(z+1)\right)$$

$$= \ln\frac{z+1}{z} - \frac{1}{z} = \ln\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x}$$

Therefore, $y_p = e^x.\ln\left(\frac{e^x+1}{e^x}\right) + e^{2x}\left[\ln\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x}\right]$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = C_1 e^x + C_2 e^{2x} + e^x \ln\left(\frac{e^x+1}{e^x}\right) + e^{2x}\left[\ln\left(\frac{e^x+1}{e^x}\right) - \frac{1}{e^x}\right]$$

iii. $y'' + 4y' + 5y = e^{-2x} \sec x$

Solution: Auxiliary equation is $m^2 + 4m + 5 = 0 \rightarrow m = -2 \pm i$.

Thus $y_c = e^{-2x}(C_1 \cos x + C_2 \sin x)$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^{-2x} \cos x$, $y_2 = e^{-2x} \sin x$

and

$$u_1 = -\int \frac{y_2 f(x)}{W} dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W} dx .$$

Here, $W = y_1 y'_2 - y_2 y'_1 = e^{-4x}$.

Thus,

$$u_1 = -\int \frac{e^{-2x} \sin x. e^{-2x} \sec x}{e^{-4x}} dx = -\int \sec x\, dx = -\ln(\sec x + \tan x)$$

Also $u_2 = \int \frac{e^{-2x} \cos x\, e^{-2x} \sec x}{e^{-4x}} dx = \int 1\, dx. = x$

Therefore, $y_p = e^{-2x}[-\cos x \ln(\sec x + \tan x) + x \sin x]$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = e^{-2x}(C_1 \cos x + C_2 \sin x) + e^{-2x}[-\cos x \ln(\sec x + \tan x) + x \sin x]$$

iv. $y'' - 4y' + 4y = e^{2x}/(1+x)$

Solution: Auxiliary equation is $m^2 - 4m + 4 = 0 \rightarrow m = 2, 2$.

Thus, $y_c = (C_1 + C_2 x) e^{2x}$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^{2x}$ and $y_2 = x e^{2x}$

and $u_1 = -\int \frac{y_2 f(x)}{W} dx$ and $u_2 = \int \frac{y_1 f(x)}{W} dx$.

Here, $W = y_1 y'_2 - y_2 y'_1 = e^{4x}$.

Thus

$$u_1 = -\int \frac{xe^{2x}.e^{2x}}{e^{4x}(1+x)} dx = -\int \frac{x}{x+1} dx = -\int \frac{(x+1)-1}{x+1} dx = -\int\left(1 - \frac{1}{x+1}\right) dx = \ln(x+1) - x$$

Also, $u_2 = \int \frac{e^{2x}.e^{2x}}{e^{4x}(1+x)} dx = \int \frac{1}{x+1} dx = \ln(x+1)$

Therefore, $y_p = e^{2x}[\ln(x+1) - x + x\ln(x+1)] = e^{2x}[\ln(x+1)(x+1) - x]$

Hence, general solution of given differential equation is:

$$y = y_c + y_p = (C_1 + C_2 x) e^{2x} + e^{2x}[\ln(x+1)(x+1) - x]$$

v. $y'' - 2y' + y = e^x \sin^{-1} x$

Solution: Auxiliary equation is $m^2-2m+1=0 \rightarrow m=1,1.$

Thus, $y_c=(C_1+C_2x)e^x$

To find y_p, we know that $y_p=u_1y_1+u_2y_2$, where: $y_1=e^x$, $y_2=xe^x$

and $u_1=-\int\frac{y_2f(x)}{W}dx$ and $u_2=\int\frac{y_1f(x)}{W}dx$.

Here, $W=y_1y'_2-y_2y'_1=e^{2x}$.

Thus

$$u_1=-\int\frac{xe^x.e^x\sin^{-1}x}{e^{2x}}dx=-\int x.\sin^{-1}x\,dx=-\sin^{-1}x.\frac{x^2}{2}+\frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}dx$$

$$=-\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int\frac{-x^2}{\sqrt{1-x^2}}dx=-\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int\frac{(1-x^2)+1}{\sqrt{1-x^2}}dx$$

$$=-\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int\frac{(1-x^2)}{\sqrt{1-x^2}}dx-\frac{1}{2}\int\frac{1}{\sqrt{1-x^2}}dx=-\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int\sqrt{1-x^2}\,dx-\frac{1}{2}\sin^{-1}x$$

$$=-\frac{(1+x^2)}{2}\sin^{-1}x-\frac{1}{2}\int\sqrt{1-x^2}\,dx=-\frac{(1+x^2)}{2}\sin^{-1}x-\frac{1}{2}\left(\frac{x\sqrt{1-x^2}}{2}-\frac{1}{2}\sin^{-1}x\right)$$

$$=-\frac{(2x^2+\sin^{-1}x)}{4}-\frac{x\sqrt{1-x^2}}{4}=-\frac{1}{4}\left(2x^2+\sin^{-1}x+x\sqrt{1-x^2}\right)$$

Also, $u_2=\int\frac{e^x.e^x\sin^{-1}x}{e^{2x}}dx=\int\sin^{-1}x\,dx=\int\sin^{-1}x.1\,dx=\sin^{-1}x.x-\int\frac{1}{\sqrt{1-x^2}}.x\,dx$

$$=x\sin^{-1}x+\frac{1}{2}\int\frac{2x}{\sqrt{1-x^2}}dx=x\sin^{-1}x-\frac{1}{2}\int(1-x^2)^{-1/2}(-2x)\,dx$$

$$=x\sin^{-1}x-\frac{1}{2}\frac{\sqrt{1-x^2}}{1/2}=x\sin^{-1}x-\sqrt{1-x^2}$$

Therefore, $y_p=-\frac{1}{4}\left(2x^2+\sin^{-1}x+x\sqrt{1-x^2}\right)e^x+xe^x[x\sin^{-1}x-\sqrt{1-x^2}]$

Hence, general solution of given differential equation is:

$$y=(C_1+C_2x)e^x-\frac{1}{4}\left(2x^2+\sin^{-1}x+x\sqrt{1-x^2}\right)e^x+xe^x[x\sin^{-1}x-\sqrt{1-x^2}]$$

vi. $y''-2y'+5y=e^x\tan 2x$

Solution: Auxiliary equation is $m^2-2m+5=0 \rightarrow m=1\pm 2i.$

Thus, $y_c=e^x(C_1\cos 2x+C_2\sin 2x)$

To find y_p, we know that $y_p=u_1y_1+u_2y_2$, where: $y_1=e^x\cos 2x$, $y_2=e^x\sin 2x$

and

$$u_1=-\int\frac{y_2f(x)}{W}dx \text{ and } u_2=\int\frac{y_1f(x)}{W}dx .$$

Here, $W=y_1y'_2-y_2y'_1=2e^{2x}$.

Thus, $u_1=-\int\frac{e^x\sin 2x.e^x\tan 2x}{2e^{2x}}dx=-\frac{1}{2}\int\frac{\sin 2x.\sin 2x}{\cos 2x}dx=-\frac{1}{2}\int\frac{\sin^2 2x}{\cos 2x}dx$

$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx = -\frac{1}{2}\int (\sec 2x - \cos 2x)\,dx = -\frac{1}{4}\ln(\sec 2x + \tan 2x) + \frac{1}{4}\sin 2x$

$= \frac{1}{4}[\sin 2x - \ln(\sec 2x + \tan 2x)]$

Also, $u_2 = \int \frac{e^x \cos 2x . e^x \tan 2x}{2e^{2x}}\,dx = \frac{1}{2}\int \sin 2x\,dx = -\frac{1}{4}\cos 2x$

Therefore, $y_P = \frac{1}{4}[\sin 2x - \ln(\sec 2x + \tan 2x)]e^x \cos 2x - \frac{1}{4}\cos 2x . e^x \sin 2x$

Hence the general solution of given differential equation is:

$y = e^x (C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{4}e^x \cos 2x\,[\sin 2x - \ln(\sec 2x + \tan 2x) - \sin 2x]$

vii. $y'' + 2y' + y = e^{-x} \ln x$

Solution: Auxiliary equation is $m^2 + 2m + 1 = 0 \rightarrow m = -1, -1.$

Thus, $y_c = (C_1 + C_2 x)\,e^{-x}$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^{-x}$, $y_2 = xe^{-x}$

and

$$u_1 = -\int \frac{y_2 f(x)}{W}dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W}dx .$$

Here, $W = y_1 y'_2 - y_2 y'_1 = e^{-2x}$.

Thus,

$u_1 = -\int \frac{xe^{-x}.e^{-x}\ln x}{e^{-2x}}dx = -\int x\ln x\,dx = \frac{x^2}{4}[1 - \ln x^2]$ [integrating by parts]

Also, $u_2 = \int \frac{e^{-x}.\,e^{-x}\ln x}{e^{-2x}}dx = \int \ln x\,dx = x[\ln x - 1]$ [integrating by parts]

Therefore, $y_P = e^{-x}\,\frac{x^2}{4}[1 - \ln x^2] + x\,e^{-x}\,.\,x[\ln x - 1]$

Hence, general solution of given differential equation is:

$y = y_c + y_p = (C_1 + C_2 x)\,e^{-x} + e^{-x}\,\frac{x^2}{4}[1 - \ln x^2] + x^2 e^{-x}\,[\ln x - 1]$

viii. $y'' + 2y' + 2y = 2e^{-x}\tan^2 x$

Solution: Auxiliary equation is $m^2 + 2m + 2 = 0 \rightarrow m = -1 \pm i.$

Thus, $y_c = e^{-x}(C_1 \cos x + C_2 \sin x)$

To find y_p, we know that $y_p = u_1 y_1 + u_2 y_2$, where: $y_1 = e^{-x}\cos x$, $y_2 = e^{-x}\sin x$

And

$$u_1 = -\int \frac{y_2 f(x)}{W}dx \text{ and } u_2 = \int \frac{y_1 f(x)}{W}dx .$$

Here, $W = y_1 y'_2 - y_2 y'_1 = e^{-2x}$.

Thus

$u_1 = -\int \frac{e^{-x}\sin x\,.\,2e^{-x}\tan^2 x}{e^{-2x}}dx = -2\int \frac{\sin x . \sin^2 x}{\cos^2 x}\,dx = 2\int \frac{1-\cos^2 x}{\cos^2 x}\,(-\sin x)dx$

Putting $z = \cos x \rightarrow dz = -\sin x\,dx$

$\therefore u_1 = 2\int \frac{1-z^2}{z^2}\,dz = 2\int \left(\frac{1}{z^2} - 1\right)dz = 2\int (z^{-2} - 1)dz = 2(-z^{-1} - z)$

$$= -2\left(\frac{1}{z}+z\right) = -2(\sec x + \cos x)$$

Also, $u_2 = \int \frac{e^{-x}\cos x . e^{-x} . 2e^{-x}\tan^2 x}{e^{-2x}} dx = 2\int \cos x \frac{\sin^2 x}{\cos^2 x} dx = 2\int \frac{(1-\cos^2 x)}{\cos x} dx$

$$= 2\int(\sec x - \cos x)dx = 2[\ln(\sec x + \tan x) - \sin x]$$

Therefore, $y_p = -2\, e^{-x} \cos x\, (\sec x + \cos x) + 2e^{-x} \sin x\, [\ln(\sec x + \tan x) - \sin x]$

Hence, general solution of given differential equation is:

$y = y_c + y_p = e^{-x}(C_1 \cos x + C_2 \sin x) - 2\, e^{-x}\, \{\cos x\, (\sec x + \cos x)$

7.6 APPLICATIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS

Although, applications of higher order differential equations are not restricted to only Electrical Engineering. Looking at the limitations and size of the book, we restrict ourselves to applications pertaining to only Electrical Simple RLC Circuits.

Here, we use two laws.

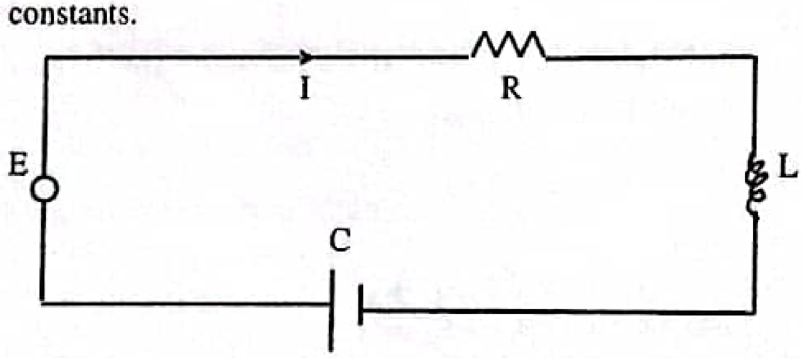
1. Ohm's law and
2. Kirchhoff's voltage law.

Ohm's law states that voltage is directly proportional to the current, that is,

$$V \propto I \rightarrow V = IR,$$

where R is the resistance

Kirchhoff's law states that sum of the voltage drops across the elements of a closed circuit are equal to total electromotive force E in the circuit. The voltage drop across a resistor is IR, across a coil of inductance is $L\cdot\frac{dI}{dt}$, and across a condenser of capacitance is $\frac{1}{C}Q\cdot$ Here, current I and the charge Q are related by $I=\frac{dQ}{dt}$, we will consider R, L, and C as constants.



The differential equation of an electric circuit containing an inductance L, a resistor R, a condenser of capacitance C, and an electromotive force $E(t)$ is therefore

$$L\frac{dI}{dt} + IR + \frac{1}{C}Q = E \qquad (1)$$

or, since $I = \frac{dQ}{dt}$ and $\frac{dI}{dt} = \frac{d^2Q}{dt^2}$, then (1) becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E, \qquad (2)$$

Example 01: If the resistor R, capacitor C and inductor L are connected with an emf E in series, find the charge and current at any time when R = 2 ohms, L = 1 Henry, C = 1 Farad and E = 5 sin t.

Solution: We have $L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=E$.

Substituting the given values, we get

$$\frac{d^2Q}{dt^2}+2\frac{dQ}{dt}+\frac{1}{1}Q=5\sin t \Rightarrow \frac{d^2Q}{dt^2}+2\frac{dQ}{dt}+Q=5\sin t \Rightarrow \left(D^2+2D+1\right)Q=5\sin t$$

Auxiliary equation is: $D^2+2D+1=0 \Rightarrow (D+1)^2=0 \Rightarrow D=-1,-1$.

The Complementary Function is: $Q_c=(c_1+c_2t)e^{-t}$.

Now the Particular Integral is

$$Q_p=\frac{1}{(D+1)^2}(5\sin t)=5\,\mathrm{Im}\left[\frac{1}{(D+1)^2}\left(e^{it}\right)\right]=5\,\mathrm{Im}\left[\frac{e^{it}}{(i+1)^2}\right]=\mathrm{Im}\left[\frac{5e^{it}}{2i}\right]=\mathrm{Im}\left[\frac{5e^{it}}{2i}\times\frac{i}{i}\right]$$

$$=-\frac{5}{2}\mathrm{Im}\left[i(\cos t+i\sin t)\right]=-\frac{5}{2}\mathrm{Im}\left[(i\cos t-\sin t)\right]=-\frac{5}{2}\cos t$$

Thus, $Q=Q_c+Q_p=(c_1+c_2t)e^{-t}-5\cos t/2$

Also, $I=\dfrac{dQ}{dt}=-e^{-t}(c_1+c_2t)+c_2e^{-t}+\dfrac{5}{2}\sin t$.

Example 02: Solve the differential equation: Q'' + R Q' + Q/C = E for charge Q and current I, where L = 1, R = 2, C = 1 and E = 5, given that Q(0) = 0 and I(0) = 0 at t = 0.

Solution: We have $L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=E$.

Substituting the given values, we get

$$\frac{d^2Q}{dt^2}+2\frac{dQ}{dt}+Q=5 \Rightarrow \left(D^2+2D+1\right)Q=5.$$

Auxiliary equation is: $m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1,-1$.

Complementary Function is: $Q_c=(c_1+c_2t)e^{-t}$.

Now, Particular Integral is: $Q_p=\dfrac{1}{(D+1)^2}\left(5e^{0t}\right)=\dfrac{5}{(0+1)}=5$.

Thus, $Q=Q_c+Q_p=(c_1+c_2t)e^{-t}+5$ (1)

Also, $I=Q'=-e^{-t}(c_1+c_2t)+c_2e^{-t}$ (2)

Applying the given conditions, that is, put t = 0, Q = 0 and I = 0, we get:

$0=c_1+5 \Rightarrow c_1=-5$.

And $0=-e^{-10}(c_1+10c_2)+c_2e^{-10} \Rightarrow 0=-e^{-10}(-5+10c_2)+c_2e^{-10} \Rightarrow c_2=5/9$.

Hence, $Q=e^{-t}\left(-5+\dfrac{5}{9}t\right)+5,\ I=-e^{-t}\left(-5+\dfrac{5}{9}t\right)+\dfrac{5}{9}e^{-t}$.

Example 03: Find the charge on the capacitor and the current in the given LRC series circuit, where L = 5/3 H, R = 10 ohms, C = 1/30 F and E = 300 V and Q(0) = 0, I(0) = 0.

Solution: We have $L\dfrac{d^2Q}{dt^2}+R\dfrac{dQ}{dt}+\dfrac{1}{C}Q=E$.

Substituting the given values, we get

$$\frac{5}{3}\frac{d^2Q}{dt^2}+10\frac{dQ}{dt}+30Q=300 \Rightarrow \frac{d^2Q}{dt^2}+6\frac{dQ}{dt}+18Q=180 \Rightarrow \left(D^2+6D+18\right)Q=180.$$

The auxiliary equation is: $m^2+6m+18=0 \Rightarrow m=\frac{-6\pm\sqrt{36-72}}{2}=\frac{-6\pm 6i}{2}=-3\pm 3i$.

The Complementary Function is: $Q_c=e^{-3t}\left(c_1\sin 3t+c_2\cos 3t\right)$.

Now the Particular Integral is: $Q_p=\frac{1}{D^2+6D+18}(180)e^{0t}=\frac{180}{18}=10$.

Thus, $Q=Q_c+Q_p=e^{-3t}\left(c_1\sin 3t+c_2\cos 3t\right)+10$.

Also, $I=\frac{dQ}{dt}=-3e^{-3t}\left(c_1\sin 3t+c_2\cos 3t\right)+e^{-3t}\left(3c_1\cos 3t-3c_2\sin 3t\right)$.

Applying the given conditions, we get

$$0=c_2+10 \Rightarrow c_2=-10 \text{ and } 0=-3(-10)+3c_1 \Rightarrow c_1=-10.$$

Hence $\quad Q=e^{-3t}\left(-10\sin 3t-10\cos 3t\right)+10.$

$$I=-3e^{-3t}\left(-10\sin 3t-10\cos 3t\right)+e^{-3t}\left(-30\cos 3t+30\sin 3t\right)$$

$$I=e^{-3t}\left(30\sin 3t+30\cos 3t-30\cos 3t+30\sin 3t\right)=60e^{-3t}\sin 3t.$$

Example 04: A circuit consists of an inductance of 0.05 henry, a resistance of 20 ohms, a condenser of capacitance 100 microfarads, and an emf of E = 100 V. Find I and Q, given the initial conditions Q = 0, I = 0 whn t = 0.

Solution: We have $\quad L\frac{d^2Q}{dt^2}+R\frac{dQ}{dt}+\frac{1}{C}Q=E.$

Substituting the given values, we get

$$(0.05)\frac{d^2Q}{dt^2}+20\frac{dQ}{dt}+\frac{Q}{100\times 10^{-6}}=100 \Rightarrow (0.05)\frac{d^2Q}{dt^2}+20\frac{dQ}{dt}+10000Q=100$$

$$\frac{d^2Q}{dt^2}+400\frac{dQ}{dt}+2,00000Q=2000 \Rightarrow \left(D^2+400D+2,00000\right)Q=2000.$$

The auxiliary equation is: $m^2+400m+2,00000=0$.

$$m=\frac{-400\pm\sqrt{160,000-8,00000}}{2}=\frac{-400\pm 800i}{2}=-200\pm 400i.$$

The Complementary Function is: $Q_c=e^{-200t}\left(c_1\sin 400t+c_2\cos 400t\right)$.

Now the Particular Integral is: $Q_p=\frac{1}{D^2+400D+2,00000}(2000)=\frac{2000}{2,00000}=0.01$.

Thus, $\quad Q=Q_c+Q_p=e^{-200t}\left(c_1\sin 400t+c_2\cos 400t\right)+0.01$.

Also,

$$I=\frac{dQ}{dt}=-200e^{-200t}\left(c_1\sin 400t+c_2\cos 400t\right)+e^{-200t}\left(400c_1\cos 400t-400c_2\sin 400t\right)$$

$$I=-e^{-200t}\left(200c_1\sin 400t+200c_2\cos 400t-400c_1\cos 400t+400c_2\sin 400t\right).$$

Applying the given conditions, we get

$$0 = c_2 + 0.01 \Rightarrow c_2 = -0.01 \text{ and } 0 = -(200c_2 - 400c_1) \Rightarrow c_1 = -0.005$$

Hence, $I = -e^{-200t}(-\sin 400t - 2\cos 400t + 2\cos 400t - 4\sin 400t) = 5e^{-200t}\sin 400t$

This is the particular solution.

WORKSHEET 07

1. Solve the following homogeneous linear differential equations:

a. $(D^2 - 3D - 4)\,y = 0$ b. $(D^3 - 7D - 6)\,y = 0$
c. $(D^3 - 9D^2 + 23D - 15)\,y = 0$ d. $(D^2 + (a+b)D + ab)\,y = 0$
e. $(D^3 - 2D^2 + 4D - 8)\,y = 0$ f. $(D^4 - 5D^2 + 4)\,y = 0$
g. $(D^2 - 4D + 1)\,y = 0$ h. $(D^4 + k^4)\,y = 0$
i. $(D^3 - D^2 - D - 2)\,y = 0$ j. $(D^4 + 2D^3 + 3D^2 + 2D + 1)\,y = 0$
NOTE: $(D^4 + 2D^3 + 3D^2 + 2D + 1)\,y = (D^2 + D + 1)^2\,y$
k. $(D^3 + 3D^2 + 3D + 1)\,y = 0$ l. $(D^4 - 7D^3 + 18D^2 - 20D + 8)\,y = 0$
m. $(D^4 - 8D^2 + 16)\,y = 0$ n. $(D^4 - k^4)\,y = 0$
o. $(D^6 - k^6)\,y = 0$ p. $(D^4 + 2D^3 - 3D^2 - 4D + 4)\,y = 0$
q. $(D^2 + 4D + 3)\,y = 0,\ y(0) = 0,\ y'(0) = 12.$
r. $(D^4 + D^2)\,y = 0;\ y(0) = y'(0) = y''(0) = 0,\ y'''(0) = 1$
s. $(D^3 + 6D^2 + 12D + 8)\,y = 0;\ y(0) = y'(0) = 0,\ y''(0) = 2$
t. $(D^3 + D^2 + 4D + 4)\,y = 0;\ y(0) = y'(0) = 0,\ y''(0) = -5$
u. $(D^3 - 6D^2 - 12D + 8)\,y = 0;\ y(0) = y'(0) = 0,\ y''(0) = 2$

2. Solve the following non-homogeneous linear differential equations:

a. $(D^2 + D + 1)\,y = e^{-x}$ b. $(D^2 - 3D + 2)\,y = e^{5x}$ c. $(D^2 - 5D + 6)\,y = \sinh 2x$
d. $(D^3 + 1)\,y = 5e^x - \cosh x$ e. $(4D^2 + 4D - 3)\,y = e^{2x}$ f. $(D^2 + D + 1)\,y = \sin 2x$
g. $(D^4 + 1)\,y = \cos x$ h. $(D^2 - 5D + 6)\,y = \sin 3x$ i. $(D^4 - 2D^2 + 1)\,y = \cos x$
j. $(D^2 + D - 6)\,y = x\cos x$ k. $(D^3 - 3D - 2)\,y = x^2$ l. $(D^3 - 13D + 12)\,y = x$
m. $(D^2 - 4)\,y = x^2$ n. $(D^2 - 2D + 4)\,y = e^x\cos x$ o. $(D^2 - 5D + 6)\,y = xe^{4x}$
p. $(D^2 + 1)\,y = x\sin 2x$ q. $(D - 1)^3\,y = 16e^{3x}$ r. $(D^3 + 2D^2 - D - 2)\,y = e^x$
s. $(D^3 - 2D^2 - 5D + 6)y = e^{3x}$ t. $(D^2 - D - 6)y = e^x\sinh 3x$ u. $(D^4 + 1)y = x^4$
v. $(5D^3 - D^2 - 6D)y = x^2$ w. $(D^3 + 1)y = 2\cos^2 x$ x. $(D^2 - 3D + 2)y = x^2e^{4x}$
y. $(D^3 - 4D^2 + 13D)y = \cos 2x$ z. $(D^3 - 7D + 10)y = e^{2x}\sin x + xe^x$

3. Solve the following Cauchy-Euler differential equations

a. $x^2y'' - xy' + y = \ln x$ b. $x^2y'' - 4xy' + 6y = x^2$
c. $x^2y'' - 2xy' - 4y = x^4$ d. $x^2y'' - 3xy' + 4y = (x+1)^2$
e. $x^2y'' - 2y = x^2 + 1/x$ f. $x^3y''' + 2x^2y'' + 2y = 10(x + 1/x)$
g. $x^2y'' + xy' + y = \ln x\sin(\ln x)$ h. $(2x+3)^2y'' - (2x+3)y' - 12y = 6x$
i. $(x+1)^2y'' + (x+1)y' + y = 4\cos[\ln(x+1)]$
j. $(x+1)^2y'' + (x+1)y' + y = \sin 2[\ln(x+1)]$
k. $x^2y''' + 3xy'' + y' = x^2\ln x$ [Hint: Multiply both sides by x]
l. $x^3y''' + 3x^2y'' + xy' + y = x + \ln x$
m. $y'' + y'/x = 12\ln x/x^2$ [Hint: Multiply by x^2]
n. $x^2y'' - 2xy' + 2y = x^2 + \sin(5\ln x)$
o. $x^3y'' + 3x^2y' + xy = \sin(\ln x)$ [Hint: Divide by x]

4. Solve the following differential equations by the method of variation of parameters:

a. $y'' + y = \operatorname{cosec} x$ b. $y'' + a^2 = \sec ax$ c. $y'' + y = \tan x$
d. $y'' + y = x\sin x$ e. $y'' - 6y' + 9y = e^{3x}/x^2$ f. $y'' - 2y' + 2y = e^x\tan x$
g. $y'' - y = e^{-2x}\sin(e^{-x})$ h. $y'' + y = \sin x$ i. $y'' - 3y' + 2y = \sin x$

j. $y'' + y = \sec x \tan x$ k. $y'' + 2y' + 2y = e^{-x} \sec^3 x$ l. $y'' + 4y = \tan 2x$

m. $y'' - 2y' + 2y = e^x \tan x$ o. $y'' + y = x - \cot x$

11. Solve RLC circuit under given conditions

a. Given: L = 10 mili H, R = 200 ohms, C = 0.1 micro F, E = 0 and at t = 0, Q = 0.01 and I = 0. Find Q

b. Given: L = 0.5 H, R = 300 ohms, C = 2 $(10)^{-6}$ F, E = 0 and at t = 0, Q = 0.01 and I = 0. Find Q and I.

c. Given: L = 0.05 H, R = 10 ohms, C = 10^{-3} F, E = 50 sin 200 t and at t = 0, Q = 0 and I = 0. Find Q.

d. Given: L = 1 H, R = 100 ohms, C = 10^{-4} F, E = 100 and at t = 0, Q = 0 and I = 0. Find Q and I.