Multivariate Calculus Manual

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1. Limits and Continuity in Higher Dimension
   - Theorem 1 : properties 1
   - Continuity 1
   - Limits of Quotients 2
   - Limits with 3 variables 4
   - Continuity for two variables 5
2. Partial Derivatives
   - Definition 6
   - Second order partial Derivatives 6
   - Mixed Derivative Theorem 6
   - partial Derivatives of still higher order 7
   - Exercises
   - Differentiating implicitly (Deduction) 17
   - Laplace Equation 18
   - The wave Equation 24
   - The Jacobian 27
     - Resiprocal property of Jacobian
3. Parametric Equations 32
4. Euler's Theorem 37
   - Homogeneous Function
   - Euler's First order Equation 37
   - Euler's Second order Equation 37
   - Euler's Theorem of Homogeneous Function 38
   - Euler's Deduction Formula 38
   - Proof of Euler's Deduction Formula 41
5. Chain Rule 43
   - Chain Rule for function of one independent variable and two intermediate variables
   - Chain Rule For function of 3-variables
     Chain Rule for function of one independent and 3-intermediate variables

- A Formula for implicit differentiation 44
- Exercises 47

6. Directional Derivative and gradient vectors 48
- Calculating Gradients 49
- Finding directional Derivative 50
- Show that $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$; $x = r\cos\theta$, $y = r\sin\theta$
- Show that $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$
- Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$ 51
- proove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$
- $\nabla \cdot (\phi \vec{u}) = \phi(\nabla \cdot \vec{u}) + (\nabla \phi) \cdot \vec{u}$

Integration

7. Evaluate Indefinite Integrals 53
- Chain Rule
- Power Rule
- U- Substitution
- Rationalization
- Trigonometric Substitution

8. Integration by parts 63
- Exercises
  - Integration by parts
  - Trigonometric Substitution
  - Tabular Integration

9. Triple Integrals in Cartesian Coordinates 88
- Triple integrals in arbitrary domains 89
- Examples: Changing the order of integration 90
- The average value of a function 92
- Triple integrals in arbitrary domains

10. Integrals in Cylindrical, spherical coordinates 87
- Integration in cylindrical coordinates
  - Review: Polar coordinates 87
  - Cylindrical coordinates in space 100
  - Triple integral in cylindrical coordinates 102

11. Triple Integrals
- Theory 108
- Meaning 108
- Rules for solving 109
- Examples 110

12. Polar cylindrical coordinates 113
- Examples 114

13. Self - Help work sheets (SHWS) 118
- problems for fun and practice
  1. Rectangular coordinates 118
  2. Iterated integrals 120
  3. Cylindrical coordinates 122
  4. Spherical Coordinates 123
  5. Cylindrical coordinats bounded by coordinate plane 125
  6. Spherical coordinates bounded by sphere 126

14. Integration Involving partial Fraction 128
- Example 1 128
- Example 2 128
- Example 3 129
- Example 4 130
- Example 5 131
- Example 6 132
- Example 7 133

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- Example 8 133
- Example 9 134

15. The definite Integrals 135
- Fundamental Theorem and properties of Definite Integrals
- Fundamental Theorem of Calculus
- Example 1 136
- Example 2 137
- Example 3 138
- Example 4 139
- Example 5
- Example 6 140
- Example 7 141

16. Applications of Definite Integral 142
- Exercises

17. Numerical Integration 148
- Approximating Definite Integrals
- Trapezoidal Rule 148
- Simpson's Rule 150

Limits and Continuity in Higher Dimensions

▸ **Theorem 1 – Properties of Limits of Functions of Two Variables**

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

1. Sum Rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)+g(x,y)] = L+M$

2. Difference Rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)-g(x,y)] = L-M$

3. Constant Multiple Rule: $\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)

4. Product Rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)\cdot g(x,y)] = L\cdot M$

5. Quotient Rule: $\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$

6. Power Rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer

7. Root Rule: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$

   n a positive integer, and if n is even, we assume that $L>0$

▸ **Continuity**

As with functions of a single variable, continuity is defined in terms of limits.

DEFINITION A function $f(x,y)$ is continuous at the point (x_0,y_0) if

1. f is defined at (x_0,y_0),

2. $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists,

3. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

A function is continuous if it is continuous at every point of its domain.

Exercises:

1) $\lim_{(x,y)\to(0,\ln 2)} e^{x-y}$ 2) $\lim_{(x,y)\to(0,0)} e^{y}\frac{\sin x}{x}$

: $e^{0-\ln 2} = e^{-\ln 2} = \boxed{0.5}$: $\frac{\sin x}{x} = 1$, $e^{y}(1) = e^{0} = \boxed{1}$

3) $\lim_{(x,y)\to(1/27,\pi^3)} \cos\sqrt[3]{xy}$

: $\cos(xy)^{1/3} = \cos\left(\frac{\pi^3}{27}\right)^{1/3} = \cos\left(\frac{\pi}{3}\right)^{3\times\frac{1}{3}} = \cos(\pi/3) = \cos(60^\circ) = \boxed{\frac{1}{2}}$

- Limits of Quotients

Find the limits in Exercises 13-24 by rewriting the fraction first.

4) $\lim_{\substack{(x,y)\to(1,1)\\ x\neq 1}} \frac{xy-y-2x+2}{x-1}$

: $\frac{xy-y}{x-1} - \frac{2x-2}{x-1} = \frac{y(x-1)}{(x-1)} - \frac{2(x-1)}{(x-1)} = y-2 = 1-2 = \boxed{-1}$

5) $\lim_{\substack{(x,y)\to(2,-4)\\ x\neq -4,\, x\neq x^2}} \frac{y+4}{x^2y-xy+4x^2-4x}$

: $\frac{y+4}{y(x^2-x)+4(x^2-4)} = \frac{(y+4)}{(y+4)(x^2-x)} = \frac{1}{x^2-x} = \frac{1}{4-2} = \boxed{\frac{1}{2}}$

6) $\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} \frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

Using rationalization technique (by denominator)

: $\frac{x-y+2\sqrt{x}-2\sqrt{y}}{\sqrt{x}-\sqrt{y}} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}} = \frac{x\sqrt{x}+x\sqrt{y}-y\sqrt{x}-y\sqrt{y}+2x+\cancel{2\sqrt{x}\sqrt{y}}-\cancel{2\sqrt{x}\sqrt{y}}-2y}{x-y}$

$= \frac{x\sqrt{x}+x\sqrt{y}-y\sqrt{x}-y\sqrt{y}+2x-2y}{x-y} = \frac{x\sqrt{x}+x\sqrt{y}+2x-y\sqrt{x}-y\sqrt{y}-2y}{x-y}$

$= \frac{x(\sqrt{x}+\sqrt{y}+2)-y(\sqrt{x}+\sqrt{y}+2)}{(x-y)} = \frac{(x-y)(\sqrt{x}+\sqrt{y}+2)}{(x-y)} = (\sqrt{x}+\sqrt{y}+2)$

$\lim_{\substack{(x,y)\to(0,0)\\ x\neq y}} = (\sqrt{x}+\sqrt{y}+2) = (\sqrt{0}+\sqrt{0}+2) = \boxed{2}$

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7) $\lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} \dfrac{x+y-4}{\sqrt{x+y}-2}$

: $\dfrac{x+y-4}{\sqrt{x+y}-2} \times \dfrac{\sqrt{x+y}+2}{\sqrt{x+y}+2} = \dfrac{2x+2y-8+x\sqrt{x+y}+y\sqrt{x+y}-4\sqrt{x+y}}{(x+y)-4}$

$= \dfrac{2(x+y-4)+\sqrt{x+y}(x+y-4)}{(x+y)-4} = \dfrac{(2+\sqrt{x+y})\cancel{(x+y-4)}}{\cancel{(x+y-4)}} = 2+\sqrt{x+y}$

$\lim\limits_{\substack{(x,y)\to(2,2)\\ x+y\neq 4}} = 2+\sqrt{2+2} = \boxed{4}$

8) $\lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$

: Using Rationalization technique with numerator

$\dfrac{\sqrt{2x-y}-2}{2x-y-4} \times \dfrac{\sqrt{2x-y}+2}{\sqrt{2x-y}+2} = \dfrac{(\sqrt{2x-y})^2-(2)^2}{(2x-y-4)(\sqrt{2x-y}+2)} = \dfrac{\cancel{(2x-y-4)}}{\cancel{(2x-y-4)}(\sqrt{2x-y}+2)}$

$= \dfrac{1}{\sqrt{2x-y}+2} = \lim\limits_{\substack{(x,y)\to(2,0)\\ 2x-y\neq 4}} = \dfrac{1}{\sqrt{2(2)-0}+2} = \dfrac{1}{\sqrt{4}+2} = \dfrac{1}{2+2} = \boxed{\dfrac{1}{4}}$

9) $\lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$

: Rationalization by Numerator

$\dfrac{\sqrt{x}-\sqrt{y+1}}{x-y-1} \times \dfrac{\sqrt{x}+\sqrt{y+1}}{\sqrt{x}+\sqrt{y+1}} = \dfrac{(\sqrt{x})^2-(\sqrt{y+1})^2}{(x-y-1)(\sqrt{x}+\sqrt{y+1})}$

$= \dfrac{x-(y+1)}{(x-y-1)(\sqrt{x}+\sqrt{y+1})} = \dfrac{x-y-1}{(x-y-1)(\sqrt{x}+\sqrt{y+1})} = \dfrac{1}{\sqrt{x}+\sqrt{y+1}}$

$= \lim\limits_{\substack{(x,y)\to(4,3)\\ x\neq y+1}} \dfrac{1}{\sqrt{4}+\sqrt{3+1}} = \dfrac{1}{2+2} = \dfrac{1}{4}$

10) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{(x^2+y^2)}$

: let $x^2+y^2 = a \rightarrow \dfrac{\sin(a)}{a}$, where $\dfrac{\sin x}{x} = 1$

So, $\dfrac{\sin(a)}{a} = 1$ and $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin a}{a} = 1 = \boxed{1}$

11) $\lim_{(x,y)\to(1,-1)} \frac{x^3+y^3}{x+y}$

: $a^3+b^3 = (a+b)(a^2-ab+b^2)$

$\lim_{(1,-1)} \frac{(x+y)(x^2-xy+y^2)}{(x+y)} = x^2-xy+y^2 = 1+1+1 = \boxed{3}$

12) $\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4}$

: Difference of two squares formula

$\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \frac{x-y}{(x^2)^2-(y^2)^2} = \frac{(x-y)}{[(x^2)-(y^2)][(x^2)+(y^2)]}$

$\lim_{(x,y)\to(2,2)} \frac{\cancel{(x-y)}}{\cancel{(x-y)}(x+y)(x^2+y^2)} = \frac{1}{(x+y)(x^2+y^2)} = \frac{1}{(4)(8)} = \boxed{\frac{1}{32}}$

• Limits with three variables

13) $\lim_{P\to(\pi,\pi,0)} (\sin^2 x + \cos^2 y + \sec^2 z)$

: $\lim_{P\to(\pi,\pi,0)} \sin^2(\pi) + \cos^2(\pi) + \sec^2(0) = 1+0+1 = 2$

14) $\lim_{P\to(-1/4,\pi/2,2)} \tan^{-1} xyz$

: $\lim_{P\to(-1/4,\pi/2,2)} \tan^{-1}\left(\frac{-1}{4} \times \frac{\pi}{\cancel{2}} \times \cancel{2}\right) = \tan^{-1}\left(\frac{-\pi}{4}\right) = \cot\left(\frac{-\pi}{4}\right) = \boxed{-1}$

15) $\lim_{P\to(\pi,0,3)} ze^{-2y}\cos 2x$

: $\lim_{P\to(\pi,0,3)} 3e^{-2(0)}\cos(2\pi) = 3(1) = \boxed{3}$

16) $\lim_{P\to(2,-3,6)} \ln\sqrt{x^2+y^2+z^2}$

: $\lim_{P\to(2,-3,6)} \ln\sqrt{4+9+36} = \ln\sqrt{49} = \boxed{\ln 7}$

- Continuity for Two variables

17) $g(x,y) = \sin \frac{1}{xy}$:

: $g(x,y)$ is discontinuous at ~~both~~ $x=0$ OR $y=0$

18) $g(x,y) = \frac{x+y}{2+\cos x}$

: $g(x,y)$ is continuous for all real values for x and y

19) $g(x,y) = \frac{x^2+y^2}{x^2-3x+2}$

: $g(x,y) = \frac{x^2+y^2}{(x-1)(x-2)}$, $g(x,y)$ is discontinuous at $x=1$ or $x=2$

20) $g(x,y) = \frac{1}{x^2-y}$

: $x^2 = y$, $g(x,y)$ is discontinuous when $\boxed{x = \sqrt{y}}$

$x = \sqrt{y}$

Partial Derivatives

DEFINITION The partial derivative of $f(x,y)$ with respect to x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$

The partial derivative of $f(x,y)$ with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

Second-Order Partial Derivatives.

When we differentiate a function $f(x,y)$ twice, we produce its second order derivatives. These derivatives are usually denoted by

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{yx}, \text{ and } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{xy}$$

The mixed derivative Theorem

If $f(x,y)$ and its partial derivates f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a,b) and are all continuous at (a,b), then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

You may have noticed that "mixed" second-order partial derivaties

$$\frac{\partial^2 f}{\partial y \partial x} \text{ and } \frac{\partial^2 f}{\partial x \partial y}$$

Partial Derivatives of Still Higher Order

Although we will deal mostly first and second-order partial derivatives, because these appear the most frequently in applications, there is no theortical limit to how many times we can differentiate a function as long as derivative involved exist. Thus, we get third and fourth-order derivatives denoted by symbols like

$$\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx} \quad , \quad \frac{\partial^3 f}{\partial x^2 \partial y^2} = f_{yyxx} \quad \text{and so on.}$$

Example: Find f_{yxyz} if $(x,y,z) = 1 - 2xy^2z + x^2y$

Solution

We first differentiate with respect to the variable y, then x, then y again, and finally with respect to z:

$$f_y = -4xyz + x^2$$
$$f_{yx} = -4yz + 2x$$
$$f_{yxy} = -4z$$
$$f_{yxyz} = -4$$

Exercises:

- Calculating First-order Partial Derivatives

Find $\partial f/\partial x$ and $\partial f/\partial y$.

1) $f(x,y) = 2x^2 - 3y - 4$

: $\frac{\partial f}{\partial x} = 2\frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[3y] - \frac{\partial}{\partial x}[4] = 4x - 0 - 0 = \boxed{4x}$

$\frac{\partial f}{\partial y} = 2\frac{\partial}{\partial y}[x^2] - 3\frac{\partial}{\partial y}[y] - \frac{\partial}{\partial y}[4] = 0 - 3 - 4 = \boxed{-3}$

2) $f(x,y) = x^2 - xy + y^2$

: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^2] - y\frac{\partial}{\partial x}[x] + \frac{\partial}{\partial x}[y^2] = 2x - y + 0 = \boxed{2x - y}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^2] - x\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}[y^2] = 0 - x + 2y = \boxed{2y - x}$

3) $f(x,y) = (x^2-1)(y+2)$

: $\frac{\partial f}{\partial x} = y\frac{\partial}{\partial x}[x^2] + 2\frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[y] - \frac{\partial}{\partial x}[2] = \boxed{2xy + 4x}$

$\frac{\partial f}{\partial y} = y\frac{\partial}{\partial y}[x^2] + 2\frac{\partial}{\partial y}[x^2] - \frac{\partial}{\partial y}[y] - \frac{\partial}{\partial y}[2] = x^2 + 0 - 1 = \boxed{x^2 - 1}$

4) $f(x,y) = 5xy - 7x^2 - y^2 + 3x - 6y + 2$

: $\frac{\partial f}{\partial x} = 5y\frac{\partial}{\partial x}[x] - 7\frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial x}[y^2] + 3\frac{\partial}{\partial x}[x] - 6\frac{\partial}{\partial x}[y] + \frac{\partial}{\partial x}[2]$

$= 5y - 14x - 0 + 3 - 0 + 0 = \boxed{5y - 14x + 3}$

$\frac{\partial f}{\partial y} = 5x\frac{\partial}{\partial y}[y] - \frac{\partial}{\partial y}[7x^2] - \frac{\partial}{\partial y}[y^2] + 3\frac{\partial}{\partial y}[x] - 6\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}[2]$

$= 5x - 0 - 2y + 0 - 6 + 0 = \boxed{5x - 2y - 6}$

5) $f(x,y) = (xy-1)^2$

: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(xy-1)^2 = 2(xy-1)^{2-1} \times \frac{\partial}{\partial x}(xy-1) = 2(xy-1)(y) = \boxed{2y(xy-1)}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy-1)^2 = 2(xy-1) \times \frac{\partial}{\partial y}(xy-1) = 2(xy-1)(x) = \boxed{2x(xy-1)}$

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6) $f(x,y) = (2x-3y)^3$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2x-3y)^3 = 3(2x-3y)^2 \times \frac{\partial}{\partial x}(2x-3y) = 3(2x-3y)^2 \times 2 = \boxed{6(2x-3y)^2}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2x-3y)^3 = 3(2x-3y)^2 \times \frac{\partial}{\partial y}(2x-3y) = 3(2x-3y)^2(-3) = \boxed{-9(2x-3y)^2}$

7) $f(x,y) = \sqrt{x^2+y^2}$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2)^{1/2} = \frac{1}{2\sqrt{x^2+y^2}} \times \frac{\partial}{\partial x}(x^2+y^2) = \frac{2x}{2\sqrt{x^2+y^2}} = \boxed{\frac{x}{\sqrt{x^2+y^2}}}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2)^{1/2} = \frac{1}{2\sqrt{x^2+y^2}} \times \frac{\partial}{\partial y}(x^2+y^2) = \frac{2y}{2\sqrt{x^2+y^2}} = \boxed{\frac{y}{\sqrt{x^2+y^2}}}$

8) $f(x,y) = (x^3 + (y/2))^{2/3}$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3 + (y/2))^{2/3} = \frac{2}{3\sqrt[3]{x^3+(y/2)}} \times \frac{\partial}{\partial x}(x^3+(y/2)) = \frac{2x^2}{\sqrt[3]{x^3+(y/2)}}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^3+(y/2))^{2/3} = \frac{2}{3\sqrt[3]{x^3+(y/2)}} \times \frac{\partial}{\partial y}(x^3+(y/2)) = \frac{1}{3\sqrt[3]{x^3+(y/2)}}$

9) $f(x,y) = \frac{1}{(x+y)}$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{x+y}\right) = \frac{(x+y)(0)-(1)(1)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{x+y}\right) = \frac{(x+y)(0)-(1)(1)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

10) $f(x,y) = \frac{x}{(x^2+y^2)}$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x}{(x^2+y^2)}\right] = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x}{(x^2+y^2)}\right] = \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

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11) $f(x,y) = \dfrac{(x+y)}{(xy-1)}$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left[\dfrac{x+y}{xy-1}\right] = \dfrac{(xy-1)(1)-(x+y)(y)}{(xy-1)^2} = \dfrac{(xy-1)-(xy+y^2)}{(xy-1)^2} = \dfrac{-y^2-1}{(xy-1)^2}$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left[\dfrac{x+y}{xy-1}\right] = \dfrac{(xy-1)(1)-(x+y)(x)}{(xy-1)^2} = \dfrac{(xy-1)-(x^2+xy)}{(xy-1)^2} = \dfrac{-x^2-1}{(xy-1)^2}$

12) $f(x,y) = \tan^{-1}(y/x)$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\tan^{-1}(y/x) = \dfrac{1}{1+(y/x)^2} \times \dfrac{\partial}{\partial x}\left(\dfrac{y}{x}\right) = \dfrac{x^2}{x^2+y^2} \times -\dfrac{y}{x^2} = \dfrac{-x^2y}{x^2(x^2+y^2)}$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\tan^{-1}(y/x) = \dfrac{1}{1+(y/x)^2} \times \dfrac{\partial}{\partial y}(y/x) = \dfrac{x^2}{x^2+y^2} \times \dfrac{1}{x} = \dfrac{x}{x^2+y^2}$

13) $f(x,y) = e^{(x+y+1)}$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x} e^{(x+y+1)} = e^{x+y+1} \times \dfrac{\partial}{\partial x}(x+y+1) = e^{x+y+1}$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y} e^{(x+y+1)} = e^{x+y+1} \times \dfrac{\partial}{\partial y}(x+y+1) = e^{x+y+1}$

14) $f(x,y) = e^{-x}\sin(x+y)$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(e^{-x}\sin(x+y)) = \sin(x+y)\cdot(-e^{-x}) + e^{-x}\cdot\cos(x+y) \times \dfrac{\partial}{\partial x}(x+y)$

$= \boxed{-e^{-x}\sin(x+y) + e^{-x}\cos(x+y)}$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}(e^{-x}\sin(x+y)) = \boxed{\sin(x+y)\cdot(-e^{-x}) + e^{-x}\cos(x+y)}$

15) $f(x,y) = \ln(x+y)$

$\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(\ln(x+y)) = \dfrac{1}{x+y} \times \dfrac{\partial}{\partial x}(x+y) = \dfrac{1}{x+y}$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}(\ln(x+y)) = \dfrac{1}{x+y} \times \dfrac{\partial}{\partial y}(x+y) = \dfrac{1}{x+y}$

16) $f(x,y) = e^{xy}\ln y$

$\therefore \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}(e^{xy}\ln y) = \ln y \dfrac{\partial}{\partial x}(e^{xy}) = \ln y\, e^{xy} \times \dfrac{\partial}{\partial x}(xy) = y\ln y\, e^{xy}$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}(e^{xy}\ln y) = \ln y\, e^{xy}(x) + e^{xy}\cdot\dfrac{1}{y} = x\ln y\, e^{xy} + \dfrac{e^{xy}}{y}$

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17) $f(x,y) = \sin^2(x-3y)$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \sin^2(x-3y) = 2\sin(x-3y) \times \cos(x-3y) \times (1) = 2\sin(x-3y)\cos(x-3y)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \sin^2(x-3y) = 2\sin(x-3y)\cos(x-3y) \times (-3) = -6\sin(x-3y)\cos(x-3y)$

18) $f(x,y) = \cos^2(3x-y^2)$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\cos^2(3x-y^2)) = 2\cos(3x-y^2) \times -\sin(3x-y^2) \times (3)$

$= -6\cos(3x-y^2)\sin(3x-y^2)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(\cos^2(3x-y^2)) = 2\cos(3x-y^2) \times -\sin(3x-y^2) \times (-2y)$

$= -4y\cos(3x-y^2)\sin(3x-y^2)$

- Find f_x, f_y, and f_z

1) $f(x,y,z) = 1 + xy^2 - 2z^2$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[1 + xy^2 - 2z^2] = 0 + y^2 - 0 = \boxed{y^2 = f_x}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[1] + x\frac{\partial}{\partial y}[y^2] - 2\frac{\partial}{\partial y}[z^2] = 0 + 2xy - 0 = \boxed{2xy = f_y}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[1] + \frac{\partial}{\partial z}[xy^2] - 2\frac{\partial}{\partial z}[z^2] = 0 + 0 - 4z = \boxed{-4z = f_z}$

2) $f(x,y,z) = xy + yz + xz$

$\therefore \frac{\partial f}{\partial x} = y\frac{\partial}{\partial x}[x] + \frac{\partial}{\partial x}[yz] + z\frac{\partial}{\partial x}[x] = y + 0 + z = \boxed{y + z = f_x}$

$\frac{\partial f}{\partial y} = x\frac{\partial}{\partial y}[y] + x\frac{\partial}{\partial y}[y] + \frac{\partial}{\partial y}[xz] = x + z + 0 = \boxed{x + z = f_y}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[xy] + y\frac{\partial}{\partial z}[z] + x\frac{\partial}{\partial z}[z] = 0 + y + x = \boxed{y + x = f_z}$

3) $f(x,y,z) = x - \sqrt{y^2 + z^2}$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x] - \frac{\partial}{\partial x}(y^2+z^2)^{1/2} = 1 - 0 = \boxed{1 = f_x}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x] - \frac{\partial}{\partial y}(y^2+z^2)^{1/2} = 0 - \frac{1}{2\sqrt{y^2+z^2}} \times \frac{\partial}{\partial y}(y^2+z^2) = \boxed{\frac{-y}{\sqrt{y^2+z^2}}}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[x] - \frac{\partial}{\partial z}[y^2+z^2]^{1/2} = 0 - \frac{1}{2\sqrt{y^2+z^2}} \times 2z = \frac{-z}{\sqrt{y^2+z^2}}$

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4) $f(x,y,z) = \sin^{-1}(xyz)$

$\therefore \frac{\partial f}{\partial x} = -\sin^{-2}(xyz)\cos(xyz)\cdot yz$, $\frac{d[\sin^{-1}(x)]}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-(xyz)^2}} \times \frac{\partial}{\partial x}[xyz] = \boxed{\frac{yz}{\sqrt{1-(xyz)^2}} = f_x}$

$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-(xyz)^2}} \times \frac{\partial}{\partial y}[xyz] = \boxed{\frac{xz}{\sqrt{1-(xyz)^2}} = f_y}$

$\frac{\partial f}{\partial z} = \frac{1}{\sqrt{1-(xyz)^2}} \times \frac{\partial}{\partial z}[xyz] = \boxed{\frac{xy}{\sqrt{1-(xyz)^2}} = f_z}$

5) $f(x,y,z) = \sec^{-1}(x+yz)$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\sec^{-1}(x+yz)) = \frac{1}{(x+yz)\sqrt{(x+yz)^2-1}} = \frac{1}{(x+yz)\sqrt{(x+yz)^2-1}}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sec^{-1}(x+yz)] = \frac{1}{(x+yz)\sqrt{(x+yz)^2-1}} \times \frac{\partial}{\partial y}(x+yz) = \frac{z}{(x+yz)\sqrt{(x+yz)^2-1}}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[\sec^{-1}(x+yz)] = \frac{1}{(x+yz)\sqrt{(x+yz)^2-1}} \times \frac{\partial}{\partial z}[x+yz] = \frac{y}{(x+yz)\sqrt{(x+yz)^2-1}}$

6) $\ln(x+2y+3z)$

$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\ln(x+2y+3z) = \frac{1}{x+2y+3z} \times \frac{\partial}{\partial x}(x+2y+3z) = \frac{1}{x+2y+3z}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\ln(x+2y+3z) = \frac{1}{x+2y+3z} \times \frac{\partial}{\partial y}(x+2y+3z) = \frac{2}{x+2y+3z}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\ln(x+2y+3z) = \frac{1}{x+2y+3z} \times \frac{\partial}{\partial z}(x+2y+3z) = \frac{3}{x+2y+3z}$

7) $yz\ln(xy)$

$\therefore \frac{\partial f}{\partial x} = yz\frac{\partial}{\partial x}[\ln(xy)] = yz\frac{1}{xy} \times \frac{\partial}{\partial x}xy = \frac{yz}{x}$

$\frac{\partial f}{\partial y} = z\frac{\partial}{\partial y}[y\ln(xy)] = z\left[\ln(xy)(1) + \frac{y}{xy}\cdot\frac{1}{x}\right] = z\ln(xy) + z$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}[z]y\ln(xy) = (1)y\ln(xy) = y\ln(xy)$

8) $f(x,y,z) = e^{-(x^2+y^2+z^2)}$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{-(x^2+y^2+z^2)}\right] = e^{-(x^2+y^2+z^2)} \times \frac{\partial}{\partial x} -(x^2+y^2+z^2) = -2x\, e^{-(x^2+y^2+z^2)}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{-(x^2+y^2+z^2)}\right] = e^{-(x^2+y^2+z^2)} \times \frac{\partial}{\partial y} -(x^2+y^2+z^2) = -2y\, e^{-(x^2+y^2+z^2)}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{-(x^2+y^2+z^2)}\right] = e^{-(x^2+y^2+z^2)} \times \frac{\partial}{\partial z} -(x^2+y^2+z^2) = -2x\, e^{-(x^2+y^2+z^2)}$

9) $f(x,y,z) = e^{-xyz}$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[e^{-xyz}\right] = e^{-xyz} \times \frac{\partial}{\partial x}[-xyz] = -yz\, e^{-xyz}$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[e^{-xyz}\right] = e^{-xyz} \times \frac{\partial}{\partial y}[-xyz] = -xz\, e^{-xyz}$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[e^{-xyz}\right] = e^{-xyz} \times \frac{\partial}{\partial z}[-xyz] = -xy\, e^{-xyz}$

10) $f(x,y,z) = \tanh(x+2y+3z)$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\tanh(x+2y+3z)\right] = \operatorname{sec^2h}(x+2y+3z) \times \frac{\partial}{\partial x}(x+2y+3z) = \operatorname{sec^2h}(x+2y+3z)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\tanh(x+2y+3z)\right] = \operatorname{sec^2h}(x+2y+3z) \times \frac{\partial}{\partial y}(x+2y+3z) = 2\operatorname{sec^2h}(x+2y+3z)$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\tanh(x+2y+3z)\right] = \operatorname{sec^2h}(x+2y+3z) \times \frac{\partial}{\partial z}(x+2y+3z) = 3\operatorname{sec^2h}(x+2y+3z)$

11) $f(x,y,z) = \sinh(xy-z^2)$

$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sinh(xy-z^2)\right] = \cos(xy-z^2) \times \frac{\partial}{\partial x}(xy-z^2) = y\cos(xy-z^2)$

$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\sinh(xy-z^2)\right] = \cos(xy-z^2) \times \frac{\partial}{\partial y}(xy-z^2) = x\cos(xy-z^2)$

$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\sinh(xy-z^2)\right] = \cosh(xy-z^2) \times \frac{\partial}{\partial z}(xy-z^2) = -2z\cosh(xy-z^2)$

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• Find the partial derivative of the function with respect to each variable.

12) $f(t,\alpha) = \cos(2\pi t - \alpha)$

: $\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}[\cos(2\pi t - \alpha)] = -\sin(2\pi t - \alpha) \times \frac{\partial}{\partial t}(2\pi t - \alpha) = -2\pi \sin(2\pi t - \alpha)$

$\frac{\partial f}{\partial \alpha} = \frac{\partial}{\partial \alpha}[\cos(2\pi t - \alpha)] = -\sin(2\pi t - \alpha) \times \frac{\partial}{\partial \alpha} = -\sin(2\pi t - \alpha) \times (-1) = +\sin(2\pi t - \alpha)$

13) $g(u,v) = v^2 e^{(2u/v)}$

: $\frac{\partial g}{\partial u} = \frac{\partial}{\partial u} v^2 e^{(2u/v)} = v^2 e^{(2u/v)} \times \frac{1}{v}\frac{\partial}{\partial u}(2u) = \frac{2}{\cancel{v}} v^{\cancel{2}} e^{2u/v} = \boxed{2v e^{2u/v}}$

$\frac{\partial g}{\partial v} = \frac{\partial}{\partial v}[v^2 e^{(2u/v)}] = e^{2u/v} \cdot 2v + v^2 \cdot e^{2u/v} \cdot -\frac{2u}{v^2} = \boxed{2v e^{2u/v} - 2u e^{2u/v}}$

14) $h(\rho, \phi, \theta) = \rho \sin\phi \cos\theta$

: $\frac{\partial h}{\partial \rho} = \frac{\partial}{\partial \rho}[\rho \sin\phi \cos\theta] = \boxed{\sin\phi \cos\theta}$

$\frac{\partial h}{\partial \phi} = \frac{\partial}{\partial \phi}[\rho \sin\phi \cos\theta] = \boxed{\rho \cos\phi \cos\theta}$

$\frac{\partial h}{\partial \theta} = \frac{\partial}{\partial \theta}[\rho \sin\phi \cos\theta] = \boxed{-\rho \sin\phi \sin\theta}$

15) $g(r, \theta, z) = r(1 - \cos\theta) - z$

$\frac{\partial g}{\partial r} = (1 - \cos\theta) \times \frac{\partial}{\partial r}(r) - \frac{\partial}{\partial r}[z] = 1 - \cos\theta - 0 = \boxed{1 - \cos\theta}$

$\frac{\partial g}{\partial \theta} = \frac{\partial}{\partial \theta}[r] - r\frac{\partial}{\partial \theta}[\cos\theta] - \frac{\partial}{\partial \theta}[z] = 0 + r\sin\theta - 0 = \boxed{r \sin\theta}$

$\frac{\partial g}{\partial z} = \frac{\partial}{\partial z} r(1 - \cos\theta) - \frac{\partial}{\partial z}[z] = 0 - 1 = \boxed{-1}$

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- Calculating Second-order partial Derivative

Find all the second-order partial derivatives of the functions

1) $f(x,y) = x + y + xy$

1: $\frac{\partial f}{\partial x} = 1 + 0 + y = \boxed{1 + y}$, $\frac{\partial f}{\partial y} = 0 + 1 + x = \boxed{1 + x}$

2: $\frac{\partial^2 f}{\partial x^2} = \boxed{0}$ $\frac{\partial f}{\partial y} = \boxed{0}$

3: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}[1+y] = \boxed{1}$; $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}[1+x] = \boxed{1}$

2) $f(x,y) = \sin xy$

1: $\frac{\partial f}{\partial x} = \cos xy \times y = \boxed{y \cos xy}$; $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[\sin xy] = \boxed{x \cos xy}$

2: $\frac{\partial^2 f}{\partial x} = \boxed{-y^2 \sin xy}$; $\frac{\partial^2 f}{\partial y} = \boxed{-x^2 \sin xy}$

3: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}[y \cos xy] = \cos xy \cdot (1) + y \cdot (-\sin xy) \cdot (x) = \boxed{\cos xy - xy \sin xy}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial}{\partial x}[x \cos xy] = \cos xy \cdot (1) + x(-\sin xy) \cdot (y) = \boxed{\cos xy - xy \sin xy}$

3) $g(x,y) = x^2 y + \cos y + y \sin x$

1: $\frac{\partial f}{\partial x} = 2xy + y \cos x$; $\frac{\partial f}{\partial y} = x^2 - \sin y + \sin x$

2: $\frac{\partial^2 f}{\partial x^2} = 2y + y \sin x$; $\frac{\partial^2 f}{\partial y^2} = 0 - \cos y + 0 = -\cos y$

3: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = 2x + \cos x$; $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = 2x + \cos x$

4) $h(x,y) = xe^y + y + 1$

1: $\frac{\partial f}{\partial x} = e^y$, $\frac{\partial f}{\partial y} = xe^y + 1$; $\frac{\partial^2 f}{\partial x^2} = 0$, $\frac{\partial^2 f}{\partial y^2} = xe^y$

2: $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial y}\left[e^y\right] = e^y$; $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = e^y$

5) $r(x,y) = \ln(x+y)$

1: $\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}[\ln(x+y)] = \frac{1}{x+y} \times \frac{\partial}{\partial x}(x+y) = \frac{1}{x+y}$

$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}[\ln(x+y)] = \frac{1}{x+y} \times \frac{\partial}{\partial y}(x+y) = \frac{1}{x+y}$

2: $\frac{\partial^2 r}{\partial x^2} = \frac{(x+y)(0)-(1)(1)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

$\frac{\partial^2 r}{\partial y^2} = \frac{(x+y)(0)-(1)(1)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

3: $\frac{\partial^2 r}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial r}{\partial x}\right] = \frac{\partial}{\partial y}\left[\frac{1}{x+y}\right] = \frac{(x+y)(0)-(1)(1)}{(x+y)^2} = \frac{-1}{(x+y)^2}$

$\frac{\partial^2 r}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial r}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{1}{x+y}\right] = \frac{-1}{(x+y)^2}$

6) $s(x,y) = \tan^{-1}(y/x)$

1: $\frac{\partial s}{\partial x} = \frac{\partial}{\partial x}[\tan^{-1}(y/x)] \quad \frac{1}{1+(y/x)^2} \times \frac{\partial}{\partial x}\left[\frac{y}{x}\right] = \frac{-y}{x^2+y^2}$

$\frac{\partial s}{\partial y} = \frac{\partial}{\partial y}\left[\tan^{-1}(y/x)\right] = \frac{1}{1+(y/x)^2} \times \frac{\partial}{\partial y}\left[\frac{y}{x}\right] = \frac{x}{x^2+y^2}$

2: $\frac{\partial^2 s}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{-y}{x^2+y^2}\right] = \frac{(x^2+y^2)(0)-(-y)(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$

$\frac{\partial^2 s}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial s}{\partial y}\right] = \frac{(x^2+y^2)(0)-(x)(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$

3: $\frac{\partial^2 s}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial s}{\partial x}\right] = \frac{(x^2+y^2)(-1)-(-y)(2y)}{(x^2+y^2)^2} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial^2 s}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial s}{\partial y}\right] = \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] = \frac{(x^2+y^2)(1)-(x)(2x)}{(x^2+y^2)^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

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• Differentiating implicitly

1) Find the value of $\partial z/\partial x$ at the point $(1,1,1)$ if the equation $xy + z^3x - 2yz = 0$ defines z as a function of the two independent variables x and y and the partial derivatives exists.

Solution

equation $\Rightarrow xy + z^3x - 2yz = 0$

Find $\frac{\partial z}{\partial x}$, where $\frac{\partial z}{\partial x} = \frac{-f_x}{f_z}$ with the help of important deduction

1: Find the derivative with respect to x

$$\frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}[xz^3] - \frac{\partial}{\partial x}[2yz] = y + z^3 = 0$$

2: Find the derivative with respect to z

$$\frac{\partial}{\partial z}[xy] + \frac{\partial}{\partial z}[xz^3] - \frac{\partial}{\partial z}[2yz] = 0 + 3xz^2 - 2y = 3xz^2 - 2y$$

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} = \boxed{\frac{-(y+z^3)}{(3xz^2-2y)}}$$

$\Rightarrow$ Using ordinary method to solve / differentiate implicitly:

Find $\frac{\partial z}{\partial x}$

$$\frac{\partial}{\partial x}[xy] + \frac{\partial}{\partial x}[z^3x] - \frac{\partial}{\partial x}[2yz] = y + (x)(3z^2)\frac{\partial z}{\partial x} + z^3(1) - 2y\frac{\partial z}{\partial x}$$

$$= y + 3xz^2\frac{\partial z}{\partial x} + z^3 - 2y\frac{\partial z}{\partial x} = y + z^3 + \frac{\partial z}{\partial x}(3xz^2 - 2y) = 0$$

$$\frac{\partial z}{\partial x}(3xz^2 - 2y) = -(y+z^3) \Rightarrow \frac{\partial z}{\partial x} = \boxed{\frac{-(y+z^3)}{3xz^2-2y}}$$

So $\frac{\partial z}{\partial x}$ at point $(1,1,1) = \frac{-(1+1)}{3-2} = \frac{-2}{1} = \boxed{-2}$

Q) Find the value of $\partial x/\partial z$ at the point $(1,-1,-3)$ if the equation

$$xz + y\ln x - x^2 + 4 = 0$$

defines x as a function of the two independent variables y and z and the partial derivatives exists.

Solution

$$\frac{\partial x}{\partial z} = \frac{-f_z}{f_x}$$

1: Find f_z, partial derivative with respect to z

$$\frac{\partial}{\partial z}[xz] + \frac{\partial}{\partial z}[y\ln x] - \frac{\partial}{\partial z}[x^2] + \frac{\partial}{\partial z}[4] = \boxed{x}$$

2: Find f_x, partial derivative w.r.t $\boxed{x}$

$$\frac{\partial}{\partial x}[xz] + \frac{\partial}{\partial x}[y\ln x] - \frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial x}[4] = \boxed{z + \frac{y}{x} - 2x}$$

Now rewrite,

$$\frac{\partial x}{\partial z} = \frac{-x}{z + \frac{y}{x} - 2x} = \frac{-x}{\frac{zx + y - 2x^2}{x}} = -x \times \frac{x}{xz + y - 2x^2} = \boxed{\frac{-x^2}{xz + y - 2x^2}}$$

Find value at $(1,-1,-3)$

$$\frac{\partial x}{\partial z} = \frac{-(1)^2}{-3-1-2} = \frac{-1}{-6} = \boxed{\frac{1}{6}}$$

• Show that each function satisfies a Laplace equation.

1) $f(x,y,z) = x^2 + y^2 - 2z^2$

Solution

Find the second-order partial derivatives w.r.t each variable, then sum the derivatives. If sum equals to zero means Laplace satisfied otherwise not.

1: $\frac{\partial}{\partial x}[x^2] + \frac{\partial}{\partial x}[y^2] - 2\frac{\partial}{\partial x}[z^2] = 2x$; $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[2x] = 2$

2: $\frac{\partial}{\partial y}[x^2] + \frac{\partial}{\partial y}[y^2] - 2\frac{\partial}{\partial y}[z^2] = 0 + 2y = 2y$; $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}[2y] = 2$

3: $\frac{\partial}{\partial z}[x^2] + \frac{\partial}{\partial z}[y^2] - 2\frac{\partial}{\partial z}[z^2] = -4z$; $\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}[-4z] = -4$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2 + 2 - 4 = 0$ Laplace satisfied

2) $f(x,y,z) = 2z^3 - 3(x^2+y^2)z$

Solution

1: $\frac{\partial}{\partial x}[2z^3] - \frac{\partial}{\partial x}[3zx^2] - \frac{\partial}{\partial x}[3zy^2] = -6zx$; $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[-6zx] = -6z$

2: $\frac{\partial}{\partial y}[2z^3] - \frac{\partial}{\partial y}[3zx^2] - \frac{\partial}{\partial y}[3zy^2] = -6zy$; $\frac{\partial^2 f}{\partial y} = \frac{\partial}{\partial y}[-6zy] = -6z$

3: $\frac{\partial}{\partial z}[2z^3] - \frac{\partial}{\partial z}[3zx^2] - \frac{\partial}{\partial z}[3zy^2] = 6z^2 - 3x^2 - 3y^2$

$\frac{\partial^2 f}{\partial z} = \frac{\partial}{\partial z}[6z^2] - \frac{\partial}{\partial z}[3x^2] - \frac{\partial}{\partial z}[3y^2] = 12z - 0 - 0 = 12z$

$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = -6z - 6z + 12z = 0$ Laplace satisfied.

3) $f(x,y) = e^{-2y}\cos 2x$

1: $\frac{\partial}{\partial x}[e^{-2y}\cos 2x] = -2e^{-2y}\sin 2x$; $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}[-2e^{-2y}\sin 2x] = -4e^{-2y}\cos 2x$

2: $\frac{\partial}{\partial y}[e^{-2y}\cos 2x] = -2e^{-2y}\cos 2x$, $\frac{\partial^2 f}{\partial y^2} = 4e^{-2y}\cos 2x$;

3: $\frac{\partial}{\partial z}[e$ | $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -4e^{-2y}\cos 2x + 4e^{-2y}\cos 2x = 0$ Laplace satisfied

4) $f(x,y) = \ln\sqrt{x^2+y^2}$

: Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

1: $\frac{\partial}{\partial x}\left(\ln\sqrt{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\ln\sqrt{x^2+y^2}\right)$

$\frac{1}{\sqrt{x^2+y^2}} \times \frac{\partial}{\partial x}(x^2+y^2) + \frac{1}{\sqrt{x^2+y^2}} \times \frac{\partial}{\partial y}(x^2+y^2)$

1: $f(x,y) = \ln\sqrt{x^2+y^2} = \frac{1}{2}\ln(x^2+y^2)$ i.e., power Rule

$f(x,y) = \frac{1}{2}\ln(x^2+y^2)$

$\frac{1}{2}\frac{\partial}{\partial x}\left(\ln(x^2+y^2)\right) + \frac{1}{2}\frac{\partial}{\partial y}\ln(x^2+y^2)$

$\frac{1}{2}\left[\frac{1}{x^2+y^2} \times \frac{\partial}{\partial x}(x^2+y^2)\right] + \frac{1}{2}\left[\frac{1}{x^2+y^2} \times \frac{\partial}{\partial y}(x^2+y^2)\right]$

$\frac{1}{\cancel{2}}\left[\frac{1}{x^2+y^2} \times \cancel{2}x\right] + \frac{1}{\cancel{2}}\left[\frac{1}{x^2+y^2} \times \cancel{2}y\right]$

$\left(\frac{x}{x^2+y^2}\right) + \left(\frac{y}{x^2+y^2}\right)$

2: Double derivative

$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$\left[\frac{(x^2+y^2)(1) - (x)(2x)}{(x^2+y^2)^2}\right] + \left[\frac{(x^2+y^2)(1) - (y)(2y)}{(x^2+y^2)^2}\right]$

$\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = \frac{\cancel{y^2}-\cancel{x^2}+\cancel{x^2}-\cancel{y^2}}{(x^2+y^2)^2} = \frac{0}{(x^2+y^2)^2} = \boxed{0}$

5) $f(x,y) = \tan^{-1}\left(\frac{x}{y}\right)$

Solution

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \times \frac{d}{dx}(x)$$

1: $$\frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{x}{y}\right)\right] + \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{x}{y}\right)\right] = 0$$

$$\left[\frac{1}{1+(x/y)^2} \times \frac{\partial}{\partial x}(x/y)\right] + \left[\frac{1}{1+(x/y)^2} \times \frac{\partial}{\partial y}(x/y)\right] = 0$$

$$\left[\frac{1}{\frac{y^2+x^2}{y^2}} \times \frac{1}{y}\right] + \left[\frac{1}{\frac{y^2+x^2}{y^2}} \times \frac{-x}{y^2}\right] = 0$$

$$\left[\frac{y^2}{x^2+y^2} \times \frac{1}{y}\right] + \left[\frac{y^2}{x^2+y^2} \times \frac{-x}{y^2}\right] = 0$$

$$\left[\frac{y}{x^2+y^2}\right] + \left[\frac{-x}{x^2+y^2}\right] = 0$$

2: Double Derivative

$$\frac{\partial}{\partial x}\left[\frac{y}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[\frac{x^2}{x^2+y^2}\right] = 0$$

$$\left[\frac{(x^2+y^2)(1) - (y)(2x)}{(x^2+y^2)^2}\right] - \left[\frac{(x^2+y^2)(2x) - (x)(2y)}{(x^2+y^2)^2}\right] = 0$$

$$\left[\frac{x^2+y^2-2xy}{(x^2+y^2)^2}\right] - \left[\frac{(x^2+y^2)-2xy}{(x^2+y^2)^2}\right] = 0$$

$$\frac{\cancel{x^2}+\cancel{y^2}-\cancel{2xy}-(\cancel{x^2}+\cancel{y^2})+\cancel{2xy}}{(x^2+y^2)^2} = 0$$

$$\boxed{0 = 0}$$

8) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$

Solution For simplicity, find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}$ Separately.

1:

a: $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times (2x) = -x(x^2+y^2+z^2)^{-3/2}$

b: $\frac{\partial}{\partial y}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times (2y) = -y(x^2+y^2+z^2)^{-3/2}$

c: $\frac{\partial}{\partial z}(x^2+y^2+z^2)^{-1/2} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \times (2z) = -z(x^2+y^2+z^2)^{-3/2}$

2: Double Derivative

a: $\frac{\partial}{\partial x}\left(-x(x^2+y^2+z^2)^{-3/2}\right) = (x^2+y^2+z^2)^{-3/2}(-1) + (-x)\left(-\frac{3}{2}(x^2+y^2+z^2)^{-5/2} \times 2x\right)$

$= (x^2+y^2+z^2)^{-3/2}(-1) + 3x^2(x^2+y^2+z^2)^{-5/2}$

Factor out the common term

$= (x^2+y^2+z^2)^{-3/2}\left(-1 + 3x^2(x^2+y^2+z^2)^{-1}\right)$

$= \frac{1}{(x^2+y^2+z^2)^{3/2}}\left(-1 + \frac{3x^2}{(x^2+y^2+z^2)}\right) = \frac{-x^2-y^2-z^2+3x^2}{(x^2+y^2+z^2)^{5/2}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}$

b: $\frac{\partial}{\partial y}\left(-y(x^2+y^2+z^2)^{-3/2}\right) = (x^2+y^2+z^2)^{-3/2}(-1) + (-y)\left(-\frac{3}{2}(x^2+y^2+z^2)^{-5/2} \times 2y\right)$

$(x^2+y^2+y^2)^{-3/2}(-1) + 3y^2(x^2+y^2+z^2) = (x^2+y^2+z^2)^{-3/2}\cdot\left(-1 + \frac{3y^2}{(x^2+y^2+z^2)}\right)$

$= \frac{1}{(x^2+y^2+z^2)^{3/2}} \times \frac{-x^2-z^2+2y^2}{(x^2+y^2+z^2)} = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)}$

c: $\frac{\partial}{\partial z}\left(-z(x^2+y^2+z^2)^{-3/2}\right) = (x^2+y^2+z^2)^{-3/2}(-1) + (-z)(x^2+y^2+z^2)^{-5/2} \times \frac{3}{2} \times 2z)$

$(x^2+y^2+z^2)^{-3/2}(-1) + 3z^2(x^2+y^2+z^2)^{-5/2}$

$= (x^2+y^2+z^2)^{-3/2}\left(-1 + 3z^2(x^2+y^2+z^2)^{-1}\right)$

$= \frac{1}{(x^2+y^2+z^2)^{3/2}} \times \left(\frac{-x^2+y^2-z^2+3z^2}{(x^2+y^2+z^2)}\right) = \frac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}}$

Laplace

$\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}} + \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{5/2}} + \frac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{5/2}} = 0$ ✓

7) $f(x,y,z) = e^{3x+4y}\cos 5z$

Solution

1: $\frac{\partial}{\partial x}(e^{3x+4y})\cos 5z = \cos 5z\, e^{3x+4y} \times \frac{\partial}{\partial x}(3x+4y) = 3\cos 5z\, e^{3x+4y}$

$\frac{\partial f}{\partial x^2} = \frac{\partial}{\partial x}(3\cos 5z\, e^{3x+4y}) = 3\cos 5z\, e^{3x+4y} \times \frac{\partial}{\partial x}(3x+4y) = \boxed{9\cos 5z\, e^{3x+4y}}$

2: $\frac{\partial}{\partial y}(e^{3x+4y})\cos 5z = \cos 5z\, e^{3x+4y} \times \frac{\partial}{\partial y}(3x+4y) = 4\, e^{3x+4y}\cos 5z$

$\frac{\partial f}{\partial y^2} = \frac{\partial}{\partial y}(4\cos 5z\, e^{3x+4y}) = 4\cos 5z\, e^{3x+4y} \times \frac{\partial}{\partial y}(3x+4y) = \boxed{16\cos 5z\, e^{3x+4y}}$

3: $\frac{\partial}{\partial z}(e^{3x+4y})\cos 5z = (e^{3x+4y}) - \sin 5z \times (5) = -5\, e^{3x+4y}\sin 5z$

$\frac{\partial f}{\partial z^2} = \frac{\partial}{\partial z}(-5e^{3x+4y}\sin 5z) = -5e^{3x+4y}\cos 5z \times \frac{\partial}{\partial z}(5z) = \boxed{-25\, e^{3x+4y}\cos 5z}$

$\Rightarrow 9\cos 5z\, e^{3x+4y} + 16\cos 5z\, e^{3x+4y} - 25e^{3x+4y}\cos 5z = 0$

The wave Equation If we stand on an ocean shore and take a snapshot of the waves, the picture shows a regular pattern of peaks and valleys in an instant of time. We see periodic vertical motion in space, with respect to distance. If we stand in the water, we can feel the rise and fall of the water as the waves go by. We see periodic vertical motion in time. In physics, this beautiful symmetry is expressed by the one-dimensional wave equation.

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}$$

where w is the wave height, x is the distance variable, t is the time variable, and c is the velocity with which the waves are propogated.

Show that the function in the following exercises all Solutions of the wave equation.

1) $w = \sin(x+ct)$

Solution

1: $\frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \sin(x+ct) = \cos(x+ct) \times c$

$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}(\cos(x+ct)\, c) = -\sin(x+ct) \times c \times c = -\sin(x+ct)\, c^2$

2: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial t}(\sin(x+ct)) = \cos(x+ct) \times c$

$\frac{\partial^2 f}{\partial t^2} = \frac{\partial}{\partial t}(\cos(x+ct) \times c) = -\sin(x+ct) \times c \times c = -\sin(x+ct)\, c^2$

2: $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(\sin(x+ct)) = \cos(x+ct) \times \frac{\partial}{\partial x}(x+ct) = \cos(x+ct)$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\cos(x+ct)) = -\sin(x+ct)$

CamScanner

$w = \sin(x+ct)$

1: $\frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \sin(x+ct)$

$\frac{\partial w}{\partial t} = \cos(x+ct) \times \frac{\partial}{\partial t}(x+ct)$

$\frac{\partial w}{\partial t} = \cos(x+ct) \times c$

2: $\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) \times \frac{\partial}{\partial t}(x+ct) \times c$

$\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) \times c \times c$

$\frac{\partial^2 w}{\partial t^2} = \boxed{-\sin(x+ct)}\, c^2$

$\frac{\partial w}{\partial x} = \frac{\partial}{\partial x} \sin(x+ct)$

$\frac{\partial w}{\partial x} = \cos(x+ct) \times \frac{\partial}{\partial x}(x+ct)$

$\frac{\partial w}{\partial x} = \cos(x+ct)$

2: $\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) \times \frac{\partial}{\partial x}(x+ct)$

$\frac{\partial^2 w}{\partial x^2} = \boxed{-\sin(x+ct)}$

$\boxed{\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} c^2}$ Hence Proved

2) $W = \cos(2x+2ct)$

1: $\frac{\partial w}{\partial t} = -\sin(2x+2ct) \times \frac{\partial}{\partial t}(2x+2ct)$

$\frac{\partial w}{\partial t} = -\sin(2x+2ct) \times 2c$

2: $\frac{\partial^2 w}{\partial t^2} = -2c \frac{\partial}{\partial t} \sin(2x+2ct)$

$\frac{\partial^2 w}{\partial t^2} = -2c \cos(2x+2ct) \times \frac{\partial}{\partial t}(2x+2ct)$

$\frac{\partial^2 w}{\partial t^2} = -2c \cos(2x+2ct) \times (2c)$

$\frac{\partial^2 w}{\partial t^2} = \boxed{-\cos(2x+2ct)} \times (2)^2 \times \boxed{4}$

$\boxed{\frac{\partial^2}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}}$

1: $\frac{\partial w}{\partial x} = -\sin(2x+2ct) \times \frac{\partial}{\partial x}(2x+2ct)$

$\frac{\partial w}{\partial x} = -\sin(2x+2ct) \times 2$

2: $\frac{\partial^2 w}{\partial x^2} = -2 \frac{\partial}{\partial x} \sin(2x+2ct)$

$\frac{\partial^2 w}{\partial x^2} = -2 \cos(2x+2ct) \times \frac{\partial}{\partial x}(2x+2ct)$

$\frac{\partial^2 w}{\partial x^2} = -2 \cos(2x+2ct) \times (2)$

$\frac{\partial^2 w}{\partial x^2} = \boxed{-4 \cos(2x+2ct)}$

3) $w = \sin(x+ct) + \cos(2x+2ct)$

1: $\frac{\partial w}{\partial t} = \cos(x+ct) \times c + \sin(2x+2ct) \times 2c$

2: $\frac{\partial^2 w}{\partial t^2} = -\sin(x+ct) \times c^2 - \cos(2x+2ct) \times 4c^2$

$\frac{\partial^2 w}{\partial t^2} = \boxed{-\sin(x+ct) - 4\cos(2x+2ct)}\, c^2$

$\boxed{\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} c^2}$ Hence proved

1: $\frac{\partial w}{\partial x} = \cos(x+ct) - \sin(2x+2ct) \times 2$

2: $\frac{\partial^2 w}{\partial x^2} = -\sin(x+ct) - \cos(2x+2ct) \times 4$

$\frac{\partial^2 w}{\partial x^2} = \boxed{-\sin(x+ct) - 4\cos(2x+2ct)}$

4) $w = \ln(2x+2ct)$

1: $\frac{\partial w}{\partial t} = \frac{1}{2x+2ct} \times \frac{\partial}{\partial t}(2x+2ct)$

$\frac{\partial w}{\partial t} = \frac{c}{x+ct}$

2: $\frac{\partial w^2}{\partial t^2} = c\left[\frac{(x+ct)(0) - (1)(c)}{(x+ct)^2}\right]$

$\frac{\partial w^2}{\partial t^2} = \frac{-c^2}{(x+ct)^2} = c^2\left[\frac{-1}{(x+ct)^2}\right]$

$\boxed{\frac{\partial w^2}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2}}$

1: $\frac{\partial w}{\partial x} = \frac{1}{2x+2ct} \times \frac{\partial}{\partial x}(2x+2ct)$

$\frac{\partial w}{\partial x} = \frac{1}{x+ct}$

2: $\frac{\partial^2 w}{\partial x^2} = \boxed{\frac{-1}{(x+ct)^2}}$

5) $w = \tan(2x-2ct)$

1: $\frac{\partial w}{\partial t} = \sec^2(2x-2ct) \times \frac{\partial}{\partial t}(2x-2ct)$

$\frac{\partial w}{\partial t} = \sec^2(2x-2ct) \times -2c$

2: $\frac{\partial^2 w}{\partial t^2} = 2\sec(2x-2ct) \times \sec(2x-2ct) \times \tan(2x-2ct) \times -2c \times -2c$

$\frac{\partial^2 w}{\partial t^2} = \boxed{2\sec(2x-2ct) \times \sec(2x-2ct) \times \tan(2x-2ct) \times 4}\, c^2$

$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} c^2$

1: $\frac{\partial w}{\partial x} = \sec^2(2x-2ct) \times \frac{\partial}{\partial x}(2x-2ct)$

$\frac{\partial w}{\partial x} = \sec^2(2x-2ct) \times 2$

2: $\frac{\partial^2 w}{\partial x^2} = 2\sec(2x-2ct) \times \sec(2x-2ct) \times \tan(2x-2ct) \times 2 \times 2$

$\frac{\partial^2 w}{\partial x^2} = \boxed{2\sec(2x-2ct) \times \sec(2x-2ct) \times \tan(2x-2ct) \times 4}$

The Jacobian in Multivariate Calculus is a matrix that captures the local rates of change of a multivariable function. It provides insights into how the function transforms and different variables, enabling analysis of optimization, transformations, and differential equations in higher dimensions.

If u and v are functions of two independent variable x and y then the determinant:

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v with respect to x, and y.

Denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$

Similarly Jacobian of u, v, and w w.r.t x, y, and z is

$$J\left(\frac{u,v,w}{x,y,z}\right) \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix}$$

- Resiprocal property of Jacobian

If $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$, then $\boxed{J \cdot J' = 1}$

- when the variable groups are in chain form:

If u, v are function of r, s and r, s are functions of x, y then:

$$u, v \rightarrow r, s \rightarrow x, y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

- If u, v are functionally related to x, y then

$\boxed{\frac{\partial(u,v)}{\partial(x,y)} = 0}$, or in 3-D $\boxed{\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0}$

1) If $x^m y^n = u$ and $x^r y^s = v$ then find $\frac{\partial(x,y)}{\partial(u,v)}$

Solution

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}[x^m y^n] = m x^{m-1} y^n \; ; \; \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}[x^m y^n] = n x^m y^{n-1}$$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}[x^r y^s] = r x^{r-1} y^s \; ; \; \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}[x^r y^s] = s x^r y^{s-1}$$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} m x^{m-1} y^n & n x^m y^{n-1} \\ r x^{r-1} y^s & s x^r y^{s-1} \end{vmatrix}$$

$$= m x^{m-1} y^n \cdot s x^r y^{s-1} - r x^{r-1} y^s \cdot n x^m y^{n-1}$$

$$= ms\, x^{r+m-1} y^{n+s-1} - nr\, x^{m+r-1} y^{s+n-1}$$

$$= (x^{r+m-1} y^{n+s-1})(ms - nr)$$

Replace $x^m y^n$ with u and $x^r y^s$ with v

$$= (x^r \cdot x^m \cdot x^{-1} \cdot y^n \cdot y^s \cdot y^{-1})(ms - nr)$$

$$= (x^r \cdot y^s \cdot x^m \cdot y^n \cdot x^{-1} \cdot y^{-1})(ms - nr)$$

$$\boxed{J\left(\frac{u,v}{x,y}\right) = \frac{(v \cdot u)(ms - nr)}{xy}}$$

Now Solve for $J\left(\frac{x,y}{u,v}\right)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\frac{(uv)(ms-nr)}{xy}} = \boxed{\frac{xy}{uv(ms-nr)}}$$

2) If $u = x^2 - y^2$, $v = 2xy$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Solution

$$\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$$

$$= (2x)(2x) - (-2y)(2y) = 4x^2 + 4y^2$$

$$J\left(\frac{u,v}{x,y}\right) = 4(x^2 + y^2)$$

3) If $u = x \sin y$, $v = y \sin x$, find $\frac{\partial(u,v)}{\partial(x,y)}$

Solution

$$\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} \sin y & x\cos y \\ y\cos x & \sin x \end{vmatrix}$$

$$J = \left(\frac{u,v}{x,y}\right) = \sin y \sin x - xy \cos x \cos y$$

4) If $u = x + \frac{y^2}{x}$, $v = \frac{y^2}{x}$ find $\frac{\partial(u,v)}{\partial(x,y)}$

Solution

$$\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 + (-y^2/x^2) & 2y/x \\ -(y/x)^2 & 2y/x \end{vmatrix}$$

$$J\left(\frac{u,v}{x,y}\right) = (1 - (y/x)^2)(2y/x) + (y/x)^2 (2y/x)$$

$$= \left(1 - \frac{y^2}{x^2}\right)\left(\frac{2y}{x}\right) + \left(\frac{y^2}{x^2}\right)\left(\frac{2y}{x}\right)$$

$$= \frac{(x^2 - y^2)\,2y}{x^3} + \frac{2y^3}{x^3} = \frac{2x^2y - \cancel{2y^3} + \cancel{2y^3}}{x^3} = \boxed{\frac{2y}{x}}$$

5) If $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ then find $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Solution

$$\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix}$$

1: $$\frac{\partial u}{\partial x} = \left[\frac{(1-xy) - (x+y)(-y)}{(1-xy)^2}\right] = \frac{1-xy+xy+y^2}{(1-xy)^2} = \boxed{\frac{1+y^2}{(1-xy)^2}}$$

2: $$\frac{\partial u}{\partial y} = \left[\frac{(1-xy) - (x+y)(-x)}{(1-xy)^2}\right] = \frac{1-xy+x^2+xy}{(1-xy)^2} = \boxed{\frac{1+x^2}{(1-xy)^2}}$$

3: $$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} + \frac{0}{1+y^2} = \boxed{\frac{1}{1+x^2}}$$

4: $$\frac{\partial v}{\partial y} = \frac{1}{1+x^2}(0) + \frac{1}{1+y^2} = \boxed{\frac{1}{1+y^2}}$$

$$J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$= \left(\frac{\cancel{1+y^2}}{(1-xy)^2}\right)\left(\frac{1}{\cancel{1+y^2}}\right) - \left(\frac{\cancel{1+x^2}}{(1-xy)^2}\right)\left(\frac{1}{\cancel{1+x^2}}\right)$$

$$J\left(\frac{u,v}{x,y}\right) = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = \boxed{0}$$

6) If $u = x + 3y^2 - z^3$, $v = 4x^2yz$ and $w = 2z^2 - xy$ then evalute $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

Solution

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y & \partial u/\partial z \\ \partial v/\partial x & \partial v/\partial y & \partial v/\partial z \\ \partial w/\partial x & \partial w/\partial y & \partial w/\partial z \end{vmatrix} = \begin{vmatrix} 1 & 6y & -3z^2 \\ 8xyz & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} 4x^2z & 4x^2y \\ -x & 4z \end{vmatrix} - 6y \begin{vmatrix} 8xyz & 4x^2y \\ -y & 4z \end{vmatrix} - 3z^2 \begin{vmatrix} 8yxz & 4x^2z \\ -y & -x \end{vmatrix}$$

$x = 1,\ y = -1,\ z = 0$

$$= \begin{vmatrix} 0 & -4 \\ -1 & 0 \end{vmatrix} + 6 \begin{vmatrix} 0 & -4 \\ 1 & 0 \end{vmatrix} + 0$$

$$J\left(\frac{u,v,w}{x,y,z}\right) = -4 + 6(4) = 20$$

Parametric Equations

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

We will be concerned with curves that are given by parametric equations

$$x = f(t) \quad , \quad y = g(t)$$

in which $f(t)$ and $g(t)$ have continuous first derivatives with respect to t. It can be proved that if $dx/dt \neq 0$, then y is a differentiable function of x, in which case the chain rule implies that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Although it is true that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

you cannot conclude that d^2y/dx^2 is the quotient of d^2y/dt^2 and d^2x/dt^2.

Example: Without eliminating the parameter, find dy/dx and d^2y/dx^2 at $(1, 1)$ and $(1, -1)$ on the semicubical parabola given by the parametric equations $x = t^2$, $y = t^3$;

Solution

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \boxed{\frac{3t^2}{2t}} \quad (t \neq 0) \; ; \quad y' = \frac{dy}{dx} \; , \; y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx}$$

$$\Rightarrow \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt} = \frac{d/dt\left(\frac{3t^2}{2t}\right)}{d/dt\,(t^2)} = \frac{d/dt\left(\frac{3t}{2}\right)}{2t} = \left(\frac{3}{4t}\right)$$

Since the point $(1, 1)$ on the curve corresponds to $t = 1$ in parametric equations, it follows from $\left.\frac{dy}{dx}\right|_{t=1}$ and $\left.\frac{d^2y}{dx^2}\right|_{t=1}$

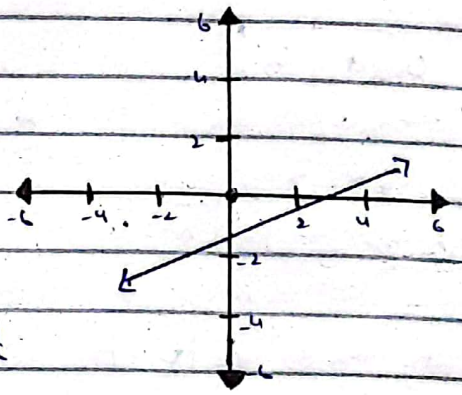
$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{3}{2} \quad \text{and} \quad \left.\frac{d^2y}{dx^2}\right|_{t=} = \frac{3}{4}$$

Make a table of values and sketch the curve, indicating the direction of your graph. Then eliminate the parameter.

1) $x = 2t + 1$ and $y = t - 1$

Solution

t	-2	-1	0	1	2
x	-3	-1	1	3	5
y	-3	-2	-1	0	1



To eliminate the parameter, solve for t

$$\frac{x-1}{2} = t \quad , \quad y + 1 = t$$

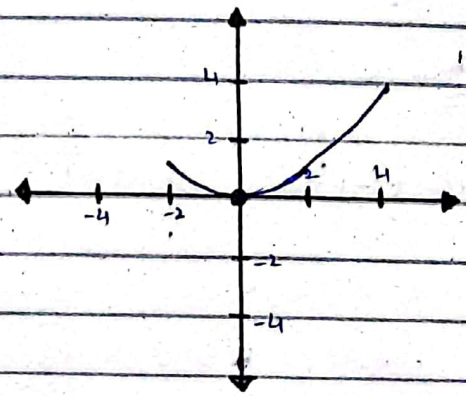
$$\frac{x-1}{2} = y + 1 \Rightarrow x - 1 = 2y + 2$$

$$\boxed{x + 3 = 2y}$$

2) $x = 2t$ and $y = t^2$, $-1 \le t \le 2$

Solution

t	-1	0	1	2
x	-2	0	2	4
y	1	0	1	4



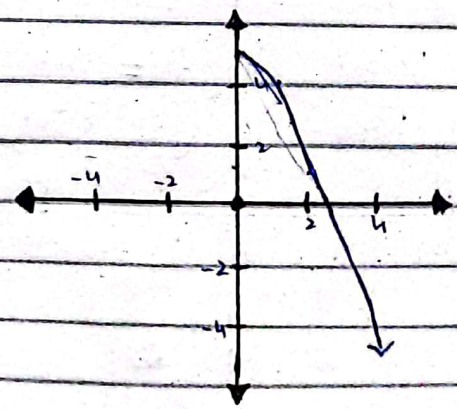
$$\frac{x}{2} = t, \quad y = t^2 \Rightarrow y = \frac{x^2}{4}$$

3) $x = \sqrt{t+2}$ and $y = 3 - t$

Solution

$x = \sqrt{t+2} \Rightarrow t \geq -2$

x	0	1	1.41	1.73	2
y	5	4	3	2	1
t	-2	-1	0	1	2



$$3 - y = t \Rightarrow x = \sqrt{3 - y + 2}$$

$$x = \sqrt{5 - y}$$

$$x^2 = 5 - y$$

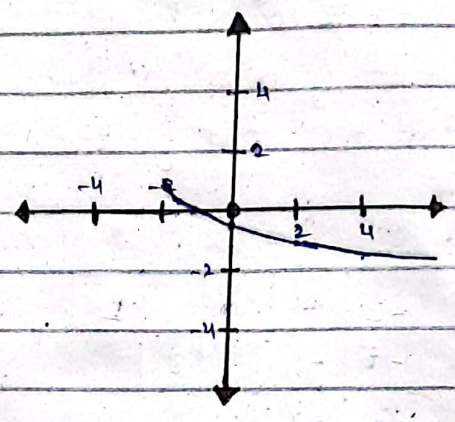
$$y = 5 - x^2$$

4) $x = t-2$ and $y = 1-\sqrt{t}$

Solution

$\boxed{t \geq 0}$

t	0	1	2
x	-2	-1	0
y	1	0	-0.41



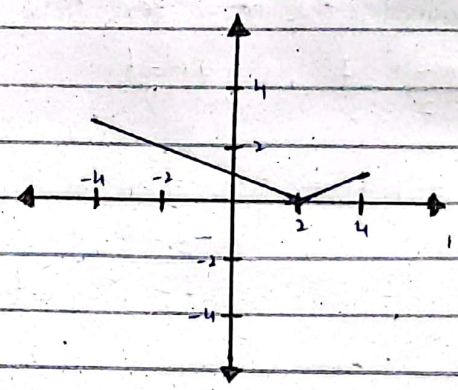
$t = x+2 \Rightarrow y = 1-\sqrt{x+2}$

$\boxed{x \geq -2}$

5) $x = 2t$ and $y = |t-1|$

Solution

t	-2	-1	0	1	2
x	-4	-2	0	2	4
y	3	2	1	0	1



$y = \left|\frac{x}{2} - 1\right|$

6) $x = 2\cos t - 1$ and $y = 3\sin t + 1$

Solution

To eliminate the parameter, solve for $\cos t$ in x's equation and $\sin t$ in y's equation.

Substitute into trigonometric identity

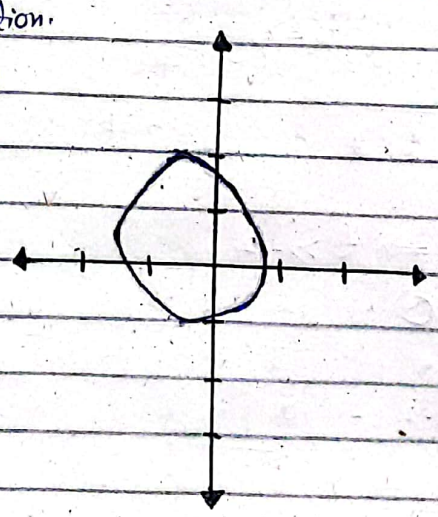
$\cos^2 t + \sin^2 t = 1$

$x = 2\cos t - 1 \qquad y = 3\sin t + 1$

$x + 1 = 2\cos t \qquad y - 1 = 3\sin t$

$\left(\frac{x+1}{2}\right) = \cos t \qquad \left(\frac{y-1}{3}\right) = \sin t$

$\cos^2 t + \sin^2 t = \left(\frac{x+1}{2}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$



t	0	$\pi/2$	π	$3\pi/2$	2π
x	1	-1	-3	-1	1
y	1	4	1	-2	1

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

1) $x = t^2,\ y = t^2 + 6t + 5$

Solution

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+6}{2t} = \boxed{\frac{t+3}{t}}$; $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(y') = \frac{dy'}{dx} = \frac{dy'/dt}{dx/dt}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{d/dt\left[(t+3)/t\right]}{d/dt\left[t^2\right]} = \left[\frac{(t)(1) - (t+3)(1)}{t^2}\right] \div 2t$

$= \frac{t - t - 3}{(t^2)\,2t} = \boxed{\frac{-3}{2t^3}}$

2) $x = t^2 + 1,\ y = 2t^3 - t^2$

Solution

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t^2 - 2t}{2t} = \frac{3t^2 - t}{t} = \boxed{3t - 1} \Rightarrow y'$

$\Rightarrow \frac{dy^2}{dx} = \frac{dy'/dt}{dx/dt} = \frac{d/dt\left[3t-1\right]}{d/dt\left[t^2+1\right]} = \boxed{\frac{3}{2t}}$

3) $x = \sqrt{t},\ y = 3t^2 + 2t$

Solution

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{6t+2}{1/2\sqrt{t}} = 2\sqrt{t}(6t+2) = \boxed{12t^{3/2} + 4\sqrt{t}}$

$\Rightarrow \frac{dy'/dt}{dx/dt} = \frac{d/dt\left[12t^{3/2} + 4\sqrt{t}\right]}{1/2\sqrt{t}} = \left[18t^{1/2} + 2/\sqrt{t}\right] 2\sqrt{t} = \boxed{36t + 4}$

4) $x = \ln t,\ y = t^2 + t$

Solution

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t+1}{1/t} = \boxed{2t^2 + t}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{4t+1}{1/t} = \boxed{4t^2 + t}$

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5) A curve C is defined by the parametric equations $x = t^2 + t - 1$, $y = t^3 - t^2$.

(a) Find $\frac{dy}{dx}$ in terms of t.

(b) Find an equation of tangent line to C at point where $t = 2$.

Solution

$\Rightarrow \frac{dy}{dx} = \frac{3t^2 + 2t}{2t + 1}$, $x = t^2 + t - 1$ $\quad y = t^3 - t^2$ $\quad (x, y)$

$x = 4 + 2 - 1 = 5$ $\quad y = 8 - 4 = 4$ $\quad \hookrightarrow (5, 4)$

$m = \frac{3(2)^2 + 2(2)}{2(2) + 1} = \frac{12 + 4}{5} = \boxed{\frac{8}{5}}$

$y - y_1 = m(x - x_1)$, $m = 8/5$, $(x_1, y_1) = (5, 4)$

$y - 4 = \frac{8}{5}(x - 5)$

Euler's Theorem

- **Homogenous Function**

A function $f(x,y)$ is said to be homogeneous function in which the power of each term is the same.

A function $f(x,y)$ is a homogeneous function or order n, if the degree of each of its terms in x and y is equal to n.

Thus $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n$ is a homogeneous function of order n.

Homogeneous Function $\rightarrow u = x^n f(y/x)$

To ~~apply~~ apply the Euler's theorem we must have the homogenous function, in case of not having the homogeneous function we can tranform a function into the homogeneous form.

- **Euler's First order Equation & Second order Equation**

Second $\Rightarrow$ $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = n(n-1)u$

First order $\Rightarrow$ $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$

(i) The function $x^3\left[1+\dfrac{y}{x}+3\left(\dfrac{y}{x}\right)^2+5\left(\dfrac{y}{x}\right)^3\right]$ is a homogeneous function of order 3.

(ii) $\dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2} = \dfrac{\sqrt{x}\left[1+\sqrt{\dfrac{y}{x}}\right]}{x^2\left[1+\left(\dfrac{y}{x}\right)^2\right]} = x^{-3/2}\dfrac{1+\sqrt{\dfrac{y}{x}}}{1+\left(\dfrac{y}{x}\right)^2}$ is a homogeneous function of order $-3/2$.

(iii) $\sin^{-1}\dfrac{\sqrt{x}+\sqrt{y}}{x^2+y^2}$ is not a homogeneous function as it cannot be written in the form of $x^n f\left(\dfrac{y}{x}\right)$ so that its degree may be pronounced.

It is a function of homogeneous expression.

Euler's Theorem on Homogeneous Function

Statement. If z is a homogeneous function of x, y of order n, then $x \cdot \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$

Note. If u is a homogeneous function of x, y, z of degree n, then $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}$

Proof. Since z is a homogeneous function of x, y of degree n, we have, by Euler's theorem,

$$x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = nz$$

Now $z = f(u)$, given

$$\therefore \quad \frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y}$$

Substituting in (1), we get

$$x f'(u) \frac{\partial u}{\partial x} + y f'(u) \frac{\partial u}{\partial y} = n f(u)$$

$$\Rightarrow \quad \boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n \frac{f(u)}{f'(u)}}$$

Euler's Deduction Formula I

See Example (4), Ex-2

Note. If $v = f(u)$ where v is a homogeneous function in x, y, z of degree n; then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = \frac{n f(u)}{f'(u)}$$

Example ① Verify Euler's Theorem for $z = \dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$

Solution Here, we have

$$z = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} = \frac{x^{1/3}\left[1 + \left(\frac{y}{x}\right)^{1/3}\right]}{x^{1/2}\left[1 + \left(\frac{y}{x}\right)^{1/2}\right]} = x^{-1/6}\,\phi\left(\frac{y}{x}\right)$$

Thus z is a homogeneous function of degree $-1/6$

By Euler's Theorem $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = -\dfrac{1}{6}z$

Example ② If $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

Solution. Here, we have, $u = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$

u is not a homogeneous function but if $z = \cos u$, then

$$u = \cos^{-1} z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x\left(1+\frac{y}{x}\right)}{\sqrt{x}\left(1+\sqrt{y/x}\right)} = x^{1/2}\frac{(1+y/x)}{(1+\sqrt{y/x})} = x^{1/2}\phi\left(\frac{y}{x}\right)$$

z is a homogeneous function in x, y of degree $1/2$.

By Euler's theorem, we have $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = \dfrac{1}{2}z$

$$x\frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} = \frac{1}{2}z$$

$$x\frac{\partial u}{\partial x}(-\sin u) + y\frac{\partial u}{\partial y}(-\sin u) = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u \Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$ Proved.

Example ③ If $u = \sin^{-1}\left[\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$

Solution we have, $u = \sin^{-1}\left[\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$

Here, u is not a homogeneous function but if $v = \sin u = \dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}$ then v is a homogeneous function in x, y, z of degree -3

By Euler's theorem $x\dfrac{\partial v}{\partial x} + y\dfrac{\partial v}{\partial y} + z\dfrac{\partial v}{\partial z} = nv$

$$x\frac{\partial v}{\partial u}\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial u}\frac{\partial u}{\partial y} + z\frac{\partial v}{\partial u}\frac{\partial u}{\partial z} = -3v$$

Putting the value of $\dfrac{\partial v}{\partial u}$ in (1), we get

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} + z\cos u\frac{\partial u}{\partial z} = -3\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\frac{\sin u}{\cos u} = -3\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + 3\tan u = 0$$

Example (4) If $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3$

Solution. we have, $u = \log_e\left(\frac{x^4+y^4}{x+y}\right)$

Here, u is not a homogeneous function but if

$$z = e^u = \frac{x^4+y^4}{x+y} = \frac{x^4\left[1+(y/x)^4\right]}{x\left[(1+(y/x))\right]} = x^3\,\phi\left(\frac{y}{x}\right)$$

Then z is a homogeneous function of degree 3.

By Euler's Deduction formula I

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = 3\frac{e^u}{e^u} = 3$$

I: Proof of Euler's Deduction Formula

If z is a homogeneous function x, y of degree n and $z = f(u)$, then

(1) $\rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nz$, Now $z = f(u)$, given

$$\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

substituting in (1), we get

$$x f'(u)\frac{\partial u}{\partial x} + y f'(u)\frac{\partial u}{\partial y} = n f(u)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

Note. If $v = f(u)$ where v is a homogeneous function in x, y, z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{n f(u)}{f'(u)}$$

Exercise

1) If $v = \frac{x^3y^3}{x^3+y^3}$, show that $x \cdot \frac{\partial v}{\partial x} + y \cdot \frac{\partial v}{\partial y} = 3v$.

Solution

1: $\frac{\partial v}{\partial x} = \left[\frac{(x^3+y^3)(3x^2y^3) - (x^3y^3)(3x^2)}{(x^3+y^3)^2}\right] = \frac{3x^5y^3 + 3x^2y^6 - 3x^5y^3}{(x^3+y^3)^2} = \frac{3x^2y^6}{(x^3+y^3)^6}$

2: $\frac{\partial v}{\partial y} = \left[\frac{(x^3+y^3)(3y^2x^3) - (x^3y^3)(3y^2)}{(x^3+y^3)^2}\right] = \frac{3x^6y^2 + 3x^3y^5 - 3x^3y^5}{(x^3+y^3)^2} = \frac{3x^6y^2}{(x^3+y^3)^2}$

$\Rightarrow x \cdot \frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = x\frac{(3x^2y^6)}{(x^3+y^3)^2} + y\frac{(3x^6y^2)}{(x^3+y^3)^2} = 3v$

$= \frac{3x^3y^6}{(x^3+y^3)^2} + \frac{3x^6y^3}{(x^3+y^3)^2} = \frac{3x^3y^6 + 3x^6y^3}{(x^3+y^3)^2} = \frac{3x^3y^3(y^3+x^3)}{(x^3+y^3)^2} = \frac{3x^3y^3}{(x^3+y^3)}$

$= \frac{3x^3y^3}{(x^3+y^3)} = \boxed{3v}$ prooved.

2) If $u = \log \frac{x^3+y^3}{x^2+y^2}$, prove that $x \cdot \frac{\partial u}{\partial x} + y \cdot \frac{\partial u}{\partial y} = 1$

Solution

1: Find the value of n, but before make the function Homogeneous.

$u = \log \frac{x^3+y^3}{x^2+y^2}$

let $z = u = \log \frac{x^3+y^3}{x^2+y^2}$

$z = e^u = \frac{x^3+y^3}{x^2+y^2}$

$z = e^u = \frac{x^3(1+(y/x)^3)}{x^2(1+(y/x)^2)}$

$z = e^u = x^1\left[\frac{1+(y/x)^3}{1+(y/x)^2}\right]$

$\boxed{n = 1}$

2: Now proove that $x \cdot \frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$ by using Euler's reduction Formula:

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$

$\boxed{f(u) = e^u}$, $f'(u) = e^u$ and $\boxed{n = 1}$

$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = (1)\frac{(e^u)}{(e^u)} = \boxed{1}$

Hence Prooved

Chain Rule

The chain Rule for functions of a single variable Says that when $w = f(x)$ is a differentiable function of x and $x = g(t)$ is a differentiable function of t, w is a differentiable function of t and dw/dt can be calculated by the formula

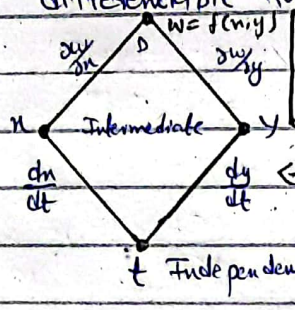
$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}$$

For functions of several variables the chain rule has more than one form, which depends on how many independent and intermediate variables are involved.

- chain rule for functions of one independent variable and two intermediate variables

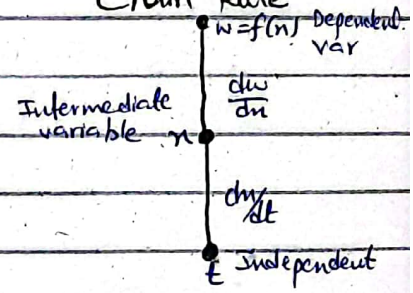
If $w = f(x,y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\boxed{\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$



← Branch Diagram

Chain Rule

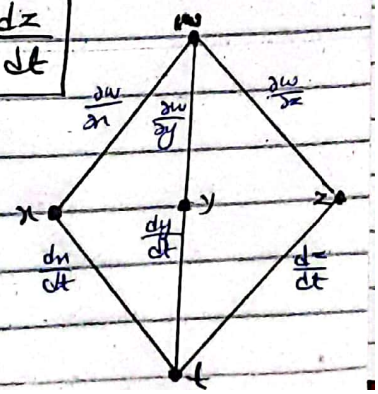


- Function of Three variables

Chain Rule for functions of one independent and Three intermediate variables.

If $w = f(x,y,z)$ is differentiable and x, y, z are differentiable functions of t, then w is a differentiable function of t and

$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}}$$



- Chain Rule for two independent variables and three Intermediate variables

Suppose that $w = f(x,y,z)$, $x = g(r,s)$, $y = h(r,s)$ and $z = k(r,s)$

If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas.

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

- A Formula for implicit Differentiation

Suppose that $F(x,y)$ is differentiable and that the equation $F(x,y)=0$ defines y as differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Exercises:

Chain Rule: One Independent Variable

Express dw/dt as a function of t,

1) $w = x^2 + y^2$, $x = \cos t$, $y = \sin t$; $t = \pi$

Solution $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$$\frac{dw}{dt} = \frac{\partial}{\partial x}(x^2+y^2)\cdot\frac{d}{dt}(\cos t) + \frac{\partial}{\partial y}(x^2+y^2)\cdot\frac{d}{dt}(\sin t)$$

$$\frac{dw}{dt} = (2x)(-\sin t) + (2y)(\cos t) = -2\cos t \sin t + 2\sin t \cos t, \; t = \pi$$

$$\frac{dw}{dt} = -2\cos(\pi)\cdot\sin(\pi) + 2\sin(\pi)\cos(\pi) = -2(-1)(0) + 2(0)(-1) = \boxed{0}$$

2) $w = x^2 + y^2$, $x = \cos t + \sin t$, $y = \cos t - \sin t$; $t = 0$

Solution $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$

$$\frac{dw}{dt} = \frac{d}{dt}(\cos t + \sin t)\cdot\frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(x^2+y^2)\cdot\frac{d}{dt}(\cos t - \sin t)$$

$$\frac{dw}{dt} = (2x)(-\sin t + \cos t) + (2y)(-\sin t - \cos t) =$$

$$\frac{dw}{dt} = 2(\cos t + \sin t)(\cos t - \sin t) + 2(\cos t - \sin t)(\cos t + \sin t)(-1)$$

$$\frac{dw}{dt} = 2(\cos^2 t - \sin^2 t) - 2(\cos^2 t - \sin^2 t)$$

$$\frac{dw}{dt} = 2\cos^2 t - 2\cos^2 t - 2\sin^2 t + 2\sin^2 t = \boxed{0}$$

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Chain Rule: Two and Three Independent Variables

Express $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ as functions of u and v.

3) $w = xy + yz + xz$, $x = u+v$, $y = u-v$, $z = uv$; $(u,v) = (1/2, 1)$

Solution

$$\Rightarrow \frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{dx}{du} + \frac{\partial w}{\partial y}\frac{dy}{du} + \frac{\partial w}{\partial z}\frac{dz}{du}$$

$$\frac{\partial w}{\partial u} = \frac{\partial}{\partial x}(xy+yz+xz)\cdot\frac{\partial}{\partial u}(u+v) + \frac{\partial}{\partial y}(xy+yz+xz)\cdot\frac{\partial}{\partial u}(u-v) + \frac{\partial}{\partial z}(xy+yz+xz)\cdot\frac{\partial}{\partial u}(uv)$$

$$\frac{\partial w}{\partial u} = (y+z)\cdot(1) + (x+z)(1) + (y+x)\frac{\partial}{\partial u}(uv)$$

$$\frac{\partial w}{\partial u} = \left((u-v)+uv\right) + (u+v) + uv + (u-v) + (u+v)\left(vu' + uv'\frac{dv}{dt}\right)$$

$$\frac{\partial w}{\partial u} = u - \cancel{v} + uv + u + \cancel{v} + uv + u - \cancel{v} + u + \cancel{v} = 4u + 2uv$$

$$\frac{\partial w}{\partial u} = (u - \cancel{v}) + uv + (u + \cancel{v}) + uv + \left[(u-v) + (u+v)\right](v)$$

$$\frac{\partial w}{\partial u} = 2uv + 2u + \left[(u - \cancel{v}) + (u + \cancel{v})\right](v) = 2uv + 2u + 2uv = 4uv + 2u$$

$$\frac{\partial w}{\partial u} = 4uv + 2uv = 4(1/2)(1) + 2(1/2) = 2 + 1 = \boxed{3}$$

$$\Rightarrow \frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$

$$\frac{\partial w}{\partial v} = \frac{\partial}{\partial x}(xy+yz+xz)\cdot\frac{\partial}{\partial v}(u+v) + \frac{\partial}{\partial y}(xy+yz+xz)\frac{\partial}{\partial v}(u-v) + \frac{\partial}{\partial z}(xy+yz+xz)\cdot\frac{\partial}{\partial v}(uv)$$

$$\frac{\partial w}{\partial v} = (y+z)(1) + (x+z)(-1) + (y+x)(u)$$

$$\frac{\partial w}{\partial v} = \left[(u-v) + uv\right] + \left[(u+v) + uv\right](-1) + \left[(u-v) + (u+v)\right](u)$$

$$= \cancel{u}v - uv + (\cancel{u} - v) - \cancel{u} - v + (2u)u = 2u^2 - 2v = 2(1/2)^2 - 2(1) = \boxed{-3/2}$$

Directional Derivative and Gradient vectors

- Calculation and Gradients

The gradient vector (gradient) of $f(x,y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del."

Algebra Rules for Gradients

1) Sum Rule: $\nabla(f+g) = \nabla f + \nabla g$
2) Difference Rule: $\nabla(f-g) = \nabla f - \nabla g$
3) Constant Multiple Rule: $\nabla(kf) = k\nabla f$
4) Product Rule: $\nabla(fg) = f\nabla g + g\nabla f$
5) Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

Exercise

Calculating Gradients

• Find the gradient of the function at the given point.

1) $f(x,y) = y - x, \ (2,1)$

Solution

$$\nabla f = \frac{\partial}{\partial x} i (y-x) + \frac{\partial}{\partial y} j (y-x) = -i + j; \ f(2,1) = -1$$

2) $f(x,y) = \ln(x^2+y^2), \ (1,1)$

Solution

$$\nabla f = \frac{\partial}{\partial x} i (\ln(x^2+y^2)) + \frac{\partial}{\partial y} j (\ln(x^2+y^2))$$

$$\nabla f = i\left(\frac{1}{x^2+y^2}\right)\cdot(2x) + j\left(\frac{1}{x^2+y^2}\right)\cdot(2y) = \frac{2xi}{x^2+y^2} + \frac{2yj}{x^2+y^2} = i + j$$

3) $g(x,y) = xy^2, \ (2,-1)$

Solution

$$\nabla g = \frac{\partial}{\partial x} i (xy^2) + \frac{\partial}{\partial y} j (xy^2) = iy^2 + 2xyj = i - 4j$$

4) $g(x,y) = \frac{x^2}{2} - \frac{y^2}{2}, \ (\sqrt{2}, 1)$

Solution

$$\nabla g = \frac{\partial}{\partial x} i \left[\frac{x^2}{2} - \frac{y^2}{2}\right] + \frac{\partial}{\partial y} j \left[\frac{x^2}{2} - \frac{y^2}{2}\right] = xi + (-y)j = \boxed{\sqrt{2}i - j}$$

5) $f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x, \ (1,1,1)$

Solution

$$\nabla f = \frac{\partial}{\partial x} i (x^2+y^2+2z^2+z\ln x) + \frac{\partial}{\partial y} j (x^2+y^2-2z^2+z\ln x) + \frac{\partial}{\partial z} k (x^2+y^2-2z^2+z\ln x)$$

$$\nabla f = i\left(2x + \frac{z}{x}\right) + (2y)j + k(-4z + \ln x)$$

$$\nabla f = i(2+1) + 2j + k(-4+0) = 3i + 2j - 4k$$

Finding Directional Derivative

6) $f(x,y) = 2xy - 3y^2$, $P_0(5,5)$, $u = 4i + 3j$

Solution

$\nabla f = \frac{\partial}{\partial x} i (2xy - 3y^2) + \frac{\partial}{\partial y} j (2xy - 3y^2) = 2y i + j(2x - 6y)$

$\nabla f(5,5) = 10i + j(-20) = 10i - 20j$

$\nabla f(5,5) = (10i - 20j) \frac{(4i + 3j)}{\sqrt{16+9}} = \frac{(40)}{5} - \frac{60}{5} = 8 - 12 = \boxed{-4}$

7) $f(x,y) = 2x^2 + y^2$, $P_0(-1,1)$, $u = 3i - 4j$

Solution

$\nabla f = \frac{\partial}{\partial x} i (2x^2 + y^2) + \frac{\partial}{\partial y} j (2x^2 + y^2) = 4x i + 2y j = -4i + 2j$

$\nabla f = (-4i + 2j) \frac{(3i - 4j)}{\sqrt{9+16}} = \frac{-12 - 8}{5} = -4$

8) $f(x,y,z) = xy + yz + zx$, $P_0(1,-1,2)$, $u = 3i + 6j - 2k$

Solution

$\nabla f = \frac{\partial}{\partial x} i (xy + yz + zx) + \frac{\partial}{\partial y} j (xy + yz + zx) + \frac{\partial}{\partial z} k (xy + yz + zx)$

$\nabla f = i(y + z) + j(x + z) + k(y + x)$

$\nabla f(1,-1,2) = i(1) + j + k(0) = i + j$

$\nabla f \cdot u = (i + j) \frac{(3i + 6j - 2k)}{\sqrt{9 + 36 + 4}} = \frac{3 + 6}{7} = \boxed{\frac{9}{3}}$

◆ Show that $\frac{\partial x}{\partial r} = \frac{\partial r}{\partial x}$; $x = r\cos\theta$, $y = r\sin\theta$

Solution

Let $x^2 = r^2\cos^2\theta$ and $y^2 = r^2\sin^2\theta$

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$x^2 + y^2 = r^2(1)$

$x^2 + y^2 = r^2$

Find $\frac{\partial x}{\partial r}$ where $x = r\cos\theta$

$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(r\cos\theta) = \cos\theta$

$\boxed{\frac{\partial x}{\partial r} = \cos\theta}$

$r^2 = x^2 + y^2$, Find $\frac{\partial r}{\partial x}$

$\frac{\partial}{\partial x} r^2 = \frac{\partial}{\partial x}(x^2 + y^2)$

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r} = \frac{(r\cos\theta)}{r} = \boxed{\cos\theta}$

$\boxed{\frac{\partial r}{\partial x} = \cos\theta}$

Hence Prooved

◆ Show that $\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = 1$

Solution

$r^2 = x^2 + y^2$

$\frac{\partial}{\partial x}(r^2) = \frac{\partial}{\partial x}(x^2 + y^2)$

$\frac{\partial r}{\partial x} = \cos\theta$

$\frac{\partial}{\partial y}(r^2) = \frac{\partial}{\partial y}(x^2 + y^2)$

$\frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$

$\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2 = \cos^2\theta + \sin^2\theta = \boxed{1}$ Hence prooved

- Show that $\nabla^2(r^n) = n(n+1)r^{n-2}$.

$$\nabla = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) ; \quad \nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

$r^2 = x^2 + y^2 \rightarrow$ circle $\qquad r^2 = x^2 + y^2 + z^2 \rightarrow$ Sphere

$$\nabla^2(r^n) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} r^n\right] + \frac{\partial}{\partial y}\left[\frac{\partial}{\partial y} r^n\right] + \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z} r^n\right]$$

$$1: \frac{\partial}{\partial x}\left[n r^{n-1}\frac{\partial r}{\partial x}\right] + \frac{\partial}{\partial y}\left[n r^{n-1}\frac{\partial r}{\partial y}\right] + \frac{\partial}{\partial z}\left[n r^{n-1}\frac{\partial r}{\partial z}\right]$$

$\Rightarrow r^2 = x^2 + y^2 + z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial}{\partial x}\left[n r^{n-1}\frac{x}{r}\right] + \frac{\partial}{\partial y}\left[n r^{n-1}\frac{y}{r}\right] + \frac{\partial}{\partial z}\left[n r^{n-1}\frac{z}{r}\right]$$

$$\frac{\partial}{\partial x}\left[n r^{n-2} x\right] + \frac{\partial}{\partial y}\left[n r^{n-2} y\right] + \frac{\partial}{\partial z}\left[n r^{n-2} z\right]$$

- Double Derivative: x, y, z are Intermediate variable of function r, so we will apply product Rule.

$$n\left[\left(x(n-2)r^{n-3}\frac{\partial r}{\partial x} + r^{n-2}\right) + \left(y(n-2)r^{n-3}\frac{\partial r}{\partial y} + r^{n-2}\right) + \left(z(n-2)r^{n-3}\frac{\partial r}{\partial z} + r^{n-2}\right)\right]$$

$$n\left[x(n-2)r^{n-3}\frac{x}{r} + r^{n-2} + y(n-2)r^{n-3}\frac{y}{r} + r^{n-2} + z(n-2)r^{n-3}\frac{z}{r} + r^{n-2}\right]$$

$$n\left[3r^{n-2} + x^2(n-2)r^{n-4} + y^2(n-2)r^{n-4} + z^2(n-2)r^{n-4}\right]$$

$$n\left[3r^{n-2} + (n-2)r^{n-4}(x^2+y^2+z^2)\right] = 3nr^{n-2} + (n-2)r^{n-4}(x^2+y^2+z^2)$$

$r^2 = x^2 + y^2 + z^2$

$$3nr^{n-2} + n(n-2)r^{n-4} \cdot r^2 = 3nr^{n-2} + n(n-2)r^{n-2}$$

$$r^{n-2}(3n + n(n-2)) = r^{n-2}(3n + n^2 - 2n) = r^{n-2}(n^2+n)$$

$$r^{n-2}(n^2+n) = \boxed{n(n+1)r^{n-2}}$$

◆ Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$

Solution $\nabla^2 f(r) = \nabla \cdot (\nabla f(r))$

1: $\nabla f(r) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) f(r)$

$$\frac{\partial}{\partial x} i\, f(r) + \frac{\partial}{\partial y} j\, f(r) + \frac{\partial}{\partial z} k\, f(r) = f'(r) \frac{\partial r}{\partial x} i + f'(r) \frac{\partial r}{\partial y} j + f'(r) \frac{\partial r}{\partial z} k$$

$$f'(r) \frac{x}{r} i + f'(r) \frac{y}{r} j + f'(r) \frac{z}{r} k = \left(\frac{x}{r} i + \frac{y}{r} j + \frac{z}{r} k\right) f'(r)$$

$$\frac{f'(r)}{r} (xi + yj + zk) \quad , \quad \vec{r} = xi + yj + zk \rightarrow \boxed{f'(r) \frac{\vec{r}}{r}}$$

$$\nabla^2 f(r) = \nabla \cdot (\nabla f(r)) = \boxed{\nabla \cdot \left(f'(r) \frac{\vec{r}}{r}\right)}$$

$$\nabla \cdot (\phi \vec{u}) = \phi (\nabla \cdot \vec{u}) + (\nabla \phi) \cdot \vec{u}$$

In this case we have $\phi = \frac{f'(r)}{r}$ and $\vec{u} = \vec{r}$

$$\nabla \cdot \left(\frac{f'(r)}{r} \vec{r}\right) = \frac{f'(r)}{r} (\nabla \cdot \vec{r}) + \left(\nabla \frac{f'(r)}{r}\right) \cdot \vec{r}$$

$$\nabla^2 f(r) = \nabla \cdot \left(\frac{f'(r)}{r} \vec{r}\right) = \frac{f'(r)}{r} (3) + \left[r \frac{f''(r) - f'(r)}{r^2}\right] \vec{r}$$

$$\nabla \cdot \vec{r} = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) (xi + yj + zk) = 3$$

$$\nabla^2 f(r) = \frac{f'(r)}{r} (3) + \vec{r}\, F(r) \qquad \text{Let } F(r) = \frac{f'(r)}{r}$$

$$\nabla^2 f(r) = \frac{3 f'(r)}{r} + \vec{r}\, F'(r) \frac{\vec{r}}{r} \qquad \text{From 1: } \nabla f(r) = \frac{f'(r)}{r} \vec{r}$$

$$\nabla^2 f(r) = \frac{3 f'(r)}{r} + \vec{r} \frac{\partial}{\partial x} [F(r)] \frac{\vec{r}}{r}$$

$$\nabla^2 f(r) = \frac{3 f'(r)}{r} + \frac{\vec{r}\vec{r}}{r} \left[\frac{\partial}{\partial x} \frac{f'(r)}{r}\right] \qquad \because F(r) = \frac{f'(r)}{r}$$

$$\nabla^2 f(r) = \frac{3 f'(r)}{r} + \frac{r^2}{r} \left[r \frac{f''(r) - f'(r)}{r^2}\right]$$

$$\nabla^2 f(r) = \frac{3 f'(r)}{r} + \frac{r^3 f''(r)}{r^3} - \frac{r^2 f'(r)}{r^3} = f'(r)\left(\frac{3}{r} - \frac{1}{r}\right) + f''(r)$$

$$= f'(r) \frac{2}{r} + f''(r)$$

Integration

1) Evaluate the Indefinite Integrals

(i) $\int (3x^2 - 2x + 1)\,dx$

$: 3\int x^2 dx - 2\int x\,dx + \int dx$

$x^3 - x^2 + x + c$

(ii) $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$

$: \int x^{1/2} dx + \int x^{-1/2} dx$

$\frac{2x^{3/2}}{3} + 2x^{1/2} + c$

(iii) $\int x(\sqrt{x} + 1)\,dx, (x>0)$

$: \int x^{3/2} dx + \int x \cdot dx$

$\frac{2}{5} x^{5/2} + \frac{x^2}{2} + c$

(iv) $\int (2x + 3)^{1/2} dx$

$: \frac{2(2x+3)^{\frac{3}{2}}}{3} + c$

(v) $\int (\sqrt{x} + 1)^2 dx, (x>0)$

$: \int (x + 2\sqrt{x} + 1)\,dx$

$\int x\,dx + 2\int x^{1/2} dx + \int dx$

$\frac{x^2}{2} + \frac{4}{3} x^{3/2} + x + c$

(vi) $\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 dx, (x>0)$

$: \int \left(x - \frac{2\sqrt{x}}{\sqrt{x}} + \frac{1}{x}\right) dx$

$\int x\,dx - 2\int dx + \int x^{-1} dx$

$\frac{x^2}{2} - 2x + (\ln x) + c$

$\frac{x^2}{2} - 2x + \ln x + c$

(vii) $\int \frac{3x + 2}{\sqrt{x}}\,dx, (x>0)$

$: 3\int \frac{x}{\sqrt{x}} dx + \int \frac{2}{\sqrt{x}} dx = 3\int x^{1/2} dx + 2\int x^{-1/2} dx$

$3 \times \frac{2}{3} x^{3/2} + 2 \times 2\, x^{1/2} + c = 2x^{3/2} + 4x^{1/2} + c$

(viii) $\int \frac{\sqrt{y}\,(y+1)}{y}\,dy, (y>0)$

$: \int \frac{y\sqrt{y}}{y} dy + \int \frac{\sqrt{y}}{y} dy$

$\frac{2}{3} y^{3/2} + 2y^{1/2} + c$

(ix) $\int \frac{(\sqrt{\theta} - 1)^2}{\sqrt{\theta}}\,d\theta \quad (\theta>0)$

$: \int \frac{(\theta - 2\sqrt{\theta} + 1)}{\sqrt{\theta}}\,d\theta$

$\int \theta^{1/2} d\theta - 2\int d\theta + \int \theta^{-1/2} d\theta$

$\frac{2}{3}\theta^{3/2} - 2\theta + 2\theta^{1/2} + c$

(x) $\int \frac{(1 - \sqrt{x})^2}{\sqrt{x}}\,dx, (x>0)$

$: \int \frac{(1 - 2\sqrt{x} + x)}{\sqrt{x}}\,dx$

$\int x^{-1/2} dx - 2\int dx + \int x^{1/2} dx$

$2x^{1/2} - 2x + \frac{2}{3} x^{3/2} + c$

(xi) $\int \frac{e^{2x} + e^x}{e^x}\,dx$

$: \int \frac{e^{2x}}{e^x} dx + \int \frac{e^x}{e^x} dx = \int e^x dx + \int dx$

$\frac{e^{1+1x}}{1+1} + x + c = e^x + x + c$

4) Evaluate

(i) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}}$, $\begin{pmatrix} x+a>0 \\ x+b>0 \end{pmatrix}$

: $\int \frac{dx}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}} = \int \frac{\sqrt{x+a}-\sqrt{x+b}\ dx}{(x+a)-(x+b)}$

$\int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}\,dx = \frac{1}{a-b}\left[\int \sqrt{x+a}\,dx - \int \sqrt{x+b}\,dx\right]$

$\frac{1}{a-b}\left[\frac{2}{3}(x+a)^{3/2} - \frac{2}{3}(x+b)^{3/2}\right] = \frac{2}{3(a-b)}\left[(x+a)^{3/2} - (x+b)^{3/2}\right] + c$

ii) $\int \frac{1-x^2}{1+x^2}\,dx$

Solution Let $u = 1+x^2$, $x^2 = u-1$, $\frac{du}{2} = x\,dx$

$\frac{1}{2}\int \frac{1-(u-1)^2}{u}\,du = \frac{1}{2}\int \frac{1-(u^2-2u+1)}{u}\,du = \frac{1}{2}\int \frac{1-u^2+2u-1}{u}\,du$

$\frac{1}{2}\int \frac{2u-u^2}{u} = \frac{1}{2}\int \frac{2u}{u}\,du - \frac{1}{2}\int \frac{u^2}{u}\,du = u - \frac{1}{4}u^2 + c = 1+x^2 - \frac{(1+x^2)^2}{4} + c$

iii) $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}}$, $(x>0, a>0)$

: $\int \frac{dx}{\sqrt{x+a}+\sqrt{x}} \times \frac{\sqrt{x+a}-\sqrt{x}}{\sqrt{x+a}-\sqrt{x}} = \int \frac{\sqrt{x+a}-\sqrt{x}}{(x+a)-x}\,dx = \frac{1}{a}\int (\sqrt{x+a}-\sqrt{x})\,dx$

$\frac{1}{a}\int \sqrt{x+a}\,dx - \frac{1}{a}\int \sqrt{x}\,dx = \frac{2}{3a}(x+a)^{3/2} - \frac{2}{3a}(x)^{3/2} + c$

iv) $\int (a-2x)^{3/2}\,dx$

Solution

: $\frac{(a-2x)^{3/2+1}}{3/2+1} = \frac{2}{5}(a-2x)^{5/2} \times \frac{1}{\frac{2}{2x}(a-2x)}$

$\frac{\cancel{2}}{5}(a-2x)^{5/2} \times \frac{1}{-\cancel{2}}$

$-\frac{1}{5}(a-2x)^{5/2}$

v) $\int \frac{(1+e^x)^3}{e^x}\,dx$

: $\int \frac{(1)^3+3e^x+3e^{2x}+e^{3x}}{e^x}\,dx$

$\int \left(\frac{1}{e^x} + 3 + 3e^x + e^{2x}\right)dx$

$-e^{-x} + 3x + 3e^x + \frac{e^{2x}}{2} + c$

vi) $\int \sin(a+b)x \, dx$

: $\sin(a+b) \int x \, dx$

$\sin(a+b) \frac{x^2}{2} + c$

vii) $\int \sqrt{1-\cos 2x} \, dx$, $(1-\cos 2x > 0)$

: $\int \sqrt{1-\cos 2x} \, dx$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\int \sqrt{\cancel{1}-(\cancel{1}-2\sin^2 x)} \, dx$

$\int \sqrt{2\sin^2 x} \, dx$

$\int \sqrt{2} \sin^{\cancel{2}\,1/2} x \, dx$

$\sqrt{2} \int \sin x \, dx$

$\boxed{-\sqrt{2}\cos x + c}$

viii) $\int (\ln x) \times \frac{1}{x} \, dx \quad (x>0)$

: Integration by parts

$u = \ln x$, $du = 1/x \, dx$;

$dv = 1/x \, dx$, $v = \ln x$

$\int u\,dv = uv - \int v\,du$

$\int \ln x \times \frac{1}{x} \, dx = (\ln x)(\ln x) - \int \ln x \times \frac{1}{x} \, dx$

Let $\int \ln x \times \frac{1}{x} \, dx = I$

$I = (\ln x)(\ln x) - I$

$2I = (\ln x)^2$

$\boxed{I = \frac{(\ln x)^2}{2}}$

ix) $\int \sin^2 x \, dx$

Solution

$\cos 2x = \boxed{\cos^2 x} - \sin^2 x$

$\cos^2 x + \sin^2 x = 1$

$\boxed{\cos^2 x} = 1 - \sin^2 x$

$\cos 2x = (1-\sin^2 x) - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$

$\int \sin^2 x \, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \, dx$

$\frac{1}{2}x - \frac{1}{2}\sin 2x \times \frac{1}{\frac{\partial}{\partial x}(2x)} + c$

$\boxed{\frac{x}{2} - \frac{\sin 2x}{4} + c}$

x) $\int \frac{1}{1+\cos x} \, dx$, $\left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$

: $\int \frac{dx}{1+\cos x} \cdot \frac{1-\cos x}{1-\cos x}$

$\int \frac{1-\cos x}{1-\cos^2 x} \, dx = \int \frac{1-\cos x}{\sin^2 x} \, dx$

$\int \frac{1}{\sin^2 x} \, dx - \int \frac{\cos x}{\sin^2 x} \, dx = \int \csc^2 x \, dx - \int \csc x \cot x \, dx$

$= -\cot x - (-\csc x) + c$

$= -\cot x + \csc x + c$

xi) $\int \frac{ax+b}{ax^2+2bx+c}\,dx$

: U- Substitution, Let $u = ax^2+2bx+c$, $du = 2ax + 2b\,dx$. $\frac{du}{2} = (ax+b)\,dx$

$$\int \frac{(ax+b)\,dx}{ax^2+2bx+c} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln u + c = \frac{1}{2}\ln|ax^2+2bx+c| + c$$

xii) $\int \cos 3x \sin 2x\,dx$

: $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$$\int \cos 3x \sin 2x\,dx = \frac{1}{2}\int 2\cos 3x \sin 2x\,dx$$

$$\frac{1}{2}\int (\sin(3x+2x) - \sin(3x-2x))\,dx = \frac{1}{2}\int (\sin(5x) - \sin(x))\,dx$$

$$\frac{1}{2}\left(-\cos(5x)\cdot\frac{1}{5} - (-\cos x)\right) + c$$

$$\frac{1}{2}\left(-\frac{\cos(5x)}{5} + \cos x\right) + c$$

xiii) $\int \frac{\cos 2x - 1}{1+\cos 2x}\,dx$,

Solution Rationalization // put, $\boxed{\cos 2x = 1 - 2\sin^2 x}$

$$\int \frac{\cos 2x - 1}{1+\cos 2x}\,dx = \int \frac{(1-2\sin^2 x) - 1}{1+(1-2\sin^2 x)}\,dx = \int \frac{-2\sin^2 x}{2-2\sin^2 x}\,dx$$

$$-\frac{2}{2}\int \frac{\sin^2 x}{1-\sin^2 x}\,dx = -\int \frac{\sin^2 x}{\cos^2 x}\,dx = -\int \tan^2 x\,dx$$

$$-\int \tan^2 x\,dx = -\int (\sec^2 x - 1)\,dx = \int (1 - \sec^2 x)\,dx = \boxed{x - \tan x + c}$$

(xiv) $\int \tan^2 x\,dx$

Solution $1 + \tan^2 x = \sec^2 x$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int dx = \boxed{\tan x - x + c}$$

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■ Evaluate the following Integrals

1) $\int \frac{-2x}{\sqrt{4-x^2}}\,dx$: Trigonometric Substitution $\sqrt{a^2-x^2}$, $x = a\sin\theta$; $\sqrt{a^2-x^2} = a\cos\theta$

Solution $\sqrt{4-x^2} = 2\cos\theta$, $x = 2\sin\theta$ or we can integrate by using u-Substitution because derivative of u is available.

Let $u = 4-x^2$ $\quad du = -2x\,dx$

$$\int \frac{-2x\,dx}{\sqrt{4-x^2}} = \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\,du = 2u^{1/2} + C = \boxed{2\sqrt{4-x^2} + C}$$

2) $\int \frac{dx}{x^2+4x+13}$

Solution

1: Complete the Square $\rightarrow x^2+4x+13 \rightarrow x^2+4x+4+9 = (x+2)^2+9$

$$\int \frac{dx}{x^2+4x+13} = \int \frac{dx}{(x+2)^2+9} \rightarrow \underset{\text{u- Substitution}}{\text{let}}\ u = x+2,\ du = dx$$

$\int \frac{du}{u^2+9}$, Trigonometric Substitution

$3^2 = a^2$ $\quad$ let $u = a\tan\theta$, $du = a\sec^2\theta\,d\theta$, where $\boxed{a=3}$

$u = 3\tan\theta$, $du = 3\sec^2\theta\,d\theta$

$$\int \frac{3\sec^2\theta\,d\theta}{(3\tan\theta)^2+9} = \int \frac{3\sec^2\theta\,d\theta}{9\tan^2\theta+9} = \frac{1}{3}\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta+1}$$

$$\frac{1}{3}\int \frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \frac{1}{3}\int d\theta = \boxed{\frac{1}{3}\theta + C}$$

Now Replace θ, where $u = 3\tan\theta \rightarrow \frac{u}{3} = \tan\theta \rightarrow \boxed{\theta = \tan^{-1}\left(\frac{u}{3}\right)}$

$\frac{1}{3}\theta + C = \frac{1}{3}\left(\tan^{-1}\left(\frac{u}{3}\right)\right) + C$, Now Replace u

$\frac{1}{3}\tan^{-1}\left(\frac{u}{3}\right) + C$, $u = x+2$ from u-substitution

$$\boxed{\frac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right) + C}$$

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3) $\int \frac{x^2}{4+x^2}\,dx$

Solution Trigonometric Substitution -

$\int \frac{x^2}{a^2+x^2}\,dx \rightarrow$ let $x = a\tan\theta = 2\tan\theta$

$dx = a\sec^2\theta = 2\sec^2\theta\,d\theta$

$\int \frac{(2\tan\theta)^2 \cdot 2\sec^2\theta\,d\theta}{2^2+(2\tan\theta)^2} = \int \frac{8\tan^2\theta\sec^2\theta\,d\theta}{4+4\tan^2\theta}$

$= \frac{8}{4}\int \frac{\tan^2\theta\sec^2\theta\,d\theta}{(1+\tan^2\theta)} = 2\int \frac{\tan^2\theta\,\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}} = 2\int \tan^2\theta\,d\theta$

$= 2\int (1-\sec^2\theta)\,d\theta = 2\int d\theta - 2\int \sec^2\theta\,d\theta$

$= 2\theta - 2\tan\theta + C \rightarrow$ Now Replace θ, where $x = 2\tan\theta$

$x = 2\tan\theta \rightarrow \frac{x}{2} = \tan\theta \rightarrow \tan^{-1}\left(\frac{x}{2}\right) = \theta$

$= 2\theta - 2\tan\theta + C = 2\tan^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right) + C = \boxed{2\tan^{-1}\left(\frac{x}{2}\right) - x + C}$

4) $\int \frac{1}{x\ln x}\,dx$

Solution let $u = \ln x$, $du = \frac{1}{x}\,dx$

$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln(\ln x) + c$

5) $\int \frac{e^x}{e^x+3}\,dx$

Solution let $u = e^x+3$, $du = e^x\,dx$

$\int \frac{e^x}{e^x+3}\,dx = \int \frac{du}{u} = \ln|u| + c = \ln|e^x+3| + c$

6) $\int \frac{x+b}{(x^2+2bx+c)^{1/2}}\,dx$

: let $u = (x^2+2bx+c)$, $du = (2x+2b)\,dx$, $du/2 = (x+b)\,dx$

$\int \frac{x+b}{(x^2+2bx+c)^{1/2}}\,dx = \int \frac{du}{2}\cdot\frac{1}{u^{1/2}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left[\frac{u^{-1/2+1}}{-1/2+1}\right] + C = u^{1/2} + C$

$u^{1/2} + C = (x^2+2bx+c)^{1/2} + C$

3) $\int \frac{x^2}{4+x^2}\,dx$

Solution Trigonometric Substitution -

$\int \frac{x^2}{2^2+x^2}\,dx \rightarrow$ let $x = a\tan\theta = 2\tan\theta$

$dx = a\sec^2\theta = 2\sec^2\theta\,d\theta$

$\int \frac{(2\tan\theta)^2 \cdot 2\sec^2\theta\,d\theta}{2^2+(2\tan\theta)^2} = \int \frac{8\tan^2\theta\sec^2\theta\,d\theta}{4+4\tan^2\theta}$

$= \frac{8}{4}\int \frac{\tan^2\theta\sec^2\theta\,d\theta}{(1+\tan^2\theta)} = 2\int \frac{\tan^2\theta\,\cancel{\sec^2\theta}\,d\theta}{\cancel{\sec^2\theta}} = 2\int \tan^2\theta\,d\theta$

$= 2\int (1-\sec^2\theta)\,d\theta = 2\int d\theta - 2\int \sec^2\theta\,d\theta$

$= 2\theta - 2\tan\theta + C \rightarrow$ Now Replace θ, where $x = 2\tan\theta$

$x = 2\tan\theta \rightarrow \frac{x}{2} = \tan\theta \rightarrow \tan^{-1}\left(\frac{x}{2}\right) = \theta$

$= 2\theta - 2\tan\theta + C = 2\tan^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right) + C = \boxed{2\tan^{-1}\left(\frac{x}{2}\right) - 2x + C}$

4) $\int \frac{1}{x\ln x}\,dx$

Solution let $u = \ln x$, $du = \frac{1}{x}\,dx$

$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{\ln x}\cdot\frac{1}{x}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln(\ln x) + c$

5) $\int \frac{e^x}{e^x+3}\,dx$

Solution let $u = e^x+3$, $du = e^x\,dx$

$\int \frac{e^x}{e^x+3}\,dx = \int \frac{du}{u} = \ln|u| + c = \ln|e^x+3| + c$

6) $\int \frac{x+b}{(x^2+2bx+c)^{1/2}}\,dx$

: let $u = (x^2+2bx+c)$, $du = (2x+2b)\,dx$, $du/2 = (x+b)\,dx$

$\int \frac{x+b}{(x^2+2bx+c)^{1/2}}\,dx = \int \frac{du}{2}\cdot\frac{1}{u^{1/2}} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\left[\frac{u^{-1/2+1}}{-1/2+1}\right] + C = u^{1/2} + C$

$u^{1/2} + C = (x^2+2bx+c)^{1/2} + C$

7) $\int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx$

: let $u = \tan x$, $du = \sec^2 x\, dx$;

$\int \frac{\sec^2 x}{\sqrt{\tan x}}\, dx = \int \frac{du}{u^{1/2}} = \int u^{-1/2} du = 2u^{1/2} + c = 2\sqrt{\tan x} + c$

8) Show that $\int \frac{dx}{\sqrt{x^2-a^2}}\, dx = \ln(x + \sqrt{x^2-a^2}) + c$

: Using Trigonometric Substitution

let $\sqrt{x^2-a^2} = a|\tan\theta|$; $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

$\int \frac{dx}{\sqrt{x^2-a^2}}\, dx = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{(a\sec\theta)^2 - a^2}} = \int \frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta - a^2}}$

$= \int \frac{\cancel{a}\sec\theta\tan\theta\, d\theta}{\cancel{a}\sqrt{(\sec^2\theta - 1)}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\tan^2\theta}} = \int \sec\theta\, d\theta$

$= \int \sec\theta \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) d\theta = \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$

U-Substitution → let $u = \sec\theta + \tan\theta$, $du = \sec\theta\tan\theta + \sec^2\theta\, d\theta$

$\int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta = \int \frac{du}{u} = \ln|u| + c$

$\ln|\sec\theta + \tan\theta| + c = \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + c$

8) $\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2} \sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$

: let $\sqrt{a^2-x^2} = a|\cos\theta|$, $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

$\int \sqrt{a^2-x^2} = \int \sqrt{a^2-(a\sin\theta)^2}\, dx = \int \sqrt{a^2 - a^2\sin^2\theta}\, dx = a\int \sqrt{1-\sin^2\theta}\, dx$

$= a\int \sqrt{\cos^2\theta}\, dx = a\int \sqrt{\cos^2\theta}\, dx = a\int \sqrt{\cos^2\theta}\,(a\cos\theta)\, d\theta$

$= a^2 \int \cos\theta\cos\theta\, d\theta = a^2 \int \cos^2\theta\, d\theta$ → Trigonometric Reduction

$\cos^2\theta - \sin^2\theta = \cos 2\theta$

$\cos^2\theta - (1-\cos^2\theta) = \cos 2\theta$

$\cos^2\theta - 1 + \cos^2\theta = \cos 2\theta$

$2\cos^2\theta = \cos 2\theta + 1$

$\boxed{\cos^2\theta = \frac{\cos 2\theta + 1}{2}}$

$= a^2 \int \cos^2\theta\, d\theta = a^2 \int \left(\frac{\cos 2\theta + 1}{2}\right) d\theta$

$= \frac{a^2}{2}\int \cos 2\theta\, d\theta + \frac{a^2}{2}\int d\theta$

$\frac{a^2}{2}\int \cos 2\theta \cdot d\theta + \frac{a^2}{2}\int d\theta$; $a\sin\theta = x$, $\theta = \sin^{-1}(x/a)$

$\sin\theta = x/a$, $\sqrt{x^2-a^2} = a|\cos\theta|$

$\frac{a^2}{2}\left[\frac{\sin 2\theta}{2}\right] + \frac{a^2}{2}\theta + c = \frac{a^2}{2\cdot 2}(\sin 2\theta) + \frac{a^2}{2}\sin^{-1}(x/a) + c$

$= \frac{a^2}{4}[2\sin\theta\cos\theta] + \frac{a^2}{2}\sin^{-1}(x/a) + c$

$= \frac{\cancel{2}a^2}{\cancel{4}}\left[\frac{x}{\cancel{a}}\cdot\frac{\sqrt{x^2-a^2}}{\cancel{a}}\right] + \frac{a^2}{2}\sin^{-1}(x/a) + c = \boxed{\frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\sin^{-1}(x/a) + c}$

Evaluate following Integrals

8) $\int \frac{dx}{(1+x^2)^{3/2}}$ Trigonometric - substitution

: $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $\theta = \tan^{-1}(x) = \arctan(x)$

$\int \frac{\sec^2\theta\, d\theta}{(1+(\tan\theta)^2)^{3/2}} = \int \frac{\sec^2\theta\, d\theta}{(\sec^2\theta)^{3/2}} = \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int \frac{1}{\sec\theta}\, d\theta = \int \cos\theta\, d\theta$

$= \int \cos\theta\, d\theta = \sin\theta + c = \sin(\tan^{-1}(x)) + c = \sin(\arctan(x)) + c$

$\sin(\arctan(x)) + c = \boxed{\frac{x}{\sqrt{x^2+1}} + c}$

10) $\int \frac{1}{(1+x^2)\tan^{-1}x}\, dx$

: Trigonometric - substitution, u-substitution

let $u = \tan^{-1}x$, $du = \frac{1}{1+x^2}\, dx$

$\int \frac{1}{(1+x^2)\tan^{-1}x}\, dx = \int \frac{1}{(1+x^2)}\, dx \cdot \frac{1}{\tan^{-1}x} = \int \frac{du}{u} = \ln|u| + c$

$= \ln(\tan^{-1}x) + c$

11) $\int \sqrt{\frac{1+x}{1-x}}\, dx$

: Rationalization

$\int \sqrt{\frac{(1+x)}{(1-x)} \cdot \frac{(1+x)}{(1+x)}}\, dx = \int \sqrt{\frac{(1+x)^2}{(1-x^2)}}\, dx = \int \frac{\sqrt{(1+x)^2}}{\sqrt{(1-x^2)}}\, dx$

$= \int \frac{1+x}{\sqrt{1-x^2}}\, dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x}{\sqrt{1-x^2}}\, dx$

$= \int \frac{1}{\sqrt{1-x^2}}\, dx + \int \frac{x}{\sqrt{1-x^2}}\, dx$, Let $u = 1-x^2$, $du = -2x\, dx$

$-du/2 = x\, dx$

$= \sin^{-1}(x) + \int -\frac{du}{2} \cdot \frac{1}{\sqrt{u}} = \sin^{-1}(x) - \frac{1}{2}\int \frac{du}{\sqrt{u}}$

$= \sin^{-1}(x) - \frac{1}{2}\left[2u^{1/2}\right] + C = \boxed{\sin^{-1}(x) - \sqrt{1-x^2} + C}$

12) $\int \frac{\sin\theta}{1+\cos^2\theta}\, d\theta$

: Let $u = \cos\theta$, $du = -\sin\theta\, d\theta \rightarrow -du = \sin\theta\, d\theta$

$\int \frac{\sin\theta\, d\theta}{1+\cos^2\theta} = \int \frac{-du}{1+u^2} = -\int \frac{1}{1+u^2}\, du \rightarrow \tan^{-1}(u) = \frac{1}{1+u^2}$

$= -\int \frac{1}{1+u^2}\, du$, trigonometric substitution let $u = \tan\theta$, $du = \sec^2\theta\, d\theta$

$= -\int \frac{\sec^2\theta\, d\theta}{1+(\tan^2\theta)} = -\int \frac{\sec^2\theta}{\sec^2\theta}\, d\theta = -\int d\theta = -\theta + C$

$\int \frac{\sin\theta\, d\theta}{1+\cos^2\theta}\, d\theta = -\int \frac{1}{1+u^2}\, du = -\tan^{-1}(u) + C$

$= -\tan^{-1}(\cos\theta) + C$

14) $\int \frac{dx}{\sqrt{7-6x-x^2}}$

: Completing the Square and trigonometric - substitution

$\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{7-(6x+x^2)}} = \int \frac{dx}{\sqrt{7-(6x+x^2+9-9)}} = \int \frac{dx}{\sqrt{16-(x+3)^2}}$

$\int \frac{dx}{\sqrt{16-(x+3)^2}} = \int \frac{dx}{\sqrt{4^2-(x+3)^2}}$, let $x+3=4\tan\theta$

$dx = 4\sec^2\theta \, d\theta$

$\int \frac{4\sec^2\theta \, d\theta}{\sqrt{4^2-(4\tan\theta)^2}} = \int \frac{4\sec^2\theta \, d\theta}{4\sqrt{1-\tan^2\theta}} = \int \frac{\sec^2\theta \, d\theta}{\sqrt{1-\tan^2\theta}}$

let $u = \tan\theta$, $du = \sec^2\theta \, d\theta$

$\int \frac{\sec^2\theta \, d\theta}{\sqrt{1-\tan^2\theta}} = \int \frac{du}{\sqrt{1-u^2}} = \int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1}(u) + C$

$\sin^{-1}(u) + C = \sin^{-1}(\tan\theta) + C = \sin^{-1}\left(\frac{x+3}{4}\right) + C$

15) $\int \frac{\cos x \, dx}{\sin x \ln \sin x}$

: let $u = \sin x$, $du = \cos x \, dx$

$\int \frac{\cos x \, dx}{\sin x \ln \sin x} = \int \frac{du}{u \ln u} = \int \frac{1}{\ln(u)} \cdot \frac{du}{u}$ let $a = \ln u$, $da = \frac{1}{u} du$

$\int \frac{du}{u \ln(u)} = \int \frac{da}{a} = \ln(a) + C = \ln(\ln(u)) + C$

$= \ln(\ln(\sin x)) + C$

16) $\int \cos x \left(\frac{\ln \sin x}{\sin x}\right) dx$

: let $u = \sin x$, $du = \cos x \, dx$

$\int du \frac{\ln(u)}{u}$, let $a = \ln(u)$, $da = \frac{du}{u}$

$\int \frac{du}{u} \cdot \ln(u) = \int \frac{du}{u} \cdot \ln(u) = \int da \cdot a = \frac{a^2}{2} + C$

$\frac{a^2}{2} + C = \frac{\ln(u)^2}{2} + C = \frac{\ln(\sin x)^2}{2} + C$

Integration By parts

$$\int u\,dv = uv - \int v\,du + c$$

→ L·I·A·T·E : To Substitute - u

L - Logarithmic Function

I - Inverse Function

A - Algebaric Function

T - Trigonometric function

E - Exponential Function

• Evaluate the following integrals by parts

① i) $\int x \sin x\,dx$

: $u = x$, $du = dx$; $dv = \sin x\,dx$, $v = -\cos x$;

$\int u\,dv = uv - \int v\,du$

$\int x \sin x\,dx = (x)(-\cos x) - \int(-\cos x)\,dx$

$= -x\cos x + \int \cos x\,dx$

$= \boxed{-x\cos x + \sin x + c}$

ii) $\int \ln x\,dx$

: $u = \ln x$, $du = \frac{1}{x}\,dx$; $dv = dx$, $v = x$;

$\int \ln x\,dx = (\ln x)(x) - \int (\cancel{x})\left(\frac{1}{\cancel{x}}\right)dx = \boxed{x\ln x - x + c}$

iii) $\int x \ln x\,dx$

: $u = \ln x$, $du = \frac{1}{x}\,dx$; $dv = x\,dx$, $v = \frac{x^2}{2}$;

$\int x\ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx$

$= \frac{x^2\ln x}{2} - \frac{1}{2}\int x\,dx = \boxed{\frac{x^2\ln x}{2} - \frac{x^2}{4} + c}$

iv) $\int x^2 \ln x\,dx$

$u = \ln x$, $du = 1/x\,dx$; $dv = x^2\,dx$, $v = x^3/3$;

$\int x^2\ln x\,dx = (\ln x)(x^3/3) - \int(x^3/3)(1/x)\,dx$

$= \frac{x^3\ln x}{3} - \frac{1}{3}\int x^2\,dx = \boxed{\frac{x^3\ln x}{3} - \frac{x^3}{9} + c}$

(v) $\int x^3 \ln x \, dx$

: $u = \ln x$, $du = 1/x \, dx$; $dv = x^3 dx$, $v = \frac{x^4}{4}$;

$$\int x^3 \ln x \, dx = (\ln x)\left(\frac{x^4}{4}\right) - \int \left(\frac{x^4}{4}\right) \frac{1}{x} \, dx$$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 \, dx$$

$$= \frac{x^4 \ln x}{4} - \frac{1}{4}\left[\frac{x^4}{4}\right] + c$$

$$= \frac{x^4}{4}\left[\ln x - \frac{1}{4}\right] + c$$

(vi) $\int x^4 \ln x \, dx$

: $u = \ln x$, $du = \frac{1}{x} dx$; $dv = x^4 dx$, $v = \frac{x^5}{5}$;

$$\int x^4 \ln x \, dx = (\ln x)\left(\frac{x^5}{5}\right) - \int \left(\frac{x^5}{5}\right)\left(\frac{1}{x}\right) dx$$

$$= \frac{x^5 \ln x}{5} - \frac{1}{5} \int x^4 \, dx$$

$$= \frac{x^5 \ln x}{5} - \frac{1}{5}\left[\frac{x^5}{5}\right] + c$$

$$= \frac{x^5}{5}\left[\ln x - \frac{1}{5}\right] + c$$

(vii) $\int \tan^{-1} x \, dx$

: $\int \tan^{-1} x \, dx = \int \frac{1}{\tan x} dx = \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$

$u = \sin x$, $du = \cos x \, dx$; → u- Substitution

$$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln(u) + c = \ln(\sin x) + c$$

→ Integrating by parts (Question vii continued)

: $\int \tan^{-1}x \, dx$

$u = \tan^{-1}x$, $du = \frac{1}{1+x^2} dx$; $dv = dx$; $v = x$;

$$\int \tan^{-1}x \, dx = (\tan^{-1}x)(x) - \int (x)\left(\frac{1}{1+x^2} dx\right)$$

$$= x\tan^{-1}x - \int \frac{x}{1+x^2} dx$$

Let $u = 1+x^2$, $du = 2x \, dx$, $du/2 = x \, dx$

$$= x\tan^{-1}x - \int \frac{du/2}{u}$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{du}{u} = x\tan^{-1}(x) - \frac{\ln(u)}{2} + c$$

$$= \boxed{x\tan^{-1}(x) - \frac{\ln(1+x^2)}{2} + c}$$

(viii) $\int x^2 \sin x \, dx$

: $u = x^2$, $du = 2x \, dx$; $dv = \sin x \, dx$, $v = -\cos x$;

$$\int x^2 \sin x \, dx = (x^2)(-\cos x) - \int (-\cos x)(2x) \, dx$$

$$= -x^2\cos x + 2\int x\cos x \, dx$$

Again Integration by parts for repeated integral

$\Rightarrow$ ~~let~~ $u = x$, $du = dx$; $dv = \cos x \, dx$, $v = \sin x$;

$$= -x^2\cos x + 2\left[(x)(\sin x) - \int \sin x \, dx\right]$$

$$= -x^2\cos x + 2\left[x\sin x - (-\cos x)\right] + c$$

$$= \boxed{-x^2\cos x + 2x\sin x + 2\cos x + c}$$

⇒ Tabular Integration

$\frac{d}{dx}f(x)$		$\int dv$
x^2	+	$\sin x$
$2x$	−	$-\cos x$
2	+	$-\sin x$
0	−	$\cos x$

$f(x) = x^2$, $\int dv = \sin x \, dx$

$+(x^2)(-\cos x) - 2x(-\sin x) + 2(\cos x) + c$

$\boxed{-x^2\cos x + 2x\sin x + 2\cos x + c}$

(ix) $\int x^2 \tan^{-1} x \, dx$

: Tabular Integration for Iterated/Repeated Integrals

$u = x^2$, $dv = \tan^{-1} x$

: We are going to solve with ordinary Integration by parts technique

$u = x^2$, $du = 2x\,dx$; $dv = \tan^{-1} x \, dx$, $v = \frac{1}{1+x^2}$;

$$\int x^2 \tan^{-1} x \, dx = (x^2)\left(\frac{1}{1+x^2}\right) - \int \left(\frac{1}{1+x^2}\right)(2x\,dx)$$

$$= \frac{x^2}{1+x^2} - 2\int \frac{x\,dx}{1+x^2}$$

Let $u = 1+x^2$, $du = 2x\,dx$, $du/2 = x\,dx$;

$$= \frac{x^2}{1+x^2} - \cancel{2}\int \frac{du}{\cancel{2}} \cdot \frac{1}{u}$$

$$= \frac{x^2}{1+x^2} - \ln(u) + c = \frac{x^2}{1+x^2} - \ln(1+x^2) + c$$

L·I·A·T·E → I: Inverse

$u = \tan^{-1} x$, $du = \frac{1}{1+x^2}dx$; $dv = x^2$, $v = \frac{x^3}{3}$;

$$\int x^2 \tan^{-1} x \, dx = \tan^{-1}(x)\left(\frac{x^3}{3}\right) - \int \frac{x^3}{3}\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \frac{x^3}{1+x^2}\, dx$$

Let $u = 1+x^2$, $du = 2x\,dx$, $\frac{du}{2} = x\,dx$;

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \frac{x \cdot x^2}{u}\, dx = \frac{x^3 \tan^{-1} x}{3} - \frac{1}{3}\int \frac{(u-1)}{u}\frac{du}{2}$$

$$= \frac{x^3 \tan^{-1} x}{3} - \frac{1}{6}\left[\int\left(du - \frac{1}{u}du\right)\right] = \frac{x^3 \tan^{-1} x}{3} - \frac{1}{6}\left[u - \ln(u)\right] + c$$

$$= \boxed{\frac{x^3 \tan^{-1} x}{3} - \frac{1}{6}\left[1+x^2 - \ln(1+x^2)\right] + c}$$

(x) $\int x \tan^{-1}x \, dx$

: $u = \tan^{-1}x$, $du = \frac{1}{1+x^2} dx$; $dv = x \, dx$, $v = \frac{x^2}{2}$;

$$\int x \tan^{-1}x \, dx = (\tan^{-1}x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \frac{x^2}{1+x^2} \, dx$$

Let $u = 1+x^2$, $u-1 = x^2$, $du = 2x \, dx$;

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\int \frac{(x^2+1)-1}{(x^2+1)} \, dx$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\left[\int \left(\frac{x^2+1}{x^2+1} - \frac{1}{(x^2+1)}\right) dx\right]$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\left[\int dx - \int \frac{1}{x^2+1} \, dx\right]$$

$$= \frac{x^2 \tan^{-1}x}{2} - \frac{1}{2}\left[x - \tan^{-1}x\right] + C$$

$$= \frac{x^2 \tan^{-1}x}{2} + \frac{\tan^{-1}x}{2} - \frac{1}{2}x + C$$

$$= \boxed{\frac{\tan^{-1}x}{2}(x^2+1) - \frac{1}{2}x + C}$$

(xi) $\int x^3 \tan^{-1}x \, dx$

: Integration by parts

Let $u = \tan^{-1}x$ (an inverse function), $du = \frac{1}{1+x^2} dx$;

$dv = x^3 \, dx$, $v = \frac{x^4}{4}$;

$\int u \, dv = uv - \int v \, du$

$$\int x^3 \tan^{-1}x \, dx = (\tan^{-1}x)\left(\frac{x^4}{4}\right) - \frac{1}{4}\int (x^4)\left(\frac{1}{1+x^2}\right) dx$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\int \frac{x^4}{1+x^2} dx$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\int \frac{x^4 - 1 + 1}{1+x^2} dx$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\int \frac{(x^2)^2 - 1 + 1}{1+x^2} dx$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\int \left(\frac{((x^2)^2 - 1)}{1+x^2} + \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\int \left(\frac{(x^2-1)\cancel{(x^2+1)}}{\cancel{1+x^2}} + \frac{1}{1+x^2}\right) dx$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\left[\int (x^2-1)\, dx + \int \frac{1}{1+x^2} dx\right]$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1}x\right] + c$$

$$= \frac{x^4 \tan^{-1}x}{4} - \frac{x^3}{12} + \frac{x}{4} - \frac{\tan^{-1}x}{4} + c$$

$$= \frac{\tan^{-1}x}{4}(x^4 - 1) - \frac{x^3}{12} + \frac{x}{4} + c$$

$$= \boxed{\frac{1}{4}\left[\tan^{-1}x(x^4-1) - \frac{x^3}{4} + x\right] + c}$$

(xii) $\int x^3 \cos x \, dx$

: Integration by parts : Tabular integration technique for iterated integral

Let $u = x^3$, and $v = \cos x \, dx$

$\frac{d}{dx}(u)$		$\int v \, dx$	
x^3	+	$\cos x$	$+x^3 \sin x - 3x^2(-\cos x) + 6x(-\sin x) - 6(\cos x) + c$
$3x^2$	−	$\sin x$	$x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x + c$
$6x$	+	$-\cos x$	$x^3 \sin x - 6x \sin x + 3x^2 \cos x - 6 \cos x + c$
6	−	$-\sin x$	$\boxed{\sin x(x^3 - 6x) + \cos x(3x^2 - 6) + c}$
0	+	$\cos x$	

(xiii) $\int \sin^{-1} x \, dx$

: let $u = \sin^{-1} x$, $du = \frac{1}{\sqrt{1-x^2}} dx$; $dv = dx$, $v = x$;

$$\int \sin^{-1} x \, dx = (\sin^{-1} x)(x) - \int (x) \left(\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

u- substitution, let $u = 1 - x^2$, $du = -2x \, dx$, $-\frac{du}{2} = x \, dx$

$$= x \sin^{-1} x - \int \left(-\frac{du}{2}\right) \left(\frac{1}{\sqrt{u}} \right)$$

$$= x \sin^{-1} x + \frac{1}{2} \int u^{-1/2} \, du$$

$$= x \sin^{-1} x + \frac{1}{2} \left[2 u^{1/2} \right] + c$$

$$= x \sin^{-1} x + \sqrt{u} + c$$

$$= \boxed{x \sin^{-1} x + \sqrt{1-x^2} + c}$$

(xiv) $\int x \sin^{-1} x \, dx$

Let $u = \sin^{-1} x$, $du = \frac{1}{\sqrt{1-x^2}} dx$; $dv = x \, dx$, $v = \frac{x^2}{2}$;

$$\int x \sin^{-1} x \, dx = (\sin^{-1} x)\left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\left(\frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}} dx$$

u-substitution, Let $u = 1-x^2$, $x^2 = 1-u$, $dx = ?$ Not possible

Trigonometric substitution Let $x = \sin\theta$; $dx = \cos\theta \, d\theta$, $\sqrt{1-x^2} = \cos\theta$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2}\int \frac{(\sin^2\theta)(\cos\theta)d\theta}{\sqrt{1-(\sin^2\theta)}}$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2}\int \frac{\sin^2\theta \cos\theta \, d\theta}{\sqrt{\cos^2\theta}}$$

$$= \frac{x^2 \sin^{-1} x}{2} - \frac{1}{2}\int \sin^2\theta \, d\theta$$

$-\sin^2\theta + \cos^2\theta = \cos 2\theta$

$-\sin^2\theta + (1 - \sin^2\theta) = \cos 2\theta$

$-\sin^2\theta + 1 + \sin^2\theta = \cos 2\theta$

$+\sin^2\theta = -\frac{(\cos 2\theta + 1)}{2}$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4}\left[\int (\cos 2\theta + 1) d\theta\right]$$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{1}{4}\left[\frac{\sin 2\theta}{2} + \theta\right] + C$$

$x = \sin\theta \rightarrow \sin^{-1}(x) = \theta$

$\sqrt{1-x^2} = \cos\theta$

$$= \frac{x^2 \sin^{-1} x}{2} + \frac{\theta}{4} + \frac{\sin 2\theta}{8} + C$$

$$= \frac{x^2 \sin^{-1}}{2} + \frac{\sin^{-1}(x)}{4} + \frac{2\sin\theta\cos\theta}{8} + C$$

$$= \frac{x^2 \sin^{-1}}{2} + \frac{\sin^{-1}(x)}{4} + \frac{\cancel{2}\sin\theta\cos\theta}{\cancel{8}\,4} + C$$

$$= \frac{x^2 \sin^{-1}}{2} + \frac{\sin^{-1}(x)}{4} + \frac{(\sin\theta)(\cos\theta)}{4} + C$$

$$= \boxed{\frac{x^2 \sin^{-1} x}{2} + \frac{\sin^{-1}(x)}{4} + \frac{x\sqrt{1-x^2}}{4} + C}$$

(xv) $\int e^x \sin x \cos x \, dx$

: $\int e^x \sin x \cos x \, dx$, $\sin(2x) = 2\sin(x)\cos(x)$

$= \int e^x \frac{\sin(2x)}{2} dx$ $\frac{\sin 2x}{2} = \sin(x)\cos(x)$

$= \frac{1}{2}\int e^x \sin(2x) \, dx$

Let $u = \sin(2x)$, $du = 2\cos(2x)\,dx$; $dv = e^x dx$, $v = e^x$;

$= \frac{1}{2}\int e^x \sin(2x) = \frac{1}{2}\left[\sin(2x)(e^x) - 2\int e^x \cos(2x)\,dx\right]$

$\frac{1}{2}\int e^x \sin(2x)\,dx = \frac{1}{2}\left[\sin(2x)\,e^x - 2\left[\cos(2x)\,e^x + 2\int e^x \sin(2x)\,dx\right]\right]$

Let $\int e^x \sin(2x) = I$

$\frac{I}{2} = \frac{1}{2}\left[\sin(2x)e^x - 2\left[\cos(2x)e^x + 2I\right]\right]$

$\frac{I}{2} = \frac{1}{2}\left[\sin(2x)e^x - 2\cos(2x)e^x - 4I\right]$

$\frac{I}{2} = \frac{e^x \sin(2x)}{2} - e^x\cos(2x) - 2I$

$2I + \frac{I}{2} = \frac{e^x \sin(2x)}{2} - e^x\cos(2x)$

$I = \boxed{\frac{e^x \sin(2x) - 2e^x\cos(2x)}{5} + C}$

(xvi) $\int x \sin x \cos x \, dx$

: $\int x \sin x \cos x \, dx$, $\sin(2x) = 2\sin(x)\cos(x)$

$\frac{\sin(2x)}{2} = \sin(x)\cos(x)$

$\frac{1}{2}\int x \sin(2x) \, dx$

Let $u = x$, $du = dx$; $dv = \sin(2x)dx$, $v = 2\cos(2x)$

Let $\frac{1}{2}\int x \sin(2x) = \frac{I}{2}$

$I = \left[(x)(2\cos(2x)) - 2\int \cos(2x) \, dx\right] \cancel{/}2$

$I = \left[2x\cos(2x) - 2\int \cos(2x) \, dx\right] \cancel{/}2$

$I = \left[4x\cos(2x) - 4\int \cos(2x) \, dx\right] /2$

$I = 4x\cos(2x) - 4\frac{\sin(2x)}{2} + c$

$I = 4x\cos(2x) - 2\sin(2x) + c$

$I = 4x\cos(2x) - 4\sin(x)\cos(x) + c$

Let $u = x$, $du = dx$; $dv = \sin(2x)\,dx$, $v = \frac{2\cos(2x)}{2}$

$\frac{1}{2}\int x \sin(2x) \, dx = \frac{1}{2}\left[\frac{(x)(\cos 2x)}{2} - \frac{1}{2}\int \cos(2x) \, dx\right]$

$= \frac{x\cos(2x)}{4} - \frac{1}{4}\int \cos(2x) \, dx$

$= \frac{x\cos(2x)}{4} - \frac{1}{4}\left[\frac{\sin(2x)}{2}\right] + c$

$= \frac{x\cos(2x)}{4} - \frac{1}{4}\left[\frac{2\sin(x)\cos(x)}{2}\right] + c$

$= \frac{x\cos(2x)}{4} - \frac{1}{4}\left[\sin(x)\cos(x)\right] + c$

$= \boxed{\frac{1}{4}\left(x\cos(x) - \sin(x)\cos(x)\right] + c}$

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(xvii) $\int x\cos^2 x\,dx$

: Let $u = x$, $du = dx$; $dv = \cos^2 x\,dx$, $v = \int \frac{\cos 2x + 1}{2}\,dx$

$$v = \frac{1}{2}\left[\int \cos 2x\,dx + \int dx\right] = \frac{1}{2}\left[\frac{\sin(2x)}{2} + x\right] = \frac{\sin 2x}{4} + \frac{x}{2} = v$$

$$\int x\cos^2 x\,dx = (x)\left(\frac{\sin(2x)}{4} + \frac{x}{2}\right) - \int\left(\frac{\sin 2x}{4} + \frac{x}{2}\right)dx$$

$$= \frac{x\sin(2x)}{4} + \frac{x^2}{2} - \left[-\frac{\cos 2x}{8} + \frac{x^2}{4}\right] + c$$

$$= \frac{x\sin(2x)}{4} + \frac{x^2}{2} + \frac{\cos(2x)}{8} - \frac{x^2}{4} + c$$

$$= \frac{x\sin(2x)}{4} + \frac{\cos(2x)}{8} + \frac{x^2}{4} + c$$

$$= \boxed{\frac{1}{4}\left(x\sin(2x) + \frac{\cos(2x)}{2} + x^2\right) + c}$$

(xviii) $\int x\sin^2 x\,dx$

: Let $u = x$, $du = dx$, $dv = \sin^2 x\,dx$, $v = \int \sin^2 x\,dx$

$v = \int \sin^2 x\,dx \rightarrow \cos^2 x - \sin^2 x = \cos 2x \rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$

$$v = \int \sin^2 x\,dx = \frac{1}{2}\int (1 - \cos 2x)\,dx = 1/2\left[x - \frac{\sin 2x}{2}\right] = v$$

$$\int x\sin^2 x\,dx = (x)\left[\frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)\right] - \int\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)dx$$

$$= \frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{x^2}{4} + \frac{(-\cos 2x)}{8} + c$$

$$= \frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{x^2}{4} - \frac{\cos 2x}{8} + c$$

$$= \boxed{\frac{1}{4}\left(x^2 - x\sin 2x - \frac{\cos 2x}{2}\right) + c}$$

(xix) $\int (\ln x)^2 \, dx$

: Let $u = (\ln x)^2$, $du = \frac{2\ln x}{x} dx$; $dv = dx$, $v = x$;

$$\int (\ln x)^2 dx = (\ln x)^2 (x) - 2\int (x) \frac{\ln x}{x} dx$$

$$= x(\ln x)^2 - 2\int \ln x \, dx$$

$$= x(\ln x)^2 - 2\left[(\ln x)(x) - \int dx\right]$$

$$= x(\ln x)^2 - 2\left[\ln(x) - x\right] + c$$

$$= x(\ln x)^2 - 2x\ln x + 2x + c$$

$$= \boxed{x\ln x(\ln x - 2) + 2x + c}$$

(xx) $\int \ln(\tan x) \sec^2 x \, dx$

: Let $u = \ln(\tan x)$, $du = \frac{1}{\tan x} \cdot \sec^2 x \, dx$; $dv = \sec^2 x \, dx$, $v = \tan x$

$$\int \ln(\tan x) \sec^2 x \, dx = \ln(\tan x)(\tan x) - \int \tan x \cdot \frac{\sec^2 x}{\tan x} dx$$

$$= \tan x \ln(\tan x) - \int \sec^2 x \, dx$$

$$= \boxed{\tan x \ln(\tan x) - \tan x + c}$$

(xxi) $\int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

: Trigonometric Substitution, Let $x = \sin\theta$, $\sin^{-1} x = \theta$, $\sqrt{1-x^2} = \cos\theta$, $dx = \cos\theta \, d\theta$

$$\int \frac{x \sin^{-1} x \, dx}{\sqrt{1-x^2}} = \int \frac{(\sin\theta)(\theta)\cos\theta \, d\theta}{\cos\theta} = \int \theta \sin\theta \, d\theta$$

Let $u = \theta$, $du = d\theta$; $dv = \sin\theta \, d\theta$, $v = -\cos\theta$;

$$\int \theta \sin\theta \, d\theta = (\theta)(-\cos\theta) + \int \cos\theta \, d\theta$$

$$= -\theta\cos\theta + \sin\theta + c$$

$$= -(\sin^{-1} x)(\sqrt{1-x^2}) + x + c$$

$$= -\sin^{-1} x\sqrt{1-x^2} + x + c$$

$$= \boxed{x - \sin^{-1} x\sqrt{1-x^2} + c}$$

2) Evaluate the following integrals

(i) $\int \tan^4 x \, dx$

: $\int \tan^4 x \, dx = \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx$

$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$

$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$

$= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx - \int dx$

Let $u = \tan x$, $du = \sec^2 x \, dx$

$= \int u^2 \, du - \int \sec^2 x - \int dx$

$= \frac{u^3}{3} - u - x + c$

$= \boxed{\frac{\tan^3 x}{3} - \tan x - x + c}$

(ii) $\int \sec^4 x \, dx$

: $\int \sec^4 x \, dx = \int \sec^2 x (\sec^2 x) \, dx = \int \sec^2 x (1 + \tan^2 x) \, dx$

$= \int \sec^2 x \, dx + \int \sec^2 x \tan^2 x \, dx$

$\Rightarrow$ Let $u = \tan x$, $du = \sec^2 x \, dx$

$= \int du + \int u^2 \, du = u + \frac{u^3}{3} + c$

$= \boxed{\tan x + \frac{\tan^3 x}{3} + c}$

(iii) $\int e^x \sin 2x \cos x \, dx$

$\therefore \int e^x (2\sin x \cos x) \cos x \, dx = 2\int e^x \sin x \cos^2 x \, dx$

$= 2\int e^x \sin x \cos^2 x \, dx = 2\int e^x \sin x (1 - \sin^2 x) \, dx$

$= 2\int e^x \sin x \, dx - 2\int e^x \sin x (\sin^2 x) \, dx$

$= 2\int e^x \sin x \, dx - 2\int e^x \sin^3 x \, dx$

$= 2\int e^x \sin x \, dx - 2\int e^x \sin x (1 - \cos^2 x) \, dx$

$= 2\int e^x \sin x \, dx - 2\int e^x \sin x \, dx + 2\int e^x \sin x \cos^2 x \, dx$

Let $\int e^x \sin x \cos^2 x \, dx = I$

$2\int e^x \sin x \cos^2 x \, dx = 2\int e^x \sin x \, dx - 2\int e^x \sin x \, dx + 2\int e^x \sin x \cos^2 x \, dx$

$2I = 2\int e^x \sin x \, dx - 2\int e^x \sin x \, dx + 2I$

$2I = 0 + 2I$

$\boxed{I = I}$

(iv) $\int \tan^3 x \sec x \, dx$

$\therefore \int \tan^3 x \sec x \, dx = \int \tan^2 x \tan x \sec x \, dx$

Let $u = \tan^2 x$, $du = 2\tan x \cdot \sec^2 x \, dx$

$\int \tan^2 x \tan x \sec x \, dx = \int u^2 \ldots$

$\int \tan^3 x \sec x \, dx = \int \tan^2 x \tan x \sec x \, dx = \int (\sec^2 x - 1) \tan x \sec x \, dx$

Let $u = \sec x$, $du = \sec x \tan x \, dx$

$\int (\sec^2 x - 1) \sec x \tan x \, dx = \int (u^2 - 1) \, du = \int u^2 \, du - \int du$

$= \frac{u^3}{3} - u + c$

$= \boxed{\frac{\sec^3 x}{3} - \sec x + c}$

(v) $\int x^3 e^{5x}\, dx$

: Tabular Integration

let $u = x^3$, $dv = e^{5x} dx$

$\frac{d}{du}(u)$		$\int dv$
x^3	+	e^{5x}
$3x^2$	−	$\frac{e^{5x}}{5}$
$6x$	+	$\frac{e^{5x}}{25}$
6	−	$e^{5x}/125$
0	+	$e^{5x}/625$

$$+x^3 \frac{e^{5x}}{5} - 3x^2 \frac{e^{5x}}{25} + 6x \frac{e^{5x}}{125} - 6 \frac{e^{5x}}{625} + c$$

$$\boxed{\frac{e^{5x}}{5}\left[x^3 - \frac{3x^2}{5} + \frac{6x}{25} - \frac{6}{125}\right] + c}$$

(vi) $\int e^{-x} \sin 2x\, dx$

: let $u = \sin 2x$, $du = (\cos 2x)2\, dx$; $dv = e^{-x} dx$, $v = -e^{-x}$;

$$\int e^{-x} \sin 2x\, dx = (\sin 2x)(-e^{-x}) - \int (-e^{-x})(2\cos 2x)\, dx$$

$$= -e^{-x} \sin 2x + 2\int e^{-x} \cdot \cos 2x\, dx$$

let $u = \cos 2x$, $du = -2\sin 2x\, dx$

$dv = e^{-x}$, $v = -e^{-x}$

$$\int e^{-x} \sin 2x\, dx = -e^{-x} \sin 2x + 2\left[(\cos 2x)(-e^{-x}) - \int (-e^{-x})(-2\sin 2x)\, dx\right]$$

$$\int e^{-x} \sin 2x\, dx = -e^{-x} \sin 2x + (2\cos 2x)(-e^{-x}) - 2\int (e^{-x})(2\sin 2x)\, dx$$

$$\int e^{-x} \sin 2x\, dx = -e^{-x} \sin 2x - 2e^{-x}\cos 2x - 4\int e^{-x} \sin 2x\, dx$$

let $I = \int e^{-x} \sin 2x\, dx$

$$I = -e^{-x} \sin 2x - 2e^{-x}\cos 2x - 4I$$

$$5I = -e^{-x} \sin 2x - 2e^{-x}\cos 2x + c$$

$$I = \frac{-e^{-x} \sin 2x - 2e^{-x}\cos 2x}{5} + c$$

$$I = \frac{-e^{-x}}{5}\left[\sin 2x + 2\cos 2x\right] + c$$

$$\int e^{-x} \sin 2x = \boxed{\frac{-e^{-x}}{5}\left[\sin 2x + 2\cos 2x\right] + c}$$

(vii) $\int e^{2x}\cos 3x\,dx$

: Let $u = \cos 3x$, $du = -3\sin 3x\,dx$;

$dv = e^{2x}dx$, $v = \frac{e^{2x}}{2}$;

$$\int e^{2x}\cos 3x\,dx = (\cos 3x)\left(\frac{e^{2x}}{2}\right) - \int\left(\frac{e^{2x}}{2}\right)(-3\sin 3x)\,dx$$

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}\cos 3x}{2} + \frac{3}{2}\int e^{2x}\sin 3x\,dx$$

Let $u = \sin 3x$, $du = 3\cos 3x\,dx$;

$dv = e^{2x}$, $v = \frac{e^{2x}}{2}$;

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}\cos 3x}{2} + \frac{3}{2}\left[(\sin 3x)\left(\frac{e^{2x}}{2}\right) - \int\left(\frac{e^{2x}}{2}\right)(3\cos 3x)\,dx\right]$$

$$\int e^{2x}\cos 3x\,dx = \frac{e^{2x}\cos 3x}{2} + \frac{3\sin 3x \cdot e^{2x}}{4} - \frac{3}{2}\cdot\frac{3}{2}\int e^{2x}\cos 3x\,dx$$

Let $I = \int e^{2x}\cos 3x\,dx$

$$I = \frac{e^{2x}\cos 3x}{2} + \frac{3\sin 3x\, e^{2x}}{4} - \frac{9}{4}I$$

$$I + \frac{9}{4}I = \frac{e^{2x}}{2}\left[\cos 3x + \frac{3\sin 3x}{2}\right] + c$$

$$\frac{4I + 9I}{4} = \frac{e^{2x}}{2}\left[\cos 3x + \frac{3\sin 3x}{2}\right] + c$$

$$I = \frac{2e^{2x}}{13}\left[\cos 3x + \frac{3\sin 3x}{2}\right] + c$$

(VIII) $\int \operatorname{cosec}^3 x \, dx$

$\therefore \int \operatorname{cosec}^3 x \, dx = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec} x \, dx$

$= \int (1 + \cot^2 x) \operatorname{cosec} x \, dx$

$= \int \operatorname{cosec} x + \int \cot^2 x \operatorname{cosec} x \, dx$

$= \int \operatorname{cosec}^2 x \, dx + \int \cot x \cdot \cot x \operatorname{cosec} x \, dx$

$\int \operatorname{cosec}^3 x \, dx = \int \operatorname{cosec} x \cdot \operatorname{cosec} x^2 \, dx$

Let $u = \operatorname{cosec} x$, $-du = +\operatorname{cosec} x \cot x \, dx$;

$dv = \operatorname{cosec}^2 x \, dx$; $v = -\cot x$;

$\int \operatorname{cosec}^3 x \, dx = (\operatorname{cosec} x)(-\cot x) - \int (-\cot x)(-\operatorname{cosec} x \cot x) \, dx$

$\int \operatorname{cosec}^3 x \, dx = -\cot x \operatorname{cosec} x - \int \cot^2 x \operatorname{cosec} x \, dx$

$\int \operatorname{cosec}^3 x \, dx = -\cot x \operatorname{cosec} x - \int (\operatorname{cosec}^2 x - 1) \operatorname{cosec} x \, dx$

$\int \operatorname{cosec}^3 x \, dx = -\cot x \operatorname{cosec} x - \int \operatorname{cosec}^3 x \, dx - \int \operatorname{cosec} x \, dx$

let $\int \operatorname{cosec}^3 x \, dx = I$

$I = -\cot x \operatorname{cosec} x - I - \int \operatorname{cosec} x \, dx$

$2I = -\cot x \operatorname{cosec} x - \int \operatorname{cosec} x \left(\frac{\operatorname{cosec} x + \cot x}{\operatorname{cosec} x + \cot x} \right) dx$

$2I = -\cot x \operatorname{cosec} x - \int \frac{\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x}{\operatorname{cosec} x + \cot x} \, dx$

let $u = \operatorname{cosec} x + \cot x$, $du = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x \, dx$

$-du = (\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x) \, dx$

$2I = -\cot x \operatorname{cosec} x - \int \frac{-du}{u}$

$2I = -\cot x \operatorname{cosec} x + \ln|u| + c$

$2I = -\cot x \operatorname{cosec} x + \ln|\operatorname{cosec} x + \cot x| + c$

$$\boxed{I = \frac{-\cot x \operatorname{cosec} x + \ln|\operatorname{cosec} x + \cot x| + c}{2}}$$

8) Show that $\int e^{ax} \sin bx \, dx = \frac{1}{\sqrt{a^2+b^2}} e^{ax} \sin\left(bx - \tan^{-1}\frac{b}{a}\right) + c$

: $\int e^{ax} \sin bx \, dx$

Let $u = \sin bx$; $du = b\cos bx \, dx$;

$dv = e^{ax} dx$, $v = \frac{e^{ax}}{a}$;

$$\int e^{ax} \sin bx \, dx = (\sin bx)\left(\frac{e^{ax}}{a}\right) - \int \left(\frac{e^{ax}}{a}\right)(b\cos bx)\, dx$$

$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int e^{ax}\cos bx \, dx$$

Let $u = \cos bx$, $du = -b\sin bx \, dx$; $dv = e^{ax} dx$, $v = \frac{e^{ax}}{a}$;

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[(\cos bx)\left(\frac{e^{ax}}{a}\right) - \int\left(\frac{e^{ax}}{a}\right)(-b\sin bx)\, dx\right]$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}\sin bx}{a} - \frac{b\cos bx\, e^{ax}}{a^2} + \frac{b}{a^2}\int e^{ax}(-b\sin bx)\, dx$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}\sin bx}{a} - \frac{b\cos bx\, e^{ax}}{a^2} - \frac{b^2}{a^2}\int e^{ax}\sin bx \, dx$$

Let $I = \int e^{ax} \sin bx \, dx$

$$I = \frac{e^{ax}\sin bx}{a} - \frac{b\cos bx\, e^{ax}}{a^2} - \frac{b^2}{a^2} I$$

$$I + \frac{b^2}{a^2} I = \frac{e^{ax}\sin bx}{a} - \frac{e^{ax} b\cos bx}{a^2} + c_1$$

$$I(a^2+b^2) = a e^{ax}\sin bx - e^{ax} b\cos bx + c_1$$

$$I = \frac{a e^{ax}\sin bx - e^{ax} b\cos bx}{(a^2+b^2)} + C$$

4) Evaluate the following indefinite integrals

(1) $\int \sqrt{a^2 - x^2}\, dx$

• Trigonometric substitution

Let $x = a\sin\theta$, $\sqrt{a^2-x^2} = a\cos\theta$, $\theta = \sin^{-1}(x/a)$, $dx = a\cos\theta\, d\theta$;

$$\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2-(a\sin\theta)^2}\, d\theta \quad \int \sqrt{a^2 - a^2\sin^2\theta}\, dx$$

$$= \int \sqrt{a^2(1-\sin^2\theta)}\, dx = a\int \sqrt{1-\sin^2\theta}\, dx$$

$$= a\int \sqrt{\cos^2\theta}\, dx = a\int \cos^2\theta\, dx$$

where, $dx = a\cos\theta\, d\theta$

$$= a\int \cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2 \int \cos^2\theta\, d\theta \quad ; \quad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$= \frac{a^2}{2}\int (\cos 2\theta + 1)\, d\theta \qquad \cos^2\theta - 1 + \cos^2\theta = \cos 2\theta$$

$$2\cos^2\theta = \cos 2\theta + 1$$

$$= \frac{a^2}{2}\left[\int \cos 2\theta\, d\theta + \int d\theta\right] \qquad \boxed{\cos^2\theta = \frac{\cos 2\theta + 1}{2}}$$

$$= \frac{a^2}{2}\left[\frac{\sin 2\theta}{2} + \theta\right] + C$$

Sum of two angles formula

$$= \frac{a^2}{2}\left[\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}} + \theta\right] + C$$

$$= \frac{a^2}{2}\left[\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + \sin^{-1}(x/a)\right] + C$$

where $x = a\sin\theta$, $\sqrt{a^2-x^2} = a\cos\theta$, $\theta = \sin^{-1}(x/a)$

$$\int \sqrt{a^2-x^2}\, dx = \boxed{\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C}$$

(ii) $\int \sqrt{x^2 - a^2}\, dx$

: $\int \sqrt{x^2-a^2}\, dx$; Trigonometric Substitution

Let $x = a\sec\theta$, $\sqrt{x^2-a^2} = a\tan\theta$

$dx = a\sec\theta\tan\theta\, d\theta$, $\theta = \sec^{-1}(x/a)$

$$\int \sqrt{x^2-a^2}\, dx = \int (a\tan\theta)(a\sec\theta\tan\theta)\, d\theta$$

$$= a^2 \int \tan^2\theta \sec\theta\, d\theta$$

$$= a^2 \int (\sec^2\theta - 1)\sec\theta\, d\theta$$

$$= a^2 \int \sec^3\theta - a^2 \int \sec\theta\, d\theta$$

$$= a^2 \int \sec^2\theta \sec\theta\, d\theta - a^2 \int \sec\theta\, d\theta$$

Let $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

$dv = \sec^2\theta\, d\theta$, $v = \tan\theta$;

$$= a^2\left[(\sec\theta)(\tan\theta) - \int (\tan\theta)\sec\theta\tan\theta\, d\theta\right] - a^2\int \sec\theta\, d\theta$$

$$= a^2\sec\theta\tan\theta - a^2\int \tan^2\theta\sec\theta\, d\theta - a^2\int \sec\theta\, d\theta$$

$$= a^2\sec\theta\tan\theta - a^2\left[\int \tan^2\theta\sec\theta\, d\theta\right] - a^2\int \sec\theta\, d\theta$$

Let

$$= a^2\sec\theta\tan\theta - a^2\left[\int (\sec^2\theta - 1)\sec\theta\, d\theta\right] - a^2\int \sec\theta\, d\theta$$

$$= a^2\sec\theta\tan\theta - a^2\int \sec^3\theta\, d\theta + a^2\int \cancel{\sec\theta}\, d\theta - a^2\int \cancel{\sec\theta}\, d\theta$$

$$= a^2\sec\theta\tan\theta - a^2\int \sec^3\theta\, d\theta$$

Let $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

$dv = \sec^2\theta\, d\theta$, $v = \tan\theta$;

$$= a^2\sec\theta\tan\theta - a^2\int \sec^2\theta\sec\theta\, d\theta$$

$$= a^2\sec\theta\tan\theta - a^2\left[uv - \int v\, du\right]$$

$$= a^2\sec\theta\tan\theta - a^2\left[(\sec\theta)(\tan\theta) - \int (\tan\theta)\sec\theta\tan\theta\, d\theta\right]$$

$$= a^2\cancel{\sec\theta\tan\theta} - a^2\cancel{\sec\theta\tan\theta} + a^2\int \tan\theta\sec\theta\tan\theta\, d\theta$$

$$a^2\int \tan^2\theta\sec\theta\, d\theta = a^2\int \tan^2\theta\sec\theta\, d\theta$$

(ii) $\int \sqrt{x^2-a^2}\, dx$

: $\int \sqrt{x^2-a^2}\, dx$; Trigonometric Substitution

Let $x = a\sec\theta$, $\sqrt{x^2-a^2} = a\tan\theta$

$dx = a\sec\theta\tan\theta\, d\theta$; $\theta = \sec^{-1}(x/a)$

$\int \sqrt{x^2-a^2}\, dx = \int (a\tan\theta)(a\sec\theta\tan\theta)\, d\theta$

$a^2 \int \tan\theta(\sec\theta\tan\theta)\, d\theta =$

$\int \sqrt{x^2-a^2}\, dx$

Let $u = \sqrt{x^2-a^2}$, $du = \dfrac{x}{\sqrt{x^2-a^2}}\, dx$;

$dv = dx$, $v = x$;

$\int \sqrt{x^2-a^2}\, dx = (\sqrt{x^2-a^2})(x) - \int (x)\dfrac{x}{\sqrt{x^2-a^2}}\, dx$

$\int \sqrt{x^2-a^2}\, dx = x\sqrt{x^2-a^2} - \int \dfrac{x^2-a^2+a^2}{\sqrt{x^2-a^2}}\, dx$

$\int \sqrt{x^2-a^2}\, dx = x\sqrt{x^2-a^2} - \int \dfrac{x^2-a^2}{\sqrt{x^2-a^2}}\, dx + a^2 \int \dfrac{-1}{\sqrt{x^2-a^2}}\, dx$

$\int \sqrt{x^2-a^2}\, dx = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\, dx + a^2 \int \dfrac{-dx}{\sqrt{x^2-a^2}}$

Let $I = \int \sqrt{x^2-a^2}\, dx$

$I = x\sqrt{x^2-a^2} - I + a^2 \int \dfrac{-dx}{\sqrt{x^2-a^2}}$

$2I = x\sqrt{x^2-a^2} + a^2 \int \dfrac{-dx}{\sqrt{x^2-a^2}}$

$$I = \boxed{\frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C}$$

(iii) $\int \sqrt{4-5x^2}\, dx$

: Trigonometric substitution

Case $\sqrt{a^2-x^2}$, $x = a\sin\theta$, also $\sqrt{a^2-x^2} = a\cos\theta$

Let $\sqrt{5}x = 2\sin\theta$, because we can write $\int\sqrt{2^2-(\sqrt{5}x)^2}$, in standard formed.

So here a is 2 and x is $\sqrt{5}x$, where $\sqrt{5}$ is an coefficient.

$\sqrt{5}x = 2\sin\theta$, $dx = \frac{2}{\sqrt{5}}\cos\theta\, d\theta$, $\theta = \sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)$, $\sqrt{4-5x^2} = 2\cos\theta$

$$\int\sqrt{4-5x^2}\,dx = \int\sqrt{4-(\sqrt{5}x)^2}\,dx = \int\sqrt{4-(2\sin\theta)^2}\,dx$$

$$= \int\sqrt{4-4\sin^2\theta}\,dx = \int 2\sqrt{1-\sin^2\theta}\,dx$$

$$= \int 2\sqrt{\cos^2\theta}\,dx = \int 2\cos\theta\,dx$$

$$= 2\int\cos\theta\left(\frac{2}{\sqrt{5}}\cos\theta\right)d\theta$$

$$= \frac{4}{\sqrt{5}}\int\cos^2\theta\,d\theta = \frac{4}{\sqrt{5}}\int\frac{\cos 2\theta+1}{2}\,d\theta$$

$$= \frac{2}{\sqrt{5}}\int(\cos 2\theta+1)\,d\theta = \frac{2}{\sqrt{5}}\left[\frac{\sin 2\theta}{2}+\theta\right]+c$$

$$= \frac{2}{\sqrt{5}}\left[\frac{2\sin\theta\cos\theta}{2}+\theta\right]+c$$

$$= \frac{2}{\sqrt{5}}(\sin\theta\cos\theta)+\frac{2}{\sqrt{5}}\theta+c$$

$\Rightarrow \sqrt{5}x = 2\sin\theta$, $2\cos\theta = \sqrt{4-5x^2}$, $\theta = \sin^{-1}(\sqrt{5}x/2)$

$$= \frac{2}{\sqrt{5}}\left(\frac{\sqrt{5}x}{2}\right)\frac{\sqrt{4-5x^2}}{2}+\frac{2}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)+c$$

$$= \boxed{\frac{x\sqrt{4-5x^2}}{2}+\frac{2}{\sqrt{5}}\sin^{-1}\left(\frac{\sqrt{5}x}{2}\right)+c}$$

(iv) $\int \sqrt{3-4x^2}\, dx$

: $\int \sqrt{3-4x^2}\, dx = \int \sqrt{(\sqrt{3})^2 - (2x)^2}\, dx$

Trigonometric substitution

Let $2x = \sqrt{3}\sin\theta$, $dx = \frac{\sqrt{3}}{2}\cos\theta\, d\theta$, $\theta = \sin^{-1}\left(\frac{2x}{\sqrt{3}}\right)$, $\sqrt{3-4x^2} = \sqrt{3}\cos\theta$

$$\int \sqrt{(\sqrt{3})^2 - (\sqrt{3}\sin\theta)^2}\, dx = \int \sqrt{(\sqrt{3})^2 - (\sqrt{3})^2\sin^2\theta}\, dx$$

$$= \int \sqrt{(\sqrt{3})^2(1-\sin^2\theta)}\, dx = \sqrt{3}\int \sqrt{1-\sin^2\theta}\, dx$$

$$= \sqrt{3}\int \sqrt{\cos^2\theta}\, dx = \sqrt{3}\int \cos\theta\, dx = \sqrt{3}\int \cos\theta\left(\frac{\sqrt{3}}{2}\cos\theta\, d\theta\right)$$

$$= \frac{3}{2}\int \cos^2\theta\, d\theta = \frac{3}{2}\int \frac{\cos 2\theta + 1}{2}\, d\theta$$

$$=$$

$$= \frac{3}{4}\int (\cos 2\theta + 1)\, d\theta = \frac{3}{4}\left[\frac{\sin 2\theta}{2} + \theta\right] + C$$

$$= \frac{3}{4}\left[\frac{2\sin\theta\cos\theta}{2} + \theta\right] + C = \frac{3}{4}(\sin\theta)(\cos\theta) + \frac{3\theta}{4} + C$$

$$= \frac{3}{4}\left(\frac{2x}{\sqrt{3}}\right)\left(\frac{\sqrt{3-4x^2}}{\sqrt{3}}\right) + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

$$= \frac{\cancel{3}}{\cancel{4}}\left(\frac{\cancel{2}x\sqrt{3-4x^2}}{\cancel{3}}\right) + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

$$= \frac{x\sqrt{3-4x^2}}{2} + \frac{3}{4}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

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(v) $\int \sqrt{x^2+4}\, dx$

: $\int \sqrt{x^2+4}\, dx = \int \sqrt{4+x^2}\, dx$

Trigonometric Substitution

$x = 2\tan\theta$, $dx = 2\sec^2\theta\, d\theta$, $\theta = \tan^{-1}\left(\frac{x}{2}\right)$, $\sqrt{x^2+4} = 2\sec\theta$,

$$\int \sqrt{x^2+4}\, dx = \int \sqrt{(2\tan\theta)^2+4}\, dx = \frac{2}{2}\int \sqrt{4\tan^2\theta+4}\, dx$$

$$= \int \sqrt{4\tan^2\theta+4}\, dx$$

$$= 2\int \sqrt{\tan^2\theta+1}\, dx = 2\int \sec\theta\, dx$$

$$= 2\int \sec\theta \cdot (2\sec^2\theta)\, d\theta$$

$$= 4\int \sec\theta \cdot \sec^2\theta\, d\theta$$

$4\int \sec\theta \cdot \sec^2\theta\, d\theta \Rightarrow$ Let $u = \sec\theta$, $du = \sec\theta\tan\theta\, d\theta$

$dv = \sec^2\theta$, $v = \tan\theta$;

$$4\int \sec^2\theta\, d\theta = (\sec\theta)(\tan\theta) - \int (\tan\theta)(\sec\theta\tan\theta)\, d\theta$$

$$4\int \sec^2\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta$$

$$4\int \sec^2\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\, d\theta$$

$$4\int \sec^2\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$$

Let $\int \sec^2\theta\, d\theta = I$

$$4I = \sec\theta\tan\theta - I + \int \sec\theta\, d\theta$$

$$5I = \sec\theta\tan\theta + \int \sec\theta\, d\theta$$

$$5I = \sec\theta\tan\theta + \int \sec\theta\left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta}\right) d\theta$$

$$5I = \sec\theta\tan\theta + \int \frac{\sec^2\theta + \sec\theta\tan\theta}{\sec\theta + \tan\theta}\, d\theta$$

Let $u = \sec\theta + \tan\theta$, $du = \sec\theta\tan\theta + \sec^2\theta\, d\theta$

$$5I = \sec\theta\tan\theta + \int \frac{du}{u}$$

$$5I = \sec\theta\tan\theta + \ln|u| + c$$

$$5I = (\sec\theta)\tan\theta + \ln|\sec\theta + \tan\theta| + c$$

$$5I = \left(\frac{\sqrt{x^2+4}}{2}\right)\left(\frac{x}{2}\right) + \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| + c$$

$$I = \frac{x\sqrt{x^2+4}}{20} + \frac{\ln}{20}\left|x + \sqrt{x^2+4}\right| + c$$

(vi) $\int x^2 e^{ax}\,dx$

: Tabular Integration

Let $u = x^2$, $dv = e^{ax}\,dx$

$\frac{d}{dx}(u)$		$\int dv$
x^2	+	e^{ax}
$2x$	−	e^{ax}/a
2	+	e^{ax}/a^2
0	−	e^{ax}/a^3

$$+\frac{x^2 e^{ax}}{a} - \frac{2x\,e^{ax}}{a^2} + \frac{2e^{ax}}{a^3} + c$$

$$\boxed{\frac{e^{ax}}{a}\left(x^2 - \frac{2x}{a} + \frac{2}{a^2}\right) + c}$$

Triple Integrals in cartesian coordinates (Sect 15.4)

- Triple integrals in arbitrary domains
- Examples: changing the order of integration
- The average value of a function in a region in space.
- Triple integrals in arbitrary domains

• Review: Triple integrals in arbitrary domains

Theorem

If $f: D \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous in the domain

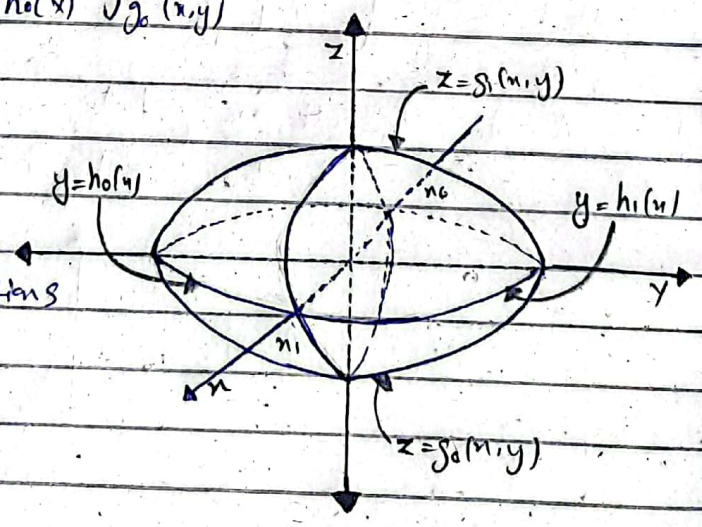
$$D = \{x \in [x_0, x_1],\ y \in [h_0(x), h_1(x)],\ z \in [g_0(x,y), g_1(x,y)]\},$$

where $g_0, g_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_0, h_1 : \mathbb{R} \to \mathbb{R}$ are continuous, then the triple integral of the function f in the region D is given by

$$\iiint_D f\, dV = \int_{x_0}^{x_1} \int_{h_0(x)}^{h_1(x)} \int_{g_0(x,y)}^{g_1(x,y)} f(x,y,z)\, dz\, dy\, dx$$

Example:

In the case that D is an ellipsoid, the figure represents the graph of functions g_1, g_0 and h_1, h_0.



- Triple Integration in Cartesian Coordinates
  - changing the order of Integration

Example

Change the order of integration in triple integral

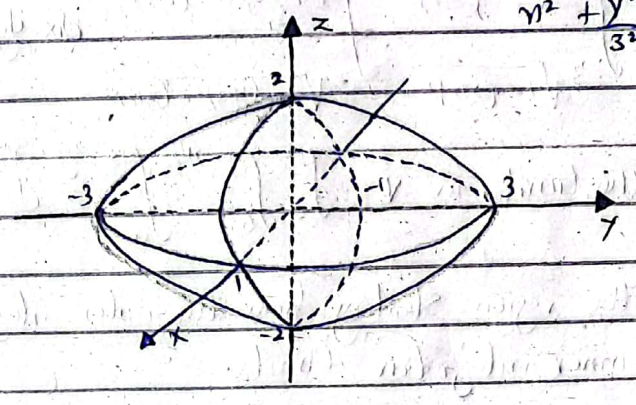
$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\,dy\,dx$$

Solution: First: Sketch the integration Region

Start from the outer integration limits to the inner limits.

- Limits in x: $x \in [-1, 1]$
- Limits in y: $|y| \le 3\sqrt{1-x^2}$, So, $x^2 + \frac{y^2}{3^2} \le 1$
- The limits in z: $|z| \le 2\sqrt{1 - x^2 - \frac{y^2}{3^2}}$, So

$$x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} \le 1$$



Example: Change the order of integration in the triple integral

$$V = \int_{-1}^{1} \int_{-3\sqrt{1-x^2}}^{3\sqrt{1-x^2}} \int_{-2\sqrt{1-x^2-(y/3)^2}}^{2\sqrt{1-x^2-(y/3)^2}} dz\, dy\, dx$$

Solution: Region $x^2 + \frac{y^2}{3^2} + \frac{z^2}{2^2} \leq 1$. we conclude:

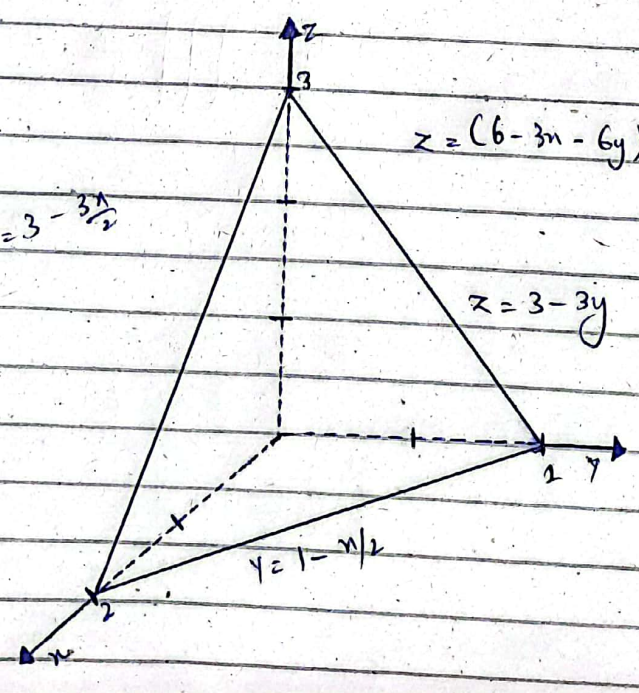
$$V = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy\, dz\, dx$$

$$V = \int_{-2}^{2} \int_{-\sqrt{1-(z/2)^2}}^{\sqrt{1-(z/2)^2}} \int_{-3\sqrt{1-x^2-(z/2)^2}}^{3\sqrt{1-x^2-(z/2)^2}} dy\, dx\, dz$$

$$V = \int_{-2}^{2} \int_{-3\sqrt{1-(z/2)^2}}^{3\sqrt{1-(z/2)^2}} \int_{-\sqrt{1-(y/3)^2-(z/2)^2}}^{\sqrt{1-(y/3)^2-(z/2)^2}} dx\, dy\, dz$$

Example: Interchange the limits in $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx$

Solution: Sketch the region starting from the outer integration limits to the inner integration limits.



- $x \in [0, 2]$
- $y \in [0, 1 - \frac{x}{2}]$ So the upper limit is $y = 1 - \frac{x}{2}$.
- $z \in \left[0, 3 - \frac{3x}{2} - 3y\right]$ So the upper limit is the plane

$z = 3 - \frac{3x}{2} - 3y$. This plane contains the points

$(2, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$.

Inter change the limits in $V = \int_0^2 \int_0^{1-x/2} \int_0^{3-3y-3x/2} dz\, dy\, dx$

Solution: The region: $x \geq 0$, $y \geq 0$, $z \geq 0$ and $6 \geq 3x + 6y + 2z$.
we conclude:

$$V = \int_0^3 \int_0^{1-\frac{z}{3}} \int_0^{2-2y-\frac{2z}{3}} dx\, dy\, dz$$

$$V = \int_0^1 \int_0^{3-3y} \int_0^{2-2y-\frac{2z}{3}} dx\, dz\, dy$$

$$V = \int_0^2 \int_0^{3-\frac{3x}{2}} \int_0^{1-\frac{x}{2}-\frac{z}{3}} dy\, dz\, dx$$

$$V = \int_0^3 \int_0^{2-\frac{2z}{3}} \int_0^{1-\frac{x}{2}-\frac{z}{3}} dy\, dx\, dz$$

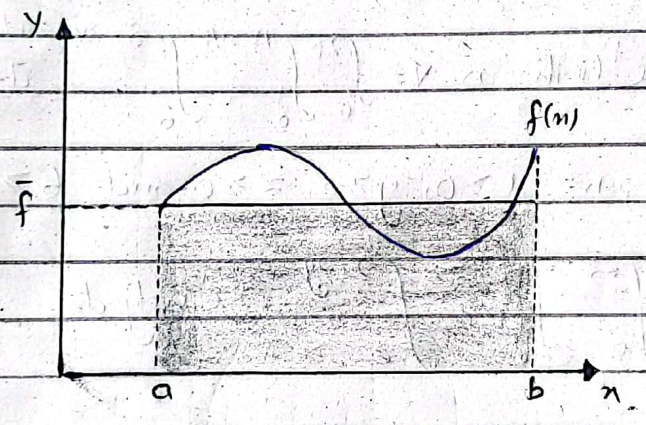
- Average value of a function in a region in space.

Review: The average value of a single function

Definition

The average of a function $f: [a,b] \to \mathbb{R}$ on the interval $[a,b]$, denoted by $\bar{f}$, is given by

$$\bar{f} = \frac{1}{(b-a)} \int_a^b f(x)\,dx$$



The average of a function $f: R \subset \mathbb{R}^3 \to$ on the region R with volume V, denoted by $\bar{f}$, is given by

$$\bar{f} = \frac{1}{V} \iiint_R f\,dV$$

- Example: Find the average of $f(x,y,z) = xyz$ in the first octant, bounded by the planes $x=1$, $y=2$, $z=3$;

Solution: The volume of the rectangular integration region is

$$V = \int_0^1 \int_0^2 \int_0^3 dz\, dy\, dx \Rightarrow V = 6$$

The average function f is:

$$\bar{f} = \frac{1}{6} \int_0^1 \int_0^2 \int_0^3 xyz \, dz\, dy\, dx$$

$$\bar{f} = \frac{1}{6} \left[\int_0^1 x\, dx \right] \left[\int_0^2 y\, dy \right] \left[\int_0^3 z\, dz \right]$$

$$\bar{f} = \frac{1}{6} \left(\frac{x^2}{6} \Big|_0^1 \right) \left(\frac{y^2}{2} \Big|_0^2 \right) \left(\frac{z^2}{2} \Big|_0^3 \right)$$

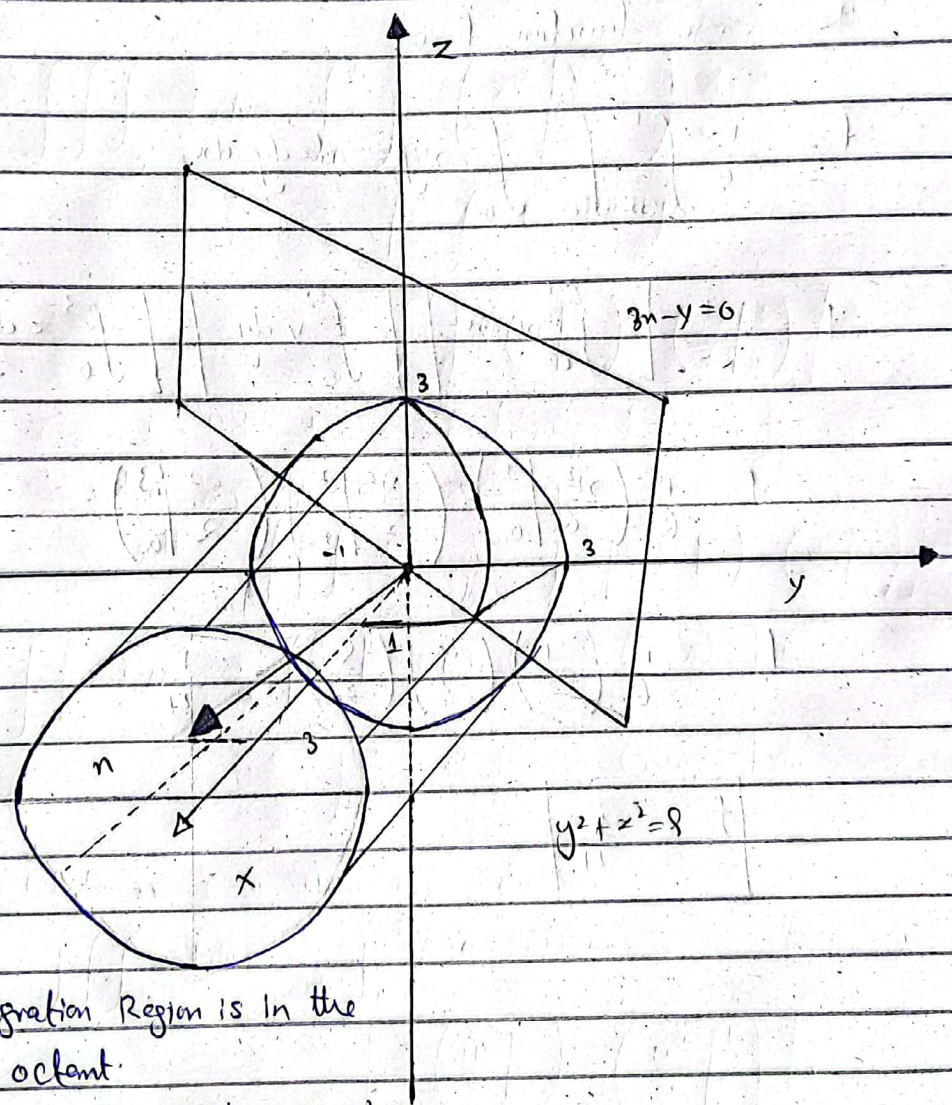
$$\bar{f} = \frac{1}{6} \left(\frac{1}{2} \right) \left(\frac{4}{2} \right) \left(\frac{9}{2} \right) = \frac{1}{4}$$

$$\boxed{\bar{f} = \frac{1}{4}}$$

- Triple integrals in arbitrary domains

Example: Compute the triple integral of $f(x,y,z)$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2+z^2$.

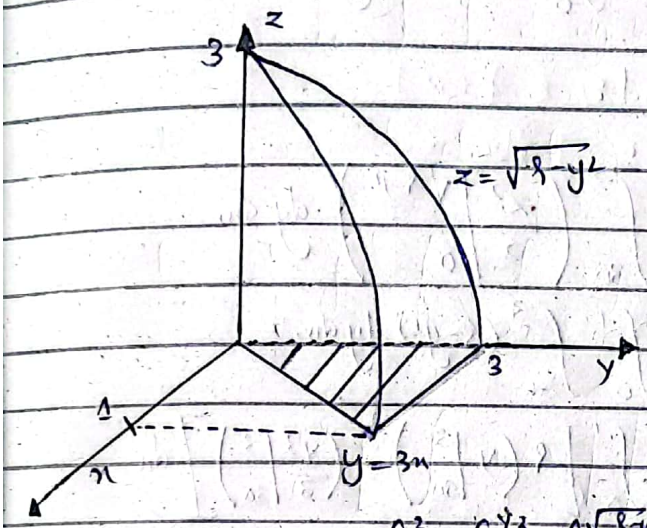
Solution : sketch the integration region.



- The Integration Region is in the first octant.
- It is inside the cylinder $y^2+z^2 \leq 9$
- It is on one side of the plane $3x-y=0$. The plane has normal vector $n = \langle 3, -1, 0 \rangle$ and containing $(0,0,0)$.

- Example: Compute the triple integral of $f(x,y,z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$.

Solution: We have found the region:



The integration limits are:

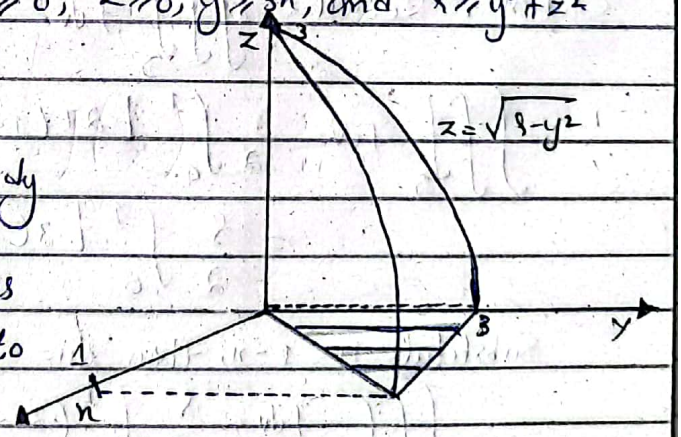
- Limits in z:
  $0 \leq z \leq \sqrt{9-y^2}$
- Limits in x: $0 \leq x \leq y/3$
- Limits in y: $0 \leq y \leq 3$

We obtain $I = \int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z \, dz \, dx \, dy$

- Example: Compute the triple integral of $f(x,y,z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $9 \geq y^2 + z^2$

Solution: Recall

$$\int_0^3 \int_0^{y/3} \int_0^{\sqrt{9-y^2}} z \, dz \, dx \, dy$$



For practice purpose only, let us change the integration order to $dz \, dy \, dx$:

The result is $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z \, dz \, dy \, dx$

- Example: Compute the triple integral of $f(x,y,z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$, and $8 \geq y^2 + z^2$.

Solution: Recall $I = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{8-y^2}} z \, dz \, dy \, dx$

We now compute the integral:

$$\iiint_D f \, dv = \int_0^1 \int_{3x}^3 \left(\frac{z^2}{2} \Big|_0^{\sqrt{8-y^2}} \right) dy \, dx,$$

$$= \frac{1}{2} \int_0^1 \int_{3x}^3 (8-y^2) \, dy \, dx,$$

$$= \frac{1}{2} \int_0^1 \left[8 \left(y \Big|_{3x}^3 \right) - \left(\frac{y^3}{3} \Big|_{3x}^3 \right) \right] dx$$

- Example: Compute the integral of $f(x,y,z) = z$ in the region bounded by $x \geq 0$, $z \geq 0$, $y \geq 3x$ and $8 \geq y^2 + z^2$

Solution: Recall: $\iiint_D f \, dv = \frac{1}{2} \int_0^1 \left[8 \left(y \Big|_{3x}^3 \right) - \left(\frac{y^3}{3} \Big|_{3x}^3 \right) \right] dx$

$$\iiint_D f \, dv = \frac{1}{2} \int_0^1 \left[27(1-x) - 8(1-x)^3 \right] dx$$

$$= \frac{8}{2} \int_0^1 \left[3(1-x) - (1-x)^3 \right] dx$$

Substitute $u = 1-x$, then $du = -dx$ so,

$$\iiint_D f \, dv = \frac{8}{2} \int_0^1 (3u - u^3) \, du$$

$$= \frac{8}{2} \left[3 \left(\frac{u^2}{2} \Big|_0^1 \right) - \left(\frac{u^4}{4} \Big|_0^1 \right) \right],$$

$$= \frac{8}{2} \left(\frac{3}{2} - \frac{1}{4} \right)$$

We conclude $\iiint_D f \, dv = \frac{45}{8}$

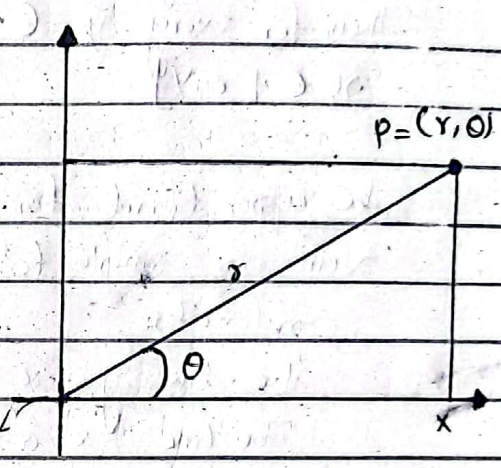
Integrals in Cylindrical, spherical Coordinates

- Integration in cylindrical coordinates
  - Review: Polar Coordinates
  - Cylindrical coordinates in space
  - Triple Integral in cylindrical coordinates

Review: Polar Coordinates in plane

Definition

The polar coordinates of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ) defined by the picture.



Theorem (Cartesian-polar transformations).

The cartesian coordinates of a point $P = (r, \theta)$ in the first quadrant are given by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

The polar coordinates of a point $P = (x, y)$ in the first quadrant are given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Recall: Polar coordinates in a plane

- Example: Express in polar coordinates the integral $I = \int_0^2 \int_0^y x \, dx \, dy$

Solution: Recall: $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

More often than not helps to sketch the integration region.

The outer integration limit: $y \in [0,2]$

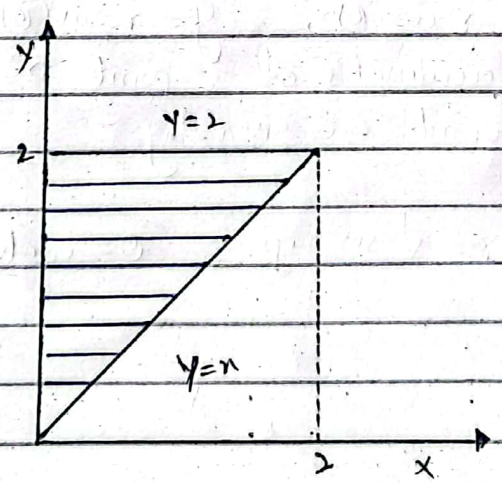
Then for every $y \in [0,2]$ the x coordinates satisfies $x \in [0,y]$

The upper limit for x is the curve $y = x$.

Now is simple to describe this domain in polar coordinates:

The line $y = x$ is $\theta_0 = \pi/4$;

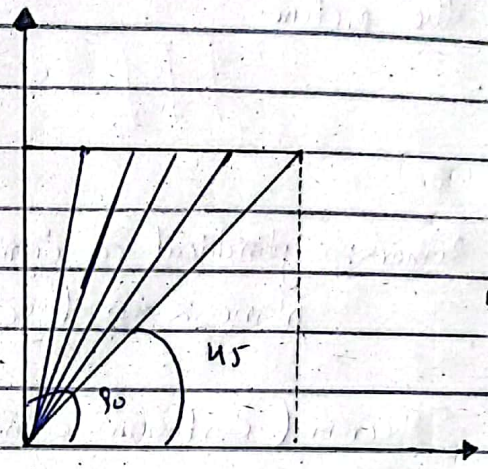
The line $x = 0$ is $\theta_1 = \pi/2$;



- Example: Express in polar coordinates the integral $I = \int_0^2 \int_0^y x \, dx \, dy$.

Solution: Recall: $x = r\cos(\theta)$, $y = r\sin\theta$, $\theta_0 = \pi/4$, $\theta_1 = \pi/2$;
The lower integration limit in r is $r = 0$
The upper integration limit is $y = 2$,
that is, $2 = y = r\sin(\theta)$.

Hence $r = 2/\sin(\theta)$.



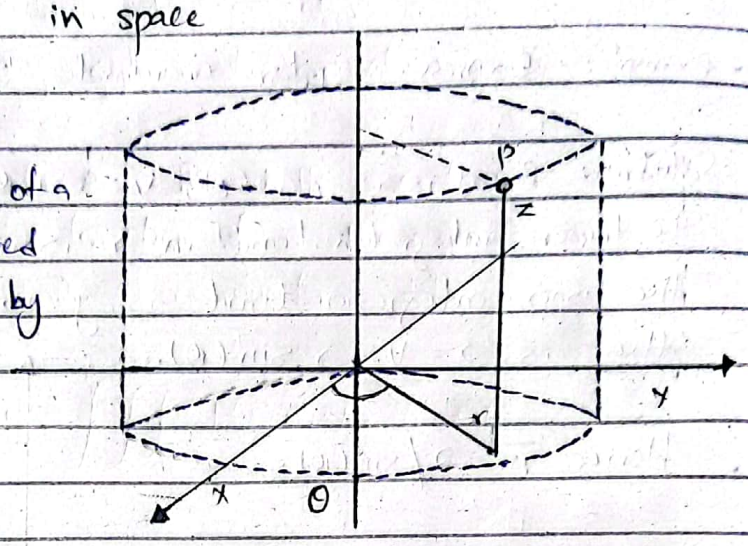
We conclude:

$$\int_0^2 \int_0^y x \, dx \, dy = \int_{\pi/4}^{\pi/2} \int_0^{2/\sin(\theta)} r\cos(\theta)\,(r\,dr)\,d\theta$$

- Cylindrical coordinates in space

Definition

The cylindrical coordinates of a point $P \in \mathbb{R}^3$ is the ordered triple (r, θ, z) defined by the picture



Remark: Cylindrical coordinates are just polar coordinates on the plane $z=0$ together with the vertical coordinate z.

Theorem (Cartesian-cylindrical tranformations)

The cartesian coordinates of a point $P=(r, \theta, z)$ in the first quadrant are given by $x = r\cos(\theta)$, $y = r\sin(\theta)$, and $z=z$.

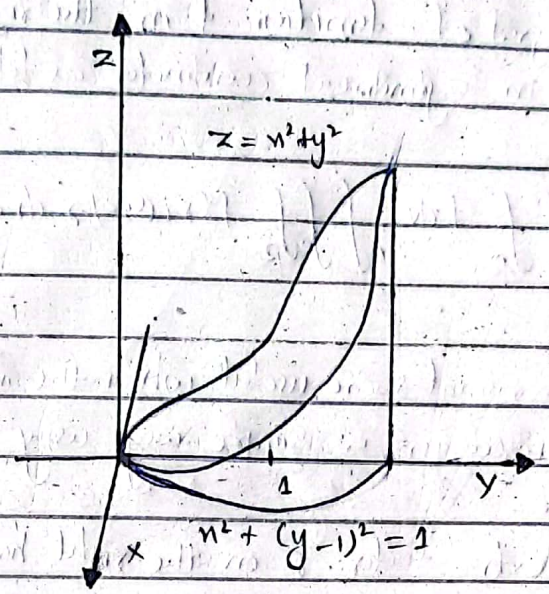
The cylindrical coordinates of a point $P=(x,y,z)$ in the first quadrant are given by $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$ and $z=z$.

- Example:

Use cylindrical coordinates to describe the region

$R = \{(x, y, z) : x^2 + (y-1)^2 \leq 1,\ 0 \leq z \leq x^2 + y^2\}$.

Solution: We first sketch the region.



The base of the region is at $z=0$, given by the disk $x^2 + (y-1)^2 \leq 1$.

The top of the region is the paraboloid $z = x^2+y^2$

In cylindrical coordinates: $z = x^2+y^2 \iff z = r^2$, and

$$x^2+y^2 - 2y + 1 \leq 1 \iff r^2 - 2r\sin(\theta) \leq 0 \iff 2\sin(\theta)$$

Hence: $R = \{(r, \theta, z) : \theta \in [0, \pi],\ r \in [0, 2\sin(\theta)],\ z \in [0, r^2]\}$.

- Triple integral in cylindrical coordinates
  - Triple integrals using cylindrical coordinates.

Theorem

If the function $f: R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in cylindrical coordinates as follows,

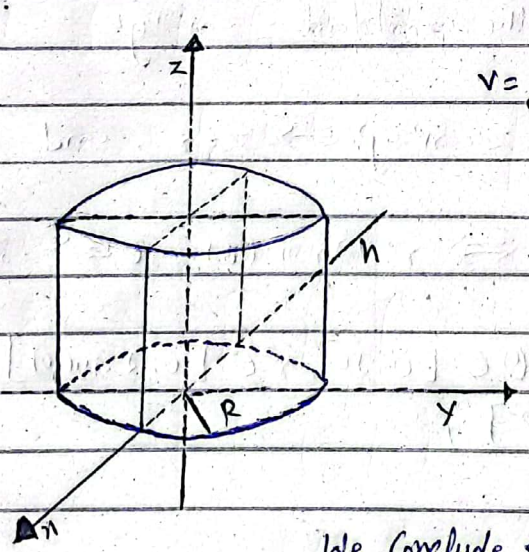
$$\iiint_R f\,dv = \iiint_R f(r,\theta,z)\, r\, dr\, d\theta\, dz$$

Remark:

- Cylindrical coordinates are useful when the integration region R is described in a simple way using cylindrical coordinates.
- Notice the extra factor r on the right-hand side.

Example: Find the volume of a cylinder of radius R and height h.

Solution: $R = \{(r,\theta,z) : \theta \in [0, 2\pi], r \in [0, 2\pi],$
$r \in [0,R], z \in [0,h]\}$.



$$V = \int_0^{2\pi}\int_0^R\int_0^h dz\,(r\,dr)\,d\theta,$$

$$= h\int_0^{2\pi}\int_0^R r\,dr\,d\theta,$$

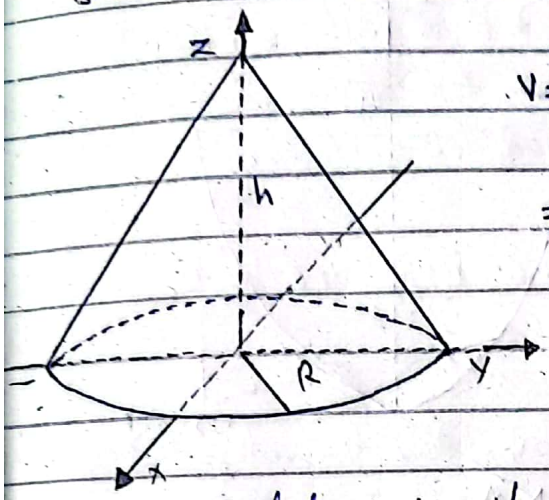
$$= h\frac{R^2}{2}\int_0^{2\pi} d\theta,$$

$$= h\frac{R^2}{2}\,2\pi,$$

We conclude: $V = \pi R^2 h$

- Example: Find the volume of a cone of base radius R and height h.

Solution: $R = \{\theta \in [0, 2\pi],\ r \in [0, R],\ z \in [0, -\frac{h}{R} r + h]\}$.



$$V = \int_0^{2\pi} \int_0^R \int_0^{h(1 - r/R)} dz\,(r\,dr)\,d\theta,$$

$$= h \int_0^{2\pi} \int_0^R \left(1 - \frac{r}{R}\right) r\,dr\,d\theta$$

$$= h \int_0^{2\pi} \int_0^R \left(r - \frac{r^2}{R}\right) dr\,d\theta$$

$$= h\left(\frac{R^2}{2} - \frac{R^3}{3R}\right) \int_0^{2\pi} d\theta = 2\pi h R^2 \frac{1}{6}$$

We conclude: $V = \frac{1}{3} \pi r^2 h$

- Example: sketch the region with volume $V = \int_0^{\pi/2} \int_0^2 \int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$

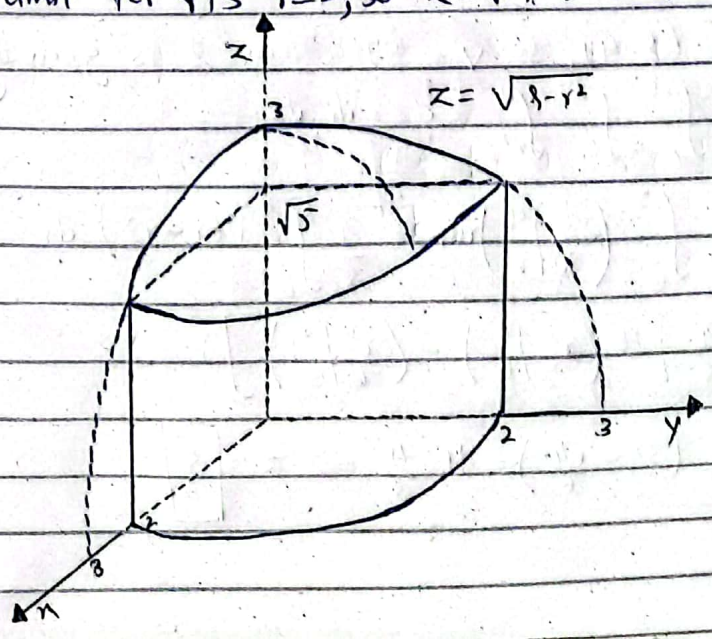
Solution: The integration region is

$R = \{(r, \theta, z) : \theta \in [0, \pi/2],\ r \in [0, 2],\ z \in [0, \sqrt{9-r^2}]\}$

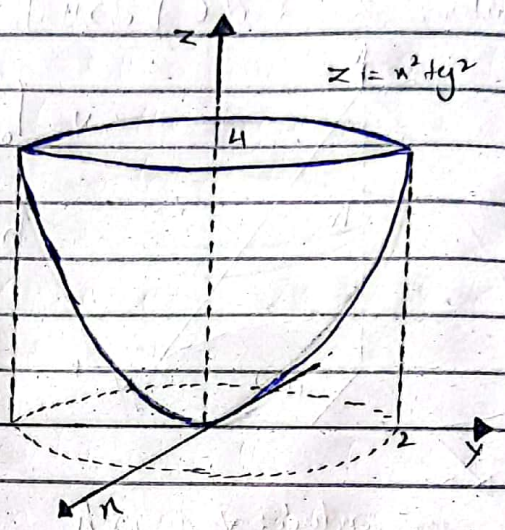
where upper boundary is a sphere, since

$$z^2 = 9 - r^2 \iff x^2 + y^2 + z^2 = 3^2$$

The upper limit for r is $r = 2$, so $z = \sqrt{9 - 2^2} = \sqrt{5}$.



- Example: Find the centroid vector $\bar{r} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region is space $R = \{(x,y,z) : x^2+y^2 \le 2^2,$
$x^2+y^2 \le z \le 4\}$.



Find the centroid vector $\bar{r} = \langle \bar{x}, \bar{y}, \bar{z} \rangle$ of the region in space
$R = \{(x,y,z) : x^2+y^2 \le 2^2,$
$x^2+y^2 \le z \le 4\}$.

The symmetry of the region implies $\bar{x} = 0$ and $\bar{y} = 0$ (we verify this result later on) we only need to compute $\bar{z}$.

Since $\bar{z} = \frac{1}{V} \iiint_R z \, dv$, we start computing the total volume V.

we use cylindrical coordinates

$$V = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 dz \, r \, dr \, d\theta = 2\pi \int_0^2 \left(z \Big|_{r^2}^4 \right) r \, dr = 2\pi \int_0^2 (4r - r^3) dr$$

$$= 2\pi \int_0^2 (4r - r^3) dr = 2\pi \left[4 \left(\frac{r^2}{2} \Big|_0^2 \right) - \left(\frac{r^4}{4} \Big|_0^2 \right) \right].$$

Hence $V = 2\pi(8-4)$, so $V = 8\pi$. Then $\bar{z}$ is given by,

$$\bar{z} = \frac{1}{8\pi} \int_0^{2\pi} \int_0^2 \int_{r^2}^4 z \, dz \, r \, dr \, d\theta$$

$$\bar{z} = \frac{2\pi}{8\pi} \int_0^2 \left(\frac{z^2}{2} \Big|_{r^2}^4 \right) r \, dr = \frac{1}{8} \int_0^2 (16r - r^5) \, dr$$

$$\bar{z} = \frac{1}{8} \left[16 \left(\frac{r^2}{2} \Big|_0^2 \right) - \left(\frac{r^6}{6} \Big|_0^2 \right) \right],$$

$$\bar{z} = \frac{1}{8} \left(32 - \frac{64}{6} \right) = 4 - \frac{4}{3} \Rightarrow \bar{z} = \boxed{\frac{8}{3}}$$

we obtained $\bar{z} = \frac{8}{3}$

It is simple to see that $\bar{x} = 0$ and $\bar{y} = 0$.

For example:

$$\bar{x} = \frac{1}{8\pi} \int_0^{2\pi} \int_0^2 \int_{r^2}^4 [r\cos(\theta)]\, dz\, r\, dr\, d\theta$$

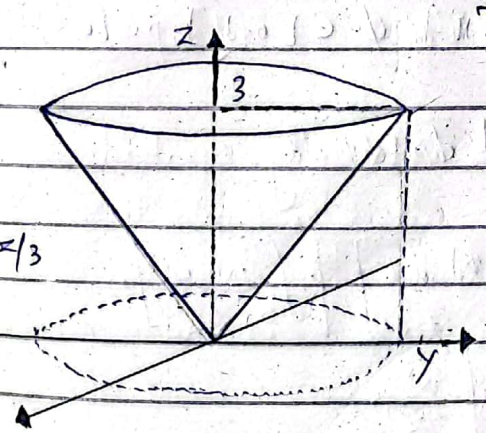
$$= \frac{1}{8\pi} \left[\int_0^{2\pi} \cos(\theta)\, d\theta \right] \left[\int_0^2 \int_{r^2}^4 dz\, r^2\, dr \right]$$

But $\int_0^{2\pi} \cos(\theta)\, d\theta = \sin(2\pi) - \sin(0) = 0$, so $\bar{x} = 0$

A similar calculation shows $\bar{y} = 0$. Hence $\bar{r} = \langle 0, 0, 8/3 \rangle$.

- Example: Change the integration order and compute the integral $I = \int_0^{2\pi} \int_0^3 \int_0^{z/3} r^3\, dr\, dz\, d\theta$

<u>Solution</u>:



$$I = \int_0^{2\pi} \int_0^1 \int_{3r}^3 dz\, r^3\, dr\, d\theta$$

$$= 2\pi \int_0^1 \left(z \Big|_{3r}^3 \right) r^3\, dr$$

$$= 2\pi \int_0^1 3(r^3 - r^4)\, dr$$

$$= 6\pi \left(\frac{r^4}{4} - \frac{r^5}{5} \right) \Big|_0^1$$

So, $I = 6\pi \frac{1}{20}$, that is $I = \frac{3\pi}{10}$

Integrals in Cylindrical, Spherical Coordinates

- Triple integral in spherical coordinates

Theorem

If the function $f: R \subset \mathbb{R}^3 \to \mathbb{R}$ is continuous, then the triple integral of function f in the region R can be expressed in spherical coordinates as follows,

$$\iiint_R f\, dv = \iiint_R f(\rho, \phi, \theta)\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

▶ Remark:

- Spherical coordinates are useful when the integration region R is described in a simple way using spherical coordinates.
- Notice the extra factor $\rho^2 \sin(\phi)$ on the right hand-side.

- Example: Find the volume of a sphere of Radius R

Solution: Sphere $S = \{\theta \in [0, 2\pi], \phi \in [0, \pi], \rho \in [0, R]\}$.

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

$$= \left[\int_0^{2\pi} d\theta\right]\left[\int_0^{\pi} \sin(\phi)\, d\phi\right]\left[\int_0^{R} \rho^2\, d\rho\right],$$

$$= 2\pi \left[-\cos(\phi)\Big|_0^{\pi}\right]\frac{R^3}{3},$$

$$= 2\pi\left[-\cos(\pi) + \cos(0)\right]\frac{R^3}{3};$$

hence: $V = \frac{4}{3}\pi R^3$

- Example: Use spherical coordinates to find the volume below the sphere $x^2+y^2+z^2=1$ and above the cone $z=\sqrt{x^2+y^2}$.

Solution: $R=\{(\rho,\phi,\theta) : \theta\in[0,2\pi], \phi\in[0,\frac{\pi}{4}], \rho\in[0,1]\}$

The calculation is simple, the region is a simple section of a sphere.

$$V=\int_0^{2\pi}\int_0^{\pi/4}\int_0^{1}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$$

$$=\left[\int_0^{2\pi}d\theta\right]\left[\int_0^{\pi/4}\sin(\phi)\,d\phi\right]\left[\int_0^{1}\rho^2\,d\rho\right],$$

$$=2\pi\left[-\cos(\phi)\Big|_0^{\pi/4}\right]\left(\frac{\rho^3}{3}\Big|_0^1\right),$$

$$=2\pi\left[-\frac{\sqrt{2}}{2}+1\right]\frac{1}{3}\Rightarrow\boxed{V=\frac{\pi}{3}(2-\sqrt{2})}$$

- Example: Find the integral of $f(x,y,z)=e^{(x^2+y^2+z^2)^{3/2}}$ in the region $R=\{x\geq 0, y\geq 0, z\geq 0, x^2+y^2+z^2\leq 1\}$ using spherical coordinates.

Solution: $R=\{\theta\in[0,\frac{\pi}{2}], \phi\in[0,\frac{\pi}{2}], \rho\in[0,1]\}$

Hence, $$\iiint_R f\,dv=\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1}e^{\rho^3}\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta,$$

$$=\left[\int_0^{\pi/2}d\theta\right]\left[\int_0^{\pi/2}\sin(\phi)\,d\phi\right]\left[\int_0^{1}e^{\rho^3}\rho^2\,d\rho\right]$$

Use substitution: $u=\rho^3$, hence $du=3\rho^2\,d\rho$, so

$$\iiint_R f\,dv=\frac{\pi}{2}\left[-\cos(\phi)\Big|_0^{\pi/2}\right]\int_0^1\frac{e^u}{3}\,du$$

$$\boxed{\iiint_R f\,dv=\frac{\pi}{6}(e-1)}$$

TRIPLE INTEGRALS

- Theory
- Meaning
- Rules for solving
- Examples

Theory

As the name implies, triple integrals are 3 successive integrations, used to calculate a volume, or to integrate in a 4th dimension, over 3 independent dimensions.

Examples:

$$\iiint_{0}^{V} f(x,y,z)\, dv$$

$$\int_{1}^{5} \int_{0}^{3x-2} \int_{\sin(x+y)}^{\cos(x+y)} (2x + e^{yz})\, dz\, dy\, dx$$

$$\int_{0}^{2\pi} \int_{0}^{2} \int_{-3}^{3} r\, dz\, dr\, d\theta$$

- To understand triple integrals, it is extremely important to have an understanding of double integrals, coordinate geometery in 3 dimensions, and polar (cylindrical) coordinates. Sums of triple integrals are based on these topics and cannot be solved without this prior knowledge.

▸ Meaning

Just as a single integral over a curve represents an area (2D), and a double integral over a curve represents a volume (3D), a triple integral represents a summation in a hypothetical 4th dimension. To understand this, imagine a slightly different scenario, where the first 3 dimensions are space, space, and time, and that the fourth dimension is the third space dimension: suppose you have an integral of z where z is a function of x, y, and t, and x and y are also functions of t. integrating z with respect to x and y gives us the volume as a function of t. If we plug in a value for t, this gives us the numerical volume at a particular instance of time

- However, as the volume is a function of t, it changes with respect to time. Thus, if we integrate this result with respect to t, we are adding up all different volumes over a period of time. For example, imagine a balloon that is being inflated. At any particular instant of time, we can use a double integral to calculate its volume. However, if we wanted sum up all the volumes over the entire inflation process, we must use a triple integral as described above.

- Triple integrals can also be used to represent a volume, in the same way that a double integral can be used to represent an area. In the triple integral

$$\int_c^d \int_a^b \int_0^z f(x,y,z)\, dz\, dy\, dx$$

If $f(x,y,z) = 1$ then this triple integral is the same as

$$\int_c^d \int_a^b z(x,y)\, dy\, dx$$

which is simply the volume under the surface represented by $z(x,y)$

▶ Rules for Solving

In general, if you are evaluating a triple integral of $f(x,y,z)$ over a volume, V, by properly choosing the limits, you can integrate with respect to the 3 variables in any order. For example, if V represents a rectangular box, then for x, y, and z, the limits of integration will all be constants and the order doesn't matter at all.

$$\int_e^f \int_c^d \int_a^b f\, dz\, dy\, dx = \int_c^d \int_a^b \int_e^f f\, dx\, dz\, dy = \dots$$

and so on, for 6 different possible orders of integration.

Now Consider the case where you want to find the triple integral

$$\iiint_V f(x,y,z)\, dV$$

where V is the volume under the plane $z+2x+3y=6$, in the first quadrant. Since V is in the first quadrant, it is bounded by the 3 planes $x=0$, $y=0$, $z=0$ in addition to the plane whose equation is given above.

Clearly, the shape of V is a tetrahedron with vertices $(3,0,0)$, $(0,2,0)$, and $(0,0,6)$, $(0,0,0)$.

One way to set up the integral is to first consider the z direction. In this direction we enter through $z=0$ and leave through $z=6-2x-3y$.
If we do this first, then we can consider the x or y direction next. If y, we enter through $y=0$, and leave through $y=-\frac{2}{3}x+2$. Thus, x must be given the limits 0 to 3, and our triple integral is:

$$\int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{6-2x-3y} f(x,y,z)\, dz\, dy\, dx$$

Consider the same volume, but now first we will go through in the x direction In this direction we enter through $x=0$ and leave through $x = \frac{1}{2}(6-z-3y)$. If we enter through, If we do this first, then we can consider the z or y direction next. If y, we enter through $y=0$, and leave through $y = -\frac{1}{3}z + 2$, and z must be from 0 to 6.

Thus the integral would be

$$\int_0^6 \int_0^{-\frac{1}{3}z+2} \int_0^{\frac{1}{2}(6-z-3y)} f(x,y,z)\,dx\,dy\,dz$$

Both these integrals are correct and either one can be used to solve the sum and obtain the same value. It is important to see that choosing the order of integration depends on how the limits are chosen. However, this is entirely independent of the function $f(x,y,z)$. In the first case, z must be integrated first, since the limits were chosen such that z is dependent on (x,y). After this, since the limits were chosen such that z is dependent on (x,y). After this since y is chosen to be dependent on x, y is integrated next. Lastly, x is the most independent variable is integrated. It is important to note that the third (outermost) integral MUST always have limits of integration that are constants, and the result should be a number. If any variables are surviving after the integral is evaluated, most probably you have made a mistake somewhere.

- POLAR (CYLINDRICAL) COORDINATES:

Triple integrals can also be used with polar coordinates in the exact same way to calculate a volume, or to integrate over a volume. For example

$$\int_0^{2\pi}\int_0^2\int_{-3}^3 r\,dz\,dr\,d\theta$$

is the triple integral used to calculate the volume of a cylinder of height 6 and radius 2. With polar coordinates, usually the easiest order of integration is z, then r then θ as shown above, though it is not necessary to do it in this order.

NOTE: it is very important to remember that in polar and cylindrical coordinates, there is an extra r in the integral, just like in double integrals. It is very easy to forget this r which will lead to a wrong answer.

Examples!

1. Integrate the function $f(x,y,z) = xy$ over the volume enclose by the planes $z = x+y$ and $z = 0$, and between the surfaces $y = x^2$ and $x = y^2$

Solution

Since z is expressed as a function of (x,y), we should integrate in the z direction first. After this we consider the xy-plane. The two curves meet at $(0,0)$ and $(1,1)$. We can integrate in x or y first. If we choose y, we can see that the region begins at $y = x^2$ and ends at $y = \sqrt{x}$, and so x is between 0 and 1. Thus the integral is:

$$\int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} f(x,y,z)\, dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\, dz\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy \left(\Big[z \Big]_{z=0}^{z=x+y} \right) dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x+y)\, dy\, dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2y + y^2x)\, dy\, dx$$

$$= \int_0^1 \Big[\frac{x^2y^2}{2} + \frac{y^3x}{3} \Big]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \frac{1}{2}\int_0^1 (x^2\sqrt{x}^2 - x^2(x^2)^2)\, dx + \frac{1}{3}\int_0^1 (\sqrt{x}^3 x - (x^2)^3 x)\, dx$$

$$= \frac{1}{2}\int_0^1 (x^3 - x^6)\, dx + \frac{1}{3}\int_0^1 (x^{5/2} - x^7)\, dx$$

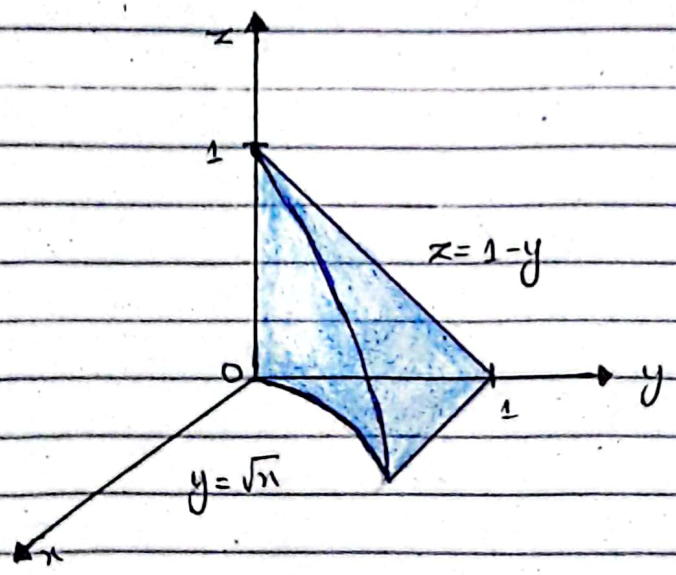
$$= \frac{1}{2}\left(\frac{1}{4} - \frac{1}{7}\right) + \frac{1}{3}\left(\frac{2}{7} - \frac{1}{8}\right) = \boxed{\frac{3}{28}}$$

2. In the following integral, exchange the order of integration of y and z:

$$\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f(x,y,z)\, dz\, dy\, dx$$

Solution:

With a problem like this, it helps to draw the figure enclose by the surfaces. From the integrals, it can be seen that z enters the volume at $z=0$ and leaves through the plane $z = 1-y$. In the xy plane, y enters through $y=\sqrt{x}$ and leaves through $y=1$. These 2 conditions alone are enough to draw the figure and see that $0 \leq x \leq 1$.



The shape is like tetrahedron with the vertices $A(0,0,0)$, $B(0,0,1)$, $C(1,1,0)$ and $D(0,1,0)$, but with a Curved Surface between vertices A, B, C.

Rewriting the equation of the plane as $y = 1 - z$, and looking at this figure, clearly, $\sqrt{x} \leq y \leq 1 - z$.
Consider the projection of this surface in the zx plane.
It is a 3-sided figure bounded by $x = 0$, $z = 0$, and $z = 1 - \sqrt{x}$.
Again, it can be seen that these conditions are sufficient to obtain the limits of x as $0 \leq x \leq 1$.
Thus, the integral is:

$$\int_0^1 \int_0^{1-\sqrt{x}} \int_{\sqrt{x}}^{1-z} f(x,y,z)\, dy\, dz\, dx$$

3. Evaluate $\iiint_E x\,dV$ where E is enclosed by $z=0$, $z=x+y+5$, $x^2+y^2=4$ and $x^2+y^2=9$.

Solution.

We will use cylindrical coordinates to easily solve this sum. Converting the given outer limits of E we get: $z=0$, $z=r\cos\theta+r\sin\theta+5$, $r=2$ and $r=3$. Since there is no limitation on the values of θ, we assume it has the values of 0 to 2π. This is also in accordance with the diagram which is obtained by drawing these surfaces. Also, we substitute $x=r\cos\theta$ in the argument of the integral.

Thus the integral is:

$$\int_0^{2\pi}\int_2^3\int_0^{r\cos\theta+r\sin\theta+5}(r\cos\theta)\,r\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_2^3(r^2\cos\theta)(r\cos\theta+r\sin\theta+5)\,dr\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{r^4}{4}\cos^2\theta+\frac{r^4}{4}\cos\theta\sin\theta+\frac{5}{3}r^3\cos\theta\right]_{r=2}^{r=3}d\theta$$

$$=\int_0^{2\pi}\left[\frac{65}{8}(1+\cos 2\theta)+\frac{65}{8}\sin(2\theta)+\frac{95}{3}\cos\theta\right]d\theta$$

$$=\frac{65}{8}\left(2\pi+0-\frac{1}{2}+0-0+\frac{1}{2}\right)+\frac{95}{3}(0)$$

$$=\boxed{\frac{65\pi}{4}}$$

Self-Help Work Sheets C41: Triple Integration

These problems are intended to give you more practice on some of the skills the chapter on Triple Integration has sought to develop. They do not cover everything so a careful review of the chapter and your class notes in also in order.

Problems for fun and practice

1) For each of the following solids give a description in rectangular coordinates in the order specified

(a) S is bounded above by $\frac{x^2}{9}+\frac{y^2}{36}+\frac{z^2}{16}=1$ and below, by $z=2$.

Descriptions interms of range on z, then y, then x.

Solution

$$\frac{x^2}{9}+\frac{y^2}{36}+\frac{z^2}{16}=1 \iff \left(\frac{x}{3}\right)^2+\left(\frac{y}{6}\right)^2+\left(\frac{z}{4}\right)^2=1$$

To find the range of x and y we substitute the value of z into the equation $\frac{x^2}{9}+\frac{y^2}{36}+\frac{z^2}{16}=1$ that will give the largest range for x and y. From inspecting the figure we see that this value is $z=2$.

For $z=2$, we have $x^2/9+y^2/36+(2)^2/16=1$ or $\frac{x^2}{9}+\frac{y^2}{36}=\frac{3}{4}$ From this we obtain:

$$2 \le z \le 4\sqrt{1-\frac{x^2}{9}-\frac{y^2}{36}}$$

$$-6\sqrt{\frac{3}{4}-\frac{x^2}{9}} \le y \le 6\sqrt{\frac{3}{4}-\frac{x^2}{9}}$$

$$-3 \le x \le 3.$$

(b) S is bounded above by $\frac{x^2}{16}+\frac{y^2}{36}+\frac{z^2}{16}=1$ and below by $\frac{y}{6}+\frac{z}{4}=1$ and below by $z=2$.

Description: in terms of range on z, then x, then y.

Solution

To find the scope of x and y we notice from the figure that the largest range of x and y in the solid occur on the plane $\frac{y}{6}+\frac{z}{4}=1$. Thus we solve this for z and substitute this into the equation $\frac{x^2}{16}+\frac{y^2}{36}+\frac{z^2}{16}=1$,

Solving $\frac{y}{6}+\frac{z}{4}=1$ for z we get $z=4\left(1-\frac{y}{6}\right)$

Substitution this value of z into $\frac{x^2}{16}+\frac{y^2}{36}+\frac{z^2}{16}=1$,

we get $\frac{x^2}{16}+\frac{y^2}{36}+\frac{16\left(1-\frac{y}{6}\right)^2}{16}=1$ or $\frac{x^2}{16}+\frac{y^2}{18}+\frac{y}{3}=1$

or $\frac{x^2}{16}+\frac{(y-3)^2}{18}=\frac{1}{2}$. From this we see that y will be maximized when $x=0$. Thus we obtain the desired description:

$$4\left(1-\frac{y}{6}\right)\le z\le 4\sqrt{1-\frac{x^2}{16}-\frac{y^2}{36}}$$

$$-2\sqrt{2}\sqrt{1-\frac{(y-3)^2}{9}}\le x\le 2\sqrt{2}\sqrt{1-\frac{(y-3)^2}{9}}$$

$$0\le y\le 6$$

2) Each of the following iterated integrals cannot be easily done in the order given. Convince yourself this is true and then convert each one to an equivalent iterated integral that can be done and evaluate it.

(a) $\int_0^2 \int_0^1 \int_y^1 \sinh(x^2)\, dz\, dy\, dx$

$= \int_0^2 \int_0^1 \int_0^x \sinh(x^2)\, dy\, dz\, dx$

$= \int_0^2 \int_0^1 x \sinh(x^2)\, dz\, dx$

$= \int_0^2 \frac{1}{2}(\cosh(1) - \cosh(0))\, dx$

$= \cosh(1) - \cosh(0)$

$= \frac{e^1 + e^{-1}}{2} - 1$

The region indicated by the integral is bounded by $x=y$, $y=0$, $x=1$, $x=0$, and $x=2$ which is indicated by the figure above. The difficulty with integrating the original triple integral is that to easily integrate $\sinh(x^2)$, we need a $x\,dx$ rather than just dx. Note that if we switch the dx and dy, we might get a x where we need it.

(b) $\int_0^2 \int_0^4 \int_z^2 yze^{x^3}\,dx\,dy\,dz$

$$= \int_0^2 \int_0^4 \int_0^x yze^{x^3}\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^4 y\frac{x^2}{2}e^{x^3}\,dy\,dx$$

$$= \int_0^2 \left(\frac{y^2}{2}\right]_0^4 \frac{x^2}{2}e^{x^3}\,dx$$

$$= 4\int_0^2 x^2e^{x^3}\,dx$$

$$= \frac{4}{3}e^{x^3}\Big]_0^2 = \frac{4}{3}(e^8-1)$$

The region is described by integral is bounded by $y=0$, $y=4$, $z=0$, $z=x$, and $x=2$. A picture of the region is indicated above. In the original integral, if we try to integrate $e^{x^3}\,dx$ we have a problem. We can easily integrate $x^2e^{x^3}$, so this suggests switching dx and dz.

3) Convert each of the following to an equivalent triple integral in a cylindrical coordinates and evaluate.

(a) $$\int_0^5 \int_0^{\sqrt{25-x^2}} \int_0^6 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2}}$$

$$= \int_0^{\pi/2} \int_0^5 \int_0^6 \frac{r\,dz\,dr\,d\theta}{r} = 15\pi$$

The region described by the integral is the set of all point (x, y, z) satisfying the three inequalities $0 \le z \le 6$, $0 \le y \le \sqrt{25-x^2}$, and $0 \le x \le 5$. It is $\frac{1}{4}$ of a cylinder as shown in the figure above.

(b) $$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\frac{x^2+y^2}{2}} \frac{z\,dz\,dy\,dx}{\sqrt{x^2+y^2}}$$

$$= \int_0^{\pi/2} \int_0^2 \int_0^{r^2/2} \frac{zr\,dz\,dr\,d\theta}{r} = \frac{8\pi}{5}$$

The region or solid is in the first octant between the paraboloid and the cylinder as indicated in the picture above. It is the set of all points (x, y, z) satisfying the three inequalities $0 \le z \le \frac{x^2+y^2}{2}$, $0 \le y \le \sqrt{4-x^2}$, and $0 \le x \le 2$.

(4) Convert each of the following to an equivalent triple integral in spherical coordinates and evaluate.

(a) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{1+x^2+y^2+z^2}$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \frac{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}{1+\rho^2}$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \sin\phi \left[1 - \frac{1}{1+\rho^2} \right] d\rho \, d\phi \, d\theta$$

$$= \left(1 - \frac{\pi}{4} \right) \int_0^{\pi/2} [\cos\phi]_0^{\pi/2} \, d\theta$$

$$= \left(1 - \frac{\pi}{4} \right) \frac{\pi}{2} = \boxed{\frac{\pi}{2} - \frac{\pi^2}{8}}$$

The solid is $\frac{1}{8}$ of the sphere with radius 1 centered at the origin as indicated in the figure.

(b) $\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} xz\,dz\,dy\,dx$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 (\rho \sin\phi \cos\theta)(\rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho^4 \cos\theta \sin^2\phi \cos\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{3^5}{5} \int_0^{\pi/2} \cos\theta \left(\frac{\sin^3\phi}{3} \right]_{\phi=0}^{\phi=\pi/2} d\theta = \boxed{\frac{3^4}{5}}$$

The solid is $\frac{1}{8}$ of the sphere with radius 3 Centered at the origin as depicted in figure in part a.

$x = \rho \sin\phi \cos\theta$

$z = \rho \cos\phi$

(5) Convert to cylindrical coordinates and evaluate the integral

(a) $\iiint_S \sqrt{x^2+y^2}\, dv$ where S is the solid in the first octant bounded by the coordinate plane, the plane $z=4$, and the cylinder $x^2+y^2=25$.

$$= \int_0^{2\pi}\int_0^5\int_0^4 r \cdot r\, dz\, dr\, d\theta = \boxed{\frac{4}{3}(5)^3 2\pi}$$

(b) $\iiint_S (x^2+y^2)^{3/2} dv$ where S is the solid bounded above by the paraboloid $z = \frac{1}{2}(x^2+y^2)$, below by the xy-plane, and laterally by the cylinder $x^2+y^2=4$.

$$= \int_0^{2\pi}\int_0^2\int_0^{r^2/2} r^3 \cdot r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r^6}{2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^7}{14}\Big]_0^2 d\theta = \left(\frac{2^6}{7}\right) 2\pi$$

(6) Convert to spherical coordinates and evaluate the integral.

(a) $\iiint_S (x^2+y^2+z^2)^{3/2}\,dv$ where S is the solid in the first octant bounded by the sphere $x^2+y^2+z^2=25$, the cone $z=\sqrt{x^2+y^2}$, and the cone $z=2\sqrt{x^2+y^2}$.

$$= \int_0^{\pi/2}\int_{\tan^{-1}(\frac{1}{2})}^{\pi/4}\int_0^5 \rho^3 \underbrace{\sin\phi\, d\rho\, d\phi\, d\theta}_{dv}$$

$$= \left(\frac{5^6}{6}\right)\int_0^{\pi/2} \cos\phi \Big]_{\tan^{-1}(1/2)}^{\pi/4} d\theta$$

$$= \left(\frac{5^6}{6}\right)\left(\frac{4-\sqrt{10}}{2\sqrt{5}}\right)\left(\frac{\pi}{2}\right)$$

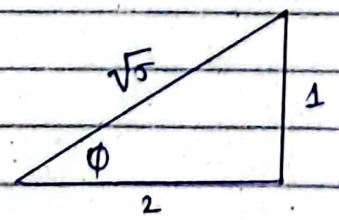
To find the range of ϕ we use the fact $\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$

For the inside $z=2\sqrt{x^2+y^2}$, so

$\tan\phi = \frac{\sqrt{x^2+y^2}}{2\sqrt{x^2+y^2}} = \frac{1}{2}$ For the outside $z=\sqrt{x^2+y^2}$,

So $\tan\phi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1.$

Aside: If $\tan\phi = \frac{1}{2}$, the $\cos\phi = \frac{2}{\sqrt{5}}$



(b) $\iiint_S \sin\sqrt{x^2+y^2+z^2}\, dv$ where S is the solid bounded above by sphere $x^2+y^2+z^2=49$ and below by the cone $z=\sqrt{x^2+y^2}$.

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^{7} (\sin\rho)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 2\pi\left(\frac{\sqrt{2}}{2}-1\right)\left(-49\cos 7 + 14\sin 7 - 2\right)$$

Using $\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$

we get $\tan\phi = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}} = 1$ or

$\phi = \frac{\pi}{4}$

Integration Involving Partial Functions

If $P(x)$, $Q(x)$ are polynomial functions and the denominator $Q(x)$ $(\neq 0)$, in the rational function $\frac{P(x)}{Q(x)}$, can be factorized into linear and quadratic (irreducible) factors, then the rational function is written as a sum of simple rational function, each of which can be integrated by methods already known to us.

Here we will give examples of the following three cases when the denominator $Q(x)$ contains

Case I Non-repeated linear factors

Case II Repeated and non-repeated linear factors.

Case III Linear and non-repeated irreducible quadratic factors or non repeated irreducible quadratic factors.

Examples of Case I

Example 1: Evaluate $\int \frac{-x+6}{2x^2-7x+6}\,dx$, $(x>2)$

Solution

The denominator $2x^2-7x+6 = (x-2)(2x-3)$,

let $\frac{-x+6}{(x-2)(2x-3)} = \frac{A}{x-2} + \frac{B}{x-3}$

or $-x+6 = A(2x-3) + B(2x-3)$ which is true for all x

putting $x=2$, we get

$-2+6 = A(4-3) + B\times 0 \Rightarrow A = 4$

and putting $x = \frac{3}{2}$, we get $-\frac{3}{2}+6 = A(0) + B\left(\frac{3}{2}-2\right)$

or $\frac{9}{2} = B\left(-\frac{1}{2}\right) \Rightarrow B = -9$

Thus $\int \frac{-x+6}{(x-2)(2x-3)}\,dx = \int \left(\frac{4}{x-2} + \frac{-2}{2x-3} \right) dx$

$= 4\int (x-2)^{-1}\, 1.dx - \frac{2}{2}\int (2x-3)^{-1}\, 2\,dx$

$= 4\ln(x-2) - \frac{2}{2}\ln(2x-3) + c, \quad (x>2)$

- Example 2: Evaluate $\int \frac{2x^3 - 9x^2 + 12x}{2x^2 - 7x + 6}$, $x > 2$

<u>Solution</u>: After performing the division by the denominator, we get

$\int \frac{2x^3 - 9x^2 + 12x}{2x^2 - 7x + 6}\,dx = \int \left(x - 1 + \frac{-x+6}{2x^2 - 7x + 6} \right) dx$

$= \int x\,dx - \int 1\,dx + \int \frac{4}{(x-2)}\,dx + \int \frac{-2}{(2x-3)}\,dx,$

$= \frac{x^2}{2} - x + 4\ln(x-2) - \frac{2}{2}(2x-3) + c, \quad (x>2)$

- Example 3: Evaluate (i) $\int \frac{2a}{x^2-a^2}\,dx$, $(x > a)$

<u>Solution</u>: The denominator $x^2 - a^2 = (x-a)(x+a)$.

Let $\frac{2a}{(x-a)(x+a)} = \frac{A}{(x-a)} + \frac{B}{(x+a)} = \frac{1}{x-a} - \frac{1}{x+a}$

(B.g Applying the method of partial fraction)

Thus $\int \frac{2a}{(x-a)(x+a)}\,dx = \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx = \int (x-a)^{-1}.1\,dx - (x+a)^{-1}.1\,dx$

$= \ln|x-a| - \ln|x+a| + c = \ln\left|\frac{x-a}{x+a}\right| + c,$

- Examples 4: Evaluate $\int \frac{7x-1}{(x-1)^2(x+1)}\,dx, \quad (x>1)$

Solution: we write

$$\frac{7x-1}{(x-1)^2(x+1)}\,dx = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

$$= \frac{2}{x-1} + \frac{3}{(x-1)^2} - \frac{2}{x+1} \quad \text{By partial fraction method}$$

Thus $\int \frac{7x-1}{(x-1)^2(x+1)} = \int \left[\frac{2}{x-1} + \frac{3}{(x-1)^2} - \frac{2}{x+1}\right] dx$

$$= 2\int (x-1)^{-1}\cdot dx + 3\int (x-1)^{-2}\cdot 1\cdot dx - 2\int (x+1)^{-1}\,dx$$

$$= 2\ln|x-1| + 3\,\frac{(x-1)^{-2+1}}{-2+1} - 2\ln|x+1| + c, \quad (x>1)$$

$$= 2\left[\ln|x-1| - \ln|x+1|\right] + 3\left[\frac{(x-1)^{-1}}{-1}\right] + c$$

$$= 2\ln\left(\frac{x-1}{x+1}\right) - \frac{3}{x-1} + c$$

Example 5: Evaluate $\int \frac{e^x(x^2+1)}{(x+1)^2}\,dx$

Solution: $\int \frac{e^x(x^2+1)}{(x+1)^2}\,dx = \int e^x\left(1 - \frac{2}{x+1} + \frac{2}{(x+1)^2}\right)dx,$

$$\Rightarrow \int \frac{e^x(x^2+1)}{(x+1)^2}\,dx = \int e^x\,dx - 2\int \frac{e^x}{x+1}\,dx + 2\int \frac{e^x}{(x+1)^2}\,dx \quad (I)$$

We integrate by parts the last integral on right side of (I)

$$\int e^x(x+1)^{-2}\,dx = e^x\frac{(x+1)^{-1}}{-1} - \int \frac{(x+1)^{-1}}{-1}\cdot e^x\,dx$$

or $$\int \frac{e^x}{(x+1)^2}\,dx = -\frac{e^x}{x+1} + \int \frac{e^x}{x+1}\,dx \quad (II)$$

Using (II), (I) becomes

$$\int \frac{e^x(x^2+1)}{(x+1)^2}\,dx = \int e^x\,dx - 2\int \frac{e^x}{x+1}\,dx + 2\left(-\frac{e^x}{x+1} + \int \frac{e^x}{x+1}\,dx\right)$$

$$= (e^x + c) - 2\int \frac{e^x}{x+1}\,dx - \frac{2e^x}{x+1} + 2\int \frac{e^x}{x+1}\,dx$$

$$= e^x - \frac{2e^x}{x+1} + c = \frac{xe^x + e^x - 2e^x}{x+1} + c$$

$$= \frac{e^x(x-1)}{x+1} + c$$

- Example 6: Evaluate $\int \frac{1}{x^3-1}\,dx$

Solution: The denominator $x^3-1 = (x-1)(x^2+x+1)$,

Let $\frac{1}{(x-1)(x^2+x+1)}\,dx = \int\left(\frac{1}{3}\cdot\frac{1}{x-1} - \frac{1}{6}\cdot\frac{2x+4}{x^2+x+1}\right)dx$

$= \int\left(\frac{1}{3}\cdot\frac{1}{x-1}\,1\cdot dx - \frac{1}{6}\,\frac{2x+1}{x^2+x+1} - \frac{1}{6}\cdot\frac{3}{x^2+x+1}\right)dx$

$= \frac{1}{3}\int (x-1)^{-1}\,dx - \frac{1}{6}\int (x^2+x+1)^{-1}(2x+1)\,dx - \frac{1}{2}\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dx$

$= \frac{1}{3}\ln(x-1) - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$

$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

Note: x^2+x+1 is positive for real values of x.

- Example 7: $\int \frac{2x}{x^6-1}\,dx$

Solution: put $x^2 = t$, then $2x\,dx = dt$ and

$$\int \frac{2x}{x^6-1}\,dx = \int \frac{1}{t^3-1}\,dt = \int \frac{1}{(t-1)(t^2+t+1)}\,dt$$

$$= \frac{1}{3}\ln|t-1| - \frac{1}{6}\ln|t^2+t+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) + c$$

$$= \frac{1}{3}\ln|x^2-1| - \frac{1}{6}\ln|x^4+x^2+1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right) + c$$

- Example 8: Evaluate $\int \frac{3}{x(x^3-1)}\,dx, \quad x \neq 0, 1$

Solution: Let $\frac{3}{x(x^3-1)} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{Cx+D}{x^2+x+1}$

$$= \frac{-3}{x} + \frac{1}{x-1} + \frac{2x+1}{x^2+x+1}$$

Let $\int \frac{3}{x(x-1)(x^2+x+1)}\,dx = \int \left(\frac{-3}{x} + \frac{1}{x-1} + \frac{2x+1}{x^2+x+1}\right) dx$

$$= -3\int (x)^{-1}\,1\cdot dx + \int (x-1)^{-1}\,1\cdot dx + \int (x^2+x+1)^{-1}(2x+1)\,dx$$

$$= -3\ln|x| + \ln|x-1| + \ln|x^2+x+1| + c$$

$$= -3\ln|x| + \ln|(x-1)(x^2+x+1)| + c$$

$$= -3\ln|x| + \ln|x^3-1| + c$$

- Example 8: Evaluate $\int \frac{2x^2+6x}{(x^2+1)(x^2+2x+3)}\,dx$

Solution:

Let $\frac{2x^2+6x}{(x^2+1)(x^2+2x+3)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2x+3}$

$= \frac{2x+1}{x^2+1} - \frac{2x+3}{x^2+2x+3}$

Thus $\int \frac{2x^2+6x}{(x^2+1)(x^2+2x+3)}\,dx = \int \left(\frac{2x+1}{x^2+1} - \frac{2x+3}{x^2+2x+3} \right) dx$

$= \int \frac{2x}{x^2+1}\,dx + \int \frac{dx}{x^2+1} - \int \frac{2x+2}{x^2+2x+3}\,dx - \int \frac{dx}{x^2+2x+3}$

$= \int (x^2+1)^{-1}(2x)\,dx + \int \frac{1}{x^2+1}\,dx - \int (x^2+2x+3)^{-1}(2x+2)\,dx$

$- \int \frac{1}{(x+1)^2+(\sqrt{2})^2}\,dx$

$= \ln(x^2+1) + \tan^{-1}(x) - \ln(x^2+2x+3) - \frac{1}{\sqrt{2}} \tan^{-1} \frac{x+1}{\sqrt{2}} + c$

The Definite Integrals

- Fundamental Theorem and properties of Definite Integrals

(a) Definite Integral

$\int_a^b f(x)\,dx$ gives the area under the curve $y = f(x)$ from $x = a$ to $x = b$ and the x-axis

(b) Fundamental Theorem of Calculus

If f is continuous on $[a, b]$ and $\phi'(x) = f(x)$, that is, $\phi(x)$ is any anti-derivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a)$$

Note that the difference $\phi(b) - \phi(a)$ is independent of the choice of anti-derivative of the function f.

(c) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(d) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx, \quad a < c < b$$

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- Example 1: Evaluate $\int_{-1}^{3} (x^3+3x^2)\,dx$ (ii) $\int_1^2 \frac{x^2+1}{x+1}\,dx$

(i) $\int_{-1}^{3} (x^3+3x^2)\,dx = \int_{-1}^{3} x^3\,dx + \int_{-1}^{3} 3x^2\,dx$

$$= \left[\frac{x^4}{4}\right]_{-1}^{3} + \left[x^3\right]_{-1}^{3} = \left[\frac{3^4}{4} - \frac{-1^4}{4}\right] + 3^3 - (-1)^3$$

$$= 20 + 28 = 48$$

(ii) $\int_1^2 \frac{x^2+1}{x+1}\,dx = \int_1^2 \frac{x^2-1+2}{x+1}\,dx$

$$= \int_1^2 \left(\frac{x^2-1}{x+1} + \frac{2}{x+1}\right)dx = \int_1^2 \left(x - 1 + \frac{2}{x+1}\right)dx$$

$$= \int_1^2 x\,dx - \int_1^2 dx + 2\left[\ln(x+1)\right]_1^2$$

$$= \left[\frac{(2)^2}{4} - \frac{(1)^2}{2}\right] - [2-1] + 2[\ln(1+1)]$$

$$= \left(2 - \frac{1}{2}\right) - 1 + 2\left[\ln 3 - \ln 2\right] = \frac{1}{2} \ln \frac{3}{2}$$

-Example 2: Evaluate (i) $\int_0^{\sqrt{3}} \frac{x^3+9x+1}{x^2+9}\,dx$

$$\int_0^{\sqrt{3}} \frac{x^3+9x+1}{x^2+9}\,dx = \int_0^{\sqrt{3}} \left(\frac{x^3+9x}{x^2+9} + \frac{1}{x^2+9} \right) dx$$

$$= \int_0^{\sqrt{3}} \left(\frac{x(x^2+9)}{x^2+9} + \frac{1}{x^2+9} \right) dx$$

$$= \int_0^{\sqrt{3}} \left(x + \frac{1}{x^2+9} \right) dx$$

$$= \int_0^{\sqrt{3}} x\,dx + \int_0^{\sqrt{3}} \frac{1}{x^2+9}\,dx$$

$$= \left(\frac{x^2}{2} \right)_0^{\sqrt{3}} + \left[\frac{1}{3} \tan^{-1} \frac{x}{3} \right]_0^{\sqrt{3}}$$

$$= \left(\frac{(\sqrt{3})^2}{2} - \frac{(0)^2}{2} \right) + \frac{1}{3} \left(\tan^{-1} \frac{\sqrt{3}}{3} - \tan^{-1} \frac{0}{3} \right)$$

$$= \left(\frac{3}{2} - 0 \right) + \frac{1}{3} \left(\tan^{-1} \frac{1}{\sqrt{3}} - \tan^{-1} 0 \right)$$

$$= \frac{3}{2} + \frac{1}{3} \left(\frac{\pi}{6} - 0 \right) = \frac{3}{2} + \frac{\pi}{18}$$

(ii) $\int_0^{\frac{\pi}{4}} \sec x(\sec x + \tan x)\,dx$

$= \int_0^{\frac{\pi}{4}} (\sec^2 x + \sec x \tan x)\,dx$

$= \int_0^{\frac{\pi}{4}} \sec^2 x\,dx + \int_0^{\frac{\pi}{4}} \sec x \tan x\,dx$

$= [\tan x]_0^{\frac{\pi}{4}} + [\sec x]_0^{\frac{\pi}{4}} = \left(\tan\frac{\pi}{4} - \tan(0)\right) + \left(\sec\frac{\pi}{4} - \sec(0)\right)$

$= (1-0) + (\sqrt{2}-1) = \sqrt{2}$

- Example 3: Evaluate $\int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{1}{1-\sin x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{(1-\sin x)(1+\sin x)}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{1-\sin^2 x}\,dx = \int_0^{\frac{\pi}{4}} \frac{1+\sin x}{\cos^2 x}\,dx$

$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right)dx = \int_0^{\frac{\pi}{4}} (\sec^2 x + \sec x \tan x)\,dx$

$= \sqrt{2}$

- Example 4: Evaluate $\int_{-1}^{2} (x+|x|)\,dx$

: $\int_{-1}^{2} (x+|x|)\,dx = \int_{-1}^{0} (x+|x|)\,dx + \int_{0}^{2} (x+|x|)\,dx$

$= \int_{-1}^{0} [x+(-x)]\,dx + \int_{0}^{2} (x+(x)]\,dx$

$= \int_{-1}^{0} 0\,dx + \int_{0}^{2} 2x\,dx = 0 + 2\int_{0}^{2} x\,dx$

$= 2\left[\frac{x^2}{2}\right]_0^2 = 2\left(\frac{4}{2} - \frac{0}{2}\right) = 4$

- Example 5: Evaluate $\int_{0}^{\sqrt{7}} \frac{3x}{\sqrt{x^2+9}}\,dx$

: Let $f(x) = x^2+9$, then $f'(x) = 2x$, so

$\int \frac{3x}{\sqrt{x^2+9}}\,dx = \int \frac{\frac{3}{2}(2x)}{\sqrt{x^2+9}}\,dx = \frac{3}{2}\int (x^2+9)^{-1/2}(2x)\,dx$

$= \frac{3}{2}\int [f(x)]^{-1/2} f'(x)\,dx$

$= \frac{3}{2}\frac{[f(x)]^{-\frac{1}{2}+1}}{-1/2+1} = 3[f(x)]^{1/2} + C = 3(x^2+9)^{1/2} + C$

Thus, $\int_{0}^{\sqrt{7}} \frac{3x}{\sqrt{x^2+9}}\,dx = \left[3(x^2+9)^{1/2}\right]_0^{\sqrt{7}} = 3\left[(7+9)^{\frac{1}{2}} - (0+9)^{\frac{1}{2}}\right]$

$= 3\left[(16)^{\frac{1}{2}} - (9)^{\frac{1}{2}}\right] = 3(4-3) = 3$

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- Example 6: Evaluate $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx, \quad x \neq -1, 1$

: Let $t = \sin^{-1}x$, then $x = \sin t$ for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

and $dx = \cos t\,dt = \sqrt{1-\sin^2 t}\,dt$

[$\because$ $\cos t$ is +ve for $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$]

$= \sqrt{1-x^2}\,dt$

or $\frac{1}{\sqrt{1-x^2}}\,dx = dt \quad (x \neq 1, 1)$

If $x = \frac{1}{2}$, then $\frac{1}{2} = \sin t \Rightarrow t = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

and if $x = \frac{\sqrt{3}}{2}$, then $\frac{\sqrt{3}}{2} = \sin t \Rightarrow t = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Thus, $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} (\sin^{-1}x)\frac{1}{\sqrt{1-x^2}}\,dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} t\,dt = \quad (\because x = \sin t \Rightarrow \sin^{-1}x = t)$

$= \left[\frac{t^2}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{1}{2}\left[\left(\frac{\pi}{3}\right)^2 - \left(\frac{\pi}{6}\right)^2\right] = \frac{1}{2}\left(\frac{\pi^2}{9} - \frac{\pi^2}{36}\right)$

$= \frac{1}{2}\left(\frac{4\pi^2 - \pi^2}{36}\right) = \frac{3\pi^2}{72} = \frac{\pi^2}{24}$

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- Example 1: Evaluate $\int_0^{\pi/6} x \cos x \, dx$

: Applying the formula

$\int f(x)\phi'(x)\,dx = f(x)\phi(x) - \int \phi(x) f'(x)\,dx$, we have

$$\int x \cos x \, dx = x \sin x - \int (\sin x)(1)\,dx$$
$$= x \sin x - [(-\cos x) + c_1)]$$
$$= x \sin x + \cos x + c, \text{ where } c = -c_1$$

Thus $\int_0^{\frac{\pi}{6}} x \cos x \, dx = [x \sin x + \cos x]_0^{\frac{\pi}{6}}$

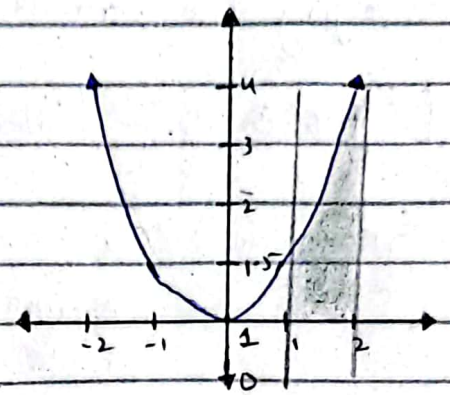
$$= \left(\frac{\pi}{6} \sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right) - (0)\sin(0) - \cos(0)$$

$$= \frac{\pi}{6} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} - (0+1) = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$

Applications of Definite Integral

1) Find the area between the x-axis and the curve $y=x^2+1$ from $x=1$ to $x=2$.

:



$$\int_1^2 (x^2+1)\,dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 dx$$

$$= \left.\frac{x^3}{3}\right]_1^2 + \left.x\right]_1^2$$

$$= \left(\frac{8}{3} - \frac{1}{3}\right) + 1$$

$$= \frac{7}{3} + 1 = \frac{10}{3}$$

2. Find the area, above the x-axis and under the $y=5-x^2$ from $x=-1$ to $x=2$

: $y = 5-x^2$

$$\int_{-1}^2 (5-x^2)\,dx = \int_{-1}^2 5\,dx - \int_{-1}^2 x^2\,dx$$

$$= 5\int_{-1}^2 dx - \int_{-1}^2 x^2\,dx$$

$$= 5\left[x\right]_{-1}^2 - \left[\frac{x^3}{3}\right]_{-1}^2$$

$$= 5\left((2)-(-1)\right) - \left[\frac{8}{3} + \frac{1}{3}\right]$$

$$= 15 - \frac{27}{9} = 18$$

3. Find the area below the curve $y = 3\sqrt{x}$ and above the x-axis between $x = +1$ and $x = 4$.

:

$$\int_1^4 (3\sqrt{x})\, dx = 3\int_1^4 \sqrt{x}\, dx$$

$$= 3\int_1^4 (x)^{1/2}\, dx$$

$$= 3\left[\frac{(x)^{\frac{1}{2}+1}}{1/2+1}\right]_1^4 = 3\cdot\frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^4$$

$$= 3\cdot\frac{2}{3}\left[(2^2)^{\frac{1}{2}} - (1)^{3/2}\right] = 3\cdot\frac{14}{3} = 14$$

4. Find the area bounded by cos function from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$

:

$$\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \cos(x)\, dx = \left[\sin x\right]_{\frac{\pi}{2}}^{-\frac{\pi}{2}} = \sin\left(-\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right)$$

$$= -\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) = -2\sin\left(\frac{\pi}{2}\right)$$

$\Rightarrow \frac{\pi}{2} = 90^\circ$ where $\sin(90^\circ) = 1$

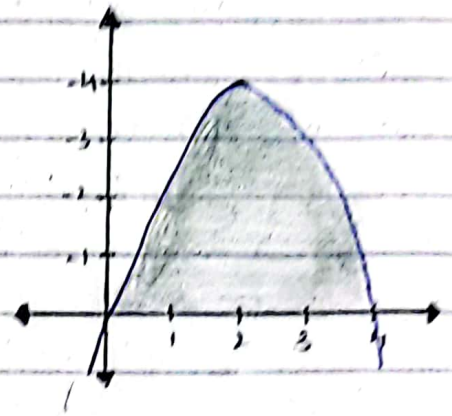
Sin

$\Rightarrow -\frac{\pi}{2} = 180^\circ$ where $\sin(180^\circ) = -1$

$$= -2\sin\left(\frac{\pi}{2}\right) = -2\sin(90^\circ) = -2(1)$$

$= -2 \Rightarrow |-2| = 2$ Area's Mod

5. Find the area between the x-axis and the curve $y = 4x - x^2$



: $\int_0^4 (4x - x^2)\,dx$

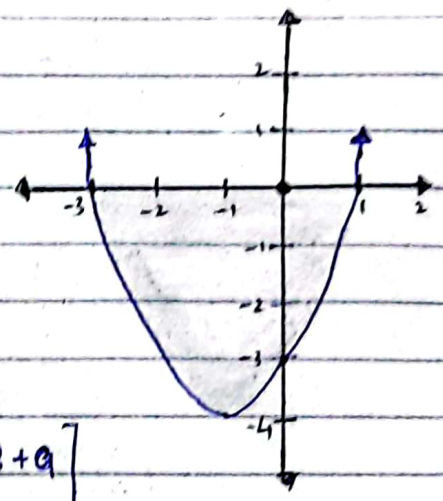
$= 4\int_0^4 x\,dx - \int_0^4 x^2\,dx$

$= 2\left[x^2\right]_0^4 - \left[\frac{x^3}{3}\right]_0^4$

$= 2[16] - \left[\frac{64}{3}\right] = 32 - \frac{64}{3}$

$= \frac{96 - 64}{3} = \boxed{\frac{32}{3}}$

6. Determine the area bounded by the parabola $y = x^2 + 2x - 3$ and the x-axis



$\therefore$

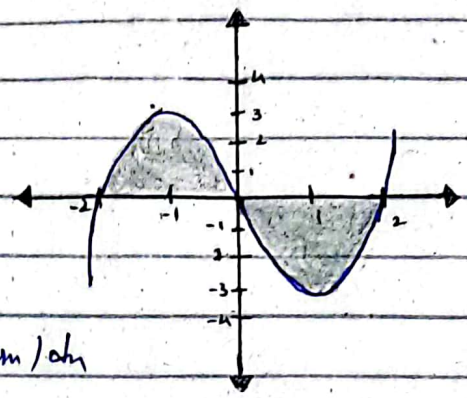
$\int_{-3}^{1} (x^2 + 2x - 3)\,dx$

$= \left[\frac{x^3}{3} + x^2 - 3x\right]_{-3}^{1}$

$= \left[\frac{1}{3} + 1 - 3(1)\right] - \left[\frac{-27}{3} + 9 + 9\right]$

$= \left[-\frac{5}{3}\right] - \left[+9\right] = \frac{-5}{3} - 9 = \frac{-5-27}{3} = \left|\frac{32}{3}\right|$

8) Find the area bounded by the curve $y = x^3 + 4x$ and the x-axis.



$$\int_{-2}^{0} (x^3 - 4x)\,dx + \int_{2}^{0} (x^3 - 4x)\,dx$$

OR $\int_{-2}^{0} f(x) - \int_{0}^{2} (x^3 - 4x)\,dx$

$\Rightarrow$

$$= \int_{-2}^{0} (x^3 - 4x)\,dx + \int_{2}^{0} (x^3 - 4x)\,dx$$

$$= \left[\frac{x^4}{4} - 2x\right]_{-2}^{0} + \left[\frac{x^4}{4} - 2x\right]_{2}^{0}$$

$$= \left[0 - (4 + 4)\right] + \left[-(4 - 8)\right] = -(8) + 0 = |-8| = 8.$$

OR $\Rightarrow$

$$\int_{-2}^{0} (x^3 - 4x)\,dx - \int_{0}^{2} (x^3 - 4x)\,dx$$

$$= \left[\frac{x^4}{4} - 2x\right]_{-2}^{0} - \left[\frac{x^4}{4} - 2x\right]_{0}^{2}$$

$$= \left[-(8)\right] - \left[(4 - 4) - 0\right] = -8 - 0 = -(8) = |-8| = \boxed{8}$$

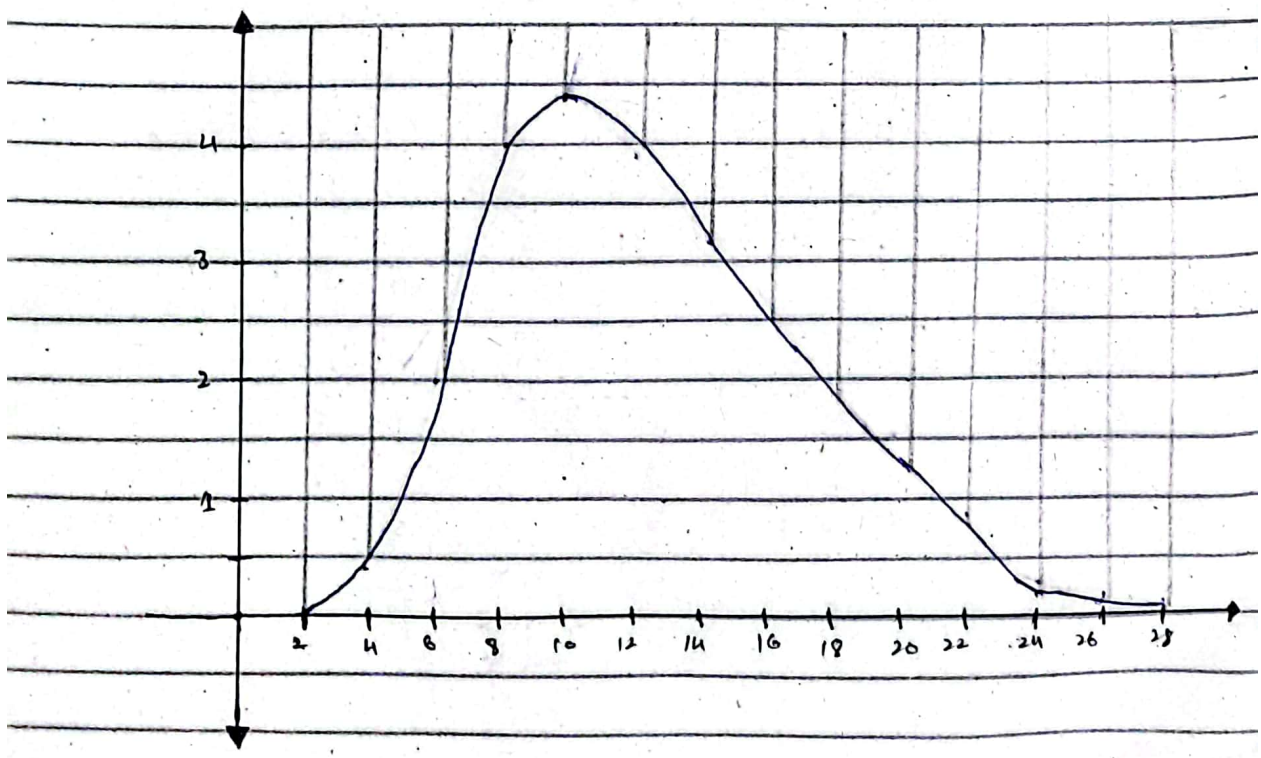
- Numerical Integration

Approximating Definite Integrals

One method of measuring cardiac output is to inject 5 to 10 milligrams (mg) of a dye into a vein leading to the heart. After making its way through the lungs, the dye returns to the heart and is pumped into the aorta, where its concentration is measured at equal time intervals. The graph of the function shows concentration of dye in a person's aorta, measured at 2-second intervals after 5mg dye have been injected. The person's cardiac output, measured in liters per minute (L/min), is computed by using the formula

$$R = \frac{60D}{\int_0^{28} c(t)\,dt}$$

where D is the quantity of dye injected



- Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2(f(x_1)) + 2f(x_2) + \dots 2f(x_{n-1}) + f(x_n) \right]$$

where $\Delta x = \frac{b-a}{n}$

The approximation generally improves with larger values of n

Example 1 Approximate the value of

$$\int_1^2 \frac{1}{x}\,dx$$

Using the Trapezoidal Rule with $n=10$. Compare this result with the exact value of the integral.

Solution Here, $a=1$, $b=2$, and $n=10$, so

$$\Delta x = \frac{b-a}{n} = \frac{1}{10} = 0.1$$

and

$x_0 = 1 \quad x_1 = 1.1 \quad x_2 = 1.2 \quad x_3 = 1.3 \quad \dots \quad x_9 = 1.9 \quad x_{10} = 2$

The Trapezoidal Rule yields

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.1}{2}\left[1 + 2\left(\frac{1}{1.1}\right) + 2\left(\frac{1}{1.2}\right) + 2\left(\frac{1}{1.3}\right) + \dots + 2\left(\frac{1}{1.9}\right) + \frac{1}{2}\right]$$

$$\approx 0.693771$$

In this case, we can easily compute the actual value of the definite integral under consideration. In fact,

$$\int_1^2 \frac{1}{x}\,dx = \ln x \Big|_1^2 = \ln 2 - \ln 1 = \ln 2$$

$$\approx 0.693147$$

Thus, the Trapezoidal Rule with $n=10$ yields a result with an error of 0.000624 to six decimal places.

- Simpson's Rule

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{n-1}) + f(x_n)\right]$$

where $\Delta x = \frac{b-a}{n}$ and n is even

In using Simpson's Rule, remember that n must be even.

- Example: Find an approximation of

$$\int_1^2 \frac{1}{x}\,dx$$

Using Simpson's Rule with $n = 10$. Compare this result with that of example 1 and also with the exact value of the integral.

Solution we have $a = 1$, $b = 2$, $f(x) = \frac{1}{x}$, and $n = 10$, so

$$\Delta x = \frac{b-a}{n} = \frac{1}{10} = 0.1$$

and

$x_0 = 1$, $x_1 = 1.1$ $x_2 = 1.2$ $x_3 = 1.3$ ... $x_9 = 1.9$ $x_{10} = 2$

Simpson's Rule yields

$$\int_1^2 \frac{1}{x}\,dx \approx \frac{0.1}{3}\left[f(1) + 4f(1.1) + 2f(1.2) + \dots + 4f(1.9) + f(2)\right]$$

$$= \frac{0.1}{3}\left[1 + 4\left(\frac{1}{1.1}\right) + 2\left(\frac{1}{1.2}\right) + 4\left(\frac{1}{1.3}\right) + 2\left(\frac{1}{1.4}\right) + 4\left(\frac{1}{1.5}\right) + 2\left(\frac{1}{1.6}\right) + 4\left(\frac{1}{1.7}\right) + 2\left(\frac{1}{1.8}\right) + 4\left(\frac{1}{1.9}\right) + \frac{1}{2}\right]$$

The trapezoidal Rule with $n=10$ yielded an approximation of 0.693771, which is 0.000624 off the value of $\ln 2 \approx 0.693147$ to six decimal places. Simpson's Rule yields an approximation with an error of 0.000003, a definite improvement over the Trapezoidal Rule.

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