Discrete Manual Structures'

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Propositional Logic

▸ Propositions

Proposition is a delarative sentence that declares a fact, that is either true or false; but not both.

	Declarative		Not Declarative
1	$1 + 1 = 2$	1	$x + 1 = 2$
2	$2 + 2 = 3$	2	$x + y = z$
3	Toronto is Capital of Canada.	3	what time is it ?
4	Islamabad is in punjab.	4	Read this carefully

- $x + 1 = 2$ is not declarative becouse x is not defined.
- Toronto is capital of Canada → True (proposition: True/False)
- what time is it ? → Interrogative
- $x + 1 = 2$, $x = 5, x = 1$ → Declarative, x is defined.

▸ Compound Propositions

Compound propositions are formed from existing propositions using logical operators/ Logical Connetives.

▸ Propositional variables

Propositional variables or statement variables, variables that represent propositions. The conventional letters used for propositional variables are $p, q, r, s, \ldots$

▸ Truth value Or False value.

Truth value of a proposition is true, denoted by T.
False value of a proposition is false, denoted by F.

- Propositional Calculus or Propositional Logic

It was first developed systematically by the Greeks Philosopher Aristotle more than 2300 years ago.

Practice Problems

1) which of these sentences are propositions? what are the truth values of those that are propositions?

a) $2+3=5$

: True

b) $5+7=10$

: False. $5+5=10$

c) $x+2=11$

: Not proposition, x is not defined.

d) Answer this question.

: Not proposition. Question isn't given

2) what is the negation of each of these propositions?

a) Today is Thursday.

: Today is not Thursday.

b) There is no pollution in new Jersey.

: There is pollution in New Jersey.

c) $2+1=3$

$2+1\neq 3$

3) Let p and q be propositions

p: I bought a lottery ticket this week.

q: I won the million dollar jackpot on Friday.

Express each of these propositions as an English sentence.

a) $\neg p$

: I did not buy a lottery ticket this week.

b) $p \vee q$

: I bought a lottery ticket this week or I won the million dollar jackpot on friday.

c) $p \rightarrow q$

: If I bought a lottery ticket this week, than I won the million dollar jackpot on Friday.

d) $p \wedge q$

: I bought a lottery ticket this week and I won the million dollar jackpot on Friday.

e) $p \leftrightarrow q$

: I bought a lottery ticket this week if and only if (or iff) I won the million dollar jackpot on Friday.

f) $\neg p \rightarrow \neg q$

: If I did not by a lottery ticket this week, then I did not win the million dollar jackpot on Friday.

▸ Precision of Logical connectives/operators

Informal version	Formal/symbolically	Order
NOT	$\neg$ / $\sim$	1st
AND/BUT	$\wedge$	2nd
OR	$\vee$	3rd
implication	$\rightarrow$	4th
Biconditional	$\leftrightarrow$	5th

4) Let p and q be the propositions

p: It is below freezing.

q: It is snowing.

Write these propositions using p and q and logical connectives.

a) It is below freezing and snowing.

: $p \wedge q$

b) It is below freezing but not snowing.

: $p \wedge \sim q$

c) It is not below freezing and it is not snowing.

: $\sim p \wedge \sim q$

d) It is either snowing or below freezing (or both)

: $(q \vee p) \vee (p \wedge q)$

e) If it is below freezing, it is also snowing.

: $p \rightarrow q$

f) It is either below freezing or it is snowing, but it is not snowing if it is below freezing.

: $(p \vee q) \wedge (p \rightarrow \sim q)$

g) That it is below freezing is necassary and sufficient for it to be snowing.

: $p \leftrightarrow q$

▶ Truth Tables

1) Construct a truth table for each of these Compound Propositions.

a) $p \wedge \sim p$

: variables = 1, $2^1 = 2$ Rows

p	~p	p ∧ ~p
T	F	F
F	T	F

b) $p \vee \sim p$

: variables = 1, $2^1 = 2$ Rows

p	~p	p ∨ ~p
T	F	T
F	T	T

c) $(p \vee \sim q) \rightarrow q$

: variables = 2, $2^2 = 4$ Rows

p	q	~q	p∨~q	(p∨~q)→q
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

d) $(p \vee q) \rightarrow (p \wedge q)$

: variables = 2, $2^2 = 4$ Rows

p	q	p∨q	p∧q	(p∨q) → (p∧q)
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

e) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$

p	q	~p	~q	p→q	~q → ~p	(p→q) ↔ (~q → ~p)
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Logical Equivalences

Name	Equivalence
Identity Laws	$p \wedge T \equiv p$ $p \vee F \equiv p$
Domination Laws	$p \vee T \equiv T$ $p \wedge F \equiv F$
Idempotent laws	$p \vee p \equiv p$ $p \wedge p \equiv p$
Double negation Law	$\neg(\neg p) \equiv p$
Commutative Laws	$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$
Associative Laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
De Morgan's Laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
Distributive Laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Absorption Laws	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$
Negation Laws	$p \wedge \neg p \equiv F$ $p \vee \neg p \equiv T$

- Logical equivalences Involving Conditional statements.

$p \rightarrow q \equiv \neg p \vee q$ $\qquad$ $p \rightarrow q \equiv \neg q \rightarrow \neg p$

$p \vee q \equiv \neg p \rightarrow q$ $\qquad$ $p \wedge q \equiv \neg(p \rightarrow \neg q)$

$\neg(p \rightarrow q) \equiv p \wedge \neg q$

$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$

$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$

- $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv \neg p \leftrightarrow \neg q$
- $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ $\qquad$ $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

▶ Propositional Equivalences

1) Tautology — Always True
2) Contradication — Always False
3) Contigency — Sometimes true, Sometimes false
4) Satisfiability — At least one true value in its Truth Table.
5) UnSatisfiability — Not even a Single True result in its Truth Table.

- Valid — Tautology. A compound proposition is valid when it is a tautology.
- Invalid — A Compound proposition is invalid when it is either Contradication or Contigency.

▶ Logical Operators NAND | and NOR ↓

$p \text{ NAND } q \equiv p \mid q \equiv \sim(p \wedge q) \equiv \sim p \vee \sim q$

$p \text{ NOR } q \equiv p \downarrow q \equiv \sim(p \vee q) \equiv \sim p \wedge \sim q$

| Sheffer stroke (H.M Sheffer stroke)

↓ Pierce arrow (C.S Pierce)

▶ Practice Problems

1) Show that each of these Conditional statements is a tautology by using truth tables and without also.

a) $(p \wedge q) \rightarrow p$

p	q	$(p \wedge q)$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

Tautology

$(p \wedge q) \rightarrow p$

$\sim(p \wedge q) \vee p$ — $\because p \rightarrow q \equiv \sim p \vee q$

$(\sim p \vee \sim q) \vee p$ — De Morgan's Law

$(\sim p \vee p) \vee \sim q$ — Associative Law

$T \vee \sim q$ — Negation Law

T — Domination Law

b) $p \rightarrow (p \vee q)$

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	F	T	T
T	T	T	T
F	F	F	T
F	T	T	T

$p \rightarrow (p \vee q)$

$\sim p \vee (p \vee q)$ — $\because p \rightarrow q \equiv \sim p \vee q$

$(\sim p \vee p) \vee q$ — Associative Law

$T \vee q$ — Negation Law

T — Domination Law

Hence Tautology

c) $\neg p \rightarrow (p \rightarrow q)$

p	q	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow (p \rightarrow q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Tautology

$\neg p \rightarrow (p \rightarrow q)$

$\neg(\neg p) \vee (\neg p \vee q)$ $\because p \rightarrow q \equiv \neg p \vee q$

$p \vee (\neg p \vee q)$ Double negation Law

$(p \vee \neg p) \vee q$ Associative Law

$T \vee q$ Negation Law

T Domination Law

Hence Tautology prooved

d) $(p \wedge q) \rightarrow (p \rightarrow q)$

p	q	$p \wedge q$	$p \rightarrow q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Tautology

$(p \wedge q) \rightarrow (p \rightarrow q)$

$\neg(p \wedge q) \vee (\neg p \vee q)$ $\because p \rightarrow q \equiv \neg p \vee q$

$(\neg p \vee \neg q) \vee (\neg p \vee q)$ De morgan's law

$(\neg p \vee \neg p) \vee (\neg q \vee q)$ Associative Law

$(\neg p) \vee (\neg q \vee q)$ Idempotent Law

$\neg p \vee T$ Negation Law

T Domination Law

e) $\neg(p \rightarrow q) \rightarrow p$

p	q	$\neg(p \rightarrow q)$	$\neg(p \rightarrow q) \rightarrow p$
T	T	F	T
T	F	T	T
F	T	F	T
F	F	F	T

Tautology

$\neg(p \rightarrow q) \rightarrow p$

$\neg[\neg(\neg p \vee q)] \vee p$ $\because p \rightarrow q \equiv \neg p \vee q$

$(\neg p \vee q) \vee p$ Double negation Law

$(\neg p \vee p) \vee q$ Associtive Law

$T \vee q$ Negation Law

T Domination Law

Hence Tautology prooved

f) $\neg(p \rightarrow q) \rightarrow \neg q$

p	q	$\neg q$	$\neg(p \rightarrow q)$	$\neg(p \rightarrow q) \rightarrow \neg q$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	T

Tautology

$\neg(p \rightarrow q) \rightarrow \neg q$

$\neg[\neg(\neg p \vee q)] \vee \neg q$ $\because p \rightarrow q \equiv \neg p \vee q$

$(\neg p \vee q) \vee \neg q$ Double negation Law

$\neg p \vee (q \vee \neg q)$ Associative Law

$\neg p \vee T$ Negation Law

T Domination Law

Hence Tautology Prooved

Quantifiers

1: Predicate : A predicate is a sentence that contains a finite number and becomes a statement when specific values are substituted for the variables

2: Truth set : The set of all elements of D (Domain set) the truth set of P(n) is the set of all such elements that make the predicate true is called the truth set of the predicate.

Domain set

$$\{x \in D \mid P(x)\}$$

element — such that — propositional function

Ex:-

Let Q(n) be the predicate "n is a factor of 8."
Find the truth set of Q(n) if

a. The domain of n is the set of all positive integers.
: The truth set of Q(n) is $\{1, 2, 4, 8\}$, these are exactly positive integers that divide 8.

b. The domain of n is set of all negative integers.
: The truth set of Q(n) is $\{-1, -2, -4, -8\}$, these are exactly negative integers that divide 8.

3: Two types of Quantifiers
a) Universal Quantifier : $\forall$: For all
b) Existential Quantifier : $\exists$: There exist

4: Counter example
A value n for which Q(n) is false, is called a counter example to the universal statement.

5: Truth and Falsity of Universal statements

a) let $D = \{1, 2, 3, 4, 5\}$ and consider the statement $\forall x \in D,\ x^2 \geq x$

: $1^2 \geq 1,\ 2^2 \geq 2,\ 3^2 \geq 3,\ 4^2 \geq 4,\ 5^2 \geq 5.$

$\forall x \in D,\ x^2 \geq x$ is valid.

Counter example: let's consider, $1/2 \in D$

$(\frac{1}{2})^2 \geq \frac{1}{2}$ False

The technique used to show the truth of universal statement is called the method of exhaustion.

- Universal Quantifier: The universal quantification of $P(x)$ is the statement "$P(x)$ for all values of x in the domain," for which $P(x)$ is true.
- Existential Quantifier: The existential quantification of $P(x)$ is the statement "There exists a value x in the domain," for which $P(x)$ is true.

words instead of there exists: there is a, we can find a, There is at least one, for some, and for at least one.

6: Formal to Informal Language

Formal version: $\forall x \in \mathbb{R},\ x^2 \geq 0$

Informal version: All real numbers have positive squares.

or: Every real number has a positive square.

or: Any real number has a positive square.

or: The square of each real number is positive.

7: Universal Conditional Statements

a) All bytes have eight bits.

: $\forall x$, if x is a byte, then x has eight bits.

b) No fire trucks are green

: $\forall x$, if x is a fire truck, then x is not green

▸ Practice Problems

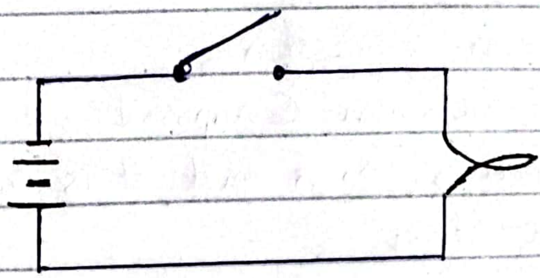
1) $\forall$ rectangles x, x is a Quadrilateral.
: All rectangles are Quadrilateral.
2) All dinosours are extinct.
: $\forall$ dinosours x, x is extinct.
3) The number 8,147,881,883 is not equal to the square of any integer.
: $\forall x$, such that $x \in \mathbb{Z}$, $x^2 \neq 8,147,581,853$
4) $\forall$ real numbers x and y, $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$
: False. Counter example $x=1$ and $y=2$.
$\sqrt{x+y} = \sqrt{x} + \sqrt{y} \rightarrow \sqrt{1+2} \neq \sqrt{1} + \sqrt{2}$
5) No irrational numbers are integers.
: $\forall$ irrational numbers x, x is not an integer.
6) No logicians are lazy.
: $\forall$ logicians x, x is not a lazy logician.
7) Some exercise have answers.
: $\exists$ Some exercises x, such that x have answers.
8) Some real numbers are irrational.
: $\exists$ an irrational number x, s.th $x \in \mathbb{R}$.
: $\exists$ an real number x, s.th x is irrational number
9) Some questions are easy.
: $\exists$ a question x, such that x is easy.
10) The sum of the angles of any triangle is 180°.
: $\forall$ triangle x, the sum of angles of x is 180°.
11) The sum of any two integers is even.
: $\forall$ even integers x and y, sum of x and y is also even.
12) The product of any two fraction is fraction.
: $\forall$ two fractions x and y, such that $x \times y$ is also a fraction.
13) Some real numbers have square 2.
: $\exists x$, such that $x \in \mathbb{R}$, x have square 2.
14) There is at least one real number whose square is 2.
: $\exists x$, such that $x \in \mathbb{R}$, x has square 2.

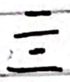
▸ Applications : Digital Logic Circuits

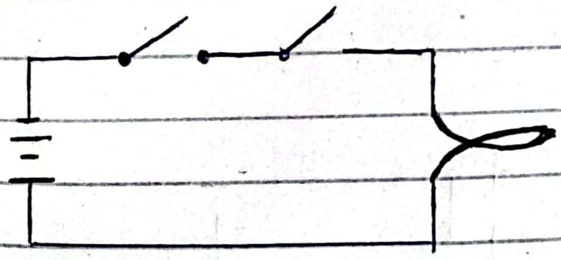
The basic electronic components of a digital system are called digital logic circuits. The word "Logic" indicates the importent role of logic in the design of such circuits, and the word digital indicates that the circuit process discrete, or separate, signals as oppossed to continuous ones.

Switches

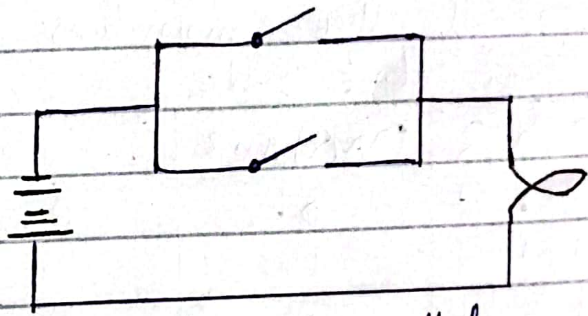
open closed



 denotes battery  denotes light bulb



switches in "series"



switches in parallel

▸ All possible behaviours of these circuits.

Switches in Series

Switches		Light bulb
P	Q	state
closed	closed	onn
closed	open	off
open	closed	off
open	open	off

Conjuction
$P \wedge Q$

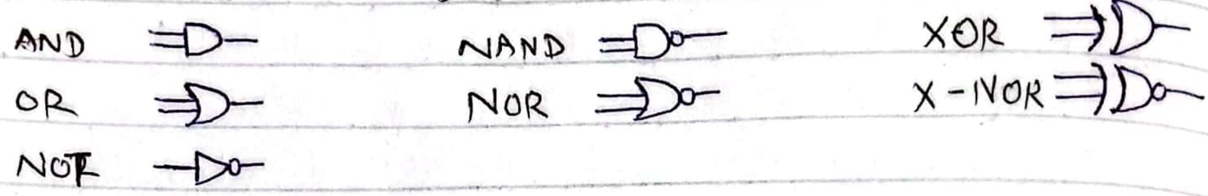
Switches in parallel

Switches		Light bulb
P	Q	state
closed	closed	~~off~~
closed	open	onn
open	closed	onn
open	open	onn

Disjunction
$P \vee Q$

▶ Logic Gates Symbollically

1) Basic Gates   2) Universal Gates   3) Arithmetic Gates



▶ Boolean variable

TRUE | FALSE

▶ Boolean Expression

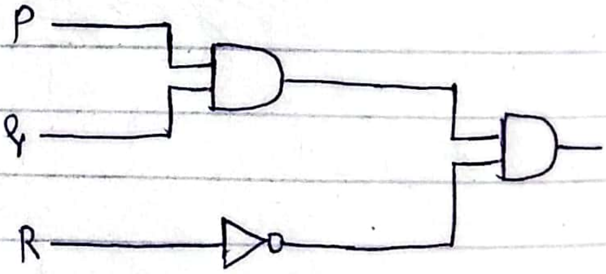
An Expression Composed of Boolean variables and the Connectives ~, ∧ and ∨ is Called a Boolean expression.

▶ Recognizer

It is a circuit that outputs a 1 for exactly one particular combination of input signals and outputs 0s for all other Combinations.

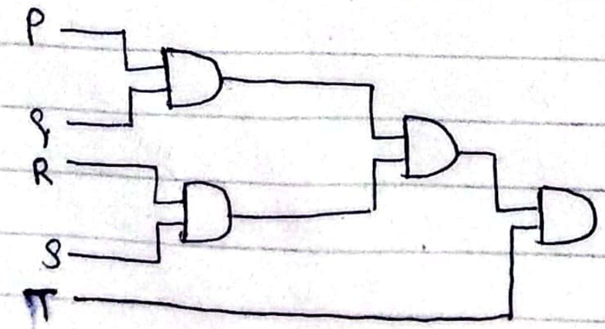
Input / output Table

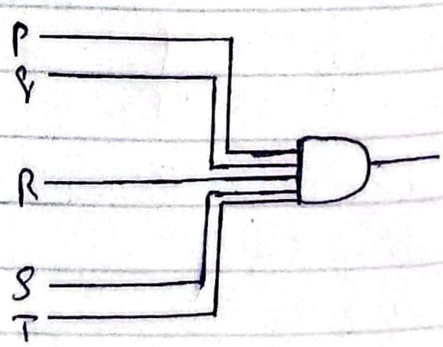
P	Q	R	$(P \wedge Q) \wedge \sim R$
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0



▶ multiple Input AND Gate

$(P \wedge Q) \wedge (R \wedge S) \wedge T \equiv P \wedge Q \wedge R \wedge S \wedge T$





▸ Finding a circuit that corresponds to a given I/O Table.

Design a circuit for the following Input/output Table

Input			Output
P	Q	R	S
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	0

→ $1 \wedge 1 \wedge 1 = (P \wedge Q \wedge R)$

→ $1 \wedge \sim 0 \wedge 1 = (P \wedge \sim Q \wedge R)$

→ $1 \wedge \sim 0 \wedge \sim 0 = (P \wedge \sim Q \wedge \sim R)$

$$\boxed{(P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R)}$$

• The above Expression which we have found, is a disjunction of terms that are themselves conjuction in which one of P or ~P, one of Q or ~Q, and one of R or ~R all appears.

▸ Disjunctive Normal Form / Sum of products Form

An expression in which one of P or ~P, one of Q or ~Q, and one of R or ~R all appear. Such expressions are said to be in disjunctive normal form or Sum of products form.

Ex:- $(P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R)$

▸ Equivalency

Two digital logic circuits are equivalent, if and only if, their Input/Output tables are identical.

Ex:- $[(P \wedge \sim Q) \vee (P \wedge Q)] \wedge Q \equiv P \wedge Q$

$[P \wedge (\sim Q \vee Q)] \wedge Q$ distributive Law

$[P \wedge (T)] \wedge Q$ Negation Law

$P \wedge Q$ Domination Law

$\boxed{P \wedge Q \equiv P \wedge Q}$ Hence proved.

▸ Practice Problems

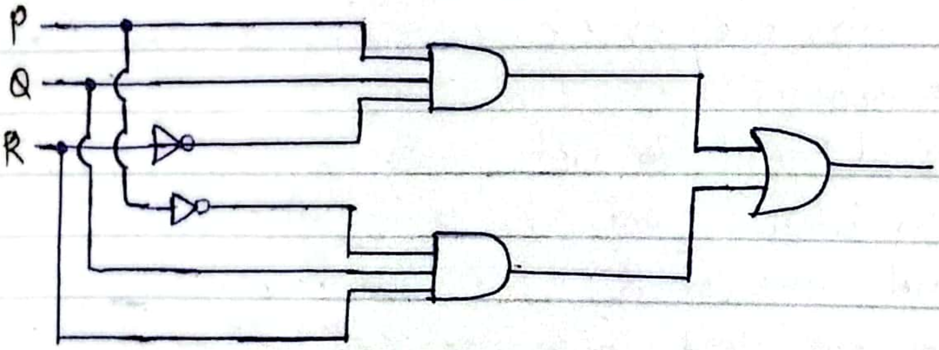
1) Construct a boolean expression having the given table as its truth table and a circuit having the given table as its Input/output table.

P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

→ $1 \wedge 1 \wedge \sim 0$ $(P \wedge Q \wedge \sim R) = 1$

→ $\sim 0 \wedge 1 \wedge 1$ $(\sim P \wedge Q \wedge R) = 1$

Expression: $(P \wedge Q \wedge \sim R) \vee (\sim P \wedge Q \wedge R)$



2)

P	Q	R	S
1	1	1	0
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	0

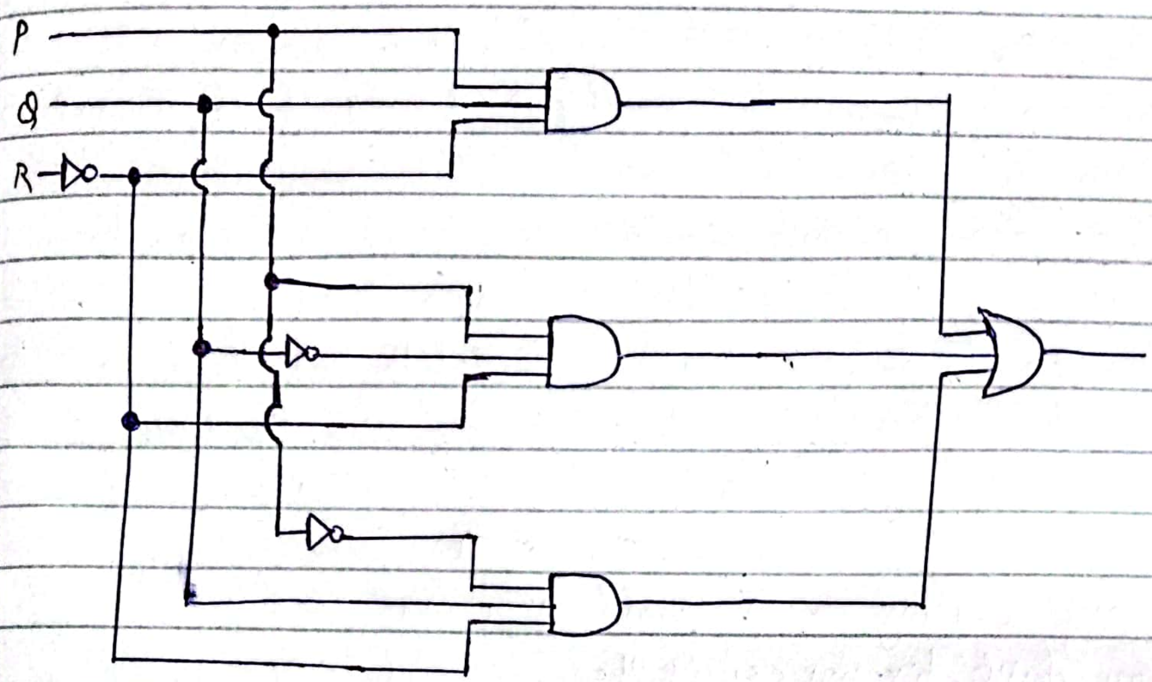
→ $1 \wedge 1 \wedge \sim 0$ $(P \wedge Q \wedge \sim R) = 1$

→ $1 \wedge \sim 0 \wedge \sim 0$ $(P \wedge \sim Q \wedge \sim R) = 1$

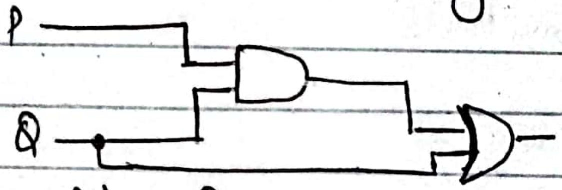
→ $\sim 0 \wedge 1 \wedge \sim 0$ $(\sim P \wedge Q \wedge \sim R) = 1$

Expression:

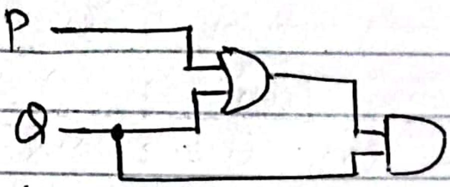
$(P \wedge Q \wedge \sim R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge Q \wedge \sim R)$



3) Proove the equivalency



$(P \wedge Q) \vee Q$



$(P \vee Q) \wedge Q$

$(P \wedge Q) \vee Q \equiv (P \vee Q) \wedge Q$

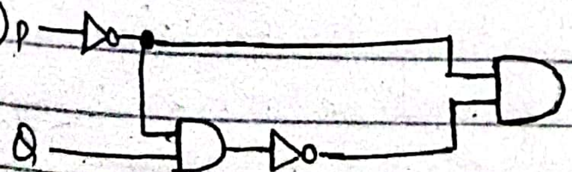
$Q \vee (P \wedge Q)$ associative law

$(Q \vee P) \wedge (Q \vee Q)$ distributive Law

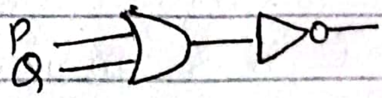
$(Q \vee P) \wedge Q$ Idempotent Law

$(P \vee Q) \wedge Q \equiv (P \vee Q) \wedge Q$

4)



$\sim P \wedge \sim(\sim P \wedge Q)$



$\sim(P \vee Q)$

$\sim P \wedge \sim(\sim P \wedge Q) \equiv \sim(P \vee Q)$

$\sim P \wedge (P \vee \sim Q)$

$(\sim P \wedge P) \vee (\sim P \wedge \sim Q)$ Distributive Law

$F \vee \sim(P \vee Q)$ Negation Law

$\sim(P \vee Q) \equiv \sim(P \vee Q)$

The Logic of Compound Statements

Conditional Statements

- Rewrite the statements in if – then form.

1) This loop will repeat exactly N times if it doesn't Contain a break or goto.

: If this loop doesn't Contain a break or goto, then this loop will repeat exactly N times.

2) I am on time for work if I catch bus at 8:05.

: If I catch the bus at 8:05 then I am on time for work.

3) Freeze or I'll shoot.

: I you don't freeze then I'll shoot.

4) Fix my ceiling or I won't pay my rent.

: If you don't fix my ceiling then I won't pay my rent.

Some Popular Types of Conditional Statements.

- Implication : $P \to Q$
- Contrapositive : $\sim Q \to \sim P$
- inverse : $\sim P \to \sim Q$
- Converse : $Q \to P$

write Contrapositives

1) If p is a square, then p is a rectangle.

: If p is not a rectangle, then p is not a Square.

2) If today is new year eve, then tomorrow is january.

: If tomorrow is not january, then today is not new year eve.

3) If n is prime, then n is odd or n is 2.

: If n is not odd n is not 2, then n is prime.

4) If n is non-negative, then n is positive or n is 0.

: If n is not positive and n is 0, then n is non-negative.

5) If n is divisible by 6, then n is divisible by 2 and n is divisible by 3.

: If n is not divisible by 2 or n is not divisible by 3, then n is not divisible by 6.

▼ Some other Conditional Statements

1) This integer is even if, and only if, it equals twice some integers.

: Biconditional Statement : $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\sim P \wedge \sim Q)$

P: This integer is even. Q: It equals twice some integers.

- If this integer is even, then it equals twice some integers and if equals twice some integer, then this integer is even.
- This integer is even and it equals twice some integers or this integer is not even and it doesn't equal twice some integer

2) The Cubs will win the pennant only if they win tomorrow's game.

: If the cubs will win the pennant, they win tomorrow's game.

3) Sam will be allowed on Signe's racing boat only if he is an expert sailor.

: If Sam will be allowed on Signe's racing boat, he is an expert sailor.

- r unless s $\equiv$ if $\sim s$, then r $\equiv \sim s \rightarrow r$

4) payment will be made on the fifth unless a new hearing is granted.

: If a new hearing is not granted, then payment will be made on the fifth.

5) Ann will go unless it rains.

: If it doesn't rain, then Ann will go.

6) This door will not open unless a security code is entered.

: If a security code is not entered, then this door will not be opened.

▶ Nested Quantifiers

1) Translate these statements into English, where the domain for each variable consists of all real numbers.

a) $\forall x \exists y (x < y)$

: For all real numbers x, we can find a y such that x is less than y.

b) $\forall x \forall y (((x \geq 0) \wedge (y \geq 0)) \rightarrow (xy \geq 0))$

: For all real numbers x and for all real numbers y, if x is a positive real number and y is a positive real number then the product of x and y is positive.

c) $\forall x \forall y \exists z (xy = z)$

: For all real numbers x, and for all real numbers y, there exists a z, such that the product of x and y is equal z.

2) Translate these statements into english, where the domain for each variable consists of all real numbers.

a) $\exists x \forall y (xy = y)$

: There exists a real number x, for all real numbers y such that the product of x and y is equal to y.

b) $\forall x \forall y (((x \geq 0) \wedge (y < 0) \rightarrow (x - y > 0))$

: For all real numbers x. and for all real numbers y, If x is a positive real number and y is a negative real number then the difference of x and y is greater than 0.

c) $\forall x \forall y \exists z (x = y + z)$

: For all real numbers x and for all real numbers y there exists a real number z such that x is equal to the sum of y and z.

3) Let $W(x, y)$ mean that student x has visited website y, where the domain for x consists of all students in your school and the domain for y consist of all websites. Express each of these statements by a simple English sentence.

a) $W($ Sarah, www.att.com$)$

: Sarah has visited www.att.com.

b) $\exists x\, W(x,$ www.imdb.org$)$

: There is student x who has visited www.imdb.org.

c) $\exists y\, W($Jose Orex$, y)$

: Jose Orex has visited the website y.

d) $\exists y\, (W($Ashok Puri$, y) \wedge (W($Cindy yoon$, y))$

: Ashok puri and Cindy yoon have visited at least one website.

e) $\exists y \forall z (y \neq ($David Belcher$) \wedge (W($David Belcher$, z) \rightarrow W(y,z)))$

: If David Belcher has visited all websites then there is a person who has visited all websites and that person is not David Belcher.

f) $\exists x \exists y \forall z ((x \neq y) \wedge (W(x,z) \leftrightarrow W(y,z)))$

: person x has visited all websites iff y has visited all websites and both x and y is not same person.

y) Let $L(x,y)$ be the statement "x Loves y" where the domain for both x and y consist of all people in the world. Use Quantifiers to express statements

a) Everybody loves Jerry.

: $\forall x\, L(x,$ Jerry$)$

b) Every body loves Somebody.

: $\forall x \exists y\, L(x,y)$

c) There is Somebody whom everybody loves.

: $\exists y \forall x\, L(x,y)$

d) Nobody Loves Everybody.

: $\forall x \exists y \neg L(x,y)$

e) There is Somebody whom Lydia doesn't love.

: $\exists y \neg L($Lydia$, y)$

f) There is Somebody whom no one loves.

$\forall x \exists y \neg L(x,y)$

Mathematical Induction

1) For each positive integer n, Let $p(n)$ be the formula

$$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution

1) $P(1)$ is true or not

$$n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(1)^2 = \frac{1(1+1)(2(1)+1)}{6}$$

$$1 = \frac{2(3)}{6} = 1 \rightarrow \boxed{1 = 1}$$

2) $P(K)$

$$1^2+2^2+\dots+K^2 = \frac{K(K+1)(2K+1)}{6}$$

3) $P(K+1)$

$$1^2+2^2+\dots+(K+1)^2 = \frac{(K+1)((K+1)+1)(2(K+1)+1)}{6}$$

$$1^2+2^2+\dots+(K+0)^2+(K+1)^2 = \frac{(K+1)(K+2)(2K+3)}{6}$$

$$1^2+2^2+\dots+K^2 = \frac{K(K+1)(2K+1)}{6}$$

$$\frac{K(K+1)(2K+1)}{6} + \frac{(K+1)^2}{1} = \frac{(K+1)(K+2)(2K+3)}{6}$$

$$K(K+1)(2K+1) + 6(K+1)^2 = (K+1)(K+2)(2K+3)$$

$$(K^2+K)(2K+1) + 6(K^2+2K+1) = (K^2+2K+K+2)(2K+3)$$

$$2K^3+2K^2+K^2+K+6K^2+12K+6 = 2K^3+6K^2+4K+3K^2+9K+6$$

$$\frac{2K^3+9K^2+13K+6}{6} = \frac{2K^3+9K^2+13K+6}{6}$$

2) $\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$

Solution

1: Find the series from the Summation.

$1(1+1) + 2(2+1) + 3(3+1) + 4(4+1) + 5(5+1) + \dots + n(n+1)$

$2 + 6 + 12 + 20 + \dots + n(n+1) = \frac{n(n-1)(n+1)}{3}$

2: $P(2) \rightarrow \boxed{a_1}$

$n(n+1) = \frac{n(n-1)(n+1)}{3}$

$2(\cancel{2}+1) = \frac{2(2-1)(\cancel{2}+1)}{3}$

$\cancel{2} = \frac{2(\cancel{3})}{\cancel{3}} \Rightarrow \boxed{P(2) = 2}$

3: $P(K)$

$2 + 6 + 12 + 20 + \dots + K(K+1) = \frac{K(K-1)(K+1)}{3}$

$2 + 6 + 12 + 20 + \dots + (K^2 + K) = \frac{K(K-1)(K+1)}{3}$

$2 + 6 + 12 + 20 + \dots + K(K-1) = \frac{K(K-1)(K+1)}{3}$

4: $P(K+1)$

$2 + 6 + 12 + 20 + \dots + (K+1)((K+1)-1) = \frac{(K+1)((K+1)-1)((K+1)+1)}{3}$

$2 + 6 + 12 + 20 + \dots + (K+1)(K) = \frac{(K+1)(K)(K+2)}{3}$

$2 + 6 + 12 + 20 + \dots + (K)(K-1) + (K+1)(K) = \frac{(K+1)(K)(K+2)}{3}$

$\underline{2 + 6 + 12 + 20 + \dots + K(K-1)} = \frac{K(K-1)(K+1)}{3}$

$\frac{K(K-1)(K+1)}{3} + \frac{(K+1)(K)}{1} = \frac{(K+1)(K)(K+2)}{3}$

$\frac{(K^2-K)(K+1) + 3(K+1)(K)}{3} = \frac{(K^2+K)(K+2)}{3}$

$\frac{K^3 - \cancel{K^2} + \cancel{K^2} - K + (3K+3)K}{3} = \frac{K^3 + K^2 + 2K^2 + 2K}{3}$

$\frac{K^3 + 3K^2 + 2K}{3} = \frac{K^3 + 3K^2 + 2K}{3}$

3) $1+3+5+\dots+(2n-1)=n^2, \quad n\geq 1$

1) P(1)

$$2(1)-1=(1)^2$$

$$\boxed{1=1}$$

2) P(K)

$1+3+5+\dots+(2K-1)=K^2$

3) P(K+1)

$1+3+5+\dots+(2(K+1)-1)=(K+1)^2$

$1+3+5+\dots+(2K+1)=(K+1)^2$

↳ Common difference + 2 (lets find previous step)

$1+3+5+\dots+((2(K+1)-2)+(2K+1)=(K+1)^2$

$\underline{1+3+5+\dots+(2K-1)}+(2K+1)=(K+1)^2$

$1+3+5+\dots+(2K-1)=K^2$

$$\boxed{K^2+(2K+1)=(K+1)^2}$$

4) $2+4+6+\dots+2n=n^2+n, \quad \forall n\in\mathbb{Z}, n\geq 1$

Salution

1: P(1)

$$2(1)=(1)^2+(1)$$

$$\boxed{2=2}$$

2: P(K)

$2+4+6+\dots+2K=K^2+K$

3: P(K+1)

$2+4+6+\dots+2(K+1)=(K+1)^2+(K+1)$

$2+4+6+\dots+(2K+2)=(K+1)^2+(K+1)$

$2+4+6+\dots+2K+(2K+2)=(K+1)^2+(K+1)$

$\underline{2+4+6+\dots+2K}=K^2+K$

$K^2+K+2K+2=(K+1)^2+(K+1)$

$K^2+3K+2=K^2+2K+1+K+1$

$$\boxed{K^2+3K+2=K^2+3K+2}$$

5) $6^n + 4$ is divisible by 5, $n \geq 0$

Solution

$$\frac{6^n + 4}{5} = m$$

1) $P(1) \rightarrow \frac{6^{(1)} + 4}{5} = \frac{6+4}{5} = 2 \rightarrow$ Defined

2) $P(K)$

$$m = \frac{6^K + 4}{5}$$

3) $P(K+1)$

$$m = \frac{6^{K+1} + 4}{5} \Rightarrow r = \frac{6^{K+1} + 4}{5}$$

3: $5r = 6^{K+1} + 4$; $5m - 4 = 6^K$

$5r = 6^K \cdot 6 + 4$

$5r = (5m - 4)6 + 4$

$5r = 30m - 24 + 4$

$5r = 30m - 20$

$\boxed{r = 6m - 4}$

6) $7^n - 2^n$ is divisible by 5

Solution

$$m = \frac{7^n - 2^n}{5}$$

1) $P(1) \rightarrow \frac{7^{(1)} - 2^{(1)}}{5} = \frac{7-2}{5} = \boxed{1} \rightarrow$ Defined

2) $P(K)$

$$m = \frac{7^K - 2^K}{5} \Rightarrow 5m = 7^K - 2^K$$

$\therefore 2^K = 7^K - 5m$ and $5m + 2^K = 7^K$

3) $P(K+1)$

$5r = 7^{K+1} + 2^{K+1}$

$5r = 7^K \cdot 7 - 2^K \cdot 2$

$5r = (5m + 2^K)7 - (7^K - 5m)2$

$5r = 35m + 7(2^K) - 2(7^K) + 10m$

$5r = 35m + 7(7^K - 5m) - 2(7^K)$

$5r = 35m + 7(7^K) - 2(7^K) - 35m$

$5r = 7^K(7-2) - 10m = \boxed{7^K(5) - 10m}$

7) $x^{2n} - y^{2n}$ is divisible by $x+y$

Solution

$$m = \frac{x^{2n} - y^{2n}}{(x+y)}$$

1) $P(1)$

$$\frac{x^{2(1)} - y^{2(1)}}{x+y} = \frac{x^2 - y^2}{x+y} = \frac{\cancel{(x+y)}(x-y)}{\cancel{(x+y)}} = (x-y)$$

Defined

2) $P(K)$

$$m = \frac{x^{2k} - y^{2k}}{x+y}$$

3) $P(K+1)$

$$r = \frac{x^{2(k+1)} - y^{2(k+1)}}{x+y}$$

$$r = \frac{x^{2k+2} - y^{2k+2}}{x+y}$$

$$x^{2k} = m(x+y) + y^{2k}$$

$$\therefore \quad r = \frac{x^{2k+2} - y^{2k+2}}{x+y} = \frac{x^{2k} \cdot x^2 - y^{2k} \cdot y^2}{x+y}$$

$$r = \frac{(m(x+y) + y^{2k}) \cdot x^2 - y^{2k} \cdot y^2}{x+y}$$

$$r = \frac{mx^2(x+y) + y^{2k}(x)^2 - y^{2k} \times y^2}{x+y}$$

$$r = \frac{mx^2(x+y) + x^2 y^{2k} - y^{2k} \cdot y^2}{x+y}$$

$$r = \frac{mx^2(x+y) + y^{2k}(x^2 - y^2)}{x+y}$$

$$\boxed{r = \frac{mx^2(x+y) + y^{2k}(x+1)(x-1)}{x+y}}$$

24) Consider the Fibonacci Series

$x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

1: $$P(1) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\cancel{1}+\sqrt{5}-\cancel{1}+\sqrt{5}}{2}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\cancel{2}\sqrt{5}}{\cancel{2}}\right] = \frac{1}{\sqrt{5}} \times \sqrt{5} = \boxed{1}$$

$$P(2) = \frac{1}{\sqrt{5}} = \left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+2\sqrt{5}+5}{4}\right) - \left(\frac{1-2\sqrt{5}+5}{4}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\cancel{1}+2\sqrt{5}+\cancel{5}-\cancel{1}+2\sqrt{5}-\cancel{5}}{4}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}+2\sqrt{5}}{4}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\frac{\cancel{4}\sqrt{5}}{\cancel{4}}\right] = \frac{1}{\sqrt{5}} \times \sqrt{5} = \boxed{1}$$

Defined

2: $P(K) = x_K = x_{K-1} + x_{K-2}$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^K - \left(\frac{1-\sqrt{5}}{2}\right)^K\right] = x_K$$

3: $$P(K+1) : \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{K+1}\right] = x_{K+1}$$

$$P(K+2) : \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{K+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{K+1}\right] = x_{K+2}$$

$$\boxed{x_{K+2} = x_{K+1} + x_K}$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]=$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]+\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}+\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}+\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}+\left(\frac{1+\sqrt{5}}{2}\right)^{k}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\left(\frac{1+\sqrt{5}}{2}\right)+1\right)-\left(\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\left(\frac{1-\sqrt{5}}{2}\right)+1\right)\right)\right]$$

$$\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}+2}{2}\right)-\left(\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}+2}{2}\right)\right)\right]$$

$$\boxed{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{\sqrt{5}+3}{2}\right)-\left(\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{3-\sqrt{5}}{2}\right)\right)\right]}$$

$\hookrightarrow x_{k+1}+x_k$

$$x_{k+2}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+\sqrt{5}}{2}\right)^{2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{1+2\sqrt{5}+5}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{1-2\sqrt{5}+5}{4}\right)\right]$$

$$=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{2\sqrt{5}+6}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{6-2\sqrt{5}}{4}\right)\right]$$

$$\boxed{x_{k+2}=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left(\frac{\sqrt{5}+3}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^{k}\left(\frac{3-\sqrt{5}}{2}\right)\right]}$$

Functions

It is a relation and usually defined on general sets.

1) Let $A = \{1, 2, 3, 4, 5\}$ and define a function.
$F: \mathcal{P}(A) \to Z$ as follows: For all sets X in $\mathcal{P}(A)$

$$F(X) = \begin{cases} 0 & \text{If } X \text{ has an even number of element} \\ 1 & \text{If } X \text{ has an odd number of elements} \end{cases}$$

Find the following

a) $F(\{1, 3, 4\})$

: $F(X) = (\{1,3,4\}) \to X = \{1,3,4\}$, number of element is 3, So the output is 1. Because X has an odd number of elements.

b) $F(\emptyset)$

$F(X) = \emptyset = \{\}$, number of elements $= 0$.
Niether even nor odd.

2) Let $J_5 = \{0, 1, 2, 3, 4\}$ and define a function
$F: J_5 \to J_5$ as follows: For each $x \in J_5$,
$F(x) = (x^3 + 2x + 4) \bmod 5$

Find the following

a) $F(0)$

: $F(0) = 0^3 + 2(0) + 4 = 4 \bmod 5 = 4$

b) $F(1)$

$F(1) = 1^3 + 2(1) + 4 = 7 \bmod 5 = 2$

c) $F(3)$

$F(3) = 3^3 + 2(3) + 4 = 37 \bmod 5 = 2$

d) $F(2)$

$F(2) = 2^3 + 2(2) + 4 = 16 \bmod 5 = 1$

e) $F(4)$

$F(4) = 4^3 + 2(4) + 4 = 76 \bmod 5 = 1$

3) Define a function $S: Z^+ \to Z^+$ as follows:
For each positive integer n,
$S(n)$ = the sum of the positive divisors of n.
Find the following:
a) $S(1)$
: $S(1) = 1$, the positive divisor of 1 is only one.
b) $S(15)$
: $S(15) = 1, 3, 5, 15 \to 1+3+5+15 =$ Sum $= 24$
c) $S(17)$
: $S(17) = 1, 17 \to 1+17 =$ Sum $= 18$
d) $S(5)$
: $S(5) = 1, 5 \to 1+5 =$ Sum $= 6$

4) Define $F: Z \times Z \to Z \times Z$ as follows:
For all ordered pairs (a, b) of integers, $F(a, b) = (2a+1, 3b-2)$
Find the following:
a) $F(4, 4)$
: $F(2(4)+1, 3(4)-2)$
$F(9, 10)$

b) $F(2, 1)$
: $F(2(2)+1, 3(1)-2)$
: $F(5, 1)$

c) $F(3, 2)$
: $F(2(3)+1, 3(2)-2)$
$F(7, 4)$

5) Let $J_5 = \{0, 1, 2, 3, 4\}$ and define functions $f: J_5 \to J_5$ and $g: J_5 \to J_5$ as follows: For each $x \in J_5$
$f(n) = (n+4)^2 \bmod 5$, and
$g(n) = (n^2+3n+1) \bmod 5$
Is $f = g$? Explain.

n	$f(n) = (n+4)^2 \bmod 5$	$g(n) = (n^2+3n+1) \bmod 5$	$f(n) = g(n)$	
			mod	$f = g$?
0	$(0+4)^2 \bmod 5 = 1$	$(0^2+3(0)+1) = 1 \bmod 5 = 1$	0	$1 = 1$
1	$(1+4)^2 = 25 \bmod 5 = 0$	$(1^2+3(1)+1) = 5 \bmod 5 = 0$	1	$0 = 0$
2	$(2+4)^2 = 36 \bmod 5 = 1$	$(2^2+3(2)+1) = 11 \bmod 5 = 1$	2	$1 = 1$
3	$(3+4)^2 = 49 \bmod 5 = 4$	$(3^2+3(3)+1) = 19 \bmod 5 = 4$	3	$4 = 4$
4	$(4+4)^2 = 64 \bmod 5 = 4$	$(4^2+3(4)+1) = 29 \bmod 5 = 4$	4	$4 = 4$

6) Hamming Distance Function

a) Find H (10101, 00011)

$$\begin{array}{r} 10101 \\ \oplus\ 00011 \\ \hline 10110 \end{array}$$

$\rightarrow 1+1+1 = 3\ (1s) = 3$ Answer

b) Find H (00110, 10111)

$$\begin{array}{r} 00110 \\ 10111 \\ \hline 10001 \end{array}$$

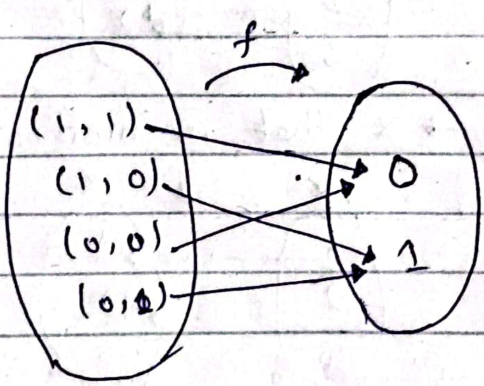
$\rightarrow 1+1 = 2$ Answer (Sum of 1s)

7) Draw arrow Diagrams for the Boolean functions defined by the following I/O table.

a)

Input P	Input Q	Output R
1	1	0
1	0	1
0	1	0
0	0	1

:-

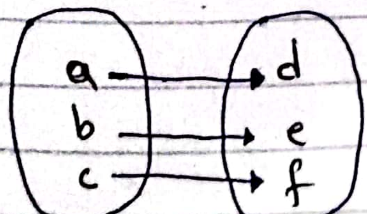


▶ One-to-one and Onto functions

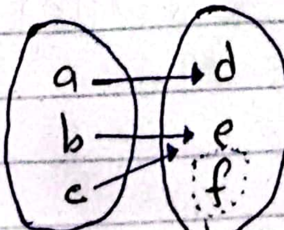
If no two arrows that start in the domain point to the same element of the co-domain then the function is called one-to-one or injection.

• Onto-functions

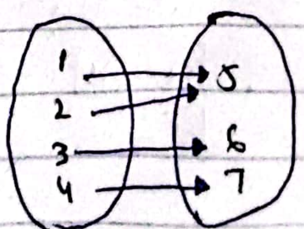
Let F be a function from a set X to a set Y. F is onto or surjective if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.



one-to-one



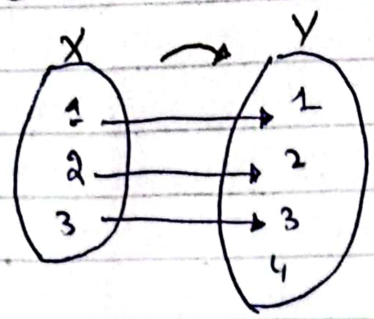
not one, onto



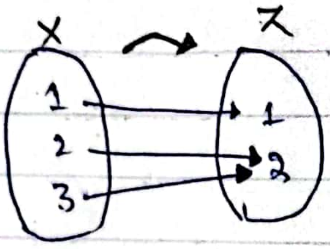
Onto

8) Let $X = \{1,2,3\}$, $Y = \{1,2,3,4\}$, and $Z = \{1,2\}$.

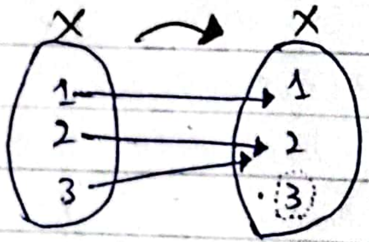
a) Define a function $f: X \to Y$ that is one-to-one but not onto



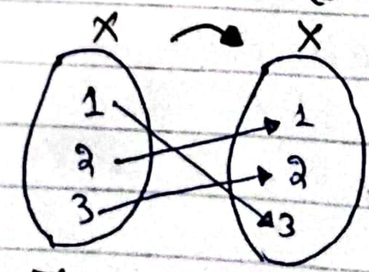
b) $g: X \to Z$ that is onto but not one-to-one



c) $h: X \to X$ that is niether one-to-one nor onto.



d) $K: X \to X$ that is one-to-one and Onto but not Identity



8) Define $g: Z \to Z$ the rule $g(n) = 4n - 5 \ \forall n \in Z$.

(i) Is g one-to-one, onto? proove or give a Counter example.

:- one-to-one

$4n - 5 = 4n - 5$

$4n = 4n$

$\boxed{n = n}$

one-to-one

Onto

$y = 4n - 5$;

$n = \frac{y+5}{4}$

$y = 4\left(\frac{y+5}{4}\right) - 5$

$y = y + 5 - 5$

$\boxed{y = y}$

Onto

▶ Relations

1) Relation on sets

- Less than relation for real numbers

$$x L y \iff x < y$$

- Congruence Modulo 2 Relation

For all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$

$$m E n \iff m - n \text{ is even}$$

- Relation on a power set

$A S B \iff$ A has at least as many elements as B.

$$A S B \iff A \geq B \text{ (in terms of elements)}$$

- Inverse of a Relation

$R = A$ to B $\qquad$ $R^{-1} = B$ to A

Can be defined by interchanging the elements of all the ordered pairs of R.

2) Properties of Relation

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$

elements in set $A = 2 = m$

elements in set $B = 3 = n$

- Relation $\rightarrow$ $2^{mn} = 2^{2\times3} = 2^6 = 64$ Subsets
- Cross product $\rightarrow$ $m \times n = 2 \times 3 = 6$ Subsets
- Power Set $\rightarrow$ $A = 2$ elements $\rightarrow$ $2^2 = 4$ Subsets

1) Binary Relation

: A Binary relation is from A to B $(A \times B)$ is a set of ordered pairs where first element is from set A and Second element is from set B is Called a Binary Relation.

$A R B$, $R: A \rightarrow B$ Relation / function

2) Reflaxive Relation

For all x in A $(x,x) \in R$

Let $A = \{1, 2\}$

$A \times A = \{ (1,1), (1,2), (2,2), (2,1) \}$

Ex:-

$R_1 = \{ (1,1), (2,4), (3,3), (4,1), (4,4) \}$ X

(2,2) is missing for reflexivity

$R_2 = \{ (1,1), (2,2), (3,3), (4,4) \}$ ✓

$R_3 = \{ \} \rightarrow$ Not Reflaxive

$R_4 = A \times A \rightarrow$ Always Reflaxive

3) Irreflaxive Relation

· For all x in A, $(x,x) \notin R$

Ex:-

$R_1 = \{ \} \rightarrow$ Irreflaxive

$R_2 = A \times A \rightarrow$ Reflaxive (Not Irreflaxive)

4) Symmetric Relation [always Reflaxive]

$(a,b) \in R \Rightarrow (b,a) \in R$.

Symmetric: iff $(1,2) \in A$ and $(2,1) \in A$

Ex:

$R_1 = \{ (1,2), (2,1), (3,4), (4,3) \}$ ✓

$R_2 = \{ (1,3), (2,3), (3,1), (4,1), (3,2) \}$ X

$R_3 = \{ \}$

$R_4 = A \times A$ Symmetric

5) Anti-Symmetric Relation [Always Reflaxive]

$(a,b) \in R \Rightarrow (b,a) \in R$

and $a = b$

$R_1 = \{ (1,2), (3,3), (4,3) \}$ X

$R_2 = \{ (1,1), (2,2) \}$ ✓

6) A- Symmetric Relation [Not Reflaxive]

iff $(a,b) \in R \Rightarrow (b,a) \notin R$

A-Symmetric: $(1,2) \in R$ but $(2,1) \notin R$

Ex:-

$R_1 = \{ (1,2), (2,3), (3,1), (2,1) \}$ ✗

$R_2 = \{ (1,2), (2,3), (3,1), (4,3) \}$ ✓

$R_3 = \{ (1,1), (2,2), (3,3), (4,4) \}$

$R_4 = \{ \}$ → A-Symmetric because not Reflaxive

$R_5 = A \times A$ → Not A-Symmetric because Reflaxive

7) Transitivity [Always Reflaxive]

For all x, y and z in A, if $(x,y) \in R$ and $(y,z) \in R$ than $(x,z) \in R$

If $(1,2) \in R$ and $(2,3) \in R$ than $(1,3) \in R$

Ex:-

$R_1 = \{ (3,1), (2,3) \}$ ✗

$R_2 = \{ (1,2), (2,1), (1,1), (2,2) \}$ ✓

$R_3 = \{ (1,1), (2,2), (3,3) \}$ ✓

$R_4 = \{ \}$ ✓

$R_5 = A \times A$ ✓

8) Inverse Relation [$A \times A$ ✗ $A \times B$ ✓]

R, inverse → R^{-1}

For all $x \in A$ and $y \in B$

$(y,x) \in R^{-1} \longleftrightarrow (x,y) \in R$

Ex:-

$R_1 = \{ (2,2), (2,6), (2,8), (3,6), (4,8) \}$

$R_1^{-1} = \{ (2,2), (6,2), (8,2), (6,3), (8,4) \}$

$R_2 = \{ \}$

$R_2^{-1} = \cup$

9) Identity Relation [Always Reflaxive]

Ex:-

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$ ✓

$R_2 = \{(1,1), (2,2)\}$ ✓

10) Complementary Relation

$R = \{(a,b) \mid (a,b) \in A \times B$ and $(a,b) \notin R\}$

Ex:-

$\bar{R} = A \times B - R$

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (3,3)\}$

$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

$\bar{R} = A \times A - R$

$\{(1,2), (1,3), (2,1), (2,3), (3,1), (3,2)\}$

11) Equivalence Relation [RST]

Reflaxive * Symmetric * Transitive

Ex:-

$R_1 = \{(a,a), (b,b), (c,c)\}$ ✓

$R_2 = \{(a,a), (b,b), (c,c), (b,a)\}$ ✗ R✓ S✗ T✗

$R_3 = \{ A \times A \rightarrow$ RST. ✓

12) POSET → Partial Order Set = RAT

Reflaxive * Anti-Symmetric * Transitive

$A = \{1, 2, 3\}$

$R_1 = \{(1,1), (2,2), (3,3)\}$ R✓ A✓ T✓ ✓

$R_2 = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$ R✓ A✗ T✓ ✗

$R_3 = \{(1,1), (2,2), (3,3), (1,3), (2,3)\}$ R✓ A✗ T✗ ✗

$R_4 = \{\}$ ✗ → because it is irreflaxive

$R_5 = A \times A$ ✓

▸ Set Theory

1) Write in words how to read each of the following out loud. Then write short hand notation for each.

a) $\{x \in U \mid x \in A$ and $x \in B\}$.

: x is an element in set U such that x belongs to and x belongs B.

short hand Notation → $A \cap B$

b) $\{x \in U \mid x \in A$ or $x \in B\}$

: x is an element in set U such that x belongs to A or x belongs to B.

Shorthand Notation → $A \cup B$

2) Let $A = \{1, 3, 5, 7, 9\}$, $B = \{3, 6, 9\}$ and $C = \{2, 4, 6, 8\}$.

Find each of the following:

a) $A \cup B$
$= \{1, 3, 5, 6, 7, 9\}$

b) $A \cap B$
$= \{3, 9\}$

c) $A \cup C$
$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

d) $A \cap C$
$= \emptyset$

e) $A - B$
$= \{1, 5, 7\}$

f) $B - A$
$= \{6\}$

g) $B \cup C$
$= \{2, 3, 4, 6, 8, 9\}$

h) $B \cap C$
$= \{6\}$

3) Let $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{b, c, e\}$

a) Find $A \cup (B \cap C)$, $(A \cup B) \cap C$, and $(A \cup B) \cap (A \cup C)$

which of these sets are equal?

: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$ By Distributive Law

4) Let $B_i = \{x \in \mathbb{R} \mid 0 \leq x \leq i\}$ for all integers $i = 1, 2, 3, 4$.

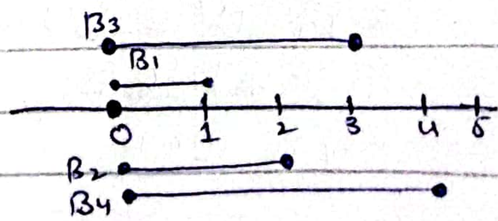
a) $B_1 \cup B_2 \cup B_3 \cup B_4 = ?$

: $B_1 = \{0 \leq x\ 1\}$ $B_1 = [0, 1]$

$B_2 = \{0 \leq x \leq 2\}$ $= [0, 2]$

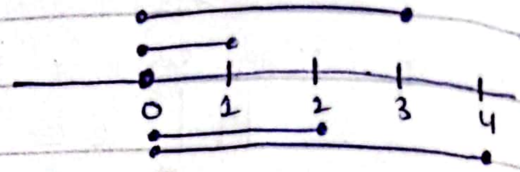
$B_3 = \{0 \leq x 3\}$ $= [0, 3]$

$B_4 = \{0 \leq x \leq 4\}$ $= [0, 4]$



$[0, 1] \cup [0, 2] \cup [0, 3] \cup [0, 4] = [0, 4]$

b) $B_1 \cap B_2 \cap B_3 \cap B_4 = ?$



: $B_1 = [0, 1]$, $B_2 = [0, 2]$

$B_3 = [0, 3]$, $B_4 = [0, 4]$

$[0,1] \cap [0,2] \cap [0,3] \cap [0,4] = [0, 1]$

5) let $C_i = \{i, -i\}$ $\forall i \in \mathbb{Z}$.

a) $\bigcup_{i=0}^{4} C_i = ?$

: $C_0 = \{0, -0\}$

$C_1 = \{1, -1\}$

$C_2 = \{2, -2\}$

$C_3 = \{3, -3\}$

$C_4 = \{4, -4\}$

$\bigcup_{i=0}^{4} C_i = C_0 \cup C_1 \cup C_2 \cup C_3 \cup C_4$

$= \{0, -0\} \cup \{1, -1\} \cup \{2, -2\} \cup \{3, -3\} \cup \{4, -4\}$

$\bigcup_{i=0}^{4} C_i = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

b) $\bigcap_{i=0}^{4} C_i = ?$

: $C_0 = \{0, -0\}$

$C_1 = \{1, -1\}$

$C_2 = \{2, -2\}$

$C_3 = \{3, -3\}$

$C_4 = \{4, -4\}$

$\bigcap_{i=0}^{4} C_i = C_0 \cap C_1 \cap C_2 \cap C_3 \cap C_4$

$= \{0, -0\} \cap \{1, -1\} \cap \{2, -2\} \cap \{3, -3\} \cap \{4, -4\}$

$\bigcap_{i=0}^{4} C_i = \{\ \}$

6) For all sets A and B, $(A \cap B) \cup (A \cap B^c) = A$

: $(A \cap B) \cup (A \cap B^c) = A$

$A \cap (B \cup B^c) = A$ By Distributive Law

$A \cap (U) = A$ By Complement Law

$A \cap U = A$ By Identity Law

7) For all sets A, B and C.

$(A - B) \cup (C - B) = (A \cup C) - B$

$(A \cap B^c) \cup (C \cap B^c) = (A \cup C) \cap B^c$ By set Difference Law

$(B^c \cap A) \cup (B^c \cap C) = (A \cup B) \cap B^c$ By Associative Law

$B^c \cap (A \cup C) = (A \cup B) \cap B^c$ By Distributive Law

8) For all sets A, B and C

$(A-B)\cap(C-B) = (A\cap C)-B$

: $(A\cap B^c)\cap(C\cap B^c) = (A\cap C)\cap B^c$

$A\cap B^c\cap C\cap B^c = (A\cap C)\cap B^c$ By Commutative Law

$A\cap C\cap B^c\cap B^c = (A\cap C)\cap B^c$

$A\cap C\cap B^c = (A\cap C)\cap B^c$

9) Let $S=\{a,b,c\}$ and for each integer $i=0,1,2,3$.
Let S_i be the set of all subsets of S that have i elements.
List the elements in S_0, S_1, S_2 and S_3. Is $\{S_0,S_1,S_2,S_3\}$ a partition of $\mathcal{P}(S)$?

: $S=\{a,b,c\}$

$S_0=\{\ \}$, $S_1=\{\{a\},\{b\},\{c\}\}$

$S_2=\{\{\ \},\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\}\}$

$S_3=\{\{\ \},\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$

$S_0\cup S_1 S_2\cup S_3=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$

$\mathcal{P}(S)=\{\emptyset,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$

$\{S_0,S_1,S_2,S_3\}$ is a partition of $\mathcal{P}(S)$.

10) Contruct an algebric proof for the given statement
For all sets A and B, $A\cup(B-A)=A\cup B$

: $A\cup(B-A) = A\cup B$

$A\cup(B\cap A^c) = A\cup B$ By set Difference Law

$(A\cup B)\cap(A\cup A^c) = A\cup B$ By Distributive Law

$(A\cup B)\cap(U) = A\cup B$ By Negation Law.

$(A\cup B) = A\cup B$ Identity Law

11) For all sets A and B, $(A-B)\cup(A\cap B)=A$

: $(A-B)\cup(A\cap B)=A$

$(A\cap B^c)\cup(A\cap B)= A$ By set Difference Law

$A\cap(B^c\cup B) = A$ By Distributive Law

$A\cap(U) = A$ By Negation Law

$A = A$ By Identity Law

Symmetric Difference

12) $A \Delta B = B \Delta A$

: $(A-B) \cup (B-A) = (B-A) \cup (A-B)$ By Commutative Law

13) $A \Delta \phi = A$

$(A-\phi) \cup (\phi - A) = A$

$(A \cap (\phi)^c) \cup (\phi \cap A^c) = A$ By Set Difference Law

$(A \cap U) \cup (\phi) = A$ By Negation + Idempotent Law

$(A) \cup \phi = A$ By Identity Law

$\boxed{A = A}$ = By Identity Law

14) $A \Delta A^c = U$

: $(A - A^c) \cup (A^c - A) = U$

$(A \cap (A^c)^c) \cup (A^c \cap A^c) = U$ By Set Difference Law

$(A \cap A) \cup (A^c) = U$ By Double Complement Law

$(A) \cup A^c = U$ By AND operator

$U = U$ By Negation Law

15) $A \Delta A = \phi$

: $(A-A) \cup (A-A) = \phi$

$(A \cap A^c) \cup (A \cap A^c) = \phi$ By Set Difference Law

$\phi \cup \phi = \phi$ By Negation Law

$\phi = \phi$ By Idempotent Law

▶ Boolean Algebra

Q.1: Prepare a truth table for the following Boolean expressions:

(i) $XYZ + \bar{X}\bar{Y}\bar{Z}$

: 3 variables → $2^n = 2^3 = 8$ Row

X	Y	Z	$\bar{X}$	$\bar{Y}$	$\bar{Z}$	$X \cdot Y \cdot Z$	$\bar{X} \cdot \bar{Y} \cdot \bar{Z}$	$XYZ + \bar{X}\bar{Y}\bar{Z}$
1	1	1	0	0	0	1	0	1
1	1	0	0	0	1	0	0	0
1	0	1	0	1	0	0	0	0
1	0	0	0	1	1	0	0	0
0	1	1	1	0	0	0	0	0
0	1	0	1	0	1	0	0	0
0	0	1	1	1	0	0	0	0
0	0	0	1	1	1	0	1	1

(ii) $(A+D)(B+C)$

: 4 variables → $2^4 = 16$ Rows

A	B	C	D	A + D	B + C	(A + D)(B + C)
1	1	1	1	1	1	1
1	1	1	0	1	1	1
1	1	0	1	1	1	1
1	1	0	0	1	1	1
1	0	1	1	1	1	1
1	0	1	0	1	1	1
1	0	0	1	1	0	0
1	0	0	0	1	0	0
0	1	1	1	1	1	1
0	1	1	0	0	1	0
0	1	0	1	1	1	1
0	1	0	0	0	1	0
0	0	1	1	1	1	1
0	0	1	0	0	1	0
0	0	0	1	1	0	0
0	0	0	0	0	0	0

Q2: Simplify the following with the help of Boolean Algebra Rules:

(i) $AB + AC + ABC$

: $AB + ABC + AC$

$AB(1 + C) + AC$

$AB + AC$ $\because 1 + X = 1$ Identity Law

$A(B + C)$ Domination Law

(ii) $AB + A(\bar{B} + C) + AB\bar{C}$

: $A(B + \bar{B} + C + B\bar{C})$

$A(C + B\bar{C})$ By Negation Law

$A(B + \bar{B} + C + B\bar{C})$

$A(B + B\bar{C} + \bar{B} + C)$ By Associative Law

$A(B(1 + \bar{C}) + \bar{B} + C)$ By Commutative Law

$A(B + \bar{B} + C)$ By Negation Law

$A(1 + C)$ By Domination Law

A

(iii) $\bar{A}BC + A\bar{B}C + ABC + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

: $\bar{A}BC + ABC + A\bar{B}C + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

$BC(\bar{A} + A) + A\bar{B}C + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

$BC + A\bar{B}C + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

$BC + A\bar{B}C + \bar{C}(AB + \bar{A}\bar{B})$

$BC + A\bar{B}C + \bar{C}(AB + \bar{A}\bar{B})$

$BC + A\bar{B}C + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

$C(B + A\bar{B}) + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

$C((B + A)\cdot(B + \bar{B})) + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

$C((B + A)) + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

$AC + BC + AB\bar{C} + \bar{A}\bar{B}\bar{C}$

Q3: minimize the following Expressions:

a) $X = W\bar{Z}(W+Y) + WY(\bar{Z}+\bar{W})$

$WW\bar{Z} + W\bar{Z}Y + WY\bar{Z} + W\bar{W}Y$

$W\bar{Z} + W\bar{Z}Y + WY\bar{Z} + Y$

$W(\bar{Z} + \bar{Z}Y) + Y(W\bar{Z} + 1)$

$W((\bar{Z}+\bar{Z})\cdot(\bar{Z}+Y)) + Y(W\bar{Z})$

$W(\bar{Z} + \bar{Z}Y) + Y(W\bar{Z})$

$W(\bar{Z}(1+Y)) + Y(W\bar{Z})$

$W(\bar{Z}) + Y(W\bar{Z})$

$W\bar{Z} + Y(W\bar{Z})$

$W\bar{Z}(1+Y)$

$\boxed{W\bar{Z}}$

b) $X = \overline{(A+B)}\,(\bar{C})$

$= (\bar{A}\bar{B}) + C$

c) $X = \overline{(A\bar{B}C + B\bar{C})}$

$= (\bar{A}+B+\bar{C})\cdot(\bar{B}+C)$

Q4: Convert the following expression to Sum-of-product form.

(1) $(A+B)(\bar{B}+C)(\bar{A}+C)$

1: Construct the Truth Table → Input/output Table

3 variables → $2^3 = 8$ Rows

A	B	C	$\bar{A}$	$\bar{B}$	A+B	$\bar{B}+C$	$\bar{A}+C$	$(A+B)(\bar{B}+C)(\bar{A}+C)$	
1	1	1	0	0	1	1	1	1	Row 1
1	1	0	0	0	1	0	0	0	
1	0	1	0	1	1	1	1	1	Row 3
1	0	0	0	1	1	1	0	0	
0	1	1	1	0	1	1	1	1	Row 5
0	1	0	1	0	1	0	1	0	
0	0	1	1	1	0	1	1	0	
0	0	0	1	1	0	1	1	0	

Row 1 $\rightarrow$ $A \cdot B \cdot C = 1 \cdot 1 \cdot 1 = ABC$

Row 3 $\rightarrow$ $A \cdot \bar{B} \cdot C = 1 \cdot \bar{0} \cdot 1 = A\bar{B}C$

Row 5 $\rightarrow$ $\bar{A} \cdot B \cdot C = 0 \cdot 1 \cdot 1 = \bar{A}BC$

SOP: Sum-of products

$(ABC) + (A\bar{B}C) + (\bar{A}BC)$

$ABC + \bar{A}BC + A\bar{B}C$

$BC(A + \bar{A}) + A\bar{B}C$

$BC(1) + A\bar{B}C$

$BC + A\bar{B}C$

$C(B + A\bar{B})$

$C((A+B)\cdot(B+\bar{B}))$

$C((A+B)\cdot(1))$

$C(A+B)$

$\boxed{AC + BC}$

ii) $(A+C)(A\bar{B} + AC)(\bar{A}\bar{B} + \bar{C})$

1: Construct an Input / Output Table

variables $= 3 \rightarrow 2^3 = 8$ Rows.

A	B	C	$\bar{A}$	$\bar{B}$	$\bar{C}$	$A\bar{B}$	AC	$\bar{A}\bar{B}$	A+C	$A\bar{B}+AC$	$\bar{A}\bar{B}+\bar{C}$	$(A+C)(A\bar{B}+AC)(A\bar{B}+$
1	1	1	0	0	0	0	1	0	1	1	0	0
1	1	0	0	0	1	0	0	0	1	0	1	0
1	0	1	0	1	0	1	1	0	1	1	0	0
1	0	0	0	1	1	1	0	0	1	1	1	1
0	1	1	1	0	0	0	0	0	1	0	0	0
0	1	0	1	0	1	0	0	0	0	0	1	0
0	0	1	1	1	0	0	0	1	1	0	1	0
0	0	0	1	1	1	0	0	1	0	0	1	0

Q5: Convert the following expression to product-of-Sum form:

(i) $A + \bar{A}B + \bar{A}C$

1: Construct the Input/Output table

3 variables = 2^3 = 8 Rows

A	B	C	$\bar{A}$	$\bar{A}B$	$\bar{A}C$	$A + \bar{A}B + \bar{A}C$	SOP	POS
1	1	1	0	0	0	1	→ ABC	null
1	1	0	0	0	0	1	$AB\bar{C}$	null
1	0	1	0	0	0	1	$A\bar{B}C$	null
1	0	0	0	0	0	1	$A\bar{B}\bar{C}$	null
0	1	1	1	1	1	1	$\bar{A}BC$	null
0	1	0	1	1	0	1	$\bar{A}B\bar{C}$	null
0	0	1	1	0	1	1	$\bar{A}\bar{B}C$	null
0	0	0	1	0	0	0		$A+B+C$

$SOP = (ABC)+(AB\bar{C})+(A\bar{B}C)+(A\bar{B}\bar{C})+(\bar{A}BC)+(\bar{A}B\bar{C})+(\bar{A}\bar{B}C)$

$POS = A+B+C$

ii) $\overline{(AB+\bar{C})} + A\bar{C}B + B$

1: Construct the Input/output Table

n = 3 variables → 2^3 = 8 Rows

A	B	C	$\bar{C}$	AB	$\overline{AB+\bar{C}}$	$A\bar{C}B$	$\overline{AB+\bar{C}}+A\bar{C}B+B$	SOP	POS
1	1	1	0	1	0	0	1	ABC	null
1	1	0	1	1	0	1	1	$AB\bar{C}$	null
1	0	1	0	0	1	0	1	$A\bar{B}C$	null
1	0	0	1	0	0	0	0	null	! null → $A+\bar{B}+\bar{C}$
0	1	1	0	0	1	0	1	$\bar{A}BC$	null
0	1	0	1	0	0	0	1	$\bar{A}B\bar{C}$	null
0	0	1	0	0	1	0	1	$\bar{A}\bar{B}C$	null
0	0	0	1	0	0	0	0	null	$A+B+C$

Product of Sum = $(A+\bar{B}+\bar{C})(A+B+C)$

Q6: Express the given function in the product of sums also draw its circuit and truth table.

$X = A\bar{B}C + \bar{A}\bar{B}\bar{C} + AB$

1: Construct a truth table:

n = 3 variables → $2^3 = 8$ Rows

A	B	C	$\bar{A}$	$\bar{B}$	$\bar{C}$	$A\bar{B}C$	$\bar{A}\bar{B}\bar{C}$	AB	$A\bar{B}C + \bar{A}\bar{B}\bar{C} + AB$	SOP	POS
1	1	1	0	0	0	0	0	1	1	ABC	null
1	1	0	0	0	1	0	0	1	1	$AB\bar{C}$	null
1	0	1	0	1	0	1	0	0	1	$A\bar{B}C$	null
1	0	0	0	1	1	0	0	0	0	null	$\bar{A}+\bar{B}+\bar{C}$
0	1	1	1	0	0	0	0	0	0	null	$\bar{A}+B+C$
0	1	0	1	0	1	0	0	0	0	null	$\bar{A}+B+\bar{C}$
0	0	1	1	1	0	0	0	0	0	null	$\bar{A}+\bar{B}+C$
0	0	0	1	1	1	0	1	0	1	$\bar{A}\bar{B}\bar{C}$	null

Sum-of-product form: $ABC + AB\bar{C} + A\bar{B}C + \bar{A}\bar{B}\bar{C}$

Product-of-Sum: $(A+\bar{B}+\bar{C})(\bar{A}+B+C)(\bar{A}+B+\bar{C})(\bar{A}+\bar{B}+C)$

Sum of product → negate the zeros

product of Sum → negate the ones

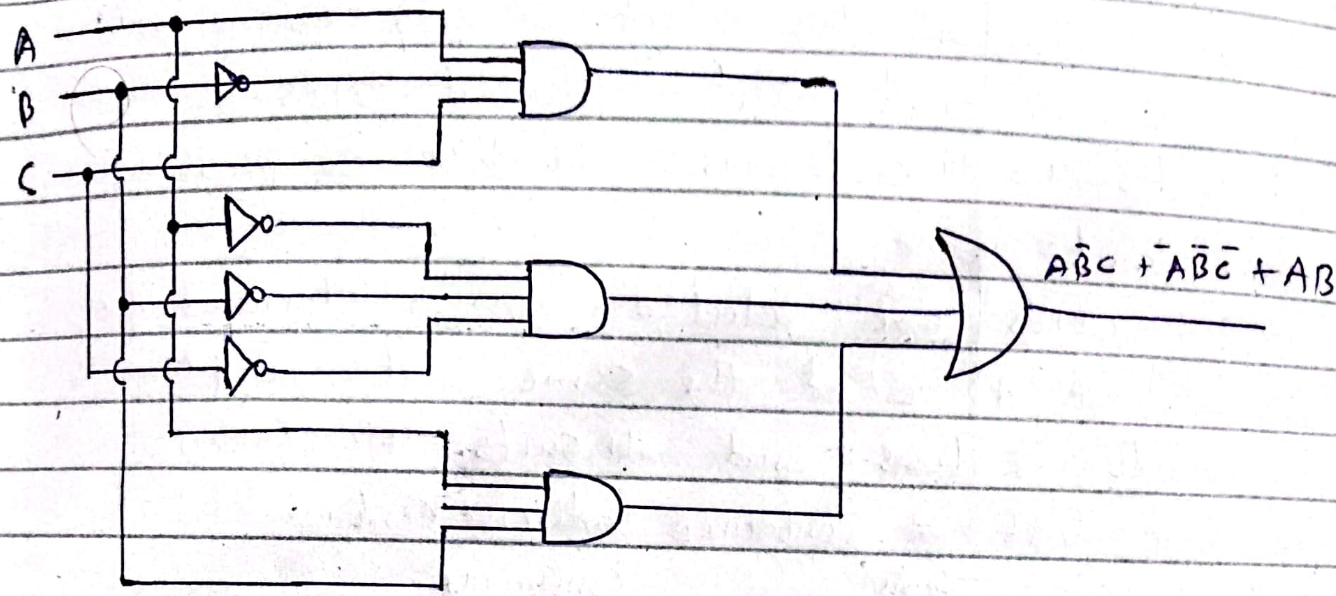
A	B	C	$\bar{A}$	$\bar{B}$	$\bar{C}$	$A\bar{B}C$	$\bar{A}\bar{B}\bar{C}$	AB	$A\bar{B}C + \bar{A}\bar{B}\bar{C} + AB$	$\bar{0}$ SOP	$\bar{1}$ POS
1	1	1	0	0	0	0	0	1	1	ABC	null
1	1	0	0	0	1	0	0	1	1	$AB\bar{C}$	null
1	0	1	0	1	0	1	0	0	1	$A\bar{B}C$	null
1	0	0	0	1	1	0	0	0	0	null	$\bar{A}+B+C$
0	1	1	1	0	0	0	0	0	0	null	$A+\bar{B}+\bar{C}$
0	1	0	1	0	1	0	0	0	0	null	$A+\bar{B}+C$
0	0	1	1	1	0	0	0	0	0	null	$A+B+\bar{C}$
0	0	0	1	1	1	0	1	0	1	$\bar{A}\bar{B}\bar{C}$	null

Sum-of-Products: $ABC + AB\bar{C} + A\bar{B}C + \bar{A}\bar{B}\bar{C}$

Product of Sums: $(\bar{A}+B+C)(A+\bar{B}+\bar{C})(A+\bar{B}+C)(A+B+\bar{C})$

$(\bar{A}+B+C)(A+\bar{B}+\bar{C})(A+\bar{B}+C)(A+B+\bar{C}) \rightarrow$ POS

Draw the Circuit of $\rightarrow$ $X = A\bar{B}C + \bar{A}\bar{B}\bar{C} + AB$



Q9: Draw a logic circuit using only NAND Gates for which output expression is $X = AC+BC$

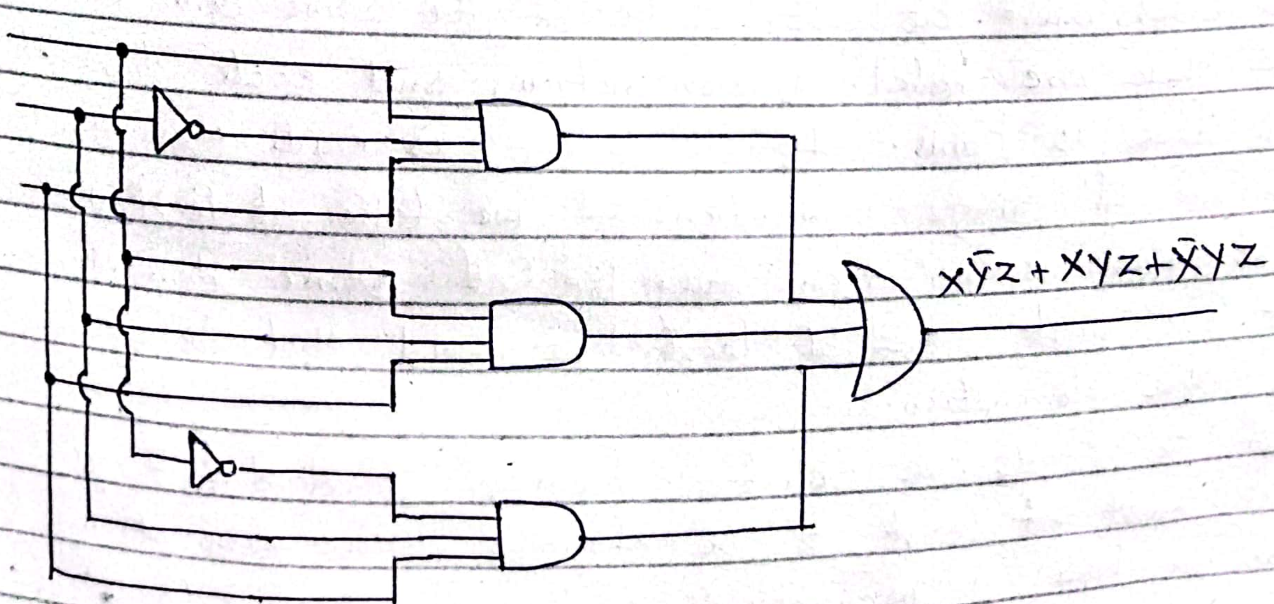
A1: $AC+BC \rightarrow$ find the negation form in product.

$\overline{(AC+BC)} = (\bar{A}+\bar{C})(\bar{B}+\bar{C}) \Rightarrow \overline{AC} \cdot \overline{BC} \Rightarrow \overline{\overline{AC} \cdot \overline{BC}}$

Q10: Draw a circuit Diagram for the following:

(i) $F = X\bar{Y}Z + XYZ + \bar{X}YZ$

:



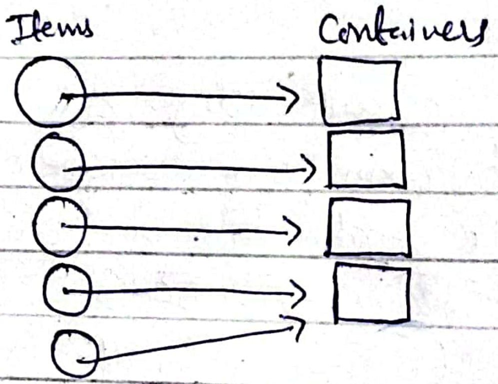
▸ Pigeonhole Principle

1) If 4 cards are selected from a standard 52-deck must atleast 2 be of the same suit why?

: No. let's consider You select 4 aces, So we have 4 card of different suit. Selected cards = 4 = 4 items. Total suits = 4 = 4 Containers. $4 \not> 4$ So pigeon hole Principle can't be applied.

2) If 5 cards are selected from a standard deck must at least 2 be of the same suit? why?

: 5 cards = 5 items and 4 suits = 4 Containers. then there exist a container which contains at least two items/ pigeons.



3) If 13 cards are selected from a standard deck of 52 cards, must atleast 2 be of the same domination? why?

: No. There are total 4 domination, and each domination can have 13 cards. If we select 13 cards than they can be of single Domination. If we take at least 14 cards, than 13 of them may be of same domination but we must have at least one card that belong to the other domination.

4) If 20 cards are selected from a standard 52-card deck, must at least 2 be of the same domination.

: Yes. we have 4 dominations and we are going to select 20 cards, and previously we had 4 domomination and now we are going to choose one card after choosing 13.

Congruence Modulo

Let a, b, and n any integers and suppose $n>1$. The following statements are all equivalent:

1) $n \mid (a-b)$
2) $a \equiv b \pmod{n}$
3) $a = b + kn$ for some integer k
4) a and b have the same (nonnegative) remainder when divided by n
5) $a \bmod n = b \bmod n$

- $\boxed{\text{Modular Arithmetic}}$

Let a, b, c, d and n be integers with $n>1$, and suppose

$$a \equiv c \pmod{n} \text{ and } b \equiv d \pmod{n}.$$

Then

1) $(a+b) \equiv (c+d) \pmod{n}$
2) $(a-b) \equiv (c-d) \pmod{n}$
3) $ab \equiv cd \bmod n$
4) $a^m \equiv c^m \bmod n$ for all integers m.

- $\boxed{\text{Corollary}}$

Let a, b, and n be integers with $n>1$. Then

$$ab \equiv [(a \bmod n)(b \bmod n)] \pmod{n}$$

or equivalently

$$ab \bmod n = [(a \bmod n)(b \bmod n)] \bmod n$$

In particular, if m is a positive integer, then

$$a^m \equiv [(a \bmod n)^m] \pmod{n}$$

1) $55 + 26 \equiv (3+2) \pmod 4$

$81 \equiv (5) \bmod 4$

$81 \equiv 5 \bmod 4 \rightarrow 4 \mid (81-5)$

$= 4 \mid (76)$ OR $4k = 81 - 5$

$76 = 4 \cdot 19$ $\quad 4k = 76$

$\boxed{k = 19}$ which in an integer

2) $55 - 26 \equiv (3-2) \pmod 4$

$29 \equiv (1 \bmod 4)$

$4 \mid (29 - 1)$

$4k = 28$

$\boxed{k = 7}$ which is true because k is an integer

3) $55 \cdot 26 \equiv (3 \cdot 2) \pmod 4$

$1430 \equiv 6 \bmod 4$

$4 \mid (1430 - 6)$

$4k = 1424$

$\boxed{k = 356}$ True because k is an integer

4) $55^2 \equiv 3^2 \bmod 4$

$3025 \equiv 9 \bmod 4$

$4 \mid (3025 - 9)$

$4k = 3016$

$\boxed{k = 754}$ True because k is an integer

5) Compute Product modulo n for $(55 \cdot 26) \bmod 4$

$(55 \cdot 26) \bmod 4 \equiv [(55 \bmod 4)(26 \bmod 4)] \bmod 4$

$\equiv [(3)(2)] \bmod 4$

$\equiv 6 \bmod 4$

$= 2$ which is true because $2 < 4$

6) Compute $a^k \bmod n$ when k is a power of 2

• Find $144^4 \bmod 713$

Find 144^4 mod 713

$144^4 \bmod 713 \equiv (144^2)^2 \bmod 713$

$= [144^2 \bmod 713]^2 \bmod 713$

$= [20736 \bmod 713]^2 \bmod 713$

$= [59]^2 \bmod 713$

$= 3481 \bmod 713$

$= 629$

7) Computing a^k mod n when k is not a power of 2

Find 12^{43} mod 713

: First write the exponent as a sum of powers of 2:

$43 = 32 + 8 + 2 + 1 = 2^5 + 2^3 + 2 + 1$.

Next compute 12^{2^k} for $k = 1, 2, 3, 4, 5$

OR Use the Corollary property.

We are going to use the corollary property

$12^{43} = 12^{32+8+2+1} = 12^{32} \cdot 12^8 \cdot 12^2 \cdot 12^1$.

Thus by Corollary

$12^{43} \bmod 713 = [(12^{32} \bmod 713)(12^8 \bmod 713)(12^2 \bmod 713)(12^1 \bmod 713)] \bmod 713$

$= [(485)(629)(144)(12)] \bmod 713$

$= 48$

8) Find 7^{-1} mod 13

: $7x = 1 \bmod 13$

$7x - 1 = 13$

$\boxed{7 \cdot 2 = 1 \bmod 13}$

$\boxed{x = 2}$

$7x = 1 \bmod 13$

$x = \frac{1}{7} \bmod 13$

$x = \frac{1 + 13}{7} \bmod 13$

$x = \frac{14}{7} = \frac{2}{1} = \boxed{2 \bmod 13}$

9) Find $13^{-1} \bmod 17$

: $13x = 1 \bmod 17$

$13 \cdot 4 = 1 \bmod 17$

$(52-1) \bmod 17 = 0$

$51 \bmod 17 = 0$

Hence prooved

$13x = 1 \bmod 17$

$x = \frac{1}{13} \bmod 17$

$x = \frac{1+17}{13} \bmod 17$

$x = \frac{18+17}{13} = \frac{35+17}{13} = \frac{52}{13} = \frac{4}{1}$

$\boxed{x = 4 \bmod 17}$

$\boxed{x = 4}$

10) Solve for x: $6x \equiv 5 \bmod 7$

$6x \equiv 5 \bmod 7$.

$6 \cdot 2 = 5 \bmod 7$

$(6 \times 2) - 5 \bmod 7 = 0$

$12 - 5 \bmod 7 = 0$

$\boxed{7 \bmod 7 = 0}$

Hence prooved

$6x = 5 \bmod 7$

$x = \frac{5}{6} \bmod 7$

$x = \frac{5+7}{6} \bmod 7$

$x = \frac{12}{6} = \frac{2}{1} = 2 \bmod 7$

$\boxed{x = 2}$

11) Solve for x: $7x \equiv 4 \bmod 11$

: $7x \equiv 4 \bmod 11$

$7x - 4 \bmod 11 = 0$

put $x = 10$

$7 \cdot 10 - 4 \bmod 11 = 0$

$70 - 4 \bmod 11 = 0$

$\boxed{66 \bmod 11 = 0}$

Hence prooved

$7x \equiv 4 \bmod 11$

$7x \equiv 4 \bmod 11$

$x = \frac{4}{7} \bmod 11$

$x = \frac{4+11}{7} = \frac{15+11}{7} = \frac{26+11}{7}$

$\frac{37+11}{7} = \frac{48+11}{7} = \frac{59+11}{7} = \frac{70}{7}$

$\frac{70}{7} = \frac{10}{1} = \boxed{10 \bmod 11}$

$\boxed{x = 10}$

12) Solve for x: $41x \equiv 5 \mod 51$

: $41x \equiv 5 \mod 51$

$x = \frac{5}{41} \mod 51$

$x = \frac{5+51}{41} = \frac{56+51}{41}$

$= \frac{107+51}{41} = \frac{158+51}{41} = \frac{209}{41}$

$= \frac{260+51}{41} = \frac{311+51}{41} = \frac{362+51}{41} = \frac{413+51}{41} = \frac{464+51}{41} = \frac{515}{41}$

$\frac{566}{41} = \frac{617+51}{41} = \frac{668+51}{41} = \frac{719+51}{41} = \frac{770}{41}$

▸ Graphs and Graph Models

Defination

A Graph $G = (V, E)$ consist of V, a non-empty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints (or adjacents), an edge is said to connect its endpoints.

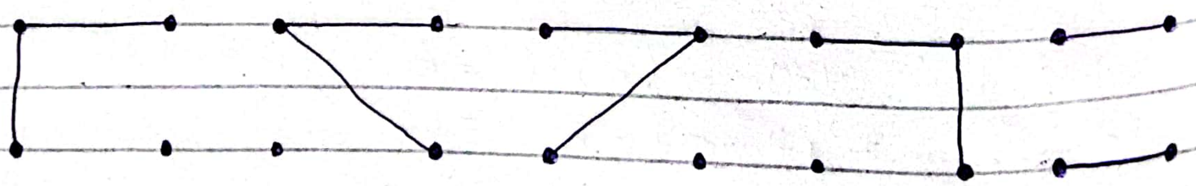
• **Finite VS Infinite**

The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set is called an infinite graph and in contrast a graph with a finite vertex set is called finite graph.

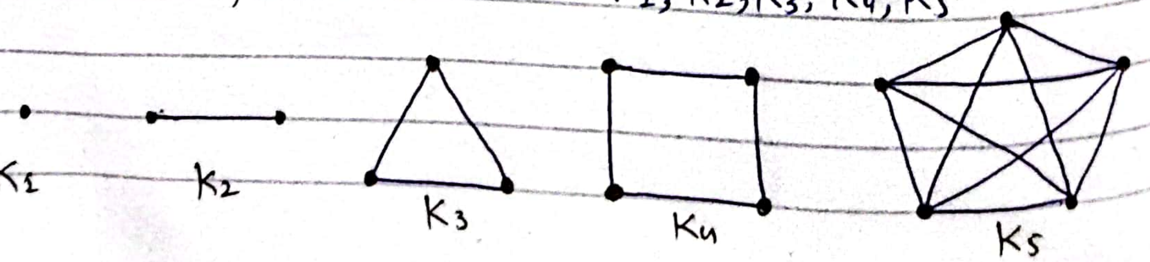
• **Graph Terminology**

	Type	Edges	multiple Edges	Loops
1)	Simple Graph	Undirected	NO	No
2)	multi Graph	Undirected	Yes	No
3)	Pseudo Graph	Undirected	Yes	Yes
4)	Simple Directed Graph	Directed	No	No
5)	Directed multi Graph	Directed	Yes	Yes
6)	Mixed Graph	Direct / Undirected	Yes	Yes

• Simple Graphs



• Complete Graphs on n vertices: K_1, K_2, K_3, K_4, K_5



- Representations of Graphs ·

The two commonly used representations of a graph are

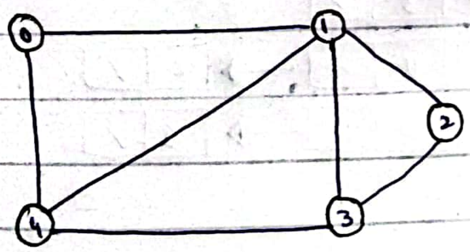
1) Adjacency Matrix (Square Array)
2) Adjacency List (Linked List)

Some others: Incidence matrix and Incidence List

- Adjacency Matrix

Adjacency matrix is a 2D array of size V×V where V is the number of vertices in a graph. Let the 2D array be adj[ ][ ], a slot adj[i][j] = 1 indicates that there is an edge from vertex i to j. The adjacency matrix for an undirected graph is always symmetric.

Ex:-



Adjacency matrix

$$
\begin{array}{c|c}
 & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\
\begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}
\end{array}
$$

Advantages of Adjacency matrix

i) Representation is easier to implement and follow.
ii) Removing an edge takes less time

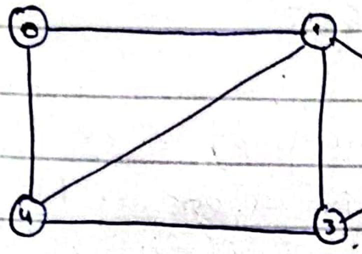
Disadvantages of Adjacency matrix

i) Consumes more space (Like zero(s) in matrix) Even if the graph is sparse (Contains less number of edges) it consumes the same space
ii) Adding a vertex takes time.

- **Adjacency List**

An array of linked lists is used. The size of array is equal to the number of vertices. Let the array be an array[ ]. An entry array[i] represents the linked list of vertices adjacent to ith vertex.

Ex:-



Adjacency List

0 [ ] → [1|] → [4|/]
1 [ ] → [0|] → [4|] → [2|] → [3|/]
2 [ ] → [1|] → [3|/]
3 [ ] → [1|] → [4|] → [2|/]
4 [ ] → [3|] → [0|] → [1|/]

- **The Handshake Theorem**

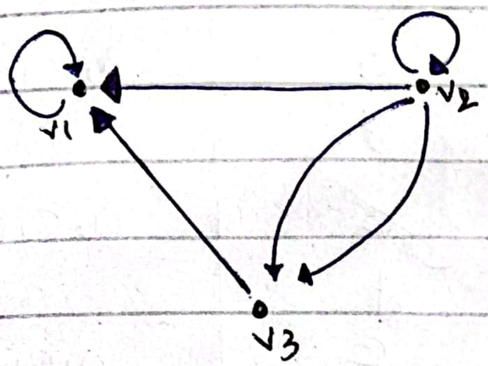
If G is any graph, then the sum of degrees of all the vertices of G equals twice the number of edges of G. Specificially, if the vertices of G are $v_1, v_2, \dots, v_n$ where n is a non-negative integer, then

The total degree of G $= deg(v_1) + deg(v_2) + \dots + deg(v_n)$
$= 2 \times$ (number of edges of G)

⇒ sharpen pencil

Matrices and Directed Graph

- Adjacency matrix of the Directed Graph

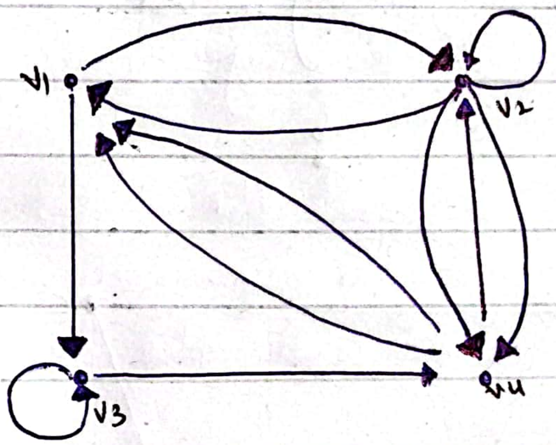


$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

out degree Concept

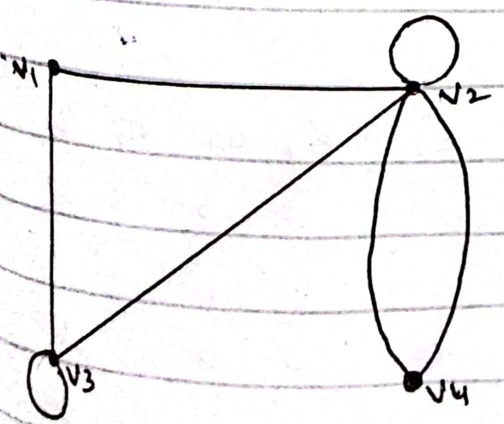
- obtaining directed Graph from a matrix

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 0 & 0 \end{bmatrix} \end{array}$$



Matrices and Undirected Graph

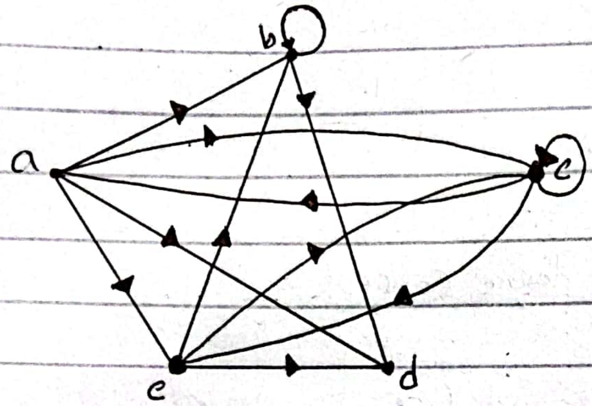
- Adjacency matrix of an Undirected Graph



$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix} \end{array}$$

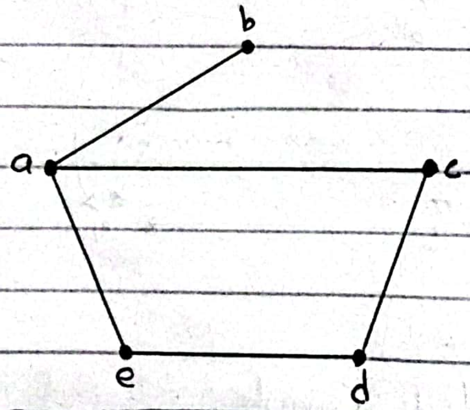
- **Adjacency List**

i) Adjacency List for a Directed Graph



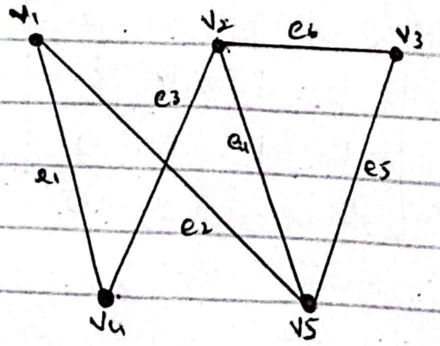
Initial vertex	Terminal vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

ii) Adjacency List for an Undirected Graph



vertex	Adjacent vertices
a	b, c, e
b	a
c	a, d
d	e, c
e	a, d

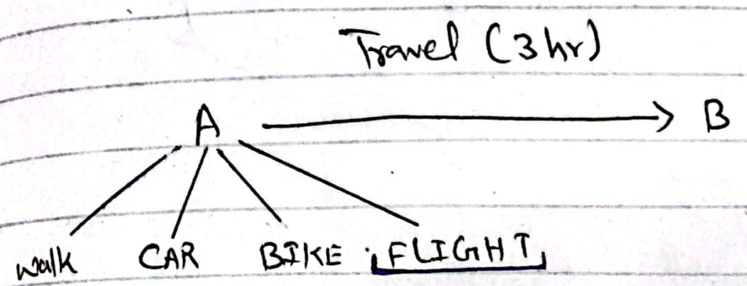
- **Incidence Matrices**



$$
\begin{array}{c}
 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0
\end{bmatrix}
\end{array}
$$

Greedy Method

- Problem Solving Method
- Used to Solve Optimization problems
  - Min or max result



Can be more than 1 → [feasible solution, because it satisfies 3 hour condition

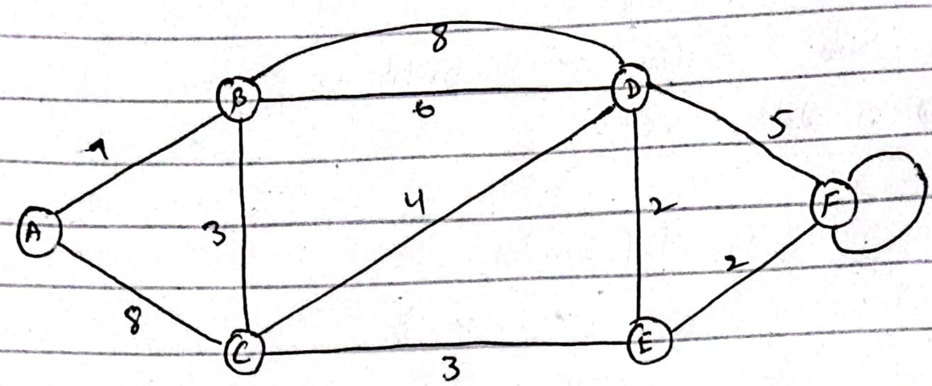
Only one ← [Optimal Solution is one of feasible Solution which gives minimum possible result.

Two other strategies for solving optimization problems
i) Dynamic programming
ii) Branch and Bound

- Feasible Solution → multiple possibilities
- Optimal solution → selection of one best choice [base]

▸ Applications of Greedy Algorithm
i) Knapsack
ii) Job sequencing
iii) Minimum Spanning tree
iv) Dijsktra
v) Prim's
vi) Kruskal
vii) Huffman encoding

Prim's Algorthim



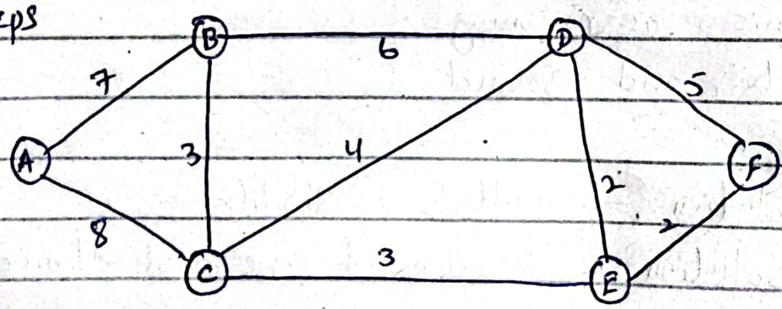
steps

1) Remove loops and parallel edges
   (keep edge with min weight)
2) Choose any arbitary node as root.

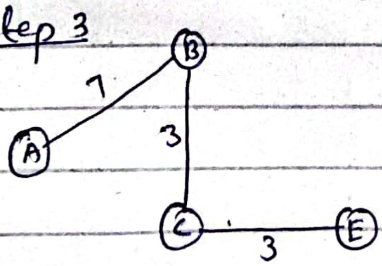
3) See all the edges from A and choose the Minimum one. Now check for edges of new node and also the previous node.

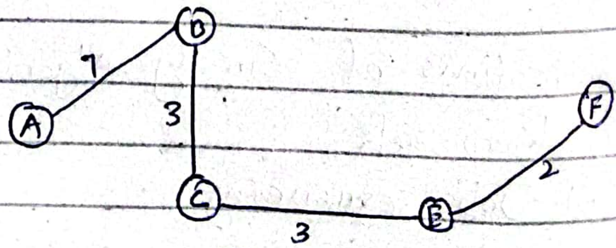
Solution

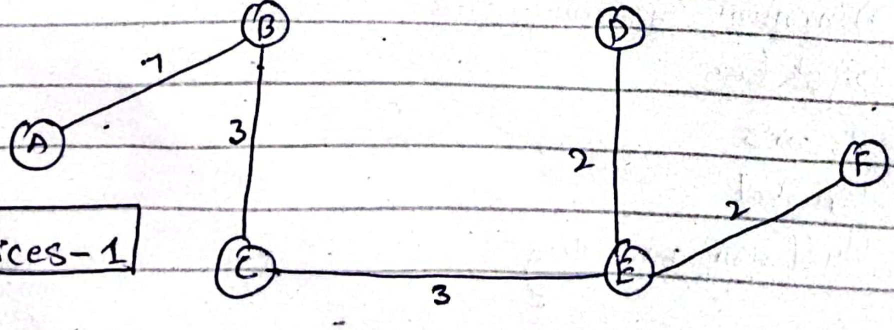
1-2 steps



Step 3

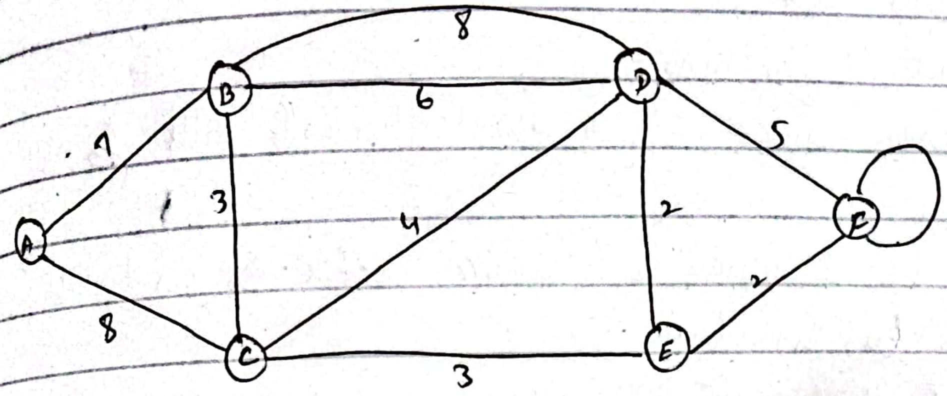






$\boxed{\text{edges} = \text{vertices} - 1}$

Kruskal Algorthim



steps

1) Remove loops and parallel edges

2) Arrange edges according to increasing order of edge weight.

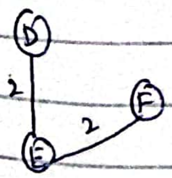
3) Choose the edge with min weight in such a way that no cycle is formed.

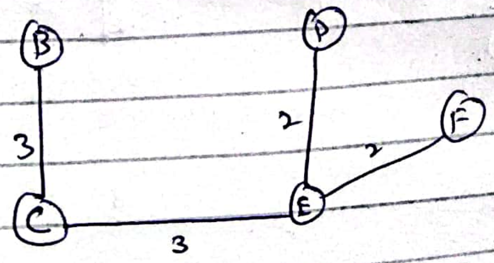
Solution

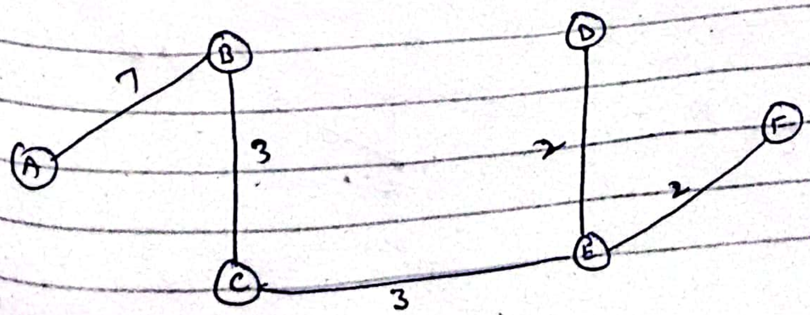
step 2

AB = 7	CD = 4	FE = 2
AC = 8	CE = 3	
BC = 3	DE = 2	
BD = 6	DF = 5	

Step 3





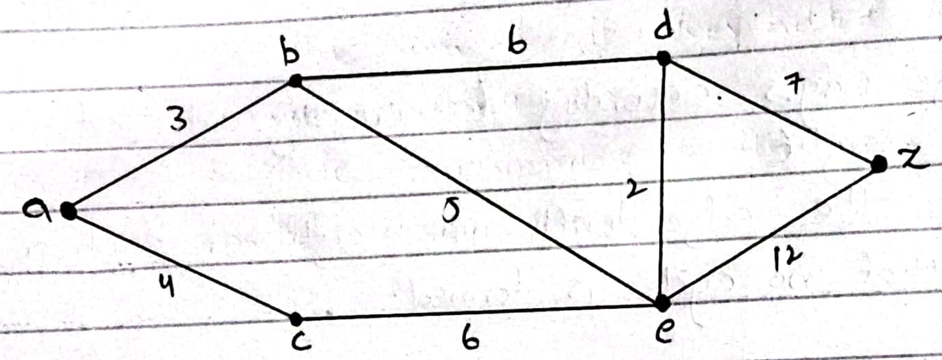


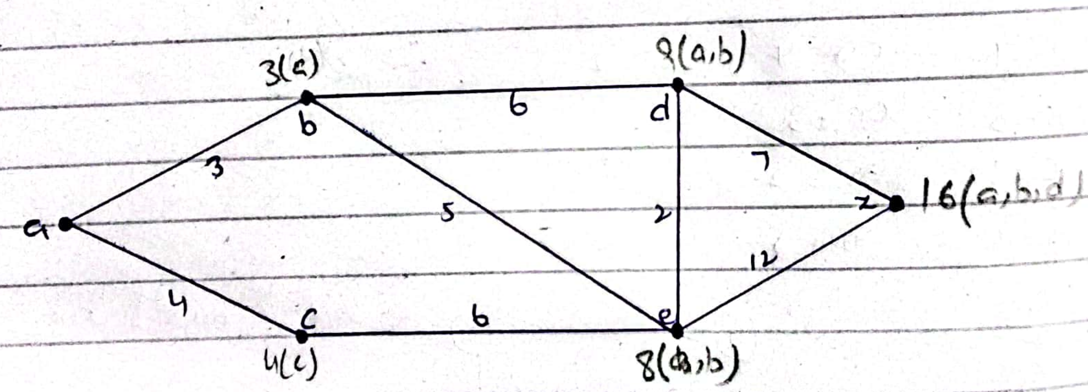
Single Source Shortest path Algorthims

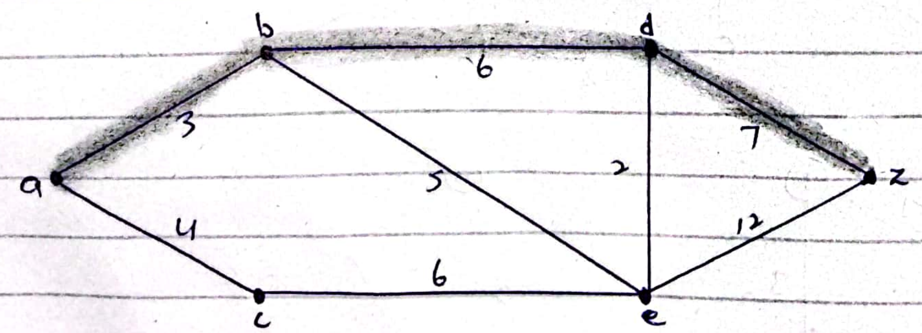
i) Dijsktra
ii) Bellmen Ford

⇒ Dijsktra Algorthim

i) from one Source, find the shortest path to all others
ii) make the distance of source vertex 0, and all others to infinity.
iii) Select the vertex with shortest path
iv) It Can not work for negative edge weights







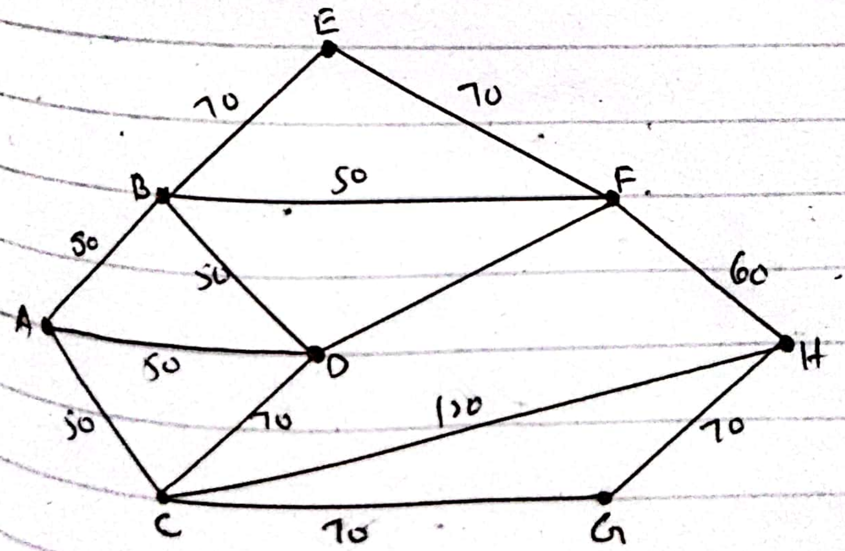
Chinese Postman Problem

- Understand the chines postman problem
- Apply an algorthum to solve problem
- Understand the importence of order of vertices of graphs

• Traversable Graph

A traversable graph is one that can be drawn without taking a pen from the paper and without retracing the same edge. In such case the graph is said to have an Eulerian trail.

In the following example a postman has to start at A, walk along all 13 streets and return to A. The numbers on each edge represent the length, in metres, of each street. The problem is to find a trail that uses all edges of a graph with minimum length.



An algorthim for finding an optimal chinese postman route is

Step 1 List all odd vertices

Step 2 List all possible pairings of odd vertices.

Step 3 For each pairing find the edges that connect the vertices with the minimum weight

Step 4 Find the pairings such that the sum of the weight is minimised.

Step 5 On the original graph add the edges that have been found in step 4.

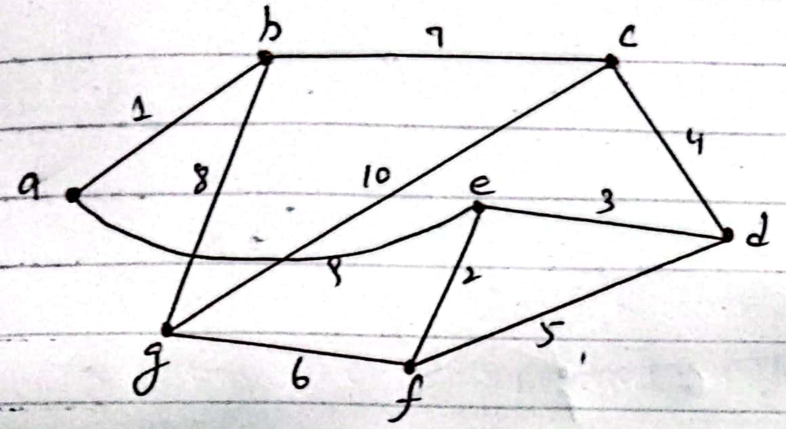
Step 6 The length of an optimal chinese postman route is the sum of all edges added to the total found in step 4.

Step 7 A route corresponding to this minimum weight can be easily found.

Algorithms with Greedy Approach

- Use Kruskal algorithm to find a minimum spanning tree.

①



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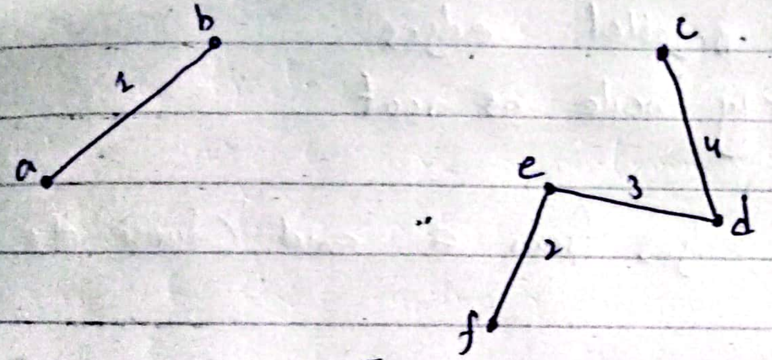
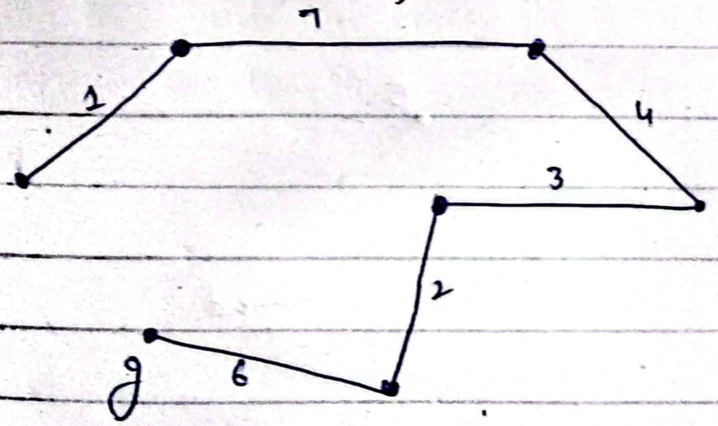
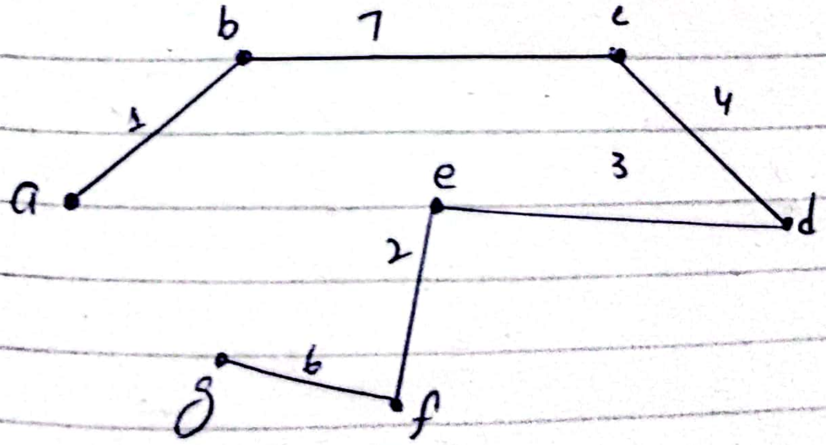
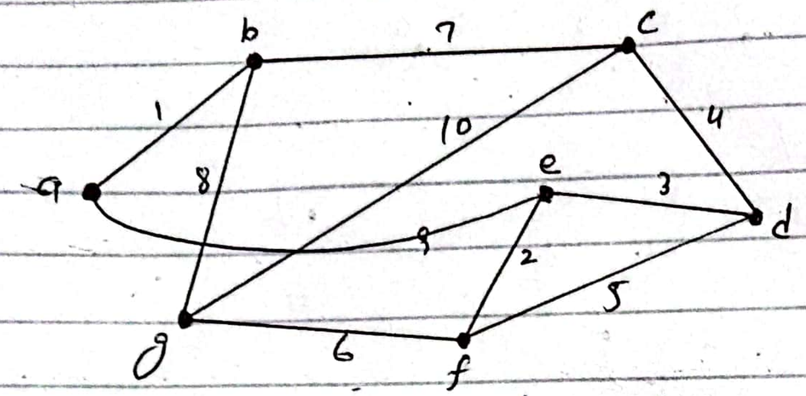




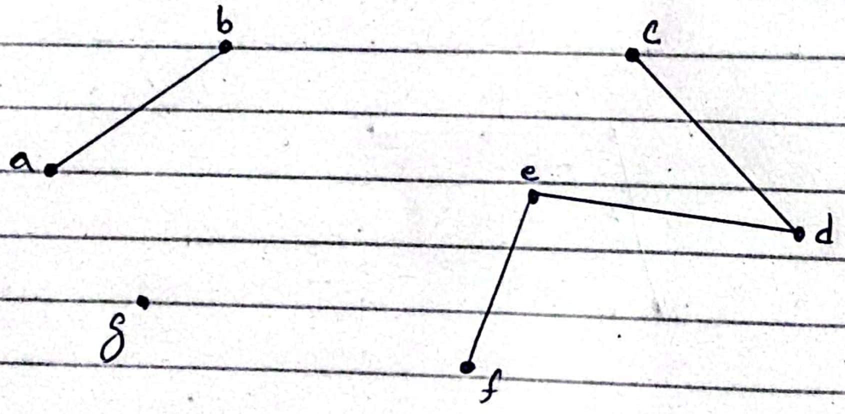


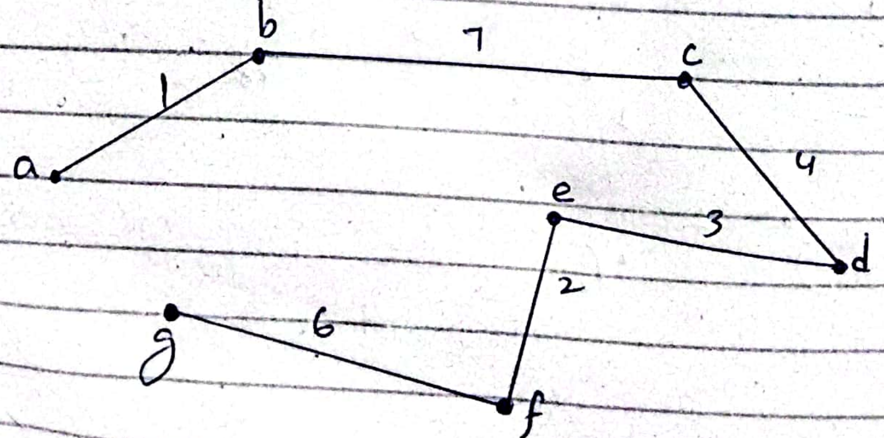
(2) Use Prim's Algorithm to Find Minimum Spanning Tree.



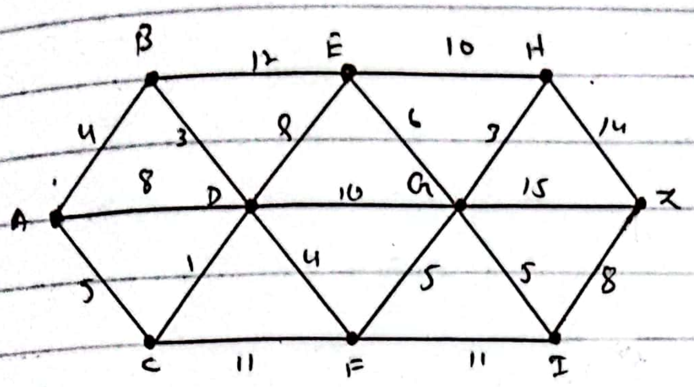
1: Remove loops and parallel edges
2: Choose an arbitrary node as root
(a)

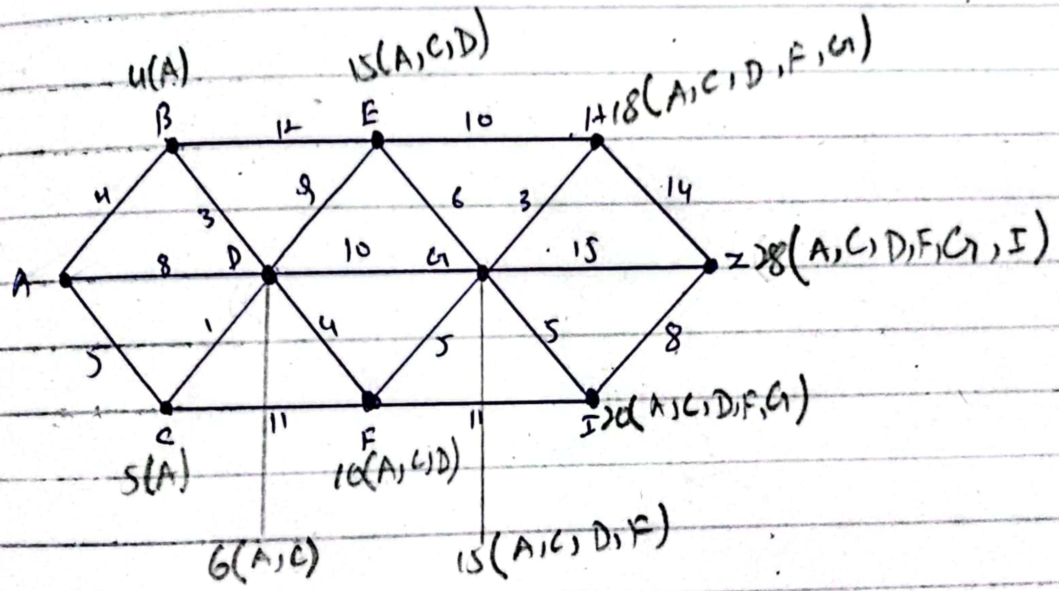
3: Now see all the edges from a and Choose the minimum one.

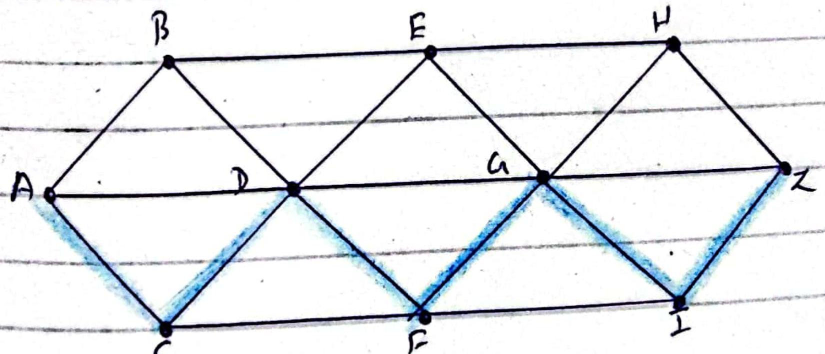




(3) Use Dijsktra Algorthim to Find Minimum Spanning Tree.



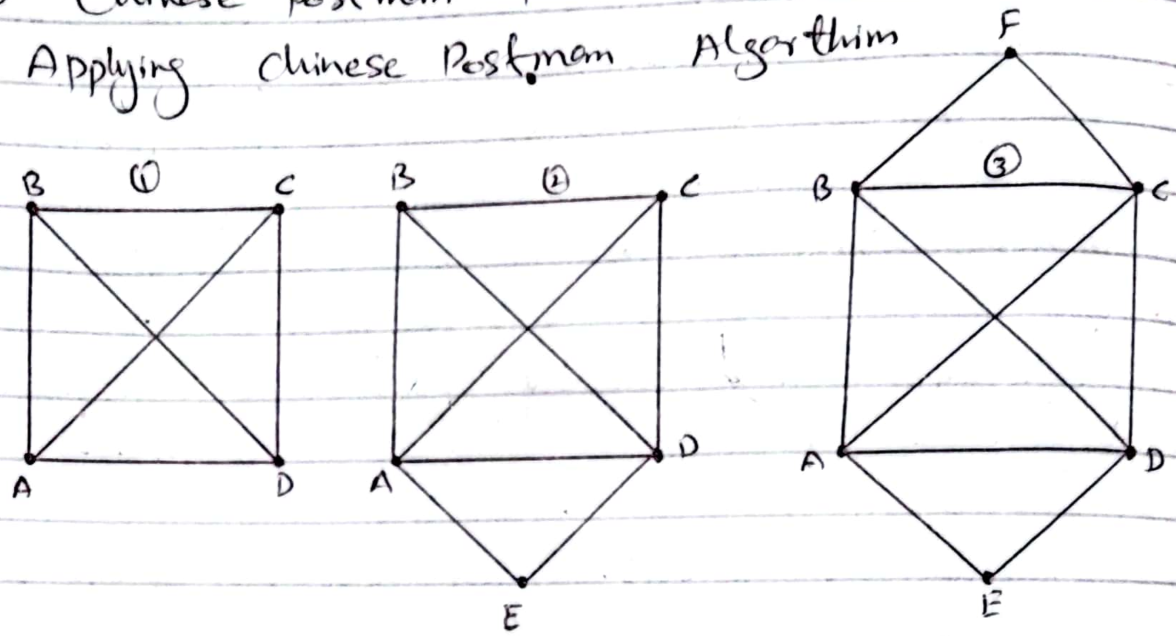




Short Distance = 28

(4) Chinese Postman Problems

- Applying Chinese Postman Algorthim



what is the difference Between the three graphs?
In order to establish the differences we must Consider the order of the vertices for each graph. we obtain the following:

Graph 1

vertex	Order
A	3
B	3
C	3
D	3

Graph 2

vertex	order
A	4
B	3
C	3
D	4
E	2

Graph 3

vertex	Order
A	4
B	4
C	4
D	4
E	2
F	2

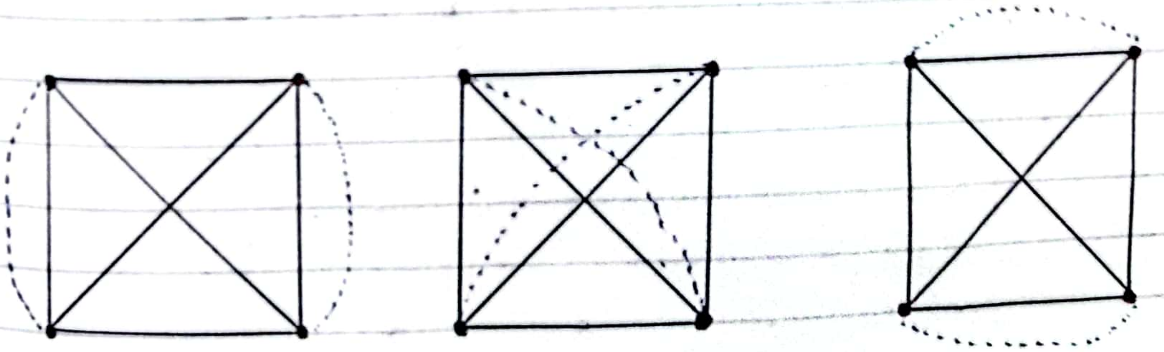
: when the Order of all vertices is even, the graph is traversable and we can draw it. when there are two odd vertices we can draw the graph but the start and end vertices are different. when there are four odd vertices the graph can't drawn without repeating an edge.

Eulerian trail: An Eulerian trail uses all the edges of a graph. For a graph to be Eulerian all the vertices must be of even order.

• Semi Eulerian: If a graph has two odd vertices then the graph is said to be semi-Eulerian. A trail can be drawn starting at one of the odd vertices and finishing at the other odd vertex.

⇒ To draw the graph with odd vertices, edges need to be repeated. To find such a trail we have to make the order of each vertex even

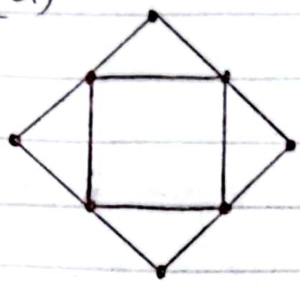
In graph 2 there are four vertices of odd order and we need to pair the vertices together by adding an extra edge to make the order of each vertex four. We can join AB and CD, or AC & BD, or AB and BC.



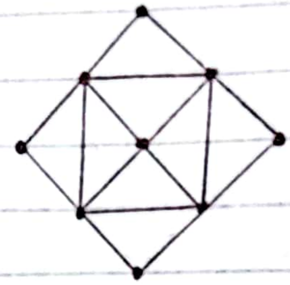
In each case the graph is now Traversable.

which of the graphs below is traversable ?

(a)



(b)



c)

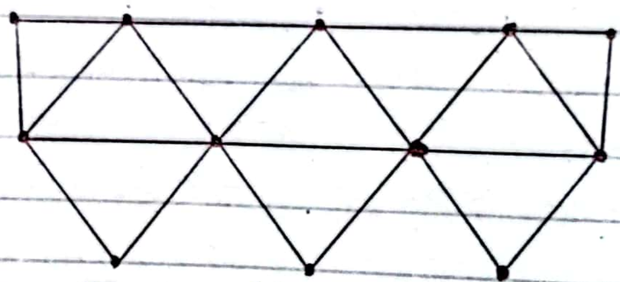
a: Traversable

b: Not Traversable

c: Traversable

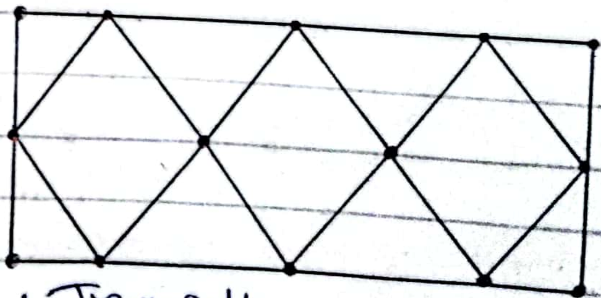
which of the graphs below are traversable?

1)



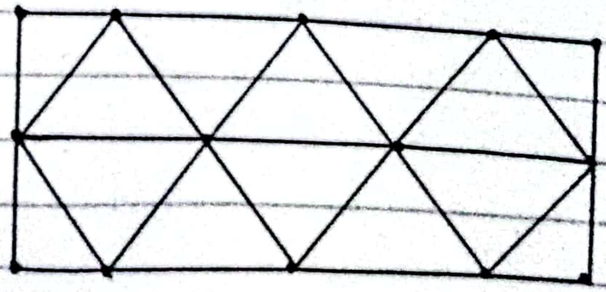
:- Traversable

2)



: Traversable

3)



: Not Traversable

- Pairing odd vertices
- If there are two odd vertices there is only one way of pairing them together
- If there are four odd vertices there are three ways of pairing them together.
- How many ways are there of pairing six or more odd vertices together?

⇒ If there are six odd vertices ABC DEF, then Consider the vertex A. It can be paired with any of the other five vertices and still leave four odd vertices. We Know that four odd vertices can be paired in three ways; therefore the number of ways of pairing six odd vertices is $5 \times 3 \times 1 = 15$.

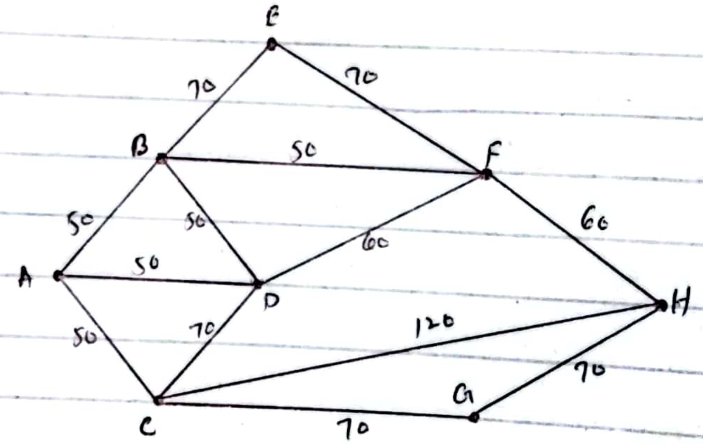
Number of odd vertices	Number of possible pairings.
2	1
4	$3 \times 1 = 3$
6	$5 \times 3 \times 1 = 15$
8	$7 \times 5 \times 3 \times 1 = 105$
10	$9 \times 7 \times 5 \times 3 \times 1 = 945$
n	$(n-1) \times (n-3) \times (n-5) \ldots \times 1$

Chinese Postman Algorthim

• To find a minimum Chinese postman route we must walk along each edge at least once and in addition we must also walk along the least pairings of odd vertices on one extra occassion.

Pattern of steps of CPA → $\boxed{\text{OPCMSL}}$

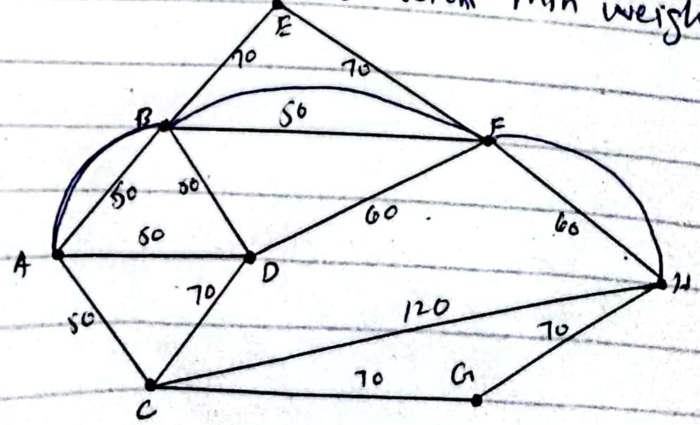
1: Order all odd vertices.
2: Pairing all odd vertices.
3: Find the edges that Connect vertices with minimum weight.
4: Minimised Sum Pairs
5: Length of route.



Solution

Pattern: OPCMSL

1: odd vertices → A and H
2: Pair odd vertices → AH
3+4: Connect vertices with min weight: AB, BF, FH, 160 length



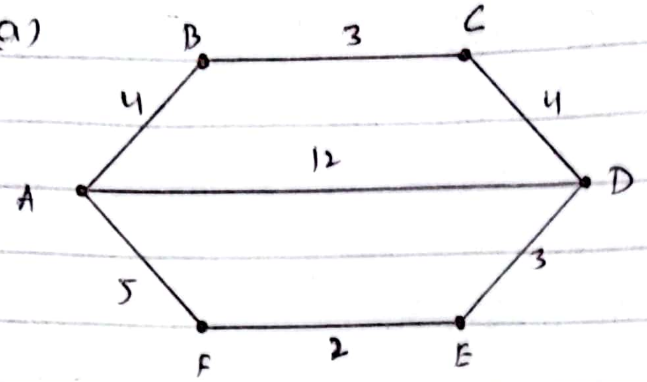
5: length: Length of the optimal chinese postman route is the sum of all edges in the original Network, which is 840m, plus the answer found in step 4, which is 160m. Hence the length of the optimal chinese postman route is 1000m.

6: One possible route:

$\boxed{ACGHCDFHFEBFBDABA}$

▶ Find the length of an optimal chinese postman rou[te] for the networks below.

(a)



Solution

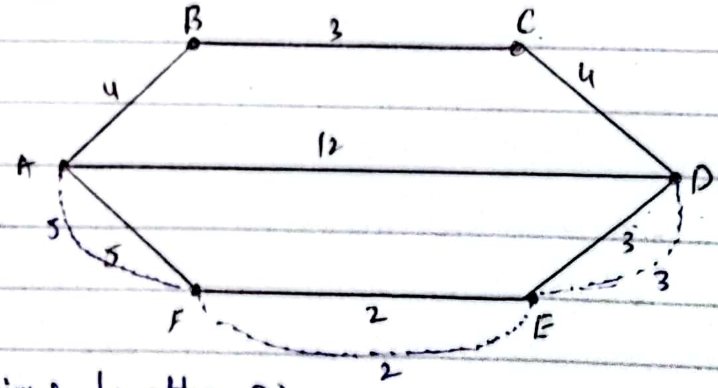
Pattern → $\boxed{\text{OPCMSL}}$

1: Odd vertices → A and D

2+3: Pair odd vertices → AF, FE, ED

4: length of Paired edges: AF+FE+ED = 10

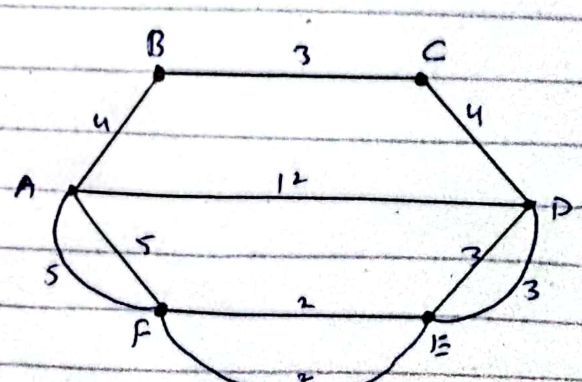
5: length of route :



Previous length = 33

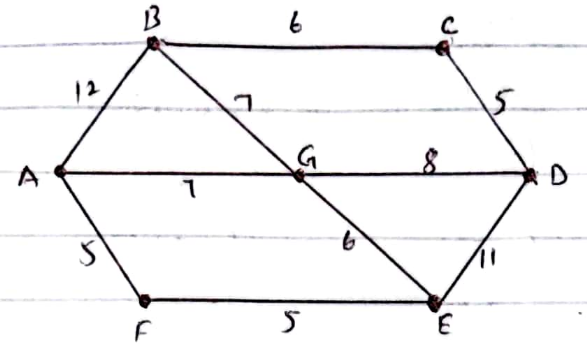
Length of new edges = 10

Total length of route = 33 + 10 = 43



One possible route : ABCDAFEDEFA

(b)



Solution

pattern : OPCMSL

1: O: odd vertices : A, B, D, E

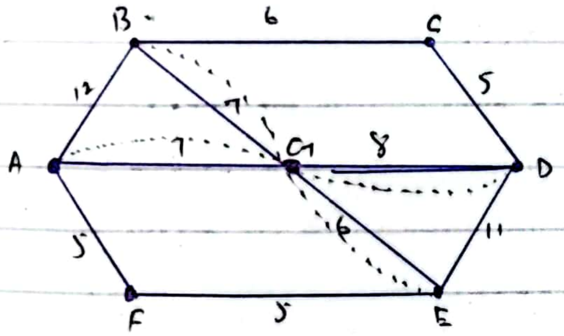
2: pair: Pairing odd vertices : AG, GD, BG, GE

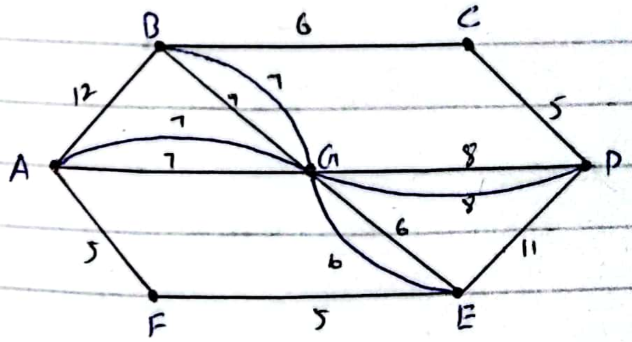
MS: Minimised Sum: $7 + 8 + 7 + 6 = 28$

L: Length of route : 72

After Applying chinese postman Algorithm

length $= 72 + 28 = 100$

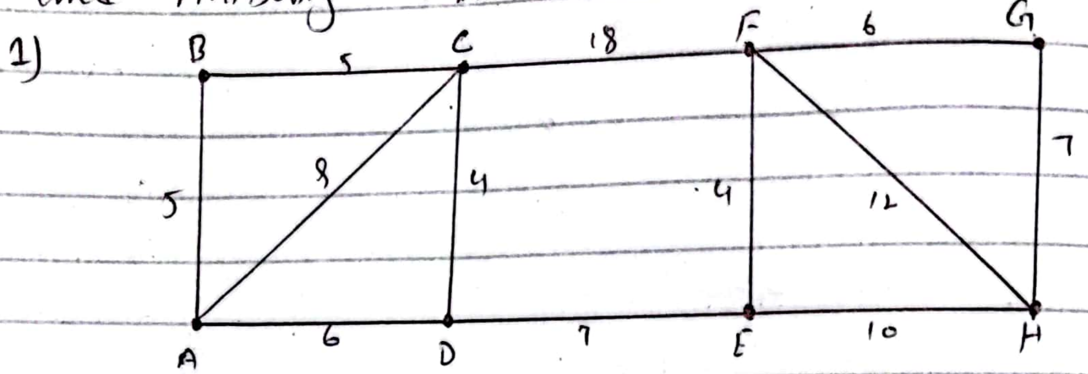




Root: One possible Route : $\boxed{\text{ABCDEFAGDGBGEGA}}$

• For each of the networks below find the length of an optimal Chinese postman route starting at A and finishing at A.

1)



Solution

Pattern: OPCMSL

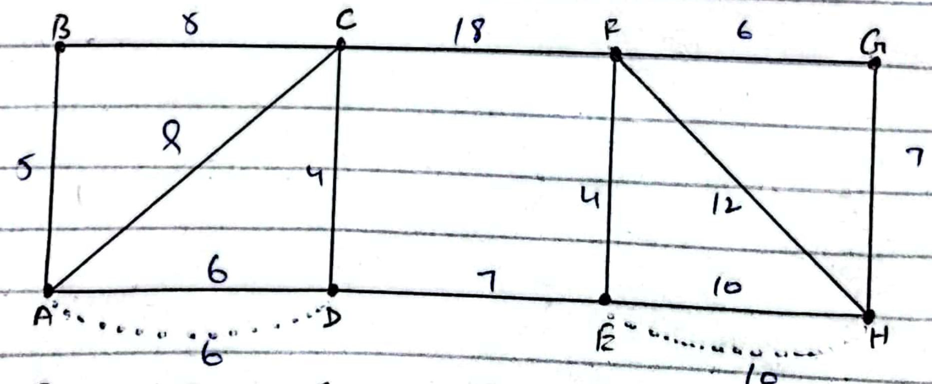
O: odd vertices → A, D, E, H

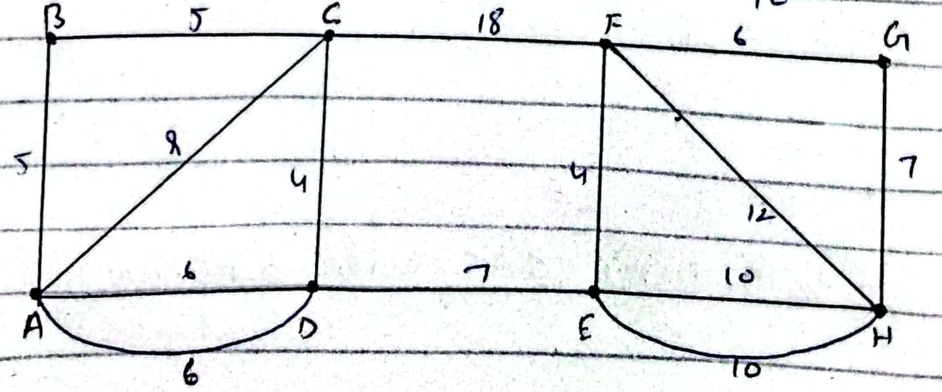
PC: Pairing odd vertices and Connecting the edges

AD, EH

MS: Minimised Sum: 6 + 10 = 16 = |AD| + |EH|

L: length of Graph: 83





One possible route: $\boxed{ABCDACFGHEFHEDA}$