Rolle's Theorem and Mean Value theorem

x — x

Rolle's Theorem	Mean Value Theorem
1: The function f is Continuous on a closed interval $[a, b]$	1: The function f is Continuous on a closed interval $[a, b]$
2: The function f is differentiable on an open interval (a, b)	2: The function f is differential on an open interval (a, b)
3: if $f(a) = f(b)$, then there exists a point "c" between a & b Such that $f'(c) = 0$	3: There exist a point "c" between a & b such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

3.2

Explain why Rolle's Theorem doesn't Apply to the function even though there exist a and b such that $f(a) = f(b)$.

1) $f(x) = \left| \frac{1}{x} \right|$, $[-1, 1]$

Solution

1st Step: check f is Continuous on a closed interval $[-1, 1]$

f is discontinuous at $\boxed{x=0}$

2nd Step: Check that f is ~~Continuous~~ differentiable on the open interval $(-1, 1)$

f is not differentiable in the open interval, because it is discontinuous and has a point of non-differentiablity.

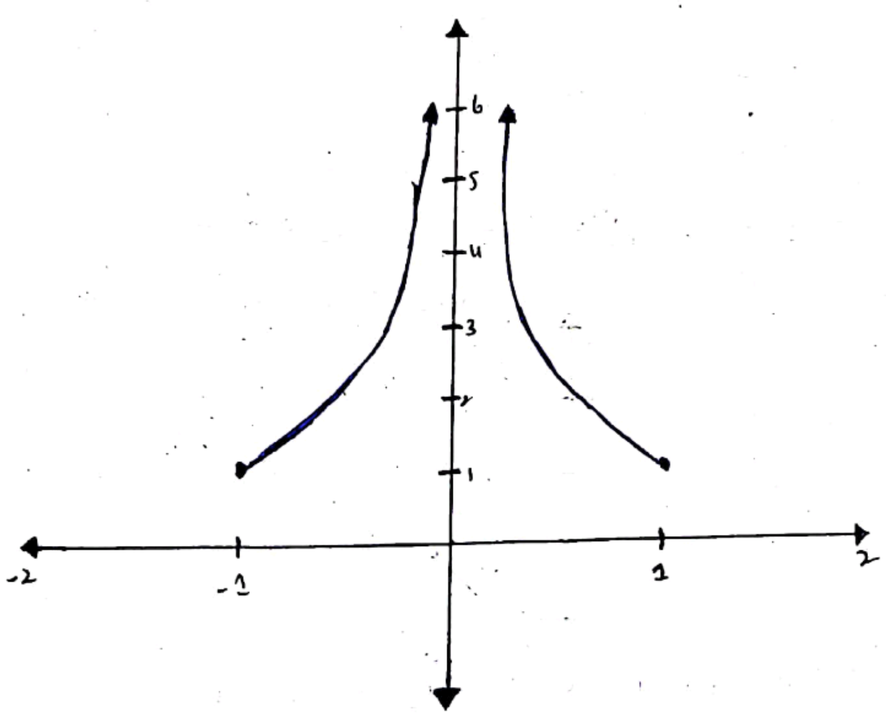
An Absolute value's graph has a point of non-differentiablity.

3rd Step

Useless, because the premises are false so the conclusion can't be true.

$$\left.\begin{aligned} f(a) &= f(-1) = \left|\frac{1}{-1}\right| = |-1| = 1 \\ f(b) &= f(1) = \left|\frac{1}{1}\right| = |1| = 1 \end{aligned}\right\} f(a) = f(b)$$

Rolle's theorem couldn't be applied because premises are FALSE.



2) $f(x) = \cot \frac{x}{2}$, $[\pi, 3\pi]$

Solution

Step 1 check the continuity

f is discontinuous at a point (near about $x=6$), examined by using graphing utility.

Step 2 f is Non-differentiable, examined using CAS.

Step 3

$f(a) = f(\pi) = \cot\frac{\pi}{2} = \cot 90^\circ = \frac{1}{\tan 90^\circ} = -0.5012$

$f(b) = f(3\pi) = \cot\frac{3\pi}{2} = \cot 270^\circ = \frac{1}{\tan 270^\circ} = -5.5816$

$f(a) \neq f(b)$.

Rolle's theorem Can't be applied, because both premises and Conclusion are FALSE.

3) $f(x) = 1 - |x-1|$, $[0, 1]$

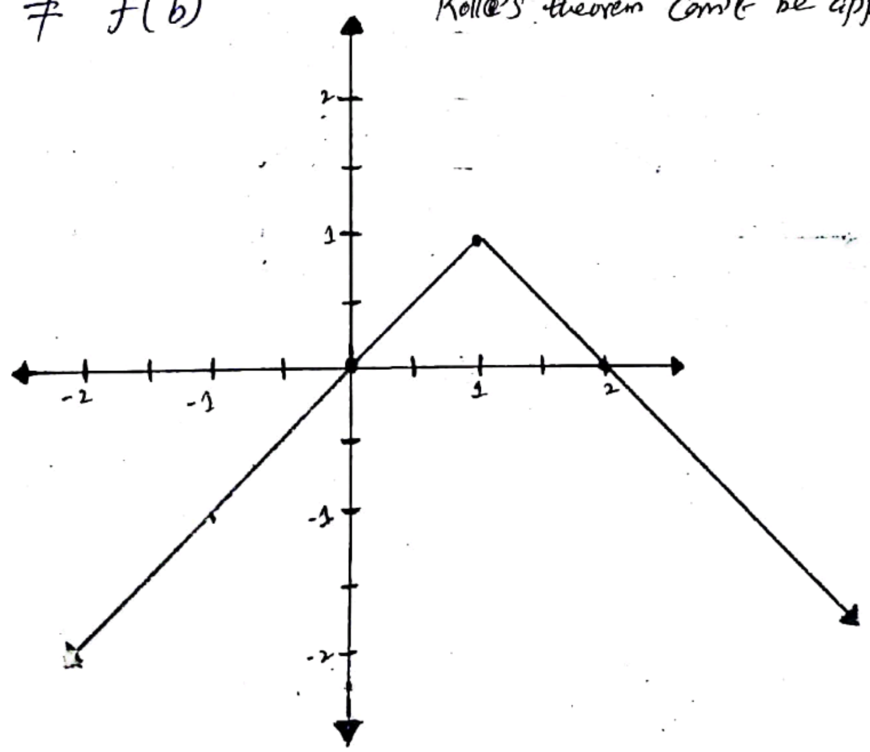
Solution

Check the Continuity

- f is Continuous but not differentiable
- f is not differentiable because it has a point of non-differentiability.
- $f(a) = f(0) = 1 - |0-1| = 1 - |-1| = 1 - (1) = 0$

  $f(b) = f(1) = 1 - |1-1| = 1 - |0| = 1 - 0 = 1$

  $f(a) \neq f(b)$ Rolle's theorem Can't be applied



1) $f(x) = \sqrt{(2-x^{2/3})^3}$, $[-1, 1]$

Solution

Step 1 $f(x)$ is continuous on the closed interval $[-1, 1]$

Step 2 f is not differentiable at $x=0$

$$f'(x) = (2-x^{2/3})\left(-\frac{1}{x^{1/3}}\right)$$

Step 3 $f(a) = f(-1) = \sqrt{(2-(-1)^{2/3})^3} = \sqrt{(2-(-1))^3} = \sqrt{(2+1)^3} = \sqrt{27}$

$f(b) = f(1) = \sqrt{(2-(1)^{2/3})^3} = \sqrt{(2-(1))^3} = \sqrt{(1)^3} = 1$

$f(a) \neq f(b)$

premises and the conclusion are false, so the Rolle's Theorem can't be applied.

Determine whether Rolle's theorem can be applied to f on the closed interval $[a, b]$. If Rolle's theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's theorem can't be applied, explain why not.

11) $f(x) = -x^2 + 3x$, $[0, 3]$

Solution

Step 1 f is continuous on the closed interval $[0, 3]$, because f is a polynomial and polynomials are always continuous.

Step 2 f is differentiable on the open interval $(0, 3)$, because it is a polynomial.

Step 3 If $f(a) = f(b)$, then there exists "c" such that $f'(c) = 0$

$f(a) = f(0) = -(0)^2 + 3(0) = 0$, $f(b) = f(3) = -(3)^2 + 3(3) = -9 + 9 = 0$

$f(a) = f(b)$

$f(x) = -x^2 + 3x$

$f'(x) = -2x + 3$

$f'(c) = -2c + 3$

$\boxed{f'(c) = 0}$

$0 = -2c + 3$

$-3 = -2c$

$\boxed{c = 3/2}$

$c = 3/2$ and the Rolle's theorem can be applied because both the premises and conclusion are TRUE.

12) $f(x) = x^2 - 5x + 4,\ [1, 4]$

Solution

- f is continuous, because f is a polynomial
- f is differentiable on the open interval $(1, 4)$
- $f(a) = f(1) = (1)^2 - 5(1) + 4 = 1 - 5 + 4 = 0$

$f(b) = f(4) = (4)^2 - 5(4) + 4 = 16 - 20 + 4 = 0$

$\boxed{f(a) = f(b)}$

$f(x) = x^2 - 5x + 4$

$f'(x) = 2x - 5$

$f'(c) = 2c - 5$

$f'(c) = 0$

$0 = 2c - 5$

$\boxed{c = 5/2}$

$c = 5/2$, Rolle's Theorem Can be Applied, because both the premises and the conclusion are TRUE.

17) $f(x) = \dfrac{x^2 - 2x - 3}{x + 2},\ [-1, 3]$

Solution

- f is continuous on the closed interval $[-1, 3]$, and is discontinuous at $x = -2$, which is not lieing in the interval.
- f is differentiable on the open interval $(-1, 3)$
- $f(a) = f(-1) = \dfrac{(-1)^2 - 2(-1) - 3}{(-1) + 2} = \dfrac{1 + 2 - 3}{1} = \dfrac{0}{1} = 0$

$f(b) = f(3) = \dfrac{(3)^2 - 2(3) - 3}{(3) + 2} = \dfrac{9 - 6 - 3}{5} = \dfrac{9 - 9}{5} = \dfrac{0}{5} = 0$

$f(a) = f(b)$, there exist a "c" b/w a & b (i.e., -1 & 3) s.th $f'(c) = 0$

$f(x) = \dfrac{x^2 - 2x - 3}{x + 2}$

$f'(x) = \left[\frac{d}{dx}(x^2 - 2x - 3) \times (x+2) - (x^2 - 2x - 3) \times \frac{d}{dx}(x + 2)\right] \div (x+2)^2$

$f'(x) = \left[\frac{d}{dx}(x^2 - 2x - 3)(x+2) - (x^2 - 2x - 3)(1 + 0)\right] \div (x+2)^2$

$f'(x) = [(2x - 2)(x+2) - (x^2 - 2x - 3)] \div (x+2)^2$

$f'(x) = \dfrac{[2(x-1)(x+2) - (x^2 - 2x - 3)]}{(x+2)^2}$

$f'(c) = [2(c-1)(c+2) - (c^2 - 2x - 3)] \div (c+2)^2$

$f'(c) = 0$

$0 = [2(c-1)(c+2) - (c^2 - 2x - 3)] \div (c+2)^2$

$0 = \left[2(c-1)(c+2) - (c^2-2c-3) \right] \div (c+2)^2$

$0 = \left[2(c^2+2c-c-2) - (c^2-2c-3) \right] \div (c+2)^2$

$0 = \left[2(c^2+c-2) - (c^2-2c-3) \right] \div (c+2)^2$

$0 = \left[2c^2+2c-4-c^2+2c+3 \right] \div (c+2)^2$

$0 \times (c+2)^2 = \left[c^2+4c-1 \right]$

$c^2+4c-1=0$

$\Delta\ b^2-4ac$; $a=1,\ b=4,\ c=-1$

$(4)^2-4(1)(-1)$

$16+4=20>0$ 2 ! equal(distinct real roots)

Using Quadratic Formula

$$c = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-(4) \pm \sqrt{(4)^2-4(1)(-1)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm 2\sqrt{5}}{2} = -2 \pm \sqrt{5}$$

$\boxed{c = -2+\sqrt{5}}$ and $\boxed{c = -2-\sqrt{5}}$

Rolle's Theorem exists and can be applied.

2) Vertical Motion. The height of a ball t Seconds after it is thrown upward from a height of 6 feet and with an initial velocity of 48 feet per Second is $f(t) = -16t^2+48t+6$

a) Verify that $f(1) = f(2)$

b) according to Rolle's theorem, what must the velocity be at Some time in the interval $(1,2)$? Find that time.

Solution

$f(1) = -16(1)^2+48(1)+6 = -16+48+6 = 38$

$f(2) = -16(2)^2+48(2)+6 = -64+96+6 = 38$

$f(1) \neq f(2)$ ✓ Rolle's theorem Can ~~not~~ be applied.

$f(n) = -16t^2+48t+6$ Because the premise $f(a) = f(b)$ is FALSE.

$f'(n) = -32t+48$

$0 = -32t+48$

$0 = -4t+6$

$\boxed{t = 3/2}$

30) Recorder Costs The ordering and transportation cost C for components used in a manufacturing process is approximated by $C(x) = 10\left(\frac{1}{n} + \frac{n}{n+3}\right)$, where C is measured in thousands of dollars and n is the order size in hundreds.

(a) Verify that $C(3) = C(6)$

(b) According to Rolle's theorem the rate of change of cost must be 0 for some order size in the interval (3,6). Find that order size.

Solution

a: $C(3) = 10\left(\frac{1}{3} + \frac{3}{3+3}\right) = 10\left(\frac{1}{3} + \frac{1}{2}\right) = 10\left(\frac{5}{6}\right) = \frac{25}{3}$

$C(6) = 10\left(\frac{1}{6} + \frac{6}{6+3}\right) = 10\left(\frac{1}{6} + \frac{2}{3}\right) = 10\left(\frac{15}{18}\right) = \frac{25}{3}$

$C(3) = C(6)$ ✓

b: $C(n) = 10\left(\frac{1}{n} + \frac{n}{n+3}\right) = 10\left(\frac{n+3+n^2}{n^2+3n}\right) = \frac{10n^2+10n+30}{n^2+3n}$

$$C'(n) = \frac{\left[(n^2+3n) \times \frac{d}{dn}(10n^2+10n+30) - (10n^2+10n+30) \times \frac{d}{dn}(n^2+3n)\right]}{(n^2+3n)^2}$$

$C'(n) = 0$

$0 \times (n^2+3n)^2 = \left[(n^2+3n) \times (20n+10) - (10n^2+10n+30)(2n+3)\right]$

$0 = \left[(20n^3+10n^2+60n^2+30n) - (20n^3+20n^2+60n+30n^2+30n+90\right.$

$0 = \left[\cancel{20n^3}+70n^2+30n-\cancel{20n^3}-20n^2-60n-30n^2-30n-90\right]$

$0 = \left[70n^2+30n-20n^2-30n^2-60n-30n-90\right]$

$0 = \left[70n^2+30n-50n^2-90n-90\right]$

$0 = \left[20n^2-60n+90\right]$, $a = 20$ $b = -60$, $c = -90$

$\Delta\, b^2-4ac$

$(-60)^2 - 4(20)(-90)$

$3600 - 80(-90)$

$\boxed{3600+7200 > 0}$

$0 = 10\left[2n^2-6n-9\right]$

$\boxed{0 = 2n^2-6n-9}$

$n = 2n^2 - 6n - 8 \rightarrow a = 2,\ b = -6,\ c = -8$

$$c = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(-8)}}{2(2)}$$

$$c = \frac{6 \pm \sqrt{36+72}}{4} = \frac{6 \pm \sqrt{108}}{4} = \frac{6 \pm \sqrt{8 \times 12}}{4}$$

$$c = \frac{6 \pm 3\sqrt{12}}{4} = \frac{6 \pm 3\sqrt{4 \times 3}}{4} = \frac{6 \pm 6\sqrt{3}}{4} = \frac{3 \pm 3\sqrt{3}}{2}$$

$\boxed{c = \frac{3 + 3\sqrt{3}}{2}}$ and $\boxed{c = \frac{3 - 3\sqrt{3}}{2}}$

11) Determine the values a, b, and c such that the function f satisfies the hypothesis of the Mean value Theorem on the interval $[0,3]$.

$$f(n) = \begin{cases} 1, & n = 0 \\ an + b, & 0 < n \le 1 \\ n^2 + 4n + c, & 1 < n \le 3 \end{cases}$$

Solution

Step 1 Check the Continuity on closed interval $[0,3]$

f is Continuous and is also a polynomial.

Step 2 Check the differentiability on open interval $(0,3)$

f is differentiable and doesn't have any point of non-differentiability

$y = 1,\ n = 0 \}$ $y = an + b,\ 0 < n \le 1 \}$ $y = n^2 + 4n + c,\ 1 < n \le 3 \}$

1: $y = y$

$an + b = 1,\ n = 0$

$a(0) + b = 1$

$\boxed{b = 1}$

2: $y = an + b$

$y' = a(1)$

$\boxed{y' = a}$

$y = n^2 + 4n + c$

$y' = 2n + 4(1) + 0$

$\boxed{y' = 2n + 4}$

$y' = y',\ n = 1$

$a = 2n + 4$

$a = 2(1) + 4$

$a = 2 + 4$

$\boxed{a = 6}$

3: $y = an + b,\ y = n^2 + 4n + c$

$a = 6,\ n = 1$ and $b = 1$,

$an + b = n^2 + 4n + c$

$an + 1 = n^2 + 4n + c$

$a(1) + 1 = (1)^2 + 4(1) + c$

$a + 1 = 1 + 4 + c$

$a = 4 + c$

$6 = 4 + c$

$\boxed{2 = c}$

4: $a = 6,\ b = 1,\ c = 2$

72) Determine the values $a, b, c,$ and d such that the function f satisfies the hypothesis of the Mean Value Theorem on the interval $[-1, 2]$

$$f(x) = \begin{cases} a, & x = -1 \\ 2, & -1 < x \leq 0 \\ bx^2 + c, & 0 < x \leq 1 \\ dx + 4, & 1 < x \leq 2 \end{cases}$$

Solution

Step 1 Check the Continuity

f is Continuous on the Closed Interval $[-1, 2]$

Step 2 Check the differentiability

f is differentiable on the open interval $[-1, 2]$, f does not have any point of non-differentiability.

Step 3

$y = a, x = -1 \}$ $y = 2, x > -1, x \leq 0 \}$ $y = bx^2 + c, 0 < x \leq 1 \}$

$y = y$

$\boxed{a = 2}$, $x = -1$ ✓

$y = y \quad , x = 0$

$2 = bx^2 + c$

$2 = b(0)^2 + c$

$\boxed{2 = c}$

$y = bx^2 + c, 0 < x \leq 1 \}$ $y = dx + 4, 1 < x \leq 2 \}$

$y' = 2bx$

$y'' = 2b$

$y' = d$

$y'' = 0$

$y'' = y'' \quad , x = 1$

$2b = 0$

$\boxed{b = 0}$

$y = y \quad , x = 1 \quad ; c = 2, b = 0$

$bx^2 + c = dx + 4$

$(0)(1)^2 + c = d(1) + 4$

$c = d + 4$

$2 = d + 4$

$\boxed{d = -2}$

$a = 2, \quad b = 0, \quad c = 2, \quad d = -2$