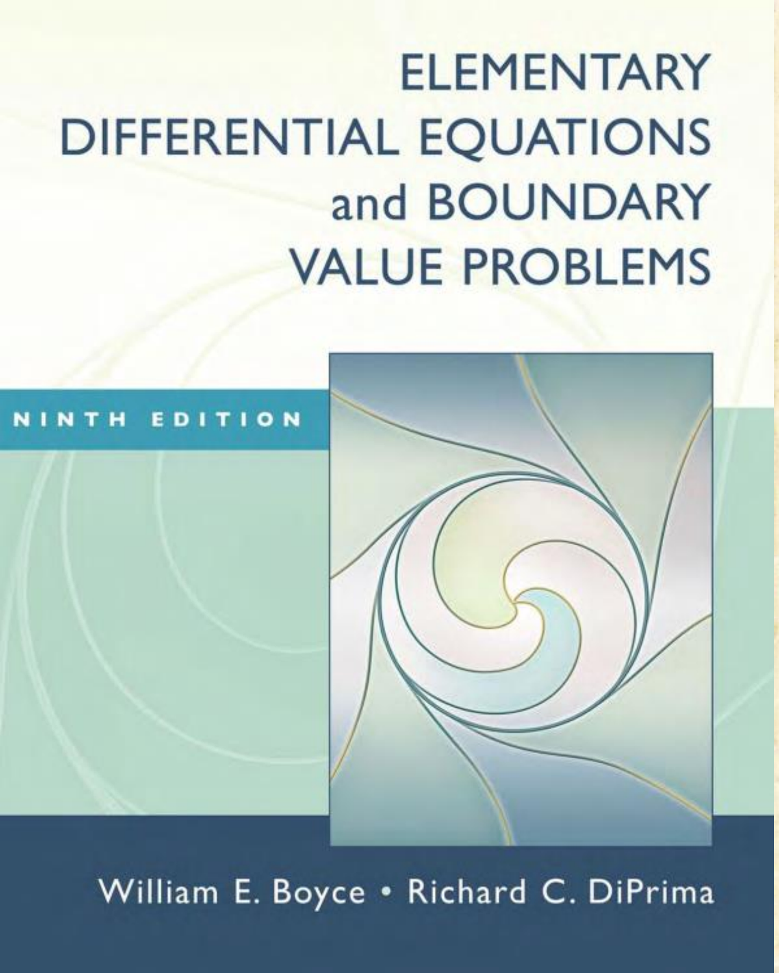
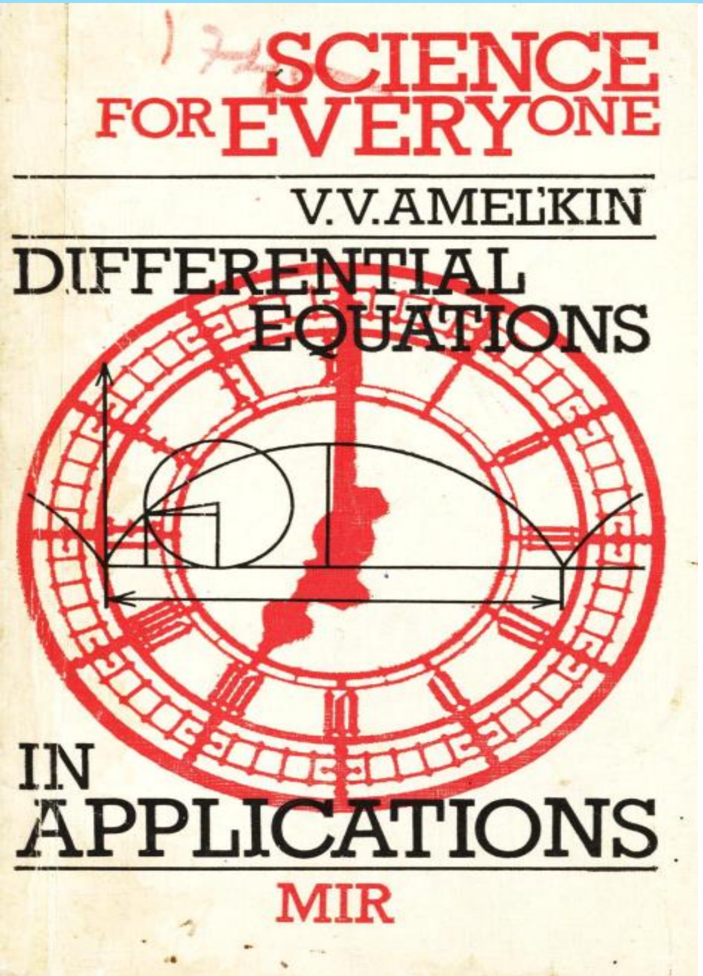
DIFFERENTIAL EQUATIONS





By

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Introduction to Differential Equations

Types of Differential Equations

Ordinary Differential Equations (ODEs)

Partial Differential Equations (PDEs)

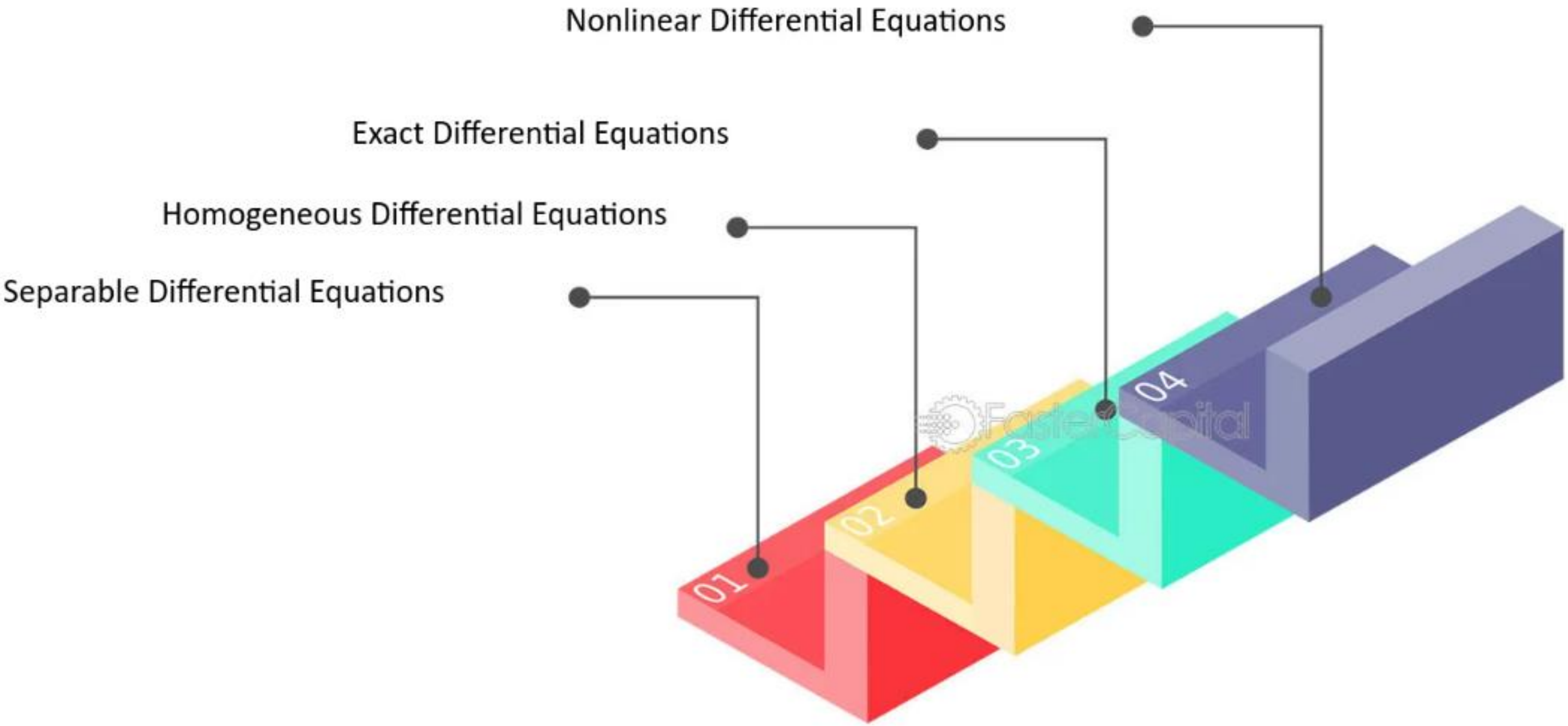
Linear Differential Equations

Nonlinear Differential Equations

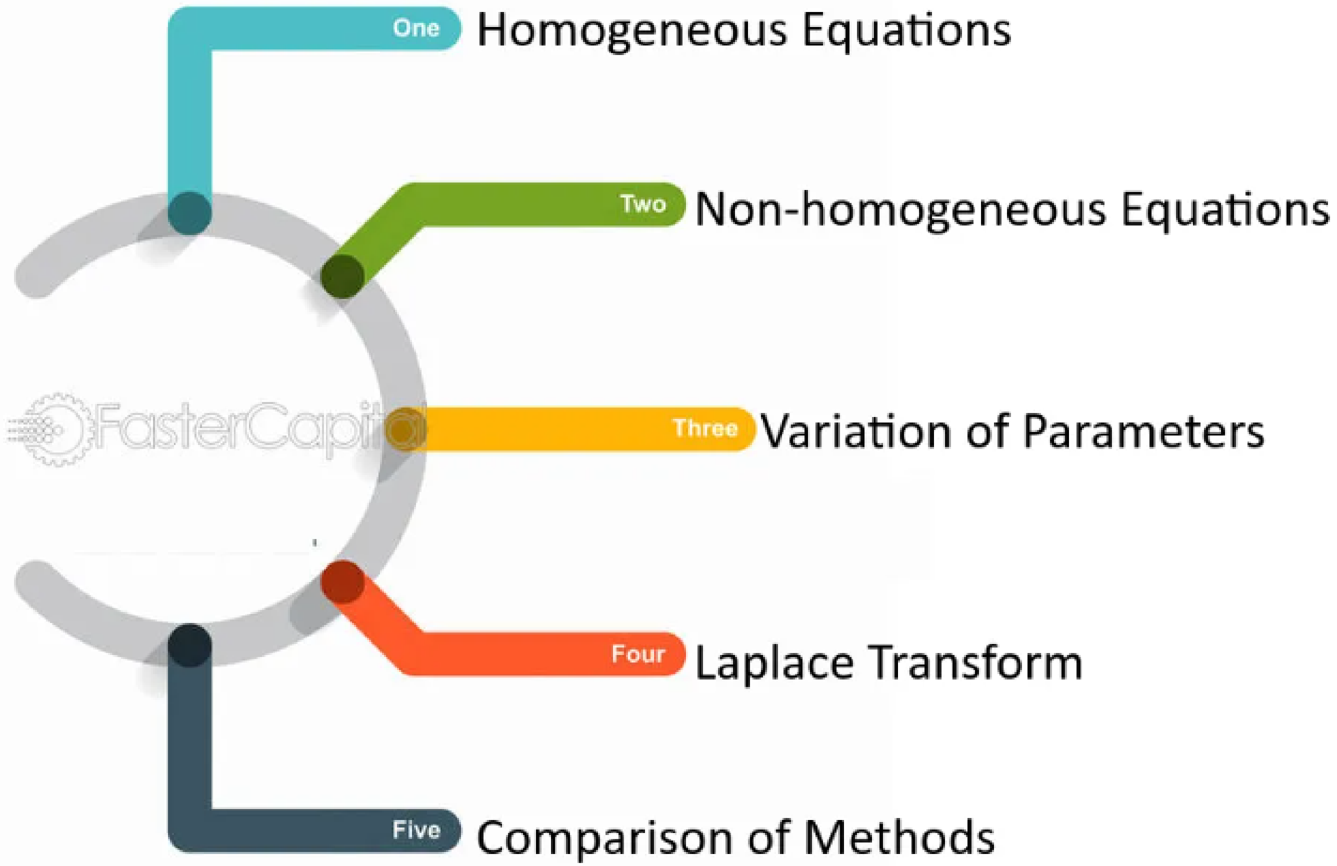
Homogeneous Differential Equations

Non-homogeneous Differential Equations

Solving First-Order Differential Equations



Solving Second-Order Differential Equations



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    - Legendre's Method (Variable coefficient)

Linear Differential Equation :-

It is in the form of,

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = R(x)$$

constant coefficient

$$\frac{d^n y}{dx^n} + (X + a_{n-1})\frac{d^{n-1}y}{dx^{n-1}} + \cdots + (X + a_1)\frac{dy}{dx} + (X + a_0 y) = R(x)$$

Vairable coefficient

Homogenous Linear D.E.

- In this R.H.S of D.E. is zero i.e

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = 0$$

Example :-

(1) $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} + y = 0$

(2) $y'' + 39y' + y = 0$

(3) $y_4 + y_3 + 3y_2 - 9y_1 = 0$

Non-homogenous Linear D.E.

- In this R.H.S of D.E. is not zero/is having $f(x)$ i.e

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = f(x)$$

Example :-

(1) $\frac{d^2y}{dx^2} + 9\frac{dy}{dx} + y = \cos x$

(2) $y'' + 39y' + y = e^x$

(3) $y_4 + y_3 + 3y_2 - 9y_1 = \log x + \sin x \cos x + x^{-2}$

Non - Linear Differential Equation

- The term homogenous and non homogenous have no meaning for non linear equation.

Examples :-

(1) $$\frac{d^2y}{dx^2} = x\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$$

(2) $$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$

Steps to solve Linear D.E.

- Identify Auxiliary Equation (A.E.) , By putting $\frac{d^n}{dx^n} = D^n$ <u>i.e.</u> $\frac{d^2y}{dx^2} = D^2y$
- Find the roots of A.E. by putting **D = m** in it and equating with it zero. i.e. **A.E. = 0**
- According o roots obtained find, Complimentary Function

  (C.F.) $= y_c$
- Find Particular Integral (P.I.) $= y_p$, from the R.H.S. of linear **Non Homogenous Equation.**
- Find complete solution / General Solution $(y) = y_c + y_p$

Auxiliary Equation (A.E.)

(1) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \sin(e^x)$

$\therefore\ D^2y + 2Dy + y = \sin(e^x)$

$\therefore\ \underbrace{\left(\boldsymbol{D^2 + 2D + 1}\right)}_{\text{A. E.}}y = \sin(e^x)$

Formulae for Finding Roots

- $a^2 \pm 2ab + b^2 = (a \pm b)^2$
- $a^3 + b^3 + 3ab(a+b) = a^3 + b^3 + 3a^2b + 3ab^2 = (\boldsymbol{a} + \boldsymbol{b})^3$
- $a^3 - b^3 - 3ab(a-b) = a^3 - b^3 - 3a^2b + 3ab^2 = (a-b)^3$
- $a^2 - b^2 = (a+b)(a-b)$
- $\boldsymbol{a^2 + b^2} \Rightarrow \boldsymbol{a^2 = -b^2}$

  $\Rightarrow \boldsymbol{a = \pm bi}$
- $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
- $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

- $\boldsymbol{a^4} - \boldsymbol{b^4} = (a^2)^2 - (b^2)^2$

  $= (a^2 - b^2)(a^2 + b^2)$

  $= (a - b)(a + b)(a^2 + b^2)$

- $\boldsymbol{a^4} + \boldsymbol{b^4} = a^4 + b^4 + 2a^2b^2 - 2a^2b^2$ **(Find Middle Term)**

  $= (a^2)^2 + 2a^2b^2 + (b^2)^2 - (2a^2b^2)$

  $= (a^2 + b^2)^2 - \left(\sqrt{2}\, ab\right)^2$

  $= (a^2 + b^2 - \sqrt{2}\, ab)(a^2 + b^2 + \sqrt{2}\, ab)$

If equation is in form of, $\boldsymbol{Ax^2 + Bx + C}$ then, $\boldsymbol{x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}}$

OR Separate the middle term (Bx) in such way that their addition or substraction be the multiple of A & C .

Solved Example

(1) Find the roots of :- $\mathbf{3y'' - y' - 2y = e^x}$

$\therefore \quad 3\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^x$

$\therefore \quad 3D^2y - Dy - 2y = e^x$

$\therefore \quad (3D^2 - D - 2)y = e^x$

Let, A.E. = 0 and put D = m

$\therefore \quad 3m^2 - m - 2 = 0$

$\therefore \quad 3m^2 - 3m + 2m - 2 = 0$

$\therefore \quad 3m(m-1) + 2(m-1) = 0$

$\therefore \quad (3m+2)(m-1) = 0$

$\therefore \quad 3m + 2 = 0 \quad$ and $\quad m - 1 = 0$

$\therefore \quad \boldsymbol{m_1 = -\frac{2}{3}} \quad$ and $\quad \boldsymbol{m_2 = 1}$

2×3 = 6

2 3

-1 = -3 +2

(2) Find the roots of : $(\boldsymbol{D^4+k^4})\boldsymbol{y}=\mathbf{0}$

Let A.E. = 0 ad put D = m

$\therefore m^4+k^4=0$

$\therefore (m^2)^2+2m^2k^2+(k^2)^2-(2m^2k^2)=0$

$\therefore (m^2+k^2)^2-\left(\sqrt{2}\,mk\right)^2=0$

$\therefore (m^2+k^2-\sqrt{2}\,mk)(m^2+k^2+\sqrt{2}\,mk)=0$

$\therefore m^2+k^2-\sqrt{2}\,mk=0$ and $m^2+k^2+\sqrt{2}\,mk=0$

$\therefore m_1=\frac{\sqrt{2}k\pm\sqrt{2k^2-4k^2}}{2}$ and $m_2=\frac{-\sqrt{2}k\pm\sqrt{2k^2-4k^2}}{2}$

$\therefore \boldsymbol{m_1}=\frac{\boldsymbol{k}}{\sqrt{2}}\pm\frac{\boldsymbol{k}}{\sqrt{2}}\boldsymbol{i}$ and $\boldsymbol{m_2}=\frac{-\boldsymbol{k}}{\sqrt{2}}\pm\frac{\boldsymbol{k}}{\sqrt{2}}\boldsymbol{i}$

Exercise

- **Find the roots of given Differential Equation :-**

(1) $(D^2 + 1)y = 0$

(2) $y''' - y'' + 100y' - 100y = 0$

(3) $\frac{d^4y}{dx^4} - \frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} - 11\frac{dy}{dx} - 4y = 0$

(4) $(D^4 + k^4)y = 0$

(5) $(D^4 - k^4)y = 0$

(6) $(D^2 + 6D + 4)y = 0$

(7) $(D^2 + 1)^3(D^2 + D + 1)^2y = 0$

(8) $y_2 - y_1 - 2y = \sinh 2x$

Complimentary Function

- *From the roots of A.E. , C.F. (y_c) of D.E. is decided. C.F. is always in terms of* $y_c = C_1 y_1 + C_2 y_2$

- If the roots are **real** & **district** (unequal), then

$$\boldsymbol{y_c = c_1\, e^{m_1 x} + c_2\, e^{m_2 x} + \cdots + c_n\, e^{m_n x}}$$

Example :- If roots are $m_1 = 2$ & $m_2 = -3$ then, $y_c = c_1 e^{2x} + c_2 e^{-3x}$

- If the roots are **real** & **equal** then,

$$\boldsymbol{y_c = \left(c_1 + c_2 x + c_3 x^2 + \cdots\right) e^{m_1 x}}$$

Example :- If roots are $m_1 = m_2 = -3$ then, $y_c = (c_1 + c_2 x)\, e^{-3x}$

- If the roots are **complex** then, i.e. roots in the form of $(\alpha \pm \beta i)$

$$\boldsymbol{y_c = e^{\alpha x}(c_1 \cos x + c_2 \sin x)}$$

Example :-

(1) If roots is $m = \frac{1}{2} \pm \sqrt{3}i$ then, $y_c = e^{\frac{1}{2}x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x)$

(2) If root is $m = \pm 3i$ then, $y_c = e^{0x}(c_1 \cos 3x + c_2 \sin 3x)$

$$= c_1 \cos 3x + c_2 \sin 3x$$

- If the roots are **complex** & **repeated** then,

$$\boldsymbol{y_c = e^{\alpha x}[(c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x]}$$

- If the roots are **complex** & **real** both then,

$$\boldsymbol{y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + e^{\alpha x}(c_3 \cos x + c_4 \sin x)}$$

NOTE :-

- If the R.H.S. $= 0$ of given D.E. i.e. for Homogenous Linear D.E. $y_p = 0$ and hence the general solution/final solution is given by, $y = y_c$

Solved Example

(1) Solve :- $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

$\rightarrow D^2x + 6Dx + 9x = 0$

$\therefore (D^2 + 6D + 9)x = 0$

Let A.E. = 0 & put D = m

$\therefore m^2 + 6m + 9 = 0$

$\therefore (m + 3)^2 = 0$

$\therefore m_1 = m_2 = -3$

- Roots are real and equal then, C.F. is given by,

$\therefore y_c = (c_1 + c_2 t)e^{-3t}$

- Here R.H.S. = 0 then, $y_p = 0$ & complete sol^n is given by,

$\therefore y = y_c = (c_1 + c_2 t)e^{-3t}$

(2) Solve :- $D^2y + 4Dy + 5y = 0$ & Find the value of c_1 & c_2 if $y = 2$ & $y_2 = y$ when $x = 0$

Solution. Here the auxiliary equation is

$$m^2 + 4m + 5 = 0$$

Its root are $-2 \pm i$

The complementary function is

$$y = e^{-2x}(A \cos x + B \sin x) \qquad \dots(1)$$

On putting $y = 2$ and $x = 0$ in (1), we get

$$2 = A$$

On putting $A = 2$ in (1), we have

$$y = e^{-2x}[2 \cos x + B \sin x] \qquad \dots(2)$$

On differentiating (2), we get

$$\frac{dy}{dx} = e^{-2x}[-2\sin x + B\cos x] - 2e^{-2x}[2\cos x + B\sin x]$$

$$= e^{-2x}[(-2B - 2)\sin x + (B - 4)\cos x]$$

$$\frac{d^2y}{dx^2} = e^{-2x}[(-2B - 2)\cos x - (B - 4)\sin x]$$

$$- 2e^{-2x}[(-2B - 2)\sin x + (B - 4)\cos x]$$

$$= e^{-2x}[(-4B + 6)\cos x + (3B + 8)\sin x]$$

But $$\frac{dy}{dx} = \frac{d^2y}{dx^2}$$

$$e^{-2x}[(-2B-2)\sin x + (B-4)\cos x] = e^{-2x}[(-4B+6)\cos x + (3B+8)\sin x]$$

On putting $x = 0$, we get

$$B - 4 = -4B + 6 \quad \Rightarrow \quad B = 2$$

(2) becomes, $$y = e^{-2x}[2\cos x + 2\sin x]$$

$$y = 2e^{-2x}[\sin x + \cos x]$$

Exercise :-

(1) $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 8\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0$ **Ans.** $y = e^x[(C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x]$

(2) $(D^8 + 6D^6 - 32D^2)y = 0$ *(A.M.I.E.T.E., Summer 2005)*

Ans. $y = C_1 + C_2x + C_3e^{\sqrt{2}x} + C_4e^{-\sqrt{2}x} + C_5\cos 2x + C_6\sin 2x$

(3) The equation for the bending of a strut is EI $\frac{d^2y}{dx^2} + Py = 0$

If $y = 0$ when $x = 0$, and $y = a$ when $x = \frac{1}{2}$, find y. **Ans.** $y = \dfrac{a\sin\sqrt{\frac{P}{EI}}\,x}{\sin\sqrt{\frac{P}{EI}}\,\frac{1}{2}}$

Methods for Finding Particular Integral

- Linear Differential eq^n with **Constant coefficient**
  - General Method
  - Shortcut Method
  - Method of Undetermined Coefficient
  - Method of Variation Parameter (Wronkian Method)
- Linear Differential eq^n with **Variable coefficient**
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General Method

Let us consider a linear differential equation of the first order

$$\frac{dy}{dx} + Py = Q \qquad \text{...(1)}$$

Its solution is $\quad ye^{\int P dx} = \int (Q\, e^{\int P dx})\, dx + C$

$$\Rightarrow \quad y = Ce^{-\int P dx} + e^{-\int P dx} \int (Qe^{\int P dx})\, dx$$

$$\Rightarrow \quad y = cu + v \text{ (say)} \qquad \text{...(2)}$$

where $u = e^{-\int P dx}$ and $v = e^{-\int P dx} \int Q\, e^{\int P dx} dx$

(*i*) Now differentiating $u = e^{-\int P dx}$ w.r.t. x, we get $\frac{du}{dx} = -Pe^{-\int P dx} = -Pu$

$$\Rightarrow \quad \frac{du}{dx} + Pu = 0 \quad \Rightarrow \quad \frac{d(cu)}{dx} + P(cu) = 0$$

which shows that $y = c.u$ is the solution of $\frac{dy}{dx} + Py = 0$

(*ii*) Differentiating $v = e^{-\int P dx} \int (Qe^{\int P dx}) dx$ with respect to x, we get

$$\frac{dv}{dx} = -Pe^{\int P dx} \int (Qe^{\int P dx}) dx + e^{-\int P dx} Qe^{\int P dx} \quad \Rightarrow \quad \frac{dv}{dx} = -Pv + Q$$

$$\Rightarrow \quad \frac{dv}{dx} + Pv = Q \text{ which shows that } y = v \text{ is the solution of } \boxed{\frac{dy}{dx} + Py = Q}$$

Solve by using general method :-

(1) $(D^2 + 3D + 2)y = e^{e^x}$

(2) $(D^2 + 1)y = \sec^2 x$

Shortcut Method

(i) $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ If $f(a) = 0$ then $\frac{1}{f(D)} \cdot e^{ax} = x \cdot \frac{1}{f'(a)} \cdot e^{ax}$

If $f'(a) = 0$ then $\frac{1}{f(D)} \cdot e^{ax} = x^2 \frac{1}{f''(a)} \cdot e^{ax}$

(ii) $\frac{1}{f(D)}x^n = [f(D)]^{-1}x^n$ Expand $[f(D)]^{-1}$ and then operate.

(iii) $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$ and $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$

If $f(-a^2) = 0$ then $\frac{1}{f(D^2)}\sin ax = x \cdot \frac{1}{f'(-a^2)} \cdot \sin ax$

(iv) $\frac{1}{f(D)}e^{ax} \cdot \phi(x) = e^{ax} \cdot \frac{1}{f(D+a)}\phi(x)$

(v) $\frac{1}{D+a}\phi(x) = e^{-ax}\int e^{ax} \cdot \phi(x)\,dx$

$$\boxed{\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}.}$$

We know that, $D.e^{ax} = a.e^{ax}$, $D^2e^{ax} = a^2.e^{ax}, \ldots\ldots\ldots, D^n\ e^{ax} = a^n\ e^{ax}$

Let $f(D)\ e^{ax} = (D^n + K_1D^{n-1} + \ldots + K_n)\ e^{ax} = (a^n + K_1a^{n-1} + \ldots + K_n)e^{ax} = f(a)\ e^{ax}$.

Operating both sides by $\dfrac{1}{f(D)}$

$$\frac{1}{f(D)}\cdot f(D)e^{ax} = \frac{1}{f(D)}\cdot f(a)e^{ax}$$

$$\Rightarrow \quad e^{ax} = f(a)\frac{1}{f(D)}\cdot e^{ax} \quad \Rightarrow \quad \frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$$

If $f(a) = 0$, then the above rule fails.

Then $\dfrac{1}{f(D)}e^{ax} = x\cdot\dfrac{1}{f'(D)}e^{ax} = x\dfrac{1}{f'(a)}e^{ax} \quad \Rightarrow \quad \boxed{\dfrac{1}{f(D)}e^{ax} = x\,.\,\dfrac{1}{f'(a)}e^{ax}}$

If $f'(a) = 0$ then $\boxed{\dfrac{1}{f(D)}e^{ax} = x^2\dfrac{1}{f''(a)}e^{ax}}$

Solved Example

(1) Solve :- $\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 9y = 5\,e^{3x}$

Solution. $(D^2 + 6D + 9)y = 5e^{3x}$

Auxiliary equation is $m^2 + 6m + 9 = 0 \quad \Rightarrow \quad (m+3)^2 = 0 \quad \Rightarrow \quad m = -3, -3,$

$$\text{C.F.} = (C_1 + C_2x)\,e^{-3x}$$

$$\text{P.I.} = \frac{1}{D^2+6D+9}.5.e^{3x} = 5\frac{e^{3x}}{(3)^2+6(3)+9} = \frac{5e^{3x}}{36}$$

The complete solution is $\quad y = (C_1 + C_2x)e^{-3x} + \frac{5e^{3x}}{36}$

(2) Solve :- $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 6e^{3x} + 7e^{-2x} - \log 2$

Solution. $(D^2 - 6D + 9)y = 6e^{3x} + 7e^{-2x} - \log 2$

A.E. is $(m^2 - 6m + 9) = 0 \quad \Rightarrow \quad (m-3)^2 = 0, \quad \Rightarrow \quad m = 3, 3$

$$\text{C.F.} = (C_1 + C_2 x)e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 6D + 9} 6e^{3x} + \frac{1}{D^2 - 6D + 9} 7e^{-2x} + \frac{1}{D^2 - 6D + 9}(-\log 2)$$

$$= x\frac{1}{2D-6} 6e^{3x} + \frac{1}{4+12+9} 7e^{-2x} - \log 2 \frac{1}{D^2 - 6D + 9} e^{0x}$$

$$= x^2 \frac{1}{2} \cdot 6 \cdot e^{3x} + \frac{7}{25} e^{-2x} - \log 2\left(\frac{1}{9}\right) = 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9}\log 2$$

Complete solution is $y = (C_1 + C_2\, x)e^{3x} + 3x^2 e^{3x} + \frac{7}{25} e^{-2x} - \frac{1}{9}\log 2$

(3) Solve :- $\frac{d^2x}{dt^2} + \frac{g}{t}x = \frac{g}{l}L$, where g, l, L are constants subjected to condition, $x = a$, $\frac{dx}{dt} = 0$ at $t = 0$.

Solution. We have, $\frac{d^2x}{dt^2} + \frac{g}{t}x = \frac{g}{l}L \quad \Rightarrow \quad \left(D^2 + \frac{g}{l}\right)x = \frac{g}{l}L$

A.E. is $m^2 + \frac{g}{l} = 0 \quad \Rightarrow \quad m = \pm\, i\sqrt{\frac{g}{l}}$

$$\text{C.F.} = C_1 \cos\sqrt{\frac{g}{l}}\,t + C_2 \sin\sqrt{\frac{g}{l}}\,t$$

$$\text{P.I.} = \frac{1}{D^2 + \frac{g}{l}} \cdot \frac{g}{l}L = \frac{g}{l}L\frac{1}{D^2 + \frac{g}{l}}e^{0t} = \frac{g}{l}L\frac{1}{0 + \frac{g}{l}} = L \qquad [D = 0]$$

$\therefore$ General solution is = C.F. + P.I.

$$x = C_1 \cos\left(\sqrt{\frac{g}{l}}\right)t + C_2 \sin\left(\sqrt{\frac{g}{l}}\right)t + L \qquad \ldots(1)$$

$$\frac{dx}{dt} = -C_1\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}\right)t + C_2\sqrt{\frac{g}{l}}\cos\left(\sqrt{\frac{g}{l}}\right)t$$

Put $t = 0$ and $\frac{dx}{dt} = 0$

$$0 = C_2\sqrt{\frac{g}{l}} \qquad \therefore \quad C_2 = 0$$

(1) becomes $$x = C_1 \cos\sqrt{\frac{g}{l}}\, t + L$$

Put $x = a$ and $t = 0$ in (2), we get

$$a = C_1 + L \quad \text{or} \quad C_1 = a - L$$

On putting the value of C_1 in (2), we get $x = (a - L)\cos\left(\sqrt{\frac{g}{l}}\right)t + L$

(3) Solve :- $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x\,e^x \sin x$

Solution. The auxiliary equation of the given differential equation is

$$m^2 - 2m + 1 = 0,$$

which yields $m = 1, 1$. Hence

$$\text{C.F.} = (c_1 + c_2 x)e^x.$$

The particular integral is

$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{(D-1)^2}xe^x \sin x$$

$$= e^x \frac{1}{(D+1-1)^2} x \sin x = e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D}\int x \sin x\,dx = e^x \frac{1}{D}(-x\cos x + \sin x)$$

$$= e^x \int (-x\cos x + \sin x)\,dx$$

$$= e^x[-x\sin x - \cos x - \cos x]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$= (c_1 + c_2 x)e^x - e^x(x\sin x + 2\cos x).$$

(4) Solve :- $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$

Solution. Given $(D^3 - 3D^2 + 4D - 2)\,y = e^x + \cos x$

A.E. is $m^3 - 3m^2 + 4m - 2 = 0$

$\Rightarrow \quad (m-1)(m^2 - 2m + 2) = 0$, *i.e.*, $m = 1,\ 1 \pm i$

$\therefore \quad \text{C.F.} = C_1 e^x + e^x (C_2 \cos x + C_3 \sin x)$

$$\text{P.I.} = \frac{1}{(D-1)(D^2-2D+2)} e^x + \frac{1}{D^3 - 3D^2 + 4D - 2} \cos x$$

$$= \frac{1}{(D-1)(1-2+2)} e^x + \frac{1}{(-1)D - 3(-1) + 4D - 2} \cos x$$

$$= \frac{1}{(D-1)} e^x + \frac{1}{3D+1} \cos x = x\,\frac{1}{1} e^x + \frac{3D-1}{9D^2-1} \cos x$$

$$= e^x . x + \frac{(-3\sin x - \cos x)}{-9-1} = e^x . x + \frac{1}{10}(3\sin x + \cos x)$$

Hence, complete solution is

$$y = C_1 e^x + e^x (C_2 \cos x + C_3 \sin x) + x\,e^x + \frac{1}{10}(3\sin x + \cos x)$$

(5) Solve :- $(D^2 - 4D + 3)y = \sin 3x \cos 2x$

Solve $(D^2 - 4D + 3)y = \sin 3x \cos 2x$.

Solution. The auxiliary equation is

$$m^2 - 4m + 3 = 0,$$

which yields $m = 3, 1$. Therefore,

$$\text{C.F.} = c_1 e^{3x} + c_2 e^x.$$

Further

$$\text{P.I.} = \frac{1}{D^2-4D+3}[\sin 3x \cos 2x]$$

$$= \frac{1}{D^2-4D+3}[\frac{1}{2} 2\sin 3x \cos 2x]$$

$$= \frac{1}{D^2-4D+3}[\frac{1}{2}(\sin 5x + \sin x)]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2-4D+3} \sin 5x + \frac{1}{2} \cdot \frac{1}{D^2-4D+3} \sin x$$

$$= \frac{1}{2}\left[\frac{1}{-25-4D+3}\sin 5x + \frac{1}{-1-4D+3}\sin x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-22-4D}\sin 5x + \frac{1}{2-4D}\sin x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2(11+2D)}\sin 5x + \frac{1}{2(1-2D)}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{11-2D}{121-4D^2}\sin 5x + \frac{1+2D}{1-4D^2}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{11-2D}{121-4(-25)}\sin 5x + \frac{1+2D}{1-4(-1)}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{11-2D}{221}\sin 5x + \frac{1+2D}{5}\sin x\right]$$

$$= \frac{1}{4}\left[-\frac{1}{221}[11\sin 5x - 2D\sin 5x]\right.$$

$$\left. + \frac{1}{5}(\sin x + 2D\sin x)\right]$$

$$= \frac{1}{4}\left[-\frac{11}{221}\sin 5x + \frac{10}{221}\cos 5x + \frac{1}{5}\sin x + \frac{2}{5}\cos x\right]$$

$$= -\frac{11}{884}\sin 5x + \frac{10}{884}\cos 5x + \frac{1}{20}\sin x + \frac{1}{10}\cos x.$$

Hence the complete solution is

$$y = c_1 e^{3x} + c_2 e^x - \frac{11}{884}\sin 5x + \frac{10}{884}\cos 5x$$

$$+ \frac{1}{20}\sin x + \frac{1}{10}\cos x.$$

(6)Solve :- $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x}\sin x$

Solution. $\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 10\frac{dy}{dx} = e^{2x}\sin x \quad \Rightarrow D^3y - 7D^2y + 10\,Dy = e^{2x}\sin x$

A.E. is

$$m^3 - 7m^2 + 10\,m = 0 \quad \Rightarrow \quad (m-2)(m^2 - 5m) = 0$$

$$\Rightarrow \quad m(m-2)(m-5) = 0 \quad \Rightarrow \quad m = 0, 2, 5$$

C.F $= C_1e^{0x} + C_2e^{2x} + C_3e^{5x}$

$$\text{P.I.} = \frac{1}{D^3 - 7D^2 + 10D} e^{2x}\sin x = e^{2x}\frac{1}{(D+2)^3 - 7(D+2)^2 + 10(D+2)}.\sin x$$

$$= e^{2x}\frac{1}{D^3 + 6D^2 + 12D + 8 - 7D^2 - 28D - 28 + 10D + 20}.\sin x$$

$$= e^{2x}\frac{1}{D^3 - D^2 - 6D}\sin x = e^{2x}\frac{1}{(-1^2)D - (-1^2) - 6D}\sin x$$

$$= e^{2x}\frac{1}{-D + 1 - 6D}\sin x = e^{2x}\frac{1}{1 - 7D}\sin x = e^{2x}\frac{1 + 7D}{1 - 49D^2}\sin x = e^{2x}\frac{1 + 7D}{1 - 49(-1^2)}\sin x$$

$$= e^{2x}\frac{1 + 7D}{50}\sin x = \frac{e^{2x}}{50}(\sin x + 7\cos x)$$

Complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = C_1 + C_2e^{2x} + C_3e^{5x} + \frac{e^{2x}}{50}(\sin x + 7\cos x)$$ **Ans.**

Exercise

(1) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = e^{2x}\sin x$ **Ans.** $y = C_1e^{2x} + C_2e^{5x} + \frac{e^{2x}}{10}(3\cos x - \sin x)$

(2) $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = e^x\cos x$ **Ans.** $y = C_1e^{-2x} + e^x(C_2\cos x + C_3\sin x) + \frac{xe^x}{20}(3\sin x - \cos x)$

(3) $(D^2 - 4D + 3)\,y = 2xe^{3x} + 3e^{3x}\cos 2x$

Ans. $y = C_1e^x + C_2e^{3x} + \frac{1}{2}e^{3x}(x^2 - x) + \frac{3}{8}e^{3x}(\sin 2x - \cos 2x)$

(4) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x^2}$ **Ans.** $y = (C_1 + C_2x)\,e^{-x} - e^{-x}\log x$

Method of Variation Parameter

- Steps to solve linear D.E.
  - Find out y_c
  - Compared with it $y_c = c_1\ y_1 + c_2\ y_2$ and find y_1 & y_2
  - Solve $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix}$
  - Find $y_p = y_1 \int \frac{w_1}{w} R(x)dx + y_2 \int \frac{w_2}{w} R(x)dx$

Solved Example :-

(1) Solve by Variation parameter method :- $\frac{d^2y}{dx^2} + y = \sec x$

Solution. The auxiliary equation for the given differential equation is $m^2 + 1 = 0$ and so $m = \pm i$. Thus

$$\text{C.F.} = c_1 \cos x + c_2 \sin x.$$

To find P.I., let

$$y_1 = \cos x \text{ and } y_2 = \sin x.$$

Then

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

Therefore,

$$\begin{aligned} \text{P.I.} &= -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx \\ &= -\cos x \int \frac{\sin x \sec x}{1} dx + \sin x \int \frac{\cos x \sec x \, dx}{1} \\ &= \cos x \log \cos x + x \sin x. \end{aligned}$$

Hence the complete solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x \\ &\quad + \cos x \log \cos x + x \sin x. \end{aligned}$$

Solve by method of variation of parameters:

$$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$ (*Uttarakhand, II Semester, June* 2007, *A.M.I.E.T.E., Summer* 2001)

(*Nagpur University, Summer* 2001)

Solution. $$\frac{d^2y}{dx^2} - y = \frac{2}{1+e^x}$$

A. E. is $(m^2 - 1) = 0$

$$m^2 = 1, \quad m = \pm 1$$

$$C.F. = C_1 e^x + C_2 e^{-x}$$

$$\therefore \quad P.I. = uy_1 + vy_2$$

Here, $y_1 = e^x, \quad y_2 = e^{-x}$

and $y_1 \cdot y'_2 - y'_1 \cdot y_2 = - e^x \cdot e^{-x} - e^x . e^{-x} = -2$

$$u = \int \frac{-y_2 X}{y_1 . y'_2 - y'_1 \cdot y_2} dx = -\int \frac{e^{-x}}{-2} \times \frac{2}{1+e^x} dx$$

$$= \int \frac{e^{-x}}{1+e^x} dx = \int \frac{dx}{e^x (1+e^x)} = \int \left(\frac{1}{e^x} - \frac{1}{1+e^x} \right) dx$$

$$= \int e^{-x} dx - \int \frac{e^{-x}}{e^{-x}+1} dx = -e^{-x} + \log(e^{-x}+1)$$

$$v = \int \frac{y_1 X}{y_1 . y'_2 - y'_1 . y_2} dx = \int \frac{e^x}{-2} \frac{2}{1+e^x} dx = -\int \frac{e^x}{1+e^x} dx = -\log(1+e^x)$$

$$\text{P.I.} = u.\, y_1 + v.\, y_2 = [-e^{-x} + \log(e^{-x}+1)]\, e^x - e^{-x} \log(1+e^x)$$

$$= -1 + e^x \log(e^{-x}+1) - e^{-x} \log(e^x+1)$$

Complete solution = $y = C_1 e^x + C_2 e^{-x} - 1 + e^x \log(e^{-x}+1) - e^{-x} \log(e^x+1)$ **Ans.**

Exercise :-

1. $\frac{d^2y}{dx^2} - 4y = e^{2x}$ **Ans.** $y = C_1 e^{2x} + C_2 e^{-2x} + \frac{x}{4} e^{2x} - \frac{e^{2x}}{16}$
2. $\frac{d^2y}{dx^2} + y = \sin x$ **Ans.** $y = C_1 \cos x + C_2 \sin x - \frac{x}{2} \cos x + \frac{1}{4} \sin x$
3. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x$ **Ans.** $y = C_1 e^x + C_2 e^{2x} + \frac{1}{10}(3\cos x + \sin x)$
4. $\frac{d^2y}{dx^2} + y = \sec x \tan x$ **Ans.** $y = C_1 \cos x + C_2 \sin x + x \cos x + \sin x \log \sec x - \sin x$

CAUCHY EULER HOMOGENEOUS LINEAR EQUATIONS

$$a_n x_n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{d x^{n-1}} + ... + a_0 y = \phi(x) \quad ... (1)$$

where $a_0, a_1, a_2, ...$ are constants, is called a homogeneous equation.

Put $\quad x = e^z, \quad z = \log_e x, \quad \frac{d}{dz} \equiv D$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} \Rightarrow x \frac{dy}{dx} = Dy$$

Again, $\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\frac{dz}{dx}$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2 y}{dz^2}\frac{1}{x} = \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) = \frac{1}{x^2}(D^2 - D)\, y; \quad x^2 \frac{d^2 y}{dx^2} = (D^2 - D)\, y$$

or $\quad \boxed{x^2 \frac{d^2 y}{dx^2} = D(D-1)\, y} \quad$ Similarly, $\quad \boxed{x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)\, y}$

Solve: $$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4$$ (*A.M.I.E. Summer* 2000)

Solution. We have, $$x^2\frac{d^2y}{dx^2}-2x\frac{dy}{dx}-4y=x^4 \quad \dots (1)$$

Putting $x=e^z$, $D\equiv\frac{d}{dz}$, $x\frac{dy}{dx}=Dy$, $x^2\frac{d^2y}{dx^2}=D(D-1)\,y$ in (1), we get

$$D(D-1)\,y-2\,Dy-4y=e^{4z} \quad \text{or} \quad (D^2-3D-4)\,y=e^{4z}$$

A.E. is $m^2-3m-4=0 \;\Rightarrow\; (m-4)(m+1)=0 \;\Rightarrow\; m=-1,\,4$

$$\text{C.F.}=C_1\,e^{-z}+C_2\,e^{4z} \quad \text{P.I.}=\frac{1}{D^2-3D-4}e^{4z} \quad \text{[Rule Fails]}$$

$$=z\frac{1}{2D-3}e^{4z}=z\frac{1}{2(4)-3}e^{4z}=\frac{z\,e^{4z}}{5}$$

Thus, the complete solution is given by

$$y=C_1\,e^{-z}+C_2\,e^{4z}+\frac{z\,e^{4z}}{5} \;\Rightarrow\; y=\frac{C_1}{x}+C_2\,x^4+\frac{1}{5}x^4\log x$$

Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x^2)$ (

Solution. We have, $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \sin(\log x^2)$ ... (1)

Let $x = e^z$, so that $z = \log x$, $D \equiv \frac{d}{dz}$

(1) becomes

$$D(D-1)y + Dy + y = \sin(2z) \quad \Rightarrow \quad (D^2+1)y = \sin 2z$$

A.E. is $m^2 + 1 = 0$ or $m = \pm i$

$$C.F. = C_1 \cos z + C_2 \sin z$$

$$P.I = \frac{1}{D^2+1}\sin 2z = \frac{1}{-4+1}\sin 2z = -\frac{1}{3}\sin 2z$$

$$y = \text{C.F.} + \text{P.I.} = C_1 \cos z + C_2 \sin z - \frac{1}{3}\sin 2z$$

$$= C_1 \cos(\log x) + C_2 \sin(\log x) - \frac{1}{3}\sin(\log x^2)$$ **Ans.**

LEGENDRE'S HOMOGENEOUS DIFFERENTIAL EQUATIONS

A linear differential equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_n y = X \qquad \ldots (1)$$

where a, b, a_1, a_2, ... a_n are constants and X is a function of x, is called Legendre's linear equation.

Equation (1) can be reduced to linear differential equation with constant coefficients by the substitution.

$$a + bx = e^z \qquad \Rightarrow \qquad z = \log (a+bx)$$

so that

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a+bx} \cdot \frac{dy}{dz}$$

$$\Rightarrow \qquad (a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\,Dy, \quad D \equiv \frac{d}{dz} \qquad \Rightarrow \quad (a+bx)\frac{dy}{dx} = b\,Dy$$

Again $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx}\cdot\frac{dy}{dz}\right)$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{(a+bx)}\cdot\frac{d^2y}{dz^2}\cdot\frac{b}{(a+bx)}$$

$$\Rightarrow \quad (a+bx)^2\frac{d^2y}{dx^2} = -b^2\frac{dy}{dz} + b^2\frac{d^2y}{dz^2}$$

$$= b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = b^2\,(D^2\,y - D\,y) = b^2 D\,(\,D-1)y$$

$$\Rightarrow \quad (a+bx)^2\frac{d^2y}{dx^2} = b^2\,D\,(D-1)$$

Similarly, $(a+bx)^3\dfrac{d^3y}{dx^3} = b^3\,D(D-1)\,(D-2)y$

....

$$(a+bx)^n\frac{d^ny}{dx^n} = b^n\,D\,(D-1)\,(D-2)\,.....\,(D-n+1)y$$

Similarly, $(a+bx)^3 \frac{d^3y}{dx^3} = b^3\, D(D-1)(D-2)y$

....

$$(a+bx)^n \frac{d^ny}{dx^n} = b^n\, D(D-1)(D-2) \ldots\ldots (D-n+1)y$$

Substituting these values in equation (1), we get a linear differential equation with constant coefficients, which can be solved by the method given in the previous section.

$$Solve\ (1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2\{\log(1+x)\}$$

Solution. We have, $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = \sin 2\{\log(1+x)\}$

Put $1 + x = e^z$ or $\log(1+x) = z$

$(1+x)\dfrac{dy}{dx} = Dy$ and $(1+x)^2 \dfrac{d^2y}{dx^2} = D(D-1)y$, where $D \equiv \dfrac{d}{dz}$

Putting these values in the given differential equation, we get

$$D(D-1)y + Dy + y = \sin 2z \quad \text{or} \quad (D^2 - D + D + 1)y = \sin 2z$$

$$(D^2 + 1)y = \sin 2z$$

A.E. is $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$C.F. = A\cos z + B\sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}\sin 2z = \frac{1}{-4+1}\sin 2z = -\frac{1}{3}\sin 2z$$

Now, complete solution is $y = \text{C.F.} + \text{P.I.}$

$$\Rightarrow \quad y = A\cos z + B\sin z - \frac{1}{3}\sin 2z$$

$$\Rightarrow \quad y = A\cos\{\log(1+x)\} + B\sin\{\log(1+x)\} - \frac{1}{3}\sin 2\{\log(1+x)\}$$

Exercise :-

1. $x^2 \dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = \dfrac{42}{x^4}$ **Ans.** $C_1 x^2 + C_2 x^3 + \dfrac{1}{x^4}$

2. $(x^2 D^2 - 3xD + 4)\, y = 2x^2$ **Ans.** $(C_1 + C_2 \log x)\, x^2 + x^2 (\log x)^2$

3. $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x$ (*AMIETE, June 2010*) **Ans.** $(C_1 + C_2 \log x)\, x + \log x + 2$

4. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}$ **Ans.** $C_1 + C_2 \log x + 2(\log x)^3$

5. $(x^2D^2 - xD - 3)\, y = x^2 \log x$ **Ans.** $\dfrac{C_1}{x} + C_2 x^3 - \dfrac{x^2}{3}\left(\log x + \dfrac{2}{3}\right)$

   (*A.M.I.E. Winter* 2001, *Summer* 2001)

6. $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^2 + \sin(5 \log x)$

   Ans. $c_1 x + c_2 x^2 + x^2 \log x + \dfrac{1}{754}[15 \cos(5 \log x) - 23 \sin(5 \log x)]$

7. $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x \dfrac{\sin(\log x) + 1}{x}$ (*AMIETE, Dec. 2009*)

   Ans. $y + C_1 x^{2+\sqrt{3}} + C_2 x^{2-\sqrt{3}} + \dfrac{1}{x}\left[\dfrac{382}{61} \cos \log x + \dfrac{54}{61} \sin(\log x) + 6 \log x \cos(\log x) + 5 \log x \sin(\log x)\right] + \dfrac{1}{6x}$


Thank you