Cauchy-Euler Equation

Linear differential equations with variable coefficient.

Same ... Same

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots$$

Degree of operator = degree of variable ⇒ Cauchy Euler

Ex: $$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

We can solve the non-homogeneous equation $ax^2y'' + bxy' + cy = g(x)$ by variation of parameters, once we have determined the complementary function y_c.

Standard form of Cauchy L.D.E

$$(x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-2} D^{n-2} + \dots + a_n)y = Q(x)$$

Ex: Given D.E $$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = 0$$

put $D = d/dx$

$x^2D^2y - xDy - 3y = 0$

$\boxed{(x^2D^2 - xD - 3)y = 0}$

→ Standard Cauchy

Substitution:

$x^4D^4 = D_1(D_1-1)(D_1-2)(D_1-3)$

$x^3D^3 = D_1(D_1-1)(D_1-2)$

$x^2D^2 = D_1(D_1-1)$

$xD = D_1$; $D_1 = \frac{d}{dx}$

$x = e^z$

log on b/s

$\log x = z$

$(x^2D^2 - xD - 3)y = 0$

$x^2D^2 - xD - 3 = 0$

$D_1(D_1-1) - D_1 - 3 = 0$

$D_1^2 - D_1 - D_1 - 3 = 0$

$\boxed{D_1^2 - 2D_1 - 3 = 0}$

→ L.D.E with constant coefficient

→ A.E

$D_1^2 - 2D_1 - 3 = 0$

$(D_1-3)(D_1+1) = 0$

$D_1 = 3,\ D_1 = -1$

C.F = $c_1e^{-x} + c_2e^{3x}$

P.I = 0

G.S = $c_1e^{-x} + c_2e^{3x}$

Ex ②: $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

$x^2D^2y - 2xDy - 4y = x^4$, $x^2D^2 = D_1(D_1-1)$; $xD = D_1$

$(D_1(D_1-1) - 2D_1 - 4)y = x^4$

Let $x = e^z$, then $z = \log x$

$[D_1(D_1-1) - 2D_1 - 4]y = e^{4z}$ → L.D.E Reduced to Constant Coefficient

$[D_1^2 - D_1 - 2D_1 - 4]y = e^{4z}$

$[D_1^2 - 3D_1 - 4]y = e^{4z}$

Auxillary Eq:

$D_1^2 - 3D_1 - 4 = 0$ $\qquad D_1 = \frac{3 \pm \sqrt{9-4(-4)}}{2} = \frac{3 \pm 5}{2} = 4, -1$

$D_1 = 4$, $D_1 = -1$

Complementary Function, $C.F = c_1e^{-z} + c_2e^{4z}$

Particular Integral, $\frac{1}{f(D)}Q(x)$

P.I $= \frac{1}{D_1^2 - 3D_1 - 4}e^{4z} = \frac{1}{(4)^2 - 3(4) - 4}e^{4z}$ → leads to zero $= \frac{1}{D_1^2 - 3D_1 - 4}e^{4z}$

$= \frac{ze^{4z}}{2D_1 - 3} = \frac{ze^{4z}}{8-3} = \boxed{\frac{ze^{4z}}{5}}$

$G.S = c_1e^{-z} + c_2e^{4z} + \frac{ze^{4z}}{5}$, Replace $x = e^z$; $z = \log x$

$= c_1x^{-1} + c_2x^4 + \frac{\log x}{5} \cdot x^4$

$$\boxed{G.S = c_1x^{-1} + c_2x^4 + \frac{x^4\log x}{5}}$$

Ex 3: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 4y = \sin(\log x^2)$

$\Rightarrow x^2D^2y + xDy + 4y = \sin(\log x^2)$

$(x^2D^2 + xD + 4)y = \sin(\log x^2) \rightarrow \sin(2\log x)$

let $x = e^z$, $z = \log x \rightarrow$ Euler Substitution

$\boxed{(x^2D^2 + xD + 4)y = \sin(2z)} \rightarrow$ Cauchy Euler D.E

A.E $\quad D_1(D_1-1) + D_1 + 4 = 0$

$D_1^2 - D_1 + D_1 + 4 = 0$

$D_1^2 + 4 = 0$

$\Bigg\} \; D_1 = \frac{-0 \pm \sqrt{0-4(4)}}{2} = \frac{\pm 4i}{2} = \pm 2i$

$\boxed{D_1 = \pm 2i} \rightarrow$ Complex and Distinct roots

Complementary Function, $C.F = e^{0x}[c_1\cos 2z + c_2 \sin 2z]$

Particular Integral: $\frac{1}{f(D)} Q(x) = \frac{1}{D_1^2+4} \sin(2z)$; $D_1^2 = -4$

$D_1^2+4 \rightarrow D_1^2 = -4$ leads to zero

$P.I = \frac{z}{2D_1} \sin(2z) = \frac{z}{2} \frac{1}{D_1}(\sin 2z) = \frac{z}{2} \int \sin 2z \, dz$

$= \frac{z}{2} \frac{\cos(2z)}{2} = \boxed{\frac{-z\cos 2z}{4}}$

$G.S = c_1\cos 2z + c_2 \sin 2z + \frac{z\cos 2z}{4}$; $z = \log x$

$$\boxed{G.S = c_1\cos(2\log x) + c_2\sin(2\log x) + \frac{\log x \cdot \cos(2\log x)}{4}}$$

Ex 4: $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = x\log x$

$x^2D^2y + 4xDy + 2y = x\log x$

$(x^2D^2 + 4xD + 2)y = x\log x$

$D_1(D_1-1) + 4D_1 + 2 = x\log x$; $x = e^z$, $z = \log x$

$D_1^2 - D_1 + 4D_1 + 2 = e^z . z$

$D_1^2 + 3D_1 + 2 = ze^z \rightarrow$ P.I : EAT [Exp, Algeb, Trig]

A.E,

$D_1^2 + 3D_1 + 2 = 0$; $D_1 = \frac{-3 \pm \sqrt{9 - 4(2)}}{2} = \frac{3 \pm 1}{2} = 2, 1$

$D_1 = 2, 1$

$C.F = c_1e^{2z} + c_2e^z$

$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D_1^2+3D_1+2} ze^z = e^z \times \frac{1}{(D_1+1)^2+3(D_1+1)+2} z$

$= e^z \times \frac{1}{D_1^2+2D_1+1+3D_1+3+2} z = e^z \times \frac{1}{D_1^2+5D_1+6} z \rightarrow$ binomial

$= e^z \times \frac{1}{6\left[1 + \frac{D_1^2+5D_1}{6}\right]} z = e^z \frac{1}{6}\left[1 + \left(\frac{D_1^2+5D_1}{6}\right)\right]^{-1} z$

$= \frac{e^z}{6} \times \left[1 + \left(\frac{D_1^2+5D_1}{6}\right) + \left(\frac{D_1^2+5D_1}{6}\right)^2 \ldots\right] z$

$= \frac{e^z}{6}\left[1 - \left(\frac{D_1^2+5D_1}{6}\right)\right] z = \frac{e^z}{6}\left[z - z\left(\frac{D_1^2+5D_1}{6}\right)\right]$

$= \frac{e^z}{6}\left[z - \left(\frac{zD_1^2+5D_1z}{6}\right)\right] = \frac{e^z}{6}\left[z - \left(\frac{0+5}{6}\right)\right] = \frac{e^z}{6}\left[z - \frac{5}{6}\right]$

$= \frac{e^z}{6}\left[\frac{6z-5}{6}\right] = \boxed{\frac{e^z}{36}(6z-5)}$

$G.S = c_1e^{2z} + c_2e^z + \frac{e^z}{36}(6z-5) = c_1x^2 + c_2x + \frac{x}{36}(6\log x - 5)$

$\boxed{G.S = c_1x^2 + c_2x + \frac{x}{36}(6\log x - 5)}$

Cauchy Euler Worksheet

① $2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0$

Step 1: Reduction to Constant Coefficient

$2x^2D^2y - xDy + y = 0$

$[2x^2D^2 - xD + 1]y = 0$

$[2D_1(D_1-1) - D_1 + 1]y = 0$

↳ Cauchy L.D.E

Step 2: A.E

$2D_1(D_1-1) - D_1 + 1 = 0$

$2D_1^2 - 2D_1 - D_1 + 1 = 0$

$D_1 = \frac{3 \pm \sqrt{9-4(2)}}{2} = \frac{3 \pm 1}{2} = 2, 1$

$\boxed{D_1 = 2, 1}$

Step 3: C.F

$\boxed{C.F = c_1e^{2z} + c_2e^{z}}$; $x = e^z$, $z = \log x$

$G.S = C.F + \overset{\nearrow 0}{P.I} = c_1e^{2z} + c_2e^{z} = \boxed{c_1x^2 + c_2x}$

② $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Step 1: Reduction

$xD^2y + Dy = 0$

multiply x on b/s

$x^2D^2y + xDy = 0$

$x^2D^2y + xDy = 0$

$\boxed{(x^2D^2 + xD)y = 0}$

↳ Cauchy Euler D.E

$z = \log x$, $x = e^z$

Step 2: A.E

$x^2D^2 + xD = 0$

$D_1(D_1-1) + D_1 = 0$

$D_1^2 - D_1 + D_1 = 0$

$D_1^2 = 0$

$\boxed{D_1 = 0, 0}$

Step 3: C.F

$C.F = e^{0z}[c_1 + c_2z]$

$\boxed{C.F = c_1 + c_2z}$

$\boxed{G.S = c_1 + c_2(\log x)}$

③ $x^2y'' + 2xy' + 2y = 0$

$[x^2D^2 + 2xD + 2]y = 0$

put $x = e^z$; $z = \log x$

$x^2D^2 + 2xD + 2 = 0$

$D_1(D_1-1) + 2D_1 + 2 = 0$

$D_1^2 - D_1 + 2D_1 + 2 = 0$

$D_1 = \frac{-1 \pm \sqrt{1-4(2)}}{2}$

$\boxed{D_1 = \frac{-1 \pm \sqrt{7}i}{2}}$

$C.F = e^{-z}\left[c_1\cos\frac{\sqrt{7}}{2}z + c_2\sin\frac{\sqrt{7}}{2}z\right]$

$P.I = 0$

$G.S = e^{-z}\left[c_1\cos\frac{\sqrt{7}}{2}z + c_2\sin\frac{\sqrt{7}}{2}z\right]$

$\boxed{G.S = x^{-1}\left[c_1\cos\left(\frac{\sqrt{7}}{2}\log x\right) + c_2\sin\left(\frac{\sqrt{7}}{2}\log x\right)\right]}$

(4) $x^2y'' - 4xy' + 6y = x^2 + \log x$

Step 1: Reduction

$x^2D^2y - 4xDy + 6y = x^2 + \log x$

$[x^2D^2 - 4xD + 6]y = x^2 + \log x$

put $x = e^z$, $z = \log x$

$$\boxed{[x^2D^2 - 4xD + 6]y = e^{2z} + z}$$

Step 2: A.E $\rightarrow x^2D^2 - 4xD + 6 = 0$

$D_1(D_1 - 1) - 4D_1 + 6 = 0$

$D_1^2 - D_1 - 4D_1 + 6 = 0$

$$D_1 = \frac{5 \pm \sqrt{25 - 4(6)}}{2} = \frac{5 \pm 1}{2} = 3, 2$$

$$\boxed{D_1 = 3, 2}$$

Step 3: $C.F = c_1e^{3z} + c_2e^{2z}$

Step 4: $P.I = \frac{1}{f(D)}Q(x) = \frac{e^{2z}}{D_1^2 - 5D_1 + 6} + \frac{z}{D_1^2 - 5D_1 + 6} = \frac{ze^{2z}}{2D_1 - 5} + \frac{z}{6\left[1 + \left(\frac{D_1^2 - 5D_1}{6}\right)\right]}$

$$= \frac{ze^{2z}}{2(2) - 5} + \frac{1}{6}\left[1 + \left(\frac{D_1^2 - 5D_1}{6}\right)\right]^{-1} z$$

$$= -ze^{2z} + \frac{1}{6}\left[1 - \left(\frac{D_1^2 + 5D_1}{6}\right) + \left(\frac{D_1^2 + 5D_1}{6}\right)^2 - \dots\right] z$$

$$= -ze^{2z} + \frac{1}{6}\left[1 - \left(\frac{D_1^2 + 5D_1}{6}\right)\right] z = -ze^{2z} + \frac{1}{6}\left[z - z\left(\frac{D_1^2 + 5D_1}{6}\right)\right]$$

$$= -ze^{2z} + \frac{1}{6}\left[z - \left(\frac{zD_1^2 + 5D_1z}{6}\right)\right] = -ze^{2z} + \frac{1}{6}\left[z - \left(\frac{0 + 5}{6}\right)\right]$$

$$= -ze^{2z} + \frac{1}{6}\left[z + \frac{5}{6}\right] = \boxed{-ze^{2z} + \frac{z}{6} + \frac{5}{36}}$$

$$G.S = c_1e^{3z} + c_2e^{2z} - ze^{2z} + \frac{z}{6} + \frac{5}{36}$$

$$= c_1x^3 + c_2x^2 - (\log x)x^2 + \frac{\log x}{6} + \frac{5}{36}$$

$$\boxed{G.S = c_1x^3 + c_2x^2 - x^2\log x + \frac{\log x}{6} + \frac{5}{36}}$$

(5) $[x^2D^2 + 4xD + 2]y = \log x$

Step 1: Reduction

$[x^2D^2 + 4xD + 2]y = \log x$

$x = e^z$; $z = \log x$

$[D_1(D_1-1) + 4D_1 + 2]y = z$

↳ Cauchy Euler

$[D_1^2 + 3D_1 + 2]y = z \rightarrow Q(x)$

Step 2: A.E

$D_1^2 + 3D_1 + 2 = 0$

$D_1 = \frac{-3 \pm \sqrt{9 - 4(2)}}{2} = \frac{-3 \pm 1}{2} = \boxed{-2, -1}$

Step 3: $\boxed{C.F = c_1e^{-2z} + c_2e^{-z}}$

Step 4: $P.I = \frac{1}{f(D)} Q(x) = \frac{1}{D_1^2 + 3D_1 + 2} z = \frac{1}{2\left[1 + \frac{(D_1^2 + 3D_1)}{2}\right]} z$

$= \frac{1}{2}\left[1 + \left(\frac{D_1^2 + 3D_1}{2}\right)\right]^{-1} z = \frac{1}{2}\left[1 - \left(\frac{D_1^2 + 3D_1}{2}\right) + \left(\frac{D_1^2 + 3D_1}{2}\right)^2 \dots\right] z$

$= \frac{1}{2}\left[1 - \left(\frac{D_1^2 + 3D_1}{2}\right)\right] z = \frac{1}{2}\left[z - \left(\frac{D_1^2 z + 3D_1 z}{2}\right)\right]$

$= \frac{1}{2}\left[z - \left(\frac{0 + 3}{2}\right)\right] = \boxed{\frac{1}{2}\left[z - \frac{3}{2}\right]}$

$G.S = c_1e^{-2z} + c_2e^{-z} + \frac{z}{2} - \frac{3}{4} = \boxed{c_1x^{-2} + c_2x^{-1} + \frac{\log x}{2} - \frac{3}{4}}$

→ Legendre's Homogeneous Differential Equations

A linear differential equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = X$$

Ex: $$(2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$$

Put $2x+1 = e^z$, $z = \log(2x+1)$

$$(2x+1)\frac{dy}{dx} = 2Dy \quad \& \quad (2x+1)^2 = 2^2 D(D-1)y$$

$$4D(D-1)y - 4Dy - 12y = 6\left[\frac{e^z - 1}{2}\right]$$

$$4D^2y - 4Dy - 4Dy - 12y = 3(e^z - 1) \Rightarrow 4m^2 - 8m - 12 = 0$$

$$m = 3, \; m = -1$$

$$C.F = C_1 e^{3z} + C_2 e^{-z}$$

$$P.I = \frac{1}{4D^2 - 8D - 12}(3e^z - 3) = \frac{3}{-16}e^z + \frac{1}{4}$$

$$y = C_1 e^{3z} + C_2 e^{-z} - \frac{3}{16}e^z + \frac{1}{4}$$

$$y = c_1(2x+1)^3 + c_2(2x+1)^{-1} - \frac{3}{16}(2x+1) + \frac{1}{4}$$

Legendre's Homogeneous D.E

$\Rightarrow (2x+1)^2 \frac{d^2y}{dx^2} - 2(2x+1)\frac{dy}{dx} - 12y = 6x$

put $(2x+1) = e^z$; $z = \log x$

$(2x+1)^2 D^2 y - 2(2x+1) Dy - 12y = 6x$

$[(2x+1)^2 D^2 - 2(2x+1)D - 12]y = 6\left(\frac{\log z - 1}{2}\right)$

$x^2D^2 \downarrow$ $\quad xD$

Derivatives

$a = (2x+1)^2$; $a' = 2(2) = 4$

$b = (2x+1)$; $b' = 2$

$[a' D_1(D_1 - 1) - 2b' D_1 - 12]y = 6\left(\frac{e^z - 1}{2}\right)$

$4D_1(D_1 - 1) - 2 \times 2D_1 - 12 = 0 \rightarrow$ A.E

$4D_1^2 - 4D_1 - 4D_1 - 12 = 0 \rightarrow 4m^2 - 8m - 12 = 0$; $m = 3, -1$

$C.F = c_1 e^{3z} + c_2 e^{-z}$

$P.I = \frac{1}{f(D)} Q(x) = \frac{1}{4D^2 - 8D - 12} \times \frac{6(e^z - 1)}{2} = \frac{3}{4D^2 - 8D - 12}(e^z - 1)$

$= \frac{3e^z}{4(1)^2 - 8(1) - 12} - \frac{3e^{0x}}{4(0)^2 - 8(0) - 12} = \frac{3e^z}{-16} + \frac{3}{12} = \boxed{\frac{+3e^z}{-16} + \frac{3}{12}}$