Formation of D.E's

Differential equations from physical phenomenon:

The problems that involve rate of change of one variable with respect to time always give rise to D.E.

→ **Growth and Decay problems**

Ex:- The rate of growth of population in a small town is proportional to present population. Find the D.E

Solⁿ Let p be the population

$$\text{rate of growth} \swarrow \frac{dP}{dt} \propto P \qquad \frac{dP}{dt} = \overset{\text{constant}}{\overset{\nearrow}{k}}P \Rightarrow \boxed{\frac{dP}{dt} - kP = 0}$$

Ex:- The rate of decrease in radio active material is proportional to the amount present. Find D.E

Solⁿ Let R be the radio active material.

$$\frac{dR}{dt} \propto R \Rightarrow \frac{dR}{dt} = \underset{\text{b/c decrease}}{-kR} \Rightarrow \frac{dR}{dt} + kR = 0$$

Ex:- The population of town was 1000 a year ago and the present population is 2000. what will be the population after the end of 4ᵗʰ year if the rate of increase in population is proportional to present population?

Solⁿ Let p be the population at time t.

initial population was 1000 when $t = 0$

So $\dfrac{dP}{dt} \propto P$

→ $\dfrac{dP}{dt} = kP \Rightarrow \int \dfrac{dP}{P} = \int k\, dt$

→ $\ln P = kt + c$

$P = e^{kt+c}$ t = -1, P = 1000

$P = e^{kt} \cdot e^{c}$ t = 0, P = 2000

So $P(t) = P_0\, e^{kt}$

when $t = 0$ $\quad P = 1000$

$P(0) = P_0\, e^{k(0)}$

$\boxed{1000 = P_0}$

$$\boxed{P(t) = 1000\, e^{kt}}$$

when $t = 1$ population $= 2000$

$P(1) = 1000\, e^{k(1)}$

$\dfrac{2000}{1000} = e^{k} \Rightarrow \boxed{2 = e^{k}}$

$\ln(2) = k \nearrow^{\text{or}}$

$$\boxed{P(t) = 1000\,(2)^{t}}$$

when $t = 4$

$P(4) = 1000\,(2)^{4} = 1000\,(16)$

$= 16000$ Ans

Ex:— The population of a certain country has grown at a rate proportional to number of people in the country. At present, the country has 80 million inhabitants. Ten years ago it had 70 million. Assuming that this trend countries find (a) an expression for the approximate number of people living in the country at time t.

(b) The approximate number of people at the end of the next ten years period.

Solⁿ

$$\frac{dp}{dt} \propto P \Rightarrow \frac{dP}{dt} = KP \Rightarrow \int \frac{1}{P} dP = \int k dt$$

$$\ln p = kt + c \Rightarrow P(t) = e^{kt} . e^{c} \Rightarrow \boxed{P(t) = P_0 e^{kt}}$$

when $t=0$, $P = 80$ million

$$P(0) = P_0 e^{k(0)} \Rightarrow \boxed{80m = P_0}$$

$$P(t) = 80 e^{kt} \longrightarrow$$

Ten year ago $\rightarrow t = t_0 - 10$, $P = 70$

$$P(-10) = 80 e^{-10k} \Rightarrow \frac{70}{80} = e^{-10k} \Rightarrow \ln\left(\frac{7}{8}\right) = -10k$$

$$k = -\frac{\ln\left(\frac{7}{8}\right)}{10} \Rightarrow \boxed{k = 0.01335}$$

a→ $$\boxed{P(t) = 80 e^{0.01335t}} \rightarrow \text{required eq.}$$

b→ $t = 10$ $p = ?$

$$P(10) = 80 e^{0.01335(10)} = 91.43 \text{ million } Ans$$

10.4: Applications of Linear Differential Equations

Recall that in Chapter 4.3, we described a very simple model for bank accounts and interest. We had a situation where an account had balance $P(t)$ at time t, and that if the interest being accumulated was compounded continuously, the function $P(t)$ satisfied the differential equation $y' = ky$, where k was the annual interest on the account. It turned out that the solution $P(t)$ was simple enough for us to understand at the time, as $P(t) = P(0)e^{kt}$.

Of course as is always the case, this model is not sufficient to describe many real situations: for instance, who actually gets a bank account and then leaves it alone? In particular, if a business is hoping to make a large enough profit to grow, it is realistic to think they would deposit money into an account every day. As such we consider the following problem:

Example 1: Adding to an account

Suppose a family is depositing money into a bank account continuously at the rate of about $\$10,000$ per year, and the account earns interest of 4% annually (compounded continuously). The family began their first year with $\$23,000$ in the account. Assuming they don't make any withdrawals, how much money is in the account after 4 years?

To solve this equation, we have to translate the text into a mathematical statement, which will turn out to be a differential equation. The equation will still account for both the fact that the money is earning interest, and the fact that additional money is being deposited regularly. Think *"change in money = interest + savings"*:

$$y' = .04y + 10000$$

This should look familiar as a first-order linear differential equation. Since we would like to account for the beginning of the account as well, we get the initial conditions $y(0) = 23000$. To solve, we follow the steps for first-order linear differential equations as prescribed in section 10.3, and get the general solution

$$P(t) = -250000 + Ce^{.04t}$$

Then we solve for C to get the particular solution

$$P(t) = -250000 + 273000e^{.04t}$$

Finally, to find the account balance after 4 years, we plug in $t = 4$ and find $P(4) \approx 70368.47$, or $\$70,368.47$. **Example 2: Adding to an account with income growth**

A small cat-toy company is growing their business at a rate which equals about $\$30,000$ of new profit each year. In their first year they make $\$50,000$. They start a business bank account earning interest of 5% per year, and deposit their profits daily. How much money should be in their account after the first 30 months?

To solve this equation, we'll use the same basic principle as before: *"change in money = interest + savings"*. To make things easier on ourselves, let $P(t)$ be the balance in thousands

of dollars. This time, however, the savings amount per year increases as the company grows. Thus, the formula for the money savings they add to the account is $50 + 30t$, and the differential equation becomes:

$$y' = .05y + 50 + 30t$$

Since we assume the company started from nothing, then we use the initial condition $y(0) = 0$. Then we solve to get:

$$P(t) = 11000 - 60t + Ce^{.05t}$$

And the initial condition $y(0) = 0$ gives us $C = 11000$, so the particular solution is:

$$P(t) = 11000 - 60t + 11000e^{.05t}$$

Thus, after 30 months, the balance should be equal to $P(2.5)$, measured in thousands of dollars. $P(2.5) \approx 1314.63298$, so we have a balance of $1,314,632.98 after 30 months.

Example 3: Paying off a loan

A similar situation is paying off a loan. Suppose that you took out college loans totaling $60,000 with interest of 7.5%. You have an online payment plan which continuously deducts money from your bank account at a rate which comes out to $15,000 per year. How long will it take you to pay off the loan?

Again, this will be a first-order linear differential equation. The change in the balance of the loan will go up as interest is accumulated, and go down as payments are made. As such, if y represents the balance of the loan at time t, then the differential equation will have a positive term for the interest, and a negative term for the payments:

$$y' = .075y - 15000$$

Again, the initial condition tells us that $y(0) = 60000$. Using the techniques from class, we get the general solution

$$P(t) = 200000 + Ce^{.075t}$$

and particular solution

$$P(t) = 200000 - 140000e^{.075t}$$

In order to find the amount of time it takes to pay off the loan, we need to solve for when the balance $P(t)$ is zero.

$$0 = 200000 - 140000e^{.075t}$$

$$200000 = 140000e^{.075t}$$

$$\frac{200000}{140000} = \frac{10}{7} = e^{.075t}$$

$$\ln\left(\frac{10}{7}\right) = .075t$$

$$t = \frac{1}{.075}\ln\left(\frac{10}{7}\right) \approx 4.76 \text{ years}$$

Example 4: Blood

A drug is introduced intravenously at a rate of .5mg/hour. On a continuous basis 2% of the drug is removed from the blood and absorbed by the body. Write a differential equation satisfied by $y(t)$, the amount of drug in the patient, and determine the equilibrium amount.

Change in drug concentration = injection amount − absorption rate

$$y' = .5 - .02y; \quad y(0) = 0$$

Solve to get $P(t) = 25 - 25e^{-.02t}$ mg of the drug at time t. The equilibrium amount is the limit at $t \to \infty$, or 25 mg.

Example 5: Population with emigration

The depressed city of Rottentown has a current population of 300 thousand people, with a growth constant of .02. Over the next 10 years, as news spreads about better opportunities elsewhere, there is to be an emigration rate of $.5 + e^{.03t}$ thousand people leaving the city per year, where t is the number of years from today. Find a function which predicts the population t years from today.

Population change in thousands = growth rate − emigration rate

$$y' = .02y - (.5 + e^{.03t}); \quad y(0) = 300$$

The general solution to this equation is $P(t) = 25 - 100e^{.03t} + Ce^{.02t}$, and if we set $P(0) = 300$, then we solve for C to get $C = 375$, so the particular solution is $P(t) = 25 - 100e^{.03t} + 375e^{.02t}$. Then plugging in $t = 10$ gives us $P(10) \approx 348$ thousand people. *Difference* *summing*

Example 6: Heat and Cooling *Rate of Temp $\propto \Delta T \Rightarrow \frac{dT}{dt} = k(T_a - T_{obj})$*

Newton's Law of Cooling says that the temperature change of an object placed in an environment is proportional to the difference between the temperature of the environment and the temperature of the object. To translate this to mathematics, remember that "proportional" means to multiply by a constant. Let $f(t)$ be the temperature of the object at time t, and $T(t)$ be the temperature of the environment.

$$f'(t) = k(T(t) - f(t))$$

It turns out that k differs, depending on the object: you might imagine that a single drop of water will heat up more quickly than an entire gallon of water if placed into a hot pot. The higher the constant k, the more easily heated the object.

Suppose that a hot pocket is placed into an old and unreliable oven, which is then turned on. The temperature of the hot pocket out of the freezer is $30°$, and the oven temperature is gradually rising from $t = 0$ to $t = 30$ (in minutes), with the temperature at time t given by $70 + 10t$. Suppose the heat constant of the hot pocket is .2. What is the temperature of the

hot pocket at the end of 30 minutes? What if the oven had been preheated and was already at 370° when the hot pocket was put in?

We solve the differential equation

$$y' = .2(70 + 10t - y); \quad y(0) = 30$$

If the oven were preheated, we would get $T(t)$ to be the constant function $T = 370$.

Example 7: <mark>Heat and cooling again</mark>

Suppose a person with a dangerously high fever of 105° were placed in a bath of cool water in order to calm the fever. The water is maintained at a temperature of 70°, and it takes 10 minutes to bring the person's temperature down to a safer 100°. What is the heating constant of this person's body?

Set up the same equation $y'' = k(70 - y)$, but we now have two sets of conditions: $y(0) = 105$ and $y(10) = 100$. We can then get an equation in terms of C and k and solve for both of these constants.