♥ RADHE x RADHE ♥

n-tuple = n-dimensional

How to determine that a function is differentiable?

1: In order for a function to be differentiable at a point, it must have a well defined derivative at that point

2: The derivative of a function can be thought of as the slope of tangent line to the function at a particular point.

3: one way to determine if a function is differentiable at a point is to check if it is continuous at that point. If a function is not continuous at a point, it cannot be differentiable at that point.

4: If a function is continuous at a point, we can use the defination of derivative to determine if it is differentiable at that point the defination of derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

If this limit exist and is finite then the function is differentiable at the point x.

5: a function with a sharp corner or a vertical tangent line will not be differentiable at that point.

6: Also, functions that contain discontinuities or singularities, such as absolute value or inverse functions may not be differentiable at that point.

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$a^2 - b^2 = (a-b)(a+b)$

$a^2 + b^2 = (a+b)^2 - 2ab$

$a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$a^3 + b^3 = (a+b)(a^2-ab+b^2)$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$

Two special trigonometric limits

1) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

2) $\lim_{x \to 0} \frac{1-\cos x}{x} = 0$

3) $\frac{\tan x}{x} \lim_{x \to 0} = 1$

4) $\frac{\cot \theta}{\theta} = 1$

5) $\frac{\sin^2 \theta}{\theta} = 0, \lim_{x \to 0}$

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a point on the termind side of an angle θ in standard position is given. find the exact value of each of the six trigonometric function of θ.

77) $(-3, 4)$

Solution

The point $(-3, 4)$ is on a circle of radius r.

$r = \sqrt{x^2 + y^2}$

$r = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = \sqrt{25} = \boxed{5}$ $\boxed{r = 5}$

for the point $(x, y) = (-3, 4)$ we have $\sin = 4 = y$

$\cos = -3 = x$ and $r = 5$ $\boxed{\text{Formed Equation is}}$

$\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$ $\boxed{x^2 + y^2 = 25 \quad 5 = r \rightarrow r^2}$

$\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{-3}{5}$ $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\tan\theta = \frac{4}{5} \div \frac{-3}{5} = \frac{4}{5} \times \frac{5}{-3} = \boxed{\frac{-4}{3}} \rightarrow \tan\theta$

$\cot\theta = \frac{-3}{4}$, $\sec\theta = \frac{-5}{3}$, $\csc\theta = \frac{5}{4}$

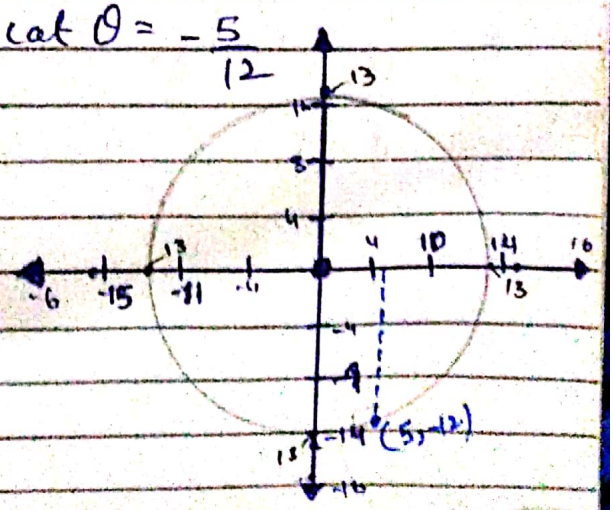
78) $(5, -12)$

$r = \sqrt{x^2 + y^2} = \sqrt{(5)^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = \boxed{13} \rightarrow r$

The formed equation is $x^2 + y^2 = r^2 \rightarrow \boxed{x^2 + y^2 = 169}$

$\sin\theta = \frac{-12}{13}$, $\cos\theta = \frac{5}{13}$, $\sec\theta = \frac{13}{5}$, $\csc\theta = \frac{13}{-12}$

$\tan\theta = \frac{-12}{13} \div \frac{5}{13} = \frac{-12}{5}$, $\cot\theta = \frac{-5}{12}$



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2) Evalute the following:

i) $\dfrac{\tan\frac{\pi}{3} - \tan\frac{\pi}{6}}{1+\tan\frac{\pi}{3}\tan\frac{\pi}{6}} = \dfrac{\tan 60^\circ - \tan 30^\circ}{1+\tan 60^\circ \tan 30^\circ}$

$\therefore \tan 30^\circ = a/b = 1/\sqrt{3}$ $\quad\therefore \tan 60^\circ = b/a = \sqrt{3}/1 = \sqrt{3}$

$$\frac{\tan 60^\circ - \tan 30^\circ}{1+\tan 60^\circ \times \tan 30^\circ} = \frac{b/a - a/b}{1+b/a \times a/b} = \frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{1+\sqrt{3}\times\frac{1}{\sqrt{3}}} = \frac{\frac{(\sqrt{3})^2-1}{\sqrt{3}}}{1+1}$$

$$\frac{3-1}{\sqrt{3}} \div 2 = \frac{2}{\sqrt{3}} \div 2 = \frac{2}{\sqrt{3}} \times \frac{1}{2} = \boxed{\frac{1}{\sqrt{3}}}$$

ii) $\dfrac{1-\tan^2 \pi/3}{1+\tan^2 \pi/3}$

$$= \frac{1-\tan^2 60^\circ}{1+\tan^2 60^\circ} = \frac{1-\left(\frac{b}{a}\right)^2}{1+\left(\frac{b}{a}\right)^2} = \frac{1-(\sqrt{3})^2}{1+(\sqrt{3})^2} = \frac{1-3}{1+3} = \frac{-2}{4} = \boxed{\frac{-1}{2}}$$

4) Find x, if $\tan^2 45^\circ - \cos^2 60^\circ = x \sin 45^\circ \cos 45^\circ \tan 60^\circ$

Solution

$\tan^2 45^\circ - \cos^2 60^\circ = x \sin 45^\circ \cos 45^\circ \tan 60^\circ$

$$(1)^2 - \left(\frac{1}{2}\right)^2 = x \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{1} = \frac{\sqrt{3}}{2}$$

$$1 - \frac{1}{4} = x\frac{\sqrt{3}}{2}$$

$$\frac{3}{4} = x\frac{\sqrt{3}}{2} \rightarrow x = \frac{3}{4} \div \frac{\sqrt{3}}{2} = \frac{3}{4} \times \frac{2}{\sqrt{3}} = \frac{3}{2\sqrt{3}}$$

$$\boxed{x = \frac{3}{2\sqrt{3}}}$$

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Graphing Trigonometric functions:- ① Sine

General Syntax → $y = A\sin(Bx + C) + D$

where

A → Amplitude (it defines the Range of function)

B → waves (it defines the Revolutions of function)

C → Horizontal shifting

D → vertical shifting

- $A > 0$ Initial point moves to upward and so on
- $A < 0$ Initial point moves to downward and so on.

▸ Range of function Sine

$D - A = M$ $\quad$ $D + A = N$ $\quad$ $[M, N]$ → Range

▸ Revolutions (Period of Sine function)

formula :- $\boxed{\dfrac{2\pi}{B}}$, where B is the coefficient of x.

1) Domain of Sine is $(-\infty, \infty+)$
2) Sine is an odd function, as the symmetry of the graph with respect to the origin
3) The Sine function is periodic, with period 2π
4) The function's Graph has absolute max & Min points.

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properties of the cosine function

General Syntax → $y = A\cos(Bx + C) + D$

A, B, C and D are same as Sine

1) Domain → $(-\infty, \infty+)$

$A > 0$ downward (moves point) $A < 0$ upward (moves point) Graph

2) period → $\boxed{\frac{2\pi}{B}}$

3) The cosine function is an even function, as the symmetry of graph with respect to the y-axis

4) Graph has Absolute Max & Min points

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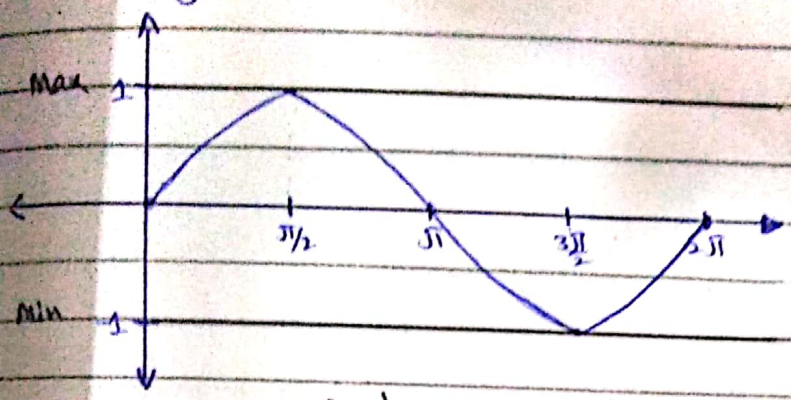
1) $y = \sin(x)$

Solution

Period $= 2\pi$, Initial point 0, Range $[-1, 1]$

$A > 0$ upward

five keypoints $\rightarrow$ $(0,0)$, $(\pi/2, 1)$, $(\pi, 0)$, $(3\pi/2, -1)$, $(2\pi, 0)$



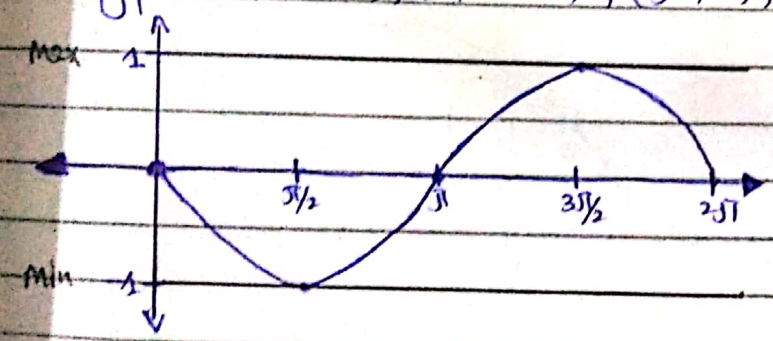
2) $y = -\sin(x)$

Solution

period $= 2\pi$, Initial point $(0,0)$, Range $[-1, 1]$

$A < 0$, Graph moves downward at initial point

5 keypoints, $(0,0)$, $(\pi/2, -1)$, $(\pi, 0)$, $(3\pi/2, 1)$, $(2\pi, 0)$

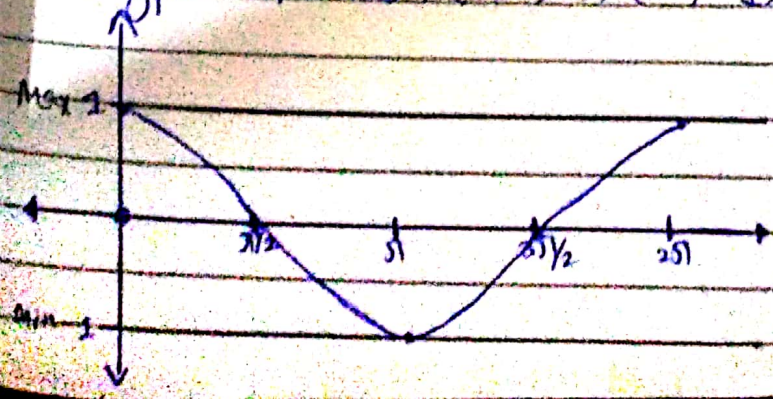


3) $y = \cos(x)$

period $= 2\pi$, Initial $(0, 1)$, Range $[-1, 1]$

$A > 0$ downward moves the point (Graph upward)

5 keypoints, $(0,1)$, $(\pi/2, 0)$, $(\pi, -1)$, $(3\pi/2, 0)$, $(2\pi, 1)$



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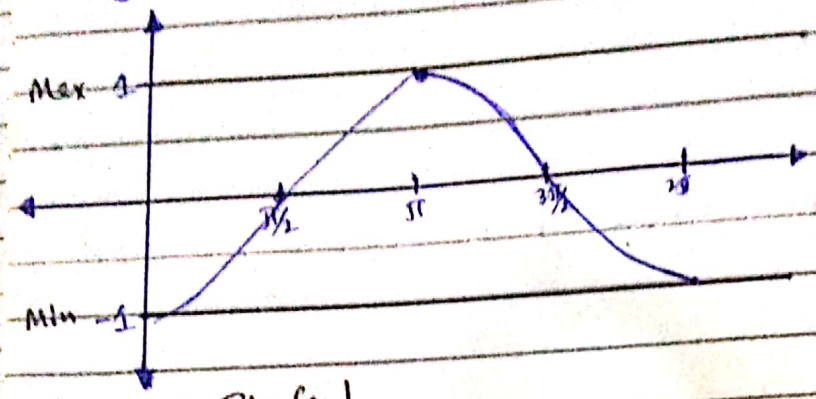
4) $y = -\cos(x)$

period = 2π, Initial $(-1, 0)$, Range $[-1, 1]$

$A < 0$, point moves to the upward (Graph downward)

5 key points, $(-1, 0)$, $(\pi/2, 0)$, $(\pi, 1)$, $(3\pi/2, 0)$, $(2\pi, -1)$

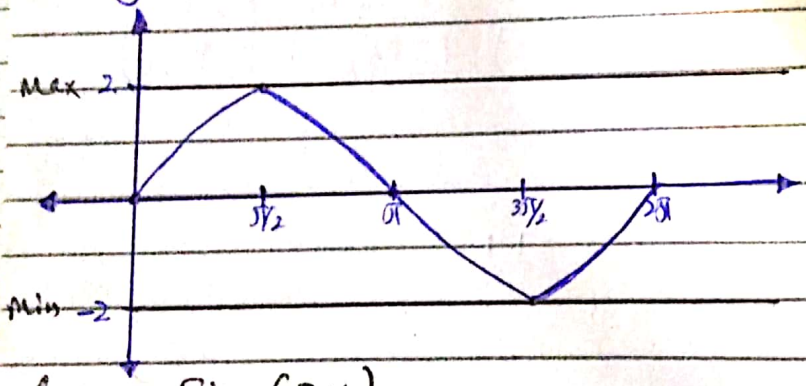


5) $y = 2\sin(x)$

period = 2π, Range → $d + A$, $d - A$ → $2, -2$ → $[-2, 2]$

Initial point $(0, 0)$, point moves upward

5 keypoints $(0, 0)$, $(\pi/2, 2)$, $(\pi, 0)$, $(3\pi/2, -2)$, $(2\pi, 0)$

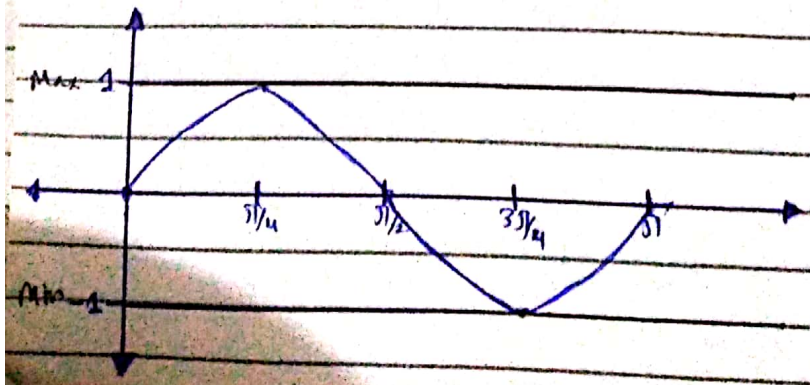


6) $y = \sin(2x)$

period = π, Initial point $(0, 0)$, Range $[-1, 1]$

$A > 0$, point moves upward

5 keypoints $(0, 0)$, $(\pi/4, 1)$, $(\pi/2, 0)$, $(3\pi/4, -1)$, $(\pi, 0)$



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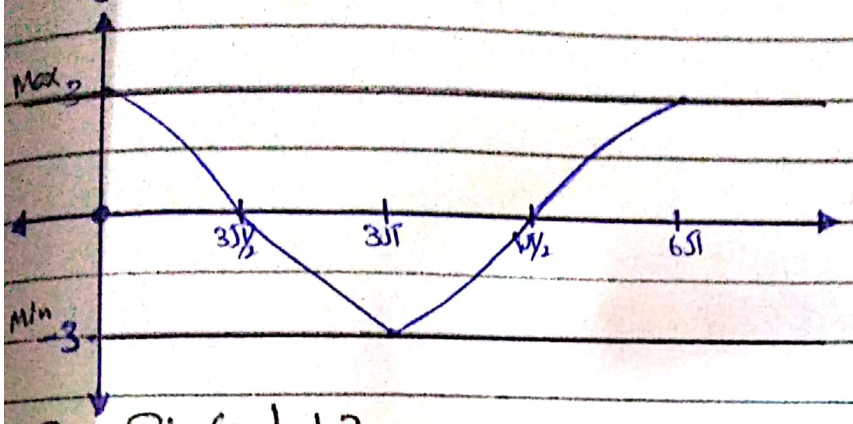
7) $y = 3\cos\left(\frac{x}{3}\right)$

Period = 6π, Range $[-3, 3]$, Initial point $(+3, 0)$

$A > 0$, Graph upward, Initial point moves toward down

5 key points $(3,0)$, $(3\pi/2, 0)$, $(3\pi, -3)$, $(9\pi/2, 0)$, $(6\pi, 3)$

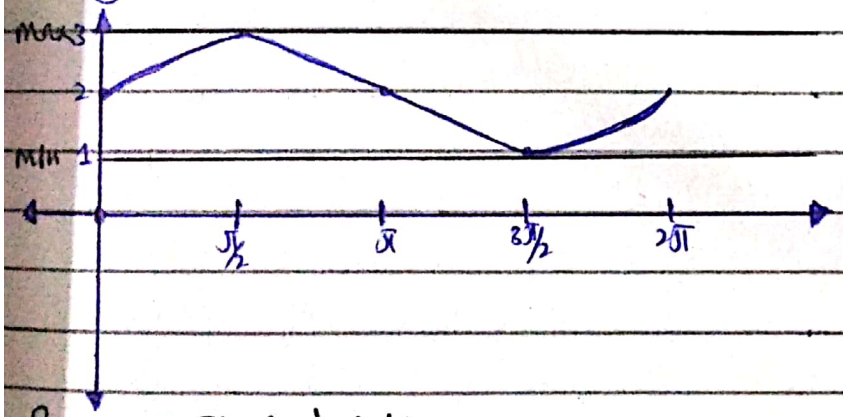


8) $y = \sin(x) + 2$

Period = 2π, Range → $C-A$, $C+A$ → 1, 3 → $[1, 3]$, Initial $(0, 2)$

$A > 0$, point moves to upward

5 key points $(0,2)$, $(\pi/2, 3)$, $(\pi, 2)$, $(3\pi/2, 1)$, $(2\pi, 2)$

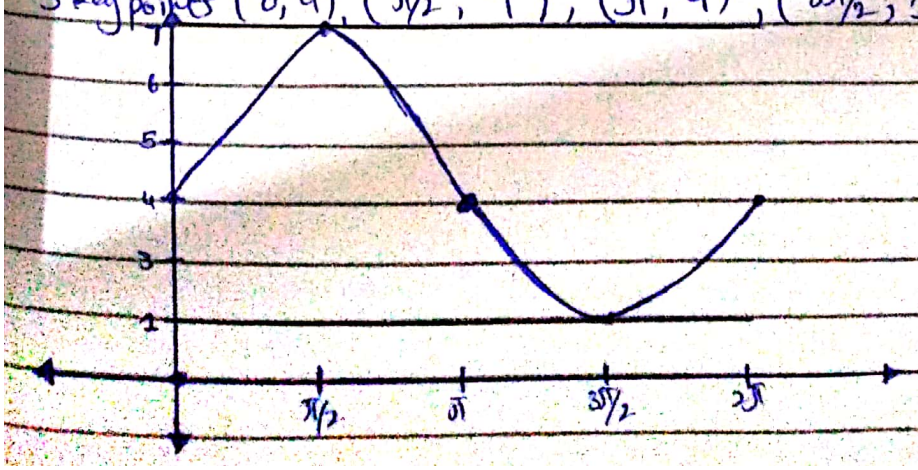


8) $y = 3\sin(x) + 4$

Period = 2π, Range $[1, 7]$, Initial point $(0, 4)$

$A > 0$, point moves to upward

5 key points $(0,4)$, $(\pi/2, 7)$, $(\pi, 4)$, $(3\pi/2, 1)$, $(2\pi, 4)$



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Arithmetic Sequence

Nigel Buckle lain

A Sequence is a set of Quantities arranged in a Definite order.

1, 2, 3, 4 ..... -1, 2, -4, 8, -16 ..... 1 1, 2, 3, 5, 8, 13 ...

when the terms of Sequence are added, we obtain a Series.

1) Arithmetic Sequence / AP
2) Geometric Sequence

Characteristics of Sequence → There is a Common Difference between Successive terms.

Ex:- 1, 3, 5, 7, 9 ..... odd numbers, common difference 2

18, 15, 12, 9, 6 ..... first term 18, Common difference -3

↳ Sequence is decreasing

formulae → $u_n = a + (n-1)d$

u_n → nth term of Sequence

a → initial point

d → Common difference

① for the sequence 7, 11, 15, 19 ...., find 20th term

$a = 7$, $n = 20$, $d = 2nd - 1st = 11 - 7 = 4 \rightarrow d = 4$

$u_n = a + (n-1)d$

$u_{20} = 7 + (20-1)4$

$u_{20} = 7 + 76 = 83 \rightarrow$ 20th term

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② An Arithmetic Sequence has a first term of 120 and a 10th term of 57. Find the 15th term.

$a = 120$, $n = 10$, $57 = u_{10}$

$u_n = a + (n-1)d$

$u_{10} = 120 + (10-1)d$

$57 = 120 + (9)d$

$-63 = 9d$

$\boxed{-7 = d}$

$d = -7$

$u_{15} = 120 + (14)-7$

$u_{15} = 120 - 98$

$\boxed{u_{15} = 22}$

③ An Arithmetic Sequence has a 7th term of 16.5 and a 12th term of 24. Find the 24th term.

$u_7 = 16.5$, $u_{12} = 24$

$u_n = a + (n-1)d$

$16.5 = a + (7-1)d$

$16.5 = a + 6d$

$a + 6d = 16.5$

$u_n = a + (n-1)d$

$24 = a + (12-1)d$

$24 = a + 11d$

$a + 11d = 24$

Using Elimination Method

$\cancel{a} + 6d = 16.5$

$\cancel{a} + 11d = 24$

$- \quad - \quad -$

$-5d = -7.5$

$\boxed{d = 1.5}$

$a + 6(1.5) = 16.5$

$a + 9 = 16.5$

$\boxed{a = 7.5}$

24th term, where $a = 7.5$, $n = 24$, $d = 1.5$

$u_{24} = 7.5 + (24-1)1.5$

$u_{24} = 7.5 + (23)1.5$

$u_{24} = 7.5 + 34.5$

$\boxed{u_{24} = 42}$

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1) (a) Show that the following Sequences are Arithmetic

(b) find the common difference

c) Define the rule that gives the nth term of the Sequence.

i) $\{2, 6, 10, 14, \ldots\}$

$a = 2, \quad d = 6 - 2 = 4$ difference is 4

nth term → $u_n = a + (n-1)d$

$u_n = 2 + (n-1)4 \rightarrow 4n - 4 + 2 \rightarrow 4n - 2$

$\boxed{u_n = 4n - 2}$ → nth term

ii) $\{20, 17, 14, 11, \ldots\}$

$a = 20, \quad \boxed{d = -3}$

$u_n = 20 + (n-1)(-3) \rightarrow 20 - 3n + 3 \rightarrow \boxed{-3n + 23}$ → nth term

iii) $\{1, -4, -9 \ldots\}$

$a = 1, \quad d = -4 - 1 = -5 \quad \boxed{d = -5}$

$u_n = 1 + (n-1)(-5) \rightarrow 1 + (-5n + 5) \rightarrow \boxed{-5n + 6}$ → nth term

iv) $\{0.5, 1.0, 1.5, 2.0, \ldots\}$

$a = 0.5, \quad d = 1.0 - 0.5 = 0.5 \quad \boxed{d = 0.5}$

$u_n = a + (n-1)d \rightarrow u_n = 0.5 + (n-1)(0.5)$

$u_n = 0.5 + 0.5n - 0.5 = \boxed{0.5n}$ → nth term

v) $\{y+1, y+3, y+5, \ldots\}$

$a = y + 1, \quad d = y + 3 - (y+1) = \cancel{y} + 3 - \cancel{y} - 1 = 2$

$u_n = y + 1 + (n-1)(2) \rightarrow y + 1 + 2n - 2 \rightarrow \boxed{y + 2n - 1}$ → nth term

vi) $\{x+2, x, x-2, \ldots\}$

$a = x + 2, \quad d = x - (x+2) = x - x - 2 = -2 \quad \boxed{d = -2}$

$u_n = x + 2 + (n-1)(-2) \rightarrow x + 2 - 2n + 2 \rightarrow x - 2n + 4$

2) find the 10th term of the Sequence whose first four terms are 8, 4, 0, −4.

$a = 8, \quad d = -4$, 10th term ?

$u_{10} = 8 + (10-1)(-4)$

$u_{10} = 8 + (9)(-4)$

$u_{10} = 8 - 36 = -28$

$\boxed{u_{10} = -28}$

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3) find the value of x and y in the arithmetic sequence $\{5, x, 13, y, \ldots\}$.

Solution

$d = x - 5$ OR $y - 13$

$u_n = a + (n-1)d$, $a = 5$, $u_3 = 13$

$13 = 5 + (3-1)(x-5)$ and $13 = 5 + (3-1)(y-13)$

$13 = 5 + (2)(x-5)$ $\quad$ $13 = 5 + (2)(y-13)$

$13 = 5 + 2x - 10$ $\quad$ $13 = 5 + 2y - 26$

$23 = 5 + 2x$ $\quad$ $39 = 5 + 2y$

$\frac{18}{2} = x = 9$ $\quad$ $\frac{34}{2} = y$

$\boxed{x = 9}$ $\quad$ $\boxed{y = 17}$

$d = x - 5 \rightarrow 9 - 5 = 4$ $\boxed{d = 4}$

4) find the value of An Arithmetic Sequence has 12 as its first term and a common difference of -5 find its 12 th term.

Solution

$a = 12$, $d = -5$ $\quad$ 12th term

$u_{12} = 12 + (12-1)(-5)$

$u_{12} = 12 + (11)(-5)$

$u_{12} = 12 - 55 \rightarrow \boxed{u_{12} = -43}$

5) An Arithmetic Sequence has -20 as its first term and a common difference of 3. find its 10th term

$a = -20$, $d = 3$ $\quad$ u_{10} ?

$u_{10} = -20 + (10-1)(3)$

$u_{10} = -20 + (9)(3)$

$u_{10} = -20 + 27$

$\boxed{u_{10} = 7}$

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6) The 14th term of an Arithmetic Sequence is 100. if the first term is 8 find the common Difference.

$a = 8, \quad u_{14} = 100 \qquad d = ?$

$u_n = a + (n-1)d \rightarrow 100 = 8 + (14-1)d$

$81 = 13d$

$d = 81/13 = 7 \quad \boxed{d = 7}$

7) The 10th term of an Arithmetic Sequence is -40. if the first term is 5, find the common Difference.

$a = 5 \qquad u_{10} = -40 \qquad d = ?$

$u_n = a + (n-1)d \rightarrow -40 = 5 + (10-1)d$

$-45 = 9d \rightarrow \boxed{d = -5}$

8) if $n+5$, $2n+1$ and $4n-3$ are three consecutive terms of an arithmetic Sequence find n.

Solution

$d = 2n+1-(n+5) = 2n+1-n-5 = \boxed{n-4} \rightarrow d$ OR

$d = 4n-3-(2n+1) = 4n-3-2n-1 = \boxed{2n-4} \rightarrow d$

$a = n+5, \quad \boxed{n = ?}$ and $u_2 = 2n+1$

$2n+1 = n+5+(2-1)(n-4)$ OR $2n+1 = n+5+(2-1)(2n-4)$

$2n+1 = n+5+(1)(n-4)$

$n = \cancel{5} + n - \cancel{4}$

$\boxed{n = n}$

$2n+1 = n+5+(1)(2n-4)$

$n = 4 + (2n-4)$

$n = \cancel{4} + 2n - \cancel{4}$

$\boxed{0 = n}$

9) The first three terms of an Arithmetic Sequence are 1, 6, 11.

a) find 8th term, b) which term will equal 151?

$a = 1, \quad d = 5$

$u_8 = 1 + (8-1)(5)$

$u_8 = 1 + 40$

$\boxed{u_8 = 41}$

$151 = 1 + (n-1)(5)$

$150 = 5n - 5$

$155 = 5n$

$\frac{155}{5} = n = 31$

$\boxed{31\text{th term}}$

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10) find x and y given that $4-\sqrt{3}$, x, y and $2\sqrt{3}-2$ are the first four terms of an Arithmetic Sequence.

$a = 4-\sqrt{3}$, $u_4 = 2\sqrt{3}-2$

$d = x-(4-\sqrt{3}) \rightarrow x-4+\sqrt{3}$ OR

$d = y\ 2\sqrt{3}-2-y \rightarrow 2\sqrt{3}-2-y$

$2\sqrt{3}-2 = 4-\sqrt{3}+(4-1)(x-4+\sqrt{3})$

$2\sqrt{3}-6 = -\sqrt{3}+(3)(x-4+\sqrt{3})$

$2\sqrt{3}-6+\sqrt{3} = 3x-12+3\sqrt{3}$

$\cancel{3\sqrt{3}}-6 = 3x-12+\cancel{3\sqrt{3}}$

$-6 = 3x-12$

$6 = 3x$ $\boxed{x=2}$

$2\sqrt{3}-2 = 4-\sqrt{3}+(4-1)(2\sqrt{3}-2-y)$.

$2\sqrt{3}-2-4+\sqrt{3} = 3(2\sqrt{3}-2-y)$

$3\sqrt{3}-\cancel{6} = 6\sqrt{3}-\cancel{6}-3y$

$3\sqrt{3} = 6\sqrt{3}-3y$

$3\sqrt{3}-6\sqrt{3} = -3y$

$\frac{-\cancel{3}\sqrt{3}}{-\cancel{3}} = y = \sqrt{3} \rightarrow \boxed{y=\sqrt{3}}$

11) for each of the following Sequences

a) $u_n = -5+2n$, $n \geq 1$ (b) $u_n = 3+4(n+1)$, $n \geq 1$

determine

i. its common difference , ii. its first term

a) $u_n = -5+2n$, $n \geq 1$

$u_1 = -3$, $u_2 = -1$

$u_3 = 1$ $u_4 = 3$

$-3, -1, 1, 3 \ldots$

$\boxed{a = -3}$

$\boxed{d = 2}$

b) $u_n = 3+4(n+1)$, $n \geq 1$

$u_n = 3+4n+4$

$u_n = 4n+7$

$u_1 = 11$, $u_2 = 15$, $u_3 = 19$

$11, 15, 19, \ldots$

$\boxed{a = 11}$

$\boxed{d = 4}$

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12) The third and fifth terms of an A.P are $x+y$ and $x-y$ respectively. find the 12th term.

$u_3 = x+y$, $u_5 = x-y$

13) The Sum of the fifth term and twice the third of an arithmetic Sequence equals the twelth term. if the Seventh term is 25 find an expression for The general term u_n.

$u_5 = a+(5-1)d$

$\boxed{u_5 = a+4d}$

$u_{12} = u_5 + u_{3\times2}$

$\boxed{u_{12} = a+11d}$

where $u_7 = 25$

$25 = a+(7-1)d$

$25 = a+6d$

$\boxed{a+6d = 25}$

$a = 25-6d$

$a = 25-6\left(\frac{10}{3}\right)$

$a = 25-20 = 5$

$\boxed{a=5}$

$u_3 = a+(3-1)d$

$u_3 = a+2d \rightarrow 2(a+2d) = 2a+4d$

$\boxed{u_{3\times2} = 2a+4d}$

$u_{12} = u_5 + u_{3\times2}$

$a+11d = a+4d+2a+4d$

$a+11d = 3a+8d$

$3a-a+8d-11d = 0$

$\boxed{2a-3d=0}$

put $a = 25-6d$

$2(25-6d)-3d = 0$

$50-12d-3d = 0$

$50-15d = 0$

$\frac{50}{15} = d$

$\boxed{d = 10/3}$

$\boxed{u_n = 5+(n-1)\frac{10}{3}}$ → General term

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Arithmetic Series

if the terms of series are added, the result is known as a Series.

The Sequence $1, 2, 3, 4, 5, 6, \ldots$

gives the Series $1+2+3+4+5+6 \ldots$

and the sequence $-1, -2, -4, -8, -16 \ldots$

gives the series $(-1)+(-2)+(-4)+(-8)+(-16)+\ldots$

[or $-1-2-4-8-16-\ldots$

The sum of the terms of a series is referred to as S_n, the sum of n terms of a Series.

for an arithmetic Series, we have

$$S_n = u_1 + u_2 + u_3 \ldots + u_n$$

$$= a + (a+d) + (a+2d) + \ldots + a + (n-1)d$$

Giving the formulae, for the sum of n terms of a Sequence :-

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

This formulae can now be used to sum large arithmetic Series:

Ex:- find the sum of 20 terms of the Series $-2+1+4+\ldots$

$a_1 = -2$, $a_5 = 10$ $\quad d = a_2 - a_1 = 1-(-2) = 3 \rightarrow \boxed{d=3}$

Then, the sum to n terms is given by $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

So that the sum to 20 terms is given by

$$S_{20} = \frac{20}{2}\left[2(-2) + (20-1)3\right]$$

$$S_{20} = 10\left[-4 + (19)\times(3)\right] = 10\left[-4 + 19\times 3\right]$$

$$= 10\left[-4 + 57\right] = 10\left[53\right] = 530$$

$$S_{20} = 530$$

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b) find the Sum of 35 terms of the Series $-\frac{3}{8} - \frac{1}{8} + \frac{1}{8} + \frac{3}{8} \ldots$

Solution

$\boxed{a = -\frac{3}{8}}$, $d = -\frac{1}{8} - \left(-\frac{3}{8}\right) = \frac{-1+3}{8} = \frac{2}{8} = \frac{1}{4}$ $\boxed{d = 1/4}$

$S_{35} = \frac{35}{2}\left[2\left(-\frac{3}{8}\right) + (35-1)\frac{1}{4}\right]$

$S_{35} = 17.5\left[-\frac{3}{4} + \frac{34}{4}\right] = 17.5\left[\frac{31}{4}\right] = \boxed{135.625}$

c) An Arithmetic Series has a third term of 0. The Sum of the first 15 term is -300. what is the first term and the Sum of the first ten terms?

$a_3 = 0$, $S_{15} = -300$, $a_1 = ?$ $S_{10} = ?$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$-300 = \frac{15}{2}[2a + (15-1)d]$

$-600 = 15[2a + 14d]$

$\boxed{-40 = 2a + 14d}$

Put $a = -2d$

$-40 = 2(-2d) + 14d$

$-40 = -4d + 14d$

$-40 = 10d$

$\boxed{d = -4}$

$a_3 = 0$

$a_n = a_1 + (n-1)d$

$0 = a + (3-1)d$

$\boxed{0 = a + 2d}$

$a = -2d$

put $d = -4$

$a = -2(-4)$

$\boxed{a = 8}$

$a_1 = 8$

$S_{10} = ?$, $a = 8$, $d = -4$

$S_{10} = \frac{10}{2}[2(8) + (10-1)(-4)]$

$S_{10} = 5[16 + (9)(-4)]$

$= 5[16 - 36]$

$= 5[-20] = -100$

$\boxed{S_{10} = -100}$

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The series itself is madeup of terms in a Sequence of So-called partial Sums often called a Sum Sequence. That is, we have that $\{S_1, S_2, S_3, \ldots\} = \{15, 33, 54, \ldots\}$ forms a Sequence.

8.1.2

1) find the Sum of the first ten terms in the arithmetic Sequences

(a) $\{1, 4, 7, 10, \ldots\}$

$a_1 = 1$, $d = 3$

$S_{10} = \frac{10}{2}[2(1) + (10-1)(3)]$

$S_{10} = 5[2 + (9)(3)] = 5[2+27] = 145$

(b) $\{3, 9, 15, 21, \ldots\}$

$a = 3$, $d = 6$

$S_{10} = 5[6 + (9)(6)]$

$S_{10} = 5[6 + 54] = 5[60] = 30$

c) $\{10, 4, -2, \ldots\}$

$a = 10$, $d = -6$

$S_{10} = 5[20 + (9)(-6)] = 5[20 + (-54)] = 5[-34] = -170$

$\boxed{S_{10} = -170}$

2) for the given arithmetic sequences, find the Sum, S_n, to to the requested number of terms

(a) $\{4, 3, 2, \ldots\}$ for $n = 12$

$a = 4$, $d = -1$

$S_{12} = \frac{12}{2}[2(4) + (12-1)(-1)] = 6[8-11] = -18$

$\boxed{S_{12} = -18}$

(b) $\{4, 10, 16, \ldots\}$ for $n = 15$

$a = 4$, $d = 6$

$S_{15} = \frac{15}{2}[8 + (14)(6)] = 7.5[8 + 84] = [7.5 \times 92] = 690$

$\boxed{S_{15} = 690}$

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c) $\{2.8, 3.6, 4.3, \ldots\}$ for $n = 11$

$a = 2.8$, $d = 0.7$

$S_{11} = \frac{11}{2}[5.8 + (10)(0.7)] = 5.5[5.8 + 7] = 5.5[12.8]$

$\boxed{S_{11} = 70.4}$

3) Find the sum of the following sequences:

a) $\{5, 4, 3, \ldots, -15\}$

$a = 5$, $d = -1$ $n = ?$, $a_n = -15$

$a_n = a_1 + (n-1)d$

$-15 = 5 + (n-1)(-1)$

$-20 = (n+1)(-1)$

$-20 = -n + 1$

$-21 = -n$

$\boxed{n = 21}$

$S_n = S_{21}$

$S_{21} = \frac{21}{2}[2(5) + (21-1)(-1)]$

$S_{21} = 10.5[10 + (-20)]$

$S_{21} = 10.5[-10]$

$\boxed{S_{21} = -105}$

b) $\{3, 9, 15, \ldots, 75\}$

$a_1 = 3$, $d = 6$, $a_n = 75$, $n = ?$

$a_n = a_1 + (n-1)d$

$75 = 3 + (n-1)(6)$

$72 = 6n - 6$

$78 = 6n$

$\boxed{13 = n}$

$S_n = S_{13}$

$S_{13} = \frac{13}{2}[2(3) + (13-1)(6)]$

$S_{13} = 6.5[6 + 72] = 507$

$\boxed{S_{13} = 507}$

c) $\{3, 5, 7, \ldots, 29\}$

$a_1 = 3$, $d = 2$, $a_n = 29$, $n = ?$

$a_n = a_1 + (n-1)d$

$29 = 3 + (n-1)(2)$

$26 = 2n - 2$

$28 = 2n$

$\boxed{14 = n}$

$S_n = S_{14}$

$S_{14} = \frac{14}{2}[2(3) + (14-1)(2)]$

$S_{14} = 7[6 + 26] = 7[32]$

$\boxed{S_{14} = 224}$

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Vectors (Second-Year-Maths_Punjab_Region)

1) Geometric interpretation of vector 7.1.1 — 1
2) Multiplication of vector by a Scalar 7.1.2
2a) Equal and parallel vectors 7.1.2 — 2
3) Addition and subtraction of two vectors 7.1.3 — 2
3a) Parallelogram of addition — 3
4) Position vector, vectors in a plane 7.1.4/5 — 4
5) Addition, Scalar multiplication 7.1.5 — 5
5b) 0, =, Position, magnitude, −, −ve 7.1.5 — 6
6) properties of magnitude of a vector 7.1.6 — 7
7) Unit vector 7.1.8 — 8
8) The Ratio Formula 7.1.9 — 10
9) vector Geometry 7.1.10 — 10
Exercise 7.1 Solution → 12 − 18 — 12-18
10) 7.2 Introduction of vector in space — 18
11) Addition and Scalar multiplication — 18-20
12) properties of vectors — 21
13) Distance between two points in space — 22
14) Direction Angles and Direction Cosines of vectors — 23
Exercise 7.2 Solution 24 − 26 — 24-26
worksheet By Sir:- Iftikhar AHMED BHUTTO 24/12/2022
15) Dot product — 28
16) Projection of one vector to another — 30
17) The cross product — 32-40
Exercise 7.4 Solution — 41-45
7.5 Scalar Triple product of vectors (examples) — 45
Exercise 7.5 Solution — 48-56
Circle worksheet → sir iftikhar ahmed Bhutto — 57-69
Conic Section
Parabolas — 70

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- $\vec{a} \cdot \vec{b} = 0$ means $\perp$
- $\vec{a} \times \vec{b} = (0,0,0)$ mean $//$

Proj a on b

$\text{Proj}_{\vec{b}} \vec{a} \rightarrow \frac{a \cdot b}{|b|^2} b$

- verification of $\perp$ vectors
- $\underline{a} \times \underline{b}$ Provide a vector $\rightarrow (i + j + k) \rightarrow$
- if $\underline{a} \cdot \underline{a} \times \underline{b} = 0$, then vectors are $\perp$, Dot prod provide constant
- if $\underline{b} \cdot \underline{a} \times \underline{b} = 0$, the vectors are $\perp$, Dot prod provides a constant
- Co-planer, when Det of $\vec{AB}, \vec{AC}, \vec{AD}$ is 0
- volume of parallelpiped $\rightarrow$ 3x3 determinant
- volume of Tetrahedron $\rightarrow \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

volume $\rightarrow$ 1/6 multiplied with the determinant of $\vec{AB}$, $\vec{AC}$ and of $\vec{AD}$ via 3x3 matrix.

- work done $\rightarrow \vec{F} \cdot \vec{AB}$ = Constant value (Dot product)
- $i \cdot i = 1$ $\quad j \cdot j = 1$ $\quad k \cdot k = 1$
- $i \cdot j = 0$ $\quad j \cdot k = 0$ $\quad k \cdot i = 0$

$\boxed{i \cdot j \neq j \cdot i}$

- orthogonal $\rightarrow \vec{a} \cdot \vec{b} = 0 \rightarrow$ prependicular | opposite direction
- Magnitude $\rightarrow \sqrt{x^2 + y^2 \ldots n^2}$
- $\underbrace{u \cdot v \times w}_{\text{Vector}} \neq \underbrace{u \times v \cdot w}_{\text{Scalar}}$?

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7.1.1 Geometric interpretation of vector

Geometrically a vector is represented by a directed line segment $\overrightarrow{AB}$ with A its initial point and B its terminal point. it is often found convenient to denote a vector by an arrow and is written either as $\overrightarrow{AB}$ or as boldface symbol like $\mathbf{v}$ or in underlined form $\underline{v}$.

(i) The magnitude or length or norm of a vector $\overrightarrow{AB}$ or $\underline{v}$, is its absolute value and is written as $|\overrightarrow{AB}|$ or simply AB or $|\underline{v}|$.

(ii) A unit vector is defined as a vector whose magnitude is unity. Unit vector of vector $\underline{v}$ is written as $\hat{\underline{v}}$ (read as $\underline{v}$ hat) and is defined by

$$\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$$

(iii) if terminal point B of a vector $|\overrightarrow{AB}|$ concides with its initial point A, then magnitude $AB=0$ and $|\overrightarrow{AB}|=0$, which is called zero or null vector.

(iv) Two vectors are said to be negative of each other if they have same magnitude but opposite direction.

if $\overrightarrow{AB} = \underline{v}$, then $\overrightarrow{BA} = -\overrightarrow{AB} = -\underline{v}$

7.1.2 Multiplication of vector by a Scalar

we use the word Scalar to mean a real number. Multiplication of a vector $\underline{v}$ by a Scalar 'K' is a vector whose magnitude is K times that of $\underline{v}$. It is denoted by $K\underline{v}$.

(i) if K is +ve, then $\underline{v}$ and $K\underline{v}$ are in same direction.

(ii) if K is -ve, then $\underline{v}$ and $K\underline{v}$ are in opposite direction.

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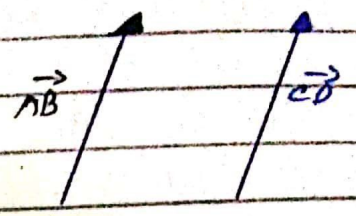
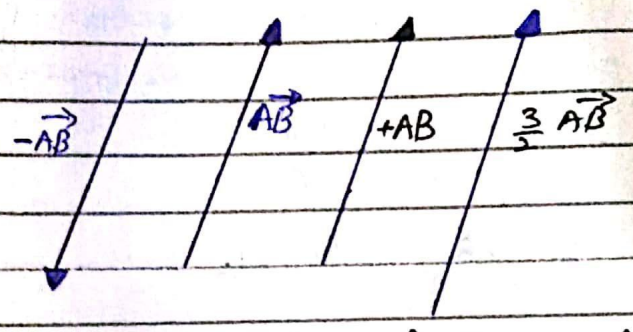
(a) Equal vectors

Two vectors $\vec{AB}$ and are Said to be Equal, if They have the same magnitude and Same direction i.e., $|\vec{AB}| = |\vec{CD}|$



(b) Parallel vectors

Two vectors are parallel if and only if they are non-zero Scalar Multiple of each other, (See figure)



7.1.3 Addition and Subtraction of Two vectors

(a) Addition of two vectors is explained by the following laws:

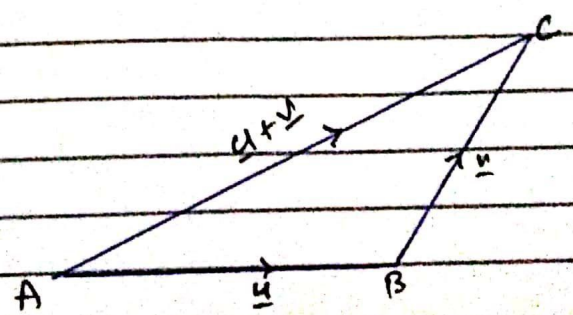
(i) Triangle Law of addition

(ii) parallelogram Law of addition

- Triangle Law of addition

if two vectors $\underline{u}$ and $\underline{v}$ are represented by the two Sides AB and BC of a triangle Such that terminal point of $\underline{u}$ Concide with the initial point of $\underline{v}$, then the third Side AC of triangle gives vector Sum $\underline{u}+\underline{v}$, that is

$$\vec{AB} + \vec{BC} = \vec{AC} \Rightarrow \underline{u}+\underline{v} = \vec{AC}$$



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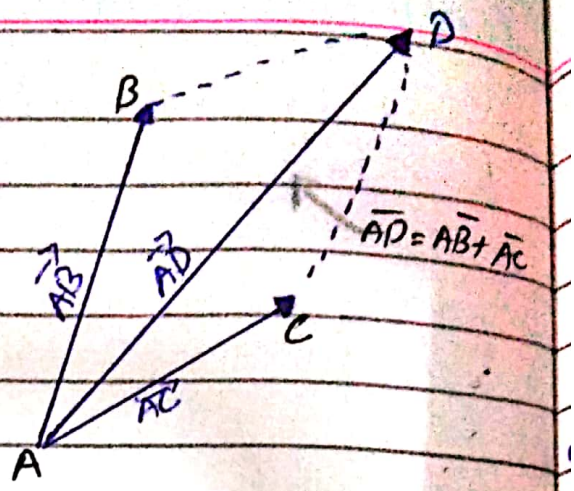
Parallelogram Law of Addition

if two vectors $\underline{u}$ and $\underline{v}$ are Represented by two adjacent Sides AB and AC of a $//^m$ as Shown in the figure, then diagonal AD give the Sum or resultant of $\vec{AB}$ and $\vec{AC}$. That is



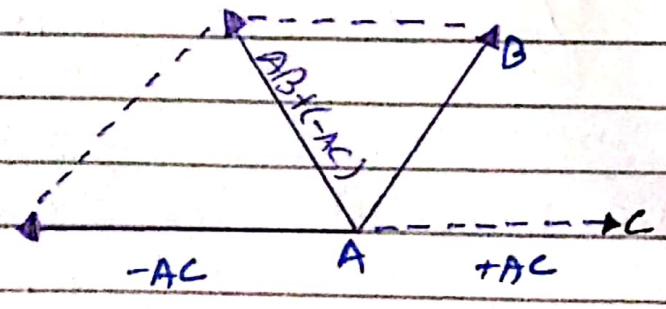
$$\vec{AD} = \vec{AB} + \vec{AC} = \underline{u} + \underline{v}$$

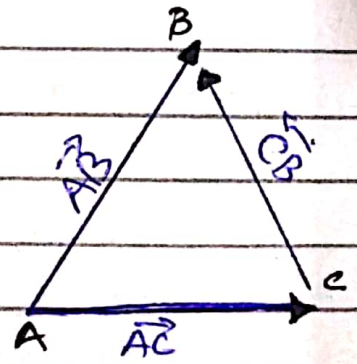
(b) Subtraction of two vectors

The difference of two vectors $\vec{AB}$ and $\vec{AC}$ is defined by

$$\vec{AB} - \vec{AC} = \vec{AB} + (-\vec{AC})$$

$$\underline{u} - \underline{v} = \underline{u} + (-\underline{v})$$





$$\vec{CB} = \vec{AB} - \vec{AC}$$

In figure, this difference is interpreted as the main Diagonal of the parallelogram with sides $\vec{AB}$ and $-\vec{AC}$. We can also interpret the Same vector difference as third side of a triangle with Sides $\vec{AB}$ and $\vec{AC}$. In this Second interpretation, The vector difference $\vec{AB} - \vec{AC} = \vec{CB}$ points the Terminal point of the vector from which we are Subtracting the Second vector.

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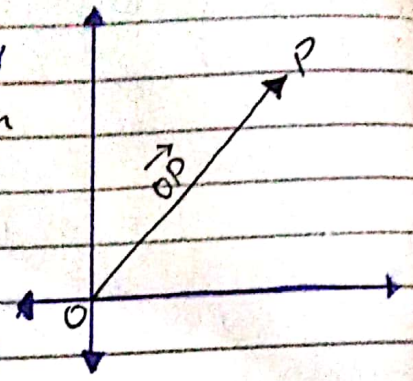
7.1.4 Position vector

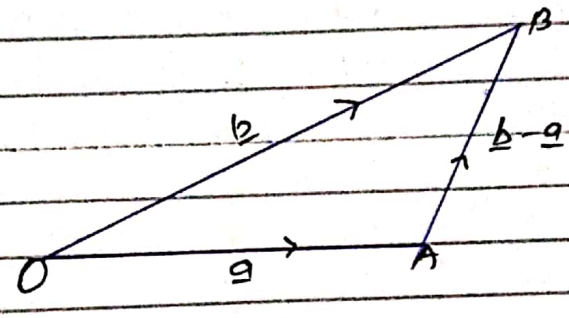
The vector whose initial point is the origin O and whose terminal point is P, is called the position vector of the point P and is written as $\vec{OP}$.



The position vectors of the points A and B relative to the origin O are defined by $\vec{OA} = \underline{a}$ and $\vec{OB} = \underline{b}$ respectively.

in the figure, by triangle law of addition,

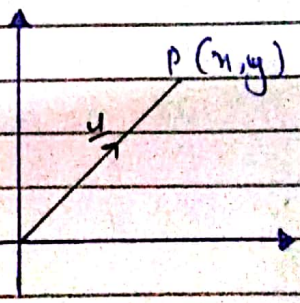
$$\vec{OA} + \vec{AB} = \vec{OB}$$
$$\underline{a} + \vec{AB} = \underline{b}$$
$$\vec{AB} = \underline{b} - \underline{a}$$



7.1.5 vectors in a plane

Let R be the set of real numbers. The Cartesian plane is defined to the $R^2 = \{(x,y) : x, y \in R\}$.

An element $(x,y) \in R^2$ represents a point $P(x,y)$ which is uniquely determined by its coordinate x and y.



Given a vector $\underline{u}$ in the plane, there exists a unique point $P(x,y)$ in the plane such that the vector $\vec{OP}$ is equal to $\underline{u}$. So we can use Rectangular coordinates (x,y) for P to associate a unique ordered pair $[x,y]$ to vector $\underline{u}$.

we define addition and Scalar multiplication in R^2 by:

P.T.O

(i) Addition

For any two vectors $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$ we have $\underline{u} + \underline{v} = [x, y] + [x', y'] = [x + x', y + y']$

(ii) Scalar Multiplication

for $\underline{u} = [x, y]$ and $\alpha \in R$, we have

$$\alpha \underline{u} = \alpha [x, y] = [\alpha x, \alpha y]$$

Definition: The set all ordered pairs $[x, y]$ of real numbers, together with the rules of addition and scalar multiplication is called the set of vectors in R^2.

For the vector $\underline{u} = [x, y]$, x and y are called the components of $\underline{u}$.

Note: The vector $[x, y]$ is an ordered pair of numbers not a point (x, y) in the plane.

- (a) Negative of a vector

in scalar multiplication (ii), if $\alpha = -1$ and $\underline{u} = [x, y]$ then

$$\alpha \underline{u} = (-1)[x, y] = [-x, -y]$$

which is denoted by $-\underline{u}$ and is called the additive inverse of $\underline{u}$ or negative vector of $\underline{u}$.

- (b) Difference of two vectors

we define

$$\underline{u} - \underline{v} \text{ as } \underline{u} + (-\underline{v})$$

if $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$, then

$$\underline{u} - \underline{v} = \underline{u} + (-\underline{v})$$

$$= [x, y] + [-x' - y'] = [x - x', y - y']$$

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(c) Zero vector

Clearly $\underline{u} + (-u) = [x, y] + [-x, -y] = [x-x, y-y] = [0,0]$
$= \underline{0}$

$\underline{0} = [0, 0]$ is called the zero (Null) vector

(d) Equal vectors

Two vectors $\underline{u} = [x, y]$ and $\underline{v} = [x', y']$ of R^2 are said to be equal if and only if they have the same components. That is,
$[x, y] = [x', y']$ if and only if $x = x'$ and $y = y'$ and we write $\underline{u} = \underline{v}$

(e) position vectors

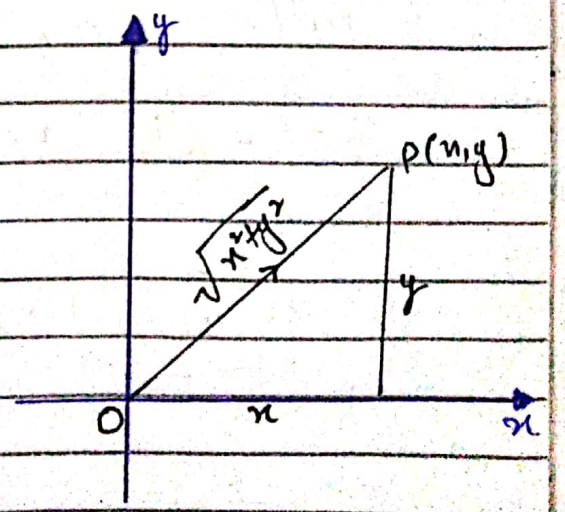
For any point $P(x, y)$ in R^2, a vector $\underline{u} = [x, y]$ is represented by a directed line segment $\overrightarrow{OP}$, whose initial point is at origin. Such vectors are called position vectors because they provide a unique correspondence between the points (positions) and vectors.

(f) magnitude of a vector

For any vector $\underline{u} = [x, y]$ in R^2 we define the magnitude or norm or length of the vector as of the point $P(x, y)$ from the origin O

$\therefore$ Magnitude of $\overrightarrow{OP}$

$\overrightarrow{OP} = |OP| = |\underline{u}| = \sqrt{x^2+y^2}$



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7.1.6 properties of Magnitude of a vector

Let $\underline{v}$ be a vector in the plane or in Space and let c be a real number, then

(i) $|\underline{v}| \geq 0$, and $|\underline{v}| = 0$ if and only if $\underline{v} = \underline{0}$

(ii) $|c\underline{v}| = |c||v|$

Proof:

(i) we write vector $\underline{v}$ in component form as

$\underline{v} = [x, y]$, then

$|\underline{v}| = \sqrt{x^2+y^2} \geq 0$ for all x and y.

further $|\underline{v}| = \sqrt{x^2+y^2} = 0$, if and only if $x=0, y=0$

in this case $\underline{v} = [0,0] = \underline{0}$

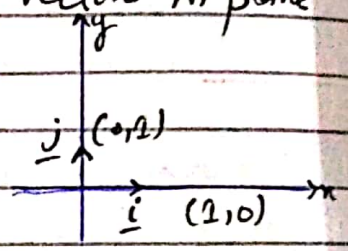
(ii)

$|c\underline{v}| = |cx, cy| = \sqrt{(cx)^2+(cy)^2} = \sqrt{c^2}\sqrt{x^2+y^2} = |c||\underline{v}|$

7.1.7 Another Notation for Representing vectors in plane

we introduce two Special vectors,

$\underline{i} = [1,0]$, $\underline{j} = [0,1]$ in R^2

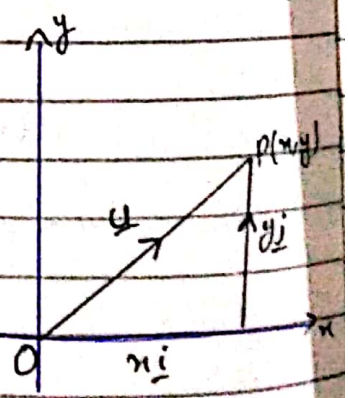


As magnitude of $\underline{i} = \sqrt{1^2+0^2} = 1$

magnitude of $\underline{j} = \sqrt{1^2+0^2} = 1$

So $\underline{i}$ and $\underline{j}$ are called unit vectors along x-axis and along y-axis respectively. Using the defination of addition and Scalar multiplication, the vector $[x,y]$ Can be written as

$$\underline{u} = [x,y] = [x,0] + [0,y]$$
$$= x[1,0] + y[0,1]$$
$$= x\underline{i} + y\underline{j}$$



Thus each vector $[x,y]$ in R^2 Can be Uniquely Represented by $x\underline{i} + y\underline{j}$.

in terms of unit vector $\underline{i}$ & $\underline{j}$, the Sum $\underline{u}+\underline{v}$ of two vectors $\underline{u} = [x,y]$ and $\underline{v} = [x', y']$ is written as

$\underline{u}+\underline{v} = [x+x', y+y'] = (x+x')\underline{i} + (y+y')\underline{j}$

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7.1.8

A unit vector in the direction of another given vector.

A vector $\underline{u}$ is called a unit vector, if $|\underline{u}| = 1$

Now we find a unit vector u in the direction of any other given vector $\underline{v}$.

we can do by the use of property (ii) of magnitude of vector, as follows:

$$\because \left| \frac{1}{|\underline{v}|} \right| = \frac{1}{|\underline{v}|} |\underline{v}| = 1$$

$\therefore$ the vector $\underline{v} = \frac{1}{|\underline{v}|} \underline{v}$ is the required unit vector

it points in the Same Direction as v, because it is positive Scalar multiple of $\underline{v}$.

Example 1:

For $\underline{v} = [1, -3]$ and $\underline{w} = [2, 5]$

(i) $\underline{v} + \underline{w} = [1, -3] + [2, 5] = [1+2, -3+5] = [3, 2]$

(ii) $4\underline{v} + 2\underline{w} = 4[1, -3] + 2[2, 5] = [4, -12] + [4, 10] = [8, -2]$

(iii) $\underline{v} - \underline{w} = [1, -3] - [2, 5] = [-1, -8]$

(iv) $\underline{v} - \underline{v} = [1, -3] - [1, -3] = [0, 0] = \underline{0}$

(v) $|\underline{v}| = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$

Example 2:-

Find the unit vector in the Same direction as the vector $\underline{v} = [3, -4]$.

Solution: $\underline{v} = [3, -4] = 3i - 4j$

$|\underline{v}| = \sqrt{(3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$

now $\underline{u} = \frac{1}{|\underline{v}|} \underline{v} = \frac{1}{5}[3, -4]$ ($\underline{u}$ is unit vector in the direction of v)

$$\underline{u} = \frac{1}{5}[3, -4] = \left[\frac{3}{5}, \frac{-4}{5}\right]$$

verification: $|\underline{u}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = \sqrt{1} = 1$

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Example 3:

Find a unit vector in the direction of the vector

(i) $\underline{v} = 2\underline{i} + 6\underline{j}$ (ii) $\underline{v} = [-2, 4]$

(i) → Solution

$\underline{v} = 2i + 6\underline{j} \rightarrow [2, 6]$

$|\underline{v}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40}$

∴ a unit vector in the direction of $\underline{v} = \dfrac{\underline{v}}{|\underline{v}|}$

$$\underline{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{2i}{\sqrt{40}} + \frac{6j}{\sqrt{40}} = \frac{2i}{2\sqrt{10}} + \frac{6j}{2\sqrt{10}} = \frac{i}{\sqrt{10}} + \frac{3j}{\sqrt{10}}$$

(ii) $\underline{v} = [-2, 4]$

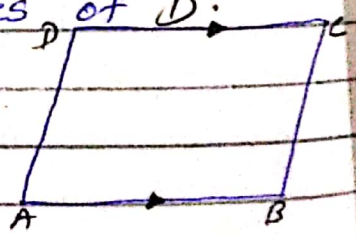
$= -2i + 4j$

$|\underline{v}| = \sqrt{(-2)^2 + (4)^2} = \sqrt{4+16} = \sqrt{20}$

A unit vector in the direction of $\underline{v} = \dfrac{\underline{v}}{|\underline{v}|}$

$$\underline{v} = \frac{\underline{v}}{|\underline{v}|} = \frac{-2i}{\sqrt{20}} + \frac{4j}{\sqrt{20}} = \frac{-2i}{2\sqrt{5}} + \frac{4j}{2\sqrt{5}} = \frac{-i}{\sqrt{5}} + \frac{2j}{\sqrt{5}}$$

Example 4: if ABCD is a parallelogram such that the points A, B and C are respectively $(-2, -3)$, $(1, 4)$ and $(0, -5)$. find the coordinates of D:



Solution:

Suppose the coordinates of D are (x, y)

As ABCD is a $//^m$

∴ $\overline{AB} = \overline{DC}$ and $\overline{AB} \parallel \overline{DC}$

$\overrightarrow{AB} = \overrightarrow{DC}$

$(x_b - x_a)i + (y_b - y_a)j = (x_c - x_d)i + (y_c - y_d)j$

$= (1 - (-2))i + (4 - (-3))j = (0 - x_d)i + (-5 + y_d)j$

$= (1+2)i + (4+3)j = (-x_d)i + (-5 - y_d)j$ (equating method)

$3i = (-x_d)i$, $7j = (-5 - y_d)j$

$\boxed{x_d = -3}$ $\boxed{y_d = -12}$ $D(-3, -12)$

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7.1.8 The Ratio Formula

Let A and B be two points whose position vectors (P.V). are $\underline{a}$ and $\underline{b}$ respectively. if a point P divides AB in the ratio $p:q$, then the positive vector of P is given by

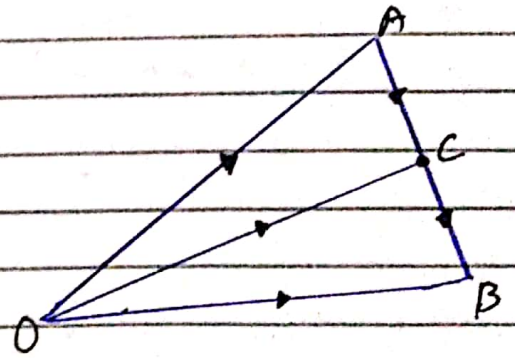
$$\underline{r} = \frac{q\underline{a} + p\underline{b}}{p+q}$$

7.1.10 Vector Geometry

Let us Now use the Concept of vectors discussed So far in proving geometrical Theorems. A few examples are being Solved here to illustrate the method

Example 5: if $\underline{a}$ and $\underline{b}$ be the P.vs of A and B respectively w.r.t origin O, and C be a point on $\overline{AB}$ Such that $\overline{OC} = \frac{a+b}{2}$, then Show That C is the mid-point of AB.

Solution: $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$ and $\overrightarrow{OC} = \frac{1}{2}(\underline{a}+\underline{b})$



Now $2\overrightarrow{OC} = \underline{a} + \underline{b}$

$\Rightarrow \overrightarrow{OC} + \overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB}$

$\Rightarrow \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OC}$

$\therefore -\overrightarrow{OA} = \overrightarrow{AO}$, $-\overrightarrow{OC} = \overrightarrow{CO}$

$\Rightarrow \overrightarrow{OC} + \overrightarrow{AO} = \overrightarrow{OB} + \overrightarrow{CO}$

$\therefore \overrightarrow{OC} = \overrightarrow{CO}$

$\overrightarrow{OC} + \overrightarrow{AO} = \overrightarrow{OB} + \overrightarrow{CO}$

$\overrightarrow{O}(C+A) = O(B+C)$

$\overrightarrow{AC} = \overrightarrow{BC}$

Thus $m\overrightarrow{AC} = m\overrightarrow{CB}$

C is equidistant from A and B, but A, B, C are Collinear. Hence C is the Mid point of AB.

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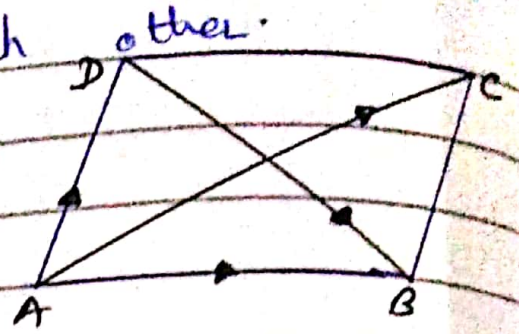
Example 6: use vectors, to prove that the Diagonals of a parallelogram bisect each other.



Solution

Let the vertices of the parallelogram be A, B, C and D

Since $\vec{AC} = \vec{AB} + \vec{AD}$, the vector from A to the Mid-point of Diagonal $\vec{AC}$ is

$$\underline{v} = \frac{1}{2}(\vec{AB} + \vec{AD})$$

Since $\vec{DB} = \vec{AB} - \vec{AD}$, the vector from A to the mid-point of Diagonal $\vec{DB}$ is

$$\underline{w} = \vec{AD} + \frac{1}{2}(\vec{AB} - \vec{AD})$$

$$\underline{w} = \vec{AD} + \frac{1}{2}\vec{AB} - \frac{1}{2}\vec{AD}$$

$$\underline{w} = \frac{1}{2}\vec{AD} + \frac{1}{2}\vec{AB} = \frac{1}{2}(\vec{AB} + \vec{AD}) = \underline{v}$$

Since $\underline{w} = \underline{v}$, there midpoints of diagonals $\vec{AC}$ and $\vec{DB}$ are the same.

Thus the Diagonals of a parallelogram bisect each other.

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Exercise 7.1

1. write the vector $\vec{PQ}$ in the form $x\underline{i} + y\underline{j}$

(i) $P(2,3)$, $Q(6,-2)$

$\vec{PQ} = [2,3] + [6,-2] = [8,+1] = 8\underline{i} + \underline{j}$

(ii) $P(0,5)$, $Q(-1,-6)$

$\vec{PQ} = [0,5] + [-1,-6] = [-1,-1] = -i - j$

2. Find the magnitude of the vector $\underline{u}$:

(i) $\underline{u} = 2i - 7j$

$\underline{u} = [2,-7]$

$|\underline{u}| = \sqrt{x^2+y^2}$

$|\underline{u}| = \sqrt{(2^2)+(-7)^2}$

$|\underline{u}| = \sqrt{4+49}$

$|\underline{u}| = \sqrt{53}$

(ii) $\underline{u} = \underline{i} + \underline{j}$

$\underline{u} = [1,1]$

$|\underline{u}| = \sqrt{x^2+y^2}$

$|\underline{u}| = \sqrt{(1)^2+(1)^2}$

$|\underline{u}| = \sqrt{2}$

(iii) $\underline{u} = [3,-4]$

$|\underline{u}| = \sqrt{x^2+y^2}$

$|\underline{u}| = \sqrt{(3)^2+(-4)^2}$

$|\underline{u}| = \sqrt{9+16}$

$|\underline{u}| = \sqrt{25}$

$\boxed{|\underline{u}| = 5}$

(iv)

3. if $\underline{u} = 2\underline{i} - 7\underline{j}$, $\underline{v} = \underline{i} - 6\underline{j}$ and $\underline{w} = -\underline{i} + \underline{j}$. find ✓

(i) $\underline{u} + \underline{v} - \underline{w}$

$\underline{u} = [2,-7]$, $\underline{v} = [1,-6]$, $\underline{w} = [-1,1]$

$\underline{u} + v = [2,-7] + [1,-6]$

$\underline{u} + v = [3,-13] - \underline{w}$

$\underline{u} + v - w = [3,-13] - [-1,1]$

$\underline{u} + \underline{v} - w = [4,-14]$

$\underline{u} + \underline{v} - w = 4\underline{i} - 14\underline{j}$

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(ii) $2\underline{u} - 3\underline{v} + 4\underline{w}$

Solution

$\underline{u} = [2, -7]$, $\rightarrow \underline{v} = [1, -6]$, $\rightarrow \underline{w} = [-1, 1]$

$\hookrightarrow 2\underline{u} = [4, -14]$, $\hookrightarrow -3\underline{v} = [-3, 18]$, $\hookrightarrow 4\underline{w} = [-4, 4]$

$2\underline{u} - 3\underline{v} = [4, -14] - [-3, 18]$

$2\underline{u} - 3\underline{v} = [7, -32]$

$2\underline{u} - 3\underline{v} = [7, -32] + 4\underline{w}$

$2\underline{u} - 3\underline{v} + 4\underline{w} = [7, -32] + [-4, 4]$

$2\underline{u} - 3\underline{v} + 4\underline{w} = [3, -28]$

$2\underline{u} - 3\underline{v} + 4\underline{w} = 3i - 28j$

(iii) $\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} + \frac{1}{2}\underline{w}$

$\underline{u} = [2, -7]$, $\underline{v} = [1, -6]$, $\underline{w} = [-1, 1]$

$\hookrightarrow \frac{1}{2}\underline{u} = [1, -7/2]$, $\hookrightarrow \frac{1}{2}\underline{v} = [1/2, -3]$, $\hookrightarrow \frac{1}{2}\underline{w} = [-1/2, 1/2]$

$\frac{1}{2}\underline{u} + \frac{1}{2}\underline{v} = [1, -7/2] + [1/2, -3]$

$= [3/2, -13/2]$

$1/2\,\underline{u} + 1/2\,\underline{v} = [3/2, -13/2] + 1/2\,\underline{w}$

$1/2\,\underline{u} + 1/2\,\underline{v} + 1/2\,\underline{w} = [3/2, -13/2] + [-1/2, 1/2]$

$1/2\,\underline{u} + 1/2\,\underline{v} + 1/2\,\underline{w} = [1, -6]$

$1/2\,\underline{u} + 1/2\,\underline{v} + 1/2\,\underline{w} = i + 6j$

4. Find the sum of the vectors $\vec{AB}$ and $\vec{CD}$, given the four points $A(1, -1)$, $B(2, 0)$, $C(-1, 3)$ and $(-2, 2)$.

Solution

$\vec{AB} = [1, -1] + [2, 0] = [3, -1]$

$\boxed{\vec{AB} = [3, -1]}$

$\vec{CD} = [-1, 3] + [-2, 2]$

$\boxed{\vec{CD} = [-3, 5]}$

$\vec{AB} + \vec{CD} = [3, -1] + [-3, 5]$

$= [0, 4]$

$\vec{AB} + \vec{CD} = 4j$

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5. Find the vector from the point A to the origin where $\vec{AB} = 4\underline{i} - 2\underline{j}$ and B is the point $(-2, 5)$.

Solution

$\vec{AB} = [4, -2]$ and $B = [-2, 5]$

"Brain Storming"

$[6, -7] \rightarrow$ let A, Now add A an B

$\vec{AB} = [6, -7] + [-2, 5]$

$= [4, -2] \rightarrow$ Prooved $\rightarrow A = [6, -7]$

Now find the vector from point $A \rightarrow [6, -7]$

$\vec{AO} = [6, -7] + [0, 0]$

$\vec{AO} = [6, -7] = 6\underline{i} - 7\underline{j}$

$\vec{AO} = 6\underline{i} - 7\underline{j}$

6. Find a unit vector in the Direction of the vector given below. (i) $2\underline{i} - \underline{j}$

Solution

Formula of unit vector $\rightarrow \underline{u} = \frac{1}{|v|} \underline{v}$

$\underline{v} = 2\underline{i} - \underline{j} = [2, -1]$

$|\underline{v}| = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-1)^2} = \sqrt{4+1} = \sqrt{5}$

$|\underline{v}| = \sqrt{5}$

$\underline{u} = \frac{1}{|\underline{v}|} \underline{v} = \frac{1}{\sqrt{5}} [2, -1]$

verification:-

$\underline{u} = \frac{1}{\sqrt{5}} [2, -1] = \left[\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right]$

$\underline{u} = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{2}{\sqrt{5}}\right)^2 + \left(\frac{-1}{\sqrt{5}}\right)^2} = \sqrt{\frac{4}{5} + \frac{1}{5}} = \sqrt{\frac{5}{5}}$

$\underline{u} = \sqrt{\frac{5}{5}} = \sqrt{1} = 1$

$\underline{u} = 1 \rightarrow$ Hence prooved

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(ii) $\underline{v} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

Solution

$|\underline{v}| = \sqrt{x^2+y^2} = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = \sqrt{\frac{4}{4}} = \sqrt{1} = 1$

$|\underline{v}| = 1$

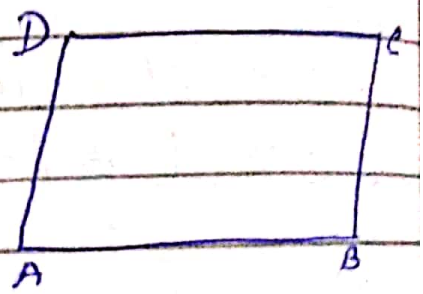
formula to find unit vector $\underline{u} = \frac{1}{|\underline{v}|}\underline{v}$

$\underline{u} = \frac{1}{1}[\frac{1}{2}, \sqrt{3}/2] = [1/2, \sqrt{3}/2]$

$\underline{u} = \frac{1}{2}\underline{i} + \frac{\sqrt{3}}{2}\underline{j}$

(iii) if A, B and C are respectively the points (2, -4), (4, 0) and (1, 6). use vector method to find the coordinates of the point D if:

(i) ABCD is a parallelogram



Solution

$A(\underset{x_1}{2}, \underset{y_1}{-4})$, $B(\underset{x_2}{4}, \underset{y_2}{0})$, $C(\underset{x_3}{1}, \underset{y_3}{6})$, $D(\underset{x_4}{x}, \underset{y_4}{y})$

$\overrightarrow{BA} = \overrightarrow{CD}$ OR $\overrightarrow{AB} = \overrightarrow{DC}$

formula to find fourth point via vector method

$(x_1 - x_2)\underline{i} + (y_1 - y_2)\underline{j} = (x_4 - x_3)\underline{i} + (y_4 - y_3)\underline{j}$

$\Rightarrow (2-4)\underline{i} + (-4-0)\underline{j} = (-1 + x_4)\underline{i} + (-6 + y_4)\underline{j}$

$(-2)\underline{i} + (-4)\underline{j} = (-1 + x_4)\underline{i} + (-6 + y_4)\underline{j}$

$(-2)\underline{i} = (-1 + x_4)\underline{i}$ and $(-4)\underline{j} = (-6 + y_4)\underline{j}$

$-2 = -1 + x_4$

$\boxed{-1 = x_4}$

$x = -1$

$-4 = -6 + y_4$

$2 = y_4$

$\boxed{y = 2}$

$D(-1, 2)$

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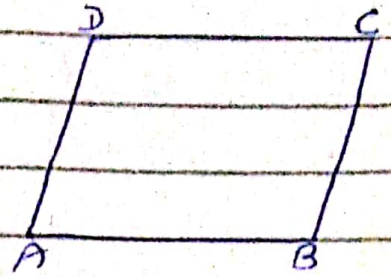
8. if B, C and D are Respectively (4, 1), (−2, 3) and (−8, 0). use vector method to find the coordinates of the point:

(i) A if ABCD is a parallelogram



Solution

A (x, y), B(4, 1), C(−2, 3), D(−8, 0)

formula to find a vertices via vector method → $(x_a - x_b)i + (y_a - y_b)j = (x_d - x_c)i + (y_d - y_c)j$

$\Rightarrow (x_a - 4)i + (y_a - 1)j = (-8 - (-2))i + (0 - 3)j$

$(x_a - 4)i + (y_a - 1)j = (-6)i + (-3)j$

$(x_a - 4)i = -6i$ and $(y_a - 1)j = -3j$

$x_a - 4 = -6$ $\qquad$ $y_a - 1 = -3$

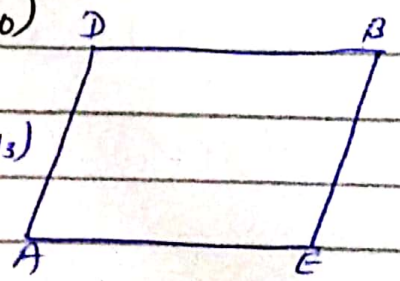
$x_a = -2$ $\qquad$ $y_a = -2$

$\boxed{x = -2}$ $\qquad$ $\boxed{y = -2}$

$\boxed{A(-2, -2)}$

(ii) E if AEBD is a parallelogram

A(−2, −2), E(x, y), B(4, 1), D(−8, 0)



Solution

$\vec{AB} = \vec{DE} \rightarrow (x_1 - x_2) + (y_1 - y_2) = (x_4 - x_3) + (y_4 - y_3)$

$(x_a - x_b) + (y_a - y_b) = (x_d - x_b) + (y_d - y_b)$

$\Rightarrow (-2 - x) + (-2 - y) = (-8 - 4) + (0 - 1)$

$(-2 - x)i + (-2 - y)j = (-12)i + (-1)j$

Comparing method

$(-2 - x)i = (-12)i$ and $(-2 - y)j = (-1)j$

$-2 - x = -12$ $\qquad$ $-2 - y = -1$

$-x = -10$ $\qquad$ $-2 - y = 0$

$\boxed{x = 10}$ $\qquad$ $\boxed{y = -2}$

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8, if O is the origin and $\vec{OP} = \vec{AB}$, find the point P when A and B are $(-3,7)$ and $(1,0)$ respectively.

Solution

$O(0,0)$, $P(x,y)$, $A(-3,7)$, $B(1,0)$

$\vec{AB} = \vec{OP}$

$(x_a - x_b)\underline{i} + (y_a - y_b)\underline{j} = (x_p - x_o)\underline{i} + (y_p - y_o)\underline{j}$

$(-3-1)\underline{i} + (7-0)\underline{j} = (x-0)\underline{i} + (y-0)\underline{j}$

$(-4)\underline{i} + (7)\underline{j} = (x)\underline{i} + (y)\underline{j}$

$x\underline{i} = -4\underline{i}$ and $y\underline{j} = 7\underline{j}$

$\boxed{x = -4}$ $\boxed{y = 7}$

$P(-4,7)$

11. if $\vec{AB} = \vec{CD}$, find the point A when points B, C, D are $(1,2)$, $(-2,5)$, $(4,11)$ respectively.

Solution

$A(x,y)$, $B(1,2)$, $C(-2,5)$, $D(4,11)$

$\vec{AB} = \vec{DC}$

$(x_a - x_b)\underline{i} + (y_a - y_b)\underline{j} = (x_d - x_c)\underline{i} + (y_d - y_c)\underline{j}$

$\Rightarrow (x-1)\underline{i} + (y-2)\underline{j} = (4-(-2))\underline{i} + (11-5)\underline{j}$

$\Rightarrow (x-1)\underline{i} + (y-2)\underline{j} = 6\underline{i} + 6\underline{j}$

$\Rightarrow (x-1)\underline{i} = 6\underline{i}$ and $(y-2)\underline{j} = 6\underline{j}$

$x - 1 = 6$ $y - 2 = 6$

$\boxed{x = 7}$ $\boxed{y = 8}$

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(2) Find the position vectors of the point of division of the line segment joining the following pair of points, in the given ratio:

(i) Point C with position vector $2\underline{i} - 3\underline{j}$ and point D with position vector $3\underline{i} + 2\underline{j}$ in the ratio 4:3.

Solution

The Ratio formula $\rightarrow \underline{r} = \dfrac{q\underline{a} + p\underline{b}}{p+q}$, when $p:q$ given

$\underline{a} = C = 2\underline{i} - 3\underline{j}$ and $\underline{b} = D = 3\underline{i} + 2\underline{j}$

$p:q = 4:3$

$$\underline{r} = \frac{3C + 4D}{4+3} = \frac{3[2\underline{i} - 3\underline{j}] + 4[3\underline{i} + 2\underline{j}]}{7}$$

$$= \frac{6\underline{i} - 9\underline{j} + 12\underline{i} + 8\underline{j}}{7} = \frac{18\underline{i} - \underline{j}}{7}$$

$$\boxed{\underline{r} = \frac{18\underline{i} - \underline{j}}{7}}$$

(ii) point E with position vector $5\underline{j}$ and point F with position vector ~~$3\underline{i} + 2\underline{j}$~~ $4\underline{i} + \underline{j}$ in ratio 2:5

Solution

$\underline{r} = \dfrac{q\underline{a} + p\underline{b}}{p+q}$, $p:q$, $p:q = 2:5$

$\underline{a} = E = 0\underline{i} + 5\underline{j}$, $\underline{b} = F = 4\underline{i} + \underline{j}$

$$\underline{r} = \frac{q\underline{a} + p\underline{b}}{p+q} = \frac{5[5\underline{j}] + 2[4\underline{i} + \underline{j}]}{p+q}$$

$$= \frac{25\underline{j} + 8\underline{i} + 2\underline{j}}{5+2} = \boxed{\frac{8\underline{i} + 27\underline{j}}{7}} \rightarrow \underline{r}$$

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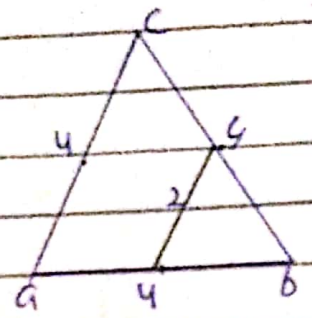
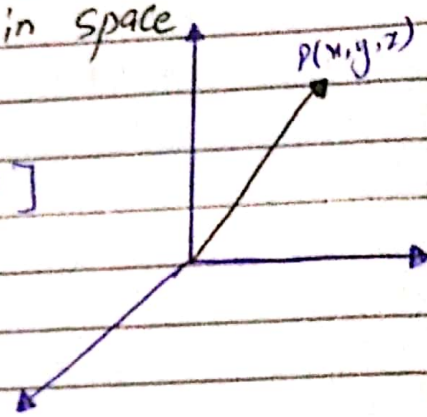
3) prove that the line segment joining the mid points of two sides of a triangle is parallel to the third side and half as long.

Solution



7.2 Introdution of vector in space

7.2.1 Concept of a vector in space



(i) Addition: For any two vectors $\underline{u} = [x, y, z]$ and $\underline{v} = [x', y', z']$ we have $\underline{u} + \underline{v}$:

$\underline{u} + \underline{v} = [x+x', y+y', z+z']$

(ii) Scalar multiplication

for $\underline{u} = [x, y, z]$ and $\alpha \in R$, we have

$$\alpha \underline{u} = \alpha [x, y, z] = [\alpha x, \alpha y, \alpha z]$$

Definition: The set of all ordered triples $[x, y, z]$ of real numbers, together with the rules of Addition and Scalar multiplication, is called the set of vectors in R^3.

a) The negative of the vector $\underline{u} = [x, y, z]$ as

$-\underline{u} = (-1)\underline{u} = [-x, -y, -z]$

b) The difference of two vectors $\underline{v} = [x', y', z']$ and $\underline{w} = [x'', y'', z'']$ as $\underline{v} - \underline{w}$:

$\underline{v} + (-\underline{w}) = [x' - x'', y' - y'', z', z'']$

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c) The zero vector as $0 = [0, 0, 0]$

d) Equality of two vectors $\underline{v} = [x', y', z']$ and $\underline{w} = [x'', y'', z'']$ by $\underline{v} = \underline{w}$ if and only if $x' = x''$, $y' = y''$, and $z' = z''$.

e) Position vector

For any point $P(x, y, z)$ in R^3, a vector $\underline{u} = [x, y, z]$ is represented by a directed line segment $\vec{OP}$, whose initial point is at origin. Such vectors are called position vectors in R^3

f) Magnitude of a vector. we define the magnitude or norm or length of a vector $\underline{u}$ in space by the Distance of the point $P(x, y, z)$ from the origin O.

$$\therefore |\vec{OP}| = |\underline{u}| = \sqrt{x^2 + y^2 + z^2}$$

Example 1: for the vectors, $\underline{v} = [2, 1, 3]$ and $\underline{w} = [-1, 4, 0]$ we have the following

(i) $\underline{v} + \underline{w} = [2, 1, 3] + [-1, 4, 0]$
$= [1, 5, 3]$

(ii) $\underline{v} - \underline{w} = [2, 1, 3] - [-1, 4, 0]$
$= [3, -3, 3]$

(iii) $2\underline{w} = 2[-1, 4, 0]$
$= [-2, 8, 0]$

(iv) $|\underline{v} - 2\underline{w}| = |[2, 1, 3] - 2[-1, 4, 0]|$
$= |[2, 1, 3] - [-2, 8, 0]|$
$= |[4, -7, 3]|$

$|[4, -7, 3]| = \sqrt{(4)^2 + (-7)^2 + (3)^2} = \sqrt{16 + 49 + 9} = \sqrt{74}$

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7.2.2 Properties of vectors

Following properties

(i) $\underline{u}+\underline{v} = \underline{v}+\underline{u}$ — Commutative

(ii) $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$ — Associative

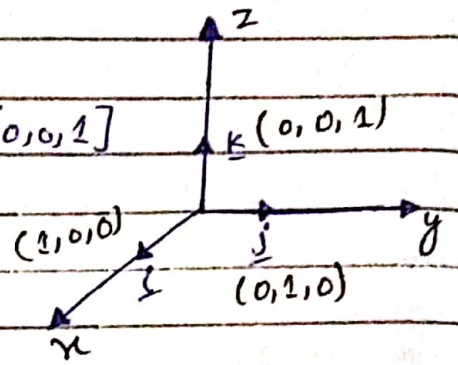
(iii) $\underline{u}-\underline{u} = \underline{u}+(-\underline{u}) = 0$ — Inverse for vector addition

(iv) $a(\underline{v}+\underline{w}) = a\underline{v}+a\underline{w}$ — Distributive

(v) $a(b\underline{u}) = (ab)\underline{u}$ — Scalar multiplication

7.2.3 Another Notation for representing vectors in space

As in plane, similarly we introduce three special vectors.



As in plane,

$\underline{i} = [1, 0, 0]$, $\underline{j} = [0, 1, 0]$, $\underline{k} = [0, 0, 1]$ in R^3.

As magnitude of $\underline{i}$, $\underline{j}$ and of $\underline{k}$

$\underline{i} = \sqrt{1^2+0^2+0^2} = 1$

$\underline{j} = \sqrt{0^2+1^2+0^2} = 1$

$\underline{k} = \sqrt{0^2+0^2+1^2} = 1$

So, $\underline{i}$, $\underline{j}$ and $\underline{k}$ are called unit vectors along x-axis, along y-axis and along z-axis respectively.

Using the definition of addition and Scalar multiplication, the vector $[x, y, z]$ can be written as

$$\underline{u} = [x, y, z] = [x, 0, 0] + [0, y, 0] + [0, 0, z]$$
$$= x[1, 0, 0] + y[0, 1, 0] + z[0, 0, 1]$$
$$= x\underline{i} + y\underline{j} + z\underline{k}$$

Thus each vector $[x, y, z]$ in R^3 can be uniquely represented by $x\underline{i}+y\underline{j}+z\underline{k}$.

in terms of unit vector $\underline{i}$, $\underline{j}$ and $\underline{k}$, the sum $\underline{u}+\underline{v}$ of two vectors

$\underline{u} = [x, y, z]$ and $\underline{v} = [x', y', z']$ is written as

$$\underline{u}+\underline{v} = [x+x', y+y', z+z']$$
$$= (x+x')\underline{i} + (y+y')\underline{j} + (z+z')\underline{k}$$

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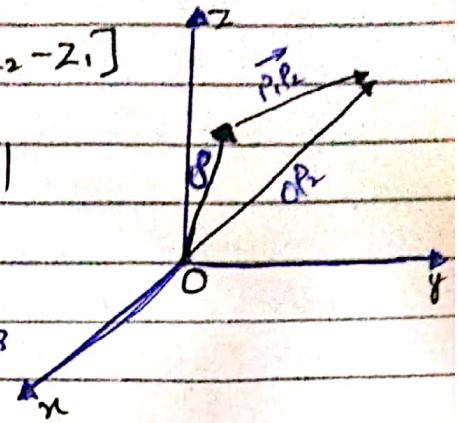
7.2.4 Distance between two points in space

if $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$ are the position vectors of the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

The vector $\overrightarrow{P_1P_2}$ is given by

$$\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$



$\therefore$ Distance between P_1 and $P_2 = |P_1P_2|$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is called distance formula between two points P_1 and P_2 in R^3

Example 2: if $\underline{u} = 2\underline{i} + 3\underline{j} + \underline{k}$, $\underline{v} = 4\underline{i} + 6\underline{j} + 2\underline{k}$ and $\underline{w} = -6\underline{i} - 8\underline{j} - 3\underline{k}$, then

(a) Find (i) $\underline{u} + 2\underline{v}$ (ii) $|\underline{u} - \underline{v} - \underline{w}|$

(b) Show that $\underline{u}$, $\underline{v}$ and $\underline{w}$ are // to each other.

Solution: (a)

(i) $\underline{u} + 2\underline{v} = 2i + 3j + k + 2(4i + 6j + 2k)$

$= 2i + 3j + k + 8i + 12j + 4k$

$= 10i + 15j + 5k$

(ii) $|\underline{u} - \underline{v} - \underline{w}| = (2i + 3j + k) - (4i + 6j + 2k) - (-6i - 8j - 3k)$

$= 2i + 3j + k - 4i - 6j - 2k + 6i + 8j + 3k$

$= 6i + 2i - 4i + 8j - 6j + 3j + 3k - 2k + k$

$= 4i + 6j + 2k$

$\therefore \underline{v} = 2\underline{u}$

$\underline{u}$ and $\underline{v}$ are parallel vectors, and have same direction

Again

$\underline{w} = -6i - 8j - 3\underline{k}$

$= -3(2i + 3j + k)$

$\underline{w} = -3\underline{u}$

$\underline{u}$ and $\underline{w}$ are parallel vectors and have opposite Direction. Hence $\underline{u}$, $\underline{v}$ and $\underline{w}$ are parallel to each other.

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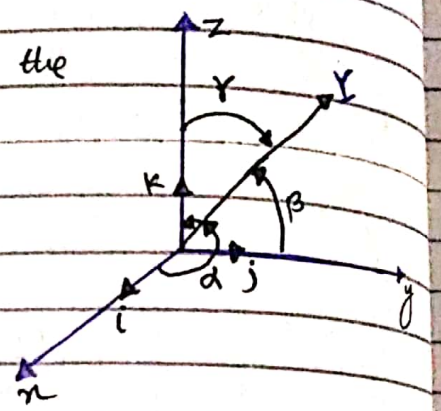
7.2.5 Direction Angles and Direction Cosines of a vector

Let $\underline{r} = \vec{OP} = x\underline{i} + y\underline{j} + z\underline{k}$ be a non zero vector, let α, β and γ denote angles formed between $\underline{r}$ and the unit Coordinate vectors $\underline{i}$, $\underline{j}$ and k respectively Such that

$$0 \le \alpha \le \pi \quad , 0 \le \beta \le \pi \text{ , and } 0 \le \gamma \le \pi$$

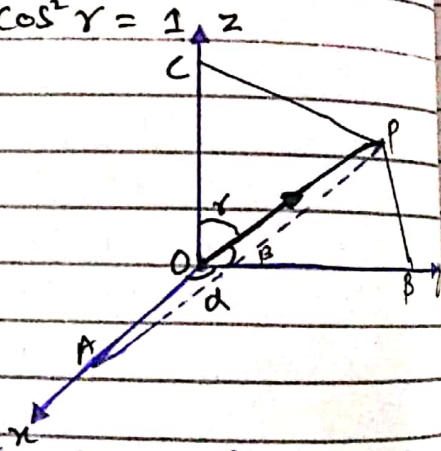


(i) the angles α, β, γ are called the Direction angles and

(ii) the number $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ are called Direction Cosines of the vector $\underline{r}$.

important Result:

prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$



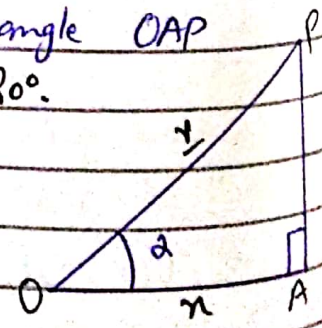
let $\underline{r} = [x, y, z] = xi + yj + zk$

$\therefore |\underline{r}| = \sqrt{x^2+y^2+z^2} = r$

then $\dfrac{\underline{r}}{|\underline{r}|} = \left[\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}\right]$

is the unit vector in the direction of the vector $\underline{r} = \vec{OP}$

it can be visualized that the triangle OAP is a right triangle with $\angle A = 90°$.



Therefore in right triangle OAP,

$\cos\alpha = \dfrac{\vec{OA}}{OP} = \dfrac{x}{r}$, Similarly

$\cos\beta = \dfrac{y}{r}$, $\cos\gamma = \dfrac{z}{r}$

The numbers $\cos\alpha = \dfrac{x}{r}$, $\cos\beta = \dfrac{y}{r}$, $\cos\gamma = \dfrac{z}{r}$ are called the direction cosines of $\vec{OP}$

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$\therefore \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{z^2}{r^2} = \frac{x^2+y^2+z^2}{r^2} = \frac{r^2}{r^2} = 1$

Exercise 7.2

1) Let $A=(2,5)$, $B=(-1,1)$ and $C=(2,-6)$, find
(i) $\vec{AB}$ (ii) $2\vec{AB} - \vec{CB}$ (iii) $2\vec{CB} - 2\vec{CA}$

(i) $\vec{AB}$

Solution

$= \vec{AB} = [2,5] + [-1,1] = [1, 6]$

$\vec{AB} = i + 6j$

(ii) $2\vec{AB} - \vec{CB} = 2(i+6j) - [2,-6] + [-1,1]$

$= 2i + 12j - [1, 7]$

$= 2i + 12j - i - 7j$

$2\vec{AB} - \vec{CB} \boxed{= i + 5j}$

(iii) $2\vec{CB} - 2\vec{CA}$

$2\vec{CB} - 2CA = 2[2,-6] + [-1,1] - 2[2,-6] + [2,5]$

$= 2[1,-5] - 2[4,-1]$

$= [2,-10] - [8,-2]$

$= [-6,-8]$

$2\vec{CB} - 2\vec{CA} = -6i + (-8j) = -6i - 8j$

2. Let $\underline{u} = i + 2j + k$, $\underline{v} = 3i - 2j + 2k$, $\underline{w} = 5i - j + 3k$
Find the indicated vector of number
(i) $\underline{u} + 2\underline{v} + \underline{w}$ (ii) $\underline{u} - 3w$ (iii) $|\underline{v} + \underline{w}|$

(i) $\underline{u} + 2\underline{v} + \underline{w}$

Solution

$\underline{u} = i + 2j - k$, $2\underline{v} = 6i - 4j + 4k$, $\underline{w} = 5i - j + 3k$

$\underline{u} + 2\underline{v} = 7i - 2j + 3k$

$\underline{u} + 2\underline{v} + \underline{w} = (7i - 2j + 3k) + 5i - j + 3k$

$= 12i - 3j + 6k$

(ii) $\underline{u} - 3w$

$\underline{u} - 3w = i + 2j - k - 15i + 3j - 9k$

$= -14i + 5j - 10k$

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(iii) $|3\underline{v} + \underline{w}|$

Solution

$3\underline{v} = 8i - 6j + 6k$, $\underline{w} = 5i - j + 3k$

$|3\underline{v} + \underline{w}| = |[8, -6, 6] + [5, -1, 3]|$

$= |[14, -7, 8]|$

$= \sqrt{14^2 + 49 + 81} = \sqrt{196 + 49 + 81} = \sqrt{326}$

3) Find the magnitude of the vector $\underline{v}$ and write the Direction Cosines of $\underline{v}$.

(i) $\underline{v} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

Solution

$\underline{v} = 2i + 3j + 4k \Rightarrow |\underline{v}| = \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{4 + 25} = \sqrt{29}$

(ii) $\underline{v} = \underline{i} - \underline{j} - \underline{k}$

Solution

$\underline{v} = \underline{i} - \underline{j} - \underline{k} \Rightarrow |\underline{v}| = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{1+1+1} = \sqrt{3}$

(iii) $\underline{v} = 4\underline{i} - 5\underline{j}$

Solution

$\underline{v} = 4\underline{i} - 5\underline{j} \Rightarrow |\underline{v}| = \sqrt{(4)^2 + (-5)^2} = \sqrt{16 + 25} = \sqrt{41}$

4) Find α, So that $|\alpha\underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3$.

Solution

$|\alpha\underline{i} + (\alpha+1)\underline{j} + 2\underline{k}| = 3 \Rightarrow \sqrt{x^2 + y^2 + z^2}$

$x = \alpha$, $y = (\alpha+1)$, $z = 2$

$\Rightarrow \sqrt{\alpha^2 + (\alpha+1)^2 + 2^2} = 3$

$\Rightarrow \sqrt{\alpha^2 + \alpha^2 + 2\alpha + 1 + 4} = 3$

$\Rightarrow \sqrt{2\alpha^2 + 2\alpha + 5} = 3$

Rising Power 2 on b/s

$\Rightarrow (2\alpha^2 + 2\alpha + 5)^{\frac{1}{2} \times 2} = 3^2$

$\Rightarrow 2\alpha^2 + 2\alpha + 5 = 9$

$2\alpha^2 + 2\alpha - 4 = 0$

$2\alpha^2 + 4\alpha - 2\alpha - 4 = 0$

$2\alpha(\alpha+2) - 2(\alpha+2) = 0$

$2\alpha(\alpha+2) - 2(\alpha+2) = 0$

$(2\alpha - 2)(\alpha + 2) = 0$

$\boxed{\alpha = 1}$

OR $\boxed{\alpha = -2}$

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3) The direction cosines of $\underline{v}$

(i) $\underline{v} = 2\underline{i} + 3\underline{j} + 4\underline{k}$

Solution

$\underline{v} = [2, 3, 4] \Rightarrow |\underline{v}| = \sqrt{4+9+16} = \sqrt{29}$

$|\underline{v}| = r$

$\cos\alpha = \frac{x}{r}, \quad \cos\beta = \frac{y}{r}, \quad \cos\gamma = \frac{z}{r}$

then $\frac{\underline{v}}{|\underline{v}|} = \left[\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right]$

$r = |\underline{v}| = \sqrt{29}$

then Direction Angles are

$\frac{\underline{v}}{|\underline{v}|} = \left[\frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right] \rightarrow$ Ans.—

(ii) $\underline{v} = \underline{i} - \underline{j} - \underline{k}$

Solution

$\underline{v} = [1, -1, -1] \Rightarrow |\underline{v}| = r = \sqrt{(1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$

$r = \sqrt{3}$

Cosine Direction

$\frac{\underline{v}}{|\underline{v}|} = \frac{\underline{v}}{r} = \left[\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right]$

(iii) $\underline{v} = 4\underline{i} - 5\underline{j}$

Solution

$\underline{v} = [4, -5] \Rightarrow \underline{v} = r = \sqrt{16+25} = \sqrt{41}$

$r = \sqrt{41}$

Cosine Direction

$\frac{\underline{v}}{|\underline{v}|} = \frac{\underline{v}}{r} = \left[\frac{4}{\sqrt{41}}, \frac{-5}{\sqrt{41}}\right]$

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5) find a unit vector in the direction of $\underline{v} = \underline{i} + 2\underline{j} - \underline{k}$

Solution

$\underline{v} = i + 2j - k \Rightarrow |\underline{v}| = \sqrt{(1)^2 + (2)^2 + (-1)^2} = \sqrt{1+4+1} = \sqrt{6}$

$\underline{v} = [1, 2, -1]$

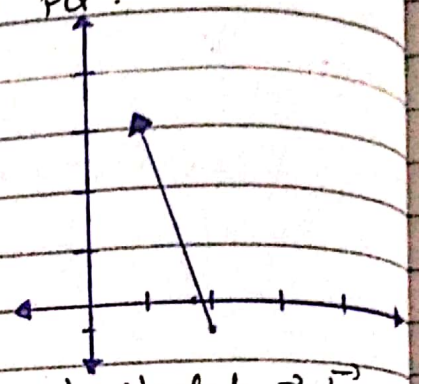
$\frac{\underline{v}}{|\underline{v}|} = \frac{1}{\sqrt{6}}[1, 2, -1] = \left[\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}\right] \Rightarrow$ unit vector

1) Let P be the point $(2,-1)$ and Q be the point $(1,4)$. Determine and sketch the vector $\overrightarrow{PQ}$.

Solution

Magnitude is $\sqrt{17}$

$\overrightarrow{PQ} = |\overrightarrow{PQ}| = \sqrt{(3+1)^2 + (1-2)^2} = \sqrt{17}$



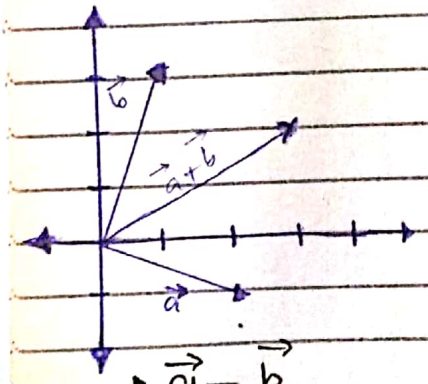
2) Let $\vec{a} = (2,-1)$ and $\vec{b} = (1,3)$, sketch $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$, $2\vec{a}$, $2\vec{a}-3\vec{b}$

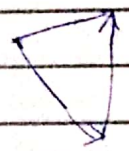
Solution

$(2,-1), (1,3)$

$\vec{a} = (2,-1)$, $\vec{b} = (1,3)$

$\vec{a} + \vec{b} = (2+1, 3-1) = (3,2)$

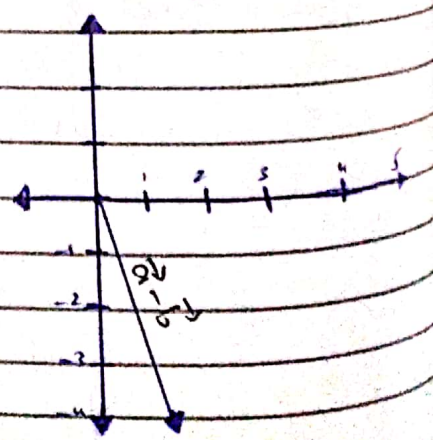




▸ $\vec{a} - \vec{b}$

Solution

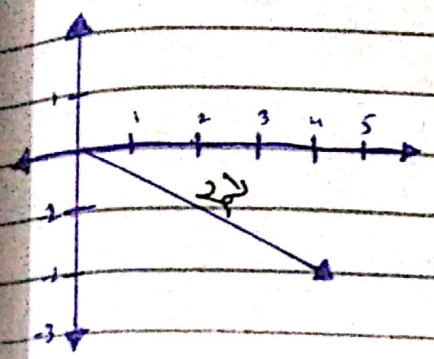
$\vec{a} - \vec{b} = (2-1, -1-3) = (1,-4)$



$2\vec{a}$

$\vec{a} = (2, -1)$

$2\vec{a} = (4, -2)$

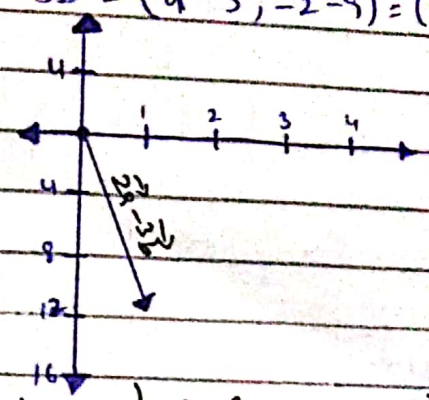


$2\vec{a} - 3\vec{b}$

Solution

$2\vec{a} = (4, -2)$, $3\vec{b} = (3, 9)$

$2\vec{a} - 3\vec{b} = (4-3, -2-9) = (1, -11)$



3) Let $\vec{a} = (3, 4, 0)$ and $\vec{b} = (-1, 4, 2)$. Determine $|\vec{a}|$, $2\vec{a} + 3\vec{b}$, $3\vec{a} - 2\vec{b}$.

Solution

- $|\vec{a}| = \sqrt{(3)^2 + (4)^2 + (0)^2} = \sqrt{9+16} = \sqrt{25} = 5$
- $2\vec{a} + 3\vec{b}$, $2\vec{a} = (6, 8, 0)$, $3\vec{b} = (-3, 12, 6)$

  $2\vec{a} + 3\vec{b} = (6, 8, 0) + (-3, 12, 6)$

  $= (3, 20, 6) = 2\vec{a} + 3\vec{b}$
- $3\vec{a} - 2\vec{b}$, $3\vec{a} = (9, 12, 0)$, $2b = (-2, 8, 4)$

  $3\vec{a} - 2\vec{b} = (9, 12, 0) - (-2, 8, 4)$

  $= (11, 4, -4)$

4) Let $\vec{a} = \vec{i} + 4\vec{j} - 8\vec{k}$ and $\vec{b} = -2\vec{i} + \vec{j} + 2\vec{k}$. Determine $|\vec{a}|$, $\vec{a} + \vec{b}$, $2\vec{a} - 3\vec{b}$.

Solution

$\vec{a} = (1, 4, -8)$, $\vec{b} = (-2, 1, 2)$

- $|\vec{a}| = \sqrt{(1)^2 + (4)^2 + (-8)^2} = \sqrt{1 + 16 + 64} = \sqrt{81} = 9$

  $|\vec{a}| = 9$
- $\vec{a} + \vec{b} = (1, 4, -8) + (-2, 1, 2) = (-1, 5, -6)$

  $\vec{a} + \vec{b} = (-1, 5, -6)$
- $2\vec{a} - 3\vec{b}$, $2\vec{a} = (2, 8, -16)$, $3\vec{b} = (-6, 3, 6)$

  $2\vec{a} - 3\vec{b} = (2, 8, -16) - (-6, 3, 6)$

  $= (8, 5, -22)$

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5) find the Normalization of the vector $\vec{a} = \vec{i} + 4\vec{j} + 8\vec{k}$

Solution

Normalization $\longrightarrow$ unit vector , $\vec{a} = (1, 4, 8)$

$\frac{1}{|\vec{a}|}\vec{a} \rightarrow$ $|\vec{a}| = \sqrt{(1)^2 + (4)^2 + (8)^2} = \sqrt{81} = 9$

$|\vec{a}| = 9$

$$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{9}[1, 4, 8] = \left[\frac{1}{9}, \frac{4}{9}, \frac{8}{9}\right]$$

6) Find the Normalization of the vector $\vec{a} = (3, 4, 0)$

Solution

Normalization $\rightarrow \frac{1}{|\vec{a}|}\vec{a}$, $\vec{a} = (3, 4, 0)$

$$|\vec{a}| = \sqrt{(3)^2 + (4)^2 + (0)^2} = \sqrt{9 + 16} = 5$$

$$\frac{1}{|\vec{a}|}\vec{a} = \frac{1}{5}[3, 4, 0] = \left[\frac{3}{5}, \frac{4}{5}, 0\right]$$

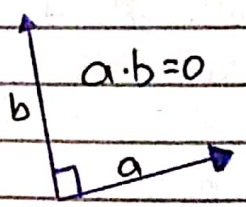
The Dot product

$\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

This product can be used to determine the angle Between the vectors and, in particular, to test whether two vectors are prependicular to Each other. if θ is The angle between two nonzero vectors $\vec{a}$ and $\vec{b}$, then

$$\boxed{\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta}$$



As a Consequence, $\vec{a}$ and $\vec{b}$ are prependicular (or orthogonal) exactly when $\cos\theta = 0$ which, in turn happens exactly when $\vec{a} \cdot \vec{b} = 0$. Thus, $\vec{a}$ and $\vec{b}$ are prependicular if and only if $\vec{a} \cdot \vec{b} = 0$.

in case when $\vec{a} = \vec{b}$, $\theta = 0$ and $\cos\theta = 1$ and The formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ becomes

$$\vec{a} \cdot \vec{a} = |\vec{a}|^2$$

which relates the dot product and the length of a vector $\vec{a}$.

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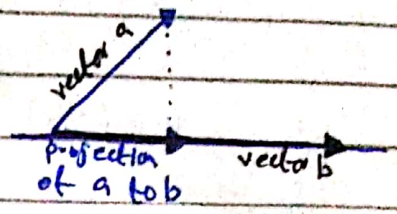
projection of one vector to another.

Let $\text{proj}_{\vec{b}}\vec{a}$ denote the projection of $\vec{a}$ onto $\vec{b}$ for given nonzero vectors $\vec{a}$ and $\vec{b}$. The projection of $\vec{a}$ onto $\vec{b}$ has the same direction and sense as vector $\vec{b}$. The length of $\text{proj}_{\vec{b}}\vec{a}$ satisfies :

$$\boxed{\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}}$$



length $= |a|\cos\theta$

Practice problems.

1) find the dot product of the vectors $\vec{a} = (1, 3, -4)$ and $\vec{b} = (-2, 3, 1)$.

Solution

$$\vec{a}\cdot\vec{b} = (1, 3, -4)\cdot(-2, 3, 1)$$
$$= (1\times(-2) + 3\times3 + (-4\times1))$$
$$= (-2 + 9 + (-4)) = -2 + 9 - 4 = 3$$

2) find the angle between the vectors $\vec{a} = (3,4)$ and $\vec{b} = (5,12)$

Solution

formula for angle $\longrightarrow$ $\boxed{\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta}$

$$\vec{a}\cdot\vec{b} = (3,4)\cdot(5,12)$$
$$\vec{a}\cdot\vec{b} = 15 + 48 = 63$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{63}{|\vec{a}||\vec{b}|}$$

$$|\vec{a}| = \sqrt{x^2+y^2} = \sqrt{(3)^2+(4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$
$$|\vec{b}| = \sqrt{x^2+y^2} = \sqrt{(5)^2+(12)^2} = \sqrt{25+144} = \sqrt{169} = 13$$
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{63}{5\cdot13} = \frac{63}{65}$$
$$\cos\theta = \frac{63}{65}$$

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3) find the angle between the vectors $\vec{a} = (3, -1, 2)$ and $\vec{b} = (2, 4, -1)$.

Solution

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

$$\vec{a} \cdot \vec{b} = (3, -1, 2) \cdot (2, 4, -1)$$

$$\vec{a} \cdot \vec{b} = 6 + (-4) + (-2) = 6 - 6 = 0$$

$$\boxed{\vec{a} \cdot \vec{b} = 0}$$

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|} = \frac{0}{|\vec{a}| \cdot |\vec{b}|} = 0$$

$\cos\theta = 0 \rightarrow$ means $\perp$ (orthogonal)

4) find the projection of $\vec{a}$ onto $\vec{b}$ if $\vec{a} = (1, -1, 0)$ and $\vec{b} = (1, 0, 1)$

Solution

formula to calculate projection $\text{proj}_{\vec{b}}\vec{a}$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b}$$

$$|\vec{b}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(1)^2 + (0)^2 + (1)^2} = \sqrt{1+1} = \sqrt{2}$$

$$\vec{a} \cdot \vec{b} = (1, -1, 0) \cdot (1, 0, 1)$$

$$\vec{a} \cdot \vec{b} = (1 + 0 + 0) = 1$$

$$\boxed{\vec{a} \cdot \vec{b} = 1}$$

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \vec{b} = \frac{1}{(\sqrt{2})^2} \vec{b} = \frac{1}{(\sqrt{2})^2} \times (1, 0, 1)$$

$$\frac{1}{(\sqrt{2})^2} \times (1, 0, 1) = \frac{1}{2} \times (1, 0, 1) = \boxed{\left(\frac{1}{2}, 0, \frac{1}{2}\right)}$$

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The Cross product

As opposed to the dot product which results in a scalar, the cross product of two vectors is again a vector. if $\vec{a}$ and $\vec{b}$ are two vectors their cross product is denoted by $\vec{a} \times \vec{b}$.
The vector $\vec{a} \times \vec{b}$ is prependicular to the plane determined by $\vec{a}$ and $\vec{b}$. This determines the direction of $\vec{a} \times \vec{b}$. The sense of $\vec{a} \times \vec{b}$ is determined by the right hand rule:

in general $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$

but $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$ is True.

if θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\boxed{|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta}$$

The cross product of $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ can be computed using the coordinates as follows.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Since $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$, $\vec{a}$ and $\vec{b}$ are parallel // exactly when $\sin\theta = 0$ which happens exactly when $\vec{a} \times \vec{b} = \vec{0}$. Thus

$\vec{a}$ and $\vec{b}$ are parallel // if and only if:

$$\boxed{\vec{a} \times \vec{b} = 0}$$

Another way to check if the two vectors are parallel is to check if one is a scalar multiple of other (i.e., if $\vec{a} = k\vec{b}$ for some k). In this case, for $\vec{b} \neq \vec{0}$, the coordinates are such that $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

ex:- $\vec{a} = (4, 8, 12)$

$\vec{b} = (2, 4, 6)$

$\vec{a} = k\vec{b} = (4, 8, 12)$, $2[2, 4, 6] = \vec{a}$

$\boxed{k} \to 2$

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▸ Practice problems.

1) Let $\vec{a} = (1, 2, 0)$ and $\vec{b} = (0, 3, 1)$. find $\vec{a} \times \vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix}$$

$= \vec{i}(2-0) - \vec{k}(1-0) + \vec{j}(3-0)$

$= 2\underline{i} - \underline{k} + 3\underline{j}$

$= [2, -1, 3]$

2) Do Same for $\vec{a} = (2\vec{i} + \vec{j} - \vec{k})$ and $\vec{b} = \vec{j} + 2\vec{k}$

Solution

$\vec{a} = [2, 1, -1]$, $\vec{b} = [0, 1, 2]$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

$\vec{a} \times \vec{b} = \underline{i}(2+1) - \underline{j}(4-0) + \underline{k}(2-0)$

$\vec{a} \times \vec{b} = 3\underline{i} - 4\underline{j} + 2\underline{k}$

$\vec{a} \times \vec{b} = [3, -4, 2]$

3) Let $\vec{a} = (-5, 3, 7)$ and $\vec{b} = (6, -8, 2)$. Determine if the vectors are parallel, prependicular or neither.

Solution

$\vec{a}$ and $\vec{b}$ parallel $\parallel$ when $\vec{a} \times \vec{b} = (0, 0, 0)$

$\vec{a}$ and $\vec{b}$ $\perp$ prep when $\vec{a} \cdot \vec{b} = 0$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5 & 3 & 7 \\ 6 & -8 & 2 \end{vmatrix}$$

$\vec{a} \times \vec{b} = \underline{i}(6 + 56) - \vec{j}(-10 - (42) + \vec{k}(+40 - 48)$

$\vec{a} \times \vec{b} = \underline{i}(62) - \vec{j}(-10 - 42) + \vec{k}(22)$

$\vec{a} \times \vec{b} = 62\underline{i} - \vec{j}(-52) + \vec{k}(22)$

$\vec{a} \times \vec{b} = 62\underline{i} + 52\vec{j} + 22\vec{k}$

$\vec{a} \times \vec{b} = 62i + 52j + 22k$

$\vec{a} \times \vec{b} = (62, 52, 22)$

$\vec{a} \times \vec{b} \neq 0$, means vectors are ~~not~~ $\parallel$

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$\vec{a}\cdot\vec{b} = 0$, means vectors are $\perp$

$\vec{a}\cdot\vec{b} = (-5, 3, 7)\cdot(6, -8, 2)$

$\vec{a}\cdot\vec{b} = -30 + (-24) + 14 = -30 - 24 + 14 = -54 + 14 = -40$

$\vec{a}\cdot\vec{b} = -40$, $\vec{a}\cdot\vec{b} \neq 0$, means vectors aren't $\perp$

4) Do the same for $\vec{a} = -\vec{i} + 2\vec{j} + 5\vec{k}$ and $\vec{b} = 3\vec{i} + 4\vec{j} - \vec{k}$.

Solution

$\vec{a}$ and $\vec{b}$ are $\parallel$ when $\vec{a}\times\vec{b} = 0$

$\vec{a}$ and $\vec{b}$ are $\perp$ when $\vec{a}\cdot\vec{b} = 0$

Solving $\vec{a}\cdot\vec{b}$

$\vec{a}\cdot\vec{b} = (-1, 2, 5)\cdot(3, 4, -1)$

$\vec{a}\cdot\vec{b} = -3 + 8 - 5 = 8 - 8 = 0$

$\vec{a}\cdot\vec{b} = 0$, means vectors are $\perp$

let's Try the Second one to see the outcome

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ -1 & 2 & 5 \\ 3 & 4 & -1 \end{vmatrix}$$

$\vec{a}\times\vec{b} = i(-2-20) - j(1-15) + k(-4-6)$

$\vec{a}\times\vec{b} = -22i + 14j - 10k$

$\vec{a}\times\vec{b} = [-22, 14, -10]$

$\vec{a}\times\vec{b} \neq 0$, means vectors $\vec{a}$ and $\vec{b}$ aren't $\parallel$

5) find a vector prependicular to the plane through the points $P(1, 0, 0)$, $Q(0, 2, 0)$ and $R(0, 0, 3)$ and find the area of triangle PQR.

Solution

first step $\rightarrow$ $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$ to calculate angle

Second step $\rightarrow$ $\Delta = \frac{1}{2}|\vec{PQ}\times\vec{PR}|$ to calculate area

Second (a) $\vec{PQ} = Q - P$

Second (b) $\vec{PR} = R - P$

Then cross product of $\vec{PQ}\times\vec{PR}$, then magnitude of $\vec{PQ}$ and $\vec{PR}$ $\rightarrow$ $|\vec{PQ}\times\vec{PR}|$ $\rightarrow$ $\frac{1}{2}|\vec{PQ}\times\vec{PR}|$

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$P(1,0,0)$, $Q(0,2,0)$, $R(0,0,3)$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}\cdot\vec{c}}{|a|\cdot|b|\cdot|\vec{c}|}$$

$\vec{a}\cdot\vec{b}\cdot\vec{c} = (1,0,0)\cdot(0,2,0)\cdot(0,0,3)$

$\vec{a}\cdot\vec{b}\cdot\vec{c} = (0+0+0) = 0$

if $\vec{a}\cdot\vec{b}\cdot\vec{c} = 0$, means vectors are $\perp$, $\boxed{\text{hence proved}}$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}\cdot\vec{c}}{|\vec{a}||\vec{b}||\vec{c}|} = \frac{\vec{P}\cdot\vec{Q}\cdot\vec{R}}{|\vec{P}||\vec{Q}||\vec{R}|} = \frac{0}{|P||Q||R|} = 0$$

$\boxed{\cos\theta = 0} \rightarrow$ vectors are prependicular

Second step

$$\Delta = \frac{1}{2}|\vec{PQ}\times\vec{PR}|$$

(a)

$\vec{PQ} = Q - P = (0,2,0) - (1,0,0)$

$\vec{PQ} = Q - P = (-1, 2, 0)$

$\boxed{\vec{PQ} = (-1, 2, 0)}$

(b)

$\vec{PR} = R - P = (0,0,3) - (1,0,0)$

$\vec{PR} = R - P = (-1, 0, 3)$

$\boxed{\vec{PR} = (-1, 0, 3)}$

$$\Delta = \frac{1}{2}|\vec{PQ}\times\vec{PR}|$$

$\vec{PQ}\times\vec{PR}$, cross product gives us a $\perp$ vector

let's Solve it

$\vec{PQ}\times\vec{PR}$, $\vec{PQ} = (-1, 2, 0)$, $\vec{PR} = (-1, 0, 3)$

$$\vec{PQ}\times\vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix}$$

$\vec{PQ}\times\vec{PR} = i(6-0) - j(-3-0) + k(0-(-2))$

$\vec{PQ}\times\vec{PR} = 6i + 3j + 2k$

$\vec{PQ}\times\vec{PR} = 6i + 3j + 2k$

$= [6, 3, 2]$

Now, find magnitude of $\vec{PQ}\times\vec{PR} \rightarrow |\vec{PQ}\times\vec{PR}|$

Prependicular $\perp$ vector is $[6, 3, 2]$

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$|\vec{PQ} \times \vec{PR}|$ to Calculate Area of Triangle

$|\vec{PQ} \times \vec{PR}| = \sqrt{x^2+y^2+z^2} = \sqrt{(6)^2+(3)^2+(2)^2}$
$= \sqrt{36+9+4} = \sqrt{49} = 7$

Now apply the Statement $\rightarrow \Delta = \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

Area of Triangle

$\Delta = \frac{1}{2}|\vec{PQ} \times \vec{PR}| = \frac{1}{2}(7) = \frac{7}{2}$

$\boxed{\Delta = 7} \rightarrow$ is the Area of Triangle

6) Do the Same for $P(0,0,0)$, $Q(1,-1,1)$ and $R(4,3,7)$.

Solution

$\cos\theta = \frac{\vec{a} \cdot \vec{b} \cdot \vec{c}}{|\vec{a}| \cdot |\vec{b}| \cdot |\vec{c}|}$

$\vec{a} \cdot \vec{b} \cdot \vec{c} = (0,0,0) \cdot (1,-1,1) \cdot (4,3,7)$
$= (0 + 0 + 0) = 0$

$\boxed{\vec{P} \cdot \vec{Q} \cdot \vec{R} = 0}$

$\cos\theta = \frac{\vec{a} \cdot \vec{b} \cdot \vec{c}}{|\vec{a}| \cdot |\vec{b}| \, |\vec{c}|} = \frac{0}{|a||b||c|} = 0$

$\boxed{\cos\theta = 0} \rightarrow$ means vectors are prependicular $\perp$

Now, move to calculate the area of Δ and find the $\perp$ vector.

Let the $\perp$ vector be with the name $\vec{N}$

So, $\vec{N}$ is $\vec{PQ} \times \vec{PR}$, the $\vec{N}$ vector will be appear when we cross product the $\vec{PQ}$ and $\vec{PR}$ vectors

$\vec{N} = \vec{PQ} \times \vec{PR}$

$\vec{PQ} = \vec{Q} - \vec{P} = (1,-1,1) - (0,0,0) = (1,-1,1)$

$\boxed{\vec{PQ} = (1,-1,1)}$

$\vec{PR} = \vec{R} - \vec{P} = (4,3,7) - (0,0,0) = (4,3,7)$

$\boxed{\vec{PR} = (4,3,7)}$

$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 4 & 3 & 7 \end{vmatrix} = i(-7-3) - j(7-4) + k(3+4)$

$= -10i - 3j + 7k$

$= [-10, -3, 7]$

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$\vec{N} = [-10, -3, 7]$

Now, calculate Area

$\Delta = \frac{1}{2} |\vec{N}| = \frac{1}{2}\sqrt{(-10)^2 + (-3)^2 + (7)^2} = \frac{1}{2}\sqrt{100+9+49}$

$\Delta = \frac{1}{2}\sqrt{158} = \frac{\sqrt{158}}{2}$

$\boxed{\Delta = \frac{\sqrt{158}}{2}}$

7.4

3, Compute the cross product $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$. check your answer by showing that each $\underline{a}$ and $\underline{b}$ is $\perp$ perpendicular to $\underline{a} \times \underline{b}$ and $\underline{b} \times \underline{a}$.

Solu: $a = 2\underline{i} + \underline{j} - \underline{k}$, $\underline{b} = \underline{i} - \underline{j} + \underline{k}$

Solution

$\underline{a} \times \underline{b}$, $\underline{a} = [2, 1, -1]$, $\underline{b} = [1, -1, 1]$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$\underline{a} \times \underline{b} = i(1-1) + j(2+1) + k(-2-1)$

$\underline{a} \times \underline{b} = i(0) + 3j - 3k$

$\underline{a} \times \underline{b} = 3j - 3k$

$\boxed{\underline{a} \times \underline{b} = [0, 3, -3]}$

$\underline{b} \times \underline{a}$, $\underline{b} = [1, -1, 1]$, $\underline{a} = [2, 1, -1]$

$$\underline{b} \times \underline{a} = \begin{vmatrix} i & j & k \\ 1 & -1 & +1 \\ 2 & 1 & -1 \end{vmatrix}$$

$\underline{b} \times \underline{a} = i(1-1) - j(-1-2) + k(1+2)$

$\underline{b} \times \underline{a} = i(0) + 3j + 3k$

$\underline{b} \times \underline{a} = 3j + 3k$

$\boxed{\underline{b} \times \underline{a} = [0, 3, 3]}$

let $\underline{a} \times \underline{b} = \vec{F}$, and $\underline{b} \times \underline{a} = \vec{S}$, then

$\underline{a} \times \underline{b} = \vec{F} = (0, 3, -3)$ and $\vec{S} = (0, 3, 3)$

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$\underline{a} \times \vec{F}$, $\underline{a} \times \vec{S}$, $\underline{b} \times \underline{F}$, and $\underline{b} \times \underline{S}$ are prependicular $\perp$?

$\underline{a} = (2, 1, -1)$, $\underline{b} = (1, -1, 1)$, $\underline{F} = (0, 3, -3)$, $\underline{S} = (0, 3, 3)$

$$\vec{a} \times \vec{f} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$\vec{a} \times \vec{f} = i(1-1) - j(2+1) + k(-2-1)$

$\vec{a} \times \vec{f} = i(0) - 3j - 3k$

$\vec{a} \times \vec{f} = [0, -3, -3] \longrightarrow$ prependicular vector

verification

if $a \cdot a \times b = 0$, then the vector is $\perp$

$a \cdot a \times b = [2, 1, -1] \cdot [0, -3, -3]$

$= 0 + (-3) + (3) = -3 + 3 = 0$

$\boxed{a \cdot a \times b = 0} \rightarrow$ The vectors are $\perp$, $\boxed{\text{Hence proved}}$

$$\vec{a} \times \vec{S} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 0 & 3 & 3 \end{vmatrix}$$

$\vec{a} \times \vec{S} = i(3+3) - j(6-0) + k(6-0)$

$\vec{a} \times \vec{S} = 6i - 6j + 6k$

$\boxed{\vec{a} \times \vec{S} = (6, -6, 6)}$

$\boxed{\text{verification:}}$

if $a \cdot a \times S = 0$, then the vectors are $\perp$

$a \cdot a \times S = (2, 1, -1) \cdot (6, -6, 6)$

$= 12 - 6 - 6 = 0$

$\boxed{a \cdot a \times S} = 0 \rightarrow \perp$ proved

$$b \times \vec{f} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 3 & -3 \end{vmatrix}$$

$\vec{b} \times \vec{f} = i(3-3) - j(-3-0) + k(3-0)$

$= i(0) + 3j + 3k$

$\boxed{\vec{b} \times \vec{f} = [0, 3, 3]}$

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$\vec{b} \times \vec{s}$

$$\vec{b} \times \vec{s} = \begin{vmatrix} i & j & k \\ 1 & -1 & 1 \\ 0 & 3 & 3 \end{vmatrix}$$

$\vec{b} \times \vec{s} = i(-3-3) - j(3-0) + k(3-0)$

$\vec{b} \times \vec{s} = -6i - 3j + 3k$

$\boxed{\vec{b} \times \vec{s} = [-6, -3, 3]}$

verification

$b \cdot b \times s = [1, -1, 1] \cdot [-6, -3, 3]$

$= (-6 + (3) + (3)) = -6 + 6 = 0$

$\boxed{b \cdot b \times s = 0} \rightarrow$ vectors are $\perp$

(ii) $\underline{a} = 3\underline{i} - 2\underline{j} + \underline{k}$, $\underline{b} = \underline{i} + \underline{j}$

Solution

$\underline{a} \times \underline{b}$, $\underline{a} = [3, -2, 1]$, $\underline{b} = [1, 1, 0]$

$$\underline{a} \times \underline{b} = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$\underline{a} \times \underline{b} = i(0-1) - j(0-1) + k(3-(-2))$

$\underline{a} \times \underline{b} = -i + j + 5k$

$\boxed{\underline{a} \times \underline{b} = [-1, 1, 5]}$

$$\underline{b} \times \underline{a} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & -2 & 1 \end{vmatrix}$$

$\underline{b} \times \underline{a} = i(1-0) - j(1-0) + k(-2-3)$

$\underline{b} \times \underline{a} = i - j - 5k$

$\boxed{\underline{b} \times \underline{a} = [1, -1, -5]}$

let $\underline{a} \times \underline{b} = \underline{f}$, and $\underline{b} \times \underline{a} = \underline{g}$

let's solve $\underline{a} \times \underline{f}$

$\underline{a} \times \underline{f}$, $\underline{a} = [3, -2, 1]$, $\underline{f} = [-1, 1, 5]$

$$\underline{a} \times \underline{f} = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ -1 & 1 & 5 \end{vmatrix} = i(-10-1) - j(15+1) + k(3-2)$$

$= -11i - 16j + k$

$= [-11, -16, 1]$

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$\underline{a} \times \underline{s}$, $a = [3, -2, 1]$, $\underline{s} = [1, -1, -5]$

$$\underline{a} \times \underline{s} = \begin{vmatrix} i & j & k \\ 3 & -2 & 1 \\ 1 & -1 & -5 \end{vmatrix} = i(10+1) - j(-15-1) + k(-3+2)$$

$$= i(11) - j(-16) + k(-1)$$

$$= 11\underline{i} + 16\underline{j} - \underline{k}$$

$$\underline{a} \times \underline{s} = [11, 16, -1]$$

verification:

$\underline{a} \cdot \underline{a} \times \underline{f} = \underline{a} \cdot \underline{f}$

$\underline{a} = [3, -2, 1]$, $\underline{f} = [-1, 1, 5]$

$\underline{a} \cdot \underline{f} = (3, -2, 1) \cdot (-1, 1, 5)$

$= -3 + (-2) + (5) = -3 - 2 + 5 = 0$

$\underline{a} \cdot \underline{f} = 0$, vectors are $\perp$, hence proved

$\underline{a} \cdot \underline{a} \times \underline{s} = \underline{a} \cdot \underline{s}$

$\underline{a} = [3, -2, 1]$, $\underline{s} = [1, -1, -5]$

$\underline{a} \cdot \underline{s} = (3, -2, 1) \cdot (1, -1, -5)$

$\underline{a} \cdot \underline{s} = 3 + 2 - 5 = 0$

$\underline{a} \cdot \underline{s} = 0$, means vectors are $\perp$, hence proved

verification:

$\underline{b} \cdot \underline{f}$, $\underline{b} = [1, 1, 0]$, $\underline{f} = [-1, 1, 5]$

$\underline{b} \cdot \underline{f} = (1, 1, 0) \cdot (-1, 1, 5)$

$\underline{b} \cdot \underline{f} = -1 + 1 + 0 = 0$

$\underline{b} \cdot \underline{f} = 0$, means vectors are $\perp$, hence proved

$\underline{b} \cdot \underline{b} \times \underline{a} = \underline{b} \cdot \underline{s}$, $\underline{b} = [1, 1, 0]$, $\underline{s} = [1, -1, -5]$

$\underline{b} \cdot \underline{s} = (1, 1, 0) \cdot (1, -1, -5)$

$\underline{b} \cdot \underline{s} = 1 - 1 + 0 = 0$

$\underline{b} \cdot \underline{s} = 0$, means vectors are $\perp$, hence prooved.

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7.24

3) Find the Area of Triangle determined by the Point $\vec{P}$, $\vec{Q}$ and $\vec{R}$.

(i) $P(0,0,0)$; $Q(2,3,2)$; $R(-1,1,4)$

(ii) $P(1,-1,-1)$; $Q(2,0,-1)$; $R(0,2,1)$

(i) $P(0,0,0)$; $Q(2,3,2)$; $R(-1,1,4)$

Solution

$\Delta = \frac{1}{2}|\vec{PQ} \times \vec{PR}|$, $\vec{PQ} = \vec{Q} - \vec{P}$, $\vec{PR} = \vec{R} - \vec{P}$

$\vec{PQ} = \vec{Q} - P$, $\vec{P} = (0,0,0)$, $\vec{Q} = (2,0,-1)$

$\vec{PQ} = \vec{Q} - \vec{P} = (2,0,-1) - (0,0,0)$

$\vec{PQ} = \vec{Q} - \vec{P} = (2,0,-1)$

$\boxed{\vec{PQ} = [2,0,-1]}$

$\vec{PR} = \vec{R} - \vec{P}$, $\vec{P} = (0,0,0)$, $\vec{R}(-1,1,4)$

$\vec{PR} = \vec{R} - \vec{P} = (-1,1,4) - (0,0,0)$

$\vec{PR} = \vec{R} - \vec{P} = (-1,1,4)$

$\boxed{\vec{PR} = (-1,1,4)}$

$\Delta = \frac{1}{2}|\vec{PQ} \times \vec{PR}|$, find $\vec{PQ} \times \vec{PR}$ (cross prod)

$\vec{PQ} = (2,0,-1)$, $\vec{PR} = (-1,1,4)$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & 0 & -1 \\ -1 & 1 & 4 \end{vmatrix}$$

$\vec{PQ} \times \vec{PR} = i(0+1) - j(8-1) + k(2-0)$

$\vec{PQ} \times \vec{PR} = i - 7j + 2k$

$\vec{PQ} \times \vec{PR} = (1,-7,2)$

$|\vec{PQ} \times \vec{PR}| = \sqrt{x^2+y^2+z^2} = \sqrt{(1)^2+(-7)^2+(2)^2} = \sqrt{54}$

$\Delta = \frac{1}{2}|\vec{PQ} \times \vec{PR}|$

$\Delta = \frac{1}{2} \times \sqrt{54} = \frac{\sqrt{54}}{2}$

Area of Triangle is $\rightarrow$ $\boxed{\Delta = \frac{\sqrt{54}}{2}}$

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(ii) $P(1,-1,-1)$; $Q(2,0,-1)$; $R(0,2,1)$

Solution

$\Delta = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$

$\vec{PQ} = \vec{Q} - \vec{P} = (2,0,-1) - (1,-1,-1)$

$\vec{PQ} = \vec{Q} - \vec{P} = (1, 1, 0)$

$\boxed{\vec{PQ} = [1, 1, 0]}$

$\vec{PR} = \vec{R} - \vec{P} = (0,2,1) - (1,-1,-1)$

$\vec{PR} = \vec{R} - \vec{P} = (-1, 3, 2)$

$\boxed{\vec{PR} = [-1, 3, 2]}$

$\Delta = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$, Now find $\vec{PQ} \times \vec{PR}$ (cross prod)

$\vec{PQ} = (1, 1, 0)$, $\vec{PR} = (-1, 3, 2)$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ -1 & 3 & 2 \end{vmatrix}$$

$\vec{PQ} \times \vec{PR} = i(2-0) - j(2-0) + k(3+1)$

$\vec{PQ} \times \vec{PR} = 2i - 2j + 4k$

$\boxed{\vec{PQ} \times \vec{PR} = [2, -2, 4]}$, Now find its magnitude

$\Delta = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$

$|\vec{PQ} \times \vec{PR}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(2)^2 + (-2)^2 + (4)^2} = \sqrt{4 + 4 + 16}$

$|\vec{PQ} \times \vec{PR}| = \sqrt{24}$

$\boxed{|\vec{PQ} \times \vec{PR}| = \sqrt{24}}$

$\Delta = \frac{1}{2} |\vec{PQ} \times \vec{PR}|$

$\Delta = \frac{1}{2} \times \sqrt{24} = \frac{\sqrt{24}}{2} = \frac{\sqrt{24}}{\sqrt{4}} = \sqrt{\frac{24}{4}} = \boxed{\sqrt{6}}$

$\Delta = \sqrt{6}$ Area of Triangle

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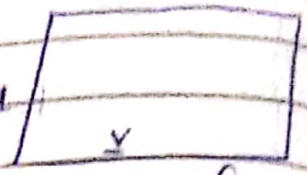
4. Find the Area of parallegram whose vertices are:

(i) $A(0,0,0)$; $B(1,2,3)$; $C(2,-1,1)$; $D(3,1,4)$
(ii) $A(1,2,-1)$; $B(4,2,-3)$; $C(6,-5,2)$; $D(8,-5,0)$
(iii) $A(-1,1,1)$; $B(-1,2,2)$; $C(-3,4,-5)$; $D(-3,5,-4)$
(i) $A(0,0,0)$; $B(1,2,3)$; $C(2,-1,1)$; $D(3,1,4)$

Solution



Area of parallegram $= |\underline{u} \times \underline{v}|$

$\underline{u} = \vec{AB}$ and $\underline{v} = \vec{AC}$

find their cross product, and the magnitude of the cross producted vector is the Area of $\|^m$.

find $\vec{AB}$ and $\vec{AC}$

$\vec{AB}$, $A = (0,0,0)$, $B(1,2,3)$

$\vec{AB} = \vec{B} - \vec{A} = (1,2,3) - (0,0,0)$

$\vec{AB} = \vec{B} - \vec{A} = (1,2,3)$

$\boxed{\vec{AB} = (1,2,3)}$

$\vec{AC}$, $A(0,0,0)$, $C = (2,-1,1)$

$\vec{AC} = \vec{C} - \vec{A} = (2,-1,1) - (0,0,0)$

$\vec{AC} = \vec{C} - \vec{A} = (2,-1,1)$

$\boxed{\vec{AC} = (2,-1,1)}$, find $\vec{AB} \times \vec{AC} = \underline{u} \times \underline{v}$

$\vec{AB} \times \vec{AC}$, $\vec{AB} = [1,2,3]$, $\vec{AC} = [2,-1,1]$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{vmatrix}$$

$\vec{AB} \times \vec{AC} = i(2+3) - j(1-6) + k(-1-4)$

$\vec{AB} \times \vec{AC} = 5i + 5j - 5k$

$\boxed{\vec{AB} \times \vec{AC} = (5, 5, -5)}$, find magnitude

$|\vec{AB} \times \vec{AC}| = \sqrt{x^2+y^2+z^2} = \sqrt{(5)^2+(5)^2+(-5)^2} = \sqrt{25+25+25}$

$|\vec{AB} \times \vec{AC}| = \sqrt{75}$

Area of $\|^m$ is $\sqrt{75}$

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(ii) $A(1, 2, -1)$; $B(4, 2, -3)$; $C(6, -5, 2)$; $D(8, -5, 0)$

Solution

Area of $\parallel^m = |u \times v| = |\vec{AB} \times \vec{AC}|$

$\vec{AB} = \vec{B} - \vec{A} = (4, 2, -3) - (1, 2, -1)$

$\vec{AB} = \vec{B} - \vec{A} = (3, 0, -2)$

$\boxed{\vec{AB} = (3, 0, -2)}$

$\vec{AC} = \vec{C} - \vec{A} = (6, -5, 2) - (1, 2, -1)$

$\vec{AC} = \vec{C} - \vec{A} = (5, -7, 3)$

$\boxed{\vec{AC} = (5, -7, 3)}$

$\square = |\vec{AB} \times \vec{AC}|$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 3 & 0 & -2 \\ 5 & -7 & 3 \end{vmatrix}$$

$= i(0 - 14) - j(9 + 10) + k(-21 - 0)$

$= -14i - 19j - 21k$

$\boxed{\vec{AB} \times \vec{AC} = (-14, -19, -21)}$, Now, find its magnitude

$|\vec{AB} \times \vec{AC}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-14)^2 + (-19)^2 + (-21)^2}$

$= \sqrt{196 + 361 + 441} = \sqrt{870}$

$\boxed{|\vec{AB} \times \vec{AC}| = \sqrt{870}} \rightarrow$ Area of $\parallel^m$

(iii) $A(-1, 1, 1)$; $B(-1, 2, 2)$; $C(-3, 4, -5)$; $D(-3, 5, -4)$

Solution

$\vec{AB} = \vec{B} - \vec{A} = (-1, 2, 2) - (-1, 1, 1)$

$\vec{AB} = \vec{B} - \vec{A} = (0, 1, 1)$

$\boxed{\vec{AB} = (0, 1, 1)}$

$\vec{AC} = \vec{C} - \vec{A} = (-3, 4, -5) - (-1, 1, 1)$

$\vec{AC} = \vec{C} - \vec{A} = (-2, 3, -6)$

$\boxed{\vec{AC} = (-2, 3, -6)}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 0 & 1 & 1 \\ -2 & 3 & -6 \end{vmatrix} = i(-6-3) - j(0+2) + k(0+2)$$

$= -9i - 2j + 2k$

$= \boxed{(-9, -2, 2)} = \vec{AB} \times \vec{AC}$

$|\vec{AB} \times \vec{AC}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-9)^2 + (-2)^2 + (2)^2}$

$= \sqrt{81 + 4 + 4} = \sqrt{89}$

Area of $\parallel^m$ is $\rightarrow \sqrt{89}$

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5) Prependicular or Parallel

(1) $\underline{u} = 5\underline{i} - \underline{j} + \underline{k}$; $\underline{v} = \underline{j} - 5\underline{k}$; $\underline{w} = -15\underline{i} + 3\underline{j} - 3\underline{k}$

Solution

$\cos\theta = \dfrac{\underline{u} \cdot \underline{v} \cdot \underline{w}}{|\underline{u}| \times |\underline{v}| \times |\underline{w}|}$, if $\underline{u} \cdot \underline{v} \cdot \underline{w} = 0$, means $\perp$

$\underline{u} \cdot \underline{v} \cdot \underline{w} = (5, -1, 1) \cdot (0, 1, -5) \cdot (-15, 3, -3)$

$= 0 + (-3) + 15 = -3 + 15 = 12$

$\boxed{\underline{u} \cdot \underline{v} \cdot \underline{w} = 12}$

$\underline{u} \cdot \underline{v} \cdot \underline{w}$ aren't $\perp$

7.5 Scalar Triple Product of vectors

Ex:- Find the volume of the parallelpiped determined by $\underline{u} = \underline{i} + 2\underline{j} - \underline{k}$, $\underline{v} = \underline{i} - \underline{j} + 3\underline{k}$, $\underline{w} = \underline{i} - 7\underline{j} - 4\underline{k}$

Solution

volume of the parallelpiped $= \underline{u} \cdot \underline{v} \times \underline{w}$

$u = [1, 2, -1]$, $\underline{v} = [1, -1, 3]$, $\underline{w} = [1, -7, -4]$

$$\underline{u} \cdot \underline{v} \times w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 3 \\ 1 & -7 & -4 \end{vmatrix}$$

$u \cdot \underline{v} \times w = 1(4 + 21) - 2(-4 - 3) - 1(-7 + 1)$

$= 25 + 14 + 6 = 45$

volume of parallelpiped is 45

2) Ex:- Prove that four points are Co-planer.

Solution

$A(-3, 5, -4)$, $B(-1, 1, 1)$, $C(-1, 2, 2)$, $D(-3, 4, -5)$

Co-planer $\rightarrow$ if volume of $\vec{AB}$, $\vec{AC}$ and of $\vec{AD}$ is 0 then A, B, C, and D are Co-planer.

first step $\rightarrow$ find $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$

- $\vec{AB} = \vec{B} - \vec{A} = (-1, 1, 1) - (-3, 5, -4)$

$\boxed{\vec{AB} = (2, -4, 5)}$

- $\vec{AC} = \vec{C} - \vec{A} = (-1, 2, 2) - (-3, 5, -4)$

$\boxed{\vec{AC} = (2, -3, 6)}$

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$\vec{AD} = \vec{D} - \vec{A} = (-3, 4, -5) - (-3, 5, -4)$

$\boxed{\vec{AD} = (0, -1, -1)}$

if the volume of $\vec{AB}$, $\vec{AC}$ and of $\vec{AD}$ is zero, then the vectors are co-planer otherwise not.

$\boxed{\text{volume of } \vec{AB}, \vec{AC}, \vec{AD}}$

$\vec{AB}(2, -4, 5)$, $\vec{AC}(2, -3, 6)$, $\vec{AD}(0, -1, -1)$

$$\begin{vmatrix} 2 & -4 & 5 \\ 2 & -3 & 6 \\ 0 & -1 & -1 \end{vmatrix} = 2(3+6) + 4(-2-0) + 5(-2-0)$$

$$= 18 - 8 - 10 = 18 - 18 = 0$$

The volume is 0, so the vectors are co-planer

Ex:- find the volume of tetrahedron whose vertices are:

$A(2, 1, 8)$, $B(3, 2, 8)$, $C(2, 1, 4)$ and $D(3, 3, 0)$

Solution

formulae $\rightarrow \frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

$\vec{AB} = \vec{B} - \vec{A} = (3, 2, 8) - (2, 1, 8)$

$\boxed{\vec{AB} = (1, 1, 1)}$

$\vec{AC} = \vec{C} - \vec{A} = (2, 1, 4) - (2, 1, 8)$

$\boxed{\vec{AC} = (0, 0, -4)}$

$\vec{AD} = \vec{D} - \vec{A} = (3, 3, 0) - (2, 1, 8)$

$\boxed{\vec{AD} = (1, 2, -8)}$

Now find the Determinat of $\vec{AB}$, $\vec{AC}$ and of $\vec{AD}$ then multiply it with 1/6.

volume of tetrahedron

$$= \frac{1}{6}\begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & -8 \end{vmatrix}$$

$$= \frac{1}{6} \times ((0+8) - (0+4) + (0-0))$$

$$= \frac{1}{6} \times (8 - 4 + 0) = \frac{1}{6} \times 4 = \frac{2}{3}$$

Volume of tetrahedron is $\rightarrow \boxed{\frac{2}{3}}$

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6) Ex:- find the work done by a constant force $F = 2i + 4j$ if its points of Application to a body moves it from $A(1, 1)$ to $B(4, 6)$.

Solution

work done $\longrightarrow$ $\underline{F} \cdot \underline{d}$, force · displacement

$F = 2i + 4j$, $d = \vec{AB} = \vec{B} - \vec{A}$

$d = (4, 6) - (1, 1)$

$\boxed{d = (3, 5)}$

Now find the Dot product of $\underline{F} \cdot \underline{d}$

$F = (2, 4)$, $d = (3, 5)$

$\vec{F} \cdot \vec{D} = (2, 4) \cdot (3, 5)$

$\vec{F} \cdot \vec{D} = 6 + 20 = 26$

$\boxed{\text{work done} = 26}$

7) The constant forces $2i + 5j + 6k$ and $-i + 2j + k$ act on a body, which is displaced from position $P(4, -3, -2)$ to $Q(6, 1, -3)$. find the total work done.

Solution

work done $= \vec{F} \cdot \underline{d}$, $\vec{d} = \vec{PQ} = \vec{Q} - \vec{P}$

Constant forces($2i + 5j + 6k$ and $-i + 2j + k$

$\vec{F} = (2i + 5j + 6k) + (-i + 2j + k)$

$\boxed{\vec{F} = i + 7j + 7k}$

$\vec{d} = \vec{PQ} = \vec{Q} - \vec{P} = (6, 1, -3) - (4, -3, -2)$

$\boxed{\vec{d} = (2, 4, -1)}$

work done $\rightarrow \vec{F} \cdot \vec{d}$

$\vec{F} \cdot \vec{D} = (1, 7, 7) \cdot (2, 4, -1)$

$= 2 + 28 - 7 = 23$

8) Ex:- find the moment about the point $M(-2, 4, -6)$ of the force represented by $\vec{AB}$, where coordinates of points A and B are $(1, 2, -3)$ and $(3, -4, 2)$ respectively.

Solution

Moment of $\vec{AB}$ about $\vec{M} = \vec{MA} \times \vec{AB}$

magnitude of moment $|\vec{MA} \times \vec{AB}|$

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$\vec{AB} = \vec{B} - \vec{A} = (3, 4, 2) - (1, 2, 3)$

$\vec{AB} = (+2, -6, +5)$

$\boxed{\vec{AB} = (2, -6, 5)}$

$\vec{MA} = \vec{A} - \vec{M} = (1, 2, -3) - (-2, 4, -6)$

$\boxed{\vec{MA} = (3, -2, 3)}$

$\vec{MA} = (3, -2, 3)$, $\vec{AB} = (2, -6, 5)$

$$\vec{MA} \times \vec{AB} = \begin{vmatrix} i & j & k \\ 3 & -2 & 3 \\ 2 & -6 & 5 \end{vmatrix}$$

$= i(-10+18) - j(15-6) + k(-18+4)$

$\boxed{\text{Moment} = 8i - 8j - 14k}$

magnitude of moment $= \sqrt{(8)^2 + (-8)^2 + (-14)^2} = \sqrt{341}$

Exercise 7.5

1) Find the volume of the parallelpiped for which the given vectors are three edges.

(i) $\underline{u} = 3\underline{i} + 2\underline{k}$; $\underline{v} = \underline{i} + 2\underline{j} + \underline{k}$; $\underline{w} = -\underline{j} + 4\underline{k}$

Solution

volume of $\underline{u}$, $\underline{v}$ and $\underline{w}$

$\underline{u} = [3, 0, 2]$, $\underline{v} = [1, 2, 1]$, $\underline{w} = [0, -1, 4]$

$$\text{volume} = \begin{vmatrix} 3 & 0 & 2 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{vmatrix} = 3(8+1) - 0(4-0) + 2(-1-0)$$

$= 27 - 0 + 2(-1)$

$= 27 - 2 = 25$

$\boxed{\text{volume of } \underline{u}, \underline{v} \text{ and } \underline{w} \text{ vectors is } 25}$

(ii) $\underline{u} = \underline{i} - 4\underline{j} - \underline{k}$; $\underline{v} = \underline{i} - \underline{j} - 2\underline{k}$; $\underline{w} = 2\underline{i} - 3\underline{j} + \underline{k}$

Solution

$\underline{u} = [1, -4, -1]$, $\underline{v} = [1, -1, -2]$, $\underline{w} = [2, -3, 1]$

$$\text{volume} = \begin{vmatrix} 1 & -4 & -1 \\ 1 & -1 & -2 \\ 2 & -3 & 1 \end{vmatrix} = 1(-1-6) + 4(1+4) - 1(-3+2)$$

$= 1(-7) + 20 - 1(-1)$

$= -7 + 20 + 1 = 14$

$\boxed{\text{volume} = 14}$

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(iii) $\underline{u} = \underline{i} - 2\underline{j} - 3\underline{k}$; $\underline{v} = 2\underline{i} - \underline{j} - \underline{k}$; $\underline{w} = \underline{j} + \underline{k}$

Solution

$\underline{u} = [1, -2, -3]$, $\underline{v} = [2, -1, -1]$, $\underline{w} = [0, 1, 1]$

$$\text{volume} = \begin{vmatrix} 1 & -2 & -3 \\ 2 & -1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 1(-1+1) + 2(2-0) + (-3)(2-0)$$

$$= 1(0) + 2(2) - 3(2)$$

$$= 4 - 6 = -2$$

$\boxed{\text{volume} = -2}$

3) Prove that vectors $\underline{i} - 2\underline{j} + 3\underline{k}$, $-2\underline{i} + 3\underline{j} - 4\underline{k}$ and $\underline{i} - 3\underline{j} + 5\underline{k}$ are coplaner.

Solution

if the volume of vectors is zero then the vectors are co-planer.

$\underline{u} = (\underline{i} - 2\underline{j} + 3\underline{k})$, $\underline{v} = (-2\underline{i} + 3\underline{j} - 4\underline{k})$, $\underline{w} = (\underline{i} - 3\underline{j} + 5\underline{k})$

$\underline{u} = [1, -2, 3]$, $\underline{v} = [-2, 3, -4]$, $\underline{w} = [1, -3, 5]$

volume of $\underline{u}$, $\underline{v}$ and of $\underline{w}$

$$\text{volume} = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix} = 1(15+12) + 2(-10+4) + 3(6-3)$$

$$= 1(3) + 2(-6) + 3(3)$$

$$= 3 - 12 + 9 = 12 - 12 = 0$$

volume of vectors $\underline{u}, \underline{v}, \underline{w}$ is 0, So the vectors are coplaner, $\boxed{\text{volume} = 0}$

4) find the constant α Such that vectors are coplaner.

Solution

$\underline{u} = \underline{i} - \underline{j} + \underline{k}$, $\underline{v} = \underline{i} - 2\underline{j} - 3\underline{k}$, $\underline{w} = 3\underline{i} - \alpha\underline{j} + 5\underline{k}$.

Such that the vectors are coplaner, means there volume will be zero.

$\underline{u} = [1, -1, 1]$, $\underline{v} = [1, -2, -3]$, $\underline{w} = [3, -\alpha, 5]$

$$\text{volume} = \begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -\alpha & 5 \end{vmatrix} = 1(-10-3\alpha) + 1(5+9) + 1(-\alpha+6)$$

$$= -10 - 3\alpha + 14 + 6 - \alpha$$

$$= 20 - 10 - 4\alpha$$

$$\text{volume} = 10 - 4\alpha$$

vectors are coplaner, and volume $= 0$, so

$10 - 4\alpha = 0$

$\boxed{\alpha = 5/2}$

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verification

$\underline{u} = [1, -1, 1]$, $\underline{v} = [1, -2, -3]$, $\underline{w} = [3, -5/2, 5]$

volume $= \begin{vmatrix} 1 & -1 & 1 \\ 1 & -2 & -3 \\ 3 & -5/2 & 5 \end{vmatrix} = 1(-10 + 15/2) + 1(5+9) + 1(-5/2 + 6)$

$= 1(-17.5) + 14 + 3.5$

$= -17.5 + 17.5 = 0$

$\boxed{\text{volume} = 0}$ → Hence prooved

6) find the volume of the tetrahedron with vertices

i) $(0,1,2)$, $(3,2,1)$, $(1,2,1)$, $(5,5,6)$

Solution

Let $\vec{A} = (0,1,2)$, $\vec{B} = (3,2,1)$, $\vec{C} = (1,2,1)$ and $\vec{D} = (5,5,6)$

volume of tetrahedron → $\frac{1}{6}[\vec{AB}\ \vec{AC}\ \vec{AD}]$

$\vec{AB} = \vec{B} - \vec{A} = (3,2,1) - (0,1,2)$

$\boxed{\vec{AB} = (3, 1, -1)}$

$\vec{AC} = \vec{C} - \vec{A} = (1,2,1) - (0,1,2)$

$\boxed{\vec{AC} = (1, 1, -1)}$

$\vec{AD} = \vec{D} - \vec{A} = (5,5,6) - (0,1,2)$

$\boxed{\vec{AD} = (5, 4, 4)}$

Now find the determinant of $\vec{AB}$, $\vec{AC}$ and of $\vec{AD}$, then multiply the value which the determinant provides with $1/6$.

$\vec{AB}\ \vec{AC}\ \vec{AD}$ $\begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ 5 & 4 & 4 \end{vmatrix} = 3(4+4) - 1(4+5) - 1(4-5)$

$= 3(8) - 1(9) - 1(-1)$

$= 24 - 9 + 1 = 16$

determinant of $\vec{AB}$, $\vec{AC}$ and of $\vec{AD}$ is 16

Now multiply 16 with → $1/6$, So

volume of tetrahedron $= \frac{1}{6} \times 16 = \frac{8}{3}$

$\boxed{\text{volume of tetrahedron} = \frac{8}{3}}$

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(ii) $(2,1,8)$, $(3,2,9)$, $(2,1,4)$ and $(3,3,10)$.

Solution

Let $A=(2,1,8)$, $B=(3,2,9)$, $C=(2,1,4)$ and $D=(3,3,10)$

$\vec{AB} = \vec{B} - \vec{A} = (3,2,9) - (2,1,8)$

$\boxed{\vec{AB} = (1, 1, 1)}$

$\vec{AC} = \vec{C} - \vec{A} = (2,1,4) - (2,1,8)$

$\boxed{\vec{AC} = (0, 0, -4)}$

$\vec{AD} = \vec{D} - \vec{A} = (3,3,10) - (2,1,8)$

$\boxed{\vec{AD} = (1, 2, 2)}$

volume of $\vec{AB}$, $\vec{AC}$, $\vec{AD}$

$$\text{volume} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 0 & -4 \\ 1 & 2 & 2 \end{vmatrix} = 1(0+8) - 1(0+4) + 1(0-0) = 8 - 4 + 0 = 4$$

$\boxed{\text{volume} = 4}$

volume of tetrahedron $= \frac{1}{6} \times 4 = \frac{4}{6} = \boxed{\frac{2}{3}} \rightarrow$ volume of tetrahedron

7) find the work done, if the point at which the constant force $\vec{F} = 4\underline{i} + 3\underline{j} + 5\vec{k}$ is applied to an object moves from $P_1(3,1,-2)$ to $P_2(2,4,6)$.

Solution

$\vec{F} = 4\underline{i} + 3\underline{j} + 5\vec{k} = [4, 3, 5]$

$\vec{P_1P_2} = P_2 - P_1 = (2,4,6) - (3,1,-2)$

$\boxed{\vec{P_1P_2} = (-1, 3, 8)}$

Now find the Dot Product of $\rightarrow$ $\vec{F} \cdot \vec{P_1P_2}$

$\vec{F} = [4, 3, 5]$, $\vec{P_1P_2} = [-1, 3, 8]$

$\vec{F} \cdot \vec{P_1P_2} = [4, 3, 5] \cdot [-1, 3, 8]$

$\vec{F} \cdot \vec{P_1P_2} = [-4 + 9 + 40] = 49 - 4 = 45$

$\vec{F} \cdot \vec{P_1P_2} = 45$

$\boxed{\text{work done} = 45}$

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8) A particle acted by content forces $4\underline{i}+\underline{j}-3\underline{k}$ and $3\underline{i}-\underline{j}-\underline{k}$ is displaced from $A(1,2,3)$ to $B(5,4,1)$. find the work done.

Solution

Constant forces $\rightarrow 4i+j-3k$ and $3i-j-k$

add these forces $\rightarrow 7i-0j-4k$

$$\boxed{\vec{F}=[7,\ 0,\ -4]}$$

Now find $\vec{AB}$:

$\vec{AB}=\vec{B}-\vec{A}=(5,4,1)-(1,2,3)$

$$\boxed{\vec{AB}=(4,2,-2)}$$

Now Solve for work done, Such that work done $=\vec{F}\cdot\vec{AB}$

$\vec{F}\cdot\vec{AB}=(7,0,-4)\cdot(4,2,-2)$

$\vec{F}\cdot\vec{AB}=28+0+8=36$

$\boxed{\vec{F}\cdot\vec{AB}=36}\rightarrow$ work done $=36$

9) A particle is displaced from the points $A(5,-5,-7)$ to the point $B(6,2,-2)$ under the action of constant forces defined by $10\underline{i}-\underline{j}+11\underline{k}$, $4\underline{i}+5\underline{j}+8\underline{k}$ and $-2\underline{i}+\underline{j}-8\underline{k}$. Show that the total work done by the forces is 102 units.

Solution

forces $\rightarrow 10\underline{i}-\underline{j}+11\underline{k}$, $4\underline{i}+5\underline{j}+8\underline{k}$, and $-2\underline{i}+\underline{j}-8\underline{k}$

addition of forces $\rightarrow 12\underline{i}+5\underline{j}+11\underline{k}$

$$\boxed{\vec{F}=[12,\ 5,\ 11]}$$

finding $\vec{AB}=\vec{B}-\vec{A}=(6,2,-2)-(5,-5,-7)$

$$\boxed{\vec{AB}=(1,7,5)}$$

$\vec{F}\cdot\vec{AB}=(12,5,11)\cdot(1,7,5)$

$\vec{F}\cdot\vec{AB}=12+35+55=102$

$\boxed{\vec{F}\cdot\vec{AB}=102 \text{ units}}\rightarrow$ Hence proved

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10) A force of magnitude 6 units acting parallel to $2i - 2j + k$ displaces, the point of application from $(1, 2, 3)$ to $(5, 3, 7)$. find the work done.

Solution

force $= 2i - 2j + k$

Points $\rightarrow$ $A(1, 2, 3)$, $B(5, 3, 7)$

finding $\vec{AB} = \vec{B} - \vec{A} = (5, 3, 7) - (1, 2, 3)$

$\boxed{\vec{AB} = (4, 1, 4)}$

$\vec{F} = (2, -2, 1)$

find work done $\rightarrow$ $\vec{F} \cdot \vec{AB}$

$\vec{F} \cdot \vec{AB} = (2, -2, 1) \cdot (4, 1, 4)$

$\vec{F} \cdot \vec{AB} = (8 - 2 + 4) = 10$

$\boxed{\vec{F} \cdot \vec{AB} = 10}$ work done $= 10$

11) A force $\vec{F} = 3i + 2j - 4k$ is applied at the point $(1, -1, 2)$. find the moment of force about the point $(2, -1, 3)$.

Solution

$\vec{F} = 3i + 2j - 4k$, $A(1, -1, 2)$, $B(2, -1, 3)$

moment $= \vec{FA} \times \vec{AB}$

finding $\vec{FA} = \vec{A} - \vec{F}$

$\vec{F} = [3, 2, -4]$, $\vec{A} = [1, -1, 2]$

$\vec{FA} = (1, -1, 2) - (3, 2, -4)$

$\boxed{\vec{FA} = (-2, -3, 6)}$

finding $\vec{AB} = \vec{B} - \vec{A} =$

$= (2, -1, 3) - (1, -1, 2)$

$\boxed{\vec{AB} = (1, 0, 1)}$

moment $\rightarrow$ $\vec{FA} \times \vec{AB}$

$\vec{FA} = (-2, -3, 6)$, $\vec{AB} = (1, 0, 1)$

$$\vec{FA} \times \vec{AB} = \begin{vmatrix} i & j & k \\ -2 & -3 & 6 \\ 1 & 0 & 1 \end{vmatrix} = i(-3 - 0) - j(-2 - 6) + k(0 + 3)$$

$= -3i + 8j + 3k$

$\boxed{\vec{FA} \times \vec{AB} = (-3, 8, 3)} \rightarrow$ Moment

magnitude of moment $\rightarrow$ $\sqrt{82}$

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(2) A force $\vec{F} = 4i - 3k$, passes through the point $A(2,-2,5)$. find the moment of $\vec{F}$ about the point $B(1,-3,1)$.

Solution

$\vec{F} = 4i - 3k = [4, 0, -3]$

$A = (2,-2,5)$, $B = (1,-3,1)$

Moment $\longrightarrow$ $\vec{FA} \times \vec{AB}$

$\vec{FA} = \vec{A} - \vec{F} = (2,-2,5) - (4,0,-3)$

$\boxed{\vec{FA} = (-2,-2,8)}$

$\vec{AB} = \vec{B} - \vec{A} = (1,-3,1) - (2,-2,5)$

$\boxed{\vec{AB} = (-1,-1,-4)}$

$$\vec{FA} \times \vec{AB} = \begin{vmatrix} i & j & k \\ -2 & -2 & 8 \\ -1 & -1 & -4 \end{vmatrix} = i(8+8) - j(8+8) + k(2-2) = 16i - 16j$$

Moment $\longrightarrow$ $16i - 16j$

Magnitude of moment $\rightarrow \sqrt{(16)^2 + (-16)^2} = \sqrt{512} \rightarrow \sqrt{512}$

(3) Give a force $\vec{F} = 2i + j - 3k$ acting at a point $A(1,-2,1)$. find the moment of $\vec{F}$ about the point $B(2,0,-2)$.

Solution

Moment of $\vec{FA}$ about $\vec{B} = \vec{BA} \times \vec{FA}$

$\vec{F} = (2, 1, -3)$, $\vec{A} = (1,-2,1)$, $\vec{B} = (2,0,-2)$

$\vec{FA} = \vec{A} - \vec{F} = (1,-2,1) - (2,1,-3)$

$\boxed{\vec{FA} = (-1,-3,4)}$

$\vec{BA} = \vec{A} - \vec{B} = (1,-2,1) - (2,0,-2)$

$\boxed{\vec{BA} = (-1,-2,3)}$

Moment $\vec{BA} \times \vec{FA}$

$$\begin{vmatrix} i & j & k \\ -1 & -2 & 3 \\ -1 & -3 & 4 \end{vmatrix} = i(-8+8) - j(-4+3) + k(3-2) = i + j + k$$

Moment $\rightarrow (1, 1, 1)$

Magnitude of Moment $\rightarrow \sqrt{3}$

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Q) find the moment About $A(1,1,1)$ of each of the Concurrent forces $i-2j$, $3i+2j-k$, $5j+2k$, where P $(2,0,1)$ is their point of concurrency.

Solution

a) $\vec{F} = [1,-2,0]$ and $[3,2,-1]$ and $[0,5,2]$

$P = (2,0,1)$

find moment $\vec{FP}$ about $A \rightarrow \vec{AF} \times \vec{FP}$

$\vec{F_1} = [1,-2,0]$, $A = [1,1,1]$, $P = [2,0,1]$

finding $\vec{AF} = \vec{F} - \vec{A} = (1,-2,0) - (1,1,1)$

$\boxed{\vec{AF} = (0,-3,-1)}$

finding $\vec{FP} = \vec{P} - \vec{F} = (2,0,1) - (1,-2,0)$

$\boxed{\vec{FP} = (1,2,1)}$

Moment $\vec{AF} \times \vec{FP}$

$\vec{AF} = [0,-3,-1]$, $\vec{FP} = [1,2,1]$

$$\vec{AF} \times \vec{FP} = \begin{vmatrix} i & j & k \\ 0 & -3 & -1 \\ 1 & 2 & 1 \end{vmatrix} = i(-3+2) - j(0+1) + k(0+3)$$

$= -i - j + 3k$

Moment $\rightarrow (-1,-1,3)$

Magnitude of moment is $\rightarrow \sqrt{11}$

b) moment of $\vec{FP}$ about $\vec{A} = \vec{AF} \times \vec{FP}$

$F = [3,2,-1]$, $\vec{P} = [2,0,1]$, $A[1,1,1]$

$\vec{AF} \times \vec{AP}$

$\vec{AF} = \vec{F} - \vec{A} = [3,2,-1] - [1,1,1]$

$\boxed{\vec{AF} = [2,1,-2]}$

$\vec{FP} = \vec{P} - \vec{F} = [2,0,1] - [3,2,-1]$

$\boxed{\vec{FP} = [-1,-2,2]}$

$\vec{AF} \times \vec{FP}$, $\vec{AF} = [2,1,-2]$, $\vec{FP} = [-1,-2,2]$

$$\vec{AF} \times \vec{FP} = \begin{vmatrix} i & j & k \\ 2 & 1 & -2 \\ -1 & -2 & 2 \end{vmatrix} = i(2-4) - j(4-2) + k(-4+1)$$

$= -2i - 2j - 3k$

Moment $\rightarrow (-2,-2,-3)$

magnitude of moment $\rightarrow \sqrt{17}$

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c) $\vec{F} = 5j + 2k$

Moment $\vec{FP}$ about $\vec{A}$ is $\vec{AF} \times \vec{FP}$

$\vec{F} = [0, 5, 2]$, $\vec{A} = [1, 1, 1]$, $\vec{P} = [2, 0, 1]$

$\vec{AF} = \vec{F} - \vec{A} = [0, 5, 2] - [1, 1, 1]$

$\boxed{\vec{AF} = [-1, 4, 1]}$

$\vec{FP} = \vec{P} - \vec{F} = (2, 0, 1) - (0, 5, 2)$

$\boxed{\vec{FP} = (2, -5, -1)}$

$\vec{AF} \times \vec{FP}$, $\vec{AF} = [-1, 4, 1]$, $\vec{FP} = [2, -5, -1]$

$$\vec{AF} \times \vec{FP} = \begin{vmatrix} i & j & k \\ -1 & 4 & 1 \\ 2 & -5 & -1 \end{vmatrix} = i(-4+5) - j(1-2) + k(5-8)$$

$$= i + j - 3k$$

$\vec{AF} \times \vec{FP} \rightarrow$ moment $\rightarrow [1, 1, -3]$

magnitude of moment $\rightarrow \sqrt{11}$

15) A Force $\vec{F} = 7i + 4j - 3k$ is applied at $P(1, -2, 3)$. Find its moment about the point $Q(2, 1, 1)$.

Solution

Moment $\vec{FP}$ about $\vec{Q}$ is $\vec{QF} \times \vec{FP}$

$\vec{QF} = \vec{F} - \vec{Q} = (7, 4, -3) - (2, 1, 1)$

$\boxed{\vec{QF} = (5, 3, -4)}$

$\vec{FP} = \vec{P} - \vec{F} = (1, -2, 3) - (7, 4, -3)$

$\boxed{\vec{FP} = (-6, -6, 6)}$

$$\vec{QF} \times \vec{FP} = \begin{vmatrix} i & j & k \\ 5 & 3 & -4 \\ -6 & -6 & 6 \end{vmatrix} = i(18-24) - j(30-24) + k(-30+18)$$

$$= -12i - 6j - 12k$$

Moment about Q is $\rightarrow (-12, -6, -12)$

Magnitude of Moment is $\rightarrow \sqrt{324} = 18$

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Exercise 13 (Assignment)

Q1:- Find the equation of the circles with the given centres and radii.

(a) $(-1, 2)$, $r = \sqrt{2}$

Solution

$(x-h)^2 + (y-k)^2 = r^2$

$\boxed{= (x+1)^2 + (y-2)^2 = 2}$

(b) $(-\sqrt{2}, -2)$, $r = \sqrt{6}$

Solution

$= (x+\sqrt{2})^2 + (y+2)^2 = 6$

$= \boxed{(x+\sqrt{2})^2 + (y+2)^2 = 6}$

(c) $(0, 0)$, $r = a$

Solution

$(x-0)^2 + (y-0)^2 = a^2$

$\boxed{x^2 + y^2 = a^2}$

(d) $(1, -3)$, $r = 3$

Solution

$\boxed{(x-1)^2 + (y+3)^2 = 9}$

Q2:- find the centres and radii of the circles with the following equations:

Solution

formulae for center $\rightarrow \left(-\frac{g}{2}, -\frac{f}{2}\right) \rightarrow$ Sada Behar

(a) $x^2 + y^2 - 6x + 6y = 0$

Solution

$x^2 + y^2 + 2gx + 2fy + c = 0$

$g = -6$, $f = 6$, $c = 0$

center

$\left(-\frac{g}{2}, \frac{-f}{2}\right) = \left(\frac{+6}{2}, \frac{-6}{2}\right)$

$\boxed{\text{center} = (3, -3)}$

radii $\rightarrow r = \sqrt{g^2 + f^2 - c}$

$g = 3$, $f = -3$, $c = 0$

$r = \sqrt{(3)^2 + (-3)^2 - 0}$

$r = \sqrt{9 + 9} = \sqrt{18}$

$r = \sqrt{2 \times 9} = 3\sqrt{2}$

$\boxed{r = 3\sqrt{2}}$

(b) $x^2 + y^2 - 4x + y - 1 = 0$

Solution

$g = -4$, $f = 1$, $c = -1$

$\left(-\frac{g}{2}, \frac{-f}{2}\right) = \left(\frac{-(-4)}{2}, \frac{-1}{2}\right)$

$\boxed{\text{Center} = (2, -1/2)}$

radii $= \sqrt{g^2 + f^2 - c}$

$g = 2$, $f = -1/2$, $c = -1$

$r = \sqrt{(2)^2 + (-1/2)^2 - (-1)}$

$r = \sqrt{4 + \frac{1}{4} + 1}$

$r = \sqrt{\frac{17}{4} + 1} = \frac{\sqrt{21}}{2}$

$\boxed{r = \frac{\sqrt{21}}{2}}$

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(c) $3x^2+3y^2-2x-6y-2=0$

Solution

$3x^2+3y^2-2x-6y-2=0$

$x^2+y^2+2gx+2fy+c=0$

So

$\frac{3x^2}{3}+\frac{3y^2}{3}-\frac{2x}{3}-\frac{6y}{3}-\frac{2}{3}=0$

$\boxed{x^2+y^2-\frac{2x}{3}-2y-\frac{2}{3}=0}$

$\left(-\frac{g}{2},-\frac{f}{2}\right)$, $g=\frac{-2}{3}$, $f=-2$

$\left(-\frac{(-2/3)}{2},\frac{-2}{-2}\right)=\left(\frac{1}{3},1\right)$

$\boxed{\text{Center}=(1/3,1)}$ → center

radii $=\sqrt{(1/3)^2+(1)^2-(-2/3)}$

$r=\sqrt{1/9+1+2/3}=\sqrt{16/9}$

$\boxed{r=4/3}$ → radius

(d) $(x+2)^2+(y-1)^2=16$

Solution

$(x-h)^2+(y-k)^2=r^2$

$(x+2)^2+(y-1)^2=16$

$(x-h)^2=(x+2)^2$

$x-h=x+2$

$\boxed{h=-2}$

$(y-k)^2=(y-1)^2$

$y-k=y-1$

$\boxed{k=1}$

Center → $(h,k)=(-2,1)$

$r^2=16$

$r^2=4^2$, $\boxed{r=4}$

$\boxed{\text{radius}=4}$

$\boxed{\text{center}=(-2,1)}$

(e) $x^2+y^2-4x+6y-12=0$

Solution

$x^2+y^2-4x+6y-12=0$

$x^2+y^2+2gx+2fy+c=0$

$g=-4$, $f=6$, $c=-12$

$\left(-\frac{g}{2},-\frac{f}{2}\right)=\left(-\frac{(-4)}{2},\frac{-6}{2}\right)$

$=\boxed{(2,-3)}$ → center

radius $=\sqrt{g^2+f^2-c}$

$r=\sqrt{(2)^2+(-3)^2-(-12)}$

$r=\sqrt{4+9+12}$

$r=\sqrt{25}=5$

$\boxed{r=5}$ → radius

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Q3: find the equations of the circles:

(a) passing through the points $(1,2), (0,-1), (-1,1)$.

Solution

put $(1,2) = (x,y)$

$x^2+y^2+2gx+2fy+c=0$

$1+4+2g+4f+c=0$

$\boxed{2g+4f+c=-5}$

Put $(0,-1) = (x,y)$

$x^2+y^2+2gx+2fy+c=0$

$0+1+0-2f+c=0$

$\boxed{-2f+c=-1}$

Put $(-1,1) = (x,y)$

$x^2+y^2+2gx+2fy+c=0$

$1+1-2g+2f+c=0$

$\boxed{-2g+2f+c=-2}$

$2g+4f+c=-5$ — (1)

$-2f+c=-1$ — (2)

$-2g+2f+c=-2$ — (3)

(2) − (1)

$-2f+\cancel{c}=-1$

$2g+4f+\cancel{c}=-5$

$-\quad -\quad -\quad +$

$\boxed{-2g-6f=4}$

$-2g-6f=4$

$g=\frac{4f+1}{2}$

$-2\left(\frac{4f+1}{2}\right)-6f=4$

$-4f-1-6f=4$

$-10f=5$

$\boxed{f=-1/2}$

(2) − (3)

$-2f+\cancel{c}=-1$

$-2g+2f+\cancel{c}=-2$

$+\quad -\quad -\quad +$

$\boxed{2g-4f=1}$

$2g-4f=1$

$2g=4f+1$

$\boxed{g=\frac{4f+1}{2}}$

Put $f=-1/2$

$g=\frac{4(-\frac{1}{2})+1}{2}=\frac{-2+1}{2}=\frac{-1}{2}$

$\boxed{g=-1/2}$

Put in (1) OR (2) or (3) No equation to obtain c.

$-2f+c=-1$

$f=-1/2$

$-2(-\frac{1}{2})+c=-1$

$1+c=-1$

$\boxed{c=-2}$

$x^2+y^2+2gx+2fy+c=0$

$g=-1/2,\ f=-1/2,\ c=-2$

$x^2+y^2+2x(-\frac{1}{2})+2y(-\frac{1}{2})-c=0$

$\boxed{x^2+y^2-x-y+2=0}$

Required equation of circle

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(b) passing through the points $(0,1)$, $(3,-3)$, $(3,-1)$

Solution

Put $(0,1)$

$x^2+y^2+2gx+2fy+c=0$

$0+1+0+2f+c=0$

$\boxed{2f+c=-1}$

Put $(3,-3)$

$x^2+y^2+2gx+2fy+c=0$

$9+9+6g-6f+c=0$

$\boxed{6g-6f+c=-18}$

Put $(3,-1)$

$x^2+y^2+2gx+2fy+c=0$

$9+1+6g-2f+c=0$

$\boxed{6g-2f+c=-10}$

$2f+c=-1$ — (1)

$6g-6f+c=-18$ — (2)

$6g-2f+c=-10$ — (3)

(2) − (1)

$6g-6f+c=-18$

$2f+c=-1$

$+$

$6g-8f=-17$

$\boxed{6g-8f=-17}$

(2) − (3)

$6g-6f+c=-18$

$6g-2f+c=-10$

$-\quad+\quad-\quad+$

$-4f=-8$

$\boxed{f=2}$

$6g-8f=-17$

put $f=2$

$6g-16=-17$

$\boxed{g=-1/6}$

$2f+c=-1$

$f=2$

$4+c=-1$

$\boxed{c=-5}$

$g=-1/6$, $f=2$, $c=-5$

$x^2+y^2+2gx+2fy+c=0$

$x^2+y^2+2x\left(-\frac{1}{6}\right)+2y(2)+(-5)=0$

$x^2+y^2-\frac{x}{3}+4y-5=0$

$3x^2+3y^2-x+12y-15=0$

$\boxed{3x^2+3y^2-x+12y-15=0}$

(c) Through $(-2,1)$, $(-4,-3)$, $(3,0)$

Put $(-2,1)$

$4+1-4g+2f+c=0$

$\boxed{-4g+2f+c=-5}$

Put $(-4,-3)$

$16+9-8g-6f+c=0$

$\boxed{-8g-6f+c=-25}$

$(3,0)$

$9+0+6g+0+c=0$

$\boxed{6g+c=-9}$

(2) − (1)

$-8g-6f+c=-25$

$-4g+2f+c=-5$

$+\quad-\quad-\quad+$

$-4g-8f=-20$

$\boxed{-2g-4f=-10}$

$-2g-4f=-10$

$-2g=4f-10$

$-g=2f-5$

$\boxed{g=5-2f}$

$\boxed{g=1/11}$

(2) − (3)

$-8g-6f+c=-25$

$6g+c=-9$

$-\quad-\quad+$

$-14g-6f=-16$

$\boxed{-7g-3f=-8}$

$g=5-2f$

$-7(5-2f)-3f=-8$

$-35+14f-3f=-8$

$11f=27$

$\boxed{f=27/11}$

$g=1/11$, $f=27/11$

$c=-105/11$

$x^2+y^2+\frac{2x}{11}+\frac{54y}{11}+\frac{105}{11}=0$

$x^2+y^2+\frac{2x}{11}+\frac{54y}{11}+\frac{105}{11}=0$

Required equation of circle.

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Qu:- find the equations of the following circles.

(a) Through the point of intersection of the lines $2x-y+7=0$ and $3x+y+8=0$ with center at origin.

Solution

Center → origin → $O(0,0)$

$2x-y+7=0$; $3x+y+8=0$

These point intersect at a point, that point also lies on circle and distance from O to that point is the radius of circle

$2x-y+7=0$; $3x+y+8=0$

$\boxed{2x+7=y}$ Put $y=2x+7$

Put $x=-3$ $3x+2x+7+8=0$

$2(-3)+7=y$ $5x+15=0$

$-6+7=y$ $\boxed{x=-3}$

$\boxed{y=1}$ $A(-3,1)$

$|OA| =$ $O(0,0)$, $A(-3,1)$

$|OA| = \sqrt{(-3-0)^2+(1-0)^2} = \sqrt{9+1} = \sqrt{10}$

radii of circle → $\sqrt{10}$ and origin $(0,0)$

$(x-h)^2+(y-k)^2=r^2$

$(x-0)^2+(y-0)^2=(\sqrt{10})^2$

$x^2+y^2=10$

$\boxed{x^2+y^2=10}$ → required equation

(b) Center at the point of intersection of the lines $x-2y+4=0$ and $2x+y-2=0$ with radius 4 units.

Solution

$x-2y+4=0$; $2x+y-2=0$

$\boxed{x=2y-4}$ Put $x=2y-4$

Put $y=2$ $2(2y-4)+y-2=0$

$x=2(2)-4$ $4y-8+y-2=0$

$x=4-4$ $5y-10=0$

$\boxed{x=0}$ $\boxed{y=2}$

Point of intersection → $(0,2)$

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$(x-h)^2+(y-k)^2=r^2$, $h=0$, $k=2$, $r=4$

$(x-0)^2+(y-2)^2=(4)^2$

$\boxed{x^2+(y-2)^2=16}$ → required equation of circle

Q6:- find the equation of following circles.

(a) Concentric with circle $x^2+y^2-6x+4y-12=0$ with radius 6 units.

Solution

$x^2+y^2-6x+4y-12=0$

$g=3$, $f=-2$ and radii $=6$

$(x-h)^2+(y-k)^2=r^2$

$(x-3)^2+(y+2)^2=(6)^2$

$\boxed{(x-3)^2+(y+2)^2=36}$

(b) Concentric with the circle $x^2+y^2-7x+8y=8/2$ and touches the y-axis.

Solution

$x^2+y^2-7x+8y-\frac{8}{2}=0$

$h=7/2$, $k=-4$, $r=\sqrt{g^2+f^2+c}$

$g=7/2$, $f=-4$, $c=-8/2$

$r=\sqrt{(7/2)^2+(-4)^2+\frac{8}{2}}$

$r=\sqrt{\frac{49}{2}+\frac{16}{1}+\frac{8}{2}}=\sqrt{\frac{49+32+8}{2}}=\sqrt{\frac{90}{2}}=\sqrt{45}=3\sqrt{5}$

$\boxed{r=3\sqrt{5}}$

$h=7/2$, $k=-4$, $r=3\sqrt{5}$

$(x-h)^2+(y-k)^2=r^2$

$(x-7/2)^2+(y+4)^2=(3\sqrt{5})^2$

$(x-7/2)^2+(y+4)^2=45$

$\boxed{(x-7/2)^2+(y+4)^2=45}$

Required equation of the circle

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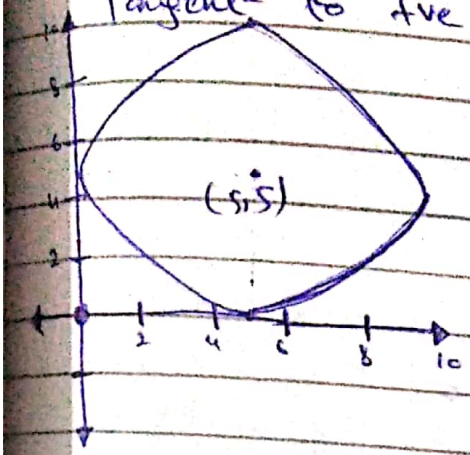
Q7:- find the equation of the following circles:

(a) which is tangent to the positive x and y-axis and radius 5 units.

Solution

Tangent to +ve x and y axes, radius = 5



center (5, 5), r = 5

$(x-h)^2+(y-k)^2=r^2$

$(x-5)^2+(y-5)^2=(5)^2$

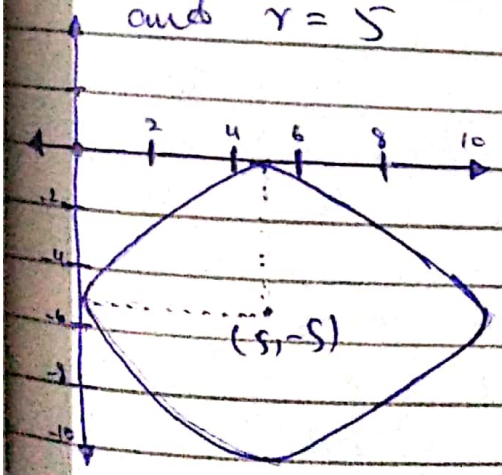
$\boxed{(x-5)^2+(y-5)^2=25}$

Required equation of circle

(b) which touches both the axes of 4th Quadrant and has a radius of 5 units.

Solution

Touches = Tangent to both axes of 4th Quadrant and r = 5



Center (5, -5), r = 5

$\boxed{(x-5)^2+(y+5)^2=25}$

Required equation of circle.

(c) whose center is (-3, 2) and passes through the center of the circle $x^2+y^2-4x+8y-16=0$

Solution

Center = (-3, 2), we can find Radius with the 2 points' distance which lies on circle, the second circle's center point lies on the first circle, so let's move to the Solution.

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$(-3, 2) \rightarrow$ center

$x^2+y^2-4x+8y-16=0$

$\rightarrow$ center $= (2, -4)$

$A(-3, 2)$, $B(2, -4)$, radii $= |AB|$

$|AB| = \sqrt{(2+3)^2+(-4-2)^2}$

$|AB| = \sqrt{25+36} = \sqrt{61}$

$h = -3$, $k = 2$, $r = \sqrt{61}$

$(x-h)^2+(y-k)^2 = r^2$

$(x+3)^2+(y-2)^2 = (\sqrt{61})^2$

$\boxed{(x+3)^2+(y-2)^2 = 61} \rightarrow$ Required Equation

Q8:- find which of two circles $x^2+y^2-3x+4y=0$ and $x^2+y^2-6x-8y=0$ is greater.

Solution

we can determine the size of circle with their Radius.

$x^2+y^2-3x+4y=0$	$x^2+y^2-6x-8y=0$
$g = 3/2$, $f = -2$, $c = 0$	$g = 3$, $f = 4$, $c = 0$
$r = \sqrt{(3/2)^2+4+0}$	$r = \sqrt{(3)^2+(4)^2-0}$
$r = \sqrt{\frac{9}{4}+\frac{4}{1}}$	$r = \sqrt{9+16} = \sqrt{25} = 5$
	$\boxed{r = 5} \rightarrow$ is the greater circle
$r = \sqrt{25/4} = 5/2$	

Q9:- find the equation of circle having:
(a) $(-2, 5)$ and $(3, 4)$ as the end points of its Diameter. find also its center and radius.

Solution

$A(-2, 5)$, $B(3, 4)$ as Diameter

So the Mid point of AB is center of circle

$M.P = \text{Center} = (1/2, 9/2) O$

$|AO|$ is the radius of circle

$|AO| = \sqrt{(1/2+2)^2+(9/2-5)^2} = \sqrt{\frac{25}{4}+\frac{1}{4}} = \frac{\sqrt{26}}{2}$

$r = \frac{\sqrt{26}}{2}$

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$h = 1/2, \quad k = 8/2, \quad r = \frac{\sqrt{26}}{2}$

$(x - 1/2)^2 + (y - 8/2)^2 = \left(\frac{\sqrt{26}}{2}\right)^2$

$\boxed{(x - 1/2)^2 + (y - 8/2)^2 = 13/4} \rightarrow$ required equation

(b) $(-3, 7)$ and $(2, -1)$ as the end points of its diameter. Find also its center and radius.

Solution

$A(-3, 7), \; B(2, -1)$

Center $= (-1/2, 3)$

Radius $= |AO|$ OR $|BO|$

$|BO| = \sqrt{(2 + 1/2)^2 + (-1 - 3)^2} = \sqrt{25/4 + 16} = \frac{\sqrt{89}}{2}$

$h = -1/2, \quad k = 3, \quad r = \sqrt{89}/2$

$(x + 1/2)^2 + (y - 3)^2 = \left(\frac{\sqrt{89}}{2}\right)^2$

$\boxed{(x + 1/2)^2 + (y - 3)^2 = 89/4} \rightarrow$ required equation

Q11: Show that the circles $x^2 + y^2 + 2x - 2y - 7 = 0$ and $x^2 + y^2 - 6x + 4y + 8 = 0$ touch externally.

Solution

The distance between the 2 centres is the same of the sum of the radius, then the two circles touch externally.

$x^2 + y^2 + 2x - 2y - 7 = 0$

$h = -1, \quad k = 1, \quad c = -7$

Center $(-1, 1)$

radius $= \sqrt{8} = 3$

$\boxed{r = 3}$

$x^2 + y^2 - 6x + 4y + 8 = 0$

$h = 3, \quad k = -2, \quad c = 8$

Center $(3, -2)$

radius $= \sqrt{4} = 2$

$r = \sqrt{(3)^2 + (-2)^2 + 8} = \sqrt{4} = 2$

$\boxed{r = 2}$

Distance of centres of circles

$A(-1, 1), \; B(3, -2)$

$|AB| = \sqrt{(3+1)^2 + (-2-1)^2}$

$|AB| = \sqrt{16 + 9} = \sqrt{25}$

$\boxed{|AB| = 5}$

Sum of radius of circles

$A = 3, \quad B = 2$

$A + B = 3 + 2 = 5$

Distance = Sum of Radius

The two circles touch externally hence proved.

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Q12:- Show that circles $x^2+y^2+2x-8=0$ and $x^2+y^2-6x+6y-46=0$ touch internally.

Solution

$x^2+y^2+2x-8=0$, $x^2+y^2-6x+6y-46=0$

The distance's and radius' difference are equal Then the circles touch internally.

$x^2+y^2+2x-8=0$

$h=-1,\ k=0,\ c=-8$

Centre $(-1,0)$

$r=\sqrt{g^2+f^2-c}$

$r=\sqrt{(1)^2+(0)^2-(-8)}$

$r=\sqrt{1+8}=\sqrt{9}=3$

$\boxed{r=3}$

$x^2+y^2-6x+6y-46=0$

$h=3,\ k=-3,\ c=-46$

Centre $(3,-3)$

$r=\sqrt{g^2+f^2-c}$

$r=\sqrt{9+9-(-46)}$

$r=\sqrt{18+46}=\sqrt{64}=8$

$\boxed{r=8}$

- Distance of circles

$A(-1,0)$, $B(3,-3)$

$|AB|=\sqrt{(3+1)^2+(-3-0)^2}$

$|AB|=\sqrt{16+9}=\sqrt{25}$

$\boxed{|AB|=5}$

Difference of two circles

$A=3,\ B=8$

$B-A=8-3=5$

$\boxed{B-A=5}$

Difference and distance of two circles is =

Hence Prooved.

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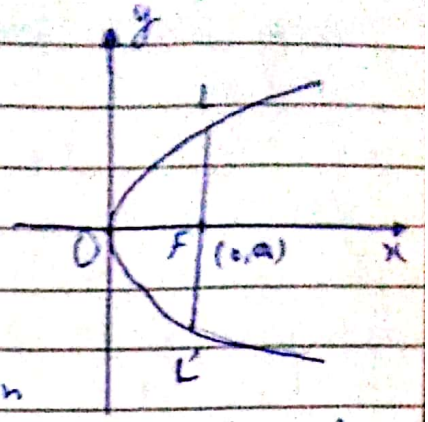
6.4.3 Graph of the parabola

$$y^2 = 4ax$$

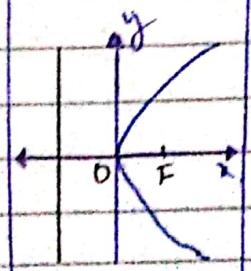
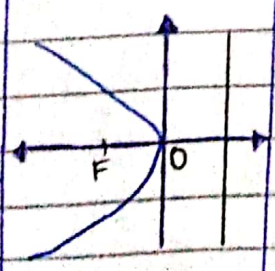
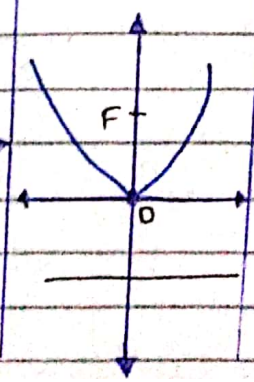
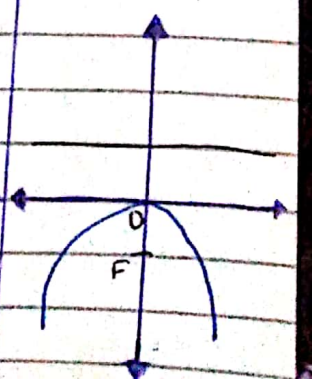


we note that corresponding to each positive value of x there are two equal and opposite values of y. Thus the curve is symmetric with respect to x-axis.

The curve passes through the origin and $x=0$ is tangent to the curve at $(0,0)$. if x is negative, then y^2 is negative and so y is imaginary. Thus no portion of the curve lies on the left of the y-axis. As x-increases, y also increases numerically so that the curve extends to infinity and lies in the first and fourth Quadrants. opening of the parabola is to The right of y-axis.

Sketching graphs of other standard parabolas is similar and is left as an exercise.

Summary of Standards parabolas

Sr.No	1	2	3	4
Equation	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Focus	$(a, 0)$	$(-a, 0)$	$(0, a)$	$(0, -a)$
Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
vertex	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0)$
Axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Latusrectum	$x = a$	$x = -a$	$y = a$	$y = -a$
				

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Example 1:- Analyze the parabola $x^2 = -16y$ and Draw its Graph.

Solution

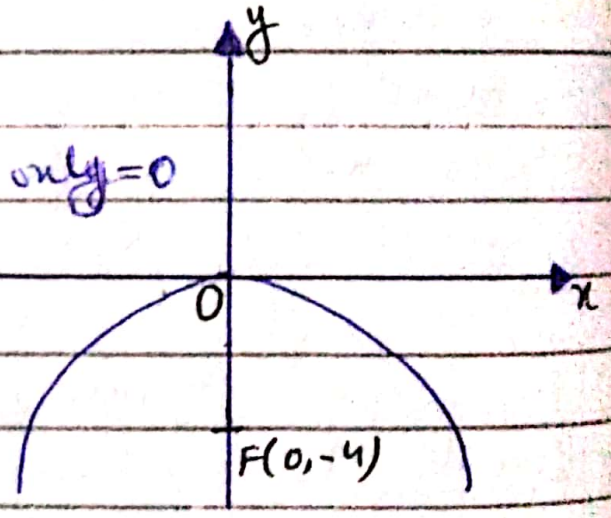
we Compare the given equation $x^2 = -4ay$

Here $4a = 16$ or $a = 4$

The focus of the parabola lies on the y-axis and its opening is downward. Coordinates of the focus is $(0, -4)$.

Equation of its axis is $x = 0$

Length of latusrectum is 16 and only $y = 0$ is tangent to the parabola at its vertex. The shape of the curve is as shown in the figure.



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Exercise 6.4

Q) Find the focus, vertex and directrix of the parabola. Sketch its graph.

(i) $y^2 = 8x$

Solution

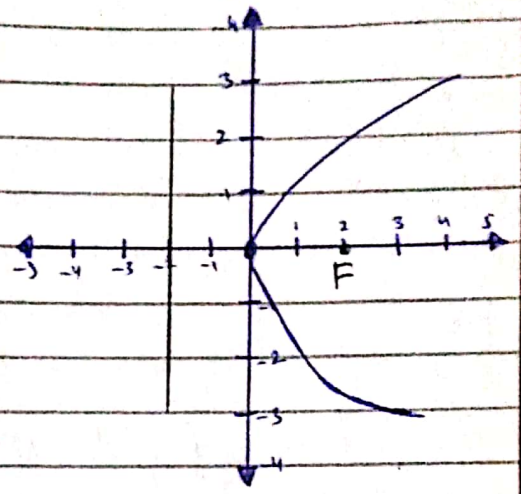
$y^2 = 8x \rightarrow$ equation , $y^2 = 4ax \rightarrow y^2 = 4(2)x$

Symmetry w.r.t x-axis $\boxed{a=2}$

focus $\rightarrow (2, 0)$ vertex $\rightarrow (0, 0)$

Directrix $\rightarrow \boxed{x = -2}$ axis $\rightarrow y = 0$

Latuslectum $\rightarrow x = a$



(ii) $x^2 = -16y$

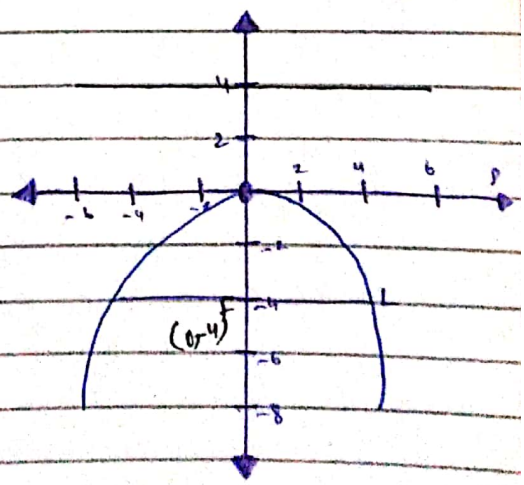
Solution

equation $\rightarrow x^2 = -16y$ Symmetry w.r.t -ve, y-axis

$x^2 = -4ay \rightarrow x^2 = -16y$

$\boxed{a=4}$ focus $\rightarrow (0, -4)$ $\boxed{y=4} \rightarrow$ Directrix

vertex $(0, 0)$, axis $\rightarrow x = 0$, latuslectum $\rightarrow y = -4$



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(iii) $x^2 = 5y$

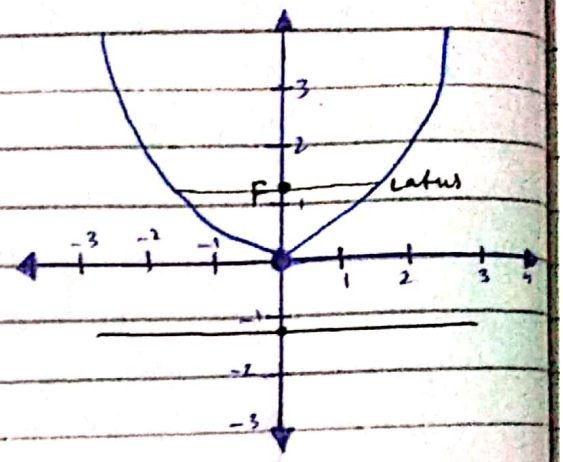
Solution

equation $\rightarrow$ $x^2 = 5y$, Symmetry w.r.t +ve y-axis

$x^2 = 4ay \rightarrow x^2 = 5y$

$\boxed{a = 5/4}$ focus $(0, 5/4)$, Directrix $y = -5/4$

vertex $= (0,0)$, Latusrectum $\rightarrow$ $y = 5/4$



(iv) $y^2 = -12x$

Solution

equation $\rightarrow$ $y^2 = -12x$,

$y^2 = -4ax \rightarrow y^2 = -12x$, $\boxed{a = 3}$

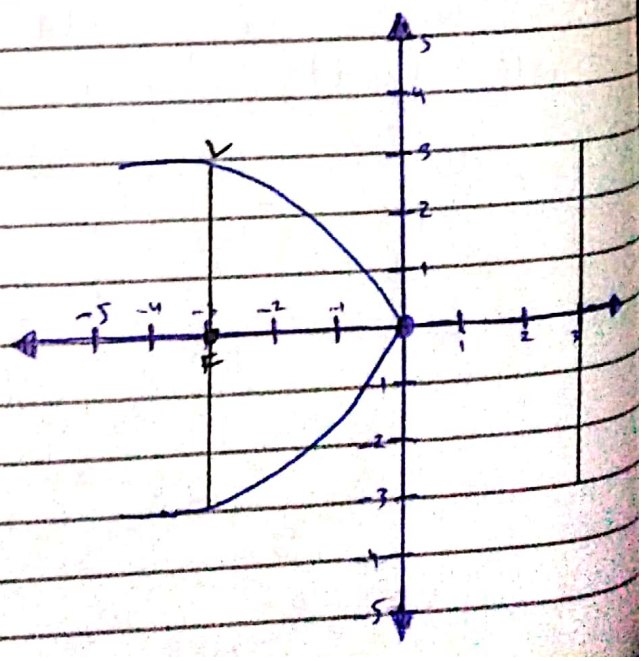
Symmetry w.r.t -ve, x-axis

vertex $(0, 0)$

focus $(-3, 0)$

Directrix $\boxed{x = 3}$

Latusrectum $\boxed{x = -3}$



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CC → Calculus

1) Cubic ; linear Simultaneously (78), 2) Asymptotes (Rational (81)
3) Area Regions (82), 4) Secant lines (84)
5) $\frac{f(a+h)-f(a)}{h}$ (87), 6) LIMIT (88)
7) Limit Graphically (88), 8) Ex: limit of Quotient, Average 195
9) Exercis (Limit) (96), 10) Exercise 1.2 Ron Larson (104)
10 :- Limit in Table form , 11) Exercise 1.3 Ron Larson (107)
12) Exercis 1.4 Continuity and one-Sided limit (116)
12: Graphically Representation (116 – 123)
13) Differentiation CHAPTER (2) (124)
13: Derivative via limit process (124), equation of tangent line(s) (131)
Complete the table (136), [find (a) Related to 1.4] (138)
• equation of tangent line (Related to 2.1), // line given (140)
• find the value of k, given the tangent line (2.2) (143)
14) 2.3: Rules → Product, Quotient, Constant Multiple, rewriting before differentiation, Derivatives of Trigonometric functions equality of Derivatives. (145)
15) Solution of 2.3 : Product and Quotient Rule(s) (148)
▸ tangent line equation, with point, via product/Quotient Rule
▸ "witches of Agnesi curve", "Serpentines curve"
▸ 73–76 horizontal tangent line

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(1)

7) $y = x^3 - 4x$; $y = -(x+2)$

$\boxed{y = -x - 2}$

:- $y = x^3 - 4x$

point of intersection of polynomial and line.

$x^3 - 4x = -x - 2$

$\boxed{x^3 - 3x + 2 = 0}$

Rational zero test $\rightarrow \frac{\pm p}{\pm q} = \frac{\pm 2}{\pm 1} = \frac{+2}{1}, \frac{+2}{-1}, \frac{-2}{1}, \frac{-2}{-1}$

$\frac{\pm p}{\pm q} = \boxed{2, -2}$

Hit and Trial method

$x = 2$, $y = x^3 - 3x + 2$	$x = -2$, $x^3 - 3x + 2 = 0$
$y = (2)^3 - 3(2) + 2$, x-int, $y=0$	$(-2)^3 - 3(-2) + 2 = 0$
$0 = 8 - 6 + 2$	$-8 + 6 + 2 = 0$
$\boxed{0 \neq 4}$	$\boxed{0 = 0}$ got-it

▶ we have got one Real root, which can be help us to find the rest roots.

• Now use the Synthetic division method.

polynomial $\rightarrow$ $x^3 - 3x + 2 = 0 \rightarrow ax^3 + bx^2 + cx + d$

$\boxed{x = -2}$

	1	0	-3	2
-2		-2	4	-2
	1	-2	1	$\boxed{0}$ → Remainder

deriving an equation of the Second degree.

$a = 1$, $b = -2$, $c = 1$ $\rightarrow$ $ax^2 + bx + c = 0$

$x^2 - 2x + 1 = 0$

$x^2 - x - x + 1 = 0$, factorization

$x(x-1) - 1(x-1) = 0$

$(x-1)(x-1) = 0$

$\boxed{x = 1}$ $\boxed{x = 1}$

2 equal real roots

points of Intersection

$(-2, 0)$ & $(1, -3)$

Roots of polynomial $y = x^3 - 3x + 2$ are $-2, 1, 1$

$x = -2$ and 1

Solve for y (or find y values)

$y = -x - 2$	$y = -x - 2$
$x = -2$	$x = 1$
$y = -(-2) - 2$	$y = -1 - 2$
$y = 2 - 2$	$\boxed{y = -3}$
$y = 0$	$\boxed{(1, -3)}$
$\boxed{(-2, 0)}$	

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$y = x^3 - 4x$ → find intercepts and sketch

$y = x^3 - 4x$, x-int, $y=0$

$x^3 - 4x = 0$, factorization

$x(x^2 - 4) = 0$

$\boxed{x=0}$ $x^2 = 2$

$\boxed{x = \pm 2}$

$x = 0, 2, -2$

3 distinct $\mathbb{R}$ roots

y-int, $x=0$

$y = -x - 2$

$y = -0 - 2$

$\boxed{y = -2}$

$y = -(x+2)$

$y = -x - 2$, x-int, $y=0$

$-x - 2 = 0$

$-x = 2$

$\boxed{x = -2}$

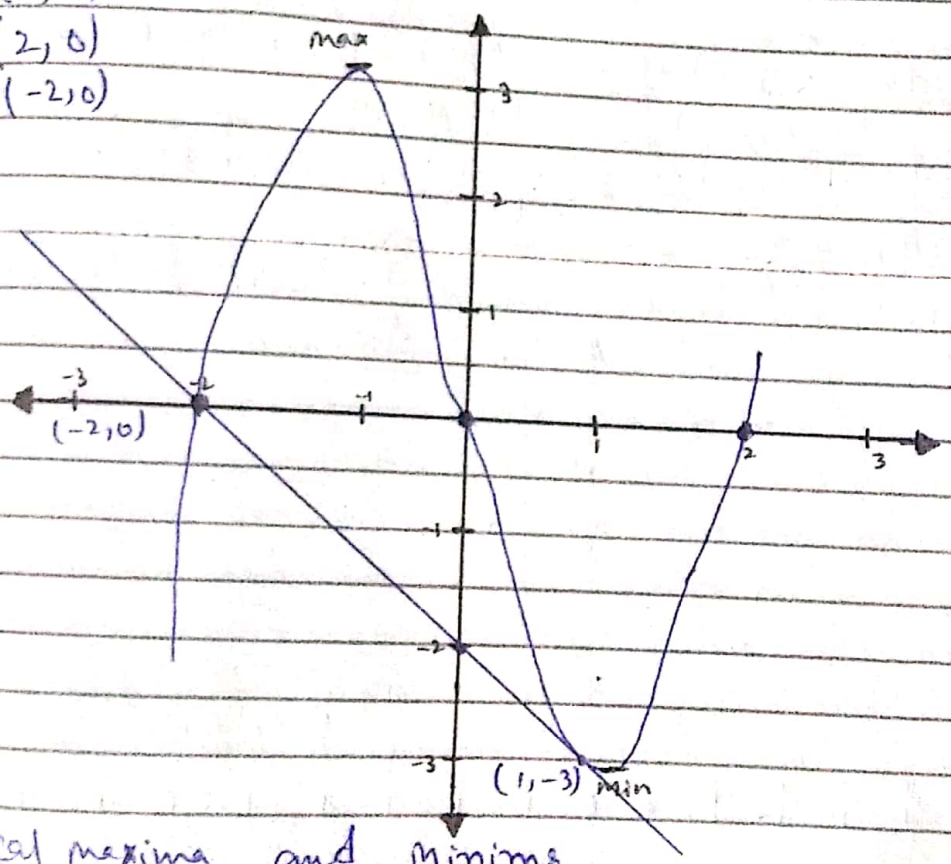
$(0, -2)$ and $(-2, 0)$

points of intersection are $(-2, 0)$ and $(1, -3)$

→ $(0, 0)$

$(2, 0)$

$(-2, 0)$



local maxima and minima

$y = x^3 - 4x$

$y = 3x^2 - 4$

$x = \pm\sqrt{4/3}$

$x = 2/\sqrt{3}, -2/\sqrt{3}$

$x = 1.15, -1.15$

$x = 1.15$	$x = -1.15$
$y = (1.15)^3 - 4(1.15)$	$y = (-1.15)^3 - 4(-1.15)$
$y = 1.52 - 4.60$	$y = -1.52 + 4.60$
$y = -3.08$	$y = 3.08$
$\boxed{(1.15, -3.08)}$	$\boxed{(-1.15, 3.08)}$
Minima	Maxima

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② Rational Function (and its Assymptots)

▶ $y = \frac{x+1}{x-1}$

:- we know that Here x must not be equal to 1. So, $\boxed{x=1}$ vertical Asymptot

1st step find inverse of y

$y = \frac{x+1}{x-1}$, Replace x with y. NO!

$(x-1)y = x+1$, just Solve for x

$xy - y = x + 1$

$xy - x = y + 1$

$x(y-1) = y+1$

$x = \frac{y+1}{y-1}$, where $y \neq 1$, Horizontal asymptote

• x-intercepts • y-intercepts • Asymptots for sketching.

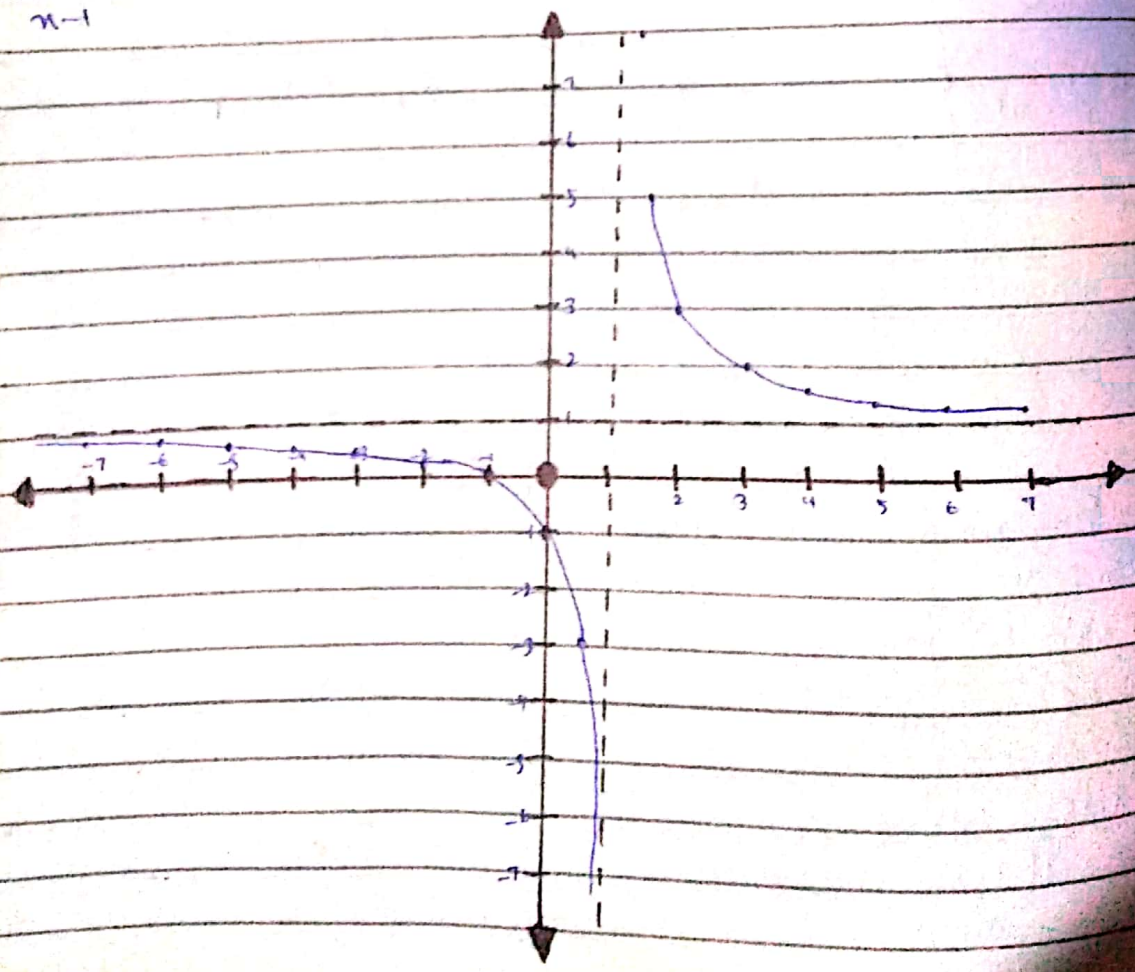
x	-3	-2	-1	0	1	2	3
y	1/2	1/3	0	-1	inf	3	2

y-intercept, $x=0$

$y = \frac{0+1}{0-1} = \frac{1}{-1} = -1 \rightarrow \boxed{(0,-1)}$

x-intercept(s), $y=0$

$\frac{x+1}{x-1} = 0 \rightarrow x+1 = 0(x-1) \rightarrow x+1=0 \rightarrow x=-1 \rightarrow \boxed{(-1,0)}$



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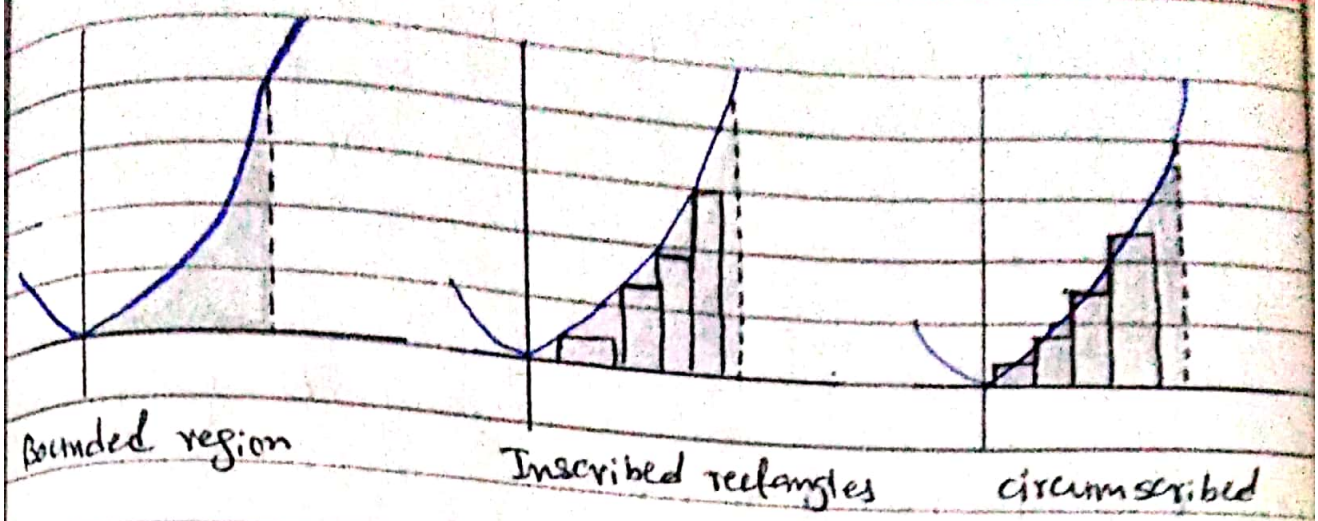
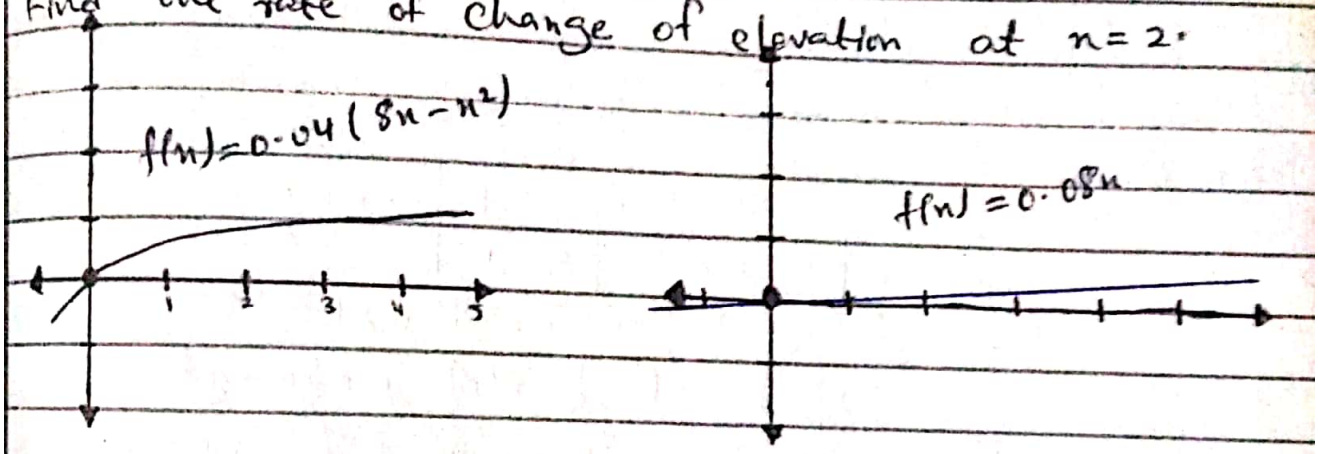
③ Bounded region, Inscribed and circumscribed rectangles.



3) A bicyclist is riding on a path modeled by the function $f(x) = 0.04(8x - x^2)$, where x and $f(x)$ are measured in miles.

Find the rate of change of elevation at $x = 2$.



Solution

we have to find the rate of change of elevation at $x=2$

$f(x) = 0.04(8x - x^2)$

$\boxed{x = 2}$

$f(2) = 0.04(8(2) - (2))$

$f(2) = 0.04(16 - 4)$

$f(2) = 0.04(12)$

$\boxed{f(2) = 0.48}$

$f(x) = 0.08x$

$f(2) = 0.08(2)$

$\boxed{f(2) = 0.16}$

$f(2) = 0.16$

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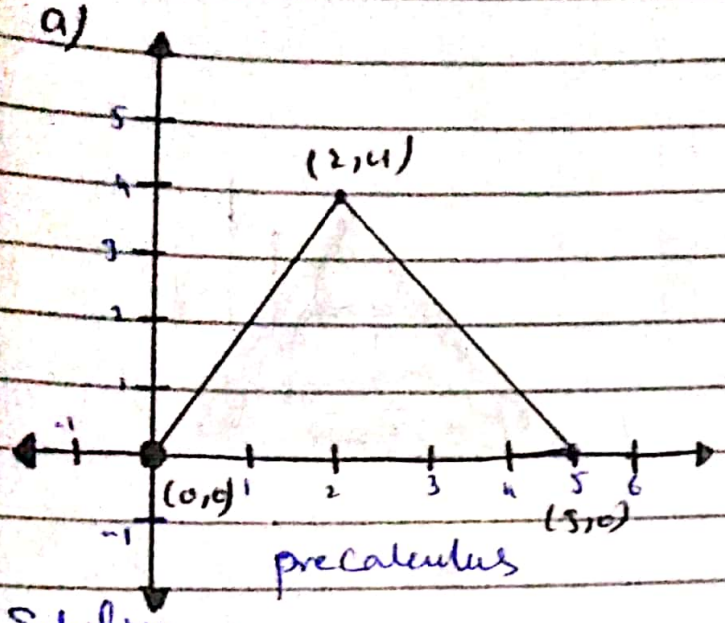
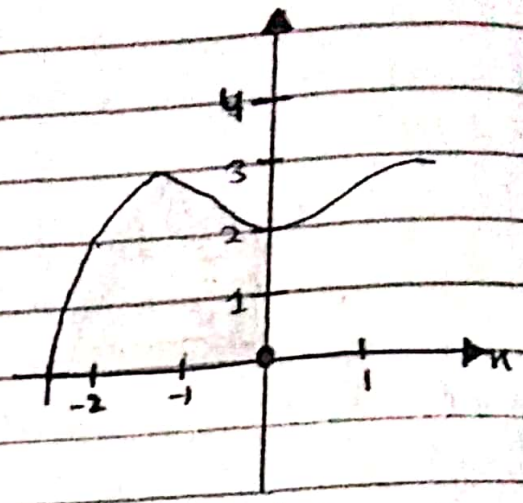
5) Find the Area of the shaded region.

a)



b)



Solution

via 3x3 Matrix Format

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \underset{x_1\ y_1}{(2,4)},\ \underset{x_2\ y_2}{(0,0)},\ \underset{x_3\ y_3}{(5,0)}$$

$$A = \frac{1}{2}\begin{vmatrix} 2 & 4 & 1 \\ 0 & 0 & 1 \\ 5 & 0 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} - 4\begin{vmatrix} 0 & 1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 0 & 0 \\ 5 & 0 \end{vmatrix}$$

$$\frac{1}{2}\left|2(0-0) - 4(0-5) + 2(0-0)\right|$$

$$\frac{1}{2}\left|(0) - 4(-5) + 2(0) = 0 + 20 + 0\right|$$

$$\frac{1}{2}(20) = 10 \rightarrow \text{Area of Triangle}$$

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6) Secant lines

function $f(x) = \sqrt{x}$ and the point $P(4,2)$

(a) Graph f and the secant lines passing through $P(4,2)$ and $Q(x, f(x))$ for x-values of 1, 3 and 5.

(b) Find the slope of each secant line

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(4,2)$. Describe how to improve your approximation of slope.

Solution

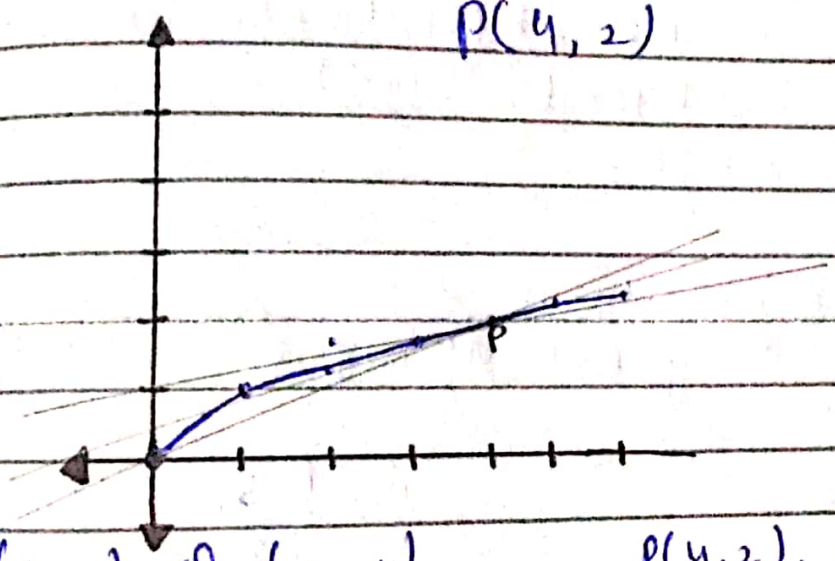
(a) $f(x) = \sqrt{x}$, $Q(x, f(x))$ x-values 1, 3 and 5

$P(4, 2)$

$Q_1(1, 1)$

$Q_2(3, 1.73)$

$Q_3(5, 2.23)$



(b) $P(4,2)$, $Q_1(1,1)$

$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-2}{1-4} = \frac{-1}{-3} = 1/3$

$P(4,2)$, $Q_2(3, 1.73)$

$\frac{\Delta y}{\Delta x} = \frac{1.73 - 2}{3 - 4} = \frac{-0.27}{-1} = \boxed{0.27}$

$P(4,2)$, $Q_3(5, 2.23)$

$\frac{\Delta y}{\Delta x} = \frac{2.23 - 2}{5 - 4} = \frac{0.23}{1} = \boxed{0.23}$

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7) Secant lines $f(x) = 6x - x^2$ and point $P(2,8)$

(a) Graph f and the secant lines passing through $P(2,8)$ and $Q(x, f(x))$ for x-values of 3, 2.5, and 1.5.

(b) Find the slope of each secant line.

(c) Use the results of part (b) to estimate the slope of the tangent line to the graph of f at $P(2,8)$. Describe how to improve your approximation of the slope.

Solution

Curve $\rightarrow 6x - x^2 = f(x) \rightarrow y = 6x - x^2$

Sketch the curve $y = -x^2 + 6x$, $a < 0$, max, $\cap$

x	-3	-2	-1	0	1	2	3
y	-27	-10	-7	0	5	8	9

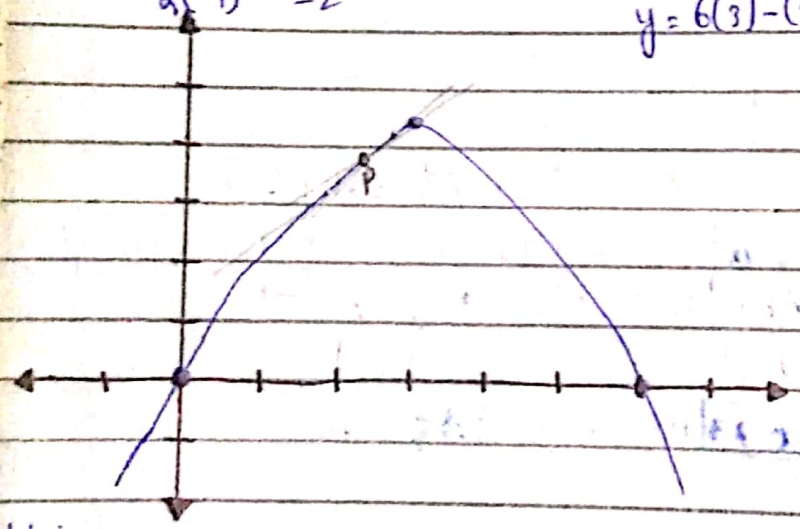
x-intercept(s), $(0,0)$, $(6,0)$

y-intercept $(0,0)$

vertex $\rightarrow x = \frac{-b}{2a}$, $a = -1$, $b = 6$

$x = \frac{-6}{2(-1)} = \frac{-6}{-2} = 3$, $\boxed{x=3}$ $y = 6x - x^2$

$y = 6(3) - (3)^2 = 18 - 9 = 9$ $(3,9)$



$Q_1(3,9)$

$Q_2(2.5, 8.75)$

$Q_3(1.5, 6.75)$

$P(2,8)$

(b)

$P(2,8)$, $Q_1(3,9)$ $\quad$ $P(2,8)$, $Q_2(2.5, 8.75)$

$\frac{\Delta y}{\Delta x} = \frac{9-8}{3-2} = \frac{1}{1} = 1$ $\quad$ $\frac{\Delta y}{\Delta x} = \frac{8.75-8}{2.5-2.0} = \frac{0.75}{0.5} = \boxed{1.5}$

$P(2,8)$, $Q_3(1.5, 6.75)$

$\frac{\Delta y}{\Delta x} = \frac{6.75-8}{1.5-2} = \frac{-1.25}{-0.5} = \boxed{2.5}$

(c) Use points

1.2 finding Limits Graphically and Numerically

▸ Complete table and sketch the graph.

1) $\lim_{x \to 4} \frac{x-4}{x^2-3x-4}$

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.2041	0.2004	0.200	0.200	0.1996	0.1961

$\therefore \frac{x-4}{(x^2-3x-4)} = \frac{x-4}{x^2-4x+x-4} = \frac{(x-4)}{x(x-4)+1(x-4)} = \frac{\cancel{(x-4)}}{(x+1)\cancel{(x-4)}} = \frac{1}{(x+1)}$

$y = \frac{1}{x+1}$, is the factorized function

$x = -1$ is the only vertical asymptote.

▸ x-intercept, $y=0$

$\frac{1}{x+1} = 0$, $1 = 0(x+1)$

No-concept of x-intercep

▸ y-intercept, $x=0$

$y = \frac{1}{x+1} = \frac{1}{0+1} = \frac{1}{1} = 1$

$(0, 1)$

▸ Intercepts

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-1/4	-1/3	-1/2	-1	∞	1	1/2	1/3	1/4	1/5	1/6

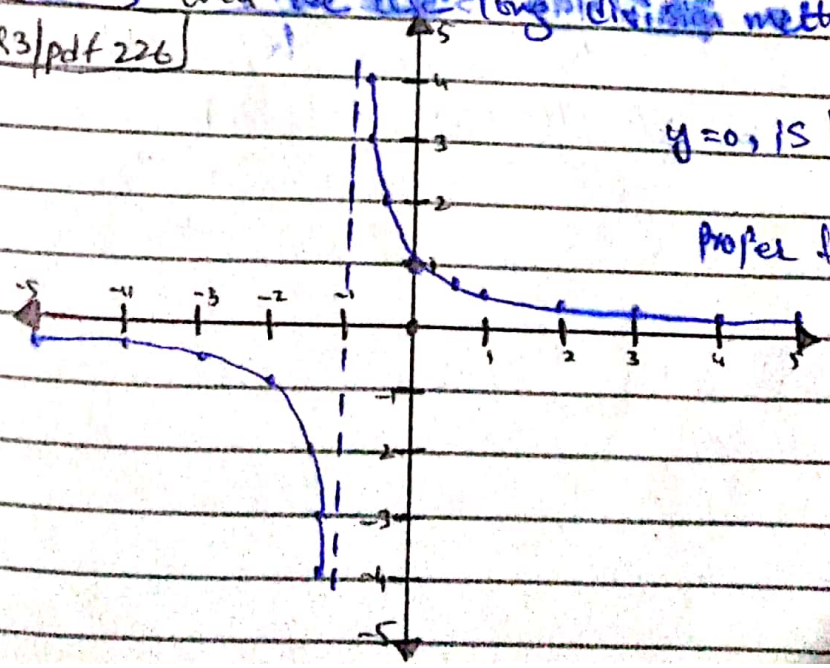
▸ Vertical asymptote : real zero of denominator.

▸ If a rational function $R(x)$ is proper, that is, if the degree of the numerator is less than the degree of the denominator.

▸ The line $y=0$ is a horizontal asymptote, if, and only if, a rational function is proper.

▸ improper : degree of numerator is > OR ≥ to the degree of denominator, then use long division method.

Sullivan 183/pdf 226



$y=0$, is horizontal asymptote

Proper function

▶ Evaluating a function (5)

$f(x) = 2x^2 + 3x - 1$

1) $f(a+h)$

$= 2(a+h)^2 + 3(a+h) - 1$

$= 2(a^2 + 2ah + h^2) + 3a + 3h - 1$

$= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1$

$= 2a^2 + 2h^2 + 4ah + 3h + 3a - 1$

2) $\dfrac{f(a+h) - f(a)}{h}$

first solve for $f(a+h)$, then for $f(a)$, at last combine the values.

$f(x) = 2x^2 + 3x - 1$ $\boxed{\text{1st step}}$

$f(a+h) = 2(a+h)^2 + 3(a+h) - 1$

$= 2(a^2 + 2ah + h^2) + 3a + 3h - 1$

$= 2a^2 + 4ah + 2h^2 + 3a + 3h - 1$

$f(a+h) = \boxed{2a^2 + 2h^2 + 4ah + 3h + 3a - 1}$

$\boxed{\text{Step 2}}$, find $f(a)$

$f(x) = 2x^2 + 3x - 1$

$f(a) = 2a^2 + 3a - 1$

$\boxed{\text{3rd Step}}$ → Combine the outputs of previous steps

$f(a+h) = 2a^2 + 2h^2 + 4ah + 3h + 3a - 1$

$f(a) = 2a^2 + 3a - 1$

$$\frac{f(a+h) - f(a)}{h} = \frac{(2a^2 + 2h^2 + 4ah + 3h + 3a - 1) - (2a^2 + 3a - 1)}{h}$$

$$= \frac{\cancel{2a^2} + 2h^2 + 4ah + 3h + \cancel{3a} - \cancel{1} - \cancel{2a^2} - \cancel{3a} + \cancel{1}}{h}$$

$$= \frac{2h^2 + 4ah + 3h}{h} = \frac{\cancel{h}(2h + 4a + 3)}{\cancel{h}\ (1)}$$

$\boxed{= 2h + 4a + 3}$ → Answer

14.1 Sullivan

LIMIT (Main chal be) (6)

Make a table to find the indicated limit.

7) $\lim_{x \to 2} (4x^3)$:-

x	1.9	1.99	1.999	2	2.001	2.01	2.1
y	27.43	31.522	31.952	32	32.0480	32.482	37.04

8) $\lim_{x \to 3} (2x^2+1)$:-

x	2.9	2.99	2.999	3	3.001	3.01	3.1
y	17.8200	18.8802	18.9880	19	19.0120	19.1202	20.2200

9) $\lim_{x \to 0} \frac{x+1}{x^2+1}$:-

x	-0.9	-0.99	-0.999	0	0.001	0.01	0.1
y	0.0552	0.5025	0.5000	1	0.5000	1.0099	1.0891

10) $\lim_{x \to 0} \frac{2-x}{x^2+4}$:-

x	-0.9	-0.99	-0.999	0	0.001	0.01	0.1	0.002
y			0.600	1/2	0.4988	0.4975	0.4738	0.4995

11) $\lim_{x \to 4} \frac{x^2-4x}{x-4}$:-

x	3.9	3.99	3.999	4	4.001	4.01	4.1
y	3.9	3.9900	4.000	4	4.000	4.0400	4.1000

:- $\frac{x^2-4x}{x-4} = \frac{x\cancel{(x-4)}}{\cancel{(x-4)}} = \boxed{x}$ $x \to 4$, $x = 4$

$\hookrightarrow 4$

$\boxed{\lim_{x \to 4} f(x) = 4}$

12) $\lim_{x \to 3} \frac{x^2-9}{x^2-3x}$:-

x	2.9	2.99	2.999	3	3.001	3.01	3.1
y	2.034	2.0033	2.0003	2	1.9997	1.9967	1.9677

:- $\frac{x^2-9}{x^2-3x} = \frac{x^2-3^2}{x(x-3)} = \frac{\cancel{(x-3)}(x+3)}{x\cancel{(x-3)}} = \frac{x+3}{x} = \frac{x}{x} + \frac{3}{x} = \boxed{1 + \frac{3}{x}}$

13) $\lim_{x \to 0} (e^x + 1)$:-

x	-0.9	-0.99	-0.999	0	0.001	0.01	0.1
y	1.4066	1.3716	1.3682	2	2.0010	2.0101	2.1052

$e^x + 1$, $x = 0 \rightarrow e^0 + 1 = 1 + 1 = 2$

14) $\lim_{x \to 0} \frac{e^x - e^{-x}}{2}$

x	-0.9	-0.99	-0.999	0	0.001	0.01	0.1
y	-2.053	-2.3196	-2.3474	0	0.0020	0.0200	0.2003
Hence →	-1.0265	-1.1598	-1.1737	0	0.0010	0.0100	0.1002

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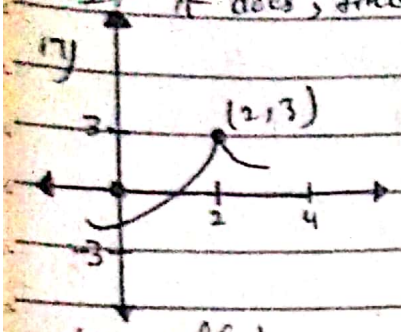
▶ Use the graph shown to determine if the limit exists (?)
If it does, find its value.

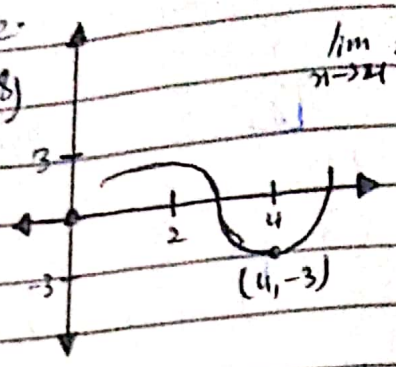
17)



$\lim_{x \to 2} f(x)$

:- $\lim_{x \to 2} f(x) = 3$

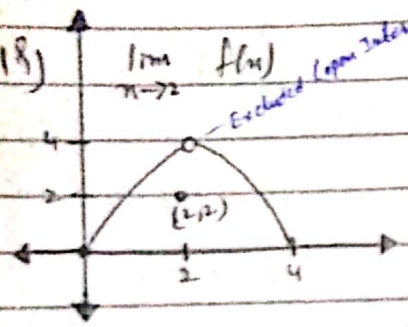
18)



$\lim_{x \to 4} f(x)$

$\lim_{x \to 4} f(x) = -3$

19)

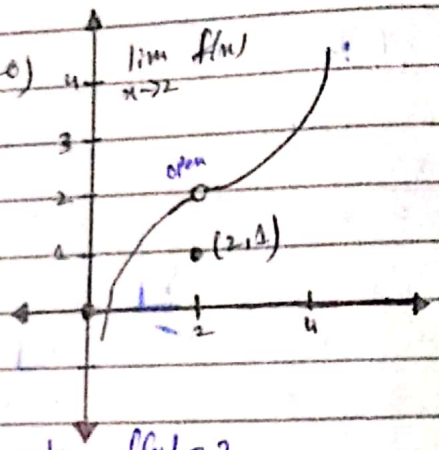


$\lim_{x \to 2} f(x)$

:- $\lim_{x \to 2} f(x) = 4$

focus on vertex, 1 on point

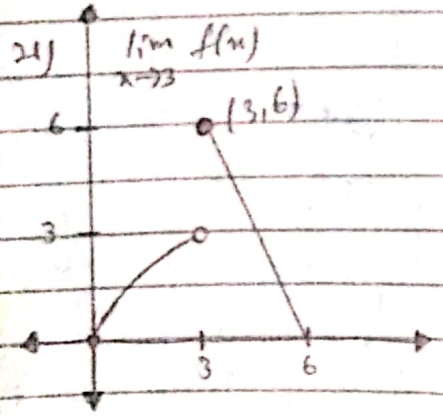
20)



$\lim_{x \to 2} f(x)$

:- $\lim_{x \to 2} f(x) = 2$

21)

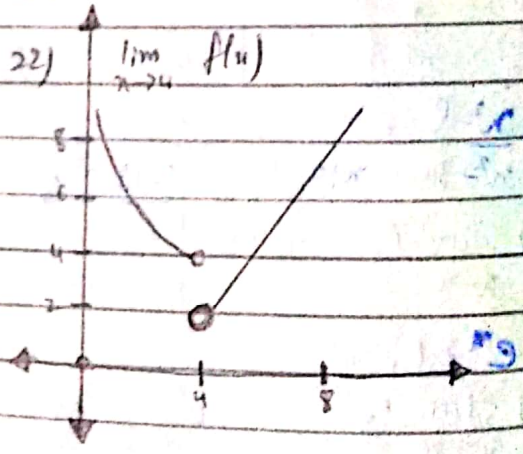


$\lim_{x \to 3} f(x)$

:- $\lim_{x \to 3} f(x) = 6$ WRONG

lim doesn't exist, because at the very point it is exclude and exclude, not possible.

22)



$\lim_{x \to 4} f(x)$

:- limit doesn't exist because at the same point(s) it is excluded, impossible.

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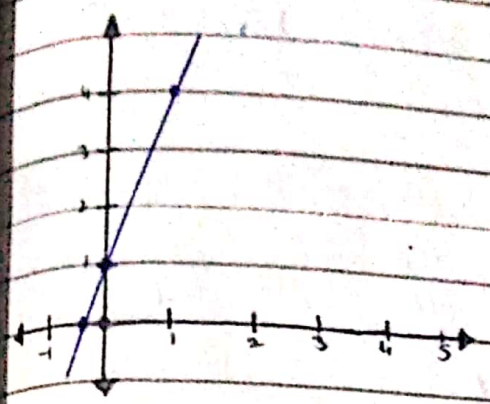
▸ Graph each function. Use graph to find the indicated limit, if it exists.

23) $\lim_{x \to 4} f(x)$, $f(x) = 3x+1$

:- $y = 3x+1$, $(0,1)$, $(-1/3, 0)$

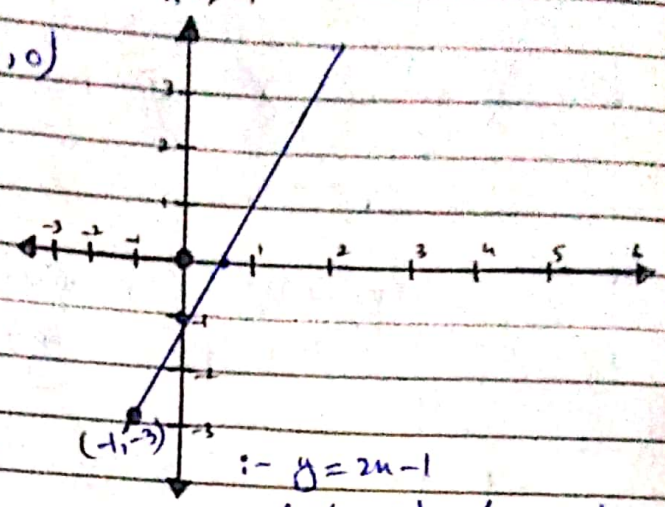


$\lim_{x \to 4} f(x) = 13$

:- $y = 3x+1$, $x = 4$

$y = 3(4)+1 = 12+1 = 13$

24) $\lim_{x \to -1} f(x)$, $f(x) = 2x-1$



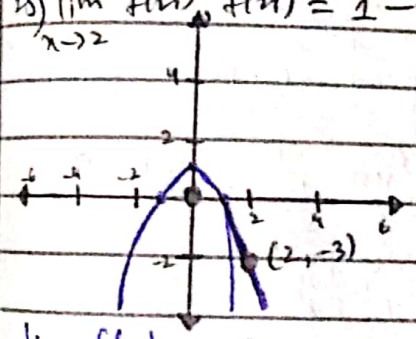
:- $y = 2x-1$

$(1/2, 0)$, $(0,-1)$

$x = -1$, $y = 2(-1)-1 = -2-1 = -3$

$\boxed{\lim_{x \to -1} f(x) = -3}$

25) $\lim_{x \to 2} f(x)$, $f(x) = 1 - x^2$

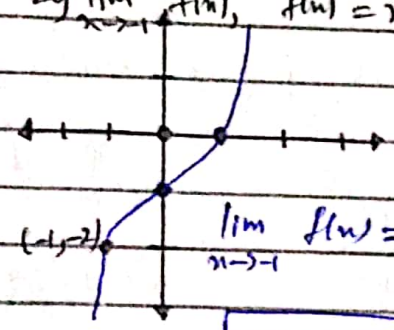


$\lim_{x \to 2} f(x) = -3$

26) $\lim_{x \to -1} f(x)$, $f(x) = x^3 - 1$

:- $x^3 - 1$

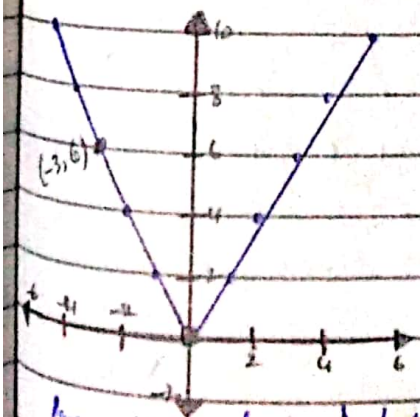
one Real zero $\boxed{1}$



$\lim_{x \to -1} f(x) = -2$

$\boxed{x \to -1 = -2}$

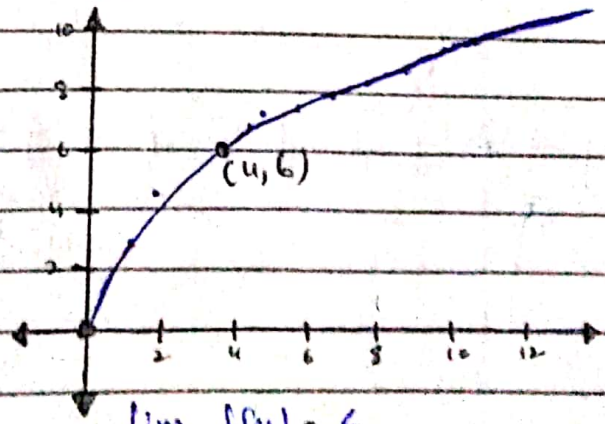
27) $\lim_{x \to -3} f(x)$, $f(x) = |2x|$



$\lim_{x \to -3} f(x) = |2(-3)| = |-6|$

$\boxed{L = 6}$

28) $\lim_{x \to 4} f(x)$, $f(x) = 3\sqrt{x}$



$\lim_{x \to 4} f(x) = 6$

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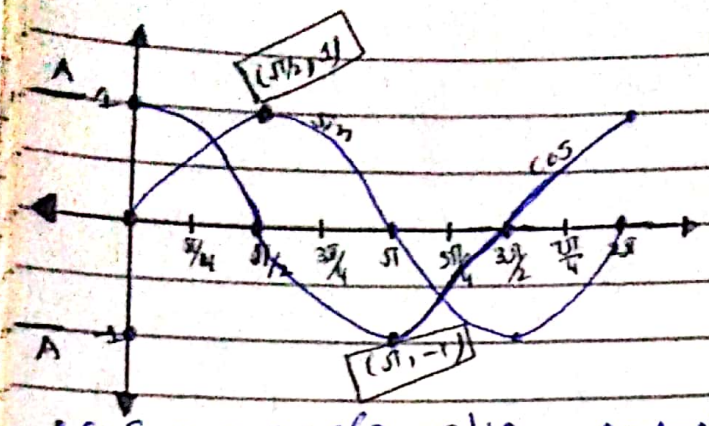
29) $\lim_{x \to \pi/2} f(x)$, $f(x) = \sin x$

30) $\lim_{x \to \pi} f(x)$, $f(x) = \cos x$



: $\sin x \equiv A \sin(Bx + C) + D$

$A = 1$, $B = 1$, $C = D = 0$

Period $\frac{2\pi}{B} = \frac{2\pi}{1} = \boxed{2\pi}$

Amplitude 1, $1 > 0$, ∼

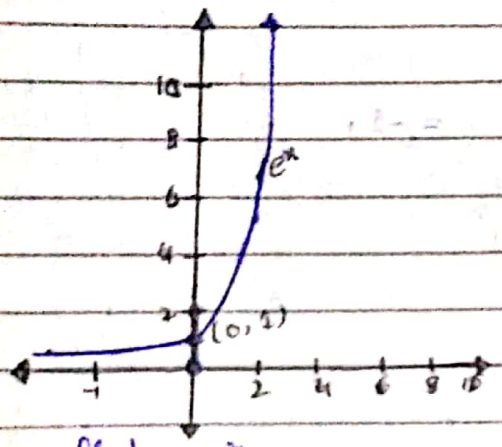
$\frac{\pi}{2} \times \frac{180^\circ}{\pi} = 90^\circ \to \sin 90^\circ = 1$

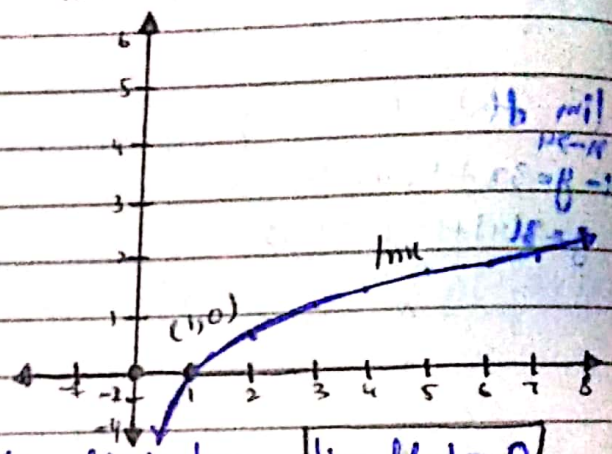
: $\cos x \equiv A\cos(Bx + C) + D$, A = Amplitude, B = period, C = Horiz, D = vert

:- $A = 1$, $B = 1$, $C = D = 0$, $A > 0$, Top to down ∪

31) $\lim_{x \to 0} f(x)$, $f(x) = e^x$

32) $\lim_{x \to 1} f(x)$, $f(x) = \ln x$





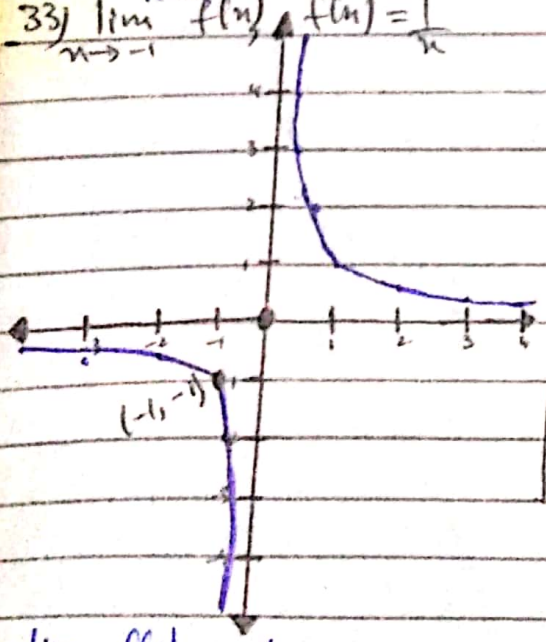
$\lim_{x \to 0} f(x) = e^x$

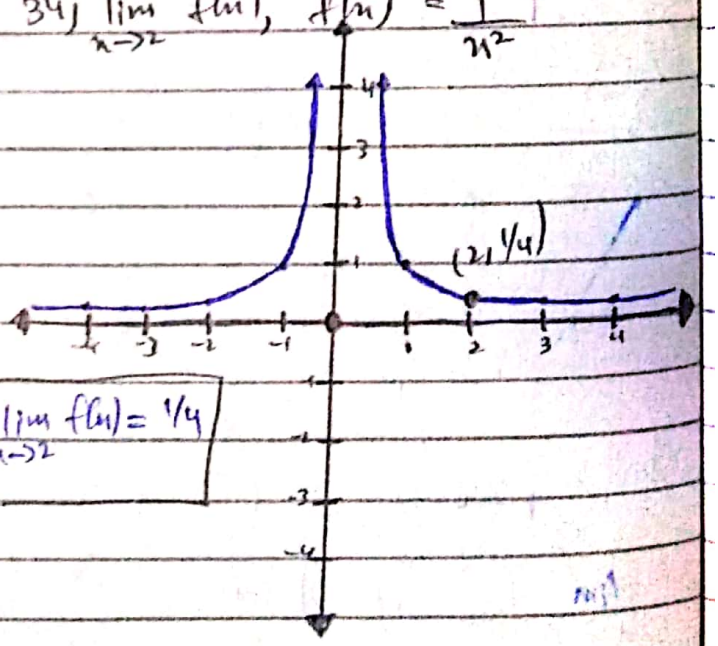
$\boxed{\lim_{x \to 0} f(x) = 1}$

$\lim_{x \to 1} f(x) = \ln x$, $\boxed{\lim_{x \to 1} f(x) = 0}$

33) $\lim_{x \to -1} f(x)$, $f(x) = \frac{1}{x}$

34) $\lim_{x \to 2} f(x)$, $f(x) = \frac{1}{x^2}$





$\boxed{\lim_{x \to 2} f(x) = 1/4}$

$\lim_{x \to -1} f(x) = -1$

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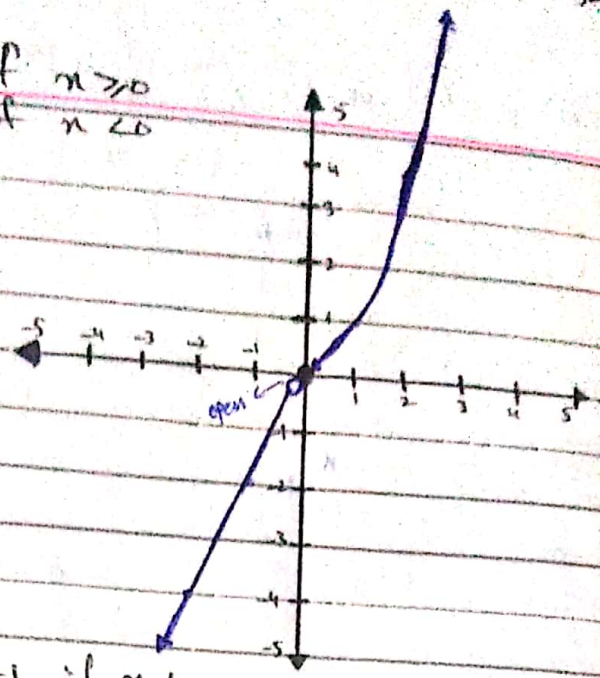
35) $\lim_{x\to 0} f(x)$, $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 3x & \text{if } x < 0 \end{cases}$

$\therefore y = x^2, \; x \geq 0$

$y = 3x, \; x < 0$

x^2	0	1	2	3	4	5
y	0	1	4	9	16	20

$3x$	1	-2	-3	-4	-5
y	-2	-4	-6	-8	-10



$\boxed{\lim_{x\to 0} f(x) = 0}$

36) $\lim_{x\to 0} f(x)$, $f(x) = \begin{cases} x-1 & \text{if } x < 0 \\ 3x-1 & \text{if } x \geq 0 \end{cases}$

$\therefore y = x-1, \; x < 0$

$y = 3x-1, \; x \geq 0$

$\Longrightarrow$ $y_{x\to 0} = x-1 = 0-1 = -1$, $\boxed{y_{x\to 0} = -1}$

$y_{x\to 0} = 3x-1, \; x=0 \rightarrow 3(0)-1 = -1$, $\boxed{y = -1}$

$\boxed{\lim_{x\to 0} f(x) = -1}$

37) $\lim_{x\to 1} f(x)$, $f(x) = \begin{cases} 3x & \text{if } x \leq 1 \\ x+1 & \text{if } x > 1 \end{cases}$

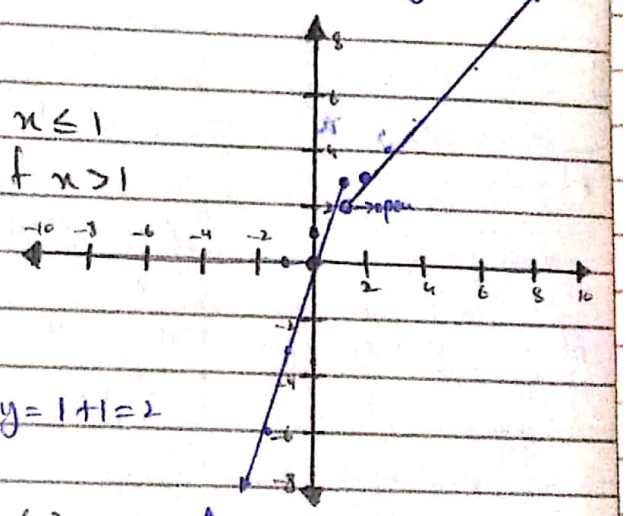
$y = 3x, \; x \leq 1$

$y = x+1, \; x > 1$

$\lim_{x\to 1} f(x) =$ doesn't exist

$x=1$, $y = 3x = 3(1) = 3$, $y = x+1$, $y = 1+1 = 2$

lim Not exist $\boxed{3 \neq 2}$



38) $\lim_{x\to 2} f(x)$, $f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ 2x-1 & \text{if } x > 2 \end{cases}$

$y = x^2, \; x \leq 2$

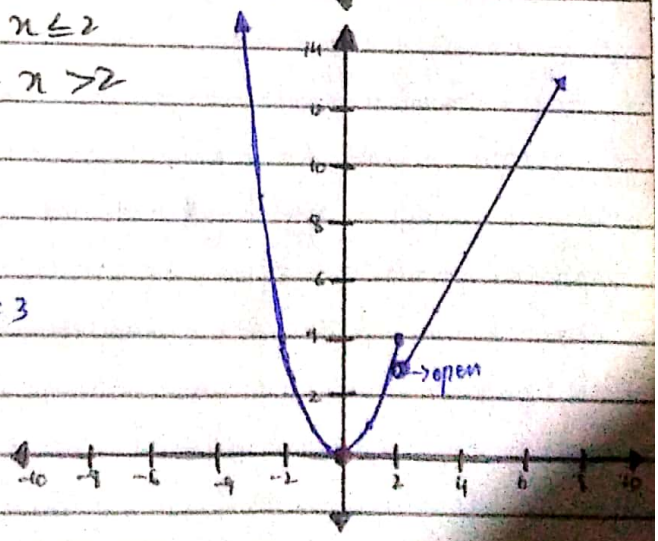
$y = 2x-1, \; x > 2$

$x=2$, $y = x^2 = (2)^2 = 4$

$y = 2x-1 = 2(2)-1 = 4-1 = 3$

$\boxed{4 \neq 3}$

$\boxed{\text{limit doesn't exist}}$



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39) $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} x & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ 3x & \text{if } x > 0 \end{cases}$

:-

① $y = x$, $x < 0$ → 0 is not less than 0

② $y = 1$, $x = 0$ → 0 is equal to 0

③ $y = 3x$, $x > 0$ → 0 is not greater than 0

② $y = 1 + 0x$, $x = 0$

$y = 1 + 0(0)$

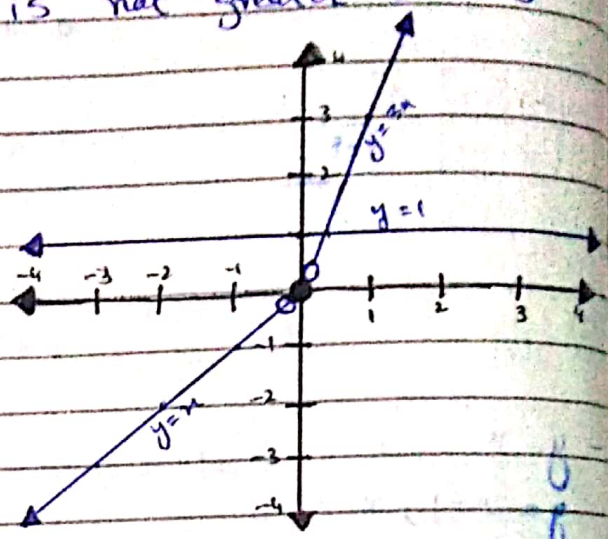
$y = 1 + 0$

$\boxed{y = 1}$

$y = x$

$\boxed{y = 0}$

$\boxed{\lim_{x \to 0} f(x) = 0}$



40) $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x > 0 \end{cases}$

:- $y = x$, $x < 0$ → $y = 1$ → $y = 1 + 0x$, $x < 0$

$y = x$, $x > 0$ → $y = -1$ → $y = -1 + 0x$, $x > 0$

- $y = 1 + 0x$, $x = 0$

  $y = 1 + 0(0)$

  $\boxed{y = 1}$

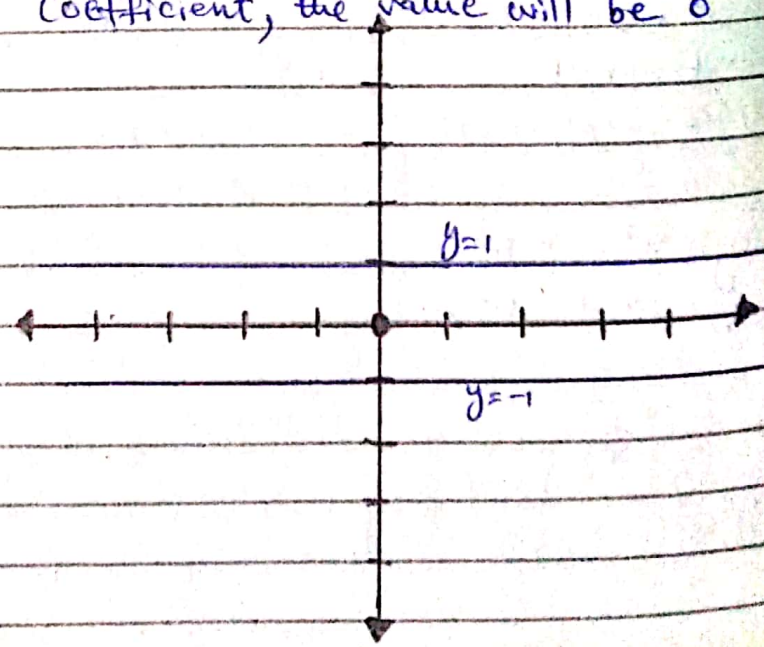
- $y = -1 + 0x$, $x = 0$

  $y = -1 + 0(0)$

  $\boxed{y = -1}$

x approaches 0, the coefficient of x is zero, if we put any real number then the output will be 0, because of coefficient, the value will be 0

$\boxed{\lim_{x \to 0} f(x) = 0}$



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1) $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} \sin x & \text{if } x \le 0 \\ x^2 & \text{if } x > 0 \end{cases}$

$y = \sin x$, $x \le 0$

$y = x^2$, $x > 0$

$y = \sin x$, $x = 0$

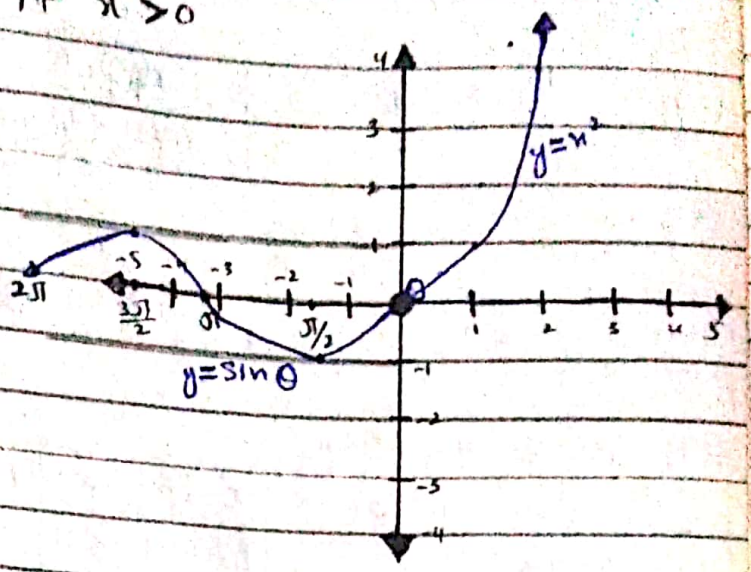
$y = \sin(0)$

$\boxed{y = 0}$

$y = x^2$, $x = 0$

$\boxed{y = 0}$

$\boxed{\lim_{x \to 0} f(x) = 0}$



2) $\lim_{x \to 0} f(x)$, $f(x) = \begin{cases} e^x & \text{if } x > 0 \\ 1 - x & \text{if } \le 0 \end{cases}$

:- $y = e^x$, $x > 0$

$y = 1 - x$, $x \le 0$

$y = e^x$, $x = 0$

$y = e^0 = 1$

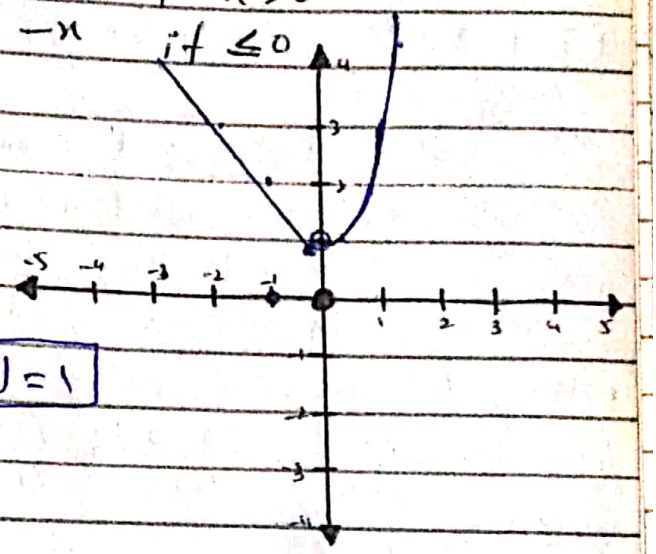
$\boxed{y = 1}$

$y = 1 - x$, $x = 0$

$y = 1 - 0 = 1$

$\boxed{y = 1}$

$\boxed{\lim_{x \to 0} f(x) = 1}$



3) $\lim_{x \to 1} \dfrac{x^3 - x^2 + x - 1}{x^4 - x^3 + 2x - 2}$

:- pleas don't bear problems for yaus self, you should use a Calculator with a set precision

x	+0.99	+0.9	1	1.1	1.01	1.001
y	0.6667	0.6632	∞	0.663	0.6667	0.6667

$\boxed{\lim_{x \to 1} f(x) = 0.6667}$

4) $\lim_{x \to -1} \dfrac{x^3 + x^2 + 3x + 3}{x^4 + x^3 + 2x + 2}$

x	-1.1	-1.01	-1.001	-1	-0.999	-0.99	-0.9
y	6.283	4.143	4	∞	4.00	3.8646	2.8876

$\boxed{\lim_{x \to -1} f(x) = 4}$

45) $\lim_{x \to 2} \frac{x^3 - 2x^2 + 4x - 8}{x^2 + x - 6}$

$\therefore \lim_{x \to 2} f(x) = 1.6$

x	1.90	1.99	1.999	2	2.001	2.01	2.1
y	1.5531	1.5852	1.6	∞	1.6	1.6048	1.6498

46) $\lim_{x \to 1} \frac{x^3 - x^2 + 3x - 3}{x^2 + 3x - 4}$

$\boxed{\lim_{x \to 1} f(x) = 0.8}$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
y	0.7776	0.7976	0.8	∞	0.8	0.8024	0.8255

47) $\lim_{x \to -1} \frac{x^3 + 2x^2 + x}{x^4 + x^3 + 2x + 2}$

$\boxed{\lim_{x \to -1} f(x) = 0}$

x	-0.9	-0.99	-0.999	-1	-1.001	-1.01	-1.1
y	-0.0708	-0.0097	0	∞	0	0.0103	-0.0824

(8) ▸ Finding the limit of Quotient

a) $\lim_{x \to 3} \frac{x^2 - x - 6}{x^2 - 9}$: Notice, that expression can be factorised.

So, why are you wasting time, factorise it.

$\frac{x^2 - x - 6}{x^2 - 9} = \frac{(x+2)\cancel{(x-3)}}{\cancel{(x-3)}(x+3)} = \frac{(x+2)}{(x+3)} \rightarrow \lim_{x \to 3} f(x) = \frac{(x+2)}{(x+3)}$

• Now put the value of $x = 3$

$\lim_{x \to 3} f(x) = \frac{(x+2)}{(x+3)} = \frac{(3+2)}{(3+3} = \frac{(5)}{(6)} = \boxed{\frac{5}{6}}$

$\boxed{\lim_{x \to 3} f(x) = 5/6}$

b) $\lim_{x \to 0} \frac{5x - \sin x}{x}$

: $\frac{5x - \sin x}{x} = \frac{5\cancel{x}}{\cancel{x}} - \frac{\sin \cancel{x}}{\cancel{x}} = (5 - \sin)$, $\boxed{\sin = 1}$

$5 - 1 = 4$

OR $\frac{5x - \sin x}{x} = \frac{\cancel{x}(5 - \sin)}{\cancel{x}} = 5 - \sin = 5 - 1 = 4$

$\boxed{\lim_{x \to 0} f(x) = 4}$

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Finding the limit of an Average Rate of change

Find the limit as x approaches 2 of the average rate of change of the function, $f(x) = x^2+3x$ from 2 to x.

Solution

$f(x) = x^2 + 3x$, Average rate of change from 2 to x. (1)

Formula for Average rate $\rightarrow \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$

Here, $f(b) = f(x) = x^3 + 3x$ and

$f(a) = f(\text{const}) = 2$

let's solve it $\frac{\Delta y}{\Delta x} = \frac{f(x) - f(\text{const})}{x - \text{const}} = \frac{(x^2+3x) - (10)}{x - 2}$

$\frac{(x^2+3x) - 2}{x - 2} \rightarrow$ limit of the average rate of change (2)

$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \frac{(x^2+3x) - 10}{x - 2} = \frac{(x+5)(x-2)}{(x-2)} = x+5$

$\lim_{x \to 2} (x+5) = (2+5) = 7$

$\boxed{\lim_{x \to 2} f(x) = 7}$

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8) Find each limit algebraically.

7) $\lim_{x \to 1} 5$

$= \lim_{x \to 1} f(c) = 5 + 0x$

$\boxed{\lim_{x \to 1} f(x) = 5}$

8) $\lim_{x \to 1} (-3)$

$= \lim_{x \to 1} f(c) = -3 + 0x$

$\boxed{= \lim_{x \to 1} f(c) = -3}$

9) $\lim_{x \to 4} x$

$\lim_{x \to 4} f(x) = x$

$\boxed{\lim_{x \to 4} f(x) = 4}$

10) $\lim_{x \to -3} x$

$= \lim_{x \to -3} f(x) = x$

$= \boxed{\lim_{x \to -3} f(x) = -3}$

11) $\lim_{x \to -2} (5x)$

$= f(x) = 5x$

$f(-2) = -10$

$\boxed{\lim_{x \to -2} f(x) = -10}$

12) $\lim_{x \to 4} (-3x)$

$= f(x) = -3x$

$f(4) = -12$

$\boxed{\lim_{x \to 4} f(x) = -12}$

13) $\lim_{x \to 2} (5x^4)$

$= f(x) = 5x^4$

$f(2) = 80$

$\boxed{\lim_{x \to 2} f(x) = 80}$

14) $\lim_{x \to -3} (2x^3)$

$f(x) = 2x^3$

$f(-3) = -54$

$\boxed{\lim_{x \to -3} f(x) = -54}$

15) $\lim_{x \to 2} (3x + 2)$

$f(x) = 3x + 2$

$f(2) = 8$

$\boxed{\lim_{x \to 2} f(x) = 8}$

16) $\lim_{x \to 3} (2 - 5x)$

$f(x) = 2 - 5x$

$f(3) = -13$

$\boxed{\lim_{x \to 3} f(x) = -13}$

17) $\lim_{x \to -1} (3x^2 - 5x)$

$f(x) = 3x^2 - 5x$

$f(-1) = 8$

$\boxed{\lim_{x \to -1} f(x) = 8}$

18) $\lim_{x \to 2} (8x^2 - 4)$

$f(x) = 8x^2 - 4$

$f(2) = 28$

$\boxed{\lim_{x \to 2} f(x) = 28}$

19) $\lim_{x \to 1} (5x^4 - 3x^2 + 6x - 8)$

$f(x) = 5x^4 - 3x^2 + 6x - 8$

$f(1) = 5(1) - 3(1) + 6(1) - 8$

$f(1) = 5 - 3 + 6 - 8$

$f(1) = -1$

$\boxed{\lim_{x \to 1} f(x) = -1}$

20) $\lim_{x \to -1} (8x^5 - 7x^3 + 8x^2 + x - 4)$

$f(x) = 8x^5 - 7x^3 + 8x^2 + x - 4$

$f(-1) = 8(-1)^5 - 7(-1)^3 + 8(-1)^2 + (-1) - 4$

$f(-1) = -8 + 7 + 8 - 5$

$f(-1) = 2$

$\boxed{\lim_{x \to -1} f(x) = 2}$

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21) $\lim_{x \to 1} (x^2+1)^3$

$f(x) = (x^2+1)^3$

$f(1) = (1^2+1)^3$

$f(1) = 8$

$\boxed{\lim_{x \to 1} f(x) = 8}$

22) $\lim_{x \to 2} (3x-4)^2$

$f(x) = (3x-4)^2$

$f(2) = (3(2)-4)^2$

$f(2) = 4$

$\boxed{\lim_{x \to 2} f(x) = 4}$

23) $\lim_{x \to 1} \sqrt{5x+4}$

$f(x) = \sqrt{5x+4}$

$f(1) = \sqrt{5(1)+4}$

$f(1) = 3$

$\boxed{\lim_{x \to 1} f(x) = 3}$

24) $\lim_{x \to 0} \sqrt{1-2x}$

$f(x) = \sqrt{1-2x}$

$f(0) = \sqrt{1-2(0)}$

$f(0) = 1$

$\boxed{\lim_{x \to 0} f(x) = 1}$

25) $\lim_{x \to 0} \frac{x^2-4}{x^2+4}$

$f(x) = \frac{x^2-4}{x^2+4}$

$f(0) = -1$

$\boxed{\lim_{x \to 0} f(x) = -1}$

26) $\lim_{x \to 2} \frac{3x+4}{x^2+x}$

$f(x) = \frac{3x+4}{x^2+x}$

$f(2) = 5/3$

$\boxed{\lim_{x \to 2} f(x) = 5/3}$

27) $\lim_{x \to 2} (3x-2)^{5/2}$

$f(x) = (3x-2)^{5/2}$

$f(2) = (4)^{5/2}$

$f(2) = \sqrt{1024} = 32$

$\boxed{\lim_{x \to 2} f(x) = 32}$

28) $\lim_{x \to -1} (2x+1)^{5/3}$

$f(x) = (2x+1)^{5/3}$

$f(-1) = (-1)^{5/3}$

$f(-1) = -1$

$\boxed{\lim_{x \to -1} f(x) = -1}$

29) $\lim_{x \to 2} \frac{x^2-4}{x^2-2x}$

$f(x) = \frac{x^2-4}{x^2-2x}$

$f(2) = \infty$

first factorise, if & then put limit value

$\frac{x^2-4}{x^2-2x} = \frac{\cancel{(x-2)}(x+2)}{x\cancel{(x-2)}} = \frac{x+2}{x}, \; x=2 \rightarrow \frac{2+2}{2} = 2$

$\boxed{\lim_{x \to 2} f(x) = 2}$ → Limit of Quotient

30) $\lim_{x \to -1} \frac{x^2+x}{x^2-1} : \frac{x\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \frac{x}{(x-1)}, \; x=-1$

$x=-1, \; \frac{x}{x-1} \rightarrow \frac{-1}{-1-1} = \frac{-1}{-2} = 1/2$ $\boxed{\lim_{x \to -1} f(x) = 1/2}$

31) $\lim_{x \to -3} \frac{x^2-x-12}{x^2-9} : \frac{\cancel{(x+3)}(x-4)}{(x-3)\cancel{(x+3)}} = \frac{(x-4)}{(x-3)}, \; x=-3$

$f(-3) = \frac{-7}{-6} = 7/6$ $\boxed{\lim_{x \to -1} f(x) = 7/6}$

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32) $\lim_{x \to -3} \frac{x^2+x-6}{x^2+2x-3} : \frac{(x-2)\cancel{(x+3)}}{(x-1)\cancel{(x+3)}} = \frac{(x-2)}{(x-1)}, x=-3 \to f(-3) = 5/4$

$\boxed{\lim_{x \to -3} f(x) = 5/4}$

33) $\lim_{x \to 1} \frac{x^3-1}{x-1} : \frac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}} = x^2+x+1, x=1$

$f(x) = x^2+x+1, x=1 \longrightarrow f(1) = (1)^2+1+1 = 3$

$\boxed{\lim_{x \to 1} f(x) = 3}$

34) $\frac{x^4-1}{x-1} \lim_{x \to 1} : \frac{((x^2)^2-1)}{(x-1)} = \frac{(x^2-1)(x^2+1)}{(x-1)}$

$\boxed{a^2-b^2 = (a-b)(a+b) \text{ and } a^2+b^2 = (a+b)^2-2ab}$

$\frac{(x^2-1)(x^2+1)}{(x-1)} = \frac{\cancel{(x-1)}(x+1)((x+1)^2-2(x)(1))}{\cancel{(x-1)}}$

$(x+1)(x^2+\cancel{2x}+1-\cancel{2x}) = (x+1)(x^2+1), x=1$

$(1+1)(1^2+1) = (2)(2) = 4$

$\lim_{x \to 1} \frac{x^4-1}{x-1} \to \boxed{\lim_{x \to 1} f(x) = 4}$

35) $\lim_{x \to -1} \frac{(x+1)^2}{x^2-1} : \frac{(x+1)(x+1)}{(x^2-1)} = \frac{(x+1)\cancel{(x+1)}}{(x-1)\cancel{(x+1)}} = \frac{(x+1)}{(x-1)}, x=-1$

$x=-1, \frac{(x+1)}{(x-1)} = \frac{(-1+1)}{(-1-1)} = \frac{0}{-2} = 0$

$\boxed{\lim_{x \to -1} f(x) = 0}$

36) $\lim_{x \to 2} \frac{x^3-8}{x^2-4} : \frac{x^3-2^3}{x^2-2^2} = \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)} = \frac{(x^2+2x+4)}{(x+2)}$

$\frac{(x^2+2x+4)}{(x+2)} = \frac{(x^2+2x+4)}{(x+2)} \to \Delta < 0$, no real roots, $\boxed{x=2}$

$\frac{((2)^2+2(2)+4)}{(2+2)} = \frac{(4+4+4)}{4} = \frac{12}{4} = 3$

$\boxed{\lim_{x \to 2} f(x) = 3}$

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37) $\lim_{x\to 1} \frac{x^3-x^2+x-1}{x^4-x^3+2x-2}$: $\frac{x^3-x^2+x-1}{x^4-x^3+2x-2} = \frac{x^2(x-1)+1(x-1)}{x^3(x-1)+2(x-1)}$

$\frac{(x^2+1)\cancel{(x-1)}}{(x^3+2)\cancel{(x-1)}} = \frac{x^2+1}{x^3+2}$, $x=1 \rightarrow \frac{(1)^2+1}{(1)^3+2} = \frac{2}{3}$

$\boxed{\lim_{x\to 1} f(x) = 2/3}$

38) $\lim_{x\to -1} \frac{x^3+x^2+3x+3}{x^4+x^3+2x+2}$: $\frac{x^2(x+1)+3(x+1)}{x^3(x+1)+2(x+1)} = \frac{(x^2+3)\cancel{(x+1)}}{(x^3+2)\cancel{(x+1)}}$

$\frac{(x^2+3)}{(x^3+2)}$, $x=-1 \rightarrow \frac{(-1)^2+3}{(-1)^3+2} = \frac{1+3}{-1+2} = \frac{4}{1} = 4$

$\boxed{\lim_{x\to -1} f(x) = 4}$

39) $\lim_{x\to 2} \frac{x^3-2x^2+4x-8}{x^2+x-6}$: $\frac{x^2(x-2)+4(x-2)}{(x-2)(x+3)} = \frac{(x^2+4)\cancel{(x-2)}}{\cancel{(x-2)}(x+3)}$

$\frac{x^2+4}{x+3}$, $x=2 \rightarrow \frac{(2)^2+4}{2+3} = \frac{4+4}{5} = 8/5$

$\boxed{\lim_{x\to 2} f(x) = 8/5}$

40) $\lim_{x\to 1} \frac{x^3-x^2+3x-3}{x^2+3x-4}$: $\frac{x^2(x-1)+3(x-1)}{(x-1)(x+4)} = \frac{(x^2+3)\cancel{(x-1)}}{\cancel{(x-1)}(x+4)}$

$\frac{x^2+3}{x+4}$, $x=1 \rightarrow \frac{(1)^2+3}{1+4} = \frac{4}{5}$

$\boxed{\lim_{x\to 1} f(x) = 4/5}$

41) $\lim_{x\to -1} \frac{x^3+2x^2+x}{x^4+x^3+2x+2}$:

-1	1	2	1	-1	1	1	2	2
		-1	-1			-1	0	-2
	1	1	0		1	0	2	0

$x^2+x \rightarrow x(x+1)$

$x=0$, $x=-1$,

x^2+x $\qquad$ $x^3+0x^2+2x = x^3+2x$

3 Real roots, (2 identitical) 0, -1, and -1

$x^3+2x = x(x^2+2) \rightarrow x=0$, $x=\pm 2$, 4 Real roots, $x=0,-1,\pm 2$

$\frac{x^3+2x^2+x}{x^4+x^3+2x+2} = \frac{(x-0)\cancel{(x+1)}(x+1)}{\cancel{(x-0)}\cancel{(x+1)}(x+2)(x-2)} = \frac{(x+1)}{(x+2)(x-2)}$, $x=-1$

$\frac{x+1}{(x+2)(x-2)} = \frac{(-1+1)}{(-1+2)(-1-2)} = \frac{0}{(1)(-3)} = \frac{0}{-3} = 0 \rightarrow \boxed{\lim_{x\to -1} f(x) = 0}$

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42) $\lim_{x \to 3} \frac{x^3 - 3x^2 + 4x - 12}{x^4 - 3x^3 + x - 3}$: $\frac{x^2(x-3) + 4(x-3)}{x^3(x-3) + 1(x-3)} = \frac{(x^2+4)\cancel{(x-3)}}{(x^3+1)\cancel{(x-3)}} = \frac{x^2+4}{x^3+1}$

$\frac{x^2+4}{x^3+1}$, $x = 3 \rightarrow \frac{(3)^2+4}{(3)^3+1} = \frac{9+4}{27+1} = \frac{13}{28}$ $\boxed{\lim_{x \to 3} f(x) = \frac{13}{28}}$

Find the limit as x approaches c of the average rate of change of each function from c to x.

43) $c = 2$; $f(x) = 5x - 3$

:- $f(x) = 5x - 3$, $f(2) = 5(2) - 3 = 10 - 3 = 7$, $f(2) = 7$

The average rate of change from 2 to x (1)

$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(2)}{x - 2} = \frac{(5x-3) - 7}{x - 2} = \frac{5x - 10}{x - 2}$

Now make a table for $\lim_{x \to 2}$ $\boxed{f(x) = \frac{5x-10}{x-2}}$

x	1.8	1.9	2	2.1	2.2
y	5	5	∞	5	5

$\boxed{\lim_{x \to 2} f(x) = 5}$

44) $c = -2$; $f(x) = 4 - 3x$

:- $f(x) = 4 - 3x$, $f(-2) = 4 - 3(-2) = 10$, $f(-2) = 10$

limit of average rate of change $\frac{f(x) - f(c)}{x - c}$

$\frac{f(x) - f(c)}{x - c} = \frac{(4 - 3x) - 10}{x - (-2)} = \frac{4 - 3x - 10}{x + 2}$

$\frac{4 - 3x - 10}{x + 2} = \frac{-3x - 6}{x + 2} \rightarrow f(x) = \frac{-3x - 6}{x + 2}$ $\lim_{x \to -2}$, make table

x	-1.8	-1.9	-2	-2.1	-2.2
y	-3	-3	∞	-3	-3

$\boxed{\lim_{x \to -2} f(x) = -3}$

45) $c = 3$; $f(x) = x^2$

:- $f(x) = x^2$, $f(c) = f(3) = 9$, $\boxed{f(3) = 9}$

Limit of average rate of change $\frac{f(x) - f(c)}{x - c}$

$\frac{f(x) - f(c)}{x - c} = \frac{x^2 - 9}{x - 3} = \frac{(x+3)(x-3)}{(x-3)}$

$\frac{(x+3)\cancel{(x-3)}}{\cancel{(x-3)}} = (x+3)$, $\lim_{x \to 3} f(x) = x + 3$, $x = 3$

$f(x) = 6$

$\boxed{\lim_{x \to 3} f(x) = 6}$

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46) $c = 3$; $f(x) = x^3$

$f(x) = x^3$, $f(c) = f(3) = 27$, $\boxed{f(c) = 27}$

Limit of average rate of change

$$\frac{f(x) - f(c)}{x - c} = \frac{x^3 - 3}{x - 3} = \frac{(x^3 - 3)}{(x-3)} = \boxed{\frac{(x-3)(x^2+3x+9)}{(x-3)(1)}}$$

$$\frac{x^3 - 3}{x - 3}$$

47) $c = -1$; $f(x) = x^2 + 2x$

$\therefore f(x) = x^2 + 2x$, $f(c) = f(-1) = (-1)^2 + 2(-1) = 1 - 2 = -1$

$$\frac{f(x) - f(c)}{x - c} = \frac{(x^2 + 2x) - (-1)}{x - (-1)} = \frac{x^2 + 2x + 1}{x + 1} = \frac{(x+1)(x+1)}{(x+1)}$$

$$\frac{(x+1)\cancel{(x+1)}}{\cancel{(x+1)}} = (x+1), \quad x = -1 \quad \boxed{\lim_{x \to -1} f(x) = 0}$$

$-1 + 1 = 0$

48) $c = -1$; $f(x) = 2x^2 - 3x$

$f(x) = 2x^2 - 3x$, $f(c) = f(-1) = 2(-1)^2 - 3(-1) = 2 + 3 = 5$

Limit of average rate of change

$$\frac{f(x) - f(c)}{x - c} = \frac{(2x^2 - 3x) - 5}{x - (-1)} = \frac{2x^2 - 3x - 5}{x + 1} = \frac{\cancel{(x+1)}(2x-5)}{\cancel{(x+1)}}$$

$\boxed{2x - 5}$, $x = -1$ $\quad \boxed{\lim_{x \to -1} f(x) = -7}$

$2(-1) - 5 = -7$

49) $c = 0$; $f(x) = 3x^3 - 2x^2 + 4$

$f(x) = 3x^3 - 2x^2 + 4$ $\quad f(c) = f(0) = 4$

$$\frac{f(x) - f(c)}{x - c} = \frac{(3x^3 - 2x^2 + 4) - 4}{x - 0} = \frac{3x^3 - 2x^2 + 4 - 4}{x}$$

$$\frac{3x^3 - 2x^2}{x} = \frac{\cancel{x}(3x^2 - 2x)}{\cancel{x}} = \boxed{3x^2 - 2x}, \quad x = 0 \quad \boxed{\lim_{x \to 0} f(x) = 0}$$

$3(0)^2 - 2(0) = 0$

50) $c = 0$; $f(x) = 4x^3 - 5x + 8$

$f(x) = 4x^3 - 5x + 8$, $f(c) = f(0) = 4(0)^3 - 5(0) + 8 = 8$

Limit of average rate of change.

$$\frac{f(x) - f(c)}{x - c} = \frac{(4x^3 - 5x + 8) - (8)}{x - 0} = \frac{4x^3 - 5x + 8 - 8}{x} = \frac{4x^3 - 5x}{x}$$

$$\frac{4x^3 - 5x}{x} = \frac{\cancel{x}(4x^2 - 5)}{\cancel{x}} = \boxed{4x^2 - 5} \to x = 0 \quad \boxed{\lim_{x \to 0} f(x) = 5}$$

$4(0)^2 - 5 = 5$

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51) $c = 1$; $f(x) = \frac{1}{x}$

$f(x) = \frac{1}{x}$, $f(c) = f(1) = \frac{1}{1} = 1$, $f(c) = 1$

Unit^{n} of average rate of^{n} change

$$\frac{f(x) - f(c)}{x - c} = \frac{\left(\frac{1}{x}\right) - 1}{x - 1} = \frac{1 - x}{x} \div x - 1 = \frac{1 - x}{x} \times \frac{1}{x - 1}$$

$$\frac{(1 - x) \times 1}{x(x - 1)} = \frac{1 - x}{x^2 - x}$$, make table

x	1.8	1.9	1	1.1	1.2
y	-0.5556	-0.526	∞	-0.9091	-0.833

$f(x) = \frac{1 - x}{x^2 - x}$, $x = -1$ $\boxed{\lim_{x \to 1} f(x) = -1}$

52) $c = 1$; $f(x) = \frac{1}{x^2}$

$f(x) = 1/x^2$, $f(c) = 1/x^2 = 1/(1)^2 = 1/1 = 1$, $f(c) = 1$

$$\frac{f(x) - f(c)}{x - c} = \frac{1/x^2 - 1}{x - 1} = \frac{1}{x^2} - 1 \div x - 1 = \frac{1 - x^2}{x^2(x - 1)} = \frac{1 - x^2}{x^3 - x^2}$$

$$\frac{1 - x^2}{x^3 - x^2} = \frac{-x^2 + 1}{x^3 - x^2} = \frac{-(x^2 - 1)}{x^3 - x^2} = \frac{-(x + 1)\cancel{(x - 1)}}{x^2\cancel{(x - 1)}} = \frac{-(x + 1)}{x^2}$$

$\frac{-(x + 1)}{x^2}$, $x = 1 \rightarrow \frac{-(1 + 1)}{(1)^2} = \frac{-(2)}{1} = -2$ $\boxed{\lim_{x \to 1} f(x) = -2}$

53)

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(10)

1.2 Ron Larson

2) $\lim_{x\to 2} \frac{x-2}{x^2-4} = \frac{\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \frac{1}{(x+2)}$

:-

x	1.9	1.99	1.999	2	2.001	2.01	2.1
y	0.2564	0.2506	0.2501	1/4	0.2499	0.2494	0.2438

3) $\lim_{x\to 0} \frac{\sqrt{x+6}-\sqrt{6}}{x}$

:- $\frac{\sqrt{x+6}-\sqrt{6}}{x} \times \frac{\sqrt{x+6}+\sqrt{6}}{\sqrt{x+6}+\sqrt{6}} = \frac{(x+6)-(\sqrt{6})^2}{x(\sqrt{x+6}+\sqrt{6})}$

: $\frac{(x+6)-6}{x(\sqrt{x+6}+\sqrt{6})} = \frac{\cancel{x}}{\cancel{x}(\sqrt{x+6}+\sqrt{6})} = \frac{1}{\sqrt{x+6}+\sqrt{6}}$

$\boxed{\lim_{x\to 0} \frac{1}{\sqrt{x+6}+\sqrt{6}}} = \frac{1}{\sqrt{0+6}+\sqrt{6}} = \frac{1}{\sqrt{6}+\sqrt{6}} = \boxed{\frac{1}{2\sqrt{6}} = 0.2041}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
y	0.2050	0.2042	0.2041	0.2041	0.2041	0.2040	0.2033

4) $\frac{\sqrt{4-x}-3}{x+5} \quad \lim_{x\to -5}$

:- $\frac{\sqrt{4-x}-3}{x+5} \times \frac{\sqrt{4-x}+3}{\sqrt{4-x}+3} = \frac{(4-x)-9}{(x+5)(\sqrt{4-x}+3)} = \frac{-x-5}{(x+5)(\sqrt{4-x}+3)}$

$\frac{-1\cancel{(x+5)}}{\cancel{(x+5)}(\sqrt{4-x}+3)} = \frac{-1}{\sqrt{4-x}+3} \longrightarrow \lim_{x\to -5} \frac{-1}{\sqrt{4-(-5)}+3} = \frac{-1}{3+3} = -\frac{1}{6}$

$\boxed{\lim_{x\to -5} f(x) = -1/6}$

x	-5.1	-5.01	-5.001	-5	-4.999	-4.99	-4.9
y	-0.1662	-0.1666	-0.1667	-1/6	-0.1667	-0.1667	-1/3

5) $\lim_{x\to 3} \frac{\frac{1}{(x+1)} - \frac{1}{4}}{x-3} = \frac{\frac{1}{x+1} - \frac{1}{4}}{x-3} = \frac{1}{x+1} - \frac{1}{4} \div (x-3) = \frac{1}{x+1} - \frac{1}{4} \times \frac{1}{(x-3)}$

$= \frac{4-(x+1)}{4x+4} \times \frac{1}{(x-3)} = \frac{4-x-1}{(4x+4)(x-3)} = \frac{(3-x)}{(4x+4)(x-3)} = \frac{-1\cancel{(x-3)}}{(4x+4)\cancel{(x-3)}}$

$\boxed{\frac{-1}{4x+4}} \rightarrow x=3 \rightarrow \frac{-1}{4(3)+4} = \frac{-1}{12+4} = \frac{-1}{16} = -0.0625$

$\boxed{\lim_{x\to 1} f(x) = -1/16}$

x	2.9	2.99	2.99	3	3.001	3.01	3.1
y	-0.0641	-0.0627	-0.0627	-0.0625	-0.0625	-0.0623	-0.0610

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6) $\lim_{x\to 4} \frac{x}{(x+1)} - \frac{4}{5} \div (x-4)$

:- $\frac{x}{(x+1)} - \frac{4}{5} \div (x-4) = \frac{5x-4x-4}{5x+5} \times \frac{1}{(x-4)} = \frac{\cancel{(x-4)}}{(5x+5)\cancel{(x-4)}}$

$\frac{1}{5x+5} \to \lim_{x\to 4} f(x) = \frac{1}{5x+5}$, $x=4 \to \frac{1}{5(4)+5} = \frac{1}{25}$

$\boxed{\lim_{x\to 4} f(x) = 1/25}$

7) $\lim_{x\to 0} \frac{\sin x}{x} \to \boxed{\frac{\sin mx}{mx} = 1} \to$ predefined

$f(x) = \frac{\sin 1x}{1(x)} = 1 \to f(x) = 1$

$\boxed{\lim_{x\to 0} f(x) = 1}$

8) $\lim_{x\to 1} \frac{x-2}{x^2+x-6} = \frac{\cancel{(x-2)}}{\cancel{(x-2)}(x+3)} = \boxed{\frac{1}{x+3}} \to \lim_{x\to 1} \to \frac{1}{x+3} = \frac{1}{1+3} = \frac{1}{4}$

$\boxed{\lim_{x\to 1} f(x) = 1/4}$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
y	0.2564	0.2506	0.2501	1/4	0.2499	0.2494	0.2439

(9) $\lim_{x\to -3} \frac{x+3}{x^2+7x+12} = \frac{(x+3)}{x^2+7x+12} = \frac{\cancel{(x+3)}}{\cancel{(x+3)}(x+4)} = \boxed{\frac{1}{x+4}}$

$\boxed{\lim_{x\to -3} f(x) = 1}$

x	-2.9	-2.99	-2.999	-3	-3.001	-3.01	-3.1
y	0.9091	0.9901	0.9990	1	1.0010	1.0101	1.1111

(10) $\lim_{x\to 1} \frac{x^4-1}{x^6-1} = \frac{(x^2)^2-1}{(x^3)^2-1} = \frac{(x^2-1)(x^2+1)}{(x^3-1)(x^3+1)}$

$\frac{\cancel{(x-1)}\cancel{(x+1)}(x^2+1)}{\cancel{(x-1)}(x^2+x+1)\cancel{(x+1)}(x^2-x+1)} = \frac{x^2+1}{(x^2+x+1)(x^2-x+1)}$

$\lim_{x\to 1} f(x) = \frac{x^2+1}{(x^2+x+1)(x^2-x+1)}$, $x=1 \to \frac{(1)^2+1}{(1^2+1+1)(1^2-1+1)} = \frac{1+1}{(3)(1)} = \frac{2}{3}$

$\boxed{\lim_{x\to 1} f(x) = 2/3}$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
y	0.7364	0.6733	0.6673	0.66	0.6660	0.6600	0.6015

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12) $\lim_{x\to -2} \frac{x^3+8}{x+2} = \frac{x^3+2^3}{x+2} = \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}} = x^2-2x+4$

$\lim_{x\to -2} f(x) = x^2-2x+4$, $x=-2$

$(-2)^2 - 2(-2)+4 = 4+4+4 = 12$

$\lim_{x\to -2} f(x) = 12$

x	-1.9	-1.99	-1.999	-2	-2.001	-2.01	-2.1
y	11.410	11.9401	11.9940	12	12.0060	12.0601	12.6100

(3) $\lim_{x\to 0} \frac{\sin 2x}{x} \rightarrow \frac{\sin m}{m} = 1$

$\frac{2}{2} \times \frac{(\sin 2x)}{x} = 2\left(\frac{\sin 2x}{2x}\right) = 2(1) = 2$ $\boxed{\lim_{x\to 0} f(x) = 2}$

(4) $\lim_{x\to 0} \frac{\tan x}{\tan 2x}$

5) $\lim_{x\to 4} f(x) = \frac{\sqrt{x+5}-3}{x-4} = \frac{\sqrt{x+5}-3}{x-4} \times \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$: \frac{(x+5)-9}{(x-4)(\sqrt{x+5}+3)} = \frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x+5}+3)} = \frac{1}{\sqrt{x+5}+3} \rightarrow \boxed{\lim_{x\to 4} f(x) = \frac{1}{\sqrt{x+5}+3}}$

$x=4$, $\frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{4+5}+3} = \frac{1}{3+3} = \frac{1}{6}$, $\boxed{\lim_{x\to 4} f(x) = \frac{1}{6}}$

$:- \boxed{x=4} \rightarrow$ is a vertical asymptote of Rational G.

5) $\frac{x-3}{x^2-4x+3} = \frac{\cancel{(x-3)}}{(x-1)\cancel{(x-3)}} = \frac{1}{x-1} \rightarrow \lim_{x\to 3} f(x) = \frac{x-3}{x^2-4x+3}$

$\lim_{x\to 3} f(x) = \frac{1}{x-1} = \frac{1}{3-1} = \boxed{\frac{1}{2}}$ $\boxed{\lim_{x\to 3} f(x) = 1/2}$

5) $\lim_{x\to 9} f(x) = \frac{x-9}{\sqrt{x}-3} \rightarrow \frac{x-9}{\sqrt{x}-3} \times \frac{\sqrt{x}+3}{\sqrt{x}+3} = (x-9)(\sqrt{x}+3)$

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11

1.3)

In excercises 1–4, use graphing utility to graph the function and visually estimate the limits.

1) $h(x) = -x^2 + 4x$

a) $\lim_{x \to 4} h(x)$

:- $x = 4$, $-x^2 + 4x$

$h(4) = -(4)^2 + 4(4)$

$= -16 + 16$

$= 0$

$\boxed{\lim_{x \to 4} h(x) = 0}$

b) $\lim_{x \to -1} h(x)$

:- $x = -1$, $-x^2 + 4x$

$h(-1) = -(-1)^2 + 4(-1)$

$h(-1) = -(1) + (-4)$

$= -1 - 4 = -5$

$\boxed{\lim_{x \to -1} h(x) = -5}$

2) $g(x) = \dfrac{12(\sqrt{x} - 3)}{x - 8}$

a) $\lim_{x \to 4} g(x)$

:- $x = 4$, $\dfrac{12(\sqrt{x} - 3)}{x - 8}$

$= \dfrac{12(\sqrt{4} - 3)}{4 - 8} = \dfrac{12(-1)}{-5} = \dfrac{12}{5}$

$\boxed{\lim_{x \to 4} g(x) = 12/5}$

b) $\lim_{x \to 0} g(x)$

:- $\dfrac{12(\sqrt{0} - 3)}{0 - 8}$

$= \dfrac{12(-3)}{-8} = \dfrac{-36}{-8} = 4$

$\boxed{\lim_{x \to 0} g(x) = 4}$

3) $\lim_{x \to 0} f(x) = x \cos x$

$\boxed{\cos 0 = 1}$

$0(\cos 0) = 0$

$\boxed{\lim_{x \to 0} f(x) = 0}$

4) $f(t) = t|t - 4|$

a) $\lim_{t \to 4} f(t)$

:- $f(4) = 4|4 - 4|$

$f(4) = 0$

$\boxed{\lim_{x \to 4} f(t) = 0}$

b) $\lim_{t \to -1} f(t)$

:- $f(-1) = -1|-1 - 4|$

$= -1(5) = -5$

$\boxed{\lim_{t \to -1} f(t) = -5}$

5) $\lim_{x \to 2} x^3$

:- $x = 2$, x^3

$= (2)^3 = 8$

$\boxed{\lim_{x \to 2} f(x) = 8}$

6) $\lim_{x \to -2} x^4$

$= x = -2$, $x^4 \to (-2)^4$

$= 16 \to \boxed{\lim_{x \to -2} f(x) = 16}$

7) $\lim_{x \to 0} (2x - 1)$

:- $x = 0$, $2x - 1$

$2(0) - 1 = 0 - 1 = -1$

$\boxed{\lim_{x \to 0} f(x) = -1}$

8) $\lim_{x \to -3} (3x + 2)$

$= 3(-3) + 2$

$= -8 + 2 = -7$

$\boxed{\lim_{x \to -3} f(x) = -7}$

9) $\lim_{x \to -3} (x^2 + 3x)$

:- $(-3)^2 + 3(-3)$

$= 8 + (-8) = 0$

$\boxed{\lim_{x \to -3} f(x) = 0}$

10) $\lim_{x \to 1} (-x^2 + 1)$

$= -(1)^2 + 1$

$-1 + 1 = 0$

$\boxed{\lim_{x \to 1} f(x) = 0}$

11) $\lim_{x \to -3} 2x^2 + 4x + 1$

:- $2(-3)^2 + 4(-3) + 1$

$= 18 - 12 + 1 = 7$

$\boxed{\lim_{x \to -3} f(x) = 7}$

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12) $\lim_{x\to 1} 3x^3-2x^2+4$

$= 3(1)^3-2(1)^2+4$

$= 3-2+4 = 5$

$\boxed{\lim_{x\to 1} f(x) = 5}$

13) $\lim_{x\to 3} \sqrt{x+1}$

$= \sqrt{3+1} = 2$

$\boxed{\lim_{x\to 3} f(x) = 2}$

14) $\lim_{x\to 4} \sqrt[3]{x+4}$

$= \sqrt[3]{4+4}$

$= \sqrt[3]{8} = (2^3)^{\frac{1}{3}} = 2$

$\boxed{\lim_{x\to 4} f(x) = 2}$

15) $\lim_{x\to -4} (x+3)^2$

$= (-4+3)^2 = 1$

$\boxed{\lim_{x\to -4} f(x) = 1}$

16) $\lim_{x\to 0} (2x-1)^3$

$= (2(0)-1)^3$

$= (0-1)^3 = -1$

$\boxed{\lim_{x\to 0} f(x) = -1}$

17) $\lim_{x\to 2} \frac{1}{x}$

$\boxed{\lim_{x\to 2} f(x) = 1/2}$

18) $\lim_{x\to -3} \frac{2}{x+2}$

$= \frac{2}{-3+2} = \frac{2}{-1} = -2$

$\boxed{\lim_{x\to -3} f(x) = -2}$

19) $\lim_{x\to 1} \frac{x}{x^2+4}$

$= \frac{1}{(1)^2+4} = 1/5$

$\boxed{\lim_{x\to 1} f(x) = 1/5}$

20) $\lim_{x\to 1} \frac{2x-3}{x+5}$

$= \frac{2(1)-3}{1+5} = \frac{-1}{6}$

$\boxed{\lim_{x\to 1} f(x) = -1/6}$

21) $\lim_{x\to 7} \frac{3x}{\sqrt{x+2}}$

$= \frac{3(7)}{\sqrt{7+2}} = \frac{21}{3} = 7$

$\boxed{\lim_{x\to 7} f(x) = 7}$

22) $\lim_{x\to 2} \frac{\sqrt{x+2}}{x-4}$

$= \frac{\sqrt{2+2}}{2-4} = \frac{2}{-2} = -1$

$\boxed{\lim_{x\to 2} f(x) = -1}$

23) $f(x) = 5-x$, $g(x) = x^3$

a) $\lim_{x\to 1} f(x)$

$= 5-(1) = 4$

$\boxed{\lim_{x\to 1} f(x) = 4}$

b) $\lim_{x\to 4} g(x)$

$:- x = 4,\ x^3$

$(4)^3 = 64$

$\boxed{\lim_{x\to 4} g(x) = 4}$

c) $\lim_{x\to 1} g(f(x))$

$g(x) = x^3,\ f(x) = 5-x$

$:- g(5-x) = (5-x)^3 = (5)^3-3(5)^2(x)+3(5)(x)^2-(x)^3$

$= 125-75x+15x^2-x^3 = 125-75+15-1 = 64$

OR $= (5-x)^3 = (5-1)^3 = (4)^3 = 64$

$\boxed{\lim_{x\to 1} f(x) = 64}$

24) $f(x) = x+7$, $g(x) = x^2$

a) $\lim_{x\to -3} f(x)$

$-3+7 = 4$

$\boxed{\lim_{x\to -3} f(x) = 4}$

b) $\lim_{x\to 4} g(x)$

$(4)^2 = 16$

$\boxed{\lim_{x\to 4} g(x) = 16}$

c) $\lim_{x\to -3} g(f(x))$

$g(x+7) = (x+7)^2,\ x = -3$

$(-3+7)^2 = (4)^2 = 16$

$\boxed{\lim_{x\to -3} = 16}$

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25) $f(x) = 4 - x^2$, $g(x) = \sqrt{x+1}$

a) $\lim_{x\to 1} f(x)$ b) $\lim_{x\to 3} g(x)$ c) $\lim_{x\to 1} g(f(x))$

$= 4 - (1)^2 = 3$ $= \sqrt{3+1} = 2$ $g(4-x^2) = \sqrt{(4-x^2)+1}$

$\boxed{\lim_{x\to 1} f(x) = 3}$ $\boxed{\lim_{x\to 3} g(x) = 2}$ $= \sqrt{5-x^2}$, $x = 1$

$\sqrt{5-(1)^2} = \sqrt{5-1} = \sqrt{4} = 2$

$\boxed{\lim_{x\to 1} g(x) = 2}$

26) $f(x) = 2x^2 - 3x + 1$, $g(x) = \sqrt[3]{x+6}$

a) $\lim_{x\to 4} f(x)$ b) $\lim_{x\to 21} g(x)$

$= f(4) = 2(4)^2 - 3(4) + 1$ $= \sqrt[3]{21+6}$ c) $\lim_{x\to 4} g(f(x))$

$= 2(16) - 12 + 1$ $= (27)^{\frac{1}{3}} = (3^3)^{\frac{1}{3}}$

$= 32 - 12 + 1 = 21$ $= 3$ $= g(2x^2 - 3x + 1) = \sqrt[3]{(2x^2-3x+1)+6}$

$\boxed{\lim_{x\to 4} f(x) = 21}$ $\boxed{\lim_{x\to 3} g(x) = 3}$ $= \sqrt[3]{2x^2 - 3x + 7}$, $x = 4$

$(2(4)^2 - 3(4) + 7)^{\frac{1}{3}}$

▸ In exercises 37–40, use the information to evalute the limits.

$(32 - 12 + 7)^{1/3} = (27)^{\frac{1}{3}}$

$(3^3)^{1/3} = 3$

37) $\lim_{x\to c} f(x) = 3$ $\lim_{x\to c} g(x) = 2$ $\boxed{\lim_{x\to 4} g(f(x)) = 3}$

(a) $\lim_{x\to c} [5g(x)]$ b) $\lim_{x\to c} [f(x) + g(x)]$

$f(x) = 3$, $g(x) = 2$

$\lim_{x\to c} [5(2)] = 10$ $\boxed{\lim_{x\to c} [3 + 2] = 5}$

$\boxed{\lim_{x\to c} [5g(x)] = 10}$

c) $\lim_{x\to c} [f(x)g(x)]$ d) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{3}{2}$

$= \lim_{x\to c} [(3)(2)] = 6$

$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{3}{2}}$

$\boxed{\lim_{x\to c} [f(x)g(x)] = 6}$

38) $\lim_{x\to c} f(x) = 3/2$ $\lim_{x\to c} g(x) = 1/2$

(a) $\lim_{x\to c} [4f(x)]$ b) $\lim_{x\to c} [f(x) + g(x)]$

$= \lim_{x\to c} [4(\frac{3}{2})]$ $= \lim_{x\to c} [\frac{3}{2} + \frac{1}{2}] = 2$

$\boxed{\lim_{x\to c} [4f(x)] = 6}$ $\boxed{\lim_{x\to c} [f(x) + g(x)] = 2}$

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c) $\lim_{x\to c} [f(x)\,g(x)]$

$= \lim_{x\to c} \left[\frac{3}{2} \times \frac{1}{2}\right] = \frac{3}{4}$

$\boxed{\lim_{x\to c} [f(x)\,g(x)] = \frac{3}{4}}$

d) $\lim_{x\to c} \frac{f(x)}{g(x)}$

$\lim_{x\to c} \frac{3}{2} \div \frac{1}{2} \to \frac{3}{2} \times 2 = 3$

$\boxed{\lim_{x\to c} \frac{f(x)}{g(x)} = 3}$

45) $\lim_{x\to -1} \frac{x^2-1}{x+1}$

$= \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}} = x-1$

$x = -1,\ x-1$

$-1-1 = -2$

$\boxed{\lim_{x\to -1} f(x) = -2}$

46) $\lim_{x\to -1} \frac{2x^2-x-3}{x+1}$

$= \frac{(2x-3)\cancel{(x+1)}}{\cancel{(x+1)}} = 2x-3$

$x = -1,\ 2x-3 = 2(-1)-3 = -2-3 = -5$

$\boxed{\lim_{x\to -1} f(x) = -5}$

47) $\lim_{x\to 2} \frac{x^3-8}{x-2}$

$= \frac{x^3-8^3}{x-2} = \frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}$

$= x^2+2x+4,\ x = 2$

$(2)^2+2(2)+4 = 12$

$\boxed{\lim_{x\to 2} f(x) = 12}$

48) $\lim_{x\to -1} \frac{x^3+1}{x+1}$

$= \frac{x^3+1^3}{x+1} = \frac{\cancel{(x+1)}(x^2+x+1)}{\cancel{(x+1)}}$

$= x^2+x+1,\ x = -1$

$= (-1)^2+(-1)+1 = 1$

$\boxed{\lim_{x\to -1} f(x) = 1}$

49) $\lim_{x\to 0} \frac{x}{x^2-x}$

$= \frac{x}{x(x-1)} = \frac{1}{x-1},\ x = 0$

$= \frac{1}{0-1} = \frac{1}{-1} = -1$

$\boxed{\lim_{x\to 0} f(x) = -1}$

50) $\lim_{x\to 0} \frac{3x}{x^2+2x}$

$3x = x(3)$

$x^2+2x = x(x+2)$

$\frac{3}{x+2},\ x = 0$

$\frac{3}{0+2} = \frac{3}{2}$

$\boxed{\lim_{x\to 0} f(x) = 3/2}$

51) $\lim_{x\to 4} \frac{x-4}{x^2-16}$

$= \frac{\cancel{(x-4)}}{\cancel{(x-4)}(x+4)} = \frac{1}{x+4},\ x = 4$

$\frac{1}{4+4} = \frac{1}{8}$

$\boxed{\lim_{x\to 4} f(x) = 1/8}$

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52) $\lim_{x\to 3} \frac{3-x}{x^2-9}$

$= \frac{3-x}{(x-3)(x+3)} = \frac{-1\cancel{(x-3)}}{\cancel{(x-3)}(x+3)}$

$= \frac{-1}{(x+3)}, \quad x \neq 3$

$= \frac{-1}{3+3} = -1/6$

$\boxed{\lim_{x\to 3} f(x) = -1/6}$

53) $\lim_{x\to -3} \frac{x^2+x-6}{x^2-9}$

$= \frac{(x-2)\cancel{(x+3)}}{(x-3)\cancel{(x+3)}} = \frac{x-2}{x-3}, \quad x \neq -3$

$= \frac{-3-2}{-3-3} = \frac{-5}{-6} = 5/6$

$\boxed{\lim_{x\to -3} f(x) = 5/6}$

54) $\lim_{x\to 4} \frac{x^2-5x+4}{x^2-2x-8}$

$= \frac{(x-1)\cancel{(x-4)}}{(x+2)\cancel{(x-4)}} = \frac{(x-1)}{(x+2)}$

$x \neq 4, \quad \frac{x-1}{x+2} = \frac{4-1}{4+2} = \frac{3}{6} = 1/2$

$\boxed{\lim_{x\to 4} f(x) = 1/2}$

55) $\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4}$

$= \frac{\sqrt{x+5}-3}{x-4} \times \frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}$

$= \frac{(x+5)-9}{(x-4)(\sqrt{x+5}+3)} = \frac{\cancel{(x-4)}}{\cancel{(x-4)}(\sqrt{x+5}+3)}$

$\frac{1}{\sqrt{x+5}+3}, \quad x \neq 4$

$\frac{1}{\sqrt{4+5}+3} = \frac{1}{3+3} = \frac{1}{6}$

$\boxed{\lim_{x\to 4} f(x) = 1/6}$

56) $\frac{\sqrt{x+1}-2}{x-3} \quad \lim_{x\to 3}$

$= \frac{\sqrt{x+1}-2}{(x-3)} \times \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$

$\frac{(x+1)-4}{(x-3)(\sqrt{x+1}+2)} = \frac{\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{x+1}+2)}$

$\frac{1}{\sqrt{x+1}+2}, \quad x \neq 3$

$\frac{1}{\sqrt{3+1}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$

$\boxed{\lim_{x\to 3} f(x) = 1/4}$

57) $\lim_{x\to 0} \frac{\sqrt{x+5}-\sqrt{5}}{x}$

$= \frac{\sqrt{x+5}-\sqrt{5}}{x} \times \frac{\sqrt{x+5}+\sqrt{5}}{\sqrt{x+5}+\sqrt{5}}$

$= \frac{(x+5)-5}{x(\sqrt{x+5}+\sqrt{5})} = \frac{\cancel{x}}{\cancel{x}(\sqrt{x+5}+\sqrt{5})}$

$= \frac{1}{\sqrt{x+5}+\sqrt{5}}, \quad x \neq 0$

$\frac{1}{\sqrt{0+5}+\sqrt{5}} = \frac{1}{\sqrt{5}+\sqrt{5}} = \frac{1}{2\sqrt{5}}$

$\boxed{\lim_{x\to 0} f(x) = \frac{1}{2\sqrt{5}}}$

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58) $\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x}$

$= \frac{\sqrt{2+x}-\sqrt{2}}{x} \times \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$

$\frac{(2+x)-(2)}{x(\sqrt{2+x}+\sqrt{2})} = \frac{\cancel{x}}{\cancel{x}(\sqrt{2+x}+\sqrt{2})}$

$\frac{1}{\sqrt{2+x}+\sqrt{2}}$, $x=0$

$\frac{1}{\sqrt{2+0}+\sqrt{2}} = \frac{1}{\sqrt{2}+\sqrt{2}} =$

$\boxed{\frac{1}{2\sqrt{2}}}$ $\boxed{\lim_{x\to 0} f(x) = \frac{1}{2\sqrt{2}}}$

59) $\lim_{x\to 0} \frac{\frac{1}{3+x}-\frac{1}{3}}{x}$

$= \frac{\frac{\cancel{3}-\cancel{3}-x}{3+3x}}{x}$

$\frac{-1(\cancel{x})}{3+0} \times \frac{1}{\cancel{x}} = \frac{-1}{3}$

$\boxed{\lim_{x\to 0} f(x) = -1/3}$

60) $\lim_{x\to 0} \frac{\frac{1}{(x+4)}-\frac{1}{4}}{x}$

$= \frac{4-x-4}{4x+16} \div x$

$= \frac{-1(\cancel{x})}{4x+16} \times \frac{1}{\cancel{x}} = \frac{-1}{4x+16}$

$x=0$, $\frac{-1}{4x+16} = \frac{-1}{4(0)+16} = \frac{-1}{16}$

$\boxed{\lim_{x\to 0} f(x) = -1/16}$

61) $\lim_{\Delta x\to 0} \frac{2(x+\Delta x)-2x}{\Delta x}$

$= \frac{\cancel{2x}+2\Delta x-\cancel{2x}}{\Delta x} = \frac{2\cancel{\Delta x}}{\cancel{\Delta x}} = 2$

$\boxed{\lim_{\Delta x\to 0} f(x) = 2}$

62) $\lim_{\Delta x\to 0} \frac{(x+\Delta x)^2-x^2}{\Delta x}$

$= \frac{\cancel{x^2}+2\Delta x^2+\Delta x^2-\cancel{x^2}}{\Delta x}$

$= \frac{3\cancel{\Delta x^2}}{\cancel{\Delta x}} = 3\Delta x$

$\boxed{\lim_{\Delta x\to 0} f(x) = 3\Delta x}$

63) $\lim_{\Delta x\to 0} (x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1) \div \Delta x$

$= (x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1) \div \Delta x$

$= \cancel{x^2}+2x(\Delta x)+(\Delta x)^2-\cancel{2x}-2\Delta x+\cancel{1}-\cancel{x^2}+\cancel{2x}-\cancel{1} \div \Delta x$

$= 2x(\Delta x)+(\Delta x)^2-2\Delta x \div \Delta x$

$\frac{2x(\Delta x)+(\Delta x)^2-2\Delta x}{\Delta x} = \frac{\cancel{\Delta x}(2x+\Delta x-2)}{\cancel{\Delta x}}$

$= 2x+\Delta x-2$, $\Delta x=0$

$2x+0-2 = 2x-2$

$\boxed{\lim_{\Delta x\to 0} f(x) = 2x-2}$

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64) $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$

$= \frac{\cancel{x^3} + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - \cancel{x^3}}{} \div \Delta x$

$= 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \div \Delta x$

$\frac{\cancel{\Delta x}(3x^2 + 3x(\Delta x) + (\Delta x)^2)}{\cancel{\Delta x}} = 3x^2 + 3x(\Delta x) + (\Delta x)^2$

$\Delta x = 0$, $3x^2 + 3x(0) + (0)^2 = 3x^2 + 0 + 0 = 3x^2$

$\boxed{\lim_{x \to 0} f(x) = 3x^2}$

▸ determine the limit of trigonometric functions

65) $\lim_{x \to 0} \frac{\sin x}{5x}$

$= \frac{\frac{1}{5}(\sin x)}{\frac{1}{5}(5x)} = \frac{\frac{1}{5}(\sin x)}{\frac{5}{5}(x)}$

$= \frac{\frac{1}{5}(\sin x)}{x} = \frac{1}{5}\left(\frac{\sin x}{x}\right)$

$\therefore \frac{\sin x}{x} = 1 \rightarrow \frac{1}{5}(1) = 1/5$

66) $\lim_{x \to 0} 3\left(\frac{1 - \cos x}{x}\right)$

$= \frac{1 - \cos x}{x} = 0 \rightarrow$ property

$3(0) = 0$

$\boxed{\lim_{x \to 0} f(x) = 0}$

67) $\lim_{x \to 0} \frac{\sin x (1 - \cos x)}{x^2}$

$= \frac{\sin x (1 - \cos x)}{x \times (x)}$

$= \frac{\sin x}{x} \times \left(\frac{1 - \cos x}{x}\right)$

$\therefore \frac{1 - \cos x}{x} = 0$, $\frac{\sin x}{x} = 1$

$1 \times (0) = 0$

$\boxed{\lim_{x \to 0} f(x) = 0}$

68) $\lim_{\theta \to 0} \frac{\cos\theta \tan\theta}{\theta}$

$= \cos\theta = \frac{\cot\theta}{\tan\theta}$

$= \frac{\cot\theta}{\cancel{\tan\theta}} \times \cancel{\tan\theta} \div \theta$

$= \frac{\cot\theta}{\theta} = 1$

$\boxed{\lim_{\theta \to 0} f(x) = 1}$

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6) $\lim_{x\to 0} \frac{\sin^2 x}{x}$ 7) $\lim_{x\to 0} \frac{\tan^2 x}{x}$ 7) $\lim_{h\to 0} \frac{(1-\cos h)^2}{h}$

$\boxed{\lim_{x\to 0} f(x) = 0}$ $\boxed{\lim_{x\to 0} f(x) = 0}$ $\therefore \frac{1-\cos x}{x} = 0$

↳ According to Book

$\frac{(1-\cos h)(1-\cos h)}{h} = 0\,(1-\cos h)$

$= 0$

85-88, find $\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$ $\boxed{\lim_{x\to 0} f(x) = 0}$

85) $f(x) = 3x-2$

• $f(x+\Delta x) = 3(x+\Delta x)-2$ → $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(3x+3\Delta x-2)-(3x-2)}{\Delta x}$

• $f(x) = 3x-2$

$\frac{\cancel{3x}+3\Delta x\cancel{-2}-\cancel{3x}\cancel{+2}}{\Delta x} = \frac{3\cancel{\Delta x}}{\cancel{\Delta x}} = 3$ $\boxed{\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = 3}$

86) $f(x) = \sqrt{x}$

• $f(x+\Delta x) = \sqrt{x+\Delta x}$ → $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$, Rationalize the

• $f(x) = \sqrt{x}$

$\frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \times \frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}} = \frac{(\cancel{x}+\Delta x)-(\cancel{x})}{\Delta x(\sqrt{x+\Delta x}+\sqrt{x})} = \frac{\cancel{\Delta x}}{\cancel{\Delta x}(\sqrt{x+\Delta x}+\sqrt{x})}$

$\frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$, $\Delta x\to 0$, $\Delta x = 0$ → $\frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$

$\boxed{\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{1}{2\sqrt{x}}}$

87) $f(x) = \frac{1}{x+3}$ → $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{1}{(x+\Delta x)+3} - \frac{1}{(x+3)} \div \Delta x$

• $f(x+\Delta x) = \frac{1}{(x+\Delta x)+3}$ $= \frac{\cancel{x}+\cancel{3}-\cancel{x}-\Delta x-\cancel{3}}{(x+3+\Delta x)(x+3)} \div \Delta x = \frac{-\Delta x}{(x+3+\Delta x)(x+3)} \div \Delta x$

$f(x) = 1/(x+3)$

$\frac{-\cancel{\Delta x}}{(x+3+\Delta x)(x+3)} \div \Delta x = \frac{-\cancel{\Delta x}}{(x+3+\Delta x)(x+3)} \times \frac{1}{\cancel{\Delta x}} = \frac{-1}{(x+3+\Delta x)(x+3)}$

$f(x) = \frac{-1}{(x+\Delta x+3)(x+3)}$, $\Delta x = 0$ → $\frac{-1}{(x+0+3)(x+3)} = \frac{-1}{(x+3)(x+3)}$

$\frac{-1}{(x+3)^2}$ → $\boxed{\lim_{\Delta x\to 0} f(x) = \frac{-1}{(x+3)^2}}$

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88) $f(x) = x^2 - 4x$

- $f(x+\Delta x) = (x+\Delta x)^2 - 4(x+\Delta x)$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{(x+\Delta x)^2 - 4(x+\Delta x) - (x^2 - 4x)}{\Delta x}$$

$$\cancel{x^2} + 2x(\Delta x) + (\Delta x)^2 - \cancel{4x} - 4\Delta x - \cancel{x^2} + \cancel{4x} \quad \div \Delta x$$

$$2x(\Delta x) + (\Delta x)^2 - 4\Delta x \quad \div \Delta x$$

$$\frac{\cancel{\Delta x}(2x + \Delta x - 4)}{\cancel{\Delta x}} = 2x + \Delta x - 4, \quad \Delta x = 0$$

$$2x + 0 - 4 = 2x - 4$$

$$\boxed{\lim_{\Delta x \to 0} f(x) = 2x - 4}$$

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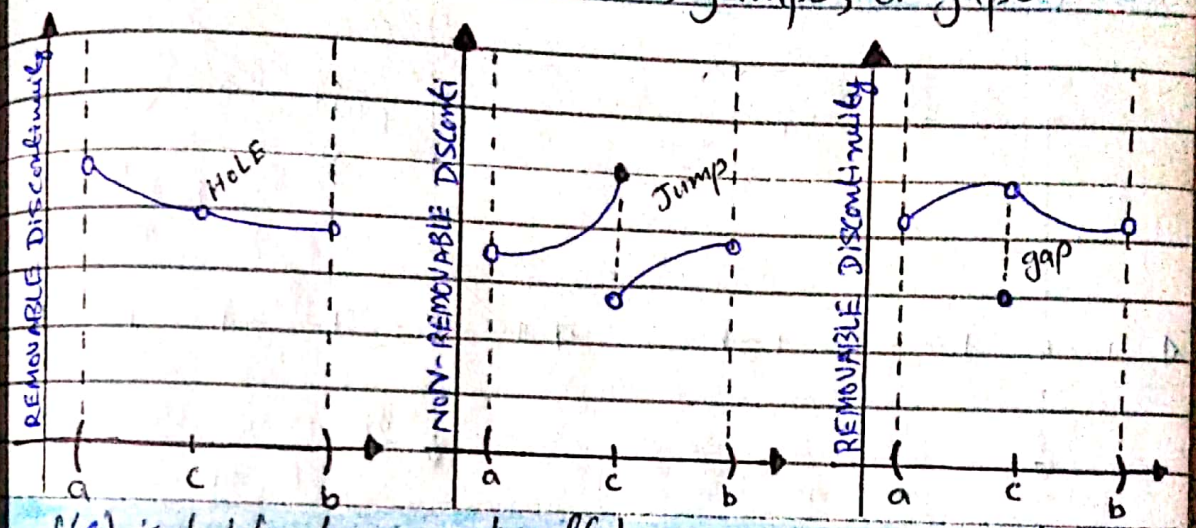
1.4 Continuity and one sided limits

▸ Continuity at a point and on an open interval.

To say that a function f is continuous at $x=c$ means that there is no interruption in the graph of f at c. That is, its graph is unbroken at c and there are NO holes, jumps, or gaps.



$f(c)$ is ! defined | $\lim_{x\to c} f(x)$ doesn't exist | $\lim_{x\to c} f(x) \neq f(c)$

▸ Continuity at $x=c$ can be destroyed by any one of the following condition.

1) The function is not defined at $x=c$.
2) The limit of $f(x)$ doesn't exist at $x=c$.
3) The limit of $f(x)$ exists at $x=c$, but not equal to $f(c)$.

▸ DEFINATION OF Continuity

Continuity at a point: A function f is continuous at c if the following three conditions are met.

1) $f(c)$ is defined.
2) $\lim_{x\to c} f(x)$ exists
3) $\lim_{x\to c} f(x) = f(c)$

Continuity on an open interval: A function is continuous on an open interval (a, b) if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, \infty)$ is everywhere continuous.

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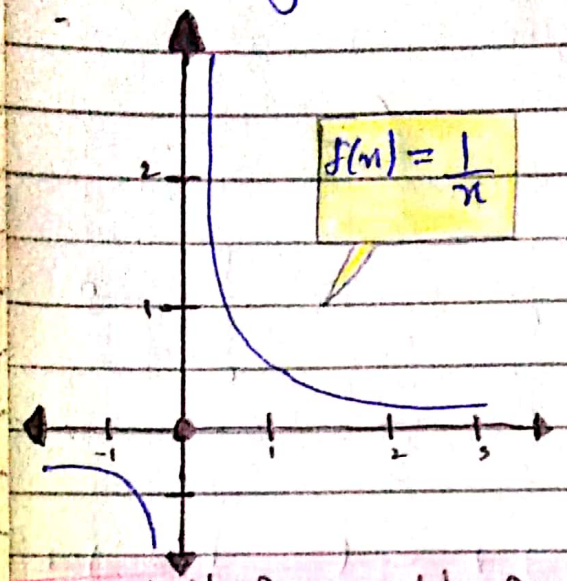
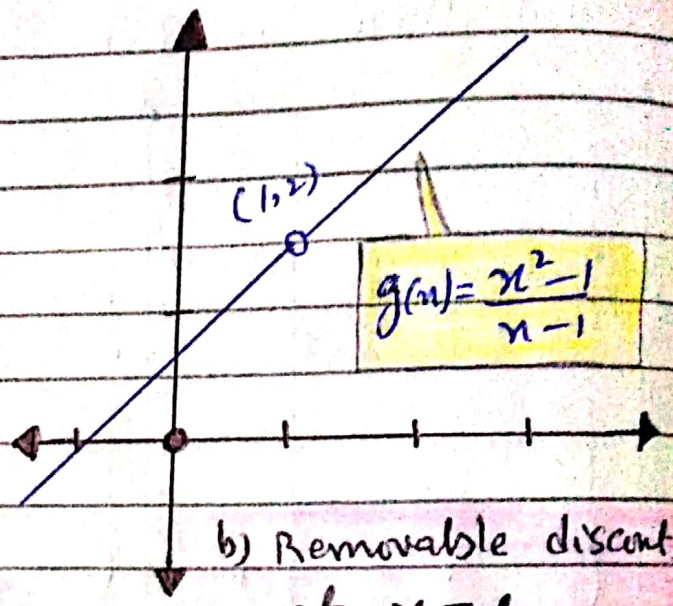
▶ Dis Continuity

1) Removable 2) NON-Removable

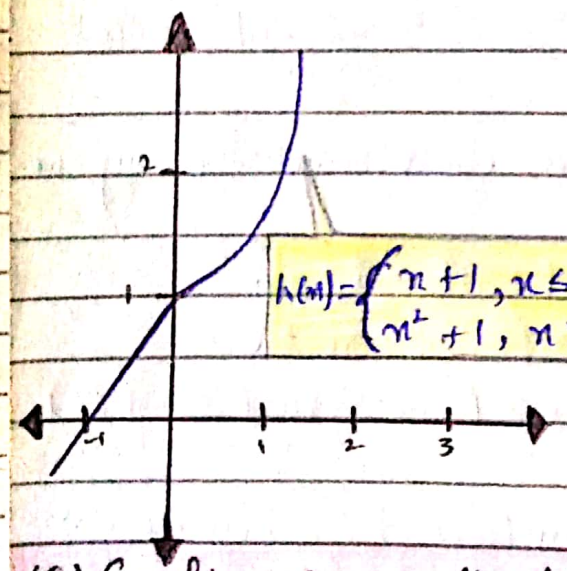
:- A discontinuity at c is called removable if f can be made continuous by appropriately defining or Redefining. f(c).



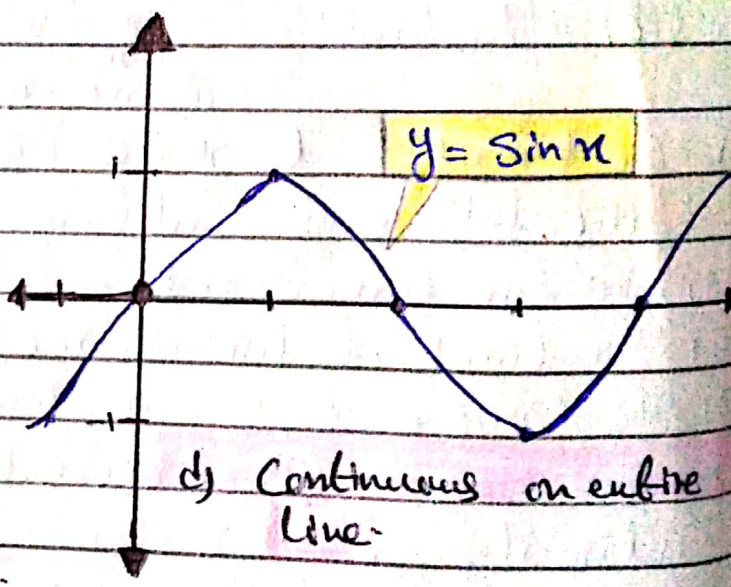
a) NON-Removable Disc, at x=0



b) Removable discont, at $x = 1$



(c) Continuous on entire line.



d) Continuous on entire line.

▶ Dis Continuity

1) Not defined — HOLE
2) Doesn't exist — Jump
3) Not — equal — Gap

▶ Continuity

1) Defined — Not-hole
2) exist — Not any jump
3) $\lim_{x \to c} f(x) = f(c)$ — Not any gap

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1.4

In Exercises 7-26, find the limit (if it exists). If it doesn't exist, explain why.

7) $\lim_{x\to 8^+} \frac{1}{x+8}$

$= x = 8$

$\frac{1}{8+8} = \boxed{\frac{1}{16}}$

8) $\lim_{x\to 5^-} \frac{-3}{x+5}$

$x = 5 \to \frac{-3}{5+5}$

$= \boxed{-3/10}$

9) $\lim_{x\to 5^+} \frac{x-5}{x^2-25}$

$= \frac{(x-5)}{(x+5)(x-5)} = \frac{1}{x+5}$

$x = 5 \to \frac{1}{5+5} = \boxed{\frac{1}{10}}$

10) $\lim_{x\to 2^+} \frac{2-x}{x^2-4}$

$= \frac{(2-x)}{(x-2)(x+2)} = \frac{-(x-2)}{(x-2)(x+2)}$

$= \frac{-1}{x+2}, x = 2 \to \boxed{\frac{-1}{4}}$

11) $\lim_{x\to -3^-} \frac{x}{\sqrt{x^2-9}}$

$x = -3 \to \frac{-3}{\sqrt{(-3)^2-9}}$

$= \frac{-3}{\sqrt{0}} = \frac{-3}{0}$

doesn't exists

12) $\lim_{x\to 9^-} \frac{\sqrt{x}-3}{x-9}$

$= \frac{\sqrt{x}-3}{x-9} \times \frac{\sqrt{x}+3}{\sqrt{x}+3}$

$= \frac{(x-9)}{(x-9)(\sqrt{x}+3)} = \frac{1}{\sqrt{x}+3}$

$= \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} = \boxed{\frac{1}{6}}$

13) $\lim_{x\to 0^-} \frac{|x|}{x}$

$x = 0 \to \frac{|0|}{0}$

lim doesn't exist at 0

$x = -1 \to \frac{|-1|}{-1} = \frac{1}{-1} = -1$

$\boxed{\lim_{x\to 0} f(x) = -1}$

14) $\lim_{x\to 10^+} \frac{|x-10|}{x-10}$

$= 10^+ \to 11$

$= \frac{|11-10|}{11-10} = \frac{|1|}{1}$

$\boxed{= 1}$

15) $\lim_{\Delta x\to 0^-} \frac{1}{x+\Delta x} - \frac{1}{x} \div \Delta x$

$= \frac{x-(x+\Delta x)}{(x+\Delta x)x} \times \frac{1}{\Delta x}$

$\frac{-\Delta x}{x(x+\Delta x)} \times \frac{1}{\Delta x} = \frac{-1}{x(x+\Delta x)}$

$\Delta x = 0 \to \frac{-1}{x(x+0)} = \frac{-1}{x(x)} = \boxed{\frac{-1}{x^2}}$

16) $\lim_{\Delta x\to 0^+} (x+\Delta x)^2 + x + \Delta x - (x^2+x) \div \Delta x$

$= x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x \div \Delta x$

$= 2x(\Delta x) + (\Delta x)^2 + \Delta x \div \Delta x$

$= \frac{\Delta x(2x + \Delta x + 1)}{\Delta x}$

$= 2x + \Delta x + 1$

$\lim_{\Delta x\to 0^+} \Delta x = 0$

$2x + 0 + 1$

$\boxed{2x+1}$

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17) $\lim_{x \to 3^-} f(x)$, where $f(x) = \begin{cases} \frac{x+2}{2}, & x \le 3 \\ \frac{12-2x}{3}, & x > 3 \end{cases}$

:- $y = \frac{x+2}{2}$, $x \le 3$ $\qquad$ $y = \frac{12-2x}{3}$, $x > 3$

let $x = 3 \to y = \frac{3+2}{2} = \frac{5}{2}$ $\qquad$ let $x = 4 \to \frac{12-2(4)}{3} = \frac{4}{3}$

let $x = 3$

$y = \frac{12-2(3)}{3} = 2$

limit exist on both sides

18) $\lim_{x \to 2} f(x)$, where $f(x) = \begin{cases} x^2 - 4x + 6, & x < 2 \\ -x^2 + 4x - 2, & x \ge 2 \end{cases}$

:- $y = x^2 - 4x + 6$, $x < 2$

let $x = 2$

$y = (2)^2 - 4(2) + 6$

$y = 4 - 8 + 6 = 2$

$\boxed{y = 2}$

:- $y = -x^2 + 4x - 2$, $x \ge 2$

$y = -(2)^2 + 4(2) - 2$

$y = -4 + 8 - 2$

$\boxed{y = 2}$

limit exists on b/s, and it is equal

19) $\lim_{x \to 1} f(x)$, where $f(x) = \begin{cases} x^3 + 1, & x < 1 \\ x + 1, & x \ge 1 \end{cases}$

:- $y = x^3 + 1$, $x < 1$ $\qquad$ $y = x + 1$, $x \ge 1$

let $x = 1$

$y = (1)^3 + 1 = 2$

$\boxed{y = 2}$

let $x = 1$

$y = 1 + 1 = 2$

$\boxed{y = 2}$

limit exists, and is equal on both sides

20) $\lim_{x \to 1^+} f(x)$, where $f(x) = \begin{cases} x, & x \le 1 \\ 1 - x, & x > 1 \end{cases}$

:- $y = x$, $x \le 1$ $\qquad$:- $y = 1 - x$, $x > 1$

let $x = 1$

$\boxed{y = 1}$

NON-Removable Discontinuity

Graph has Gap at 1, 0

let $x = 1$

$y = 1 - 1 = 0$ $\quad$ $\boxed{y = 0}$

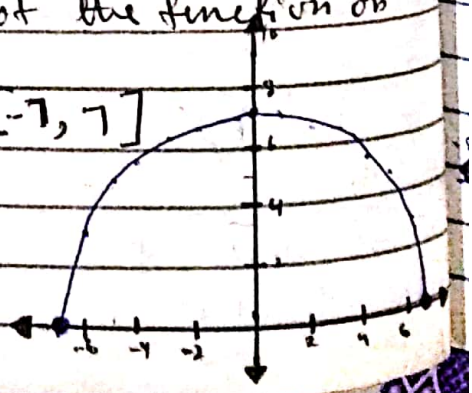
• 31 – 34, discuss the Continuity of the function on the Closed interval.

31) $g(x) = \sqrt{48 - x^2}$ $\qquad$ $[-7, 7]$

:- $\sqrt{48 - (-7)^2} = \sqrt{48 - 49} = 0, (-7, 0)$

$\sqrt{48 - (7)^2} = \sqrt{48 - 49} = \sqrt{0} = 0, (7, 0)$

Continuous function $\longrightarrow$



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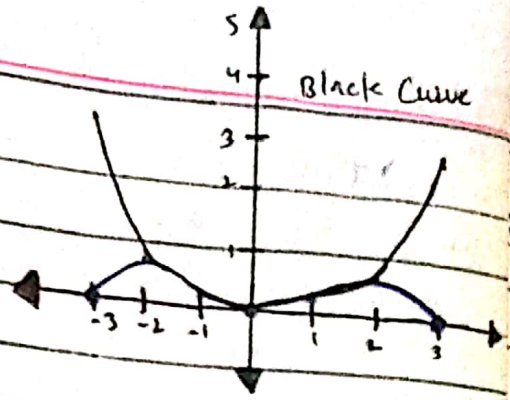
32) $f(t) = 3 - \sqrt{9 - t^2}$, $[-3, 3]$

:- $= 3 - \sqrt{9 - (-3)^2} = 3 - \sqrt{0} = 3$

$(-3, 3)$

$= 3 - \sqrt{9 - (3)^2} = 3 - \sqrt{0} = 3$

$(3, 3)$

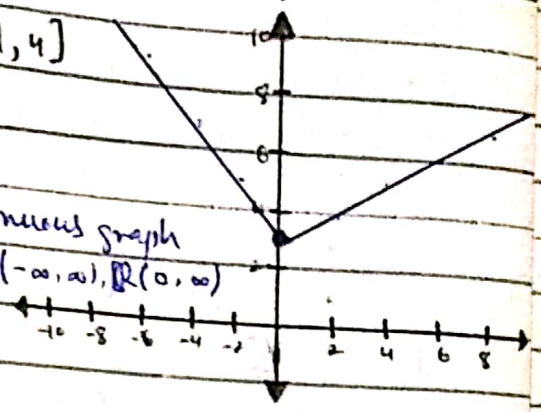


33) $f(x) = \begin{cases} 3 - x, & x \le 0 \\ 3 + \frac{x}{2}, & x > 0 \end{cases}$ $[-4, 4]$

:- $y = 3 - x$, $x \le 0$

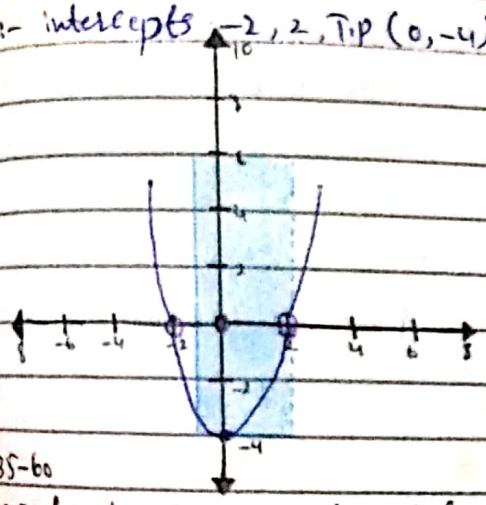
$y = \frac{6 + x}{2}$, $x > 0$

Continuous graph at $D(-\infty, \infty)$, $R(0, \infty)$



34) $g(x) = \frac{1}{x^2 - 4}$, $[-1, 2]$

:- intercepts $-2, 2$, T.P $(0, -4)$:- $\frac{1}{(x-2)(x+2)}$, $x \to$ undefined at 2 and -2 on x-axis. And it does not lie in $[-1, 2]$ closed interval, but it lies in open interval $[-1, 2)$ means, Semi open.

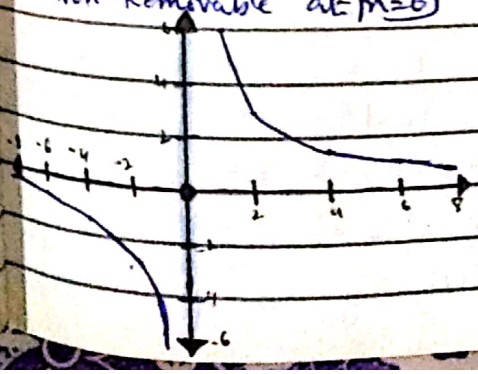


35-60

35) find, the x-values (if any) at which f is not continuous. which of the discontinuities are removable.

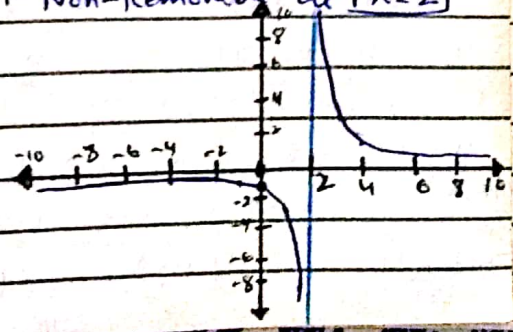
35) $f(x) = \frac{6}{x}$

:- Non-Removable at $\boxed{x = 0}$



36) $f(x) = \frac{3}{x - 2}$

:- Non-Removable at $\boxed{x = 2}$



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37) $f(x) = x^2 - 9$

:- x - [redacted] continuous at $x = -3$ and 3

Continuous at all $\mathbb{R}$.

38) $\frac{1}{4 - x^2}$

:- Discontinuous at $x = 2$ and $x = -2$

Non-removable Discontinuity.

38) $x^2 - 2x + 1$

:- $x^2 - x - x + 1$

$= x(x-1) - 1(x-1)$

$(x-1)(x-1)$ $\quad x = -1, -1$

2 equal $\mathbb{R}$ roots $\boxed{x = -1}$

Continuous at all $\mathbb{R}$ numbers

40) $f(x) = \frac{1}{x^2+1}$:- $x^2 + 1 = 0 \rightarrow x^2 = -1 \rightarrow x = \pm\sqrt{-1}$

$x = \pm i$

:- $f(x)$ is Continuous at all real numbers x.

41) $f(x) = 3x - \cos x$

:- $-\cos x + 3x$

$y = A\cos(Bx + c) + D$

$y = -1\cos(1(x) + 0) + 3x$

$y = -1\cos(x) + 3x$

$y = -\cos x + 3x$

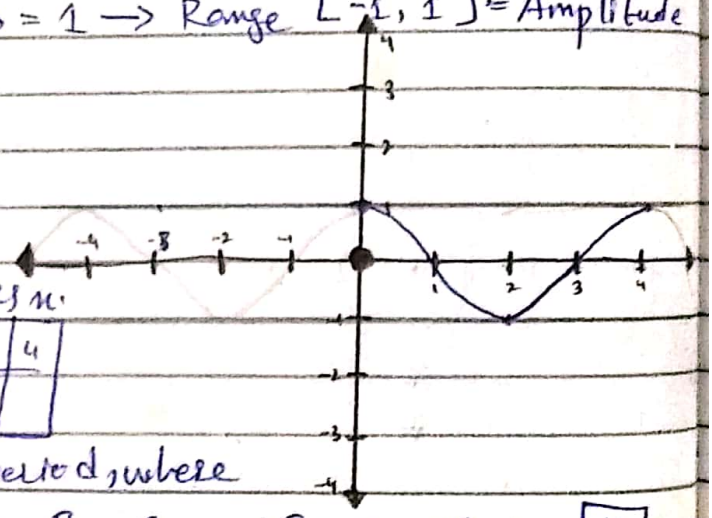
$A = -1, B = 1$

$A = +1$, $A > 0$, Upward

General Syntax $\rightarrow y = A\cos(Bx + c) + D$

$A = 1$, $B = \pi/2$

$A = 1 \rightarrow$ Range $[-1, 1]$ = Amplitude



42) $f(x) = \cos \frac{\pi x}{2}$

:- $f(x)$ is Continuous for all real numbers x.

x	-4	-3	-2	-1	0	1	2	3	4
y									

first of all find its period, where

period $\rightarrow \frac{2\pi}{B}$, Here $B = \frac{\pi}{2}$ So:- $2\pi \div \frac{\pi}{2} = 2\pi \times \frac{2}{\pi} = \boxed{4}$

$\boxed{\text{Period} = 4}$, As though $A > 0$, So the graph will start from the upperbound of the closed interval $\rightarrow [-1, 1]$, upper means $\boxed{\text{positive}}$ $\rightarrow$ that is 1 ✓, and it moves to downward. So, that the period is 4, it will have 4 Major points in each period. (with 5 steps include origin)

$\cos \frac{\pi(0)}{2} = 1$, $\cos \frac{\pi(1)}{2} = 0$, $\cos \frac{\pi(2)}{2} = -1$, $\cos \frac{\pi(3)}{2} = 0$, $\cos \frac{\pi(4)}{2} = 1$

$(0, 1)$ $\quad (1, 0)$ $\quad (2, -1)$ $\quad (3, 0)$, $\quad (4, 1)$,

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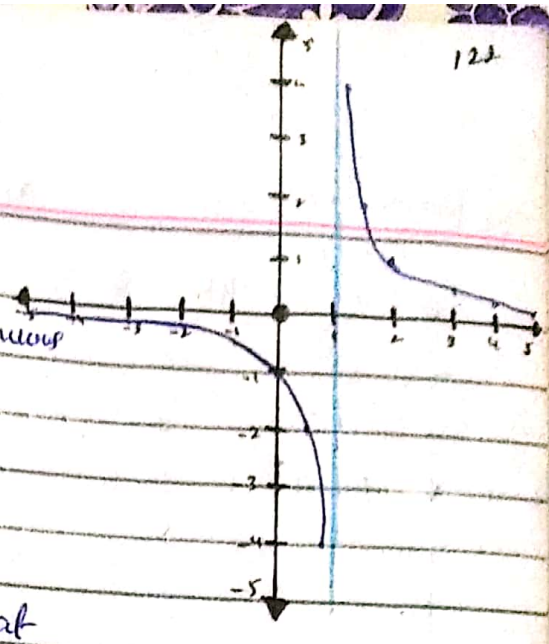
43) $f(x) = \dfrac{x}{x^2 - x}$

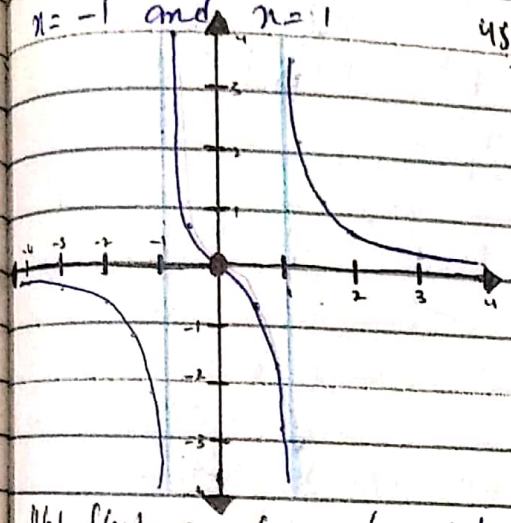
$\dfrac{x}{x(x-1)} = \dfrac{1}{x-1}$, $f(x)$ is discontinuous at $\boxed{x = 1}$

Non-Removable Discontinuity

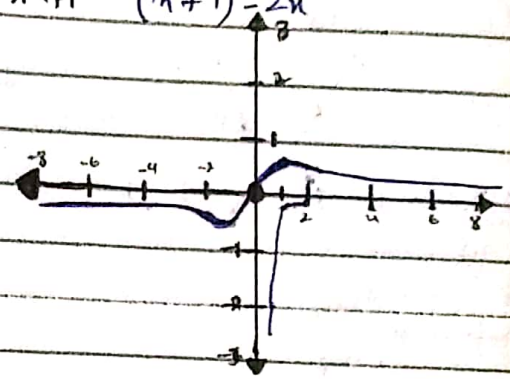


44) $f(x) = \dfrac{x}{x^2 - 1} = \dfrac{x}{(x-1)(x+1)}$

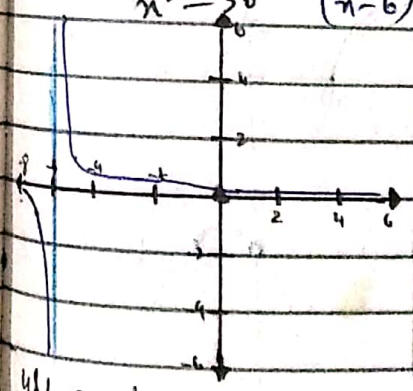
Non-Removable discontinuity at $x = -1$ and $x = 1$



45) $f(x) = \dfrac{x}{x^2+1} = \dfrac{x}{(x^2+1)^{\frac{1}{2}} - 2x}$ Continuous



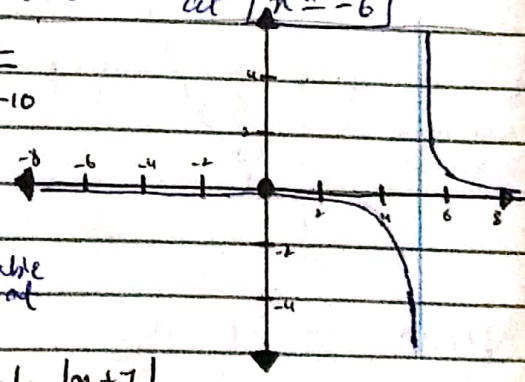
46) $f(x) = \dfrac{x-6}{x^2 - 36} = \dfrac{(x-6)}{(x-6)(x+6)} = \dfrac{1}{x+6}$, Non-Removable dis Continuity at $\boxed{x = -6}$



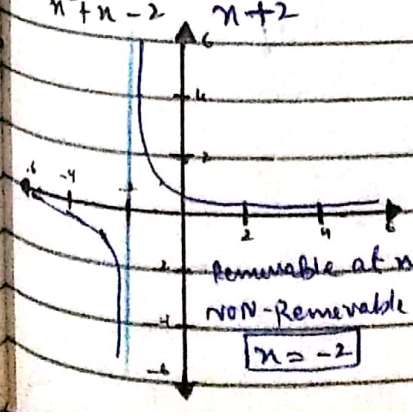
47) $f(x) = \dfrac{x+2}{x^2 - 3x - 10}$

$= \dfrac{1}{x-5}$

Non-Removable discontinuity at $\boxed{x = 5}$



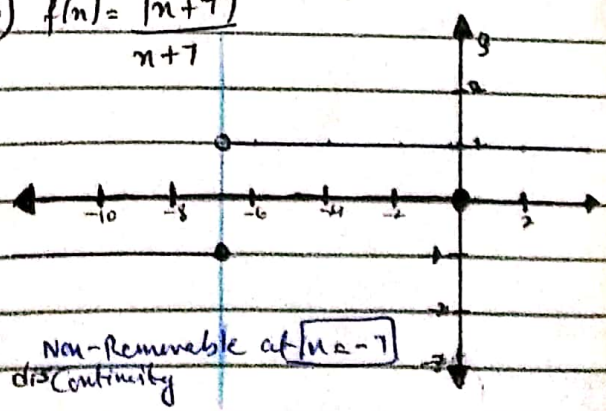
48) $\dfrac{x-1}{x^2 + x - 2} = \dfrac{1}{x+2}$



Removable at $x = 1$

NON-Removable at $\boxed{x = -2}$

49) $f(x) = \dfrac{|x+7|}{x+7}$



Non-Removable at $\boxed{x = -7}$ discontinuity

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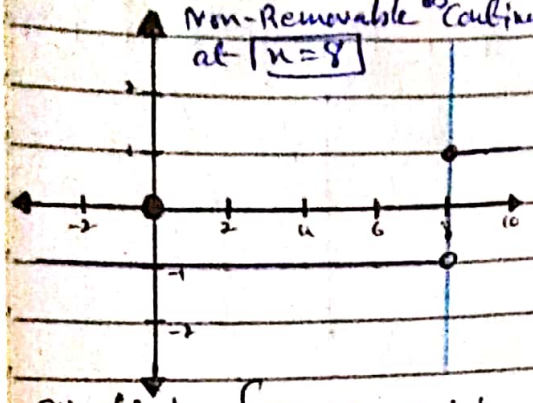
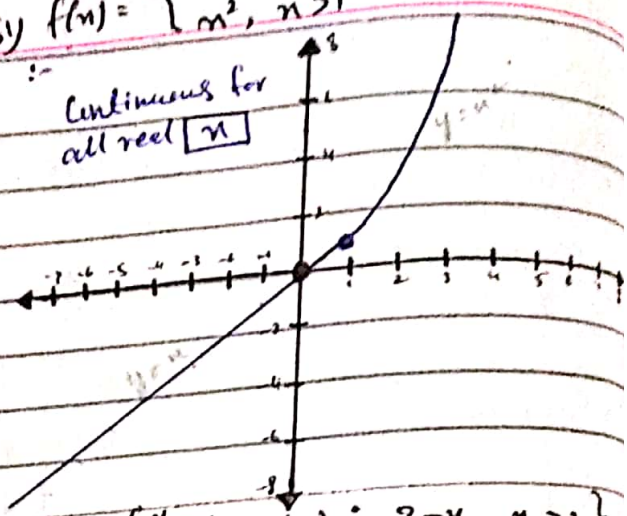
50) $f(x) = \frac{|x-8|}{x-8}$

Non-Removable discontinuity at $\boxed{x=8}$

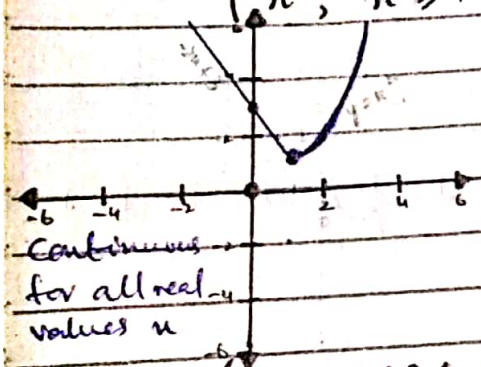


51) $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

:- Continuous for all real $\boxed{x}$



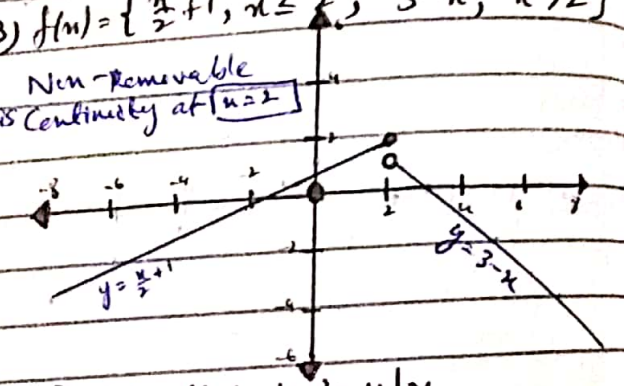
52) $f(x) = \begin{cases} -2x+3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$



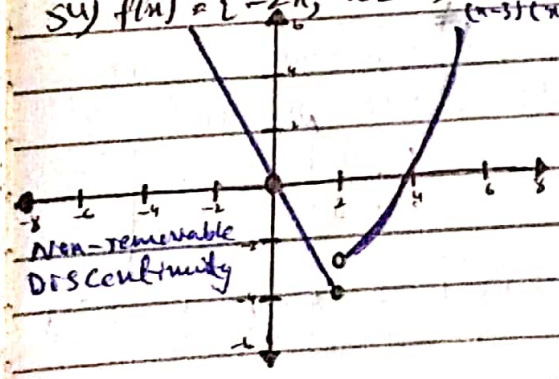
Continuous for all real values x

53) $f(x) = \{ \frac{x}{2}+1, x \le 2 ; 3-x, x > 2 \}$

Non-Removable discontinuity at $\boxed{x=2}$



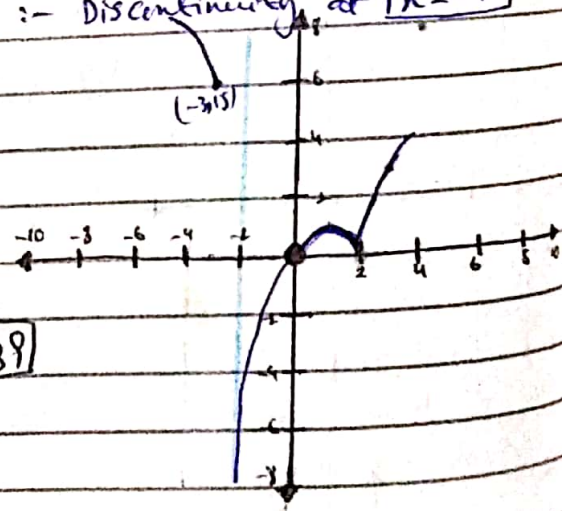
54) $f(x) = \{ -2x, x \le 2 ; \underset{(x-3)(x-1)}{x^2-4x+1}, x > 2 \}$



Non-Removable Discontinuity

61) $f(x) = \frac{|x^2-4|x}{x+2}$

:- Discontinuity at $\boxed{x=-2}$



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(13)

2.1 Differentiation

5-10, find the slope of the tangent line to the graph of the function at the given point.

5) $f(x) = 3 - 5x$, $(-1, 8)$

: formula to Calculate the slope :

$$\boxed{\lim_{\Delta x \to 0} = \frac{f(c + \Delta x) - f(c)}{\Delta x}}$$

:- $f(c + \Delta x) = 3 - 5(-1 + \Delta x)$, $c = -1$

$f(c) = 3 - 5(-1)$

$f(c + \Delta x) = 3 + 5 - 5\Delta x = 8 - 5\Delta x$

$f(c) = 3 + 5 = 8$

$$\lim_{\Delta x \to 0} = \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

$$\lim_{\Delta x \to 0} = \frac{3 - 5(c + \Delta x) - (3 - 5c)}{\Delta x}, \quad c = -1$$

$$\lim_{\Delta x \to 0} = 3 - 5(-1 + \Delta x) - (3 - 5(-1)) \div \Delta x$$

$$\lim_{\Delta x \to 0} = 3 + 5 - 5\Delta x - (3 + 5) \div \Delta x$$

$$= \cancel{8} - 5\Delta x - \cancel{8} \div \Delta x$$

$$\frac{-5\cancel{\Delta x}}{\cancel{\Delta x}} = -5$$

$$\boxed{\lim_{\Delta x \to 0} = -5}$$

6) $\frac{3}{2}x + 1$, $(-2, -2)$

:- $f(c + \Delta x) - f(c) \div \Delta x$, $c = -2 = x = -2$

$= 3/2\, x + 1 - (3/2\, x + 1) \div \Delta x$

$= 3/2\,(-2 + \Delta x) + 1 - (3/2\,(-2) + 1) \div \Delta x$

$= (-3 + 3\Delta x + 1) - (-3 + 1) \div \Delta x$

$= (-2 + 3\Delta x) - (-2) \div \Delta x$

$= (-2 + 3\Delta x) + 2 \div \Delta x$

$$\frac{3\cancel{\Delta x}}{\cancel{\Delta x}} = 3 \qquad \boxed{m = 3}$$

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7) $g(x) = x^2 - 8$, $(2, -5)$

:- $f(c + \Delta x) - f(c) \div \Delta x$, $c = 2$

$= \dfrac{[(c+\Delta x)^2 - 8] - (x^2 - 8)}{\Delta x}$

$= [(2 + \Delta x)^2 - 8] - (2^2 - 8) \div \Delta x$

$= [(4 + 4\Delta x + (\Delta x)^2) - 8] - (-5) \div \Delta x$

$= [((\Delta x)^2 + 4\Delta x - 5)] + 5 \div \Delta x$

$= \Delta x^2 + 4\Delta x \cancel{-8} \cancel{+8} \div \Delta x$

$= \dfrac{(\Delta x)^2 + 4\Delta x}{\Delta x} = \dfrac{\cancel{\Delta x}(\Delta x + 4)}{\cancel{\Delta x}} = \Delta x + 4$

$\lim_{\Delta x \to 0} = \Delta x + 4 = 0 + 4 = 4$

8) $g(x) = 6 - x^2$, $(1, 5)$

$\lim_{\Delta x \to 0} = \dfrac{f(c + \Delta x) - f(c)}{\Delta x}$, $\boxed{c = 1}$

$\lim_{\Delta x \to 0} = \dfrac{[6 - (c + \Delta x)^2] - (6 - c^2)}{\Delta x}$

$\lim_{\Delta x \to 0} = [6 - (1 + \Delta x)^2] - (6 - (1)^2) \div \Delta x$

$= [6 - (1 + 2\Delta x + (\Delta x)^2] - (6 - 1) \div \Delta x$

$= [6 - (1 + 2\Delta x + (\Delta x)^2] - 5 \div \Delta x$

$= [6 - 1 - 2\Delta x - (\Delta x)^2] - 5 \div \Delta x$

$= [\cancel{5} - 2\Delta x - (\Delta x)^2 - \cancel{5}] \div \Delta x$

$\dfrac{-2\Delta x - (\Delta x)^2}{\Delta x} = \dfrac{\cancel{\Delta x}[-2 - \Delta x)}{\cancel{\Delta x}} = -2 - \Delta x$

$\lim_{\Delta x \to 0} = -2 - \Delta x = -2 - (0) = -2$

$\boxed{\lim_{\Delta x \to 0} = -2}$

9) $3t - t^2$, $(0, 0)$

:- $\lim_{\Delta x \to 0} = \dfrac{f(c + \Delta x) - f(c)}{\Delta x}$, $c = 0$

$= [3(c + \Delta x) - (c + \Delta x)^2] - (3c - c^2) \div \Delta x$, $c = 0$

$= [3(0 + \Delta x) - (0 + \Delta x)^2] - (3(0) - (0)^2] \div \Delta x$

$= [3\Delta x - (\Delta x)^2] - (0) \div \Delta x$

$\lim_{\Delta x \to 0} = \dfrac{\cancel{\Delta x}(3 - \Delta x)}{\cancel{\Delta x}} = 3 - \Delta x = 3 - 0 = \boxed{3}$

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10) $h(t) = t^2+3$, $(-2, 7)$

$$\lim_{\Delta x \to 0} = \frac{f(c+\Delta x) - f(c)}{\Delta x}, \quad c = -2$$

$$\lim_{\Delta x \to 0} = \left[(c+\Delta x)^2 + 3\right] - (c^2+3) \div \Delta x$$

$$= \left[(-2+\Delta x)^2 + 3\right] - ((-2)^2+3) \div \Delta x$$

$$= \left[(-2)^2 + 2(-2)(\Delta x) + (\Delta x)^2 + 3\right] - (4+3) \div \Delta x$$

$$= \left[4 + (-4\Delta x) + (\Delta x)^2 + 3\right] - 7 \div \Delta x$$

$$= \left[7 - 4\Delta x + (\Delta x)^2\right] - 7 \div \Delta x$$

$$= \frac{-4\Delta x + \Delta x^2}{\Delta x} = \frac{\cancel{\Delta x}(-4+\Delta x)}{\cancel{\Delta x}} = -4 + \Delta x$$

$$\lim_{\Delta x \to 0} = -4 + \Delta x = -4 + 0 = -4$$

$$\boxed{\lim_{\Delta x \to 0} = -4}$$

In Exercises 11–24, find the derivative by the limit process.

11) $f(x) = 7$

$f(x) = 7 + 0x$

$$\lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = 7 + 0(x+\Delta x) - (7+0x) \div \Delta x$$

$$= \frac{7-7}{\Delta x} = \frac{0}{\Delta x} = 0$$

$$\boxed{\lim_{\Delta x \to 0} = 0}$$

12) $g(x) = -3$

$g(x) = -3 + 0x$

$$\lim_{\Delta x \to 0} = \frac{-3 + 0(x+\Delta x) - (-3+0x)}{\Delta x} = \frac{-3+0-(-3)}{\Delta x} = \frac{-3+3}{\Delta x} = 0$$

$$\boxed{\lim_{\Delta x \to 0} = 0}$$

13) $f(x) = -10x$

$$\lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-10(x+\Delta x) - (-10x)}{\Delta x}$$

$$= \frac{\cancel{-10x} - 10\Delta x + \cancel{10x}}{\Delta x} = \frac{-10\cancel{\Delta x}}{\cancel{\Delta x}} = -10$$

$$\boxed{\lim_{\Delta x \to 0} = -10}$$

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14) $f(x) = 3x + 2$

:- $\lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{[3(x+\Delta x)+2] - (3x+2)}{\Delta x}$

$\lim_{\Delta x \to 0} = \frac{[3x + 3\Delta x + 2] - 3x - 2}{\Delta x} = \frac{\cancel{3x} + 3\Delta x + \cancel{2} - \cancel{3x} - \cancel{2}}{\Delta x}$

$\lim_{\Delta x \to 0} = \frac{3\cancel{\Delta x}}{\cancel{\Delta x}} = 3$ $\boxed{\lim_{\Delta x \to 0} = 3}$

15) $h(s) = 3 + \frac{2}{3}s$

:- $\lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = 3 + \frac{2}{3}(x+\Delta x) - (3 + \frac{2}{3}x) \div \Delta x$

$\lim_{\Delta x \to 0} = 3 + \frac{2}{3}(x+\Delta x) - (3 + \frac{2}{3}x) \div \Delta x$

$\lim_{\Delta x \to 0} = \frac{9 + 2(x+\Delta x)}{3} - (\frac{9 + 2x}{3}) \div \Delta x$

$\lim_{\Delta x \to 0} = \frac{9 + 2x + 2\Delta x - 9 - 2x}{3} \div \Delta x$

$\lim_{\Delta x \to 0} = \frac{2\Delta x}{3} \div \Delta x = \frac{2\cancel{\Delta x}}{3} \times \frac{1}{\cancel{\Delta x}} = \frac{2}{3}$

$\boxed{\lim_{\Delta x \to 0} = 2/3}$

(8) $f(x) = x^3 - 12x$

:- $\lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{[(x+\Delta x)^3 - 12(\Delta x + x)] - (x^3 - 12x)}{\Delta x}$

$\lim_{\Delta x \to 0} = \frac{[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x - 12x] - (x^3 - 12x)}{\Delta x}$

$\lim_{\Delta x \to 0} = [\cancel{x^3} + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x - \cancel{12x} - \cancel{x^3} + \cancel{12x}] \div \Delta x$

$= 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - 12\Delta x \div \Delta x$

$= \frac{\cancel{\Delta x}(3x^2 + 3x(\Delta x) + (\Delta x)^2 - 12)}{\cancel{\Delta x}}$

$\lim_{\Delta x \to 0} = 3x^2 + 3x(\Delta x) + (\Delta x)^2 - 12$

$= 3x^2 + 3x(0) + (0)^2 - 12$

$\boxed{\lim_{\Delta x \to 0} = 3x^2 - 12}$

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10) $f(x) = x^3 + x^2$

$$\therefore \lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\left[(x+\Delta x)^3 + (x+\Delta x)^2\right] - (x^3+x^2)}{\Delta x}$$

$$\lim_{\Delta x \to 0} = \frac{\left[x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x(\Delta x) + (\Delta x)^2 - x^3 - x^2\right]}{\Delta x}$$

$$\lim_{\Delta x \to 0} = \frac{\left[\cancel{x^3} + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + \cancel{x^2} + 2x(\Delta x) + (\Delta x)^2 - \cancel{x^3} - \cancel{x^2}\right]}{\Delta x}$$

$$\lim_{\Delta x \to 0} = \frac{3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 2x(\Delta x) + (\Delta x)^2}{\Delta x}$$

$$\lim_{\Delta x \to 0} = \frac{\cancel{\Delta x}\left(3x^2 + 3x(\Delta x) + (\Delta x)^2 + 2x + (\Delta x)\right)}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} = 3x^2 + 3x(0) + (0)^2 + 2x + 0$$

$$\boxed{\lim_{\Delta x \to 0} = 3x^2 + 2x}$$

11) $f(x) = \dfrac{1}{x-1}$

$$\therefore \lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{1}{(x+\Delta x)-1} - \left(\frac{1}{x-1}\right) \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{1}{x+\Delta x-1} - \frac{1}{x-1} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{\cancel{x} - \cancel{1} - \cancel{x} - \Delta x + \cancel{1}}{x^2 + (\Delta x)x - x - x - \Delta x + 1} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{-\cancel{\Delta x}}{x^2 + x(\Delta x) - 2x - \Delta x + 1} \times \frac{1}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} = \frac{-1}{x^2 + x(\Delta x) - 2x - \Delta x + 1} = \frac{-1}{x^2 + x(0) - 2x - (0) + 1}$$

$$\lim_{\Delta x \to 0} = \frac{-1}{x^2 + 0 - 2x + 1} = \frac{-1}{x^2 - 2x + 1} = \frac{-1}{(x-1)(x-1)}$$

$$\boxed{\lim_{\Delta x \to 0} = \frac{-1}{(x-1)^2}}$$

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22) $f(x) = \frac{1}{x^2}$

$$:- \lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \left[\frac{1}{(x+\Delta x)^2} - \left(\frac{1}{x^2}\right)\right] \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{1}{(x+\Delta x)^2} - \frac{1}{x^2} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{x^2 - (x+\Delta x)^2}{x^2 (x+\Delta x)^2} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{x^2 - (x^2 + 2x(\Delta x) + (\Delta x)^2)}{x^2 (x^2 + 2x(\Delta x) + (\Delta x)^2)} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{\cancel{x^2} - \cancel{x^2} - 2x(\Delta x) - (\Delta x)^2}{x^4 + 2x^3(\Delta x) + x^2(\Delta x)^2} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{\cancel{\Delta x}(-2x - \Delta x)}{x^4 + 2x^3(\Delta x) + x^2(\Delta x)^2} \times \frac{1}{\cancel{\Delta x}}$$

$$\lim_{\Delta x \to 0} = \frac{-2x - \Delta x}{x^4 + 2x^3(\Delta x) + x^2(\Delta x)^2} = \frac{-2x - (0)}{x^4 + 2x^3(0) + x^2(0)^2}$$

$$\lim_{\Delta x \to 0} = \frac{-2x}{x^4 + 0 + 0} = \frac{-2x}{x^4} = \frac{-2}{x^3}$$

$$\boxed{\lim_{\Delta x \to 0} = \frac{-2}{x^3}}$$

23) $f(x) = \sqrt{x+4}$

$$:- \lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\left(\sqrt{(x+\Delta x)+4}\right) - \left(\sqrt{x+4}\right)}{\Delta x}$$

$$\lim_{\Delta x \to 0} = \frac{\sqrt{x+\Delta x+4} - \left(\sqrt{x+4}\right)}{\Delta x} \times \frac{\sqrt{x+\Delta x+4} + \sqrt{x+4}}{\sqrt{x+\Delta x+4} + \sqrt{x+4}}$$

$$\lim_{\Delta x \to 0} = \frac{\left[\sqrt{x+\Delta x+4} - \left(\sqrt{x+4}\right)\right] \times \left(\sqrt{x+\Delta x+4} + \sqrt{x+4}\right)}{\Delta x\left(\sqrt{x+\Delta x+4} + \sqrt{x+4}\right)}$$

$$\lim_{\Delta x \to 0} \frac{\left(\sqrt{x+\Delta x+4}\right)^2 - \left(\sqrt{x+4}\right)^2}{\Delta x\left(\sqrt{x+\Delta x+4} + \sqrt{x+4}\right)} = \frac{x+\Delta x+4-(x+4)}{\Delta x\left(\sqrt{x+\Delta x+4} + \sqrt{x+4}\right)}$$

$$\lim_{\Delta x \to 0} = \frac{\cancel{x} + \Delta x + \cancel{4} - \cancel{x} - \cancel{4}}{\Delta x\left(\sqrt{x+\Delta x+4} + \sqrt{x+4}\right)} = \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x+4} + \sqrt{x+4}\right)}$$

$$\lim_{\Delta x \to 0} = \frac{1}{\sqrt{x+0+4} + \sqrt{x+4}} = \frac{1}{\sqrt{x+4} + \sqrt{x+4}} = \frac{1}{2\sqrt{x+4}}$$

$$\boxed{\lim_{\Delta x \to 0} = \frac{1}{2\sqrt{x+4}}}$$

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04) $f(x) = \frac{4}{\sqrt{x}}$

$$\lim_{\Delta x \to 0} = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{4}{\sqrt{x+\Delta x}} - \frac{4}{\sqrt{x}} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}} \div \Delta x$$

$$\lim_{\Delta x \to 0} = \frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{(\sqrt{x}\sqrt{x+\Delta x})\Delta x} \times \frac{4\sqrt{x} + 4\sqrt{x+\Delta x}}{4\sqrt{x} + 4\sqrt{x+\Delta x}}$$

$$\lim_{\Delta x \to 0} = \frac{(4\sqrt{x})^2 - (4\sqrt{x+\Delta x})^2}{\Delta x(\sqrt{x}\sqrt{x+\Delta x}) \times 4\sqrt{x} + 4\sqrt{x+\Delta x}}$$

$$\lim_{\Delta x \to 0} = \frac{16x - (16(x+\Delta x))}{\Delta x(5\sqrt{x} + 5\sqrt{x+\Delta x})}$$

$$\lim_{\Delta x \to 0} = \frac{16x - (16x + 16\Delta x)}{\Delta x(4\sqrt{x}(\sqrt{x}\sqrt{x+\Delta x}) + 4\sqrt{x+\Delta x}(\sqrt{x} \times \sqrt{x+\Delta x}))}$$

$$\lim_{\Delta x \to 0} = \frac{\cancel{16x} - \cancel{16x} - 16\Delta x}{\Delta x(5\sqrt{x} \times 4\sqrt{x}\sqrt{x+\Delta x} + 4\sqrt{x}\sqrt{x+\Delta x} \times 5\sqrt{x+\Delta x}}$$

$$\lim_{\Delta x \to 0} = \frac{-16\cancel{\Delta x}}{\cancel{\Delta x}(5\sqrt{x} \times 4\sqrt{x}\sqrt{x+\Delta x} + 4\sqrt{x}\sqrt{x+\Delta x} \times 5\sqrt{x+\Delta x}}$$

$$\lim_{\Delta x \to 0} = \frac{-16}{(5\sqrt{x} \times 4\sqrt{x}\sqrt{x+\Delta x} + 4\sqrt{x} \times \sqrt{x+\Delta x} \times 5\sqrt{x+\Delta x}}$$

$$\lim_{\Delta x \to 0} = \frac{-16}{5\sqrt{x} \times 4\sqrt{x}\sqrt{x+0} + 4\sqrt{x} \times \sqrt{x+0} \times 5\sqrt{x+0}}$$

$$\lim_{\Delta x \to 0} = \frac{-16}{(5\sqrt{x} \times 4\sqrt{x} \times \sqrt{x}) + (4\sqrt{x} \times \sqrt{x} \times 5\sqrt{x})}$$

$$\lim_{\Delta x \to 0} = \frac{-16}{(10\sqrt{x}) + (10\sqrt{x})}$$

$$\lim_{\Delta x \to 0} = \frac{-16}{20\sqrt{x}} = \frac{-4}{5\sqrt{x}}$$

$$\boxed{\lim_{\Delta x \to 0} = \frac{-4}{5\sqrt{x}}}$$

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▶ 25–32 (a) find an equation of the tangent line to the graph of f at the given point:

25) $f(x) = x^2 + 3$, $(1,4)$

1st step → find slope (derivative by limit process)

2nd step → put slope in point $(1,4)$, find an equation

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{[(c+\Delta x)^2+3] - (c^2+3)}{\Delta x}, \boxed{c=1}$$

$$\lim_{\Delta x \to 0} \frac{[(1+\Delta x)^2+3] - (1^2+3)}{\Delta x} = \frac{[1+2\Delta x+(\Delta x)^2+3] - 4}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{(\Delta x)^2 + 2\Delta x + \cancel{4} - \cancel{4}}{\Delta x} = \frac{(\Delta x)^2 + 2\Delta x}{\Delta x} = \Delta x + 2$$

$$\lim_{\Delta x \to 0} \Delta x + 2 = 0 + 2 = 2 \quad \boxed{\lim_{\Delta x \to 0} = 2}$$

$\boxed{m = 2}$ $(1,4)$ find an equation by using point slope form.

$$y_2 - y_1 = m(x_2 - x_1) \rightarrow y_2 - 4 = (2)(x-1)$$

$$\boxed{y = 2x + 2}$$

(b) Sketch curve and line

$y = x^2 + 3$

$b^2 - 4ac < 0$

$\boxed{0 - 12 < 0}$

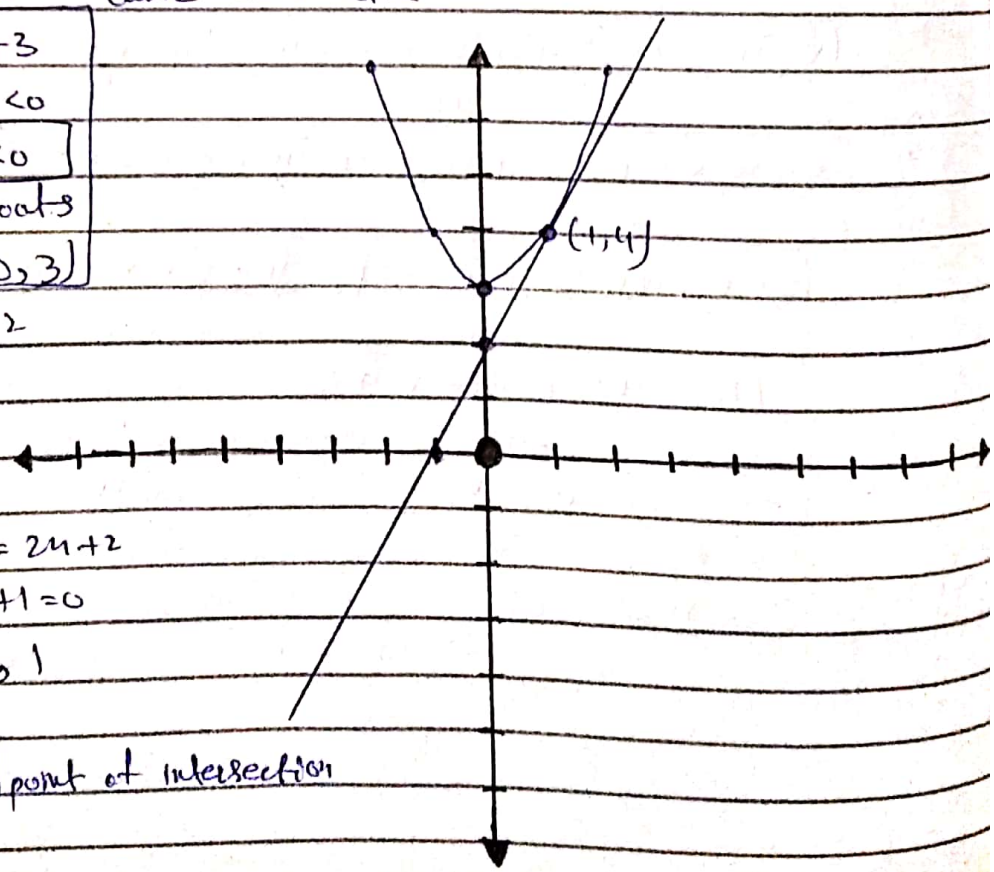
No real roots

Vertex $(0,3)$

$y = 2x + 2$

$(0,2)$

$(-1,0)$



$x^2 + 3 = 2x + 2$

$x^2 - 2x + 1 = 0$

$x = 1, 1$

$y = 4$

$\boxed{(1,4)}$ point of intersection

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26) $f(x) = x^2+3x+4, \quad (-2, 2)$

$\because f(c+\Delta x) = (c+\Delta x)^2 + 3(c+\Delta x) + 4$

$\Rightarrow f(-2+\Delta x) = (-2+\Delta x)^2 + 3(-2+\Delta x) + 4$

$f(c) = f(-2) = (-2)^2 + 3(-2) + 4$

$= \cancel{4} - 6 + \cancel{4} = +2$

$\boxed{f(c) = -6}$

$\frac{f(c+\Delta x) - f(c)}{\Delta x} = \frac{[(-2+\Delta x)^2 + 3(-2+\Delta x) + 4] - (2)}{\Delta x}$

$= [(4 + (-2)(2)(\Delta x) + (\Delta x)^2 + 3(-2+\Delta x) + 4] - (2) \div \Delta x$

$= [(4 - 4\Delta x + (\Delta x)^2 - 6 + 3\Delta x + 4] - 2 \div \Delta x$

$= [\Delta x^2 - \Delta x + 2] - 2 \div \Delta x$

$= [\Delta x^2 - \Delta x + 2 - 2] \div \Delta x$

$= \frac{[\Delta x^2 - \Delta x]}{\Delta x} = \frac{\Delta x(\Delta x - 1)}{\Delta x} = \Delta x - 1 = \boxed{-1}$

$m = -1, \quad (-2, 2) \quad y - 2 = -1(x+2)$

$\boxed{y = -x}$

27) $f(x) = x^3, \quad (2, 8)$

$\because \bullet f(c+\Delta x) = f(2+\Delta x) = (2+\Delta x)^3$

$\bullet f(c) = f(2) = (2)^3 = 8$

$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = [(2+\Delta x)^3] - (8) \div \Delta x$

$= [2^3 + 3(2)^2(\Delta x) + 3(2)(\Delta x)^2 + (\Delta x)^3] - 8 \div \Delta x$

$= [8 + 12\Delta x + 6(\Delta x)^2 + (\Delta x)^3] - 8 \div \Delta x$

$= (6(\Delta x)^2 + 12\Delta x + (\Delta x)^3) \div \Delta x$

$\cancel{\Delta x}(\Delta x^2 + 6\Delta x + 12) \div \cancel{\Delta x}$

$(0)^2 + 6(0) + 12 = 12$

$m = 12, \quad (2, 8) \rightarrow y - 8 = 12(x-2) \Rightarrow \boxed{y = 12x - 16}$

28) $f(x) = x^3 + 1, \quad (1, 2)$

$\because f(c+\Delta x) - f(c) \div \Delta x = [(1+\Delta x)^3 + 1] - (2) \div \Delta x$

$\frac{dy}{dx} = \lim_{\Delta x \to 0} [(1)^3 + 3(1)^2(\Delta x) + 3(1)(\Delta x)^2 + (\Delta x)^3 + 1] - 2 \div \Delta x$

$= \lim_{\Delta x \to 0} [1 + 3\Delta x + 3(\Delta x)^2 + (\Delta x)^3 + 1] - 2 \div \Delta x$

$\lim_{\Delta x \to 0} [(\Delta x)^3 + 3(\Delta x)^2 + 3\Delta x + \cancel{2}] - \cancel{2} \div \Delta x$

$\lim_{\Delta x \to 0} [(\Delta x)^2 + 3(\Delta x) + 3] \div \cancel{\Delta x}$

$= [(0)^2 + 3(0) + 3] = \boxed{3}$

$m = 3, \quad (1, 2)$

$y - 2 = 3(x-1)$

$\boxed{y = 3x - 1}$

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28) $f(x) = \sqrt{x}$, $(1,1)$

:- $f(c+\Delta x) - f(c) \div \Delta x = f(1+\Delta x) - f(1) \div \Delta x$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left[\left(\sqrt{1+\Delta x}\right) \right] - \left(\sqrt{1}\right) \div \Delta x$$

$$\lim_{\Delta x \to 0} \left[\left(\sqrt{1+\Delta x}\right) \right] - 1 \div \Delta x$$

$$\lim_{\Delta x \to 0} \left(\sqrt{1+\Delta x}\right) - 1 \div \Delta x$$

$$= \frac{\sqrt{1+\Delta x} - 1}{\Delta x} \times \frac{\sqrt{1+\Delta x} + 1}{\sqrt{1+\Delta x} + 1}$$

$$= \frac{\left(\sqrt{1+\Delta x}\right)^2 - (1)^2}{\Delta x\left(\sqrt{1+\Delta x} + 1\right)} = \frac{\cancel{1} + \Delta x - \cancel{1}}{\cancel{\Delta x}\left(\sqrt{1+\Delta x} + 1\right)} = \frac{1}{\sqrt{1+\Delta x} + 1}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{1+\Delta x} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{\sqrt{1} + 1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

$m = 1/2$, $(1,1)$ $(y-1) = \frac{1}{2}(x-1) \rightarrow 2y - 2 = x - 1$

tangent line $\rightarrow$ $\boxed{2y = x + 1}$

30) $f(x) = \sqrt{x-1}$, $(5,2)$

:- $$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \left(\sqrt{(5+\Delta x) - 1}\right) - \left(\sqrt{5-1}\right) \div \Delta x$$

$$= \left(\sqrt{5+\Delta x - 1}\right) - (2) \div \Delta x$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\left(\sqrt{5+\Delta x - 1}\right) - 2}{\Delta x} \times \frac{\left(\sqrt{5+\Delta x - 1}\right) + 2}{\left(\sqrt{5+\Delta x - 1}\right) + 2}$$

$$= \frac{\left(\sqrt{5+\Delta x - 1}\right)^2 - (2)^2}{\Delta x\left(\sqrt{5+\Delta x - 1}\right) + 2)} = \frac{\left(\sqrt{5+\Delta x - 1}\right)^2 - 4}{\Delta x\left(\sqrt{5+\Delta x - 1}\right) + 2)} = \frac{5 + \Delta x - 1 - 4}{\Delta x\left(\sqrt{5+\Delta x - 1} + 2\right)}$$

$$= \frac{5 + \Delta x - 1 - 4}{\Delta x\left(\sqrt{5+\Delta x - 1} + 2\right)} = \frac{\cancel{\Delta x}}{\cancel{\Delta x}\left(\sqrt{5+\Delta x - 1} + 2\right)} = \frac{1}{\sqrt{5+\Delta x - 1} + 2}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{5+\Delta x - 1} + 2} = \frac{1}{\sqrt{5+0-1} + 2} = \frac{1}{\sqrt{4} + 2} = \boxed{\frac{1}{4}}$$

$\Rightarrow m = 1/4$, $(5,2) \rightarrow 4(y-2) = 1(x-5)$

$\boxed{4y = x + 3} \rightarrow$ tangent line

31) $f(x) = x + \frac{4}{x}$, $(4,5)$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \left[\frac{(4+\Delta x)^2 + 4}{(4+\Delta x)}\right] - (5) \div \Delta x$$

$$= \left[\frac{16 + 8(\Delta x) + (\Delta x)^2 + 4}{4 + \Delta x} - 5\right] \div \Delta x$$

$$= \left[\frac{16 + 8(\Delta x) + (\Delta x)^2 + 4 - 5(4+\Delta x)}{4 + \Delta x}\right] \div \Delta x$$

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$$= \left[\frac{\cancel{20} + 8(\Delta x)x + (\Delta x)^2 - \cancel{26} - 5\Delta x}{4 + \Delta x}\right] \div \Delta x$$

$$= \left[\frac{(\Delta x)^2 + 3\Delta x - 4}{4 + \Delta x}\right] \div \Delta x$$

$$= \left[\frac{(\Delta x)^2 + 4\Delta x - \Delta x - 4}{4 + \Delta x}\right] \div \Delta x \rightarrow \left[\frac{\Delta x(\Delta x + 4) - 1(\Delta x + 4)}{(4 + \Delta x)}\right] \div \Delta x$$

$$= \left[\frac{(\Delta x - 1)\cancel{(\Delta x + 4)}}{\cancel{(4 + \Delta x)}}\right] \div \Delta x \rightarrow \frac{(\Delta x - 1)}{\Delta x} = \frac{\Delta x}{\Delta x} - \frac{1}{\Delta x} = \boxed{1 - \frac{1}{\Delta x}}\ ?$$

$\cancel{\Delta x}(8x + \Delta x - 5) \div \cancel{\Delta x}$

$\boxed{8x - 5}$

32) $f(x) = \dfrac{1}{x+1}$, $(0, 1)$

$$= \frac{f(c + \Delta x) - f(c)}{\Delta x} = \left[\frac{1}{[(0 + \Delta x) + 1]}\right] - (1) \div \Delta x$$

$$= \left(\frac{1}{\Delta x + 1} - 1\right) \div \Delta x \rightarrow \frac{1 - (\Delta x + 1)}{(\Delta x + 1)} \div \Delta x$$

$$= \frac{-\Delta x}{\Delta x + 1} \div \Delta x \rightarrow \frac{-\cancel{\Delta x}}{\Delta x + 1} \times \frac{1}{\cancel{\Delta x}} = \frac{-1}{\Delta x + 1} = \frac{-1}{0 + 1} = \boxed{-1}$$

$\boxed{m = -1}$, $(0, 1) \rightarrow y - 1 = -1(x - 0)$

$\boxed{y = 1 - x}$

33 – 38 find an equation of the line that is tangent to the graph of f and parallel to given line.

33) $f(x) = x^2$, $2x - y + 1 = 0$

1: derivative of curve x^2, derivative of line $y = 2x + 1$

$x^2 \rightarrow \dfrac{dy}{dx} = \boxed{2x}$ $\qquad \dfrac{dy}{dx} = \boxed{2}$

$\boxed{y = 2x}$ and $y = 2x + 1$

2: equating method, Compare both derivative and slope // line

$2x = 2 \rightarrow$ slope of line

$\boxed{x = 1}$

3: put $x = 1$ in curve's function

$y = x^2$, $x = 1$ $\quad$ 4: put $(1, 1)$ in point-slope form

$y = (1)^2 = 1$ $\quad \boxed{m = 2}$ $\quad y - 1 = 2(x - 1)$

$\boxed{(1, 1)}$ $\quad \rightarrow$ // line $\quad \boxed{y = 2x - 1}$ ✓

$\boxed{34 \text{ to } 38 \text{ on page no } 140}$

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Exercise 3.2 Basic differentiation and Rates of change

▸ Find the derivative of function.

3. $y = 12$
$= 0$

4. $f(x) = -9$
$= 0$

5. $y = x^7$
$= 7x^6$

6. $y = x^{16}$
$= 16x^{15}$

7. $y = \frac{1}{x^5} = \frac{1}{x^5} = x^{-5} = -5x^{-5-1} = -5x^{-6} = \frac{-5}{x^6}$

8. $y = \frac{1}{x^8} = x^{-8} = -8x^{-8-1} = -8x^{-9} = \frac{-8x}{x^9}$

9. $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}} = \frac{x^{\frac{1}{5}-1}}{5} = \frac{x^{-\frac{4}{5}}}{5} = \frac{1}{5x^{4/5}}$

10. $g(x) = \sqrt[4]{x} = x^{\frac{1}{4}} = \frac{x^{\frac{1}{4}-1}}{4} = \frac{x^{-\frac{3}{4}}}{4} = \frac{1}{4x^{3/4}}$

11. $f(x) = x+11 = 1+0 = 1$

12. $g(x) = 3x-1 = 3(1)-0 = 3$

13. $f(t) = -2t^2+3t-6 = -4t+3(1)+0 = -4t+3$

14. $y = t^2+2t-3 = 2t+2(1)+0 = 2t+2$

15. $g(x) = x^2+4x^3 = 2x+12x^2$

16. $y = 8-x^3 = 0-3x^2 = -3x^2$

17. $s(t) = t^3+5t^2-3t+8 = 3t^2+10t-3(1)+0 = 3t^2+10t-3$

18. $f(x) = 2x^3-x^2+3x = 6x^2-2x+3(1) = 6x^2-2x+3$

19. $y = \frac{\pi}{2}\sin\theta - \cos\theta = \frac{\pi}{2}\cos\theta - (-\sin\theta) = \frac{\pi}{2}\cos\theta + \sin\theta$

20. $g(t) = \sqrt{\pi}\cos t = \sqrt{\pi}(-\sin)t$

21. $y = x^2 - \frac{1}{2}\cos x = \frac{4x^2}{2} - (-\sin x) = \frac{2x^2}{2} + \sin x = \frac{4x}{2} + \sin x$

22. $y = 7+\sin x = 7+\cos x$

23. $y = \frac{1}{x} - 3\sin x = \frac{1}{x} - 3(\cos x) = 1 - 3\cos x$

$\frac{1}{x} - 3\sin x = x^{-1} - 3\sin x = -1x^{-2} - 3(\cos x) = \boxed{\frac{-1}{x^2} - 3\cos x}$

24. $y = \frac{5}{(2x)^3} + 2\cos x = \frac{5(8x^{-3})}{8} + 2(-\sin x) = \frac{-15x^{-4}}{8} - \sin x = \frac{-15}{8x^4} - \sin x$

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In Exercises 25–30 Complete the table

Original	Rewrite	Differentiate	Simplify
25. $y = \frac{5}{2x^2}$	$\frac{5x^{-2}}{2}$	$\frac{-10x^{-3}}{2}$	$\frac{-5}{x^3}$
26. $y = \frac{2}{3x^2}$	$\frac{2x^{-2}}{3}$	$\frac{-4x^{-3}}{3}$	$\frac{-4}{3x^3}$
27. $y = \frac{6}{(5x)^3}$	$\frac{6x^{-3}}{125}$	$\frac{-18x^{-4}}{125}$	$\frac{-18}{125x^4}$
28. $y = \frac{\pi}{(3x)^2}$	$\frac{\pi x^{-2}}{9}$	$\frac{-2\pi x^{-3}}{9}$	$\frac{-2\pi}{9x^3}$
29. $y = \frac{\sqrt{x}}{x}$	$\sqrt{x} \times x^{-1}$ $x^{\frac{1}{2}} \times x^{-1}$	$\frac{-1\sqrt{x} \times x^{-2}}{x^{-\frac{1}{2}}}$	$\frac{-\sqrt{x}}{x^2}$ $\frac{1}{\sqrt{x}}$
30. $\frac{\sqrt{x}}{x}$	$\frac{1}{2}x^{\frac{1}{2}-1} \times x^{-1}$	$\frac{x^{-\frac{1}{2}} \times -x^{-1-1}}{2}$	$\frac{-x^{-5/2}}{2}$ $\frac{-1}{2x^{5/2}}$
30. $\frac{4}{x^{-3}}$	$4x^3$	$12x^2$	$\frac{12}{x^{-2}}$

31–38 find the slope of graph of function at given point.

31. $f(x) = \frac{8}{x^2}$, $(2,2)$

$8x^{-2} = -16x^{-3} = \frac{-16}{x^3}$

$x = 2$

$\frac{-16}{(2)^3} = \frac{-16}{8} = \boxed{-2}$

32. $3 - \frac{3}{5t}$, $(3/5, 2)$

$0 - \frac{3t^{-1}}{5} = -\frac{(-3)t^{-2}}{5} = \frac{3}{5t^2}$

$t = 3/5$

$\frac{3}{5\left(\frac{3}{5}\right)^2} = \frac{3}{5\left(\frac{9}{25}\right)} = 3 \div \frac{9}{5} = \boxed{\frac{5}{3}}$

33. $f(x) = -\frac{1}{2} + \frac{7}{5}x^3$, $(0, -1/2)$

$0 + \frac{3(7)x^{3-1}}{5} = \frac{21x^2}{5}$

$x = 0$

$\frac{21(0)^2}{5} = \frac{0}{5} = 0$

34. $y = 3x^3 - 10$ $(2, 14)$

$3(3x^{3-1}) - 0 = 9x^2$

$x = 2 \rightarrow 9(2)^2 = 9(4) = 36$

35. $y = (4x+1)^2$, $(0, 1)$

$16x^2 + 8x + 1$: $32x + 8 + 0 = 32x + 8$

$x = 0 \rightarrow 32(0) + 8 = 0 + 8 = \boxed{8}$

36. $f(x) = 3(5-x)^2$, $(5, 0)$

$3(25 - 10x + x^2) = 75 - 30x + 3x^2$

$x = 5 \rightarrow 75 - 30(5) + 3(5)^2$

$75 - 150 + 75 = \boxed{0}$

37. $4\sin\theta - \theta$, $(0, 0)$

$= 4(\cos\theta) - \theta = 4(\cos 0) - 0$

$= 4(1) = 4$

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38. $g(t) = -2\cos t + 5$, $(\pi, 1)$

$: -2(-\sin t) + 5 = 2\sin t + 5 = 2\sin\pi + 5 = 2(1) + 5 = \boxed{7}$

▶39–54 find derivative of function

39. $f(x) = x^2 + 5 - 3x^{-2}$

$: 2x + 0 - (-6)x^{-3}$

$2x + 6x^{-3}$

$= 2x + \frac{6}{x^3}$

40. $f(x) = x^2 - 3x - 3x^{-2}$

$: 2x - 3 - 3(-2)x^{-3}$

$2x - 3 + 6x^{-3}$

$= 2x - 3 + \frac{6}{x^3}$

41. $g(t) = t^2 - \frac{4}{t^3}$

$: 2t - 4(-3)t^{-4}$

$2t + \frac{12}{t^4}$

42. $f(x) = x + \frac{1}{x^2}$

$: 1 + 1(-2)x^{-3}$

$1 - \frac{2}{x^3}$

43. $\frac{4x^3 + 3x^2}{x}$

$: \frac{12x^2 + 6x}{x} = (12x^2 + 6x)x^{-1}$

$: \frac{-12x^2 + 6x}{x^2}$

44. $\frac{x^3 - 6}{x^2}$

$: (3x^2 - 0)(-2)x^{-2-1}$

$(3x^2)(-2)x^{-3}$

$\frac{-6x^2}{x^3}$

45. $f(x) = \frac{x^3 - 3x^2 + 4}{x^2}$

$: (3x^2 - 6x + 0)x^{-2} \Rightarrow (3x^2 - 6x)(-2)x^{-2-1}$

$(3x^2 - 6x)(-2)x^{-3} \Rightarrow \frac{-6x^2 + 12x}{x^3} = \frac{12x - 6x^2}{x^3}$

46. $y = 3x(6x - 5x^2)$

$= 18x^2 - 15x^3 \Rightarrow \boxed{36x - 45x^2}$

47. $x(x^2 + 1)$

$= x^3 + x$

$\boxed{3x^2 + 1}$

48. $f(x) = \sqrt{x} - 6\sqrt[3]{x}$

$: \frac{1}{2}x^{\frac{1}{2}-1} - 6\left(\frac{1}{3}x^{\frac{1}{3}-1}\right) = \frac{x^{-\frac{1}{2}}}{2} - 6\left(\frac{x^{-\frac{2}{3}}}{3}\right) = \frac{1}{2x^{1/2}} - \frac{6}{x^{2/3}}$

$= \frac{1}{2x^{1/2}} - \frac{6}{3x^{2/3}} = \frac{1}{2\sqrt{x}} - \frac{6}{3x^{2\times\frac{1}{3}}} = \frac{1}{2\sqrt{x}} - \frac{\cancel{6}^{2}}{\cancel{3}_{1}\sqrt[3]{x^2}} = \boxed{\frac{1}{2\sqrt{x}} - \frac{2}{\sqrt[3]{x^2}}}$

50. $f(x) = \sqrt[3]{x} + \sqrt[5]{x}$

$: x^{1/3} + x^{1/5}$

$\frac{1}{3}(x^{\frac{1}{3}-1}) + \left(\frac{1}{5}x^{\frac{1}{5}-1}\right)$

$\frac{x^{-2/3}}{3} + \frac{x^{-4/5}}{5}$

$\boxed{\frac{1}{3\sqrt[3]{x^2}} - \frac{1}{5\sqrt[5]{x^4}}}$

51. $h(s) = s^{4/5} - s^{2/3}$

$= \frac{4}{5}s^{\frac{4}{5}-1} - \frac{2}{3}s^{\frac{2}{3}-1} = \frac{4}{5}s^{-\frac{1}{5}} - \frac{2s^{-\frac{1}{3}}}{3}$

$: \frac{4}{5s^{\frac{1}{5}}} - \frac{2}{3s^{1/3}} = \frac{4}{5\sqrt[5]{s}} - \frac{2}{3\sqrt[3]{s}}$

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52. $f(t) = t^{2/3} - t^{1/3} + 4$

$= \frac{2}{3} t^{\frac{2}{3}-1} - \frac{1}{3} t^{\frac{1}{3}-1} + 0$

$= \frac{2}{3} t^{-\frac{1}{3}} - \frac{1}{3} t^{-\frac{2}{3}}$

$= \frac{2}{3\sqrt[3]{t}} - \frac{1}{3\sqrt[3]{t^2}}$

53. $f(x) = 6\sqrt{x} + 5\cos x$

$= 6\left(\frac{1}{2} x^{\frac{1}{2}-1}\right) + 5(-\sin x)$

$= 3x^{-\frac{1}{2}} - 5\sin x$

$\frac{3}{\sqrt{x}} - 5\sin x$

45) $f(x) = \frac{x^3 - 3x^2 + 4}{x^2}$

$= \frac{(x^2)(3x^2 - 6x) - (x^3 - 3x^2 + 4)(2x)}{x^4}$

$\frac{3x^4 - 6x^3 - 2x^4 - 6x^3 + 8x}{x^4} = \frac{x^4 - 12x^3 + 8x}{x^4}$

$\frac{x(x^3 - 12x^2 + 8)}{x(x^3)} = \frac{x^3 - 12x^2 + 8}{x^3}$

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☆ Related to (1.4)

63) $f(x) = \begin{cases} 3x^2, & x \geq 1 \\ ax-4, & x < 1 \end{cases}$

$\because 3x^2 = ax-4$

put $x = 1$

$3(1)^2 = a(1)-4$

$3 = a-4$

$\boxed{7 = a}$

$\boxed{\begin{matrix} 3x^2, & x \geq 1 \\ 7x-4, & x < 1 \end{matrix}}$

64) $f(x) = \begin{cases} 3x^3, & x \leq 1 \\ ax+5, & x > 1 \end{cases}$

$\because 3x^3 = ax+5$

put $x = 1$

$3(1)^3 = a(1)+5$

$3 = a+5$

$\boxed{a = -2}$

$\boxed{\begin{matrix} 3x^3, & x \leq 1 \\ x+5, & x > 1 \end{matrix}}$

65. $f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2, & x > 2 \end{cases}$

$x^3 = ax^2$

Put $x = 2$

$(2)^3 = a(2)^2$

$8 = 4a$

$\boxed{2 = a}$

66. $g(x) = \begin{cases} \dfrac{4\sin x}{x}, & x < 0 \\ a-2x, & x \geq 0 \end{cases}$

$\because \dfrac{4\sin x}{x} = a-2x$ put, $\boxed{x=0}$

$4\left(\dfrac{\sin x}{x}\right) = a-2x \rightarrow 4(1) = a-2x$

$4 = a-2x$, put $x=0$

$4 = a-2(0)$

$\boxed{a = 4}$

67. $\begin{cases} 2, & x \leq -1 \\ ax+b, & -1 < x < 3 \\ -2, & x \geq 3 \end{cases}$

① $y = 2, x \leq -1$ ② $y = ax+b, -1 < x < 3$, ③ $y = -2, x \geq 3$

$y = 2$, $y = ax+b$

$2 = ax+b$, equating

Domain of ② is $-1 < x < 3$ and -1 is also of ① both are equal, so solve it

$\boxed{2 = ax+b} \rightarrow$ equation (i)

Put $\boxed{x = -1}$

$\boxed{b-a=2} \rightarrow$ equation (1)

② $y = ax+b, -1 < x < 3$

③ $y = -2, x \geq 3$

3 is in the Domain of both ②'s and ③, so, Solve it

$y = y \rightarrow ax+b = -2$

Put $x = 3$

$\boxed{3a+b=-2} \rightarrow$ equation 2

$3a+b = -2$

$-a+b = 2$

$+ \quad - \quad -$

$4a = -4$

$\boxed{a = -1}$

Put $a = -1$ in any one equation

$b-a = 2$, $a = -1$

$b-(-1) = 2$

$b+1 = 2$

$\boxed{b = 1}$

$(a,b) = (-1,1)$

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find an equation of the line that is tangent to the graph of f and parallel to given line.

34) $2x^2$; $4x + y + 3 = 0$ $\boxed{\text{Related to Q.1}}$

1: derivative of curve and slope of line

$y = 2x^2$ $\quad 4x + y + 3 = 0$

$\frac{dy}{dx} = 4x$ $\quad y = -4x - 3$

$\boxed{m = -4}$

2: equating method on derivative and slope

$4x = -4 \Leftrightarrow \frac{dy}{dx} = m$

$\boxed{x = -1}$

3: put $x = -1$ in curve's function

$y = 2x^2$

$y = 2(-1)^2 = 2$

$\boxed{\text{point } (-1, 2)}$

$m = -4$

4: point slope form and an equation

$y - y_1 = m(x - x_1)$, $(-1, 2)$, $m = -4$

$y - 2 = -4(x + 1)$

$y - 2 = -4x - 4$

$y = -4x - 2$

equation of tangent line $\longrightarrow$ $\boxed{y + 4x + 2}$

35) $f(x) = x^3$, $3x - y + 1 = 0$

1: derivative of curve and slope of line

$y = x^3 \rightarrow \frac{dy}{dx} = 3x^2$ $\quad$ Slope of line is $\boxed{3}$

2: Comparing derivative and gradient of line

$3x^2 = 3$

$x^2 = 1 \rightarrow \boxed{x = \pm 1}$

3: put $x = \pm 1$ in curve's function

i) $y = x^3$, $x = 1$

$\boxed{y = 1}$

$(1, 1)$

ii) $y = x^3$, $x = -1$

$y = (-1)^3$

$\boxed{y = -1}$

$(-1, -1)$

4: we have two points means these are two tangent lines.

$(1, 1)$, $m = 3$ and $(-1, -1)$, $m = 3$

$\Rightarrow y - 1 = 3(x - 1)$

$\boxed{y = 3x - 2} \rightarrow$ line (1)

$\Rightarrow y + 1 = 3(x + 1)$

$\boxed{y = 3x + 2} \rightarrow$ line (2)

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36) $f(x) = x^3+2$, $3x-y-4=0$

1: derivative of curve and slope of line

$y = x^3+2$ $\qquad$ $3x-4=y$

$\frac{dy}{dx} = 3x^2$ $\qquad$ $\boxed{m=3}$

2: Comparing derivative and slope

$3x^2 = 3$

$x^2 = 1 \rightarrow \boxed{x = \pm 1}$

3: Put $x = \pm 1$ ~~and~~ curve's function

$f(x) = x^3+2$ $\qquad$ $f(-1) = (-1)^3+2$

$f(1) = 1+2 = 3$ $\qquad$ $f(-1) = 1$

$\boxed{(1, 3)}$ $\qquad$ $\boxed{(-1, 1)}$

4: points $(1,3)$, $m=3$ and $(-1,1)$, $m=3$

$y-1 = 3(x-3)$ $\qquad$ $y-1 = 3(x+1)$

$y-1 = 3x-9$ $\qquad$ $y-1 = 3x+3$

$\boxed{y = 3x-8}$ $\qquad$ $\boxed{y = 3x+4}$

37) $f(x) = \frac{1}{\sqrt{x}}$ $\qquad$ $x+2y-6=0$

1: derivative of $1/\sqrt{x}$ and slope of line

$\frac{dy}{dx} = \frac{-1}{2\sqrt[2]{x^3}}$ $\qquad$ slope $= -1/2$

2: Comparing derivative and slope

$\frac{-1}{2\sqrt[2]{x^3}} = \frac{-1}{2} \rightarrow -2 = -2\sqrt[2]{x^3}$

$\sqrt[2]{x^3} = +1 \rightarrow x^{3/2} = +1 \rightarrow$ rising power 2/3 on both side

$x^{\frac{3}{2}\times\frac{2}{3}} = +1^{\frac{2}{3}}$

$x = (+1)^{2/3}$

$\boxed{x = +1}$

3: put $x = +1$ in $\frac{1}{\sqrt{x}}$

$y = \frac{1}{\sqrt{+1}} = \frac{1}{\sqrt{1}} = 1$ $\qquad$ $(1, 1)$

4: point $(1,1)$ and $m = -1/2$ find tangent line

$y-1 = -\frac{1}{2}(x-1)$

$y = \frac{-x}{2} + \frac{1}{2} + 1$

$\boxed{y = -\frac{1}{2}x + 3/2}$

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38) $\frac{1}{\sqrt{x-1}}$; $x+2y+7=0$

1: derivative of $\frac{1}{\sqrt{x-1}}$ and slope of $x+2y+7=0$

$y=\frac{1}{\sqrt{x-1}}$, $\frac{dy}{dx}=\frac{-1}{2\sqrt[2]{x^3}}$, $\boxed{m=-1/2}$

2: Comparing derivative and slope

$\frac{-1}{2\sqrt[2]{x^3}}=\frac{-1}{2}$ → $2=2\sqrt[2]{x^3}$

$\sqrt[2]{x^3}=1$ (Rising power 2/3 on both

$\boxed{x=1}$ ← we get)

3: Put $x=1$ in $1/\sqrt{x-1}$

$\frac{1}{\sqrt{1-1}}=\frac{1}{\sqrt{0}}=\frac{1}{0}=\infty$, not any tangent line.

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Continue

Exercise 2.3

65–70, $\boxed{\text{find } k}$ such that the line is tangent to the graph of the function.

65. $f(x) = x^2 - kx$ $\qquad y = 5x - 4$

1: Use equating method, and compare the equations

$y = x^2 - kx$; $y = 5x - 4$

$\boxed{y = y}$

$x^2 - kx = 5x - 4$

$x^2 - kx - 5x + 4 = 0$

$x^2 + x(-k-5) + 4 = 0$

$\boxed{a = 1,\ b = (-k-5),\ c = 4}$

2: Use discriminant method to find the value of k.

$b^2 - 4ac = 0$, $\Delta = 0$, tangent line

$(-k-5)^2 - 4(1)(4) = 0$

$+k^2 + 10k + 25 - 16 = 0$

$k^2 + k + 9k + 9 = 0$

$k(k+1) + 9(k+1) = 0$

$(k+9)(k+1) = 0$

$\boxed{k = -9}$ $\boxed{k = -1}$

66. $f(x) = k - x^2$ $\qquad y = -6x + 1$

1: Use equating method, and compare the both equations

$y = k - x^2$; $y = -6x + 1$

$k - x^2 = -6x + 1$

$x^2 - 6x + 1 - k = 0$

$a = 1,\ b = -6,\ c = 1 - k$

2: Use discriminant method to find the value of k.

$b^2 - 4ac = 0$, $\Delta = 0$, tangent line

$(-6)^2 - 4(1)(1-k) = 0$

$36 - 4 + 4k = 0$

$4k = -32$, $\boxed{k = -8}$

67. $f(x) = \frac{k}{x}$; $y = -\frac{3}{4}x + 3$

1: Use equating method, and compare both equations

$y = \frac{k}{x}$; $y = -\frac{3}{4}x + 3$

$\boxed{y = y}$

$\frac{k}{x} = \frac{-3}{4}x + \frac{3}{1}$

$\frac{k}{x} = \frac{-3x + 12}{4}$

$4k = -3x^2 + 12x$

$a = -3,\ b = 12,\ c = -4k$

2: Discriminant formula

$(12)^2 - 4(-3)(-4k) = 0$

$144 + 12(-4k) = 0$

$144 + 48k = 0$

$-48k = -144$

$\boxed{k = 3}$

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68. $f(x) = k\sqrt{x}$; $y = x+4$

1: equating

$k\sqrt{x} = x+4$

$\boxed{k\sqrt{x} - x - 4 = 0}$

$y = mx + c$

$y = x + 4$

$\boxed{m = 1}$

$\boxed{k\sqrt{x} = 1}$

2: derivative of $k\sqrt{x}$

$\frac{k}{2\sqrt{x}} = 1 \longrightarrow \boxed{k = 2\sqrt{x}}$

3: put $k = 2\sqrt{x}$ in original Question's derived equal equations.

$k\sqrt{x} - x - 4 = 0$

$(2\sqrt{x})\sqrt{x} - x - 4 = 0$

$2x - x - 4 = 0 \longrightarrow \boxed{x = 4}$

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2.3 Product and Quotient rules and Higher-order derivatives

1. derivative of function using product rule
2. derivative of function using Quotient rule
3. derivative of trigonometric function.
4. higher order derivative of function

▸ **RULES**

1- The product rule → $\boxed{V \times U' + UV'}$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$\boxed{\text{First function} \times \text{derivative of Second} + \text{Second} \times \text{derivative of first}}$

Ex:-

• $h(x) = (3x - 2x^2)(5+4x)$

$(3x - 2x^2)(4) + (5+4x)(3 - 4x)$

$12x - 8x^2 + 15 + 12x - 20x - 16x^2$

$\boxed{-24x^2 + 4x + 15}$

• $f(x) = x^3 \cos x$

:- $UV' + UV'$, $U = x^3$, $V = \cos x$, $U' = 3x^2$, $V' = -\sin x$

$\boxed{(x^3)(-\sin x) + (\cos x)(3x^2)}$ ✓

$VU' + UV' = UV' + VU'$

$\boxed{(\cos x)(3x^2) + (x^3)(-\sin x)}$ ✓

$(x^3)(-\sin x) + ((\cos x)(3x^2))$

$x^2(x(-\sin x) + ((\cos x)(3))$

$\boxed{x^2(x - \sin x + 3\cos x)}$

• $f(x)$: $3x^2 \sin x$

$U = 3x^2$, $U' = 6x$, $V = \sin x$, $V' = \cos x$

$U \cdot V' + V \cdot U'$

$(3x^2)(\cos x) + (\sin x)(6x)$

$3x(x(\cos x) + (\sin x)2)$

$\boxed{3x(x\cos x + 2\sin x)}$

▸ Product Rule in 3 dimensions

$\boxed{U \cdot V \cdot W = (U' \times V \times W) + (U \times V' \times W) + (U \times V \times W')}$

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▸ Product Rule + Constant multiple Rule

find the derivative of $y = 2x \cdot \cos x - 2\sin x$

$$\frac{dy}{dx} \Rightarrow \underbrace{2x \times \cos x}_{\text{product rule}} \underbrace{- 2\sin x}_{\text{Constant multiple rule}}$$

$U = 2x,\ V = \cos x,\ U' = 2,\ V' = (-\sin x),\ -2(\cos x)$

$UV' + VU' - KW$

$[(2x)(-\sin x) + (\cos x)(2)] - 2(\cos x)$ derivative of sinx

$(2x)(-\sin x) + (\cos x)(2) - 2(\cos x)$

$(2x)(-\sin x) + 2(\cancel{\cos x}) - 2(\cancel{\cos x})$

$\boxed{2x(-\sin x)}$

2. Qoutient Rule → $\frac{U}{V} = \frac{VU' - UV'}{V^2},\ V \neq 0$

Ex:- find the derivative of $y = \frac{5x-2}{x^2+1}$

$U = 5x - 2,\ U' = 5$

$V = x^2 + 1,\ V' = 2x$

formula → $(VU' - UV') \div V^2,\ V \neq 0$

$[(x^2+1)(5) - (5x-2)(2x)] \div (x^2+1)^2$

$\frac{5x^2 + 5 - 10x^2 - 4x}{(x^2)^2 + 2(x^2)(1) + (1)^2} = \boxed{\frac{-5x^2 - 4x + 5}{x^4 + 2x^2 + 1}}$ Answer

▸ Rewriting before Differentiating

$f(x) = \frac{3 - (1/x)}{x+5}$, find an equation of tangent line at $(-1, 1)$

: Rewritten → $\left(3 - \frac{1}{x}\right) \div (x+5) \rightarrow \frac{3x - 1}{x(x+5)} \rightarrow \frac{3x-1}{x^2+5x}$

: Qoutient Rule

$(VU' - UV') \div V^2,\ V \neq 0,\ \frac{U}{V}$

$V = x^2 + 5x,\ U = 3x - 1,\ V' = 2x + 5,\ U' = 3$

$[(x^2+5x)(3) - (3x-1)(2x+5)] \div (x^2+5x)^2 \rightarrow \boxed{f(x)}$

$[3x^2 + 15x - (6x^2 - 2x + 15x - 5)] \div (x^2+5x)^2$

$[3x^2 + 15x - 6x^2 - 13x + 5] \div (x^4 + 10x^3 + 25x^2) \rightarrow \boxed{f(x)'}$

Put $x = -1$ in $f'(x)$, simply find $\boxed{f'(-1)}$

$[-3x^2 + 2x + 5] \div [x^4 + 10x^3 + 25x^2]$

$\frac{-3(-1)^2 + 2(-1) + 5}{(-1)^4 + 10(-1)^3 + 25(-1)^2} = \frac{-3 - 2 + 5}{1 - 10 + 25} = \frac{-5+5}{16} = \frac{0}{16} = \boxed{0}$

Point $(-1, 1)$
$m = 0$

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▸ Constant multiple rule [more convinient]

How to determine the function can be solved by Constant multiple rule?

: one possible answer, the denominator is monomial.

a. $\frac{x^2+3x}{6}$ $y=\frac{1}{6}(x^2+3x)$ $y'=\frac{1}{6}(2x+3)$ $y'=\frac{2x+3}{6}$

original — rewritten — differentiate — Simplify

b. $y=\frac{5x^4}{8}$ $y=\frac{5}{8}(x^4)$ $y'=\frac{5}{8}(4x^3)$ $y'=\frac{20x^3}{8}$

c. $y=\frac{-3(3x-2x^2)}{7x}$ $y=\frac{-9x+6x^2}{7x}$ $y'=\frac{-9+12x}{7}$ $y'=\frac{-9+12x}{7}$ ✗

$y=\frac{-3}{7}\frac{(3x-2x^2)}{(x)}$ $y'=\frac{-3}{7}\frac{(3-4x)}{(1)}$ $y'=\frac{-9+12x}{7}$ ✗

$y=\frac{-3x(3-2x)}{7x}$ $y'=\frac{-3}{7}(-2)$ $\boxed{y'=\frac{6}{7}}$ ✓

d. $y=\frac{9}{5x^2}$ $y=\frac{9}{5}(x^{-2})$ $y'=\frac{-18(x^{-3})}{5}$ $y'=\frac{-18}{5x^3}$

3. Derivates of Trigonometric functions

1. $\frac{d}{dx}[\tan x] = \sec^2 x$

2. $\frac{d}{dx}[\cot x] = -\csc^2 x$

3. $\frac{d}{dx}[\sec x] = \sec x \tan x$

4. $\frac{d}{dx}[\csc x] = -\csc x \cot x$

Proof:- $\tan x \rightarrow \sin x$

$\frac{d}{dx}=[\tan x]\ \frac{\sin x}{\cos x} \rightarrow$ Qoutient rule $(vu'-uv')\div v^2$, $v\neq 0$

$=(\cos x)(\cos x)-(\sin x)(-\sin x)\div \cos^2 x$

$\frac{\cos x^2+\sin x^2}{\cos^2 x}=\frac{1}{\cos^2 x}=\boxed{\sec^2 x}$ prooved

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• Differentiating Trigonometric functions

a. $y = x - \tan x$: $y' = 1 - \sec^2 x$

b. $y = x \sec x$: $y' = (\sec x)(1) + x(\sec x \tan x)$

$y = x \times \sec x$ $\boxed{y' = \sec x(1 + x\tan x)}$

▸ Differentiate forms of a Derivative

Differentiate both terms of $y = \dfrac{1-\cos x}{\sin x} = \csc x - \cot x$

1-

• first form: $y = \dfrac{1-\cos x}{\sin x} \rightarrow \dfrac{U}{V} \rightarrow (VU' - UV') \div V^2$, $v \neq 0$

$y' = [(\sin x)(0-(-\sin x)) - (1-\cos x)(\cos x)] \div \sin^2 x$

$y' = [(\sin x)(\sin x) - (1-\cos x)(\cos x)] \div \sin^2 x$

$y' = (\sin^2 x - (\cos x - \cos^2 x)) \div \sin^2 x$

$(\sin^2 x - \cos x + \cos^2 x) \div \sin^2 x$

$(\sin^2 x + \cos^2 x - \cos x) \div \sin^2 x$

$y' = \boxed{\dfrac{1 - \cos x}{\sin^2 x}}$

• Second form: $y = \csc x - \cot x$

$y' = (-\csc x \cot x) - (-\csc^2 x)$

$\boxed{y' = -\csc x \cot x + \csc^2 x}$

To show that two derivatives are equal, you can write

$$\boxed{\frac{1-\cos x}{\sin^2 x} = -\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right) + \frac{1}{\sin^2 x}}$$

4. Higher order Derivatives

$s(t)$ position function

$v(t) = s'(t)$ velocity function

$a(t) = v'(t) = s''(t)$ Acceleration function

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Exercise 2.3

1–6 Use product rule to differentiate

1. $g(x) = (x^2+3)(x^2-4x)$

$U \cdot V = VU' + UV'$

$V = x^2-4x, \quad U = x^2+3$

$V' = 2x-4, \quad U' = 2x$

$(x^2-4x)(2x) + (x^2+3)(2x-4)$

$(2x^3-8x^2) + (2x^3+6x-4x^2-12)$

$\boxed{4x^3-12x^2+6x-12} \rightarrow g(x)'$

2. $f(x) = (6x+5)(x^3-2)$

$U \cdot V = VU' + UV'$

$U = 6x+5, \quad U' = 6$

$V' = 3x^2, \quad V = x^3-2$

$(x^3-2)(6) + (6x+5)(3x^2)$

$\boxed{6x^3-12 + 18x^3+15x^2} \rightarrow f(x)'$

3. $h(t) = \sqrt{t}\,(1-t^2)$

$: U \cdot V = VU' + UV'$

$U = \sqrt{t}, \quad U' = \boxed{-1/2\sqrt[3]{t^3}} \; 1/2\sqrt{t}$

$V = 1-t^2, \quad V' = -2t$

$(1-t^2)\left(\frac{1}{2\sqrt{t}}\right) + \sqrt{t}(-2t)$

$\frac{-t^2+1}{2\sqrt{t}} + \frac{\sqrt{t}(-2t)}{1}$

$\frac{-t^2+1 + \sqrt{t}(-2t) \times 2\sqrt{t}}{2\sqrt{t}}$

$\frac{(-t^2+1) + (-2t) \times 2\sqrt{t} \times \sqrt{t}}{2\sqrt{t}}$

$\frac{(-t^2+1) + (-2t) \times 2t}{2\sqrt{t}}$

$\frac{-t^2+1 + (-4t^2)}{2\sqrt{t}}$

$\boxed{\frac{-5t^2+1}{2\sqrt{t}}} \rightarrow h'(t)$

4. $g(s) = \sqrt{s}\,(s^2+8)$

$: U \cdot V = VU' + UV'$

$U = \sqrt{s}, \quad U' = 1/2\sqrt{s}$

$V = s^2+8, \quad V' = 2s$

$: (s^2+8)\left(\frac{1}{2\sqrt{s}}\right) + (\sqrt{s})(2s)$

$: \frac{s^2+8}{2\sqrt{s}} + \frac{2s(\sqrt{s})}{1}$

$: \frac{(s^2+8) + (2s)(\sqrt{s})(2\sqrt{s})}{2\sqrt{s}}$

$: \frac{(s^2+8) + (2s)(2s)}{2\sqrt{s}}$

$: \frac{s^2+8+4s^2}{2\sqrt{s}}$

$: \boxed{\frac{5s^2+8}{2\sqrt{s}}} \rightarrow g(s)'$

5. $x^3 \cos x$

$U \cdot V = VU' + UV'$

$(\cos x)(3x^2) + (x^3)(-\sin x)$

$3x^2(\cos x) + (-\sin x)x^3$

$x^2(3(\cos x) + (-\sin x)x)$

$\boxed{x^2(3\cos x - x\sin x)} \rightarrow$

6. $g(x) = \sqrt{x}\,\sin x$

$: U \cdot V = VU' + UV'$

$U = \sqrt{x}, \quad V = \sin x, \quad U' = 1/2\sqrt{x}, \quad V' = \cos x$

$(\sin x)\left(\frac{1}{2\sqrt{x}}\right) + (\sqrt{x})(\cos x)$

$\frac{\sin x}{2\sqrt{x}} + \frac{\sqrt{x}(\cos x)}{1} = \boxed{\frac{\sin x + 2x\cos x}{2\sqrt{x}}} \; g(x)'$

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7-12 Use the Quotient Rule to differentiate function.

7. $f(x) = \dfrac{x}{x^2+1} \rightarrow \dfrac{U}{V} = (VU' - UV') \div V^2, \quad V \neq 0$

$U = x, \quad U' = 1, \quad V = x^2+1, \quad V' = 2x$

$= \dfrac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2} = \dfrac{x^2+1-2x^2}{(x^2+1)^2} = \boxed{\dfrac{1-x^2}{(x^2+1)^2}} \rightarrow f(x)'$

8. $g(t) = \dfrac{t^2+4}{5t-3} \rightarrow \dfrac{U}{V} = (VU' - UV') \div V^2, \quad V \neq 0$

$\dfrac{(5t-3)(2t) - (t^2+4)(5)}{(5t-3)^2} = \dfrac{10t^2 - 6t - 5t^2 - 20}{(5t-3)^2} = \boxed{\dfrac{5t^2-6t-20}{(5t-3)^2}}$

9. $h(x) = \dfrac{\sqrt{x}}{x^3+1} \rightarrow \dfrac{U}{V} = (VU' - UV') \div V^2, \quad V \neq 0$

$U = \sqrt{x}, \quad U' = 1/2\sqrt{x}, \quad V = x^3+1, \quad V' = 3x^2$

$\left[(x^3+1)\left(\dfrac{1}{2\sqrt{x}}\right) - (\sqrt{x})(3x^2)\right] \div (x^3+1)^2$

$\left[\left(\dfrac{x^3+1}{2\sqrt{x}}\right) - \dfrac{(\sqrt{x})(3x^2)}{1}\right] \div (x^3+1)^2$

$\left[\dfrac{(x^3+1)}{2\sqrt{x}} \times \dfrac{1}{1} - (\sqrt{x})(3x^2)\right] \div (x^3+1)^2$

$\left[\dfrac{x^3+1 - (3x^5+3x^2)\sqrt{x}}{2\sqrt{x}}\right] \times \dfrac{1}{(x^3+1)^2}$

$\left[\dfrac{x^3+1 - (3x^5+3x^2)\sqrt{x}}{2\sqrt{x}\,(x^3+1)^2}\right]$

$\left[\dfrac{x^3+1}{2\sqrt{x}} - \dfrac{\sqrt{x}\,(3x^2)}{1}\right] \div (x^3+1)^2$

$= \dfrac{(x^3+1) - 2\sqrt{x}\,(3x^2)\sqrt{x}}{2\sqrt{x}} \times \dfrac{1}{(x^3+1)^2} \quad h(x)'$

$= \dfrac{x^3+1 - 2x(3x^2)}{2\sqrt{x}\,(x^3+1)^2} = \dfrac{x^3+1-6x^3}{2\sqrt{x}\,(x^3+1)^2} = \boxed{\dfrac{-5x^3+1}{2\sqrt{x}\,(x^3+1)^2}}$

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10. $h(s) = \dfrac{s}{\sqrt{s}-1} \rightarrow \dfrac{U}{V} = (VU' - UV') \div V^2, \; V \neq 0$

$U = s, \; U' = 1, \; V = \sqrt{s} - 1, \; V' = 1/2\sqrt{s}$

$: \dfrac{(\sqrt{s}-1)(1) - (s)\left(\frac{1}{2\sqrt{s}}\right)}{} \div (\sqrt{s}-1)^2$

$\left(\dfrac{(\sqrt{s}-1)}{1} - \dfrac{s}{2\sqrt{s}}\right) \div (\sqrt{s}-1)^2$

$\dfrac{2\sqrt{s}(\sqrt{s}-1) - s}{2\sqrt{s}} \times \dfrac{1}{(\sqrt{s}-1)^2} = \dfrac{2s - 2\sqrt{s} - s}{2\sqrt{s}(\sqrt{s}-1)^2}$

$h(s)' \; \boxed{\dfrac{s - 2\sqrt{s}}{2\sqrt{s}(\sqrt{s}-1)^2}}$

11. $g(x) = \dfrac{\sin x}{x^2} \rightarrow \dfrac{U}{V} = (VU' - UV') \div V^2, \; V \neq 0$

$U = \sin x, \; U' = \cos x, \; V = x^2, \; V' = 2x$

$: \dfrac{(x^2)(\cos x) - (\sin x)(2x)}{(x^2)^2} = \dfrac{x^2 \cos x - 2x \sin x}{(x^2)^2}$

$: \dfrac{\cancel{x}(x\cos x - 2\sin x)}{\cancel{x}(x^3)} = \boxed{\dfrac{x\cos x - 2\sin x}{x^3}} \rightarrow g(x)'$

12. $\dfrac{\cos t}{t^3} \rightarrow \dfrac{U}{V} = (VU' - UV') \div V^2, \; V \neq 0$

$U = \cos t, \; U' = -\sin t, \; V = t^3, \; V' = 3t^2$

$: \dfrac{(t^3)(-\sin t) - (\cos t)(3t^2)}{(t^3)^2} = \dfrac{(t^3)\sin t - 3t^2 \cos t}{t^6}$

$\dfrac{-t^3 \sin t - 3t^2 \cos t}{t^6} = \dfrac{\cancel{t^2}(-t\sin t - 3\cos t)}{\cancel{t^2}(t^4)}$

$\boxed{\dfrac{-t\sin t - 3\cos t}{t^4}} \rightarrow f(t)'$

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In Exercise 13-18, find $f'(x)$ and $f'(c)$

13. $f(x) = (x^3+4x)(3x^2+2x-5)$ $\quad c=0$

: Product rule, for differentiation $\rightarrow U \cdot V = (VU' + UV')$

$(3x^2+2x-5)(3x^2+4) + (x^3+4x)(6x+2)$

$(9x^4 + 6x^3 - 15x^2 + 12x^2 + 8x - 20) + 6x^4 + 24x^2 + 2x^3 + 8x$

$9x^4 + 6x^3 - 3x^2 + 8x - 20 + 6x^4 + 2x^3 + 24x^2 + 8x$

$f(x)' = 15x^4 + 8x^3 + 21x^2 + 16x - 20$

$f(c)' = f(0)' = -20$

14. $f(x) = (x^2-2x+1)(x^3-1)$ $\quad c=1$

: Product rule $\quad U \cdot V = VU' + UV'$

$U = x^2-2x+1$, $U' = 2x-2$, $V = x^3-1$, $V' = 3x^2$

$(x^3-1)(2x-2) + (x^2-2x+1)(3x^2)$

$(2x^4 - 2x - 2x^3 + 2) + 3x^4 - 6x^3 + 3x^2$

$\boxed{f(x)' = 6x^4 - 8x^3 + 3x^2 - 2x + 2}$

$f'(c) = f'(1) = 1$

15. $f(x) = \dfrac{x^2-4}{x-3}$; $\quad c=1$

Quotient rule $\dfrac{U}{V} = (VU' - UV') \div V^2$, $V \neq 0$

$U = x^2-4$, $U' = 2x$, $V = x-3$, $V' = 1$

$[(x-3)(2x) - (x^2-4)(1)] \div (x-3)^2$

$\dfrac{[2x^2-6x - x^2+4]}{(x-3)^2} = \dfrac{x^2-6x+4}{(x-3)^2}$

$f'(x) = \dfrac{x^2-6x+4}{(x-3)^2}$

$f'(c) = f'(1) = \dfrac{1-6+4}{(1-3)^2} = \dfrac{-1}{4} = \boxed{-1/4}$

16. $f(x) = \dfrac{x+5}{x-5}$; $\quad c=4$

Quotient rule

$\dfrac{(x-5)(1) - (x+5)(1)}{(x-5)^2} = \dfrac{(x-5) - x - 5}{(x-5)^2} = \dfrac{-10}{(x-5)^2} \rightarrow \boxed{f'(x) = \dfrac{-10}{(x-5)^2}}$

$f'(c) = f'(4) = \dfrac{-10}{(4-5)^2} = \dfrac{-10}{(-1)^2} = \dfrac{-10}{1} = \boxed{-10}$

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17. $f(x) = x\cos x$ $\quad c = \pi/4$

:- Product rule $U \cdot V = VU' + UV'$

$U = x,\ U' = 1,\ V = \cos x,\ V' = -\sin x$

$(\cos x)(1) + (x)(-\sin x)$

$\cos x + x(-\sin x)$

$\boxed{\cos x - x\sin x}$

$f'(\pi/4) = \cos(45^\circ) - x\sin(45^\circ)$

$= \frac{1}{\sqrt{2}} - x\left(\frac{1}{\sqrt{2}}\right)$

$\frac{\sqrt{2}}{2} - \frac{x\sqrt{2}}{2} = \frac{\sqrt{2}}{2} - \frac{(\pi/4)\sqrt{2}}{2}$

$\frac{\sqrt{2}}{2} - \frac{\pi\sqrt{2}}{4} \div 2 \rightarrow \frac{4\sqrt{2} - \pi\sqrt{2}}{8}$

$\boxed{\frac{\sqrt{2}(4-\pi)}{8}} \rightarrow f'(\pi/4)$

18. $f(x) = \frac{\sin x}{x}$; $c = \pi/6$

Quotient rule $\rightarrow U/V = (VU' - UV') \div V^2,\ V \neq 0$

:- $U = \sin x,\ U' = \cos x,\ V = x,\ V' = 1$

$\frac{(x)(\cos x) - (\sin x)(1)}{(x)^2} = \boxed{\frac{x\cos x - \sin x}{(x)^2}} \rightarrow f'(x)$

$f'(c) = f'(\pi/6)$

$\frac{\frac{\pi}{6}\cos(30^\circ) - \sin(30^\circ)}{(x)^2} = \frac{\pi/6\,(\sqrt{3}/2) - (1/2)}{(\pi/6)^2}$

$\left(\frac{\pi}{6} \times \frac{\sqrt{3}}{2}\right) - \left(\frac{1}{2}\right) \div \left(\frac{\pi}{6}\right)^2 \rightarrow \left(\frac{\pi\sqrt{3}}{12} - \frac{1}{2}\right) \div \left(\frac{\pi}{6}\right)^2$

$\frac{2\pi\sqrt{3} - 12}{24} \times \frac{36}{(\pi)^2} = \frac{3(2\pi\sqrt{3} - 12)}{2(\pi)^2} = \frac{6\pi\sqrt{3} - 36}{2\pi^2}$

$\left(6\left(\frac{22}{7}\right)\sqrt{3} - 36\right) \div 2\left(\frac{22}{7}\right)^2 \rightarrow \left(\frac{132\sqrt{3}}{7} - \frac{36}{1}\right) \Big| 2\left(\frac{484}{49}\right)$

$\left(\frac{132\sqrt{3} - 216}{7}\right) \div \frac{968}{49} \rightarrow \frac{132\sqrt{3} - 216}{7} \times \frac{49}{968} = \frac{(132\sqrt{3} - 216)7}{968}$

$f'(\pi/6) = \frac{(132\sqrt{3} - 216)7}{968}$

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19-24, Complete the table without using Quotient rule.

Function	Rewrite	Differentiate	Simplify
19. $\frac{x^2+3x}{7}$	$y=\frac{1}{7}(x^2+3x)$	$y'=\frac{1}{7}(2x+3)$	$y'=\frac{2x+3}{7}$
20. $y=\frac{5x^2-3}{4}$	$y=\frac{1}{4}(5x^2-3)$	$y'=\frac{1}{4}(10x)$	$y'=\frac{10x}{4}=\frac{5x}{2}$
21. $y=\frac{6}{7x^2}$	$y=\frac{6}{7}(x^{-2})$	$y'=\frac{-12}{7}(x^{-3})$	$y'=\frac{-12}{7x^3}$
22. $y=\frac{10}{3x^3}$	$y=\frac{10}{3}(x^{-3})$	$y'=\frac{-30}{3}(x^{-4})$	$y'=\frac{-10}{x^4}$
23. $y=\frac{4x^{3/2}}{x}$	$y'=\frac{3}{2}\left(\frac{4x^{\frac{3}{2}-1}}{1}\right)$	$y'=$	
	$y=4(x^{3/2}\cdot x^{-1})$	$y'=4(x^{1/2})$	$y'=2(x^{-1/2})=\frac{2}{\sqrt{x}}$
24. $y=\frac{5x^2-8}{11}$	$y=\frac{5}{11}(x^2-8/5)$	$y'=\frac{5}{11}(2x)$	$y'=\frac{10x}{11}$

25 - 38 → find derivative of algebric function.

25. $f(x)=\frac{4-3x-x^2}{x^2-1}$ → Quotient Rule $\frac{U}{V}=(VU'-UV')/V^2,\ V\neq 0$

$\therefore U=(4-3x-x^2),\ U'=-2x-3,\ V=x^2-1,\ V'=2x$

$[(x^2-1)(-2x-3)-(4-3x-x^2)(2x)]\div(x^2-1)^2$

$[-2x^3+2x-3x^2+3-(4+3x+x^2)(2x)]\div(x^2-1)^2$

$[\cancel{-2x^3}-3x^2+2x+3-8x+6x^2+\cancel{2x^3}]\div(x^2-1)^2$

$[3x^2-6x+3]\div(x^2-1)^2$

$\frac{3x^2-6x+3}{(x^2-1)^2}=\frac{(3x-3)(x-1)}{(x-1)^2(x+1)^2}=\frac{(3x-3)}{(x-1)(x+1)^2}=\boxed{\frac{3x-3}{(x-1)(x+1)^2}}$

26. $f(x)=\frac{x^3+5x+3}{x^2-1}$ → Quotient rule $\frac{U}{V}=(VU'-UV')/V^2,\ V\neq 0$

$[(x^2-1)(3x^2+5)-(x^3+5x+3)(2x)]\div(x^2-1)^2$

$[3x^4-3x^2+5x^2-5-(2x^4+10x^2+6x)]\div(x^2-1)^2$

$[3x^4+2x^2-5-2x^4-10x^2-6x]\div(x^2-1)^2$

$\frac{[x^4-8x^2-6x-5]}{(x^2-1)^2}\div\boxed{\frac{x^4-8x^2-6x-5}{x^4-2x^2+1}}$

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27. $f(x) = x\left(1 - \frac{4}{x+3}\right)$

:- $x\left(\frac{x+3-4}{x+3}\right) = x\left(\frac{(x-1)}{(x+3)}\right) = \left(\frac{x^2-x}{x+3}\right)$

$\frac{U}{V} = (V \times U' - UV')/V^2, V \neq 0,$

$U = x^2 - x, \quad U' = 2x-1, \quad V = x+3, \quad V' = 1$

$\frac{(x+3)(2x-1) - (x^2-x)(1)}{(x+3)^2} = \frac{2x^2+6x-x^2+x-x-3}{(x+3)^2}$

$\frac{x^2+6x+3}{(x+3)^2} = \boxed{\frac{x^2+7x-3}{(x+3)^2}}$

28. $f(x) = x^4\left(1 - \frac{2}{x+1}\right)$

:- $x^4\left(\frac{x+1-2}{x+1}\right) = x^4\left(\frac{x-1}{x+1}\right) = \boxed{\frac{x^5-x^4}{x+1}} \rightarrow \frac{U}{V}$

$U = x^5 - x^4, \quad U' = 5x^4 - 4x^3, \quad V = x+1, \quad V' = 1$

$[(x+1)(5x^4-4x^3) - (x^5-x^4)(1)] \div (x+1)^2$

$[5x^5 - 4x^4 + 5x^4 - 4x^3 - x^5 + x^4] \div (x+1)^2$

$[5x^5 + x^4 - 4x^3 - x^5 + x^4] \div (x+1)^2$

$\boxed{\frac{[4x^5 + 2x^4 - 4x^3]}{(x+1)^2}}$

29. $f(x) = \frac{3x-1}{\sqrt{x}}$

$U = 3x-1, \quad U' = 3, \quad V = \sqrt{x}, \quad V' = 1/2\sqrt{x}$

$[(\sqrt{x})(3) - (3x-1)(1/2\sqrt{x})] \div (\sqrt{x})^2$

$\left[\frac{3\sqrt{x}}{1} - \frac{3x-1}{2\sqrt{x}}\right] \div x$

$\left[\frac{(3\sqrt{x})(2\sqrt{x}) - 3x+1}{2\sqrt{x}}\right] \div x \rightarrow \frac{6x-3x+1}{2\sqrt{x}} \times \frac{1}{x}$

$\frac{6x-3x+1}{x(2\sqrt{x})} = \frac{3x+1}{2x\sqrt{x}} = \frac{3x+1}{2 \cdot x^1 \cdot x^{\frac{1}{2}}} = \boxed{\frac{3x+1}{2x^{3/2}}}$

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30. $f(x) = \sqrt[3]{x}\,(\sqrt{x}+3)$ → product rule

$U \cdot V = (V \times U' + U \times V')$

$U = \sqrt[3]{x}$, $U' = 1/3\sqrt[3]{x^2}$, $V = \sqrt{x}+3$, $V' = 1/2\sqrt{x}$

$$\left((\sqrt{x}+3) \times \frac{1}{3\sqrt[3]{x^2}}\right) + 3\sqrt{x} \times (\sqrt{x}+3)$$

$$\frac{\sqrt{x}+3}{3\sqrt[3]{x^2}} + \frac{3x+9\sqrt{x}}{1} = \frac{\sqrt{x}+3+3\sqrt[3]{x^2}\,(3x+9\sqrt{x})}{3\sqrt[3]{x^2}}$$

$$\frac{\sqrt{x}+3+9x\sqrt[3]{x^2}+27\sqrt[3]{x^2}\times\sqrt{x}}{3\sqrt[3]{x^2}} = \boxed{\frac{\sqrt{x}+3+9x\sqrt[3]{x^2}+27x^{5/3}}{3\sqrt[3]{x^2}}}$$

31. $h(s) = (s^3-2)^2$

$= (s^3)^2 - 2(s^3)(2) + (2)^2$

$= s^6 - 4s^3 + 4$

$\frac{dy}{dx} = 6s^5 - 12s^2 + 0$

$\boxed{6s^2(s^3-2)}$

32. $(x^2-1)^2$

$= (x^2)^2 - 2(x^2)(1) + 1^2$

$= x^4 - 2x^2 + 1$

$\frac{dy}{dx} = 4x^3 - 4x + 0$

$\boxed{4x(x^2-1)}$

33. $f(x) = \left(2 - \frac{1}{x}\right) \div (x-3)$

$\therefore \frac{2x-1}{x} \times \frac{1}{(x-3)} = \frac{2x-1}{x(x-3)} = \frac{2x-1}{x^2-3x} \rightarrow \frac{U}{V}$, $(VU'-UV')/V^2, V \neq 0$

$U = 2x-1$, $U' = 2$, $V = x^2-3x$, $V' = 2x-3$

$$\frac{(x^2-3x)(2) - (2x-1)(2x-3)}{(x^2-3x)^2} = \frac{(2x^2-6x)-(4x^2-2x-6x+3)}{(x^2-3x)^2}$$

$$= \frac{2x^2-\cancel{6x}-4x^2+2x+\cancel{6x}+3}{(x^2-3x)^2} = \boxed{\frac{-2x^2+2x+3}{(x^2-3x)^2}}$$

34. $f(x) = x^2\left(\frac{2}{x} - \frac{1}{x+1}\right)$

$$\therefore x^2\left(\frac{2x+2-1}{x^2+x}\right) = \frac{x^2(2x+1)}{x(x+1)} = \frac{x(2x+1)}{x+1} = \boxed{\frac{2x^2+x}{x+1}}$$

$U = 2x^2+x$, $U' = 4x+1$, $V = x+1$, $V' = 1$

$[(x+1)(4x+1) - (2x^2+x)(1)] \div (x+1)^2$

$[4x^2+4x+\cancel{x}+1-2x^2-\cancel{x}] \div (x+1)^2$

$$\boxed{\frac{2x^2+4x+1}{(x+1)^2}}$$

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35. $f(x) = (2x^3+5x)(x-3)(x+2)$

:- $(2x^3+5x)(x^2+2x-3x-6)$

$(2x^3+5x)(x^2-x-6) = 2x^5 - 2x^4 - 12x^3 + 5x^3 - 5x^2 - 30x$

$= 2x^5 - 2x^4 - 7x^3 - 5x^2 - 30x$

:- $(U \times V \times W) = (U' \times V \times W) + (U \times V' \times W) + (U \times V \times W')$

$U = 2x^3+5x$, $U' = 6x^2+5$, $V = x-3$, $V' = 1$, $W = x+2$, $W' = 1$

$[(6x^2+5)(x-3)(x+2)] + [(2x^3+5x)(1)(x+2)] + [(2x^3+5x)(x-3)(1)]$

$\boxed{[(6x^2+5)(x-3)(x+2)] + [(2x^3+5x)(x+2)] + [(2x^3+5x)(x-3)]}$

36. $f(x) = (x^3-x)(x^2+2)(x^2+x-1)$

:- $U \times V \times W = (U' \times V \times W) + (U \times V' \times W) + (U \times V \times W')$

$U = x^3-x$, $U' = 3x^2-1$, $V' = 2x$, $V = x^2+2$, $W = x^2+x-1$, $W' = 2x+1$

$U \times V \times W = [(3x^2-1)(x^2+2)(x^2+x-1)] + [(x^3-x)(2x)(x^2+x-1)] + [(x^3-x)(x^2+2)(2x+1)]$

$= [(3x^2-1)(x^2+2)(x^2+x-1)] + [(x^3-x)(2x)(x^2+x-1)] + [(x^3-x)(x^2+2)(2x+1)]$

38–54 - Find the Derivative of trigonometric functions

38. $f(t) = t^2 \sin t$

Product Rule

$(\sin t)(2t) + (t^2)(\cos t)$

$2t \sin t + t^2 \cos t$

$t(2\sin t + t\cos t)$

40. $f(\theta) = (\theta+1)\cos\theta$

: Product Rule

$\cos\theta(1) + (\theta+1)(-\sin\theta)$

$\cos\theta + (\theta+1)(-\sin\theta)$

41. $f(t) = \dfrac{\cos t}{t}$

:- $\dfrac{(t)(-\sin t) - (\cos t)(1)}{t^2}$

: $\boxed{\dfrac{-t\sin t - \cos t}{t^2}}$

42. $f(x) = \dfrac{\sin x}{x^3}$: $(VU' - UV')/V^2$

:- $(x^3)(\cos x) - (\sin x)(2x^2) \div (x^3)^2$

: $x^3\cos x - 2x^2 \sin x$

x^6

45. $g(t) = 4\sqrt[4]{t} + 6\csc t$

:- $\sqrt[4]{t} \to$ differentiation $\to \dfrac{1}{4\sqrt[4]{t^3}}$

$\csc \to -\csc x \cot x$

$6\csc t \to$ differentiation $\to -6\csc t \cot(t)$

$\dfrac{1}{4\sqrt[4]{t^3}} - 6\csc t \cot t$

43. $f(x) = -x + \tan x$

:- $-(x) + \tan x$

: $-1 + \sec^2 x$

44. $y = x + \cot x$

:- $x + \cot x$

:- $1 + (-\csc^2 x)$

$\boxed{1 - \csc^2 x}$

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46. $\frac{1}{x} - 12\sec x$

$\frac{1}{x} \rightarrow$ differentiation $-1/x^2$

$\sec x \rightarrow$ differentiation $\sec x \tan x$

$-\frac{1}{x^2} - 12 \sec x \tan x$

48. $\frac{\sec x}{x} \rightarrow$ Quotient Rule

$\frac{u}{v} = (v \cdot u' - u v')/v^2$

$[(x)(\sec x \tan x) - (\sec x)(1)]/(x)^2$

$[x \sec x \tan x - \sec x]/x^2$

$\frac{\sec x [x \tan x - 1]}{x^2}$

$\boxed{\frac{(x \tan - 1)}{x^2 \cos^2 x}}$ $\because \cos x = \frac{1}{\sec x}$

49. $y = -\csc x - \sin x$

$\csc x \rightarrow$ differentiation $\Rightarrow (-\csc x \cot x)$

$-(-\csc x \cot x) - (\cos x)$

$\csc x \cot x - \cos x$

$\frac{1}{\sin x} \times \frac{\cos x}{\sin x} - \frac{\cos x}{1}$

$\frac{\cos x}{\sin^2 x} - \frac{\cos x}{1}$

$\frac{\cos x - \cos x \sin^2 x}{\sin^2 x}$

$\frac{\cos x(1 - \sin^2 x)}{\sin^2 x}$

$\frac{\cos x}{\sin^2 x} (\cos^2 x) = \left(\frac{1}{\sin x} \times \frac{\cos x}{\sin x}\right)(\cos^2 x)$

$\frac{\cos x (\cos^2 x)}{\sin^2 x} = \cos x \times \frac{\cos^2 x}{\sin^2 x}$

$\boxed{\cos x \cot^2 x}$ Answer

47. $y = 3(1 - \sin x)/2\cos x$

$\therefore$ Quotient rule: $(v \cdot u' - u \cdot v')/v^2$

$: \frac{(2\cos x)(-3\cos x) - (3 - 3\sin x)(-2\sin x)}{4\cos^2 x}$

$: \frac{-6\cos^2 x + 6\sin x - 6\sin^2 x}{4\cos^2 x}$

$: \frac{6}{4}\left(\frac{(-1\cos^2 x + \sin x - \sin^2 x)}{\cos^2 x}\right)$

$: \frac{3}{2}\left(\frac{-(\cos^2 x + \sin^2 x) + \sin x}{\cos^2 x}\right)$

$: \frac{3}{2}\left(\frac{-1 + \sin x}{\cos^2 x}\right)$

$: \frac{3}{2}\left(\frac{-1}{\cos^2 x} + \frac{\sin}{\cos^2 x}\right) = \frac{3}{2}\left(\frac{-\sec^2 x + \tan x}{\cos x}\right)$

$: \frac{3}{2}(-\sec^2 x + \tan x \sec x)$

So. $y = x\sin x + \cos x$

$x(\cos x) + (-\sin x)$

Product Rule OR Not?

$(v \cdot u' + u v') = u \cdot v$

$(\sin x)(1) + (x)(-\sin x) + (-\sin x)$

$\sin x + (-x\sin x) + (-\sin x)$

$\sin x - x\sin x - \sin x$

$\boxed{\cos x - \sin x(x + 1)}$

51. $x^2 \tan x \rightarrow$ Product Rule

$: - (v \cdot u' + u v')$

$\tan x(2x) + (x^2)(\sec^2 x)$

$2x \tan x + x^2 \sec^2 x$

$\boxed{x(2\tan x + x\sec^2 x)}$

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52. $f(x) = \sin x \cos x$

Product Rule $(v \cdot u' + u \cdot v')$

$(\cos x)(\cos x) + (\sin x)(-\sin x)$

$\cos^2 x + (-\sin^2 x)$

$\cos^2 x - \sin^2 x$

$\cos^2 x - (1 - \cos^2 x)$

$\cos^2 x - 1 + \cos^2 x$

$\boxed{2\cos^2 x - 1}$

53. $y = 2x \sin x + x^2 \cos x$

:- Present Condition $(u \cdot v) + (u \cdot v)$

$= (v \cdot u' + u \cdot v') + (v \cdot u' + u \cdot v')$

$= (\sin x)(2) + (2x)(\cos x) + x^2 \cos x$

$2\sin x + 2x \cos x + (\cos x)(2x) + (x^2)(-\sin x)$

$2\sin x + 2x \cos x + 2x \cos x - x^2 \sin x$

$2x \cos x + 2x \cos x + 2\sin x - x^2 \sin x$

$4x \cos x + 2\sin x - x^2 \sin x$

$4x \cos x + \sin x (2 - x^2)$

54. $5\theta \sec\theta + \theta \tan\theta$

:- $(u \cdot v) + (u \cdot v)$

$5\theta \sec\theta + \theta \tan\theta$

$[(\sec\theta)(5) + (5\theta)(\sec\theta \tan\theta)] + [(\tan\theta)(1) + (\theta)(\sec^2\theta)]$

$[5\sec\theta + 5\theta \sec\theta \tan\theta] + [\tan\theta + \theta \sec^2\theta]$

$5(\sec\theta + \theta \sec\theta \tan\theta) + \tan\theta + \theta \sec^2\theta$

$5\sec\theta(1 + \tan\theta) + \tan\theta + \sec^2\theta$

55–58, Use CAS: Computer Algebra System to Differentiate

55. $g(x) = \left(\frac{x+1}{x+2}\right)(2x-5)$

Present Condition $\left(\frac{u}{v}\right) \cdot v =$

$\left(\frac{u}{v}\right) \cdot w = \left(\frac{v \cdot u' - u \cdot v'}{v^2}\right) \cdot w = w\left(\frac{v \cdot u' - u \cdot v'}{v^2}\right)' + \left(\frac{v \cdot u' - u \cdot v'}{v^2}\right) \times w'$

$\left[\frac{(x+2)(1) - (x+1)(1) \times (2x-5)}{(x+2)^2}\right] + \left[\frac{(x+2)(1) - (x+1)(1) \times (2)}{(x+2)^2}\right]$

$\left[\frac{(x+2) - (x+1) \times (2x-5)}{(x+2)^2}\right] + \left[\frac{(x+2) - (x+1) \times (2)}{(x+2)^2}\right]$

$\left[\frac{(2x-5) \times (1)}{(x+2)^2}\right] + \left[\frac{(2)(1)}{(x+2)^2}\right] = \left[\frac{2x-5}{(x+2)^2} + \frac{2}{(x+2)^2}\right]$

$= \left[\frac{(2x-5) + 2}{(x+2)^2}\right] = \left[\frac{2x-3}{(x+2)^2}\right]$

INCORRECT

USE $\boxed{\text{CAS}}$

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CORRECT: Via Computer Algebra System

55. $g(n) = \left(\frac{n+1}{n+2}\right)(2n-5)$ → Product Rule ONLY

$U = \frac{n+1}{n+2}$, $U' = \left[\frac{(n+2)(1) - (n+1)(1)}{(n+2)^2}\right]$, $V = 2n-5$, $V' = 2$

$U \cdot V = [VU' + UV']$

$(2n-5)\left[\frac{(n+2)(1) - (n+1)(1)}{(n+2)^2}\right] + \left(\frac{n+1}{n+2}\right)(2)$

$(2n-5)\left[\frac{n+2-n-1}{(n+2)^2}\right] + \frac{2n+2}{n+2}$ → There was ± placed

$(2n-5)\left[\frac{1}{(n+2)^2}\right] + \frac{2n+2}{n+2} = \frac{(2n-5)}{(n+2)^2} + \frac{2n+2}{n+2}$ → Mistake

$\frac{2n-5}{(n+2)^2} + \frac{2(n+1)}{(n+2)} = \frac{(2n-5)}{(n+2)^2} + \frac{2}{1} = \frac{(2n-5) + 2(n+2)^2}{(n+2)^2}$

$\frac{(2n-5) + 2(n^2+4n+4)}{(n+2)^2} = \frac{2n-5+2n^2+8n+8}{(n+2)^2} = \frac{2n^2+10n-3}{(n+2)^2}$

57. $\frac{\theta}{1-\sin\theta}$ → Quotient Rule $(V \cdot U' - UV')/V^2$

$[(1-\sin\theta)(1) - (\theta)(0-(\cos\theta)] \div (1-\sin\theta)^2$

$[(1-\sin\theta) - (\theta)(-\cos\theta)] \div (1-\sin\theta)^2$

$[1 - \sin\theta + \theta\cos\theta] / (1-\sin\theta)^2$

58. $f(\theta) = \frac{\sin\theta}{1-\cos\theta}$ → Quotient Rule $(V \cdot U' + U \cdot V')/V^2$

$[(1-\cos\theta)(\cos\theta) - (\sin\theta)(0-(-\sin\theta)] \div (1-\cos\theta)^2$

$[\cos\theta(1-\cos\theta) - \sin\theta(\sin\theta)] \div (1-\cos\theta)^2$

$[\cos\theta - \cos^2\theta - \sin^2\theta] \div [1-\cos\theta]^2$

$[\cos\theta - (\cos^2\theta + \sin^2\theta)] \div (1-\cos\theta)^2$

$[\cos\theta - (1)] \div (1-\cos\theta)^2$

$\frac{\cos\theta - 1}{(1-\cos\theta)^2} = \frac{-(1-\cos\theta)}{(1-\cos\theta)(1-\cos\theta)} = \frac{-1\cancel{(1-\cos\theta)}}{\cancel{(1-\cos\theta)}(1-\cos\theta)} = \boxed{\frac{-1}{1-\cos\theta}}$

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SP-62 Evalute the derivative of function at the given point. Use a graphing utility to verify your result

Sol. $y = \frac{1 + \csc x}{1 - \csc x}$, Point $\left(\frac{\pi}{6}, -3\right)$

1: find the derivative of y

$$\frac{1 + \csc x}{1 - \csc x} = \frac{(1 - \csc x)(0 + (-\csc x \cot x)) - (1 + \csc x)(0 - (-\csc x \cot x))}{(1 - \csc x)^2}$$

$$: \frac{(1 - \csc x)(-\csc x \cot x) - (1 + \csc x)(\csc x \cot x)}{(1 - \csc x)^2}$$

$$\frac{-\csc x \cot x + \cancel{\csc^2 x \cot x} - \csc x \cot x - \cancel{\csc^2 x \cot x}}{(1 - \csc x)^2}$$

$$\boxed{\frac{-2 \csc x \cot x}{(1 - \csc x)^2}}$$

2: put $x = \pi/6 = 30^\circ$

$$\frac{-2 \csc(30^\circ) \cot(30^\circ)}{(1 - \csc(30^\circ))^2} = \frac{-2(2)(\sqrt{3})}{(1 - 2)^2} = \frac{-4\sqrt{3}}{(-1)^2} = \boxed{-4\sqrt{3}}$$

60. $f(x) = \tan x \cot x$ $(1, 1)$

1: derivative

$(\cot x)(\sec^2 x) + (\tan x)(-\csc^2 x)$

$\cot x \sec^2 x + (\tan x)(-\csc^2 x)$

$\cot x \sec^2 x - \tan x \csc^2 x$

2: $\cot(1) \sec^2(1) - \tan(1) \csc^2(1)$, $x = 1$

$$\left(\frac{\cos(1)}{\sin(1)} \times \frac{1}{\cos^2(1)}\right) - \frac{\sin(1)}{\cos(1)} \times \frac{1}{\sin^2(1)}, \quad \sin 1 = 0.841, \quad \cos 1 = 0.540$$

$$\left(\frac{0.540}{0.841} \times \frac{1}{(0.540)^2}\right) - \left(\frac{0.841}{0.540}\right) \times \frac{1}{(0.841)^2}$$

$$\left(\frac{1}{(0.841)(0.540)}\right) - \left(\frac{1}{(0.540)(0.841)}\right) = 2.201 - 2.201 = 0$$

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61. $\frac{\sec t}{t}$ $(\pi, -1/\pi)$

1: $[(t)(\sec t \tan t) - (\sec t)(1)] \div (t)^2$

$[t \sec t \tan t - \sec t] \div t^2$

$\boxed{\frac{\sec t\,(t \tan t - 1)}{t^2}}$

2: $\frac{\sec(\pi)((\pi)(\tan \pi) - 1)}{(\pi)^2}$, $t = \pi$

$y' = \frac{-1(\pi(0) - 1)}{\pi^2} = \frac{-1(-1)}{\pi^2} = \boxed{\frac{1}{\pi^2}}$

62. $f(x) = \sin x(\sin x + \cos x)$, $(\pi/4, 1)$

1: $\sin^2 x + \sin x \cos x$

$(\sin^2 x)' + ((\cos x)(\cos x) + (\sin x)(-\sin x))$

$(\sin^2 x)' + (\cos^2 x - \sin^2 x)$

$\cancel{\sin^2 x} + \cos^2 x - \cancel{\sin^2 x}$

$\boxed{f(x)' = \cos^2 x}$, put $x = \pi/4$ OR $45°$

2: $f(x)' = \cos^2 x$.

$f(\pi/4)' = \cos^2(\pi/4) = \cos^2(45°)$

$= \cos(45°)\cos(45°)$

$= \left(\frac{1}{\sqrt{2}}\right) \times \left(\frac{1}{\sqrt{2}}\right) = \boxed{\frac{1}{2}}$

▶ 63–68 find an equation of tangent line

63. $(x^3 + 4x - 1)(x - 2)$, $(1, -4)$

1: function is in product form, use product rule and find derivative then, put $x = 1$, you will get $\boxed{m}$

: $(x-2)(3x^2+4) + (x^3+4x-1)(1)$

$3x^3 + 4x - 6x^2 - 8 + x^3 + 4x - 1$

$4x^3 + 8x - 6x^2 - 9$, put $x = 1$

$4(1)^3 + 8(1) - 6(1)^2 - 9 = -3$

$\boxed{m = -3}$, point $(1, -4)$

2: $y + 4 = -3(x - 1)$.

$\boxed{y = -3x - 1}$

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64. $f(x) = (x+3)(x^2-2)$, $(-2, 2)$

1: Product Rule (Differentiation)

$(x^2-2)(1) + (x+3)(2x)$

$(x^2-2) + 2x^2+6x$

$3x^2+6x-2$, $\boxed{\text{Put } x=-2}$

2: $3(-2)^2+6(-2)-2$

$12-12-2=-2$

$y+2=-2(x+2)$

$y-2=-2x-4$

$\boxed{y=-2x-2}$

65. $f(x) = \dfrac{x}{x+4}$, $(-5, 5)$

1: Quotient Rule (Differentiation)

$[(x+4)(1)-(x)(1)] \div (x+4)^2$

$[(x+4)-x] \div (x+4)^2$

$\boxed{\dfrac{4}{(x+4)^2}} \rightarrow f(x)'$

2: Put $x=-5$, in $f(x)'$

$f(x)' = \dfrac{4}{(x+4)^2} \rightarrow f(-5)' = \dfrac{4}{(-5+4)^2} = \dfrac{4}{(-1)^2} = 4 \rightarrow \boxed{m=4}$

Point $(-5, 5)$, $m=4$

$y-5=4(x+5) \rightarrow \boxed{y=4x+25}$ Tangent line

66. $f(x) = \dfrac{(x-1)}{(x+1)}$, $(2, 1/3)$

1: $[(x+1)(1) - (x-1)(1)] \div (x+1)^2$

$[(x+1)-(x-1)] \div (x+1)^2$

$[x+1-x+1] \div (x+1)^2$

$\boxed{f(x)' = \dfrac{2}{(x+1)^2}}$, put $x=2$

2: $f(2)' = \dfrac{2}{(2+1)^2} = \dfrac{2}{9}$

3: Point $(2, 1/3)$, $m = 2/9$

$y - \dfrac{1}{3} = \dfrac{3}{9}(x-2)$

$3y-1=(x-2)$

$\boxed{3y=x-1}$

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67. $f(x) = \tan x, \quad (\pi/4, 1)$

1: Differentiation

$(\tan x)' = \sec^2 x = \dfrac{1}{\cos^2 x}$

$\pi/4 = 45^0$

$\dfrac{1}{\cos^2(45^0)} = \dfrac{1}{(1/\sqrt{2})^2} = 1 \div \left(\dfrac{1}{\sqrt{2}}\right)^2 = 1 \times (\sqrt{2})^2 = \boxed{2} \quad \boxed{m = 2}$

2: Point $(\pi/4, 1)$, $m = 2$

$y - 1 = 2\left(x - \dfrac{\pi}{4}\right)$

$y - 1 = 2\left(\dfrac{4x - \pi}{4}\right) \Rightarrow y - 1 = \dfrac{4x - \pi}{2} \Rightarrow \boxed{2y - 2 = 4x - \pi}$

68. $f(x) = \sec x, \quad (\pi/3, 2)$

1- $(\sec x)' = \sec x \tan x$, $\pi/3 = 60^0$ [Differentiation]

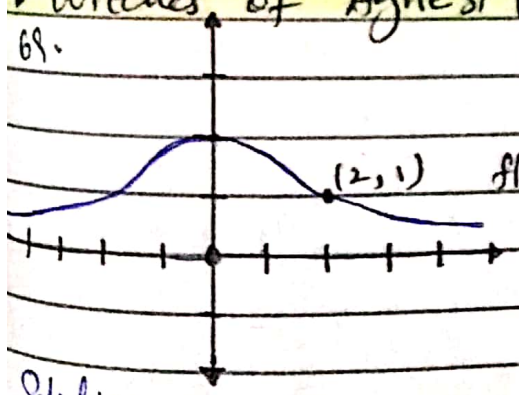
$\sec x \tan x \Rightarrow \sec(60^0)\tan(60^0) = (2)(\sqrt{3}) = \boxed{2\sqrt{3}} \rightarrow m$

2- Point $(\pi/3, 2)$, $m = 2\sqrt{3}$

$y - 2 = 2\sqrt{3}\left(x - \dfrac{\pi}{3}\right) \Rightarrow y - 2 = 2\sqrt{3}\left(\dfrac{3x - \pi}{3}\right)$

$\boxed{3y - 6 = 2\sqrt{3}(3x - \pi)} \rightarrow$ equation of tangent line

- Witches of Agnesi (Curves name(s))

69. 

find an equation of tangent line.

$f(x) = \dfrac{8}{x^2+4}$

Solution

1: Differentiation [Quotient Rule] $\rightarrow [v \cdot u' - u \cdot v'] \div v^2 \Leftrightarrow u/v$

$f(x) = \dfrac{8}{x^2+4} \Rightarrow [(x^2+4)(0) - (8)(2x)] \div (x^2+4)^2$

$[(0) - 16(x)] \div (x^2+4)^2$

$f(x)' = \dfrac{-16x}{(x^2+4)^2}$, $\boxed{\text{put } x = 2}$

2: $f(2)' = \dfrac{-16(2)}{((2)^2+4)^2} = \dfrac{-32}{64} = \boxed{\dfrac{-1}{2}}$

$\boxed{m = -1/2}$

3: Point $(2, 1)$, $m = -1/2$

$y - 1 = \dfrac{-1}{2}(x - 2)$

$2y - 2 = -x + 2$

$\boxed{2y + x = 4} \rightarrow$ Tangent line

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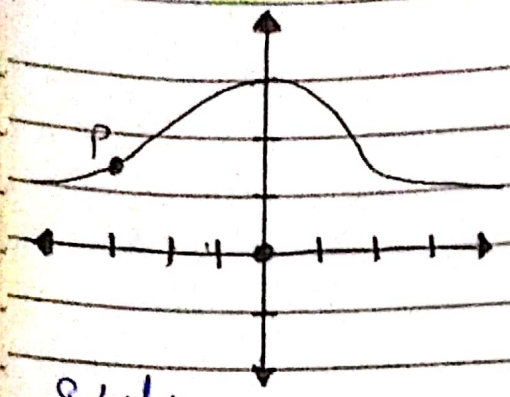
70. witches of Agnesi



Find an equation, $f(x) = \frac{27}{x^2+9}$ of tangent line, where Point P is $P(-3, 3/2)$

Solution

1: Differentiation [Quotient Rule]

$f(x) = \frac{27}{x^2+9} \rightarrow [V \cdot U' - U \cdot V'] \div V^2 \Rightarrow \frac{U}{V}$

$\therefore [(x^2+9)(0) - (27)(x^2+9)] \div (x^2+9)^2$

$[0 - (27x^2 + 243)] \div (x^2+9)^2$

$\boxed{f(x)' = \frac{-27x^2 - 243}{(x^2+9)^2}}$ $\boxed{\text{Put } x = -3}$

2: $f(-3)' = \frac{-27(-3)^2 - 243}{((-3)^2+9)^2} = \frac{-27(-27) - 243}{(18)^2} = \frac{729 - 243}{324} = \frac{486}{324}$

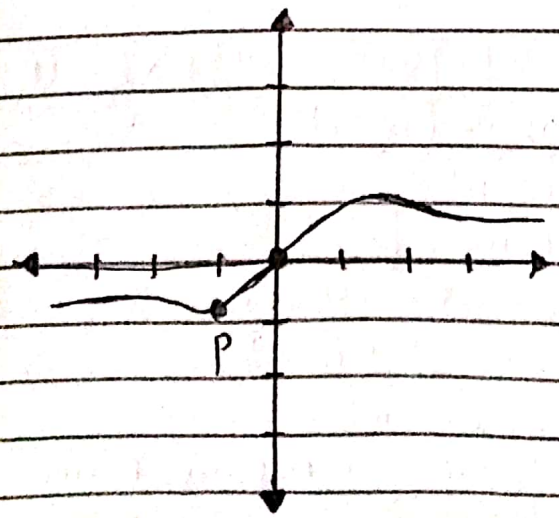
$\frac{486}{324} = \boxed{\frac{3}{2}} \rightarrow \boxed{m = 3/2}$

3: Point $P(-3, 3/2)$, $m = 3/2$

$y - \frac{3}{2} = \frac{3}{2}(x+3)$

$\frac{2y-3}{2} = \frac{3x+9}{2} \Rightarrow 2y - 3 = 3x + 9 \Rightarrow \boxed{2y = 3x + 12}$

71 — 72 Serpentines



Find an equation of tangent line, given the point $P(-2, -8/5)$ and $f(x) = \frac{16x}{x^2+16}$

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1: Differentiation [Quotient Rule] → $[v \cdot u' - u \cdot v'] \div v^2$

$= [(x^2+16)(16) - (16x)(2x)] \div (x^2+16)^2$

$f(x)' = [16x^2 + 256 - 32x^2] \div (x^2+16)^2$

$f(x)' = [-16x^2 + 256] \div (x^2+16)^2$

$f(x)' = \dfrac{-16x^2+256}{(x^2+16)^2}$, $\boxed{\text{put } x=-2}$

2: $f(-2)' = \dfrac{-16(-2)^2+256}{((-2)+16)^2} = \dfrac{-16(4)+256}{(14)^2} = \dfrac{-64+256}{196} = \dfrac{192}{196}$

2: $f(-2)' = \dfrac{-16(-2)^2+256}{((-2)^2+16)^2} = \dfrac{-64+256}{(4+16)^2} = \dfrac{192}{400} = \dfrac{96}{200} = \dfrac{48}{100} = \boxed{\dfrac{12}{25}}$

$\boxed{m = 12/25}$

3: Point $P(-2, -8/5)$, $\boxed{m = 12/25}$

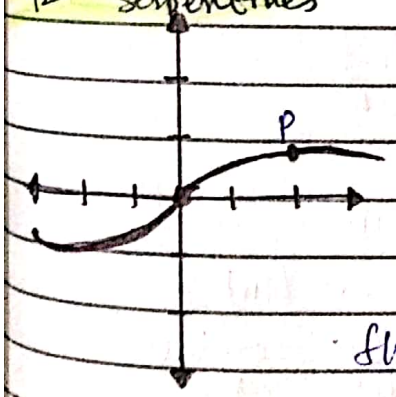
$y - \left(-\dfrac{8}{5}\right) = \dfrac{12}{25} \cdot (x - (-2))$

$\dfrac{5y+8}{5} = \dfrac{12x+24}{25} \Rightarrow 5(5y+8) = 12x+24$

$25y + 40 = 12x + 24$

$\boxed{25y - 12x + 16 = 0}$

12. Serpentines



$P(2, 4/5)$, $f(x) = \dfrac{4x}{x^2+6}$

1: Differentiation [Quotient Rule]

$[(x^2+6)(4) - (4x)(2x)] \div (x^2+6)^2$

$[4x^2+24 - 8x^2] \div (x^2+6)^2$

$f(x)' = \dfrac{-4x^2+24}{(x^2+6)^2}$, $\boxed{\text{put } x=2}$

2: $f(2)' = \dfrac{-4(2)^2+24}{((2)^2+6)^2}$

$= \dfrac{-16+24}{100} = \dfrac{8}{100} = \boxed{\dfrac{2}{25}}$

$\boxed{m = 2/25}$

3: Point $P(2, 4/5)$, $m = 2/25$

$y - \dfrac{4}{5} = \dfrac{2}{25}(x-2)$

$\dfrac{5y-4}{5} = \dfrac{2x-4}{25} \Rightarrow 5(5y-4) = 2x-4$

$5(5y-4) = 2x-4$

$25y - 20 = 2x - 4$

$\boxed{25y = 2x + 16}$

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73–76, Determine the point(s) at which the graph of the function has a horizontal tangent line.

73. $f(x) = \frac{2x-1}{x^2}$

1: Differentiation [Quotient Rule]

$[(x^2)(2) - (2x-1)(2x)] \div (x^2)^2$

$[2x^2 - (4x^2 - 2x)] \div (x^4)$

$[2x^2 - 4x^2 + 2x] \div (x^4)$

$\frac{[-2x^2 + 2x]}{x^4} = \frac{x(-2x+2)}{x^4} = \frac{-2x+2}{x^3} \Rightarrow \boxed{f(x)' = \frac{-2x+2}{x^3}}$

2: Let $f(x)' = 0$

$f(x)' = \frac{-2x+2}{x^3}$

$0 = \frac{-2x+2}{x^3}$

$0 = -2x+2$

$\boxed{x = 1}$

3: Put $x = 1$ in $f(x)$

$f(1) = \frac{2(1)-1}{(1)^2} = \frac{2-1}{1} = \frac{1}{1} = 1$

$x = 1, \; y = 1$

There is a point at which the graph has a horizontal line and that point is $(1, 1)$

74. $f(x) = \frac{x^2}{x^2+1}$

1: Differentiation [Quotient Rule]

$[(x^2+1)(2x) - (x^2)(2x)] \div (x^2+1)^2$

$\frac{[\cancel{2x^3} + 2x - \cancel{2x^3}]}{(x^2+1)^2} = f(x)' = \frac{2x}{(x^2+1)^2}$

2: Let $f(x)' = 0$

$\frac{2x}{(x^2+1)^2} = 0$

$2x = 0$

$x = \frac{0}{2}$

$\boxed{x = 0}$

3: Put $x = 0$ in $f(x)$

$f(0) = \frac{(0)^2}{(0)^2+1} = \frac{0}{1} = 0$

The graph of $f(x)$ has a point $P(0, 0)$ at which the graph has a Horizontal tangent line.

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75. $f(x) = \frac{x^2}{x-1}$

1: Differentiation [Quotient Rule]

$[(x-1)(2x) - (x^2)(1)] \div (x-1)^2$

$[2x^2 - 2x - x^2] \div (x-1)^2$

$\boxed{f(x)' = \frac{x^2 - 2x}{(x-1)^2}}$

2: let $f(x)' = 0$

$0 = \frac{x^2 - 2x}{(x-1)^2}$

$x^2 - 2x = 0$

$x(x-2) = 0$

$\boxed{x = 2}$

3: Put $x=2$ in original function

$f(2) = \frac{(2)^2}{2-1} = \frac{4}{1} = \boxed{4} \rightarrow y = 4$

The graph of $f(x)$ has a horizontal line at point $\boxed{(2, 4)}$

76. $f(x) = \frac{x-4}{x^2-7}$

1: Differentiation [Quotient Rule]

$[(x^2-7)(1) - (x^2-4)(2x)] \div (x^2-7)^2$

$\frac{[\cancel{x^2} - \cancel{7} - \cancel{2x^2} + 8x]}{(x^2-7)^2} = \frac{-x^2 + 8x - 7}{(x^2-7)^2} = \boxed{f(x)' = \frac{-x^2 + 8x - 7}{(x^2-7)^2}}$

2: Let $f(x)' = 0$

$\frac{-x^2 + 8x - 7 = 0}{(x^2-7)^2}$

$-x^2 + 8x - 7 = 0$

$-x^2 + x + 7x - 7 = 0$

$-x(x-1) + 7(x-1) = 0$

$(-x+7)(x-1) = 0$

$\boxed{x=7}$ $\boxed{x=1}$

3: Put $x=7$ and $x=1$ in $f(x)$

$f(1) = \frac{1-4}{(1)^2-7} = \frac{-3}{1-7} = \frac{-3}{-6} = \boxed{\frac{1}{2}}$

$(1, 1/2)$

$f(7) = \frac{7-4}{(7)^2-7} = \frac{3}{49-7} = \frac{3}{42} = \frac{1}{14}$

$(7, 1/14)$

▸ The graph of $f(x)$ has two horizontal line one at point $(1, 1/2)$ and the second at point $(7, 1/14)$

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77. Tangent lines Find equations of the tangent lines

to the graph of $f(x) = (x+1)/(x-1)$ that are parallel to the line $2y+x=6$. Then graph the equation (function) and the tangent lines.

1: $f(x) = \frac{(x+1)}{(x-1)}$, Parallel line $2y+x=6$

▸ Similar Question on page No (140)

1: find derivative of curve and slope of line

$f(x) = \frac{x+1}{x-1}$ → Quotient Rule

$[(x-1)(1) - (x+1)(1)] \div (x-1)^2$

$\frac{[x-1-x-1]}{(x-1)^2} = \frac{-2}{(x-1)^2} \Rightarrow \boxed{f(x)' = \frac{-2}{(x-1)^2}}$

: Slope of line

$2y+x=6$

$2y=-x+6$

$y=\frac{-1}{2}x+3$

$\boxed{m=-1/2}$

2: Use equating method and compare the derivative $f(x)'$ and m.

$\frac{-2}{(x-1)^2} = \frac{-1}{2}$

$-4 = -(x-1)^2$

$x^2-2x+1=4$

$x^2-2x-3=0$, $x=-1$, $x=3$

3: Put $x=-1$, and $x=3$ in $f(x)$ in Original function

$f(-1) = \frac{-1+1}{-1-1} = \frac{0}{-2} = 0$, $\boxed{(-1,0)}$

$f(3) = \frac{3+1}{3-1} = \frac{4}{2} = 2$, $\boxed{(3,2)}$

} Points, and slope is $-1/2$ because lines are // parallel

4: $A(-1,0)$ and $B(3,2)$ and $m=-1/2$ for each point

$y-0 = \frac{-1}{2}(x-(-1))$

$2y=-x+1$

$\boxed{2y+x=1}$ → Line ①

$y-2 = \frac{-1}{2}(x-3)$

$2y-4=-x+3$

$\boxed{2y+x=7}$ → Line → ②

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2.4 chain Rule $\rightarrow$ $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \equiv \frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

① Using chain Rule

(1) $y = (x^2+1)^3$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3(x^2+1)^2 \times \frac{du}{dx}(x^2+1)$$

$$3(x^2+1)^2 \times (2x) = \boxed{6x(x^2+1)^2}$$

(2) Involving Radicals

$f(x) = \sqrt[3]{(x^2-1)^2}$ for which $f'(x) = 0$

: $(x^2-1)^{2/3}$ Rewriting

$$\frac{2}{3}(x^2-1)^{-1/3} \times \frac{du}{dx}(x^2-1)$$

$$\frac{2}{3}(x^2-1)^{-1/3} \times (2x) = \frac{4x}{3(x^2-1)^{1/3}} = \frac{4x}{3\sqrt[3]{x^2-1}}$$

(3) Differentiating Quotients with Constant(s) Numerators

$$g(t) = \frac{-7}{(2t-3)^2}$$

: $-7(2t-3)^{-2}$ Rewriting

$$(-7)(-2)(2t-3)^{-3} \times \frac{du}{dx}(2t-3)$$

$$14(2t-3)^{-3} \times (2) = 28(2t-3)^{-3} = \frac{28}{(2t-3)^3}$$

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171.

7-36 find the derivative of the function.

7. $y = (4x-1)^3$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3(4x-1)^2 \times \frac{du}{dx}(4x-1)$

$3(4x-1)^2 \times (4) = 12(4x-1)^2$

8. $y = 2(6-x^2)5$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (2)(5)(6-x^2)^4 \times \frac{du}{dx}(6-x^2)$

$10(6-x^2)^4 \times (-2x) = -20x(6-x^2)^4$

9. $g(x) = 3(4-9x)^4$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = (3)(4)(4-9x)^3 \times \frac{du}{dx}(4-9x)$

$12(4-9x)^3 \times (-9) = -108(4-9x)^3$

10. $f(t) = (9t+2)^{2/3}$

$\frac{dy}{dx} = \left(\frac{dy}{du} \times \frac{du}{dx}\right) = \frac{2}{3}(9t+2)^{-1/3} \times \frac{du}{dx}(9t+2)$

$\frac{2}{3}(9t+2)^{-1/3} \times (9) = 6(9t+2)^{-1/3}$

11. $f(t) = \sqrt{5-t}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}(5-t)^{-1/2} \times \frac{du}{dx}(5-t)$

$= \frac{1}{2\sqrt{5-t}} \times (-1) = \frac{-1}{2\sqrt{5-t}}$

12. $g(x) = \sqrt{9-4x}$

$\therefore \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}(9-4x)^{-1/2} \times \frac{du}{dx}(9-4x)$

$= \frac{1}{2\sqrt{9-4x}} \times (-4) = \frac{-4}{2\sqrt{9-4x}} = \frac{-2}{\sqrt{9-4x}}$

13. $y = \sqrt[3]{6x^2+1}$

$= \frac{1}{3}(6x^2+1)^{-2/3}(12x) = (4x)(6x^2+1)^{-2/3} = \frac{4x}{\sqrt[3]{(6x^2+1)^2}}$

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14. $g(x) = \sqrt{x^2 - 2x + 1}$

$\frac{1}{2}(x^2 - 2x + 1)^{-1/2} \times (2x - 2) = \frac{(x-1)}{\sqrt{x^2 - 2x + 1}}$

15. $y = 2\sqrt[4]{8 - x^2}$

$= \frac{1}{2}(8 - x^2)^{-3/4} \times (-2x) = \frac{(-x)}{\sqrt[4]{(8 - x^2)^3}}$

16. $f(x) = -3\sqrt[4]{2 - 8x}$

$\frac{-3}{4}(2 - 8x)^{-3/4} \times (-8) = \frac{24}{4}(2 - 8x)^{-3/4} = \frac{24}{4\sqrt[4]{(2 - 8x)^3}}$

17. $y = \frac{1}{x - 2} \rightarrow$ Quotient Rule $\rightarrow (vu' - uv') \div v^2$

$\frac{(x-2)(0) - (1)(1)}{(x-2)^2} = \frac{-1}{(x-2)^2}$

18. $s(t) = \frac{1}{t^2 + 3t - 1} \rightarrow$ Quotient

$\frac{(t^2 + 3t - 1)(0) - (1)(2t + 3)}{(t^2 + 3t - 1)^2} = \frac{-2t + 3}{(t^2 + 3t - 1)^2}$

19. $f(t) = \left(\frac{1}{t - 3}\right)^2 \rightarrow$ Chain Rule $\rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 2\left(\frac{1}{t - 3}\right) \times \left(\frac{(t-3)(0) - (1)(1)}{(t-3)^2}\right) = \frac{-2}{(t-3)^3}$

20. $y = \frac{-5}{(t+3)^3} \rightarrow$ Quotient + Chain

$= \frac{(t+3)^3(0) - (-5)[5(t+3)^2 \times (1)]}{(t+3)^5} = \frac{15(t+3)^2}{(t+3)^5} = \frac{15}{(t+3)^3}$

21. $y = \frac{1}{\sqrt{x+2}} \rightarrow$ Quotient + Chain

$= (\sqrt{x+2})(0) - (1)\left[\frac{1}{2}(x+2)^{-1/2} \times (1)\right] \div (\sqrt{x+2})^2$

$= 0 - \left[\frac{1}{2\sqrt{x+2}}\right] \div (x+2) = \frac{-1}{(2\sqrt{x+2})(x+2)}$

$= \frac{-1}{2(x+2)^{3/2}} = \boxed{\frac{-1}{2\sqrt{(x+2)^3}}}$

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M3

22. $g(t) = \sqrt{\frac{1}{t^2-2}}$

$= \left(\frac{1}{t^2-2}\right)^{1/2} = \frac{1}{2}\left(\frac{1}{t^2-2}\right)^{-1/2} \times \left[\frac{(t^2-2)(0)-(1)(2t)}{(t^2-2)^2}\right]$

$= \frac{1}{2} \times \frac{1}{\sqrt{1/t^2-2}} \times \left[\frac{0-2t}{(t^2-2)^2}\right] = \frac{-2t}{(2\sqrt{1/t^2-2})(t^2-2)^2}$

23. $f(x) = x^2(x-2)^4$

$x^2 \times [4(x-2)^3 \times (1)] = x^2 \times [4(x-2)^3]$

$4(x-2)^3(2x) + (x^2)[12(x-2)^2 \times (1)]$

$8x(x-2)^3 + x^2(12(x-2)^2]$

$8x(x-2)^3 + 12x^2(x-2)^2$

$(x-2)^2[8x(x-2) + 12x^2(1)]$

$(x-2)^2[8x^2 - 16x + 12x^2]$

$(x-2)^2[20x^2 - 16x]$ INCORRECT ↑ CORRECT ↓

: $f(x) = x^2(x-2)^4$ → Product + chain Rule

$(x-2)^4(2x) + (x^2)[4(x-2)^3]$

$2x(x-2)^4 + 4x^2(x-2)^3$

$2x(x-2)^3[(x-2) + 2x]$

$2x(x-2)^3(3x-2)$

24. $f(x) = x(3x-8)^3$

: $(3x-8)^3(1) + (x)[3(3x-8)^2 \times (3)]$

$(3x-8)^3 + (x)[8(3x-8)^2]$

$(3x-8)^3 + 8x(3x-8)^2$

$(3x-8)^2[(3x-8) + 8x]$

$(3x-8)^2[12x-8) = (3x-8)^2(12x-8)$

25. $y = x\sqrt{1-x^2}$ → Product + chain Rule

: $(\sqrt{1-x^2})(1) + (x)[\frac{1}{2}(1-x^2)^{-1/2} \times (-2x)]$

: $(\sqrt{1-x^2}) + (x)[(-x)(1-x^2)^{-1/2}]$

: $(\sqrt{1-x^2}) + x\left[\frac{-x}{\sqrt{(1-x^2)}}\right] = \frac{\sqrt{1-x^2}}{1} + \frac{x^2}{\sqrt{1-x^2}}$

$\frac{(\sqrt{1-x^2})^2 - x^2}{\sqrt{1-x^2}} = \frac{1-x^2-x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}$

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26. $y = \frac{1}{2} x^2 \sqrt{16-x^2}$

$: (\sqrt{16-x^2})(x) + \frac{x^2}{2}\left[\frac{1}{2}(16-x^2)^{-1/2} \times (-2x)\right]$

$: x\sqrt{16-x^2} + \frac{x^2}{2}\left[\frac{1}{2\sqrt{16-x^2}} \times (-2x)\right]$

$: \frac{x\sqrt{16-x^2}}{1} + \frac{-2x^3}{4\sqrt{16-x^2}} = \frac{(x\sqrt{16-x^2})(4\sqrt{16-x^2}) - 2x^3}{4\sqrt{16-x^2}}$

$: \frac{4x(\sqrt{16-x^2})^2 - 2x^3}{4\sqrt{16-x^2}} = \frac{4x(16-x^2) - 2x^3}{4\sqrt{16-x^2}}$

$: \frac{64x - 4x^3 - 2x^3}{4\sqrt{16-x^2}} = \frac{64x - 6x^3}{4\sqrt{16-x^2}} = \frac{32x - 3x^3}{2\sqrt{16-x^2}}$

27. $y = \frac{x}{\sqrt{x^2+1}}$ → Quotient + Chain Rule.

$: (\sqrt{x^2+1})(1) - (x)\left[\frac{1}{2}(x^2+1)^{-1/2} \times (2x)\right] \div (\sqrt{x^2+1})^2$

$(\sqrt{x^2+1}) - x\left[\frac{2x}{2\sqrt{x^2+1}}\right] \div (\sqrt{x^2+1})^2$

$\frac{\sqrt{x^2+1}}{1} - \frac{\cancel{2}x^2}{\cancel{2}\sqrt{x^2+1}} \div (x^2+1)$

$\frac{(\sqrt{x^2+1})(\sqrt{x^2+1})}{(\sqrt{x^2+1})} - x^2 \times \frac{1}{x^2+1} = \frac{(x^2+1) - x^2}{(\sqrt{x^2+1})(x^2+1)}$

$\frac{1}{(x^2+1)^{1+1/2}} = \boxed{\frac{1}{(x^2+1)^{3/2}}}$

28. $y = \frac{x}{\sqrt{x^4+4}}$

$: (\sqrt{x^4+4}) \times (1) - (x)\left[\frac{1}{2}(x^4+4)^{-1/2} \times (4x^3)\right] \div (x^4+4)$

$\frac{(\sqrt{x^4+4})}{1} - x\left[\frac{2x^3}{\sqrt{x^4+4}}\right] \div (x^4+4)$

$\frac{(\sqrt{x^4+4})(\sqrt{x^4+4})}{(\sqrt{x^4+4})(x^4+4)} - 2x^4 = \frac{(x^4+4) - 2x^4}{\sqrt{(x^4+4)^3}} = \frac{x^4+4-2x^4}{\sqrt{(x^4+4)^3}}$

$= \boxed{\frac{4-x^4}{\sqrt{(x^4+4)^3}}}$

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28. $g(x) = \left(\frac{x+5}{x^2+2}\right)^2$

$: 2\left(\frac{x+5}{x^2+2}\right) \times \frac{du}{dx}\left[\frac{(x^2+2)(1) - (x+5)(2x)}{(x^2+2)^2}\right]$

$\left(\frac{2x+10}{x^2+2}\right) \times \left[\frac{x^2+2-2x^2-10x}{(x^2+2)^2}\right]$

$\left(\frac{2x+10}{x^2+2}\right) \times \left[\frac{2-x^2-10x}{(x^2+2)^2}\right] = \boxed{\frac{(2x+10)(2-x^2-10x)}{(x^2+2)^3}}$

30. $h(t) = \left(\frac{t^2}{t^3+2}\right)^2$

$: 2\left(\frac{t^2}{t^3+2}\right) \times \left[\frac{(t^3+2)(2t) - (t^2)(3t^2)}{(t^3+2)^2}\right]$

$\frac{2t^2}{(t^3+2)} \times \left[\frac{2t^4+4t-3t^4}{(t^3+2)^2}\right]$

$\boxed{\frac{(2t^2)(4t-t^4)}{(t^3+2)^3}}$

31. $f(v) = \left(\frac{1-2v}{1+v}\right)^3$

$: 3\left(\frac{1-2v}{1+v}\right)^2 \times \left[\frac{(1+v)(-2) - (1-2v)(1)}{(1+v)^2}\right]$

$3\left(\frac{1-2v}{1+v}\right)^2 \times \left[\frac{-2-\cancel{2v}-1+\cancel{2v}}{(1+v)^2}\right]$

$\frac{3(1-2v)^2}{(1+v)^2} \times \frac{(-3)}{(1+v)^2} = \frac{-9(1-2v)^2}{(1+v)^4}$

32. $\left(\frac{3x^2-2}{2x+3}\right)^3$

$: 3\left(\frac{3x^2-2}{2x+3}\right)^2 \times \left[\frac{(2x+3)(6x) - (3x^2-2)(2)}{(2x+3)^2}\right]$

$: \frac{3(3x^2-2)^2}{(2x+3)^2} \times \left[\frac{12x^2+18x-6x^2+4}{(2x+3)^2}\right]$

$: \frac{3(3x^2-2)^2 \times (6x^2+18x+4)}{(2x+3)^4} = \boxed{\frac{6(3x^2-2)^2(3x^2+9x+2)}{(2x+3)^4}}$

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33. $f(x) = ((x^2+3)^5 + x)^2$

$: 2((x^2+3)^5+x) \times \frac{du}{dx}\left[5(x^2+3)^4 \times \frac{du}{dx}(x^2+3) + 1\right]$

$: 2(x^2+3)^5+x) \times \frac{du}{dx}\left[5(x^2+3)^4 \times (2x) + 1\right]$

$(2(x^2+3)^5+x)\left[10x(x^2+3)^4+1\right]$

$\boxed{\left[2\left[(x^2+3)^5+x\right]\left[10x(x^2+3)^4+1\right]\right]}$

34. $g(x) = (2 + (x^2+1)^4)^3$

$: 3(2+(x^2+1)^4)^2 \times \frac{du}{dx}\left[0 + 4(x^2+1)^3\right]$

$: 3(2+(x^2+1)^4)^2(4(x^2+1)^3)$

$12(2+(x^2+1)^4)^2(x^2+1)^3$

35. $f(x) = \sqrt{2+\sqrt{2+\sqrt{x}}}$ → chain Rule

$: (2+\sqrt{2+\sqrt{x}})^{1/2} = (2+(2+\sqrt{x})^{1/2})^{1/2} = (2+(2+x^{1/2})^{1/2})^{1/2}$

$(2+(2+x^{1/2})^{1/2})^{1/2}$

$\frac{1}{2}(2+(2+x^{1/2})^{1/2})^{-1/2} \times \frac{du}{dx}\left(0 + \frac{1}{2}(2+x^{1/2})^{-1/2}\right)$

INCORRECT

$\frac{1}{2\sqrt{2+\sqrt{2+\sqrt{x}}}} \times \left(\frac{1}{2\sqrt{2+\sqrt{x}}}\right) = \frac{1}{4(\sqrt{2+\sqrt{2+\sqrt{x}}})(\sqrt{2+\sqrt{x}})}$

CORRECT:

$: (2+(2+x^{1/2})^{1/2})^{1/2}$

$\frac{1}{2}(2+(2+x^{1/2})^{1/2})^{-1/2} \times \frac{du}{dx}(2+(2+x^{1/2})^{1/2})$

$\frac{1}{2\sqrt{2+\sqrt{2+\sqrt{x}}}} \times \left(0 + \frac{1}{2}(2+x^{1/2})^{-1/2} \times \frac{du}{dx}(2+x^{1/2})\right)$

$\frac{1}{2\sqrt{2+\sqrt{2+\sqrt{x}}}} \times \left(\frac{1}{2\sqrt{2+\sqrt{x}}}\right) \times \left(0 + \frac{1}{2\sqrt{x}}\right)$

$= \frac{1}{8(\sqrt{2+\sqrt{2+\sqrt{x}}})(\sqrt{2+\sqrt{x}})(\sqrt{x})}$

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36. $\sqrt{\sqrt{t+1}+1}$

: $\left(\sqrt{t+1}+1\right)^{1/2}$

$\frac{1}{2}\left(\sqrt{t+1}+1\right)^{-1/2} \times \frac{dy}{dx}\left(\sqrt{t+1}+1\right)$

$\frac{1}{2}\left(\sqrt{t+1}+1\right)^{-1/2} \times \left(\frac{1}{2}(t+1)^{-1/2}+0\right)$

: $\frac{1}{2}\left(\frac{1}{\sqrt{\sqrt{t+1}+1}}\right) \times \left(\frac{1}{2} \times \frac{1}{\sqrt{t+1}}\right)$

: $\frac{1}{4\left(\sqrt{\sqrt{t+1}+1}\right)\left(\sqrt{t+1}\right)}$

45. $y = \cos 4x$

$u = \cos 4x$

$(-\sin 4x)(4)$

$y' = \boxed{-4\sin 4x}$

46. $y = \sin \pi x$

$(\cos \pi x)(\pi)$

$y' = \pi \cos \pi x$

47. $g(x) = 5\tan 3x$

$5(\sec^2 3x)(3)$

$y' = 15\sec^2 3x$

48. $h(x) = \sec x^2$

$(\sec x^2 \tan x^2)(2x)$

$2x \sec x \tan x^2$

49. $y = \sin(\pi x)^2$

$(\cos(\pi x)^2)(2(\pi x))(\pi)$

$2\pi^2 x \cos(\pi x)^2$

50. $y = \cos(1-2x)^2$

: $(-\sin(1-2x)^2)(2(1-2x))(-2)$

: $(-\sin(1-2x)^2(-4)(1-2x)$

$-4+8x-\sin(1-2x)^2$

51. $h(x) = \sin 2x \cos 2x$

: $(\cos 2x)(2)(\cos 2x) + \sin 2x(-\sin 2x)(2)$

$2\cos 2x \cos 2x + (-2\sin^2 2x)$

$2\cos^2 2x - 2\sin^2 2x$

$2(\cos^2 2x - \sin^2 2x)$

53. $f(x) = \frac{\cot x}{\sin x}$ Quotient Rule

: $\frac{\sin x(-\csc^2 x) - \cot x(\cos x)}{\sin^2 x} = \frac{-\sin x \csc^2 x}{\sin^2 x} - \frac{\cot x \cos x}{\sin^2 x}$

$\frac{-\sin x}{\sin^2 x} \times \frac{1}{\sin^2 x} - \frac{\cos x}{\sin x} \times \frac{\cos x}{\sin^2 x} = \frac{-\sin x}{\sin^4 x} - \frac{\cos^2 x}{\sin^3 x}$

$= \frac{-1}{\sin^3 x} - \frac{\cos^2 x}{\sin^3 x} = \boxed{\frac{-1-\cos^2 x}{\sin^3 x}}$

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54. $g(v) = \frac{\cos v}{\csc v}$ → Quotient Rule

: $\frac{(\csc v)(-\sin v) - (\cos v)(-\csc v \cot v)}{\csc^2 v} = \frac{-\csc v \sin v}{\csc^2 v} + \frac{\csc^2 v \cot v}{\csc^2 v}$

$= \frac{-\csc v}{\csc^2 v} \times \frac{1}{\csc v} + \frac{\csc^2 v}{\csc^2 v} \times \cot v = \frac{-1}{\csc^2 v} + \cot v = \boxed{-\sin^2 v + \cot v}$

55. $4 \sec^2 x$ → chain Rule

: $4(\sec x)^2 = (2)4(\sec x)(\sec x \tan x) = \boxed{8 \sec^2 x \tan x}$

56. $g(t) = 5\cos^2 \pi t$

: $5(\cos \pi t)^2 = (2)5(\cos \pi t)(-\sin \pi t) = -10 \cos \pi t \sin \pi t$

57. $f(\theta) = \tan^2 5\theta$ → chain Rule

: $(\tan 5\theta)^2 = 2(\tan 5\theta)(\sec^2 5\theta) = 10 \tan 5\theta \sec^2 5\theta$

: $(\tan 5\theta)^2 = 2(\tan 5\theta)(\sec^2 5\theta)(5) = 10 \tan 5\theta \sec^2 5\theta$

58. $g(\theta) = \cos^2 8\theta$ → Trigonometric functions derivative provided

: $(\cos 8\theta)^2 = 2(\cos 8\theta)(-\sin 8\theta)(8) = -16 \cos 8\theta \sin 8\theta$

59. $f(\theta) = \frac{1}{4} \sin^2 2\theta$

: $\frac{1}{4}(\sin 2\theta)^2 = \frac{(2)1}{4}(\sin 2\theta)(\cos 2\theta)(2) = \boxed{\sin 2\theta \cos 2\theta}$

60. $h(t) = 2\cot^2(\pi t + 2)$

: $2(\cot(\pi t + 2))^2 = 4(\cot(\pi t + 2))(-\csc(\pi t + 2)\cot(\pi t + 2))(\pi)$

: $-4\pi \cot^2(\pi t + 2)\csc(\pi t + 2)$

61. $f(t) = 3\sec^2(\pi t - 1)$

: $3(\sec(\pi t - 1))^2 = 6(\sec(\pi t - 1))(\sec(\pi t - 1)\tan(\pi t - 1))(\pi)$

$= 6\pi(\sec^2(\pi t - 1))(\tan(\pi t - 1))$

$= \frac{6\pi \tan(\pi t - 1)}{\cos^2(\pi t - 1)} = \boxed{\frac{6\pi \sin(\pi t - 1)}{\cos^3(\pi t - 1)}}$

62. $y = 3x - 5\cos(\pi x)^2$

: $y' = 3 - 5(-\sin(\pi x)^2)(2\pi x) = \boxed{3 + 10\pi x \sin(\pi x)^2}$

63. $y = \sqrt{x} + \frac{1}{4}\sin(2x)^2$ → chain + Trigonometric

: $\frac{1}{2\sqrt{x}} + \frac{1}{4}\cos(2x)^2 \times 2(2x)(2) = \frac{1}{2\sqrt{x}} + 2x\cos(2x)^2$

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64. $y = \sin\sqrt[3]{x} + \sqrt[3]{\sin x}$ → chain + Trigonometric

: $\sin(x)^{1/3} + (\sin x)^{1/3} = \cos(x)^{1/3} \times \frac{1}{3}(x)^{-2/3}(1) + \frac{1}{3}(\sin x)^{-2/3}(\cos$

$\frac{\cos\sqrt[3]{x}}{3\sqrt[3]{x^2}} + \frac{\cos x}{3\sqrt[3]{\sin^2 x}}$

66. $y = \cos\sqrt{\sin(\tan\pi x)}$

: $-\sin\left(\sqrt{\sin(\tan\pi x)}\right)\left(\frac{1}{2\sqrt{\sin(\tan\pi x)}}\right)$

65. $y = \sin(\tan 2x)$

$\cos(\tan 2x)(\sec^2 2x)(2)$: $\frac{-\sin\left(\sqrt{\sin(\tan\pi x)}\right) \times \cos(\sec^2\pi x) \times \pi}{2\sqrt{\sin(\tan\pi x)}}$

$2\cos(\tan 2x)(\sec^2 2x)$

$2(\sec^2 2x)\cos(\tan 2x)$

$\frac{-\pi\sin\left(\sqrt{\sin(\tan\pi x)}\right)\left(\cos(\sec^2\pi x)\right)}{2\sqrt{\sin(\tan\pi x)}}$

▶ 67–74 Evalute the derivative at given point.

67. $s(t) = \sqrt{t^2+6t-2}$, $(3, 5)$ → chain Rule

$\frac{1}{2}(t^2+6t-2)^{-1/2} \times \frac{d}{dx}(t^2+6t-2) = \frac{1}{2\sqrt{t^2+6t-2}} \times (2t+6)$

$= \frac{2(t+3)}{2\sqrt{t^2+6t-2}} = \frac{t+3}{\sqrt{t^2+6t-2}}$, put $t=3 \Rightarrow \frac{6}{\sqrt{9+18-2}} = \frac{6}{\sqrt{25}} = \boxed{\frac{6}{5}}$

68. $\sqrt[5]{3x^3+4x}$, $(2, 2)$ → chain Rule

: $(3x^3+4x)^{1/5} \Rightarrow \frac{1}{5}(3x^3+4x)^{-4/5} \times \frac{d}{dx}(3x^3+4x)$

$= \frac{1}{5\sqrt[5]{(3x^3+4x)^4}} \times (9x^2+4) = \frac{9x^2+4}{5\sqrt[5]{(3x^3+4x)^4}}$, put $x=2$

$= \frac{9(2)^2+4}{5\sqrt[5]{(3(2)^3+4(2))^4}} = \frac{40}{5\sqrt[5]{(24+8)^4}} = \frac{40}{5(32)^{4/5}} = \frac{8}{(2^5)^{4/5}} = \frac{8}{2^4} = \boxed{\frac{1}{2}}$

69. $f(x) = \frac{5}{(x^3-2)}$, $(-2, -1/2)$ → Quotient

: $\frac{(x^3-2)(0)-(5)(x^3-2)}{(x^3-2)^2} = \frac{-5(x^3-2)}{(x^3-2)^2}$, put $x=-2$

: $\frac{-5((-2)^3-2)}{(-2^3-2)^2} = \frac{-5(-10)}{(-10)^2} = \frac{50}{100} = \frac{1}{2}$

$(vu'-uv') \div v^2$

$\frac{(x^3-2)(0)-(5)(3x^2)}{(x^3-2)^2} = \frac{-15x^2}{(x^3-2)^2} = \frac{-15(-2)^2}{(-8-2)^2} = \frac{-60}{100} = \boxed{\frac{-3}{5}}$

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70. $f(x) = \dfrac{1}{(x^2-3x)^2}$, $(4, 1/16)$ → Quotient Rule + Chain Rule

$$\frac{[(x^2-3x)^2(0) - (1)\times[2(x^2-3x))\times(2x-3)]}{[(x^2-3x)^2]^2}$$

$$\frac{(2x^2-6x)(2x-3)}{(x^2-3x)^4} \Rightarrow \text{put } x=4 \rightarrow \frac{(2(4)^2-6(4))(2(4)-3)}{(4^2-3(4))^4} = \frac{8(5)}{256} = \boxed{\frac{5}{32}}$$

71. $f(t) = \dfrac{3t+2}{t-1}$, $(0, -2)$ → Quotient Rule

$$= \frac{(t-1)(3) - (3t+2)(1)}{(t-1)^2} = \frac{(3t-3)-3t-2}{(t-1)^2} = \frac{-5}{(t-1)^2} \Rightarrow \text{put } t=0$$

$$= \frac{-5}{(0-1)^2} = \frac{-5}{1} = \boxed{-5}$$

72. $f(x) = \dfrac{x+1}{2x-3}$, $(2, 3)$ → Quotient Rule

$$= \frac{(2x-3)(1) - (x+1)(2)}{(2x-3)^2} = \frac{(2x-3)-(2x+2)}{(2x-3)^2}, \text{ put } x=2$$

$$\frac{(2(2)-3)-(2(2)+2)}{(2(2)-3)^2} = \frac{-5}{(1)^2} = \boxed{-5}$$

73. $y = 26 - \sec^3 4x$, $(0, 25)$

$y' = 0 - (\sec^3 4x \tan^3 4x)$, put $x=0$ ✗

$y' = 0 - 3(\sec 4x)^2 \times (\sec 4x \tan 4x)(4)$

$y' = -12(\sec^2 4x) \times (\sec 4x \tan 4x)$

$y' = -12(\sec^3 4x)(\tan 4x)$, put $x=0$

$y' = -12(1/4)(0) = 0$

74. $\dfrac{1}{x} + \sqrt{\cos x}$, $\left(\dfrac{\pi}{2}, \dfrac{2}{\pi}\right)$

$$\frac{-1}{x^2} + \frac{1}{2\sqrt{\cos x}} \times (-\sin x) = \frac{-1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}, \quad x = \pi/2 = 90^\circ$$

$$= \frac{-4}{\pi^2} - \frac{1}{2(\sqrt{0})} = \boxed{\frac{-4}{\pi^2}}$$

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▶ 75–82 find an equation of tangent line

75. Function $f(x) = \sqrt{2x^2-7}$ | Point $(4,5)$

: Chain Rule

$(2x^2-7)^{1/2}$

$= \frac{1}{2}(2x^2-7)^{-1/2}$

$y' = \frac{1}{2\sqrt{2x^2-7}} \times 4x$

$y' = \frac{4x}{2\sqrt{2x^2-7}}$, put $x=4$

$y' = \frac{16}{2\sqrt{32-7}} = \frac{8}{\sqrt{25}} = \boxed{\frac{8}{5}} \to m$

$5(y-5) = 8(x-4)$

$5y - 25 = 8x - 32$

$\boxed{5y = 8x - 7}$

76. $f(x) = \frac{1}{3} \times \sqrt{x^2+5}$; $(2,2)$

: Chain + Product Rule

$= (\sqrt{x^2+5})(1/3) + (x/3)\left[\left(\frac{1}{2\sqrt{x^2+5}}\right) \times (2x)\right]$

$= \left(\frac{\sqrt{x^2+5}}{3}\right) + \frac{x}{3}\left(\frac{2x}{2\sqrt{x^2+5}}\right) = \frac{\sqrt{x^2+5}}{3} + \frac{2x^2}{6\sqrt{x^2+5}}$, Put $x=2$

$\frac{\sqrt{4+5}}{3} + \frac{8}{6\sqrt{9}} = \frac{1}{1} + \frac{4}{9} = \frac{9+4}{9} = \boxed{\frac{13}{9}} \to m$, $P(2,2)$

77. $y = (4x^3+3)^2$; $(-1,1)$

: Chain Rule

$= 2(4x^3+3) \times \frac{dy}{dx}(12x^2) = (24x^2)(4x^3+3)$

$96x^5 + 72x^2$, put $x=-1$

$-96 + 72 = \boxed{-24} \to m$

$y - 1 = -24(x+1)$

$\boxed{y + 24x + 23 = 0}$

78. $f(x) = (9-x^2)^{2/3} \to$ chain Rule $(1,4)$

: $\frac{2}{3}(9-x^2)^{-1/3} \times \frac{dy}{dx}(-2x) = \frac{-4x}{3\sqrt[3]{9-x^2}}$, put $x=1$

: $\frac{-4(1)}{3\sqrt[3]{9-(1)^2}} = \frac{-4}{3\sqrt[3]{8}} = \frac{-4}{3(8)^{1/3}} = \frac{-4}{3(2^3)^{1/3}} = \frac{-4}{3(2)} = \boxed{-\frac{2}{3}} \to m$

$y - 4 = -\frac{2}{3}(x-1)$

$3y - 12 = -2x + 2 \Rightarrow \boxed{3y + 2x = 14}$

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79. $f(x) = \sin 2x$ $(\pi, 0)$

: Trigonometric functions Differentiation [parenthesis Rule]

$(\cos(2x))(2)$

$y' = 2\cos 2x$

put $x = \pi$

$y' = 2\cos 2\pi$

$y' = 2(1)$

$\boxed{m = 2}$

$y = 2(x - \pi)$

$\boxed{y = 2x - 2\pi}$

80. $y = \cos 3x$; $(\pi/4, -\sqrt{2}/2)$

$y' = (-\sin(3x))(3)$

$y' = -3\sin 3x$

$x = \pi/4 / 45°$

$m = -3\sin 3(45°)$

$m = -3(3)$

$\boxed{m = -9}$

$y + \frac{\sqrt{2}}{2} = -9\left(x - \frac{\pi}{4}\right)$

$\frac{2y + \sqrt{2}}{2} = \frac{-9(4x - \pi)}{4}$

$2(2y + \sqrt{2}) = +9\pi - 36x$

$\boxed{4y + 2\sqrt{2} = 9\pi - 36x}$

81. $f(x) = \tan^2 x$; $(\pi/4, 1)$

: $(\sec^2 x)$

$(\sec x)^2$

$(\sec 45°)^2$

$(\sqrt{2})^2$

$\boxed{m = 2}$

$y - 1 = 2(x - \pi/4)$

$y - 1 = 2\left(\frac{4x - \pi}{4}\right)$

$2(y - 1) = 4x - \pi$

$2y - 2 = 4x - \pi$

$2y = 4x + 2 - \pi$

82. $y = 2\tan^3 x$; $(\pi/4, 2)$

$y' = 2(\sec x)^3$

$y = 2(\sec 45°)^3$

$y = 2(\sqrt{2})^3$

$m = \boxed{2^{5/2}}$

$y - 2 = 2^{5/2}(x - \pi)$

$\boxed{y = 2^{5/2}(x - \pi) + 2}$

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▶ Derivative Test + Stationary point + Critical points + Concavity + Interval.

(1) $y = x^4 - 6x^2 + 8x + 10$

1: First Derivative

$\boxed{y' = 4x^3 - 12x + 8}$ → First Derivative

2: Solve $f(x)'$ and set critical points

1	4	0	-12	8
	×	4	4	-8
	4	4	-8	0

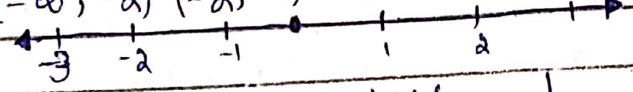
→ $4x^2 + 4x - 8 = 0$

$x^2 + x - 2 = 0$

CRITICAL Points → $\boxed{x = 1,\ x = -2,\ x = 1}$

3: Find Intervals in which graph Increase / Decrease

$(-\infty, -2)$ $(-2, 1)$ $(1, \infty)$

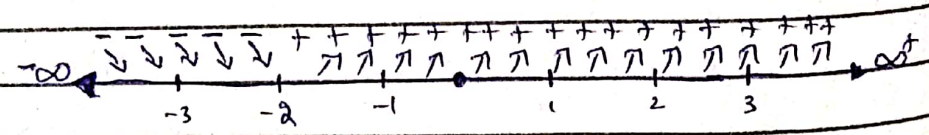


$(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$

Put Nearest points of Interval(s) In first Derivative's function

$f(x)' = 4x^3 - 12x + 8$

$f(-3)' = 4(-27) - 12(-3) + 8$	$f(0)' = 4(0)^3 - 12(0) + 8$	$f(2)' = 4(2)^3 - 12(2) + 8$
$= -108 + 36 + 8$	$f(0)' = 8$	$f(2)' = 32 - 24 + 8$
$= -108 + 44$	$\boxed{8 > 0}$	$f(2)' = 16$
$\boxed{-64 < 0}$	Increase	$\boxed{16 > 0}$
Decrease		Increase



4: Check Concavity → find Second derivative

$f(x) = x^4 - 6x^2 + 8x + 10$

$f(x)' = 4x^3 - 12x + 8$

$\boxed{f(x)'' = 12x^2 - 12}$ → Second Derivative

Simplifying :- $y = 12x^2 - 12$

$x^2 - 1 = 0$

$\boxed{x = \pm 1}$ → Put these points In Original Func

$f(x) = x^4 - 6x^2 + 8x + 10$

$f(1) = (1)^4 - 6(1)^2 + 8(1) + 10$	$f(-1) = (-1)^4 - 6(-1)^2 + 8(-1) + 10$
$f(1) = 1 - 6 + 8 + 10$	$f(-1) = 1 - 6 - 8 + 10$
$f(1) = 13$	$f(-1) = -3$
$(1, 13)$	$(-1, -3)$

Stationary Points $\longrightarrow$ $(1, 13)$ and $(-1, -3)$

[put critical points in Second derivative]

$f(x)'' = 12x^2 - 12x$,

Critical points $\longrightarrow$ $1, 1, -2$

$f(1)'' = 12(1)^2 - 12(1)$	$f(-2)'' = 12(-2)^2 - 12(-2)$
$f(1)'' = 12 - 12$	$f(-2)'' = 48 + 24$
$f(1)'' = 0$	$f(-2)'' = 72$
Neither increase nor decrease	$\boxed{72 > 0}$
Stopped	Concave up

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② $y = x^3 - 5x^2 + 7x$

1: first derivative

$\boxed{y' = (3x^3 - 10x + 7)}$ → first Derivative

2: Solve for critical points

via hit and try method, we get one root → $\boxed{1 = x}$

$y' = 3x^3 - 10x + 7$

1	3	0	-10	7
	x	3	3	-7
	3	3	-7	0

→ $\boxed{3x^2 + 3x - 7 = 0}$

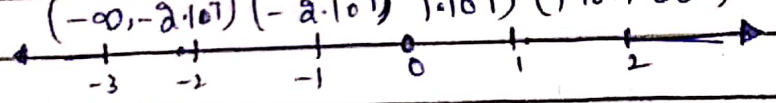
$a = 3, \quad b = 3, \quad c = -7$

$x = \frac{-3 \pm \sqrt{9 - 4(3)(-7)}}{6} = \frac{-3 \pm \sqrt{9 + 84}}{6} = \frac{-3 \pm \sqrt{93}}{6}$

$\frac{-3 \pm 9.64}{6}$ → $\boxed{x = 1.107, \quad x = -2.107}$ → critical points

3: find Intervals in which graph increase / Decreas

$(-\infty, -2.107) \quad (-2.107, 1.107) \quad (1.107, \infty^+)$

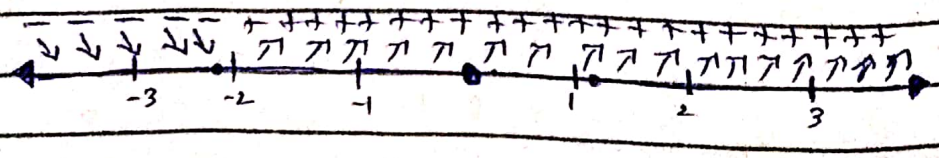


$\boxed{(-\infty, -2.107) \cup (-2.107, 1.107) \cup (1.107, \infty^+)}$

Put Nearest points of interval in first Derivative

$y' = 3x^3 - 10x + 7$

$f(-3)' = 3(-3)^3 - 10(-3) + 7$	$f(0)' = 3(0)^3 - 10(0) + 7$	$f(2)' = 3(2)^3 - 10(2) + 7$
$f(-3)' = 3(-27) + 30 + 7$	$f(0)' = 7$	$f(2)' = 3(8) - 10(2) + 7$
$f(-3)' = -81 + 37$	$\boxed{7 > 0}$	$f(2)' = 24 - 20 + 7$
$f(-3)' = -44$	Increase	$f(2)' = 4$
$\boxed{-44 < 0}$		$\boxed{11 > 0}$
Decrease		Increase



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4: check concavity and find stationary points

: Concavity → find second derivative

$f(x)'' = 9x^2 - 10$

put critical points in second derivative

critical points → 1, 1.107, −2.107

$f(1)'' = 9(1)^2 - 10$	$f(1.107)'' = 9(1.107)^2 - 10$	$f(-2.107)'' = 9(-2.107)^2 - 10$
$\boxed{-1 < 0}$	$f(1.107)'' = 1.029$	$f(-2.107)'' = 39.95 - 10$
Concave down	$\boxed{1.029 > 0}$	$\boxed{29.95 > 0}$
	Concave up	$\boxed{\text{concave up}}$

$f(x)'' = 9x^2 - 10$ Solve for x → solving for stationary points

$9x^2 - 10 = 0$

$x^2 = \frac{10}{9} \Rightarrow \boxed{x = \pm \frac{\sqrt{10}}{3}}$

put x-values in original function

$f(x) = x^3 - 5x^2 + 7x$;

$f(\sqrt{10}/3) = \left(\frac{\sqrt{10}}{3}\right)^3 - 5\left(\frac{\sqrt{10}}{3}\right)^2 + 7\left(\frac{\sqrt{10}}{3}\right)$	$f(-\sqrt{10}/3) = \left(\frac{-\sqrt{10}}{27}\right)^3 - 5\left(\frac{-\sqrt{10}}{3}\right)^2 + 7\left(\frac{-\sqrt{10}}{3}\right)$
$f(\sqrt{10}/3) = \frac{(\sqrt{10})^3}{27} - \frac{50}{9} + \frac{7\sqrt{10}}{3}$	$f(-\sqrt{10}/3) = -1.171 + 5.55 + (-7.37)$
$f(\sqrt{10}/3) = 1.171 - 5.55 + 7.37$	$f(-\sqrt{10}/3) = -1.171 + 5.55 - 7.37$
$2.881 = y = 2.881$	-2.98
$\boxed{(\sqrt{10}/3, 2.881)}$	$\boxed{(-\sqrt{10}/3, -2.98)}$

↑ Stationary points ↑

Defination:-

Concave up:- If f is differentiable on an open interval, and f' is decreasing on that interval.

Concave down:- If f' is decreasing on that interval.

Theorem: Let f be twice differentiable on an open interval.

(a) If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.

(b) If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval.

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04/04/2020

2.5 Implicit Differentiation

1. Distinguish between functions written in implicit form & explicit form
2. Use implicit differentiation to find derivative of function

▶ Implicit and Explicit Functions | defination:-

1. Explicit [Parent function] → basically 6

: A function in which y is separate from x-values

$y = x$, $y = x^2$, $y = x^3$, $y = 1/x$, $y = |x|$

$y = e^x$, $y = a^x$, $y = \sqrt{x^2 - 1}$

$$\boxed{|x| = \sqrt{x^2}}$$

$$e^x = e^x$$

$$a^x = \ln a$$

: Solved for y functions are known as explicit functions

2. Implicit Functions

: y can't be separate from x-values.
Solved for x function(s)

Basic form:- $f(x, y) = 0$

: uses chain rule frequently

$y^2 + 2xy + y^3 = 0$

Implicit form	Explicit Form	Derivative
$xy = 1$	$y = \frac{1}{x}$	$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$

: This strategy works whenever you can solve for function explicitly

Consider this one:- $x^2 - 2y^3 + 4y = 2$

: where it is difficult to express y as a function of x explicitly? To do this, you can use implicit differentiation

:- To understand how to find dy/dx implicitly, you must realize that the differentiation is taking place with respect to x.

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04/04/2025 :

This means that when you differentiate terms involving x alone, you can differentiate as usual. However, when you differentiate terms involving y, you must apply the chain rule, because you are assuming that y is defined implicitly as a differentiable function of "x."

Ex:- Differentiating with respect to x.

a. $\frac{d}{dx}[x^3] = 3x^2$ variables agree : use simple power Rule

variables agree

: in other words → $\frac{d}{dx} x^3 = \frac{d(x)}{dx} 3x^{3-1} = \frac{dx}{dx} 3x^2 = 3x^2$

b. $\frac{d}{dx}[y^3] = 3y^2 \frac{dy}{dx}$ variables disagree : use chain Rule

variables disagree

chain Rule

→ $\frac{d}{dx}[y^3] = \frac{d(y)}{dx}[3y^2] \times (1) = \frac{dy}{dx} 3y^2 = \boxed{3y^2 \frac{dy}{dx}}$

c. $\frac{d}{dx}[x + 3y] = 1 + 3\frac{dy}{dx}$ → chain rule

: $\frac{d(x)}{dx}(1) + \frac{d}{dx} 3y = \frac{dx}{dx} + \frac{d(y)}{dx} 3 = \boxed{1 + 3\frac{dy}{dx}}$

d. xy^2

:- Applying product rule → $\boxed{vu' + uv'}$

$v = y^2$, $v' = 2y$, $u = x$. $u' = 1$

$= y^2 \times \frac{d}{dx}(x) + x \times \frac{d}{dx}(y^2)$ Product rule

agree dis agree chain rule

$= y^2 \times \frac{d(x)}{dx} \times (1) + x \times \frac{d(y)}{dx}(2y) \times (1)$

$= y^2 \times \frac{dx}{dx} + 2xy \frac{dy}{dx}$

$= \boxed{y^2 + 2xy \frac{dy}{dx}}$ Simplify

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▶ Guide Lines for implicit differentiation.

1. Differenticele both sides of the equation w.r.t x.
2. Collect all terms involving dy/dx on the "left side" of the equation. and move all other terms to the "right side" of the equation.
3. Factor dy/dx out of the left side of the equation.
4. Solve for dy/dx.

Example (2) Implicit Differentiation

Find dy/dx given that $y^3 + y^2 - 5y - x^2 = -4$

Solution

1: Differentiate both sides of equation w.r.t x.

$$\frac{d}{dx}[y^3 + y^2 - 5y - x^2] = \frac{d}{dx}[-4]$$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[y^2] - 5\frac{d}{dx}[y] - \frac{d}{dx}[x^2] = \frac{d}{dx}[-4]$$

$$\frac{d(y)}{dx}[3y^2] + \frac{d(y)}{dx}[2y] - 5\frac{d(y)}{dx}[1] - \frac{d(x)}{dx}[2x] = \frac{d(0)}{dx}(0)$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - \frac{dy}{dx}5 - 2x\frac{dx}{dx} = 0$$

$$3y^2\frac{dy}{dx} + 2y\frac{dy}{dx} - 5\frac{dy}{dx} - 2x = 0$$

2: $\frac{dy}{dx}[3y^2 + 2y - 5] = 2x$

3: $\boxed{\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}}$

Slope at $A(2,0)$, $B(1,-3)$, $C)\ x = 0$

A) $\frac{dy}{dx} = \frac{2(2)}{3(0)^2 + 2(0) - 5} = \boxed{\frac{-4}{5}}$

B) $\frac{dy}{dx} = \frac{2(1)}{3(-3)^2 + 2(-3) - 5} = \frac{2}{27 - 6 - 5} = \frac{2}{16} = \boxed{\frac{1}{8}}$

C) $\frac{dy}{dx} = \frac{2(0)}{3y^2 + 2y - 5} = \frac{0}{3y^2 + 2y - 5} = \boxed{0}$

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A short hand way: very useful when dealing with complex

$y^3 + y^2 - 5y - x^2 + 4 = 0$

$\therefore \frac{dy}{dx} = \frac{-F_x}{F_y}$

1: $-F_x$, put $y=0$ in the original function, multiply 0 with all constants

$y^3 + y^2 - 5y - x^2 + 4 = 0$

$(0)^3 + (0)^2 - 5(0) - x^2 + 0 = 0$

$0 + 0 - 0 - x^2 + 0 = 0$

2: $\frac{dy}{dx}$ of $-x^2$: $\boxed{-2x}$

$\boxed{-F_x = -2x}$

3: $F_y = y^3 + y^2 - 5y - x^2 + 4 = 0$

put $x=0$, and differentiate the function simply

$\frac{dy}{dx} F_y = 3y^2 + 2y - 5 - 0 + 0 = 0$

$F_y = 3y^2 + 2y - 5 = 0$

$\boxed{F_y = 3y^2 + 2y - 5}$

$\frac{dy}{dx} F_x = -2x \qquad \frac{dy}{dx} F_y = 3y^2 + 2y - 5$

$\frac{-F_x}{F_y} = \frac{-(-2x)}{3y^2 + 2y - 5} = \boxed{\frac{2x}{3y^2 + 2y - 5}}$

Example ③ Find slope of graph implicitly

Determine the slope of the tangent line to the graph of $x^2 + 4y^2 = 4$ at point $(\sqrt{2}, -1/\sqrt{2})$

Solution

$x^2 + 4y^2 = 4$

$\frac{d}{dx}[x^2] + \frac{d}{dx}[4y^2] = \frac{d}{dx}[4]$

$\frac{dx}{dx}[2x] + \frac{d(y)}{dx}[8y] = 0$

$\frac{dy}{dx} = \frac{-2x}{8y} = \frac{-x}{4y}$

at $(\sqrt{2}, -1/\sqrt{2})$ slope is:

$\frac{dy}{dx} = \frac{-x}{4y} = \frac{-(\sqrt{2})}{4\left(\frac{-1}{\sqrt{2}}\right)} = \frac{-\sqrt{2}}{\frac{-4}{\sqrt{2}}}$

$= \cancel{-}\sqrt{2} \div \frac{\cancel{-}4}{\sqrt{2}}$

$= \sqrt{2} \times \frac{\sqrt{2}}{4} = \frac{2}{4} = \boxed{\frac{1}{2}}$

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4/4/23

Example (4) Finding a tangent line to a graph

Find the tangent line to the graph given by $x^2(x^2+y^2)=y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$

Solution

1: $x^2(x^2+y^2)=y^2$

$x^4+x^2y^2=y^2$

$\boxed{x^4+x^2y^2-y^2=0}$

Apply d/dx operator

$\frac{d}{dx}[x^4+x^2y^2-y^2]=0$

$\frac{d}{dx}[x^4]+\frac{d}{dx}[x^2y^2]-\frac{d}{dx}[y^2]=0$

product rule

$\frac{d(x)}{dx}[4x^3]+\left[(y^2)\times\frac{d}{dx}(x^2)+x^2\times\frac{d}{dx}(y^2)\right]-\frac{d(y)}{dx}[2y]=0$

$4x^3+\left[y^2\times\frac{d(x)}{dx}(2x)+x^2\times\frac{d(y)}{dx}(2y)\right]-\frac{dy}{dx}(2y)=0$

$4x^3+\left[2xy^2+\frac{dy}{dx}2x^2y\right]-2y\frac{dy}{dx}=0$

$4x^3+2xy^2+2x^2y\frac{dy}{dx}-2y\frac{dy}{dx}=0$

$4x^3+2xy^2+\frac{dy}{dx}(2x^2y-2y)=0$

$2\frac{dy}{dx}(x^2y-y)=-4x^3+2xy^2$

$2\frac{dy}{dx}(x^2y-y)=-2(2x^3+xy^2)$

$\frac{dy}{dx}=\frac{-(2x^3+xy^2)}{x^2y-y}=\frac{-x(2x^2+y^2)}{y(x^2-1)}$

at point $(\sqrt{2}/2, \sqrt{2}/2)$ the slope is

$\frac{dy}{dx}=\frac{-(\sqrt{2}/2)(2(\sqrt{2}/2)^2+(\sqrt{2}/2)^2)}{(\sqrt{2}/2)((\sqrt{2}/2)^2-1)}=\frac{-(\sqrt{2}/2)(1+1/2)}{(\sqrt{2}/2)(1/2-1)}$

$m=\left(-\frac{\sqrt{2}}{2}\times\frac{3}{2}\right)\div\left(\frac{\sqrt{2}}{2}\times-\frac{1}{2}\right)=\left(\frac{-3\sqrt{2}}{4}\right)\div\left(-\frac{\sqrt{2}}{4}\right)=\frac{-3\sqrt{2}}{4}\times\frac{4}{-\sqrt{2}}=3$

$\boxed{m=3}$ Point $((\sqrt{2}/2), \sqrt{2}/2)$

$y-\frac{\sqrt{2}}{2}=3(x-\sqrt{2}/2) \Rightarrow \boxed{y=3x-\sqrt{2}}$

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Exercise 2.5 Implicit/Explicit Differentiation 4/4/23

In Exercises 1–16, find dy/dx by implicit Differentiation.

1. $x^2 + y^2 = 8$

$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[8]$

$\frac{d(x)}{dx}[2x] + \frac{d(y)}{dx}[2y] = 0$

$2x + \frac{dy}{dx}[2y] = 0$

$dy = \frac{2x}{2y} = \boxed{\frac{-x}{y}}$

2. $x^2 - y^2 = 25$

$\frac{d}{dx}[x^2] - \frac{d}{dx}[y^2] = \frac{d}{dx}[25]$

$\frac{d(x)}{dx}[2x] - \frac{d(y)}{dx}[2y] = 0$

$2x - \frac{dy}{dx}[2y] = 0$

$\frac{dy}{dx} = \frac{2x}{2y} = \boxed{\frac{x}{y}}$

3. $x^{1/2} + y^{1/2} = 16$

$\frac{d}{dx}[x^{1/2}] + \frac{d}{dx}[y^{1/2}] = \frac{d}{dx}[16]$, Chain Rule

$\left[\left(\cancel{\frac{d(x)}{dx}}\right)\left(\frac{1}{2}(x)^{-1/2}\right) \times (1)\right] + \left[\frac{d(y)}{dx} \times \left(\frac{1}{2}(y)^{-1/2} \times (1)\right)\right] = 0$

$\left[\frac{1}{2\sqrt{x}}\right] + \left[\frac{d(y)}{dx} \times \left(\frac{1}{2\sqrt{y}}\right)\right] = 0$

$\frac{dy}{d(x)} = \left(\frac{-1}{\cancel{2}\sqrt{x}}\right) \div \left(\frac{1}{\cancel{2}\sqrt{y}}\right) = \frac{-1}{\sqrt{x}} \times \sqrt{y} = \frac{-\sqrt{y}}{\sqrt{x}} = \boxed{-\sqrt{\frac{y}{x}}}$

4. $x^3 + y^3 = 64$

$\frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] = \frac{d}{dx}[64]$

$\frac{d(x)}{dx}[3x^2] + \frac{d(y)}{dx}[3y^2] = 0$

$3x^2 + 3y^2\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{3x^2}{3y^2} = \boxed{\frac{-x^2}{y^2}}$

5. $x^3 - xy + y^2 = 7$

$\frac{d}{dx}[x^3] - \frac{d}{dx}[xy] + \frac{d}{dx}[y^2] = \frac{d}{dx}[7]$

$3x^2 - [y(1) + \frac{dy}{dx}(1)] + \frac{dy}{dx}[2y] = 0 \rightarrow 3x^2 - y - x\frac{dy}{dx} + \frac{dy}{dx}[2y] = 0$

$3x^2 - y - \frac{dy}{dx}(x - 2y) = 0 \rightarrow \frac{dy}{dx} = (3x^2 - y)/(x - 2y)$

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6. $x^2y + y^2x = -2$

$\therefore \frac{d}{dx}[x^2y] + \frac{d}{dx}[y^2x] = \frac{d}{dx}[-2]$

$dx \leftarrow$ product rule $\quad dx \leftarrow$ product rule

$U.V = VU' + UV'$

$[(y) \times \frac{d}{dx}(x^2) + (x^2) \times \frac{d}{dx}(y)] + [(x) \times \frac{d}{dx}(y^2) + (y^2) \times \frac{d}{dx}(x)] = 0$

$[y \times \frac{d(x)}{dx}(2x) + x^2 \times \frac{d(y)}{dx}(1)] + [x \times \frac{d(y)}{dx}(2y) + y^2 \times \frac{d(x)}{dx}(1)] = 0$

$[y \times (2x) + x^2 \times \frac{dy}{dx}] + [2xy \times \frac{dy}{dx} + y^2] = 0$

$2xy + x^2\frac{dy}{dx} + 2xy\frac{dy}{dx} + y^2 = 0$

$\frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$

$\boxed{dy/dx = (-2xy - y^2)/(x^2 + 2xy)}$

7. $x^3y^3 - y = x$

$\therefore \frac{d}{dx}[x^3y^3] - \frac{d}{dx}[y] = \frac{d}{dx}[x]$

$[(y^3) \times \frac{d}{dx}[x^3] + (x)^3 \times \frac{d}{dx}[y^3]] - \frac{d(y)}{dx}(1) = 1$

$[(y^3) \times \frac{d(x)}{dx}(3x^2) + x^3 \times \frac{d(y)}{dx}(3y^2)] - \frac{dy}{dx} = 1$

$3x^2y^3 + 3x^3y^2\frac{dy}{dx} - \frac{dy}{dx} = 1$

$\frac{dy}{dx}(3x^3y^2 - 1) = (1 - 3x^2y^3)$

$dy/dx = (1 - 3x^2y^3)/(3x^3y^2 - 1)$

8. $\sqrt{xy} = x^2y + 1$

$\therefore \frac{d}{dx}(\sqrt{xy}) = \frac{d}{dx}(x^2y) + \frac{d}{dx}(1)$

$\frac{1}{2}(xy)^{-1/2} \times \frac{d}{dx}(xy) = [y \times \frac{d}{dx}(x^2) + (x^2) \times \frac{d}{dx}(y)] + 0$

$\left(\frac{1}{2\sqrt{xy}}\right) \times [y \times \frac{d}{dx}(x) + (x) \times \frac{d}{dx}(y)] = 2xy + x^2\frac{dy}{dx}$

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$$\left(\frac{1}{2\sqrt{xy}}\right)\left[y + x\frac{d(y)}{dx}\right] = 2xy + x^2\frac{dy}{dx}$$

$$\frac{y}{2\sqrt{xy}} + \frac{x}{2\sqrt{xy}}\frac{dy}{dx} - x^2\frac{dy}{dx} = 2xy$$

$$\frac{dy}{dx}\left(\frac{x}{2\sqrt{xy}} - x^2\right) = 2xy - \frac{y}{2\sqrt{xy}}$$

$$\boxed{\frac{dy}{dx} = \left(2xy - \frac{y}{2\sqrt{xy}}\right) \div \left(\frac{x}{2\sqrt{xy}} - x^2\right)}$$

9. $x^3 - 3x^2y + 2xy^2 = 12$

$$\therefore \frac{d}{dx}[x^3] - \frac{d}{dx}[3x^2y] + \frac{d}{dx}[2xy^2] = \frac{d}{dx}[12]$$

$$\frac{d(x)}{dx}[3x^2] - \left[y \times \frac{d}{dx}(3x^2) + (3x^2) \times \frac{d}{dx}(y)\right] + \frac{d}{dx}[2xy^2] = 0$$

$$3x^2 - \left[y \times \frac{d(x)}{dx}(6x) + 3x^2 \times \frac{d(y)}{dx}\right] + \left[(y^2) \times \frac{d}{dx}(2x) + (2x) \times \frac{d}{dx}(y^2)\right] = 0$$

$$3x^2 - \left[6xy + 3x^2\frac{dy}{dx}\right] + \left[2xy^2 + \overset{2\times1=2}{2x} \times \frac{d(y)}{dx}(2y)\right] = 0$$

$$3x^2 - 6xy - 3x^2\frac{dy}{dx} + 2xy^2 + 4xy\frac{d(y)}{dx} = 0$$

$$4xy\frac{dy}{dx} - 3x^2\frac{dy}{dx} = 6xy - 2xy^2 - 3x^2$$

$$\therefore \frac{dy}{dx}(4xy - 3x^2) = 6xy - 2xy^2 - 3x^2$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{6xy - 2xy^2 - 3x^2}{4xy - 3x^2}} = \frac{x(6y - 2y^2 - 3x)}{x(4y - 3x)} = \frac{6y - 2y^2 - 3x}{4y - 3x}$$

10. $4\cos x \sin y = 1$

$$\therefore \frac{d}{dx}[(4\cos x)(\sin y)] = \frac{d}{dx}(1) \rightarrow \text{product rule}$$

$$\left[(\sin y) \times \frac{d}{dx}(4\cos x) + 4\cos x \times \frac{d}{dx}(\sin y)\right] = 0$$

$$(\sin y)(4)(-\sin x) + 4\cos x \times \frac{dy}{dx}(\cos y) = 0$$

$$\boxed{\begin{array}{l} +4\sin y \sin x \\ 4\cos y \cos x \end{array} \frac{dy}{dx}}$$

10. $\sin x + 2\cos 2y = 1$

1: $\frac{d}{dx}[\sin x] + 2\frac{d}{dx}[\cos 2y] = \frac{d}{dx}[1]$

2: $\frac{d(x)}{dx}[\cos x] + 2\frac{d(y)}{dx}[-\sin 2y] = 0$

$\cos x - 2\sin 2y \frac{dy}{dx} = 0$

3: $\boxed{\frac{dy}{dx} = \frac{\cos x}{2\sin 2y}}$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

1: $(\sin \pi x)^2 + 2(\sin \pi x)(\cos \pi y) + (\cos \pi y)^2 = 2$

$\frac{d}{dx}(\sin \pi x)^2 + 2\frac{d}{dx}(\sin \pi x)(\cos \pi y) + \frac{d}{dx}(\cos \pi y)^2 = \frac{d}{dx}(2)$

Chain Rule + Product Rule + Chain Rule = Power Rule

$2(\sin \pi x)(\cos \pi x) + 2\frac{d}{dx}(\sin \pi x)(\cos \pi y) + \frac{dy}{dx}(2\cos \pi y)(-\sin \pi y) = 0$

$2(\sin \pi x)(\cos \pi x) + 2\left[\cos \pi y \times \frac{d}{dx}(\sin \pi x) + \sin \pi x \times \frac{d}{dx}(\cos \pi y)\right] + \frac{dy}{dx}(-2\cos \pi y \sin \pi y)$

$2(\sin \pi x)(\cos \pi x) + 2\left[\cos \pi y \times \frac{d(x)}{dx}(\cos \pi x) + \sin \pi x \times \frac{d(y)}{dx}(-\sin \pi y)\right] + \frac{dy}{dx}(-2\cos \pi y \sin \pi y) = 0$

$2\sin \pi x \cos \pi x + 2\left[\cos \pi x \cos \pi y + \sin \pi x \sin \pi y \frac{dy}{dx}\right] + \frac{dy}{dx}(-2\cos \pi y \sin \pi y) = 0$

$2\sin \pi x \cos \pi x + 2\cos \pi x \cos \pi y - 2\sin \pi x \sin \pi y \frac{dy}{dx} + \frac{dy}{dx}(-(2\cos \pi y \sin \pi y)) = 0$

$2\sin \pi x \cos \pi x + 2\cos \pi x \cos \pi y = 2\sin \pi x \sin \pi y \frac{dy}{dx} + 2\cos \pi y \sin \pi y \frac{dy}{dx}$

$\frac{2\sin \pi x \cos \pi x + 2\cos \pi x \cos \pi y}{2\sin \pi x \sin \pi y + 2\cos \pi y \sin \pi y} = \frac{dy}{dx}$

$\boxed{\frac{dy}{dx} = \frac{\sin \pi x \cos \pi x + \cos \pi x \cos \pi y}{\sin \pi x \sin \pi y + \cos \pi y \sin \pi y}}$

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13: $\sin x = x(1 + \tan y)$

1: $\frac{d}{dx}(\sin x) = \frac{d}{dx}[x(1+\tan y)]$

$\cos x = (1+\tan y) \times \frac{d}{dx}(x) + (x) \times \frac{d}{dx}(1+\tan y)$

$\cos x = (1+\tan y) \times \frac{d(x)}{dx}(1) + x \times \left(0 + \frac{d(y)}{dx}(\sec^2 y)\right)$

2: $\cos x = 1 + \tan y + (x)(\sec^2 y)\frac{dy}{dx}$

3: $\boxed{\frac{\cos x - 1 - \tan y}{x\sec^2 y} = \frac{dy}{dx}}$

14: $\cot y = x - y$

1: $\frac{d}{dx}[\cot y] = \frac{d}{dx}[x] - \frac{d}{dx}[y]$

$\frac{d(y)}{dx}[-\csc^2 y] = \frac{d(x)}{dx} \times (1) - \frac{d(y)}{dx} \times [1]$

$\frac{dy}{dx}(-\csc^2 y) = 1 - \frac{dy}{dx}$

$-\csc^2 y \frac{dy}{dx} + \frac{dy}{dx} = 1 \rightarrow \frac{dy}{dx}(-\csc^2 y + 1) = 1 \rightarrow \boxed{\frac{dy}{dx} = \frac{1}{-\csc^2 y + 1}}$

15: $y = \sin xy$

1: $\frac{d}{dx}[y] = \frac{d}{dx}[\sin xy]$

$\frac{d(y)}{dx} =$

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17 – 20 (a) find two functions by solving equation for y in terms of x.
(b) Differentiate the explicit functions
(c) find dy/dx and show that result is equivalent to that of part (b).

17. $x^2 + y^2 = 64$

a: $x^2 + y^2 = 64$

$y^2 = 64 - x^2$

$y = \pm\sqrt{64 - x^2}$

$y_1 = \sqrt{64 - x^2}$

$y_2 = -\sqrt{64 - x^2}$

b: $y_1' = \frac{1}{2}(64 - x^2)^{-1/2} \times \frac{d}{dx}(64 - x^2)$

$y_1' = \frac{1}{2\sqrt{64 - x^2}} \times (0 - 2x)$

$y_1' = \frac{-x}{\sqrt{64 - x^2}}$, $y = \sqrt{64 - x^2}$

$\therefore \boxed{y' = \frac{-x}{y}}$

$y_2' = -\sqrt{64 - x^2}$

$= -(64 - x^2)^{1/2}$

$= -\frac{1}{2}(64 - x^2)^{-1/2} \times \frac{d}{dx}(64 - x^2)$

$= \frac{-1}{2\sqrt{64 - x^2}} \times (0 - 2x)$

$= \frac{2x}{2\sqrt{64 - x^2}} = \boxed{\frac{x}{\sqrt{64 - x^2}}}$

where $y_2 = -\sqrt{64 - x^2} = y$

$\therefore y_2' = \frac{x}{\sqrt{64 - x^2}} = -\frac{x}{y}$

$\boxed{y' = \frac{-x}{y}}$

c: $x^2 + y^2 = 64$

find dy/dx, and show

1: $\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[64]$

$\frac{d(x)}{dx}[2x] + \frac{d(y)}{dx}[2y] = 0$

$2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{2y} = \boxed{\frac{-x}{y}}$

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15. $x^2+y^2-4x+6y+8=0$

a: $y^2+6y = 4x-x^2-8$

$\boxed{y(y+6) = 4x-x^2-8}$

b: Differentiate

$\frac{d}{dx}[y(y+6)] = \frac{d}{dx}[4x] - \frac{d}{dx}[x^2] - \frac{d}{dx}[8]$

$\frac{d}{dx}[y^2+6y] = \frac{d(x)}{dx}[4] - \frac{d(x)}{dx}[2x] - 0$

$\frac{d}{dx}[y^2] + \frac{d}{dx}[6y] = 4-2x$

$\frac{d(y)}{dx}[2y] + \frac{d(y)}{dx}[6] = 4-2x$

$2y\frac{dy}{dx} + 6\frac{dy}{dx} = 4-2x$

$\frac{dy}{dx}(2y+6) = 4-2x$

$\frac{dy}{dx} = \frac{4-2x}{2y+6} = \boxed{\frac{2-x}{y+3}}$

18. $16x^2+25y^2 = 400$

a: $25y^2 = 400-16x^2$

$y^2 = \frac{400-16x^2}{25}$

$y = \pm\sqrt{\frac{400-16x^2}{5}}$

$y_1 = \frac{\sqrt{400-16x^2}}{5}$

$y_2 = -\frac{\sqrt{400-16x^2}}{5}$

$y_1 = \sqrt{16(25-x^2)} \div 5$

$\boxed{y_1 = \frac{4\sqrt{25-x^2}}{5}}$

$\boxed{y_2 = \frac{-4}{5}\sqrt{25-x^2}}$

b: Differentiate explicit function

$y_1' = \frac{(400-16x^2)^{1/2}}{5}$

$y_1' = \frac{(400-16x^2)^{-1/2}}{10} \times (0-32x) = \frac{-32x}{10\sqrt{400-16x^2}}$

$y_1' = \frac{-32x}{10\sqrt{16(25-x^2)}} = \frac{-32x}{40\sqrt{25-x^2}} = \frac{-4x}{5\sqrt{25-x^2}}$

multiply 4/5 with Num + Denom

$\left(-4x \times \frac{4}{5}\right) \div \left(5\sqrt{25-x^2} \times \frac{4}{5}\right)$

$\left(\frac{-16x}{5}\right) \div \left(4\sqrt{25-x^2}\right)$

Divide 5 with Num and Denom

$\left(\frac{-16x}{5} \div 5\right) \div \left(\frac{4\sqrt{25-x^2}}{5}\right)$

$\left(\frac{-16x}{25}\right) \div \boxed{y' = \frac{-16x}{25y}}$

$\boxed{\because y = \pm\frac{4\sqrt{25-x^2}}{5}}$

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C: $16x^2 + 25y^2 = 400$

$$\frac{d}{dx}[16x^2] + \frac{d}{dx}[25y^2] = \frac{d}{dx}[400]$$

$$\frac{d(x)}{dx}[32x] + \frac{d(y)}{dx}[50y] = 0$$

$$32x + 50y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-32x}{50y} = \boxed{\frac{-16x}{25y}}$$

20. $16y^2 - x^2 = 16$

a: $16y^2 = 16 + x^2$

$y^2 = \frac{16+x^2}{16}$

$\boxed{y = \pm\frac{\sqrt{16+x^2}}{4}}$

b: Differentiate explicit function(s)

① $y = \frac{\sqrt{16+x^2}}{4}$

$y' = \frac{(16+x^2)^{-1/2}}{8} \times (2x) = \frac{x}{4\sqrt{16+x^2}}$

Divide 16 with both Denominator and Numerator

$$\left(\frac{x}{16}\right) \div \left(\frac{\cancel{4}\sqrt{16+x^2}}{\cancel{16}}\right) = \left(\frac{x}{16}\right) \div \left(\frac{\sqrt{16+x^2}}{4}\right)$$

$$\frac{x}{16} \div y = \boxed{\frac{x}{16y} = y'} \quad \because y = \pm\frac{\sqrt{16+x^2}}{4}$$

② $y = -\frac{\sqrt{16+x^2}}{4}$

$y' = -\frac{(16+x^2)^{-1/2}}{8} \times (0+2x)$

$y' = \frac{-2x}{8\sqrt{16+x^2}} = \frac{-x}{4\sqrt{16+x^2}}$

$y' = \frac{-x}{4\sqrt{16+x^2}}$

Divide Num and denom with 16

$$\left(\frac{-x}{16}\right) \div \left(\frac{4\sqrt{16+x^2}}{16}\right)$$

$$\left(\frac{-x}{16}\right) \div \left(\frac{\sqrt{16+x^2}}{16}\right) = \frac{-x}{16} \div y = \frac{-x}{-16y} = \frac{x}{16y}$$

$$\left\{\because y = -\frac{\sqrt{16+x^2}}{4}\right.$$

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c: Differentiate the original function Implicitly

$16y^2 - x^2 = 16$

1: $\frac{d}{dx}[16y^2] - \frac{d}{dx}[x^2] = \frac{d}{dx}[16]$

$\frac{d(y)}{dx}[32y] - \frac{d(x)}{dx}[2x] = 0$

$32y\frac{dy}{dx} - 2x = 0 \longrightarrow \frac{dy}{dx} = \frac{2x}{32y} = \boxed{\frac{x}{16y}}$

▸ 21 - 28 find dy/dx by implicit differentiation and evalute the derivative at given point.

21. $xy = 6, \ (-6, -1)$

:- first of all product rule

$U \cdot V = VU' + UV'$

$\frac{d}{dx}[xy] = \frac{d}{dx}[6]$

$[y \times \frac{d}{dx}[x] + (x) \times \frac{d}{dx}[y]] = 0$

$y \times \frac{d(x)}{dx}(1) + x \times \frac{d(y)}{dx}[1] = 0$

$y + x\frac{d(y)}{dx} = 0$

$\boxed{\frac{dy}{dx} = \frac{-y}{x}}$

$\frac{dy}{dx} = \frac{-y}{x}, \ (-6, -1)$

$x = -6, \ y = -1$

$\frac{dy}{dx} = \frac{-(-1)}{(-6)} = \boxed{\frac{1}{-6}}$

$\boxed{m = -1/6 \text{ at } (-6, -1)}$

22. $x^2 - y^3 = 0, \ (1, 1)$

1: $\frac{d}{dx}[x^2] - \frac{d}{dx}[y^3] = 0$

$\frac{d(x)}{dx}[2x] - \frac{d(y)}{dx}[3y^2] = 0$

2: $2x - 3y^2\frac{dy}{dx} = 0$

3: $\boxed{\frac{dy}{dx} = \frac{2x}{3y^2}}$

4: $\frac{dy}{dx} = \frac{2x}{3y^2}$, point $(1, 1)$

$x = 1, \ y = 1$

$\frac{dy}{dx} = \frac{2(1)}{3(1)^2} = \boxed{\frac{2}{3}}$

$\boxed{m = 2/3 \text{ at } (1, 1)}$

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Q3. $y^2 = \frac{x^2-49}{x^2+49}$, $(7,0)$

1: $\frac{d}{dx}[y^2] = \frac{d}{dx}\left[\frac{x^2-49}{x^2+49}\right]$ → Quotient Rule $\frac{U}{V} = \frac{VU'-UV'}{V^2}$, $V \neq 0$

$2y\frac{dy}{dx} = \left[\frac{(x^2+49)(2x) - (x^2-49)(2x)}{(x^2+49)^2}\right]$

$2y\frac{dy}{dx} = \left[\frac{(2x^3+98x)-(2x^3-98x)}{(x^2+49)^2}\right]$

$2y\frac{dy}{dx} = \left[\frac{\cancel{2x^3}+98x-\cancel{2x^3}+98x}{(x^2+49)^2}\right]$

2: $2y\frac{dy}{dx} = \left[\frac{196x}{(x^2+49)^2}\right]$

3: $\frac{dy}{dx} = \frac{\cancel{196}x}{\cancel{2}y(x^2+49)^2} = \boxed{\frac{98x}{y(x^2+49)^2}} \rightarrow \frac{dy}{dx}$

4: $\frac{dy}{dx} = \frac{98x}{y(x^2+49)^2}$, $x=7$, $y=0$

$\frac{dy}{dx} = \frac{98(7)}{0(x^2+49)^2} = \frac{98\times7}{0} \rightarrow$ $\boxed{\text{Undefined}}$

Q4. $(x+y)^3 = x^3+y^3$, $(-1,1)$

: $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

1: $x^3+3x^2y+3xy^2+y^3 = x^3+y^3$

$\frac{d}{dx}[x^3]+\frac{d}{dx}[3x^2y]+\frac{d}{dx}[3xy^2]+\frac{d}{dx}[y^3] = \frac{d}{dx}[x^3]+\frac{d}{dx}[y^3]$

$3x^2+3\left[y\times\frac{d}{dx}[x^2]+(x^2)\times\frac{d}{dx}(y)\right]+3\left[y^2\times\frac{d}{dx}(x)+(x)\times\frac{d}{dx}(y^2)\right]+\frac{dy}{dx}(3y^2) =$

$3x^2+3\left[(y)(2x)+(x^2)\times\frac{dy}{dx}\times(1)\right]+3\left[y^2+x\times\frac{dy}{dx}(2y)\right]+3y^2\frac{dy}{dx} =$

$3x^2+3\left[2xy+x^2\frac{dy}{dx}\right]+3\left[y^2+2xy\frac{dy}{dx}\right]+3y^2\frac{dy}{dx} = 3x^2+\frac{dy}{dx}\times(3y^2)$

$3x^2+6xy+3x^2\frac{dy}{dx}+3y^2+6xy\frac{dy}{dx}+3y^2\frac{dy}{dx} = 3x^2+3y^2\frac{dy}{dx}$

$2\frac{dy}{dx}\left[3x^2+6xy+\cancel{3y^2}+\cancel{3y^2}\right] = \cancel{3x^2}-\cancel{3x^2}-6xy-6xy-3y^2$

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$$\frac{dy}{dx}\left[3x^2+6xy\right] = -12xy - 3y^2$$

$$\therefore \frac{dy}{dx} = \frac{-12xy - 3y^2}{3x^2+6xy} = \boxed{\frac{-4xy - y^2}{x^2+2xy}}$$

$$\frac{dy}{dx} = \frac{-4xy - y^2}{x^2+2xy}, \quad (-1, 1), \quad x = -1,\ y = 1$$

$$\frac{dy}{dx} = \frac{-4(-1)(1) - (1)^2}{(-1)^2 + 2(-1)(1)} = \frac{4-1}{1-2} = \frac{3}{-1} = \boxed{-3} \to m \text{ at } (-1, 1)$$

Q5: $x^{2/3} + y^{2/3} = 5, \quad (8, 1)$

$$\therefore \frac{d}{dx}[x]^{2/3} + \frac{d}{dx}[y]^{2/3} = 5$$

Chain Rule

Bak' Der

Baqi is Deriv

$$\boxed{\frac{2}{3}[x]^{-1/3} + \frac{2}{3}\frac{d(y)}{dx}(y')} \quad \frac{2}{3}\frac{d(x)}{dx}[x]^{-1/3} \times (1) + \frac{2}{3}\frac{d(y)}{dx}[y]^{-1/3} \times (1) = \frac{d}{dx}[5]$$

$$\frac{2}{3}[x]^{-1/3} + \frac{2}{3}\frac{dy}{dx}[y]^{-1/3} = 0$$

$$\frac{2}{3x^{1/3}} + \frac{2}{3y^{1/3}}\frac{dy}{dx} = 0$$

$$\frac{2}{3\sqrt[3]{x}} + \frac{2}{3\sqrt[3]{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\left(\frac{2}{3\sqrt[3]{x}}\right) \div \left(\frac{2}{3\sqrt[3]{y}}\right)$$

$$\frac{dy}{dx} = -\left(\frac{2}{3\sqrt[3]{x}}\right) \div \left(\frac{2}{3\sqrt[3]{y}}\right)$$

$$x = 8, \quad y = 1$$

$$\frac{dy}{dx} = -\left(\frac{2}{3\sqrt[3]{8}}\right) \div \left(\frac{2}{3\sqrt[3]{1}}\right)$$

$$\frac{dy}{dx} = -\left(\frac{2}{3(2^3)^{1/3}}\right) \div \left(\frac{2}{3(1)}\right)$$

$$\frac{dy}{dx} = -\left(\frac{2}{6}\right) \times \frac{3}{2} = \boxed{-\frac{1}{2}} \to \boxed{m = -1/2} \text{ at } (8, 1)$$

$$\frac{dy}{dx} = -\left(\frac{2}{3\sqrt[3]{x}}\right) \div \left(\frac{2}{3\sqrt[3]{y}}\right) = \frac{-\cancel{2}}{3\sqrt[3]{x}} \times \frac{3\sqrt[3]{y}}{\cancel{2}} = \frac{-3\sqrt[3]{y}}{3\sqrt[3]{x}}$$

$$\frac{dy}{dx} = \frac{-\cancel{3}\sqrt[3]{y}}{\cancel{3}\sqrt[3]{x}} = \frac{-\sqrt[3]{y}}{\sqrt[3]{x}} = \boxed{-\sqrt[3]{\frac{y}{x}}}$$

$$\boxed{\frac{dy}{dx} = -\sqrt[3]{\frac{y}{x}}}$$

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26. $x^3+y^3 = 6xy - 1$, $(2,3)$

1: $\frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] = 6\frac{d}{dx}[xy] - \frac{d}{dx}[1]$

$\frac{d(x)}{dx}[3x^2] + \frac{d(y)}{dx}[3y^2] = 6\frac{d}{dx}[xy] - 0$

$3x^2 + \frac{dy}{dx}[3y^2] = 6\left[y \times \frac{d}{dx}[x] + [x] \times \frac{d}{dx}[y]\right]$

$3x^2 + 3y^2\frac{dy}{dx} = 6\left[xy + xy\frac{d(y)}{dx}\right]$

2: $x^2+y^2 = 2xy + 2xy\frac{d(y)}{dx}$

3: $\boxed{\frac{dy}{dx} = \frac{x^2+y^2-2xy}{2xy}}$ ▸ $x=2$, $y=3$

4: $\frac{dy}{dx} = \frac{(2)^2+(3)^2-2(2)(3)}{2(2)(3)} = \frac{4+9-12}{12} = \boxed{\frac{1}{12}}$ at $(2,3)$ (m)

27. $\tan(x+y) = x$, $(0,0)$

1: $\tan x + \tan y = x$

$\frac{d}{dx}[\tan x] + \frac{d}{dx}[\tan y] = \frac{d}{dx}[x]$

$\frac{d(x)}{dx}[\sec^2 x] + \frac{d(y)}{dx}[\sec^2 y] = 1$

$\sec^2 x + \frac{dy}{dx}[\sec^2 y] = 1$

$\frac{dy}{dx} = \frac{1-\sec^2 x}{\sec^2 y} = \frac{(1-\sec x)(1+\sec x)}{(1+\tan^2 y)} = \frac{(1-\sec x)(1+\sec x)}{1+\tan^2 y}$

$\because 1+\tan^2\theta = \sec^2\theta$

3: $\frac{dy}{dx} = \frac{(1-\sec x)(1+\sec x)}{1+\tan^2 y}$, $x=0$, $y=0$

4: $\frac{dy}{dx} = m = \frac{(1-\sec(0))(1+\sec(0))}{1+\tan^2(0)} = \frac{(1-(1/\cos(0)))(1+(1/\cos(0)))}{1}$

$(1-(1/1))(1+(1/1)) = (1-1)(1+1) = (0)(2) = 0$

$\boxed{m=0}$ at $(0,0)$

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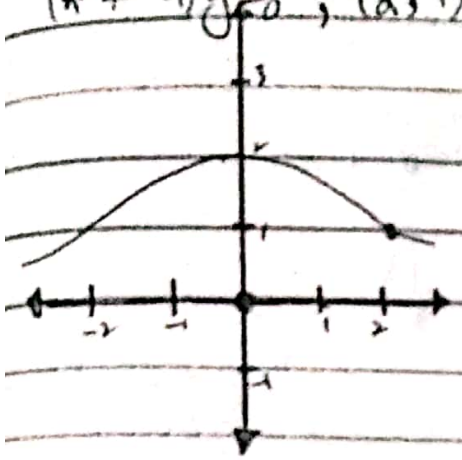
▸ Famous Curves 28-32 find slope of tangent line to the graph at the given point.

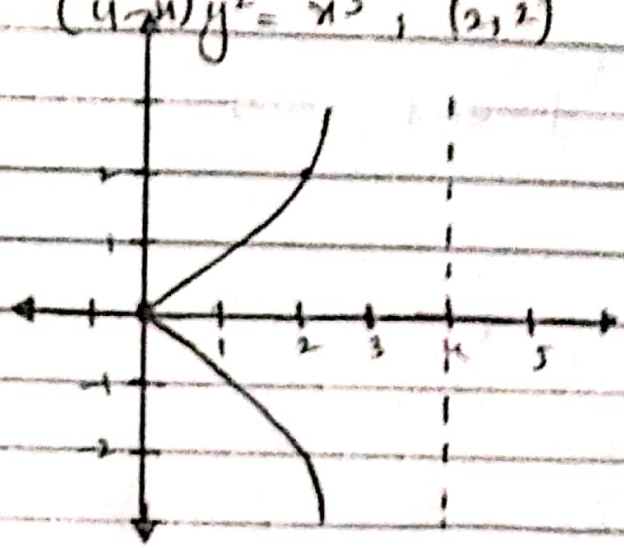
28. Witch of Agnesi:

$(x^2+4)y=8$, $(2,1)$



30. Cissoid:

$(4-x)y^2=x^3$, $(2,2)$



:- $(x^2+4)y=8$

$\frac{d}{dx}[(x^2+4)y]=\frac{d}{dx}[8]$

$[y\times\frac{d}{dx}[x^2+4]+[x^2+4]\times\frac{d}{dx}[y]=0$

$[y\times[2x+0]+[x^2+4]\times\frac{dy}{dx}(1)]=0$

$[2xy+[x^2+4]\frac{dy}{dx}]=0$

$\boxed{\frac{dy}{dx}=\frac{-2xy}{x^2+4}}$

Put $x=2$, $y=1$

$\frac{dy}{dx}=\frac{-2(2)(1)}{(2)^2+4}=\frac{-4}{8}=-\frac{1}{2}$

$\boxed{m=-1/2}$ at $(2,1)$

: $(4-x)y^2=x^3$

: $\frac{d}{dx}[(4-x)y^2]=\frac{d}{dx}[x^3]$

: $[y^2\times\frac{d}{dx}[4-x]+[4-x]\times\frac{d}{dx}[y^2]=3x^2$

$[y^2\times[0-1]+[4-x]\times\frac{dy}{dx}[2y]=3x^2$

$[-y^2+[8y-2xy]\frac{dy}{dx}]=3x^2$

$\boxed{\frac{dy}{dx}=\frac{3x^2+y^2}{8y-2xy}}$ put $x=2$, $y=2$

$\frac{dy}{dx}=\frac{3(2)^2+(2)^2}{8(2)+2(2)(2)}=\frac{12+4}{16+8}=\frac{16}{8}=2$

$\boxed{m=2}$ at $(2,2)$

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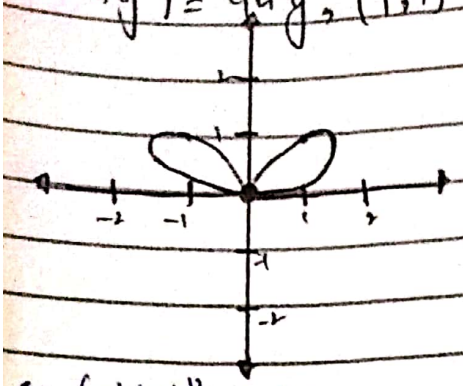
31. Bifolium

$(x^2+y^2)^2 = 4x^2y$, $(1,1)$



1: $(x^2+y^2)^2 = 4x^2y$

$x^4 + 2x^2y^2 + y^4 = 4x^2y$

$\frac{d}{dx}[x^4] + 2\frac{d}{dx}[x^2y^2] + \frac{d}{dx}[y^4] = \frac{d}{dx}[4x^2y]$

$4x^3 + 2\frac{d}{dx}[x^2y^2] + \frac{dy}{dx}[4y^3] = \frac{d}{dx}[4x^2y]$

$4x^3 + 2\left[y^2 \times \frac{d}{dx}[x^2] + [x^2] \times \frac{d}{dx}[y^2]\right] + \frac{dy}{dx}[4y^3] = \dots$

$4x^3 + 2\left[y^2 \times (2x) + (x^2) \times 2y\frac{dy}{dx}\right] + 4y^3\frac{dy}{dx} \dots$

$$\boxed{4x^3 + 4xy^2 + 4x^2y\frac{dy}{dx} + 4y^3\frac{dy}{dx} = \frac{d}{dx}[4x^2y]}$$

$\dots = 4\left[y \times \frac{d}{dx}[x^2] + [x^2] \times \frac{d}{dx}[y]\right]$

$\dots = 4\left[2xy + x^2\frac{dy}{dx}\right]$

$\dots = 8xy + 4x^2\frac{dy}{dx}$

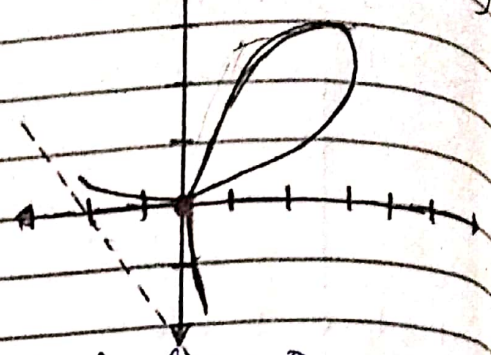
$4x^3 + 4xy^2 + \frac{dy}{dx}[4x^2y + 4y^3 - 4x^2] = 8xy$

$$\boxed{\frac{dy}{dx} = \frac{8xy - 4x^3 - 4xy^2}{4x^2y + 4y^3 - 4x^2}}$$

$\frac{dy}{dx} = \frac{4xy - x^3 - xy^2}{x^2y + y^3 - x^2}$

32. Folium of Descartes

$x^3 + y^3 - 6xy = 0$, $(4/3, 8/3)$



1: $x^3 + y^3 - 6xy = 0$

$\frac{d}{dx}[x^3] + \frac{d}{dx}[y^3] - 6\frac{d}{dx}[xy] = \frac{d}{dx}[0]$

$3x^2 + \frac{dy}{dx}[3y^2] - 6\left[y(1) + x \times \frac{dy}{dx}\right] = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 6\left[y + x\frac{dy}{dx}\right] = 0$

$3x^2 + 3y^2\frac{dy}{dx} - 6y - 6x\frac{dy}{dx} = 0$

$x^2 + y^2\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} = 0$

$\frac{dy}{dx}(y^2 - 2x) = 2y - x^2$

$$\boxed{\frac{dy}{dx} = \frac{2y - x^2}{y^2 - 2x}}$$

put $x = 4/3$, $y = 8/3$

$\frac{dy}{dx} = \left[2\left(\frac{8}{3}\right) - \left(\frac{4}{3}\right)^2\right] \div \left[\left(\frac{8}{3}\right)^2 - 2\left(\frac{4}{3}\right)\right]$

$= \left[\frac{16}{3} - \frac{16}{9}\right] \div \left[\frac{64}{9} - \frac{8}{3}\right]$

$= \left[\frac{48 - 16}{9}\right] \div \left[\frac{64 - 24}{9}\right]$

$= \frac{32}{9} \times \frac{9}{40} = \frac{32}{40} = \boxed{\frac{4}{5}}$

$\boxed{m = 4/5}$ at $(4/3, 8/3)$

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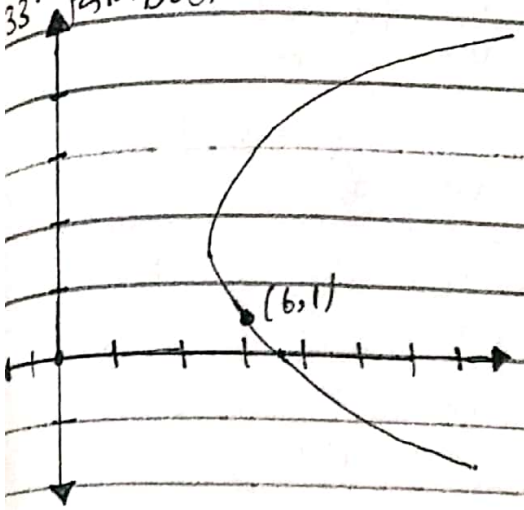
Famous Curves 33 – 40 find an equation of the tangent line to the graph at the given point.

33. Parabola



$(y-3)^2 = 4(x-5)$

$y^2 - 6y + 9 = 4x - 20$

$y^2 - 6y - 4x + 29 = 0$

$\frac{d}{dx}[y^2] - 6\frac{d}{dx}[6y] - 4\frac{d}{dx}[x] + \frac{d}{dx}[29] = 0$

$\frac{dy}{dx}[2y] - 6\frac{dy}{dx}[1] - 4\cancel{\frac{dx}{dx}}[1] + 0 = 0$

$\frac{dy}{dx}[2y-6] = 4$

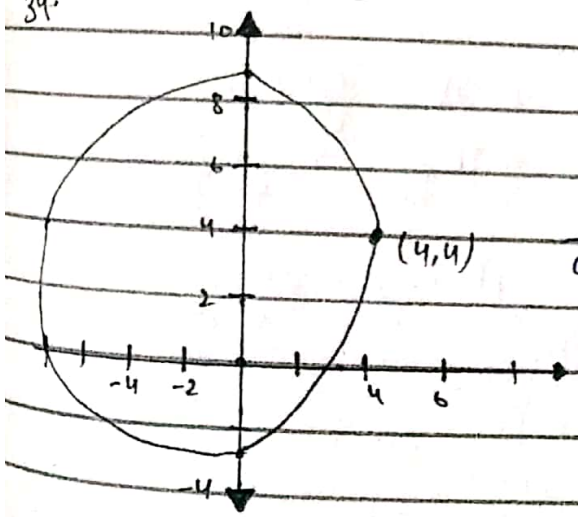
$\frac{dy}{dx} = \frac{4}{2y-6} = \boxed{\frac{2}{y-3}}$

$\frac{dy}{dx} = \frac{2}{y-3}$, $y = 1$

$\frac{dy}{dx} = \frac{2}{1-3} = \frac{2}{-2} = \boxed{-1}$, $m = -1$, Point $(6,1)$

$y - 1 = -1(x-6) \Rightarrow \boxed{y + x = 7}$

34.



$(x+2)^2 + (y-3)^2 = 37$

$x^2 + 4x + 4 + y^2 - 6y + 9 = 37$

$\frac{d}{dx}[x^2] + \frac{d}{dx}[4x] + \frac{d}{dx}[y^2] - 6\frac{d}{dx}[6y] = \frac{d}{dx}[24]$

$2x + 4 + \frac{dy}{dx}[2y] - 6\frac{dy}{dx}[1] = 0$

$\frac{dy}{dx}[2y-6] = -2x - 4$

$\frac{dy}{dx} = \frac{-2x-4}{2y-6} = \boxed{\frac{-x-4}{y-3}}$

$\frac{dy}{dx} = \frac{-x-4}{y-3}$, $x = 4$, $y = 4$

$\frac{dy}{dx} = \frac{-(4)-4}{4-3} = \frac{-8}{1} = \boxed{-8}$

$m = -8$, $P(4,4)$

$y - 4 = -8(x-4)$

$\boxed{y + 8x = 28}$

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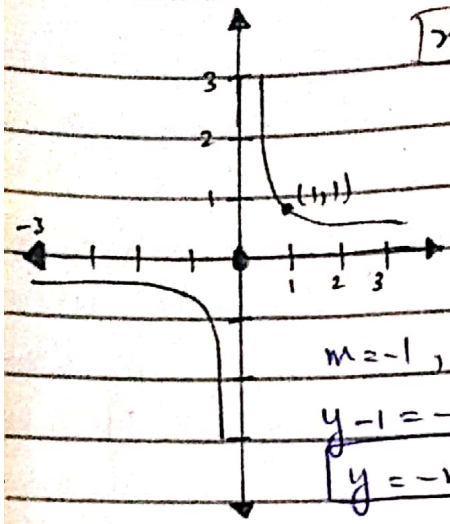
35. Rotated parabola



$\boxed{xy = 1}$

$\therefore\ y = \frac{1}{x} \rightarrow \frac{d}{dx}[y] = \frac{d}{dx}[x^{-1}]$

$\frac{dy}{dx} = \frac{dx}{dx} - 1[x^{-2}]$

$\boxed{\frac{dy}{dx} = \frac{-1}{x^2}}$, $x = 1$

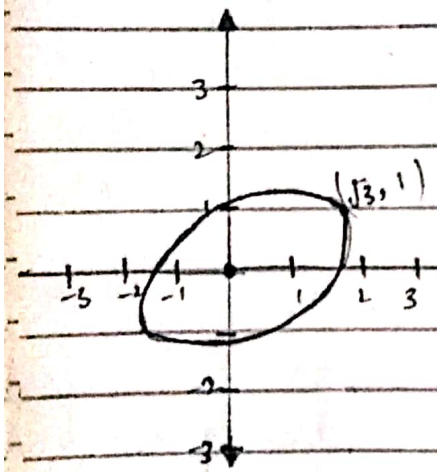
$\frac{dy}{dx} = \frac{-1}{(1)^2} = \frac{-1}{1} = \boxed{-1}$

$m = -1$, $(1,1)$

$y - 1 = -1(x - 1)$

$\boxed{y = -x + 2}$

36. Rotated ellipse



$7x^2 - 6\sqrt{3}\,xy + 13y^2 - 16 = 0$

$\therefore\ 7\frac{d}{dx}[x^2] - 6\sqrt{3}\frac{d}{dx}[xy] + 13\frac{d}{dx}[y^2] = \frac{d}{dx}[16]$

$7\frac{dx}{dx}[2x] - 6\sqrt{3}\frac{d}{dx}[xy] + 13\frac{dy}{dx}[2y] = 0$

$14x - 6\sqrt{3}\left[y \times (1) + (x) \times \frac{d}{dx}(y)\right] + 26y\frac{dy}{dx} = 0$

$14x - 6\sqrt{3}\left[y + x\frac{dy}{dx}\right] + 26y\frac{dy}{dx} = 0$

$14x - 6\sqrt{3}y - 6\sqrt{3}\,x\frac{dy}{dx} + 26y\frac{dy}{dx} = 0$

$14x - 6\sqrt{3}y - \frac{dy}{dx}\left[6\sqrt{3}x - 26y\right] = 0$

$\frac{14x - 6\sqrt{3}y}{6\sqrt{3}x - 26y} = \frac{dy}{dx}$, $x = \sqrt{3}$, $y = 1$

$\frac{dy}{dx} = \frac{14(\sqrt{3}) - 6\sqrt{3}(1)}{6\sqrt{3}(\sqrt{3}) - 26(1)} = \frac{14\sqrt{3} - 6\sqrt{3}}{18 - 26} = \frac{8\sqrt{3}}{-8} = \boxed{-\sqrt{3}}$

$m = -\sqrt{3}$ $P(\sqrt{3}, 1)$

$y - 1 = -\sqrt{3}(x - \sqrt{3})$

$y - 1 = -\sqrt{3}x + 3$

$\boxed{y + \sqrt{3}x = 4}$

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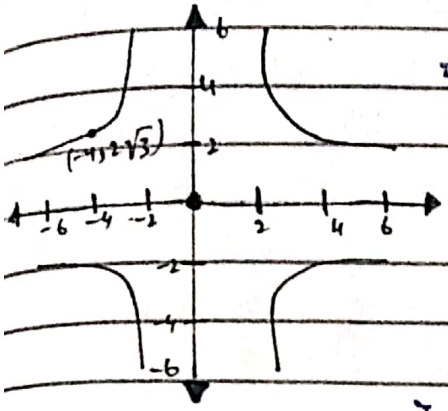
37. Cruciform



$$x^2y^2 - 9x^2 - 4y^2 = 0$$

$$\frac{d}{dx}[x^2y^2] - 9\frac{d}{dx}[x^2] - 4\frac{d}{dx}[y^2] = 0$$

$$d/dx[xy]^2 - 9\frac{dx}{dx}[2x] - 4\frac{dy}{dx}[2y] = 0$$

$$2(xy)\times\frac{d}{dx}[xy] - 18x - 8y\frac{dy}{dx} = 0$$

$$2xy\times[y\times(1) + x\times\frac{d}{dx}[y]] - 18x - 8y\frac{dy}{dx} = 0$$

$$2xy[y + x\frac{dy}{dx}] - 18x - 8y\frac{dy}{dx} = 0$$

$$2xy^2 + 2x^2y\frac{dy}{dx} - 18x - 8y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[2x^2y - 8y] = 18x - 2xy^2$$

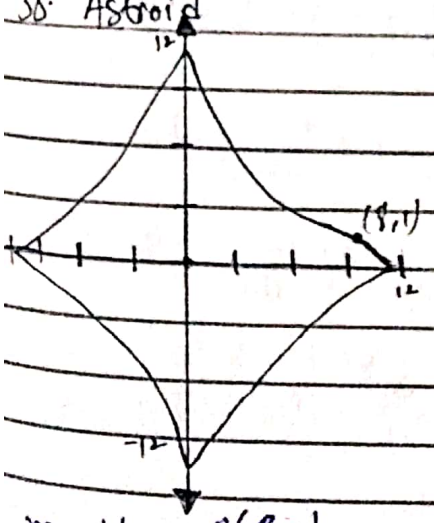
$$\boxed{\frac{dy}{dx} = \frac{-18x + 2xy^2}{-2x^2y + 8y}} \qquad x = -4,\ y = 2\sqrt{3}$$

$$\frac{dy}{dx} = \frac{-18(-4) + 2(-4)(2\sqrt{3})^2}{-2(-4)^2(2\sqrt{3}) + 8(2\sqrt{3})} = \frac{+72 + 8(12)}{-32(2\sqrt{3}) + 16\sqrt{3} - 64\sqrt{3} + 16\sqrt{3}} = \frac{-\cancel{24}^{1}}{-\cancel{48}_{2}\sqrt{3}}$$

$$\boxed{m = \frac{1}{2\sqrt{3}}}, P(-4, 2\sqrt{3})$$

$$y - 2\sqrt{3} = \frac{1}{2\sqrt{3}}(x+4) = \boxed{y = \frac{(x+4)}{2\sqrt{3}} + 2\sqrt{3}}$$

38. Astroid



$$x^{2/3} + y^{2/3} = 5$$

$$\frac{d}{dx}[x^{2/3}] + \frac{d}{dx}[y^{2/3}] = 5$$

$$\frac{2}{3}[x^{-1/3}]\times\frac{d}{dx}[x] + \frac{2}{3}[y^{-1/3}]\times\frac{dy}{dx}[y] = \frac{d}{dx}[5]$$

$$\frac{2}{3\sqrt[3]{x}}\times\frac{d(x)}{dx} + \frac{2}{3\sqrt[3]{y}}\times\frac{d(y)}{dx}(1) = 0$$

$$\frac{2}{3\sqrt[3]{x}} + \frac{2}{3\sqrt[3]{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\cancel{2}}{3\sqrt[3]{x}}\times\frac{3\sqrt[3]{y}}{\cancel{2}} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} = -\frac{\sqrt[3]{1}}{\sqrt[3]{8}} = \frac{-(1)^{1/3}}{(2^3)^{1/3}} = \boxed{\frac{-1}{2}}$$

$m = -1/2$, $P(8,1)$

$$y - 1 = -\frac{1}{2}(x - 8)$$

$$2y - 2 = -x + 8$$

$$\boxed{2y + x = 10}$$

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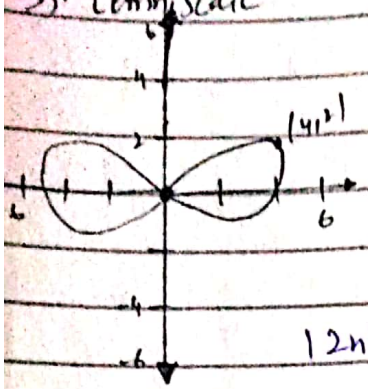
39. Lemniscate



$$3(x^2+y^2)^2 = 100(x^2-y^2)$$

$$3\frac{d}{dx}[x^2+y^2]^2 = 100\frac{d}{dx}[x^2-y^2]$$

$$6(x^2+y^2)\times\frac{d}{dx}(x^2+y^2) = 100\left[\frac{d}{dx}[x^2]-\frac{d}{dx}[y^2]\right]$$

$$(6x^2+6y^2)\left[(2x)+\frac{dy}{dx}2y\right] = 100\left[2x-\frac{dy}{dx}2y\right]$$

$$12x^3+12xy^2+2y(6x^2+6y^2)\frac{dy}{dx} = 200x-200y\frac{dy}{dx}$$

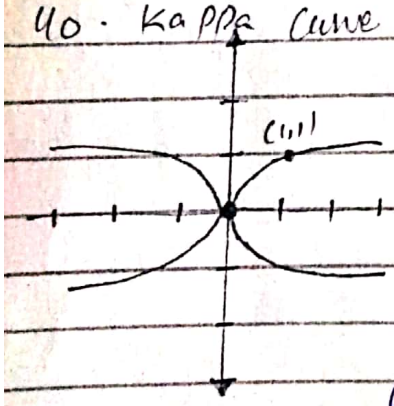
$$2y(6x^2+6y^2)\frac{dy}{dx}+200y\frac{dy}{dx} = 200x-12x^3-12xy^2$$

$$\frac{dy}{dx} = \frac{200x-12x^3-12xy^2}{2y(6x^2+6y^2)+200y} = \frac{200(4)-12(4)^3-12(4)(2)^2}{2(2)[6(4)^2+6(2)^2]+200(2)} = -\frac{2}{11}$$

$m = -2/11$, $P(4,2)$

$$y-2 = \frac{-2}{11}(x-4) \Rightarrow \boxed{y = \frac{-2x}{11}+\frac{30}{11}}$$

40. Kappa Curve



$$y^2(x^2+y^2) = 2x^2$$

$$\frac{d}{dx}[y^2(x^2+y^2)] = \frac{d}{dx}[2x^2]$$

$$(x^2+y^2)\times\frac{d}{dx}[y^2]+y^2\times\frac{d}{dx}(x^2+y^2) = 4x$$

$$(x^2+y^2)2y\frac{dy}{dx}+y^2\left(2x+\frac{dy}{dx}2y\right) = 4x$$

$$(x^2+y^2)2y\frac{dy}{dx}+2xy^2+2y^3\frac{dy}{dx} = 4x$$

$$\frac{dy}{dx} = \frac{4x-2xy^2}{(x^2+y^2)2y+2y^3} = \frac{4(1)-2(1)(1)^2}{(1^2+1^2)2(1)+2(1)^3} = \boxed{\frac{2}{6}}$$

$m = 1/3$, $P(1,1)$

$$y-1 = \frac{1}{3}(x-1)$$

$$3y-3 = x-1$$

$$\boxed{3y = x+2}$$

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11. Use implicit differentiation to find an equation of the tangent line to ellipse $\frac{x^2}{2} + \frac{y^2}{8} = 1$ at $(1,2)$

$$\frac{x^2}{2} + \frac{y^2}{8} = 1$$

$$8x^2 + 2y^2 = 16$$

$$8\frac{d}{dx}[x^2] + 2\frac{d}{dx}[y^2] = \frac{d}{dx}[16]$$

$$8\frac{d(x)}{dx}(2x) + 2\frac{d(y)}{dx}[2y] = 0$$

$$16x + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-16x}{4y} = \boxed{\frac{-4x}{y}}$$

$$\frac{dy}{dx} = \frac{-4x}{y}, \quad x = 1, \; y = 2$$

$$\frac{dy}{dx} = \frac{-4(1)}{2} = \boxed{-2}, \; (1,2)$$

$$y - 2 = -2(x-1)$$

$$y = -2x + 2 + 2$$

$$\boxed{y = -2x + 4}$$

(b) Show that the equation of the tangent line to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_0, y_0) is $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

$$\frac{1}{a^2}\frac{d}{dx}[x^2] + \frac{1}{b^2} \times \frac{d}{dx}[y^2] = \frac{d}{dx}[1]$$

$$\frac{1}{a^2} \times [2x] + \frac{1}{b^2} \times \frac{dy}{dx}[2y] = 0$$

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y} = \boxed{\frac{-2xb^2}{2a^2y}}$$

$$\boxed{\frac{dy}{dx} = \frac{-xb^2}{a^2y}}$$

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

$$\therefore \frac{yy_0}{b^2} + \frac{xx_0}{a^2} = 1 \quad \text{and}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$m = \frac{-xb^2}{a^2y}, \; (x_0, y_0)$$

$$y - y_0 = \frac{-x_0b^2}{a^2y_0}(x - x_0)$$

$$y - y_0 = \frac{-xx_0b^2 + x_0^2b^2}{a^2y_0^2}$$

$$y - y_0 = \frac{b^2(-xx_0 + x_0^2)}{a^2y_0^2}$$

$$yy_0 - y_0^2 = \frac{b^2(-xx_0 + x_0^2)}{a^2}$$

$$\frac{yy_0 - y_0^2}{b^2} = \frac{(-xx_0)}{a^2} + \frac{x_0^2}{a^2}$$

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{-xx_0}{a^2} + \frac{x_0^2}{a^2}$$

$$\boxed{\frac{yy_0}{b^2} + \frac{xx_0}{a^2}} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

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In Exercises 45–50, find d^2y/dx^2 in terms of x and y.

Second derivative implicitly

45. $x^2+y^2=4$

Find the second derivative implicitly.

:- $x^2+y^2=4$, find d^2y/dx^2

$\frac{d}{dx}[x^2]+\frac{d}{dx}[y^2]=\frac{d}{dx}[4]$

$2x+2y\frac{dy}{dx}=0$

$\boxed{\frac{dy}{dx}=\frac{-x}{y}}$

Differentiating a second time with respect to x yields

$\frac{d^2y}{dx^2}=\left(y\times(-1)-(-x)\frac{d}{dx}[y]\right)\div y^2$

$\frac{d^2y}{dx^2}=\frac{-y+x(dy/dx)}{y^2}$, $dy/dx=-x/y$

$\frac{-y+x\left(\frac{-x}{y}\right)}{y^2}=\frac{-y-\frac{x^2}{y}}{y^2}=\frac{-y^2-x^2/y}{y^2}=\frac{-x^2-y^2}{y}\times\frac{1}{y^2}=\frac{-x^2-y^2}{y^3}$

$\frac{-x^2-y^2}{y^3}=\frac{-(x^2+y^2)}{y^3}=\frac{-(4)}{y^3}=\boxed{\frac{-4}{y^3}}$ $\because x^2+y^2=4$

46. $x^2y^2-2x=3$

:- $\left(y^2(2x)+(x^2)\times\frac{d}{dx}[y^2]\right)-2\frac{d}{dx}[x]=\frac{d}{dx}[3]$

$\left(2xy^2+2x^2y\frac{dy}{dx}\right)-2=0$ $\quad\frac{dy}{dx}=\frac{2-2xy^2}{2x^2y}=\boxed{\frac{(1-xy^2)}{x^2y}}$

differentiate a second time w.r.t x yields ↗

$\frac{d^2y}{dx^2}=\left[(x^2y)\times\frac{d}{dx}[1-xy^2]-(1-xy^2)\times\frac{dv}{dx}[x^2y]\right]\div(x^2y)^2$

$=\left[(x^2y)\times-1[y^2(1)+(x)\frac{dy}{dx}[2y]]-(1-xy^2)\times\frac{d}{dx}[x^2y]\right]\div(x^2y)^2$

$=[x^2y\times-[y^2+2xy\frac{dy}{dx}]-(1-xy^2)\times[y(2x)+(x^2)\frac{dy}{dx}]\div(x^2y)^2$

$=[x^2y\times-[y^2+2xy\frac{dy}{dx}]-(1-xy^2)[2xy+x^2\frac{dy}{dx}]\div(x^2y)^2$

$=[-x^2y^3-2x^3y^2\frac{dy}{dx}]-(1-xy^2)[2xy+x^2\,dy/dx]\div(x^2y)^2$

$=-x^2y^3-2x^3y^3\frac{dy}{dx}-[(2xy-2x^2y^3)+x^2-x^3y^2\,dy/dx]\div(x^2y)^2$

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$$\left[-x^2y^3 - 2x^3y^3\frac{dy}{dx} - 2xy + 2x^2y^3 - x^2 + x^3y^2\, dy/dx\right] \div (x^2y)^2$$

$\frac{dy}{dx} = \frac{(1-xy^2)}{x^2y}$ $\therefore$:-

$$\left[-x^2y^3 - 2x^3y^3\left(\frac{1-xy^2}{x^2y}\right) - 2xy + 2x^2y^3 - x^2 + x^3y^2\left(\frac{1-xy^2}{x^2y}\right)\right] \div (x^2y)^2$$

$$\left[-x^2y^3 - 2xy^2(1-xy^2) - 2xy + 2x^2y^3 - x^2 + xy(1-xy^2)\right] \div (x^2y)^2$$

$$\left[-x^2y^3 - 2xy^2 + 2x^2y^4 - \cancel{2xy} + 2x^2y^3 - x^2 + \cancel{xy} - \cancel{x^2y^3}\right] \div (x^2y)^2$$

$$\left[2x^2y^4 - 2xy^2 - xy - x^2\right] \div (x^2y)^2$$

$$\boxed{\frac{xy\left[2xy^3 - 2y - 1\right] - x^2}{x^4y^2} = \frac{d^2y}{dx^2}}$$

47. $x^2 - y^2 = 36$

:- ① $\frac{d}{dx}[x^2] - \frac{d}{dx}[y^2] = \frac{d}{dx}[36]$

$2x - 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = \frac{-2x}{-2y} = \boxed{\frac{x}{y}}$

② $\frac{d^2y}{dx^2} = \left(y - x\frac{dy}{dx}\right) \div y^2$

but $\frac{dy}{dx} = \frac{x}{y}$

$\frac{d^2y}{dx^2} = \left[y - x\left(\frac{x}{y}\right)\right] \div y^2$

$= \left[y - \frac{x^2}{y}\right] \div y^2$

$= \frac{y^2 - x^2}{y} \times \frac{1}{y^2}$

$\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$

Second time differentiation

$\frac{d^2y}{dx^2}\left[\frac{x}{y}\right]$ ②

$= \left[(y)(1) - (x) \times \frac{d}{dx}[y]\right] \div [y^2]$

$= \left[y - x\frac{d(y)}{dx}\right] \div y^2$

④ $\frac{d^2y}{dx^2} = \frac{y^2 - x^2}{y^3}$, $\boxed{x^2 - y^2 = 36}$

$= \frac{-(x^2 - y^2)}{y^3} = \frac{-36}{y^3}$

$$\boxed{\frac{d^2y}{dx^2} = \frac{-36}{y^3}}$$

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48. $1 - xy = x - y$

1: first derivative

$$\frac{d}{dx}[1] - \frac{d}{dx}[xy] = \frac{d}{dx}[x] - \frac{d}{dx}[y]$$

$$0 - \left[y(1) + (x)\frac{dy}{dx}\right] = 1 - \frac{d(y)}{dx} \times (1)$$

$$-\left[y + x\frac{dy}{dx}\right] = 1 - \frac{dy}{dx}$$

$$-y - x\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - x\frac{dy}{dx} = 1 + y$$

$$\boxed{\frac{dy}{dx} = \frac{1+y}{1-x}}$$

2: Second time differentiation [Qoutient Rule]

$$\frac{d^2y}{dx^2} = \left[(1-x) \times \frac{d}{dx}(1+y) - (1+y) \times \frac{d}{dx}(1-x)\right] \div (1-x)^2$$

$$= \left[(1-x) \times \frac{d(y)}{dx} \times (1) - (1+y) \times \frac{d(x)}{dx} \times (0-1)\right] \div (1-x)^2$$

$$= \left[(1-x) \times \frac{dy}{dx} - (1+y) \times -(1)\right] \div (1-x)^2$$

$$= \left[(1-x)\frac{dy}{dx} + (1+y)\right] \div (1-x)^2$$

3: put $\frac{dy}{dx} = \frac{1+y}{1-x}$

$$= \left[\cancel{(1-x)}\left(\frac{1+y}{\cancel{1-x}}\right) + (1+y)\right] \div (1-x)^2$$

$$= \left[(1+y) + (1+y)\right] \div (1-x)^2$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{2+2y}{(1-x)^2}}$$

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8. $y^2 = x^3$

1: first differentiation

$\frac{d}{dx}[y^2] = \frac{d}{dx}[x^3]$

$\frac{d(y)}{dx}[2y] = \frac{d(x)}{dx} \times [3x^2]$

$\boxed{\frac{dy}{dx} = \frac{3x^2}{2y}}$

2: Second time differentiation

$\frac{d^2y}{dx^2} = \frac{3x^2}{2y}$, Quotient rule

$\frac{d^2y}{dx^2} = [(2y)(6x) - (3x^2) \times \frac{d}{dx}(2y)] \div (2y)^2$

$= [12xy - 3x^2 \times 2\frac{d(y)}{dx}] \div 4y^2$

$\frac{d^2y}{dx^2} = \left[12xy - 6x^2\frac{dy}{dx}\right] \div 4y^2$ (1)

put $\frac{dy}{dx} = 3x^2/2y$

$\frac{d^2y}{dx^2} = \left[12xy - 6x^2\left(\frac{3x^2}{2y}\right)\right] \div 4y^2$

$= \left[\frac{12xy}{1} - \frac{18x^4}{2y}\right] \div 4y^2$

$\frac{d^2y}{dx^2} = \frac{24xy^2 - 18x^4}{8y^3}$

$\frac{d^2y}{dx^2} = \frac{24xy^2 - 18x^4}{8y^3}$, $y^2 = x^3$

$= \frac{24x(x^3) - 18x^4}{8y(x^3)} = \frac{6x^4}{8yx^3}$

$= \frac{3x}{4y} \rightarrow \boxed{\frac{d^2y}{dx^2} = \frac{3x}{4y}}$

So. $y^2 = 10x$

1: $\frac{d}{dx}[y^2] = \frac{d}{dx}[10x]$

$\boxed{\frac{dy}{dx} = \frac{10}{2y}}$

2: Second time differentiation

$\frac{d^2y}{dx^2} = \frac{10}{2y} \rightarrow \left[2y \times (0) - 10 \times \frac{d}{dx}[2y]\right] \div (2y)^2$

$= \left[0 - 10 \times 2 \times \frac{dy}{dx} \times (1)\right] \div 4y^2$

$= -20\frac{dy}{dx} \div 4y^2$

put $\frac{dy}{dx} = \frac{10}{2y}$

$= -20\left(\frac{10}{2y}\right) \times \frac{1}{4y^2} = \frac{-200}{8y^3}$

$= \frac{-200}{8y^3} = \frac{-25}{y^3}$, $y^2 = 10x$

$= \frac{-25}{y(10x)} = \frac{-5}{2xy}$

$\boxed{\frac{d^2y}{dx^2} = \frac{-5}{2xy}}$

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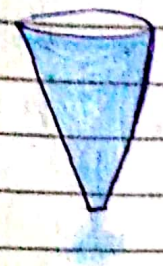
2.6 Related Rates → Multivariable Calculus

- Find a related area
- Use related rates to solve real-life problemes.

:- You have seen how chain can be used to find dy/dx implicitly. Another important use of the chain Rule is to find the rates of change of two or more related variables that are changing with respect to time.



Ex:- when water drained out of a conical tank the volume V, the radius r, and the height h of the water level are all functions of time t.

Knowing that these variables are related by the equation.

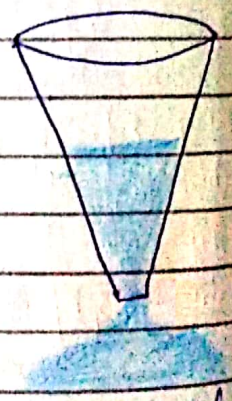
$$V = \frac{\pi}{3} r^2 h \quad \boxed{\text{Original equation}}$$

you can differentiate implicitly with respect to t to obtain the related-rate equation

$$\frac{d}{dt}(V) = \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right)$$

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$

$$= \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right]$$



From this equation you can see that the rate of change of V is related to the rates of change of both h and r.

Ex 1 :- Two rates that are related

Suppose x and y are both differentiable functions of t and are related by the equation $\boxed{y = x^2+3}$ Find $\boxed{dy/dt}$ when $\boxed{x=1}$, given that $\boxed{dx/dt = 2}$ when $\boxed{x=1}$

:- $y = x^2 + 3$

$$\frac{d}{dt}[y] = \frac{d}{dt}[x^2] + \frac{d}{dt}[3]$$

$$dy/dt = dx/dt(2x) + 0$$

$$\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\frac{dy}{dt} = 2(1)(2) = 4$$

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2.6 Exercise

1–4, assume that x and y are both differentiable functions of t and find the required values of dy/dt and dx/dt.

	Equation	Find	Given
1.	$y = \sqrt{x}$	(a) $\frac{dy}{dt}$ when $x = 4$	$\frac{dx}{dt} = 3$
		(b) $\frac{dx}{dt}$ when $x = 25$	$\frac{dy}{dt} = 2$

a:– $\frac{d}{dt}[y] = \frac{d}{dt}[\sqrt{x}]$

$$\frac{dy}{dt} = \frac{dx}{dt}\left(\frac{1}{2\sqrt{x}}\right) = 3\left(\frac{1}{2\sqrt{4}}\right) = \frac{3}{4}$$

b:– $\frac{d}{dt}[y] = \frac{d}{dt}[\sqrt{x}]$

$$\frac{dy}{dt} = \frac{dx}{dt}\left(\frac{1}{2\sqrt{x}}\right) \rightarrow 2 \div \left(\frac{1}{2\sqrt{x}}\right) = \frac{dx}{dt} = 2(2\sqrt{x}) = 2(2\sqrt{25})$$
$$= 2(10) = \boxed{20}$$

2. $y = 4(x^2 - 5x)$ (a) $\frac{dy}{dt}$ when $x = 3$ $\quad \frac{dx}{dt} = 2$

(b) $\frac{dx}{dt}$ when $x = 1$ $\quad \frac{dy}{dt} = 5$

a:– $\frac{d}{dt}[y] = 4\frac{d}{dt}[x^2] - 20\frac{d}{dt}[5x]$

$$\frac{dy}{dt} = 8x\frac{dx}{dt} - 20\frac{dx}{dt} = \frac{dx}{dt}(8x - 20) = \frac{dx}{dt}(8(1) - 20)$$

$$\frac{dy}{dt} = \frac{dx}{dt}(-12) \Rightarrow \frac{dx}{dt} = \frac{dy}{dt} \div (-12) = (5)/-12 = \boxed{\frac{-5}{12}}$$

b:– $\frac{dy}{dt} = \frac{dx}{dt}(8x - 20) \Rightarrow \frac{dy}{dt} = 2(8(3) - 20) = \boxed{8}$

3. $xy = 4$ (a) $\frac{dy}{dt}$ when $x = 8$ $\quad \frac{dx}{dt} = 10$

(b) $\frac{dx}{dt}$ when $x = 1$ $\quad \frac{dy}{dt} = -6$

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a:- $\frac{d}{dt}[y] = 4\frac{d}{dt}\left[\frac{1}{x}\right]$

$\frac{dy}{dt} = 4\frac{dx}{dt}\left(-\frac{1}{x^2}\right) = 4(10)\left(-\frac{1}{8^2}\right) = \frac{-40}{64} = \frac{-10}{16} = \boxed{\frac{-5}{8}}$

b:- $\frac{dy}{dt} = 4\frac{dx}{dt}\left(-\frac{1}{x^2}\right)$

$\frac{dx}{dt} = \frac{dy}{dt} \div 4\left(-\frac{1}{x^2}\right) = -6 \div 4\left(-\frac{1}{1^2}\right) = \frac{-6}{-4} = \frac{3}{2}$

4. $x^2 + y^2 = 25$ (a) $\frac{dy}{dt}$ when $x=3$, $y=4$ $\frac{dx}{dt} = 8$

:- $\frac{d}{dt}[x^2] + \frac{d}{dt}[y^2] = \frac{d}{dt}[25]$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \rightarrow 2(3)(8) + 2(4)\frac{dy}{dt} = 0$

$48 + 8\frac{dy}{dt} = 0 \rightarrow \frac{dy}{dt} = \frac{-48}{8} = \boxed{-6}$

(b) $\frac{dx}{dt}$ when $x=4$, $y=3$ $\frac{dy}{dt} = -2$

:- $\frac{d}{dt}[x^2] + \frac{d}{dt}[y^2] = \frac{d}{dt}[25]$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \rightarrow \frac{dx}{dy} = \left(-2y\frac{dy}{dt}\right) \div 2x$

$\frac{dx}{dt} = \left(-2(3)(-2)\right) \div 2(4) \rightarrow (12)/8 = \boxed{3/2}$

5-8 a point is moving along the graph of the given function such that $\boxed{dx/dt}$ is $\boxed{2}$ centimeters per second. Find dy/dt for the given values of x.

5. $y = 2x^2 + 1$ (a) $x = -1$ (b) $x = 0$ (c) $x = 1$

a:- $\frac{dy}{dt} = 4x\frac{dx}{dt}$ b:- $\frac{dy}{dx} = 4x\frac{dx}{dt}$ c:- $\frac{dy}{dx} = 4x\frac{dx}{dt}$

$\frac{dy}{dt} = 4(-1)(2) = \boxed{-8}$ $= 4(0)(2)$ $\boxed{\frac{dy}{dx} = 0}$ $\frac{dy}{dx} = 4(1)(2) = \boxed{8}$

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6: $y = \frac{1}{1+x^2}$ (a) $x = -2$ (b) $x = 0$ (c) $x = 2$

a: $\frac{dy}{dt} = \left[(1+x^2)(0) - (1)\frac{d}{dt}[1+x^2] \right] \div (1+x^2)^2$

$= \left[0 - 2x\frac{dx}{dt} \right] \div (1+x^2)^2 = \boxed{-2x\frac{dx}{dt} \times \frac{1}{(1+x^2)^2}}$

$\frac{dy}{dt} = \frac{-2(-2)(2)}{(1+(-2)^2)^2} = \boxed{\frac{8}{25}}$

b:- $\frac{dy}{dt} = -2x\frac{dx}{dt} \times \frac{1}{(1+x^2)^2} = \frac{-2(0)(2)}{(1+0^2)^2} = \boxed{0}$

c:- $\frac{dy}{dt} = -2x\frac{dx}{dt} \times \frac{1}{(1+x^2)^2} = \frac{-2(2)(2)}{(1+2^2)^2} = \boxed{\frac{-8}{25}}$

7: $y = \tan x$ (a) $x = -\pi/3$ (b) $x = -\pi/4$ (c) $x = 0$

a:- $\frac{dy}{dt} = \sec^2 x = \sec^2\left(-\frac{\pi}{3}\right) = \sec^2(-60^\circ)\frac{dx}{dt}$

b:- $\frac{dy}{dt} = \frac{dc}{dt}\tan x = \frac{dx}{dt}\sec^2 x = 2\sec^2 x$

$= 2\sec^2(-\pi/3) = \frac{2}{\cos^2(-\pi/3)} = \frac{2}{\cos^2(-60^\circ)} = \frac{2}{(-\frac{1}{2})^2} = 2 \div \frac{1}{4} = \boxed{8}$

c:- $\frac{dy}{dt} = \frac{2}{\cos^2(-\pi/4)} = \frac{2}{\cos^2(-45^\circ)} = \frac{2}{(-1/\sqrt{2})^2} = 2 \div \frac{1}{2} = 2 \times 2 = \boxed{4}$

d:- $\frac{dy}{dt} = \frac{2}{\cos^2 x} = \frac{2}{\cos^2(0)} = \frac{2}{1} = \boxed{2}$

8: $y = \cos x$ (a) $x = \pi/6$ (b) $x = \pi/4$ (c) $x = \pi/3$

a:- $\frac{dy}{dt} = \frac{dx}{dt}(-\sin x) = -(2)\sin(\pi/6) = -2\sin(30^\circ) = -2(1/2) = \boxed{-1}$

b:- $\frac{dy}{dt} = \frac{dx}{dt} \times (-\sin x) = -2\sin(\pi/4) = -2\sin(45^\circ) = -2\left(\frac{1}{\sqrt{2}}\right) = \boxed{-2/\sqrt{2}}$

c:- $\frac{dy}{dt} = \frac{dx}{dt} \times (-\sin x) = -2\sin(\pi/3) = -2\sin(60^\circ) = -2(\sqrt{3}/2) = \boxed{-\sqrt{3}}$

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13. Area — The radius r of a circle is increasing at a rate of 4 centimeters per minute. Find the rates of change of the area when (a) $r = 8$ centimeters and (b) $r = 36$ cm.

:- To calculate the area of circle → $A = \pi r^2$

rate of r changes 4 cm per minute

equation we have → $A = \pi r^2$ → Area $= \pi (\text{radius})^2$

a:- $A = \pi r^2$, $dr/dt = 4$, $r = 8$

$$\frac{dA}{dt} = \pi \times 2r \times \frac{dr}{dt} = \pi \times 2(8) \times (4) = \boxed{64\pi}$$

b:- $$\frac{dA}{dt} = \pi \times 2r \times \frac{dr}{dt} = \pi \times 2(32) \times (4) = \boxed{256\pi}$$

14. Area — Let A be the area of a circle of radius r that is changing with respect to time. If dr/dt is constant, is dA/dt constant? Explain.

$A = \pi r^2$

$$\frac{d}{dt}(A) = \pi \times \frac{d}{dt}(r^2)$$

$$\frac{dA}{dt} = \pi 2r \frac{dr}{dt}$$

If dr/dt is constant then dA/dt must be constant because dA/dt is dependent on the realated variable(s) either r or dr/dt.

16. Volume — The radius r of a sphere is increasing at a rate of 3 inches per minute.

(a) Find the rates of change of the volume when $r = 9$ and $r = 36$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

a:- given $\frac{dr}{dt} = 3$ inches per min $r = 9$ and $r = 36$ find $\frac{dV}{dt}$

To calculate the volume of sphere → $\boxed{V = \frac{4}{3}\pi r^3}$

$$\frac{d}{dt}(V) = \frac{4\pi}{3} \times \frac{d}{dt}(r^3)$$

$$\frac{dV}{dt} = \frac{4\pi}{3} \times 3r^2 \times \frac{dr}{dt}$$

$$= (4\pi r^2)(3)$$

$$= (4\pi (9)^2)(3) = 972\pi$$

$$\frac{dV}{dt} = (4\pi r^2)(3)$$

$$= (4\pi (36)^2)(3)$$

$$= 15552\pi$$

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17) Volume — A spherical ballon is inflated with gas at the rate of 800 cubic centi meters per minute. How fast is the radius of balloon increasing at the instant the radius is (a) 30 cm and (b) 60 cm?

:- Given rate : $\frac{dv}{dt} = 800$ cubic cmpm

Find : $\frac{dr}{dt}$ when $r = 30, r = 60$

To Calculate the volume → equation of sphere → $\boxed{V = \frac{4}{3}\pi r^3}$

:- $\frac{d}{dt}(V) = \frac{4\pi}{3}\frac{d}{dt}[r^3] = \frac{4\pi}{3} \times 3r^2 \times \frac{dr}{dt} = (4\pi r^2)(dr/dt)$.

$\frac{dV}{dt} = (4\pi r^2)(dr/dt) \rightarrow \frac{800}{4\pi r^2} = \frac{dr}{dt} = \frac{200}{\pi r^2} = \frac{200}{900\pi} = \boxed{\frac{2}{9\pi}}$

b:- $\frac{dr}{dt} = \frac{dv}{dt} \div (4\pi r^2) = \frac{800}{4\pi r^2} = \frac{800}{4\pi(60)^2} = \frac{200}{3600\pi} = \boxed{\frac{1}{18\pi}}$

20) Volume — The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Find the rates of change of the volume if dr/dt is 2 inches per minute and $h = 3r$ when (a) $r = 6$ inches and (b) $r = 24$ inches.

:- Equation of volume of cone → $V = \frac{1}{3}\pi r^2 h$

Find : dv/dt when $r = 6, 24$ given : $dr/dt = 2$ inch per mint

a:- $V = \frac{1}{3}\pi r^2 h$, $r = 6$, $h = 3r$, $dr/dt = 2$

$\frac{d}{dt}(V) = \frac{\pi}{3} \times \frac{d}{dt}[r^2 h] = \frac{\pi}{3} \times \frac{d}{dt}[r^2(3r)] = \frac{\pi}{3} \times \frac{d}{dt} \times [3r^3] = \frac{\pi}{3} \times \frac{dr}{dt} \times 9r^2$

$\frac{dV}{dt} = \frac{\pi}{3} \times \frac{dr}{dt} \times 9r^2 = \pi r^2 \times \frac{dr}{dt} = \pi r^2 \times (2) = \boxed{2\pi r^2}$

b:- $\frac{dv}{dt} = \frac{\pi}{3} \times \frac{d}{dt} \times [r^2(3r)] = \frac{\pi}{3} \times (2) \times \frac{d}{dt}(3r^3) = \boxed{8\pi r^2}$

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CHAPTER 3 Application of Differentiation

3.1 Extrema on an Interval

- Understand the defination of extrema of a function on an interval
- Understand defination of relative extrema of function on an open interval
- Find extrema on a closed interval.

▶ Defination Of Extrema :-

Let f be defined on an interval I containing c.

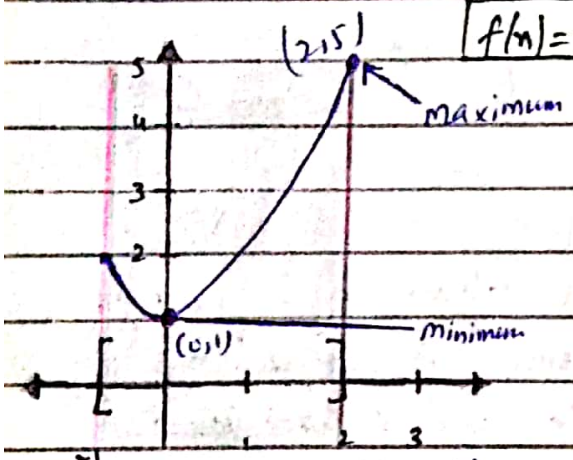
1. $f(c)$ is minimum of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is maximum of f on I if $f(c) \geq f(x)$ for all x in I.

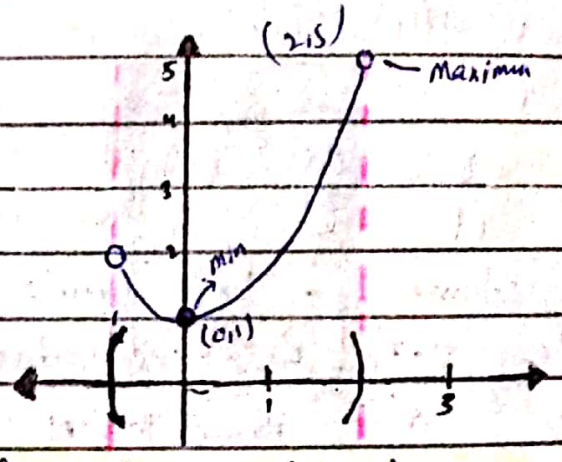
extreme value: or extrema (or singular extremum)

minimum: or absolute minimum / global minimum

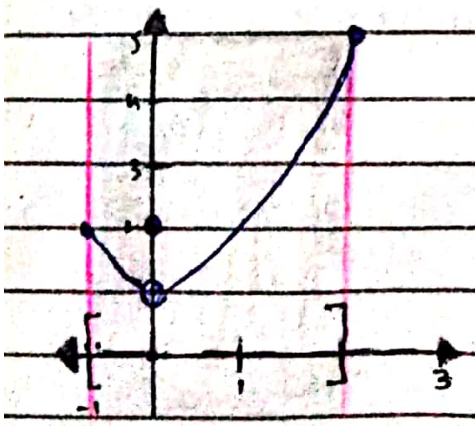
maximum: absolute maximum / global maximum

$\boxed{f(x) = x^2+1}$





(a) f is continuous, $[-1,2]$ is closed (b) f is continuous, $(-1,2)$ is open



$$g(x) = \begin{cases} x^2+1 & , x \neq 0 \\ 2 & , x = 0 \end{cases}$$

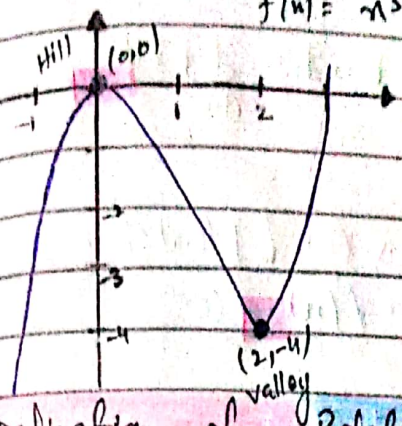
(c) g is not continuous, $[-1,2]$ is closed

: Extrema can occur at interior points or endpoints of an interval.

- Endpoint extrema: Extrema that occur at the points are called endpoint extrema

Relative Extrema and Critical Numbers

$f(x) = x^3 - 3x^2$



: The graph of $f(x) = x^3 - 3x^2$ has a relative maximum at the point $(0,0)$ and a relative minimum at $(2,-4)$

Defination of Relative Extrema

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of f, or you can say that f has a relative maximum at $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of f, or you can say that f has a relative minimum at $(c, f(c))$.

Relative extrema: maximum (plural: maxima), Minimum (plural: minima)

Relative maximum = local maximum

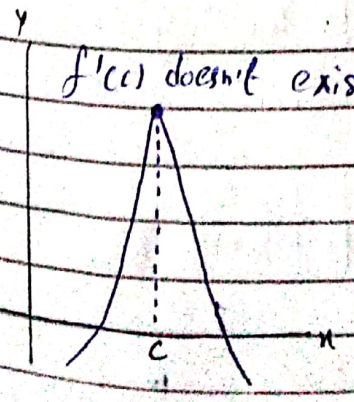
Relative minimum = local minimum

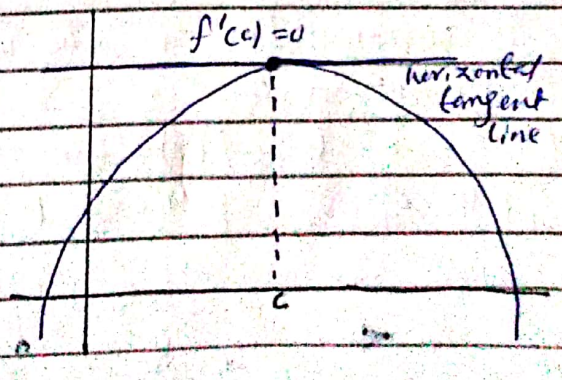
Defination of Critical Number → $f'(c) = 0$, c = roots of $f(x)$

Let f be defined at c. If $f'(c) = 0$ or f is not differentiable at c, then c is a critical number

Theorem:- Relative EXTREMA occur only at CRITICAL Number

If f has a relative maximum or relative minimum at $x = c$ then c is a critical number of f.





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CRITICAL Numbers + Increasing and Decreasing

Ex:- $y = 2x^3 + 3x^2 - 12x + 8$

1: Differentiation

$$\frac{d}{dx}[y] = 2\frac{d}{dx}[x^3] + 3\frac{d}{dx}[x^2] - 12\frac{d}{dx}[x] + \frac{d}{dx}[8]$$

$$\frac{dy}{dx} = 6x^2 + 6x - 12 + 0$$

$\boxed{m = 6x^2 + 6x - 12}$ → first derivative $[f'(x)]$

2: check existension, put $dy/dx = m = f'(x) = 0$

$f'(x) = 6x^2 + 6x - 12$

$6x^2 + 6x - 12 = 0$

$6(x^2 + x - 2) = 0$

$x^2 + x - 2 = 0/6$

$x^2 + 2x - x - 2 = 0$

$x(x+2) - 1(x+2) = 0$

$(x-1)(x+2) = 0$

$\boxed{x = 1}$ $\boxed{x = -2}$

→ CRITICAL Numbers

$x = 1$ and $x = -2$ are CRITICAL points

$\boxed{\text{Increasing or Decreasing intervals}}$

3: put the critical numbers in original function and you will get the y-values

$c_1 = x_1 = 1$, $c_2 = x_2 = -2$

$f(c_1) = 2 + 3 - 12 + 8 = 13 - 12 = 1$

$f(c_1) = 1 \longrightarrow \boxed{(1, 1)}$ → Extreme point ①

$f(c_2) = 2(-2)^3 + 3(-2)^2 - 12(-2) + 8$

$= 2(-8) + 3(4) + 24 + 8$

$= -16 + 12 + 32$

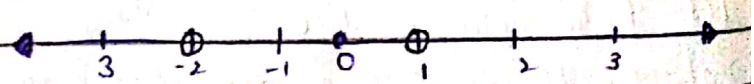
$f(c_2) = 28$

$\boxed{(-2, 28)}$

→ Extreme point ②

4: Increasing or decreasing Intervals given by first derivative

Increase: $f'(x) > 0$, Decrease: $f'(x) < 0$



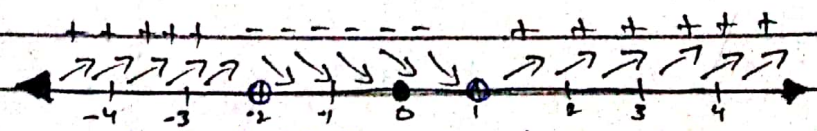
$(-\infty, -2), (-2, 1), (1, \infty)$

:- The only condition to check Increment and decrement in graph of f is to put the nearest points of open intervals in the $f'(x)$, first derivative, and check the Incrementation and Decrementation in each interval.

$f'(x) = 6x^2 + 6x - 12$

$f'(-3) = 24$ ① $f'(0) = -12$ $f'(2) = 24$

$\boxed{24 > 0}$ → ① $\boxed{-12 < 0}$ → ② $\boxed{24 > 0}$ → ③



$(-\infty, -2) \cup (-2, 1) \cup (1, \infty)$

① ② ③

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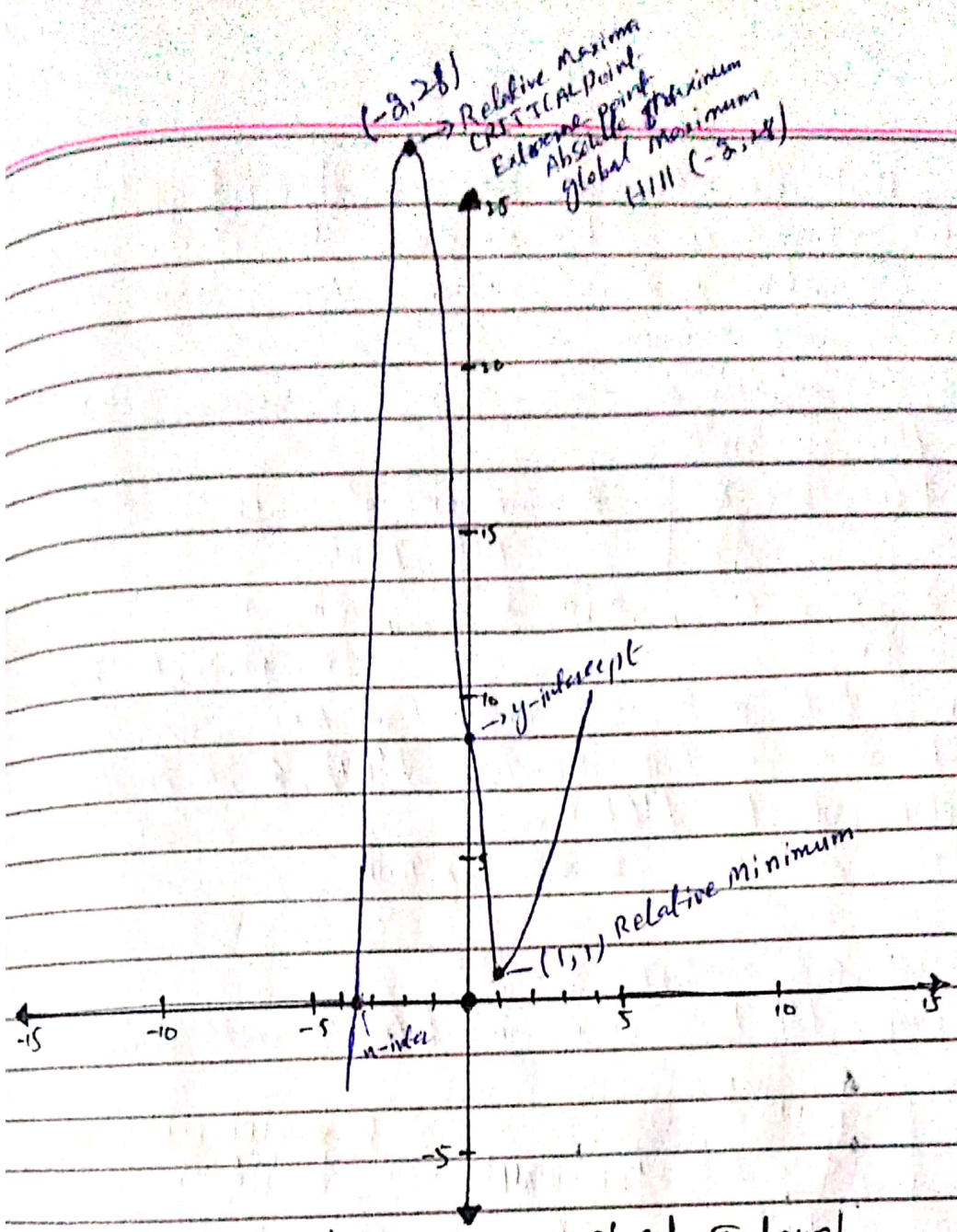


Ex ② :- Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 3x^4 - 4x^3$ on interval $[-1, 2]$

1: Differentiate original function

$f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

2: Find Critical Numbers, put $f'(x) = 0$

$12x^3 - 12x^2 = f'(x)$

$12x^3 - 12x^2 = 0$

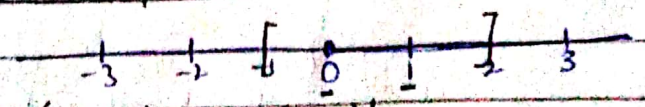
$12x^2(x-1) = 0$

$\boxed{x = 0, 1}$

CRITICAL Numbers

3:

Left Endpoint	CRITICAL Number	CRITICAL Number	Right Endpoint
$f(-1) = 7$	$f(0) = 0$	$f(1) = -1$ Minimum	$f(2) = 16$ Maximum



Min(1,-1), Max(2,16)

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note that the critical number $x=0$ does not yield a relative minimum or relative maximum. The critical number of a function need not produce relative extrema.

Ex ③:- Finding Extrema on a Closed Interval

Find the extrema of $f(x) = 2x - 3x^{2/3}$ on interval $[-1, 3]$

:- $f(x) = 2x - 3x^{2/3}$

$\boxed{1:\ f'(x) = 2 - \frac{2}{\sqrt[3]{x}}}$

2: Find Critical points

Put $f'(x) = 0$

$2 - \frac{2}{\sqrt[3]{x}} = 0 \rightarrow 2\sqrt[3]{x} - 2 = 0 \rightarrow \sqrt[3]{x} = 1$

$\boxed{x = 1} \rightarrow$ Critical number

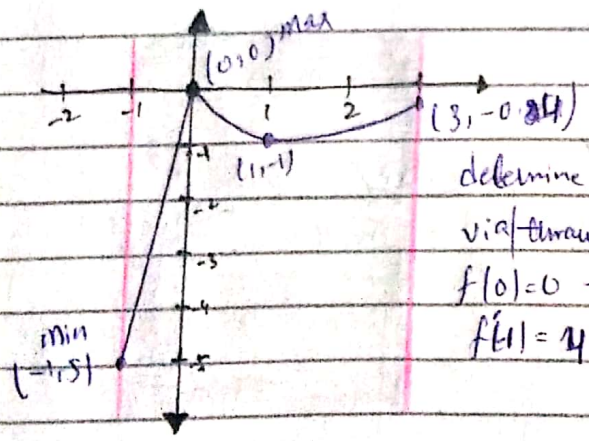
verify that $x=1$ is a critical number

: Put $x=1$ in first derivative $f'(1)$, 1 is critical Number Only if $f'(1) = 0$

$f'(1) = 2 - \frac{2}{\sqrt[3]{1}} = 2 - 2 = 0 \rightarrow$ True

CRITICAL Numbers $\rightarrow x = 0; 1$

because $f'(1)$ is 0, and $f'(0)$ is not defined



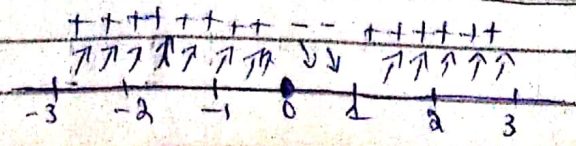
$(-\infty, 0) \cup (0, 1)\ (1, \infty)$

determine Incrementation & decrementation via/through critical numbers

$f(0) = 0 \rightarrow$ Neither increase nor decrease

$f(1) = 1 \rightarrow$ point lies on interval

Left end point	Critical Number	Critical number	Right endpoint
$f(-1) = -5$	$f(0) = 0$	$f(1) = -1$	$f(3) = -0.24$
Minimum	Maximum		



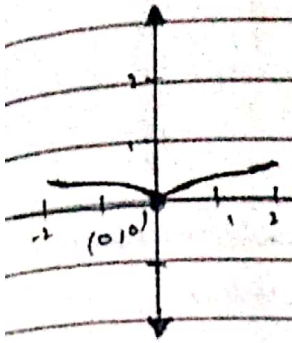
Exercise 3.1

1-6, find the value of derivative (if it exists) at each indicated extremum.

1. $f(x) = x^2/x^2+4$: $f(x)' = \left[(x^2+4)(2x) + (x^2)(2x)\right] \div (x^2+4)^2$

$= \left[2x^3+8x + 2x^3\right] \div (x^2+4)^2$

$f'(x) = \dfrac{4x^3+8x}{(x^2+4)^2} \rightarrow f'(0) = \dfrac{4(0)^3+8(0)}{(0^2+4)^2} = 0$



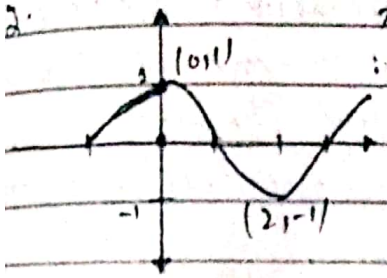
$\boxed{f'(0) = 0}$

2. $f(x) = \cos \pi x/2$

: $f'(x) = -\sin \pi x/2 \rightarrow f'(0) = -\sin \dfrac{\pi(0)}{2} = 0$

$f'(0) = 0$

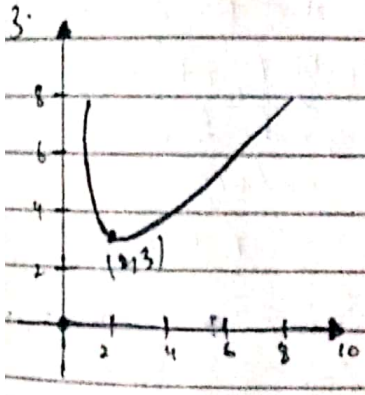
$f'(2) = -\sin \dfrac{\pi(2)}{2} = -\sin \pi = -0.054$



3. $g(x) = x + \dfrac{4}{x^2}$

: $g'(x) = 1 - \dfrac{8}{x^3}$

$g'(2) = 1 - \dfrac{8}{2^3} = 1 - \dfrac{8}{8} = 1 - 1 = 0$



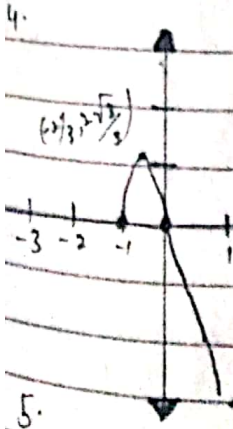
$\boxed{g'(2) = 0}$

4. $f(x) = -3x\sqrt{x+1}$

: $f'(x) = 3\sqrt{x+1} - \dfrac{3x}{2\sqrt{x+1}} = 3\sqrt{-\frac{2}{3}+1} - \dfrac{3(-\frac{2}{3})}{2\sqrt{\frac{-2}{3}+1}}$

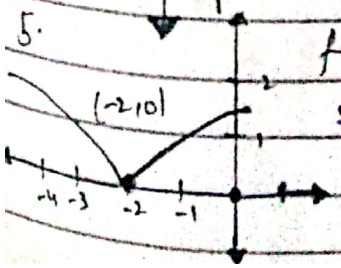
$= 3\sqrt{1/3} + \dfrac{2}{2\sqrt{1/3}} = \dfrac{3\sqrt{1/3}}{1} + \dfrac{1}{\sqrt{1/3}} = \dfrac{3(\frac{1}{3})+1}{\sqrt{1/3}}$

$f'(-2/3) = (1+1) \div (\sqrt{1/3}) = \boxed{2/\sqrt{1/3}}$



5. $f(x) = (x+2)^{2/3}$

: $f'(x) = \dfrac{2}{3\sqrt[3]{x+2}} \rightarrow f'(-2) = \dfrac{2}{3\sqrt[3]{-2+2}} = \dfrac{2}{3\sqrt[3]{0}}$



$\boxed{f'(-2) \text{ is not defined}}$

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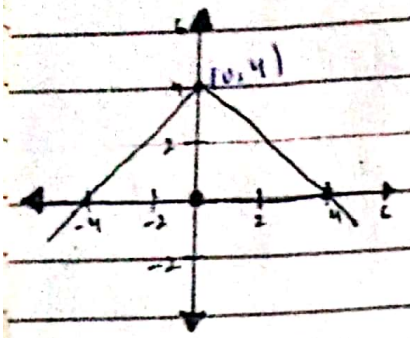
227

6. $f(x) = 4 - |x|$



$\therefore f(x) = 4 - |x| = 4 - \sqrt{x^2}$

$\therefore |x| = \sqrt{x^2}$

$f'(x) = 0 - \frac{1}{2}(x^2)^{-\frac{1}{2}} \times \frac{d}{dx}[x^2]$

$= -\frac{1}{2}(x)^{-1} \times 2x \times \frac{d}{dx}[x]$

$= -\frac{1}{2x} \times 2x \times (1) = -\frac{2x}{2x} = \boxed{-}$

▶ In Exercise 7-10, approximate the critical points/numbers of the function shown in graph. Determine wether the function has a relative maximum, a relative minimum, absolute maximum, an absolute minimum or none of these at each critical number on the interval shown.

▶ In exercises 11-16, find any critical numbers of function

11) $f(x) = x^3 - 3x^2$

1: differentiate $f(x)$

$f'(x) = 3x^2 - 6x$

2: Solve for x, $f'(x) = 0$

$0 = 3x(x-2)$

$\boxed{x=0}, \boxed{x=2}$

Critical numbers $\rightarrow x = 0, 2$

Relative Extrema occur only at critical numbers

3: put $x = 0, 2$ in original function

$f(0) = (0)^3 - 3(0)^2 = 0 \rightarrow (0,0)$

$f(2) = 2^3 - 3(2)^2 = 8 - 12 = -4 \rightarrow (2,-4)$

(12) $g(x) = x^4 - 4x^2$

1: differentiate $g(x)$

$g'(x) = 4x^3 - 8x$

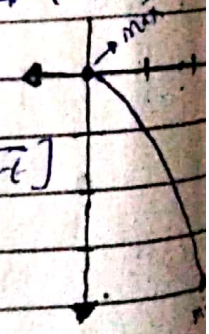
2: Solve for x, $g'(x) = 0$

$0 = 4x(x-2)$

$\boxed{x = 2, 0, 0}$

↳ CRITICAL Numbers

13) $g(t) = t\sqrt{4-t}$, $t < 3$



1: Differentiate $g(t)$

$g'(t) = (\sqrt{4-t})(1) + (t) \times \frac{d}{dx}[\sqrt{4-t}]$

$g'(t) = (\sqrt{4-t}) + (t) \times \frac{-1}{2\sqrt{4-t}}$

$g'(t) = \sqrt{4-t} - \frac{t}{2\sqrt{4-t}}$

$g'(t) = \frac{2(\sqrt{4-t})(\sqrt{4-t}) - t}{2\sqrt{4-t}} = \frac{2(4-t)-t}{2\sqrt{4-t}}$

2: Solve for t, put $g'(t) = 0$

$0 = \frac{2(4-t)-t}{2\sqrt{4-t}}$

$8 - 2t - t = 0$

$\boxed{t = 8/3}$ and $t < 3$ satisfied

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14) $f(x) = \frac{4x}{x^2+1}$: 1: differentiate $f(x)$

$f'(x) = \left[(x^2+1)(4) - (4x)\frac{d}{dx}(x^2+1)\right] \div (x^2+1)$

2: put $f'(x) = 0$

$= \left[4x^2+4 - (4x)(2x)\right] \div (x^2+1)^2$

$f'(x) = \frac{8x^2}{(x^2+1)^2}$

$= \frac{[16x^2 - 8x^2]}{(x^2+1)^2} = \frac{8x^2}{(x^2+1)^2}$

$0 = \frac{8x^2}{(x^2+1)^2} \rightarrow 8x^2 = 0$

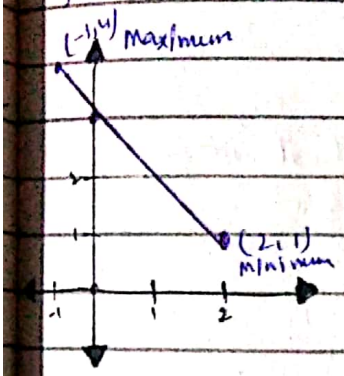
$\boxed{x = 0}$

▸ 17-36 locate the absolute extrema of function on closed interval

17) $f(x) = 3 - x$, $[-1, 2]$

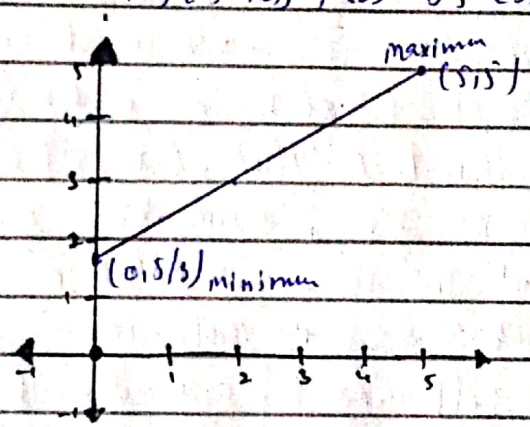
1:- $f(-1) = 4$, $(-1, 4)$

$f(2) = 1$, $(2, 1)$



18) $f(x) = \frac{2x+5}{3}$, $[0, 5]$

$f(0) = 5/3$, $(0, 5/3)$, $f(5) = 5$, $(5, 5)$



18) $g(x) = x^2 - 2x$, $[0, 4]$

1: $f(0) = 0$, $(0,0)$, $f(4) = 8$, $(4,8)$

2: differentiate $g(x)$

$g'(x) = 2x - 2$

3: find critical numbers

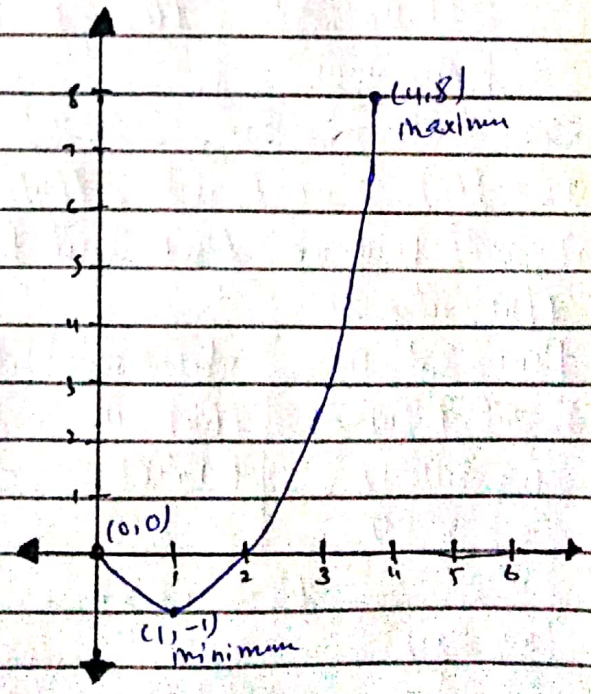
$2x - 2 = 0 \rightarrow \boxed{x = 1}$

4: Find extrema, put $x = 1$ in $g(x)$ i.e, original function

$g(1) = (1)^2 - 2(1)$

$g(1) = -1 \rightarrow \boxed{(1, 1)}$

Left endpoint	Critical	Right endpoint
0	1	4
$f(0) = 0$	$f(1) = -1$	$f(4) = 8$
	Minimum	Maximum



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20) $h(x) = -x^2+3x-5$, $[-2,1]$

1: $h(-2) = -(-2)^2+3(-2)-5$, $h(1) = -(1)^2+3(1)-5$

$= -4-6-5$ $\qquad = -1+3-5$

$= -15 \rightarrow (-2,-15)$ $\qquad = -3 \rightarrow (1,-3)$

2: Differentiate $h(x)$, find critical number and extrem poin

$\boxed{h'(x) = -2x+3}$, put $h'(x)=0$

$0 = -2x+3$ $\qquad h(3/2) = -\left(\frac{3}{2}\right)^2+3\left(\frac{3}{2}\right)-5 = -\frac{9}{4}+\frac{9}{2}-\frac{5}{1} = -\frac{11}{4}$

$\boxed{x = 3/2}$

$(3/2, -11/4)$	Left endpoint	Right endpoint	Critical numb
$-2 < 1 < 3/2$	$f(-2) = -15$	$f(1) = -3$	$f(3/2) = -11/$
$-15 < -3 < -11/4$	Minimum		Maximum

21) $f(x) = x^3 - \frac{3}{2}x^2$, $[-1,2]$

1: $f(-1) = -1 - \frac{3}{2} = -5/2 \rightarrow (-1, -5/2)$

$f(2) = 8 - 3(2) = 2 \rightarrow (2,2)$

2: Differentiate $f(x)$, find critical number, find extreme points

$f(x) = x^3 - \frac{3}{2}x^2 \rightarrow f'(x) = 3x^2 - \frac{3}{2}(2x) = \boxed{3x^2-3x}$

$f'(x) = 3x^2-3x$, put $f'(x)=0$

$0 = 3x^2-3x \rightarrow 3x(x-1) = 0 \rightarrow \boxed{x=0,\ x=1}$

Left endpoint	Critical number	Critical number	Right endpoint
$f(-1) = -5/2$	$f(0) = 0$	$f(1) = -1/2$	$f(2) = 2$
Minimum			Maximum

22) $f(x) = x^3 - 12x$, $[0,4]$

1: $f(0) = 0 \rightarrow (0,0)$, $f(4) = (4)^3 - 12(4) = 16 \rightarrow (4,16)$

2: Differentiate $f(x)$ and find critical number(s)

$f(x) = x^3 - 12x$

$f'(x) = 3x^2 - 12$

3: put $f'(x) = 0$

$0 = 3x^2 - 12$

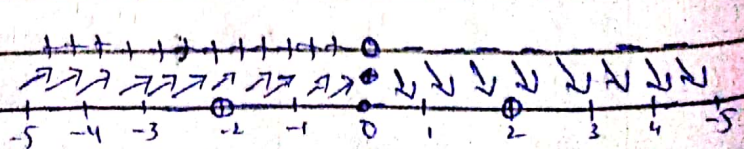
$0 = x^2 - 4$

$x^2 = 4$

$\boxed{x = \pm 2}$

critical num	left endpoint	Critical num	Right endpo
$f(-2) = 16$	$f(0) = 0$	$f(2) = -16$	$f(4) = 16$
$(-2, 16)$	$(0,0)$	$(2,-16)$	$(4,16)$
Maximum		Minimum	maximum

$f'(-3) = 63 > 0$,



$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

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3) $y = 3x^{2/3} - 2x$, $[-1, 1]$

1: $f(-1) = -5$, $f(1) = 1$

2: Differentiate y, then $y = 0$

$y' = 3 \times \frac{2}{3} \times x^{-\frac{1}{3}} - 2(1)$

$y' = \frac{2}{\sqrt[3]{x}} - \frac{2}{1}$

$y' = \frac{2 - 2\sqrt[3]{x}}{\sqrt[3]{x}}$

3: $y' = 0$

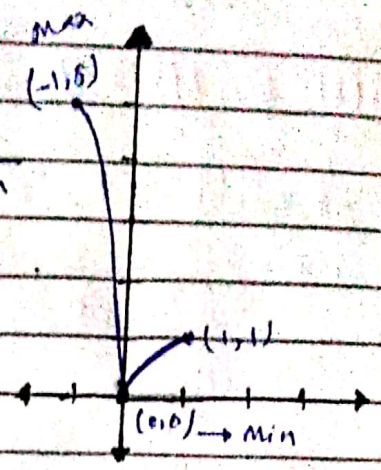
$0 = 2 - 2\sqrt[3]{x}$

$2\sqrt[3]{x} = 2$

$\sqrt[3]{x} = 1$

$x^{1/3} = 1$

$\boxed{x = 1}$

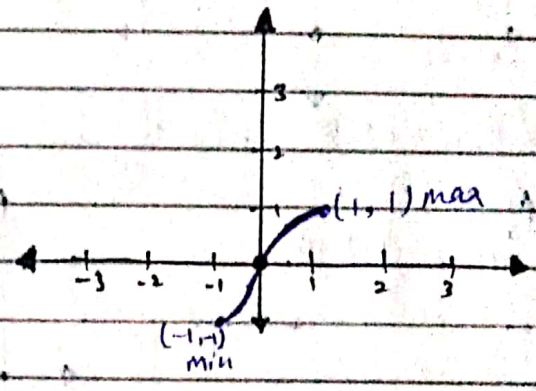


4) $g(x) = \sqrt[3]{x}$, $[-1, 1]$

1: $g(-1) = -1$, $g(1) = 1$

2: Differentiate $g(x)$

$g'(x) = \frac{1}{3\sqrt[3]{x^2}}$



5) $g(t) = \frac{t^2}{t^2+3}$, $[-1, 1]$

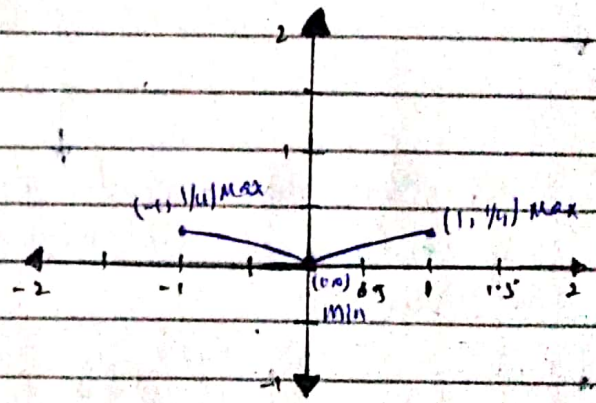
1: $g(-1) = 1/4$, $g(1) = 1/4$

2: $g'(t)$

$g'(t) = \frac{(t^2+3)(2t) - (t^2)(2t)}{(t^2+3)^2}$

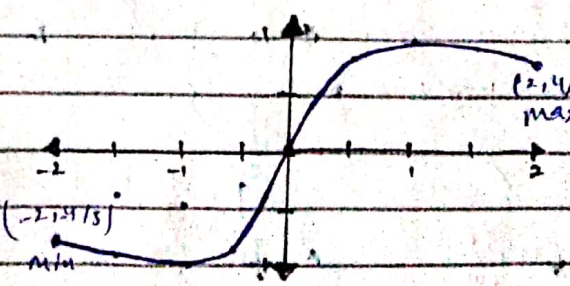
$g'(t) = \frac{2t^3 + 6t - 2t^3}{(t^2+3)^2}$

$g'(t) = \frac{6t}{(t^2+3)^2}$



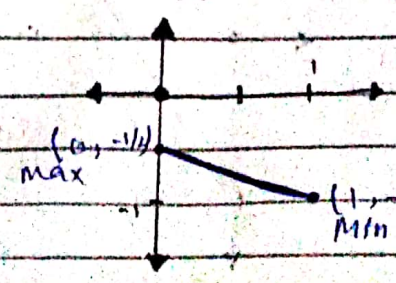
6) $f(x) = \frac{2x}{x^2+1}$, $[-2, 2]$

1: $f(-2) = -4/5$, $f(2) = 4/5$



7) $h(s) = \frac{1}{s-2}$, $[0, 1]$

1: $h(0) = -1/2$, $h(1) = -1$



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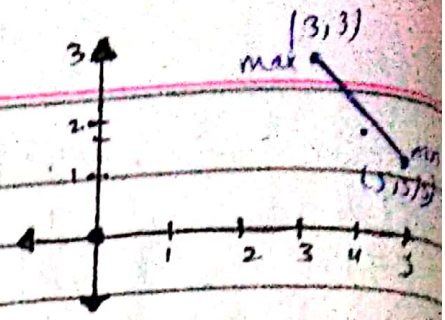
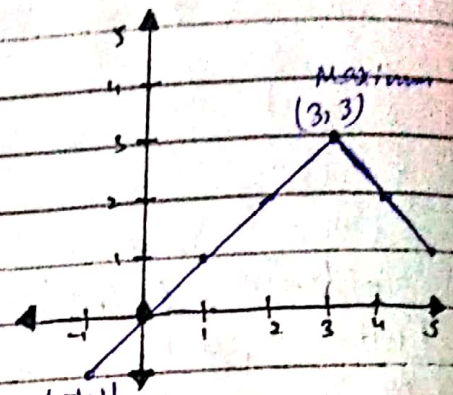
28) $h(t) = \dfrac{t}{t-2}$, $[3,5]$

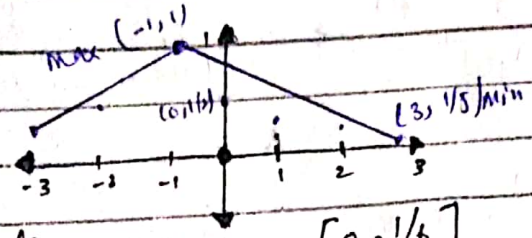
: $h(3) = 3$, $h(5) = 5/3$



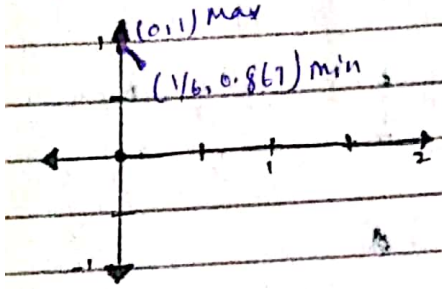
29) $y = 3 - |t-3|$, $[-1,5]$

30) $f(x) = \dfrac{1}{1+|x+1|}$, $[-3,3]$

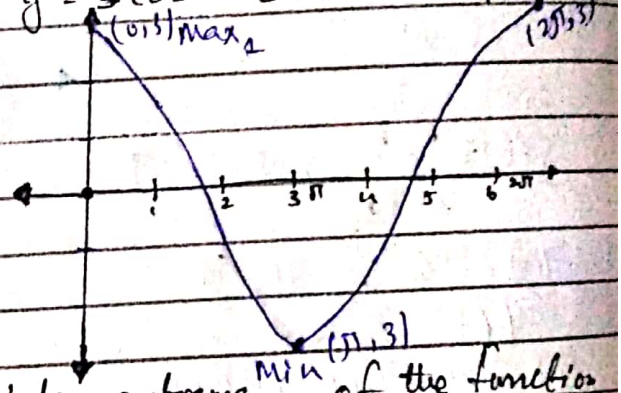




33) $f(x) = \cos \pi x$, $[0, 1/6]$



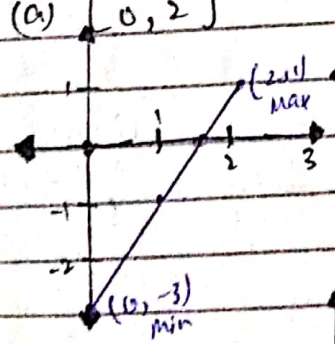
35) $y = 3\cos x$ $[0, 2\pi]$



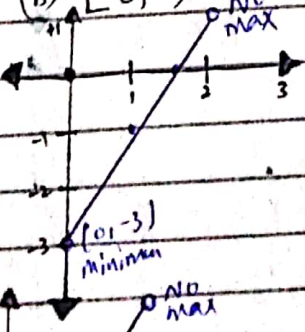
▶ 37–40, locate the absolute extrema of the function (if any exist) over each interval.

37) $f(x) = 2x - 3$

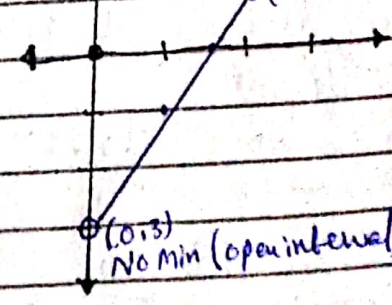
(a) $[0,2]$



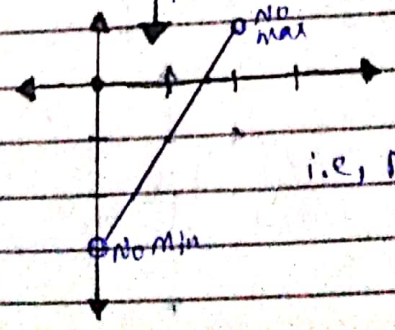
(b) $[0,2)$



(c) $(0,2]$



d) $(0,2)$



i.e, No Extremes

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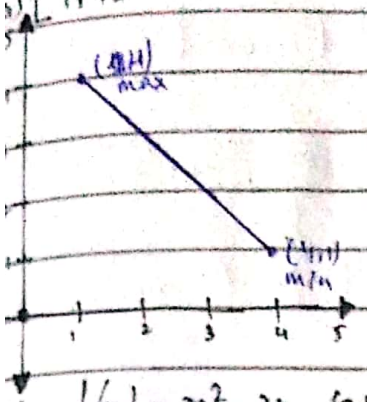
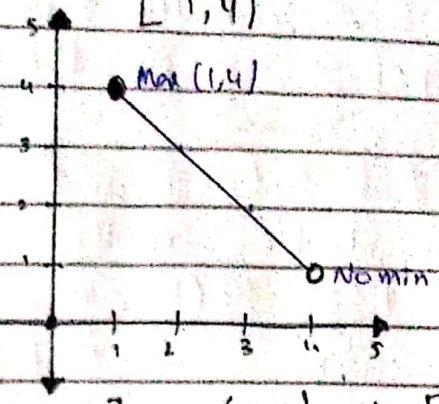
38) $f(x) = 5 - x$

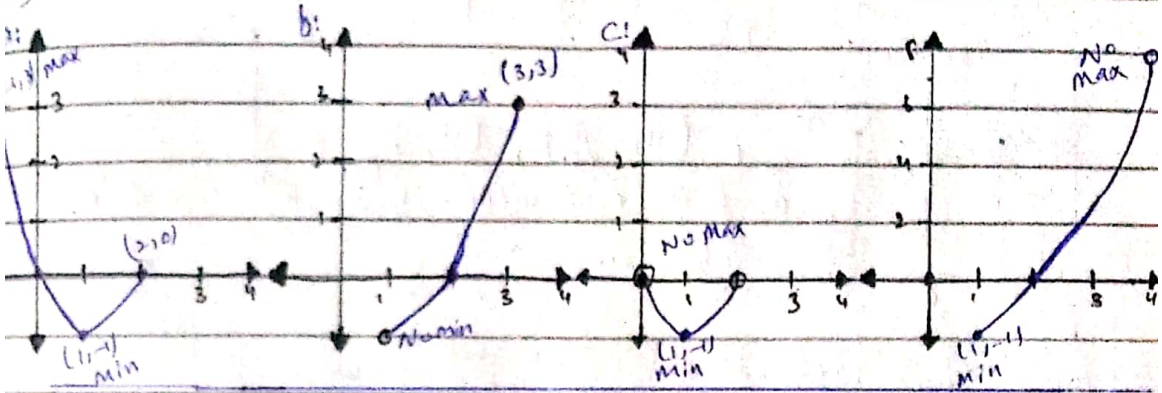
a) $[1,4]$



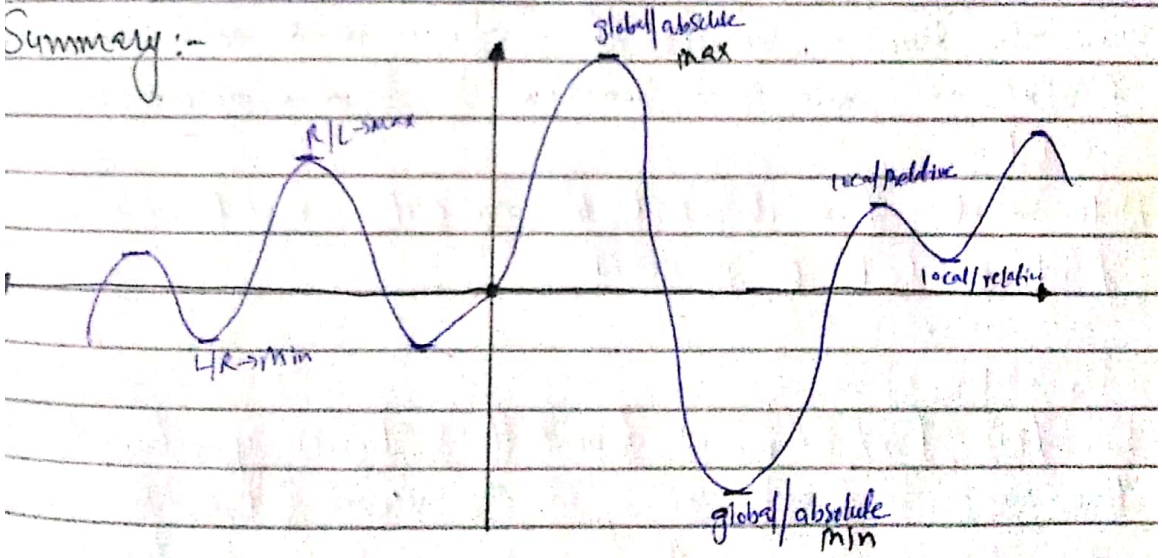
(b) $[1,4)$



39) $f(x) = x^2 - 2x$ (a) $[-1,2]$ (b) $(1,3]$ (c) $(0,2)$ (d) $[1,4)$



Summary :-



Global/Absolute → max/min → only one

Local/Relative → max/min → can be multiple

CRITICAL Numbers (mean critical points): Can occur at both local or global extremes.

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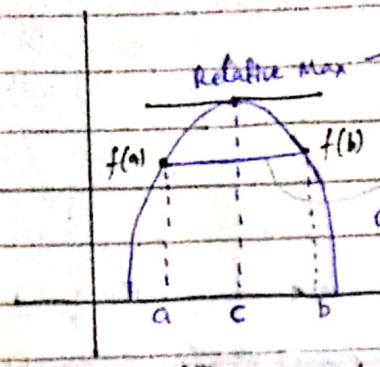
3.2 Rolle's Theorem and the mean value Theorem

- Understand and use Rolle's Theorem.
- Understand and use the Mean Value Theorem (MVT)

Rolle's Theorem

1: Let f be continuous on the closed interval $[a, b]$, and
2: Differentiable on the open interval (a, b)
3: If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.



(x_1, y_1), (x_2, y_2)
as f is continuous on $[a, b]$ and differentiable on (a, b).

Example ① Illustrating Rolle's Theorem

Find the two x-intercepts of $f(x) = x^2 - 3x + 2$ and show that $f'(x) = 0$ at some point between the two x-intercepts.

:- Solution

Note that f is differentiable on entire real line. Setting $f(x)$ equal to 0 produces

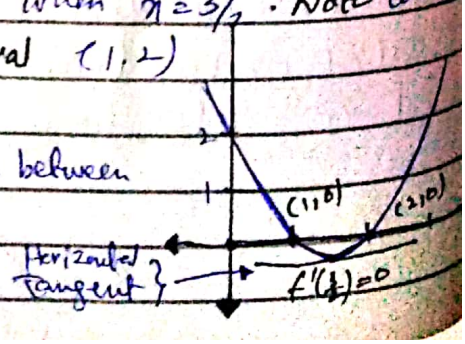
$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

So, $f(1) = f(2) = 0$, and from Rolle's Theorem you know that there exist at least one c in the interval $(1, 2)$ such that $f'(c) = 0$. To find such a c, you can solve the equation

$f'(x) = 2x - 3 = 0 \rightarrow$ set $f'(x)$ equal to 0

and determine that $f'(x) = 0$ when $x = 3/2$. Note that this x-value lies in open interval $(1, 2)$



: The x-value for which $f'(x) = 0$ is between the two x-intercepts

:- Rolle's Theorem states that if f satisfies the conditions of the theorem, there must be at least one point between a and b at which the derivative is 0. There may of course be more than one such point, as shown below:

Example ②

Let $f(x) = x^4 - 2x^2$. Find all values of c in the interval $(-2, 2)$ such that $f'(c) = 0$.

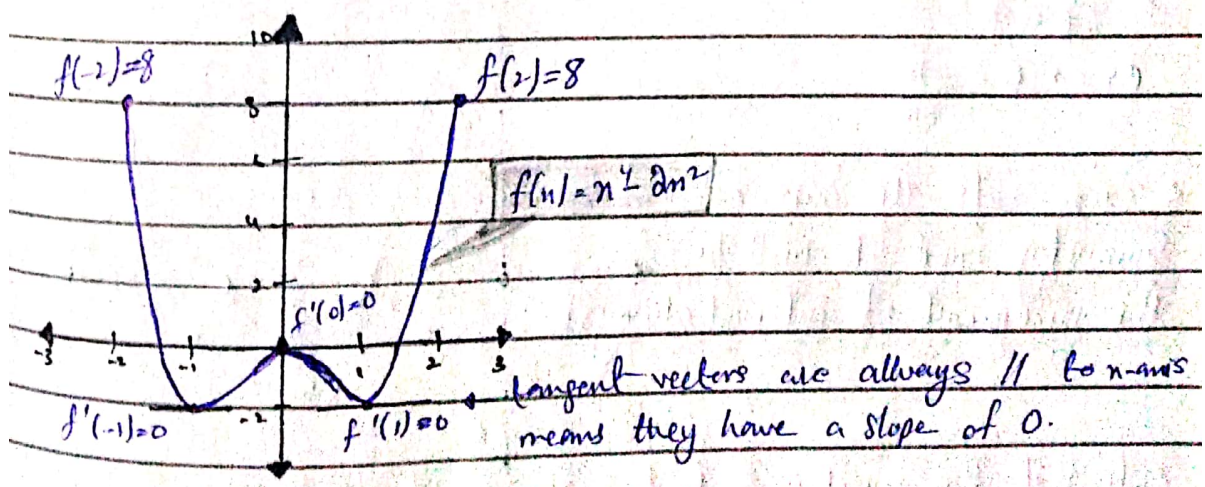
:- To begin, note that the function satisfies the conditions of Rolle's theorem. That is, f is continuous on interval $[-2, 2]$ and differentiable on the interval $(-2, 2)$. Moreover because $f(-2) = f(2) = 8$, you can conclude that there exists at least one c in $(-2, 2)$ such that $f'(c) = 0$. Setting the derivative equal to 0 produces

$$f'(x) = 4x^3 - 4x = 0$$

$$4x(x-1)(x+1) = 0$$

$$x = 0, 1, -1 \quad \text{x-values for which } f'(x) = 0$$

So, in the interval $(-2, 2)$, the derivative is zero at three different values of x, as shown below



$f'(x) = 0$ for more than one x-value in interval $(-2, 2)$

Example (3) given by Sir Abdul Rehman Soomro on (11/04/2023)

$f(x) = x^2 - 6x + 8$, $[2, 4]$

:- 1: Let f be Continuous on closed interval $[2,4]$ → True

2: Differentiable in open interval $(2, 4)$

3: If $f(a) = f(4)$, there is at least a number c in $(2,4)$ such that $f'(c) = 0$.

1: $f(x) = x^2 - 6x + 8$ is Continuous + defined on $[2,4]$

2: Differentiate

$f'(x) = 2x - 6$

f is differentiable in $(2, 4)$, if it would not be then Just write the statement and explain.

3: $f(2) = f(4)$

$f(2) = 2^2 - 6(2) + 8 = 4 - 12 + 8 = 0$

$f(4) = 4^2 - 6(4) + 8 = 16 - 24 + 8 = 0$

Find c in $(2,4)$ such that $f'(c) = 0$

$f(x) = x^2 - 6x + 8$

$f'(x) = 2x - 6$

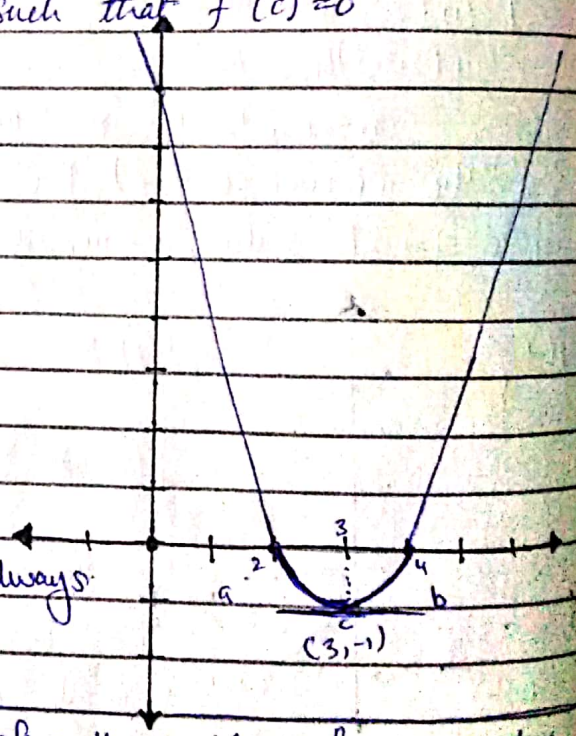
$f'(c) = 2c - 6$, $f'(c) = 0$

$0 = 2c - 6$

$\boxed{c = 3}$

or → $c = \frac{-b}{2a}$

c represents the axis of symmetry and it will be the midpoint of interval always.



Note that the function satisfies the Condition of Mean value Theorem. That is, f is Continuous on $[1, 4]$ and differentiable on $(1,4)$. So there exists a c in $(1,4)$, s.th $f'(c) = 1$. Solving the equation $f'(x) = 1$ yields

$f'(x) = \frac{4}{x^2} = 1$, which implies that $x = \pm 2$. So, in $(1,4)$, you can conclude that $c = 2$ as shown in graph →

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The Mean Value Theorem

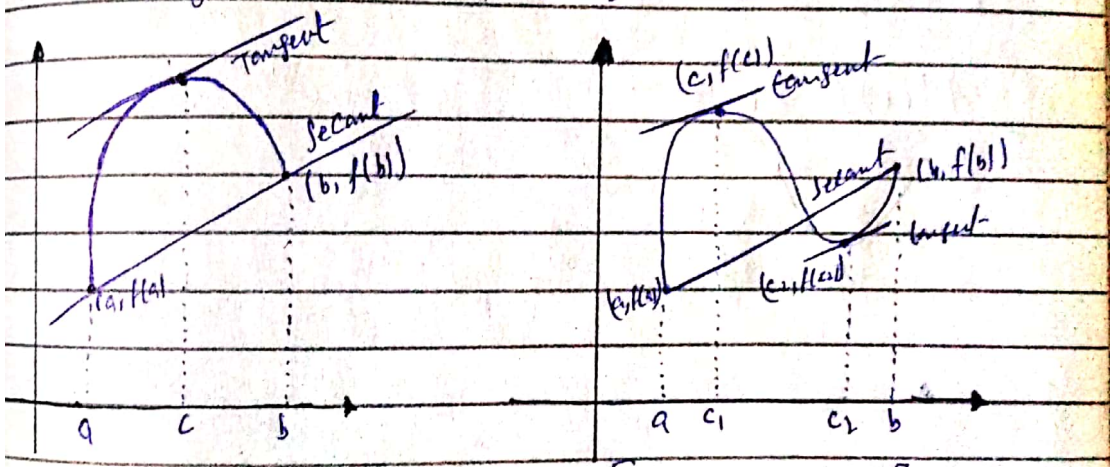
If f is continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The mean value theorem (MVT) claims the existance of a part "c" at which tangent line is // (parallel) to the secant line joining the $(a, f(a))$ and $(b, f(b))$.

Rolles: tangent line is // to x-axis.

MVT: tangent line is // to secant line.



Tangent lines are // to secant $[a, c_1, c_2, b]$

1: f is continuous

2: Differentiable on open interval

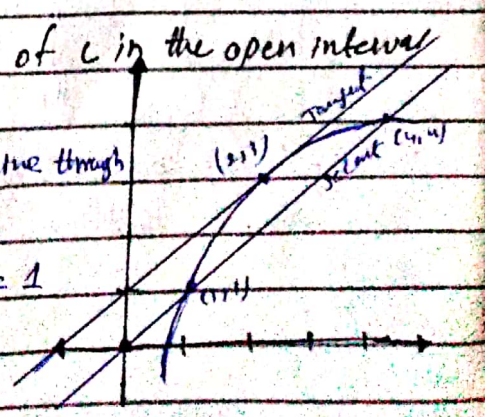
3: There exist a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Ex① Finding a tangent line

Given $f(x) = 5 - (4/x)$, find all values of c in the open interval $(1,4)$ such that

$f'(c) = \frac{f(4) - f(1)}{4-1}$. The slope of secant line through $(1, f(1))$ and $(4, f(4))$ is $\frac{f(4) - f(1)}{4-1} = \frac{4-1}{4-1} = 1$



Ex:- (5) $f(x) = x^2 - 3x + 2$, $[-2, 3]$

1: Continuous on $[-2, 3]$ ✓

2: Differentiable on $(2, 3)$

$f(x) = x^2 - 3x + 2$

$\boxed{f'(x) = 2x - 3}$

3: $f(c) = c^2 - 3c + 2$

$f'(c) = 2c - 3$

$f(a) = f(2) = (2)^2 - 3(2) + 2 = 4 + 6 + 2 = 12$

$f(b) = f(3) = (3)^2 - 3(3) + 2 = 9 - 9 + 2 = 2$

4: $\dfrac{f(b) - f(a)}{b - a} = \dfrac{2 - 12}{3 - (-2)} = \dfrac{-10}{5} = -2$

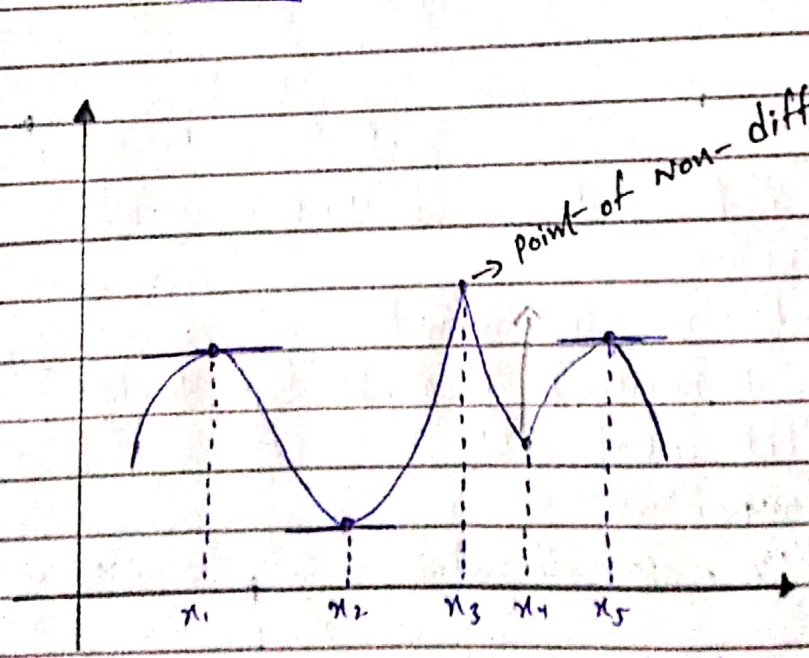
$f'(c) = \dfrac{f(b) - f(a)}{b - a} = -2$

$\Rightarrow f'(c) = 2c - 3$

$-2 = 2c - 3$

$2c = -2 + 3$

$\boxed{c = 1/2}$



points: x_1, x_2, x_3, x_4, x_5 are critical points

: x_1, x_2 and x_5 are stationary points

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Exercise 3.2

1-4, Explain why Rolle's Theorem doesn't apply to the function even though there exist a and b such that $f(a) = f(b)$

1) $f(x) = \left|\frac{1}{x}\right|$ $[-1, 1]$

: steps of Rolle's Theorem

1: f is continuous on closed interval $[a, b]$

2: Differentiable on open interval (a, b)

3: $f(a) = f(b)$, then there exists at least one number c in (a, b) such that $f'(c) = 0$.

Solution

1: $f(x) = \left|\frac{1}{x}\right|$ is discontinous at $x=0$, Rolle's Theorem can't be applied to $f(x)$.

2) $f(x) = \cot \frac{x}{2}$, $[\pi, 3\pi]$

1: f is continous on closed interval $[\pi, 3\pi]$

2: Differentiable on open interval $(\pi, 3\pi)$

$f'(x) = -\csc^2 \frac{x}{2}$

$f(a) = f(\pi) = \cot \pi/2 = \cot 90° = \frac{\cos 90}{\sin 90} = -0.50$

$f(b) = f(3\pi) = \cot 3\pi/2 = \cot 270° = \frac{\cos 270°}{\sin 270°} = -5.58$

3: $f(a) = f(b) \rightarrow \boxed{-0.501 \neq -5.58}$

Rolle's Theorem doesn't satisfy.

3) $f(x) = 1 - |x-1|$, $[0, 2]$

1: $f(x)$ is continuous on closed

2: f is not differentiable, because it is a absolute value function/ modulus function, it has a point of non-differentiable

4: $f(x) = \sqrt{(2 - x^{2/3})^3}$ $[-1, 1]$

1: f is defined but not continous, has a point of non-differentiablity and $f(a) \neq f(b)$ w.r.t "desmos utility"

5-8 find the two x-intercepts of the function f and show that $f'(x)=0$ at some point between the two x-intercepts.

5) $f(x)=x^2-x-2$

1: find two x-intercepts; 2: show that $f'(x)=0$ between $[-1,2]$

$x^2-x-2=0$

$x^2-2x+x-2=0$

$x(x-2)+1(x-2)=0$

$(x+1)(x-2)=0$

$\boxed{[-1,2]}$

→ Interval found

$f(x)=x^2-x-2$

$f'(x)=2x-1$

$0=2x-1$

$\boxed{x=1/2}$ → between $[-1,2]$

$\boxed{f'(x)=0 \text{ at } x=1/2}$ OR

$f'(1/2)=0$

6) $f(x)=x(x-3)$

1: find intercepts 2: show that $f'(x)=0$ in between $[0,3]$

$0=x(x-3)$

$\boxed{0=x}, \boxed{3=x}$

$[0,3]$

$f(x)=x^2-3x$

$f'(x)=2x-3$

$0=2x-3$

$\boxed{x=3/2}$ $\boxed{f'(3/2)=0}$

7) $f(x)=x\sqrt{x+4}$

1: find intercepts (two) 2: show that $f'(x)=0$, $[0,-4]$

$x\sqrt{x+4}=0$

$\sqrt{x+4}=0$, $\boxed{x=0}$

$x+4=0^2$

$\boxed{x=-4}$

$[0,-4]$

$f(x)=x\sqrt{x+4}$

$f'(x)=(\sqrt{x+4})(1)+(x)\left(\frac{1}{2\sqrt{x+4}}\right)$

$0=\frac{\sqrt{x+4}}{1}+\frac{x}{2\sqrt{x+4}}$

$2\sqrt{x+4}\,(0)=2(\sqrt{x+4})(\sqrt{x+4})+x$

$0=2(x+4)+x$

$\boxed{x=-8/3}$ $\boxed{f'(-8/3)=0}$

8) $f(x)=-3x\sqrt{x+1}$

1: two intercepts 2: show that $f'(x)=0$, in between $[-1,0]$

$0=-3x\sqrt{x+1}$

$\boxed{-3x=0}$, $\sqrt{x+1}=0$

$x=0$, $\boxed{x=-1}$

$[-1,0]$

$f(x)=-3x\sqrt{x+1}$

$f'(x)=(\sqrt{x+1})(-3)+(-3x)(1/2\sqrt{x+1})$

$0=-3\sqrt{x+1}+3x/2\sqrt{x+1}$

$2(\sqrt{x+1})(0)=-3(\sqrt{x+1})\times 2(\sqrt{x+1})-3x$

$0=-6(x+1)-3x$

$\boxed{x=-2/3}$ $\boxed{f'(-2/3)=0}$

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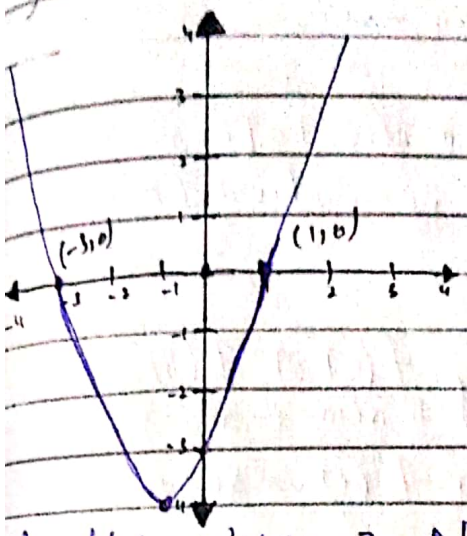
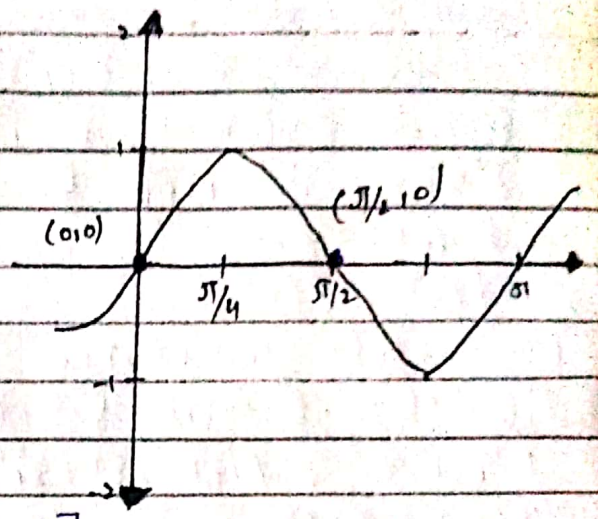
Rolle's Theorem → 8 and 10, the graph of f is shown.

Apply Rolle's Theorem and find all values of c such that $f'(c)=0$ at some point between the labeled intercepts

8) $f(x) = x^2+2x-3$ 10) $f(x) = \sin 2x$





8: $f(x) = x^2+2x-3 \rightarrow [-3, 1]$

1: $f(x)$ is continuous on closed interval $[-3, 1]$

2: $f(x)$ is differentiable on open interval $(-3, 1)$

$f(x) = x^2+2x-3 \rightarrow f'(x) = 2x+2$

3: $f(a) = f(-3) = -3^2+2(-3)-3 = 9-6-3 = 0$

$f(b) = f(1) = 1^2+2(1)-3 = 1+2-3 = 0$

$f(a) = f(b)$ satisfied.

$f'(x) = 0$ all three conditions of Rolle's theorem satisfied

$0 = 2x+2$ and we have point where $f'(x)=0$.

$\boxed{x=-1}$ $\boxed{f'(-1)=0}$

10: $f(x) = \sin 2x \rightarrow [0, \pi/2]$

1: $f(x)$ is continuous on closed interval $[0, \pi/2]$

2: $f(x)$ is differentiable on open interval $(0, \pi/2)$

$f'(x) = \cos 2x$

3: $f(a) = f(0) = \sin 2(0) = \sin(0) = 0$

$f(b) = f(\pi/2) = \sin(90°) = 1$

$f(a) \neq f(b) \rightarrow$ theorem doesn't satisfy

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11–24 determine whether Rolle's Theorem can be applied to f on the closed interval $[a, b]$. If Rolle's theorem can be applied, find all values of c in the open interval (a, b) such that $f'(c) = 0$. If Rolle's Theorem cannot be applied, explain why not.

11) $f(x) = -x^2 + 3x$, $[0, 3]$

1:- $f(x)$ is continuous on closed interval $[0, 3]$

2: $f'(x)$ is differentiable on open interval $(0, 3)$

$f'(x) = -2x + 3$

3: $f(a) = f(b)$

$f(a) = f(0) = -(0)^2 + 3(0) = 0$

$f(b) = f(3) = -3^2 + 3(3) = 9 + 9 = 18$

$f(a) \neq f(b)$ Theorem doesn't satisfy

$f'(c) = 0$, $f(c) = -c^2 + 3c \rightarrow f'(c) = -2c + 3$

4: $f'(c) = -2c + 3$

$0 = -2c + 3$

$\boxed{c = 3/2}$

$f'(3/2) = 0$

12) $f(x) = x^2 - 5x + 4$, $[1, 4]$

1: $f(x)$ is continuous on closed interval $[-1, 4]$

2: $f'(x)$ is differentiable open interval. $(-1, 4)$

$f'(x) = 2x - 5$

3: $f(a) = f(b)$

$f(a) = f(-1) = -1^2 - 5(-1) + 4 = 1 + 5 + 4 = 10$

$f(b) = f(4) = 4^2 - 5(4) + 4 = 16 - 20 + 4 = 0$

$f(a) \neq f(b)$

4: $f'(c) = 0 \rightarrow f(c) = c^2 - 5c + 4 \rightarrow f'(c) = 2c - 5$

$f'(c) = 2c - 5$

$0 = 2c - 5 \rightarrow \boxed{c = 5/2}$ $\boxed{f'(5/2) = 0}$

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$\sin 1 \neq \sin 1^\circ$

Trigonometric Identities

① $\sin^2\theta + \cos^2\theta = 1$ ② $1 + \tan^2\theta = \sec^2\theta$

③ $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

Fundamental Law

① $\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$

Deduction from fundamental Law

① $\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

② $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

③ $\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$

④ $\tan(\alpha + \beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$

⑤ $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$

⑥ $\cot(\alpha + \beta) = \dfrac{\cot\alpha \cot\beta - 1}{\cot\alpha + \cot\beta}$

⑦ $\cot(\alpha - \beta) = \dfrac{\cot\alpha \cos\beta + 1}{\cot\alpha - \cot\beta}$

▸ **Componendo and Dividendo theorem**

if $\dfrac{p}{q} = \dfrac{a}{b}$ then $\dfrac{p+q}{p-q} = \dfrac{a+b}{a-b}$

$a = 1,\ b = \sqrt{3},\ c = 2,\ d = \sqrt{2}$

	30°	60°	45°
Sin	a/c	b/c	a/d
Cos	b/c	a/c	a/d
tan	a/b	b/a	a
Cot	b/a	a/b	a
Sec	c/b	c/a	d
csc	c/a	c/b	d

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Product to Sum formula

1) $\sin\alpha \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

2) $\cos\alpha \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$

$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

3) $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$

$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$

4) $\sin\alpha\sin\beta = -\frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]$

$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

Sum to product formula:-

1) $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

2) $\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

3) $\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

4) $\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

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Periods of trigonometric functions

2π is the period of

$\sin\theta$, $\cos\theta$, $\sec\theta$, $\operatorname{cosec}\theta$

π is the period of

$\tan\theta$, $\cot\theta$

2.7 Combining functions pre-james-text-11 pg 210

Sums, differences, products, and Quotients.

- ALGEBRA of Functions

Let f and g be functions with Domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows.

1) $(f+g)(x) = f(x) + g(x)$

2) $(f-g)(x) = f(x) - g(x)$

3) $(fg)(x) = f(x)g(x)$

4) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$\boxed{(f \circ g)(x) = f(g(x))}$

Continuity of a Composite func

▸ Example 1

Let $f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x}$

(a) find the functions $f+g$, $f-g$, fg and f/g.

- $(f+g)(x) = f(x) + f(g) = \frac{1}{x-2} + \sqrt{x}$
- $(f-g)(x) = f(x) - f(g) = \frac{1}{x-2} - \sqrt{x}$
- $(fg)(x) = (f \circ g)(x) = f(x)g(x) = \frac{1}{x-2} \times \sqrt{2} = \frac{\sqrt{2}}{x-2}$
- $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{1}{x-2} \div \sqrt{x} = \frac{1}{(x-2)\sqrt{x}}$

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(b) Each of these values exist because $x=4$ is in the domain of each function:

$f(x) = \frac{1}{x-2}$ and $g(x) = \sqrt{x}$

- $(f+g)(4) = f(4) + g(4) = \frac{1}{(4)-2} + \sqrt{4} = \frac{1}{2} + 2 = \frac{5}{2}$
- $(f-g)(4) = f(4) - g(4) = \frac{1}{(4)-2} - \sqrt{4} = \frac{1}{2} - 2 = \frac{3}{2}$
- $(f \circ g)(4) = f(4)g(4) = \frac{1}{(4)-2} \times \sqrt{4} = \frac{1}{2} \times 2 = 2 = 1$
- $\left(\frac{f}{g}\right)(4) = \frac{f(4)}{g(4)} = \frac{1}{(4)-2} \div \sqrt{4} = \frac{1}{2} \div 2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Finding Asmptots of Rational function.

let r be the Rational function

$$r(x) = \frac{a_n x^n + a_{n-1} x^{n-1} \cdots\cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} \cdots\cdots + b_0}$$

1) The vertical asymptote(s) of r are the lines $x=a$, where a is a zero of the Denominator.

2) (a) if $\boxed{n<m}$, then r has horizonal asymptote $y=0$

(b) if $\boxed{n=m}$, then r has horizontal asymptote $y = \frac{a_n}{b_m}$

(c) if $\boxed{n>m}$, then r has no horizontal asymptote.

Example 4 :- Asymptote of a Rational Function

Find the vertical and Horizontal asymptotes of

$$r(x) = \frac{3x^2-2x-1}{2x^2+3x-2}$$

Solution

$$r(x) = \frac{3x^2-2x-1}{2x^2+3x-2} = \frac{3x^2-2x-1}{(2x-1)(x+2)}$$

factors = vertical asymptote = $\boxed{x = 1/2}$ and $\boxed{x=-2}$

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Horizontal Asymptote:- The degrees of numerator and Denominator are the same,

$$r(x) = \frac{3x^2-2x-1}{(2x^2+3x-2)} = \frac{a_n x^n + a_{n-1}x^{n-1} \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} \dots + b_0}$$

if $n = m$, then $\rightarrow$ Horizontal asymptote $= \frac{a_n}{b_n}$

$a_n = 3$, $b_n = 2$, Horizontal Asymptote $= 3/2$

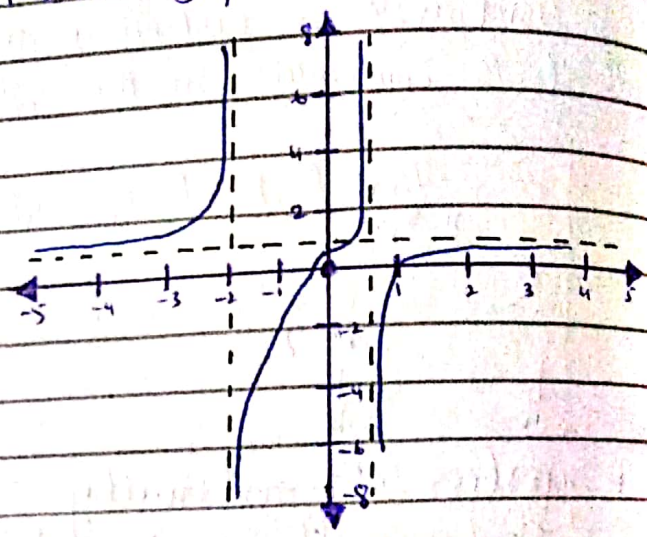
Graph

x-intercept $= 1/2$ and -2

y-intercept $= 1/2$

V-asymptote $= 1/2$ and -2

H-asymptote $= 3/2$



▸ Properties of LIMITS

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

1. Scalar multiple : $\lim_{x \to c} [b f(x)] = bL$

2. Sum or difference : $\lim_{x \to c} [f(x) \pm g(x)] = L \pm K$

3. product : $\lim_{x \to c} [f(x) g(x)] = LK$

4. Quotient : $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{K}$, $K \neq 0$

5. Power : $\lim_{x \to c} [f(x)]^n = L^n$

▸ Limit of Trigonometric functions

1. $\lim_{x \to c} \sin x = \sin c$
2. $\lim_{x \to c} \cos x = \cos c$
3. $\lim_{x \to c} \tan x = \tan c$
4. $\lim_{x \to c} \cot x = \cot c$
5. $\lim_{x \to c} \sec x = \sec c$
6. $\lim_{x \to c} \csc x = \csc c$

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The existence of a LIMIT

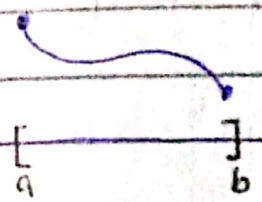
The limit of $f(x)$ as x approaches c is L iff

$$\lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L$$

Defination of Continuity on a closed Interval

A function f is continuous on the closed interval $[a, b]$ if it is continuous on the open interval (a, b) and

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x \to b^-} f(x) = f(b)$$



Properties of Continuity

1. Scalar multiple : bf
2. Sum or difference : $f \pm g$
3. Product : fg
4. Quotient : $\frac{f}{g}$, if $g(c) \neq 0$

Continuity of a Composite function

$$(f \circ g)(x) = f(g(x))$$

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Square root

$$\sqrt{27} = \sqrt{25+2} \rightarrow 5 + \frac{2}{2\times 5} = 5 + \frac{2}{10} = 5 + \frac{1}{5} = \frac{26}{5}$$

$$\sqrt{17} = \sqrt{16+1} = 4 + \frac{1}{2\times 4} = 4 + \frac{1}{2\times 4} = 4 + \frac{1}{8} = \frac{33}{8}$$

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جو لوگ سوچیں گے وہ کم سوچیں تو
لوگ کیا سوچیں گے۔

ਕਤ ਇਤ ਲਗੀ ਮਾਂ ਕਿਛ ਰਵਤਾ
ਤਬ ਲਗ ਗਾਵਤ ਖੇਲ ਮਾਂ ਡੂਬਤਾ

ਕਮ ਬਮਤ ਹੈ ਪਾਛੇ ਪਾਏ, ਏਕ ਬਮਤ ਕਾਵਲ ਸਿਖਰ ਮਹਾਏ
ਏਕਹਿ ਕਾ ਹੈ ਕਮਤਿ ਗਤਿ ਹੈ, ਤਬ ਮੂਰਤ ਰਤ ਰਾ ਰਹੈ
ਜਿਮ ਠਾਕੁਰ ਸਿਰ ਲਾਗੀ ਚਾਤ, ਤਾ ਕੇ ਰੀਸੈ ਮਰਾ ਕਮਮਰਾਤ

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