ES 691
Mathematics for Machine Learning

with

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@

GIKI - FES

So Here's the Story So Far…

We ventured into the Multiverse of Mathematics…

Multiverse & Mathematics

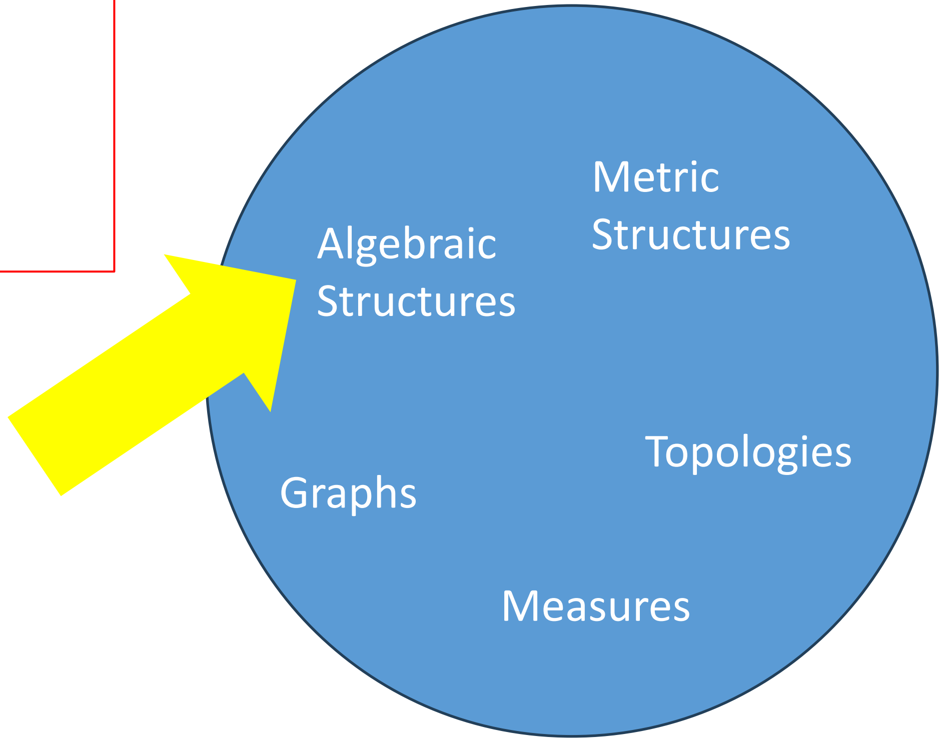
"Multiverse" – many universes, each with its own set of natural laws and objects.

Mathematics also likes to create its own "universes", each with set[s] of objects and "laws" (axioms)

...and got interested in a sub-Multiverse of "Algebraic Structures"

Some Mathematical Multiverses ("Structures")

Mathematical Structure = A Set of Objects with Some Features/Laws (operation, relation, distance etc.)



Algebraic Structure

Which were…

- A Non-Empty Set,
- With a Selection of Operations on the Set,
- And a Finite Set of Axioms ("laws") that the Operations must follow.

Each choice leads to a different structure ("universe")

Set
A

Operations
+, ×

Axioms
Commutative, Associative, Existence of Identity…

Set
R

Operations
$*, \otimes$

Axioms
Distributive, Associative, Existence of Inverse…

We defined a particular Algebraic Structure…

Algebraic Structure – Group

A Set (of objects) **with a Set** (of rules)…

Axioms

Definition 2.7 (Group). Consider a set $\mathcal{G}$ and an operation $\otimes : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ defined on $\mathcal{G}$. Then $G := (\mathcal{G}, \otimes)$ is called a *group* if the following hold:

1. *Closure* of $\mathcal{G}$ under $\otimes$: $\forall x, y \in \mathcal{G} : x \otimes y \in \mathcal{G}$
2. *Associativity:* $\forall x, y, z \in \mathcal{G} : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3. *Neutral element:* $\exists e \in \mathcal{G}\, \forall x \in \mathcal{G} : x \otimes e = x$ and $e \otimes x = x$
4. *Inverse element:* $\forall x \in \mathcal{G}\, \exists y \in \mathcal{G} : x \otimes y = e$ and $y \otimes x = e$, where e is the neutral element. We often write x^{-1} to denote the inverse element of x.

If, in addition, Commutative Property also holds, then G is called an **Abelian Group**.

$$\forall x, y \in \mathcal{G} : x \otimes y = y \otimes x,$$

But the Algebraic Structure we were really interested in was…

The Algebraic Structure We Are Interested In…

Vector Space

A Set (of vectors) **with a Set** (of inner rules)
and a Set (of outer rules)…

Boromir warned us…

So we took a detour and played…

A Game of Knowns (and Unknowns)!

Revisiting critical notions such as…

- Systems of Linear Equations
  - Overdetermined vs. Underdetermined
  - Consistent vs. Inconsistent
  - Unique Solution vs. Infinite Solutions vs. No [Unique] Solution
  - Linear Independence and True Worth (kind of “rank”) of a System
  - Some Ways of Deciding Between Multiple Solutions (Least Squares etc.)
- Concepts of Linear Combinations and Euclidean Distance
- Matrix Formulation of Linear Equations and
  - Matrix operations
  - Identities
  - Inverse

And now we feel finally ready…

Algebraic Structure – Vector Space

Algebraic Structure – Vector Space

A Set (of vectors) **with a Set** (of inner rules) **and a Set** (of outer rules)…

Algebraic Structure – Vector Space

A Set (of vectors) **with a Set** (of inner rules) **and a Set** (of outer rules)…

Definition 2.9 (Vector Space). A real-valued *vector space* $V = (\mathcal{V}, +, \cdot)$ is a set $\mathcal{V}$ with two operations

(inner operation +) $$+ : \mathcal{V} \times \mathcal{V} \to \mathcal{V} \tag{2.62}$$

(outer operation ·) $$\cdot : \mathbb{R} \times \mathcal{V} \to \mathcal{V} \tag{2.63}$$

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1. $(\mathcal{V}, +)$ is an Abelian group
2. Distributivity:
   1. $\forall \lambda \in \mathbb{R}, \boldsymbol{x}, \boldsymbol{y} \in \mathcal{V} : \lambda \cdot (\boldsymbol{x} + \boldsymbol{y}) = \lambda \cdot \boldsymbol{x} + \lambda \cdot \boldsymbol{y}$
   2. $\forall \lambda, \psi \in \mathbb{R}, \boldsymbol{x} \in \mathcal{V} : (\lambda + \psi) \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x} + \psi \cdot \boldsymbol{x}$

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Q. Does this definition require any closure under scalar multiplication?

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Implies closure under outer operation also.

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Vector Space is where Linear Algebra Lives…

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…With Some Rules

Algebraic Structure – Vector Space

Example 1

- $\mathcal{V} = \mathbb{R}^n, n \in \mathbb{N}$ is a vector space with operations defined as follows:
  - Addition: $\boldsymbol{x}+\boldsymbol{y} = (x_1, \ldots, x_n)+(y_1, \ldots, y_n) = (x_1+y_1, \ldots, x_n+y_n)$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$
  - Multiplication by scalars: $\lambda\boldsymbol{x} = \lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$ for all $\lambda \in \mathbb{R}, \boldsymbol{x} \in \mathbb{R}^n$

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$\boldsymbol{x} \in V$ are called *vectors*.

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Additive Inverse for $(V, +)$

$$\forall\, \boldsymbol{x} \in \boldsymbol{V}$$
$$\exists -\boldsymbol{x} \in \boldsymbol{V}:$$
$$\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$$

Since, by definition, values $x_1, x_2 \ldots \in \boldsymbol{R}^n$

Algebraic Structure – Vector Space

Example 2

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HW:
- What is the identity element?
- Does this require inverses to exist?

Algebraic Structure – Vector Space

Example 3

Could the set P be a Vector Space?

$$P := \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \, a, b \geq 0 \right\}$$

Algebraic Structure – Vector Space

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E.g., for any $\begin{bmatrix} x \\ y \end{bmatrix} \in P$, we have $\begin{bmatrix} -x \\ -y \end{bmatrix} \notin P$

Algebraic Structure – Vector Space

Example 4

Btw, Vector Spaces are Not Limited to Vectors

Set of All Differentiable Functions

$$\left\{ f : \Re \to \Re \;\middle|\; \frac{d}{dx} f \text{ exists} \right\}$$

Algebraic Structure – Vector Space

Example 4

Btw, Vector Spaces are Not Limited to Vectors

Set of All Differentiable Functions

$$\left\{ f : \mathfrak{R} \rightarrow \mathfrak{R} \;\middle|\; \frac{d}{dx} f \text{ exists} \right\}$$

Closure:

Algebraic Structure – Vector Space

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Neutral Element:

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$$\left(\frac{d}{dx}(cf) = c\frac{d}{dx} f\right).$$

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HW:
Show that the rest of the vector space properties are inherited from addition and scalar multiplication in $\boldsymbol{R}$

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Vector Subspace

A **Subset** of a Vector Space that Qualifies as a Vector Space

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Vector Space is where Linear Algebra Lives (with Some Rules)

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Vector Space is where Linear Algebra Lives (with Some Rules)

Rooms in a Vector Space where all the Rules of a Vector Space are Followed.

Vector **Subspace**

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Vector Space is where Linear Algebra Lives (with Some Rules)

Unlike real rooms, these can be overlapping.

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Vector Space is where Linear Algebra Lives (with Some Rules)

Unlike real rooms, these can be overlapping.

Rooms in a Vector Space where all the Rules of a Vector Space are Followed.

Then there are rooms (subsets) where rules of Vector Space are not followed, and these are not a Subspace.

Vector **Subspace**

A **Subset** of a Vector Space that Qualifies as a Vector Space

Definition 2.10 (Vector Subspace). Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{U} \subseteq \mathcal{V}$, $\mathcal{U} \neq \emptyset$. Then $U = (\mathcal{U}, +, \cdot)$ is called *vector subspace* of V (or *linear subspace*) if U is a vector space with the vector space operations $+$ and $\cdot$ restricted to $\mathcal{U} \times \mathcal{U}$ and $\mathbb{R} \times \mathcal{U}$. We write $U \subseteq V$ to denote a subspace U of V.

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HW:
Show that properties such as Associativity, Commutativity, Distributivity would already hold for $\boldsymbol{U}$ (hint: keep in mind that every $\boldsymbol{x} \in \boldsymbol{U}$ also satisfies $\boldsymbol{x} \in \boldsymbol{V}$).

Vector Subspace

A **Subset** of a Vector Space that Qualifies as a Vector Space

As long as $\boldsymbol{U} \subseteq \boldsymbol{V}$ and $\boldsymbol{V}$ is a Vector Space, we only need to show the following to prove that $\boldsymbol{U}$ is a Subspace of $\boldsymbol{V}$.

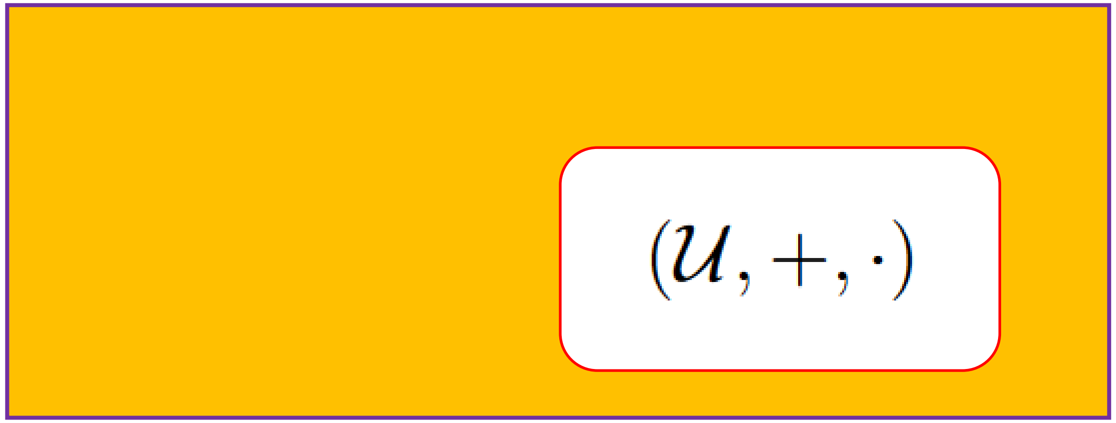
Vector Subspace

A **Subset** of a Vector Space that Qualifies as a Vector Space

As long as $\boldsymbol{U} \subseteq \boldsymbol{V}$ and $\boldsymbol{V}$ is a Vector Space, we only need to show the following to prove that $\boldsymbol{U}$ is a Subspace of $\boldsymbol{V}$.

1. $\mathcal{U} \neq \emptyset$, in particular: $\mathbf{0} \in \mathcal{U}$

$(\mathcal{V}, +, \cdot)$



Vector Subspace

A **Subset** of a Vector Space that Qualifies as a Vector Space

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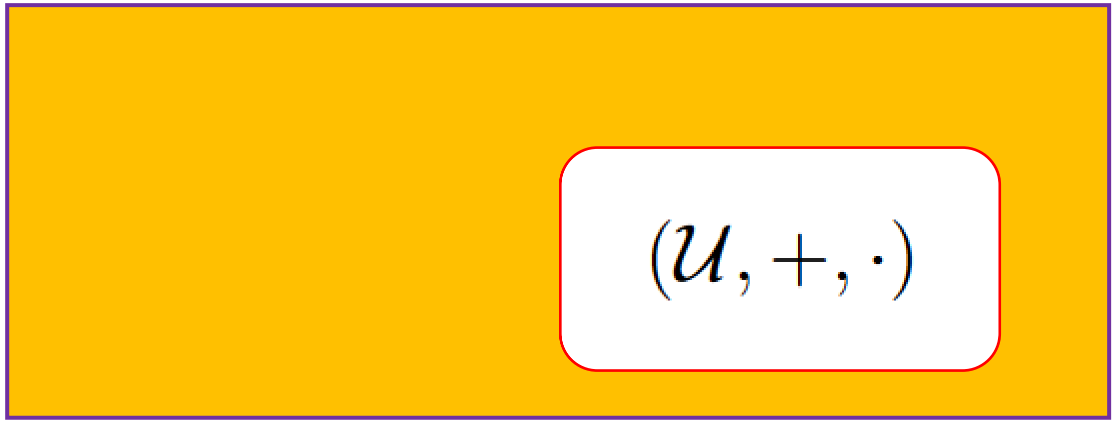

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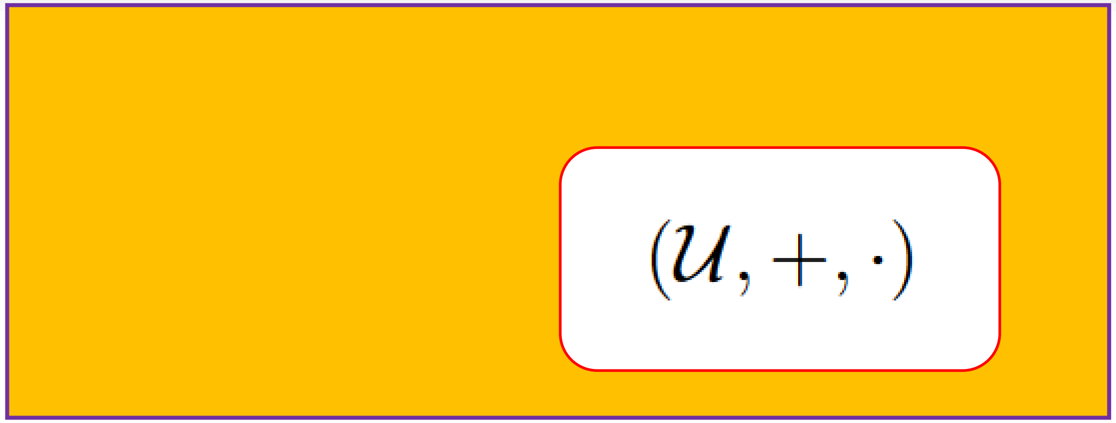

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This also ensures existence of inverse element $-\boldsymbol{x} \in \boldsymbol{U}, \forall\, \boldsymbol{x} \in \boldsymbol{U}$

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Subspaces are our key tool of analysis, determining solutions sets, transformations, and compression in working with matrices and vectors!

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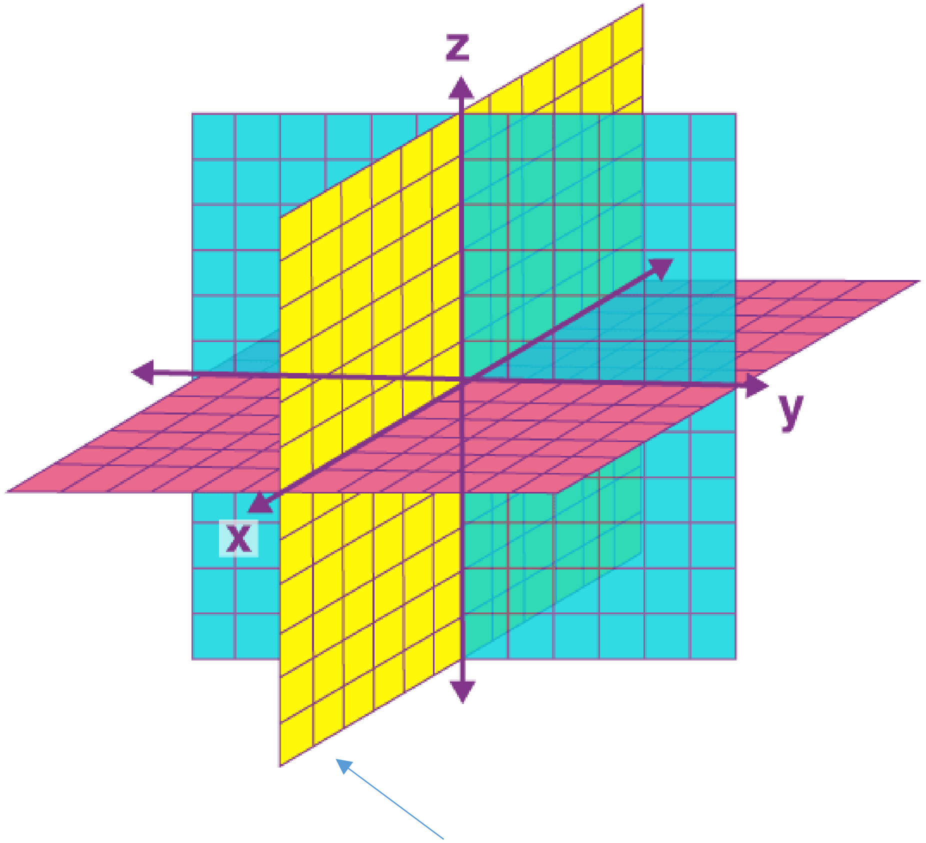

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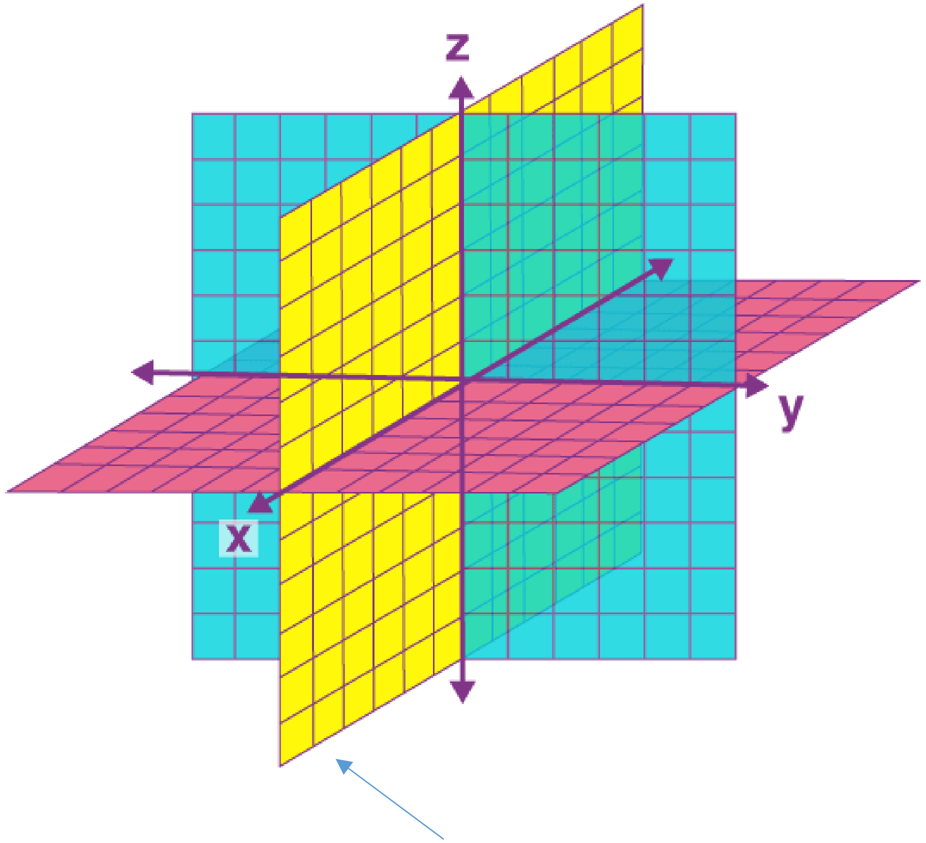

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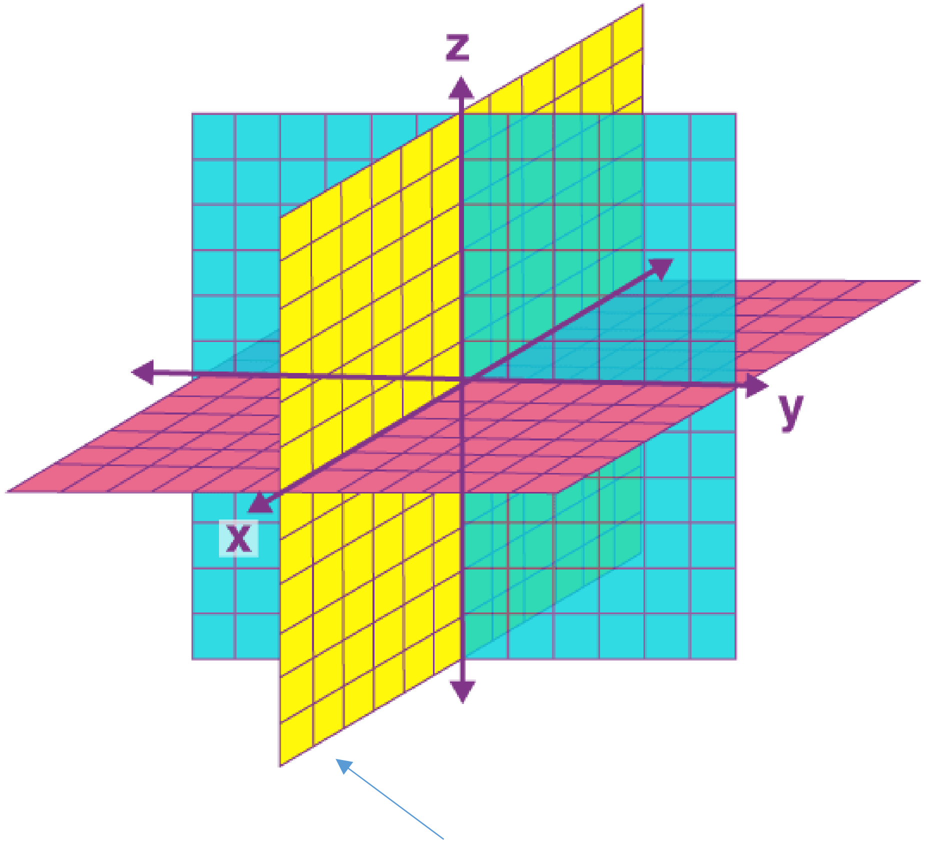

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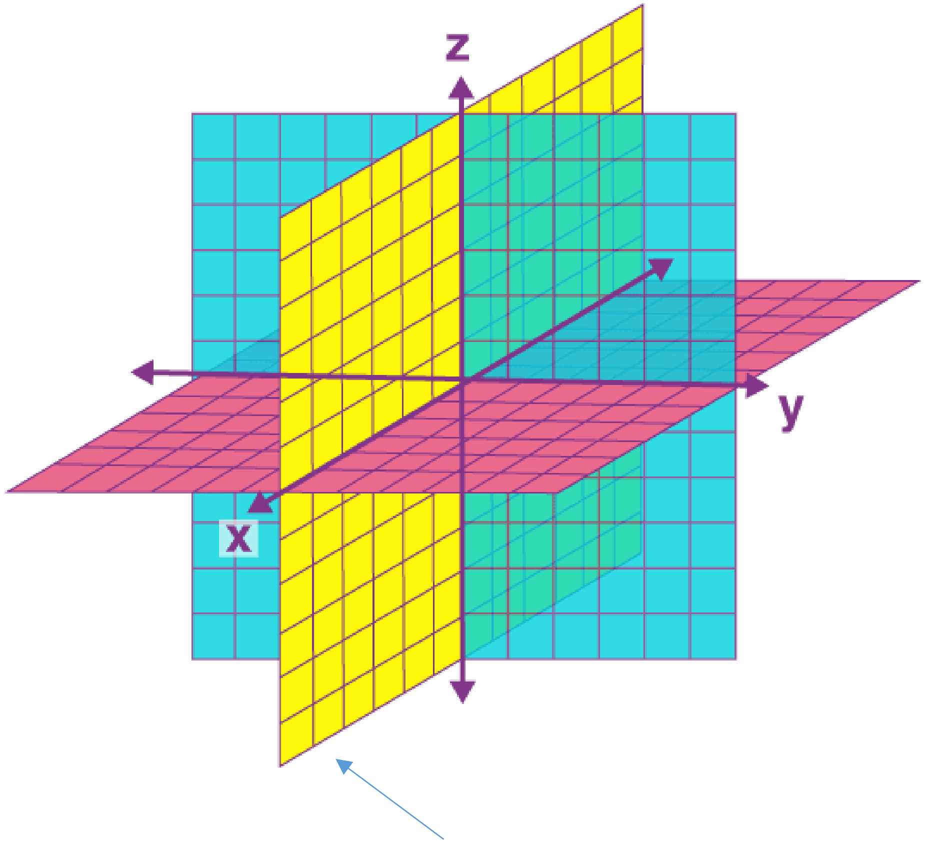

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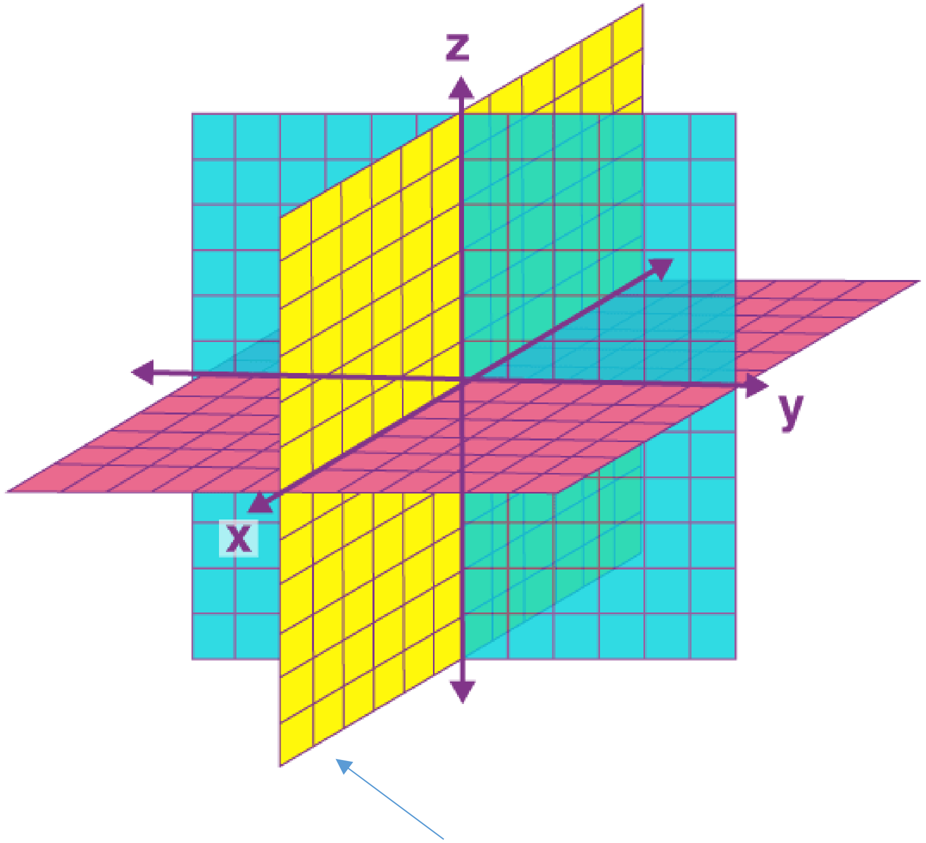

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- Also, we know now that if $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are solutions, then so are $4\boldsymbol{x_1} + 5\boldsymbol{x_2}$, $10\boldsymbol{x_2}$ etc. (by properties of subspace).

Vector Subspace

Example 1

Verify that the set of all real solutions to the following linear system is a subspace of $\mathbb{R}^3$:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0, \\ 2x_1 + 5x_2 - 4x_3 &= 0. \end{aligned}$$

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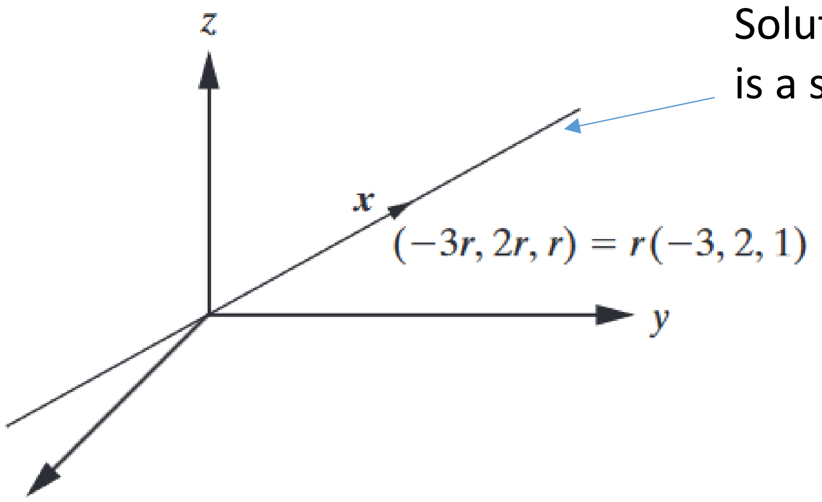
Using tools of basic linear algebra (e.g., Gaussian Elimination) it should be possible to show that the solution set is

$$S = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} = (-3r, 2r, r),\ \ r \in \mathbb{R}\},$$



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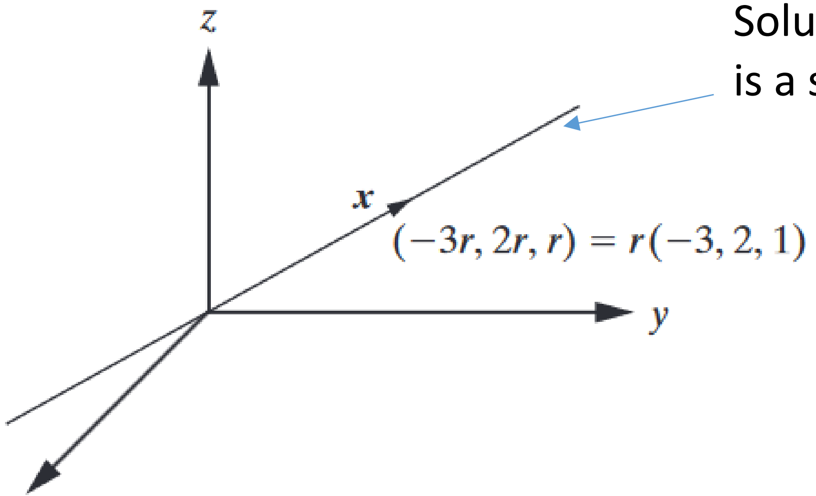
HW:
Show this set meets all requirements of being a subspace of $\boldsymbol{R}^3$.



Solution set which is a subspace of $\boldsymbol{R}^3$

Vector Subspace

Example 2

The set $\{0\}$ containing only the zero vector

Vector Subspace

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The set $\{0\}$ containing only the zero vector is a subspace of $\mathbb{R}^n$:

Vector Subspace

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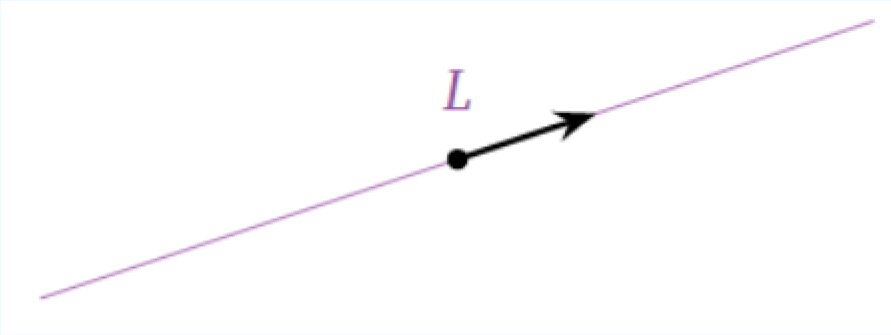
The set $\{0\}$ containing only the zero vector is a subspace of $\mathbb{R}^n$:

it contains zero, and if you add zero to itself or multiply it by a scalar, you always get zero.

Vector Subspace

Example 3

A line L through the origin is a subspace.



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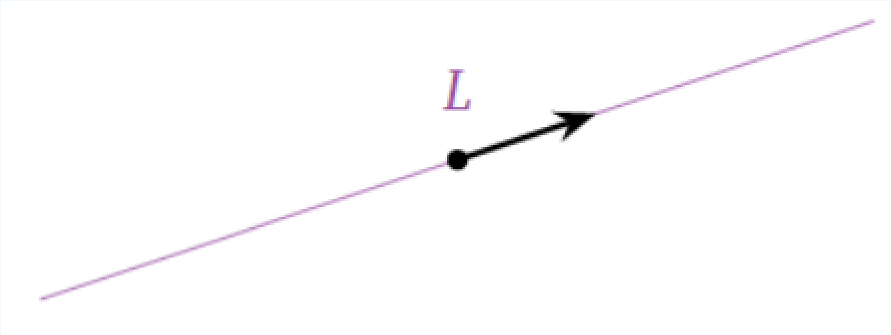


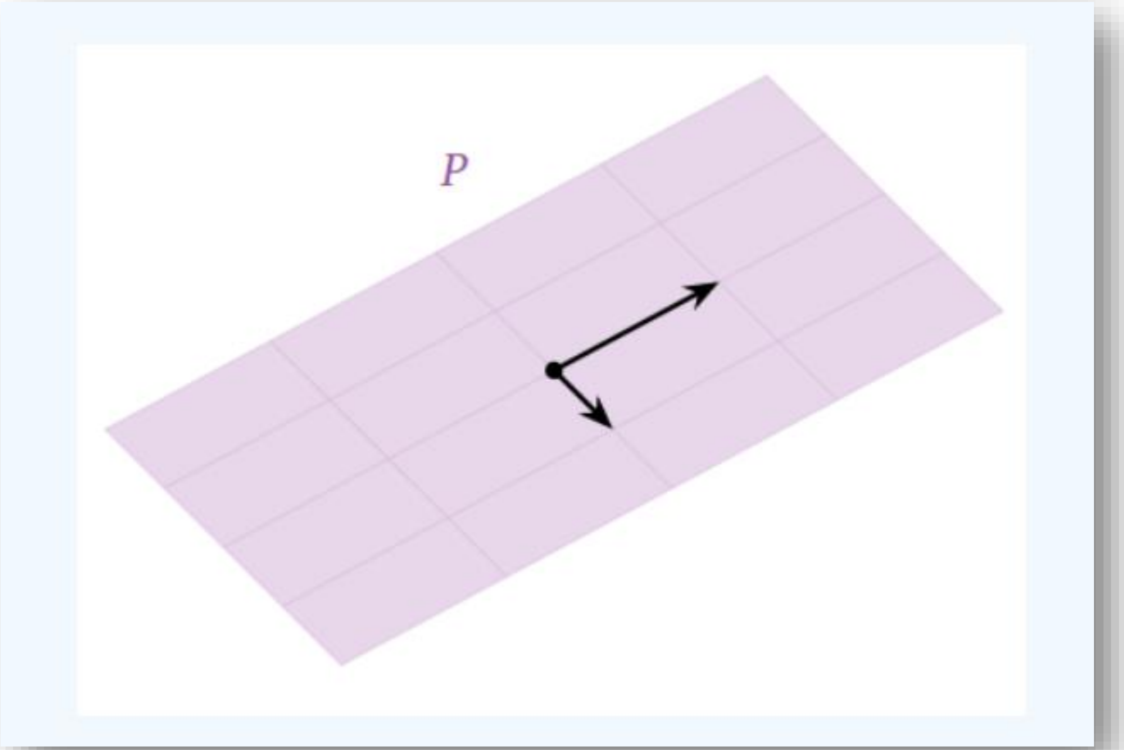
Figure 2.6.2

Indeed, L contains zero, and is easily seen to be closed under addition and scalar multiplication.

Vector Subspace

Example 4

Infinite Plane Through Origin



Vector Subspace

Example 4

Infinite Plane Through Origin

A plane P through the origin is a subspace.

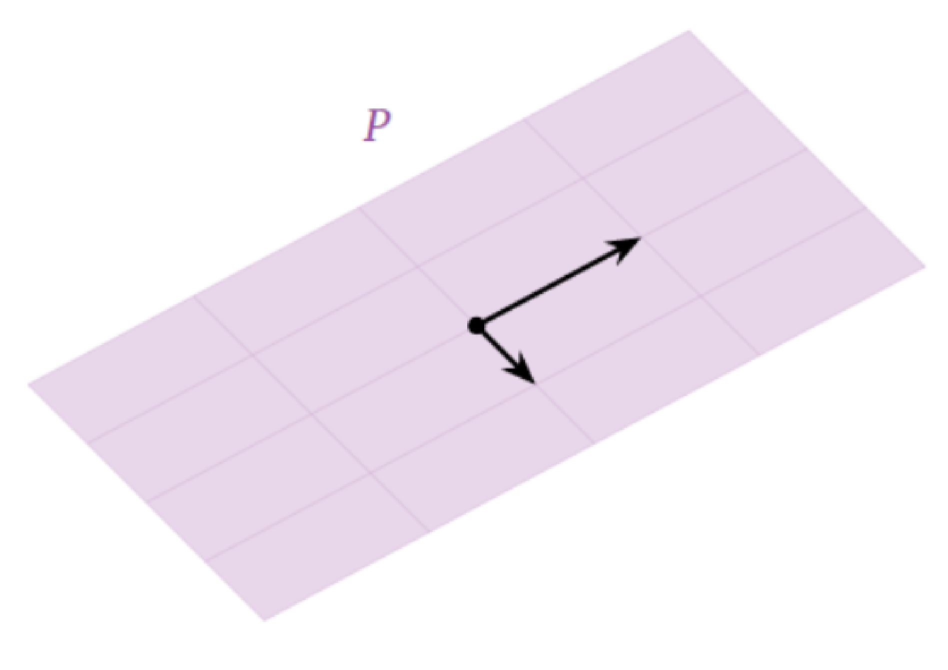


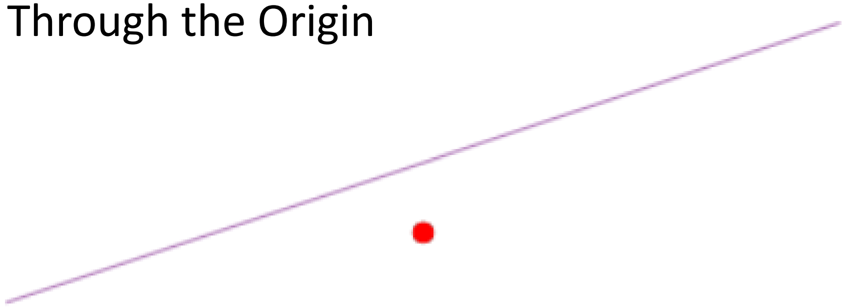
Figure 2.6.3

Indeed, P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.

Vector Subspace

Example 5

A Line Not Through the Origin

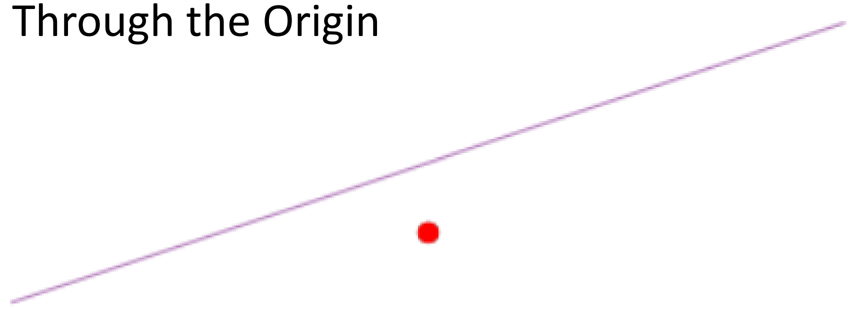


Subspace of $\boldsymbol{R}^n$ or not?

Vector Subspace

Example 5

A Line Not Through the Origin



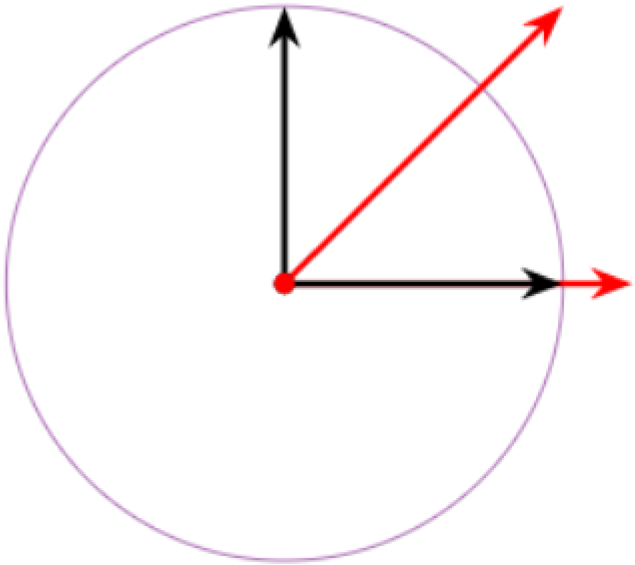
Subspace of $\boldsymbol{R}^n$ or not?

Hint: Does it contain additive identity (0)?

Vector Subspace

Example 6

Points on a Unit Circle (not inside it)



Subspace of $\boldsymbol{R}^n$ or not?

Vector Subspace

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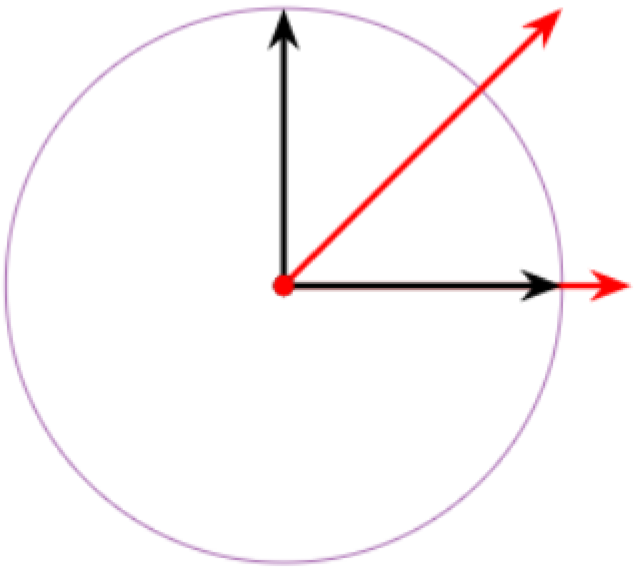


Hints:

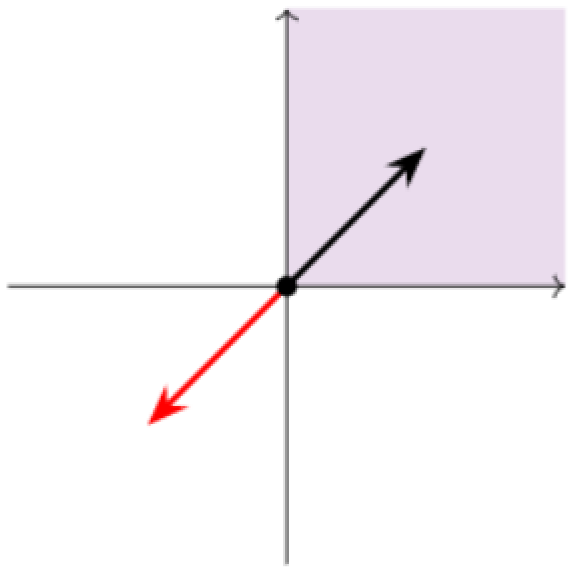
- Does it contain additive identity (0)?
- Do linear combinations also lie on the unit circle?

Subspace of $\boldsymbol{R}^n$ or not?

Vector Subspace

Example 7

First Quadrant



Subspace of $\boldsymbol{R}^n$ or not?

Vector Subspace

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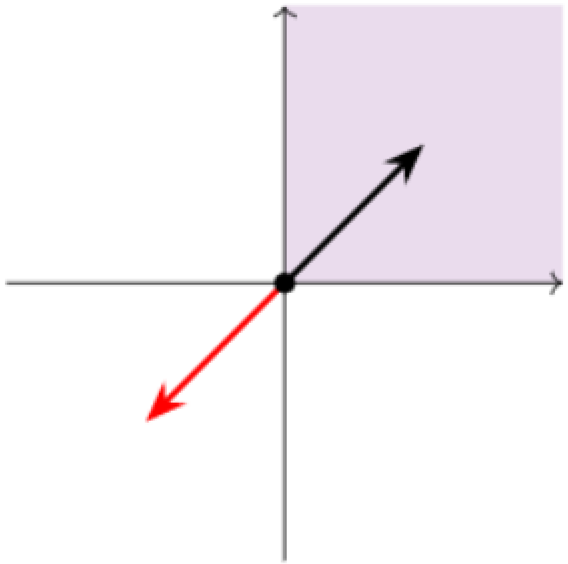

Hints:
- Is it the subset of a Space?
- Does it contain additive identity (0)?
- Do sums of vectors in the first quadrant also lie within the first quadrant?
- Do scalar multiples of vectors in the first quadrant also lie within the first quadrant?

Subspace of $\boldsymbol{R}^n$ or not?

Vector Subspace

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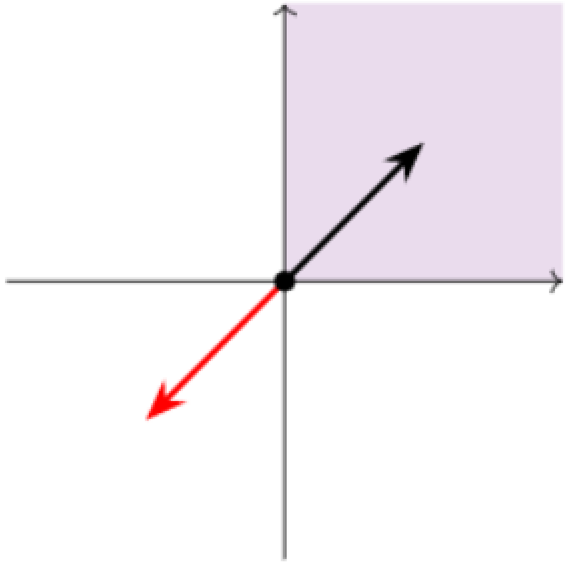


Hints:

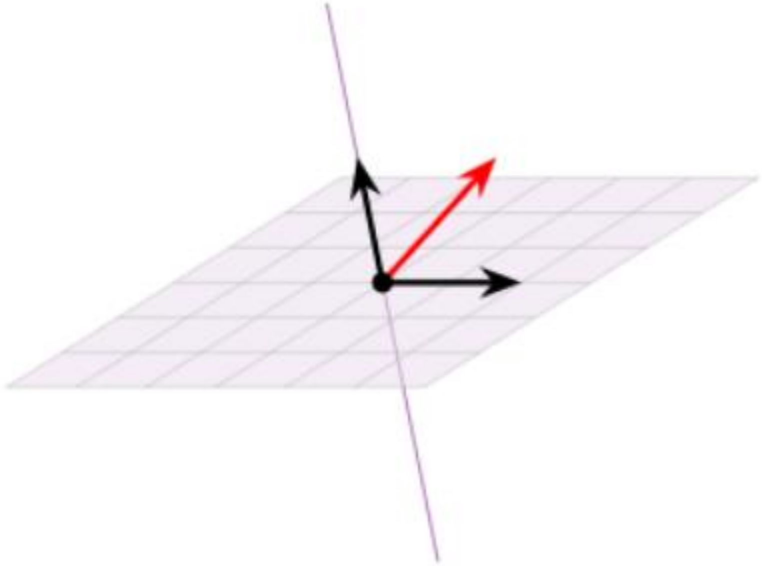
- Is it the subset of a Space? **Yes**
- Does it contain additive identity (0)? **Yes**
- Do sums of vectors in the first quadrant also lie within the first quadrant? **Yes**
- Do scalar multiples of vectors in the first quadrant also lie within the first quadrant? **No**

Subspace of $\boldsymbol{R}^n$ or not?

Vector Subspace

Example 8

Union of shown line and plane
(both through origin)



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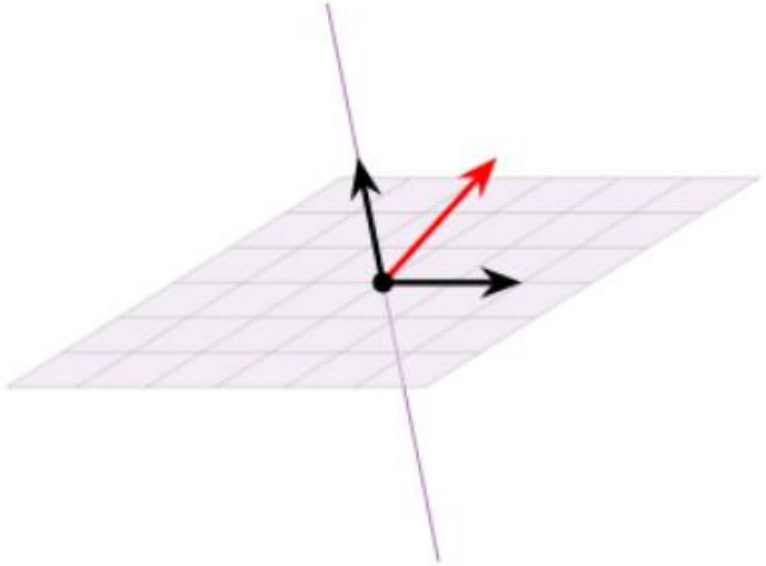

Subspace of $\boldsymbol{R}^n$ or not?

Hint:

- Note that the sum of vectors on the line and on the plane fall neither on the line nor the plane.

And now, to the **most important** (*probably, arguably, quite definitely*) **concept in Linear Algebra** (and its Spaces/Subspaces)

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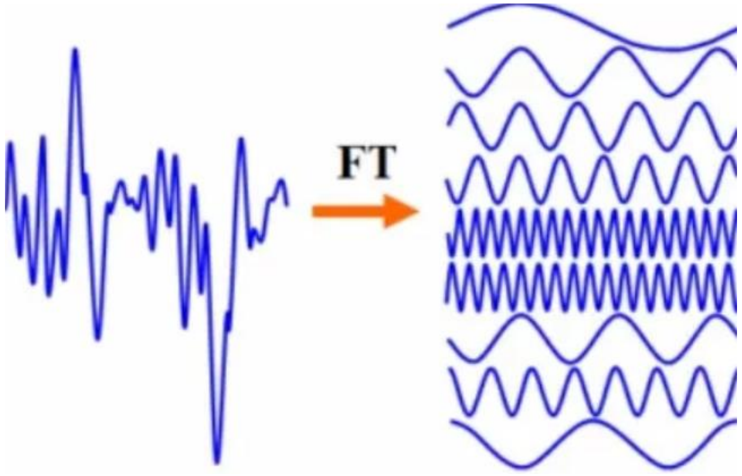
BASIS

To familiarize ourselves with the concept,

Let’s Visit an Old Friend…

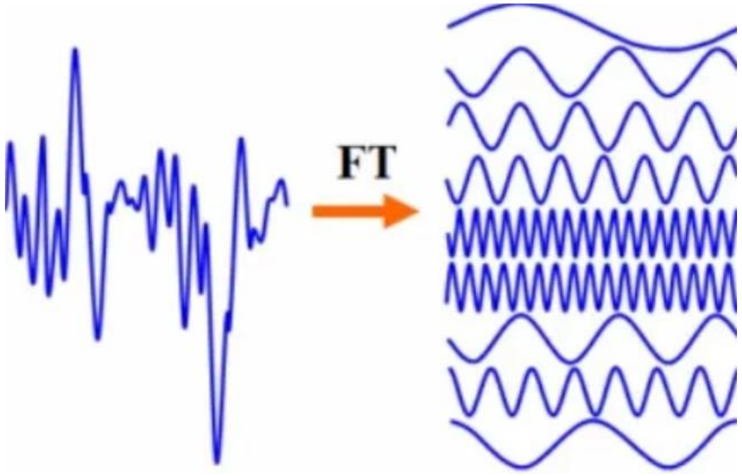
Fourier Transform…

Fourier Transform…



Writing $x(t)$ as a (continuous) linear combination of complex sinusoids.

Fourier Transform…



$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

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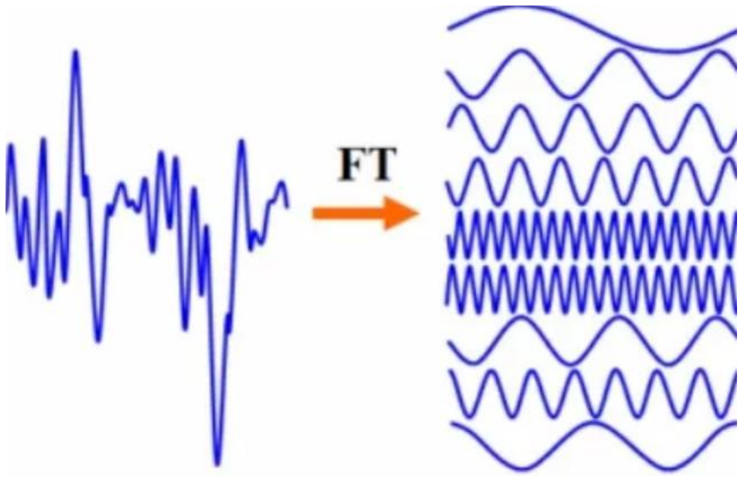

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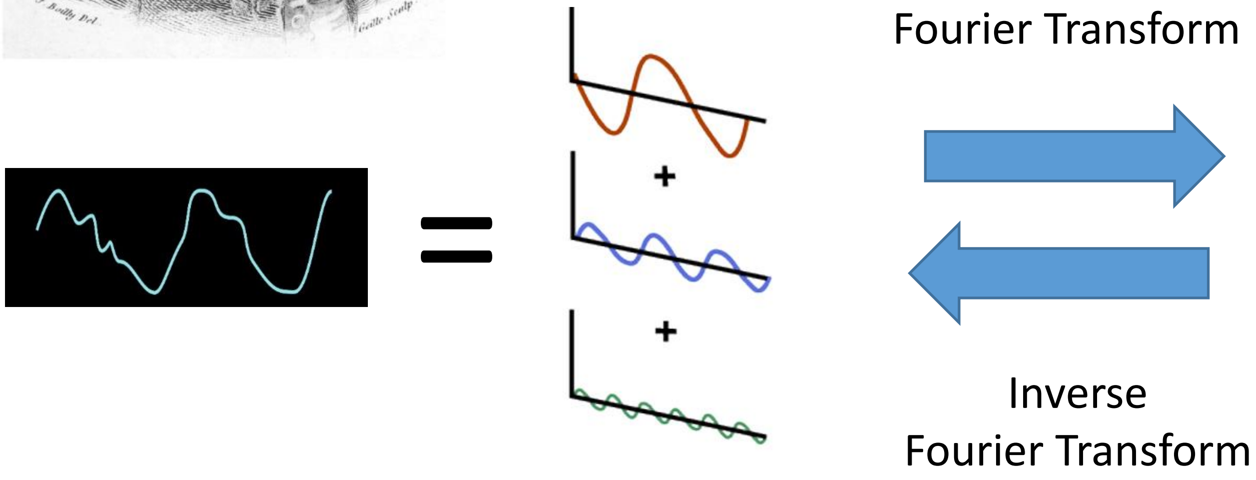
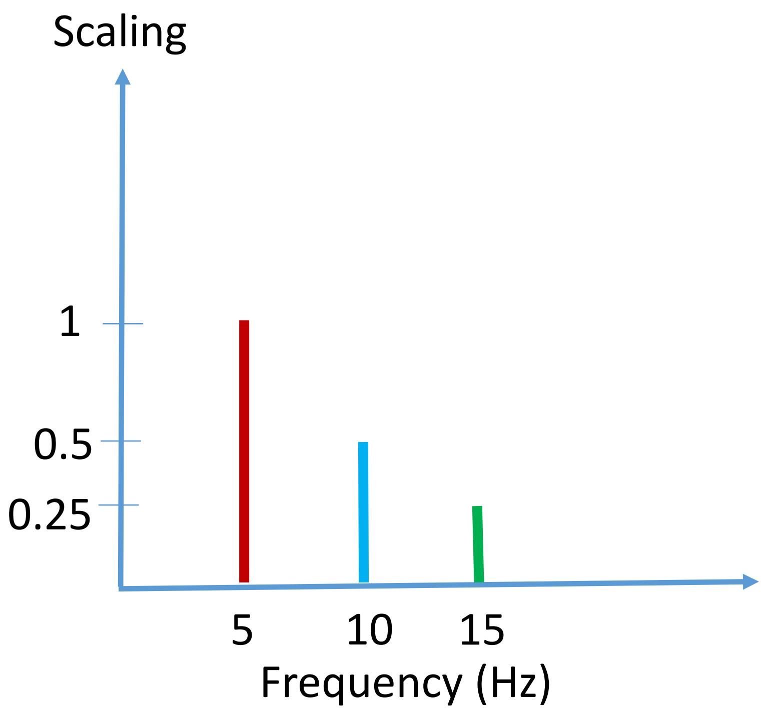
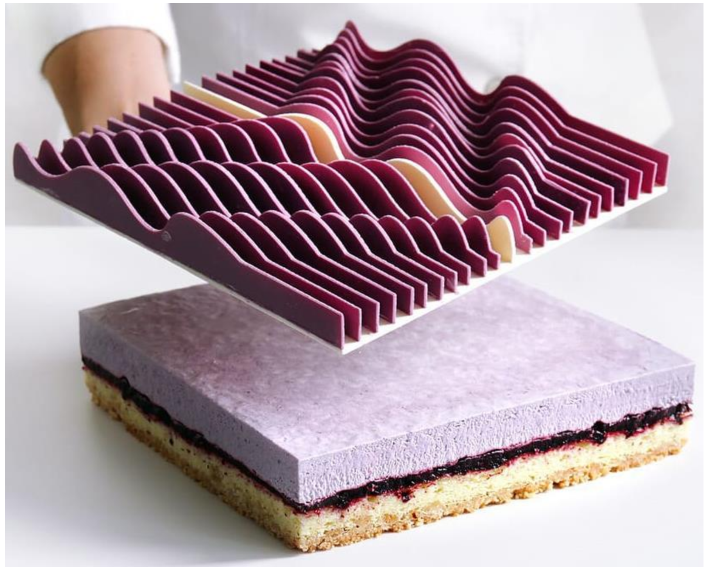




I Like to Call it “Baking a Fourier Cake”



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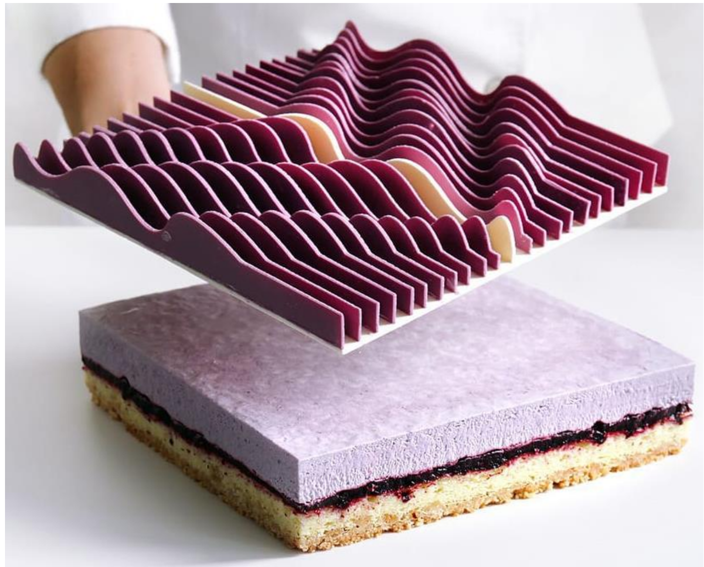

- **Given**: Signal shape (time-domain)

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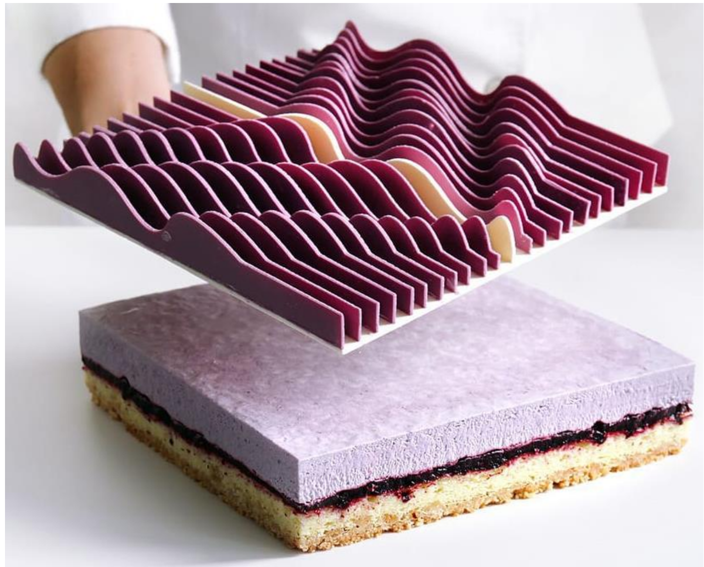

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- **Given**: Signal shape (time-domain)
- **Ingredients**: Sinusoids of different frequencies and phases

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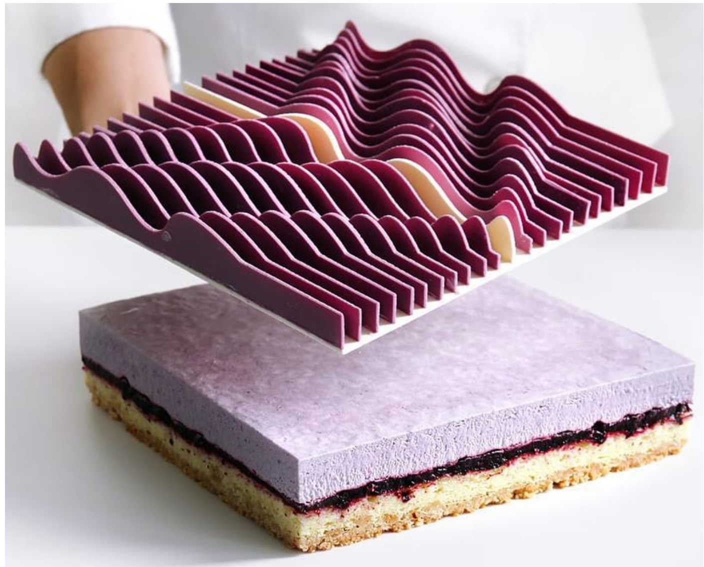

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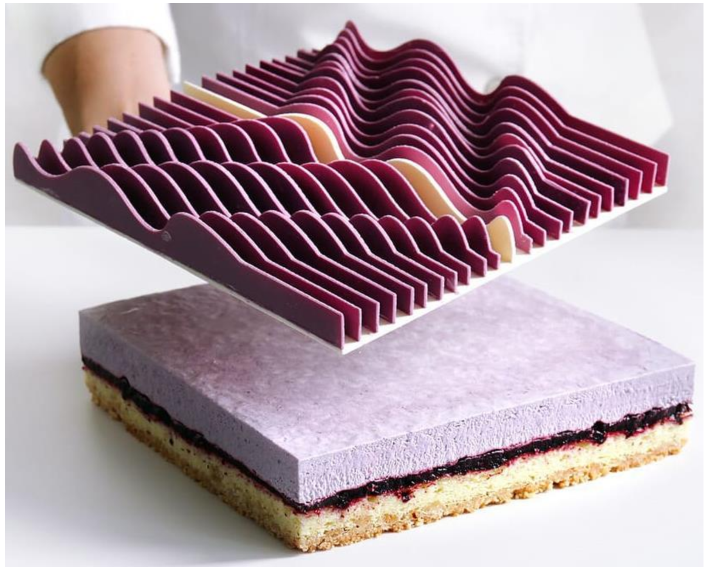
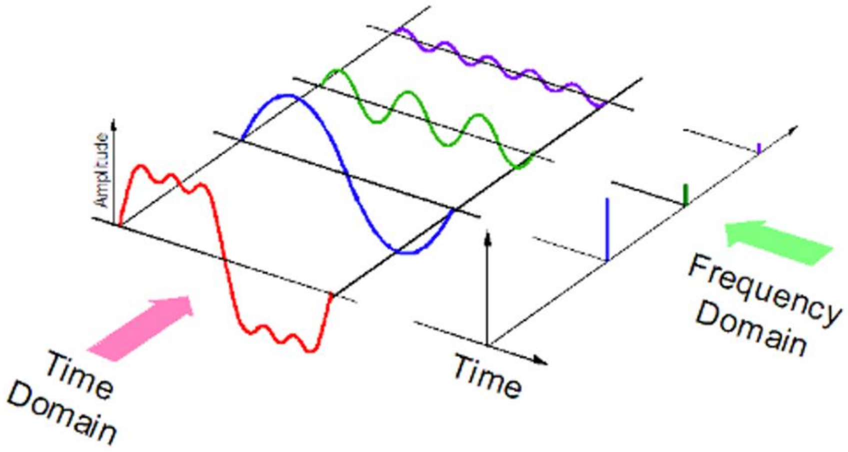


- **Given**: Signal shape (time-domain)
- **Ingredients**: Sinusoids of different frequencies and phases
- **Choose**: How much of the each ingredient (sinusoid) to use?
- **Blend**: Make linear combinations of the chosen ingredients to get the cake.

Fourier Transform…



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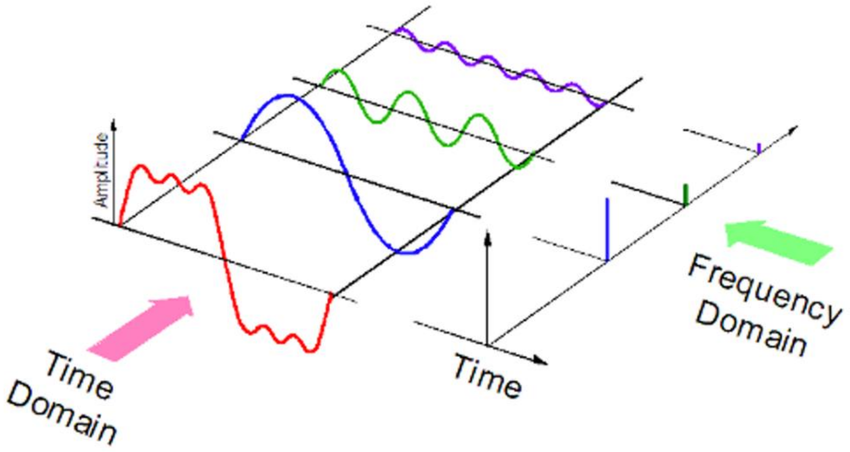

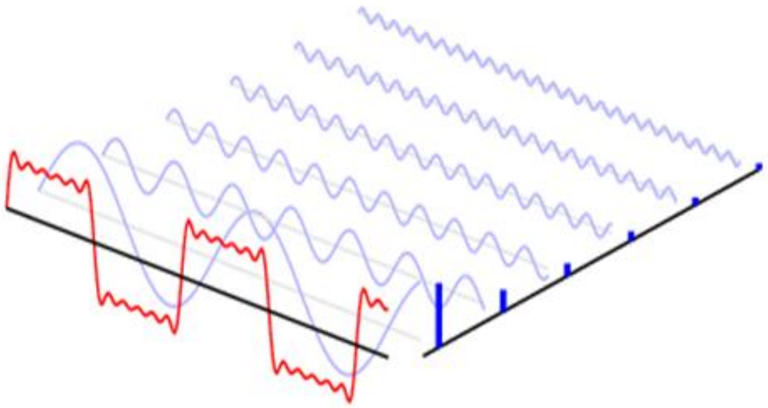

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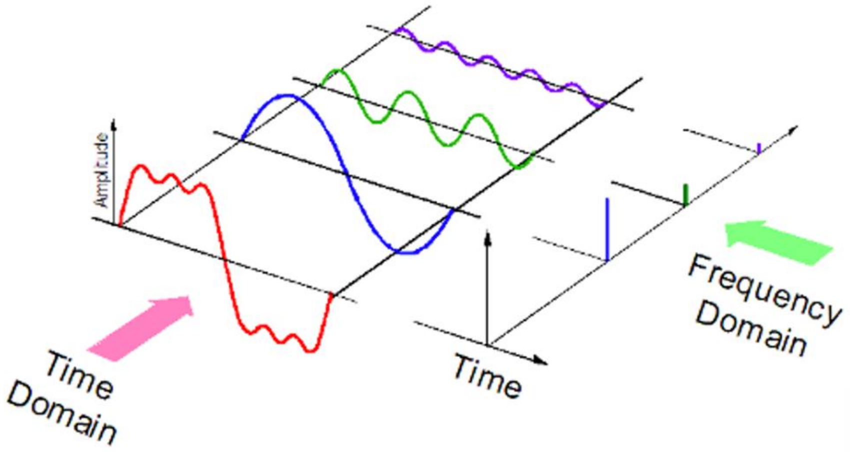
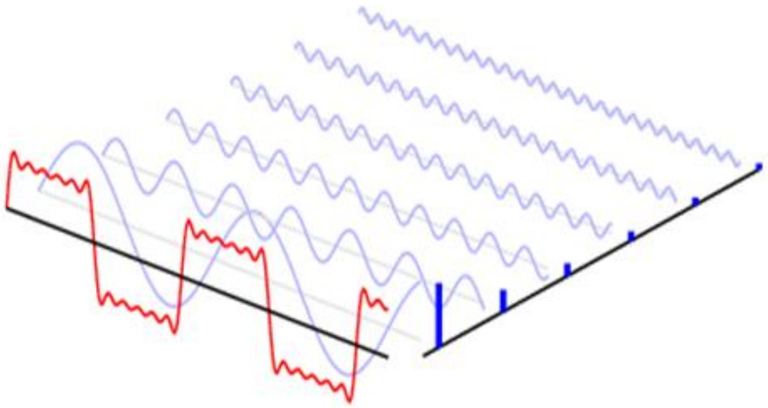
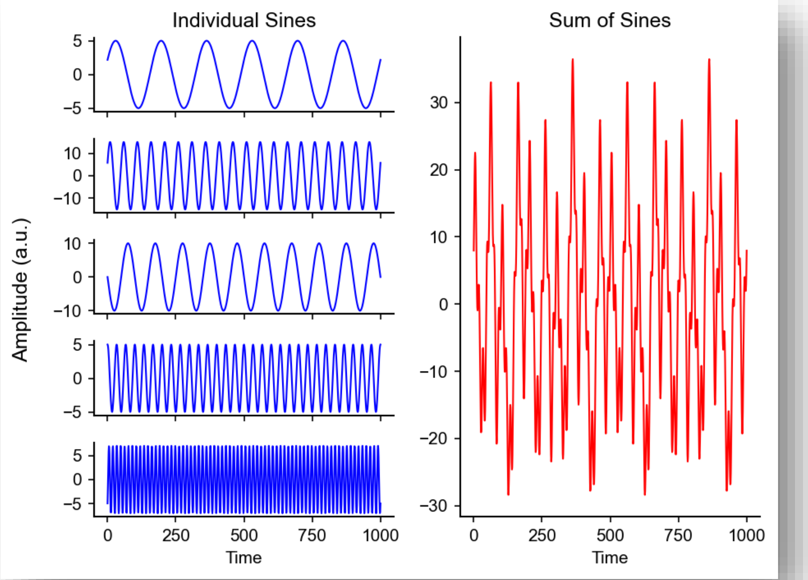
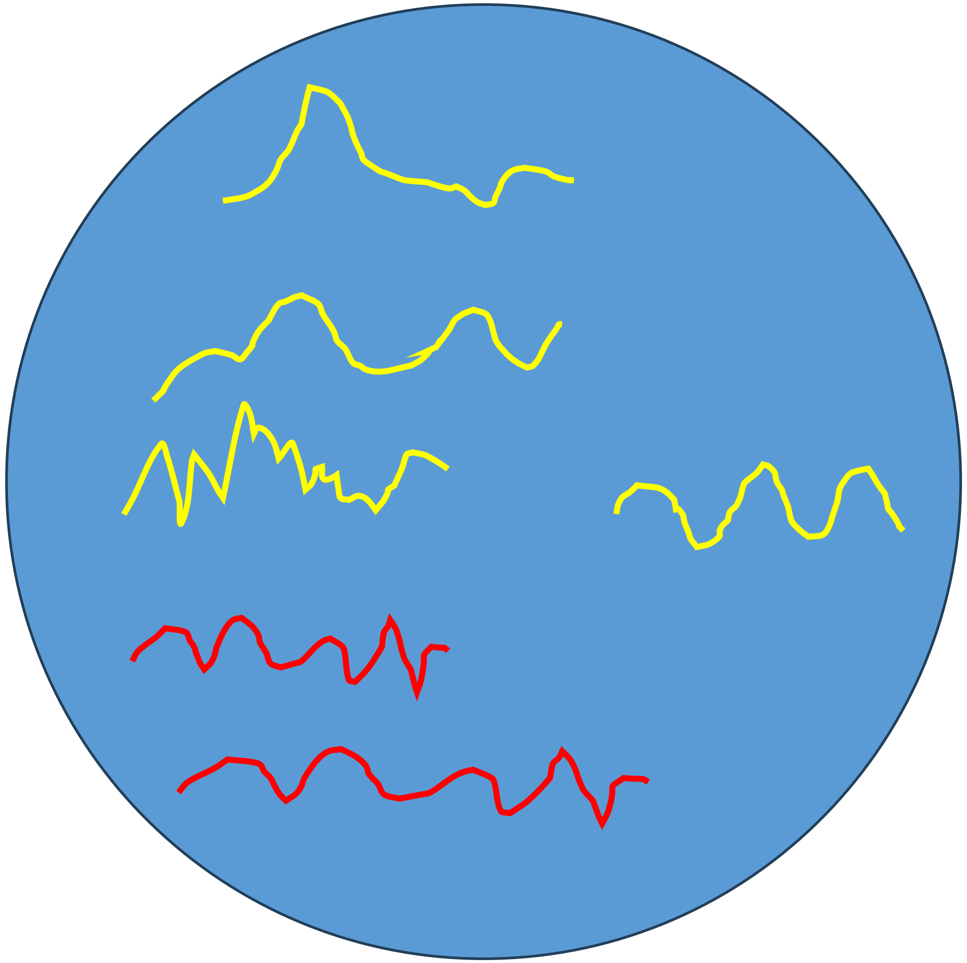






Key Takeaways...

A Broad Set of Continuous Functions



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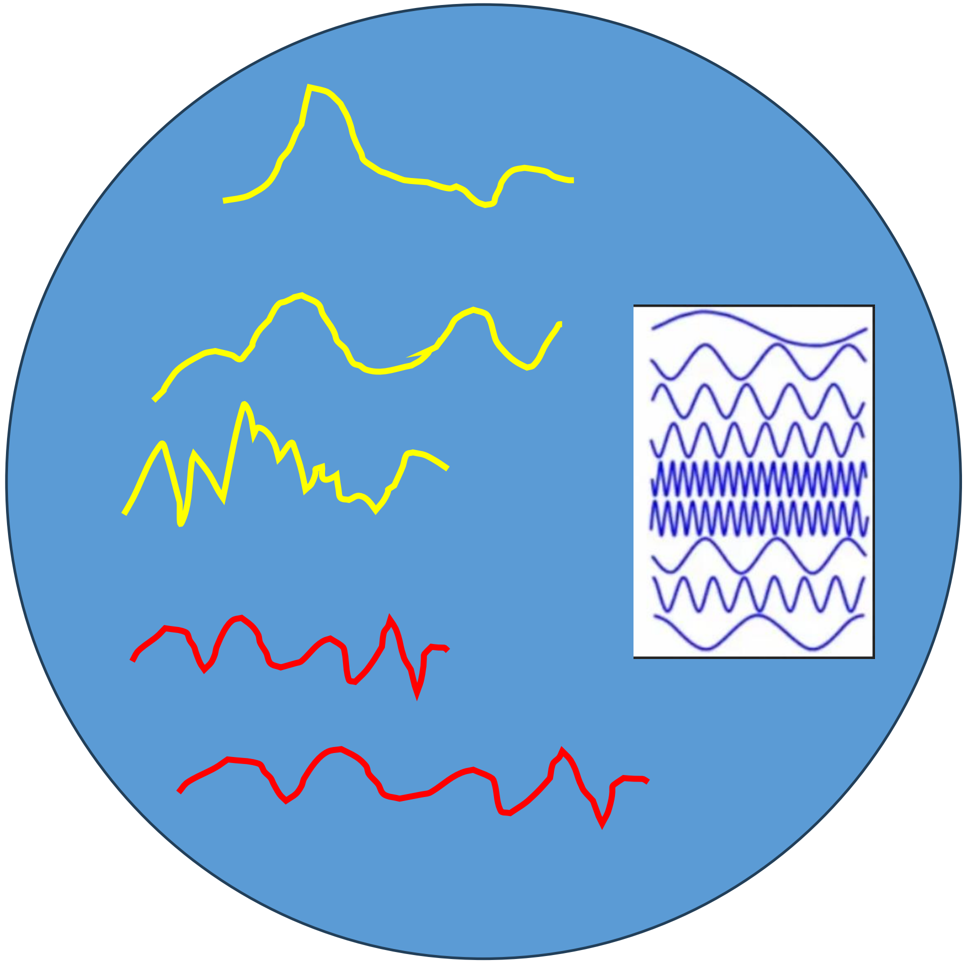

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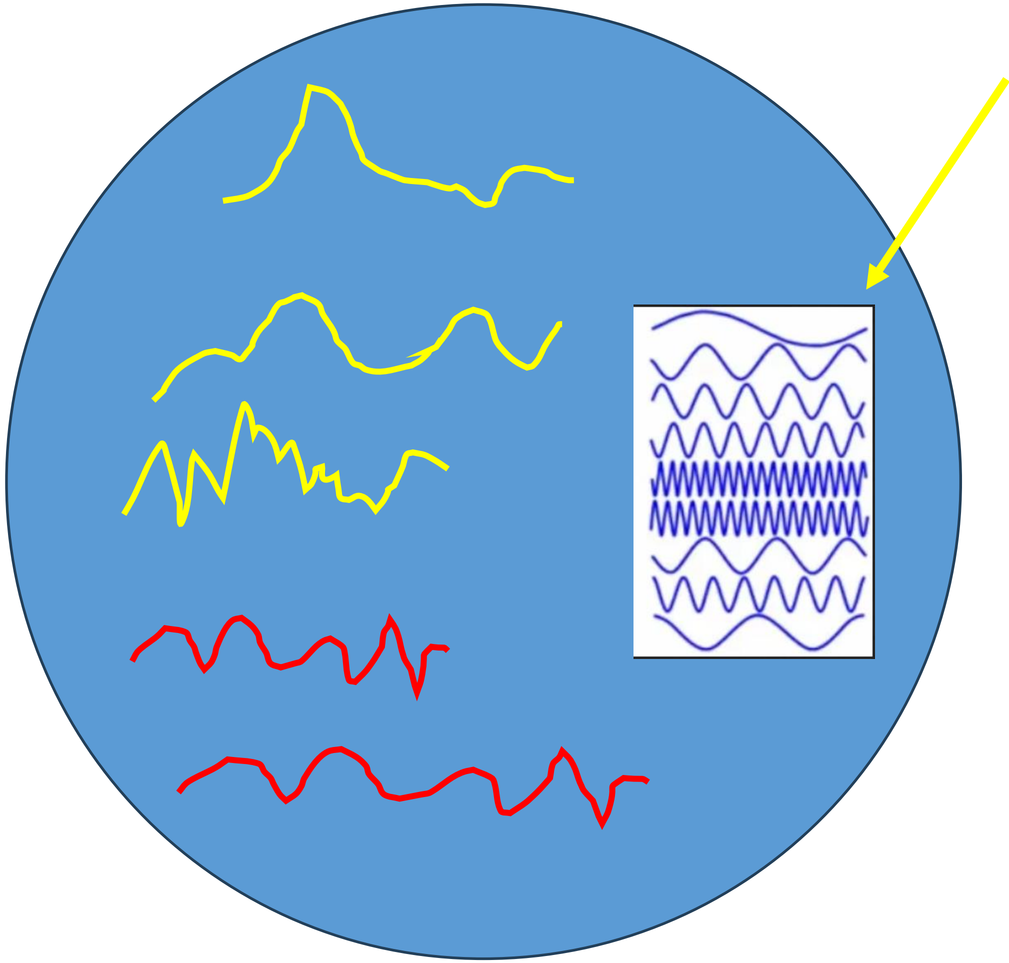

- A **subset** comprising of [complex] sinusoids ***whose linear combinations can represent all the remaining functions in the set***.

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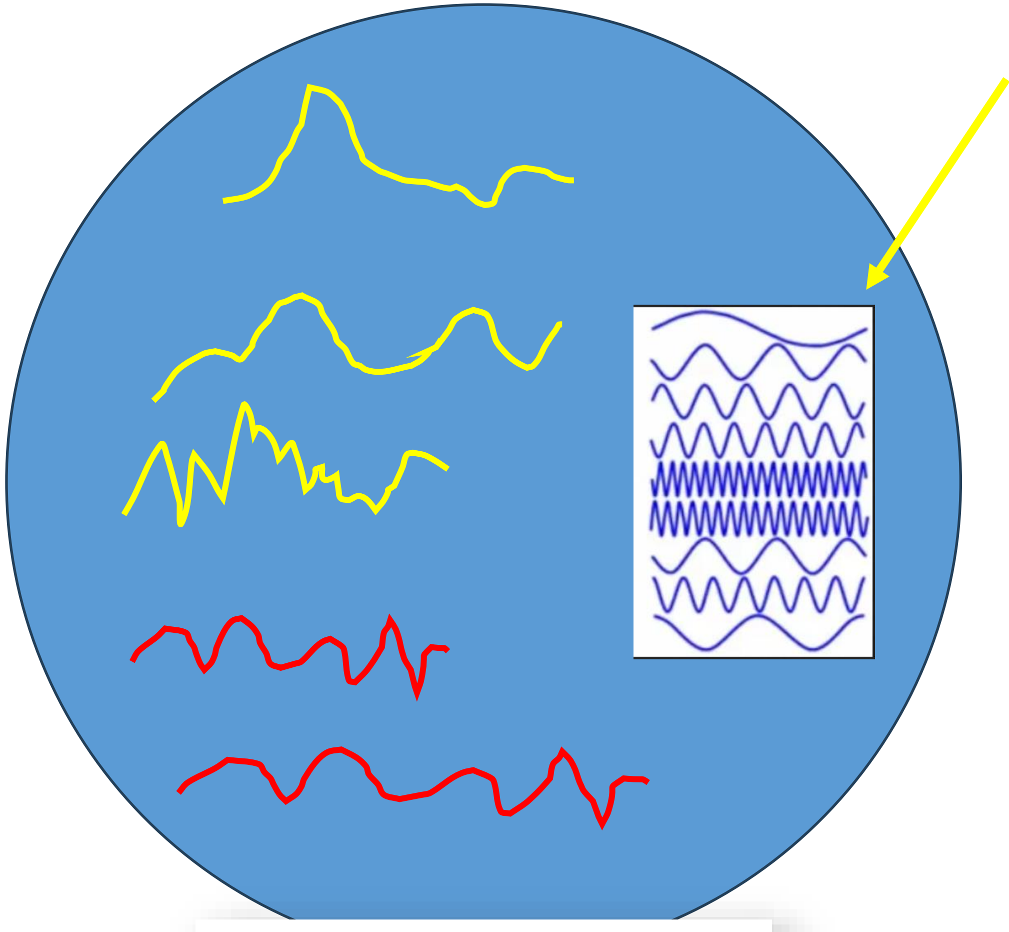

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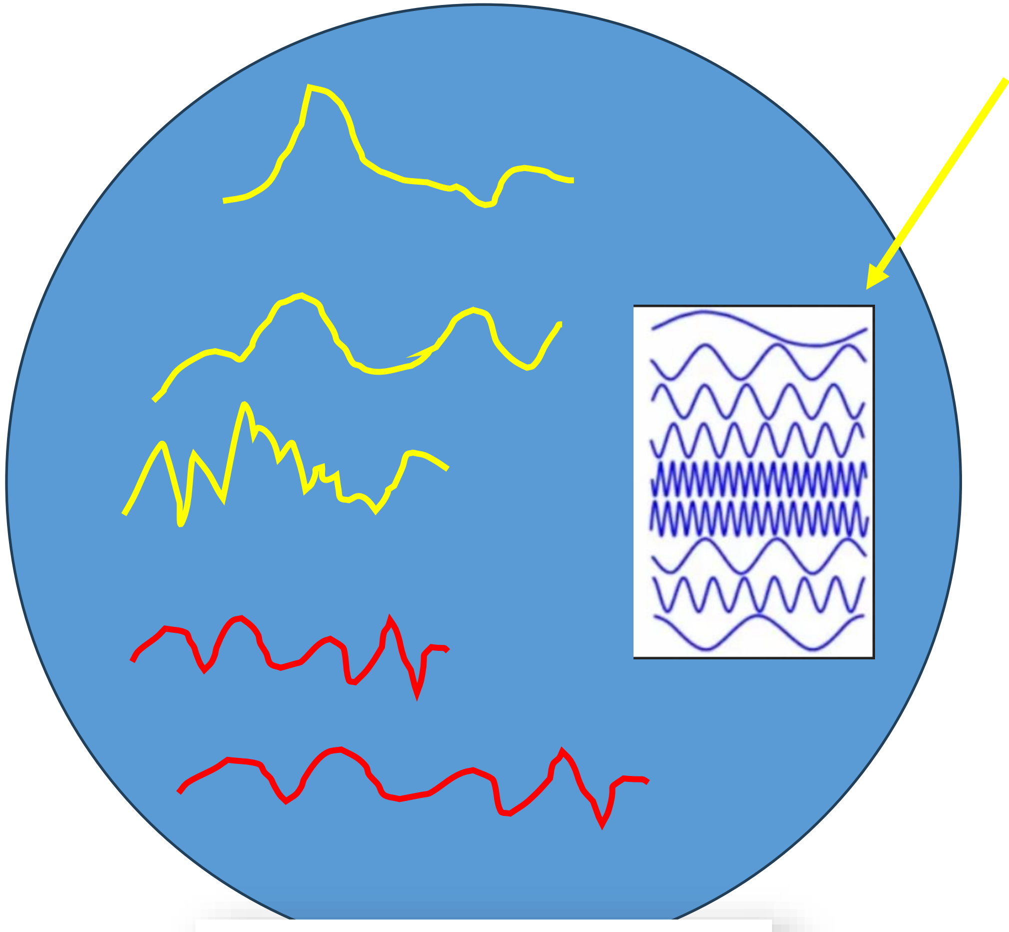

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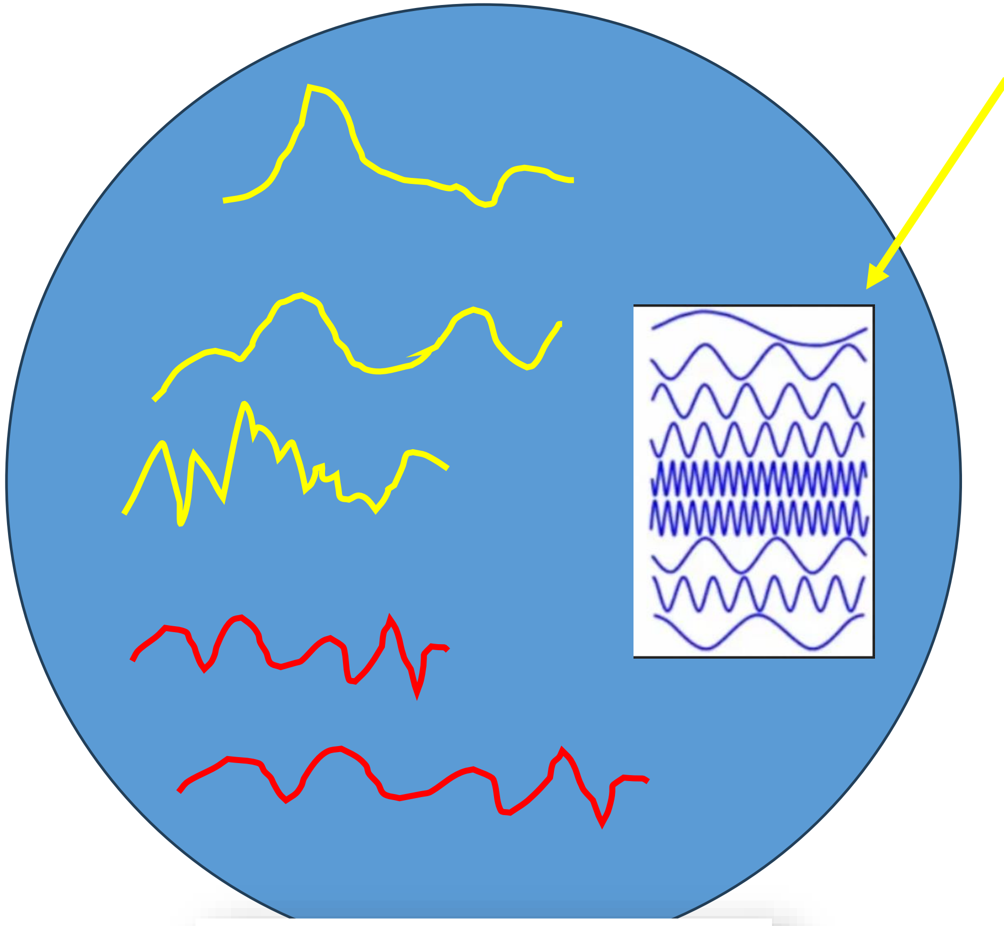
- A **subset** comprising of [complex] sinusoids ***whose linear combinations can represent all the remaining functions in the set***.

- If we remove scaled versions, and choose only sinusoids of different frequencies and phases, these form the **minimum subset** that can represent the entire set.

Key Takeaways...

A Broad Set of Continuous Functions



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

- A **subset** comprising of [complex] sinusoids ***whose linear combinations can represent all the remaining functions in the set***.

- If we remove scaled versions, and choose only sinusoids of different frequencies and phases, these form the **minimum subset** that can represent the entire set.

    - No member of this minimal subset can be formed by any linear combination or scaling of the other members (i.e., **all are linearly independent**).

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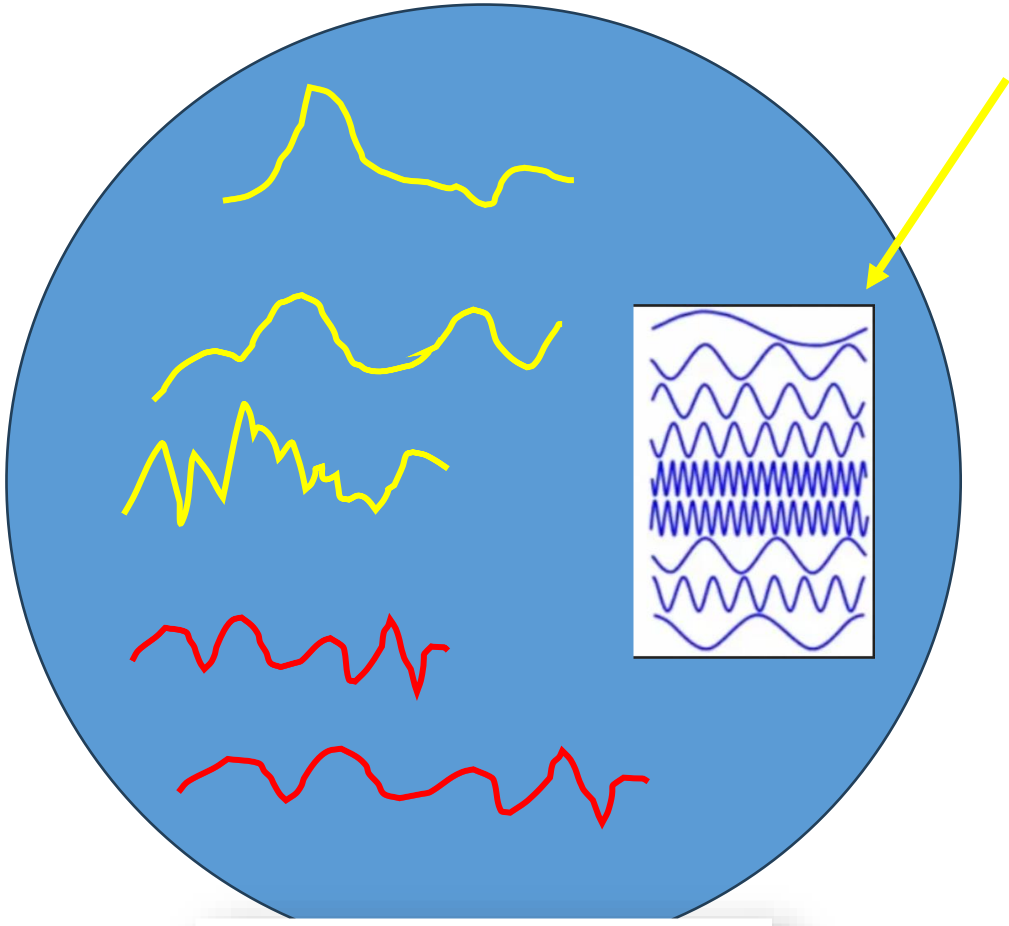

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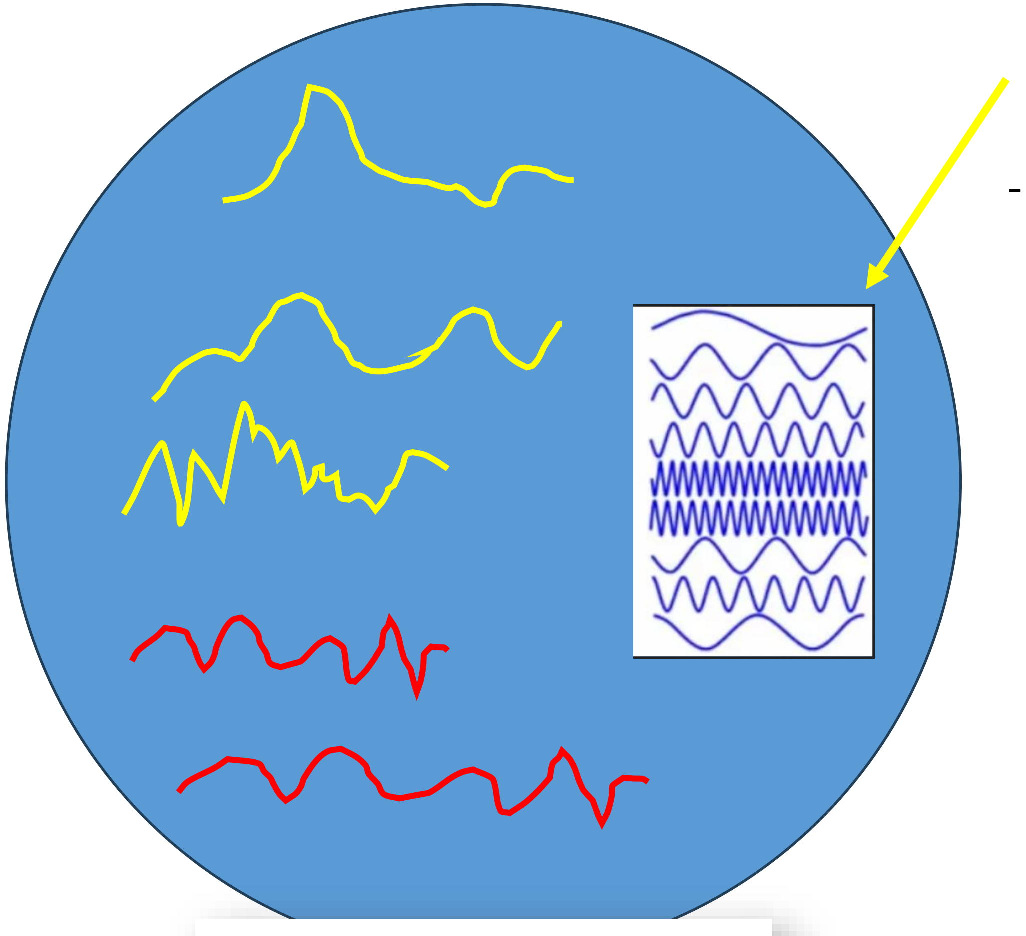

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    - In a way, this minimal subset forms a **basis** for making the rest of the set.

    - Another way of saying this is that this minimal set **spans** the whole set by representing any member as a linear combination.

OK, But What’s the Use?

Glad you asked…

*The Power of **Basis** …*

Efficient
Representation/Compression

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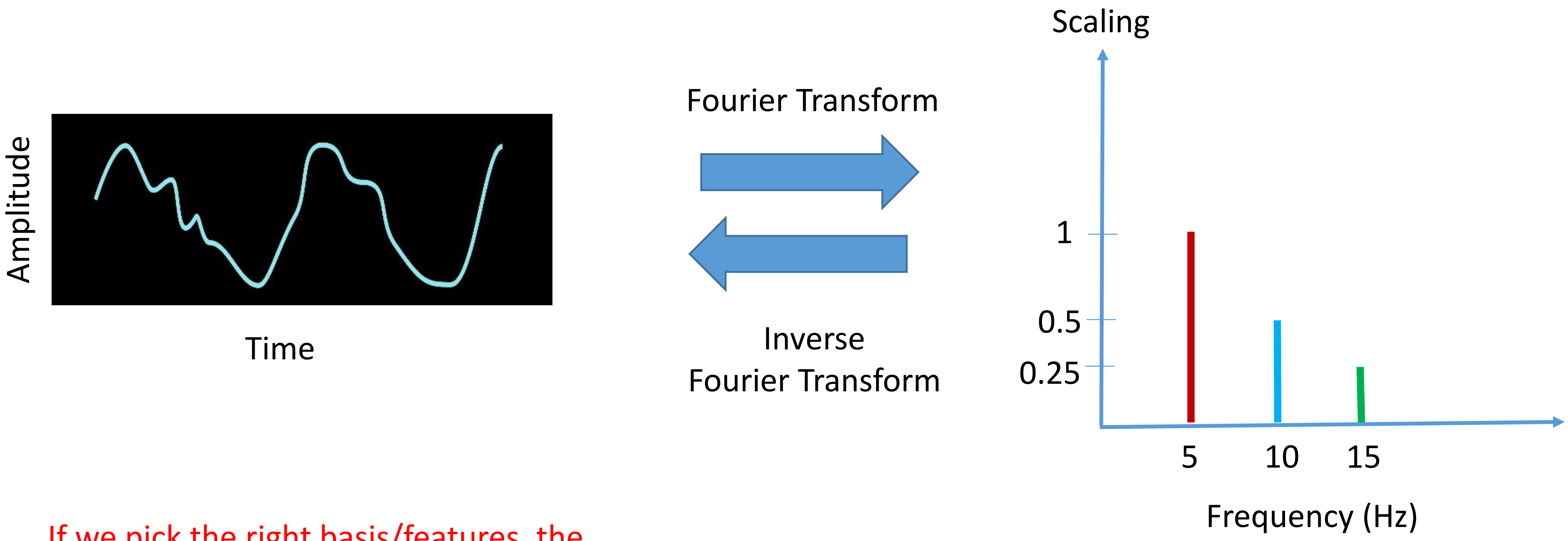

If we pick the right basis/features, the problem could become very sparse!

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Analysis/Comparison/Synthesis Easier When All Member Viewed in Terms of Same Basis.

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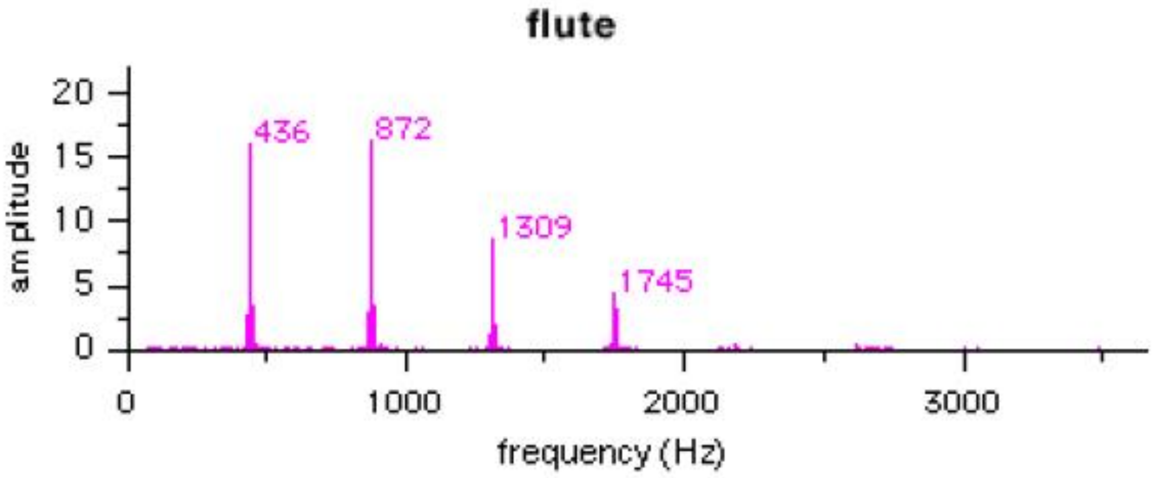

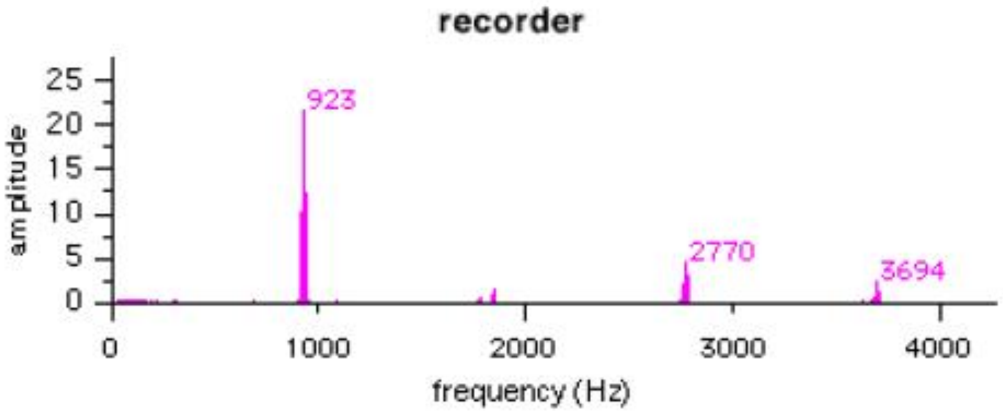

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$$x(t) \longrightarrow \overset{h(t)}{\boxed{\boldsymbol{S}\{\cdot\}}} \longrightarrow y(t)$$

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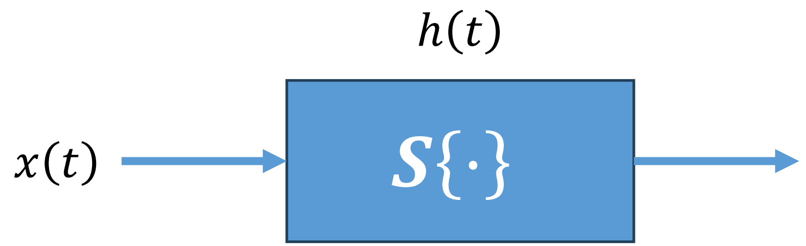

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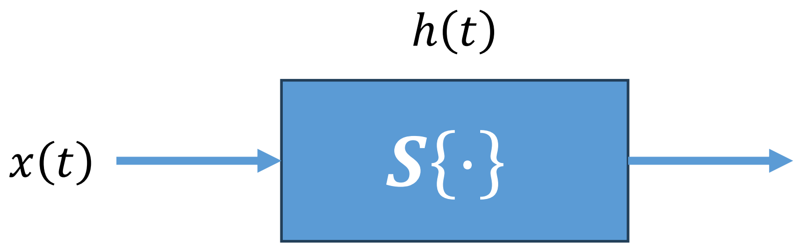

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$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Convolution – a rather complicated computation.

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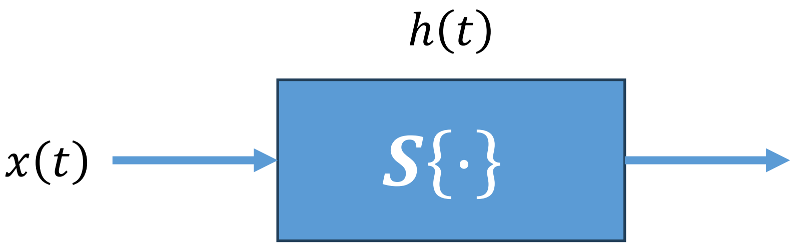

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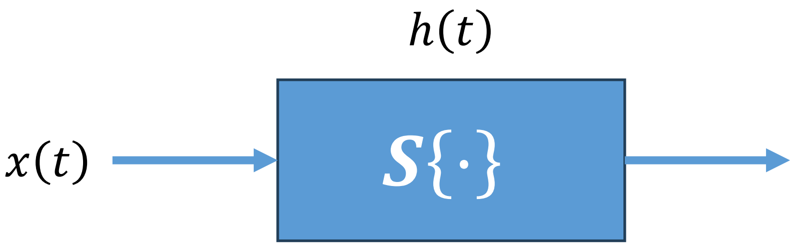

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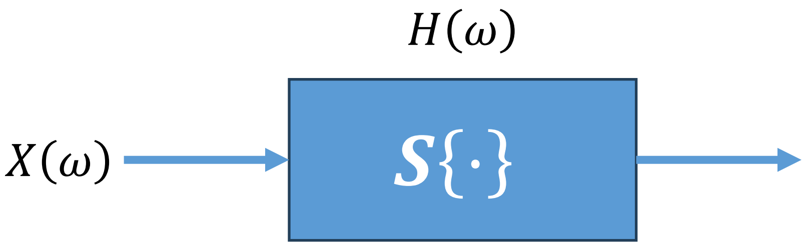
Then, we find (provable) that the output becomes a simple product



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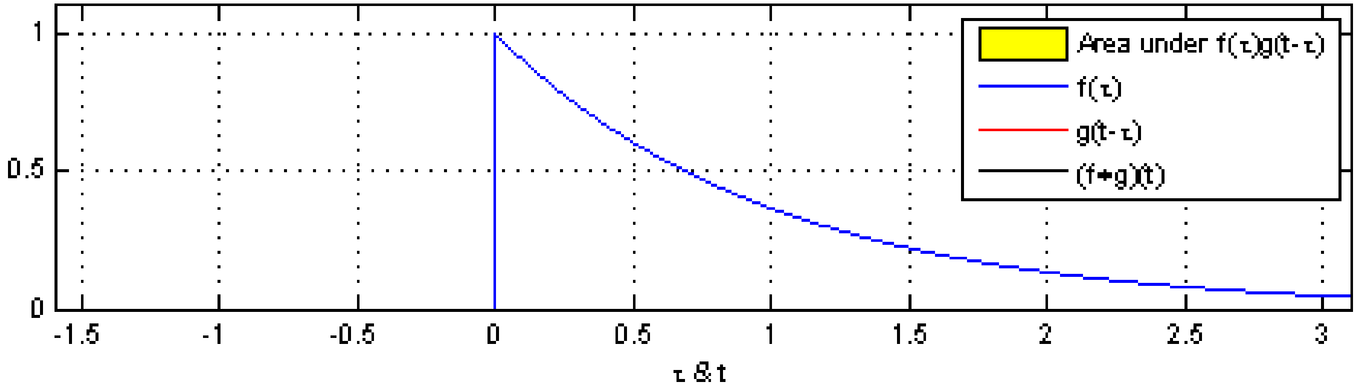

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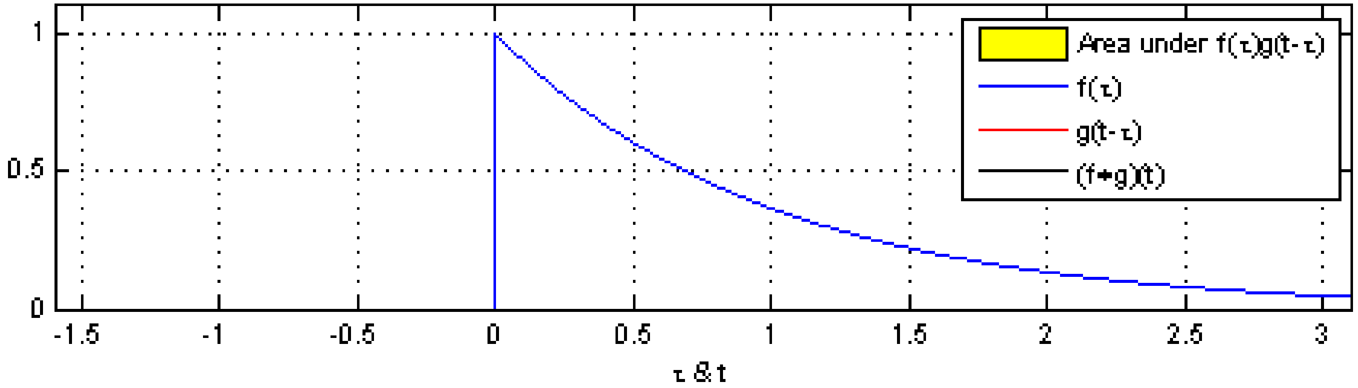

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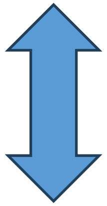

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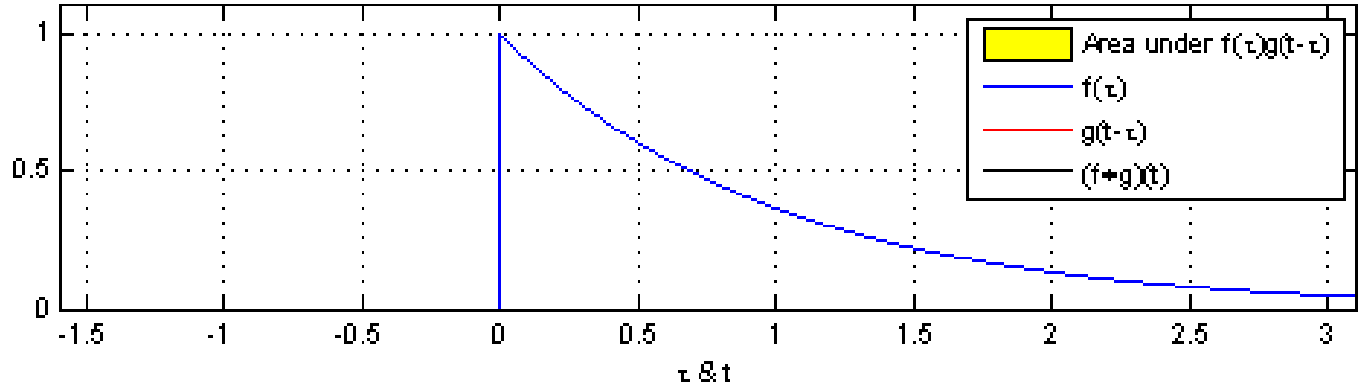

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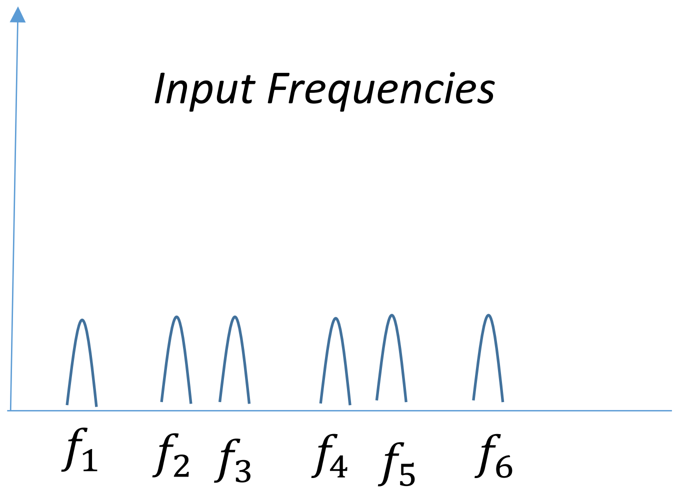

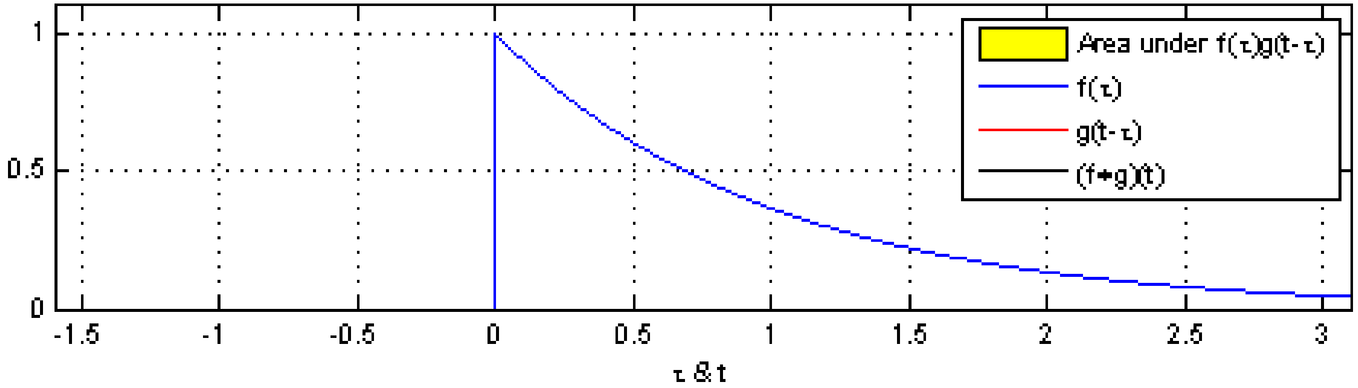

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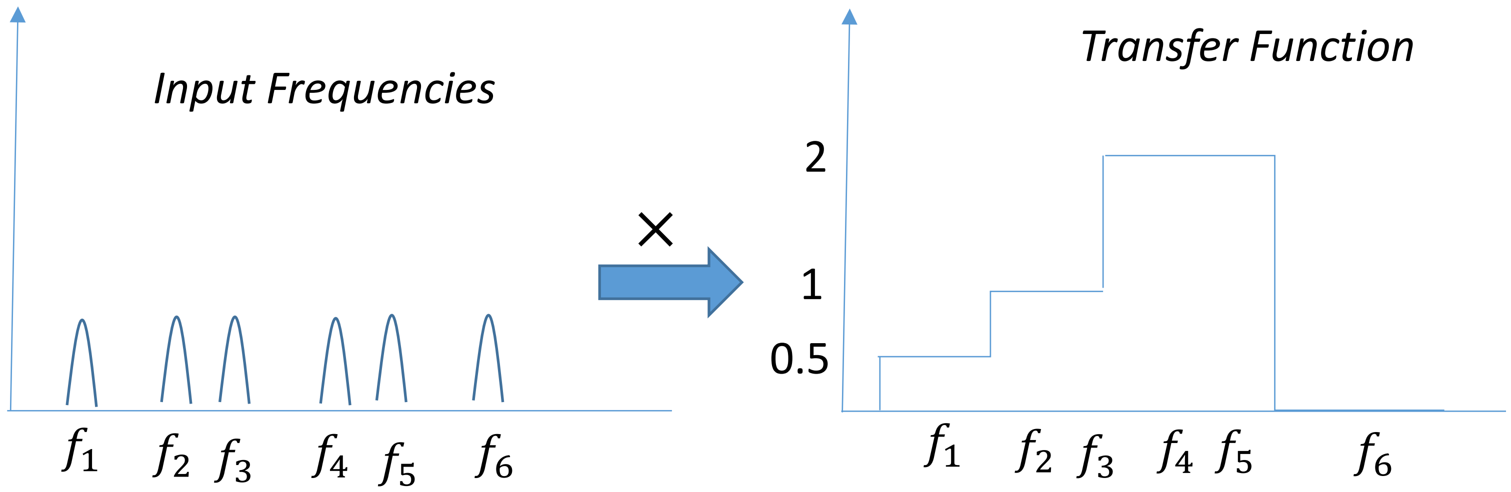

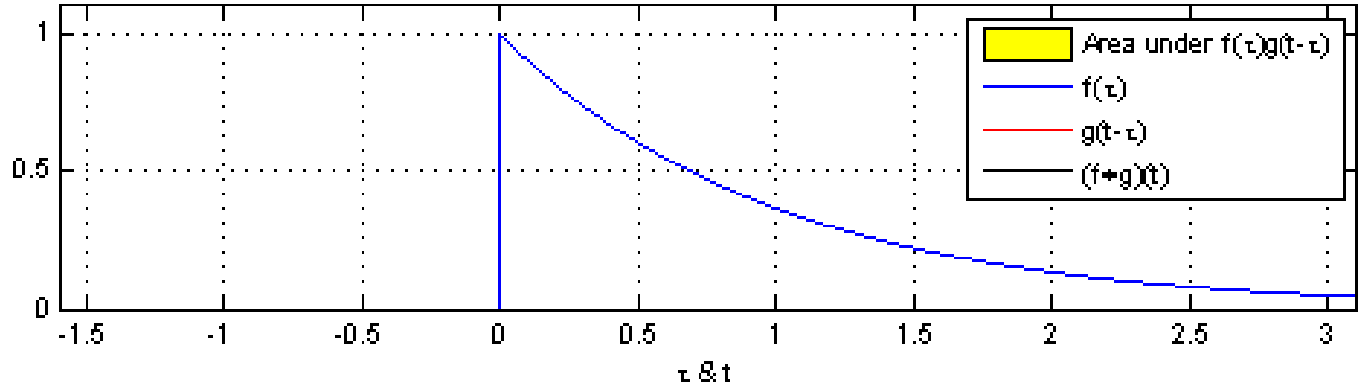

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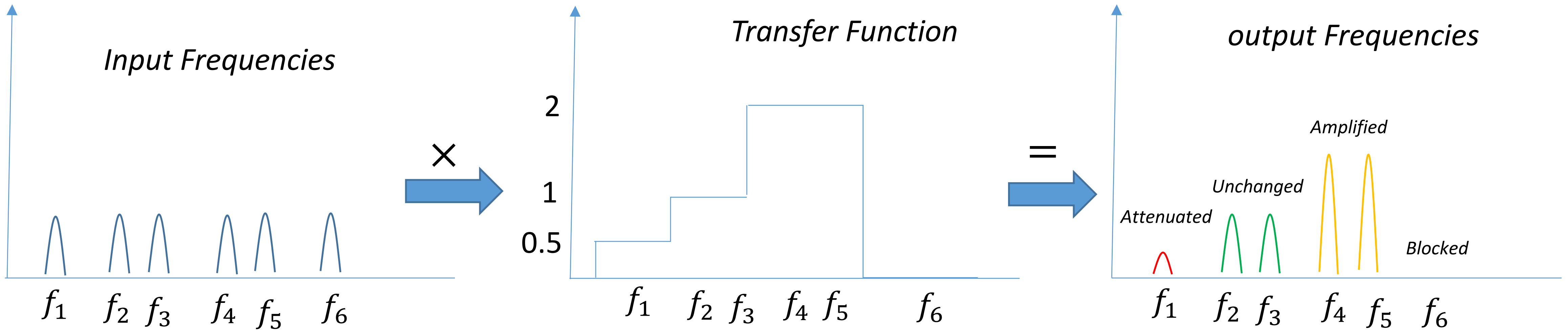

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Calculus → Algebra

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$$
\begin{tikzcd}
\ddot{y}+3\dot{y}+2y=u(t) \arrow[r, "\text{solve using direct methods}"] \arrow[d, "\text{Laplace transform}"'] & y(t)=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t} \\
\frac{1}{s^2+3s+2}\,\frac{1}{s} \arrow[r, "\text{algebraic manipulation}"'] & \frac{1}{s^3+3s^2+2s}=\frac{1}{2s}-\frac{1}{s+1}+\frac{1}{2(s+2)} \arrow[u, "\text{inverse Laplace transform}"']
\end{tikzcd}
$$

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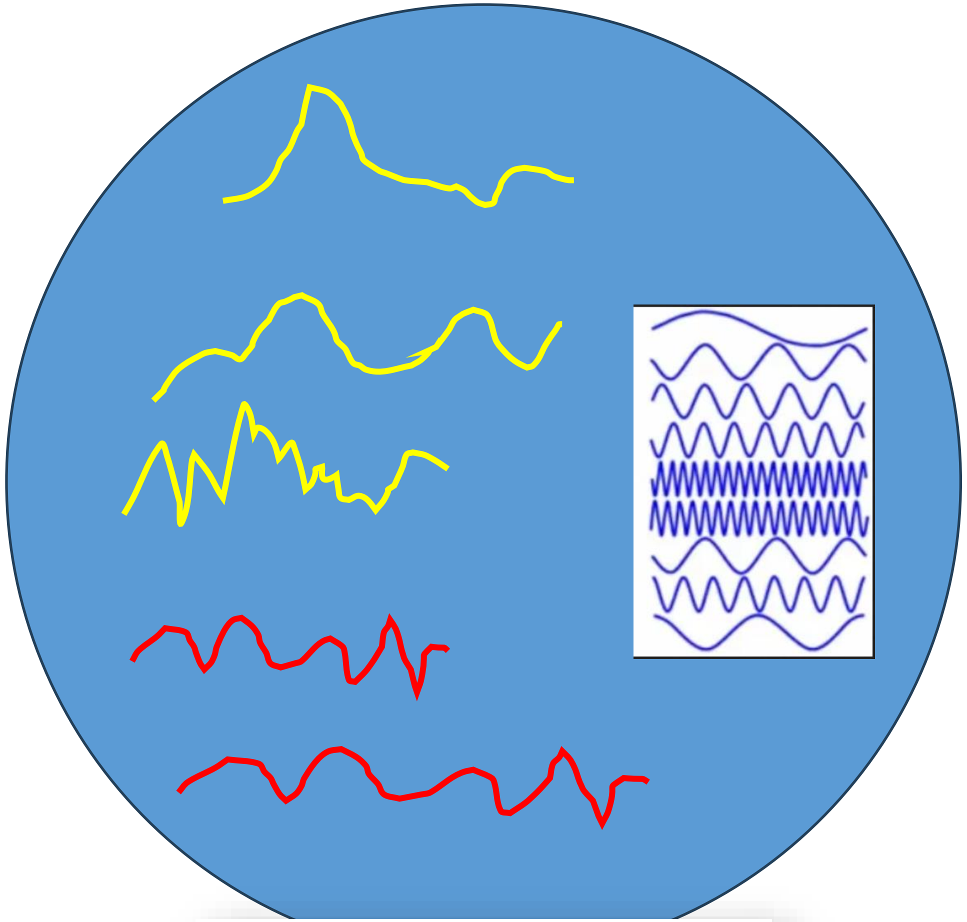

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Linearly Independent Subset

Forming

Basis that Span the Whole Set

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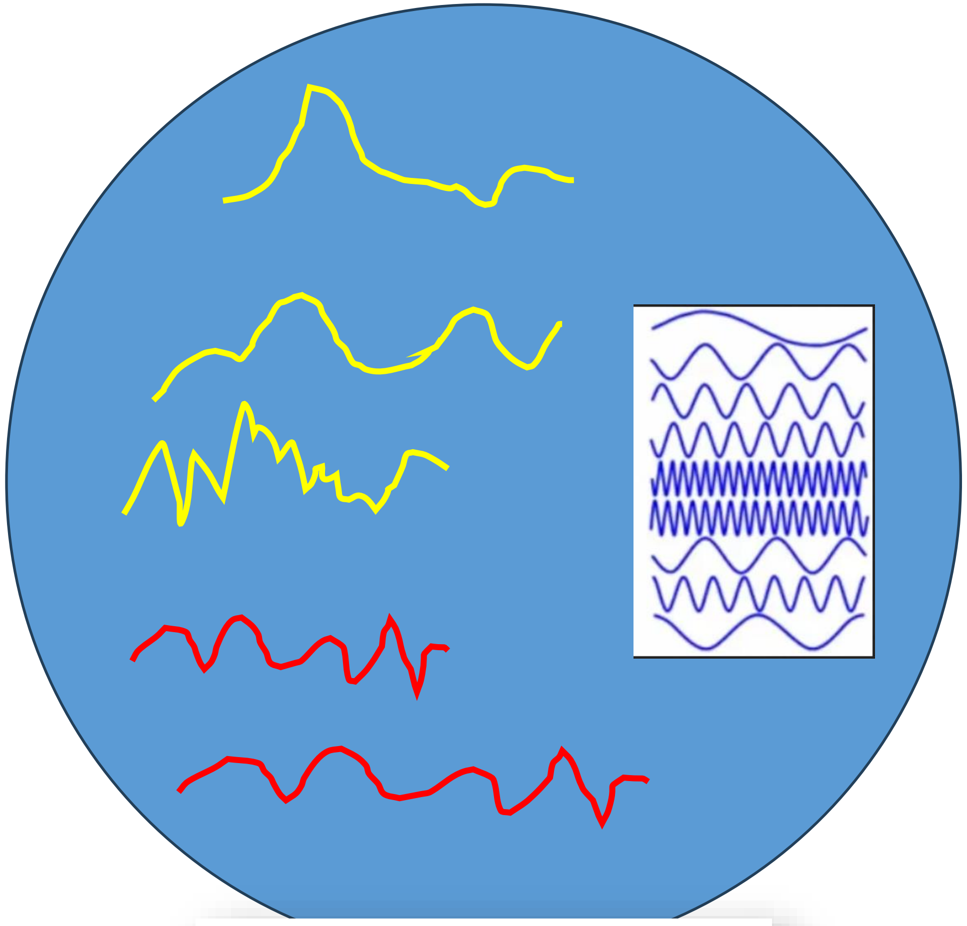

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Now let's take these key concepts back to Linear Algebra

Linearly Independent Subset

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Scaled and Summed Versions of Members of Some Set/Space.

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$$\boldsymbol{v} = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_k \boldsymbol{x}_k = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i \in V \tag{2.65}$$

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e.g., given vectors $\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}$, some of their linear combinations are

$$v_1 = \mathbf{4}\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} + \mathbf{5}\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{4} \\ \mathbf{5} \end{bmatrix}, v_2 = -\mathbf{3}\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} + \mathbf{2}\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} -\mathbf{3} \\ \mathbf{2} \end{bmatrix}, v_r = \boldsymbol{r}_1\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} + \boldsymbol{r}_2\begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{bmatrix}; \boldsymbol{r}_1, \boldsymbol{r}_2 \in \boldsymbol{R}$$

Linear Independence

When no member can be written as a linear combination of the others.

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Definition 2.12 (Linear (In)dependence)**.** Let us consider a vector space V with $k \in \mathbb{N}$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k \in V$. If there is a non-trivial linear combination, such that $\mathbf{0} = \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i$ with at least one $\lambda_i \neq 0$, the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are *linearly dependent*. If only the trivial solution exists, i.e., $\lambda_1 = \ldots = \lambda_k = 0$ the vectors $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are *linearly independent*.

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e.g., $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent as none can be written as linear combination of these elements.

But $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ are linearly dependent since $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

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Several Ways. One Discussed Here.

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Is the set

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Example 1

Solution

Equivalently, we are asking if the homogeneous vector equation

$$x\begin{pmatrix}1\\1\\1\end{pmatrix} + y\begin{pmatrix}1\\-1\\2\end{pmatrix} + z\begin{pmatrix}3\\1\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

has a nontrivial solution. We solve this by forming a matrix and row reducing

Solution other than all zeros.

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$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This says $x = -2z$ and $y = -z$. So there exist nontrivial solutions: for instance, taking $z = 1$ gives this equation of linear dependence:

$$-2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

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HW:

- **Revise Row Echelon Forms.**
- **Revise How Gaussian Elimination and Gauss-Jordan Elimination Help Get These Forms.**

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Example 2

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This says $x = y = z = 0$, i.e., the only solution is the trivial solution. We conclude that the set is linearly independent.

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Let's Consider the Real Number Line $\boldsymbol{R}$ (a subspace of $\boldsymbol{R}^n$)

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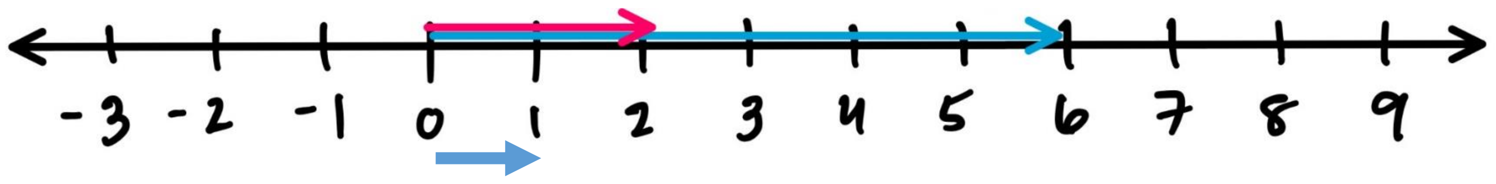

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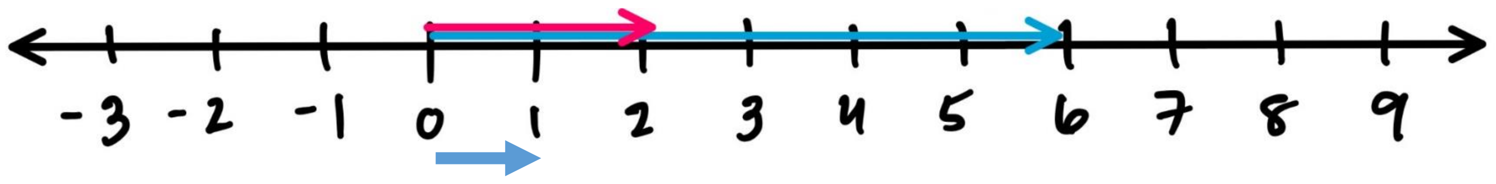

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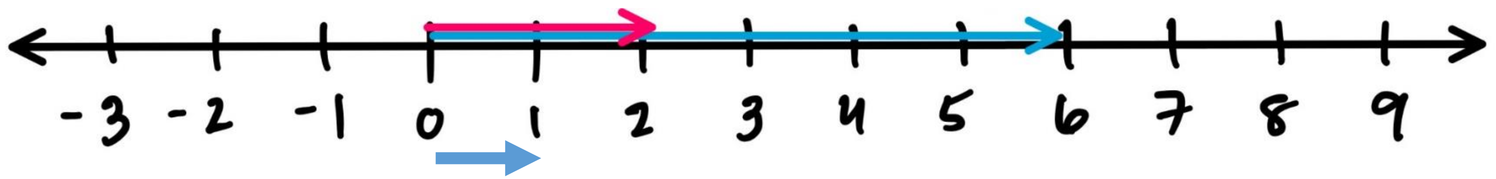

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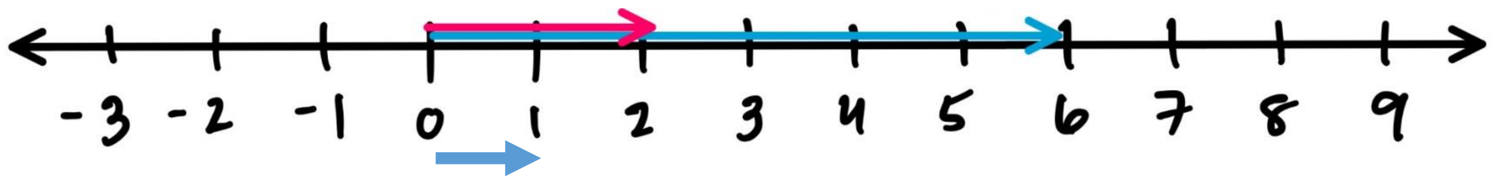

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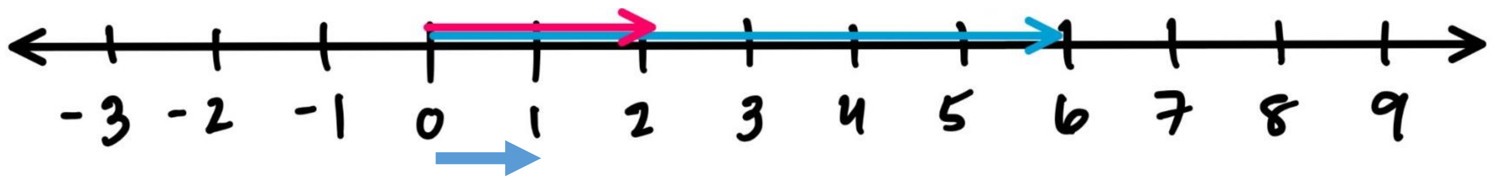
A. Only one. It can be $[\mathbf{1} \ \ \mathbf{0}]^{\boldsymbol{T}}$. Then all others can be formed by simple scaling (a form of linear combination). E.g., $[\mathbf{5} \ \ \mathbf{0}]^{\boldsymbol{T}} = \mathbf{5} \times [\mathbf{1} \ \ \mathbf{0}]^{\boldsymbol{T}}$

Real Line

Let's Consider the Real Number Line $\boldsymbol{R}$ (a subspace of $\boldsymbol{R}^n$)



Each vector on this line can be represented as

$$\begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{0} \end{bmatrix}, \boldsymbol{x} \in \boldsymbol{R}$$

Q. How many linearly independent vectors do we need such that their linear combinations can represent ALL the vectors on the real number line?

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Q. Is this basis vector unique?

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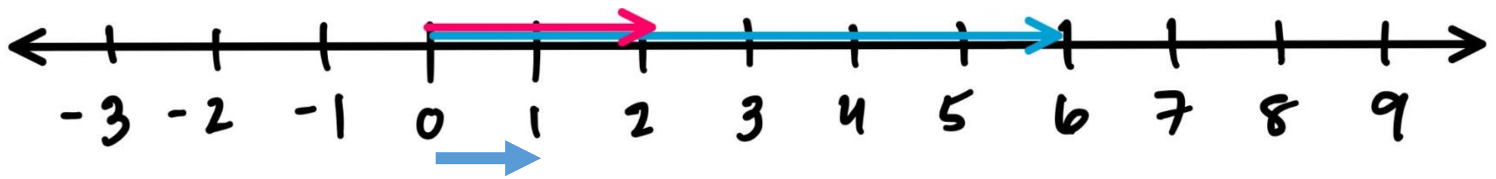

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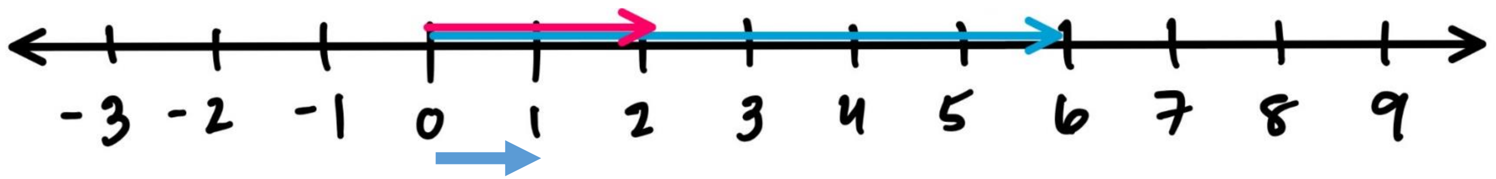

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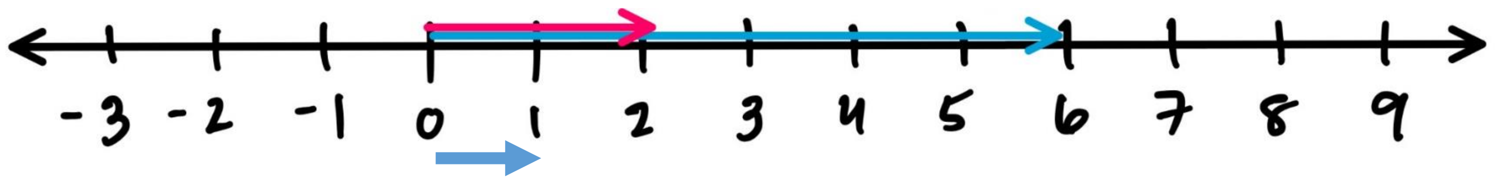

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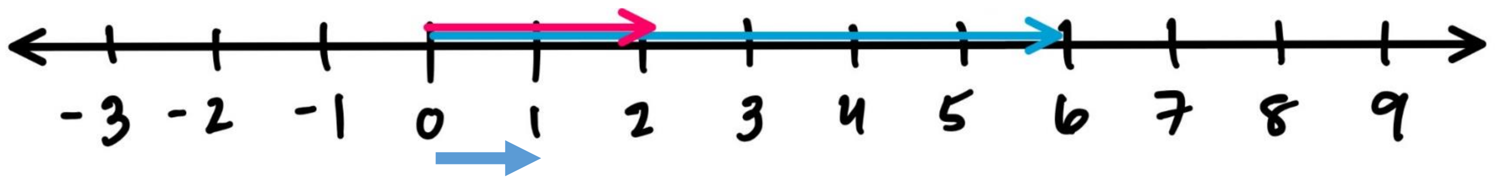

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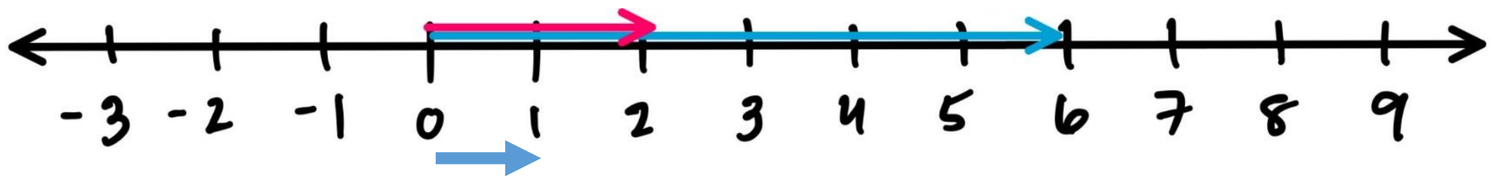

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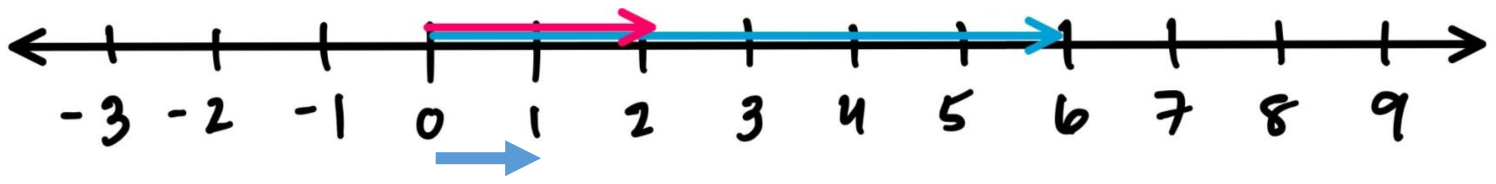

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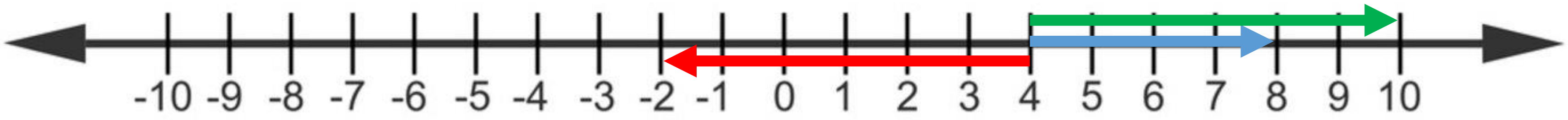
Real Line

A Curious Case…



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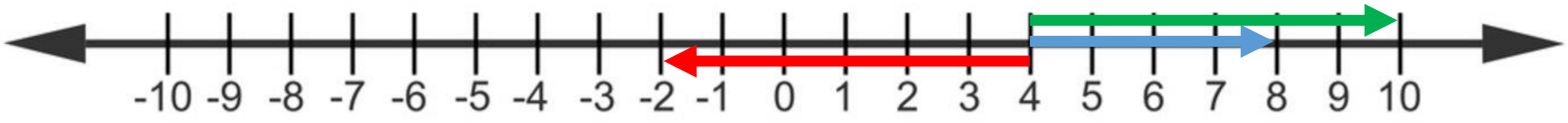

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What if we want to represent a special set of vectors that all start at $[\mathbf{4} \;\; \mathbf{0}]^{\boldsymbol{T}}$ instead of the conventional origin $[\mathbf{0} \;\; \mathbf{0}]^{\boldsymbol{T}}$?

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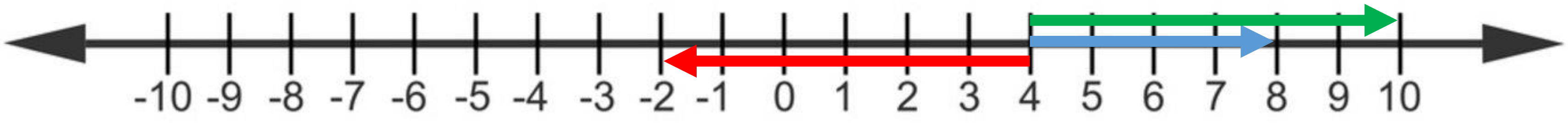

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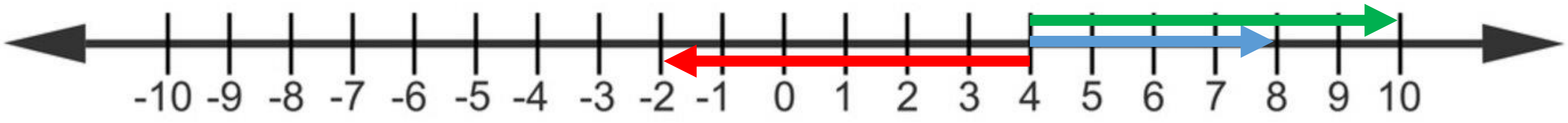

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Original Subspace

"Translation" Vector from within the Original Subspace

Real Line

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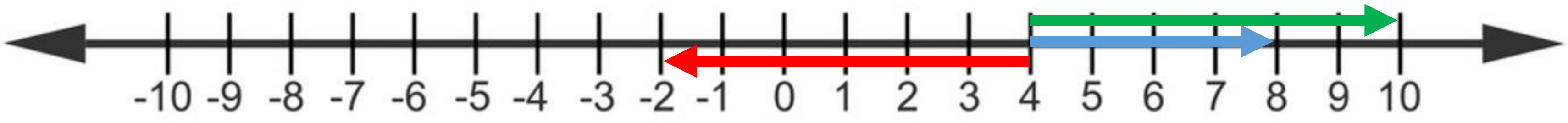
Later we will define this special translated variant of a space (that does not *always* qualify as space) as **Affine Subspace**



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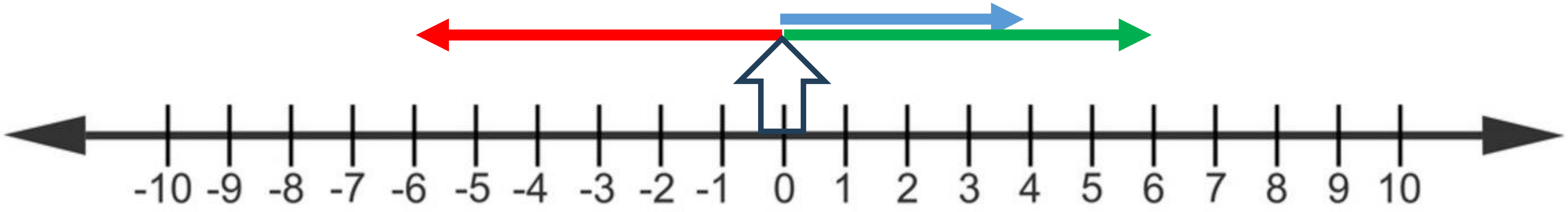

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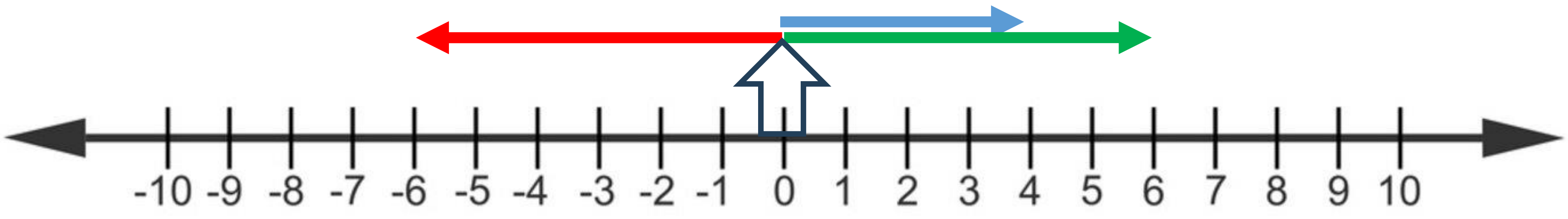

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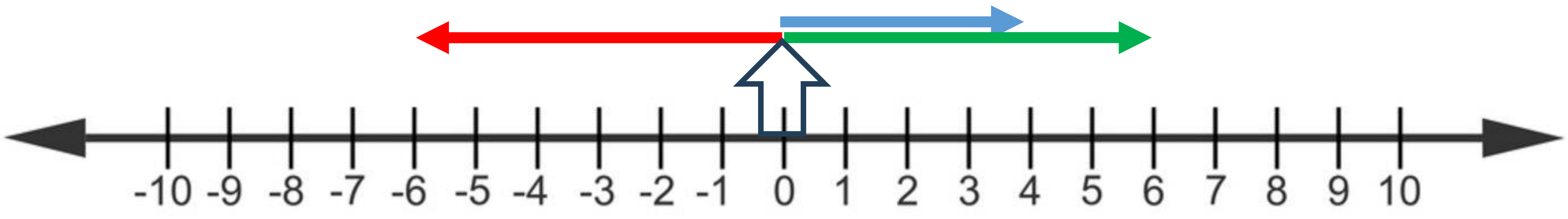

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- Firstly, such a set would not qualify as a Subspace (or Space) as it would fail the identity element axiom (since it does not contain $[\mathbf{0} \;\; \mathbf{0}]^{\boldsymbol{T}}$)

- However, it can still be represented as a translated version of the subspace $\boldsymbol{R}$.

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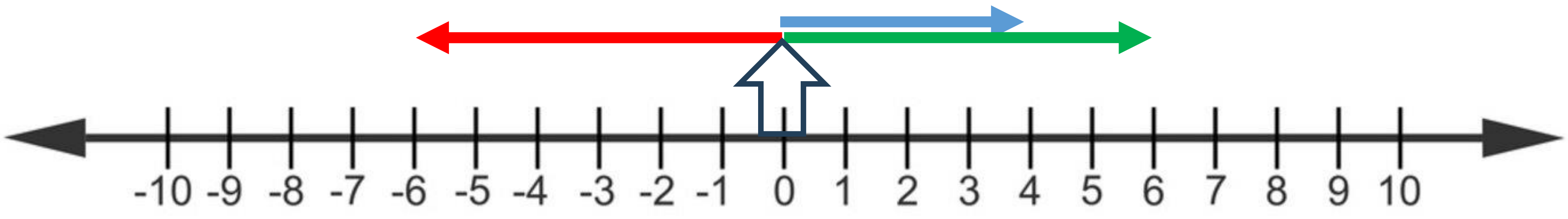

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Typically $\boldsymbol{R}$ or $\boldsymbol{C}$

$$\operatorname{span}(S) = \left\{ \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_n \mathbf{v}_n \;\middle|\; n \in \mathbb{N},\ \mathbf{v}_1, \ldots \mathbf{v}_n \in S,\ \lambda_1, \ldots \lambda_n \in K \right\}$$

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e.g., Let $S = \{\begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$, and $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}\}$ then the span of $\boldsymbol{S}$ is the whole 3D space $\boldsymbol{R}^3$.

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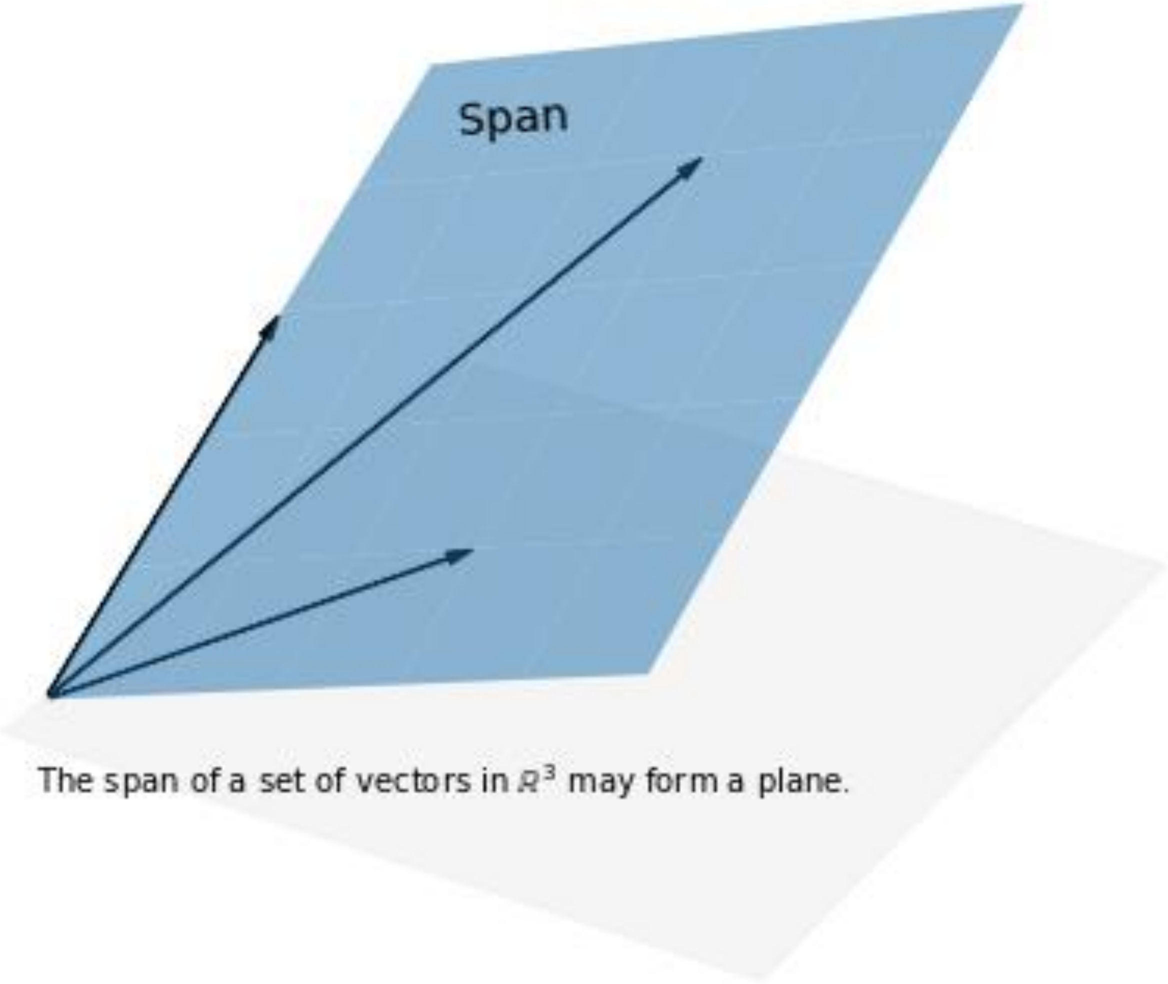

The span of a set of vectors in R^3 may form a plane.

Generating Set

The set of vectors that *spans* a given vector space

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Definition 2.13 (Generating Set and Span)**.** Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and set of vectors $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subseteq \mathcal{V}$. If every vector $\boldsymbol{v} \in \mathcal{V}$ can be expressed as a linear combination of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$, $\mathcal{A}$ is called a *generating set* of V. The set of all linear combinations of vectors in $\mathcal{A}$ is called the *span* of $\mathcal{A}$. If $\mathcal{A}$ spans the vector space V, we write $V = \mathrm{span}[\mathcal{A}]$ or $V = \mathrm{span}[\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k]$.

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e.g., $S = \{\begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix}$, and $\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}\}$ is a generating set of vector space $\boldsymbol{R}^3$.

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May contain redundancies.

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Example

Both $\boldsymbol{S_1} = \left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix} \right\}$ and $\boldsymbol{S_2} = \left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix} \right\}$ are Generating Sets of $\boldsymbol{R^3}$,

only $\boldsymbol{S_1}$ forms a "Basis" as it is minimal and necessarily linearly independent.

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Note that the Basis of a Vector Space are not unique.

E.g., $\boldsymbol{S_3} = \left\{ \begin{bmatrix} \mathbf{2} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{3} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{4} \end{bmatrix} \right\}$ also forms a Basis for $\boldsymbol{R^3}$

Basis

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Example

Both $\boldsymbol{S_1} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\boldsymbol{S_2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\}$ are Generating Sets of $\boldsymbol{R}^3$,

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E.g., $\boldsymbol{S_3} = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$ also forms a Basis for $\boldsymbol{R}^3$

However, since basis are necessarily minimal generating sets, they are always of the same size (e.g., any basis $\boldsymbol{S}$ of a $\boldsymbol{R}^3$ will always contain three linearly independent vectors, no more no less).

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For a given vector space, the number of basis vectors is fixed and is called "Dimension of the Space"

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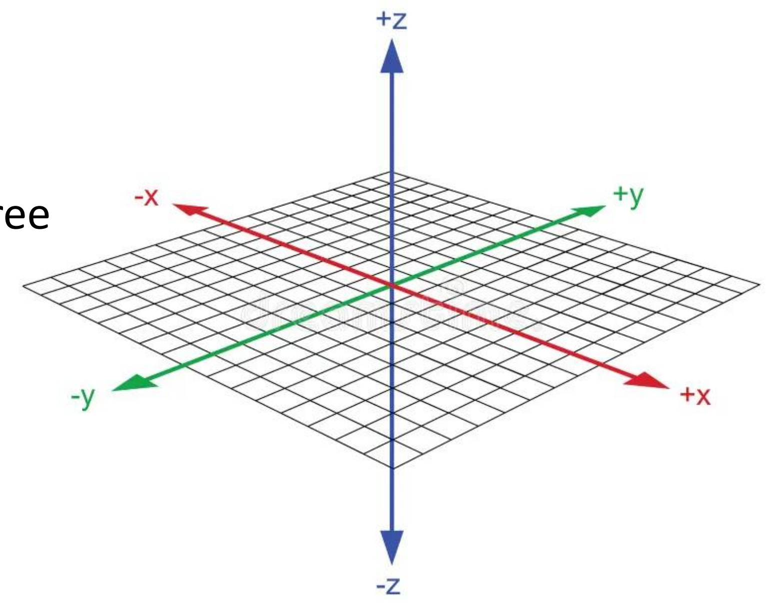
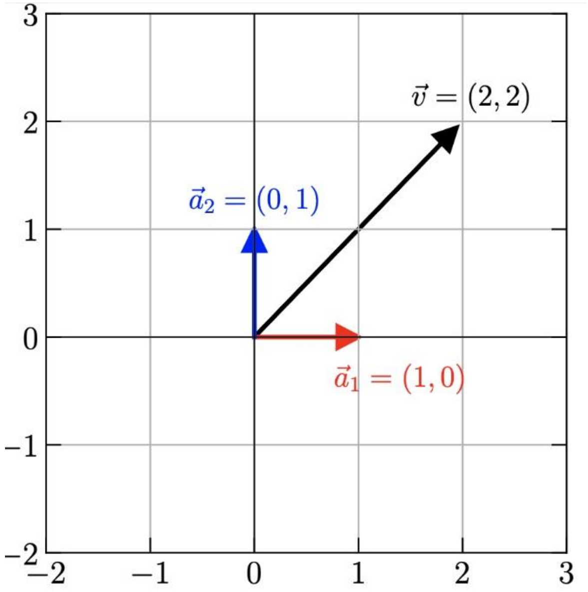
e.g., Dimension of $\boldsymbol{R}^3$ is 3 as its basis are always three linearly independent vectors.



Cartesian Plane

Let us now consider $\boldsymbol{R}^2$ (a subspace of $\boldsymbol{R}^n$)



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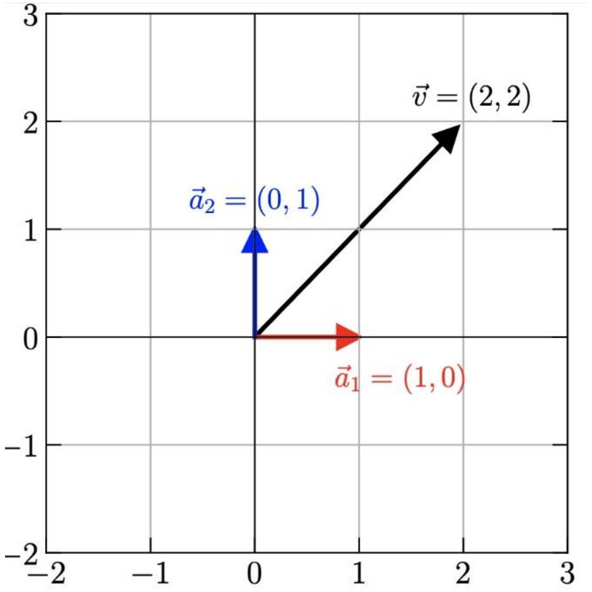

Each vector on this plane can be represented as

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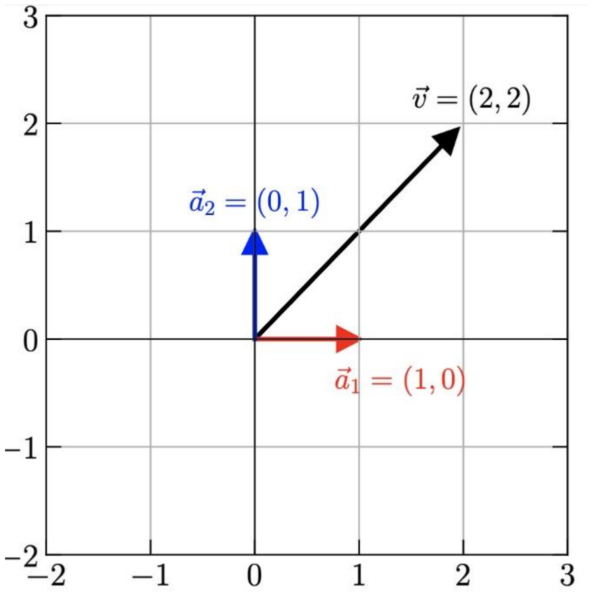

Q. How many vectors do we need to represent all vectors in $\boldsymbol{R}^2$ as their linear combinations?

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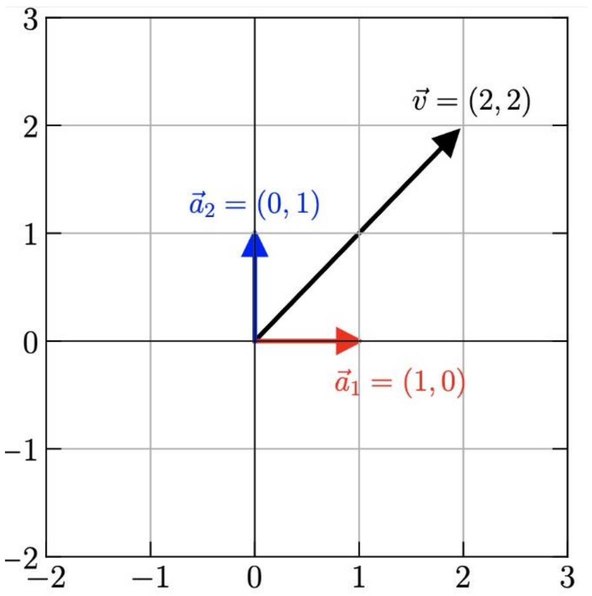

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Q. How many vectors do we need to represent all vectors in $\boldsymbol{R}^2$ as their linear combinations?

A. Only two. It could be $[\mathbf{1} \;\; \mathbf{0}]^{\boldsymbol{T}}$ and $[\mathbf{0} \;\; \mathbf{1}]^{\boldsymbol{T}}$

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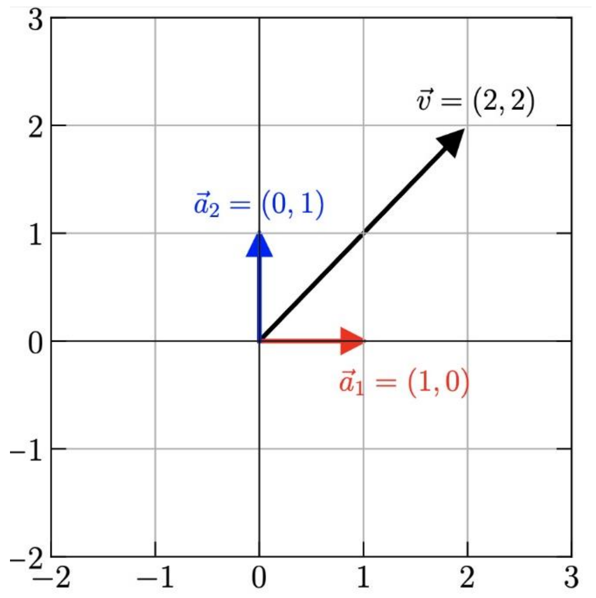

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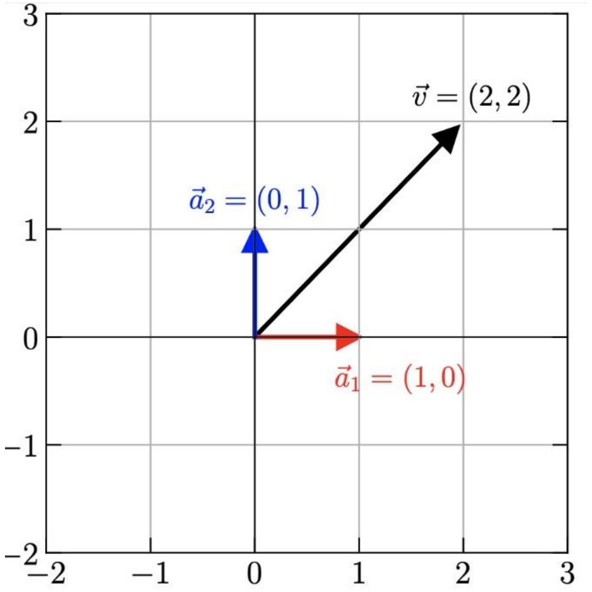
Cartesian Plane

Let us now consider $\boldsymbol{R}^2$ (a subspace of $\boldsymbol{R}^n$)



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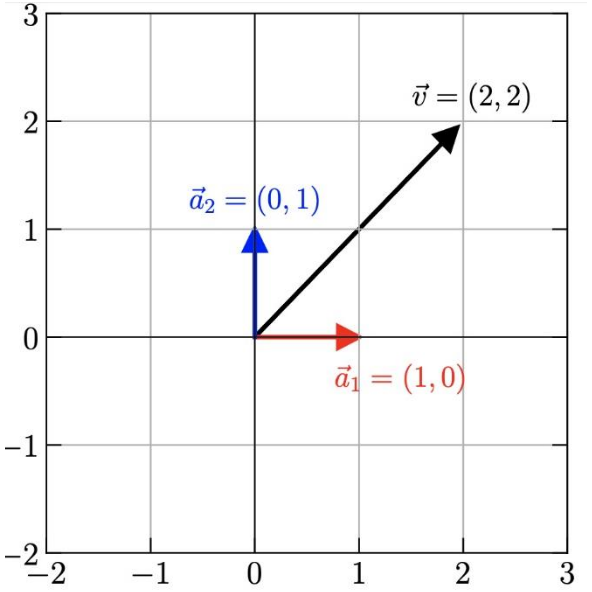

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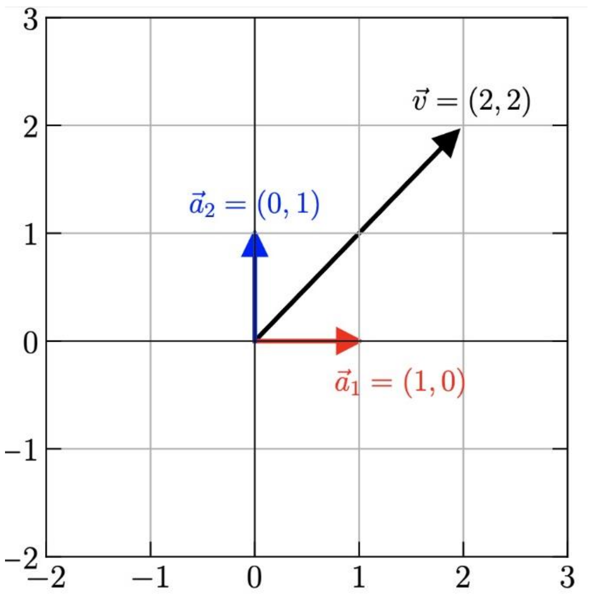

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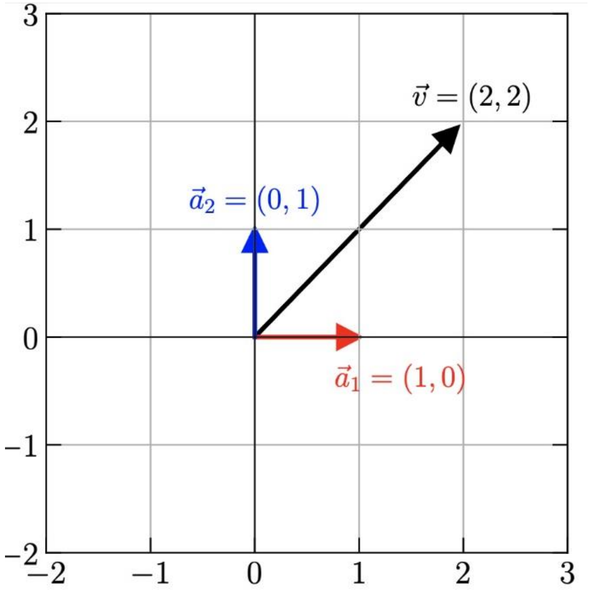

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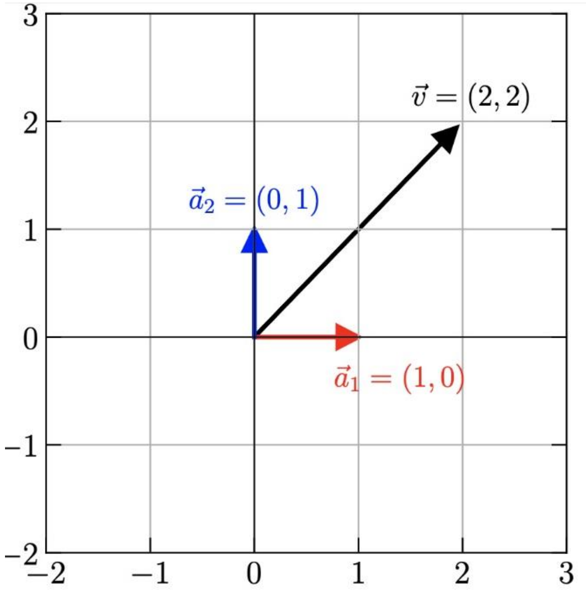

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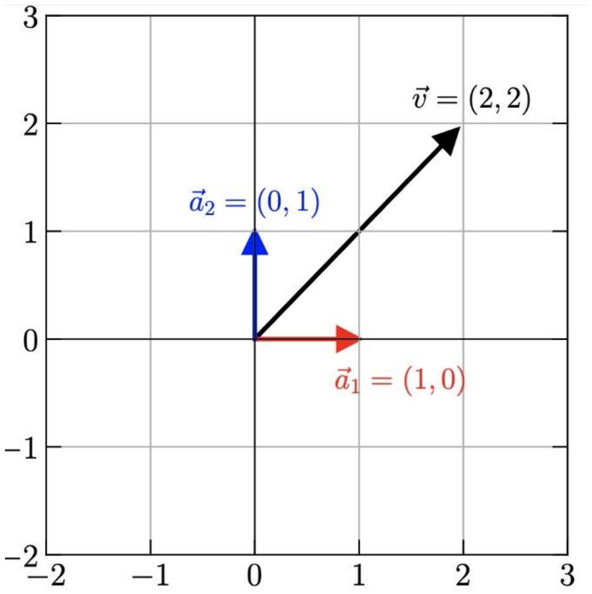

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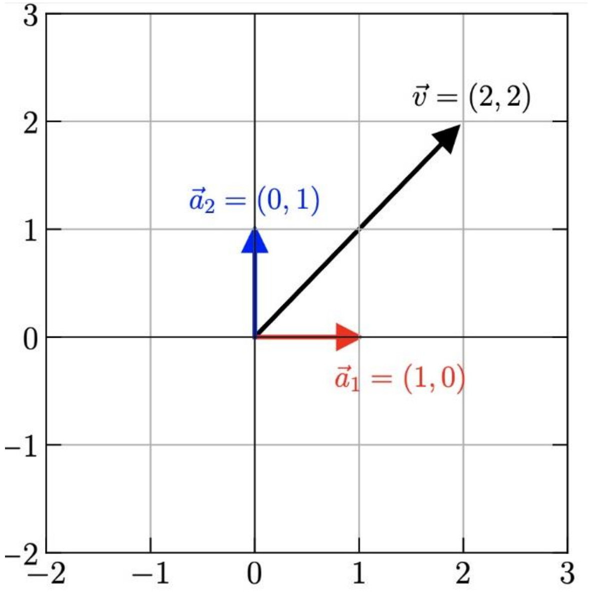

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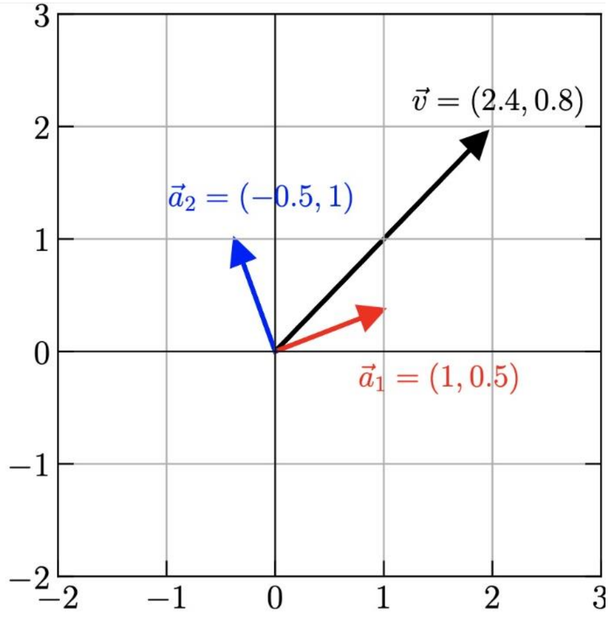

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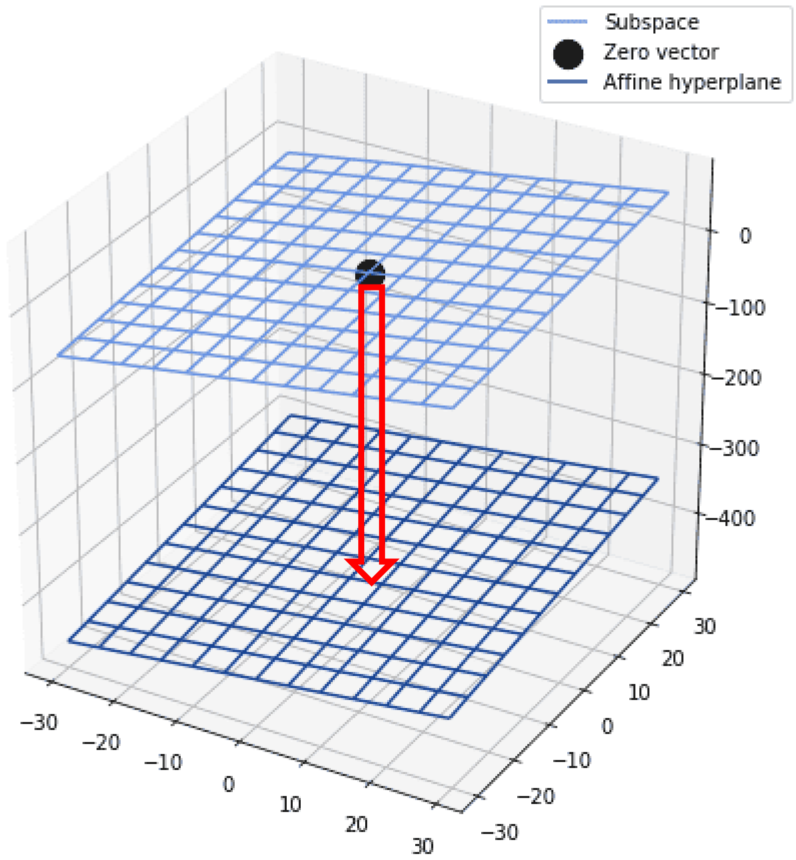

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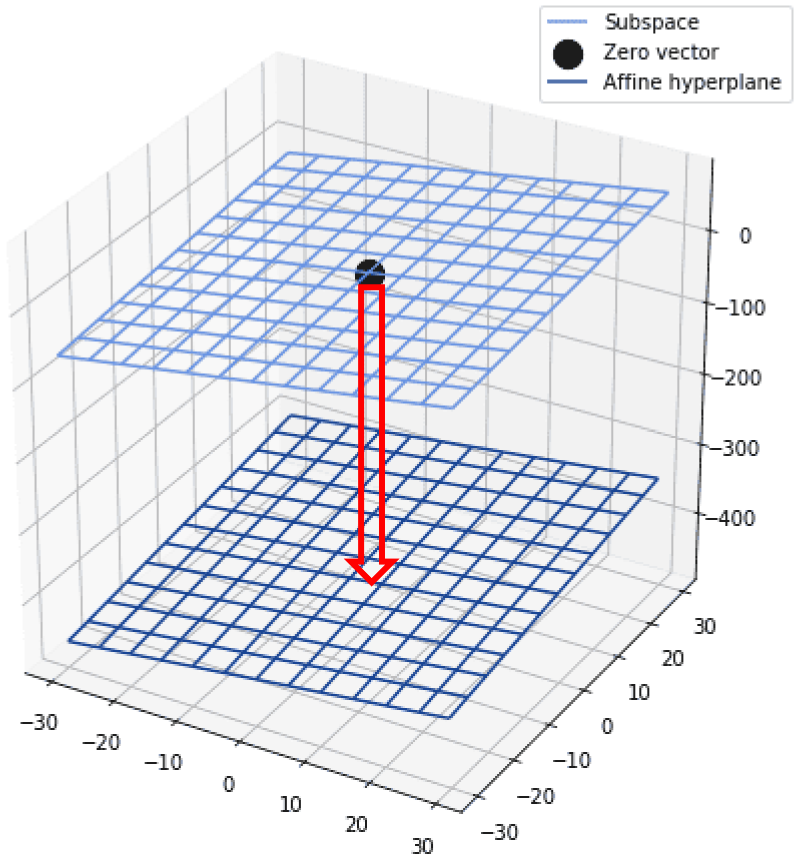

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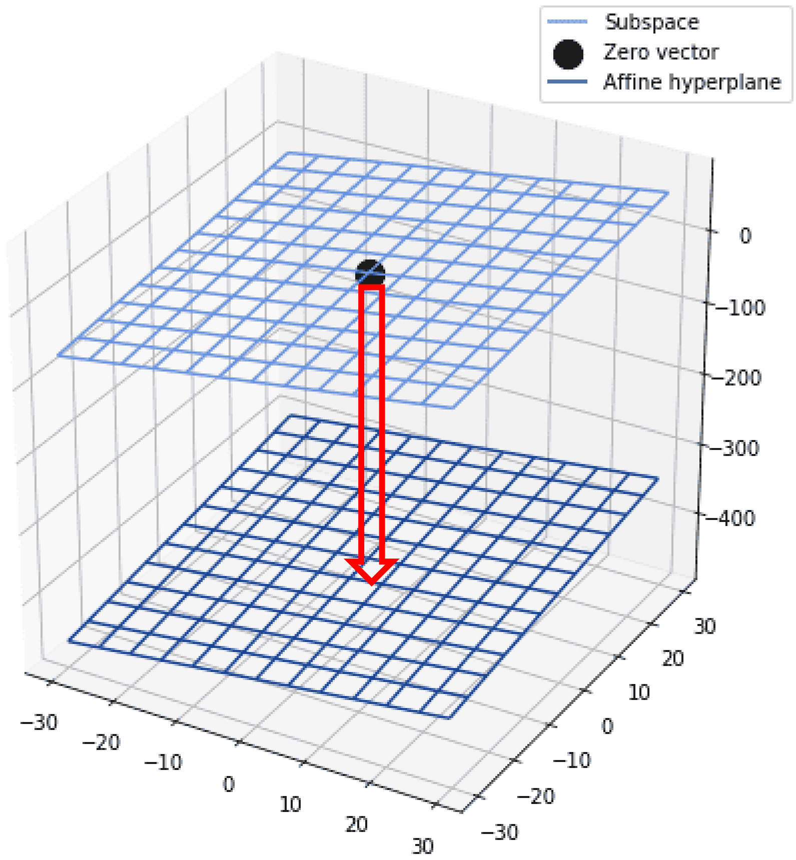

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- However, note that it would not qualify as a subspace if the Translation Vector is not from within the original Space $\boldsymbol{R}^2$ (as it would not contain $[\mathbf{0}\ \ \mathbf{0}]^T$)
- In any case, we call such a translated version **Affine Subspace** or **Linear Manifold**.

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Some Useful Results on Generating Sets, Span, Linear Independence, and Basis

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No such result for *tall* matrices.

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Here $\begin{bmatrix}\mathbf{2}\\\mathbf{0}\end{bmatrix}$ is in the span of the other two.

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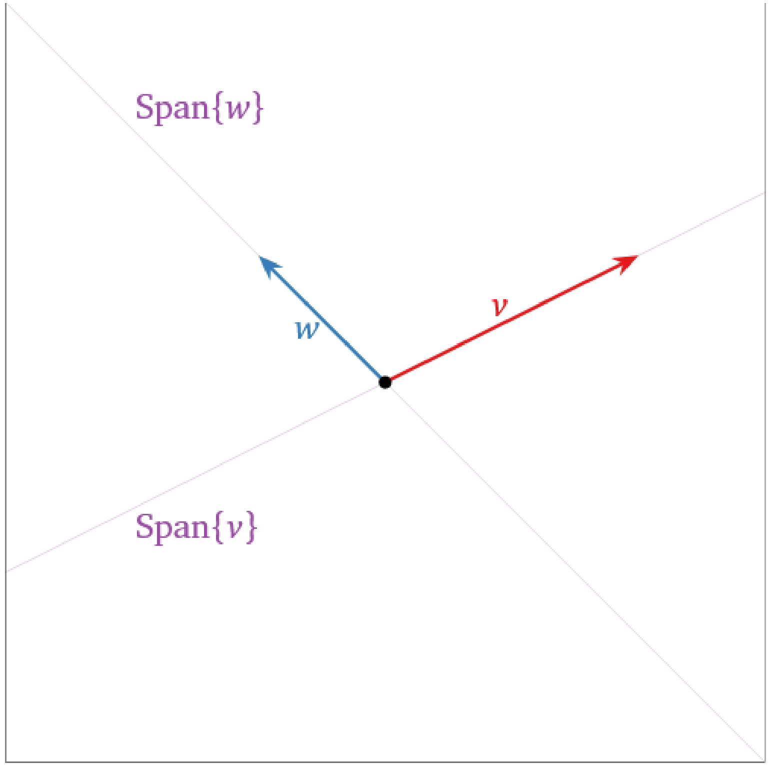
e.g., $span(v)$ is a just a line (shown). $span(w)$ is also just a line (though different one).
Together their span is a plane!



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For now, a simple trailer….

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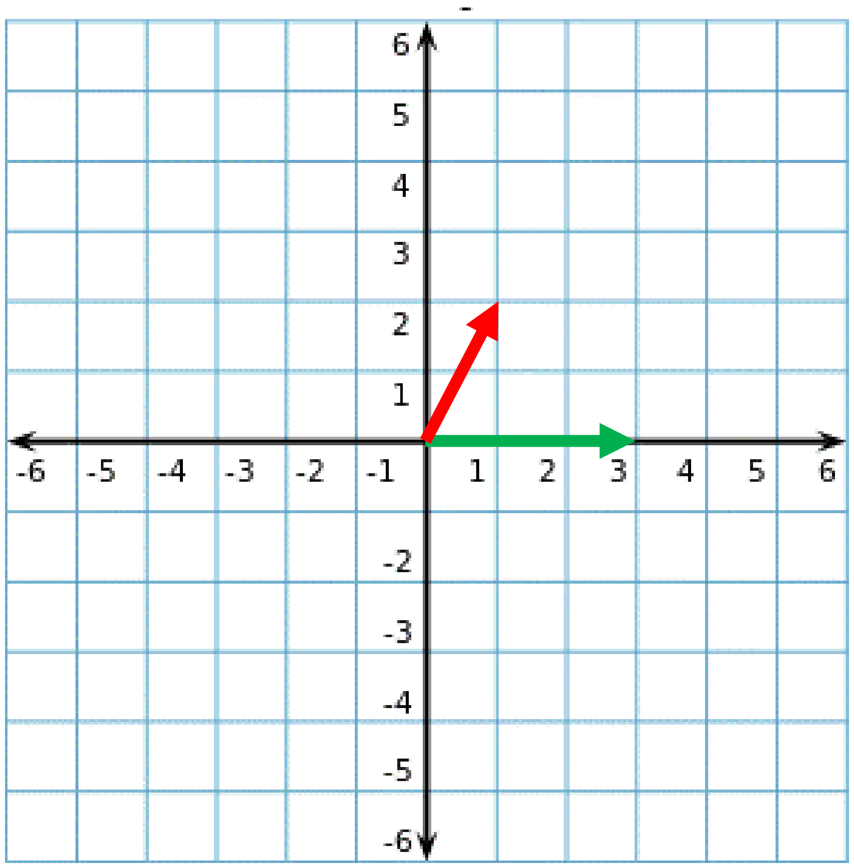

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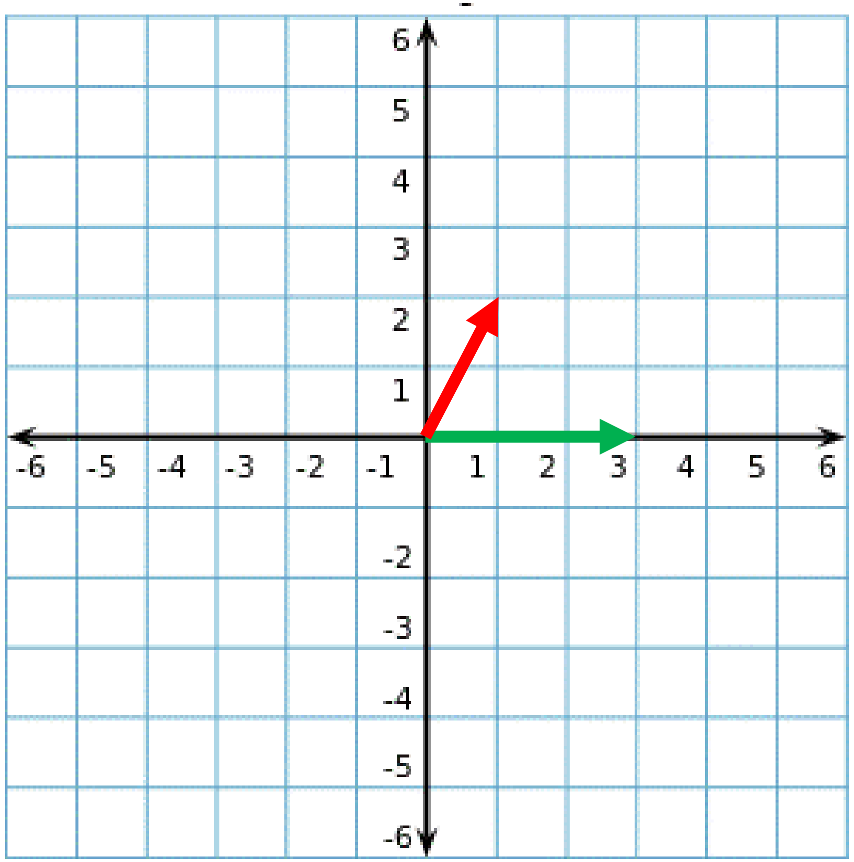
What if we could write these in terms of an alternate set of basis where $\boldsymbol{A}$ is diagonal? Such as $[\mathbf{1} \;\; \mathbf{0}]^{\boldsymbol{T}}$ and $[-\mathbf{1} \;\; \mathbf{1}]^{\boldsymbol{T}}$.

You can check that these are linearly independent.



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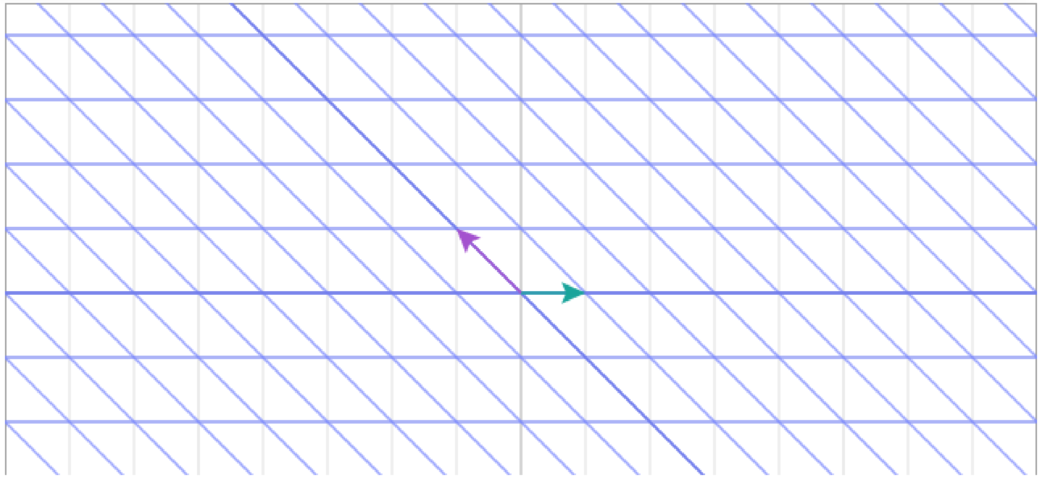

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https://www.3blue1brown.com/lessons/eigenvalues

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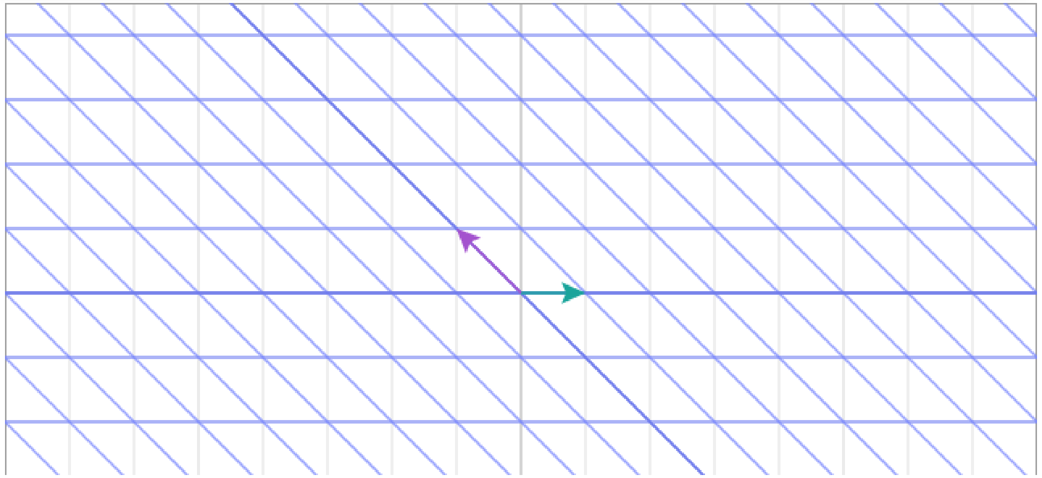

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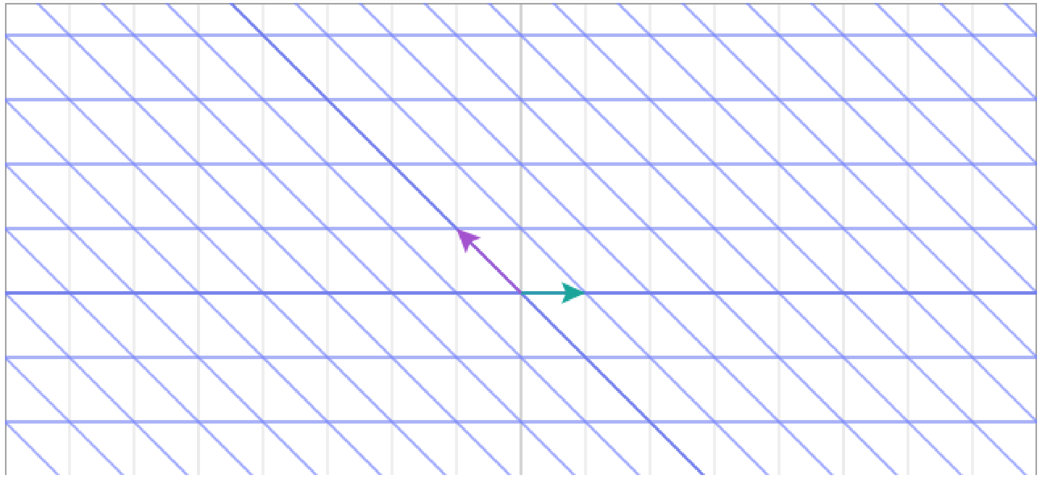

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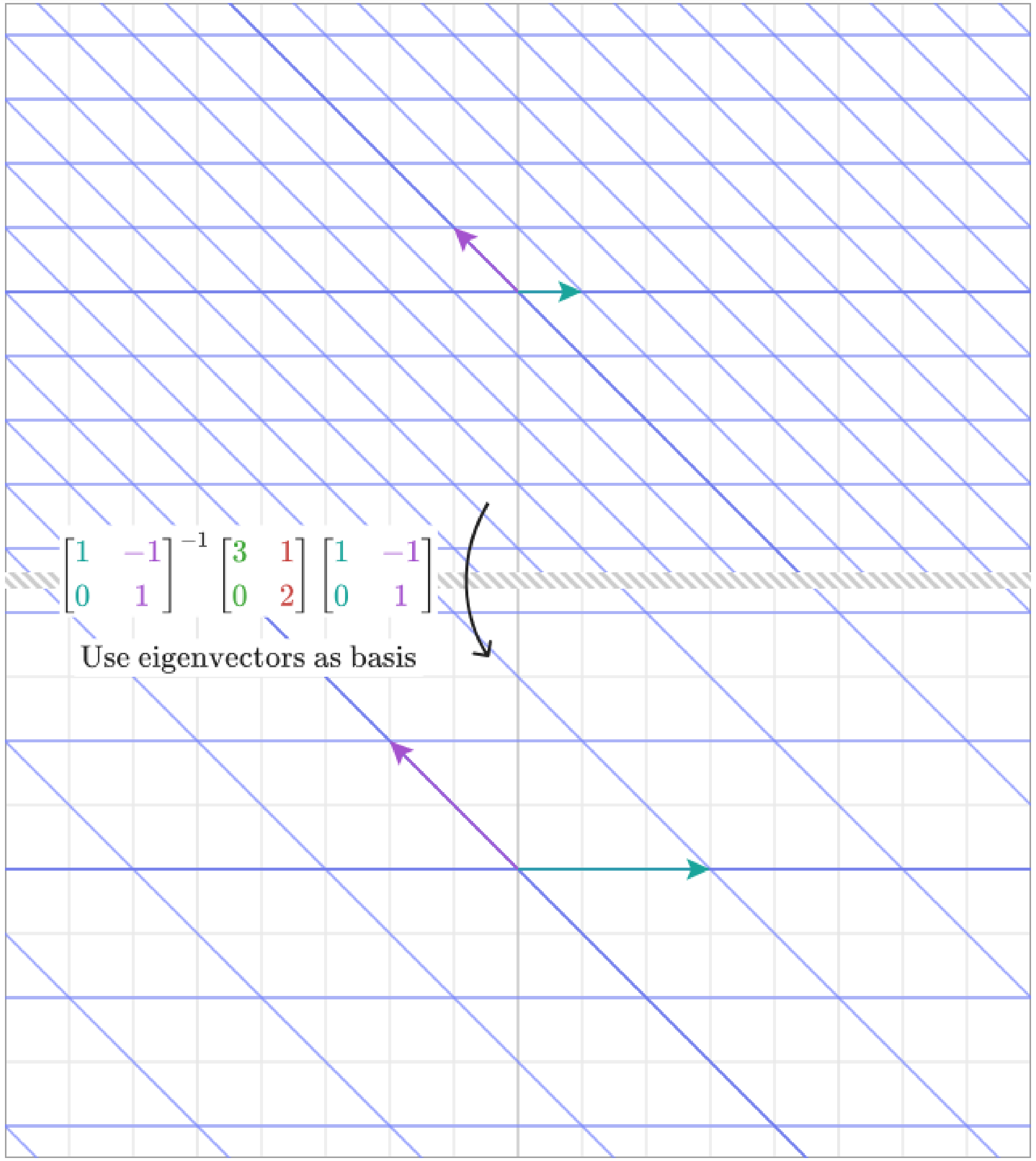
Details later.

https://www.3blue1brown.com/lessons/eigenvalues

Given Matrix A, Find A^{100}



The new basis can effectively be seen as forming a new grid.

What if there were a procedure to carry out the change of basis? (there actually is.)

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New representation of $\boldsymbol{A}$.

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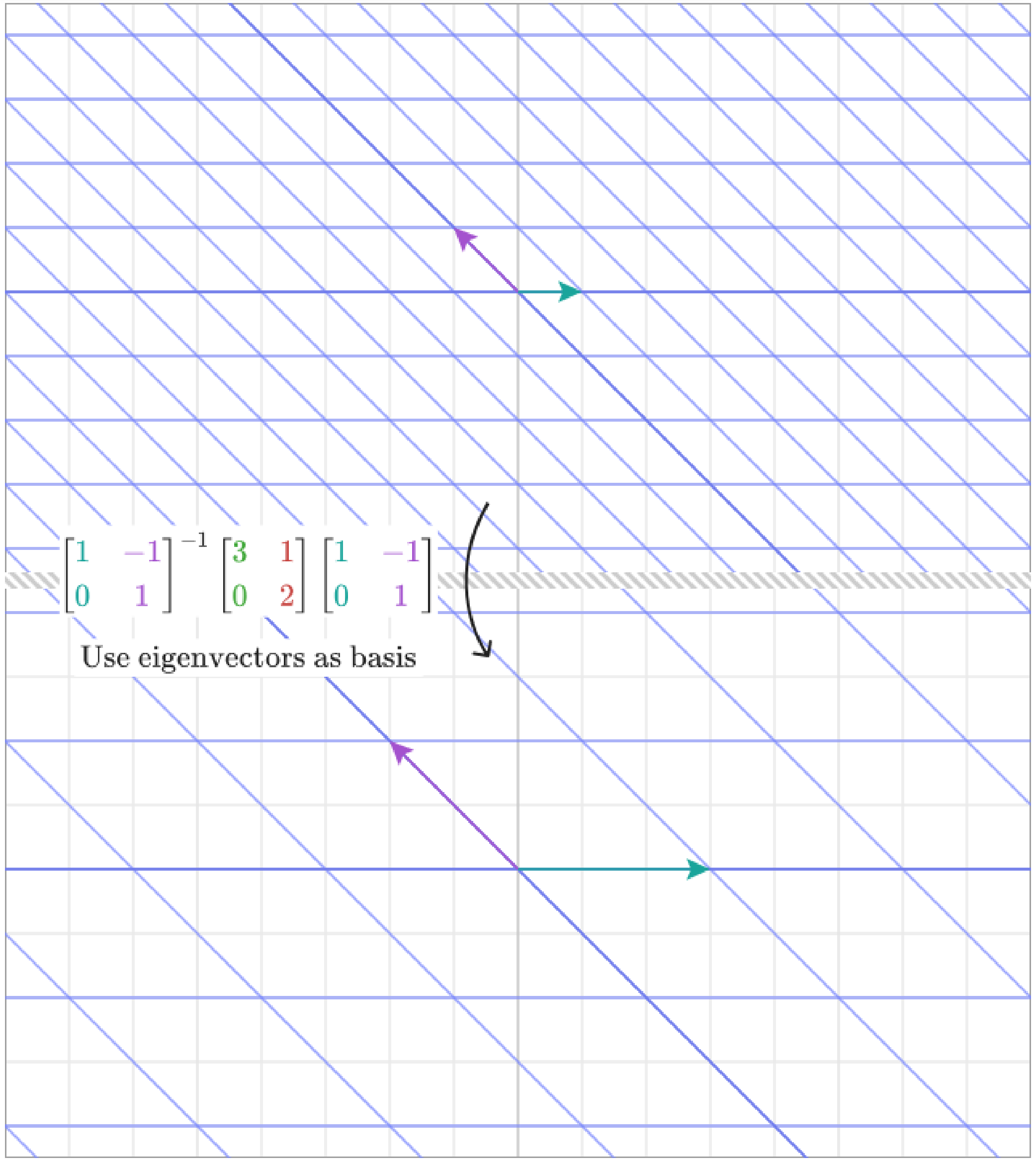

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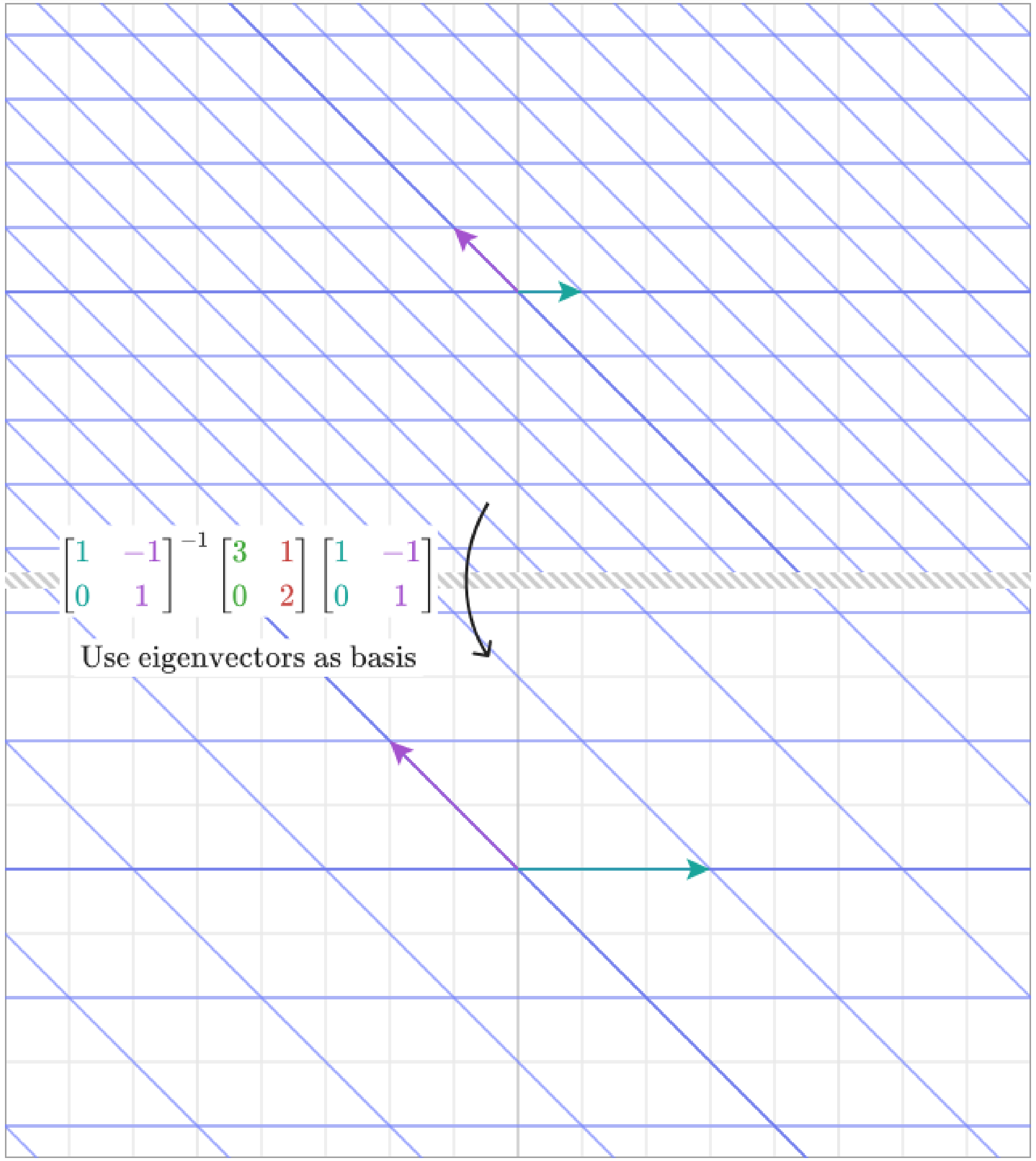

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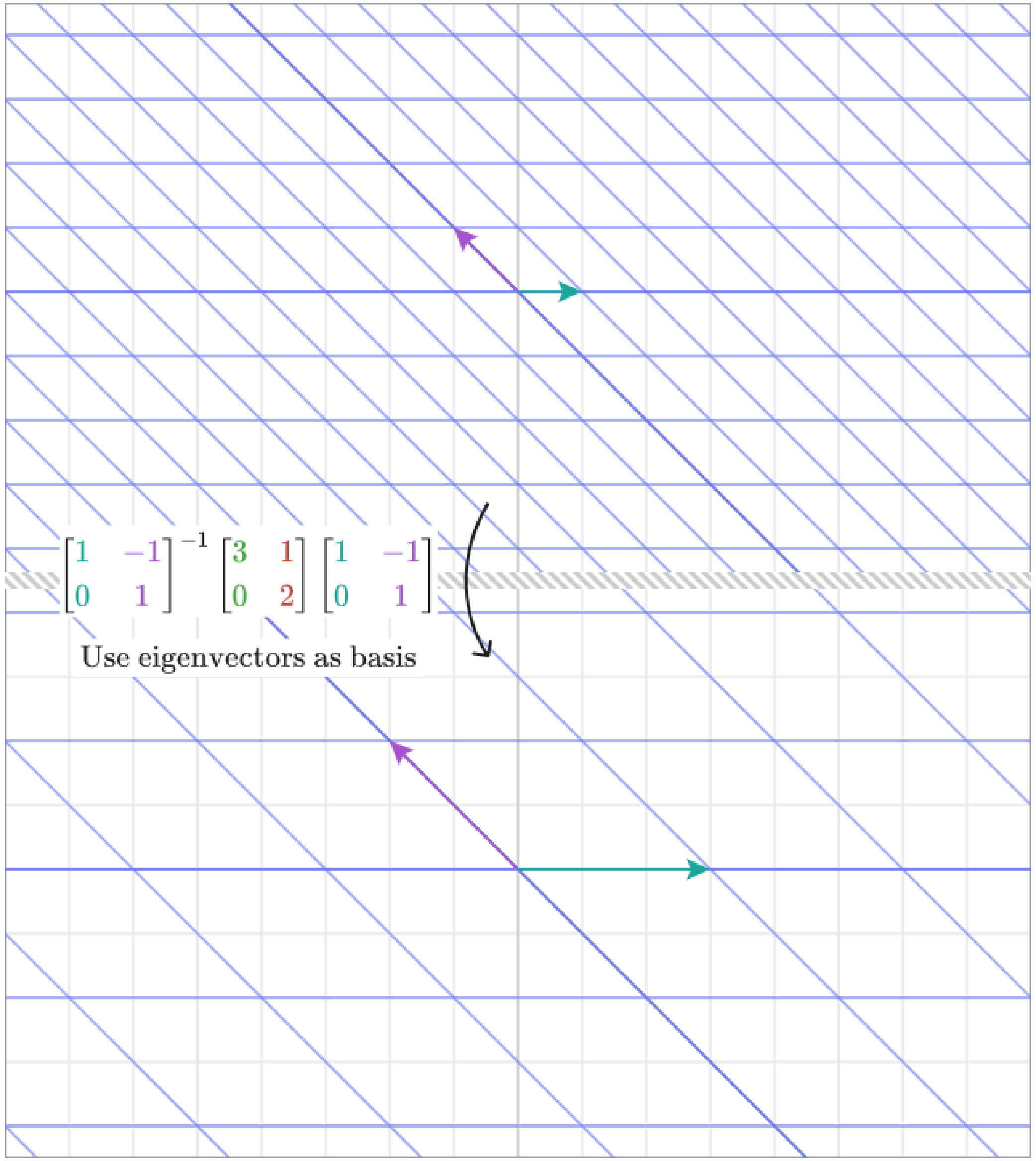

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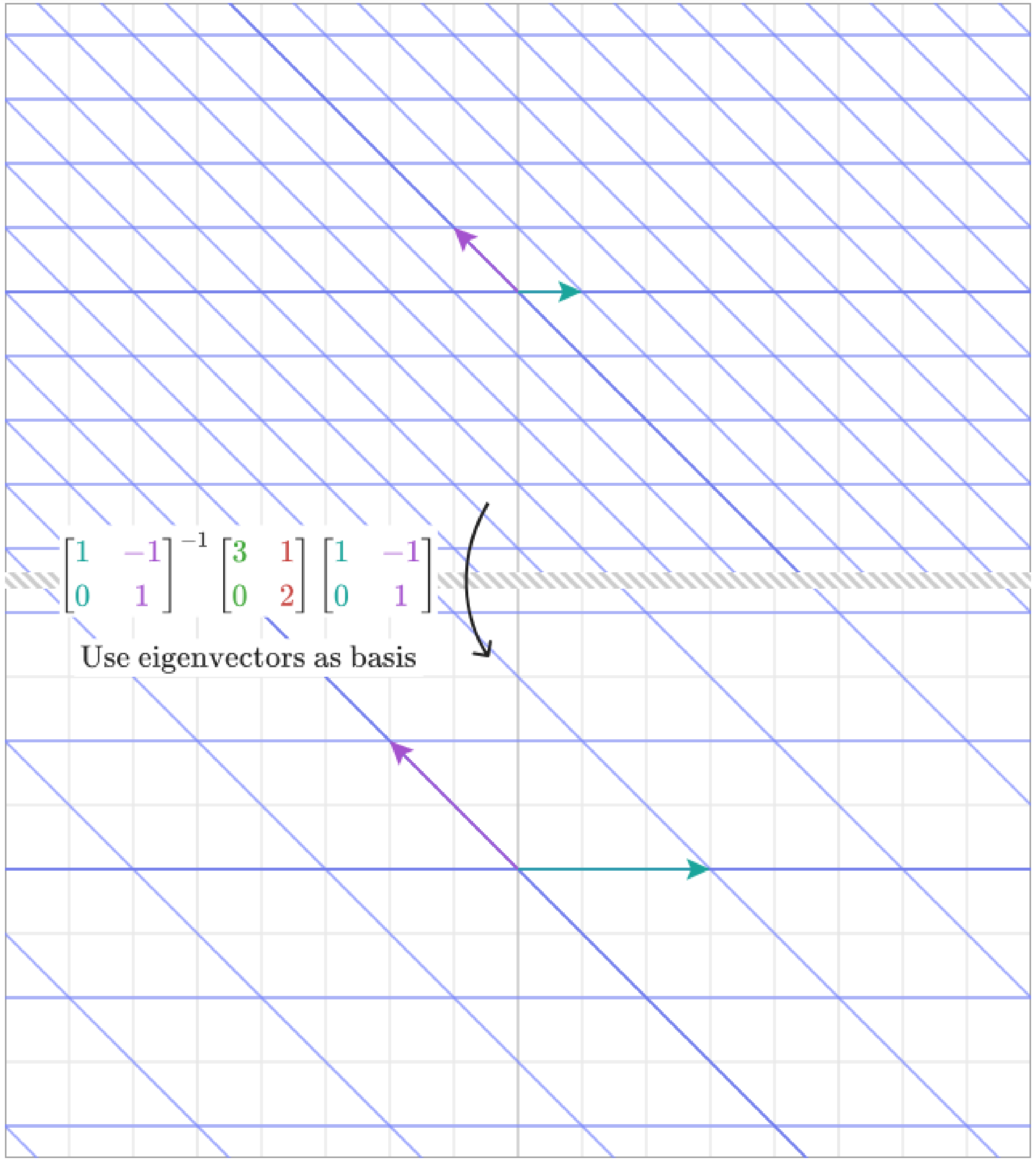

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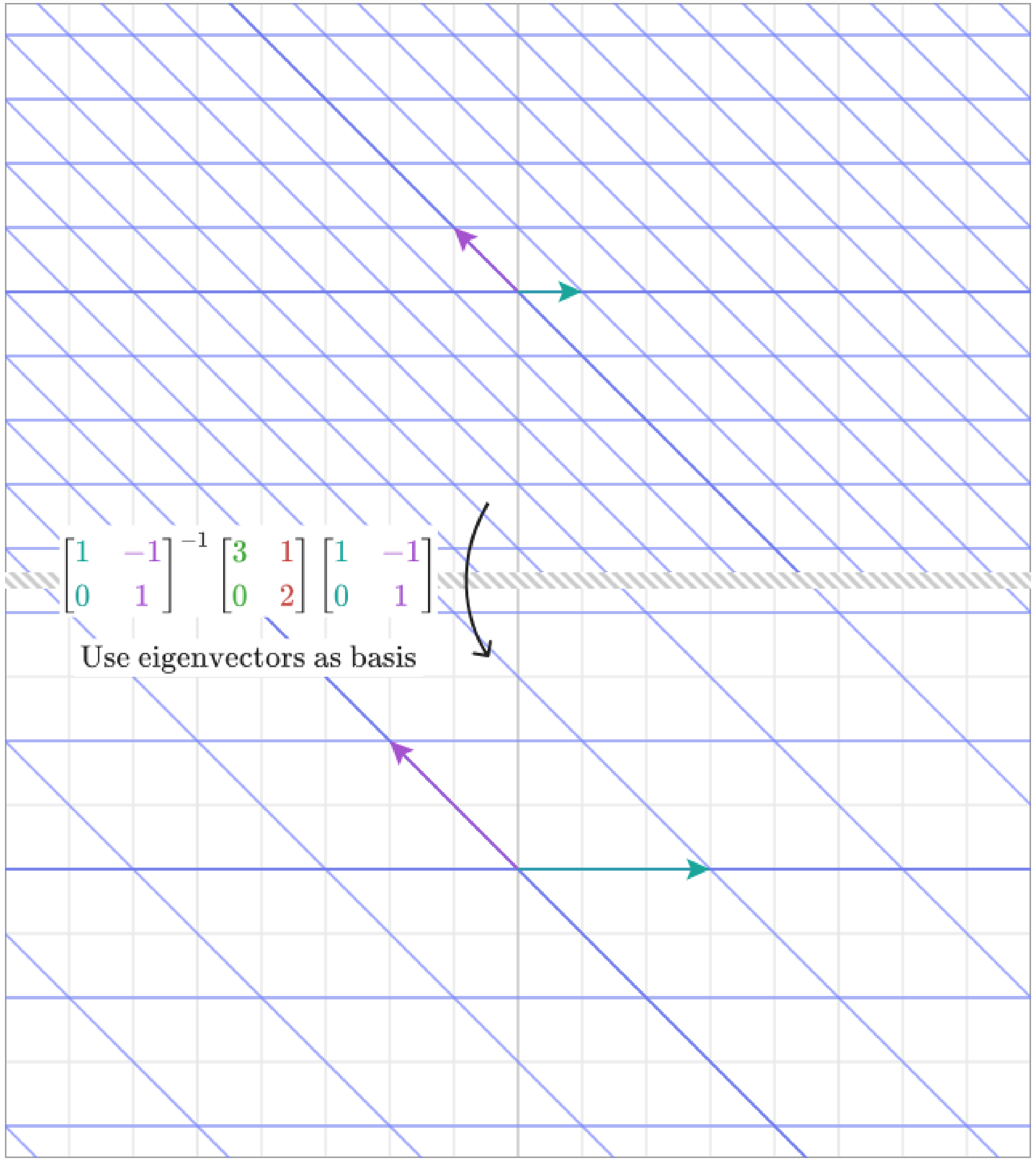

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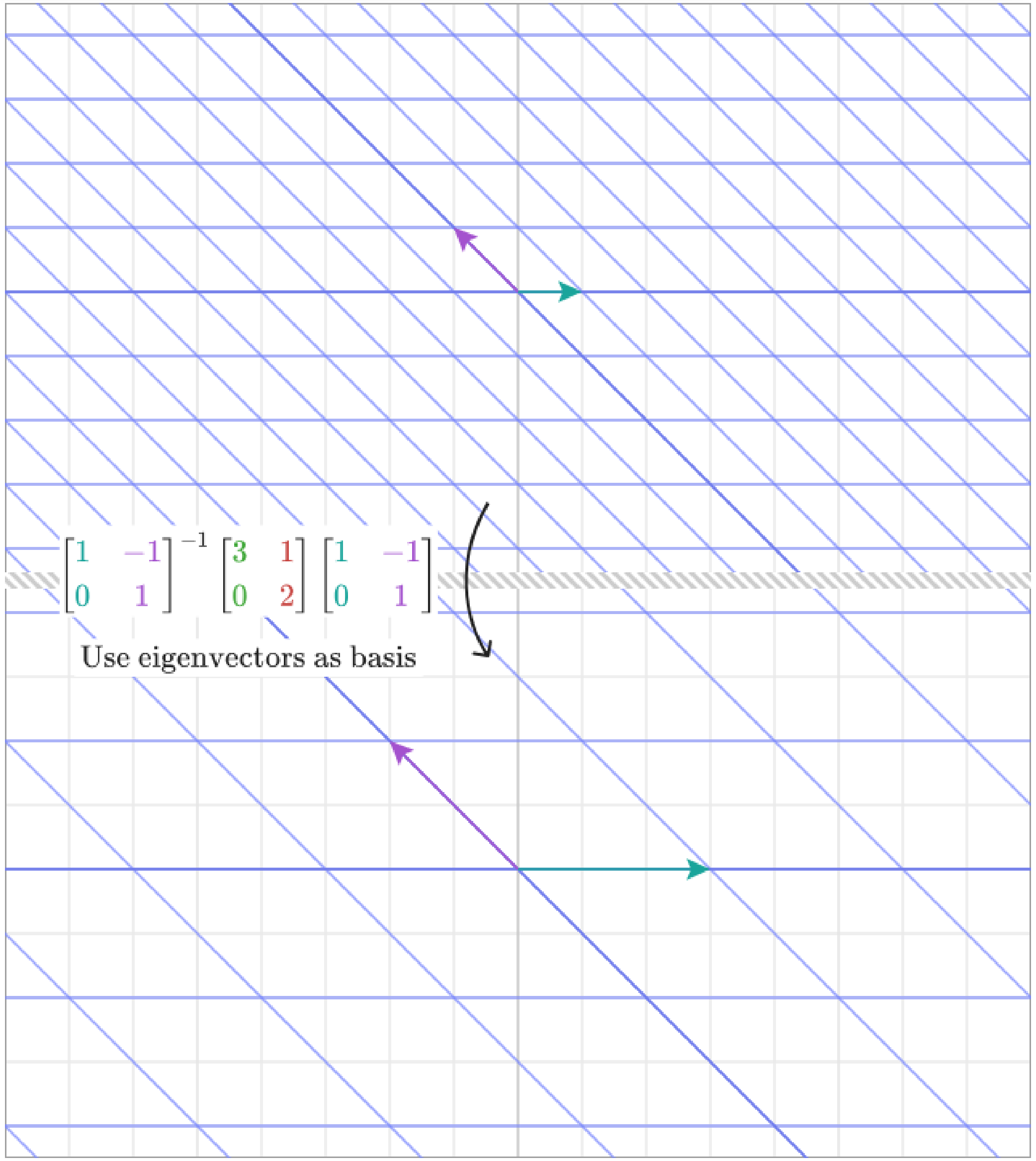

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Answers coming soon when we visit *Eigenvectors*.

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Transformations

Changing One Vector Space to [Possibly] Another

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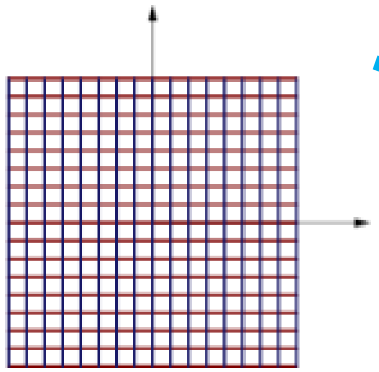
$f\Big($  $\Big)$

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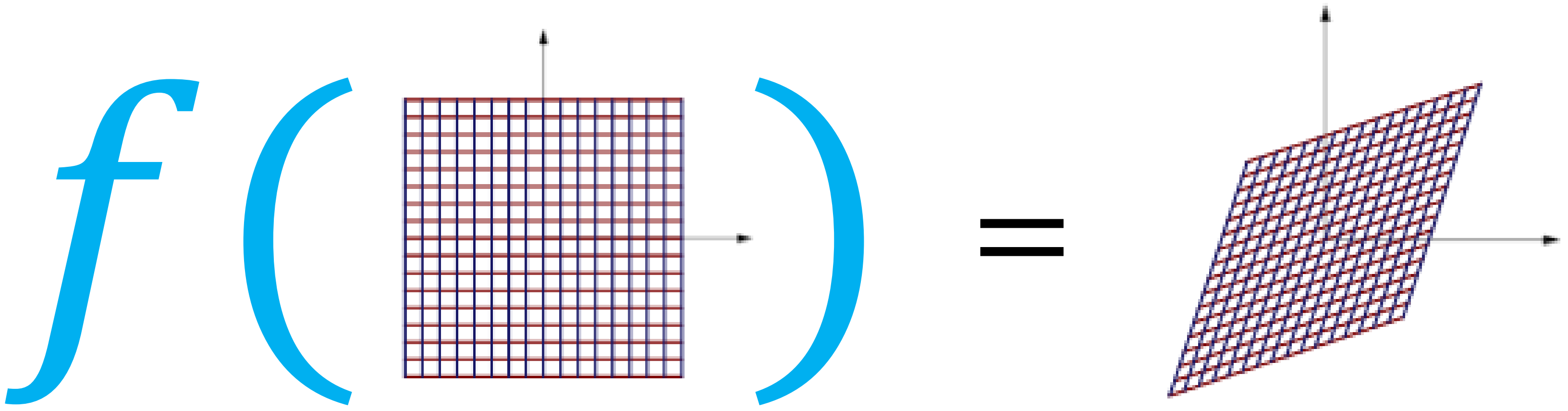

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$\boldsymbol{W}$
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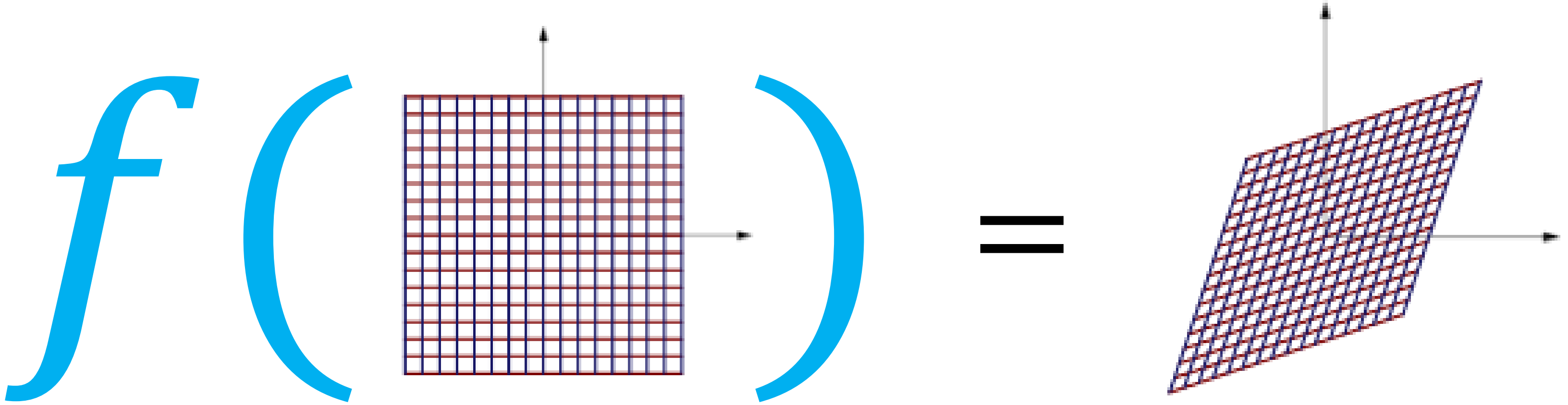
Questions we may like to ask

- Is the output also a vector space?
- Is the transformation invertible?
- How does the transformation affect various attributes of input vector space (basis, dimension, origin, etc.)
- Is the transformed version more beneficial for us?



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For that, let’s dial back a bit…

Stay tuned…

Questions?? Thoughts??