Assignment 1

1 F frame Coordinates in terms of N frame. DC matrix is

$$\begin{Bmatrix} x_F \\ y_F \\ z_F \end{Bmatrix} = \underbrace{\begin{bmatrix} \frac{3}{4} & -\frac{2}{4} & \frac{\sqrt{3}}{4} \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{4} & -\frac{2\sqrt{3}}{4} & -\frac{1}{4} \end{bmatrix}}_{A} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix} \quad \text{--- } (1)$$

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B frame coordinates in terms of N frame:
DC Matrix is

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \underbrace{\begin{bmatrix} +\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{4}{3\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{bmatrix}}_{B} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix} \quad \text{--- } (2)$$

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The DC matrix of B wrt F is

from (1)

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = B \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix} \quad \& \quad \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix} = A^{-1} \begin{Bmatrix} x_F \\ y_F \\ z_F \end{Bmatrix} \quad \text{--- } (3)$$

using (3)

$$\therefore \begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = BA^{-1} \begin{Bmatrix} x_F \\ y_F \\ z_F \end{Bmatrix}$$

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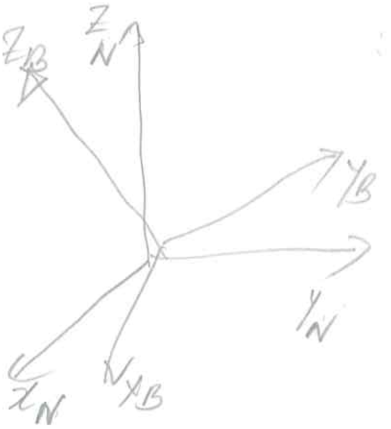
$$= BA^{T} \begin{Bmatrix} x_F \\ y_F \\ z_F \end{Bmatrix}$$

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} -0.372 & -0.744 & -0.555 \\ -0.0474 & 0.6124 & -0.7892 \\ 0.92702 & -0.2673 & -0.2631 \end{bmatrix} \begin{Bmatrix} x_F \\ y_F \\ z_F \end{Bmatrix}$$

Good

#2

(a) $$\begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix}$$



$$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix} \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix}$$

$x_N y_N z_N$

$z_N \downarrow 30^\circ (\phi)$

$x_1 y_1 z_1$

$y_1 \downarrow -45^\circ (\theta)$

$x_2 y_2 z_2$

$x_2 \downarrow 60 \quad (\psi)$

$x_B y_B z_B$

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & S\psi \\ 0 & -S\psi & C\psi \end{bmatrix} \begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix}$$

3-2-1 rotation matrix is

$$\therefore \begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & S\psi \\ 0 & -S\psi & C\psi \end{bmatrix} \begin{bmatrix} C\theta & 0 & -S\theta \\ 0 & 1 & 0 \\ S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix}$$

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$$= \begin{bmatrix} C\theta & 0 & -S\theta \\ S\psi S\theta & C\psi & S\psi C\theta \\ C\psi S\theta & -S\psi & C\psi C\theta \end{bmatrix} \begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix}$$

$$= \begin{bmatrix} C\theta C\phi & C\theta S\phi & -S\theta \\ C\phi S\psi S\theta - S\phi C\psi & S\psi S\theta S\phi + C\psi C\phi & S\psi C\theta \\ C\phi C\psi S\theta + S\psi S\phi & S\phi C\psi S\theta - S\psi C\phi & C\psi C\theta \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix} \quad -(1)$$

When $\phi = 30^\circ$, $\theta = -45^\circ$ & $\psi = 60^\circ$

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} 0.61237 & 0.35355 & 0.70711 \\ -0.78033 & 0.12683 & 0.61237 \\ 0.12683 & -0.92678 & 0.35355 \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix} \quad -(2)$$

called $\bar{C}$ matrix qed

(b) Let $\theta = +90^\circ$ ① becomes

$$\begin{Bmatrix} x_B \\ y_B \\ z_B \end{Bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ C\phi S\psi - S\phi C\psi & S\psi S\phi + C\psi C\phi & 0 \\ C\phi C\psi + S\psi S\phi & S\phi C\psi - S\psi C\phi & 0 \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ S(\psi-\phi) & C(\psi-\phi) & 0 \\ C(\psi-\phi) & -S(\psi-\phi) & 0 \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix}$$ 5

$\therefore$ When $\theta = +90^\circ$ or -90°, we will not be able to differentiate and uniquely identify ψ or ϕ only $\psi - \phi$ can be computed.

$\therefore$ The unsymmetric sequence 3-2-1 has a singularity when $\theta = \pm \pi/2$.

(c) $$\phi = \text{Euler Angle} = \cos^{-1}\left\{ \frac{\text{trace } C_{21} - 1}{2} \right)$$

$$\phi = 87.3418^\circ$$ 5

$$\bar{a} = \{a_1 \quad a_2 \quad a_3\}^T$$

$$= \left\{ \frac{C_{23} - C_{32}}{2\sin\phi}, \frac{C_{31} - C_{13}}{2\sin\phi}, \frac{C_{12} - C_{21}}{2\sin\phi} \right\}^T$$

$$= \{0.77040, -0.29045, 0.56755\}$$

qed.

#3 (a) $\bar{C}\bar{C}^T = \bar{C}^T\bar{C} = I$ can be seen easily (used matlab) 4

(b) Let us take C_{13} — 4

Cofactor of $C_{13} = (-1)^4\, 0.7071 = 0.7071$

(c) $\det(\bar{C}) = 1$ (matlab) 4 ; (d) col(1) · Col(2) $= 6.9 \times 10^{-17}$ orthogonal — 4

Row(1) · Row(2) $= -5.55 \times 10^{-17}$ orthogonal

The orthonormality conditions are <u>same</u> for both column & rows.

(e) There are nine terms in $\bar{C}$. But 6 of the parameters (column/row wise) are orthogonal and therefore are constrained. Therefore only 3 of them are independent.

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#4. $\bar{a} = \frac{1}{\sqrt{3}}\{1 \quad 1 \quad 1\}^T$

Euler angle $= \Phi = 45^\circ$; 3-2-1 (y-p-r)?

Rotated frame as seen from Inertial frame

is $\bar{C}_{NB} = \cos\Phi I + (1-\cos\Phi)\bar{a}\bar{a}^T - \sin\Phi\,\bar{a}^\times$

$$= \begin{bmatrix} 0.80474 & 0.50588 & -0.31062 \\ -0.31062 & 0.80474 & 0.50588 \\ 0.50588 & -0.31062 & 0.80474 \end{bmatrix}$$

used Matlab

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For 3-2-1 rotation:

From eqn (1) of problem (2)

$2 = \text{Pitch} = \theta = -\sin^{-1}\{C_{13}\} = 18.096^\circ$

$1 = \text{Roll}\ \psi = \tan^{-1}\left\{\frac{C_{23}}{C_{33}}\right\} = 32.155^\circ$

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$3 = \text{Yaw}\ \phi = \tan^{-1}\left\{\frac{C_{12}}{C_{11}}\right\} = 32.155^\circ$

Substituting in eqn (1) & reverified.

#5. $$\begin{Bmatrix} x_F \\ y_F \\ z_F \end{Bmatrix} = \begin{bmatrix} 0.892539 & 0.157379 & -0.422618 \\ -0.275451 & 0.932257 & -0.234570 \\ 0.357073 & 0.325773 & 0.875426 \end{bmatrix} \begin{Bmatrix} x_N \\ y_N \\ z_N \end{Bmatrix}$$

#3

$$\eta = \frac{\pm\{1+\text{trace}[C]\}^{1/2}}{2} = \pm 0.961798$$

$$\bar{e} = \left\{\frac{c_{23}-c_{32}}{4\eta}, \frac{c_{31}-c_{13}}{4\eta}, \frac{c_{12}-c_{21}}{4\eta}\right\}^T$$

$$= \{-0.1456499, 0.2026649, 0.1125054\}^T$$

Verification: $e_1^2+e_2^2+e_3^2+\eta^2 = 0.9999918$
≈ 1

Fine!

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(b)
3(d) Co-factor Eqns:

P.C. Hughes

$$\bar{C} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$c_{11} = c_{22}c_{33} - c_{23}c_{32}$$
$$c_{21} = c_{32}c_{13} - c_{33}c_{12}$$
$$c_{31} = c_{12}c_{23} - c_{13}c_{22}$$
$$c_{12} = c_{23}c_{31} - c_{21}c_{33}$$
$$c_{22} = c_{33}c_{11} - c_{31}c_{13}$$
$$c_{32} = c_{13}c_{21} - c_{11}c_{23}$$
$$c_{13} = c_{21}c_{32} - c_{22}c_{31}$$
$$c_{23} = c_{31}c_{12} - c_{32}c_{11}$$
$$c_{33} = c_{11}c_{22} - c_{12}c_{21}$$

9 relations among the elements of $\bar{C}$ are not independent.

Orthonormality conditions: (rows of $\bar{C}$ are orthonorm[al])

$$\Rightarrow \bar{C}\bar{C}^T = I \quad \Rightarrow \quad c_{11}^2+c_{12}^2+c_{13}^2 = 1;$$
$$c_{21}^2+c_{22}^2+c_{23}^2 = 1;$$
$$c_{31}^2+c_{32}^2+c_{33}^2 = 1;$$

Also $c_{11}c_{21}+c_{12}c_{22}+c_{13}c_{23} = 0$
$c_{21}c_{31}+c_{22}c_{32}+c_{23}c_{33} = 0$
$c_{31}c_{11}+c_{32}c_{12}+c_{33}c_{13} = 0$

All these 6 eqn[s] are n[ot] indepen[dent]
pendent

III^{ly} column orthonormality $C^T C = I \implies$

$$c_{11}^2 + c_{21}^2 + c_{31}^2 = 1 \qquad c_{11}c_{12} + c_{21}c_{22} + c_{31}c_{32} = 0$$

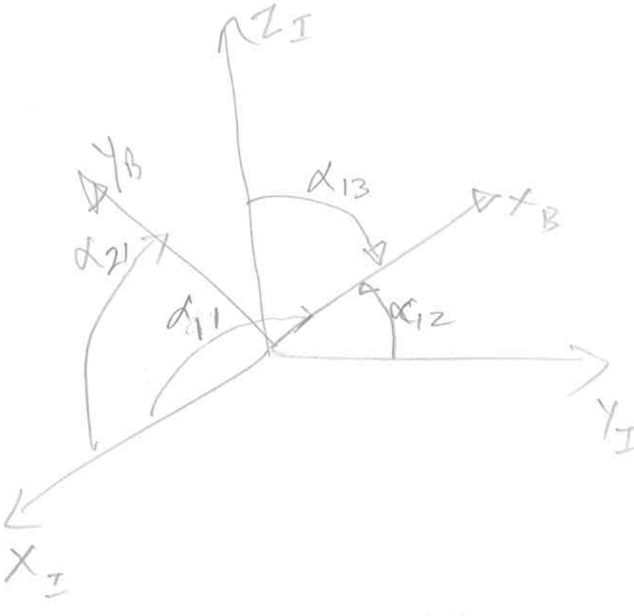
$$c_{12}^2 + c_{22}^2 + c_{32}^2 = 1 \qquad c_{12}c_{13} + c_{22}c_{23} + c_{32}c_{33} = 0$$

$$c_{13}^2 + c_{23}^2 + c_{33}^2 = 1 \qquad c_{13}c_{11} + c_{23}c_{21} + c_{33}c_{31} = 0$$

These 6 eqns are not independent.

∴ We know that out of nine dependent parameters of cofactor eqns, 6 of them are constraine by row/column orthonormality; therefore only 3 parameters are basic necessity for defining attitude; however, DC matrix is a over-parameterized attitude representation system.

(e)



$\cos\alpha_{11} = c_{11}$

$\cos\alpha_{12} = c_{12}$

$\cos\alpha_{13} = c_{13}$

III^{ly} $\cos\alpha_{21} = c_{21}$

$\cos\alpha_{22} = c_{22}$

$\cos\alpha_{23} = c_{23}$

& $\cos\alpha_{31} = c_{31}$

$\cos\alpha_{32} = c_{32}$

$\cos\alpha_{33} = c_{33}$

Suppose we are given

c_{11}, c_{12} & c_{33} $\implies$ Can be uniquely determine $X_B Y_B Z_B$ wrt $X_I Y_I Z_I$?

Though we have 3 parameters, we cannot define attitude matrix; Therefore, non unique

Hence the proof.