

HOMWORK 3

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1 Regression problem

Pipe diameters	Cost per length
80	424
100	570
125	767
150	977
200	1431
250	1924
300	2451
350	3008
400	3591
450	4198
500	4828
600	6149
700	7545
750	8269

Table 1: Pipe cost

The costs per length of the pipe seems to be increasing almost linearly with the pipe diameter. Hence a least squared linear regression solution would give a near approximate correlation between them.

1. Let the error be e for between the actual costs y_i and the predicted cost y_p for corresponding diameters x_i . The following objective function is to be minimized. (n - number of diameters considered)

$$e^2 = \sum (Y - y_i)^2 \quad (1.1)$$

$$y_p = mx_i + b \quad (1.2)$$

2. Combining (1.1) and (1.2),

$$e^2 = \sum_i (y_i - mx_i - b)^2 = \sum_i (m^2 x_i^2 + 2mbx_i - 2mx_i y_i + b^2 - 2by_i + y_i^2)$$

3. The error is a quadratic function of \mathbf{m} and \mathbf{b} . This convex function and has a least optimal solution when,

$$\frac{de^2}{dm} = 0 = 2m \sum x_i^2 + 2b \sum x_i - 2 \sum (x_i y_i) \quad (1.3)$$

and

$$\frac{de^2}{db} = 0 = 2m \sum x_i + 2 \sum b - 2 \sum y_i \quad (1.4)$$

4. Solving (1.3) & (1.4) for \mathbf{m} and \mathbf{b} ,

$$m = \frac{n(\sum x_i y_i) - (\sum x_i)(\sum y_i)}{n(\sum x_i^2) - (\sum x_i)(\sum x_i)} \quad (1.5)$$

$$b = \frac{\sum y_i - m \sum x_i}{n} \quad (1.6)$$

Sl.no	x_i	y_i	$x_i y_i$	x_i^2	x_i^3	x_i^4	$x_i^2 y_i$
1	80	424	33920	6400	512000	40960000	2713600
2	100	570	57000	10000	1000000	100000000	5700000
3	125	767	95875	15625	1953125	244140625	11984375
4	150	977	146550	22500	3375000	506250000	21982500
5	200	1431	286200	40000	8000000	1600000000	57240000
6	250	1924	481000	62500	15625000	3906250000	120250000
7	300	2451	735300	90000	27000000	8100000000	220590000
8	350	3008	1052800	122500	42875000	15006250000	368480000
9	400	3591	1436400	160000	64000000	25600000000	574560000
10	450	4198	1889100	202500	91125000	41006250000	850095000
11	500	4828	2414000	250000	125000000	62500000000	1207000000
12	600	6149	3689400	360000	216000000	129600000000	2213640000
13	700	7545	5281500	490000	343000000	240100000000	3697050000
14	750	8269	6201750	562500	421875000	316406250000	4651312500
\sum	4955	46132	23800795	2394525	1361340125	844716350625	14835407975

Table 2: Calculations required for \mathbf{m} and \mathbf{b} , $n=14$

- $m = \frac{14*23800795 - 46132*4955}{14*2394525 - 4955*4955} = 11.6624$

- $b = \frac{46132 - 11.6624*4955}{14} = -832.5093$

- The approximate correlation between the pipe diameter \mathbf{x} and the cost per unit length \mathbf{y} is given by (2),

$$y = 11.6624x - 832.5093 \quad (1.7)$$

Similarly a quadratic fit can be can be obtained for reducing the error,

$$e^2 = \sum (Y - Ax_i^2 - Bx_i - C)^2 \quad (1.8)$$

The minimum is obtained for the conditions,

$$\frac{de^2}{dA} = 0 \quad (1.9)$$

$$\frac{de^2}{dB} = 0 \quad (1.10)$$

$$\frac{de^2}{dC} = 0 \quad (1.11)$$

This boils down to solving the following,

$$\begin{bmatrix} \sum x_i^4 & \sum x_i^3 & \sum x_i^2 \\ \sum x_i^3 & \sum x_i^2 & \sum x_i \\ \sum x_i^2 & \sum x_i & n \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} \sum x_i^2 y_i \\ \sum x_i y_i \\ \sum y_i \end{bmatrix} \quad (1.12)$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 844716350625 & 1361340125 & 2394525 \\ 1361340125 & 2394525 & 4955 \\ 2394525 & 4955 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 14835407975 \\ 23800795 \\ 46132 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0.00517256 \\ 7.51463691 \\ -249.20335197 \end{bmatrix}$$

$$y = 0.00517256x^2 + 7.51463691x - 249.20335197 \quad (1.13)$$

The following table gives a comparison of the predicted costs for linear and quadratic regressions.

D	actual cost	Linear	quadratic
80	424	100	385
100	570	334	554
125	767	625	771
150	977	917	994
200	1431	1500	1461
250	1924	2083	1953
300	2451	2666	2471
350	3008	3249	3015
400	3591	3832	3584
450	4198	4416	4180
500	4828	4999	4801
600	6149	6165	6122
700	7545	7331	7546
750	8269	7914	8296

Table 3: Comparison of predictions from linear and quadratic regressions with actual costs

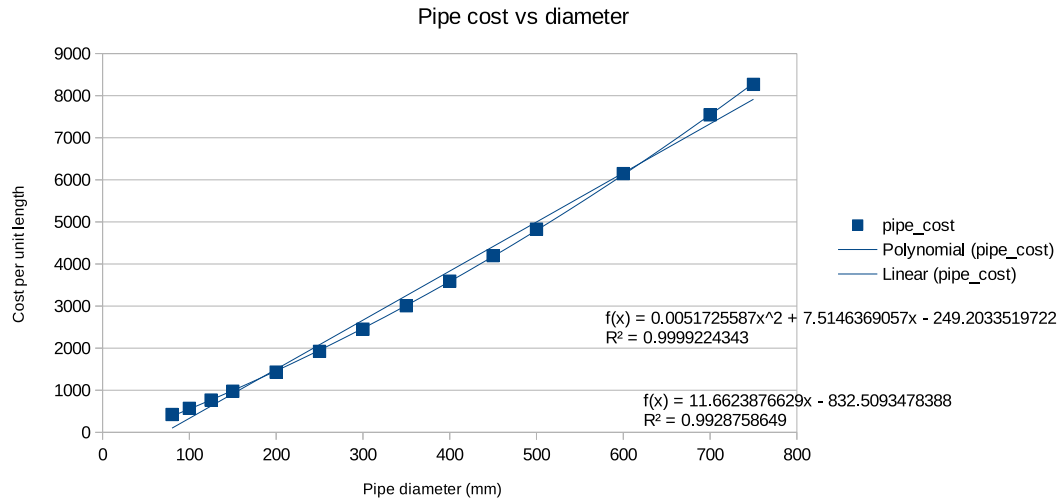


Figure 1.1: The regression equations verified on Libre office

2 Pipe optimization problem

Roll number AE14B021

Enter your roll number in the box below and press the button titled Click

Minimum pressure at node B =79.5 m
Minimum pressure at node C=89.5 m
Minimum pressure at node D=84.5m

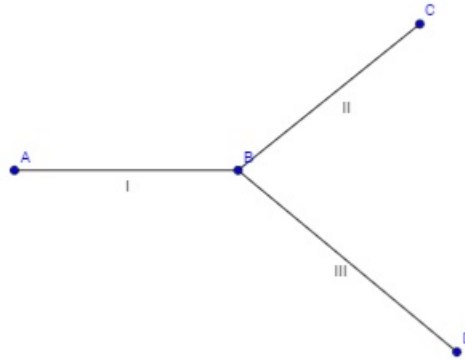


Figure 2.1: Network from HW1

The optimum diameters obtained were:

D_1	324.80mm	$9m^3/min$
D_2	225.27mm	$3m^3/min$
D_3	162.69mm	$2 m^3/min$

From HW(1) the original cost function is,

$$C = 1.2654 [300D_1^{1.327} + 500D_2^{1.327} + 400D_3^{1.327}]$$

Now in order to use the available diameters, each section of pipe is split such that the nearest available diameters are used. The length is kept the same

For AB, $L^I = 300 = L_1^I + L_2^I$ & nearest diameters are 300mm and 350mm.

For BC, $L^{II} = 500 = L_1^{II} + L_2^{II}$ & nearest diameters are 200mm and 250mm.

For BD, $L^{III} = 400 = L_1^{III} + L_2^{III}$ & nearest diameters are 150mm and 200mm.

- The new cost function is given by

$$\begin{aligned}
 C' &= 1.2654 [L_1^I(300)^{1.327} + L_2^I(350)^{1.327} + L_1^{II}(200)^{1.327} + L_2^{II}(250)^{1.327} + L_1^{III}(150)^{1.327} + L_2^{III}(200)^{1.327}] \\
 &= 2451.1378L_1^I + 3007.5034L_2^I + 1431.1812L_1^{II} + 1924.3950L_2^{II} + 977.0143L_1^{III} + 1431.1812L_2^{III} \\
 &= 2451.1378L_1^I + 3007.5034(300 - L_1^I) \\
 &\quad + 1431.1812L_1^{II} + 1924.3950(500 - L_1^{II}) \\
 &\quad + 977.0143L_1^{III} + 1431.1812(400 - L_1^{III}) \\
 &= 2436921 - 556.3656L_1^I - 493.2138L_1^{II} - 454.1669L_1^{III}
 \end{aligned}$$

This is equivalent to maximizing the objective function:

$$C = 556.3656L_1^I + 493.2138L_1^{II} + 454.1669L_1^{III} \quad (2.1)$$

- Let X, Y & Z be the joints in AB, BC & BD connecting pipes of different diameters. For the minimum conditions, $H_C = 89.5m$ & $H_D = 84.5m$. The condition on H_B is redundant and thus $89.5 \leq H_B < 100$.

$$H_A - H_B = \Delta H_{AB} = 4.457 \times 10^8 L_D^{\frac{9^{185}}{300^{4.87}}}$$

$$\begin{aligned} \Delta H_{AB} &= \Delta H_{XB} + \Delta H_{AX} = 4.457 \times 10^8 L_1^I \frac{9^{1.85}}{300^{4.87}} + 4.457 \times 10^8 L_2^I \frac{9^{185}}{350^{4.87}} = 100 - H_B \\ H_B &= 100 - 2.2429 * 10^{-2} L_1^I - 1.0587 * 10^{-2} (300 - L_1^I) \\ &= 96.8239 - 1.1842 * 10^{-2} L_1^I \geq 89.5 \end{aligned}$$

This condition is redundant for $0 < L_1^I < 300$.

$$\begin{aligned} \Delta H_{BC} &= \Delta H_{YC} + \Delta H_{BY} = 4.457 * 10^8 L_1^{II} \frac{3^{1.85}}{200^{4.87}} + 4.457 * 10^8 L_2^{II} \frac{3^{1.85}}{250^{4.87}} = H_B - H_C \\ H_B - H_C &= 2.1169 * 10^{-2} L_1^{II} + 0.7141 * 10^{-2} (500 - L_1^{II}) \\ H_C &= 96.8239 - 1.1842 * 10^{-2} L_1^I - 1.4028 * 10^{-2} L_1^{II} - 3.5705 \\ &= 93.2534 - 1.1842 * 10^{-2} L_2^I - 1.4028 * 10^{-2} L_2^{II} \geq 89.5 \end{aligned}$$

$$1.1842 L_1^I + 1.4028 L_1^{II} \leq 375.34$$

$$\begin{aligned} \Delta H_{BD} &= \Delta H_{ZD} + \Delta H_{BZ} = 4.457 * 10^8 L_1^{III} \frac{2^{1.85}}{150^{4.87}} + 4.457 * 10^8 L_2^{III} \frac{2^{1.85}}{200^{4.87}} = H_B - H_D \\ H_B - H_D &= 4.0587 * 10^{-2} L_1^{III} + 0.9998 * 10^{-2} (400 - L_1^{III}) = -3.0589 * 10^{-2} L_1^{III} + 3.9992 \\ H_D &= 93.2713 - 1.1842 * 10^{-2} L_1^I - 3.0589 * 10^{-2} L_1^{III} + 3.9992 \\ &= 77.0365 - 1.1842 * 10^{-2} L_1^I - 3.0589 * 10^{-2} L_2^{III} \geq 84.5 \end{aligned}$$

$$1.1842 L_1^I + 3.0589 L_1^{III} \leq 832.47$$

- The problem reformulated is,

$$\max(C) = \max \left\{ \begin{bmatrix} 556.3656 & 493.2138 & 454.1669 \end{bmatrix} \begin{bmatrix} L_1^I \\ L_1^{II} \\ L_1^{III} \end{bmatrix} \right\} \quad (2.2)$$

subject to,

$$1.1842 L_1^I + 1.4028 L_1^{II} \leq 375.34$$

$$1.1842 L_1^I + 3.0589 L_1^{III} \leq 832.47$$

$$L_2^I, L_2^{II}, L_2^{III} \geq 0$$

Hence the problem can be reduced to the form,

$$\min(c^T z)$$

$$\text{subject to, } Az = B \quad z_i \geq 0$$

Slack variables $s_1, s_2 \geq 0$ are introduced to convert the inequality constraints to equality constraints. Initial basic variables are s_1, s_2 . **The remaining inactive variables are set to zero to get the basic solution.**

$$1.1842L_1^I + 1.4028L_1^{II} + s_1 = 375.34$$

$$1.1842L_1^I + 3.0589L_1^{III} + s_2 = 832.47$$

$$-556.3656L_1^I - 493.2138L_1^{II} - 454.1669L_1^{III} + C = 0$$

Basic solutions : $s_1 = 375.34/1$, $s_2 = 832.47/1$, $L_1^I = L_1^{II} = L_1^{III} = 0$ [Starting point (0,0,0)]

Basic		L_1^I	L_1^{II}	L_1^{III}	s_1	s_2	RHS	Solution	Ratio test
s_1	1	1.1842	1.4028	0	1	0	375.34	$s_1 = 375.34/1$	$\frac{375.34}{1.1842}$
s_2	1	1.1842	0	3.0589	0	1	832.47	$s_2 = 832.47/1$	$\frac{832.47}{1.1842}$
C		-556.3656	-493.2138	-454.1669	0	0	0	C=0	

Next iteration: From the determined pivot, s_1 is eliminated and L_1^I enters. Pivot column is cleared.

Basic		L_1^I	L_1^{II}	L_1^{III}	s_1	s_2	RHS	Solution	Ratio test
L_1^I	556.3656	1	1.1846	0	0.8445	0	316.9625	$L_1^I = \frac{375.34}{1.1842} = 316.96$	$\frac{375.34}{0}$
s_2	1	0	-1.4028	3.0588	-1	1	457.1044	$s_2 = \frac{457.13}{1}$	$\frac{457.14}{3.0589}$
C		0	165.8712	-454.1669	469.8290	0	176347.0378	C=214770.65	

Next iteration: L_1^{III} becomes active and s_2 becomes inactive. Pivot column is cleared.

Basic		L_1^I	L_1^{II}	L_1^{III}	s_1	s_2	RHS	Solution	Ratio test
L_1^I	556.37	1	1.1846	0	0.8445	0	316.9625	$L_1^I = \frac{375.34}{1.1842} = 316.96$	$\frac{316.9625}{1.1846} = 267$
L_1^{III}	454.17	0	-0.4586	1	-0.3269	0.3269	149.4379	$L_1^{III} = \frac{457.13}{3.0589} = 149.43$	$\frac{149.44}{0.46} = 323.91$
C		0	-42.4159	0	321.3514	148.4775	244216.7715	C=214770.65	

Next iteration: L_1^{II} becomes active and L_1^I becomes inactive. Pivot column is cleared.

Basic		L_1^I	L_1^{II}	L_1^{III}	s_1	s_2	RHS	Solution
L_1^{II}	493.2138	0.8441	1.0000	0.0000	0.7129	0.0000	267.5634	$L_1^{II} = 267.56$
L_1^{III}	454.1669	0.3871	0.0000	1.0000	0.0000	0.3269	272.1461	$L_1^{III} = 272.15$
C		35.8053	0.0000	0.0000	351.5876	148.4775	255565.7097	

With all the coefficients of C positive, termination is reached.

Optimal solution is $C_{max} = 255565.71$

L_1^I	0
L_2^I	$300 - 0 = 300$
L_1^{II}	267.56
L_2^{II}	$500 - 267.56 = 232.44$
L_1^{III}	272.15
L_2^{III}	$400 - 272.15 = 127.85$

The minimum cost for the pipe network is:

$$C' = 2436921 - 556.3656 * 0 - 493.2138 * 267.56 - 454.1669 * 272.15 = 2181356.2$$