Note :- For easy I will write # of trials n is m because that is making a lot of confusion. → Ask her

1.) Let $X_1, \dots, X_n$ be an i.i.d sample from Binomial (n, p)

a.) Use Method of moments to find point estimators for parameters n and p.

$\frac{117}{120}$

$= 97.5$

b.) Assume ~~X_1~~ n is known. Find the MLE of p

Soln

given $X_1, \dots, X_n \overset{iid}{\sim}$ Binomial (n, p)

First population moment:

$$E(X) = np$$

First sample moment

$$\frac{\sum X_i}{n} = \overline{X}$$

Second population moment :

$$E(X^2) = Var(X) + \{[E(X)]\}^2$$

$$\Rightarrow E(X^2) = npq + \{np\}^2$$

$$= np(1-p) + n^2p^2$$

# of trials

Second sample moment

$$\frac{\sum X_i^2}{n}$$

no. of r.v

By MME

$$np = \overline{X} \quad \text{——— ①}$$

$\&$

$$np(1-p) + (np)^2 = \frac{\sum X_i^2}{n}$$

solving

from eqn ①

$$\hat{p} = \frac{\overline{X}}{n} \quad \text{—} ⊛$$

from eqn ②

l of trials

$$np(1-p) + (np)^2 = \frac{\sum X_i^2}{n}$$ no. r.v

$$\Rightarrow np(1-p) + \overline{X}^2 = \frac{\sum X_i^2}{n} \quad \text{(using ①)}$$

$$\Rightarrow np(1-p) = \frac{\sum X_i^2}{n} - \overline{X}^2$$

$$\Rightarrow np(1-p) = \frac{1}{n} \sum (X_i - \overline{X} + \overline{X})^2 - \overline{X}^2$$

$$\Rightarrow np(1-p) = \frac{1}{n} \sum \left[(X_i - \overline{X})^2 + \overline{X}^2 + 2(X_i - \overline{X})\overline{X} \right] - \overline{X}^2$$

$$\Rightarrow np(1-p) = \frac{1}{n} \left[\sum (X_i - \overline{X})^2 + \sum \overline{X}^2 + \cancelto{0}{2\overline{X} \sum (X_i - \overline{X})} \right] - \overline{X}^2$$

$$\Rightarrow np(1-p) = \frac{1}{n} \left[\sum (X_i - \overline{X})^2 + n\overline{X}^2 \right] - \overline{X}^2$$

$$\Rightarrow np(1-p) = \frac{\sum (X_i - \overline{X})^2}{n} + \overline{X}^2 - \overline{X}^2$$

$$\Rightarrow np(1-p) = \frac{\sum (X_i - \overline{X})^2}{n}$$

$$\Rightarrow n\frac{\overline{X}}{n}\left(1-\frac{\overline{X}}{n}\right) = \frac{\sum (X_i - \overline{X})^2}{n}$$

$$\Rightarrow \overline{X}\left(1-\frac{\overline{X}}{n}\right) = \frac{\sum (X_i - \overline{X})^2}{n}$$

$$\Rightarrow \overline{X}\left(1-\frac{\overline{X}}{n}\right) = \frac{n-1}{n} \frac{\sum (X_i - \overline{X})^2}{n-1}$$

$$\Rightarrow \overline{X}\left(1-\frac{\overline{X}}{n}\right) = \frac{n-1}{n} s^2$$

$$\Rightarrow \left(1-\frac{\overline{X}}{n}\right) = \frac{(n-1)s^2}{n\overline{X}}$$

# of trials

no. of term

Note

$$1-\frac{\overline{X}}{n} = \frac{(n-1)s^2}{n\overline{X}}$$

$$\frac{\overline{X}}{n} = 1 - \frac{(n-1)s^2}{n\overline{X}}$$

$$\frac{\overline{X}}{n} = \frac{n\overline{X} - (n-1)s^2}{n\overline{X}}$$

$$\Rightarrow \frac{n}{\overline{X}} = \frac{n\overline{X}}{n\overline{X} - (n-1)s^2}$$

$$\therefore \hat{n} = \frac{n\overline{X}^2}{n\overline{X} - (n-1)s^2}$$

Since $\hat{p} = \frac{\overline{X}}{n}$

$$= \overline{X} \cdot \frac{n\overline{X} - (n-1)s^2}{n\overline{X}^2}$$

$$= \frac{n\overline{X} - (n-1)s^2}{n\overline{X}}$$

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b.)

Given $X_1, \ldots, X_n \overset{i.i.d}{\sim} BIN(n, p)$

since n is given known in the question so only unknown parameter is p.

by MLE :

$$L(P) = \prod_{i=1}^{n} \binom{n}{x_i} p^{x_i} (1-p)^{n-x_i}$$

Taking ln on both side

$$\ln[L(P)] = \sum_{i=1}^{n} \left[\ln\left[\binom{n}{x_i}\right] + \ln p^{x_i} + \ln\left[(1-p)^{n-x_i}\right] \right]$$

$$= \sum_{i=1}^{n} \left[\ln\left[\binom{n}{x_i}\right] + x_i \ln(p) + (n-x_i)\ln(1-p) \right]$$

$$= \sum \ln\left[\binom{n}{x_i}\right] + \sum x_i \ln(p) + \sum (n-x_i)\ln(1-p)$$

Now,

$$\frac{\partial [L(P)]}{\partial p} = 0 + \sum x_i \cdot \frac{1}{p} + \sum (n-x_i) \cdot \frac{1}{1-p} (-1)$$

$$= \frac{\sum x_i}{p} - \frac{\sum (n-x_i)}{1-p}$$

Now

$$\frac{\sum x_i}{p} - \frac{\sum (n - x_i)}{1-p} = 0$$

$$\Rightarrow \frac{\sum x_i}{p} - \frac{n^2 - \sum x_i}{1-p} = 0$$

$$\Rightarrow \frac{\sum x_i}{p} = \frac{n^2 - \sum x_i}{1-p}$$

$$\Rightarrow \frac{1-p}{p} = \frac{n^2 - \sum x_i}{\sum x_i}$$

$$\Rightarrow \left(\frac{1}{p} - 1\right) = \frac{n^2 - \sum x_i}{\sum x_i}$$

$$\Rightarrow \left(\frac{1}{p} - 1\right) = \frac{n - \frac{\sum x_i}{n}}{\frac{\sum x_i}{n}}$$

$$\Rightarrow \left(\frac{1}{p} - 1\right) = \frac{n - \bar{X}}{\bar{X}}$$

$$\Rightarrow \frac{1}{p} = \frac{n - \bar{X}}{\bar{X}} + 1$$

$$\Rightarrow \frac{1}{p} = \frac{n - \overline{x} + \overline{x}}{\overline{x}}$$

$$\Rightarrow \frac{1}{p} = \frac{n}{\overline{x}}$$

$$\therefore \hat{p} = \frac{\overline{x}}{n}$$

ask her?

For checking [I guess optional ?]

Yes.

$$\frac{\partial^2 L(p)}{\partial p^2} = -\frac{\sum x_i}{p^2} - \frac{\sum (n - x_i)}{(1-p)^2} < 0$$

Suppose $X_1, \ldots, X_n$ is an iid sample from Uniform $(\theta, \theta+|\theta|)$, where $\theta \neq 0$

a) Use Method of moments to find an estimator for θ

b) Find the MLE of θ

Soln

given $X_1, \ldots, X_n \overset{iid}{\sim}$ Uniform $(\theta, \theta+|\theta|)$

For MME, we have

first population moment:

$$E(X) = \frac{\theta + \theta + \theta}{2} \quad \text{when } \theta > 0$$

$$= \frac{3\theta}{2}$$

&

$$E(X) = \frac{\theta + \theta - \theta}{2} \quad \text{when } \theta < 0$$

$$= \frac{\theta}{2}$$

first sample moment is

$$\frac{\sum x_i}{n} = \overline{X}$$

By MME

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$$\frac{3\theta}{2} = \bar{X} \qquad \text{if } \theta > 0$$

$$\Rightarrow \hat{\theta} = \frac{2\bar{X}}{3} \qquad \text{if } \theta > 0$$

&

$$\frac{\theta}{2} = \bar{X}$$

$$\Rightarrow \hat{\theta} = 2\bar{X} \qquad \text{if } \theta < 0$$

i) given $x_1, \dots, x_n \overset{iid}{\sim} \text{uniform}(\theta, \theta + |\theta|)$

Soln

since $\theta \neq 0$ given, so there can be two cases

i) $\theta > 0$

When $\theta > 0$, then

$x_1, \dots, x_n \overset{iid}{\sim} \text{uniform}(\theta, 2\theta)$ (look all interval is +ve)

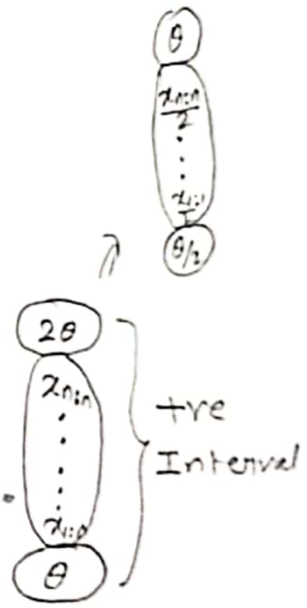
For MLE

$$L(\theta) = f(x_1|\theta) \dots f(x_n|\theta)$$

$$= \frac{1}{2\theta - \theta} \cdots \frac{1}{2\theta - \theta} I(\theta < x_1, \dots, x_n < 2\theta)$$

$$= \left(\frac{1}{2\theta - \theta}\right)^n I(\theta < x_1, \dots, x_n < 2\theta)$$

$$= \left(\frac{1}{\theta}\right)^n I(\theta < x_1, \dots, x_n < 2\theta)$$



Now at this point our main goal is to find out the value of θ that maximize $L(\theta)$

so need to make θ as small as possible and keep the indicator as 1

$$\Rightarrow \hat{\theta} = \frac{1}{2} x_{n:n}$$

2) When $\theta < 0$

$x_1, \ldots, x_n \overset{iid}{\sim}$ uniform$(\theta, 0)$ +ve interval

For MLE

$$L(\theta) = f(x_1|\theta) \cdots f(x_n|\theta)$$

$$= \frac{1}{0-\theta} \cdots \frac{1}{0-\theta} \; I(\theta < x_1, \ldots, x_n < 0)$$

$$= \frac{1}{-\theta} \cdots \frac{1}{-\theta} \; I(\theta < x_1, \ldots, x_n < 0)$$

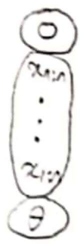
$$= \left(\frac{1}{-\theta}\right)^n I(\theta < x_1, \ldots, x_n < 0)$$

→ This -ve sign makes sense to me becoz, θ is also -ve

Now at this point our main goal is to maximize $L(\theta)$

θ can not be larger than the smallest x_i

$$\Rightarrow \hat{\theta} = x_{1:n}$$



5) Let $X_1, \dots, X_n$ be an iid sample from the following pdf,

$$f_X(x;\theta) = \theta x^{\theta-1}, \text{ for } 0<x<1, \text{ and } 0 \text{ otherwise}. \ \theta>0$$

a) Use method of moments to find an estimator of θ.

b) Find the MLE of θ.

c) Show that variance of the MLE goes to 0 as $n \to \infty$

d) Find the MLE of $\tau(\theta) = P(X > 1/2)$ using the invariant property of MLE.

Soln

given $X_1, \dots, X_n$ are iid

$$\text{pdf}: \ f_X(x;\theta) = \theta x^{\theta-1}$$

For MME:

First population moment

$$E(X) = \int_0^1 x f(x)\, dx$$

$$= \int_0^1 x \, \theta x^{\theta-1}\, dx$$

$$= \int_0^1 \theta x^{\theta}\, dx$$

$$= \theta \left. \frac{x^{\theta+1}}{\theta+1} \right]_0^1$$

उसरी त limit 0 to ∞ नै हुनु पर्ने हो but here x is only defined on $0<x<1$ so, outside of it is zero हुने भएकाले ignore other part

$$= \frac{\theta}{\theta+1}\left[x^{\theta+1}\right]_0^1$$

$$= \frac{\theta}{\theta+1}\left[1^{\theta+1} - 0^{\theta+1}\right]$$

$$= \frac{\theta}{\theta+1}(1-0)$$

$$= \frac{\theta}{\theta+1}$$

First sample Moment

$$\frac{\sum x_i}{n} = \bar{X}$$

by MME

$$\frac{\theta}{\theta+1} = \bar{X}$$

$$\Rightarrow \frac{\theta+1}{\theta} = \frac{1}{\bar{X}}$$

$$\Rightarrow 1 + \frac{1}{\theta} = \frac{1}{\bar{X}}$$

$$\Rightarrow \frac{1}{\theta} = \frac{1}{\bar{X}} - 1$$

$$\Rightarrow \frac{1}{\theta} = \frac{1-\bar{X}}{\bar{X}}$$

$$\therefore \hat{\theta} = \frac{\bar{X}}{1-\bar{X}}$$

For MLE

b.) For MLE :

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

Taking natural log on both side

$$\ln[L(\theta)] = \sum_{i=1}^{n} \ln[\theta x_i^{\theta-1}]$$

$$= \sum_{i=1}^{n} \left[\ln(\theta) + \ln(x_i^{\theta-1})\right]$$

$$= \sum_{i=1}^{n} \left[\ln(\theta) + (\theta-1)\ln(x_i)\right]$$

$$= \sum \ln(\theta) + (\theta-1)\sum \ln(x_i)$$

$$= n\ln(\theta) + (\theta-1)\sum \ln(x_i)$$

Now

$$\frac{\partial \ln[L(\theta)]}{\partial \theta} = \frac{n}{\theta} + \sum \ln(x_i)$$

Now set $= 0$

$$\therefore \frac{n}{\theta} + \sum \ln(x_i) = 0$$

$$\Rightarrow \frac{n}{\theta} = -\sum \ln(x_i)$$

$$\Rightarrow \hat{\theta} = \frac{-n}{\sum \ln(x_i)}$$

$\left(\because \ln(x_i) < 0 \text{ If } x_i \in (0,1) \Rightarrow \hat{\theta} > 0 \right)$

Yes, I know $\hat{\theta}$ should be +ve and this completely makes sense to me because natural log of any number between 0 & 1 is -ve so n must should have -ve to make $\hat{\theta}$ as a whole +ve.

$\Rightarrow \quad \hat{\theta}_{MLE} = \dfrac{-n}{\sum \ln(x_i)}$

$$Var\left(\frac{-n}{\sum \ln(x_i)}\right)$$

$$\Rightarrow n^2 \, Var\left(\frac{1}{-\sum \ln(x_i)}\right) \quad — \; (*)$$

Now at this point I need to figure it out that what will be the distribution of $\sum \ln(x_i)$, For which first I need to figure it out the distribution of x_i's.

So, $P(-\ln x_i \le y)$

$= P(\ln x_i \ge -y)$

$= P(x_i \ge e^{-y})$

$= 1 - \underbrace{P(x_i < e^{-y})}_{CDF}$ — Look at this point I know the pdf of x_i, so will use that to get CDF

So I need CDF

$CDF = \int pdf$

$= \int_0^x \theta t^{\theta-1} \, dt = \theta \cdot \frac{1}{\theta} \, t^{\theta} \Big]_0^x = x^{\theta}$

$\therefore F(x) = x^{\theta}$

$= 1-(e^{-y})^{\theta}$

$= 1-e^{-y\theta}$, which is the CDF of $\exp(\theta)$

$\Rightarrow -\ln x_1, -\ln x_2, \dots, -\ln x_n \sim \exp(\theta)$. Then

$$S = \sum -\ln x_i \sim \text{gamma}(n,\theta)$$

$$\Rightarrow S \sim \text{gamma}(n,\theta)$$

$$\Rightarrow \frac{1}{S} \sim \text{inverse gamma}(n,\theta)$$

For which $\text{var}\left(\frac{1}{S}\right) = \text{Var}\left(\frac{1}{-\sum \ln x_i}\right) = \frac{\theta}{(n-1)^2(n-2)}$

Hence ⊛ becomes.

$$n^2 \frac{\theta^2}{(n-1)^2(n-2)}$$

$$= \frac{\cancel{n^2}\,\theta^2}{\cancel{n^2}\left(1-\frac{1}{n}\right)^2(n-2)}$$

So as $n \longrightarrow \infty$

$$\frac{\theta^2}{\left(1-\frac{1}{n}\right)^2(n-2)} \longrightarrow 0 \quad \square$$

i.e variance of MLE goes to 0.

1) We are given

$$\tau(\theta) = P(X > 1/2)$$

$$= 1 - P(X \leq 1/2)$$

$$= 1 - \left(\frac{1}{2}\right)^{\theta}$$ — Already calculated CDF as $F(x) = x^{\theta}$.

$$= 1 - \frac{1}{2^{\theta}}$$

by invariant property of MLE, we know that

$$\hat{\tau}(\theta) = \tau(\hat{\theta})$$

$$= 1 - \frac{1}{2^{\hat{\theta}}}$$

$$= 1 - \frac{1}{2^{\frac{-n}{\sum \ln(x_i)}}}$$

$$= 1 - \frac{1}{2^{\frac{n}{-\sum \ln(x_i)}}}$$

4/0.

1) — c)

Solving part c), with the hint what she gives me at the end of this class. [can be taken as another approach to to find the distribution of $-\ln(x_i)$, when I know that my x_i are iid with pdf $f_X(x;\theta)$

$$\hat{\theta}_{MLE} = \frac{n}{-\sum \ln(x_i)}$$

$$Var\left(\frac{n}{-\sum \ln(x_i)}\right)$$

$$= n^2 \, var\left(\frac{1}{-\sum \ln(x_i)}\right) \quad \text{———} \quad (*)$$

At this point I am sure that I need to know the distribution of $-\ln(x_i)$ to move forward. So to know the distribution of $-\ln(x_i)$, I will first ~~its~~ its pdf.

So using transformation to find pdf of $y_i = -\ln x_i$ (say)

$$y_i = -\ln(x_i)$$

solving for orginal variable

$$\ln(x_i) = -y_i$$

$$x_i = e^{-y_i}$$

Jacobian $J = \left|\frac{\partial x_i}{\partial y_i}\right|$

$$= \left|\frac{\partial e^{-y_i}}{\partial y_i}\right| = e^{-y_i}$$

So, pdf of y_i is

$$f_Y(y_i) = f_X(x_i)\,|J|$$

$$= f_X(e^{-y_i}) * e^{-y_i}$$

$$= \theta\,(e^{-y_i})^{\theta-1} * e^{-y_i}$$

$$= \theta\,(e^{-y_i(\theta-1)} * e^{-y_i})$$

$$= \theta\, e^{-y_i\theta}$$

$\hookrightarrow$ which is the pdf of $\exp(\frac{1}{\theta})$

$\Rightarrow$ $y_i = -\ln(x_i) \sim \exp(\frac{1}{\theta})$

So, $-\sum \ln(x_i) \sim \text{gamma}(n, 1/\theta)$

$\Rightarrow$ $\dfrac{1}{-\sum \ln(x_i)} \sim$ Inverse gamma (n, θ)

So finally, I know the distribution for $\dfrac{1}{-\sum \ln(x_i)}$

So $\text{Var}\left(\dfrac{1}{-\sum \ln(x_i)}\right) = \dfrac{\theta}{(n-1)^2 (n-2)}$

Hence ⊛ becomes

$$= n^2 \frac{\theta}{(n-1)^2 (n-2)}$$

$$= \frac{\theta}{(1-\frac{1}{n})^2 (n-2)} \longrightarrow 0 \text{ as } n \to \infty. \;\square$$

i.e variance of MLE goes to 0.

A) Suppose $Y_1, \dots, Y_n$ satisfy

$$Y_i = \beta x_i + \varepsilon_i$$

Where $x_1, \dots, x_n$ are fixed constants and $\varepsilon_1, \dots, \varepsilon_n$ are iid Normal $(0, \sigma^2)$

a) Find The MLE of β

b) show that the MLE of β has expected value equal to β (which means the MLE is and unbiased estimator of β).

Soln. We are given

$$Y_i = \beta x_i + \varepsilon_i$$

$\left\{ \text{Since } \varepsilon_i \sim N(0, \sigma^2) \text{ so their linear combination } Y_i \text{ is also Normally distributed.} \right\}$

The MLE of β is :

$$L(\beta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(Y_i - \beta x_i)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\Sigma(Y_i - \beta x_i)^2}$$

Taking log on both side

$$\ln(L(\beta)) = \ln\left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n - \frac{1}{2\sigma^2}\Sigma(Y_i - \beta X_i)^2$$

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$$= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum (Y_i - \beta x_i)^2$$

Now

$$\frac{\partial L(\beta)}{\partial \beta} = 0 - \frac{1}{2\sigma^2} 2 \sum (Y_i - \beta x_i)(-x_i)$$

set $= 0$

$$\therefore \; -\frac{1}{2\sigma^2} 2 \sum (Y_i - \beta x_i)(-x_i) = 0$$

$$\Rightarrow \sum (Y_i - \beta x_i)(x_i) = 0$$

$$\Rightarrow \sum (Y_i x_i - \beta x_i^2) = 0$$

$$\Rightarrow \sum Y_i x_i - \beta \sum x_i^2 = 0$$

$$\Rightarrow \hat{\beta} = \frac{\sum Y_i x_i}{\sum x_i^2}$$

check to verify

$$\frac{\partial^2 L(\beta)}{\partial \beta^2} = \frac{x_i}{\sigma^2} \cdot (-x_i)$$

$$= -\frac{x_i^2}{\sigma^2} < 0$$

(ii) checking for whether the estimator for β is biased or unbiased.

$$E\left[\frac{\sum Y_i x_i}{\sum x_i^2}\right]$$

Since x_i are constant, so

$$= \frac{1}{\sum x_i^2} E[\sum Y_i x_i]$$

where is the summation?

$$E(\sum Y_i X_i)$$

$$= \frac{1}{\sum x_i^2} x_i E[\sum Y_i]$$

$$= \sum E(Y_i X_i)$$

$$= \sum X_i E(Y_i)$$

$$= \sum X_i \cdot \beta X_i = \beta \sum X_i^2 .$$

$$= \frac{1}{\sum x_i^2} x_i \sum [E(Y_i)]$$

$\because Y_i = \beta x_i + \varepsilon_i$

$$E(Y_i) = \beta E(x_i) + E[\varepsilon_i]$$

$$= \beta \cdot x_i + 0$$

$$= \beta x_i$$

$$= \frac{1}{\sum x_i^2} x_i \sum (\beta x_i)$$

where are the summations?

$$= \frac{\beta n x_i^2}{n x_i^2}$$

$\because x_i$ are constant

& $\sum 1 = n$

$$= \beta$$

So $\hat{\beta}$ is unbiased estimator. $\square$

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5.) Suppose X_1 and X_2 are independent Binomial random variables.

$X_1 \sim Binom(n_1, p_1)$ and $X_2 \sim Binom(n_2, p_2)$

a) Find the MLE of $\theta = (P_1, P_2)$

b) Find the restricted MLE of θ subject to the constraint $p_1 = p_2$.

Soln

For MLE

$$L(\hat{\theta}) = f(x_1, P_1) * f(x_2, P_2)$$

$$= \binom{n_1}{x_1} P_1^{x_1} (1-P_1)^{n-x_1} * \binom{n_2}{x_2} P_2^{x_2} (1-P_2)^{n-x_2}$$

Taking natural log on both side

$$\ln[L(\theta)] = \ln\left[\binom{n_1}{x_1}\right] + \ln(P^{x_1}) + \ln(1-P_1)^{n-x_1} + \ln\binom{n_2}{x_2} + \ln(P_2^{x_2})$$

$$+ \ln(1-P_2)^{n-x_2}$$

$$= \ln\left[\binom{n_1}{x_1}\right] + x_1 \ln(P_1) + (n_1 - x_1)\ln(1-P_1) + \ln\left[\binom{n_2}{x_2}\right] + x_2 \ln(P_2)$$

$$+ (n_2 - x_2)\ln(1-P_2).$$

Now, need

$$\frac{\partial \ln[L(\theta)]}{\partial P_1} = 0 \quad \& \quad \frac{\partial \ln[L(\theta)]}{\partial P_2} = 0$$

Here $\dfrac{\partial \ln[L(\theta)]}{\partial P_1} = \dfrac{x_1}{P_1} + \dfrac{(n_1 - x_1)}{(1-P_1)}(-1)$

and

& $$\frac{\partial \ln[L(\theta)]}{\partial P_2} = \frac{x_2}{P_2} + \frac{(n_2-x_2)}{1-P_2}(-1)$$

So, we have

$$\frac{x_1}{P_1} - \frac{n_1-x_1}{1-P_1} = 0 \quad \& \quad \frac{x_2}{P_2} - \frac{(n_2-x_2)}{1-P_2} = 0$$

$$\Rightarrow \frac{x_1}{P_1} = \frac{n_1-x_1}{1-P_1}$$

$$\Rightarrow \frac{1-P_1}{P_1} = \frac{n_1-x_1}{x_1}$$

$$\Rightarrow \frac{1}{P_1} - 1 = \frac{n_1-x_1}{x_1}$$

$$\Rightarrow \frac{1}{P_1} = \frac{n_1-x_1}{x_1} + 1$$

$$\Rightarrow \frac{1}{P_1} = \frac{n_1-x_1+x_1}{x_1}$$

$$\Rightarrow \frac{1}{P_1} = \frac{n_1}{x_1}$$

$$\therefore \hat{P}_1 = \frac{x_1}{n_1}$$

and

$$\frac{x_2}{p_2} - \frac{(n_2 - x_2)}{1 - p_2} = 0$$

$$\Rightarrow \frac{1 - p_2}{p_2} = \frac{(n_2 - x_2)}{x_2}$$

$$\Rightarrow \frac{1}{p_2} - 1 = \frac{n_2 - x_2}{x_2}$$

$$\Rightarrow \frac{1}{p_2} = \frac{n_2 - x_2}{x_2} + 1$$

$$\Rightarrow \frac{1}{p_2} = \frac{n_2 - x_2 + x_2}{x_2}$$

$$\Rightarrow \frac{1}{p_2} = \frac{n_2}{x_2}$$

$$\Rightarrow \hat{p}_2 = \frac{x_2}{n_2}$$

Now checking

$$\frac{\partial^2 \ln[L(\theta)]}{\partial p_1^2} = \frac{-x_1}{p_1^2} - \frac{(n_1 - x_1)}{(1-p_1)^2} < 0$$

&

$$\frac{\partial^2 \ln[L(\theta)]}{\partial p_2^2} = \frac{-x_2}{p_2^2} - \frac{(n_2 - x_2)}{(1-p_2)^2} < 0$$

Also $$\frac{\partial^2 \ln[L(P)]}{\partial P_1 \partial P_2} = 0$$

so $$\frac{\partial^2 \ln[L(P)]}{\partial P_1^2} * \frac{\partial^2 \ln[L(P)]}{\partial P_2^2} > \left(\frac{\partial^2 \ln[L(\theta)]}{\partial P_1 \partial P_2}\right)^2$$

Hence $\hat{\theta} = \left(P_1 = \frac{x_1}{n_1},\ P_2 = \frac{x_2}{n_2}\right)$ is MLE for θ.

ii) we are given constraint $P_1 = P_2$

$$\Rightarrow P_1 - P_2 = 0$$

using lagrange's multipliers λ {but she told me that +,-ve doesn't matter alot [I can choose any, I choose -ve]}

$$\ln[L(\theta)] \pm \lambda(P_1 - P_2)$$

$$\Rightarrow \ln\left[\binom{n_1}{x_1}\right] + x_1 \ln(P_1) + (n_1 - x_1)\ln(1-P_1) + \ln\left[\binom{n_2}{x_2}\right] +$$

$$x_2 \ln(P_2) + (n_2 - x_2)\ln(1-P_2) - \lambda(P_1 - P_2)$$

{I choose -ve}

Now need differentiating w.r.t P_1, we get

$$\frac{x_1}{P_1} + \frac{(n_1 - x_1)}{1-P_1}(-1) = \lambda$$

differentiating w.r.t P_2, we get

$$\frac{x_2}{P_2} + \frac{(n_2 - x_2)}{(1-P_2)}(-1) = -\lambda$$

Now, let $P_1 = P_2 = P$. Then

this is the same as differentiating with respect to λ and set to 0.

$$\frac{x_1}{P_1} - \frac{(n_1 - x_1)}{(1-P_1)} = \frac{(n_2 - x_2)}{(1-P_2)} - \frac{x_2}{P_2}$$

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$$\frac{x_1(1-p) - (n_1-x_1)p}{p(1-p)} = \frac{p(n_2-x_2) - (1-p)x_2}{p(1-p)}$$

$$\Rightarrow x_1(1-p) - (n_1-x_1)p = p(n_2-x_2) - (1-p)x_2$$

$$\Rightarrow x_1 - \cancel{x_1 p} - n_1 p + \cancel{x_1 p} = n_2 p - \cancel{p x_2} - x_2 + \cancel{p x_2}$$

$$\Rightarrow x_1 - n_1 p = n_2 p - x_2$$

$$\Rightarrow n_1 p + n_2 p = x_1 + x_2$$

$$\Rightarrow p(n_1+n_2) = x_1 + x_2$$

$$\Rightarrow \hat{p} = \frac{x_1+x_2}{(n_1+n_2)}$$

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Thus, MLE of θ is $\hat{\theta} = \frac{x_1+x_2}{n_1+n_2}$. $\square$