Chapter-9

(point estimation)

→ हामीले orginal population को distribution जान्दैं but तिनिहरूको parameters हरु जान्दैनौं like θ, μ, σ^2, K ... etc.

→ हाम्रो यस chapter को main goal भनेको sample population लाई हेरेर orginal population का parameters हरुलाई estimate गर्नु हो ।

→ मतलब हामीले sample distribution लाई हेरेर orginal population का parameters (unknown) लाई estimate गर्नु पर्ने हो । यसका लागि we have two methods in this chapter

1) Methods of moments

2) Maximum likelihood Estimator (MLE)

1) Methods of moments :- (MME)

Main idea :-

- calculate population moment & sample moment and equate them.
- No. of moments to be calculated depends upon no. of unknown parameters needed to be estimated.

Important Algebra

$$\frac{1}{n}\sum X_i^2 - \bar{X}$$

$$= \frac{1}{n}\sum (X_i - \bar{X} + \bar{X})^2 - \bar{X}^2$$

$$= \frac{1}{n}\sum \left[(X_i - \bar{X})^2 + \bar{X}^2 + 2\cdot(X_i - \bar{X})\bar{X}\right] - \bar{X}^2$$

$$= \frac{1}{n}\left\{\sum (X_i - \bar{X})^2 + n\bar{X}^2 + 2\bar{X}\underbrace{\sum (X_i - \bar{X})}_{\to 0}\right\} - \bar{X}^2$$

$$= \frac{\sum (X_i - \bar{X})^2}{n} + \bar{X}^2 - \bar{X}^2 \Rightarrow \frac{\sum (X_i - \bar{X})^2}{n}. \square$$

$2\bar{X}(\sum X_i - n\bar{X})$

$2\bar{X}(\sum X_i - n\bar{X})$

$2\bar{X}(n\bar{X} - n\bar{X})$

$= 0$

• $E(S^2) = \sigma^2$ shown in chapter-8.

• Biased and unbiased estimator :

कुनै estimator लाई unbiased estimator तब भनिन्छ यदी If it estimates (it estimate) the unknown parameter what it was supposed to estimate without any biasness.

• कुनै estimator bias छ कि छैन भनेर कसरी जान्ने?

कुनै estimator bias छ की छैन जान्नका लागी त्यसको expected value निकाल्ने and यदी त्यो expected value gives what it was supposed to estimate भने त्यो unbiased estimator हो।

• एक good estimator का properties भनेको त्यो unbiased हुनु पर्छ and त्यसको variance small हुनु पर्छ।

Maximum Likelihood Method (MLE) :-

Main Idea :-

Likelyhood function is defined as the product of individual pdf

$$L(\theta) = f(x_1/\vec{\theta})\, f(x_2/\vec{\theta}) \dots f(x_n/\vec{\theta})$$

हाम्रो main goal भनेको $L(\theta)$ लाई maximize गर्नु हो। त्यसको लागी पहिले यसलाई critical point चाहिन्छ for that put first derivative w.r.t θ of $L(\theta)$ to 0 and get expected values from them. And If we calculate second derivative it will be negative.

• $\hat{\theta}$ लाई MLE of θ हुनका लागी, it should have to maximize $L(\theta)$ (or $\ln[L(\theta)]$)

MLE को question solve गर्दा अनि and if there is need to use natural log अनि without doing any calculation first मा product $\prod_{i=1}$ लाई summation मा convert गर।

Eg. $$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}$$

$$\ln[L(\theta)] = \sum_{i=1}^{n} \left[\ln\left\{\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}\right\}\right]$$

- Invariance property of MLE :-

If $\hat{\theta}$ is the MLE of θ then $\hat{\tau} = \tau(\hat{\theta})$ is the MLE of $\tau(\theta)$ for any function $\tau(\theta)$.

यदी तिमीलाई θ को MLE $\hat{\theta}_{MLE}$ थाह छ भने, θ को जुन सुकै function लाई पनि एकै चोटीमा estimate गर्न सकिन्छ just by replacing θ with $\hat{\theta}$.

eg. If $\hat{\theta}_{MLE}$ is MLE of θ. Then The MLE of $e^{-\theta}$ will be $e^{-\hat{\theta}_{MLE}}$.

- Mean squared error (MSE)

$$MSE(T) = E[(T-\tau(\theta))^2]$$

where T is an estimator for $\tau(\theta)$.

$$= Var(T) + [Bias(T)]^2$$

where $Var(T) = E[(T-E(T))^2]$

& $Bias(T) = E(T) - \tau(\theta)$.

Bias → The value it was supposed to estimate − actual value.

- T is an unbiased estimator if $E(T) = \tau(\theta)$.

* UMVUE (Uniformly minimum variance unbiased estimator) *

T^* लाई चाहीं UMVUE of $\tau(\theta)$ भनिन्छ यदी

1) $\text{Bias}(T^*) = 0 \Rightarrow E(T^*) = \tau(\theta) \quad \forall\, \theta \in \Omega$

2) Variance of T^* is smallest among all unbiased estimator of $\tau(\theta)$

i.e $Var(T^*) \leq Var(T) \quad \forall \theta$

where T is any other unbiased estimator of $\tau(\theta)$.

* so कुनै unbiased estimator लाई UMVUE भन्नको लागी, त्यसको variance smallest हुनु पर्छ, मतलब we need the lowerbound for variances of all the unbiased estimators. And त्यो lowerbound पत्ता लगाउनको लागी we have the theorem CRLB.

* Cramer-Rao lower bound.

For any unbiased estimator T of $\tau(\theta)$ को लागी,

$$Var(T) \geq \frac{[\tau'(\theta)]^2}{n \cdot I(\theta)}$$

Where $I_1(\theta) = E\left[\frac{\partial}{\partial\theta} \ln f(X|\theta)^2\right]$

or $I_1(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \ln f(X|\theta)\right]$

* यदी $X_i \sim exp(\theta)$ छ भने, then

$\Sigma X_i \sim gamma(\theta, n)$

* CRLB is only the lower bound of unbiased estimator, so it can be possible that all the unbiased estimator may have variance more than CRLB. It may be the case that UMVUE may also has the variance > CRLB.

Now my question is
since UMVUE has variance lowest among all unbiased estimator it can be little greater than CRLB? It means CRLB only gives the limit of lower bound of unbiased estimator. It is not necessary that it should be the variance of one of the unbiased estimator.?

* So the Big question is कहिले unbiased estimator (let T) को variance ले CRLB attain गर्न सक्छ (means $Var(T) = CRLB$)। त्यो सक्नो लागि we are given one corollary called 'Attainment of CRLB'

* Corollary [Attainment of CRLB]
कुनै unbiased estimator T of $\tau(\theta)$ को variance $var(T)$ ले CRLB लाई attain गर्छ $[Var(T) = \tau(\theta)]$ iff score function is linear function of T (or T is linear function of score function)

* अब कुरा उठो what is score function?
score function $S = \frac{\partial}{\partial \theta} \ln L(\theta)$ where $L(\theta)$ is likelyhood fun^n.

? मानौं मैले एउटा unbiased estimator पत्ता लगाएँ T of $\tau(\theta)$, यदी त्यसको variance यदी lowest छ among all T of $\tau(\theta)$. Then त्यसले CRLB attain गरेन भने Isn't it UMVUE? (Sthg weired written in the note).

As → Yes.

- Score function बाट पनि लागिले estimator पत्ता लगाउन सक्दैँ, अनि त्यो estimator biased हुने कि unbiased हुने त्यो gurantee यहाँ केही भन्न सकिन्न ।

- MSE Consistancy (Stonger)

$\{T_n\}$ is MSE consistent if and only if it is asymptotically unbiased ($Bias \to 0$, as $n \to \infty$)

and $\lim_{n\to\infty} Var(T_n) = 0$.

- मतलब कुनै estimator लाई MSE consistant देखाउनु छ भने त्यसलाई Asymptotically unbiased देखाउने i.e $Bias \to 0$ as $n \to \infty$ and त्यसको varionce tends to 0 (i.e varionce $\to 0$) as $n \to \infty$ देखाउने ।
  - $Bias \to 0$ as $n \to \infty$ ⇒ unbiased देखाउने
  - $var \to 0$ as $n \to \infty$ देखाउने

- Asymptotic unbiased
  estimator is being unbiased as $n \to \infty$ i.e $\lim_{n\to\infty} E[T_n] = \tau(\theta)$

उसलाई त unbiased छ (i.e for finite n) $n \to \infty$ मा पनि unbiased हुनैं भयो ।

But we can find some estimator which are biased when n is finite but becomes unbiased as $n \to \infty$. Eg $T = \frac{n-1}{n}\bar{x}$ is suppose a biased estimator of $\tau(\theta)$ when n is finite.

But $\lim_{n\to\infty} E(T) = \lim_{n\to\infty} E\left(\frac{n-1}{n}\bar{x}\right) = \lim_{n\to\infty} \left(\frac{n-1}{n}\right) E(\bar{x})$

$$= \lim_{n\to\infty} \left(\frac{n-1}{n}\right) \tau(\theta) = \tau(\theta).$$

- So basically Asymptotic unbiased भनेको अहिले biased भए पनि $n\to\infty$ हुँदा unbiased हुनु पर्यो।

- मलाई simple consistent देखाउ भनेर question आयो भने म MSE consistent देखाई दिन्छु। यदी कुनै चिज MSE consistent छ भने त simple consistent त भइहाल्छ।

- $\hat{\theta}_n$ यहाँ MLE of $\theta = \tau(\theta)$ छ भने, under regulatarity condition, we can write

$$\hat{\theta}_n \sim AN(\theta, CRLB) \quad \text{i.e.}\ \hat{\theta}_n \sim AN(\tau(\theta), CRLB)$$

☆ Ask Her? के सबै MLE हरु MSE consistent हुन्छन् र?

☆ Homework question मा सिधै CRLB attain गर्छ कि गर्दैन check गरी दिने, by using score function wala concept. If it attains then variance calculate गर्नु कुरा भयो (☆ Ask Her)

Asymptotic Efficiency

$\{T_n\}$ and $\{T_n^*\}$ are 2 sequence of asymptotically unbiased estimator of $\tau(\theta)$.

relative efficiency $RE = \dfrac{Var(T_n^*)}{Var(T_n)} = RE(T_n, T_n^*)$

Asymptotic relative efficiency (ARE)

$$ARE = \lim_{n \to \infty} \frac{Var(T_n^*)}{Var(T_n)}$$

If the $ARE(T_n, T_n^*) = \lim_{n \to \infty} \dfrac{Var(T_n^*)}{Var(T_n)} \leq 1$

for all other $\{T_n\}$. Then $\{T_n^*\}$ is called asymptotically efficient.

- यदी हामीलाई asymptotically efficient हेर्नु छ भने, पहिले त यो estimator को variance calculate गर्ने हो and then CRLB calculate गर्ने हो। Later on both को ratio as $n \to \infty$ ≤ 1 देखाउने हो i.e

$$\lim_{n \to \infty} \frac{Var\{T_n^*\}}{CRLB} \leq 1 \qquad \text{CRLB भए, equal भए नि हुन्छ।}$$

- MLE चाहि always asymptotically efficient हुन्छन
  जसको मतलब $n \to \infty$ मा $Var(\hat{\theta}_{MLE}) = CRLB$ हुन्छ for sure.

Remember ★ के कुरा हो भने हामीले finite n को लागि पनि $Var(\hat{\theta}_{MLE}) = CRLB$ लेख्न मिल्दैन