1. (20) $X_1, X_2, \cdots X_n$ are i.i.d EXP(1) random variables.

(a) Show that $U = \sum_{i=1}^{n} X_i$ has a Gamma distribution. identify the corresponding shape and scale parameters.
(b) Use Central Limit Theorem to find the asymptotic distribution of $U = \sum_{i=1}^{n} X_i$ when n is large.
(c) Let $\bar{X}$ be the sample mean, then approximate $P(1.1 < \bar{X} < 1.2)$ for $n = 100$.

2. (30pt) Consider a random sample $X_1, X_2 \cdots X_n$ from CDF $F(x) = 1 - 1/x$ for $x \in [1, \infty)$ and zero otherwise.

(a) Find the limiting distribution of $X_{1:n}$, the smallest order statistic.
(b) Find the limiting distribution of $X_{1:n}^n$.
(c) Find the limiting distribution of $n \ln X_{1:n}$.

3. (20pt) Suppose that $X_1, X_2, \cdots X_n$ is an i.i.d sample from a population with following pdf,

$$f_X(x) = e^{-(x-\mu)}, \text{ for } x > \mu \text{ and 0 otherwise.}$$

(a) Find the cumulative distribution function of the minimum order statistics $X_{1:n}$.
(b) Show that $X_{1:n}$ converges to a degenerate distribution at μ.

4. (20pt) Stirling's Formula, which gives approximation for factorials, can be derived using CLT. In this problem, we will derive Stirling's Formula.

(a) Suppose that $X_1, X_2, \cdots, X_n$ is an i.i.d. sample from Exp(1). Use CLT to show $T_n = \frac{\bar{X}_n - 1}{1/\sqrt{n}}$ converges in distribution to a standard normal random variable Z, i.e.

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} < x\right) \to P(Z < x).$$

(b) Show

$$\frac{\sqrt{n}}{\Gamma(n)}(x\sqrt{n} + n)^{n-1} e^{-(x\sqrt{n}+n)} \approx \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

by differencing both sides of the approximation in part a. Then set $x = 0$ and write out the final formula. The result is Stirling's Formula.

5. (20pt) Let $X_1, X_2, \cdots X_n$ be a i.i.d. sample from Bernoulli(p).

(a) Show $\sqrt{n}(\hat{p} - p) \xrightarrow{d} N(0, p(1-p))$.
(b) (Mandatory for Graduate Student. Extra credit for undergrad.) Let $Y_n = \sum_{i=1}^{n}(X_i - p)/n$. Show that Y_n converges to a degenerate distribution at 0 as $n \to \infty$.