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INSTRUCTIONS

1. You are allowed **1 hour 15 minutes** for this exam.
2. Show **ALL** work on this sheet to receive partial credit or full credit.
3. Neat presentation of work is required. Answers which are not legible will be assumed incorrect
4. Make sure your mobile phone is switched off and place it in your bag together with any books and materials not allowed on this test
5. Leave the exam hall quickly and quietly. Remember to take all your belongings with you.
6. No calculators during the exam.
7. **Answer ALL questions!**

Answer the following questions [2 point each 14 total points]

1. If matrix $\mathbf{A}$ is positive definite, $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$.

(a) True (b) False

2. **(Select all that apply)** If the covariance between two variables is zero, this implies :

a) The variables are independent.
b) The variables are not correlated.
c) The mean of the variables is zero.
d) The variables have a normal distribution.

3. What can be said about the eigenvalues and eigenvectors of $\mathbf{\Sigma}^{-1}$ relative to those of $\mathbf{\Sigma}$?

a) $\mathbf{\Sigma}^{-1}$ and $\mathbf{\Sigma}$ have the same eigenvalues and eigenvectors
b) The eigenvalues of $\mathbf{\Sigma}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{\Sigma}$, and the eigenvectors are the same
c) The eigenvalues of $\mathbf{\Sigma}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{\Sigma}$, and the eigenvectors are negatives of each other
d) The eigenvalues of $\mathbf{\Sigma}^{-1}$ are the squares of the eigenvalues of $\mathbf{\Sigma}$, and the eigenvectors are the same.

4. What is the distribution of a single linear combination $\mathbf{a}'\mathbf{X}$ when $\mathbf{X}$ is multivariate normal?

a) Multivariate normal
b) Univariate normal
c) Wishart
d) Chi-squared

5. Which of the following is true about the multivariate normal distribution?

a) Zero correlation implies independence
b) Linear combinations are normally distributed
c) The conditional distributions are normal
d) All of the above

6. Which measure is used to summarize the relationship between two continuous variables?

a) Mean
b) Variance
c) Correlation
d) Median

7. **Select the best answer.** If a multivariate dataset follows a normal distribution in every possible linear combination of variables, it is said to be:

a) Marginally normal
b) Jointly normal
c) Unconditionally normal
d) Fully normal

$$
\begin{array}{c|cc|cc}
 & x_1 & x_3 & x_2 & x_4 \\
\hline
x_1 & 3 & 2 & 0 & 2 \\
x_3 & 2 & 9 & 1 & -2 \\
\hline
x_2 & 0 & 1 & 1 & 0 \\
x_4 & 2 & -2 & 0 & 4
\end{array}
$$

SECTION B

In what follows, only perform matrix multiplication if the answer is needed for a follow up question. If not, leave your answers as a product of matrices.

1. **[13 pts]** Suppose the mean vector and covariance matrix of $\mathbf{X} = (X_1, X_2, X_3, X_4)$ is given by:

$$\mu = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{bmatrix}.$$

$$\mu_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\mu_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Let

$$\mathbf{X}^{(1)} = \begin{bmatrix} X_1 \\ X_3 \end{bmatrix} \quad \mathbf{X}^{(2)} = \begin{bmatrix} X_2 \\ X_4 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$$

Find

(a) [4 pts] $E(\mathbf{AX}^{(1)})$ and $E(\mathbf{BX}^{(2)})$

$$E(AX^{(1)}) = A\,E(X^{(1)}) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= 8$$

$$E(BX^{(2)}) = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

(b) [4 pts] $\text{Cov}(\mathbf{AX}^{(1)})$ and $\text{Cov}(\mathbf{BX}^{(2)})$

$$\text{Cov}(AX^{(1)}) = A\,\Sigma_{11}\,A^T$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\text{Cov}(BX^{(2)}) = B\,\Sigma_{22}\,B^T$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

(c) [2 pts] $\text{Cov}(\mathbf{X}^{(1)}, \mathbf{X}^{(2)})$

$$\text{cov}(X^{(1)}, X^{(2)}) = \Sigma_{12} = \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$$

(d) [3 pts] $\text{Cov}(\mathbf{AX}^{(1)}, \mathbf{BX}^{(2)})$

$$\text{cov}(AX^{(1)}, BX^{(2)}) = A\,\text{cov}(X^{(1)}, X^{(2)})\,B^T$$

$$= \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$$

2. [8 pts] The density function of a multivariate normal distribution can be written in the form

$$f(\mathbf{x}) = \frac{\sqrt{|\mathbf{A}|}}{(2\pi)^{p/2}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{b})^T \mathbf{A}(\mathbf{x}-\mathbf{b})} \quad -\infty < \mathbf{x} < \infty \quad \text{———} \quad (*)$$

Find $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{p}$ for the random vector $\mathbf{x}$ with density function:

$$f(\mathbf{x}) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1^2 + x_2^2 + 4x_1 - 6x_2 + 13)\right]$$

We are given

$$f(x) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1^2 + x_2^2 + 4x_1 - 6x_2 + 13)\right]$$

$$= \frac{1}{2\pi} \exp\left[-\frac{1}{2}(x_1^2 + 4x_1 + 4 - 4 + x_2^2 - 6x_2 + 9 - 9 + 13)\right]$$

$$= \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left((x_1+2)^2 + (x_2-3)^2\right)\right]$$

$$= \frac{1}{2\pi} \exp\left[-\frac{1}{2}\left((x-b)^T A (x-b)\right)\right]$$

Where $(x-b)^T A (x-b)$

$$= \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\}$$

$\therefore$ Clearly $p = 2$, $b = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ & $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Such that $|A| = 1$.

3. **[10 pts]** Suppose f(x,y) is a bivariate normal density with $\mu_{X_1} = 0, \mu_{X_2} = 0$, $\sigma^2_{X_1} = 4, \sigma^2_{X_2} = 1, \rho_{X_1 X_2} = 0.8$

(a) **[4 pts]** Write the marginal distributions of X_1 and X_2.

Marginal distribution of X_1 and X_2 both is Univariate Normal

so $X_1 \sim N(\mu_{X_1}, \sigma_{X_1}^2)$

i.e $X_1 \sim N(0, 4)$

& $X_2 \sim N(\mu_{X_2}, \sigma_{X_2}^2)$

i.e $X_2 \sim N(0, 1)$

(b) **[6 pts]** What is the marginal distribution of $X_1 + 2X_2$?

① $E(X_1 + 2X_2) = \mu_1 + 2\mu_2 = 0 + 2\cdot(0) = 0$

② $Var(X_1 + 2X_2) = Var(X_1) + 4\,Var(X_2) + 4\,cov(X_1, X_2)$

$= 4 + 4\cdot(1) + 4\cdot(1.6)$

$= 14.4$

$\{ \because \sigma_{12} = \rho_{12}\sqrt{\sigma_1^2}\sqrt{\sigma_2^2}$
$= 0.8\cdot(2)\cdot(1)$
$= 1.6$

So The marginal distribution of $X_1 + 2X_2$ is univariate Normal

i.e $X_1 + 2X_2 \sim N\left(\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1.6 \\ 1.6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

4. **[5 pts]** Let $\mathbf{X} \sim N_p(\mu, \Sigma)$. Show that the random variable $\mathbf{Z} = (\mathbf{X} - \mu)^T \Sigma^{-1} (\mathbf{X} - \mu)$ has a $\chi^2_{(p)}$ distribution.

given $\underset{\sim}{X} \sim N_p(\mu, \Sigma)$.

so $\underset{\sim}{X} - \underset{\sim}{\mu} \sim N_p(\underset{\sim}{0}, \Sigma)$

Let $Z = (\underset{\sim}{X} - \underset{\sim}{\mu}) \Sigma^{-1/2} \sim N_p(\underset{\sim}{0}, I)$. Then

$$Z Z^T = [(\underset{\sim}{X} - \underset{\sim}{\mu}) \Sigma^{-1/2}][(\underset{\sim}{X} - \underset{\sim}{\mu}) \Sigma^{-1/2}]^T$$

$$= (\underset{\sim}{X} - \underset{\sim}{\mu}) \Sigma^{-1/2} \cdot \Sigma^{-1/2} (\underset{\sim}{X} - \underset{\sim}{\mu})^T$$

$$= (\underset{\sim}{X} - \underset{\sim}{\mu}) \Sigma^{-1} (\underset{\sim}{X} - \underset{\sim}{\mu}) \sim \chi^2_{(p)}$$

5. **[4 pts]** A quadratic form $\mathbf{x}^T \mathbf{A} \mathbf{x}$ for a matrix $\mathbf{A}$ is given as $3x_1^2 + 3x_2^2 - 2x_1x_2$. Is the matrix $\mathbf{A}$ positive definite?

We are given, for martrix A, The quadratic form

$$x^T A x = 3x_1^2 + 3x_2^2 - 2x_1x_2$$

$$= 2x_1^2 + 2x_2^2 + x_1^2 + x_2^2 - 2x_1x_2$$

$$= 2x_1^2 + 2x_2^2 + (x_1 - x_2)^2 > 0$$

so Matrix A is positive definite (provide $\underset{\sim}{x}$ should be non-zero vector)

6. [7 pts] Find the spectral decomposition of the matrix $\mathbf{A} = \begin{bmatrix} 1 & r \\ r & 1 \end{bmatrix}$

$$Ax = \lambda x \quad \text{—— ①} \Rightarrow (A-\lambda I)x = 0$$

The characteristic Eqn of ① is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & r \\ r & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - r^2 = 0$$

$$\Rightarrow (1-\lambda) = \pm r$$

$\Rightarrow \lambda = 1+r$ or $1-r$ are the Eigen Values

$\boxed{\text{For } \lambda = 1+r}$

$$\begin{bmatrix} -r & r \\ r & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -rx_1 + rx_2 &= 0 \\ rx_1 - rx_2 &= 0 \end{aligned}\right\} \text{identical}$$

Let $x_1 = 1$, then $x_2 = 1$

$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is eigen vector

so Normalized eigen vector is $e_1 = \begin{bmatrix} \frac{1}{\sqrt{1^2+1^2}} \\ \frac{1}{\sqrt{1^2+1^2}} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\boxed{\text{For } \lambda = 1 - r}$$

$$\begin{bmatrix} r & r \\ r & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} r x_1 + r x_2 &= 0 \\ r x_1 + r x_2 &= 0 \end{aligned}\right\} \text{identical}$$

Let $x_1 = 1$, then $x_2 = -1$.

$\therefore$ Eigen vector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow$ Normalized eigen vector $\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

Finally, spectral decomposition of A is

$$A = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T$$

$$= (1+r) \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + (1-r) \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

7. **[5 pts]** Given the data matrix

$$\mathbf{X} = \begin{bmatrix} 9 & 1 \\ 5 & 1 \\ 1 & 2 \end{bmatrix},$$

Find the covariance and correlation between $-X_1 + 2X_2$ and $2X_1 + 3X_2$. (*Use the unbiased covariance matrix of X*).

$\underset{\sim}{b}^T \underset{\sim}{x} = [-1\ 2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $\underset{\sim}{c}^T \underset{\sim}{x} = [2\ 3]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$S = \frac{1}{(n-1)} \underline{X}^T \left(I - \frac{1}{n} \mathbb{1}\mathbb{1}^T\right) \underline{X}$$

$$= \frac{1}{2}\begin{bmatrix} 9 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 9 & 1 \\ 5 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 9 & 5 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -1/3 \\ 0 & -1/3 \\ -4 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -2 \\ -2 & 1/3 \end{bmatrix}$$

covariance

$$\mathrm{cov}(\underset{\sim}{b}^T \underset{\sim}{x}, \underset{\sim}{c}^T \underset{\sim}{x})$$

$$= [-1\ 2]\begin{bmatrix} 16 & -2 \\ -2 & 1/3 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$= [-1\ 2]\begin{bmatrix} 26 \\ -3 \end{bmatrix}$$

$$= -32$$

Variance

$$\mathrm{Var}(-X_1 + 2X_2) = 16 + 4\left(\frac{1}{3}\right) - 4(-2)$$

$$= 16 + \frac{4}{3} + 8 = \frac{76}{3}$$

Page	2	3	4	5	6	7	8	9	10	Total
Points earned										