Homework-2 — Sagar Kalani

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Please staple your work

2.6) Let $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$

1	1	4
2	16	18
Proof 1	3	3
Proof 2	3	3
	23	28

23/28

a.) Is A symmetric

b.) show that A is positive definite

soln

a.) A squre matrix is said to be symmetric if it is equals to its transpose.

Mathematicall

$$\forall a_{ij} \in A, \quad a_{ij} = a_{ji} \quad , \text{where } A \text{ is a squre matrix}$$

$\Rightarrow$ A is symmetric matrix

1

$\therefore$ Matrix $A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$ is a symmetric matrix. $\square$

Problem 2?

b.) First we will find eigen value and eigen vector for the symmetric matrix A.

we have $\quad AX = \lambda X$

$$\Rightarrow (A - \lambda I) X = 0 \quad \text{—— ①}$$

where $I \rightarrow$ identity matrix

$X \rightarrow$ unknown vector

$0 \rightarrow$ zero vector

To find eigen value, we will solve the equation

$$|A - \lambda I| = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (9-\lambda)(6-\lambda) - 4 = 0$$

$$\Rightarrow 54 - 9\lambda - 6\lambda + \lambda^2 - 4 = 0$$

$$\Rightarrow 50 - 15\lambda + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - 15\lambda + 50 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 10\lambda + 50 = 0$$

$$\Rightarrow \lambda(\lambda - 5) - 10(\lambda - 5) = 0$$

$$\Rightarrow (\lambda - 10)(\lambda - 5) = 0$$

$\Rightarrow \lambda = 5, 10$ $\longrightarrow$ So these are our eigen values.

Now crossponding to these eigen value, we will find the eigen vector.

For $\lambda = 5$,

from eqn ①, we have

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 4x_1 - 2x_2 = 0 \quad \& \quad -2x_1 + x_2 = 0$

$\Rightarrow \quad 2x_1 = x_2 \quad \& \quad 2x_1 = x_2$

Since both of the equation's are same so we have infinitely many solutions

let $x_1 = 1$. Then $x_2 = 2$

$$\Rightarrow X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow e_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$ → eigen vector for eigen value $\lambda = 5$

where did $\sqrt{5}$ come from?

Ⓘ For $\lambda = 10$

from eqn ①, we have

$$\begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x_1 - 2x_2 = 0$ and $-2x_1 - 4x_2 = 0$

$\Rightarrow \quad x_1 = -2x_2 \quad \& \quad x_1 = -2x_2$

$\Rightarrow$ Since both the equation are same so we have infinitly many solutions

let $x_1 = 2$, then $x_2 = -1$

$$\Rightarrow X = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

where from $\sqrt{5}$?

$$\Rightarrow e_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -1/\sqrt{5} \end{bmatrix}$$ $\rightarrow$ eigen vector for eigen value $\lambda = 10$.

Now the Spectral decomposition of symmetric matrix A is given as

$$A = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T$$

Note:

Using the spectral decomposition, we can easily show that a symmetric matrix A is a positive definite matrix if and only if every eigenvalue of A is positive

your

let's show that, for that let consider $x' = [x_1, x_2]$ be any non-zero vector

$x^T = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ $(x_1 \quad x_2)$

Label your work appropriately.

he premultiplication and postmultiplication by this vector and its transpose to Matrix A will give

$$\underset{1\times2}{x'}\underset{2\times2}{A}\underset{2\times1}{x} = x'(\lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T)x$$

Using spectral decomposition of A

$$= x'(5e_1e_1^T + 10e_2e_2^T)x$$

$$= 5x'e_1 e_1^T x + 10x'e_2 e_2^T x$$

This does not match what you said on previous page.

let $y_1 = x'e_1$. Then $(e_1^T x)^T = x'e_1 = y_1$

& $y_2 = x'e_2$. Then $(e_2^T x)^T = x'e_2 = y_2$

$\therefore$ $x'Ax = 5y_1^2 + 10y_2^2 \geq 0$

$\Rightarrow$ A is positive definite. $\square$

-3

2.7
Let A be as given in exerise 2.6

a) Determine the eigenvalues and eigenvectors of A.

b) Write the spectral decomposition of A

c) Find A^{-1}

d) Find the eigenvalues and eigenvector of A^{-1}.

solu

ⓐ To find a eigen value and eigen vector, we have

$$AX = \lambda X$$

$$\Rightarrow (A - \lambda I)X = 0 \quad \text{—①}$$

To find eigen value, we solve the eqn

$$|A - \lambda I| = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 9-\lambda & -2 \\ -2 & 6-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (9-\lambda)(6-\lambda) - 4 = 0$$

$\Rightarrow \lambda = 5, 10 \longrightarrow$ So these are our eigen values

Now crossponding to these eigen value, we will find the eigen vector

① For $\lambda = 5$

from eqn ①, we have

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 4x_1 - 2x_2 = 0 \quad \& \quad -2x_1 + x_2 = 0$

$\Rightarrow \quad 2x_1 = x_2 \qquad \& \quad 2x_1 = x_2$ (same eqn, infinitly many sol^ns)

let $x_1 = 1$, then $x_2 = 2$

$\Rightarrow X = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \Rightarrow e_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ → eigen vector crossponding to eigen value $\lambda = 5$

where from $\sqrt{5}$

② for $\lambda = 10$

similarly, we can get

$e_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ → as our eigen vector crossponding to eigen value $\lambda = 10$

[complete step-by-step, I already solved in last question] ok

Now
given

Now the spectral decomposition of symmetric matrix A is given as

$$A = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2'$$

$$= 5 \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} + 10 \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix}$$

(c) We are given matrix

$$A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

The inverse of given matrix A is denoted by A^{-1} and given by

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

Here

$$|A| = 9 * 6 - (-2)\cdot(-2)$$

$$= 54 - 4$$

$$= 50$$

& $$\operatorname{adj}(A) = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{50}\begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 6/50 & 2/50 \\ 2/50 & 9/50 \end{bmatrix}$$

To find eigen value, we solve eqn

$$A^{-1}X = \lambda X$$

$$\Rightarrow (A^{-1} - \lambda I)X = 0 \quad \text{——} \; (1)$$

The characteristic eqn is

$$|A^{-1} - \lambda I| = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 6/50 & 2/50 \\ 2/50 & 9/50 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 6/50 - \lambda & 2/50 \\ 2/50 & 9/50 - \lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \left(\frac{6 - 50\lambda}{50}\right)\cdot\left(\frac{9 - 50\lambda}{50}\right) - \frac{2}{50}\cdot\frac{2}{50} = 0$$

$$\Rightarrow 2500\lambda^2 - 750\lambda + 50 = 0$$

$\Rightarrow (50\lambda - 5)(50\lambda - 10) = 0$

$\Rightarrow \lambda = \frac{1}{5}, \frac{1}{10}$ $\longrightarrow$ So these are our eigen values

Now, finding the Eigen value crossponding to these eigen Values

i) for $\lambda = \frac{1}{5}$

From eqn ①, we have (Just above ①)

$$\begin{bmatrix} 6/50 & 2/50 \\ 2/50 & 9/50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow \frac{6x_1}{50} + \frac{2x_2}{50} = \frac{x_1}{5}$ & $\frac{2x_1}{50} + \frac{9x_2}{50} = \frac{x_2}{5}$

$\Rightarrow 6x_1 + 2x_2 = 10x_1$ & $2x_1 + 9x_2 = 10x_2$

$\Rightarrow 4x_1 = 2x_2$ & $2x_1 = x_2$ (same eqn, infinitly many solns)

let $x_1 = 1$ then $x_2 = 2$. so

$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow e_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ $\rightarrow$ eigen vector for eigen value $\lambda = 1/5$

where from $\sqrt{5}$?

Let X

For $\lambda = \frac{1}{10}$

We have

$$\begin{bmatrix} 6/50 & 2/50 \\ 2/50 & 9/50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow \frac{6x_1}{50} + \frac{2x_2}{50} = \frac{x_1}{10}$ & $\frac{2x_1}{50} + \frac{9x_2}{50} = \frac{x_2}{10}$

$\Rightarrow 6x_1 + 2x_2 = 5x_1$ & $2x_1 + 9x_2 = 5x_2$

$\Rightarrow x_1 = -2x_2$ & $x_1 = -2x_2$ (same eqn, infinitely many soln)

let $x_1 = 2$, then $x_2 = -1$. So

$X = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \Rightarrow e_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \rightarrow$ eigen vector for eigen value $\lambda = 1/10$

$\sqrt{5}$ From?

OBSERVATION :-

If matrix A has eigen value λ_1 & λ_2. and eigen vectors e_1 and e_2 then A^{-1} will have eigen value $1/\lambda_1$ & $1/\lambda_2$ with same eigen vector. $\square$

for problem 2 or 3?

1. Let X have covariance matrix

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find a) Σ^{-1}

b) The eigenvalue and eigenvector of Σ

c) The eigenvalues and eigenvector of Σ^{-1}.

Solu

We are given

$$\Sigma = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can calculate inverse of Σ by using formulae

$$\Sigma^{-1} = \frac{1}{|\Sigma|} \operatorname{adj}(\Sigma)$$

But since Σ is a diagonal matrix it inverse can be easily calculated as

$$\Sigma^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/1 \end{bmatrix}$$

Mathematically

inverse of diagonal matrix A is denoted by A^{-1} and given by, we

$\forall\ x_{ii} \in A,\ x'_{ii} = \frac{1}{x_{ii}} \in A^{-1}$ & $x_{ij} = 0$ for $i \neq j$

b.> To find Eigen value & Eigen vector, we have

$$\Sigma X = \lambda X$$ where $X \rightarrow$ unknown vector

The chracteristic eqn is $\Rightarrow (\Sigma - \lambda I)X = 0$ — ① $\lambda \rightarrow$ unknown value

$\Sigma \rightarrow$ given matrix

$$|\Sigma - \lambda I| = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 4-\lambda & 0 & 0 \\ 0 & 9-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow (4-\lambda)\begin{vmatrix} 9-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0 + 0 = 0$$

$$\Rightarrow (4-\lambda)(9-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 1, 4, 9 \longrightarrow \text{our Eigen values}$$

For

Now we will find eigen vectors crossponding to these eigen vectors

1) For $\lambda = 1$

we have

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$3x_1 = 0$, $8x_2 = 0$ & $0x_1 + 0x_2 + 0x_3 = 0$

$\Rightarrow x_1 = 0$, $x_2 = 0$ & x_3 is a free variable (let $x_3 = 1$)

$\therefore X = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to$ eigen vector for eigen value $\lambda = 1$

2) for $\lambda = 4$

we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 0x_1 + 0x_2 + 0x_3 = 0$, $5x_2 = 0$ & $-3x_3 = 0$

$\Rightarrow x_2 = 0$, $x_3 = 0$ & x_1 is free variable (let $x_1 = 1$)

$\therefore X = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to$ eigenvector for eigen value $\lambda = 4$

3) for $\lambda = 9$

$$\begin{bmatrix} -5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \; -5x_1 = 0, \quad -8x_3 = 0 \;\&\; 0x_1 + 0x_2 + 0x_3 = 0$

$\Rightarrow \; x_1 = 0, \; x_3 = 0 \;\&\; x_2$ is free variable (let $x_2 = 1$)

$$\therefore \; X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ → eigen vector for eigen value $\lambda = 9$.

c) From our Last OBSERVATION, we can easily say that eigen values of Σ^{-1} are $1, 1/4, 1/9$ with same eigen vectors e_1, e_2, e_3. $\square$

Extra Work

$$\Sigma^{-1} X = \lambda X$$

$$\Rightarrow (\Sigma^{-1} - \lambda I) X = 0 \quad \text{— (1)}$$

Characteristic eqn is

$$|\Sigma^{-1} - \lambda I| = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\Rightarrow \det\left(\begin{bmatrix} 1/4-\lambda & 0 & 0 \\ 0 & \frac{1}{9}-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = 0$$

$$\Rightarrow \left(\frac{1}{4}-\lambda\right)\begin{vmatrix} 1/9-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0 + 0 = 0$$

$$\Rightarrow \left(\frac{1}{4}-\lambda\right)\left(\frac{1}{9}-\lambda\right)(1-\lambda) = 0$$

$$\Rightarrow \lambda = 1, \frac{1}{4}, \frac{1}{9} \longrightarrow \text{our Eigen values}$$

So, the eigen vector for these eigen values are

i) For $\lambda = 1$

$$\begin{bmatrix} -3/4 & 0 & 0 \\ 0 & -8/9 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$\because$ x_3 is free variable
so, let $x_3 = 1$

$\Rightarrow \quad e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Similarly, For $\lambda = \frac{1}{4}$ $\quad e_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $\quad x_1$ was free variable

& $\lambda = \frac{1}{9}$, $e_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, x_2 was free variable. $\square$

So our OBSERVATION is cross-checked. $\square$

** THE END **

Prove $S^2 = \dfrac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n}$ is biased estimator of σ^2.

Solu.

Suppose that we are given $X_1, X_2, \ldots, X_n$ iid random sample from any distribution with mean $E(X_i) = \mu$ and variance, $Var(X_i) = \sigma^2 < \infty$

~~[illegible]~~, we know that

$$\overline{X} \sim D\left(\mu, \frac{\sigma^2}{n}\right)$$

where D is any distribution.

- To prove s^2 is biased estimator of σ^2, we need to show $E(s^2) \neq \sigma^2$.

So Here

$$E(S^2)$$

$$= E\left(\frac{\sum (X_i - \overline{X})^2}{n}\right)$$

$$= \frac{1}{n} E\left(\sum (X_i - \overline{X})^2\right)$$

$$= \frac{1}{n} E\left(\sum (X_i^2 + \bar{X}^2 - 2X_i\bar{X})\right)$$

$$= \frac{1}{n}\left[E\left(\sum X_i^2 + \sum \bar{X}^2 - 2\bar{X}\sum X_i\right)\right]$$

Since we know that $\bar{X} = \frac{\sum X_i}{n}$

$$\Rightarrow \sum X_i = n\bar{X}$$

$$= \frac{1}{n}\left[E\left(\sum X_i^2 + n\bar{X}^2 - 2n\bar{X}^2\right)\right]$$

$$= \frac{1}{n}\left[E\left(\sum X_i^2 - n\bar{X}^2\right)\right]$$

Now I am going to use sum of expectation is equal to Expectation of Sum

$$\cancel{= \frac{1}{n}\left[\sum\left(E(X_i^2)\right) - n\bar{X}^2\right.}$$

$$= \frac{1}{n}\left[\sum\left(\sum\right.\right.$$

$$= \frac{1}{n}\left[\sum\left(E(X_i^2)\right) - E(n\overline{X}^2)\right] \quad \text{—— } \star$$

Expected value goes on $n\overline{X}^2$ term also because it also contain random variables X_i, means $n\overline{X}^2$ is not constant term

Now we will us few relations, lets build them

we know

$$Var(X_i) = E(X_i^2) - \{E(X_i)\}^2$$

$$\Rightarrow \quad \sigma^2 = E(X_i^2) - \mu^2$$

$$\Rightarrow \quad E(X_i^2) = \sigma^2 + \mu^2 \quad \text{—— (I)}$$

Also since we have

$$\overline{X} \sim \cancel{D}\left(\mu, \frac{\sigma^2}{n}\right)$$

~~where D is any distribution~~

not true.

$$\text{so} \quad Var(\overline{X}) = E(\overline{X}^2) - \{E(\overline{X})\}^2$$

$$\Rightarrow \quad \frac{\sigma^2}{n} = E(\overline{X}^2) - \mu^2$$

$$\Rightarrow \quad E(\overline{X}^2) = \frac{\sigma^2}{n} + \mu^2 \quad \text{—— (II)}$$

Using (I) & (II) in $\star$, we get

$$= \frac{1}{n}\left[\Sigma(\sigma^2+\mu^2) - n\,E(\bar{X}^2)\right]$$

$$= \frac{1}{n}\left[\Sigma(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right]$$

$$= \frac{1}{n}\left[n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2}\right]$$

$$= \frac{1}{n}(n-1)\sigma^2$$

$\neq \sigma^2$, Hence s^2 is biased estimator of σ^2

when $s^2 = \frac{\Sigma(X_i-\bar{X})^2}{n}$

And from above reasult we can clearly see that

$s^2 = \frac{\Sigma(X_i-\bar{X})^2}{n-1}$ will be unbiased estimator

of σ^2. $\square$

Prove $Var(aX_1+bX_2) = a^2 Var(X_1) + b^2 Var(X_2) + 2ab\, cov(X_1, X_2)$

Sol^n

So we have on LHS

$$Var(aX_1 + bX_2)$$

$$= E\left[\left((aX_1+bX_2) - E(aX_1+bX_2)\right)^2\right] \quad \text{by def}^n$$

$$= E\left[\left(aX_1 + bX_2 - aE(X_1) - bE(X_2)\right)^2\right]$$

$$= E\left[\left(a(X_1 - E(X_1)) + b(X_2 - E(X_2))^2\right)^2\right]$$

$$= E\Big[a^2(X_1 - E(X_1))^2 + b^2(X_2 - E(X_2))^2$$
$$+ 2a(X_1 - E(X_1)) \cdot b(X_2 - E(X_2))\Big]$$

$$= E\Big[a^2(X_1 - E(X_1))^2 + b^2(X_2 - E(X_2))^2$$
$$+ 2ab\,(X_1 - E(X_1))(X_2 - E(X_2))\Big]$$

$$= a^2 E\left[(X_1 - E(X_1))^2\right] + b^2 E\left[(X_2 - E(X_2))^2\right]$$

$$+ 2ab\, E\left[(X_1 - E(X_1))(X_2 - E(X_2))\right]$$

$$= a^2 \operatorname{Var}(X_1) + b^2 \operatorname{Var}(X_2) + 2ab \operatorname{Cov}(X_1, X_2). \square$$