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Stat 480B HomeWork #1

1.) Solu

@ $X_1, X_2, \ldots, X_n \overset{iid}{\sim} EXP(1)$. "exponential distribution with rate parameter 1" OR exponential distribution with $\lambda = 1$.

95/100

Then CDF of X_i:

$$F_i(X) = 1 - e^{-\lambda x}$$

$\lambda = 1$ is given so

$$= 1 - e^{-x} \quad [= P(X \leq x)]$$

let $U = \sum_{i=1}^{n} X_i$

(summition as mean को term मा आउदा सामेल mgf technique use गर्नु पर्छ)

Now mgf of U: $M_U(t) = E(e^{tU})$

$$= E(e^{t(X_1 + \cdots + X_n)})$$

$$= E(e^{tX_1} \cdots e^{tX_n})$$

X_i हरु independent भएको हुनाले तिनीहरूको linear combination पनि independent होला

$$= E(e^{tX_1}) \cdots E(e^{tX_n})$$

$$= \left(\frac{\lambda}{\lambda - t}\right) \cdots \left(\frac{\lambda}{\lambda - t}\right), \text{ for } t < \lambda$$

mgf of exponential distribution $M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)$

$$= \left(\frac{\lambda}{\lambda - t}\right)^n$$

But $\lambda = 1$ is given, so

$$= \left(\frac{1}{1 - t}\right)^n$$

$$= (1-t)^{-n}$$

This is mgf for gamma distribution

so,

$U = \Sigma X_i \sim \text{gamma}(\alpha, \beta)$ where $\alpha = n$

& we know

$\beta = \frac{1}{\lambda} = \frac{1}{1} = 1$

so $U = \Sigma X_i \sim \text{gamma}(n, 1)$

For gamma distribution $\alpha (=n)$ is known as shape parameter and $\beta (=1)$ is reffered as the scale parameter.

$n = k$ shape

given

$X_1,$

exponential distribution with
$\lambda = 1$ or rate parameter 1.

given

$X_1, \ldots, X_n$ are iid EXP(1). So

mean $= E(X_i) = \frac{1}{\lambda} = \frac{1}{1} = 1$ & $\longrightarrow \mu = 1 \quad \theta = 1$

variance $= \frac{1}{\lambda^2} = \frac{1}{(1)^2} = 1 \longrightarrow \sigma^2 = 1 \quad \theta^2 = 1^2 = 1$

Using CLT :

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

$$\frac{\bar{X} - 1}{1/\sqrt{n}} \xrightarrow{d} N(0,1)$$

$$\frac{n\bar{X} - n}{\sqrt{n}} \xrightarrow{d} N(0,1)$$

$\because \bar{X} = \frac{\sum X_i}{n}$

$$\frac{\sum X_i - n}{\sqrt{n}} \xrightarrow{d} N(0,1) \qquad \Rightarrow \sum X_i = n\bar{X}$$

$$\therefore \sum X_i \sim AN(\overset{\text{mean}}{n}, \overset{\text{variance}}{n})$$

given $X_1, X_2, \dots, X_n$ are i.i.d EXP(1), so

mean $= E(X_i) = \frac{1}{\lambda} = \frac{1}{1} = 1 \longrightarrow \mu = 1, \mu = \theta = 1$

$var(X_i) = \frac{1}{\lambda^2} = \frac{1}{1^2} = 1 \longrightarrow \sigma^2 = 1, \sigma^2 = \theta^2 = 1^2 = 1$

Using the fact: (not a fact!).

The sample mean of i.i.d exponential random variable is also an exponential random variable

so,

$\cancel{\bar{X} \sim EXP(1)}$.

Since sample mean $\bar{X}$ follows Exponential distribution, it CDF is

$F(X) = 1 - e^{-\lambda X}$ $\quad P(1.1 < X < 1.2)$

Here,

$P(1.1 < \bar{X} < 1.2)$

$= F(1.2) - F(1.1)$

$\because$ $\bar{X}$ = mean of $X_1, \dots, X_n$

$\Rightarrow \bar{X} = \frac{X_1 + \dots + X_n}{n}$

$$E(\bar{X}) = E\left(\frac{X_1 + \dots + X_n}{n}\right)$$
$$= \frac{1}{n} E(X_1) + \dots + E(X_n)$$
$$= \frac{1}{n}(n * 1)$$
$$= 1$$

$$var(\bar{X}) = var\left(\frac{X_1 + \dots + X_n}{n}\right)$$
$$= \frac{1}{n^2} var(X_1 + \dots + X_n)$$
$$= \frac{1}{n^2}(n * 1^2)$$
$$= \frac{1}{n}$$

PTO

Ignore

$$= \left[1 - e^{-\frac{1.2}{100}}\right] - \left[1 - e^{-\frac{1.1}{100}}\right]$$

$$= \left[1 - e^{0.012}\right] - \left[1 - e^{0.011}\right]$$

$$\approx 0.0933$$

$\because$ we know that rate parameter $(\lambda) = \frac{1}{\text{sample size}}$

i.e $\lambda = \frac{1}{n}$

So the approximate probability that the sample mean falls between 1.1 and 1.2 for $n=100$ is approximately 0.0933. $\square$

* * Remaining part * *

$$= F(1.2) - F(1.1)$$

$$= \Phi\left(\frac{1.2 - 1}{1/\sqrt{100}}\right) - \Phi\left(\frac{1.1 - 1}{1/\sqrt{100}}\right)$$

$$= \Phi\left(\frac{0.2}{1/10}\right) - \Phi\left(\frac{0.1}{1/10}\right)$$

$$= \Phi(2) - \Phi(1)$$

$$= 0.9772 - 0.8413$$

$$= 0.1359$$

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(2.) Solⁿ Given a random sample $X_1, X_2, \dots, X_n$ with

CDF $$F(x) = 1 - \frac{1}{x} \quad \text{for } x \geq 1 \text{ \& zero otherwise.}$$
$$= P(X \leq x)$$

Let $Y_n = X_{1:n} \rightarrow$ (smallest order statistics)

We want limiting distribution of Y_n.

For that,

CDF of Y_n : $F_{Y_n}(y)$

$$= P(Y_n \leq y)$$

$$= P(X_{1:n} \leq y)$$

$$= 1 - P(X_{1:n} > y)$$

random variable हरु भएको को smallest y भन्दा ठूलो छ भने, सबै y भन्दा ठूलो होलान

$$1 - F(x) = 1 - \left(1 - \frac{1}{x}\right) = \frac{1}{x}$$

$$= 1 - P(X_1 > y, \dots, X_n > y)$$

$$= 1 - [P(X_1 > y) \dots P(X_n > y)]$$

$$= 1 - \left[\frac{1}{y} \cdots \frac{1}{y}\right]$$

For support set $x \geq 1$ & $y = x$ $\Rightarrow y \geq 1$

$$= 1 - \frac{1}{y^n} \quad , \; y \geq 1$$

जता define गरेको छैन त्यता directly zero ~~लेख्ने~~

~~$F_{Y_n}(y) \longrightarrow$~~

$n \to \infty \quad y \in \mathbb{R}$

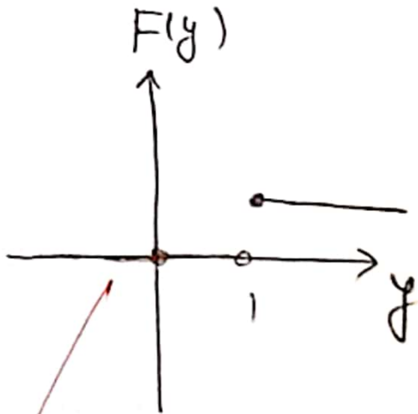
$$F_{Y_n}(y) \longrightarrow F(y) = \begin{cases} 0 & , y = 1 \\ 1 & , y > 1 \\ 0 & , y < 1 \end{cases}$$

$$n \to \infty, \quad F(y) = \begin{cases} 1, & y > 1 \\ 0, & y \leq 1 \end{cases}$$

not a valid CDF

Valid CDF

$$F(y) = \begin{cases} 1 & , y \geq 1 \\ 0 & , y < 1 \end{cases}$$



Y has a degenerated distribution at $y = 1$

(1) Given a random sample $X_1, X_2, \dots, X_n$ with

CDF $\quad F(x) = 1 - \frac{1}{x}$ for $\boxed{x \geq 1}$ & zero otherwise
$= P(X \leq x)$

let $\boxed{Y_n = X_{1:n}^n} \rightarrow$ nth power of smallest order statistics

we want limiting distribution of Y_n $\quad \sqrt[n]{Y} = x^n$

For that $\quad \sqrt{Y}$

$x^2 = y$

CDF of Y_n : $F_{Y_n}(y)$ $\quad x = y^{1/2}$ $\quad \boxed{x \geq 1}$

$= P(Y_n \leq y)$ $\quad y^{1/n} \geq 1$ $\quad x = y^{1/n}$

$= P(X_{1:n}^n \leq y)$ $\quad ((y)^{1/n})^n \geq 1^n$

$y^{1/n} \geq 1$

$= P(X_{1:n} \leq y^{1/n})$ $\quad \boxed{y \geq 1}$

$= 1 - P(X_{1:n} > y^{1/n})$

random variable हरु सबै को smallest y भन्दा ठुलो छ भने सबै y भन्दा ठुलो होलान

$= 1 - P(X_1 > y^{1/n}, \dots, X_n > y^{1/n})$

$= 1 - P(X_1 > y^{1/n}) \dots P(X_n > y^{1/n})$

$= 1 - \left(\frac{1}{y^{1/n}} \cdots \frac{1}{y^{1/n}}\right)$

$$= 1 - \left\{ \frac{1}{y^{1/n}} \right\}^n$$

$$= 1 - \frac{1}{y}, \quad y \geq 1$$

For support

$x \geq 1$

$Y_n = x^n$

$\Rightarrow x = y^{1/n}$

$y^{1/n} \geq 1$

$y \geq 1$

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So $F(y) = \begin{cases} 1 - \frac{1}{y} & y > 1 \\ 0 & y \leq 1 \end{cases}$

IS not valid CDF

this is actualy ok since ~~f(1)~~ f(y) is continuous at y=1 from both side.

So, $F(y) = \begin{cases} 1 - \frac{1}{y}, & y \geq 1 \\ 0 & y < 1 \end{cases}$

Valid CDF.

2.(c) Given a random sample $X_1, X_2, \ldots, X_n$ with

CDF $\quad F(x) = 1 - \frac{1}{x}$ for $\boxed{x \geq 1}$ & zero otherwise

$= P(X \leq x)$

let $Y_n = n \ln X_{1:n}$

we want limiting distribution of Y_n.

for that

CDF of Y_n : $F_{Y_n}(y)$

$$= P(Y_n \leq y)$$

$$= P(n \ln X_{1:n} \leq y)$$

$$= P(\ln X_{1:n} \leq y/n)$$

$$= P(X_{1:n} \leq e^{y/n})$$

$$= 1 - P(X_{1:n} > e^{y/n})$$

random variable हरु मध्यको सबै भन्दा सानो $e^{y/n}$ भन्दा ~~ठुलो~~ छ भने सबै $e^{y/n}$ भन्दा ठुलो हुनु पर्छ

$$= 1 - P(X_1 > e^{y/n}, \ldots X_n > e^{y/n})$$

$x \geq 1$

~~Y_n~~

$Y = n \ln x$

$\frac{y}{n} = \ln x$

$e^{y/n} = x$

$(e^{y/n})^n \geq (1)^n$

$e^y \geq 1$

$y \geq \ln(1)$

$y \geq 0$

$= 1 - P(X_1 > e^{y/n}) \dots P(X_n > e^{y/n})$

$= 1 - \left[\frac{1}{e^{y/n}} \dots \frac{1}{e^{y/n}} \right]$

$= 1 - \left(\frac{1}{e^{y/n}} \right)^n$

$= 1 - \left(e^{-y/n} \right)^n$

$= 1 - e^{-y} \, , \; y \geq 0$

for support set

$x \geq 1$

$n \ln x = y$

$\ln x = \frac{y}{n}$

$x = e^{y/n}$

$e^{y/n} \geq 1$

$\Rightarrow e^{y} \geq 1$

$= \; y \geq \ln(1)$

$y \geq 0$

so,
$$F(y) = \begin{cases} 0 & , \; y \leq 0 \\ 1 - e^{-y} & , \; y > 0 \end{cases}$$

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3(a)

3(a) Given $X_1, X_2, \ldots, X_n$ is an i.i.d sample from a population with following pdf

$$f_X(x) = e^{-(x-\mu)}, \text{ for } x > \mu \text{ and } 0 \text{ otherwise.}$$

Using the given pdf, we can integrate to get the CDF of each X_i in the random sample.

The CDF of random variable X with pdf $f(x)$ is given by $\boxed{F(x) = \int_{-\infty}^{x} f(t)\,dt}$

But since given pdf is for $x > \mu$ and 0 otherwise, so CDF is

$$F_i(x) = \int_{\mu}^{x} e^{-(t-\mu)}\,dt$$

$$= \int_{\mu}^{x} e^{-t} \cdot e^{\mu}\,dt$$

$$= e^{\mu} \int_{\mu}^{x} e^{-t}\,dt$$

$$= e^{\mu} \left. \frac{e^{-t}}{(-1)} \right]_{\mu}^{x}$$

$$= e^{\mu} \left. e^{-t} \right]_{x}^{\mu}$$

$$\Rightarrow e^{\mu} \left(e^{-\mu} - e^{-x} \right)$$

$$\Rightarrow e^{\mu} . e^{-\mu} - e^{\mu} . e^{-x}$$

$$= 1 - e^{-(x-\mu)}$$

Thus, $F_i(x) = 1 - e^{-(x-\mu)}$, $x > \mu$

let $Y_n = X_{1:n} \longrightarrow$ smallest order statistics

Then CDF of Y_n :

$$F_{Y_n}(y) = P(Y_n \leq y)$$

$$= P(X_{1:n} \leq y)$$

$$= 1 - P(X_{1:n} > y)$$

Random variable हरु मध्यको सब भन्दा सानो, y भन्दा ठुलो छ भने, सबै y भन्दा ठुलो होलान

$$= 1 - P(X_1 > y, \ldots, X_n > y)$$

$$= 1 - P(X_1 > y) \dots P(X_n > y)$$

$$= 1 - [e^{-(y-\mu)} \dots e^{-(y-\mu)}]$$

$$= 1 - [e^{-(y-\mu)}]^n$$

for support set

$x > \mu$

& $y = x$

$\Rightarrow y > \mu$

$$= 1 - e^{-n(y-\mu)}, \quad y > \mu$$

CDF for smallest order statistics

$$F(y) = 1 - e^{-n(y-\mu)}$$

$$= -e^{-n(y-\mu)}(-n)$$

$$= ne^{-n(y-\mu)}$$

Given $X_1, \dots, X_n$ is an i.i.d sample from a population with pdf

$$f_X(x) = e^{-(x-\mu)}, \text{ for } x>\mu \text{ and } 0 \text{ otherwise}$$

So their CDF is

$$F_i(x) = 1 - e^{-(x-\mu)}, \quad x>\mu$$ as calculated in part ⓐ

let $Y_n = X_{1:n} \rightarrow$ (smallest order statistics)

we want limiting distribution of Y_n. So for that

CDF of Y_n: $F_{Y_n}(y) = P(Y_n \leq y)$

$$= P(X_{1:n} \leq y)$$

$$= 1 - P(X_{1:n} > y)$$

$$= 1 - P(X_1>y, \dots, X_n>y)$$

$$= 1 - P(X_1>y) \dots P(X_n>y)$$

$$= 1 - [e^{-(y-\mu)} \dots e^{-(y-\mu)}]$$

$$= 1 - [e^{-(y-\mu)}]^n$$

$$= 1 - e^{-n(y-\mu)}, \quad y>\mu$$

as $n \to \infty$, $F_{Y_n}(y) \to F(y) = \lim_{n \to \infty} 1 - e^{-n(y-\mu)}$ i)(a)

&

$$F(y) = \begin{cases} 1 & , y > \mu \\ 0 & , y \leq \mu \end{cases}$$

Is not a valid CDF

$$F(y) = \begin{cases} 1 & , y \cancel{\geq} \mu \\ 0 & , y < \mu \end{cases}$$

Is valid CDF

Hence, degenerate distribution at $y = \mu$.

$y < \mu$ मा सीडी zero लेखले

because it is not defined one

(ii) by CLT, we have

$$Z_n = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$$

$$\overline{X} = \frac{\sum X_i}{n}$$

$$= \frac{n\overline{X} - n\mu}{\sqrt{n} \cdot \sigma} = \frac{\sum X_i - n\mu}{\sigma \cdot \sqrt{n}}$$

Since $X_1, X_2, \ldots, X_n \overset{i.i.d}{\sim} Exp(1)$

So mean $E(X_i) = \frac{1}{\lambda} = \frac{1}{1} = 1 \rightarrow \underline{\mu = 1}$

variance $Var(X_i) = \frac{1}{\lambda^2} = \frac{1}{(1)^2} = 1 \rightarrow \underline{\sigma^2 = 1}$

$$T_n = \frac{(\sum X_i - n \cdot 1)}{1 \cdot \sqrt{n}} \qquad (\because \mu = 1 = \sigma)$$

$$= \frac{(\sum X_i - n)\frac{1}{n}}{\sqrt{n} \cdot \frac{1}{n}}$$

$$= \frac{\frac{\sum X_i}{n} - 1}{1/\sqrt{n}} \xrightarrow{d} N(0,1)$$

(b) $P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} < x\right)$

$= P\left(\bar{X}_n - 1 < x \cdot (1/\sqrt{n})\right)$

$= P\left(\bar{X}_n - 1 < x (n)^{-1/2}\right)$

$= P\left(\bar{X}_n < x (n)^{-1/2} + 1\right)$

$= P\left(\frac{\sum X_i}{n} < x (n)^{-1/2} + 1\right)$

$= P\left(\sum X_i < x (n)^{1/2} + n\right)$ → This will be CDF for gamma distribution

$P(Z < x)$ Normal

$\sum X_i$ — बदलना distribution को

If $X_1, X_2, \dots, X_n \overset{iid}{\sim} EXP(1)$. Then

$\sum X_i \sim GAM(n, 1)$

pdf of gamma distribution will be derivative of its CDF ← show why

$pdf = \frac{\sqrt{n}}{\Gamma n} (x\sqrt{n} + n)^{n-1} e^{-(x\sqrt{n} + n)}$ → integration is gone because of derivative

$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ → This is pdf for Normal distribution ~~[illegible]~~ $P(Z < x)$

If we put $x = 0$

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$$\Rightarrow \frac{\sqrt{n}}{\Gamma n} (n)^{n-1} \cdot e^{-n} = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \frac{\sqrt{n}}{n(n-1)!} n^n e^{-n} = \frac{1}{\sqrt{2\pi}}$$

$\left(\because \Gamma n = (n-1)! \quad n^{-1} = \frac{1}{n}\right)$

$$\Rightarrow \frac{\sqrt{n}}{n!} n^n e^{-n} = \frac{1}{\sqrt{2\pi}}$$

$(\because n(n-1) = n!)$

$$\Rightarrow \cancel{n! = (\sqrt{2\pi})\, n^{1/2} \cdot (n)^{-n} e^{-n}}$$

$$\Rightarrow n! = (2\pi)^{1/2} (n)^{1/2} \cdot n^n e^{-n}$$

$$\Rightarrow n! = (2\pi)^{1/2} (n)^{n+1/2} \cdot e^{-n}$$

$$\Rightarrow n! = (2\pi)^{1/2} (n/e)^n \cdot n^{1/2}$$

$$\therefore n! = (2\pi n)^{1/2} (n/e)^n$$

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5(a) Given $X_1, X_2, \ldots, X_n$ be a i.i.d sample from Bernoulli$(p) \approx BIN(1, p)$

So $E(X_i) = p \longrightarrow$ mean $(\mu) = p$

$Var(X_i) = p(1-p) \longrightarrow$ variance $(\sigma^2) = p(1-p)$

by CLT:

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

$$\frac{\bar{X}_n - p}{\sqrt{p(1-p)}/\sqrt{n}} \xrightarrow{d} N(0,1)$$

$$\frac{n.\bar{X}_n - np}{\sqrt{n}\left(\sqrt{p(1-p)}\right)} \xrightarrow{d} N(0,1)$$

$$\frac{\sqrt{n}(\bar{X}_n - p) - 0}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1)$$

$$\bar{X}_n = \frac{\sum X_i}{n} \rightarrow \text{sample propration of sucess}$$

$$= \hat{p}$$

$$\frac{\sqrt{n}\,(\hat{p}-p)}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1)$$

$$\frac{\sqrt{n}\,(\hat{p}-p)-0}{\sqrt{p(1-p)}} \xrightarrow{d} N(0,1)$$

Standardize गरे पछि यस्तो मा आउँछ भने पहिले mean 0 हो + variance $p(1-p)$ होला।

$$\therefore \sqrt{n}\,(\hat{p}-p) \xrightarrow{d} N(0,\, p(1-p))$$

given $X_1, X_2, \ldots, X_n$ be i.i.d sample from Bernoulli(p)

so, mean $E(X_i) = p \longrightarrow \mu = p$

variance $Var(X_i) = p(1-p) \longrightarrow \sigma^2 = p(1-p)$

let $Y_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - p)$

So the mean of Y_n is :

$$E(Y_n) = \frac{1}{n} \sum E(X_i - p)$$

$$= \frac{1}{n} * \sum (p - p) \qquad \because E(X_i) = p$$

$$= 0$$

The variance of Y_n is

$$Var(Y_n) = \frac{1}{n^2} * \sum Var[X_i - p]$$

$$= \frac{1}{n^2} * np(1-p)$$

$$= \frac{p(1-p)}{n}$$

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As n approaches infinity, the variance of Y_n approaches 0. This means that the distribution of Y_n becomes more and more peaked around 0, and the distribution converges to a degenerate distribution at 0, which means the probability of observing any value other than 0 is 0.

Techicaly correct.

but this proof involves theorem that we did not cover in class.

i.e. the theorem:

$E(X_n) \to \mu$, and $Var(X_n) \to 0$

then $X_n \xrightarrow{P} \mu$.

can be proved using cheby-chev. inequality.

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