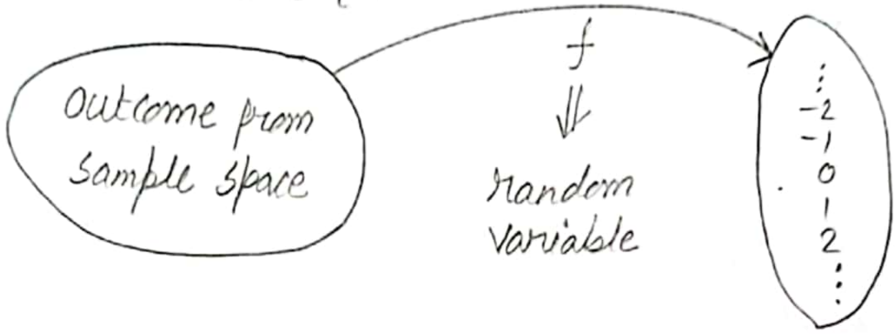
Random Variable — Chapter-②

→ एटो die लाई एक चोटी roll गर्ने हो भने r.v X भनेको number of dots on the die हो



Random Vector

Random vector भनेको vector हो whose elements are random variable.

Example:-

Biology random vector f उदाहरण random variables are weight (X_1), Height (X_2), blood-pressure (X_3)

Marginal parameters

1) $$E(X_i) = \begin{cases} \int_{-\infty}^{\infty} x_i f(x_i)\,dx_i & \text{If continouse} \\ \sum_{\text{all } x} x_i P_i(x_i) & \text{If discrete} \end{cases}$$

2) $Var(X_i) = E[(X_i - \mu_i)^2] = \sigma_{ii} = \sigma_i^2$

3) For two random variable X_i, X_j

$$Cov(X_i, X_j) = E[(X_i - \mu_i)(X_j - \mu_j)] = \sigma_{ij} \quad , i \neq j$$

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So, $\boxed{Cov(X_i, X_i) = Var(X_i)}$ → आफैं सँगको covariance भनेको variance हो।

σ_{ij} → This denotes covariance between r.v X_i & X_j
$= Cov(X_i, X_j)$

σ_{ii} → Variance of the variable (X_i)

- $Cov(X_i, X_j)$ measures the linear association between the variable X_i, X_j.
- If there is any other relation then linear, covariance will not detect it
- यदी X_i and X_j independent छन् भने, $cov(X_i, X_j) = 0$
- कुनै पनि pair of random variable X_i & X_j छ भने ~~तिनीहरुको~~ behaviour is described by their joint pdf.

1.) population Mean Vector $\mu = E(X) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_p) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix}$

2.) population Variance Covariance matrix

$$\Sigma = E[(\underset{\sim}{X} - \underset{\sim}{\mu})(\underset{\sim}{X} - \underset{\sim}{\mu})^T]$$

$$= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

- p Variables भएको dataset मा, there will be p variances (along diagonal) and $\frac{p(p-1)}{2}$ distint covariances.

3) population correlation matrix

$$\rho = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1p} \\ \rho_{21} & 1 & \cdots & \rho_{2p} \\ \vdots & & & \\ \rho_{p1} & \rho_{p2} & \cdots & 1 \end{bmatrix},$$

- correlation is defined as measure of linear relations between variables

$$\operatorname{COR}(X_i, X_j) = \rho_{ij} = \frac{\operatorname{cov}(X_i, X_j)}{\sqrt{\operatorname{Var}(X_i)}\sqrt{\operatorname{Var}(X_j)}}$$

- Covariance of r.v X_i and X_j को उनके respective standard deviation (squre root of variance) से devide करें तो हमें we can get the correlation between random variable X_i and X_j

- So If we are given variance covariance matrix Σ, con can easily find correlation matrix as

If $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$. Then

$$\rho = \begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \frac{\sigma_{13}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{33}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \frac{\sigma_{23}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{33}}} \\ \frac{\sigma_{31}}{\sqrt{\sigma_{33}}\sqrt{\sigma_{11}}} & \frac{\sigma_{32}}{\sqrt{\sigma_{33}}\sqrt{\sigma_{22}}} & \frac{\sigma_{33}}{\sqrt{\sigma_{33}}\sqrt{\sigma_{33}}} \end{bmatrix}$$

$$\Rightarrow \rho = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{21} & 1 & \rho_{23} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix}$$

→ Note:- This ρ will be Identity matrix when all random variables are independent.

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4) Standard deviation matrix

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \sqrt{\sigma_{pp}} \end{bmatrix}.$$

- The important Relation bet^n correlation matrix, standard deviation matrix and covariance matrix is :

$$\rho = V^{-1/2} \, \Sigma \, V^{-1/2}$$

→ Σ बाट ρ पाउँन, Σ लाई $V^{-1/2}$ ले both side बाट multiply गर्दा, which makes sense also

$$\Rightarrow \boxed{\rho = (V^{1/2})^{-1} \, \Sigma \, (V^{1/2})^{-1}}$$

conversly $\boxed{\Sigma = (V^{1/2}) \, \rho \, (V^{1/2})}$

- V. imp note

यदी distribution discrete छ भने expected value is replaced by summation so need to multiply with crossponding probability also

Eg:- $\sigma_{11} = E[(X_1 - \mu_1)^2] = \sum_{\text{all } x_1} (X_1 - \mu_1)^2 P_1(x_1)$

or $\sigma_{12} = E[(X_1 - \mu_1)(X_2 - \mu_2)] = \sum_{\substack{\text{all pairs} \\ (X_1, X_2)}} (x_1 - \mu_1)(x_2 - \mu_2) P_{12}(x_1, x_2)$

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Random Matrix

A random matrix is a matrix whose elements are random variables.

$$\underline{X}_{n\times p} = \{x_{ij}\} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1p} \\ \vdots & & & \\ X_{n1} & X_{n2} & \cdots & X_{np} \end{bmatrix}$$

a) $E(\underline{X}) = \{E(X_{ij})\}$ if each expectation exists.

b) यदी $\underline{X}$ and $\underline{Y}$ are random matrix of same dimension and let $\underline{A}$ and $\underline{B}$ be two comformable matrix of constants

$$E(\underline{X} + \underline{Y}) = E(\underline{X}) + E(\underline{Y})$$

$$E(\underset{k\times p}{\underline{A}}\ \underset{p\times m}{\underline{X}}\ \underset{m\times l}{\underline{B}}) = \underbrace{\underline{A}\ E(\underline{X})\ \underline{B}}_{k\times l}$$

Partitioning random Vectors :

हामीलाई Large dataset दिइएको छ भने, characteristics measured on individual could naturally fall into two or more groups.

Example

Consumption and Income.

- To handle situation like these, where distinct groups of characteristics arise naturally, a useful approach is to consider these groups as subset of larger collection of characteristics.

- If the total collection is represented by a $p \times 1$ dimension random vector $\underset{\sim}{X}_{p \times 1}$. We can regard sub-set as component of $\underset{\sim}{X}$ and can be delt with by partitioning

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_q \\ \hline X_{q+1} \\ X_{q+2} \\ \vdots \\ X_p \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} X_1 \\ X_2 \\ \vdots \\ X_q \end{matrix}}\right\} q \times 1 \\ \left.\vphantom{\begin{matrix} X_1 \\ X_2 \\ \vdots \\ X_q \end{matrix}}\right\} (p-q) \times 1 \end{matrix} = \begin{bmatrix} X^{(1)} \\ X^{(2)} \end{bmatrix}$$

a) population mean vector ($\underset{\sim}{\mu}$)

$$\underset{\sim}{\mu} = E(\underset{\sim}{X}) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_q) \\ \hdashline E(X_{q+1}) \\ \vdots \\ E(X_p) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_q \\ \hdashline \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \underset{\sim}{\mu}^{(1)} \\ \underset{\sim}{\mu}^{(2)} \end{bmatrix}$$

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b) population Variance covariance Matrix

$$\Sigma_{p\times p} = E\left[(\underset{\sim}{X}-\underset{\sim}{\mu})(\underset{\sim}{X}-\underset{\sim}{\mu})^T\right] \longrightarrow \text{by def}^n$$

$$= \begin{bmatrix} \underset{q\times q}{\Sigma_{11}} & \underset{q\times(p-q)}{\Sigma_{12}} \\ \underset{(p-q)\times q}{\Sigma_{21}} & \underset{(p-q)\times(p-q)}{\Sigma_{22}} \end{bmatrix}$$

Where Σ_{11} = Variance covariance matrix for $X^{(1)}$

Σ_{22} = variance covariance matrix for $X^{(2)}$

$$\Sigma_{12} = E\left[(\underset{\sim}{X}^{(1)}-\underset{\sim}{\mu}^{(1)})(\underset{\sim}{X}^{(2)}-\underset{\sim}{\mu}^{(2)})^T\right] = \Sigma_{21}^T$$

Linear Combinations of Random Vectors

- यदी X_1 भनेको single r.v हो with mean μ_1 & variance σ_{11} and c be any constant number. Then

a) $E(cX_1) = c\mu_1$

b) $Var(cX_1) = c^2 Var(X_1) = c^2\sigma_{11}$

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- यदि X_1 and X_2 two random variable [illegible] हुन भने with their respective mean and variances μ_1, μ_2 & σ_{11}, σ_{22} and their covariance as $\mathrm{COV}(X_1, X_2) = \sigma_{12}$

a) $E(aX_1 + bX_2) = aE(X_1) + bE(X_2)$

$= a\mu_1 + b\mu_2.$

b) $\mathrm{COV}(aX_1, bX_2) = ab\,\mathrm{Cov}(X_1, X_2) = ab\,\sigma_{12}$

c) $\mathrm{Var}(aX_1 + bX_2) = a^2\,\mathrm{Var}(X_1) + b^2\,\mathrm{Var}(X_2) + 2ab\,\mathrm{Cov}(X_1, X_2)$

$= a^2\sigma_{11} + b^2\sigma_{22} + 2ab\,\sigma_{12}$

Note

$$\boxed{\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\,\mathrm{cov}(X_1, X_2)}$$

For Matrix Representation

- Linear combination $aX_1 + bX_2$ can be written as

$$aX_1 + bX_2 = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underset{\sim}{C}^T X$$

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$$E(aX_1 + bX_2) = E\left([a\ b] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right)$$

$$= [a\ b] \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

$$= \underset{\sim}{C}^T \mu$$

- If we let $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$. Then

Then $Var(aX_1 + bX_2) = Var(\underset{\sim}{C}^T \underset{\sim}{X})$

$$= var\left([a\ b] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \right)$$

$$= \underset{\sim}{C}^T \Sigma (\underset{\sim}{C}^T)^T$$

$$= \underset{\sim}{C}^T \Sigma \underset{\sim}{C}$$

$$= [a\ b] \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= a^2 \sigma_{11} + 2ab\, \sigma_{12} + b^2 \sigma_{22}$$

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- We can extend this result for a linear combination of p variables

$$\underset{\sim}{C}^T \underset{\sim}{X} = C_1 X_1 + C_2 X_2 + \cdots + C_p X_p \text{ . Then}$$

a.) $E(\underset{\sim}{C}^T \underset{\sim}{X}) = \underset{\sim}{C}^T \underset{\sim}{\mu}$ $\qquad \underset{\sim}{\mu} = E(\underset{\sim}{X})$

Keep in mind that Transpose भनेका row matrix हुन, all others are column matrix (without transpose)

b.) $$Var(\underset{\sim}{C}^T \underset{\sim}{X}) = \underset{\sim}{C}^T \Sigma (\underset{\sim}{C}^T)^T$$

$$= \underset{\sim}{C}^T \Sigma \underset{\sim}{C}$$

where Σ = variance covariance matrix of X.

- Even more generally, we can consider a linear combination of p random variables $X_1, \ldots, X_p$

$$Z_1 = C_{11} X_1 + C_{12} X_2 + \cdots + C_{1p} X_p$$

$$Z_2 = C_{21} X_1 + C_{22} X_2 + \cdots + C_{2p} X_p$$

$$\vdots$$

$$Z_q = C_{q1} X_1 + C_{q2} X_2 + \cdots + C_{qp} X_p \text{ . So that}$$

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$$\underset{\sim}{Z} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & & \ddots & \vdots \\ c_{q1} & c_{q2} & \cdots & c_{qp} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix} = \underline{C}\,\underset{\sim}{X}$$

a) population mean of Z

$$E(Z) = E(\underline{C}\,\underset{\sim}{X})$$

$$= \underline{C}\,E(\underset{\sim}{X})$$

b) variance covariance of Z

$$\Sigma_Z = \mathrm{cov}(Z) = \mathrm{Cov}(\underline{C}\,X)$$

$$= \underline{C}\,\Sigma_X\,\underline{C}^T$$

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Multi-variate Analysis | Chapter-2

↳ Multivariate data analysis deals with datasets containing multiple variables that interact and influnce one another.

↳ each data लाई छुट्टै seperately analyze गर्ने हो भने, we can miss key features and interesting patterns inherent in the multivariate data.

↳ Multivariate Data Analysis is Important

1) Data reduction or structural simplification Eg GPA
2) Sorting and grouping. Eg bank → consumer → 'good credit' & 'bad credit'
3) Investigation of the dependence among variables. Eg Consultant hiring
4) Prediction
5) Hypothesis construction and testing.

↳ Sample mean

$$\bar{x}_K = \frac{1}{n}\sum_{j=1}^{n} x_{jK}, \quad K = 1, 2, \ldots, P$$

↳ sample variance

$$S_K^2 = S_{KK} = \frac{1}{n}\sum_{j=1}^{n}(x_{jK} - \bar{x}_K)^2, \quad K = 1, 2, \ldots, P$$

↳ Sample standard deviation $= \sqrt{S_{KK}}$

↳ The sample covariance measures the degree to which two variables in a sample change together (co-vary).

$$S_{iK} = \frac{1}{n}\sum_{j=1}^{n}(x_{ji} - \bar{x}_i)(x_{jK} - \bar{x}_K)$$

↳ If large value from one variable occur with small values for the other variable, sample covariance will be negative

↳ If large values for one variable are observed in conjunction with Large values for the other variable, and smaller values also occur together, then sample covariance will be positive

↳ If there is no particular linear association between the values for the two variable, then sample covariance will be zero. {we are just talking about linear relationship, there might be other relationship like non-linear

↳ The correlation coefficient measures the strength and direction of the linear relationship between two variable

$$r_{ik} = \frac{S_{ik}}{\sqrt{S_{ii}}\sqrt{S_{kk}}} \quad \text{i.e.} \quad S_{ik} = \frac{Cov(X_i, X_k)}{\sqrt{Var(X_i)}\sqrt{Var(X_k)}}$$

↳ Signs of the sample correlation and the sample covariance are the same, but correlation is ordinarily easier to interperet because its magnitude is bounded.

Interpertation

$r = 0 \rightarrow$ No linear relationship between variable

$r < 0 \rightarrow$ -ve linear relationship betn variable

$r > 0 \rightarrow$ +ve linear relationship betn variable

$r \simeq 0 \Rightarrow r$ close to zero $\rightarrow$ Weak linear relationship betn the two variable.

$r \simeq 1$ or $r \simeq -1$ i.e r is close to 1 or -1 $\rightarrow$ very strong linear relationship between the two variable.

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Interpretation of the Covariance :

→ The positive value of the covariance ($\simeq 65.22$) indicates that as the hours of study increases for a student, there their corresponding test score tend to increase as well. This suggest a positive relationship between hours of study and test scores. Student who study more hours tend to achive higher test score.

• Interpretation of the correlation coefficient :

→ correlation value of 0.8352 indicates that there is a strong positive linear relationship between hours of study and test score in the dataset

• It suggest that the two variables are closely aligned in a linear fashion, where an increase in one variable crossponds to a substantial increase in other.

Maxtrix Algebra :-

→ The sum of two vectors emanating from the origin is the diagonal of the paralelogram formed with the two original vectors as adjacent sides.

→ Inner product of vector x and y is $x'y = x_1y_1 + x_2y_2$

→ A pair of vectors x and y of the same dimension is said to be Linearly dependent if there exist constants c_1 and c_2, both not zero, such that $c_1x + c_2y = 0$

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↳ Linear dependence implies that at least one vector in the set can be written as linear combination of the other vectors

Matrix multiplication Rule

↳ When A is $(n \times K)$ and B is $(K \times P)$, so that the number of elements in a row of A is the same as the number of elements in a column of B, we can form a matrix product AB.

मतलब
A भन्ने matrix को row मा कति element छन (i.e no. of column in matrix A) should be equal to number of element in the column of matrix B (i.e no. of rows in matrix. B)

Symmetric Matrix

→ A squre matrix is said to be symmetric if $A = A^T$ or $a_{ij} = a_{ji}$ $\forall i, j$.

Inverse matrix:-

The technical condition that an inverse exists is that the K columns $a_1, a_2, \ldots, a_K$ of A are linearly independent. That is, The existance of A^{-1} is equivalent to

$$c_1 a_1 + c_2 a_2 + \cdots + c_K a_K = 0 \text{ only if } c_1 = \cdots = c_K = 0$$

↳ कुनै पनि matrix को inverse exist गर्नका लागी त्यसका सबै columns हरु linearly independent हुनु पर्छ।

↳ And linearly Independent हुनका लागि Linear combination zero

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only if all constant is zero हुन्थ्यो

Orthogonal matrix

→ A matrix Q is said to be Orthogonal If

$$QQ^T = Q^TQ = I \quad or \quad Q^T = Q^{-1}$$

Eigen Value and Eigen Vector :-

→ A squre matrix A is said to have an eigenvalue λ, with corresponding eigenvector $x \neq 0$, if

$$\boxed{Ax = \lambda x}$$

→ Eigen vector

→ Eigen value.

↳ eigen vector भनेको यस्तो vector हो जसलाई कुनै matrix ले multiply गर्दा जे आउछ same result can be obtained by just multiplying by any simple number to that vector.

→ let A be a K × K squre matrix. Then A has pairs of eigenvalues and normalized Eigen vector namely,

$$\lambda_1, e_1 \qquad \lambda_2 e_2 \qquad \dots \qquad \lambda_K e_K$$

→ Two Eigenvector are unique unless two or more eigenvalues are equal.

→ Eigen value & Eigen vector हुनकालागि condition, A squre matrix मात्र हो।

only if all constant is zero हुन्छ्यो

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→ let A be a $K \times K$ squre matrix. Then A has pairs of eigenvalues and normalized Eigen vector namely,

$$\lambda_1, e_1 \qquad \lambda_2 e_2 \quad \ldots \quad \lambda_K e_K$$

→ Two Eigenvector are unique unless two or more eigenvalues are equal.

→ Eigen value & Eigen vector निकाल्नलाई condition, A squre matrix चाहिन्छ हो।

→ कुनै vector लाई कुनै matrix ले multiply गर्दा आउने result is equa to vector लाई number ले multiply गर्दा आउने result छ भने, त्यो vector लाई त्यही eigen vector भनिन्छ and त्यो value लाई त्यही Eigen value भनिन्छ, crossponding to that matrix.

Positive Definite Matrix :-

→ A squre matrix is positive definite if pre-multiplying and post-multiplying it by the same vector always give a positive number as a result, independently of how we choose the vector.

→ Positive Definite symmetric matrix have the property that all their eigenvalues are positive.

→ The spectral decomposition of a $K \times K$ symmetric matrix A is given by

$$A = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T + \dots + \lambda_K e_K e_K^T$$

→ The characteristic eqn of matrix A is $|A - \lambda I| = 0$

→ To show A is positive definite matrix

$$A = \lambda_1 e_1 e_1^T + \lambda_2 e_2 e_2^T$$

$$x^T A x = \lambda_1 x^T e_1 e_1^T x + \lambda_2 x^T e_2 e_2^T x$$

put $y_1 = x^T e_1$ + $y_2 = x^T e_2$

equ → The behavior of any pair of random variable such as X_i and X_K is described by their joint probability function and the measure of the linear association between them is provided by the covariance.

→ Joint probability can be written as the product of individual probability, then the random variable are said to be indepen-dent.

→ The independent condition become $f_{iK}(x_i, x_K) = f_i(x_i) f_K(x_K)$ for all pair (x_i, x_K)

→ If X_i and X_K are independent $\Rightarrow Cov(X_i, X_K) = 0$

but In general $COV(X_i, X_K) = 0$ does not emply X_i and X_K are independent

→ Independent হলে ধরলে covariance surly zero হবে but covariance zero হলেই ধরতেপারি independent ~~হবেনা~~ ধরা ~~যাবেনা~~ always

→ p distind variance & $\frac{p(p-1)}{2}$ distanct covariance in $p \times p$ variance covariance matrix.

- $\Sigma = E[(\underset{\sim}{X} - \underset{\sim}{\mu})(\underset{\sim}{X} - \underset{\sim}{\mu})^T]$

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→ We note that the computation of mean, variances and covariances for discrete random variable involves summation, while analogous computations for continous random variable involve integration

→ The multivariate normal distribution is completely specified once the mean vector μ and variance-covariance matrix Σ are given.

→ $$\rho_{ik} = \frac{\sigma_{ik}}{\sqrt{\sigma_{ii}}\sqrt{\sigma_{kk}}}$$ → population correlation coefficient

→ The correlation coefficient measures the amount of linear association betn random variable X_i and X_k.

→ ρ निकाल्न Σ मात्र चाहिन्छ (becoz Σ given भए $V^{1/2}$ easily calculate गर्न पाइयो from that) but Σ निकाल्न $V^{1/2}$ and ρ both चाहिन्छ।

Partitioning the Covariance Matrix:

→ Σ_{12} is a covariance matrix of partition $X^{(1)}$ and $X^{(2)}$ and This matrix is not necessarily symmetric or even square

→ Note : $\Sigma_{12} = \Sigma_{21}^T$

↳ Covariance matrix of $X^{(1)}$ is Σ_{11}

↳ Covariance matrix of $X^{(2)}$ is Σ_{22}.

↳ And $Cov(X^{(1)}, X^{(2)}) = \Sigma_{12}$.

↳ $Var(cX_1) = c^2 Var(X_1) = c^2 \sigma_{11}$

↳ $Cov(aX_1, bX_2) = ab\, Cov(X_1, X_2) = ab\,\sigma_{12}$

↳ $Var(aX_1 + bX_2) = a^2 Var(X_1) + b^2 Var(X_2) + 2ab\, Cov(X_1, X_2)$

$$= a^2\sigma_{11} + b^2\sigma_{22} + 2ab\,\sigma_{12}$$

↳ $Var(\underset{\sim}{c}'\underset{\sim}{X}) = \underset{\sim}{c}'\underset{\sim}{\Sigma}\, c$

& $\Sigma_Z = Cov(\underline{C}Z) = \underline{C}\,\Sigma_Z\,\underline{C}'$

↳ $\Sigma_Z = \begin{bmatrix} \sigma_{11} - 2\sigma_{12} + \sigma_{22} & \sigma_{11} - \sigma_{22} \\ \sigma_{11} - \sigma_{22} & \sigma_{11} + 2\sigma_{12} + \sigma_{22} \end{bmatrix}$ for $Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$

$Z_1 = X_1 - X_2$

$Z_2 = X_1 + X_2$

↳ Note:-

If $\sigma_{11} = \sigma_{22}$ that is, if X_1 and X_2 have equal variances-

→ The off-diagonal terms in Σ_Z vanish. This demonstrates the well-known result that SUM $(X_1 + X_2)$ and difference $(X_1 - X_2)$ of two random variable with same variance are uncorrelated.

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FROM Class Note:-

→ Multivariate normal distribution को case मा मात्र $cov(X_1, X_2) = 0 \iff X_1$ and X_2 independent hold from both side.

Otherwise independent हुँदा covariance zero त हुन्छ but covariance zero हुँदैमा independent हुन्छ पाइदैन।

* Chapter 2, THE END *

Q.) Find μ_Z and Σ_Z for $\underset{\sim}{Z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ If

$$Z_1 = X_1 - X_2$$

$$Z_2 = X_1 + X_2 \quad . \quad \text{Also } \underset{\sim}{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \underset{\sim}{\mu}_X = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

$$\Sigma_X = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

Solution

Here we have

$$Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= \underline{C} \underset{\sim}{X}$$

So, $\mu_Z = E(Z) = E(\underline{C} \underset{\sim}{X})$

$$= \underline{C}\, E(\underset{\sim}{X})$$

$$= \underline{C}\, \underset{\sim}{\mu}_X$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$$

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$$= \begin{bmatrix} \mu_1 - \mu_2 \\ \mu_1 + \mu_2 \end{bmatrix}$$

and $Var(Z) = \Sigma_z$

$$= Var(\underline{C}\, \underset{\sim}{X})$$

$$= \underline{C}\, \Sigma_x\, \underline{C}^T$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{21} & \sigma_{12} - \sigma_{22} \\ \sigma_{11} + \sigma_{21} & \sigma_{12} + \sigma_{22} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - \sigma_{21} - \sigma_{12} + \sigma_{22} & \sigma_{11} - \sigma_{21} + \sigma_{12} - \sigma_{22} \\ \sigma_{11} + \sigma_{21} - \sigma_{12} - \sigma_{22} & \sigma_{11} + \sigma_{21} + \sigma_{12} + \sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \sigma_{11} - 2\sigma_{12} + \sigma_{22} & \sigma_{11} - \sigma_{22} \\ \sigma_{11} - \sigma_{22} & \sigma_{11} + 2\sigma_{12} + \sigma_{22} \end{bmatrix}$$

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Q) Computing the Covariance Matrix

Find the covariance matrix for two random variables X_1 and X_2, when their joint pmf is $P_{12}(x_1, x_2)$, represented by the entries in the body of the following table:

$x_1 \backslash x_2$	0	1	$P_1(x_1)$
-1	0.24	0.06	0.3
0	0.16	0.14	0.3
1	0.40	0.00	0.4
$P_2(x_2)$	0.8	0.2	1

Solu

$$E(X_1) = \sum_{\text{all } x_1} x_1 P_1(x_1)$$

$$= (-1)(0.3) + (0)(0.3) + (1)(0.4)$$

$$= 0.1$$

$$E(X_2) = \sum_{\text{all } x_2} x_2 P_2(x_2)$$

$$= (0)(0.8) + (1)(0.2)$$

$$= (0.2)$$

$$= 0.2$$

Thus $E(X_1) = \mu_1 = 0.1$ & $E(X_2) = \mu_2 = 0.2$

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$$\sigma_{11} = E[(X_1-\mu_1)^2]$$

$$= \sum_{\text{all } x_1} (x_1 - 0.1)^2 P_1(x_1)$$

$$= (-1-0.1)^2 (0.3) + (0-0.1)^2 (0.3) + (1-0.1)^2 \cdot (0.4)$$

$$= 0.69$$

$$\sigma_{22} = E[(x_2-\mu_2)^2]$$

$$= \sum_{\text{all } x_2} (x_2 - 0.2)^2 P_2(x_2)$$

$$= (0-0.2)^2 (0.8) + (1-0.2)^2 (0.2)$$

$$= 0.16$$

$$\sigma_{12} = \sigma_{21} = E[(X_1-\mu_1)(X_2-\mu_2)]$$

$$= \sum_{\substack{\text{all pair} \\ (x_1, x_2)}} (x_1 - 0.1)(x_2 - 0.2) P_{12}(x_1, x_2)$$

$$= (-1-0.1)(0-0.2)(0.24) + (-1-0.1)(1-0.2)(0.06)$$
$$+ (0-0.1)(0-0.2)(0.16) + (0-0.1)(1-0.2)(0.14)$$
$$+ (1-0.1)(0-0.2)(0.4) + (1-0.1)(1-0.2)(0.00)$$

$$= -0.08$$

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So, $\underset{\sim}{\mu} = E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$

and

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 0.69 & -0.08 \\ -0.08 & 0.16 \end{bmatrix}$$

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Q) computing expected values for discrete random variable

Suppose $p=2$ and $n=1$, and consider the random vector $X' = [X_1, X_2]$. Let the discrete random variable X_1 have the following probability function

x_1	-1	0	1
$P_1(x_1)$	0.3	0.3	0.4

Then $$E(X_1) = \sum_{\text{all } x_1} x_1 P_1(x_1)$$

$$= (-1)(0.3) + (0)(0.3) + (1)(0.4)$$

$$= 0.1$$

Similarly, Let the discrete random variable X_2 has the probability function

x_2	0	1
$P_2(x_2)$	0.8	0.2

Then $$E(X_2) = \sum_{\text{all } x_2} x_2 P_2(x_2)$$

$$= (0)(0.8) + (1)(0.2)$$

$$= 0.2$$

Thus $$E(X) = \begin{bmatrix} E(X_1) \\ E(X_2) \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$$

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