Problem - I

Given the data matrix $X = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$

1	6	6
2	8	8
3	11	11
4	6	6
Proof 5	3	3
	34	34

Find the

i) Sample mean vector

ii) Sample Covariance matrix, S_n

iii) The Sample Correlation matrix, R.

Solu

Given data Matrix $\underline{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$

(i) Sample mean vector

$$\underset{3\times 1}{\bar{\underset{\sim}{X}}} = \frac{1}{n} \underline{X}^T \underset{3\times 1}{\mathbb{1}}$$

$$= \frac{1}{3} \begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

(ii) Sample Covariance matrix, S_n

Finally

$$S_n = \frac{1}{n} \underline{X}^T \left(I - \frac{1}{n} \mathbb{1} \mathbb{1}^T \right) \underline{X}$$

$$= \frac{1}{3} \begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right\} \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix}$$

Tried working with decimal point, got error because of rounding

(iii) Sample Correlation matrix, R

we know sample standard devation matrix

$$D^{1/2} = \begin{bmatrix} \sqrt{32/3} & 0 \\ 0 & \sqrt{2/3} \end{bmatrix}$$

$$D^{-1/2} = \begin{bmatrix} 1/\sqrt{32/3} & 0 \\ 0 & 1/\sqrt{2/3} \end{bmatrix}$$

Finally sample correlation matrix

$$R = D^{-1/2} S_n D^{-1/2}$$

$$= \begin{bmatrix} 1/\sqrt{32/3} & 0 \\ 0 & 1/\sqrt{2/3} \end{bmatrix} \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{32/3} & 0 \\ 0 & 1/\sqrt{2/3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{32/3}{\sqrt{32/3}\sqrt{32/3}} & \dfrac{-4/3}{\sqrt{32/3}\sqrt{2/3}} \\[2ex] \dfrac{-4/3}{\sqrt{2/3}\sqrt{32/3}} & \dfrac{2/3}{\sqrt{2/3}\sqrt{2/3}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{32/3}{32/3} & \dfrac{-4/3}{8/3} \\[2ex] \dfrac{-4/3}{8/3} & \dfrac{2/3}{2/3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

3.15.

Repeat Exercise 3.14 using the data matrix

$$X = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

and the linear combination

$$b'X = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

and

$$c'X = \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Solu Given data matrix is $\underline{X} = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$

and linear combination

$$\underset{\sim}{b}'\underset{\sim}{X} = X_1 + X_2 + X_3$$

$$\underset{\sim}{c}'\underset{\sim}{X} = X_1 + 2X_2 - 3X_3$$

(a) Observation on these linear combination are obtained by replacing X_1, X_2 and X_3 with their observed value

$$\begin{aligned} \underset{\sim}{b}'\underset{\sim}{X}_1 &= x_{11} + x_{12} + x_{13} \\ &= 1 + 4 + 3 \\ &= 8 \end{aligned}$$

what is this? ok. understood

$$\underset{\sim}{b}'\underset{\sim}{X}_2 = x_{21} + x_{22} + x_{23}$$
$$= 6 + 2 + 6$$
$$= 14$$

$$\underset{\sim}{b}'\underset{\sim}{X}_3 = x_{31} + x_{32} + x_{33}$$
$$= 8 + 3 + 3$$
$$= 14$$

?? what is this exactly.
ok - understood

The sample mean and variance of these values are

$$\text{Sample mean} = \frac{8+14+14}{3} = 12$$

$$\text{Sample Variance} = \frac{(8-12)^2 + (14-12)^2 + (14-12)^2}{3-1}$$
$$= \frac{16+4+4}{2}$$
$$= 12$$

Now the observation of $\underset{\sim}{c}'\underset{\sim}{X}$ are

$$\underset{\sim}{c}'\underset{\sim}{X}_1 = x_{11} + 2x_{12} - 3x_{13}$$
$$= 1 + 2(4) - 3(3)$$
$$= 0$$

$$\underset{\sim}{c}'\underset{\sim}{X}_2 = x_{21} + 2x_{22} - 3x_{23}$$
$$= 6 + 2(2) - 3(6)$$
$$= -8$$

$$\underset{\sim}{c}'\underset{\sim}{X}_3 = x_{31} + 2x_{32} - 3x_{33}$$

$$= 8 + 2(3) - 3(3)$$

$$= 5$$

The sample mean and variance of these values are

$$\text{Sample mean} = \frac{0 + (-8) + 5}{3}$$

$$= -1$$

$$\text{Sample Variance} = \frac{(0+1)^2 + (-8+1)^2 + (5+1)^2}{3-1}$$

$$= \frac{1 + 49 + 36}{2}$$

$$= 43$$

And, the sample covariance computed from the pair of observation $(\underset{\sim}{b}'\underset{\sim}{X}_1, \underset{\sim}{c}'\underset{\sim}{X}_1)$, $(\underset{\sim}{b}'\underset{\sim}{X}_2, \underset{\sim}{c}'\underset{\sim}{X}_2)$ and $(\underset{\sim}{b}'\underset{\sim}{X}_3, \underset{\sim}{c}'\underset{\sim}{X}_3)$ is

$$= \frac{(8-12)(0+1) + (14-12)(-8+1) + (14-12)(5+1)}{3-1}$$

$$= \frac{(-4)(1) + (2)(-7) + (2)(6)}{2}$$

$$= -3$$

(b) Using the Matrix approach to find sample mean, sample var. and sample covariance for the observation $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X}$

To find sample mean, sample variance and sample covariance of linear combination, we will use sample mean vector $\underset{\sim}{\bar{X}}_{3\times1}$ and sample covariance matrix S obtained from the orginal data matrix $\underline{X}$

Here

$$\underset{\sim}{\bar{X}}_{3\times1} = \frac{1}{n}\underline{X}^T \underline{\mathbb{1}}_{3\times1}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 3 \\ 3 & 6 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 15 \\ 9 \\ 12 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

and

$$S = \frac{1}{(n-1)}\underline{X}^T\left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)\underline{X}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 3 \\ 3 & 6 & 3 \end{bmatrix}\left\{\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}\right\}\begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 26 & -5 & 3 \\ -5 & 2 & -3 \\ 3 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix}$$

Now, so

sample mean of $\underset{\sim}{b}'\underset{\sim}{X} = \underset{\sim}{b}'\underset{\sim}{\bar{X}}_{3\times1}$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$= 12$ (same)

sample mean of $\underset{\sim}{c}'\underset{\sim}{X} = \underset{\sim}{c}'\underset{\sim}{\bar{X}}_{3\times1}$

$$= \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$= -1$ (same)

sample variance of $\underset{\sim}{b}'\underset{\sim}{X} = \underset{\sim}{b}' S \underset{\sim}{b}$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 12 \qquad \text{(same)}$$

sample variance of $\underset{\sim}{c}'\underset{\sim}{X} = \underset{\sim}{c}' S \underset{\sim}{c}$

$$= \begin{bmatrix} 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 7/2 & 4 & -21/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= 43 \qquad \text{(same)}$$

and finally sample covariance of $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X}$ is

$$\mathrm{cov}(\underset{\sim}{b}'\underset{\sim}{X}, \underset{\sim}{c}'\underset{\sim}{X}) = \underset{\sim}{b}' S \underset{\sim}{c}$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$= -3$ (same)

Conclusion:

The results in part (a) and (b) are same

18. Energy consumption in 2001, by state, from the major source

x_1 = petroleum $\quad$ x_2 = natural gas

x_3 = hydroelectric power $\quad$ x_4 = nuclear power

is recorded in quadrillions (10^{15}) of BTUs.

The resulting mean and covariance matrix are

$$\bar{\underset{\sim}{X}} = \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix} \quad S = \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix}$$

a.) using the summary statistics, determine the sample mean and variance of a sate's total energy consumption of these major sources

b.) Determine the sample mean and variance of the excess of petroleum consumption over natural gas consumption. Also find the sample covariance of this variable with the total variable in part a.

Solu

(a) Total Energy consumption:

$$\text{Total: } \underset{\sim}{b}'\underset{\sim}{X} = x_1 + x_2 + x_3 + x_4$$

$$= [1 \; 1 \; 1 \; 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Sample mean of Total : $\underset{\sim}{b}'\underset{\sim}{X}$ is $\underset{\sim}{b}'\underset{\sim}{\bar{X}}_{4\times 1}$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix}$$

$$= 0.766 + 0.508 + 0.438 + 0.161$$

$$= 1.873$$

Sample variance of total : $\underset{\sim}{b}'\underset{\sim}{X}$ is $\operatorname{Var}(\underset{\sim}{b}'\underset{\sim}{X})$

$$= \underset{\sim}{b}' \, S \, \underset{\sim}{b}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.76 & 1.397 & 0.511 & 0.245 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 3.913$$

(b) Excess of petroleum consumption over natural gas : $\underset{\sim}{c}'\underset{\sim}{X} = X_1 - X_2$

so Excess : $\underset{\sim}{c}'\underset{\sim}{X} = X_1 - X_2 + 0X_3 + 0X_4$

$$= [1\ -1\ 0\ 0]\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Sample mean of Excess : $\underset{\sim}{c}'\underset{\sim}{X}$ is $\underset{\sim}{c}'\bar{\underset{\sim}{X}}$

$$= [1\ -1\ 0\ 0]\begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix}$$

$$= 0.258$$

Sample variance of Excess : $\underset{\sim}{c}'\underset{\sim}{X}$ is $Var(\underset{\sim}{c}'\underset{\sim}{X})$

$$= \underset{\sim}{c}' S \underset{\sim}{c}$$

$$= [1\ -1\ 0\ 0]\begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 0.154$$

Finally, sample covariance of excess & Total

$$Cov(\underset{\sim}{c}'\underset{\sim}{X}, b'\underset{\sim}{X}) = \underset{\sim}{c}' S\, b$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.221 & 0.067 & 0.045 & 0.029 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 0.362$$

Q.20 Long question from book (page 148). To find Sample mean & sample covariance of $x_2 - x_1$. (Edited)

Solu

$\text{mean}(X_1) = \mu_1 = 9.42$

$\text{var}(X_1) = \sigma_{11} = 13.5736$

All in Excel

$\text{mean}(X_2) = \mu_2 = 19.272$

$\text{var}(X_2) = \sigma_{22} = 59.74922$

$\text{cov}(X_1, X_2) = \sigma_{12} = 12.93376$

NOW let $Y = X_2 - X_1$ = let $b'X$, where $b' = [1 \ -1]$

sample mean $(Y) = E(X_2 - X_1)$

$= E(X_2) - E(X_1)$

$= 9.852$

Bonus problem

so we have

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 13.5736 & 12.93376 \\ 12.93376 & 59.74922 \end{bmatrix}$$

Sample variance of $Y : b'X$ is

$b'Sb$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 13.5736 & 12.93376 \\ 12.93376 & 59.74922 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.63984 & -46.81546 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 47.4553$$

(b) For part (b) I first calculated $x_2 - x_1$ in excel and then find mean $(x_2 - x_1) = 9.852$ (same)

&

$var(x_2 - x_1) = 47.4553$ (same)

Excel

Conclusion:

Value of mean and variance from both part are same. □

X−Y = [1.2, 2, 9.4, 2, 4.1, 5.2, 23.1, 5.3, 4.8, 15.4, 11.7, 11.5, 22.5, 14.2, 8.8, 7.1, 5.5, 4.8, 22.6, 6.5, 24.8, 7, 9.8, 4.4, 12.6]

HomeWork proof

Q.> Show that when X_1 and X_2 are uncorrelated, the joint density function for bivariate normal can be written as the product of 2 univariate normal densities

Soln

The density function for the bivariate normal is given by

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}(1-\rho_{12}^2)}} \exp\left\{\frac{-1}{2(1-\rho_{12})^2}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right) - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right]\right\}$$

Now, when X_1 and X_2 are uncorrelated, then $\rho_{12} = 0$

Substitute $\rho = 0$, we get

$$f(x_1, x_2) = \frac{1}{2\pi\sqrt{\sigma_{11}\sigma_{22}}} \exp\left\{\frac{-1}{2}\left[\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2 + \left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right]\right\}$$

This can be written as

$$f(x_1, x_2) = \frac{1}{\sqrt{2\pi\sigma_{11}}\sqrt{2\pi\sigma_{22}}} \cdot \exp\left\{\frac{-1}{2}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)^2\right\} \cdot \exp\left\{\frac{-1}{2}\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma_{11}}} \exp\left\{\frac{-1}{2}\left(\frac{x_1-\mu_1}{\sqrt{\sigma_{11}}}\right)\right\} \times \frac{1}{\sqrt{2\pi\sigma_{22}}} \exp\left\{\frac{-1}{2}\left(\frac{x_2-\mu_2}{\sqrt{\sigma_{22}}}\right)\right\}$$

$$= f(x_1) \cdot f(x_2)$$

$\square$

HomeWork #3

Problem-1

Given data matrix is $\underline{X} = \begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$

i) Sample mean Vector

$$\underset{\sim\,3\times1}{\overline{X}} = \frac{1}{n}\underline{X}^T \mathbb{1}_{3\times1}$$

$$= \frac{1}{3}\begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 15 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

becoz here I find Sn, so my R will also be on that basis

ii) Sample covariance matrix, Sn

$$S_n = \frac{1}{n}\underline{X}^T\left(I - \frac{1}{n}\mathbb{1}\mathbb{1}^T\right)\underline{X}$$

$$= \frac{1}{3}\begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}\left\{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\right\}\begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 9 & 5 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 4 & 0 & -4 \\ -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 9 & 1 \\ 5 & 3 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 32 & -4 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 10.667 & -1.333 \\ -1.333 & 0.667 \end{bmatrix}$$

Tried to solve with points, got Rounding error, so trying number itself.

iii.) Sample correlation matrix.

We have $D^{1/2} = \begin{bmatrix} \sqrt{32/3} & 0 \\ 0 & \sqrt{2/3} \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{32/3} & 0 \\ 0 & \sqrt{2/3} \end{bmatrix}$$

So $D^{-1/2} = \begin{bmatrix} 1/\sqrt{32/3} & 0 \\ 0 & 1/\sqrt{2/3} \end{bmatrix}$

Finally $R = D^{-1/2} S_n D^{-1/2}$

$$= \begin{bmatrix} 1/\sqrt{32/3} & 0 \\ 0 & 1/\sqrt{2/3} \end{bmatrix} \begin{bmatrix} 32/3 & -4/3 \\ -4/3 & 2/3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{32/3} & 0 \\ 0 & 1/\sqrt{2/3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{32/3}{\sqrt{32/3}} & \dfrac{(-4/3)}{\sqrt{32/3}} \\ \dfrac{(-4/3)}{\sqrt{2/3}} & \dfrac{(2/3)}{\sqrt{2/3}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{32/3} & 0 \\ 0 & 1/\sqrt{2/3} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{32/3}{\sqrt{32/3}\cdot\sqrt{32/3}} & \dfrac{(-4/3)}{\sqrt{\frac{32}{3}}\cdot\sqrt{2/3}} \\ \dfrac{(-4/3)}{\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{32}{3}}} & \dfrac{(2/3)}{\sqrt{2/3}\cdot\sqrt{2/3}} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{32/3}{32/3} & \dfrac{-4/3}{8/3} \\ \dfrac{-4/3}{8/3} & \dfrac{2/3}{2/3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}. \square$$

5.)

$$\underline{X} = \begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix} \qquad \underset{\sim}{b}'\underset{\sim}{X} = [1 \;\; 1 \;\; 1] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\underset{\sim}{c}'\underset{\sim}{X} = [1 \;\; 2 \;\; -3] \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$\underset{\sim}{b}'\underset{\sim}{X} = X_1 + X_2 + X_3$$

$$\underset{\sim}{c}'\underset{\sim}{X} = X_1 + 2X_2 - 3X_3$$

(a)

observation on these linear combination are obtained by replacing X_1, X_2 and X_3 with their observed value.

$$\underset{\sim}{b}'\underset{\sim}{X_1} = x_{11} + x_{12} + x_{13}$$

$$= 1 + 4 + 3 = 8$$ first observation বাদ দিয়ে আসো

$$\underset{\sim}{b}'\underset{\sim}{X_2} = x_{21} + x_{22} + x_{23}$$

$$= 6 + 2 + 6 = 14$$ second observation বাদ দিয়ে আসো

$$\underset{\sim}{b}'\underset{\sim}{X_3} = x_{31} + x_{32} + x_{33}$$

$$= 8 + 3 + 3$$

$$= 14$$ third observation বাদ দিয়ে আসো

The sample mean and variance of these values are,

$$\text{Sample mean} = \frac{8 + 14 + 14}{3} = 12$$

$$\text{Sample variance} = \frac{(8-12)^2 + (14-12)^2 + (14-12)^2}{3-1}$$

$$= \frac{16 + 4 + 4}{2} = \frac{24}{2} = 12$$

In the similar mannar, the $n=3$ observation of $\underset{\sim}{c}'\underset{\sim}{X}$ are

$$\underset{\sim}{c}'\underset{\sim}{X}_1 = x_{11} + 2x_{12} - 3x_{13}$$
$$= 1 + 2\cdot(4) - 3\cdot(3) = 0$$

first observation of $\underset{\sim}{c}'\underset{\sim}{X}$ বাৰ পৰা আহিছে

$$\underset{\sim}{c}'\underset{\sim}{X}_2 = x_{21} + 2x_{22} - 3x_{23}$$
$$= 6 + 2\cdot(2) - 3\cdot(6)$$
$$= 6 + 4 - 18 = -8$$

Second observation on $\underset{\sim}{c}'\underset{\sim}{X}$ বাৰ পৰা আহিছে

$$\underset{\sim}{c}'\underset{\sim}{X}_3 = x_{31} + 2x_{32} - 3x_{33}$$
$$= 8 + 2\cdot(3) - 3(3)$$
$$= 8 + 6 - 9 = 5$$

Third observation of $\underset{\sim}{c}'\underset{\sim}{X}$ বাৰ পৰা আহিছে

The sample mean and variance of these values are

$$\text{Sample mean} = \frac{0+(-8)+5}{3} = -1$$

$$\text{sample variance} = \frac{(0+1)^2 + (-8+1)^2 + (5+1)^2}{3-1}$$
$$= \frac{1+49+36}{2} = \frac{86}{2} = 43$$

And, the sample covariance computed from the pair of observation $(\underset{\sim}{b}'\underset{\sim}{X}_1, \underset{\sim}{c}'\underset{\sim}{X}_1)$, $(\underset{\sim}{b}'\underset{\sim}{X}_2, \underset{\sim}{c}'\underset{\sim}{X}_2)$ and $(\underset{\sim}{b}'\underset{\sim}{X}_3, \underset{\sim}{c}'\underset{\sim}{X}_3)$ is

$$= \frac{(8-12)(0+1) + (14-12)(-8+1) + (14-12)(5+1)}{3-1} =$$

$$= \frac{(-4)(1) + (2)\cdot(-7) + (2)\cdot(6)}{2}$$

$$= \frac{-4-14+12}{2} = \frac{-6}{2} = -3$$

ⓑ Now using (3-36)→from book, we will find sample mean, sample variance and sample covariance for the observations $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X}$

To find sample mean, sample variance and sample covariance of linear combination, we will use sample mean vector $\underset{\sim 3\times1}{\bar{X}}$ and sample covariance matrix S obtained from the orginal data matrix $\underline{X}$

Here

$$\underset{\sim 3\times1}{\bar{X}} = \frac{1}{n} \underline{X}^T \underline{\mathbb{1}}_{3\times1}$$

$$= \frac{1}{3}\begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 3 \\ 3 & 6 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 15 \\ 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

and

$$S = \frac{1}{(n-1)} \underline{X}^T \left(I - \frac{1}{n} \mathbb{1}\mathbb{1}^T\right) \underline{X}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 6 & 8 \\ 4 & 2 & 3 \\ 3 & 6 & 3 \end{bmatrix}\left\{\begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}\right\}\begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

$$\frac{2}{3} - \frac{6}{3} - \frac{8}{3}$$
$$\frac{2-6-8}{3}$$

$$= \frac{1}{2}\begin{bmatrix} -4 & 1 & 3 \\ 1 & -1 & 0 \\ -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 & 3 \\ 6 & 2 & 6 \\ 8 & 3 & 3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 26 & -5 & 3 \\ -5 & 2 & -3 \\ 3 & -3 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix}$$

Now, so

sample mean of $\underset{\sim}{b}'\underset{\sim}{X} = \underset{\sim}{b}'\underset{\sim}{\overline{X}}_{3\times 1}$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$= 12$$

(same)

Sample mean of $\underset{\sim}{c}'\underset{\sim}{X} = \underset{\sim}{c}'\underset{\sim}{\overline{X}}_{3\times 1}$

$$= \begin{bmatrix} 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

$$= 5 + 6 - 12$$

$$= -1$$

(same)

sample variance of $\underset{\sim}{b}'\underset{\sim}{X} = b' S b$

$$= [1\ 1\ 1]\begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= [12\ -3\ 3]\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 12 - 3 + 3$$

$$= 12 \qquad \text{(same)}$$

sample variance of $\underset{\sim}{c}'\underset{\sim}{X} = \underset{\sim}{c}' S \underset{\sim}{c}$

$$= [1\ 2\ -3]\begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= [7/2\ \ 4\ \ -21/2]\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= 43 \qquad \text{(same)}$$

and finally sample covariance of $\underset{\sim}{b}'\underset{\sim}{X}$ and $\underset{\sim}{c}'\underset{\sim}{X}$ is

$$\text{cov}(\underset{\sim}{b}'\underset{\sim}{X}, \underset{\sim}{c}'\underset{\sim}{X}) = \underset{\sim}{b}' S \underset{\sim}{c}$$

$$= [1\ 1\ 1]\begin{bmatrix} 13 & -5/2 & 3/2 \\ -5/2 & 1 & -3/2 \\ 3/2 & -3/2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

$$= -3 \qquad \text{(same)}$$

Conclusion :

The results in part ⓐ and ⓑ are same.

5.18

given $\underset{4\times1}{\bar{\underset{\sim}{X}}} = \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix}$ $S = \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix}$

a) Total Energy consumption:

Total: $\underset{\sim}{b}' \underset{\sim}{X} = x_1 + x_2 + x_3 + x_4$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$

Sample mean of Total: $\underset{\sim}{b}' \underset{\sim}{X}$ is $\underset{\sim}{b}' \underset{4\times1}{\bar{\underset{\sim}{X}}}$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix}$

$= 0.766 + 0.508 + 0.438 + 0.161$

$= 1.873$

Sample variance of total: $\underset{\sim}{b}' \underset{\sim}{X}$ is $Var(\underset{\sim}{b}' \underset{\sim}{X})$

$= b' S b$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 1.76 & 1.397 & 0.511 & 0.245 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= 3.913$$

(b) Excess of petroleum consumption over natural gas : $\underset{\sim}{c}'\underset{\sim}{X} = X_1 - X_2$

So Excess : $\underset{\sim}{c}'\underset{\sim}{X} = X_1 - X_2 + 0X_3 + 0X_4$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

Sample mean of Excess : $\underset{\sim}{c}'\underset{\sim}{X}$ is $\underset{\sim}{c}'\underset{\sim}{\bar{X}}_{2\times 1}$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.766 \\ 0.508 \\ 0.438 \\ 0.161 \end{bmatrix}$$

$$= 0.258$$

Sample variance of Excess : $\underset{\sim}{c}'\underset{\sim}{X}$ is $Var(\underset{\sim}{c}'\underset{\sim}{X})$

$$= \underset{\sim}{c}' S \underset{\sim}{c}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= -0.1617$$

Find, sample covariance of Total & excess i.e

$$\text{Cov}(\underset{\sim}{b}'\underset{\sim}{X},\ \underset{\sim}{c}'\underset{\sim}{X})$$

$$= \underset{\sim}{b}' S \underset{\sim}{c}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.856 & 0.635 & 0.173 & 0.096 \\ 0.635 & 0.568 & 0.128 & 0.067 \\ 0.173 & 0.127 & 0.171 & 0.039 \\ 0.096 & 0.067 & 0.039 & 0.043 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1.76 & 1.397 & 0.511 & 0.245 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$= 0.363$$

I personally didn't like this matrix approach to solve these question ⓐ and ⓑ so I am solving them using definition.

ⓐ

Total: $\underset{\sim}{b}'\underset{\sim}{X} = X_1 + X_2 + X_3 + X_4$

sample mean of Total $= E(\text{Total})$

$$= E(X_1) + E(X_2) + E(X_3) + E(X_4)$$

$$= \mu_1 + \mu_2 + \mu_3 + \mu_4$$

$$= 0.766 + 0.508 + 0.438 + 0.161$$

$$= 1.873$$

Sample variance of total

$$\begin{aligned}
Var(total) &= Var(X_1 + X_2 + X_3 + X_4) \\
&= Var(X_1) + var(X_2 + X_3 + X_4) + 2cov(X_1, X_2 + X_3 + X_4) \\
&= Var(X_1) + Var(X_2) + Var(X_3 + X_4) + 2cov(X_2, X_3 + X_4) \\
&\quad + 2cov(X_1, X_2 + X_3 + X_4) \\
&= Var(X_1) + Var(X_2) + Var(X_3) + Var(X_4) + 2cov(X_3, X_4) \\
&\quad + 2cov(X_2, X_3) + 2cov(X_2, X_4) + 2cov(X_1, X_2) \\
&\quad + 2cov(X_1, X_3) + 2cov(X_1, X_4) \\
&= \sigma_{11} + \sigma_{22} + \sigma_{33} + \sigma_{44} + 2\sigma_{34} + 2\sigma_{23} + 2\sigma_{24} \\
&\quad + 2\sigma_{12} + 2\sigma_{13} + 2\sigma_{14} \\
&= 0.856 + 0.568 + 0.171 + 0.043 + 2[0.039 + 0.128 \\
&\quad + 0.067 + 0.635 + 0.173 + 0.096] \\
&= 1.638 + 2(1.138) \\
&= 1.638 + 2.276 \\
&= 3.914
\end{aligned}$$

(same)

Excess : $X_1 - X_2$

$$\text{mean (Excess)} = E(X_1 - X_2)$$
$$= E(X_1) - E(X_2)$$
$$= \mu_1 - \mu_2$$
$$= 0.766 - 0.508$$
$$= 0.258$$

$$\text{Var (Excess)} = \text{var}(X_1 - X_2)$$
$$= \text{Var}(X_1) + \text{var}(X_2) + 2\,\text{cov}(X_1, -X_2)$$
$$= \sigma_{11} + \sigma_{22} - 2\sigma_{12}$$
$$= 0.856 + 0.568 - 2(0.635)$$
$$= 1.424 - 1.27$$
$$= 0.154 \longrightarrow$$

$\boxed{\text{Ask Her (Why I get different) by matrix approach : } -0.617 \text{ by defn approach : } 0.154}$

finally

$$\text{cov}(X_1 + X_2 + X_3 + X_4,\ X_1 - X_2)$$
$$= \text{cov}(X_1, X_1) - \text{cov}(X_1, X_2) + \text{cov}(X_2, X_1) - \text{cov}(X_2, X_2) + \text{cov}(X_3, X_1) - \text{cov}(X_3, X_2)$$
$$+ \text{cov}(X_4, X_1) - \text{cov}(X_4, X_2)$$
$$= \sigma_{11} - \cancel{\sigma_{12}} + \cancel{\sigma_{21}} - \sigma_{22} + \sigma_{31} - \sigma_{32} + \sigma_{41} - \sigma_{42}$$
$$= 0.856 - 0.568 + 0.173 + 0.096 - 0.067$$
$$= 0.49$$

$\rightarrow$ different (ask why)