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100/100

1.) Assume X_1 and X_2 are iid normal random variables with mean μ variance σ^2. let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - X_2$. Show that Y_1 and Y_2 are independent and find their distribution.

Soln: we are given

$$X_1, X_2 \overset{iid}{\sim} N(\mu, \sigma^2)$$

given

$$\begin{aligned} Y_1 &= X_1 + X_2 \\ Y_2 &= X_1 - X_2 \end{aligned} \quad \xRightarrow[\text{original variable}]{\text{Solving for}} \quad \begin{aligned} X_1 &= \frac{Y_1 + Y_2}{2} \\ X_2 &= \frac{Y_1 - Y_2}{2} \end{aligned}$$

The Jacobian

$$J = \det \begin{vmatrix} \frac{\partial X_1}{\partial Y_1} & \frac{\partial X_1}{\partial Y_2} \\ \frac{\partial X_2}{\partial Y_1} & \frac{\partial X_2}{\partial Y_2} \end{vmatrix}$$

$$= \det \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}$$

$$= -\frac{1}{4} - \frac{1}{4} \qquad \therefore |J| = 1/2$$

Now Joint pdf of Y_1 and Y_2 is

$$f_{Y_1,Y_2}(y_1,y_2) = f_{X_1,X_2}\left(\frac{y_1+y_2}{2}, \frac{y_1-y_2}{2}\right) \cdot |J|$$

But since X_1 & $X_2 \overset{iid}{\sim} N(\mu, \sigma^2)$ given

So,

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

$$= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2} \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_2-\mu}{\sigma}\right)^2} \quad (*)$$

Indep चिरकों के होने, Joint pdf is the product of individual pdf.

Now using this concept in (*), we get

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}\left(\frac{y_1+y_2}{2}-\mu\right)^2} \cdot \frac{1}{\sigma\sqrt{2\pi}}$$

$$e^{-\frac{1}{2\sigma^2}\left(\frac{y_1-y_2}{2}-\mu\right)^2} \cdot \frac{1}{2} \text{ — Jacobian}$$

$$= \frac{1}{2} \cdot \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2\sigma^2}\left[\left(\frac{y_1+y_2}{2}-\mu\right)^2 + \left(\frac{y_1-y_2}{2}-\mu\right)^2\right]}$$

$$= \frac{1}{2} \cdot \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2\sigma^2}\left[\left(\frac{y_1+y_2}{2}\right)^2 + \mu^2 - \cancel{2}\cdot\left(\frac{y_1+y_2}{\cancel{2}}\right)\cdot\mu \right.}$$

$$\left. + \left(\frac{y_1-y_2}{2}\right)^2 + \mu^2 - \cancel{2}\cdot\left(\frac{y_1-y_2}{\cancel{2}}\right)\mu \right]$$

$$= \frac{1}{2}\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2\sigma^2}\left[\frac{y_1^2}{4} + \frac{y_2^2}{4} + \cancel{\frac{y_1y_2}{2}} + \mu^2 \right.}$$

$$\left. - \mu\left[y_1+y_2+y_1-y_2\right] + \frac{y_1^2}{4} + \frac{y_2^2}{4} - \cancel{\frac{y_1y_2}{2}} + \mu^2\right]$$

$$= \frac{1}{2}\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2\sigma^2}\left[\frac{y_1^2}{2} + \frac{y_2^2}{2} + 2\mu^2 - 2\mu y_1\right]}$$

$$= \frac{1}{2}\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2\sigma^2}\left[\frac{y_1^2}{2} - 2\mu y_1 + 2\mu^2\right]} \cdot e^{-\frac{1}{2\sigma^2}\left[\frac{y_2^2}{2}\right]}$$

$$y_1^2 - 4\mu y_1 + 4\mu^2$$

$$= \frac{1}{\underset{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}}{\underset{\|}{2}}}\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^2 e^{-\frac{1}{2\sigma^2}\left(\frac{(y_1-2\mu)^2}{2}\right)} \cdot e^{-\frac{1}{2\sigma^2}\left(\frac{y_2^2}{2}\right)}$$

$$= \frac{1}{\underline{\sqrt{2}\sigma}\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_1-2\mu}{\sqrt{2}\sigma}\right)^2} \cdot \frac{1}{\sqrt{2}\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_2}{\sqrt{2}\sigma}\right)^2}$$

$\sim N(2\mu, 2\sigma^2)$ $\qquad\qquad$ $\sim N(0, 2\sigma^2)$

Also, given

$Y_1 = X_1 + X_2$ & $Y_2 = X_1 - X_2$

$E(Y_1) = \underbrace{E(X_1 + X_2)}_{\text{Indep variables}}$

$= E(X_1) + E(X_2)$

$= \mu + \mu$

$= 2\mu$

$E(Y_2) = E(X_1 - X_2)$

$= E(X_1) - E(X_2)$

$= \mu - \mu$

$= 0$

$Var(Y_1) = Var(X_1 + X_2)$

$= 1^2 Var(X_1) + 1^2 Var(X_2)$

$= Var(X_1) + Var(X_2)$

$= \sigma^2 + \sigma^2$

$= 2\sigma^2$

& $Var(Y_2) = Var(X_1 - X_2)$

$= 1^2 Var(X_1) + (-1)^2 Var(X_2)$

$= Var(X_1) + Var(X_2)$

$= \sigma^2 + \sigma^2$

$= 2\sigma^2$

Since, sum of normal R.V.s (indep) is Normal.

$Y_1 \sim N(2\mu, 2\sigma^2)$ &

$Y_2 \sim N(0, 2\sigma^2)$. $\square$

20.

$\Rightarrow$ $X_1, X_2, \ldots, X_n$ is an i.i.d sample from $N(\mu, \sigma^2)$. Let s^2 denotes the sample variance. Find $E(s^2)$ and $Var(s^2)$.

Soln: we are given

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$$

sample variance is defined as

$$s^2 = \frac{1}{n-1} \Sigma (X_i - \bar{X})^2$$

we know that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

For χ^2 dist. $E(X)$ = degree of freedom

$V(X) = 2 \times$ degree of freedom

$$E\left[\frac{(n-1)s^2}{\sigma^2}\right] = (n-1)$$

$$\Rightarrow \frac{(n-1)}{\sigma^2} E(s^2) = (n-1)$$

$$\Rightarrow E(s^2) = \frac{(n-1)\sigma^2}{(n-1)}$$

$$\Rightarrow E(s^2) = \sigma^2.$$

...riance of sample variance

We know that

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2_{n-1} \quad \text{---} (*)$$

Also, for chi-squared distribution, we know that

mean = degree of freedom

& variance = 2 * degree of freedom.

Normal population

$$Var\left[\frac{(n-1)s^2}{\sigma^2}\right] = \cancel{Var}\ 2(n-1) \qquad \text{from } (*)$$

$$\Rightarrow \frac{(n-1)^2}{\sigma^4} Var(s^2) = 2(n-1)$$

$$\Rightarrow \quad Var(s^2) = \frac{2(n-1)\sigma^4}{(n-1)^2}$$

$$\therefore \quad Var(s^2) = \frac{2\sigma^4}{(n-1)}$$

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(Can be Ignored)

Q.) $X_1, X_2, \ldots, X_n$ is an i.i.d sample from $N(\mu, \sigma^2)$. Let s^2 denotes the sample variance. Find $E(s^2)$ and $Var(s^2)$.

Soln We are given

$$X_1, X_2, \ldots, X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$$

We know

$$s^2 = \frac{1}{n-1} \Sigma (X_i - \overline{X})^2$$

So,

$$E(s^2) = E\left(\frac{1}{n-1} \Sigma (X_i - \overline{X})^2\right)$$

$$= \frac{1}{n-1} E\left(\Sigma (X_i - \overline{X})^2\right)$$

Here $\Sigma (X_i - \overline{X})^2 = \Sigma (X_i^2 + \overline{X}^2 - 2X_i\overline{X})$

indep [illegible]

$$\overline{X} = \frac{\Sigma X_i}{n}$$

$$\Rightarrow \Sigma X_i = n\overline{X}$$

$$= \Sigma X_i^2 + \Sigma \overline{X}^2 - 2 \Sigma X_i \overline{X}$$

$$= \Sigma X_i^2 + n\overline{X}^2 - 2\overline{X} \Sigma X_i$$

$$= \Sigma X_i^2 + n\overline{X}^2 - 2n\overline{X}^2$$

$$= \sum X_i^2 - n\bar{X}^2$$

So

$$E(s^2) = \frac{1}{n-1}\left(E(\sum X_i^2) - nE(\bar{X}^2)\right)$$ indep theorem

Here $E(\sum X_i^2) = \sum E(X_i^2)$

$$= \sum \{ Var(X_i) + (E(X_i))^2 \}$$

$$= n\sigma^2 + n\mu^2$$

& $E(\bar{X}^2) = Var(\bar{X}) + [E(\bar{X})]^2$

$$= \frac{\sigma^2}{n} + \mu^2$$

Hence,

$$E(s^2) = \frac{1}{n-1}\left[n\sigma^2 + n\mu^2 - n\left(\frac{\sigma^2}{n} + \mu^2\right)\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2}\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 - \sigma^2\right]$$

$$= \frac{(n-1)\sigma^2}{(n-1)} = \sigma^2$$

correct.

5) Let $Y_1, Y_2, \ldots, Y_5$ be a random sample of size 5 from a normal population with mean 0 and variance 1 and let $\bar{Y} = \frac{1}{5}\sum_{i=1}^{5} Y_i$. Let Y_6 be another independent observation from the same population. Find the distribution of the following and explain.

a) $W = \sum_{i=1}^{5} Y_i^2$

Y_i random variables हरु Normally distribute गरिएका हुन with mean $\mu = 0$ and variance $\sigma^2 = 1$.

Soln

we are given

$$Y_1, Y_2, \ldots, Y_5 \sim N(0,1)$$

so

$$Y_i^2 \sim \chi^2(1)$$

Normal random variable हरुलाई squre गरेर add गर्दा हो भने तिनिहरुले chi-squared distribution बनाउँछन् and degree of freedom is equal to no. of squared term added.

$$\Rightarrow \sum_{i=1}^{n} Y_i^2 \sim \chi^2(n)$$

n terms (normal) हरुलाई squre गरेर add गर्दा हो भने they form chi-squared distribution with degree of freedom n.

so

$$\sum_{i=1}^{5} Y_i^2 \sim \chi^2(5)$$

i.e

$$W \sim \chi^2(5)$$

b.) $U = \sum_{i=1}^{5} (Y_i - \bar{Y})^2 \Rightarrow (Y_1 - \bar{Y})^2 + \dots + (Y_5 - \bar{Y})^2$

Solu

We are given that

Normally distribute होता है with mean $\mu = 0$ & varionce $\sigma^2 = 1$.

$Y_1, Y_2, \dots, Y_5 \sim N(0,1)$

If S^2 is the sample variation of the population. Then

$$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

We Know the result that

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{(n-1)}$$

Chi-squared with $(n-1)$ degree of freedom.

$$\Rightarrow \frac{(n-1)}{\sigma^2} * \frac{1}{(n-1)} \sum_{i=1}^{n} (Y_i - \bar{Y})^2 \sim \chi^2_{(n-1)}$$

But Here we have $n = 5$, & varionce $\sigma^2 = 1$. so

$$\frac{\cancel{(5-1)}}{1} * \frac{1}{\cancel{(5-1)}} \sum_{i=1}^{5} (Y_i - \bar{Y})^2 \sim \chi^2_{(5-1)}$$

$$\sum_{i=1}^{5} (Y_i - \bar{Y})^2 \sim \chi^2_{(4)}$$

i.e. $U \sim \chi^2(4)$

i) $U + Y_6^2$

Soln: from part (b) of this question, we know that

$$U \sim \chi^2(4)$$

and since Y_6 is also the another random variable from the same population so

$$Y_6^2 \sim \chi^2(1)$$

normal random variable छ भने त्यसलाई square गर्ने हो भने त्यो chi-squared distribution मा जान्छ

So clearly,

$$U + Y_6^2 \sim \chi^2(5)$$

$\Rightarrow 2(5\bar{Y}^2 + Y_6^2)/U$

~~Soln~~ from part ⓑ, we know that

$$U \sim \chi^2(4) \quad \text{— (x)}$$

We want the distribution of

$$\frac{2(5\bar{Y}^2 + Y_6^2)}{U} \sim ?$$

~~let~~
$$\frac{\frac{2(5\bar{Y}^2 + Y_6^2)}{4}}{\frac{U}{4}}$$

degree of freedom मीलाउन both लाई 4 ले divide गरे

$$= \frac{\frac{1}{2}(5\bar{Y}^2 + Y_6^2)}{\frac{U}{4}}$$

Suppose $W = (5\bar{Y}^2 + Y_6^2)$ — We want distribution of this quantity

$$= \left(5\left(\frac{\Sigma Y_i}{5}\right)^2 + Y_6^2\right)$$

$$= \left(\frac{1}{5}(\Sigma Y_i)^2 + Y_6^2\right)$$

$\because \quad Y_1, Y_2, \ldots, Y_5 \sim N(0,1)$

$$\sum Y_i \sim N(0,5)$$

$$\Rightarrow \frac{1}{\sqrt{5}} \sum Y_i \sim N(0,1)$$

$$\Rightarrow \left(\frac{1}{\sqrt{5}} \sum Y_i\right)^2 \sim \chi^2(1)$$

$$\Rightarrow \frac{1}{5} \left(\sum Y_i\right)^2 \sim \chi^2(1)$$

Also, given $Y_6 \sim N(0,1)$

$$\Rightarrow Y_6^2 \sim \chi^2(1)$$

Hence $\left(\frac{1}{5}\left(\sum Y_i\right)^2 + Y_6^2\right) \sim \chi^2(1+1) = \chi^2(2)$

$$\Rightarrow \left(\frac{1}{5}\left(\sum Y_i\right)^2 + Y_6^2\right) \sim \chi^2(2)$$

$$\Rightarrow W \sim \chi^2(2)$$

So, our orginal question becomes

$$\frac{\frac{1}{2}W}{\frac{1}{4}V}$$

$$= \frac{\frac{W}{2}}{\frac{V}{4}} = \frac{\frac{\text{chi-squared}}{\text{degree of freedom}}}{\frac{\text{chi-squared}}{\text{degree of freedom}}} \sim F(2,4)$$

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Suppose that $X \sim \chi^2(m)$, $S = X+Y \sim \chi^2(m+n)$, and X and Y are independent. Use MGF to show that $S-X \sim \chi^2(n)$.

Soln: We are given

$$X \sim \chi^2(m)$$ X यांही chi-squared distribute असतो ह with m degree of freedom.

$$S = X+Y \sim \chi^2(m+n)$$ X+Y यांही chi-squared distribute असतो ह with (m+n) degree of freedom.

Also X & Y are independent.

Here

mgf of X is

$$M_X(t) = \underbrace{(1-2t)}_{\text{mgf of chi-squared distribution}}{}^{-\frac{m}{2}}$$

$\frac{m}{2}$ → degree of freedom

mgf of S is

$$M_S(t) = M_{X+Y}(t) = (1-2t)^{-\frac{m+n}{2}}$$

$\frac{m+n}{2}$ → degree of freedom

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Also since X & Y are indep

$$= E(e^{tS}) = E(e^{t(X+Y)}) = E(e^{tX} \cdot e^{tY}) = E(e^{tX}) \cdot E(e^{tY}) = M_X(t) M_Y(t)$$

$$M_S(t) = M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$\Rightarrow M_Y(t) = \frac{M_S(t)}{M_X(t)} = \frac{(1-2t)^{-\frac{m+n}{2}}}{(1-2t)^{-m/2}}$$

$$= (1-2t)^{-n/2}$$ → mgf of chi-squared distribution

$$\therefore Y \sim \chi^2(n)$$

Suppose that independent samples (of size n_i) are taken from each of k populations and that population i is normally distributed with mean μ_i and variance σ^2, $i = 1, 2, \dots, k$. That is, all populations are normally distributed with the same variance but with (possibly) different means. Let $\overline{X}_i$ and S_i^2, $i = 1, 2, \dots, k$ be the respective sample means and variances. Let $\theta = c_1\mu_1 + c_2\mu_2 + \dots + c_k\mu_k$, where $c_1, c_2, \dots, c_k$ are given constants.

ⓐ Give the distribution of $\hat{\theta} = c_1\overline{X}_1 + c_2\overline{X}_2 + \dots + c_k\overline{X}_k$.

Soln: we are given $\theta = c_1\mu_1 + c_2\mu_2 + \dots + c_k\mu_k$

we have to find out the distribution of $\hat{\theta}$

Since any linear combination of Normal distribution is distributed Normal so $\hat{\theta}$ is also distributed Normal with

$$E(\hat{\theta}) = E(c_1\overline{X}_1 + c_2\overline{X}_2 + \dots + c_k\overline{X}_k)$$

$$= E(c_1\overline{X}_1) + \dots + E(c_k\overline{X}_k)$$

$$= c_1E(\overline{X}_1) + \dots + c_kE(X_k)$$

$$= c_1\mu_1 + \dots + c_k\mu_k$$

$$= \theta$$

(given)

Also,

$$Var(\hat{\theta}) = Var(c_1\bar{X}_1 + c_2\bar{X}_2 + \cdots + c_K\bar{X}_K)$$

$$= Var(c_1\bar{X}_1) + \cdots + Var(c_K\bar{X}_K)$$

$$= c_1^2 \frac{\sigma^2}{n_1} + \cdots + c_K^2 \frac{\sigma^2}{n_K}$$

$$= \sigma^2\left(\frac{c_1^2}{n_1} + \frac{c_2^2}{n_2} + \cdots + \frac{c_K^2}{n_K}\right)$$

$$= \sigma^2 \sum_{i=1}^{K} \frac{c_i^2}{n_i}$$

$$\therefore \hat{\theta} \sim N\left(\theta, \sigma^2 \sum_{i=1}^{K} \frac{c_i^2}{n_i}\right) \qquad \left(\theta, \sigma^2 \sum_{i=1}^{K} \frac{c_i^2}{n_i}\right)$$

b.) Give the distribution of

$$\frac{SSE}{\sigma^2}, \text{ where } SSE = \sum_{i=1}^{K}(n_i-1)s_i^2$$

Soln

We have to find out the distribution of $\frac{SSE}{\sigma^2}$

where $SSE = \sum_{i=1}^{K}(n_i-1)s_i^2$

Here $$\frac{SSE}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^{K}(n_i-1)s_i^2$$

$$= \sum_{i=1}^{K} \frac{(n_i - 1) s_i^2}{\sigma^2} \sim \chi^2\left(\sum_{i=1}^{K} (n_i - 1)\right)$$

$$= \frac{(n_1 - 1) s_1^2}{\sigma^2} + \frac{(n_2 - 1) s_2^2}{\sigma^2} + \cdots + \frac{(n_K - 1) s_K^2}{\sigma^2}$$

$$= \sim \chi^2(n_1 - 1) + \quad \cdots \quad + \sim \chi^2(n_K - 1)$$

$$= \sim \chi^2\left((n_1 - 1) + \cdots + (n_K - 1)\right)$$

$$= \sim \chi^2\left(\sum_{i=1}^{K} (n_i - 1)\right)$$

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