Q1. (a) mgf of u. $m_u(t) = E(e^{tu}) = E(e^{t\sum_1^n X_i})$

$$= E\left(\prod_{i=1}^n e^{tX_i}\right) = \prod_{i=1}^n E(e^{tX_i}) \quad \text{by indep.}$$

$$= \left[E(e^{tX_1})\right]^n \quad \text{by identical}$$

$$= \left(\frac{1}{1-t}\right)^n \leftarrow \text{mgf of gamma}(n, 1)$$

n ↑ shape, 1 ↑ scale.

(b) by CLT: $\dfrac{\bar{X}-1}{1/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$.

$$\frac{n\bar{X}-n}{\sqrt{n}} \xrightarrow{d} Z \sim N(0,1), \quad n\bar{X} = \sum X_i = u$$

So $u \sim AN(n, n)$.

(c). by CLT $\dfrac{\bar{X}-1}{1/\sqrt{100}} \xrightarrow{d} Z \sim N(0,1)$, or $\bar{X} \sim AN\left(1, \frac{1}{100}\right)$.

$$P(1.1 < \bar{X} < 1.2) \approx P\left(\frac{1.1-1}{1/\sqrt{100}} < Z < \frac{1.2-1}{1/\sqrt{100}}\right)$$

$$= P(1 < Z < 2) = 0.9772 - 0.8413 = 0.1359$$

Q2. (a) CDF of $X_{1:n}$.

$$F_{X_{1:n}}(x) = P(X_{1:n} < x) = 1 - P(X_{1:n} > x)$$
$$= 1 - P(X_1 > x, X_2 > x, \cdots, X_n > x)$$
$$= 1 - [P(X_1 > x)]^n \quad \text{by i.i.d.}$$
$$= 1 - [1 - P(X_1 < x)]^n$$
$$= \begin{cases} 1 - [1 - (1 - \frac{1}{x})]^n = 1 - (\frac{1}{x})^n, & x \in [1, \infty) \\ 1 - [1 - 0]^n = 0, & \text{o.w.} \end{cases}$$

As $n \to \infty$ $F_{X_{1:n}}(x) \to \begin{cases} 1 & x > 1 \\ 0 & x = 1 \\ 0 & x < 1 \end{cases}$ ← $X_{1:n}$ converge to a degenerate distribution at 1.

(b) CDF of $X_{1:n}^n$

$$F_{X_{1:n}^n}(x) = P(X_{1:n}^n < x) = P(X_{1:n} < x^{\frac{1}{n}})$$
$$= 1 - P(X_{1:n} > x^{\frac{1}{n}}) = 1 - P(X_1 > x^{\frac{1}{n}}, \cdots, X_n > x^{\frac{1}{n}})$$
$$= 1 - [P(X_1 > x^{\frac{1}{n}})]^n = \begin{cases} 1 - \left(\frac{1}{x^{\frac{1}{n}}}\right)^n = 1 - \frac{1}{x}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

As $n \to \infty$.
$$F_{X_{1:n}^n}(x) \to \begin{cases} 1 - \frac{1}{x}, & x \geq 1 \\ 0, & x < 1 \end{cases}$$

(c) CDF of $n \ln X_{1:n}$.

$$F(x) = P(n \ln X_{1:n} < x) = P(\ln X_{1:n} < \frac{x}{n})$$
$$= P(X_{1:n} < e^{x/n}) = 1 - P(X_{1:n} > e^{x/n})$$
$$= \begin{cases} 1 - \left(\frac{1}{e^{x/n}}\right)^n, & e^{x/n} \geq 1 \Rightarrow x \geq 0 \\ 0, & e^{x/n} < 1 \Rightarrow x < 0. \end{cases}$$

$$= \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & x < 0. \end{cases} \longrightarrow \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

exponential (1).

Q3 (a). $F_{X_{1:n}}(x) = P(X_{1:n} < x) = 1 - P(X_{1:n} > x)$

$$= 1 - P(X_1 > x, \cdots, X_n > x)$$

$$= 1 - [P(X_1 > x)]^n$$

$$P(X_1 > x) = \int_x^\infty e^{-(t-\mu)}\,dt = -e^{-(t-\mu)}\Big]_x^\infty$$

$$= e^{-(x-\mu)}$$

$$F_{X_{1:n}}(x) = 1 - [e^{-(x-\mu)}]^n = \begin{cases} 1 - e^{-n(x-\mu)}, & x > \mu \\ 0, & \text{otherwise} \end{cases}$$

(b). as $n \to \infty$, $F_{X_{1:n}}(x) \to \begin{cases} 1 & x > \mu \\ 0 & x \le \mu \end{cases}$

$$X_{1:n} \xrightarrow{d} F(x) = \begin{cases} 1 & x \ge \mu \\ 0 & x < \mu \end{cases}$$ ← degenerate dist. at μ.

Q4 (a). by CLT

$$T_n = \frac{\bar{X}_n - 1}{1/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$$

According to definition of converge in distribution (CDF converges)

$$F_{T_n}(x) = P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} < x\right) \longrightarrow F_Z(x) = P(Z < x), \text{ as } n \to \infty$$

(b) left-hand side

$$P\left(\frac{\bar{X}_n - 1}{1/\sqrt{n}} < x\right) = P\left(\frac{\sum X_i - n}{\sqrt{n}} < x\right) = P(\sum X_i < n + \sqrt{n}x)$$

$$= F_{\sum X_i}(n + \sqrt{n}x)$$

$\boxed{\text{plug-in } x = 0. \quad \frac{\sqrt{n}}{\Gamma(n)} n^{n-1} e^{-n} = \frac{1}{\sqrt{2\pi}}}$

$$\frac{d}{dx} F_{\sum X_i}(n + \sqrt{n}x) = F'_{\sum X_i}(n+\sqrt{n}x) \cdot \sqrt{n} = \underbrace{f_{\sum X_i}(n+\sqrt{n}x)}_{\text{gamma pdf}} \cdot \sqrt{n}$$

< note $\sum X_i \sim \text{gamma}(1, n)$ (Q1) > $= \frac{1}{\Gamma(n)} (n + \sqrt{n}x)^{n-1} e^{-n+\sqrt{n}x} \cdot \sqrt{n}$

$$= \frac{\sqrt{n}}{\Gamma(n)} (x\sqrt{n} + n)^{n-1} e^{-x\sqrt{n} + n}$$

Right hand side:

$$\frac{d}{dx} F_Z(x) = F'(x) = f(x) = \frac{1}{\ \ } e^{-x^2/2} \quad Z \sim N(\ \ )$$

Q5 (a) by CLT

$$\frac{\bar{X}-p}{\sqrt{p(1-p)}/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$$

$$\frac{\sqrt{n}(\bar{X}-p)}{\sqrt{p(1-p)}} \xrightarrow{d} Z \quad \Rightarrow \sqrt{n}(\bar{X}-p) \xrightarrow{d} N(0, p(1-p))$$

note $\hat{p} = \frac{\sum X_i}{n} = \bar{X}$ here.

(b). $Y_n = \frac{\sum X_i - pn}{n}$

CDF: $F_{Y_n}(y) = P(Y_n \le y)$

$$= P\left(\frac{\sum X_i - pn}{n} \le y\right)$$

$$= P\left(\bar{X} \le y + p\right)$$

$$= P\left(\frac{\bar{X}-p}{\sqrt{\frac{p(1-p)}{n}}} \le \frac{y}{\sqrt{\frac{p(1-p)}{n}}}\right)$$

$$\lim_{n\to\infty} F_{Y_n}(y) = \lim_{n\to\infty} P\left(Z \le \frac{y}{\sqrt{\frac{p(1-p)}{n}}}\right)$$ by CLT $\frac{\bar{X}-p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} Z$, $Z \sim N(0,1)$

$$= \begin{cases} P(Z \le -\infty) & y<0 \\ P(Z \le +\infty) & y>0 \\ P(Z \le 0) & y=0 \end{cases} = \begin{cases} 0 & y<0 \\ 0.5 & y=0 \\ 1 & y>0 \end{cases}$$

$$Y_n \xrightarrow{d} Y \sim F(y) = \begin{cases} 1 & y \ge 0 \\ 0 & y<0 \end{cases}$$

^degenerate at 0.