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Instruction:
Write you answers clearly on **separate sheets** of paper, which means one question per sheet. Show all your steps. You may not use notes, your textbook, etc. You are to work completely independently on this exam. You have 75 minutes to complete the exam. Good luck.

Points

Q1 (20 points total): 20

Q2 (15 points total): 13

Q3 (20 points total): 8

Q4 (15 points total): 15

Q5 (20 points total): 20.

Total (100 points): 86

given

1. $X_1, X_2, \cdots, X_n$ are i.i.d N(0, 1) random variables. Let

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (Z_i + 1/n) = \frac{1}{\sqrt{n}} \Sigma(Z_i + 1/n)$$

Find the limiting distribution of Y_n. (Hint: using MGF.) $= \frac{1}{\sqrt{n}}(\Sigma Z_i + 1)$

2. Suppose $X_1, X_2, \cdots, X_n$ is an i.i.d sample with mean μ_1 and variance σ_1^2 and $Y_1, Y_2, \cdots, Y_n$ is an i.i.d sample with mean μ_2, and variance σ_2^2. The X's and Y's are independent. Let $\bar{X}$ and $\bar{Y}$ denote the corresponding sample means. Find the asymptotic distribution of $\bar{X} - \bar{Y}$.

random variable $Y_n \sim \chi^2(n)$

3. Suppose that Y_n is an ~~iid sample~~ from $\chi^2(n)$ distribution. Give a normal approximation of Y_n use CLT, when n is large. State the mean and variance of your limiting normal distribution.

4. X_1 and X_2 are i.i.d $\chi^2(2)$ random variables. Recall the pdf a $\chi^2(\nu)$ random variable is

$$f_X(x) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \text{ for } x > 0 \text{ and } 0 \text{ otherwise.}$$

(a) $U_1 = X_1 + X_2$ and $U_2 = X_1/X_2$. What is the marginal distribution of U_1 and U_2?
(b) Let Z be a standard normal random variable that is independent of X_1 and X_2. What is the distribution of $U_3 = 2Z/\sqrt{X_1 + X_2}$?
(c) What is the distribution of $U_4 = U_3^2$?

5A. (This is for undergraduate student) Suppose that $X_1, X_2, \cdots, X_n$ is an i.i.d sample with CDF $F(x) = 1 - (1 + x)^{-1}$, for $x > 0$. Find the limiting distribution of $Y_n = nX_{1:n}$.
Note: $\lim_{n\to\infty}(1 + \frac{a}{n})^n = e^a$

5B. (This is for graduate student) Suppose that $X_1, X_2, \cdots, X_n$ is an i.i.d sample with CDF $F(x) = (1 + e^{-x})^{-1}$, for $x > 1$. Find the limiting distribution of $Y_n = X_{n:n} - \ln(n)$.
Note: $\lim_{n\to\infty}(1 + \frac{a}{n})^n = e^a$

$mgf_{Y_n} = E(e^{tY_n})$
$= \left(\frac{1}{1-2t}\right)^{n/2}$

1.) given $X_1, X_2, \ldots, X_n \overset{iid}{\sim} N(0,1)$

$$Y_n = \frac{1}{\sqrt{n}} \Sigma(Z_i + 1/n)$$

$$= \frac{1}{\sqrt{n}} (\Sigma Z_i + 1).$$

mgf of Y_n is

$$M_{Y_n}(t) = E(e^{tY_n})$$

$$= E\left(e^{t \cdot \frac{1}{\sqrt{n}}(\Sigma Z_i + 1)}\right)$$

$$= E\left(e^{\frac{t}{\sqrt{n}} \Sigma Z_i} \cdot e^{t/\sqrt{n}}\right)$$

$$= e^{t/\sqrt{n}} E\left(e^{\frac{t}{\sqrt{n}} \Sigma Z_i}\right)$$

$$= e^{t/\sqrt{n}} E\left(e^{\frac{t}{\sqrt{n}} Z_1} \ldots e^{\frac{t}{\sqrt{n}} Z_n}\right)$$

$$= e^{t/\sqrt{n}} \left\{ e^{\frac{1}{2}(t/\sqrt{n})^2} \right\}^n$$

$$= e^{t/\sqrt{n}} \cdot \left\{ e^{\frac{t^2}{2n}} \right\}^n$$

$$= e^{t/\sqrt{n}} \cdot e^{\frac{t^2}{2}}$$

as $n \to \infty$ $M_{Y_n}(t) \to e^{\frac{t^2}{2}}$

mgf of $N(0,1)$

2.) $X_1, X_2, \ldots, X_n \overset{iid}{\sim} D(\mu_1, \sigma_1^2)$ where $D \to$ distribution (any)

$Y_1, Y_2, \ldots, Y_n \overset{iid}{\sim} D(\mu_2, \sigma_2^2)$

$$\bar{X} \sim D\left(\mu_1, \frac{\sigma_1^2}{n}\right)$$

$$\bar{Y} \sim D\left(\mu_2, \frac{\sigma_2^2}{n}\right)$$

$$\bar{X} - \bar{Y} \sim D\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}\right)$$

CLT: $X_i - Y_i \overset{iid}{\sim} D(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$

then $\overline{X_i - Y_i} \xrightarrow{d} N\left(\mu_1 - \mu_2, \frac{\sigma_1^2 + \sigma_2^2}{n}\right)$

not how CLT states!

$$\frac{\frac{\bar{X} - \bar{Y}}{1} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} \longrightarrow N(0,1) \times$$

$$\frac{\bar{X} - \bar{Y} - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}}} \longrightarrow N(0,1)$$

$$\therefore \bar{X} - \bar{Y} \sim AN\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n}\right)$$

) $X_1, X_2 \overset{iid}{\sim} \chi^2(2)$

$\mu = 2, \sigma^2 = 2 \times 2$
$= 4$

given $U_1 = X_1 + X_2$ $\quad U_2 = \frac{X_1}{X_2}$

(a)

$U_1 = X_1 + X_2 \sim \chi^2(4)$ — marginal distributions

Chi-squared

$U_2 = \frac{X_1}{X_2} = \frac{\frac{X_1}{2}}{\frac{X_2}{2}} \sim F(2,2)$ → F-distribution

(b)

$U_3 = \frac{2Z}{\sqrt{X_1 + X_2}}$

$= \frac{Z}{\frac{1}{2}\sqrt{X_1 + X_2}}$

$= \frac{Z}{\sqrt{\frac{X_1 + X_2}{4}}} \sim t(4)$ t-distribution with degree of freedom 4.

(c) $U_4 = U_3^2 = \left(\frac{Z}{\sqrt{\frac{X_1+X_2}{4}}}\right)^2 = \frac{Z^2}{\frac{X_1+X_2}{4}} = \frac{\frac{Z^2}{1}}{\frac{X_1+X_2}{4}} \sim F(1,4)$

5.) $F(x) = (1+e^{-x})^{-1}, \quad x > 1$

let $Y_n = X_{n:n} - \ln(n)$

CDF of Y_n :

$$F_{Y_n}(y) = P(Y_n \leq y)$$

$$= P(X_{n:n} - \ln(n) \leq y)$$

$$= P(X_{n:n} \leq y + \ln(n))$$

$$= P(X_1 \leq y + \ln(n), \ldots, X_n \leq y + \ln(n))$$

$$= P(X_1 \leq y + \ln(n)) \ldots P(X_n \leq y + \ln(n))$$

$$= (1 + e^{-(y+\ln(n))})^{-1} \ldots (1 + e^{-(y+\ln(n))})^{-1}$$

$$= (1 + e^{-(y+\ln(n))})^{-n}$$

$$= (1 + e^{-y} \cdot e^{-\ln(n)})^{-n}$$

$$= (1 + e^{-y} \cdot e^{\ln(1/n)})^{-n}$$

$$= \left(1 + e^{-y} \cdot \frac{1}{n}\right)^{-n}$$

$$= \left(1 + \frac{e^{-y}}{n}\right)^{-n}$$

as $n \longrightarrow \infty$ $\quad F_{Y_n}(y) \longrightarrow e^{-e^{-y}}$, $\quad y > 1 - \ln(n)$.

$$= \exp(-e^{-y}) = F(y)$$

For support set

dommay notation

$y = x - \ln(n)$

given $x > 1$

so, $x = y + \ln(n)$

$\Rightarrow y + \ln(n) > 1$

$\Rightarrow y > 1 - \ln(n)$

o.k.

3) $Y_n \overset{iid}{\sim} \chi^2(n)$

$\mu = n \longrightarrow$ mean

& $\sigma^2 = 2n \longrightarrow$ varience

$Y_n \sim \chi^2(n)$

$\Rightarrow Y_1 \sim \chi^2(1)$

$Y_2 \sim \chi^2(2)$

$\vdots$

$Y_n \sim \chi^2(n)$

$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \chi^2(1)$

$\sum X_i \sim \chi^2(n)$.

Then, the normal approximation of Y_n is

$$\frac{Y_n - \mu}{\sigma} \xrightarrow{d} N(0,1)$$

$$\Rightarrow \frac{Y_n - n}{2n} \xrightarrow{d} N(0,1)$$

$$\Rightarrow Y_n \sim N(n, 4n^2)$$ X, $n \to \infty$?

Where mean $= n$

Variance $= 4n^2$