// #Homework

Prove $\bar{X}$ and S^2 are Independent

To prove $\bar{X}$ and S^2 are Independent, we will first prove $\bar{X}$ and $X_i - \bar{X}$ are independent where X_i's are iid random variables.

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Consider

$$Y_1 = \bar{X} \qquad\qquad X_1 = Y_1 - (Y_2 + \dots + Y_n)$$

$$Y_2 = X_2 - \bar{X} \quad \Rightarrow \quad X_2 = Y_2 + Y_1$$

$$\vdots \qquad\qquad \vdots$$

$$Y_n = X_n - \bar{X} \qquad\qquad X_n = Y_n + Y_{n-1}$$

$\because$

$$Y_2 + \dots + Y_n = X_2 + \dots + X_n - (n-1)\bar{X}$$

$$= \sum_{i=2}^{n} X_i + n\bar{X} + \bar{X}$$

$$= \sum_{i=2}^{n} X_i + \sum_{i=1}^{n} X_i + \bar{X} \qquad \left(\because \bar{X} = \frac{\Sigma X_i}{n}\right.$$

$$= -X_1 + \bar{X}$$

$$\therefore \quad X_1 = \bar{X} - (Y_2 + \dots + Y_n)$$

$$\Rightarrow X_1 = Y_1 - (Y_2 + \dots + Y_n)$$

Now the Joint distribution of Y_i based on X_i is

$$f_{Y_1,Y_2,\ldots,Y_n}(y_1,\ldots,y_n) = f_{X_1,X_2,\ldots,X_n}\left(y_1-(y_2+\cdots+y_n),\ Y_2+Y_1,\ldots\ Y_n+Y_{n-1}\right)\cdot|J|$$

Where

$$|J| = \det\begin{vmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 1 & 0 & 0 & \cdots & 1 \end{vmatrix} = n$$

So $f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$ $\quad(\because \text{iid})$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\bar{x}+\bar{x}-\mu)^2\right\}$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\left(\sum(x_i-\bar{x})^2+\sum(\bar{x}-\mu)^2 + \underbrace{2\sum(x_i-\bar{x})(\bar{x}-\mu)}_{\to 0}\right)\right\}$$

$$\left(\because \sum 2(x_i-\bar{x})(\bar{x}-\mu) = 2(\bar{x}-\mu)(n\bar{x}-n\bar{x}) = 0\right)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\left(\Sigma(X_i-\overline{X})^2 + n(\overline{X}-\mu)^2\right)\right\}$$

$$\therefore \quad f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left\{\frac{-1}{2\sigma^2}\left[\{-(y_2+\cdots+y_n)^2 + y_2^2 + \cdots + y_n^2\} + n(y_1-\mu)^2\right]\right\} \cdot n$$

$$= C \exp\left[-\frac{1}{2\sigma^2}\{-(y_2+\cdots+y_n)^2 + y_2^2+\cdots+y_n^2\}\right]$$

$$* \exp\left(-\frac{1}{2\sigma^2}\{n(y_1-\mu)^2\}\right)$$

Joint kernel pdf of $Y_2,\ldots Y_n$

kernel pdf of Y_1

So Y_1 indep of $Y_2, Y_3, \ldots Y_n$

given $\overline{X}$ independent of $X_2-\overline{X}, \ldots, X_n-\overline{X}$

also, Now using Lemma

Y_1 independent of function of $(Y_2, \ldots, Y_n)$

{ Lemma:
If U and V be a independent random variable. Then $g(U)$ and $g(V)$ are independent }

$\Rightarrow$ Y_1 indep of $X_1-\overline{X}$ also

Thus $\overline{X}$ is independent of $X_i-\overline{X}$ $\forall i=1,2,\ldots,n$

Also, since sample variance

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \overline{x})^2}{n-1}$$ which is also the function of $x_i - \overline{x}$

Hence $\overline{x}$ and s^2 are independent. $\square$

not for grading

Ask her.: Does this works?

To prove $\overline{X}$ and S are Independent

To prove this I will use Lemma

Lemma:
If U and V be a independent random variable. Then $g(U)$ and $g(V)$ are independent

Let us suppose $U = (X_2 - \overline{X}, X_3 - \overline{X}, \dots, X_n - \overline{X})$

$$V = \overline{X}$$

To prove our theorem using this Lemma, we need to show two things

1.> U and V are independent

2.> Sample Variance $S^2 = g(U)$ i.e sample variance is function of U only.

Step-I

To Show $S^2 = g(U)$.

we Know

$$S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1}$$

$$= \frac{1}{n-1}\left[(X_1 - \overline{X})^2 + \sum_{i=2}^{n} (X_i - \overline{X})^2\right] \quad \text{---} \ *$$

also we Know

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{1}{n}\left(X_1 + \sum_{i=2}^{n} X_i\right)$$

$$\Rightarrow X_1 = n\overline{X} - \sum_{i=2}^{n} X_i$$

$$\Rightarrow X_1 - \overline{X} = n\overline{X} - \sum_{i=2}^{n} X_i - \overline{X}$$

$$= (n-1)\overline{X} - \sum_{i=2}^{n} X_i$$

$$= \sum_{i=2}^{n} \overline{X} - \sum_{i=2}^{n} X_i \qquad \left\{ \because \sum_{i=2}^{n} \overline{X} = (n-1)\overline{X} \right\}$$

$$= \sum_{i=2}^{n} (\overline{X} - X_i)$$

So

$$(X_1 - \overline{X})^2 = \left[\sum_{i=2}^{n} (X_i - \overline{X}) \right]^2 \qquad (\because (-y)^2 = y^2$$

Thus eq^n * becomes

$$S^2 = \frac{1}{n-1} \left\{ \left[\sum_{i=2}^{n} (X_i - \overline{X}) \right]^2 + \sum_{i=2}^{n} (X_i - \overline{X})^2 \right\}$$

$$= g(X_2 - \overline{X}, X_3 - \overline{X}, \ldots, X_n - \overline{X})$$

Hence S^2 is function of U i.e $S^2 = g(U)$.

Step-2

To show $\bar{X}$ and $X_i-\bar{X}$ are independent

For showing $\bar{X}$ and $X_i-\bar{X}$ are independent, we need to show

$$Cov(\bar{X}, X_i-\bar{X})=0$$

Since $$Cov(U,V)=E[(U-E(U))(V-E(V))]$$
$$=E(UV)-E(U)E(V)$$

So $$Cov(\bar{X}, X_i-\bar{X}) = E[\bar{X}(X_i-\bar{X})] - \cancel{E(\bar{X})E(X_i-\bar{X})}^{0}$$

$$= E[\bar{X}(X_i-\bar{X})]$$ $(\because E(X_i)=\mu$ & $E(\bar{X})=\mu)$

$$= E(\bar{X}X_i) - E(\bar{X}^2) \quad \text{——} **$$

For $i \neq j$, all X_i and X_j are independent

$\boxed{X_i\text{'s are indep given}}$

So $$E(\bar{X}X_i) = \frac{1}{n}\left[(n-1)\mu^2 + E(X_i^2)\right]$$

$$= \frac{1}{n}\left[(n-1)\mu^2 + \sigma^2 + \mu^2\right]$$ $\because E(X_i^2)=\sigma^2+\mu^2$ by $Var(X_i)=E(X_i^2)-\{E(X_i)\}^2$

$$= \frac{1}{n}[n\mu^2+\sigma^2]$$

$$= \mu^2 + \frac{\sigma^2}{n}$$

And $E(\bar{X}^2) = Var(\bar{X}) + \{E(\bar{X})\}^2$

$= \frac{\sigma^2}{n} + \mu^2$

$\because E(\bar{X}) = \mu$

$Var(\bar{X}) = Var\left(\frac{\sum X_i}{n}\right)$

$= \frac{1}{n^2} \sum (Var X_i)$

$= \frac{1}{n^2} n . \sigma^2$

$= \frac{\sigma^2}{n}$

So from eqn $**$

$$Cov(\bar{X}, X_i - \bar{X}) = \mu^2 + \frac{\sigma^2}{n} - \left(\frac{\sigma^2}{n} + \mu^2\right)$$

$$= 0$$

$\Rightarrow$ $\bar{X}$ and $X_i - \bar{X}$ are Independent.

Thus, $\bar{X}$ and S^2 are Independent (by Lemma). $\square$