Structural Models

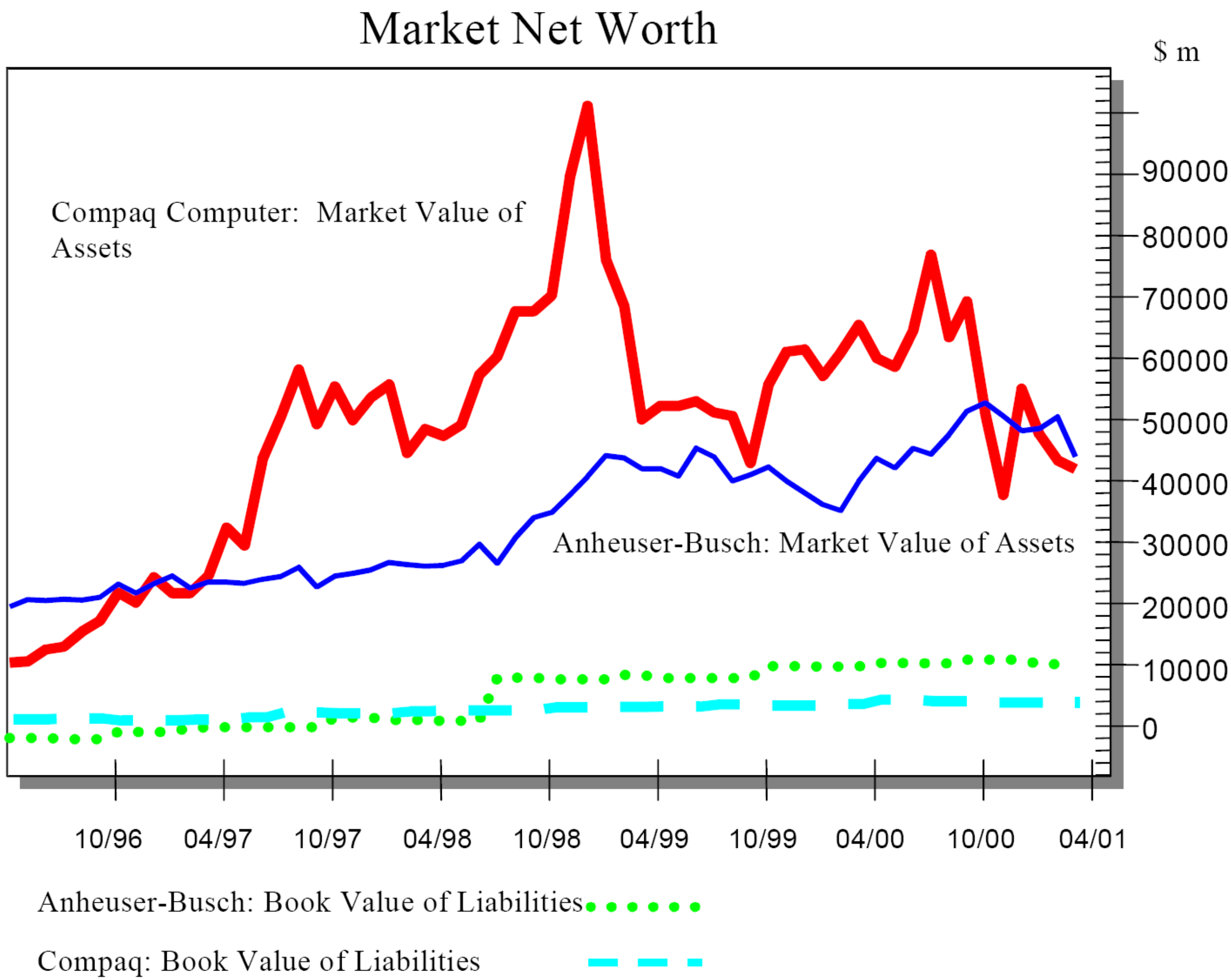


FIGURE 2 Evolution of asset values and default points for Compaq and Anheuser-Busch

Source: Moody's-KMV

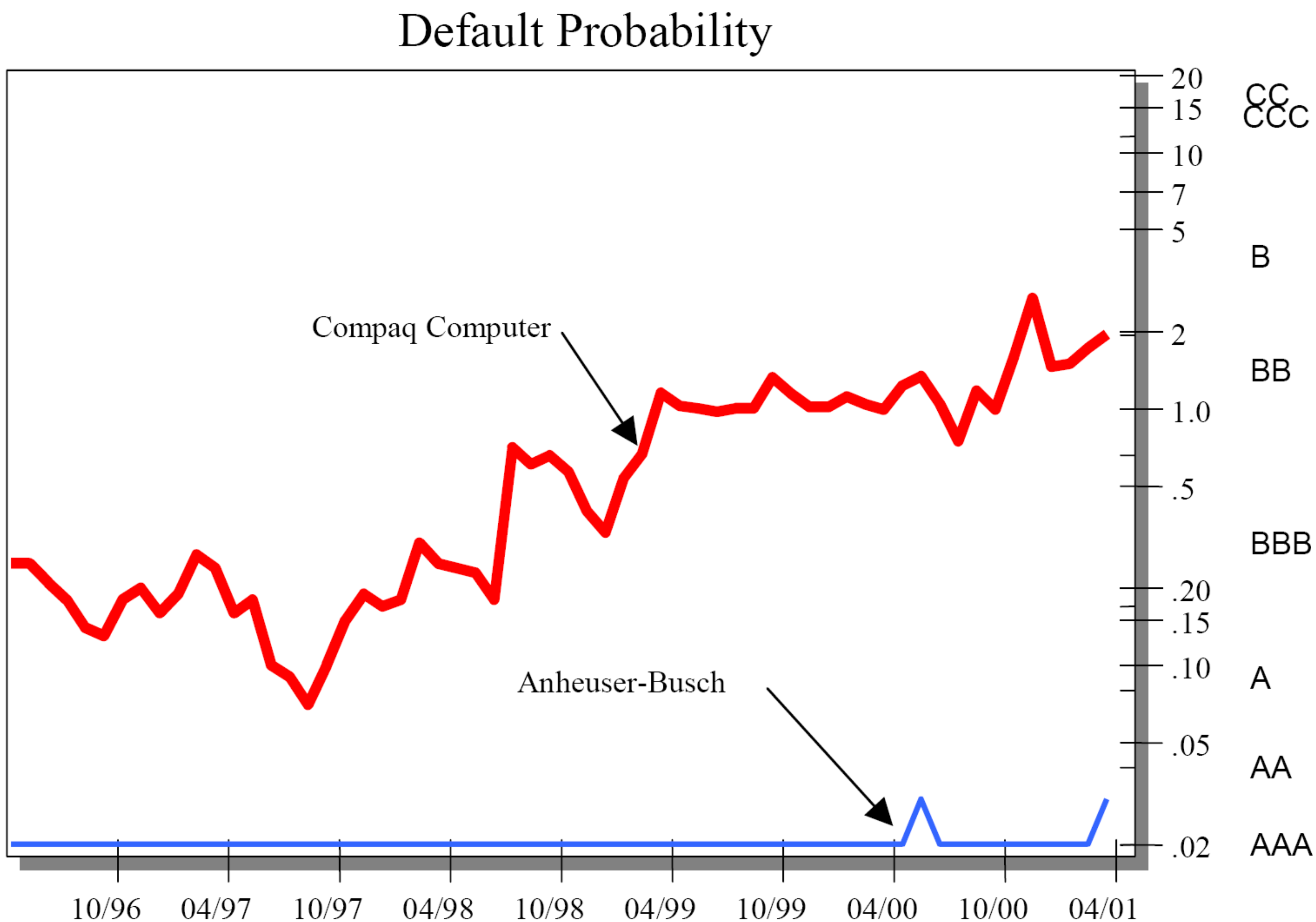


FIGURE 3 Corresponding evolution of the annual default probabilities

Source: Moody's-KMV

What do we learn from these plots?

- The volatility of a firm's assets is a major determinant of its risk of default
- But how do we estimate it?
- We will show how an understanding of option pricing models will be directly applicable in finding default probabilities ...
- Such an approach is known as "structural"

Firm Value and Volatility

- Structural models depend on ***value*** and ***volatility*** of ***firm assets***
  - ✓ neither is directly observable

- Value of ***equity*** = stock price *x* number of shares

- Problem is the value of ***debt***
  - ✓ which ***debt to include***? (e.g., should some short-term debt be netted against short-term assets)
  - ✓ what is ***value of debt***? only part of total debt will be traded.
  - ✓ total borrowings observed only in periodic ***accounts***.

The Black-Scholes-Merton Option Pricing Model

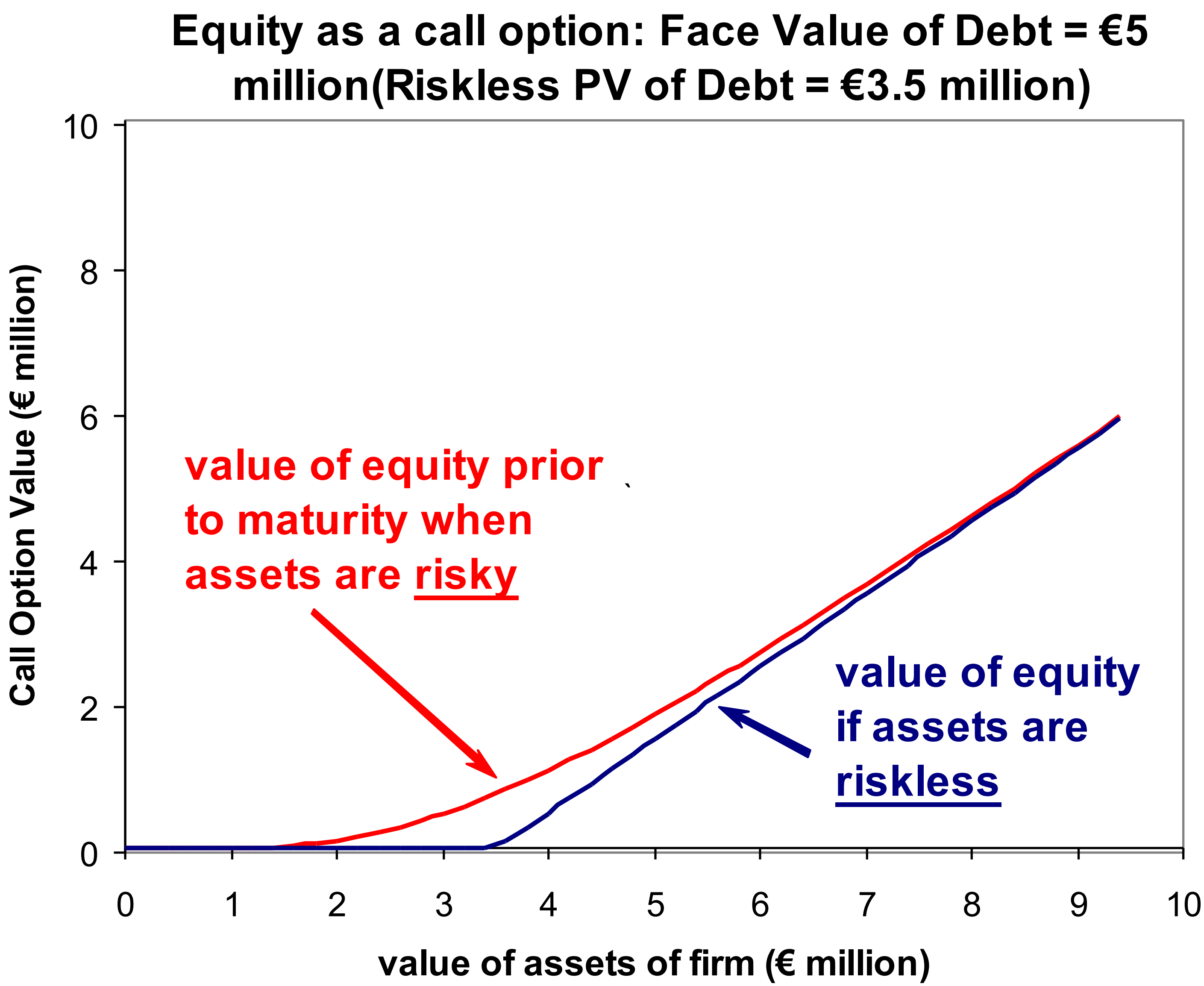
"… options are specialized and relatively unimportant financial securities …".

Robert Merton – Nobel prize winner for work on option pricing – in 1974 seminal paper on option pricing:

- Great hope for the new theory was the valuation of corporate liabilities, in particular
  - ✓ equity
  - ✓ **corporate debt**

Equity is a call option on the firm

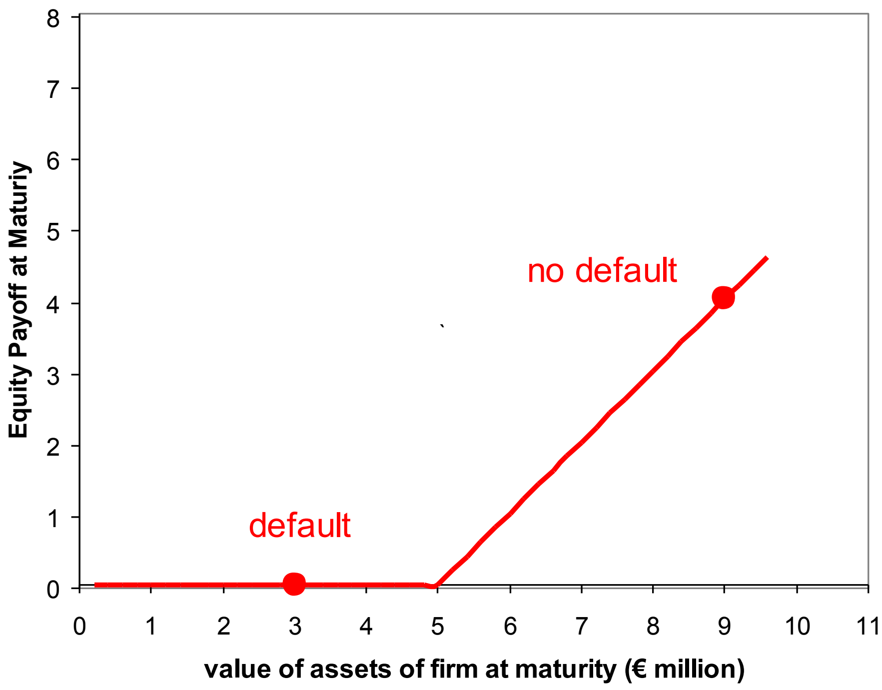
- Suppose a firm has borrowed ***€5 million*** (zero coupon) and that at the time the loan (5 years, say) is due
- ***Scenario I:*** the ***assets of the firm are worth €9 million***:
  - ✓ lenders get €5 million (paid in full)
  - ✓ equity holders get residual: €9 - €5 = ***€4 million***
- ***Scenario II:*** the ***assets are worth, say, €3 million***
  - ✓ firm defaults, lenders take over assets and get ***€3 million***
  - ✓ equity holders receive zero (Limited liability)
- Payments to equity holders are those of a call option written on the assets of the firm with a strike price of ***€5 million***, the face value of the debt


Equity as a call option: Face Value of Debt = €5 million(Riskless PV of Debt = €3.5 million)
Call Option Value (€ million)
0
2
4
6
8
10
0
1
2
3
4
5
6
7
8
9
10
value of assets of firm (€ million)
value of equity prior to maturity when assets are risky
value of equity if assets are riskless

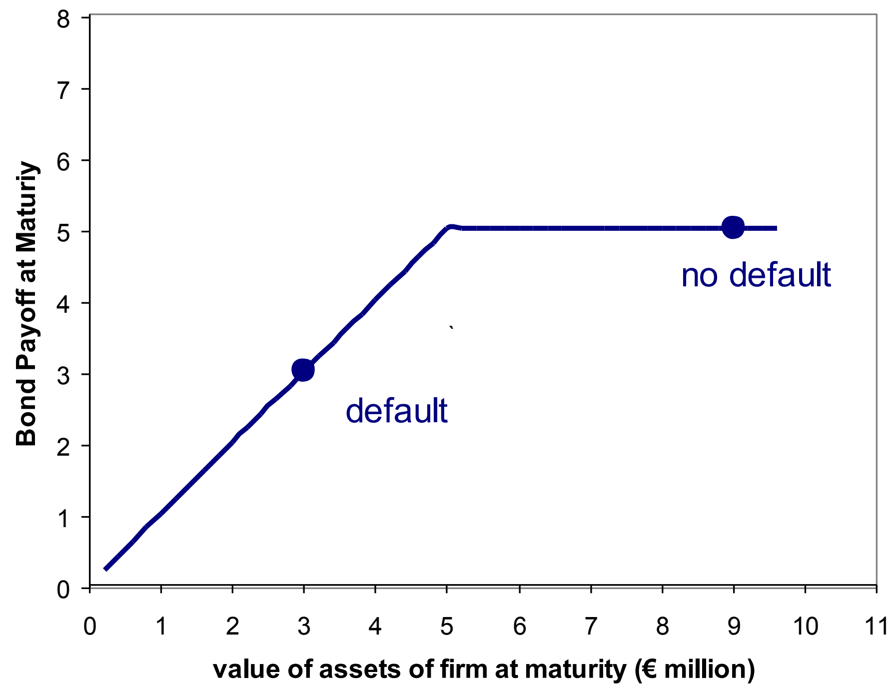
Payoffs to Debt and Equity *at Maturity*

- Firm has single 5-year zero-coupon bond outstanding with face value *B=5 (million)*

Equity is a ***call option*** on the ***assets of the firm***

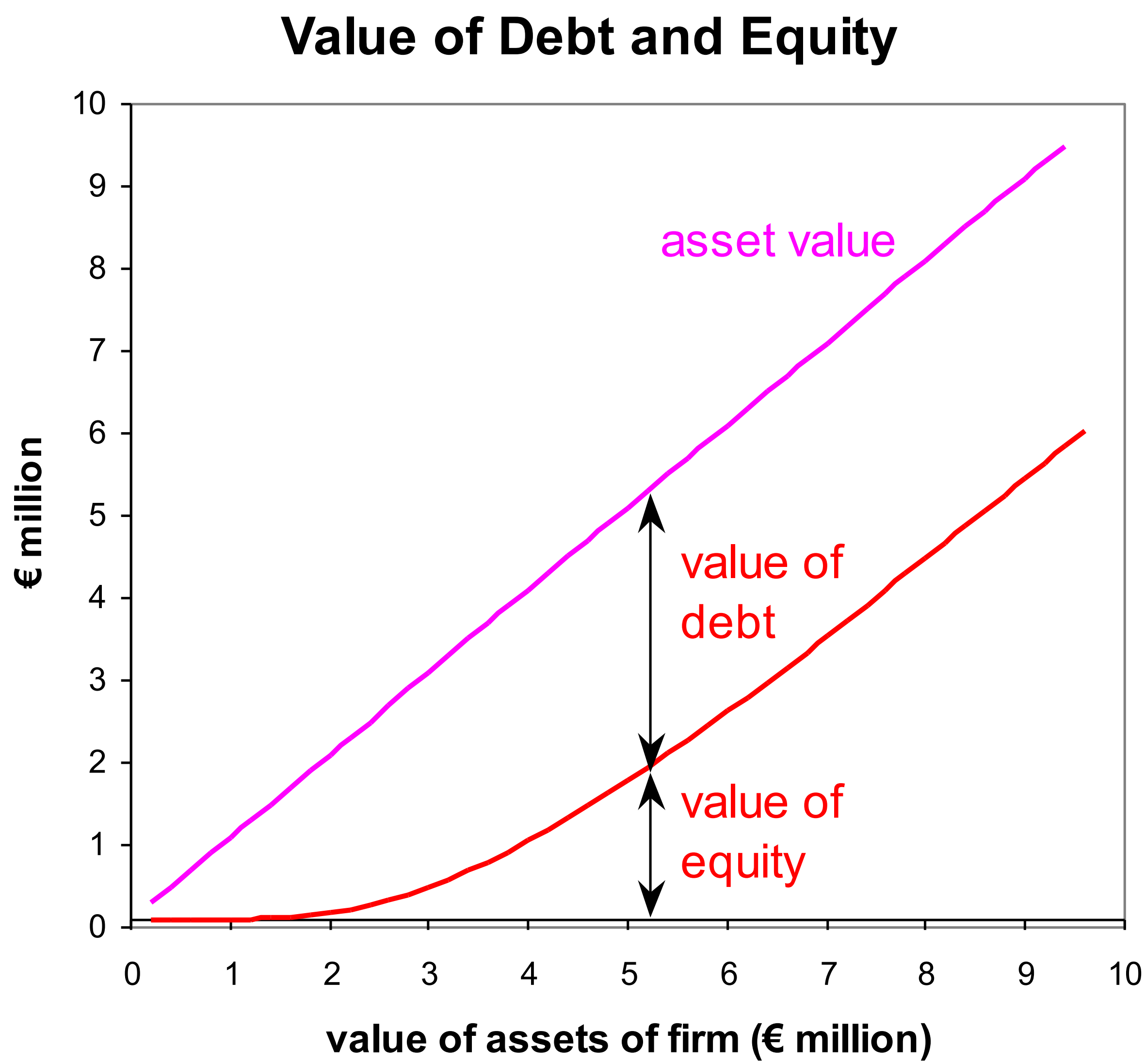


Payoff on ***risky debt*** looks like this

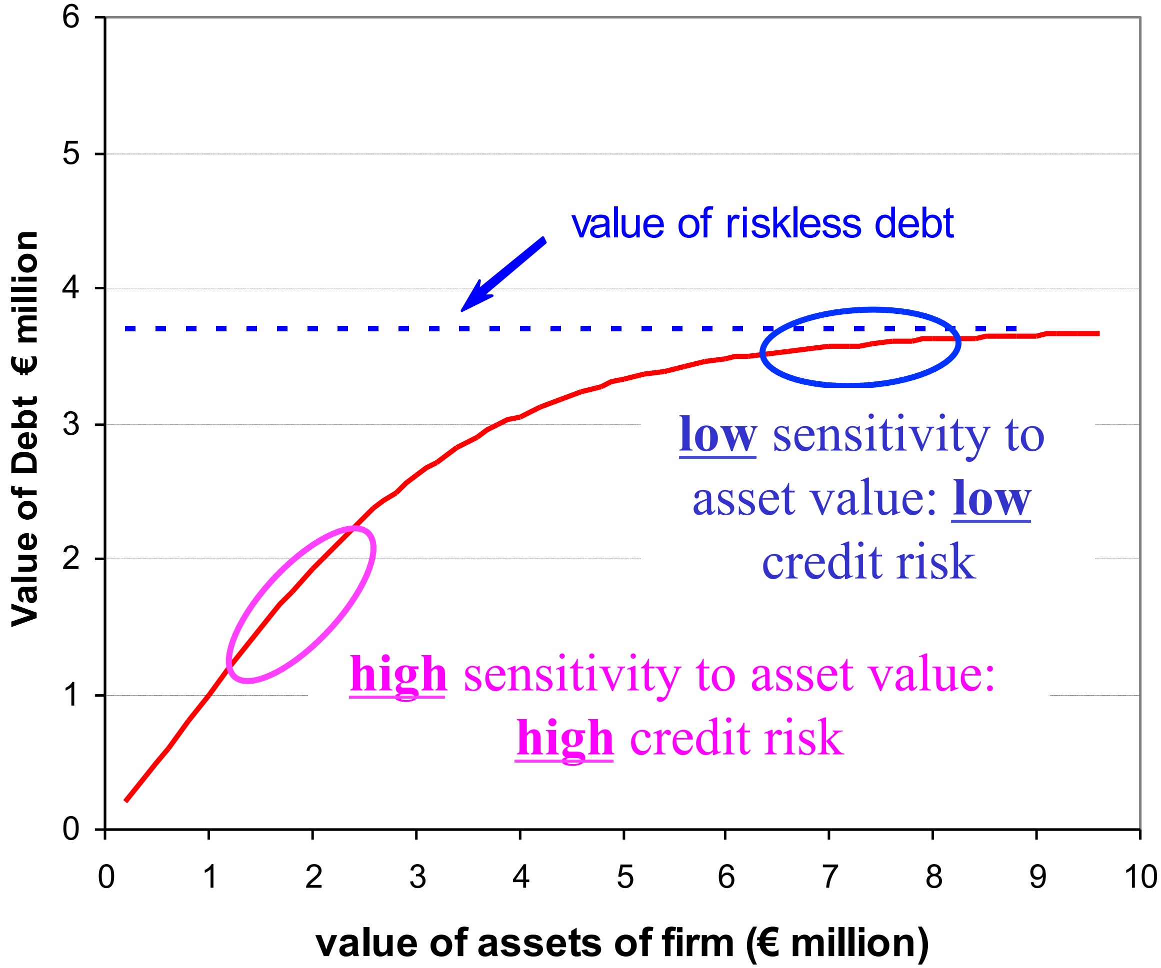


Prior to maturity …

- value of the debt is value of firm's assets less the value of the equity (a call)



Value of Debt and Credit Exposure



What is the price discount on credit risky debt?

put-call parity	underlying asset	=	riskless bond	-	put option	+	call option
Modigliani-Miller	value of firm assets	=	bond value			+	equity value

- Since equity is a ***call option***

value of risky debt	=	riskless bond	-	value of put option on assets

- ***Merton model*** uses Black-Scholes to value the (default) put.

Limited Liability and the "Default Put"

- ***Limited liability*** of equity means that no matter how bad things get, equity holders can walk away from firm's debt in exchange for payoff of zero

- ***Limited liability*** equivalent to equity holders:
  - ✓ issuing ***riskless debt***

  <u>BUT</u>

  - ✓ lenders giving equity holders a ***put*** on the ***firm's assets*** with a ***strike*** price equal to the ***face amount*** of the debt (***"default put"***)

Two Approaches

- Merton model (direct application of Black-Scholes) to valuing zero-coupon risky debt
  - ✓ Default only at Maturity

- MKMV Approach
  - ✓ A sketch of how the approach works
  - ✓ Examples

The Merton Model

Risky debt as an option

A key insight of the Merton model is that risky debt in the firm may be valued by treating it as an option on economic firm value. This may be analyzed formally.

On date T, when the debt matures, debt holders are repaid the face value of their bonds, i.e. the amount D. However, if the assets of the firm are insufficient to meet this amount, then they only receive the value of the assets, i.e. the amount V_T. Hence, at maturity, debt holders receive

$$\begin{cases} D, & \text{if } V_T \geq D \\ V_T, & \text{otherwise} \end{cases} \tag{32.1}$$

Expressing this in shorthand notation, debt holders receive $\min\{V_T, D\}$. Equivalently, this may be rewritten as

$$D - \max\{D - V_T, 0\}.$$

That is, it is as if debt holders hold a portfolio that is

- *Long* a default-risk-free bond paying D at time T (the first part of the formula above).
- *Short* a put option on the firm's assets with strike D and maturity T (the second part of the equation above).

Implications

Thus, the economic value of the firm's debt may be determined by identifying the value of the risk-free bond (denoted B, say, with a zero-coupon based principal in the amount D) and the put (denoted P, say, with a strike price of D, based on the underlying firm value V).

In particular, the *spread* on the risky debt is determined by the value of the put P: A higher value of P increases the price difference between the risky and riskless bonds, increasing the spread. The elegance of the model derives from the ease with which we can make statements about the underlying drivers of credit spreads. Thus, for example:

- The spread increases as volatility of the firm value process increases. This happens because the value of the put P increases as volatility increases. Therefore, if the managers of the firm indulge in riskier business strategies, thereby raising firm volatility, the value of the put P will increase, with a corresponding decrease in debt value.

- The spread decreases with an increase in the risk-free rate of interest. This follows from the fact that in the Black-Scholes model, put values are inversely related to interest rates. Therefore, the Merton model suggests that interest rates and credit spreads in the economy should be negatively correlated.

Valuing the default put

Suppose as in Merton (1974) that the Black and Scholes (1973) conditions hold, that is:

1. V_t follows a geometric Brownian motion process with constant volatility σ.
2. The risk-free rate of interest is a constant r.

Under these assumptions, any difference in prices between the risky and riskless bonds is solely on account of default and recovery risk. In the Black-Scholes setting, where compounding and discounting are undertaken in continuous time, the value of the risk-free bond at date t is

$$B = e^{-r(T-t)}D.$$

Moreover, using the Black-Scholes formula, the value of the put option is:

$$P = e^{-r(T-t)}D \cdot N\left(-d + \sigma_v\sqrt{T-t}\right) - V_t \cdot N(-d),$$

where

$$d = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{1}{L}\right) + \frac{1}{2}\sigma^2(T-t)\right],$$

and

$$L = \frac{e^{-r(T-t)}D}{V_t}$$

Value of Risky Debt

Now, let B^* denote the price of the risky bond. Since $B^* = B - P$

$$B^* = e^{-r(T-t)}D \cdot N\left(d - \sigma\sqrt{T-t}\right) + V_t \cdot N(-d)$$

$$\begin{aligned} B^* &= e^{-r(T-t)}D \cdot \left[N\left(d - \sigma\sqrt{T-t}\right) + \frac{1}{L} \cdot N(-d)\right], \\ L &= [e^{-r(t-t)}D]/V_t \end{aligned}$$

The value of debt declines as leverage (L) increases.

Example

Input Variable	Value
Firm Value (V_t)	100
Face value of zero-coupon debt (D)	60
Maturity ($T-t$)	1 (year)
Volatility of firm value (σ)	0.30
Riskfree interest rate for the given maturity (r)	0.10

Riskless debt $$B = D.\exp[-r(T-t)] = e^{-0.10\times 1} = 54.29025.$$

Risky debt $$B^* = e^{-r(T-t)}D \cdot N\left(d - \sigma\sqrt{T-t}\right) + V_t \cdot N(-d)$$

$$B^* = 54.12146.$$

The difference is the value of the put option.

Term Structure of Credit Spreads

Let R denote the continuously compounded yield on the risky bond:

$$R = -\left(\frac{1}{T-t}\right)[\ln(B^*)].$$

The yield on the riskless bond is, of course, just the riskless rate r.

Using (32.7), it is easily seen that the *credit risk premium* $R-r$ is given by

$$R - r = -\left(\frac{1}{T-t}\right) \cdot \ln\left[N\left(d - \sigma\sqrt{T-t}\right) + \frac{1}{L} \cdot N(-d)\right].$$

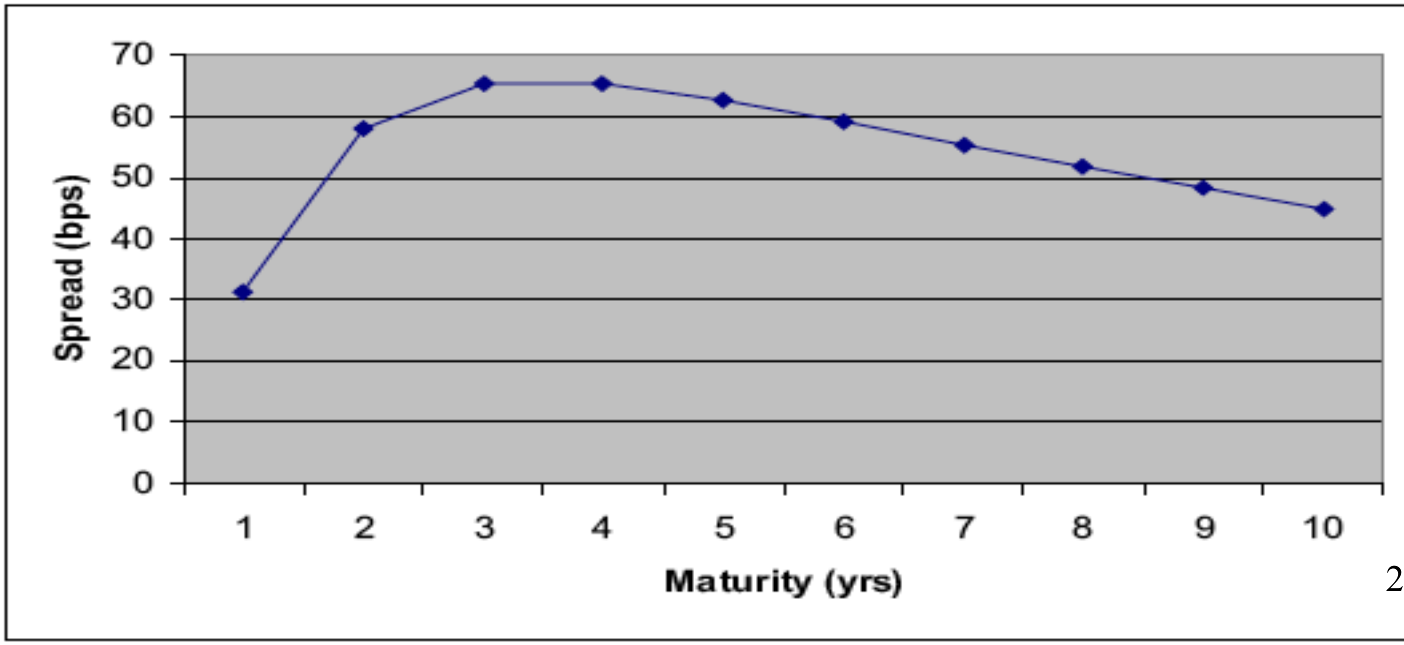
Spreads depend on:

1. The time to maturity $T-t$.
2. The volatility of firm value σ.
3. The ratio $L = [e^{-r(T-t)}D]/V_t$ (and, therefore, on r, D, and V_t).

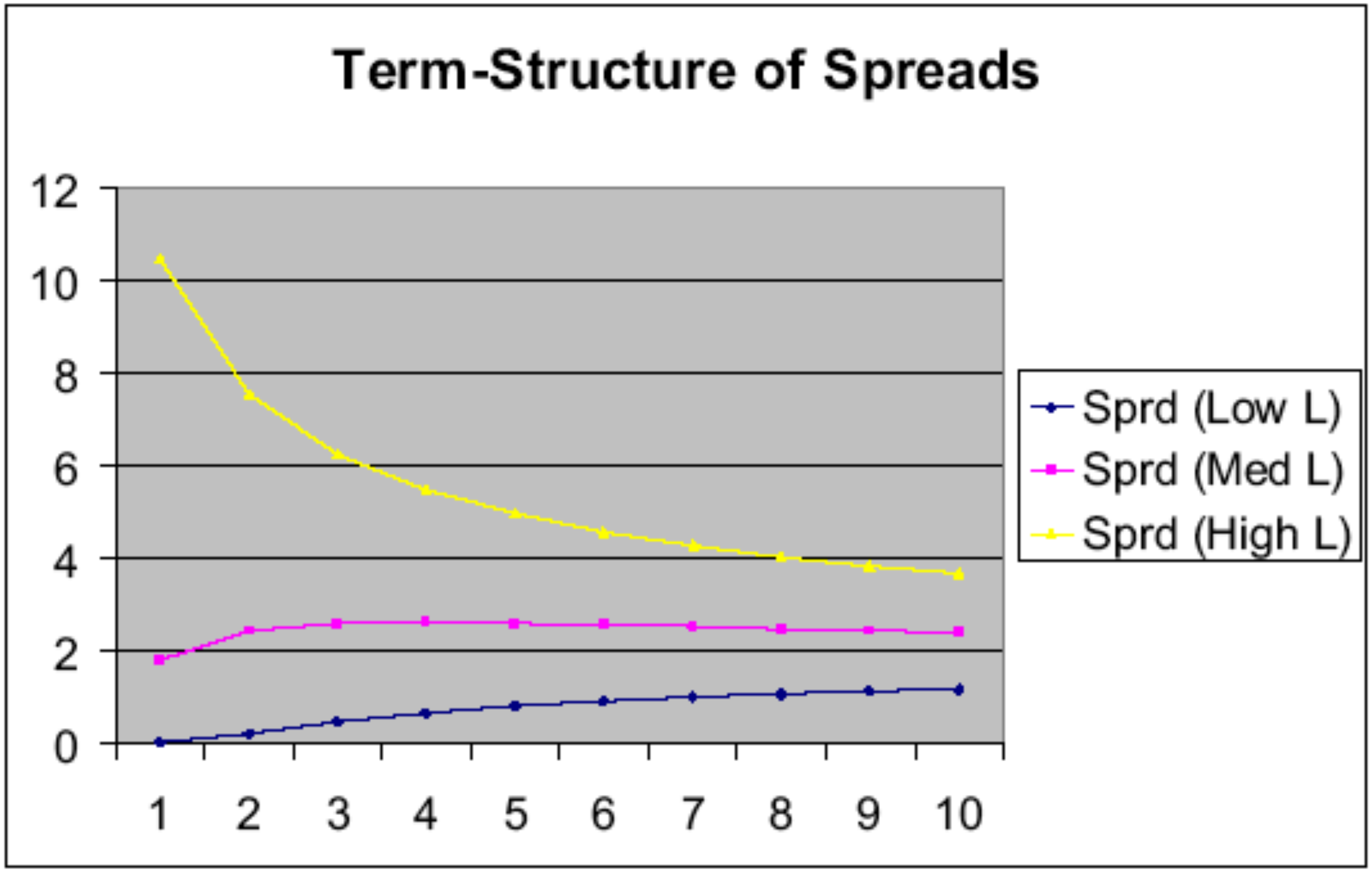
Term Structure of Spreads (data as before)

T	Riskless Debt	Risky Debt	Spread (bps)
1	54.2902	54.1215	31.1387
2	49.1238	48.5562	58.1090
3	44.4491	43.5873	65.2647
4	40.2192	39.1835	65.2249
5	36.3918	35.2708	62.5788
6	32.9287	31.7827	59.0387
7	29.7951	28.6639	55.2948
8	26.9597	25.8687	51.6363
9	24.3942	23.3590	48.1810
10	22.0728	21.1021	44.9705

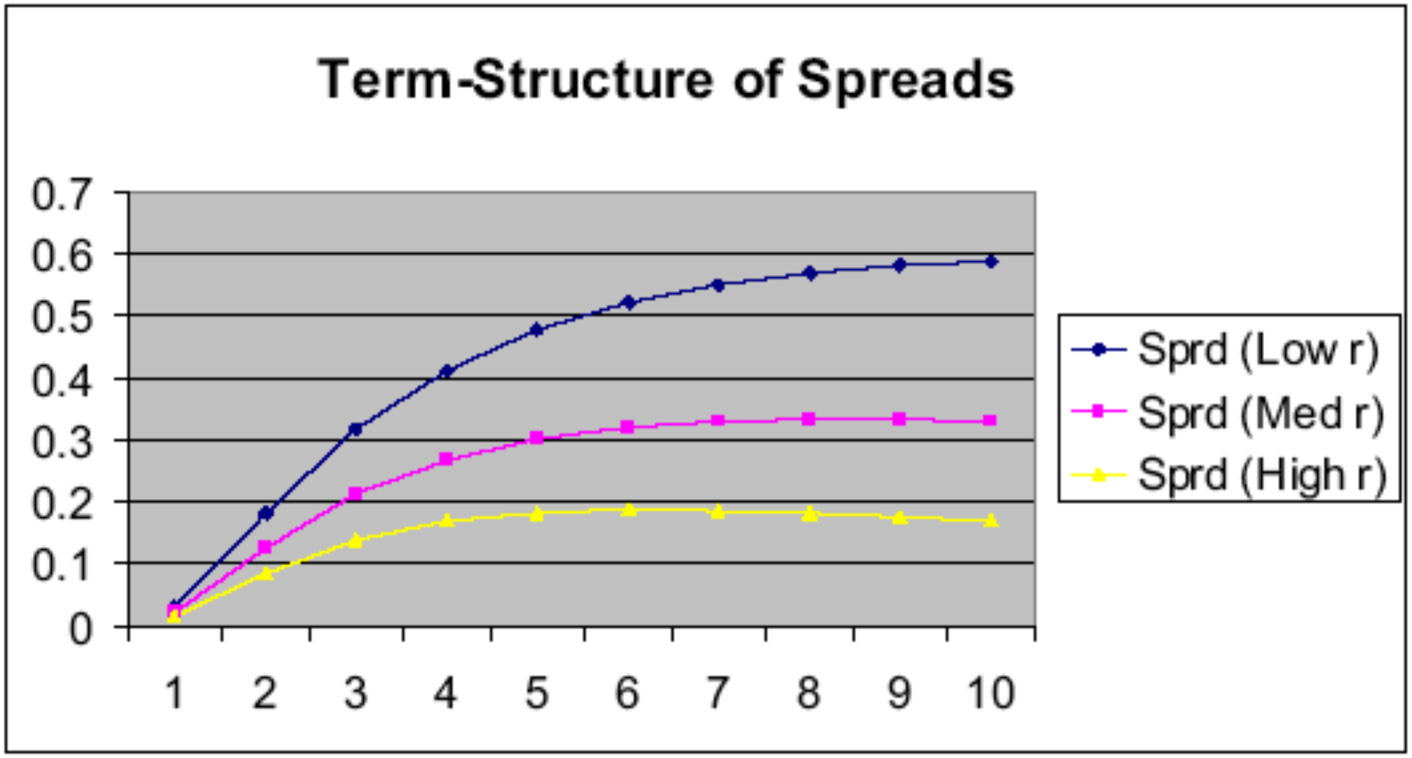
The term structure of credit spreads is plotted against maturity in the following figure.



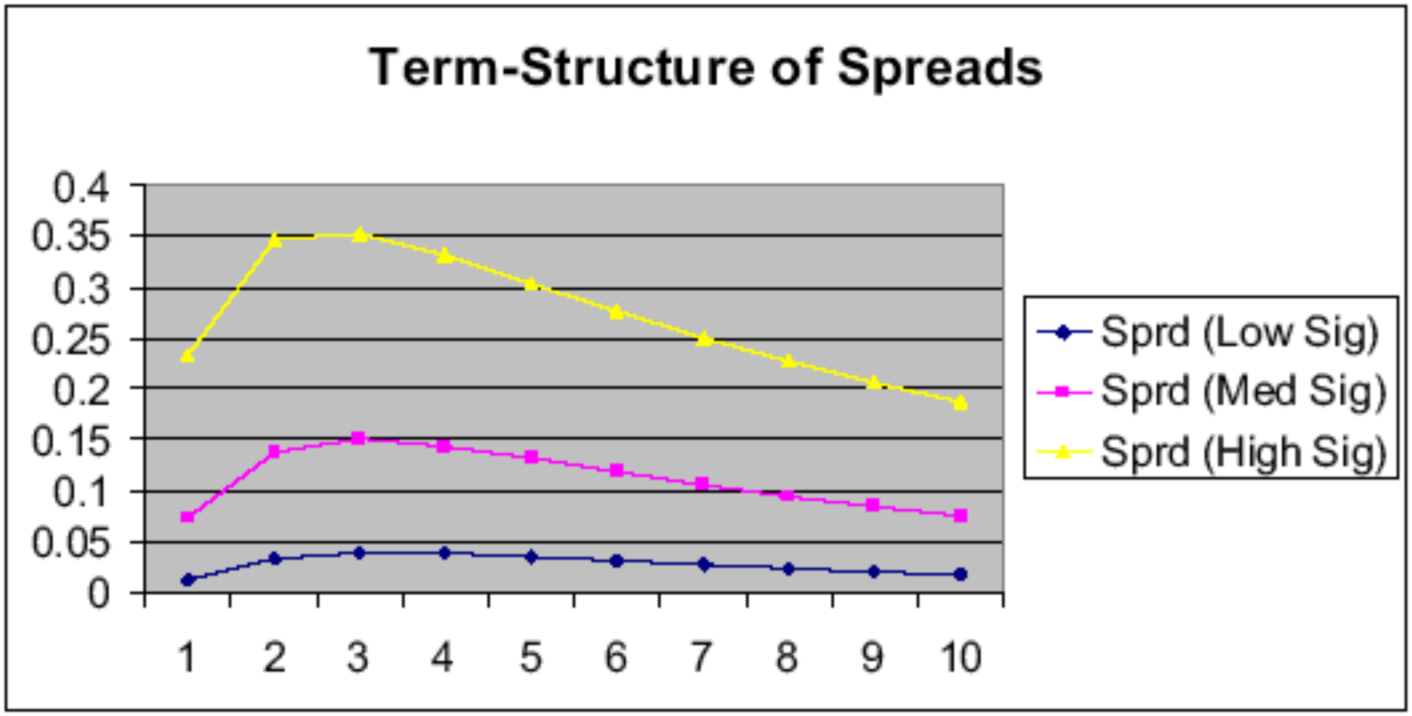
Leverage


Term-Structure of Spreads
12
10
8
6
4
2
0
1
2
3
4
5
6
7
8
9
10
Sprd (Low L)
Sprd (Med L)
Sprd (High L)

Interest rates



Volatility


Term-Structure of Spreads
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5
6
7
8
9
10
Sprd (Low Sig)
Sprd (Med Sig)
Sprd (High Sig)

Implementation Issues

1. V and σ are *unobservable*. Both values were assumed to be known in the numerical examples.

2. The model assumes that the firm has a single issue of zero-coupon debt outstanding. No doubt, this is a trivialization of reality. When undertaking practical implementation, we would like to extend this to complex debt structures.

Issue 1: solve 2 equations

$$E_t = f(V_t, \sigma) \qquad f(V_t,\sigma) = V_t \cdot N(d) - e^{-r(T-t)} D \cdot N\left(d - \sigma\sqrt{T-t}\right),$$

$$\sigma_E = g(V_t, \sigma). \qquad g(V_t,\sigma) = \sigma V_t \cdot \frac{f_V}{f}$$

Example:

$$E_t = 45.88, \quad \sigma_E = 0.6445.$$

$$V_t = 100, \text{ (firm value per share)}$$
$$\sigma = 0.30.$$

$$B_t^* = V_t - E_t = 100 - 45.88 = 54.12. \qquad B = e^{-r(T-t)} = 54.29$$

$$\text{Spread} = R - r = -\frac{1}{T-t}\ln\left[\frac{B_t^*}{B_t}\right] = 0.003114, \quad (31.14 \text{ bps}).$$

Risk-neutral default probability

$$\mathsf{Prob}[V_T < D] = \mathsf{Prob}\left\{V_t \exp\left[(r - \frac{1}{2}\sigma^2)(T-t) + \sigma\epsilon\sqrt{T-t}\right]\right\}$$

$$= \mathsf{Prob}\left\{\epsilon < \frac{\ln\left(\frac{D}{V_t}\right) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(}T-t)}\right\}$$

$$= N\left\{\frac{\ln\left(\frac{D}{V_t}\right) - (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(}T-t)}\right\}$$

$$\mathsf{Prob}[V_T < 60] = N\left\{\frac{\ln\left(\frac{60}{100}\right) - (0.10 - \frac{1}{2}0.3^2)(1)}{0.3\sqrt{(}1)}\right\} = 0.029642.$$

This results in a *risk-neutral* default probability of 2.96%.

Default probability under the real-world measure

Suppose the growth rate of the firm is 20%, i.e. $\mu = 0.2$. If we replace r with μ in the equation above we may compute the actual default probability:

$$\text{Prob}[V_T < 60] \;\; (\text{physical measure}) = N\left\{ \frac{\ln\left(\frac{D}{V_t}\right) - (\mu - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \right\}$$

$$\text{Prob}[V_T < 60] \;\; (\text{physical measure}) = 0.013229.$$

Hence, the true physical default probability is lower, and is equal to 1.32%, less than half that under the risk-neutral probability.

Finally, where do we obtain the parameter for the return on firm value? This is extracted using the equity return (r_E, determined from the usual models), and then unlevering it to determine firm value returns. The usual unlevering equation applies, i.e. $\mu = r_E \times \frac{E}{V}$. We note that this unlevering is only approximate; with more detailed financial statement information it may be made more precise, and some versions of the unlevering equation will also account for tax effects. For example, an extended equation that accounts for the cost of debt r_D and corporate taxes τ_c is as follows: $\mu = r_E \frac{E}{V} + r_D(1 - \tau_c)\frac{D}{V}$.

Issue 2: More complex capital structures

The Delianedis-Geske model

We now describe the DG model formally. There are two tranches of zero-coupon debt, face value D_1 with maturity T_1 and face value D_2 with maturity T_2, and $T_2 > T_1$. The values of debt are denoted B_1 and B_2 respectively, and we are interested in computing the prices of these two tranches of debt.

At the first maturity date T_1, the firm is solvent if

$$V_{T_1} > D_1 + B_{2,T_1},$$

and it can then refinance the first tranche of debt with equity. The same model may be implemented by assuming that refinancing is not permitted, but this would be less realistic, as it would adversely impact the second tranche of debt. Thus, the condition above defines a critical cut-off value V^* for firm value at T_1, which is analogous to the strike price of the first option in the compound option. Therefore,

$$V^* = D_1 + B_{2,T_1}.$$

The strike price for the second option at date T_2 is just the face value of the second debt tranche, i.e. D_2.

Delianedis-Geske Solution

$$\begin{aligned} E_t &= V_t N_2[d_1 + \sigma\sqrt{T_1 - t}, d_2 + \sigma\sqrt{T_2 - t}; \rho] \\ &\quad - D_2 e^{-r(T_2-t)} N_2[d_1, d_2; \rho] - D_1 e^{-r(T_1-t)} N(d_1), \\ \rho &= \sqrt{\frac{T_1 - t}{T_2 - t}}, \end{aligned}$$

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{V_t}{V^*}\right) + (r + \frac{1}{2}\sigma^2)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \\ d_1 &= \frac{\ln\left(\frac{V_t}{D_2}\right) + (r + \frac{1}{2}\sigma^2)(T_2 - t)}{\sigma\sqrt{T_2 - t}}. \end{aligned}$$

$$\sigma_E = \frac{\partial E}{\partial V}\frac{V}{E}\sigma.$$

DG provide the three *risk-neutral* probabilities from their model as follows:

$$\begin{aligned} \text{Total default probability} &= TRNPD = 1 - N_2[d_1, d_2; \rho] \\ \text{Short term probability} &= RNPD_1 = 1 - N(d_1) \\ \text{Long term probability} &= RNPD_2 = 1 - \frac{N_2[d_1, d_2; \rho]}{N(d_1)} \end{aligned}$$

Example of the DG model

$$E_t = 54.73, \quad \sigma_E = 0.5425.$$

$$D_1 = 30$$

The long-term debt is of maturity five years, and is of face value $D_2 = 30$

The risk free interest rate is $r = 0.10$.

Solution:

$$\begin{aligned} V_t &= 100 \\ \sigma &= 0.30 \\ V^* &= 49.57689. \end{aligned}$$

$$\begin{aligned} TRNPD &= 1 - N_2[d_1, d_2; \rho] = 0.0186 \\ RNPD_1 &= 1 - N(d_1) = 0.0058 \\ RNPD_2 &= 1 - \frac{N_2[d_1, d_2; \rho]}{N(d_1)} = 0.0129. \end{aligned}$$

Solution 2: Simplifying Reality

set the zero-coupon equivalent level D to be a point that is the sum of

- the face-value of all short-term liabilities, and
- a fraction of the face-value of all longer-term liabilities.

Distance to default: volatility normalized measure of leverage

$$\delta = \frac{V_t - D}{\sigma V_t}.$$

1. Market Value of Equity: \$33.9 billion.
2. Book Value of Liabilities: \$7.2 billion.
3. Estimated Market Value of Firm: \$41.3 billion.
4. Estimated Volatility of Firm Value: 20%.
5. Estimated Default Point: \$5.7 billion.

$$\frac{41.3 - 5.7}{0.20 \times 41.3}$$

Given this data, we compute:

1. The distance to default is \$41.3 - \$5.7 billion. Therefore, the distance to default = \$35.6 billion = 4.3 standard deviations. This is computed as follows

The MKMV Model

The MKMV Model

- The MKMV model uses a four-step procedure to track changes in credit risk for publicly-traded firms:

1. Identify the ***default point*** B to be used in the computation.
   - Set to Short-term liabilities + 0.5 * Long-term liabilities
2. Use the default point in conjunction with the firm's equity value and equity volatility to identify the firm value V and the firm volatility $\sigma_{V.}$
3. Given these quantities, identify the number of standard deviation moves that would result in the firm value falling below B. This is the firm's ***distance-to-default*** .
4. * Use its database to identify the proportion of firms with distance-to-default who actually defaulted within a year. This is the ***expected default frequency*** or EDF.

The Distance-to-Default

$$\text{default value for } \varepsilon = \frac{\ln(B/V) - (\mu - \frac{1}{2}\sigma_V^2)T}{\sigma_V \sqrt{T}}$$

- In the Merton model, default occurs when the "surprise" term, ε, is large enough (typically a large ***negative*** number). ***What does this number mean?***
- In the numerator, *ln(B/V)* is the ***actual*** continuously compounded return on the assets that is necessary to lead to default.
  - ✓ if $V > B$, this return is negative (i.e., the asset value must fall to lead to default).
- The term $(\mu - \sigma^2/2)T$ is the ***expected value*** of the continuously compounded return (usually positive)
- Thus the numerator is the difference between the actual continuously compounded rate of return required for default and the expected value of the return, i.e., it is the "***surprise***", or unexpected component of the rate of return necessary for default.
- The ***denominator*** is the standard deviation of the rate of return

The Distance-to-Default, contd.

$$\text{default value for } \varepsilon\% = \frac{\ln(B/V) - (\mu - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$$

- Therefore, the ***ratio*** (again typically negative) measures the number of standard deviations of return necessary to lead to default at time T
- ***The negative of this ratio (a positive number) is called the distance-to-default***

$$\text{Distance-to-Default} = \frac{\ln(V/B) + (\mu - \frac{1}{2}\sigma_V^2)T}{\sigma_V\sqrt{T}}$$

* Note: Sometimes, the term “distance to default” is applied to other, closely related, quantities

Merton Model Distance to Default and Default Probabilities

- Distance to default is smaller (and default probability higher) when volatility is higher and maturity is longer

Distance-to-Default*

		V			
Vol	***T***	***150***	***100***	***80***	***60***
20%	1	5.89	3.87	2.75	1.31
20%	20	3.02	2.56	2.31	1.99
40%	1	2.80	1.78	1.23	0.51
40%	20	0.84	0.61	0.49	0.33

Default Probabilities*

		V			
Vol	***T***	***150***	***100***	***80***	***60***
20%	1	0.00%	0.01%	0.30%	9.48%
20%	20	0.13%	0.52%	1.03%	2.31%
40%	1	0.26%	3.73%	11.03%	30.65%
40%	20	20.11%	27.06%	31.34%	37.24%

*Note: Assumptions - expected return on assets = 10%; face value of debt = 50

Distance-to-Default and EDF

- In principle, we should be able to compute default frequencies using

  - the distance-to-default, and

  - the probability distribution governing the evolution of V.

- However, it turns out that under the usual assumptions, this method ***underpredicts*** defaults by a large margin.

  - It is typical to assume ***normality*** in returns and value distribution.

  - reality is ***fat-tailed*** (leptokurtic) and extreme events are far more likely.

- Using the empirical database improves default predictions enormously.

Distance-to-Default and EDF (Cont'd)

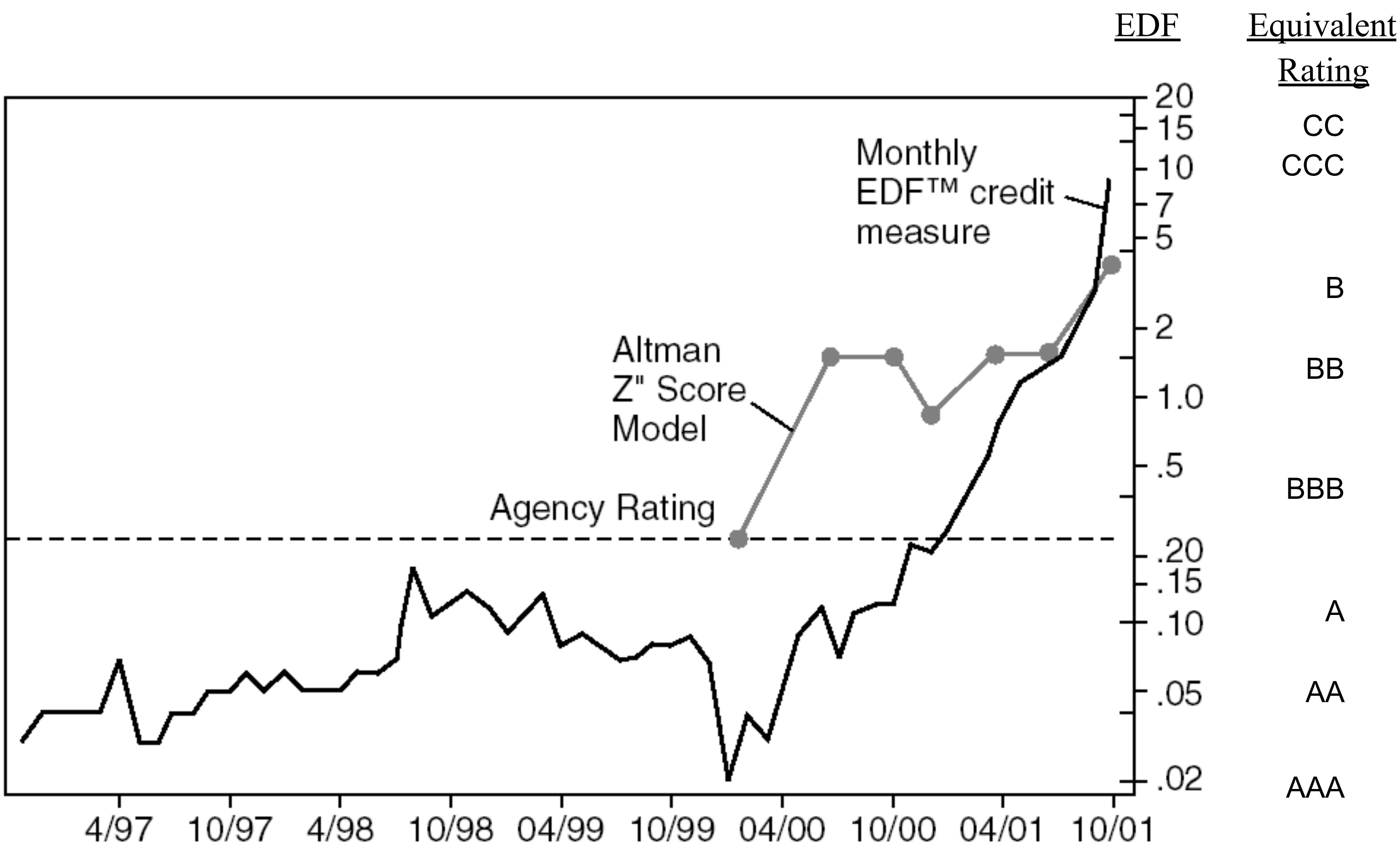
- For instance

  - For distance-to-default of 4.3, the EDF using KMV's database was 0.03%. Had we used normality, it would have been 0.00086%!

  - For distance-to-default of 3.2, the EDF using KMV's database was 0.25%. Using normality, it would have been 0.069%!

- In fact, the default probabilities predicted by the Merton Model-which is based on normality-would imply that well over 50% of all US companies are AAA or better!
- So MKMV uses Merton model to rank credit risk of firms in a relative sense but does not use its absolute probability

Enron Credit Risk Measures



Source: A. Saunders and L. Allen, Credit Risk Measurement; J. Wiley, 2002

A direct alternative method

- Alternatively
  - ✓ use market value for equity (*E*) and equity volatility
  - ✓ book value for *total* debt (D^*) and market value of *traded* debt for debt volatility
  - ✓ calculate $V = E + D^*$ and calculate leverage ratio
  - ✓ Calculate asset volatility from debt and equity volatility, correlation and leverage:

$$R_V = \frac{E}{V} R_E + \frac{D^*}{V} R_D$$

$$\sigma_V^2 = \left(\frac{E}{V}\right)^2 \sigma_E^2 + \left(\frac{D^*}{V}\right)^2 \sigma_D^2 + 2\left(\frac{E}{V}\right)\left(\frac{D^*}{V}\right) \operatorname{cov}(R_E, R_D)$$

The Volatility of Corporate Assets

	All	AAA	AA	A	BBB	BB	B
			Quasi-Market Leverage				
Mean	0.34	0.10	0.21	0.32	0.37	0.50	0.66
Std.Dev.	0.21	0.08	0.19	0.20	0.17	0.23	0.22
			Equity Volatility				
Mean	0.32	0.25	0.29	0.31	0.33	0.42	0.61
Std.Dev.	0.13	0.06	0.10	0.11	0.13	0.19	0.19
			Estimated Asset Volatility				
Mean	0.22	0.22	0.22	0.21	0.22	0.23	0.28
Std.Dev.	0.08	0.05	0.07	0.08	0.08	0.08	0.08

- L_{jt} = Quasi-market leverage ratio of firm j, time t

$$\frac{\text{Book Value of Debt (Compustat items 9 and 34)}}{\text{Book Value of Debt + Market Value of Equity}}$$

- Estimated asset volatility

$$\sigma^2_{Ajt} = (1 - L_{jt})^2 \sigma^2_{Ejt} + L^2_{jt} \sigma^2_{Djt} + 2L_{jt}(1 - L_{jt})\sigma_{ED,jt}$$