Correlated Default Models

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Products

- Basket default products: *n*-th to default baskets.
- CDOs (Collateralized Default Obligations)

Modeling Approaches

- **Bottom-up**: where each name in the pool of credits is modeled and the pool is the aggregate of these with appropriate default correlation.
- **Top-down**: where the loss level in the aggregate pool is directly modeled.

Simple Correlated Default Math

Default Indicators

$$D_i = \begin{cases} 1 & \text{if firm } i \text{ defaults} \\ 0 & \text{if firm } i \text{ does not default} \end{cases}$$

Probability of default

$$E(D_i) = [1 \times p_i] + [0 \times (1 - p_i)] = Pr[D_i = 1] = p_i$$

$$\text{Var}(D_i) = p_i(1 - p_i)$$

Correlation of Joint default

$$\begin{aligned} \rho_{ij} \equiv Corr(D_i, D_j) &= \frac{E(D_i D_j) - E(D_i)E(D_j)}{\sqrt{Var(D_i)Var(D_j)}} \\ &= \frac{E(D_i D_j) - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} \end{aligned}$$

Probability of joint default

$$\begin{aligned} Pr[D_i \cap D_j) &= E(D_i D_j) \\ &= p_i p_j + \rho_{ij}\sqrt{p_i(1-p_i)p_j(1-p_j)} \end{aligned}$$

$$Pr[D_i \cap D_j] = Pr(D_i) + Pr(D_j) - Pr[D_I \cup D_j],$$

$$E(D_i D_j) = E(D_i) + E(D_j) - Pr[D_I \cup D_j]$$

$$\rho_{ij} = \frac{p_i + p_j - Pr[D_I \cup D_j] - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}}$$

Be careful with default correlation …. It cannot be arbitrarily specified:

$$p_i p_j + \rho_{ij}\sqrt{p_i p_j (1-p_i)1-p_j)} \leq \min\{p_i, p_j\}$$

$$p_i = \min\{p_i, p_j\} \Longrightarrow \rho_{ij} \leq \sqrt{\frac{p_i(1-p_j)}{p_j(1-p_i)}}$$

if $p_i = 0.01$ and $p_j = 0.10$, then

$$\rho_{ij} \leq \sqrt{\frac{(0.01)(0.90)}{(0.10)(0.99)}} \approx 0.3015$$

ρ_{IJ}	$Pr(D_I \cap D_J)$
0.000	0.00100
0.025	0.00175
0.050	0.00249
0.075	0.00324
0.100	0.00398
0.125	0.00473
0.150	0.00548
0.175	0.00622
0.200	0.00697
0.225	0.00772
0.250	0.00846
0.275	0.00921
0.300	0.00995

Assume that $p_i = 0.01$ and $p_j = 0.10$. The following table provides a comparison of the probability of joint default and the correlation of default. Note that the maximum possible correlation in this case is roughly 0.3015.

Conditional Default Correlation

$$\begin{aligned} Pr[D_i \cap D_j] &= Pr[D_j|D_i]\ Pr(D_i) \\ &= Pr[D_i|D_j]\ Pr(D_j) \end{aligned}$$

Let $p_i = 0.01$ and $p_j = 0.10$, and $Pr[D_j|D_i] = 0.8$ (if firm i defaults, then j defaults 80% of the time, signifying an increase in p_j after conditioning). This implies that

$$Pr[D_i \cap D_j] = 0.8 \times Pr(D_i) = 0.8 \times 0.01 = 0.008$$

1. What is the default correlation given the conditional probability of default? In this example, we compute the default correlation as follows:

$$\begin{aligned} \rho_{ij} = Corr[D_i \cap D_j] &= \frac{Pr(D_i \cap D_j) - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-P_j)}} \\ &= \frac{0.008 - (0.01)(0.10)}{\sqrt{(0.01)(0.99)(0.10)(0.90)}} \\ &= 0.2345 \end{aligned}$$

2. What is the reverse conditional probability of default, i.e., $Pr[D_i|D_j]$? This may again be computed applying Bayes' theorem as follows:

$$\begin{aligned} Pr[D_i|D_j] &= \frac{Pr[D_i \cap D_j]}{Pr(D_j)} \\ &= \frac{0.008}{0.100} = 0.08 = 8\% \end{aligned}$$

■

Basket Default Contracts

First-to-default baskets (any one default):

$$\begin{aligned} Pr[D_i \cup D_j] &= Pr(D_i) + Pr(D_j) - Pr[D_i \cap D_j] \\ &= p_i + p_j - Pr[D_j|D_i]\, p_i. \end{aligned}$$

This results in two polar cases:

1. When there is perfect conditional default equals, i.e. $Pr[D_j|D_i] = 1$, then

$$Pr[D_i \cup D_j] = p_i + p_j - 1.\ p_i = p_j.$$

2. When there is zero conditional default, i.e. $Pr[D_j|D_i] = 0$, then

$$Pr[D_i \cup D_j] = p_i + p_j - 0.\ p_i = p_i + p_j.$$

Correlation works inversely for 1st-to-default and 2nd-to-default baskets.

Connection between correlation of default and conditional default

$$
\begin{aligned}
\rho_{ij} &= \frac{Pr[D_i \cap D_j] - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} \\
&= \frac{Pr[D_j|D_i]p_i - p_i p_j}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} \\
&= \frac{p_i(Pr[D_j|D_i] - p_j)}{\sqrt{p_i(1-p_i)p_j(1-p_j)}} \\
&= \frac{\sqrt{p_i}(Pr[D_j|D_i] - p_j)}{\sqrt{(1-p_i)p_j(1-p_j)}} \equiv \frac{\sqrt{p_j}(Pr[D_i|D_j] - p_i)}{\sqrt{p_i(1-p_i)(1-p_j)}}
\end{aligned}
$$

Re-arranging we also have the expression for conditional default:

$$
Pr(D_j|D_i) = \frac{1}{p_i}\left[\rho_{ij}\sqrt{p_i(1-p_i)p_j(1-p_j)} + p_i p_j\right].
$$

Structural Model based on Asset Values

Starting point: distance-to-default (DTD)

$$d_i = DTD_i = \frac{A_i - F_i}{\sigma_i \, A_i}.$$

$$\text{Prob of default} = p_i = N(-d_i) = 1 - N(d_i),$$

$$d_i = N^{-1}[1 - p_i].$$

Simulate

$$x_i \sim N(0,1),$$

$$D_i = \begin{cases} 1 & \text{if} \quad x_i \leq -d_i \\ 0 & \text{if} \quad x_i > -d_i \end{cases}$$

Correlated default comes from:

$$Corr(A_i, A_j) = \rho_{ij}$$

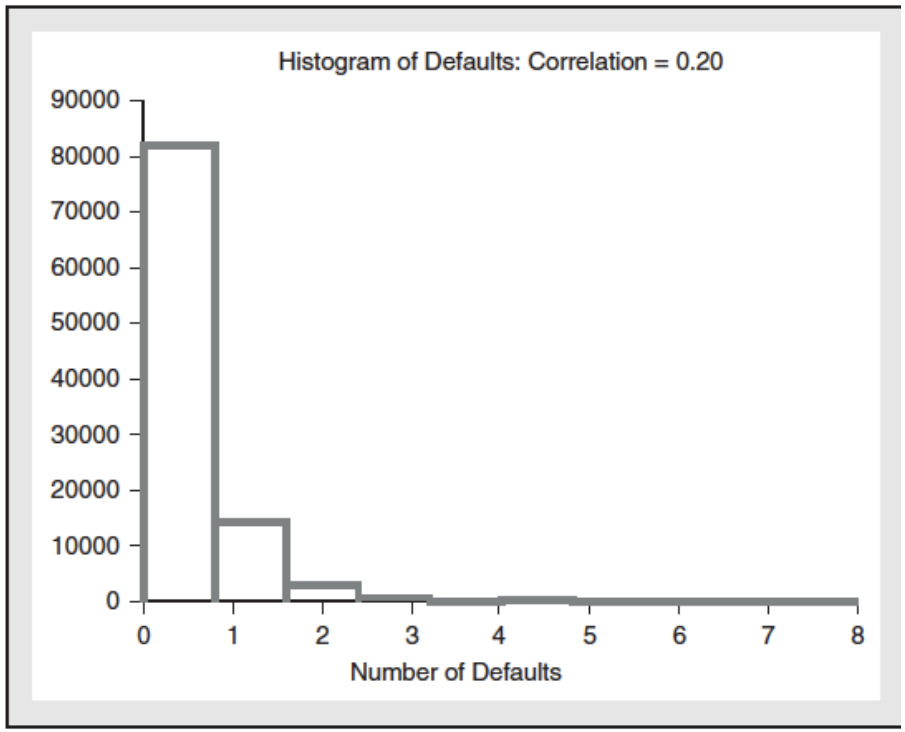
$$\begin{bmatrix} x_i \\ x_j \end{bmatrix} \sim MVN \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 1 & \rho_{ij} \\ \rho_{ij} & 0 \end{pmatrix} \right]$$

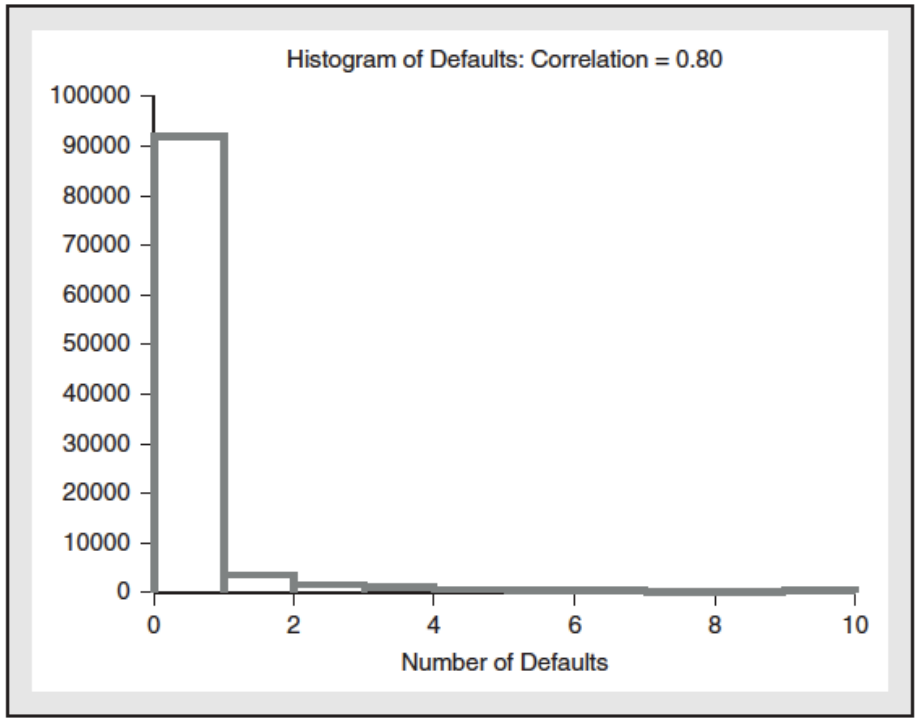
More general: N firms

$$\mathbf{x} \sim MVN[\mathbf{0}; \mathbf{R}]$$

```
%Program to simulate defaults in asset values model
%Input Variables
n = 10;     %number of firms
d = 2;     %dtd
rho = 0.8;    %avg correlation across firms
R = rho*ones(n,n);
for i=1:n;
    R(i,i) = 1;
end;
%Generate many correlated defaults
iter = 100000;
x = mvn_rnd(R,iter);
numdef = zeros(iter,1);
for i=1:iter;
    numdef(i) = length(find(x(i,:)<=-d));
end;

grid;
title('Histogram of Defaults: Rho = 0.80');
hist(numdef);
```

Default Frequencies for Varied Correlation





The histogram on the right has fatter tails
(Both left and right tails.)

Reducing dimension to M << N

The simulation based on asset values is easy to implement, as we have seen. But the speed at which the implementation works is misleading for small problems such as basket contracts. When the same implementation needs to be undertaken for baskets numbering in the hundreds, the computation can become very time consuming, and handling a correlation matrix of a size such as $1,000 \times 1,000$ can quickly become infeasible. In such situations, reducing the dimensionality of the problem becomes an urgent necessity.

We assume that the firms in the basket or CDO collateral may be subdivided into M "sectors". These sectors may be based on any chosen classification, for example, an industry breakdown. The number of firms within each sector j is denoted N_j. We denote the correlation matrix of each aggregate sector asset values as $\mathbf{R}$. Therefore $\mathbf{R}$ is of dimension $M \times M$. We then draw a random vector $\mathbf{y} \in \mathbb{R}^M$ from this system, i.e.

$$\mathbf{y} \sim MVN[\mathbf{0}; \mathbf{R}].$$

The values in this vector are for the average firm from each sector. Such a firm does not exist in reality, but the random vector $\mathbf{y}$ will be used to support the simulation for each individual firm as well. For now, this vector $\mathbf{y}$ is the default variable for each sector. Since the correlation matrix $\mathbf{R}$ is used, this approach ensures that across sector default correlation is obtained. It also ensures that the dimensionality of the system is low, for it is restricted to the number of sectors that are stipulated.

The next step is to draw the default variables for each individual firm. This is done sector by sector, using the within sector correlation matrix, which we denote $\mathbf{R}_j$ for each j. This correlation matrix is of dimension $N_j \times N_j$. Using this, we draw a random vector $\mathbf{x}'_j$ with the following distribution:

$$\mathbf{x}'_j \sim MVN[\mathbf{0}; \mathbf{R}_j].$$

This is done for each industry, i.e. for all j. Finally, the default variable is determined for each individual firm within an industry as follows:

$$\mathbf{x}_j = \mathbf{x}'_j + y_j \mathbf{1}, \quad \forall j$$

where $\mathbf{1}$ is a unit vector of dimension N_j.

Reducing dimension with a factor model

We denote the return on the firm's assets as R_i

$$\left(\frac{A_T}{A_0} - 1\right)$$

$$R_i = f_i(\mathbf{X};\ \boldsymbol{\beta}_i, \theta_i)$$

$$R_i = \beta_i' \mathbf{X} + \epsilon_i, \quad \forall i \qquad \sigma_i^2 = \boldsymbol{\beta}_i' \boldsymbol{\Sigma} \boldsymbol{\beta}_i'$$

1. Draw a sample of the factors using the multivariate factor distribution. For example, if this is multivariate normal, then we have

$$\mathbf{X} \sim MV\,N[\boldsymbol{\mu}; \boldsymbol{\Sigma}].$$

2. The factor vector is used to compute the values of returns for all the firms, which may then be used to determine if default has occurred. This is done by comparing the normalized return $x_i = \frac{R_i}{\sigma_i}$ with the distance to default d_i. Thus, default occurs if

$$x_i < -d_i, \quad \forall i.$$

Explanation of step 2

The last step here requires some further explanation, and we can show how to link it to the discussion on distance to default. Recall that the distance to default for firm i is as follows:

$$d_i = \frac{1}{\sigma_i} \left[\frac{A_i - F_i}{A_i} \right].$$

Assuming that at the outset $A_i > F_i$, the value in square brackets above is the return by which the firm value needs to fall to trigger default. Therefore default occurs if

$$R_i < - \left[\frac{A_i - F_i}{A_i} \right].$$

Dividing both sides by σ_i, we see that default gets triggered when

$$x_i \equiv \frac{R_i}{\sigma_i} < -\frac{1}{\sigma_i} \left[\frac{A_i - F_i}{A_i} \right] \equiv -d_i.$$

Reduced-Form: Doubly Stochastic Correlated Default

$$s_i(t) = \exp\left[-\int_0^t \lambda_i(u)\, du\right] \qquad p_i(t) = 1 - s_i(t)$$

In doubly stochastic models, there are two sources of default correlation.

1. Correlation amongst the default intensities.

2. Conditional on default intensities, there is correlation amongst the default events as well. If this latter correlation is zero, we call the Cox process "conditionally independent".

Example

Suppose we have 10 issuers in a basket of credits. The constant default intensity for each issuer is $\lambda_i = 0.1,\ \forall i = 1..10$. If the horizon for the analysis is one month (i.e. $t = 1/12$), then the one month default probability is

$$p_i = 1 - \exp(-\lambda_i t) = 0.0082987.$$

Now, we assume that default is conditionally independent, i.e. we sample 10 independent uniform random variables x_i and check if $x_i \leq p_i,\ \forall i$. For example, we generated 10 random numbers from $x_i \sim U[0,1]$ and obtained:

```
0.8807160   0.6384585   0.5407283   0.1113356   0.2733150
0.1002537   0.9708890   0.9470903   0.6617928   0.0022545
```

Only one of these values is less than 0.0082987, the last one in the vector above. Hence, one of the ten firms defaults in this random draw.

Multi-period Correlated Default

1. Generate the default intensity from a stochastic model for all firms (if the framework is reduced form). Or, generate the return on the firm's assets if the framework is structural. During this process, ensure that the correlations are factored in while simulating the new values.

2. Conditional on these values, determine which firms default within the first year. Unless conditional independence is imposed on the model, make sure that the default variables used to check for default are generated with appropriate correlations.

3. Using the values of default intensity or firm value (as the case may be) at the end of the first year, we proceed back to step 1 above and repeat the process for the next year. We continue to do this loop until all the years are accounted for. We make sure to record all defaults that occur in each year.

This method is general but inefficient. In some cases, it is better to work in default time space than in default event space.

Working with default times

Example

Assume again a set of 10 firms for whom we have generated default intensities, which are constant and equal, i.e. $\lambda_i = 0.10$. We are interested in simulating defaults over the interval of time $(0,t)$. Conditional on these intensities, we need to generate stopping times τ_i, $\forall i = 1...10$. Since default arrival is Poisson, the time between Poisson arrivals is exponentially distributed. Our procedure for generating stopping times is as follows.

1. Generate a default variable $x_i \sim U[0,1]$, $\forall i$. This generates 10 uniform random numbers between 0 and 1.

2. Transform these uniform random numbers into default times using the exponential distribution, i.e. set

$$x_i = \exp(-\lambda_i \tau_i).$$

The transformation gives the stopping time τ_i as follows:

$$\tau_i = -\frac{1}{\lambda_i}\ln(x_i).$$

3. Compare the stopping time with horizon t, and establish default as follows:

$$D_i = \begin{cases} 1 & \text{if } \tau_i \leq t \\ 0 & \text{if } \tau_i > t \end{cases}$$

```
octave:1> x = rand(10,1)
x =
  0.086135
  0.449259
  0.307056
  0.243939
  0.125794
  0.912623
  0.664363
  0.732090
  0.731622
  0.458583

octave:2> lambda = 0.1; tau = -log(x)/lambda
tau =
  24.51845
   8.00156
  11.80724
  14.10837
  20.73112
   0.91433
   4.08927
   3.11851
   3.12491
   7.79614

octave:3> t=1; def_firm_no = find(tau<t)
def_firm_no = 6
```

Now, if the horizon was 10 years, i.e. $t = 10$, then more firms would be in default in the single sample draw depicted above, as can be seen from the results of the following Octave code.

```
octave:10> t=10; def_firm_nos = find(tau<t)'
def_firm_nos =
   2   6   7   8   9  10
```

Fast Calculation of Credit Loss Distributions (the recursion model of Andersen, Sidenius & Basu (2003)

Step 1: generate state-dependent default probabilities for all issuers in the portfolio

Assume a one-factor model:

$$Y_k = \sqrt{\rho_k}\, X + \sqrt{1-\rho_k}\, Z_k$$

Y_k is the default variable, $X \sim N(0,1)$, $Z_k \sim N(0,1)$

Correlation is ρ_k. The probability of default is:

$$p_k = \text{Prob}[Y_k < -d_k] = \text{Prob}[\sqrt{\rho_k}\, X - \sqrt{1-\rho_k}\, Z_k < -d_k]$$

Since Z_k is normally distributed, we can then denote p_k, *conditional on* X as

$$p_k|X = N\left[\frac{-d_k - \sqrt{\rho_k}\, X}{\sqrt{1-\rho_k}}\right]$$

Step 2: generate the loss distribution conditional on X:

Our credit portfolio comprises the K issuers: $k = 1, \ldots, K$. Let w_k be the loss level of the k-th name. This is fixed, and hence, implies that conditional on the issuer defaulting, the dollar loss amount is prespecified and is constant. The set of loss levels (in round dollars) is denoted as

$$l = \{0, 1, \ldots, l_{max}\}$$

The probability of a total loss level of l if the first k firms is considered is denoted $P^k(l)$.

Build up the loss distribution using the following recursion:

$$P^{k+1}(l) = P^k(l)[1 - p_{k+1}] + P^k(l - w_{k+1})p_{k+1}$$

We call this the Andersen, Sidenius, Basu (ASB) recursion.

ASB Implementation

1. In the first iteration only two possible outcome exist, name $l = 1$ either fails or not, i.e.

$$\begin{aligned} P^1(0) &= 1 - p_1 \\ P^1(2) &= p_1 \end{aligned}$$

2. For the second iteration, we need all combinations of losses from the first two names, i.e.

$$\begin{aligned} P^2(0) &= P^1(0)[1 - p_2] = (1 - p_1)(1 - p_2) \\ P^2(1) &= P^1(1)[1 - p_2] + P^1(1 - 1)p_2 = 0 + (1 - p_1)p_2 \\ P^2(2) &= P^1(2)[1 - p_3] + P^1(2 - 1)p_2 = p_1(1 - p_2) + 0 \\ P^2(3) &= P^1(3)[1 - p_2] + P^1(3 - 1)p_2 = 0 + p_1 p_2 \end{aligned}$$

Therefore, after the second firm, the maximum loss level is 3, given that firm 1 has a loss level of 2 and the second one has a loss level of 1. After $k = 2$, there are 4 possible loss levels, $\{0, 1, 2, 3\}$ and these have the four probabilities given above. The probabilities sum up to 1.

Iteration 3

3. Lets now do iteration 3. Here the third firm may default as well, and so the maximum loss level is 10. We can write down a few of the probabilities for the possible loss levels as follows:

$$
\begin{aligned}
P^3(0) &= P^2(3)[1-p_3] = (1-p_1)(1-p_2)(1-p_3) \\
P^3(1) &= P^2(1)[1-p_3] + P^2(1-7)p_3 = (1-p_1)p_2(1-p_3) + 0 \\
P^3(2) &= P^2(2)[1-p_3] + P^2(2-7)p_3 = p_1(1-p_2)(1-p_3) + 0 \\
P^3(3) &= P^2(3)[1-p_3] + P^2(3-7)p_3 = p_1p_2(1-p_3) + 0 \\
P^3(7) &= P^2(7)[1-p_3] + P^2(7-7)p_3 = 0 + (1-p_1)(1-p_2)p_3 \\
P^3(8) &= P^2(8)[1-p_3] + P^2(8-7)p_3 = 0 + (1-p_1)p_2p_3
\end{aligned}
$$

Likewise, you can write down $P^3(4)$, $P^3(5)$, $P^3(6)$, $P^3(9)$ and $P^3(10)$. This is left as an exercise.

Repeat this K times until all issuers have been processed.

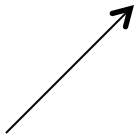
Recursion Program (Example with 4 issuers)

```
%INPUTS
%PROGRAM: asb_recusrsion.m
w = [2,1,3,7];  %Loss weights
p = [0.1, 0.05, 0.03, 0.2]; %Loss probabilities

%BASIC SET UP
N = length(w);
maxloss = sum(w);
bucket = [0:maxloss];
LP = zeros(N,maxloss+1);  %probability grid over losses

%DO FIRST FIRM
LP(1,1) = 1-p(1);
LP(1,w(1)+1) = p(1);

%LOOP OVER REMAINING FIRMS
for i=2:N;
    for j=1:maxloss+1;
        LP(i,j) = LP(i-1,j)*(1-p(i));
        if bucket(j)-w(i) >= 0; LP(i,j) = LP(i,j) + LP(i-1,j-w(i))*p(i); end;
    end;
end;

%SHOW HISOTOGRAM OF LOSS DISTRIBUTION
lossprobs = LP(N,:);
fprintf('CHECK: Sum of final probs = %10.6f \n',sum(lossprobs));
[bucket'  LP']
```

Program Execution

```
octave-2.9.17:4> asb_recursion
CHECK: Sum of final probs =   1.000000
ans =

    0.00000    0.90000    0.85500    0.82935    0.66348
    1.00000    0.00000    0.04500    0.04365    0.03492
    2.00000    0.10000    0.09500    0.09215    0.07372
    3.00000    0.00000    0.00500    0.03050    0.02440
    4.00000    0.00000    0.00000    0.00135    0.00108
    5.00000    0.00000    0.00000    0.00285    0.00228
    6.00000    0.00000    0.00000    0.00015    0.00012
    7.00000    0.00000    0.00000    0.00000    0.16587
    8.00000    0.00000    0.00000    0.00000    0.00873
    9.00000    0.00000    0.00000    0.00000    0.01843
   10.00000    0.00000    0.00000    0.00000    0.00610
   11.00000    0.00000    0.00000    0.00000    0.00027
   12.00000    0.00000    0.00000    0.00000    0.00057
   13.00000    0.00000    0.00000    0.00000    0.00003
```

Loss level



probability

Copulas

Copulas are useful in modeling multivariate distributions very generally with flexible dependence structure.

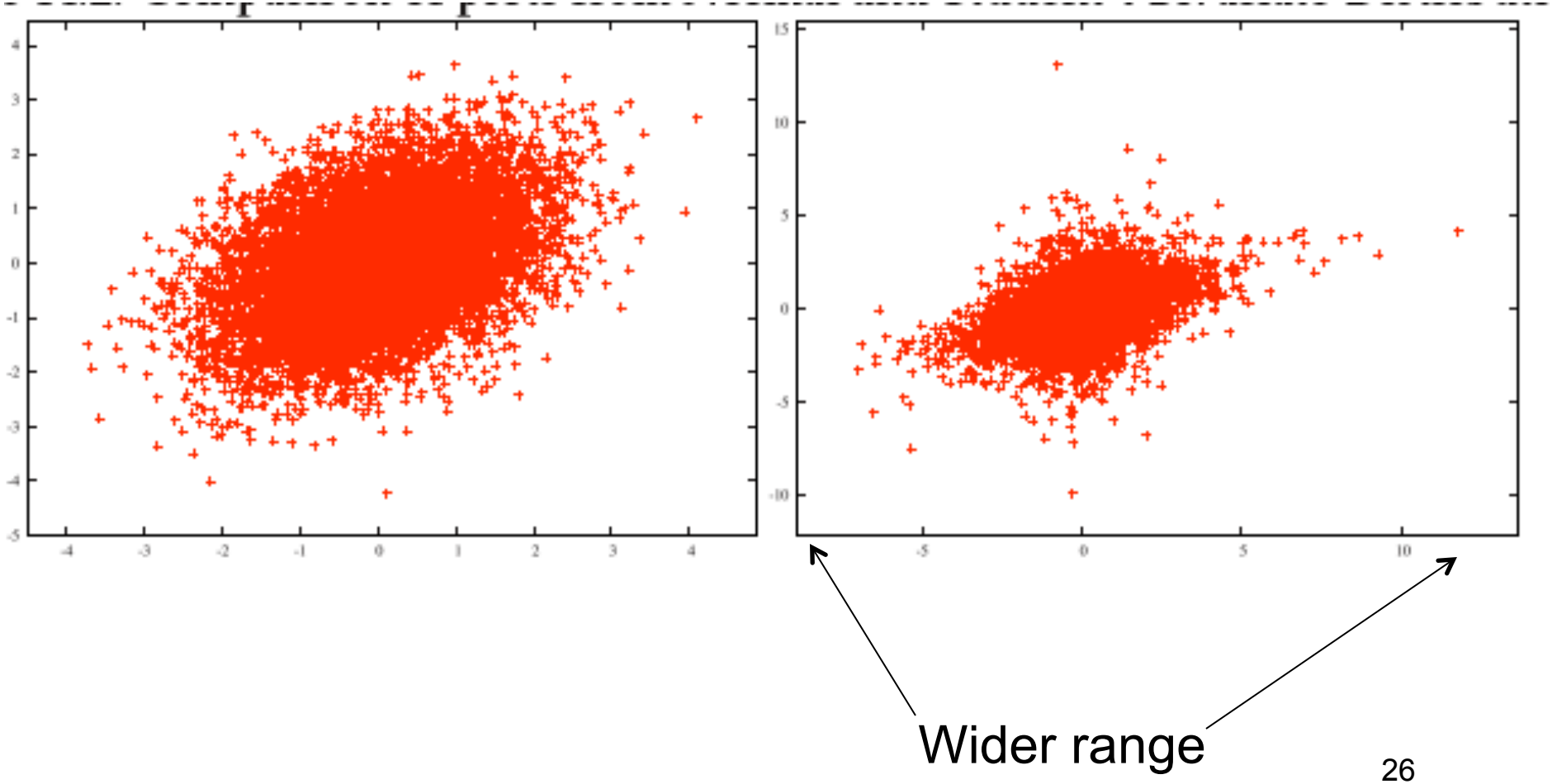
$$F_{X,Y}(x,y) = C[F_X(x), F_Y(y)] = C(u,v)$$

$$f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = \frac{\partial^2 C[F_X(x), F_Y(y)]}{\partial x \partial y} = \frac{\partial^2 C(u,v)}{\partial x \partial y}$$

$$= \frac{\partial^2 C(u,v)}{\partial u \partial v} \frac{\partial u}{\partial x} \frac{\partial v}{\partial y} = c(u,v)\, f_X(x)\, f_Y(y)$$

Idea of "divide and estimate" -- separate joint distribution from the marginals.

Plots from normal and student-T distributions



Rank Correlations (essential ingredient for copulas)

Kendall's Tau

$$\tau = Pr[(X_2 - X_1)(Y_2 - Y_1) > 0] - Pr[(X_2 - X_1)(Y_2 - Y_1) < 0].$$

If $(X_2 - X_1)(Y_2 - Y_1) > 0$, then the pair of random draws is *concordant*, else it is *discordant*. If we define c as the number of concordant pairs, and d as the number of discordant pairs, then we can also define the rank correlation as:

$$\begin{aligned} \tau &= \frac{c-d}{c+d} = \frac{c-d}{\frac{n(n-1)}{2}} \\ &= \frac{2}{n(n-1)} \sum_{i<j} \text{sign}[(X_i - X_j)(Y_i - Y_j)] \end{aligned}$$

Properties:

- $\tau \in [-1, 1]$.
- If X, Y are independent, then $\tau = 0$.
- τ is invariant under strictly monotonic transformations, that is if $f(X), g(Y)$ are strictly increasing or decreasing functions. Then, $\tau[f(X), g(Y)] = \tau(X, Y)$.

$$\tau[\Phi(X), \Phi(Y)] = \tau[X, Y] = \frac{2}{\pi} \arcsin(\rho(X, Y))$$

Normal copula

The Gaussian copula is a popular one for scenario generation because of its parsimony and ease of use.

1. Compute the matrix of rank correlation coefficients prior to the simulation. This may be done from the data or it may be derived from an economic model. Denote this matrix as τ.

2. Convert this matrix into the normal correlation matrix, by computing values from the inverse of the transform equation used above, i.e.

$$\rho_{ij} = \sin\left[\frac{\pi}{2}\tau_{ij}\right].$$

3. Generate multivariate random normal numbers with mean zero and correlation matrix ρ, to get a random vector $(Z_1, Z_2, ..., Z_m)'$.

4. Let $u_i = \Phi(Z_i),\ i = 1..m$. Here $\Phi(Z_i)$ is the cumulative normal distribution function over Z_i.

5. Finally, we get $x_i = F_i^{-1}(u_i),\ i = 1..m$, where F_i is the marginal distribution and may be different for each i.

A similar approach may be used to generate samples from a student-T distribution using copulas. Or for that matter, any other distribution.

Example

1. Convert the rank correlation into a linear correlation parameter, i.e.

$$\rho_{12} = \sin\left[\frac{\pi}{2}\tau_{12}\right] = \sin(\frac{\pi}{2} \times 0.6) = 0.80902.$$

2. We generate 1000 correlated pairs of random normal values using ρ_{12}, and the code is as follows:

```
octave:2> z = mvn_rnd([1 0.80902; 0.80902 1],1000);
```

3. Convert these z values into $[0, 1]$ variates using the normal CDF, i.e.

```
octave:3> u1 = normal_cdf(z(:,1)); u2 = normal_cdf(z(:,2));
```

4. Generate the random values by parsing the u_i through their respective marginal distributions. This is done by inverting the CDF for each marginal distribution.

```
octave:4> x1 = exponential_inv(u1,0.1); x2 = chisquare_inv(u2,3);
octave:5> corrcoef([x1  x2])
ans =
  1.00000  0.78936
  0.78936  1.00000
octave:6> ktau(x1,x2)
ans = 0.59775
```

Rank correlation is preserved!

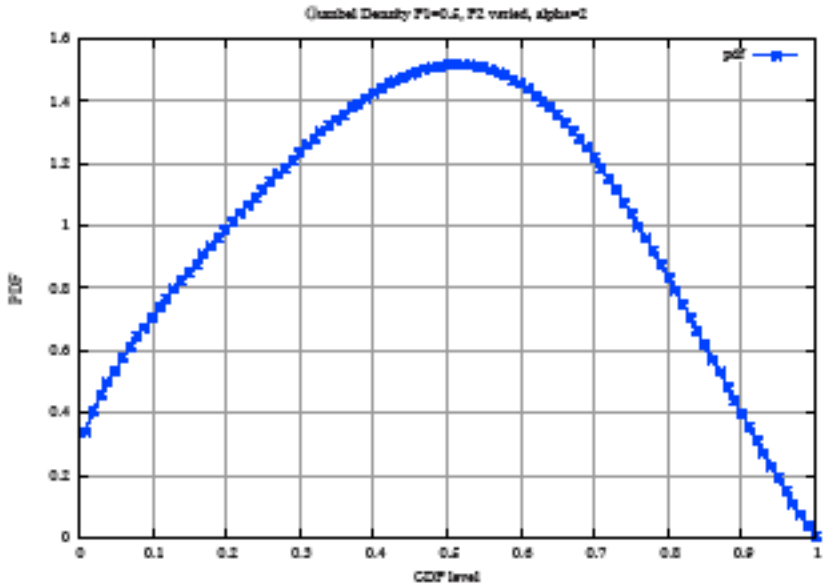
Archimedian Copulas

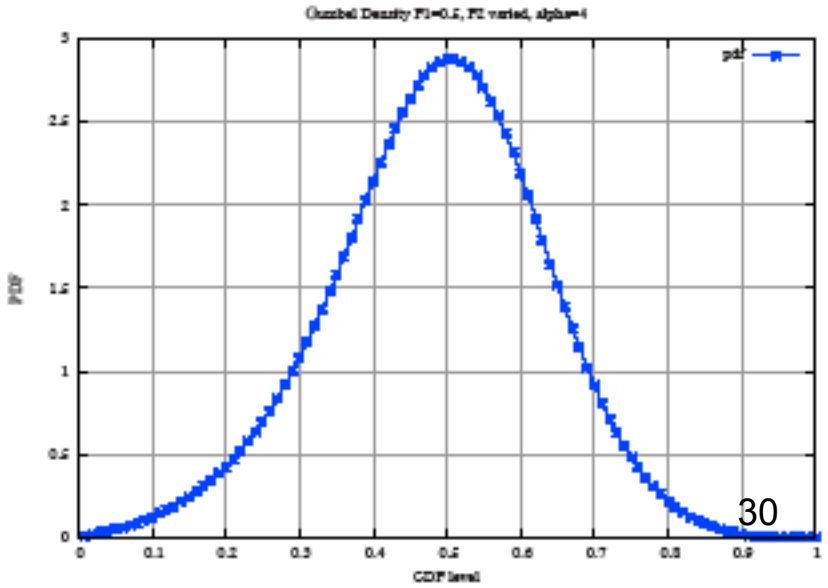
$$C(F_1, ..., F_m) = \Omega^{-1}[\Omega(F_1) + ... + \Omega(F_m)]$$

The function $\Omega(F)$ is known as the "generator" of the copula.

Gumbel copula: $$C(F_1, F_2) = \Omega^{-1}[(-\ln F_1)^\alpha + (-\ln F_2)^\alpha]$$

This graph shows how the density function varies as the tail dependence parameter α is varied. We hold $F_1 = 0.5$, and then compute the density function as F_2 varies from 0 to 1. When $\alpha = 2$, as in the left side plot, the graph is less leptokurtic than when $\alpha = 4$ as in the right side plot.





Implied Correlation from Copulas

- Implied correlation is analogous to implied volatility.
- It is the correlation that generates the loss distribution that gives the value of a tranche of a CDO.
- The normal copula is applied in practice for this calculation.
- Each tranche has a different implied correlation.
- Base correlation is the implied correlation for cumulative tranche values. For the second tranche it is the correlation that matches the value of the first and second lowest priority tranches.

Top-Down Models

- Model the aggregate loss process for the credit portfolio directly.
- The distinct names in the portfolio are not modeled.
- The aggregate loss process may be modeled as a function of multiple factors.
- Example: the model of Longstaff and Rajan (2008).

Longstaff-Rajan Model

Instantaneous proportional loss on the credit portfolio:

$N_0=0$

$$\frac{dL}{1-L} = \gamma \, dN(\lambda)$$

Cumulative losses:

$$\int_0^t \frac{dL}{1-L} = \int_0^t \gamma \, dN$$

$$L_t = 1 - e^{-\gamma \, N_t(\lambda_t)}$$

Generalization to *n* factors:

$$\frac{dL}{1-L} = \sum_{k=1}^{n} \gamma_k \, dN_k \qquad L_t = 1 - \exp\left[-\sum_{k=1}^{n} \gamma_k N_{tk}(\lambda_k)\right]$$

Losses may be simulated or obtained in closed-form if tractable, as in the Longstaff-Rajan model.

Binomial TreeTop-Down Model (Hull & White 2006)

Cumulative survival function: $S(t) = \exp\left[-\int_0^t \lambda(u)\, du\right]$

Changes in hazard rate: $\Delta\lambda = \mu\, h + J$

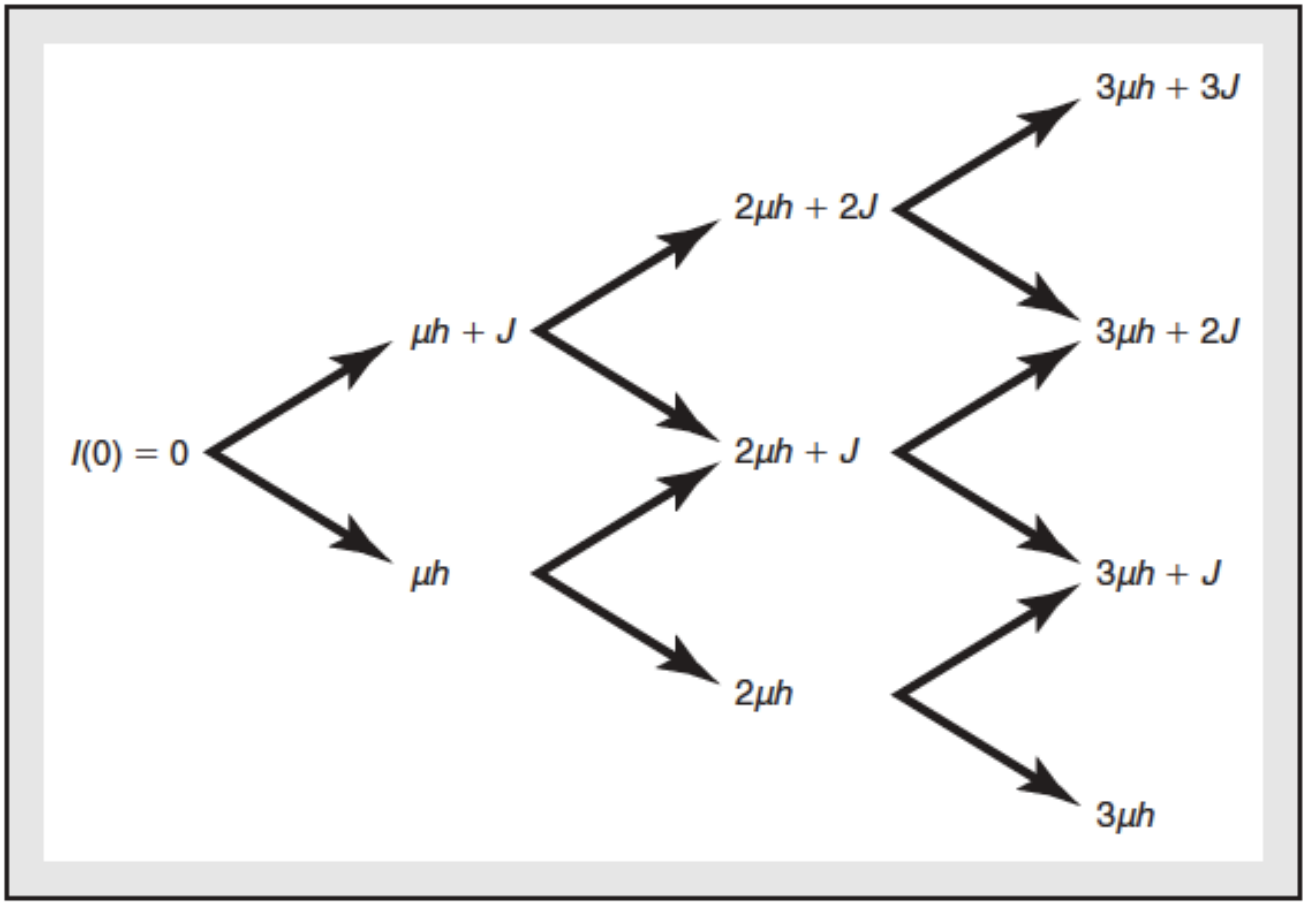
[h: time step, J is a jump process, with prob (1-p)]

I(t) : accumulated default intensity upto time t.
After two periods, there are three cases:

$$I(t_2) = \{2\mu\, h + 2J, \quad 2\mu\, h + J, \quad 2\mu\, h\}$$

with probabilities $\{p^2, \quad p(1-p), \quad (1-p)^2\}$

Tree of accumulated intensity



Survival function at each node: $S = e^{-I}$

With cumulative prob of default: $q = 1 - S$

Credit Portfolios on the Tree

Let the normalized credit portfolio have value $1.
N identical issuers with face value 1/N each.
Constant recovery on default ϕ

Probability that n of N names default at any node:

$$P(n) = \frac{N!}{n!(N-n)!} q^n (1-q)^{N-n}, \quad \forall n = 0, 1, \ldots, N$$

Expected principal balance at the node, E(B) is:

$$E(B) = \sum_{n=1}^{N} P(n)[1 - (n/N)(1-\phi)]$$

Credit derivatives on B may be priced using the tree and corresponding probabilities *(q, p)*.

Example 1:
A derivative that pays rate s on the principal of the portfolio. Use B, q to get the expected value E(B) at each node. Then discount these on the tree using the tree probabilities p.

Example 2:
A tranched security with attachment points: $(x_l, x_u) \leq 1$
Expected tranche principal balance at each node:

$$E(B) = \sum_{n=1}^{N} P(n)T(n)$$

$$T(n) = \begin{cases} 1 & \text{if } (n/N)(1-\phi) \leq x_l \\ \frac{x_u - (n/N)(1-\phi)}{x_u - x_l} & \text{if } x_l < (n/N)(1-\phi) \leq x_u \\ 0 & \text{if } (n/N)(1-\phi) > x_u \end{cases}$$

Any function of these cashflows may be used to construct a tranched derivative.

Chapter Summary

- Modeling default correlations is more complex than modeling return correlations.
- Simulation of correlated default (or default times) is possible in both structural and reduced-form models.
- The one-factor Gaussian copula is widely applied and makes simulation easy, but has flaws.
- Credit portfolios may be modeled bottom-up or top-down.
- Fast recursion may be used for bottom-up models.
- Closed-form solutions available for top-down models.
- Binomial tree methods are possible for top-down models.

Supplementary Material from related research

Application: Das and Geng (JOIM 2002)

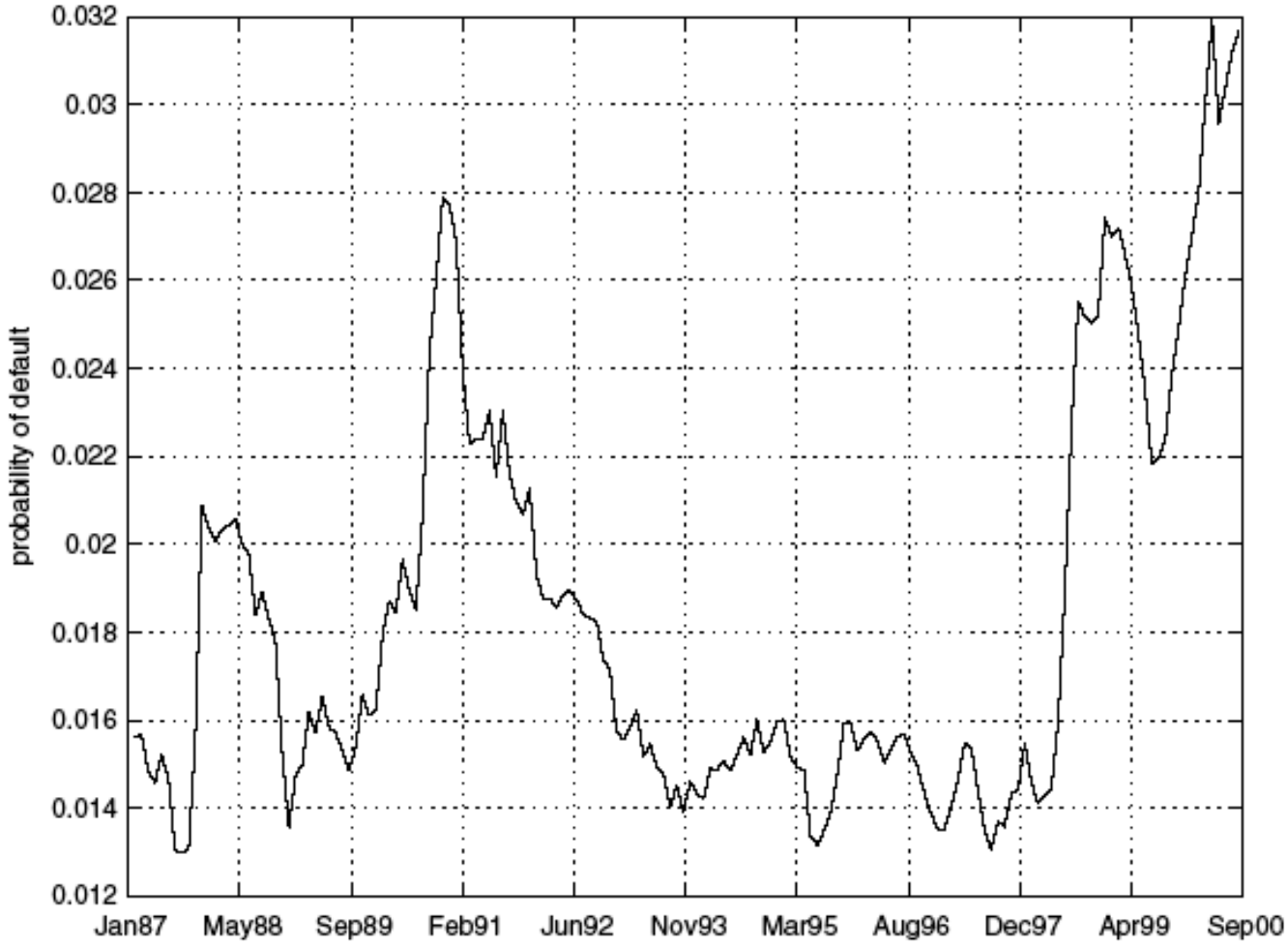


Figure 1 Time series of average PDs. This figure depicts the average level of default probabilities in the data set. The data shows the presence of two regimes, one in which PDs were high, as in the early and latter periods of the data. In the other regime, PDs were much lower, less than half of those seen in the high PD regime. Complementary analysis of correlated default on the same data set by subperiod in this same time frame is presented in DFGK (2001a) from where this figure is taken.

Table 2 Descriptive statistics for PDs: This table presents the summary of the time series of average PDs for each rating class. Each time series represents a diversified portfolio of issuers within each rating class, equally weighted. The probabilities are in percentages.

Rating category	PD in levels		PD in changes	
	Mean	Std Dev.	Mean	Std Dev.
1	0.0728	0.1051	0.0012	0.0441
2	0.1701	0.4884	0.0036	0.1468
3	0.6113	1.6808	0.0044	0.7317
4	1.4855	2.6415	0.0066	1.2694
5	3.8494	4.6401	0.0138	2.3576
6	6.5389	4.7731	0.0165	3.2075

Table 3 Kendall's τ for probabilities of default: In this table we report Kendall's τ, which is a measure of the dependence between any two time series. The upper right triangle below reports the measure for PD levels. The lower left triangle reports the correlation for PD changes.

	Correlations					
Rating	1	2	3	4	5	6
1	1.0000	0.6508	0.2724	0.2868	0.2987	0.2762
2	0.4254	1.0000	0.4671	0.4569	0.4052	0.1794
3	0.1849	0.2312	1.0000	0.6416	0.4483	0.1080
4	0.1447	0.1393	0.0543	1.0000	0.5382	0.1708
5	0.0378	0.0877	0.0352	0.1245	1.0000	0.2911
6	0.0362	−0.0429	0.0238	−0.0276	0.1032	1.0000

Choice of Copula

Normal copula: The normal copula of the n-variate normal distribution with correlation matrix ρ, is defined as

$$C_\rho(u_1, \ldots, u_n) = \Phi_\rho^n(F^{-1}(u_1), \ldots, F^{-1}(u_n))$$

Student's t copula: Let $T_{\rho,\nu}$ be the standardized multivariate Student's t distribution with ν degrees of freedom and correlation matrix ρ. The multi-variate Student's t copula is then defined as follows:

$$C(u_1, \ldots, u_n; \rho, \nu) = T_{\rho,\nu}(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_n))$$

where t_ν^{-1} is the inverse of the cumulative distribution function of a univariate Student's t distribution with ν degree of freedom.

Gumbel copula: This copula was first introduced by Gumbel (1960) and can be expressed as follows:

$$C(u_1, \ldots, u_n) = \exp\left[-\left(\sum_{i=1}^{n}(-\ln u_i)^\alpha\right)^{1/\alpha}\right] \quad (2)$$

where α is the parameter determining the tail-dependence of the distribution.

Clayton copula: This copula, introduced in Clayton (1978), is as follows:

$$C(u_1, \ldots, u_n) = \left[\sum_{i=1}^{n} u_i^{-\alpha} - n + 1\right]^{-1/\alpha} \quad (3)$$

Again, $\alpha > 1$ is a tail-dependence parameter.

$$\Delta\lambda_k(t) = \kappa_k[\theta_k - \lambda_k(t)]\Delta t + x_k(t)\sqrt{\lambda_k(t)\Delta t}$$

$$x_k(t) = \epsilon_k(t) + J_k(t)L_k(q_k, t)$$

$$\epsilon_k(t) \sim N[0, \sigma_k^2]$$

$$J_k(t) \sim U[a_k, b_k]$$

$$L_k(q_k, t) = \begin{cases} 1 & \text{w/prob} \quad q_k \\ 0 & \text{w/prob} \quad 1 - q_k \end{cases}$$

Jump case

Table 5 Estimation of the average rating process: in this table we report the estimation results of the stochastic process for the mean of the intensities of each rating class. Numbers below the estimates are the t-statistics from the estimation, which is undertaken by maximum-likelihood.

	Rating class					
Parameter	1	2	3	4	5	6
θ_k	0.0524	0.1336	0.3499	1.1550	2.3812	5.4848
	7.73	16.74	2.75	4.01	1.99	7.12
κ_k	0.4813	0.0018	0.6680	0.4811	0.4305	2.8350
	2.84	0.01	1.88	1.87	2.01	4.19
σ_k	0.0487	0.1050	0.2518	0.2633	0.2872	0.5743
	14.17	13.39	12.48	11.44	8.51	5.85
b_k	0.0839	0.1316	0.1241	0.2070	0.2684	1.1231
	12.39	5.64	1.94	1.58	1.52	5.50
a_k	−0.0707	−0.0644	0.1241	0.2069	0.2684	0.0771
	−6.33	−3.46	1.96	1.61	2.37	0.21
q_k	0.1016	0.0777	0.1514	0.0950	0.2461	0.4374
	3.87	3.18	2.94	2.07	2.08	2.07
Log-likelihood	−622.16	−462.95	−205.48	−133.56	−32.82	−35.03

Regime-Switching Case

$$\Delta\bar{\lambda}_r(t) = \kappa_r[\theta_r - \bar{\lambda}_r(t)]\Delta t + \sigma_r\sqrt{\bar{\lambda}_r(t)\Delta t}\epsilon(t) \qquad \begin{bmatrix} p_{LO} & 1-p_{LO} \\ 1-p_{HI} & p_{HI} \end{bmatrix}$$

$$r = \{HI, LO\} \quad (\qquad p_r = \exp(\alpha_r)/(1+\exp(\alpha_r)), r \in \{LO, HI\}$$

Table 6 Estimation of the regime-switching model across all issuers: this table provides estimation results for regimes estimated on the average intensity process $\left(\bar{\lambda} = \sum_{i=1}^{N} \lambda_i\right)$ across all issuers in the data set. We designated periods where $\bar{\lambda} \leq 2\%$ as the "low-PD" regime, and periods in which $\bar{\lambda} > 2\%$ as the "high-PD" regime. The two regimes are indexed by r, which is either HI or LO. κ_r is the rate of mean-reversion. The mean value within the regime is θ_r and σ_r is the volatility parameter. The regimes were bifurcated exogenously, as described, and we estimated the values of θ_r by simply averaging the hazard rate within each regime. The probability of switching between regimes comes from a logit model based on a transition matrix: $\begin{bmatrix} p_{LO} & 1-p_{LO} \\ 1-p_{HI} & p_{HI} \end{bmatrix}$, where $p_r = \frac{\exp(\alpha_r)}{1+\exp(\alpha_r)}$, $r \in \{LO, HI\}$. Estimation is undertaken using maximum-likelihood.

Parameter	Low-PD regime	High-PD regime
θ	1.68%	2.40%
κ	1.5043	1.7746
t-stat	2.4053	2.0558
σ	0.1681	0.2397
t-stat	14.2093	7.9849
α	4.1286	3.3644
t-stat	4.6649	2.5428
Loglik	−195.5948	

$$\Delta\bar{\lambda}_{k,r}(t) = \kappa_{k,r}[\theta_{k,r} - \bar{\lambda}_{k,r}(t)]\Delta t + \sigma_{k,r}\sqrt{\bar{\lambda}_{k,r}(t)\Delta t}\,\epsilon(k,t),$$
$$r = \{HI, LO\}, \quad k = 1, \ldots, 6.$$

Table 7 Estimation of the regime-switching model for the mean of each rating class: this table presents parameters for the regime-switching model applied to the mean process for each rating class, i.e. $\bar{\lambda}_k$, $k = 1, \ldots, 6$.

	Rating class					
Parameter	1	2	3	4	5	6
θ_{LO}	0.0559	0.1625	0.5678	1.3588	3.6367	6.3138
t-stat	0.40	7.06	11.90	14.22	45.24	74.32
θ_{HI}	0.0991	0.3757	0.9818	2.2358	5.1216	7.8295
t-stat	0.36	6.55	4.91	19.29	18.74	8.07
κ_{LO}	1.3535	0.5094	1.5855	1.1317	2.2365	6.2456
t-stat	0.23	1.31	2.86	2.25	3.49	6.45
κ_{HI}	0.6152	0.1775	1.1717	2.8567	2.0833	1.8862
t-stat	0.35	1.74	0.95	2.52	1.67	1.81
σ_{LO}	0.0568	0.1224	0.3214	0.2986	0.3338	0.6798
t-stat	17.63	17.27	16.19	16.57	16.92	16.02
σ_{HI}	0.1103	0.2214	0.3275	0.3305	0.3681	0.8336
t-stat	8.41	7.83	7.84	7.93	8.15	8.24

Table 9 Metric for best correlated default model (normal and Gumbel copulas): this table presents the summary statistic for the asymmetric correlation metric to determine the best simulation model. For each model we generate the asymmetric correlation plot and then compute the distance metric. The asymmetric correlations (the metric q) are computed for the following seven cases: normal distribution, Student-t distribution, skewed double-exponential distribution, the combination of the best distributions based on Kolmogorov criterion, the combination of the best distributions based on the Anderson and Darling statistic, the combination based on the L^1 and L^2 norms. Our metric comprises the average squared point-wise difference between the empirical exceedance correlation plot and the simulated one. This a natural distance metric. The points in each plot that are used are for the combination of rating class (from 1 to 6) and exceedance levels (from -1.5 to $+1.5$). Hence, there are a total of 48 points in each plot which are used for computing the metric. Define the points in the empirical plot as $h_{k,x}$, where k indexes the rating class and x indexes the exceedance levels. The corresponding points in the simulated plot are denoted $h'_{k,x}$. The metric q (a RMSE statistic) is as follows: $q = \sqrt{\frac{1}{48}\sum_{k=1}^{6}\sum_{x=-1.5}^{1.5}[h_{k,x} - h'_{k,x}]^2}$. We report the results for both models, the jump-diffusion setup and the regime-switching one, and two copulas, the Gaussian and Gumbel copulas.

	Gaussian copula		Gumbel copula	
Model	Jump-diffusion	Regime-shifting	Jump-diffusion	Regime-shifting
Normal	0.0508	0.0276	0.0486	0.0462
Student's t	0.0375	0.0327	0.0416	0.0515
Double-exponential	0.0752	0.0355	0.0700	0.0381
Kolmogorov	0.0802	0.0387	0.0743	0.0364
Anderson–Darling	0.0447	0.0315	0.0452	0.0450
L^2	0.0764	0.0367	0.0710	0.0376
L^1	0.0667	0.0318	0.0625	0.0401

Table 10 Metric for best correlated default model (Clayton and Student's t copulas): this table presents the summary statistic for the asymmetric correlation metric to determine the best simulation model. For each model we generate the asymmetric correlation plot and then compute the distance metric. The asymmetric correlations (the metric q) are computed for the following seven cases: normal distribution, Student-t distribution, skewed double-exponential distribution, the combination of the best distributions based on Kolmogorov criterion, the combination of the best distributions based on the Anderson and Darling statistic, the combination based on the L^1 and L^2 norms. Our metric comprises the average squared pointwise difference between the empirical exceedance correlation plot and the simulated one. This a natural distance metric. The points in each plot that are used are for the combination of rating class (from 1 to 6) and exceedance levels (from -1.5 to $+1.5$). Hence there are a total of 48 points in each plot which are used for computing the metric. Define the points in the empirical plot as $h_{k,x}$, where k indexes the rating class and x indexes the exceedance levels. The corresponding points in the simulated plot are denoted $h'_{k,x}$. The metric q (a RMSE statistic) is as follows: $q = \sqrt{\frac{1}{48}\sum_{k=1}^{6}\sum_{x=-1.5}^{1.5}[h_{k,x} - h'_{k,x}]^2}$. We report the results for both models, the jump-diffusion set up and the regime-switching one, and two copulas, the Clayton and Student's t copulas.

	Clayton copula		Student's t copula	
Model	Jump-diffusion	Regime-shifting	Jump-diffusion	Regime-shifting
Normal	0.0484	0.0395	0.0446	0.0506
Student's t	0.0415	0.0473	0.0370	0.0564
Double-exponential	0.0694	0.0304	0.0672	0.0433
Kolmogorov	0.0738	0.0293	0.0714	0.0404
Anderson–Darling	0.0445	0.0397	0.0418	0.0514
L^2	0.0703	0.0296	0.0681	0.0423
L^1	0.0622	0.0324	0.0596	0.0452

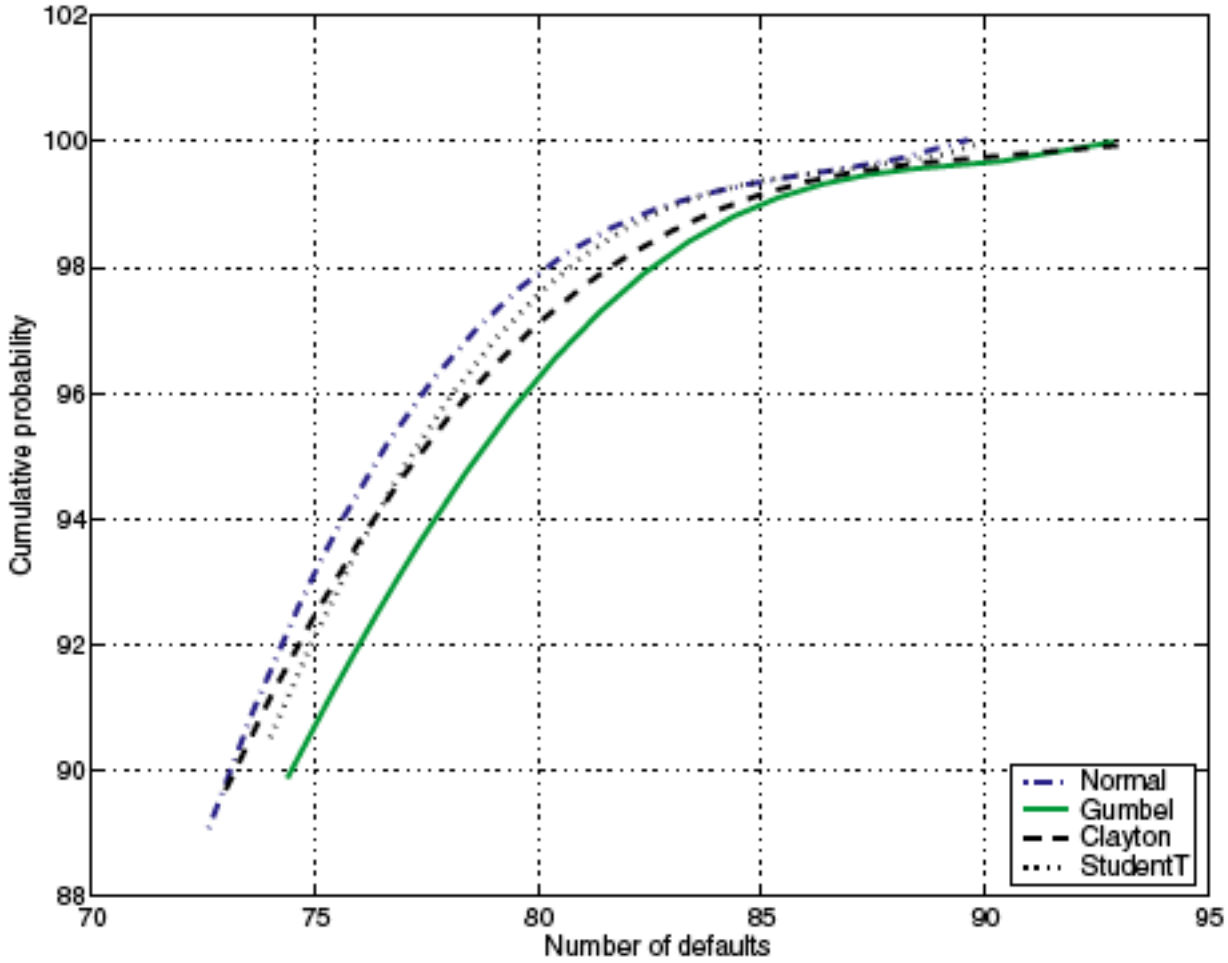
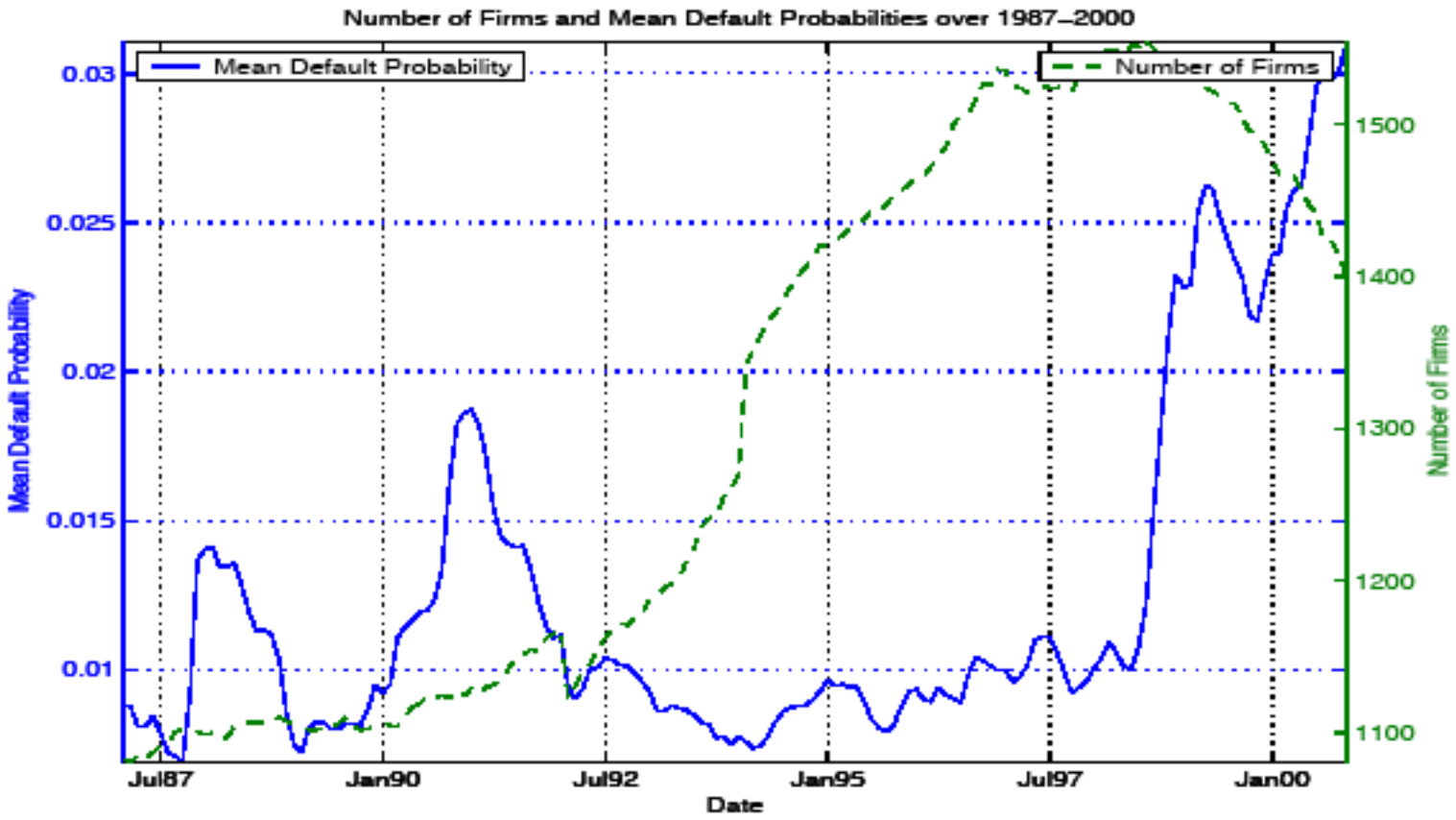


Figure 5 Comparing copula tail loss distributions: this figure presents plots of the tail loss distributions for four copulas, when the marginal distribution is normal. The *x*-axis shows the number of losses out of more than 600 issuers, and the *y*-axis depicts the percentiles of the loss distribution. The simulation runs over a horizon of 5 years and accounts for regime shifts as well. The copulas used are: normal, Gumbel, Clayton, Student's *t*.

Contagion
(Das, Duffie, Kapadia, Saita: Journal of Finance, 2007)



Cox Process Framework

- Key ingredient in a "intensity" model (as opposed to "structural" models).
- Intensity is based on exogenous state variables $\mathbf{X}$.
- Default arrival by jump N_t such that $\lambda(X_t)$ is the $\mathcal{F}_t$-intensity of N.
- Also known as a "doubly stochastic" process.
  Default depends on a 2-step process:
  1. Intensity.
  2. Poisson arrival conditional on intensity.

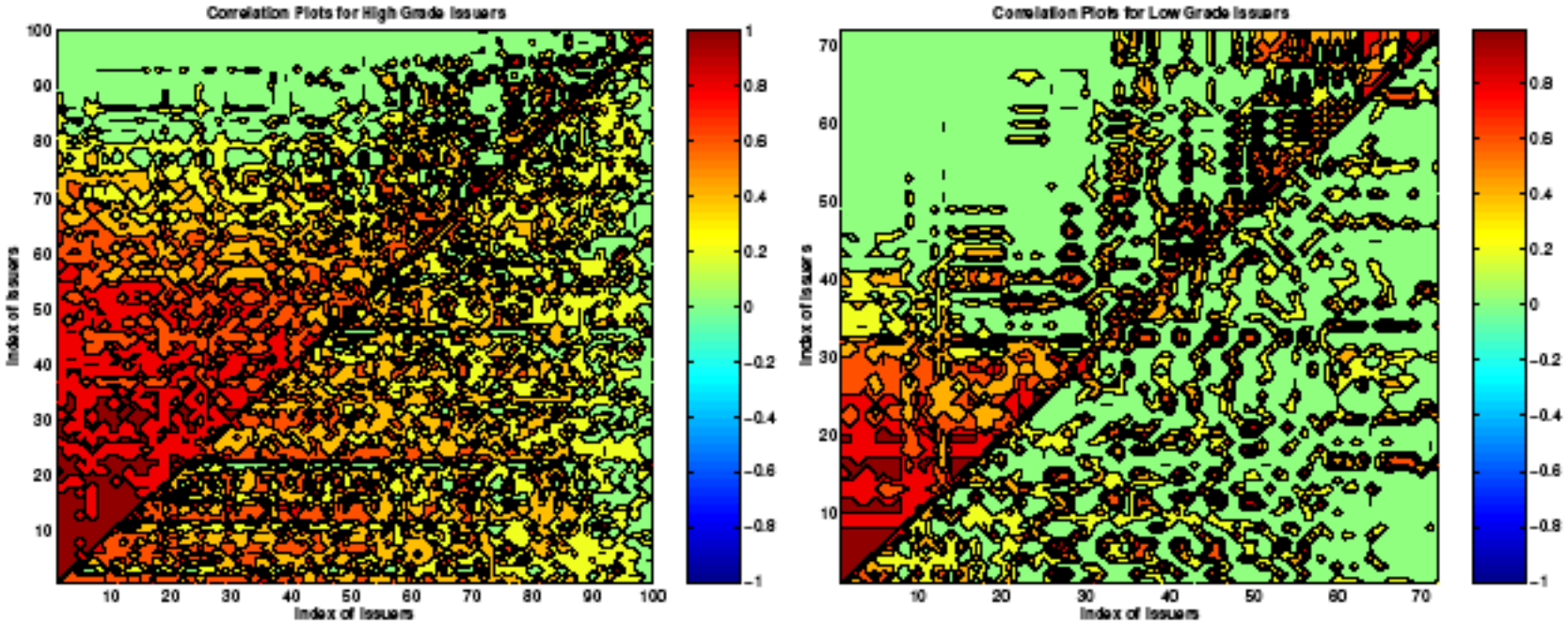
  Hence correlation could arise in either step.

Doubly Stochastic Process

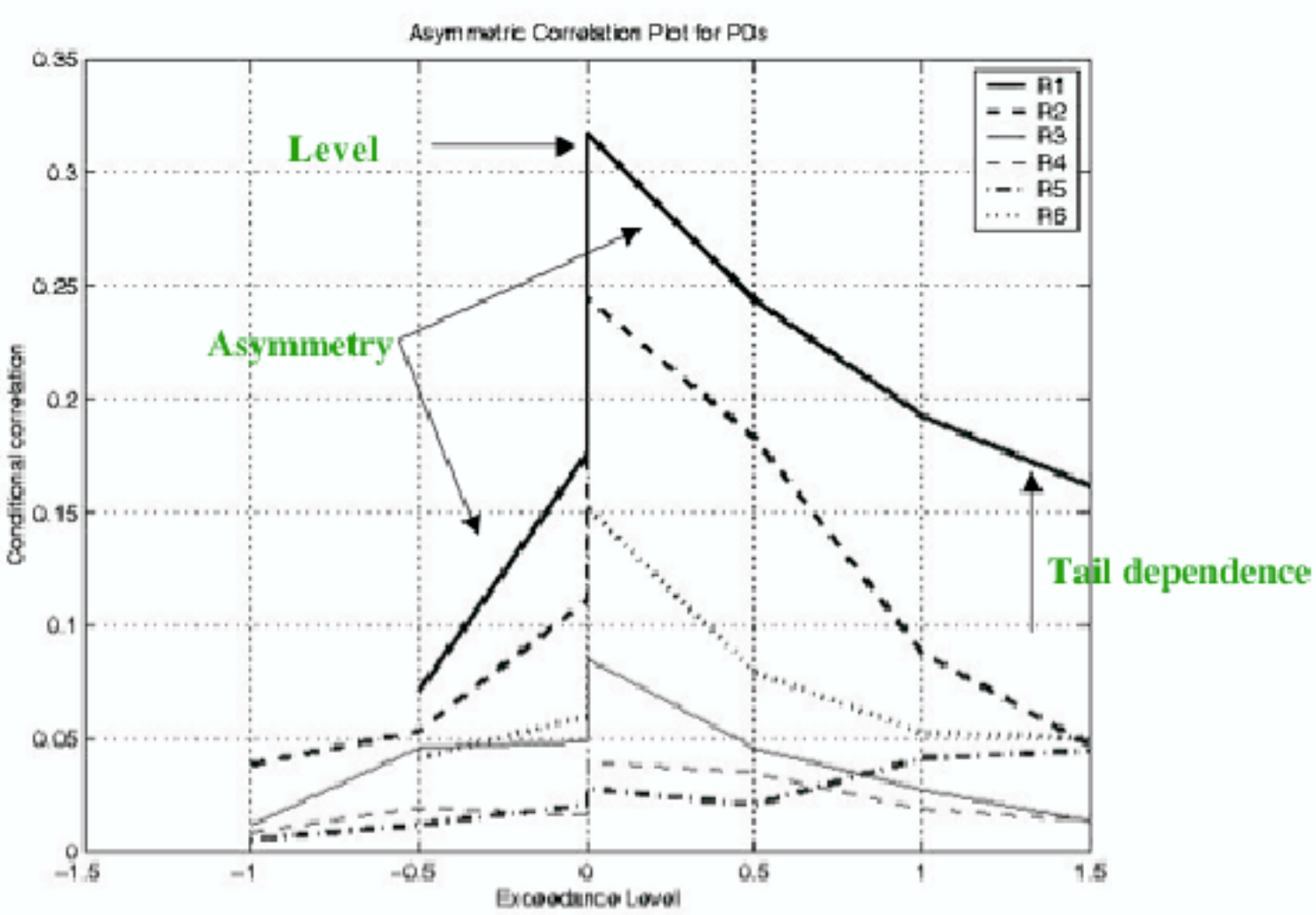
- Two processes:
  - Process 1: Default intensity: λ.
  - Process 2: Conditional default probability: $Pr[D_i = 1|\lambda],\ \forall i$.
- Doubly stochastic assumption is that processes in 2 are independent $\Longrightarrow$ defaults are Poisson after conditioning.
- Plenty of data for Process 1, much less for Process 2.
- Strong evidence that Process 1 evidences correlation across intensities.
- Nothing known about Process 2.

How Do Defaults Cluster?

- Factor-based: systematic default risk.
- Frailty: excess clustering in intensities after default arrivals.
- Contagion: excess clustering in default arrivals.

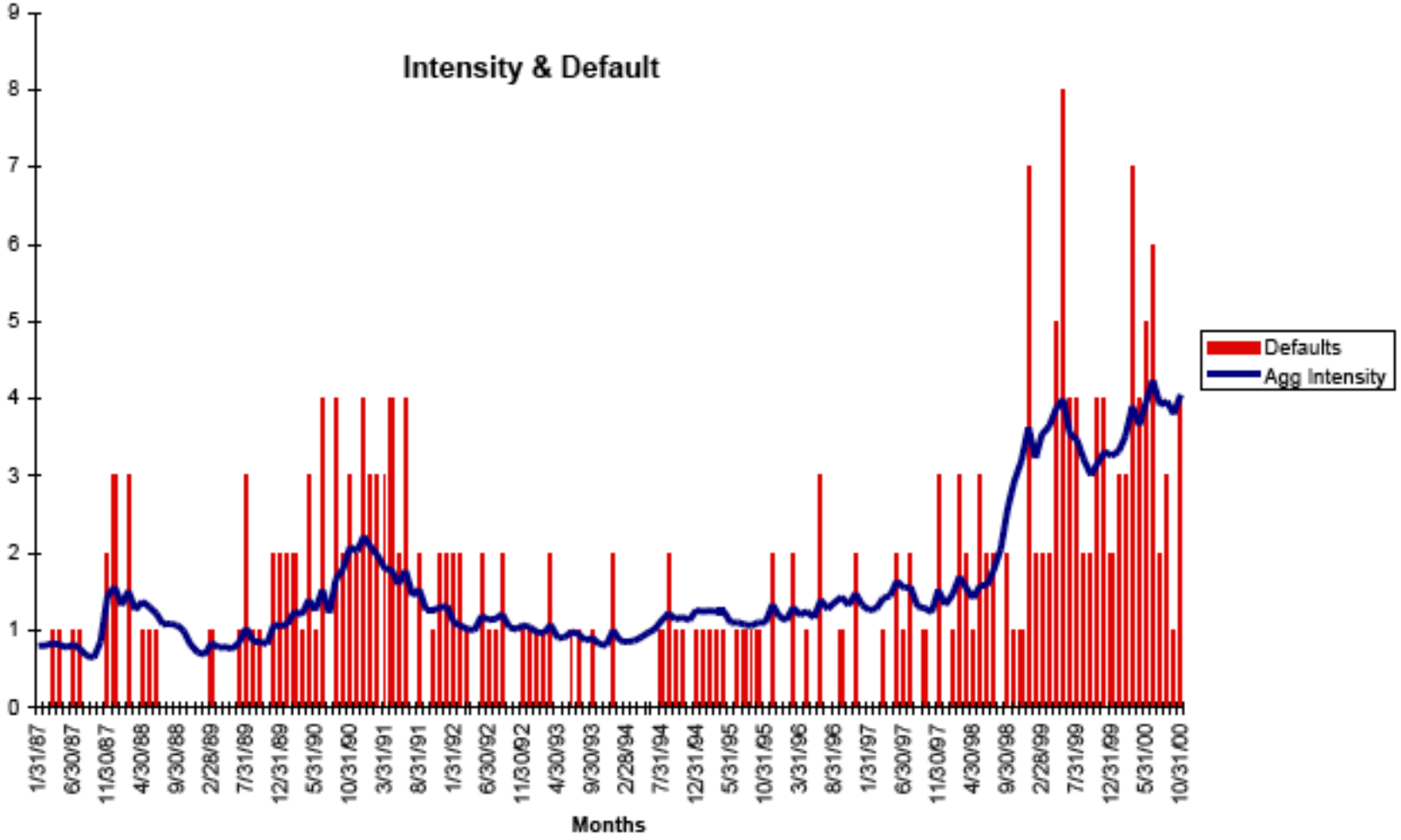


Heat Maps


Asymmetric Correlation Plot for PDs
Level
Asymmetry
Tail dependence
R1
R2
R3
R4
R5
R6
Conditional correlation
Exceedance Level
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−1.5
−1
−0.5
0
0.5
1
1.5

Question

- Qualitatively: Is correlation in default intensities sufficient to account for the degree of default clustering?
- Technically: Conditional on the path of risk factors (intensities), are defaults independent Poisson arrivals?



Intensities based on the model of Duffie, Saita, Wang (2005)

Rescaling to Intensity Time

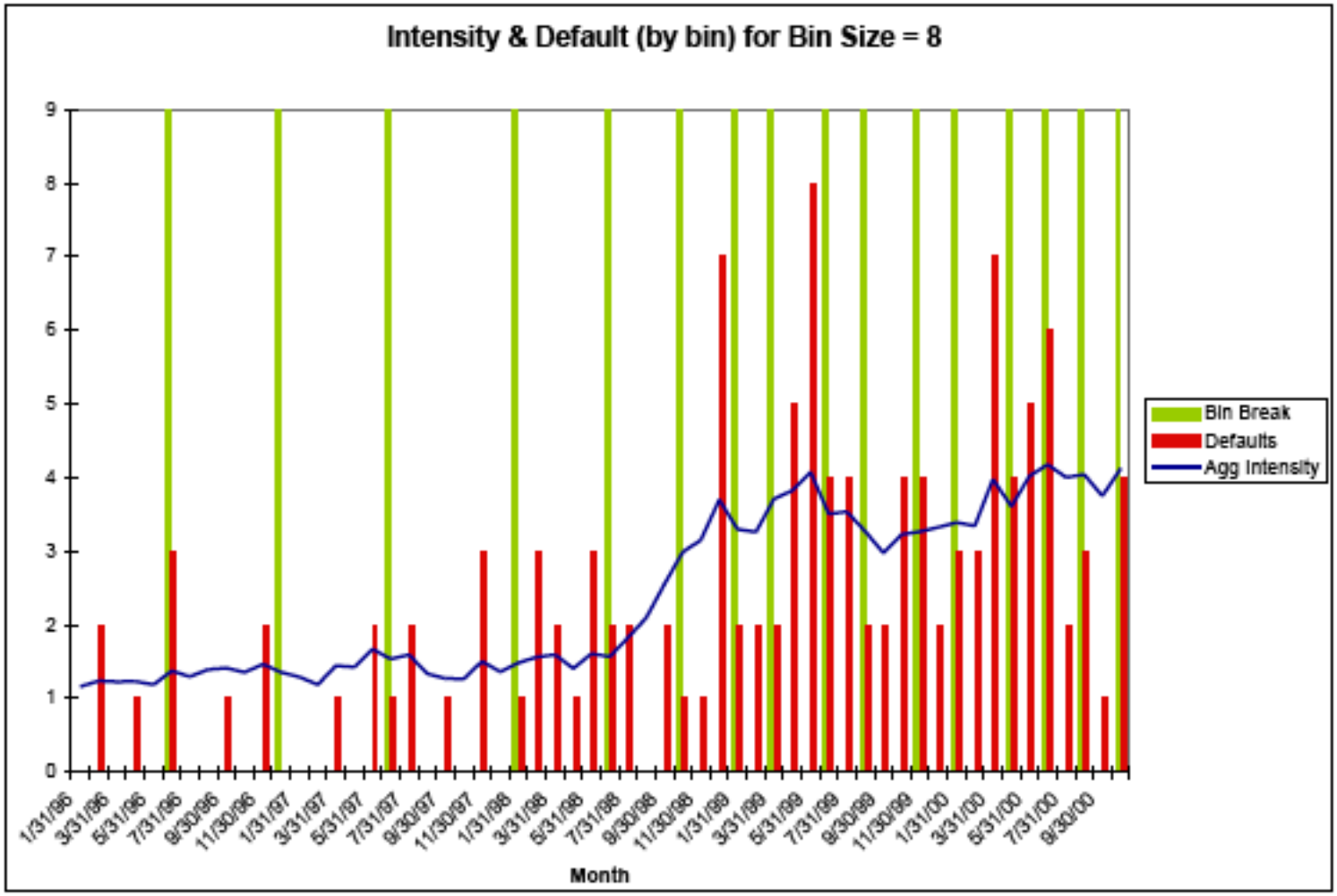
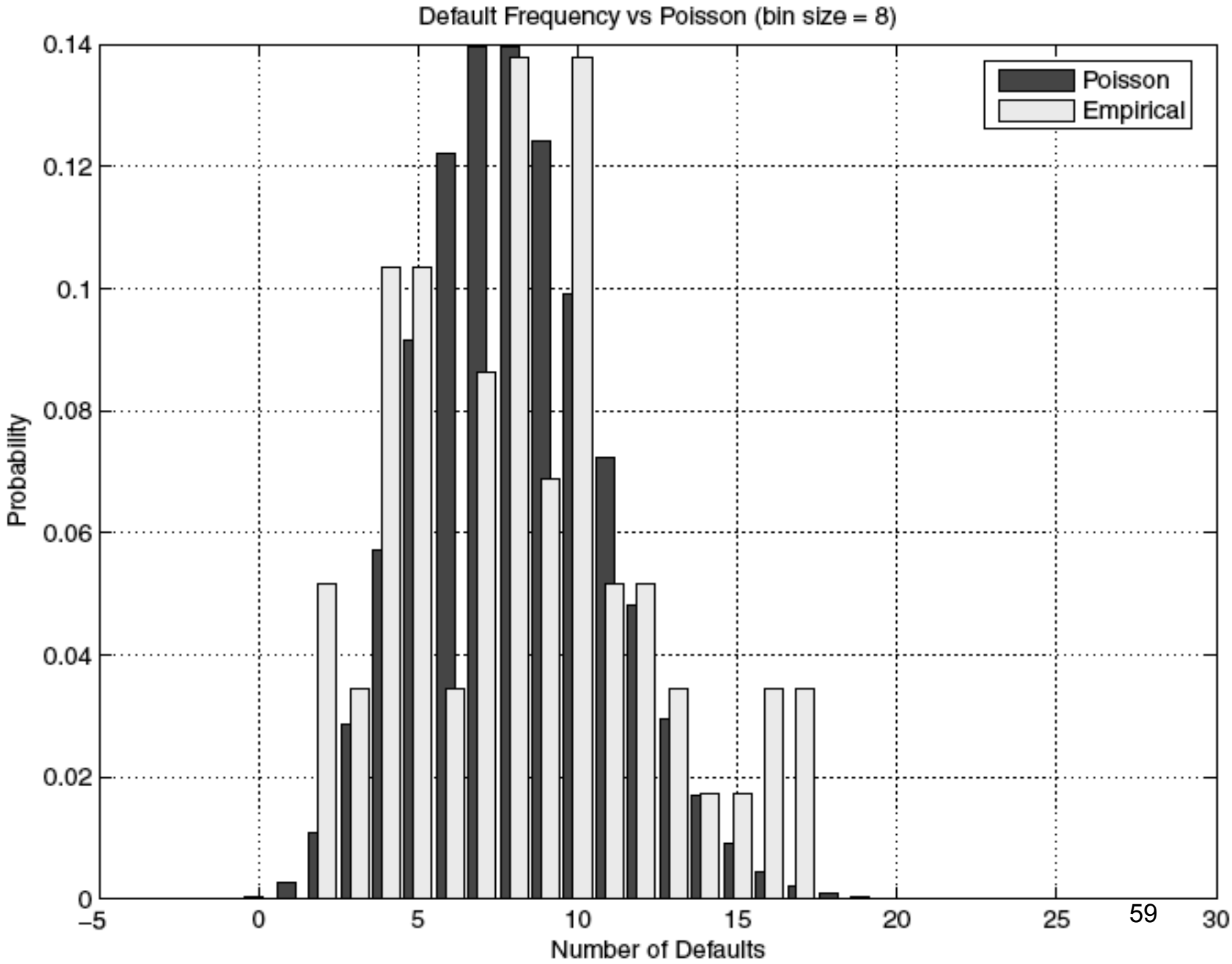


Table 1

Empirical and Theoretical Moments

This table presents a comparison of empirical and theoretical moments for the distribution of defaults per bin. The number K of bin observations is shown in parentheses under the bin size. The upper-row moments are those of the theoretical Poisson distribution under the doubly stochastic hypothesis; the lower-row moments are the empirical counterparts.

Bin Size	Mean	Variance	Skewness	Kurtosis
2	2.04	2.04	0.70	3.49
(230)	2.12	2.92	1.30	6.20
4	4.04	4.04	0.50	3.25
(116)	4.20	5.83	0.44	2.79
6	6.04	6.04	0.41	3.17
(77)	6.25	10.37	0.62	3.16
8	8.04	8.04	0.35	3.12
(58)	8.33	14.93	0.41	2.59
10	10.03	10.03	0.32	3.10
(46)	10.39	20.07	0.02	2.24


Default Frequency vs Poisson (bin size = 8)
Poisson
Empirical
Probability
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
−5
0
5
10
15
20
25
30
Number of Defaults

Fisher's Dispersion Test

Fixing the bin size c, under the null,

$$W = \sum_{i=1}^{K} \frac{(X_i - c)^2}{c},$$

is χ^2 with $K - 1$ degrees of freedom.

Fisher's Dispersion Test

The table presents Fisher's dispersion test for goodness of fit of the Poisson distribution with mean equal to bin size. Under the joint hypothesis that default intensities are correctly measured and the doubly stochastic property, the statistic W is χ^2-distributed with $K - 1$ degrees of freedom, and is provided in equation (2).

Bin Size	K	W	p-Value
2	230	336.00	0.0000
4	116	168.75	0.0008
6	77	132.17	0.0001
8	58	107.12	0.0001
10	46	91.00	0.0001

Upper Tail Tests

For a given bin size c, suppose there are K bins. We let M denote the sample mean of the upper quartile of the empirical distribution of distribution of $X_1, \ldots, X_K$. By Monte Carlo simulation, we generated 10,000 data sets, each consisting of K iid Poisson random variables with parameter c. We then compute the fraction p of the simulated data sets whose sample upper-quartile size (mean or median) is above the actual sample mean M. Under the null hypothesis that the distribution of the actual sample is Poisson with parameter c, the p-value would be approximately 0.5.

Mean and Median of Default Upper Quartiles

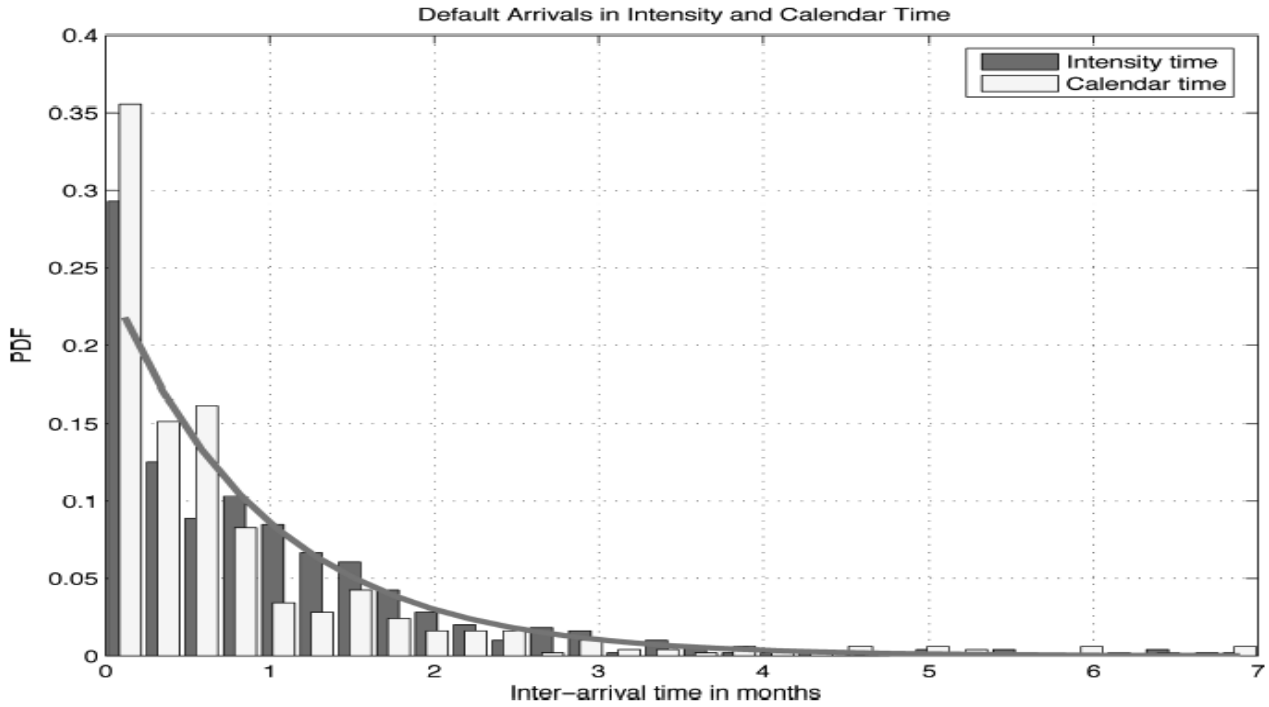
This table presents tests of the mean and median of the upper quartile of defaults per bin against the associated theoretical Poisson distribution. The last row of the table, "All," indicates the estimated probability, under the hypothesis that time-changed default arrivals are Poisson with parameter 1, that there exists at least one bin size for which the mean (or median) of number of defaults per bin exceeds the corresponding empirical mean (or median).

	Mean of Tails			Median of Tails		
Bin Size	Data	Simulation	*p*-Value	Data	Simulation	*p*-Value
2	4.00	3.69	0.00	4.00	3.18	0.00
4	7.39	6.29	0.00	7.00	6.01	0.00
6	9.96	8.95	0.02	9.00	8.58	0.06
8	12.27	11.33	0.08	11.50	10.91	0.19
10	16.08	13.71	0.00	16.00	13.25	0.00
All			0.0018			0.0003

Moments of the Distribution of Inter-default Times

This table presents selected moments of the distribution of inter-default times. Under the joint hypothesis of doubly stochastic defaults and correctly measured default intensities, the inter-default times in intensity-based time units are exponentially distributed. The inter-arrival time empirical distribution is also shown in calendar time, after a linear scaling of time that matches the first moment, mean inter-arrival time.

Moment	Intensity Time	Calendar Time	Exponential
Mean	0.95	0.95	0.95
Variance	1.17	4.15	0.89
Skewness	2.25	8.59	2.00
Kurtosis	10.06	101.90	6.00



Prahls (1999) Test of Clustered Defaults (across bin sizes)

Prahl's test statistic is based on the fact that, in the new time scale under which default arrivals are those of a Poisson process (with rate parameter 1), the inter-arrival times $Z_1, Z_2, \ldots$ are iid exponential of mean 1.

Letting C^* denote the sample mean of $Z_1, \ldots, Z_n$, Prahl shows that

$$M = \frac{1}{n} \sum_{\{Z_k < C^*\}} \left(1 - \frac{Z_k}{C^*}\right). \tag{15}$$

is asymptotically (in n) normal with mean $e^{-1} - \alpha/n$ and variance β^2/n,

$$\alpha \simeq 0.1839, \quad \beta \simeq 0.2431.$$

Using our data, for $n = 495$ default times,

$$M = 0.4055$$

$$\mu_n = \frac{1}{e} - \frac{\alpha}{n} = 0.3675$$

$$\sigma_n = \frac{\beta}{\sqrt{n}} = 0.0109.$$

The test statistic M measured from our data is 3.48 standard deviations from the asymptotic mean associated with the null hypothesis of i.i.d. exponential inter-default times (in the new time scale), indicating some evidence of default clustering in excess of that associated with the default intensities under the doubly stochastic model. (In the calendar time scale, the same test statistic M is 11.53 standard deviations from the mean μ_n under the null of exponential inter-default times.)

Residual Gaussian Copula Correlation

Using a Gaussian copula for intensity-conditional default times and equal pairwise correlation r for the underlying normal variables, we estimate by Monte Carlo the mean of the upper quartile of the empirical distribution of the number of defaults per bin, according to an algorithm described in the Appendix. We set in bold the correlation parameter r at which the Monte Carlo-estimated mean best approximates the empirical counterpart. (Under the null hypothesis of correctly measured intensity and the doubly stochastic assumption, the theoretical residual Gaussian copulation r is zero.)

Bin Size	Mean of Upper Quartile (data)	Mean of Simulated Upper Quartile Copula Correlation				
		$r = 0.00$	$r = 0.01$	$r = 0.02$	$r = 0.03$	$r = 0.04$
2	4.00	3.87	**4.01**	4.18	4.28	4.48
4	7.39	6.42	6.82	7.15	**7.35**	7.61
6	9.96	8.84	9.30	9.74	**10.13**	10.55
8	12.27	11.05	11.73	**12.29**	12.85	13.37
10	16.08	13.14	14.01	14.79	15.38	**16.05**