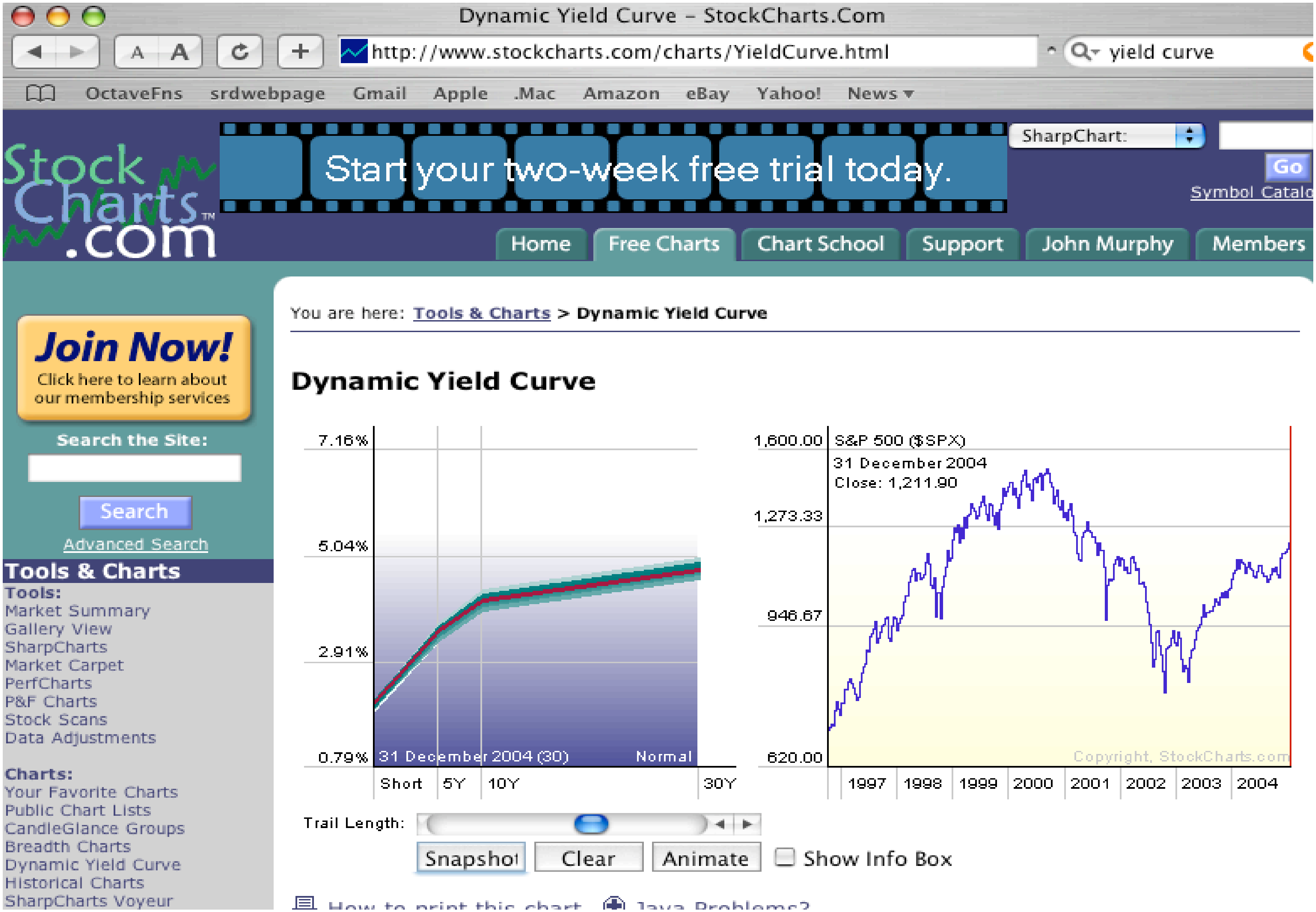
Risk Free Debt Markets

Bond Mathematics

Yield to Maturity: internal rate of return on a bond



U.S. Treasury – Daily Treasury Yield Curve

http://www.treasury.gov/offices/domestic-finance/debt-management/interest-rate/yield.html

bpage Gmail Apple .Mac Amazon eBay Yahoo! News

INTEREST RATE STATISTICS

Daily Treasury Yield Curve Rates

Historical Data

December 2004

Date	1 mo	3 mo	6 mo	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr	20 yr
12/01/04	2.06	2.22	2.40	2.60	3.01	3.28	3.72	4.08	4.38	5.04
12/02/04	2.06	2.22	2.41	2.62	3.04	3.30	3.75	4.10	4.40	5.07
12/03/04	2.06	2.21	2.39	2.58	2.94	3.19	3.61	3.96	4.27	4.95
12/06/04	2.09	2.25	2.44	2.60	2.93	3.17	3.59	3.94	4.24	4.92
12/07/04	2.08	2.25	2.43	2.60	2.95	3.19	3.60	3.94	4.23	4.91
12/08/04	2.08	2.24	2.42	2.59	2.91	3.12	3.53	3.85	4.14	4.80
12/09/04	2.07	2.24	2.42	2.59	2.93	3.15	3.54	3.87	4.19	4.84
12/10/04	2.07	2.25	2.44	2.61	2.95	3.15	3.52	3.85	4.16	4.83
12/13/04	2.04	2.24	2.50	2.66	2.98	3.18	3.54	3.85	4.16	4.81
12/14/04	2.01	2.21	2.48	2.65	2.99	3.17	3.53	3.83	4.14	4.79
12/15/04	1.98	2.21	2.47	2.64	2.97	3.14	3.48	3.78	4.09	4.72
12/16/04	1.93	2.20	2.47	2.66	3.01	3.21	3.58	3.89	4.19	4.84
12/17/04	1.95	2.20	2.48	2.67	3.03	3.22	3.59	3.91	4.21	4.85
12/20/04	1.97	2.22	2.54	2.72	3.06	3.23	3.59	3.91	4.21	4.84
12/21/04	1.92	2.20	2.54	2.72	3.05	3.22	3.57	3.89	4.18	4.82
12/22/04	1.84	2.18	2.53	2.71	3.04	3.21	3.57	3.91	4.21	4.85
12/23/04	1.83	2.19	2.54	2.70	3.02	3.21	3.58	3.92	4.23	4.86
12/27/04	1.90	2.26	2.63	2.78	3.07	3.26	3.65	3.99	4.30	4.95
12/28/04	1.88	2.25	2.62	2.77	3.08	3.27	3.66	4.00	4.31	4.94
12/29/04	1.76	2.22	2.60	2.77	3.12	3.32	3.69	4.03	4.33	4.96
12/30/04	1.68	2.22	2.59	2.76	3.10	3.27	3.64	3.97	4.27	4.92
12/31/04	1.89	2.22	2.59	2.75	3.08	3.25	3.63	3.94	4.24	4.85

* 30-year Treasury constant maturity series was discontinued as of 2/18/02. See Long-Term Average Rate for more information.

- Daily Treasury Yield Curve Rates
- Daily Treasury Long-Term Rates
- Daily Treasury Real Yield Curve Rates
- Daily Treasury Real Long-Term Rates
- Weekly Aa Corporate Bond Index

Term premia

Coupon	Month	Year	Ask Price	Type	T (mos)	Ask YTM
2.375	8	2006	99.14	n	36	2.57
6.5	10	2006	111.13	n	38	2.67
3.5	11	2006	102.13	n	39	2.71
3.375	1	2007	108.04	i	41	0.93
6.25	2	2007	111.08	n	42	2.81
6.625	5	2007	112.26	n	45	2.95
4.375	5	2007	104.3	n	45	2.96
3.25	8	2007	100.18	n	48	3.09
6.125	8	2007	111.09	n	48	3.08
3	11	2007	99.06	n	51	3.2
3.625	1	2008	109.27	i	53	1.31
3	2	2008	98.2	n	54	3.33
5.5	2	2008	109.04	n	54	3.29
2.625	5	2008	96.16	n	57	3.43
5.625	5	2008	109.17	n	57	3.42
3.25	8	2008	98.23	n	60	3.53
4.75	11	2008	105.13	n	63	3.6
8.75	11	2008	101.22		63	0.85
3.875	1	2009	111.17	i	65	1.62
5.5	5	2009	109.06	n	69	3.7
9.125	5	2009	105.24		69	1.03
6	8	2009	111.17	n	72	3.82
10.375	11	2009	110.21		75	1.47
4.25	1	2010	114.07	i	77	1.87
6.5	2	2010	114.12	n	78	3.96
11.75	2	2010	114.1		78	1.8
5.75	5	2010	109.31	n	81	4.09
10	5	2010	113.19		81	1.89
12.75	11	2010	122.22		87	2.19
3.5	1	2011	109.26	i	89	2.06
5	2	2011	105.01	n	90	4.21
13.875	5	2011	129.22		93	2.49
5	8	2011	104.2	n	96	4.31
14	11	2011	134.24		99	2.64
3.375	1	2012	108.27	i	101	2.21
4.875	2	2012	103.13	n	102	4.39
3	7	2012	105.29	i	107	2.26
4.375	8	2012	99.13	n	108	4.46
4	11	2012	96.09	n	111	4.49
10.375	11	2012	128		111	3.22
3.875	2	2013	95.05	n	114	4.51
3.625	5	2013	93.15	n	117	4.46
1.875	7	2013	95.29	i	119	2.34
4.25	8	2013	97.24	n	120	4.53
12	8	2013	138.12		120	3.51
13.25	5	2014	149.1		129	3.62
12.5	8	2014	146.15		132	3.73
11.75	11	2014	143.18		135	3.81
11.25	2	2015	157.06		138	4.73
10.625	8	2015	152.19		144	4.8
9.875	11	2015	145.28		147	4.85
9.25	2	2016	140.09		150	4.89
7.25	5	2016	121.13		153	4.96
7.5	11	2016	123.27		159	5.01

Sample Treasury Rates

Type: n, i

Quotes in 1/32

Compounding convention

Consider a bond with a cash flow of c_i in t_i years, $i = 1, \ldots, n$. If the ytm is expressed with annual compounding, it is the value of y that satisfies

$$P = \sum_{i=1}^{n} \frac{c_i}{(1+y)^{t_i}}. \tag{1.1}$$

More generally, if we use a convention in which we compound k times a year, the ytm is that value of y which satisfies

$$P = \sum_{i=1}^{n} \frac{c_i}{(1+y/k)^{kt_i}}. \tag{1.2}$$

In particular, semi-annual compounding ($k = 2$) is a popular convention in the many parts of the world (including the US) where sovereign bonds pay semi-annual coupons. In this case, (1.2) becomes

$$P = \sum_{i=1}^{n} \frac{c_i}{(1+y/2)^{2t_i}}. \tag{1.3}$$

Progression of compounds

$$1 + 0.12 \; = \; 1.12.$$

$$1.06 \; \times \; \left(1 + \frac{0.12}{2}\right) \; = \; (1.06)^2 \; = \; 1.1236.$$

$$1.04 \; \times \; \left(1 + \frac{0.12}{3}\right) \; \times \; \left(1 + \frac{0.12}{3}\right) \; = \; (1.04)^3 \; = \; 1.124864.$$

$$\left(1 + \frac{0.12}{k}\right)^k$$

$$\left(1 + \frac{r}{k}\right)^{kt}$$

- 1.12 if interest is compounded annually ($k = 1$).
- 1.1236 if interest is compounded semi-annually ($k = 2$).
- 1.1255 if interest rate is compounded quarterly ($k = 4$).
- 1.1275 if interest is compounded continuously ($k = \infty$)

If $k \to \infty$, e^{rt}

$e = 2.71828$

How do you convert from one frequency to another?

Continuous compounding

$$P = \sum_{i=1}^{n} c_i e^{-y t_i}. \qquad (1.4)$$

The compounding frequency under which ytm is expressed is very important. For example, consider a bond with cash flows of \$5 in 6 months and \$105 in one year. Suppose the current price of the bond is \$101. If we express the bond's ytm under semi-annual compounding, its ytm is the value of y that satisfies

$$101 = \frac{5}{1+y/2} + \frac{105}{(1+y/2)^2},$$

which is roughly 8.93%. However, if we express the bond's ytm under continuous compounding, its ytm is the value of y that satisfies

$$101 = 5\,e^{-y/2} + 105\,e^{-y},$$

which is about 8.74%. Thus, in translating financial data on ytm's into bond prices, it is important to know the compounding frequency under which the ytm's have been computed.

Non-fungeability of YTM

1/2 yr bond

$$P = \frac{102.5}{1+y/2} = \frac{102.5}{1.025} = 100.$$

$\longrightarrow$ x_1

1 yr bond

$$100 = \frac{3}{1+y/2} + \frac{103}{(1+y/2)^2}$$

Ytm = 0.06

$\longrightarrow$ x_2

$$\begin{aligned} 102.5\,x_1 + 3\,x_2 &= 4 \\ 103\,x_2 &= 104 \end{aligned}$$

Replication

$$P = \frac{4}{1.03} + \frac{104}{(1.03)^2} == 101.9135.$$

Correct?

$$x_1 = 0.009471939 \qquad x_2 = 1.009708738$$

$\longleftrightarrow$

$$100\,x_1 + 100\,x_2 = 101.9181$$

Not Equal!

Discount function (d)

$$P = \sum_{i=1}^{n} c_i d(t_i).$$

$$\begin{aligned} P(0.5) &= 102.5\, d(0.5) &&= 100 \\ P(1) &= 3\, d(0.5) + 103\, d(1) &&= 100. \end{aligned}$$

These equations are easily solved for $d(0.5)$ and $d(1)$:

$$d(0.5) = 0.97561 \qquad d(1) = 0.942458.$$

We can use these values to price any bond that has cash flows at $t = 0.5$ and $t = 1$. For example, the price of a one-year 8% coupon bond is

$$4\, d(0.5) + 104\, d(1) = 101.9181,$$

Zero-coupon or Spot rate: $r^{(k)}(t)$

$$d(t) \;=\; \frac{1}{(1+(r^{(k)}(t)/k))^{kt}} \qquad\qquad r^{(k)}(t) \;=\; k \times ([d(t)]^{-1/kt} - 1)$$

$$d(0.5) \;=\; 0.97561 \qquad\qquad d(1) \;=\; 0.942458.$$

$$\begin{aligned} r^{(2)}(0.5) &= 2[d(0.5)^{-1} - 1] &= 0.05 \\ r^{(2)}(1) &= 2[d(1)^{-1/2} - 1] &= 0.060151. \end{aligned}$$

Let us use these zcr's to value a one-year bond with a coupon of 8%. The bond has two cash flows: a cash flow of 4 after 6 months and a cash flow of 104 after one year. Given the zcr's, the present values of these cash flows are

$$\frac{4}{1+r^{(2)}(0.5)/2} \;=\; \frac{4}{1.025} \;=\; 3.902439$$

and

$$\frac{104}{(1+r^{(2)}(1)/2)^2} \;=\; \frac{104}{(1.030075)^2} \;=\; 98.0157.$$

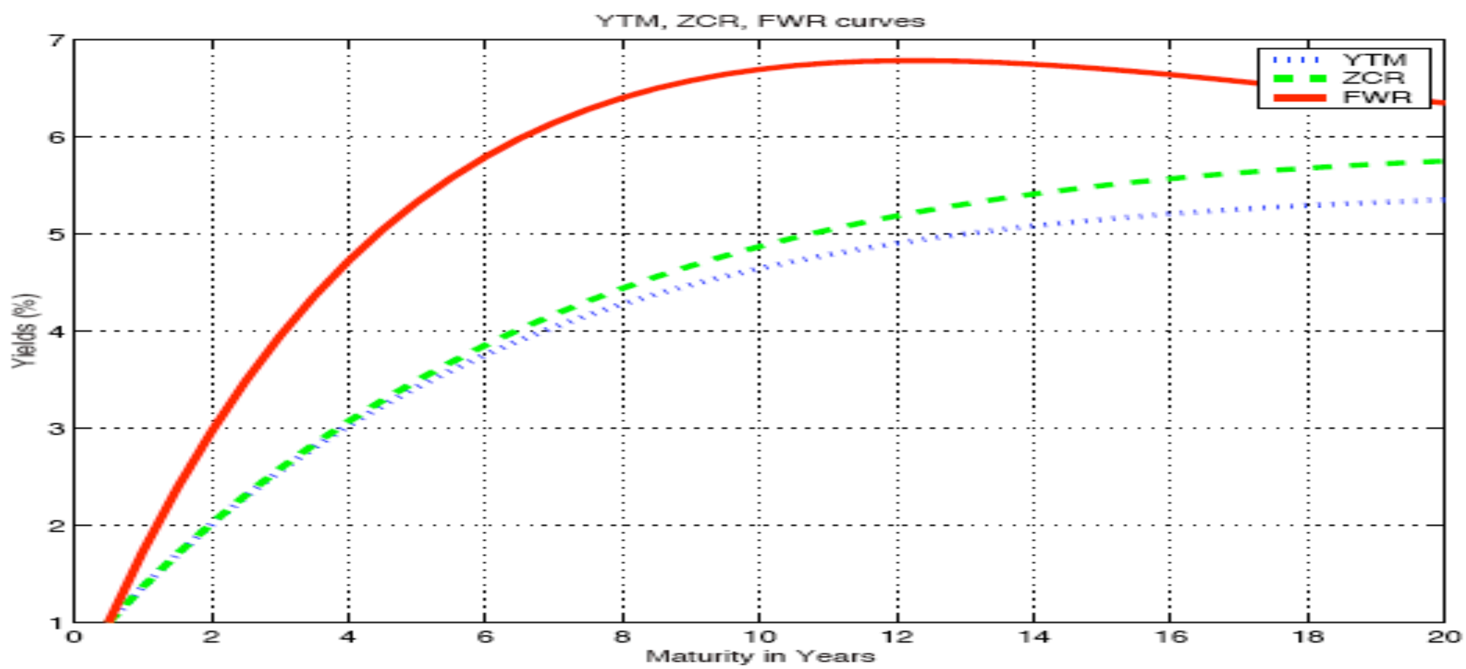
Adding these values, we obtain 101.9181, which is exactly the bond value obtained by replication or using the discount functions.

Forward Rates: $f(t_1,t_2)$

Compounding $$e^{f(t_1,t_2)\,(t_2-t_1)} = \frac{d(t_1)}{d(t_2)}$$

$$f(t_1,t_2) = \frac{1}{t_2-t_1}\ln\left(\frac{d(t_1)}{d(t_2)}\right) = \frac{\ln d(t_1) - \ln d(t_2)}{t_2-t_1}$$

$$f(t_1,t_2) = \frac{t_2 r(t_2) - t_1 r(t_1)}{t_2 - t_1}$$



Bootstrapping

Standard convention is semi-annual in the U.S.

Bond	Price
6–month zero	0.95959
1–year zero	0.91851
18–month 8% coupon	0.98857
24–month 9% coupon	1.00127

The 18–month 8% bond has cash flows of 0.04 at the 6-month and one-year points, and a cash flow of 1.04 at the 18-month point. Expressing its price in terms of these cash flows and the discount function, we have

$$0.98857 \;=\; (0.04)\,d(0.50) + (0.04)\,d(1) + (1.04)\,\boxed{d(1.50)}. \tag{2.2}$$

$$d(1.50) \;=\; 0.87831$$

One more period

$$1.00127 = (0.045)\,d(0.50) + (0.045)\,d(1) + (0.045)\,d(1.50) + (1.045)\,\boxed{d(2)}$$

$$d(2) = 0.83946.$$

$$\begin{aligned} d(0.50) &= 0.95959 \\ d(1.00) &= 0.91851 \\ d(1.50) &= 0.87831 \\ d(2.00) &= 0.83946 \end{aligned}$$

Translate into spot rates

$$\begin{aligned} r^{(2)}(0.50) &= 0.084223 \\ r^{(2)}(1.00) &= 0.086835 \\ r^{(2)}(1.50) &= 0.088402 \\ r^{(2)}(2.00) &= 0.089440 \end{aligned}$$

Cashflows in-between dates?

Does the set of chosen bonds matter?

Fitting d by regression!

Homework Exercise

You are provided the following data from the market. The market convention is semi-annual coupons and semi-annual compounding. Please answer the questions that follow the table.

Maturity (yrs)	YTM	Coupon rate
0.5	0.06	0.062
1.0	0.07	0.072
1.5	0.08	0.082
2.0	0.09	0.092

1. Plot the yield curve from this table.
2. What are the prices of the four bonds?
3. Using bootstrapping, step-by-step, compute the discount function for each maturity.
4. Convert the discount function into zcrs.
5. Plot the zcr curve, on the same graph as the ytm curve. Which of the curves lies above the other? Why do you think this is so?
6. Compute the forward rate curve (fwr) as well.
7. Plot the fwr curve. Where does it lie in relation to the other two curves? Why?

Splines

(N+1) intervals $[T_0, T_1], [T_1, T_2], \ldots, [T_N, T_{N+1}]$

Spline function $g_n(t)$ for each interval:

$$d(t) \;=\; \begin{cases} g_0(t), & \text{if } t \in [T_0, T_1) \\ \vdots & \vdots \\ g_k(t), & \text{if } t \in [T_k, T_{k+1}) \\ \vdots & \vdots \\ g_N(t), & \text{if } t \in [T_N, T_{N+1}] \end{cases}$$

Types:
(a) Polynomial
(b) Exponential

$$g_k(T_k).$$

However, T_k is also the upper-endpoint of the time-interval $[T_{k-1}, T_k)$, and on this interval, the discount function is specified by $g_{k-1}(\cdot)$. Thus, the discount function at T_k may also be taken to be

$$g_{k-1}(T_k).$$

With a continuous yield-curve, these values should coincide, so we must have

$$g_{k-1}(T_k) \;=\; g_k(T_k). \tag{2.10}$$

Thus, the functions g_{k-1} and g_k must be "knotted" together at their common endpoints, which explains the term "knot-points" to describe $T_1, \ldots, T_N$.

Polynomial Splines

$$g_k(t) \;=\; a_k + b_k t.$$

$$g_k(t) \;=\; a_k + b_k t + c_k t^2$$

If $\ell = 3$, we have *cubic splines*: each g_k is of the form

$$g_k(t) \;=\; a_k + b_k t + c_k t^2 + d_k t^3.$$

1. *Condition 0:* The present value of \$1 due immediately is evidently \$1. Thus, we should have $d(0) = 1$, which means

$$g_0(0) \;=\; 1. \tag{2.11}$$

2. *Condition 1: Continuity of the Discount Function.* This requires, as we have seen, that at the knot points we must have

$$g_k(T_{k+1}) \;=\; g_{k+1}(T_{k+1}), \quad k = 1, \ldots, N. \tag{2.12}$$

   This places N restrictions on the parameters.

3. *Condition 2: Continuity of the Forward Curve.* From equation (1.17), the instantaneous forward rate is related to the discount function via

$$f(t) \;=\; -\frac{1}{d(t)}\, d'(t).$$

Thus, if we want a *continuous* forward curve, we need $d'(\cdot)$ to be continuous. Thus, the values of $d'(\cdot)$ also need to be equated at the knot-points. That is, we must have:

$$g_k'(T_{k+1}) \;=\; g_{k+1}'(T_{k+1}), \quad \text{for } k = 1, \ldots, N. \tag{2.13}$$

This places a further N restrictions on the parameters.

4. *Smoothness of the Forward Curve:* Further, if we want the forward curve to be smooth and not jagged, we also need $f'(\cdot)$ to be continuous. This means the second-derivative $d''(\cdot)$ of the discount function must be continuous. Equating these second-derivatives at the knot-points, we obtain

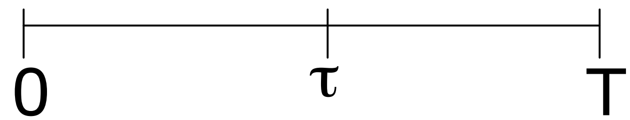
$$g_k''(T_{k+1}) \;=\; g_{k+1}''(T_{k+1}), \quad \text{for } k = 1, \ldots, N. \tag{2.14}$$

This too places N restrictions on the parameters.

cubic splines, or splines of a higher order than 3, all the conditions are relevant, and the number of parameters to be estimated is $4(N+1) - (3N+1) = N+3$.

Reduced Form

Let the single knot point be denoted τ



$$d(t) = \begin{cases} g_0(t), & t \in [0, \tau) \\ g_1(t), & t \in [\tau, T^*] \end{cases}$$

where

$$\begin{aligned} g_0(t) &= a_0 + b_0 t + c_0 t^2 + d_0 t^3 \\ g_1(t) &= a_1 + b_1 t + c_1 t^2 + d_1 t^3 \end{aligned}$$

In shorthand notation, we can write the discount function as

$$d(t) = g_0(t) + I_{t \geq \tau}(g_1(t) - g_0(t)), \tag{2.17}$$

where $I_{t\geq\tau}$ is the *indicator function* on $t \geq \tau$, i.e., the function that takes on the value 1 if $t \geq \tau$, and is zero otherwise. Writing the full forms of g_0 and g_1, this is

$$\begin{aligned} d(t) &= a_0 + b_0 t + c_0 t^2 + d_0 t^3 \\ &\quad + I_{t\geq\tau}\left[(a_1 - a_0) + (b_1 - b_0)\, t + (c_1 - c_0)\, t^2 + (d_1 - d_0)\, t^3\right]. \end{aligned} \tag{2.18}$$

There are 8 parameters in total: (a_0, b_0, c_0, d_0) and (a_1, b_1, c_1, d_1). However, there are four restrictions imposed on these parameters as described above:

- Condition 0: $d(0) = 1$. This means $g_0(0) = 1$ so, from (2.16), we must have $a_0 = 1$.

- Condition 1: At the knot point τ, we must have $g_0(\tau) = g_1(\tau)$. Substituting for g_0 and g_1 from (2.16), this results in

$$(a_1 - a_0) + (b_1 - b_0)\tau + (c_1 - c_0)\tau^2 + (d_1 - d_0)\tau^3 \;=\; 0. \tag{2.19}$$

- Condition 2: At the knot point τ, we must also have $g_0'(\tau) = g_1'(\tau)$. From expression (2.16),

$$g_i'(t) \;=\; b_i + 2c_i\,t + 3d_i\,t^2, \qquad i = 1,2$$

so this means

$$(b_1 - b_0) + 2(c_1 - c_0)\,\tau + 3(d_1 - d_0)\,\tau^2 \;=\; 0. \tag{2.20}$$

- Condition 3: Finally, at the knot point τ, we must also have $g_0''(\tau) = g_1''(\tau)$. From expression (2.16),

$$g_i''(t) \;=\; 2c_i + 6d_i\,t, \qquad i = 1,2$$

so

$$(c_1 - c_0) + 3(d_1 - d_0)\,\tau \;=\; 0. \tag{2.21}$$

Solving (2.19)–(2.21), we obtain:

$$a_1 - a_0 = -(d_1 - d_0)\tau^3$$

$$b_1 - b_0 = 3(d_1 - d_0)\,\tau^2$$

$$c_1 - c_0 = -3(d_1 - d_0)\,\tau$$

$$e_0 = d_1 - d_0$$

$$\Big\downarrow$$

$$\begin{aligned} d(t) &= a_0 + b_0 t + c_0 t^2 + d_0 t^3 \\ &\quad + I_{t \geq \tau}\left[(a_1 - a_0) + (b_1 - b_0)\,t + (c_1 - c_0)\,t^2 + (d_1 - d_0)\,t^3\right]. \end{aligned} \tag{2.18}$$

$$\Big\downarrow$$

$$d(t) = a_0 + b_0 t + c_0 t^2 + d_0 t^3 + e_0 I_{t \geq \tau}\,(t - \tau)^3. \tag{2.25}$$

Expression (2.25) is the reduced-form representation of the discount function.

Estimating the Parameters by OLS

Given a discount function, the predicted price of a bond is simply the sum of the cash flows from the bond weighted by the appropriate discount factors. Consider a bond which has cash flows of ξ_t at times t. With the discount function given by (2.25), the theoretical price of such a bond is

$$\begin{aligned} \widehat{P} &= \textstyle\sum_t \xi_t \, d(t) \\ &= \textstyle\sum_t \xi_t \left[1 + b_0 t + c_0 t^2 + d_0 t^3 + e_0 \, I_{(t \geq \tau)} \, (t - \tau)^3\right] \end{aligned} \tag{2.26}$$

Rearranging, we obtain

$$\widehat{P} - \sum_t \xi_t \;=\; b_0 X_1 + c_0 X_2 + d_0 X_3 + e_0 X_4 \tag{2.27}$$

where

$$\begin{aligned} X_1 &= \textstyle\sum_t \xi_t t \\ X_2 &= \textstyle\sum_t \xi_t t^2 \\ X_3 &= \textstyle\sum_t \xi_t t^3 \\ X_4 &= \textstyle\sum_t \xi_t I_{(t \geq \tau)} \, (t - \tau)^3 \end{aligned}$$

The four free parameters b_0, c_0, d_0, and e_0 may now be estimated by regressing $(P - \sum \xi_t)$ on X_1, X_2, X_3 and X_4. This is a simple OLS regression.

Exponential Splines

Exponential splines use exponential functions of time as drivers. In an exponential spline, each function g_k takes on the form

$$g_k(t) \;=\; a_k + b_k(1 - e^{-mt}) + c_k(1 - e^{-2mt}) + d_k(1 - e^{-3mt}) + \cdots \tag{2.28}$$

Here, $m > 0$ is an additional free parameter. This last parameter has a nice interpretation: if $f(t)$ denotes the forward curve generated by the splined discount function (2.28), then it turns out that

$$m \;=\; \lim_{t\to\infty} f(t),$$

so m is the "long forward rate." Once again, the most popular form is *cubic* exponentials, i.e., to have each g_k of the form

$$g_k(t) \;=\; a_k + b_k(1 - e^{-mt}) + c_k(1 - e^{-2mt}) + d_k(1 - e^{-3mt}). \tag{2.29}$$

How many knot points? Smoothness vs fit

Nelson-Siegel Curve Fitting

$$f(t) \;=\; \beta_0 + \beta_1 \exp\left(-\frac{t}{\theta}\right) + \beta_2 \left(\frac{t}{\theta}\right) \exp\left(-\frac{t}{\theta}\right)$$

- β_0: As $t \to \infty$, the forward rate $f(t)$ given by (2.30) goes to β_0. Thus, β_0 is just the long forward rate. Moreover, a change in β_0 results in a parallel shift in the forward curve. Hence, β_0 is also called the "level" parameter.

- β_1: At $t = 0$, the forward rate $f(t)$ under (2.30) is equal to $\beta_0 + \beta_1$. Thus, $\beta_0 + \beta_1$ is the short forward rate. This means β_1 is the difference between the short and long forward rates. This is called the "slope" of the curve.

- β_2: This determines the magnitude and direction of the hump in $f(t)$. If $\beta_2 > 0$, then the $f(t)$ curve has a hump at θ. If $\beta_2 < 0$, then the $f(t)$ curve has a U-shape at θ. As such, β_2 is sometimes referred to as the "curvature" parameter.

- θ: This determines the location of the hump or U-Shape. The last term of the NS function has two countervailing terms, (t/θ) which increases in t, and $\exp(-t/\theta)$, which decreases in t. As t increases (provided $\theta > 0$), the curve rises initially on account of the first part that increases in t, and then the exponential decay of the second part gathers greater influence, and drives the curve downward. Where this cut-over occurs depends on the size of θ, and hence it determines the location of the hump.

Nelson-Siegel-Svensson

$$\begin{aligned} f(t) &= \beta_0 + \beta_1 \exp(-t/\theta) + \beta_2 [t/\theta] \exp(-t/\theta) \\ &\quad + \beta_3 [t/\nu] \exp(-t/\nu) \end{aligned}$$

additional term

1. Select a vector of starting parameters. This is the initial guess before the numerical search for the best parameters can begin.
2. Compute the spot rate curve and discount function corresponding to these initial parameters.
3. Using the discount function, determine theoretical (or model) coupon bond prices (i.e., prices under chosen parameters).
4. Compute the difference between predicted and actual prices.
5. Minimize the squared difference using a numerical procedure.

Bootstrapping by Matrix Inversion

$$P^k = \sum_{i=1}^{n} c_i^k d(t_i)$$

$$\begin{bmatrix} P^1 \\ \vdots \\ P^n \end{bmatrix} = \begin{bmatrix} c_1^1 & \dots & c_n^1 \\ \vdots & \vdots & \vdots \\ c_1^n & \dots & c_n^n \end{bmatrix} \begin{bmatrix} d(t_1) \\ \vdots \\ d(t_n) \end{bmatrix}$$

$$P = C \cdot d \longrightarrow d = C^{-1} \cdot P.$$

$$P = \begin{bmatrix} 0.95959 \\ 0.91851 \\ 0.98857 \\ 1.00127 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.04 & 0.04 & 1.04 & 0 \\ 0.045 & 0.045 & 0.045 & 1.045 \end{bmatrix} \qquad d = \begin{bmatrix} d(0.50) \\ d(1.00) \\ d(1.50) \\ d(2.00) \end{bmatrix}$$

Inverting the matrix C, we obtain

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.0385 & -0.0385 & 0.9615 & 0 \\ -0.0414 & -0.0414 & -0.0414 & 0.9569 \end{bmatrix}$$

so

$$d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.0385 & -0.0385 & 0.9615 & 0 \\ -0.0414 & -0.0414 & -0.0414 & 0.9569 \end{bmatrix} \begin{bmatrix} 0.95959 \\ 0.91851 \\ 0.98857 \\ 1.00127 \end{bmatrix} = \begin{bmatrix} 0.95959 \\ 0.91851 \\ 0.87831 \\ 0.83946 \end{bmatrix}$$