Chapter 30
HJM/LMM Models

Derivatives: Principles and Practice

Sundaram & Das

Outline

- Quick recap of concepts.
- One-factor HJM model.
- Two-factor HJM model.
- Deriving risk-neutral adjustments to set up the HJM model.
- Libor market models (LMM).
- Calibration and pricing vanilla interest-rate options in the LMM.
- Swap Market Model (SMM).

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Recap 1: Swaps, Caps/Floors Parity Relationships

Swap (Pay Fixed X/Rec Flo L)

= FRN(L) – Fixed Rate Bond(X)

= Cap(L,X) – Floor(L,X)

Uses:

– Synthetic Financing

– Reconfiguring securities (e.g. YCANs)

Recap 2: Differences in Modeling Interest-Rate Options vs Equity Options

The following three features of equity processes are not true of interest-rate processes:

1. The price S_t of the underlying security follows a lognormal distribution.
2. The interest-rate r is known and constant.
3. The volatility σ of the security's returns is constant.

Interest-rate trees are richer in structure.

It is not as easy to make sure that there is no-arbitrage on the tree. Here is an example of what might happen if you are not careful.

Recap 3: Arbitrage Violations Example

Arbitrarily moving the term structure up and down often leads to arbitrage. Here, we see that even a simple parallel shift of the term structure is fraught with error.

$$Z_u = \begin{pmatrix} 0.11 \\ 0.12 \\ 0.13 \end{pmatrix}$$

$$Z_0 = \begin{pmatrix} 0.10 \\ 0.11 \\ 0.12 \end{pmatrix}$$

$$Z_d = \begin{pmatrix} 0.09 \\ 0.10 \\ 0.11 \end{pmatrix}$$

the price of the two-year zero at this point will be either

$$\frac{100}{1.11} = 90.10$$

$$\frac{100}{1.09} = 91.74$$

initial price of the two-year zero is

$$\frac{100}{1.11^2} = 81.16$$

Therefore, the amount borrowed at inception under the strategy is 81.16. This is a one-year borrowing and so takes place at the one-year rate of 10%. At the end of one year, the amount to be repaid is

$$81.16 \times 1.10 = 89.276$$

Since this is less than either possible price of the two-year zero at the end of one year (90.10 or 91.74), the strategy leads to a cash inflow at the end of one year with no outflows. That is, the model specification admits arbitrage. ■

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HJM vs Factor Models

- Some factor models (e.g., Vasicek 1977) are single-factor only. HJM is multi-factor.
- The functional forms are generally specified and not tied to specific stochastic processes as in models like CIR (1985).
- Factor models usually model the *spot* curve, HJM models the *forward* curve.
- Risk-neutral drifts are purely functions of volatilities in HJM; in factor models a separate specification of risk premiums is required.
- HJM models are consistent with the initial term structures of interest rates and volatilities.

One-Factor HJM Model

The one-factor model assumes that forward rates of all maturities move up together or down together, albeit by different amounts, depending on the volatility of each forward rate.

The forward rate at time t for a *one-period* borrowing or investment at time s (where $s \geq t$ and $s \leq n-1$) is denoted $f(t,s)$. Note that $f(t,s)$ is quoted at time t but applicable to the period from s to $s+1$. All interest rates are quoted in continuously compounded and annualized terms. Since the time interval between s and $s+1$ is h years, this means \$1 invested at time s at the rate $f(t,s)$ will grow by time $s+1$ to

$$\exp\{f(t,s)\cdot h\}$$

Let $P(t,s)$ denote the time-t price of a zero-coupon bond maturing at time s and with a face value of \$1. The usual spot-forward parity arguments (see Section 26.14) tell us that we must have

$$P(t,s) = \exp\left\{-\sum_{i=t}^{s-1} f(t,i)\cdot h\right\}$$

The Goal: Evolve a tree of arbitrage-free forward rate curves.

Numerical Example

We illustrate the mechanics of setting up the HJM tree with a five-period model. The initial forward-rate term structure is given in the table below. All forward rates have the same volatility.

T	Forward Rate	Value
0	$f(0,0)$	0.10
1	$f(0,1)$	0.11
2	$f(0,2)$	0.12
3	$f(0,3)$	0.13
4	$f(0,4)$	0.14

$$\sigma = 0.015$$

$$P(0,5) = \exp\left[-\sum_{i=0}^{4} f(0,i) \times 1\right] = 0.548812$$

Evolution of tree and nodes

$$\mathbf{f}_u(t+1,T) = \begin{bmatrix} f_u(t+1,t+1) \\ f_u(t+1,t+2) \\ \vdots \\ f_u(t+1,n-1) \end{bmatrix} \qquad \mathbf{f}_d(t+1,T) = \begin{bmatrix} f_d(t+1,t+1) \\ f_d(t+1,t+2) \\ \vdots \\ f_d(t+1,n-1) \end{bmatrix}$$

To complete the specification of the model, we must explain how the curves $\mathbf{f}_u$ and $\mathbf{f}_d$ are related to $\mathbf{f}$. Let X be a random variable that takes on the value $+1$ with probability q and -1 with probability $1-q$. Then, for each s, we assume that

$$f(t+1,s) = f(t,s) + \alpha(t,s)\,h + \sigma(t,s)X\sqrt{h}$$

This means that for each s, we have

$$f_u(t+1,s) - f(t,s) = \alpha(t,s)h + \sigma(t,s)\sqrt{h}$$
$$f_d(t+1,s) - f(t,s) = \alpha(t,s)h - \sigma(t,s)\sqrt{h}$$

The probabilities q and $1-q$ represent the risk-neutral probabilities of up and down moves in the model. For convenience, we choose $q = \frac{1}{2}$. This is what HJM also assume, but the assumption is for expositional simplicity only and is not analytically necessary.

Note: *the tree is binomial, and the* ***vector*** *of forward rates is shifted up and down.*

Tree Branching

$f(0,0)$	0.10
$f(0,1)$	0.11
$f(0,2)$	0.12
$f(0,3)$	0.13
$f(0,4)$	0.14

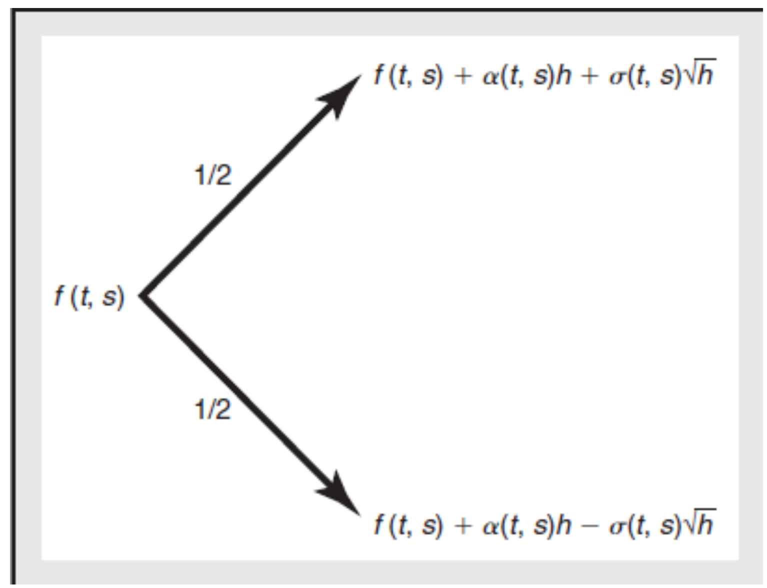
σ = 0.015

$$\mathbf{f}_u(t+h,T) = \begin{bmatrix} f_u(1,1) \\ f_u(1,2) \\ f_u(1,3) \\ f_u(1,4) \end{bmatrix} = \begin{bmatrix} f(0,1)+\alpha(0,1)+\sigma(0,1) \\ f(0,2)+\alpha(0,2)+\sigma(0,2) \\ f(0,3)+\alpha(0,3)+\sigma(0,3) \\ f(0,4)+\alpha(0,4)+\sigma(0,4) \end{bmatrix}$$

$$\sigma(t,s) = 0.15 \qquad \text{for all } (t,s)$$

$$\mathbf{f}_d(t+h,T) = \begin{bmatrix} f_d(1,1) \\ f_d(1,2) \\ f_d(1,3) \\ f_d(1,4) \end{bmatrix} = \begin{bmatrix} f(0,1)+\alpha(0,1)-\sigma(0,1) \\ f(0,2)+\alpha(0,2)-\sigma(0,2) \\ f(0,3)+\alpha(0,3)-\sigma(0,3) \\ f(0,4)+\alpha(0,4)-\sigma(0,4) \end{bmatrix}$$

$$\begin{bmatrix} 0.10 \\ 0.11 \\ 0.12 \\ 0.13 \\ 0.14 \end{bmatrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \begin{bmatrix} 0.125+\alpha(0,1) \\ 0.135+\alpha(0,2) \\ 0.145+\alpha(0,3) \\ 0.155+\alpha(0,4) \end{bmatrix} \\ \\ \begin{bmatrix} 0.095+\alpha(0,1) \\ 0.105+\alpha(0,2) \\ 0.115+\alpha(0,3) \\ 0.125+\alpha(0,4) \end{bmatrix} \end{matrix}$$



Identifying Risk-Neutral Drifts

$$P(0,2) = \exp[-(0.10 + 0.11)]$$

$$f_u(1,1) = 0.125 + \alpha(0,1)$$

$$f_d(1,1) = 0.095 + \alpha(0,1)$$

$$P_u(1,2) = \exp[-(0.125 + \alpha(0,1))]$$

$$P_d(1,2) = \exp[-(0.095 + \alpha(0,1))]$$

$$\begin{aligned} P(0,2) &= \exp(-0.10) \times \frac{1}{2}[P_u(1,2) + P_d(1,2)] \\ &= \exp(-0.10) \times \frac{1}{2}\{\exp[-(0.125 + \alpha(0,1))] + \exp[-(0.095 + \alpha(0,1))]\} \end{aligned}$$

$$\alpha(0,1) = 0.000112$$

$$\begin{bmatrix} 0.10 \\ 0.11 \\ 0.12 \\ 0.13 \\ 0.14 \end{bmatrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \begin{bmatrix} 0.125112 \\ 0.135 + \alpha(0,2) \\ 0.145 + \alpha(0,3) \\ 0.155 + \alpha(0,4) \end{bmatrix} \\ \\ \begin{bmatrix} 0.095112 \\ 0.105 + \alpha(0,2) \\ 0.115 + \alpha(0,3) \\ 0.125 + \alpha(0,4) \end{bmatrix} \end{matrix}$$

Here, we have solved for the first forward rate after one period on the tree, at both, up and down nodes, by finding the risk-neutral, arbitrage-free drift of the forward rate in the second period.

Solving for α(0,2)

We now move on to solving for $\alpha(0, 2)$. To do this, we use a three-year zero-coupon bond. After one year, the price of this bond takes one of two values, depending on whether we are in the up state or down state. In the up state, the price of the bond is

$$P_u(1, 3) = \exp[-(0.125112 + (0.135 + \alpha(0, 2)))]$$

And in the down state, the price is

$$P_d(1, 3) = \exp[-(0.095112 + (0.105 + \alpha(0, 2)))]$$

At time zero, this bond's price must be

$$\begin{aligned} P(0, 3) &= \exp(-0.10) \times \frac{1}{2}(P_u(1, 3) + P_d(1, 3)) \\ &= \exp(-0.10) \times \frac{1}{2}(\exp[-(0.125112 + (0.135 + \alpha(0, 2)))] \\ &\quad + \exp[-(0.095112 + (0.105 + \alpha(0, 2)))]) \end{aligned}$$

which should be equal to

$$P(0, 3) = \exp[-(0.10 + 0.11 + 0.12)]$$

Solving for $\alpha(0, 2)$, we obtain

$$\alpha(0, 2) = 0.000337$$

Here we have solved for the forward rate for the third period at the up and down nodes.

$$\begin{bmatrix} 0.10 \\ 0.11 \\ 0.12 \\ 0.13 \\ 0.14 \end{bmatrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \begin{bmatrix} 0.125112 \\ 0.135337 \\ 0.145 + \alpha(0, 3) \\ 0.155 + \alpha(0, 4) \end{bmatrix} \\ \\ \begin{bmatrix} 0.095112 \\ 0.105337 \\ 0.115 + \alpha(0, 3) \\ 0.125 + \alpha(0, 4) \end{bmatrix} \end{matrix}$$

Finishing up

Our procedure here is a "bootstrapping" one. We first solved for $\alpha(0, 1)$ and then used this value to solve for $\alpha(0, 2)$. We then proceed by using the values of $\alpha(0, 1)$ and $\alpha(0, 2)$ to solve for $\alpha(0, 3)$ in the same way. Thus, maturity after maturity, we solve for the risk-neutral drifts in each period of the model. Rather than repeat the same calculations again, we simply state the solutions for the remaining alphas:

$$\alpha(0, 3) = 0.000562$$

$$\alpha(0, 4) = 0.000787$$

This finalizes the first period evolution on the HJM tree, which is represented as follows:

$$\begin{array}{ccc} & & \begin{bmatrix} 0.125112 \\ 0.135337 \\ 0.145562 \\ 0.155787 \end{bmatrix} \\ \begin{bmatrix} 0.10 \\ 0.11 \\ 0.12 \\ 0.13 \\ 0.14 \end{bmatrix} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \\ & & \begin{bmatrix} 0.095112 \\ 0.105337 \\ 0.115562 \\ 0.125787 \end{bmatrix} \end{array}$$

Growing the tree to the next period

Upper subtree

$$\begin{bmatrix}0.10\\0.11\\0.12\\0.13\\0.14\end{bmatrix} \begin{matrix}\nearrow \\ \\ \searrow\end{matrix} \begin{matrix}\begin{bmatrix}0.125112\\0.135337\\0.145562\\0.155787\end{bmatrix} \begin{matrix}\nearrow \begin{bmatrix}0.150337+\alpha(1)\\0.160562+\alpha(2)\\0.170787+\alpha(3)\end{bmatrix} \\ \searrow \begin{bmatrix}0.120337+\alpha(1)\\0.130562+\alpha(2)\\0.140787+\alpha(3)\end{bmatrix}\end{matrix} = \begin{matrix}\begin{bmatrix}0.150449\\0.160899\\0.171349\end{bmatrix} \\ \begin{bmatrix}0.120449\\0.130899\\0.141349\end{bmatrix}\end{matrix} \\ \\ \begin{bmatrix}0.095112\\0.105337\\0.115562\\0.125787\end{bmatrix}\end{matrix}$$

$$P(1,3) = \exp[-(0.125112 + 0.135337)]$$

$$P(1,3) = \exp(-0.125112) \times \frac{1}{2}[P_u(2,3) + P_d(2,3)]$$

$$P_u(2,3) = \exp[-(0.150337 + \alpha(1))]$$
$$P_d(2,3) = \exp[-(0.120337 + \alpha(1))]$$

$$\alpha(1) = 0.000112$$
$$\alpha(2) = 0.000337$$
$$\alpha(3) = 0.000562$$

Using the same procedure as before, we derive the term structures of forward rates after two periods.

Lower sub-tree

An identical calculation as done on the previous slides gives the lower sub-tree after two periods.

$$
\begin{bmatrix} 0.10 \\ 0.11 \\ 0.12 \\ 0.13 \\ 0.14 \end{bmatrix}
\begin{matrix} \nearrow \\ \searrow \end{matrix}
\begin{matrix}
\begin{bmatrix} 0.125112 \\ 0.135337 \\ 0.145562 \\ 0.155787 \end{bmatrix} \\
\begin{bmatrix} 0.095112 \\ 0.105337 \\ 0.115562 \\ 0.125787 \end{bmatrix}
\end{matrix}
$$

$$
\begin{bmatrix} 0.095112 \\ 0.105337 \\ 0.115562 \\ 0.125787 \end{bmatrix}
\begin{matrix} \nearrow \\ \searrow \end{matrix}
\begin{matrix}
\begin{bmatrix} 0.120337 + \alpha(1) \\ 0.130562 + \alpha(2) \\ 0.140787 + \alpha(3) \end{bmatrix} \\
\begin{bmatrix} 0.090337 + \alpha(1) \\ 0.100562 + \alpha(2) \\ 0.110787 + \alpha(3) \end{bmatrix}
\end{matrix}
=
\begin{matrix}
\begin{bmatrix} 0.120449 \\ 0.130899 \\ 0.141349 \end{bmatrix} \\
\begin{bmatrix} 0.090449 \\ 0.100899 \\ 0.111349 \end{bmatrix}
\end{matrix}
$$

$$
\begin{aligned}
\alpha(1) &= 0.000112 \\
\alpha(2) &= 0.000337 \\
\alpha(3) &= 0.000562
\end{aligned}
$$

Completing the Second Period

$$
\begin{array}{ccccc}
 & & & & \begin{bmatrix}0.150449\\0.160899\\0.171349\end{bmatrix} \\
 & & \begin{bmatrix}0.125112\\0.135337\\0.145562\\0.155787\end{bmatrix} & \nearrow & \\
 & \nearrow & & \searrow & \\
\begin{bmatrix}0.10\\0.11\\0.12\\0.13\\0.14\end{bmatrix} & & & & \begin{bmatrix}0.120449\\0.130899\\0.141349\end{bmatrix} \\
 & \searrow & & \nearrow & \\
 & & \begin{bmatrix}0.095112\\0.105337\\0.115562\\0.125787\end{bmatrix} & \searrow & \\
 & & & & \begin{bmatrix}0.090449\\0.100899\\0.111349\end{bmatrix}
\end{array}
$$

$\sigma(t, s) = \sigma(s)$ for all s. (Recombination condition)

See that the middle node after two periods is the same, irrespective of whether it was derived from the upper sub-tree or lower sub-tree.

Bond Option Pricing

We use the two-period arbitrage-free tree we computed to illustrate interest-rate derivative pricing. We shall price options on a five-year coupon bond with a coupon of 13% per year. In particular, we price the one-year call option on this bond at a strike (ex-coupon) of 100.

In order to price this option, we need to determine the price of the five-year bond at the end of one year. There are two states at the end of one year, and at each node there is a forward curve with four rates, one for each of the remaining years. These forward curves may be used to price the bond at the end of one year. In the up state, the forward curve is

$$\begin{bmatrix} 0.125112 \\ 0.135337 \\ 0.145562 \\ 0.155787 \end{bmatrix}$$

The price of the bond (coupon = 13%) is obtained by discounting all coupons and the final principal (remember that $h = 1$):

$$\begin{aligned} P_u &= 13\exp[-(0.125112)] \\ &\quad +13\exp[-(0.125112+0.135337)] \\ &\quad +13\exp[-(0.125112+0.135337+0.145562)] \\ &\quad +113\exp[-(0.125112+0.135337+0.145562+0.155787)] \\ &= 94.58296 \end{aligned}$$

Since this price is less than the call strike of 100, the call will not be exercised, and its value in the up state is zero.

Option Value

An identical calculation may be undertaken for the down state price, which turns out to be

$$P_d = 104.5823$$

In this state, the call option is in-the-money and generates a payoff of 4.5823. The time-0 price of the option may now be easily computed by taking expected values (when $q = \frac{1}{2}$) and discounting back to the beginning of year 1,

$$\text{Call price} = \exp(-0.10)\left[\frac{1}{2} \times 0 + \frac{1}{2} \times 4.5823\right] = 2.0731$$

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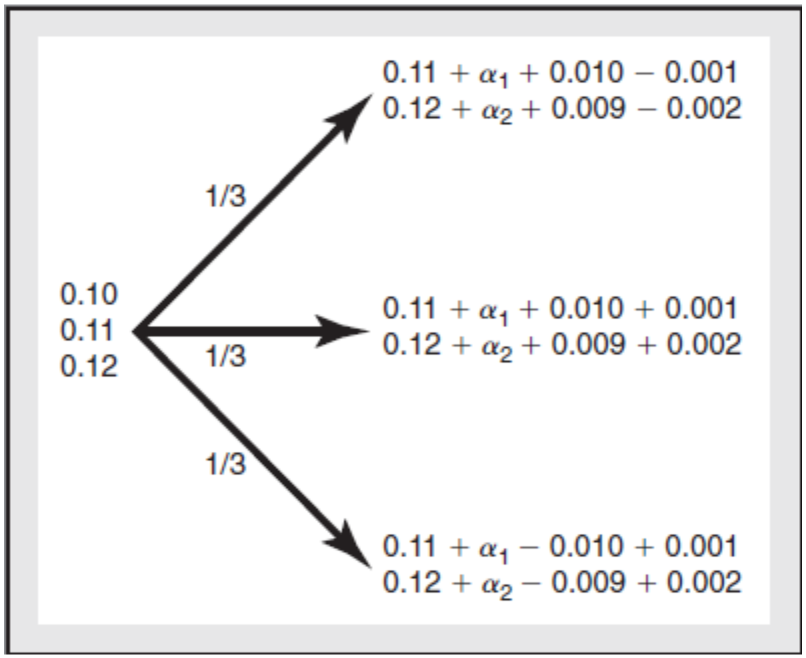
Two-Factor HJM

(in a three-period setting)

Problem setup:

$$f(0,0) = 0.10 \qquad \sigma_1(1) = 0.010 \qquad \sigma_2(1) = -0.001$$
$$f(0,1) = 0.11 \qquad \sigma_1(2) = 0.009 \qquad \sigma_2(2) = -0.002$$
$$f(0,2) = 0.12$$

$$f(t+1,s)) = \begin{cases} f(t,s) + \alpha(s)h + \sigma_1(s)\sqrt{h} + \sigma_2(s)\sqrt{h} & \text{with prob } 1/3 \\ f(t,s) + \alpha(s)h + \sigma_1(s)\sqrt{h} - \sigma_2(s)\sqrt{h} & \text{with prob } 1/3 \\ f(t,s) + \alpha(s)h - \sigma_1(s)\sqrt{h} - \sigma_2(s)\sqrt{h} & \text{with prob } 1/3 \end{cases}$$



Solving for α(1), α(2)

We solve for α_1 and α_2 in two steps. First, using the two-period bond, we solve for the value of α_1. Then, using a three-period bond and the value of α_1, we solve for the value of α_2. We undertake the first step in the following equation.

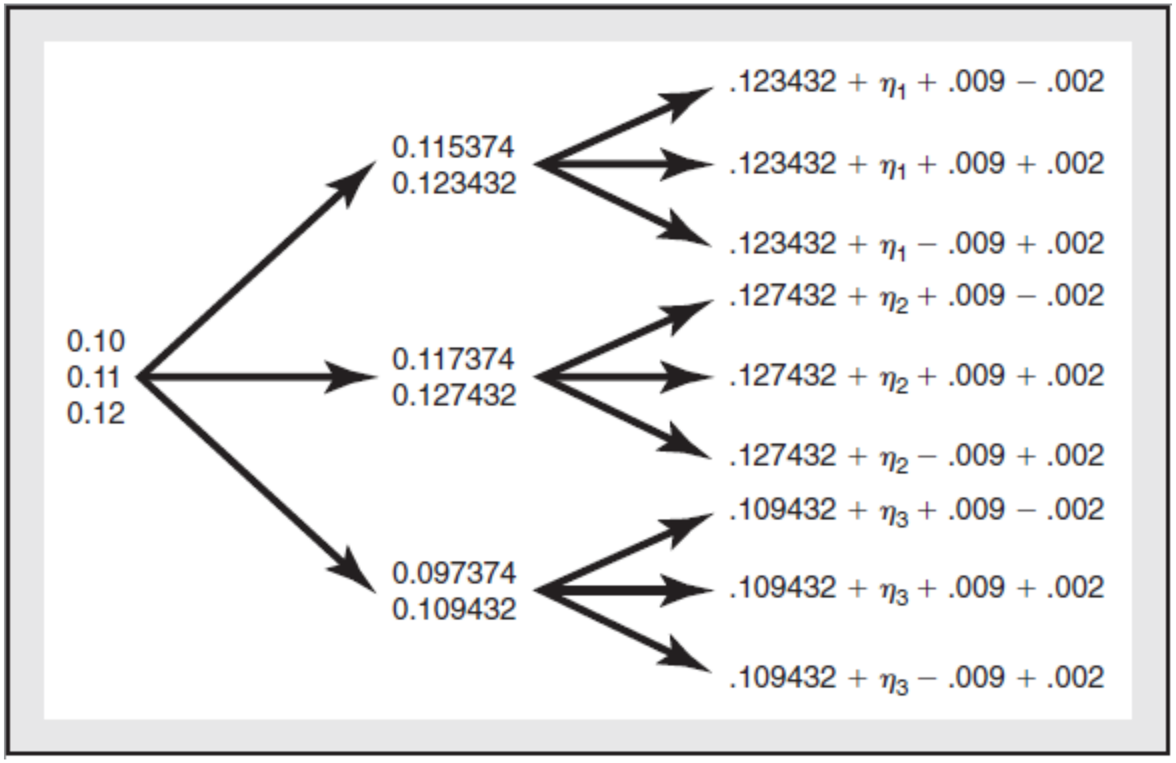
$$\begin{aligned}\exp[-(0.10)] \times \exp[-(0.11)] = \exp[-(0.10)] \times \frac{1}{3} &\times \exp[-(0.11 + \alpha_1 + 0.010 - 0.001)] \\ &+ \frac{1}{3} \times \exp[-(0.11 + \alpha_1 + 0.010 + 0.001)] \\ &+ \frac{1}{3} \times \exp[-(0.11 + \alpha_1 - 0.010 + 0.001)]\end{aligned}$$

$$\alpha_1 = -0.00362614$$

$$\begin{aligned}&\exp[-(0.11 + 0.12)] \\ &= \frac{1}{3} \times \exp[-(0.11 + \alpha_1 + 0.010 - 0.001)]\ \exp[-(0.12 + \alpha_2 + 0.009 - 0.002)] \\ &\quad + \frac{1}{3} \times \exp[-(0.11 + \alpha_1 + 0.010 + 0.001)]\ \exp[-(0.12 + \alpha_2 + 0.009 + 0.002)] \\ &\quad + \frac{1}{3} \times \exp[-(0.11 + \alpha_1 - 0.010 + 0.001)]\ \exp[-(0.12 + \alpha_2 - 0.009 + 0.002)]\end{aligned}$$

$$\alpha_2 = -0.00356759$$

Completed Tree: First Period

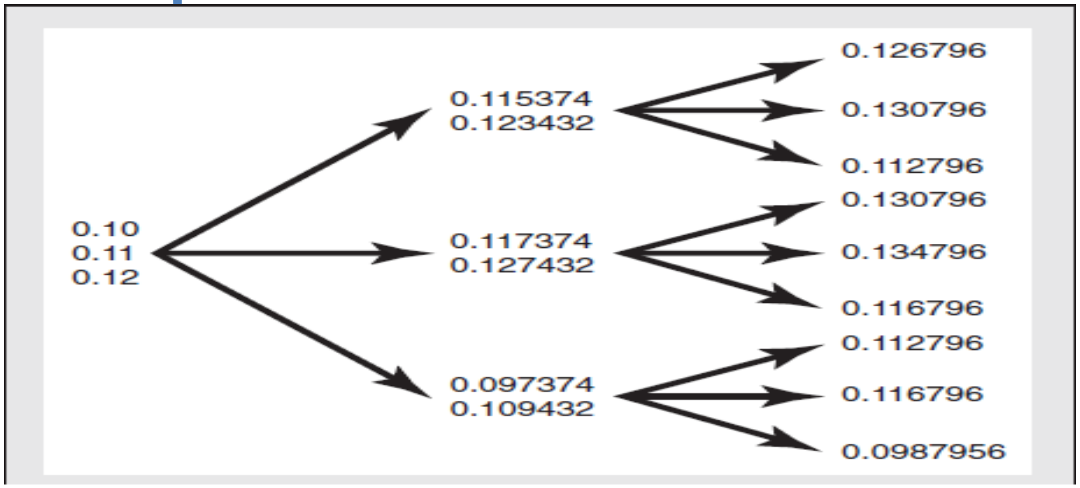


Solve upper sub-tree (for example); similar for the middle and lower sub-trees:

$$\begin{aligned}
&\exp[-0.123432] = \\
&\exp[-(0.123432 + \eta_1 + 0.009 - 0.002)] \\
&+ \exp[-(0.123432 + \eta_1 + 0.009 + 0.002)] \\
&+ \exp[-(0.123432 + \eta_1 - 0.009 + 0.002)]
\end{aligned}$$

$$\begin{aligned}
\eta_1 &= -0.00363685 \\
\eta_2 &= -0.00363685 \\
\eta_3 &= -0.00363685
\end{aligned}$$

Complete Two-Period Tree



Consider a coupon bond with a maturity of three years and a coupon of 12%. We price a one-year call option on this bond at a strike price of 100. The strike price is the ex-coupon strike, i.e., we compare the call strike price with the value of the bond excluding the coupon.

Using the tree, we compute the possible values of the underlying bond at the end of one year. There are two remaining cash flows at the end of one year, and the forward curve may be used to determine the prices. The three nodes at time 1 gives us the following three values:

$$12\exp[-(0.115374)] + 112\exp[-(0.115374 + 0.123432)] = 98.8999$$
$$12\exp[-(0.117374)] + 112\exp[-(0.117374 + 0.127432)] = 98.3509$$
$$12\exp[-(0.097374)] + 112\exp[-(0.097374 + 0.109432)] = 101.962$$

Finally the option value is computed.

$$\begin{aligned}\text{Option Value} &= \left[\frac{1}{3}\max(0, 98.8999 - 100) + \frac{1}{3}\max(0, 98.3509 - 100)\right.\\ &\quad \left. + \frac{1}{3}\max(0, 101.962 - 100)\right]\exp[-0.10]\\ &= 0.591898\end{aligned}$$

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Algebraic Derivation of RN drifts

HJM models lead to a beautiful closed-form expression for risk-neutral drifts!

$$f(0,0):\quad f(0,1):\quad f(0,2):$$

$$\begin{bmatrix} f(0,1)+\alpha_1 h+\sigma_1\sqrt{h} \\ f(0,2)+\alpha_2 h+\sigma_2\sqrt{h} \end{bmatrix}$$

$$\begin{bmatrix} f(0,1)+\alpha_1 h-\sigma_1\sqrt{h} \\ f(0,2)+\alpha_2 h-\sigma_2\sqrt{h} \end{bmatrix}$$

We solve for α_1 and α_2 exactly as we did in the numerical examples. Starting with the two-period \$1 zero-coupon bond as the basis, we compute its two possible prices at the end of the first period, i.e., in the up state and down state. These prices are, respectively,

$$\exp[-(f(0,1)+\alpha_1 h+\sigma_1\sqrt{h})h]$$
$$\exp[-(f(0,1)+\alpha_1 h-\sigma_1\sqrt{h})h]$$

$$P(0,2)=\exp[-(f(0,0)+f(0,1))h]\cdot\frac{1}{2}\left(\exp[-(\alpha_1 h+\sigma_1\sqrt{h})h]+\exp[-(\alpha_1 h-\sigma_1\sqrt{h})h]\right)$$

$$P(0,2)=\exp[-(f(0,0)+f(0,1))h]$$

$$1=\frac{1}{2}\left(\exp[-(\alpha_1 h+\sigma_1\sqrt{h})h]+\exp[-(\alpha_1 h-\sigma_1\sqrt{h})h]\right)$$

First period

$$\begin{aligned}\alpha_1&=\frac{1}{h^2}\ln\left[\frac{1}{2}\left(\exp[\sigma_1\sqrt{h}h]+\exp[-\sigma_1\sqrt{h}h]\right)\right]\\&=\frac{1}{h^2}\ln\left[\frac{1}{2}\left(\exp[\sigma_1 h^{\frac{3}{2}}]+\exp[-\sigma_1 h^{\frac{3}{2}}]\right)\right]\\&\boxed{=\frac{1}{h^2}\ln\left[\cosh\left(\sigma_1 h^{\frac{3}{2}}\right)\right]}\end{aligned}$$

Second Period Algebraic Drifts

$$\exp[-(f(0,1)+\alpha_1 h+\sigma_1\sqrt{h}+f(0,2)+\alpha_2 h+\sigma_2\sqrt{h})h]$$
$$\exp[-(f(0,1)+\alpha_1 h-\sigma_1\sqrt{h}+f(0,2)+\alpha_2 h-\sigma_2\sqrt{h})h]$$

$$\begin{aligned} P(0,3) = \exp[-f(0,0)h] \times \\ &\frac{1}{2}\Big(\exp[-(f(0,1)+\alpha_1 h+\sigma_1\sqrt{h}+f(0,2)+\alpha_2 h+\sigma_2\sqrt{h})h] \\ &+\exp[-(f(0,1)+\alpha_1 h-\sigma_1\sqrt{h}+f(0,2)+\alpha_2 h-\sigma_2\sqrt{h})h]\Big) \end{aligned}$$

$$P(0,3) = \exp[-(f(0,0)+f(0,1)+f(0,2))h]$$

$$1 = \frac{1}{2}\left(\exp[-(\alpha_1 h+\alpha_2 h+\sigma_1\sqrt{h}+\sigma_2\sqrt{h})h]+\exp[-(\alpha_1 h+\alpha_2 h-\sigma_1\sqrt{h}-\sigma_2\sqrt{h})h]\right)$$

which may be rewritten as

$$\begin{aligned} \alpha_1+\alpha_2 &= \frac{1}{h^2}\ln\left[\frac{1}{2}\left(\exp[\sigma_1\sqrt{h}h+\sigma_2\sqrt{h}h]+\exp[-\sigma_1\sqrt{h}h-\sigma_2\sqrt{h}h]\right)\right] \\ &= \frac{1}{h^2}\ln\left[\frac{1}{2}\left(\exp[(\sigma_1+\sigma_2)h^{\frac{3}{2}}]+\exp[-(\sigma_1+\sigma_2)h^{\frac{3}{2}}]\right)\right] \\ &= \frac{1}{h^2}\ln\left[\cosh\left((\sigma_1+\sigma_2)h^{\frac{3}{2}}\right)\right] \end{aligned}$$

$$\boxed{\sum_{t=1}^{T}\alpha_t = \frac{1}{h^2}\ln\left[\cosh\left(\left(\sum_{t=1}^{T}\sigma_t\right)h^{\frac{3}{2}}\right)\right]}$$

A recursive expression for drifts of all maturities.

HJM One-Node Program

```
% hjm(f0,sig0,h)
% Program to generate the HJM Tree
% The program takes in a fwd curve and vol curve and returns
% the next periods up and down fwd curves
%f0 : initial forward rate curve
%sig0 : forward rate volatilities (for this node)

function u = hjm(f0,sig0,h);
   n = length(f0);
   m = n-1;
   fu = f0(2:n);
   fd = f0(2:n);
   sigma = sig0(2:n);
   alpha = zeros(m,1);
   for j=[1:m];
       if (j==1);
           alpha(j) = log(0.5*(exp(-sigma(j)*h*sqrt(h)) + ...
               exp(sigma(j)*h*sqrt(h))))/h^2;
       end;
       if (j>1);
          alpha(j) = log(0.5*(exp(-sum(sigma(1:j))*h*sqrt(h)) + ...
              exp(sum(sigma(1:j))*h*sqrt(h))))/h^2-sum(alpha(1:j-1));
       end;
   end;
   fu = fu+alpha*h+sigma*sqrt(h);
   fd = fd+alpha*h-sigma*sqrt(h);
   u = [fu  fd];
```

```
octave:1> f0 = [0.10; 0.11; 0.12; 0.13; 0.14];
octave:2> sig0 = 0.015*ones(5,1);
octave:3> h=1;
octave:4> hjm(f0,sig0,h)
ans =

  0.125112  0.095112
  0.135337  0.105337
  0.145562  0.115562
  0.155787  0.125787
```

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Libor Market Models (LMMs)

Three landmark papers in 1997:

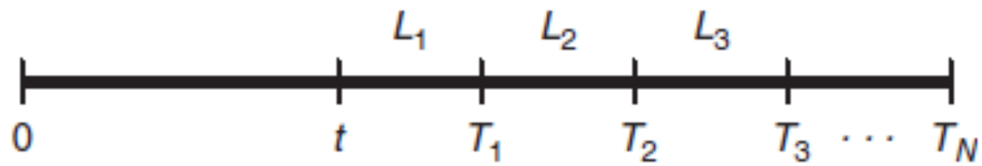
1. Brace, Gatarek, Musiela
2. Jamshidian
3. Miltersen, Sandmann, Sondermann

FEATURES:

1. Also based on forward rate and volatilities.
2. More particular then HJM models, and calibrate to observable Libor rates directly.
3. Based on lognormal Libor rates, so that they yield Black-Scholes looking formulae for options.
4. Ideal for pricing caps and floors.
5. They come in two flavors: LMM and SMM (Swap Market Model)

Notation

$$T_1 = t + \delta, \quad T_2 = t + 2\delta, \quad \ldots \quad , T_N = t + N\delta$$



$$dL_i(t) = \mu_i(t)L_i(t)\,dt + \sigma_i L_i(t)\,dW_i(t), \qquad i = 1, \ldots, N$$

$$\begin{aligned}
P(t, t) &= P(t, T_1)[1 + \delta L_1] \\
P(t, T_1) &= P(t, T_2)[1 + \delta L_2] \\
P(t, T_2) &= P(t, T_3)[1 + \delta L_3] \\
&\vdots \\
P(t, T_{i-1}) &= P(t, T_i)[1 + \delta L_i] \\
&\vdots \\
P(t, T_{N-1}) &= P(t, T_N)[1 + \delta L_N]
\end{aligned}$$

$$\delta L_i = \frac{P(t, T_{i-1})}{P(t, T_i)} - 1, \quad \text{for all } i$$

Libor rates are martingales under the "forward" probability measure.

But each Libor rate is a martingale under a different martingale measure.

Risk-Neutral Pricing in LMM

As before, Libor rates are assumed to be lognormally distributed, i.e.,

$$\frac{dL_i}{L_i} = \mu_i \, dt + \sigma_i \, dW_i, \qquad i = 1, \dots, N$$

It is simplest to choose the discount bond of maturity T_N as the numeraire asset. Based on this, we will compute the drift terms $\mu_1, \mu_2, \dots, \mu_N$ such that $\frac{L_1}{P(t,T_N)}, \frac{L_2}{P(t,T_N)}, \dots, \frac{L_N}{P(t,T_N)}$ are martingales with respect to a probability measure over correlated Brownian motions $W_1, W_2, \dots, W_N$. The numeraire is no longer a money market account as in the case of the HJM model. Instead, it is a bond maturing in the future, and, hence, the probability measure is known as the "forward" measure. From a nomenclature point of view, pricing in the HJM model is undertaken under the "spot" measure.

Deriving the martingale process for $\mathbf{L_3}$

We begin with Libor rate L_3 and determine its martingale process when the numeraire is $P(t, T_3)$. From the previous section, we restate the functional form for L_3

$$\delta \cdot L_3 = \frac{P(t, T_2)}{P(t, T_3)} - 1$$

The right-hand side of the equation contains an expression for the price of the two-period bond $P(t, T_2)$ normalized by numeraire $P(t, T_3)$. By assumption, this must be a martingale under the probability measure based on the Brownian motion W_3. The expected change of the right-hand side of the equation is therefore zero. Given this, the expected change of the left-hand side is also zero, implying that L_3 is a martingale. Hence, if L_3 is already a martingale, its drift must be zero, i.e., $\mu_3 = 0$. We may then specify the stochastic process for L_3 as follows:

$$dL_3 = \sigma_3 L_3 \, dW_3$$

Note that L_3 is lognormal and it is a martingale.

L_2

Deriving the Martingale Process for L_2

Turning to L_2, we write down its relation to discount bonds:

$$\delta \cdot L_2 = \frac{P(t, T_1)}{P(t, T_2)} - 1 = \frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)}$$

Since the normalizing asset on the right-hand side of the equation is $P(t, T_2)$, not the numeraire $P(t, T_3)$, the process for L_2 is not a martingale with respect to the required numeraire, and, hence, we need to make a change of probability measure to convert it into a process with respect to the chosen numeraire, $P(t, T_3)$. To do this, multiply both sides of the equation by $P(t, T_2)$ and divide both sides by $P(t, T_3)$. This results in the following:

$$\delta \cdot L_2 \times \frac{P(t, T_2)}{P(t, T_3)} = \frac{P(t, T_1) - P(t, T_2)}{P(t, T_3)}$$

In the absence of arbitrage, all assets of any shape or form, normalized by $P(t, T_3)$, must be martingales. Hence, looking at the left-hand side of the equation above, the asset $L_2 P(t, T_2)$ normalized by $P(t, T_3)$ is also a martingale with respect to Q_3. On the right-hand side of the equation above, an asset defined as the difference of two bonds, i.e., $P(t, T_1) - P(t, T_2)$ normalized by $P(t, T_3)$ is also, by construction, a martingale. Defining $Z_2 \equiv \frac{P(t,T_2)}{P(t,T_3)}$, we have

$$\delta \cdot L_2 Z_2 = \frac{P(t, T_1) - P(t, T_2)}{P(t, T_3)}$$

L_2
continued

A_2 is a martingale, as is Z_2, an asset normalized by the numeraire. Therefore, taking expectations on both sides with respect to the martingale probability measure results in

$$0 = 0 + E\left[\frac{dL_2}{L_2}\right] + E\left[\frac{dZ_2}{Z_2}\frac{dL_2}{L_2}\right]$$

Noting that $E\left[\frac{dL_2}{L_2}\right] = \mu_2\,dt$, we get, via a series of simplifying steps, the expression for the drift that makes L_2 a martingale with respect to the chosen numeraire.

$$\begin{aligned}\mu_2\,dt &= -E\left[\frac{dZ_2}{Z_2}\frac{dL_2}{L_2}\right]\\ &= -E\left[dZ_2\frac{dL_2}{L_2}\right]\frac{1}{Z_2}\end{aligned}$$

L_2 ….continued

$$
\begin{aligned}
\mu_2\,dt &= -E\left[\frac{dZ_2}{Z_2}\frac{dL_2}{L_2}\right] \\
&= -E\left[dZ_2\frac{dL_2}{L_2}\right]\frac{1}{Z_2} \\
&= -\frac{1}{Z_2}E\left[d(1+\delta L_3)\frac{dL_2}{L_2}\right] \\
&= -\frac{\delta L_3}{Z_2}E\left[\frac{dL_3}{L_3}\frac{dL_2}{L_2}\right] \\
&= -\frac{\delta L_3}{Z_2}E\left[\sigma_2\sigma_3\,dW_2\,dW_3\right] \\
&= -\frac{\delta L_3}{Z_2}\rho_{23}\sigma_2\sigma_3\,dt \\
&= -\frac{\delta L_3}{1+\delta L_3}\rho_{23}\sigma_2\sigma_3\,dt
\end{aligned}
$$

Note that in lines 3 and 7 of the derivation above, we exploited the fact that

$$Z_i = \frac{P(t, T_i)}{P(t, T_{i+1})} = 1 + \delta L_{i+1}, \quad \forall i$$

Thus, we have derived the drift term for Libor rate L_2 to make it a martingale with respect to numeraire $P(t, T_3)$. The final result is

$$\boxed{\mu_2 = -\frac{\delta L_3}{1+\delta L_3}\rho_{23}\sigma_2\sigma_3}$$

L_1

Deriving the Martingale Process for L_1

The calculations for L_1 are only slightly more complicated and lead on directly to the general case of many periods. Therefore, it is instructive to work through this last period of the model in detail.

Turning to L_1, we write down its relation to discount bonds:

$$\delta \cdot L_1 = \frac{P(t,t)}{P(t,T_1)} - 1 = \frac{P(t,t) - P(t,T_1)}{P(t,T_1)}$$

Since the normalizing asset on the right-hand side of the equation is $P(t, T_1)$, not the numeraire $P(t, T_3)$, the process for L_1 is not a martingale with respect to the required numeraire, and, hence, we need to make a change in the equation to convert it into a process with respect to the chosen numeraire, $P(t, T_3)$. To do this, multiply both sides of the equation by $P(t, T_1)$ and divide both sides by $P(t, T_3)$. This results in the following

$$\delta \cdot L_1 \times \frac{P(t,T_1)}{P(t,T_3)} = \frac{P(t,t) - P(t,T_1)}{P(t,T_3)}$$

We modify the left-hand side of the equation a little bit as follows:

$$\delta \cdot L_1 \times \frac{P(t,T_1)}{P(t,T_2)} \frac{P(t,T_2)}{P(t,T_3)} = \frac{P(t,t) - P(t,T_1)}{P(t,T_3)}$$

which may also be written as

$$\delta \cdot L_1 Z_1 Z_2 = \frac{P(t,t) - P(t,T_1)}{P(t,T_3)}$$

Let $A_1 = L_1 Z$ where $Z = Z_1 Z_2$ and using Ito's lemma, we get the following:

$$dA_1 = Z\,dL_1 + L_1\,dZ + dL_1\,dZ$$

Dividing both sides by A_1, we have

$$\frac{dA_1}{A_1} = \frac{dL_1}{L_1} + \frac{dZ}{Z} + \frac{dL_1}{L_1}\frac{dZ}{Z}$$

L_1 contd

Because A_1 and Z are martingales, we must have that, after taking expectations on both sides

$$0 = E\left[\frac{dL_1}{L_1}\right] + 0 + E\left[\frac{dL_1}{L_1}\frac{dZ}{Z}\right]$$

Noting that $Z = Z_1 Z_2$ and that Ito's lemma gives $\frac{dZ}{Z} = \frac{dZ_1}{Z_1} + \frac{dZ_2}{Z_2} + \frac{dZ_1}{Z_1}\frac{dZ_2}{Z_2}$, the equation above may be written as

$$0 = \mu_1\, dt + 0 + E\left[\frac{dL_1}{L_1}\left(\frac{dZ_1}{Z_1} + \frac{dZ_2}{Z_2} + \frac{dZ_1}{Z_1}\frac{dZ_2}{Z_2}\right)\right]$$

Simplifying, and noting that the third power term $dL_1\, dZ_1\, dZ_2 = 0$, we get

$$\mu_1\, dt = -E\left[\frac{dL_1}{L_1}\frac{dZ_1}{Z_1} + \frac{dL_1}{L_1}\frac{dZ_2}{Z_2}\right] = -E\left[\frac{dL_1}{L_1}\frac{dZ_1}{Z_1}\right] - E\left[\frac{dL_1}{L_1}\frac{dZ_2}{Z_2}\right]$$

Let's simplify each term on the right-hand side separately.

$$\begin{aligned}
-E\left[\frac{dL_1}{L_1}\frac{dZ_1}{Z_1}\right] &= \frac{1}{1+\delta L_2}E\left[d(1+\delta L_2)\frac{dL_1}{L_1}\right] \\
&= -\frac{\delta L_2}{1+\delta L_2}E\left[\frac{dL_2}{L_2}\frac{dL_1}{L_1}\right] \\
&= -\frac{\delta L_2}{1+\delta L_2}\,\sigma_1\sigma_2 E(dW_1\, dW_2) \\
&= -\frac{\delta L_2}{1+\delta L_2}\,\rho_{12}\sigma_1\sigma_2\, dt
\end{aligned}$$

$$\begin{aligned}
-E\left[\frac{dL_1}{L_1}\frac{dZ_2}{Z_2}\right] &= \frac{1}{1+\delta L_3}E\left[d(1+\delta L_3)\frac{dL_1}{L_1}\right] \\
&= -\frac{\delta L_3}{1+\delta L_3}E\left[\frac{dL_3}{L_3}\frac{dL_1}{L_1}\right] \\
&= -\frac{\delta L_3}{1+\delta L_3}\,\sigma_1\sigma_3 E(dW_1\, dW_3) \\
&= -\frac{\delta L_3}{1+\delta L_3}\,\rho_{13}\sigma_1\sigma_3\, dt
\end{aligned}$$

General Form of RN Drift

$$\mu_1 = -\left[\frac{\delta L_2}{1+\delta L_2}\rho_{12}\sigma_1\sigma_2 + \frac{\delta L_3}{1+\delta L_3}\rho_{13}\sigma_1\sigma_3\right]$$

By simple analogy, we may write down the result for the general case immediately:

$$\mu_i = -\sum_{j=i+1}^{N}\left[\frac{\delta L_j}{1+\delta L_j}\rho_{ij}\sigma_i\sigma_j\right], \quad \forall i < N \qquad \textbf{(30.13)}$$

and when $i = N$, $\mu_i = 0$. Equation (30.13) is the main result of the Libor market model derivation above and may be then used in all further computations. It provides the drift terms that are then substituted back into the stochastic processes for Libor rates. The risk-neutral Libor dynamics under which no-arbitrage pricing may be undertaken with respect to numeraire $P(t, T_N)$ may now be stated as follows:

$$\boxed{\frac{dL_i}{L_i} = -\sum_{j=i+1}^{N}\left[\frac{\delta L_j}{1+\delta L_j}\rho_{ij}\sigma_i\sigma_j\right] dt + \sigma_i\, dW_i, \quad \forall i < N}$$

As in the HJM model, we note that the risk-neutral drifts of the LMM Libor processes are also functions of the volatilities and correlations. If instead of separate Brownian motions W_i, we have a simple one-factor LMM model, then $\rho_{ij} = 1$ in the equation above.

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Simulation & Calibration

The continuous-time process may be written in discrete form to be simulated over time steps of length *h*.

$$L_i(t+h) = L_i(t)\exp\left[\left(-\sum_{j=i+1}^{N}\left[\frac{\delta L_j}{1+\delta L_j}\,\rho_{ij}\sigma_i\sigma_j\right] - \frac{1}{2}\sigma_i^2\right)h + \sigma_i\sqrt{h}\,\cdot W_i\right], \quad \forall i$$

Caplets & Floorlets

$$C_0 = P(0,T_i)\,[L_{i+1}(0)\,N(d_1) - X\,N(d_2)]\,\delta$$

$$d_1 = \frac{1}{\sigma_{i+1}\sqrt{T_i}}\left[\ln\left(\frac{L_{i+1}(0)}{X}\right) + \frac{1}{2}\sigma_{i+1}^2 T_i\right]$$

$$d_2 = d_1 - \sigma_{i+1}\sqrt{T_i}$$

$$F_0 = P(0,T_i)\,[-L_{i+1}(0)\,N(-d_1) + X\,N(-d_2)]\,\delta$$

Because caps and floors may be priced in closed-form, volatility parameters may be directly calibrated using the formula here.

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Swap Market Models (SMMs)

Direct calibration to the swaps and swaptions market is obtained. Recall, Libor rates are related to discount bonds as follows:

$$\delta L_l(t) = \frac{P(t, T_{l-1})}{P(t, T_l)} - 1$$

$$E_{Q_l}\left[\frac{P(t, T_{l-1})}{P(t, T_l)} - 1\right] = E_{Q_l}\left[\frac{P(t, T_{l-1}) - P(t, T_l)}{P(t, T_l)}\right] = 0$$

In other words, in the LMM setting, Libor rates are martingales with respect to numeraire $P(t, T_l)$.

Swap Floating leg:

A swap comprises an exchange of fixed-for-floating payments. Each floating payment is of the amount δL_l and at time T_l has the following present value:

$$\delta\ L_l(T_{l-1})\ P(T_{l-1}, T_l) = P(T_{l-1}, T_{l-1}) - P(T_{l-1}, T_l)$$

This is obtained this by rearranging equation (30.16) and setting $t = T_{l-1}$. A swap is nothing but a collection of such floating payments in exchange for fixed payments. We focus on the floating payment first. Since Libor rates are martingales, we can see that the present value at time t of any floating payment is simply obtained by moving the equation above to time t so that the expected present value of the entire floating side of the swap is

$$\sum_{l=1}^{N} \delta\ L_l(t)\ P(t, T_l) = \sum_{l=1}^{N} [P(t, T_{l-1}) - P(t, T_l)] = P(t, T_1) - P(t, T_N)$$

Fair Swap Rate

Fixed side of the swap:

The present value of the fixed side of the swap is the present value of payments made at the fixed rate S:

$$\delta S \sum_{i=1}^{N} P(t, T_i)$$

Equating the floating and fixed sides of the swap we get the fair swap rate:

$$\boxed{S(T_1, T_N) = \frac{P(t, T_1) - P(t, T_N)}{\delta \sum_{i=1}^{N} P(t, T_i)}}$$

where $S \equiv S(T_1, T_N)$ is written to indicate that the swap covers the cash flow periods that end in the range (T_1, T_N). From equation (30.16), it follows that because $[P(t, T_1) - P(t, T_N)]$ is a martingale, then S is a martingale with respect to the numeraire

$$\sum_{i=1}^{N} P(t, T_i)$$

which is the present value of the sum of unit payments each swap period. Compare this to the numeraire in the LMM, which is just $P(t, T_i)$. Thus, the essential difference between the LMM and SMM models boils down to a specification of numeraire.

Swaptions

Swap floating and fixed cashflows:

$$\text{Floating side: } A = \sum_{i=1}^{N} \delta\ L_i(t)\ P(t, T_i)$$

$$\text{Fixed side: } B = \delta S \sum_{i=1}^{N} P(t, T_i)$$

where S is the fixed rate that is set on the swap that underlies the swaption. As swap rates change in the market, the fair swap rate $\mathcal{S}$ will always be such that floating and fixed sides are equal, i.e., that $\sum_{i=1}^{N} \delta\ L_i(t)\ P(t, T_i) = \delta \mathcal{S} \sum_{i=1}^{N} P(t, T_i)$. The ratio of the fair-price floating-side value to the fixed side value is

$$\frac{A}{B} = \frac{\delta \mathcal{S} \sum_{i=1}^{N} P(t, T_i)}{\delta S \sum_{i=1}^{N} P(t, T_i)} = \frac{\mathcal{S}}{S}$$

Since S is constant, if we assume that $\mathcal{S}$ is lognormal with volatility σ, then we are in the setting of the Black and Scholes (1973) model. If the ratio A/B is lognormal, then the swaption that is the option to exchange the floating side (receive) in return for the fixed side (pay) will be an option to exchange one side for the other and may be valued using the well-known formula of Margrabe (1978). The value of a swaption at time t to receive floating Libor and pay fixed-rate S for maturity $T \le T_1$ is as follows:

$$\boxed{\begin{aligned} &\text{Swaption(receive floating, pay fixed)} = A\ N(d_1) - B\ N(d_2) \\ &d_1 = \frac{\ln(A/B) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ &d_2 = d_1 - \sigma\sqrt{T} \end{aligned}}$$

Correspondingly, the value of the swaption to pay floating and receive fixed is as follows:

$$\boxed{\begin{aligned} &\text{Swaption(pat floating, receive fixed)} = B\ N(-d_2) - A\ N(-d_1) \\ &d_1 = \frac{\ln(A/B) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ &d_2 = d_1 - \sigma\sqrt{T} \end{aligned}}$$

A final point to note is that nothing in the derivation above was particular to the market model framework. The same derivation applies to models such as HJM and to other interest-rate models where the underlying swap rates are assumed to be lognormal.

Summary

- HJM models describe the arbitrage-free movement of the entire forward curve. LMMs describe the movement of forward Libor.
- Drifts of the forward rates are functions of volatilities – identification of risk premiums not required.
- Both frameworks admit multiple factors.
- Lognormal versions of these models results in closed-form, Black-Scholes type option formulae.
- Both models are easy to calibrate to the interest rate markets, but LMMs allow direct calibration.