Factor Models of the Term Structure

Raghu Sundaram and Sanjiv Das

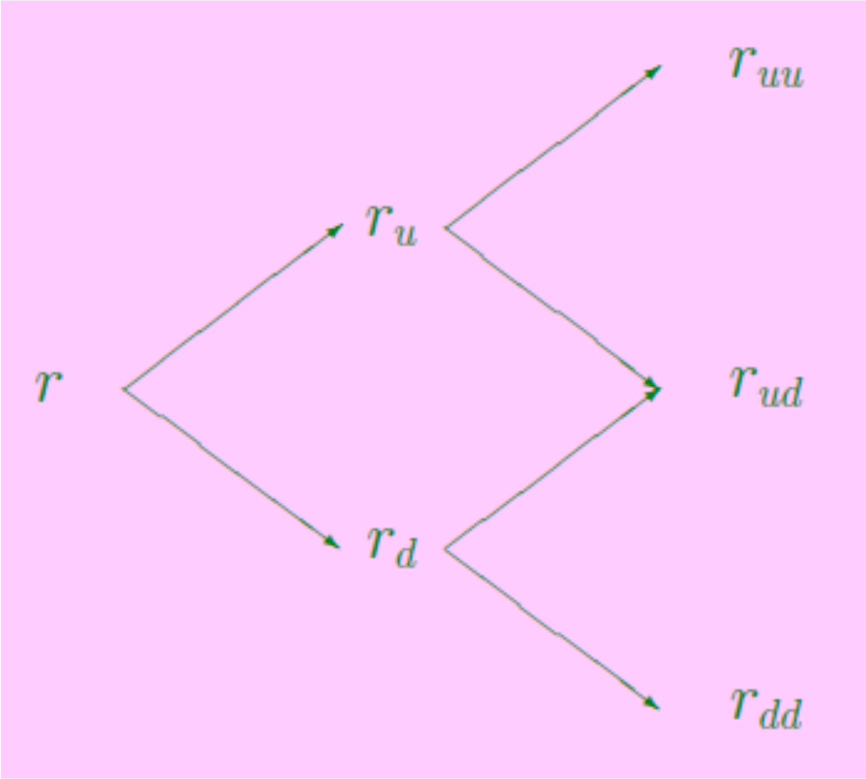
Overview of the BDT Model

- The Black-Derman-Toy (henceforth, BDT) model has as its objective the construction of a model of interest rate evolution that satisfies two important properties:
  - The model is arbitrage-free.
  - The model is consistent with the current term-structure of interest rates and volatilities.

Main Assumptions

- To achieve these ends, BDT posit a single-factor, discrete-time model of the interest rate process.
- All movements in the yield curve are derived in an arbitrage-free manner from movements in the short rate.
- The short rate process itself is assumed to follow a lognormal distribution on a Binomial tree.
- It is further assumed that the volatility of the short rate depends only on time.
- It is assumed throughout that the risk-neutral probabilities of up and down moves are each 1/2.

The Short Rate Tree



The volatility of the short rate is given by: $\sigma = \frac{1}{2} \ln \left(\frac{r_u}{r_d} \right)$

Tree probabilities are ½ and ½ for going up and down, respectively.

Implications of the BDT Structure

- Single factor model => All bond price movements are perfectly correlated over each period.

- Lognormal short rate => Negative interest rates are precluded.

BDT Procedure

- The evolution of short rates in the binomial tree is determined using two inputs:
  - 1. The current term structure of interest rates.
  - 2. The volatilities of yields of different maturities.

- Bootstrapping is employed:
  - The initial short-rate r and a two-year zero-coupon bond are used to determine the possible short rates r_u and r_d after one period.
  - The rates r, r_u, and r_d are used with a three-year zero-coupon bond to determine the possible short rates r_{uu}, r_{ud}, and r_{dd} after two periods.
  - Inductively, the procedure is completed with each step relying on all the previous steps and the price and yield volatility of an appropriate-maturity zero-coupon bond.

Example

We will illustrate the BDT procedure in a three period example in which each period represents one year. Consider the following data:

Maturity (Years)	Zero-Coupon Rate (%)	Volatility (%)
1	10	20
2	11	19
3	12	18

This data is the first three years of the five-year input data used in the original BDT paper. We use this data to construct the tree out to three years. Interest rates in the model are quoted in discrete terms with annual compounding.

Basic Calculations

As a first step, we use the zero-coupon rates to calculate the prices of zero-coupon bonds of various maturities. The current price of a one-year zero-coupon bond is

$$\frac{100}{1.10} = 90.909$$

The current price of a two-year zero is

$$\frac{100}{(1.11)^2} = 81.162$$

The current price of a three-year zero is

$$\frac{100}{(1.12)^3} = 71.178$$

The following notation is used to denote the evolution of the short rate: r will denote the initial short rate; r_u and r_d will denote the possible values of the short rate one year hence and r_{uu}, r_{ud}, and r_{dd} possible values of the short rate two years hence. We are given $r = 10\%$. The remaining values are to be identified.

Identifying rates after one period

After one year, a two-year zero becomes a one-year zero. Its value at this time is the face value of 100 discounted back by the prevailing one-year rate at this point (which is either r_u or r_d):

$$\frac{100}{1+r_u} \quad \text{and} \quad \frac{100}{1+r_d}$$

Therefore, the gross expected return (under the risk-neutral probabilities) from investing in a two-year zero for one year is

$$1 + \text{return} \;=\; \frac{1}{81.162}\left[\frac{1}{2}\left(\frac{100}{1+r_u}\right) + \frac{1}{2}\left(\frac{100}{1+r_d}\right)\right]$$

This must equal the one-year risk-free rate of 10%, so:

$$1.10 \;=\; \frac{1}{81.162}\left[\frac{1}{2}\left(\frac{100}{1+r_u}\right) + \frac{1}{2}\left(\frac{100}{1+r_d}\right)\right]$$

This is one equation in the two unknowns r_u and r_d. A second equation is required. For this, we use the information given that the two-year yield volatility is 19%. A two-year zero is a one-year zero after one year. At this point, its yield is either r_u or r_d. Therefore, its yield volatility is $\frac{1}{2}\ln(r_u/r_d)$, and we obtain as our second equation:

$$\frac{1}{2}\ln\left(\frac{r_u}{r_d}\right) \;=\; 0.19$$

Solving the two equations, we obtain

$$\boxed{r_u \;=\; 14.32\% \quad \text{and} \quad r_d = 9.79\%}$$

Rates after two periods

In the second step, we use r, r_u, and r_d with a three-year zero-coupon bond to identify r_{uu}, r_{ud}, and r_{dd}. As a first step, we compute the value of the three-year bond after two years. At this point, the three-year zero has become a one-year zero, so its possible values are

$$
\begin{aligned}
B_{uu} &= \frac{100}{1 + r_{uu}} \\
B_{ud} &= \frac{100}{1 + r_{ud}} \\
B_{dd} &= \frac{100}{1 + r_{dd}}
\end{aligned}
$$

A year before this, the original three-year zero was a two-year zero. Let B_u and B_d denote its two possible values at this stage. From the risk-neutral pricing principle, these values are the discounted expectation of its future values, so:

$$
\begin{aligned}
B_u &= \frac{1}{1 + r_u} \left(\frac{1}{2} B_{uu} + \frac{1}{2} B_{ud} \right) \\
B_d &= \frac{1}{1 + r_d} \left(\frac{1}{2} B_{ud} + \frac{1}{2} B_{dd} \right)
\end{aligned}
$$

Taking expectations of these values under the risk-neutral probability and discounting back at the risk-free rate of 10%, we should obtain the initial price of the three-year zero:

$$
71.178 = \frac{1}{1.10} \left[\frac{1}{2} B_u + \frac{1}{2} B_d \right]
$$

Solving for the rates

We can now substitute (a) first for B_u and B_d in terms of B_{uu}, B_{ud}, and B_{dd} from the earlier expressions, and then (b) for B_{uu}, B_{ud}, and B_{dd} in terms of r_{uu}, r_{ud}, and r_{dd}, to obtain a single (large and unwieldy) expression involving the three unknowns r_{uu}, r_{ud}, and r_{dd}:

$$71.178 = \frac{1}{1.10}\left\{\frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}.\frac{100}{1+r_{uu}} + \frac{1}{2}.\frac{100}{1+r_{ud}}\right)\right]\right.$$
$$\left. + \frac{1}{2}\left[\frac{1}{2}\left(\frac{1}{2}.\frac{100}{1+r_{ud}} + \frac{1}{2}.\frac{100}{1+r_{dd}}\right)\right]\right\}$$

We need two more equations to identify the three unknowns. For a second equation, we appeal again to the volatility equation. Consider the three-year zero again. If state u occurs after one period, the three-year zero is worth B_u. At maturity, it is worth 100. Therefore, the yield of the three-year zero in state u will be

$$y_u = \left[\sqrt{(100/B_u)}\right] - 1$$

The square root is used since there are still two periods remaining on the three-year bond at this point.

Similarly, the yield of the three-year zero in state d is

$$y_d = \left[\sqrt{(100/B_d)}\right] - 1$$

Since the current volatility of the three-year yield is given to be 0.18, we obtain our second equation:

$$\frac{1}{2}\ln\left(\frac{y_u}{y_d}\right) = 0.18$$

Solution

Finally, recall the assumption that the volatility of the short rate can at most depend on time. This means we must have

$$\frac{1}{2}\ln\left(\frac{r_{uu}}{r_{ud}}\right) = \frac{1}{2}\ln\left(\frac{r_{ud}}{r_{dd}}\right)$$

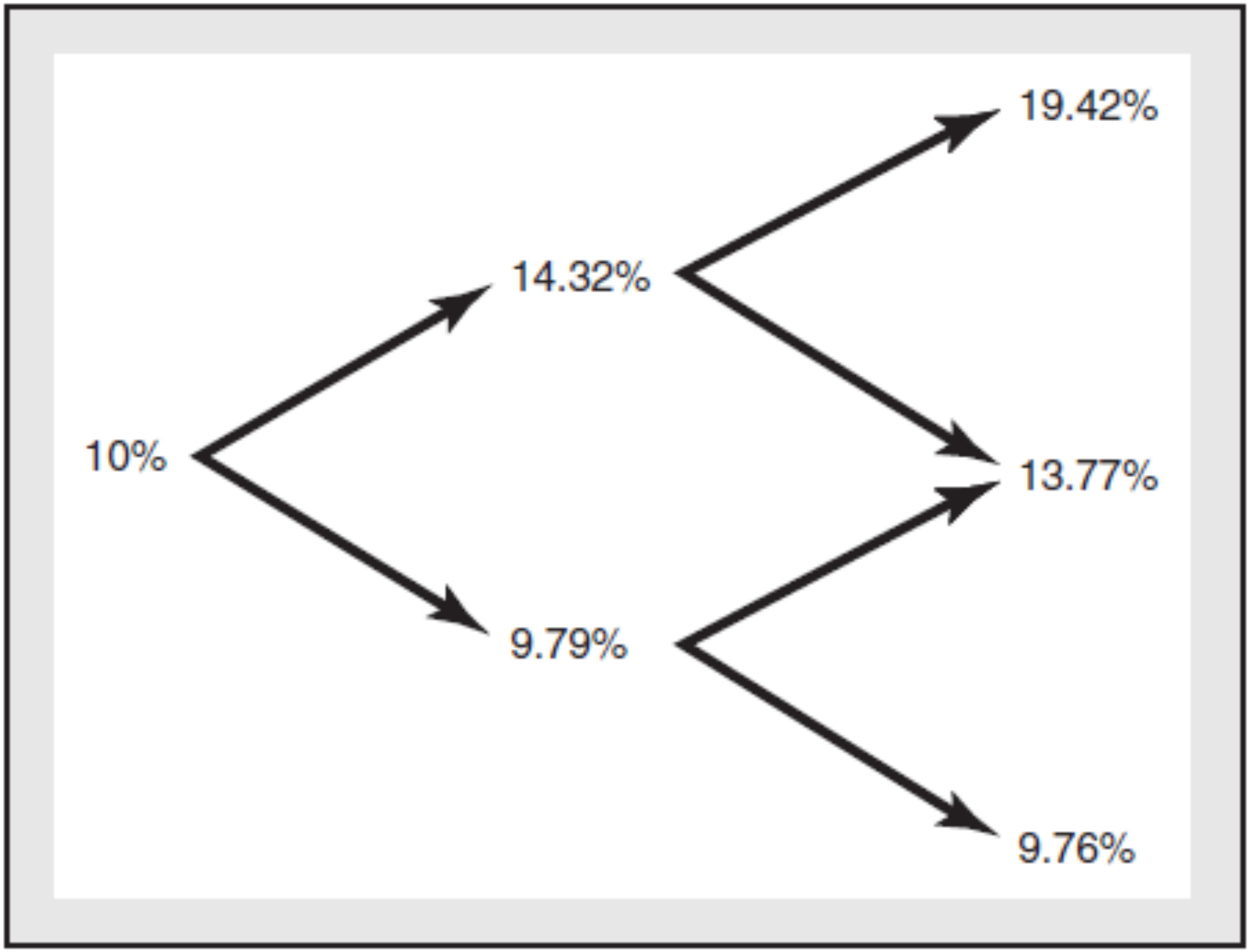
This provides us with our final equation in the three unknowns:

$$(r_{ud})^2 = r_{dd} \cdot r_{uu}$$

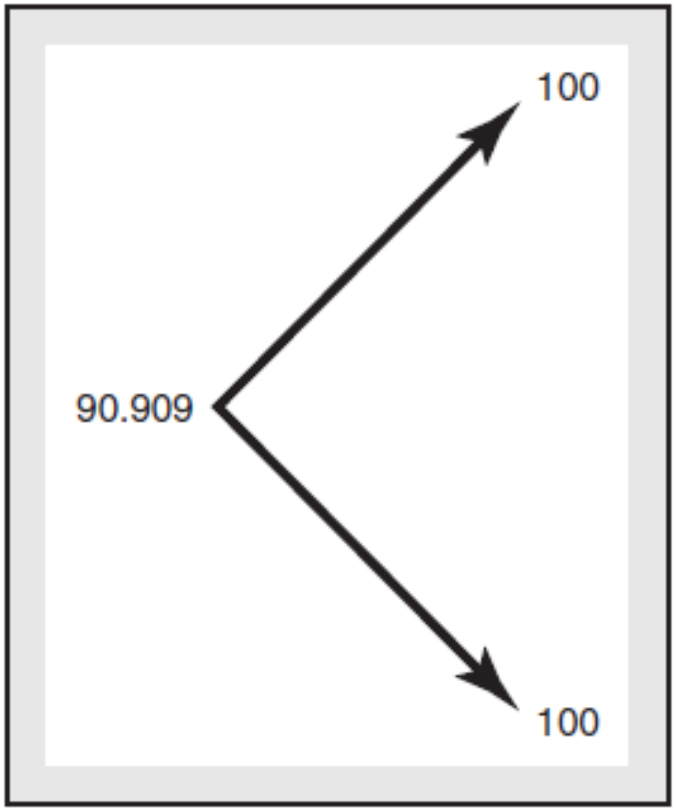
Solving these equations, we finally obtain

$$\boxed{\begin{aligned} r_{uu} &= 19.42\% \\ r_{ud} &= 13.77\% \\ r_{dd} &= 9.76\% \end{aligned}}$$

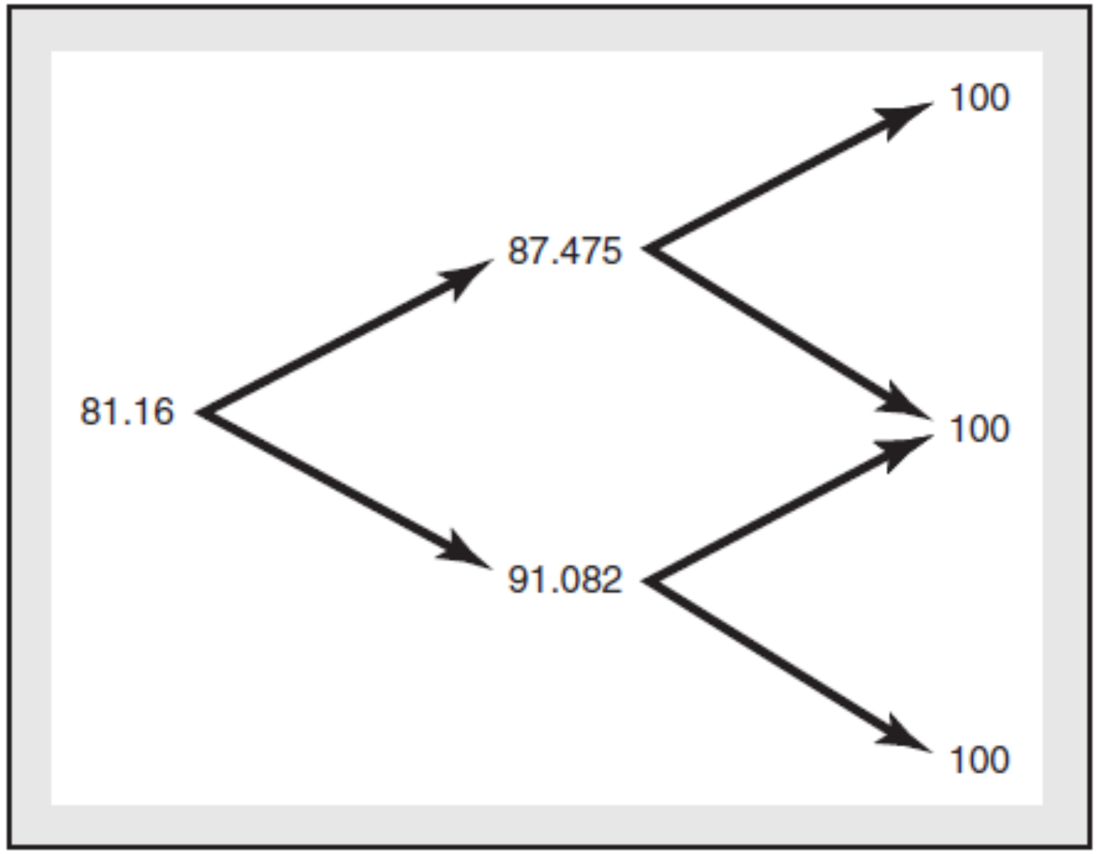
Tree of rates



Trees of Bond Prices

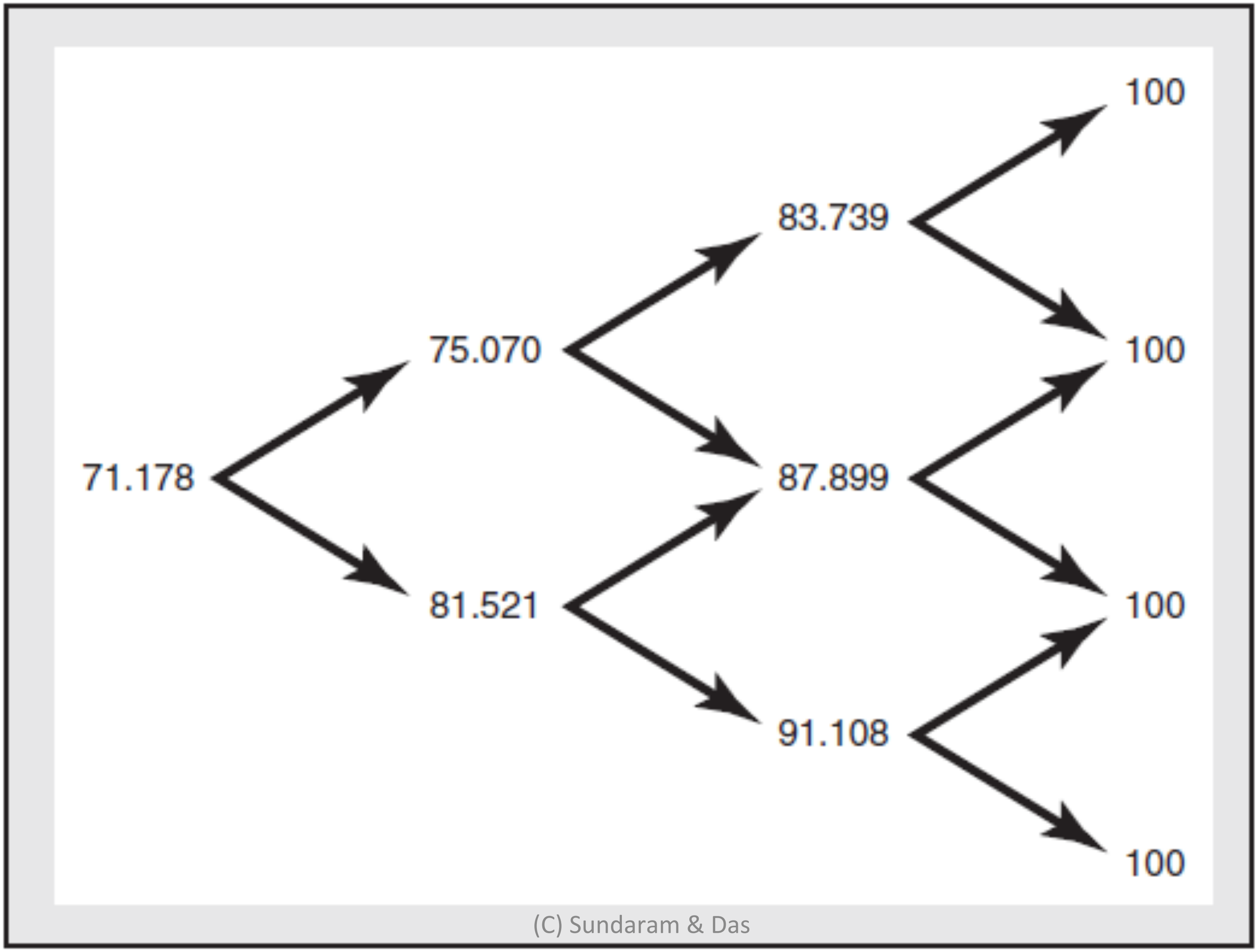


One-period bond

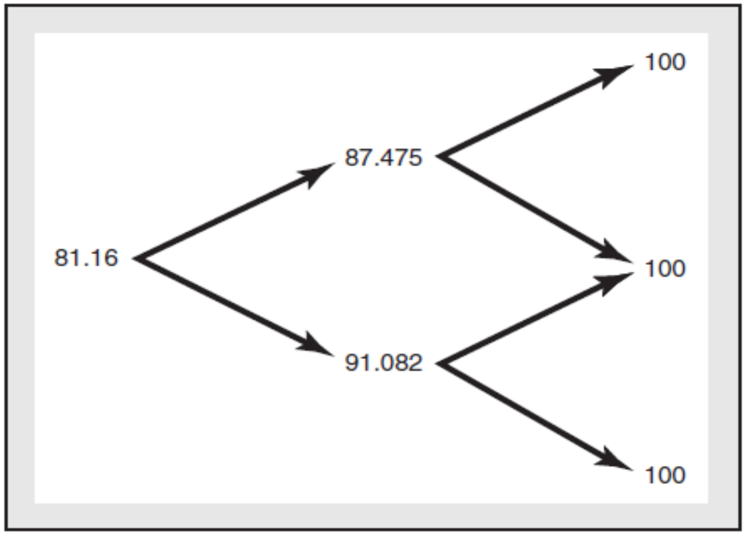


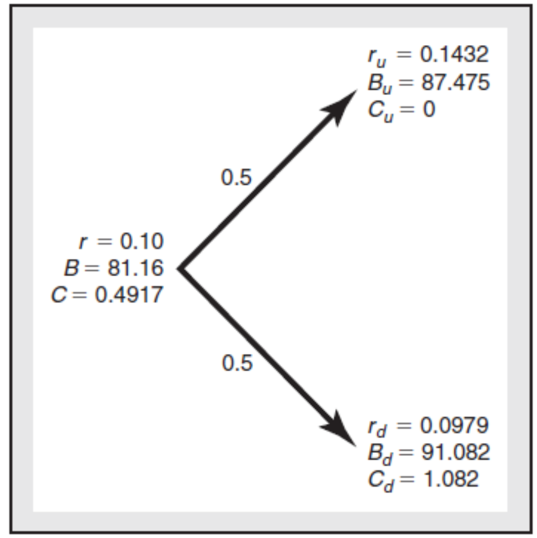
Two-period bond

Tree for the three-period bond



Pricing example - 1





Consider a call option that gives its holder the right to buy a one-year zero in one year for \$90. The underlying asset in this case (on which the call is written) is a two-year zero since a two-year zero will be a one-year zero after one year.

The possible prices of the two-year zero after one year are $B_u = 87.475$ and $B_d = 91.082$. Therefore, the possible values of the call after one year are $C_u = 0$ and $C_d = 1.082$.

As usual, the arbitrage-free price of the option may be obtained by taking the discounted expectation of the option payoffs under the risk-neutral probability. The risk-neutral probability of an up move is 1/2. The rate of interest over the one-year period is 10%. Therefore, the current value of the option is

$$C = \frac{1}{1.10}\left[\frac{1}{2}(0) + \frac{1}{2}(1.082)\right]$$

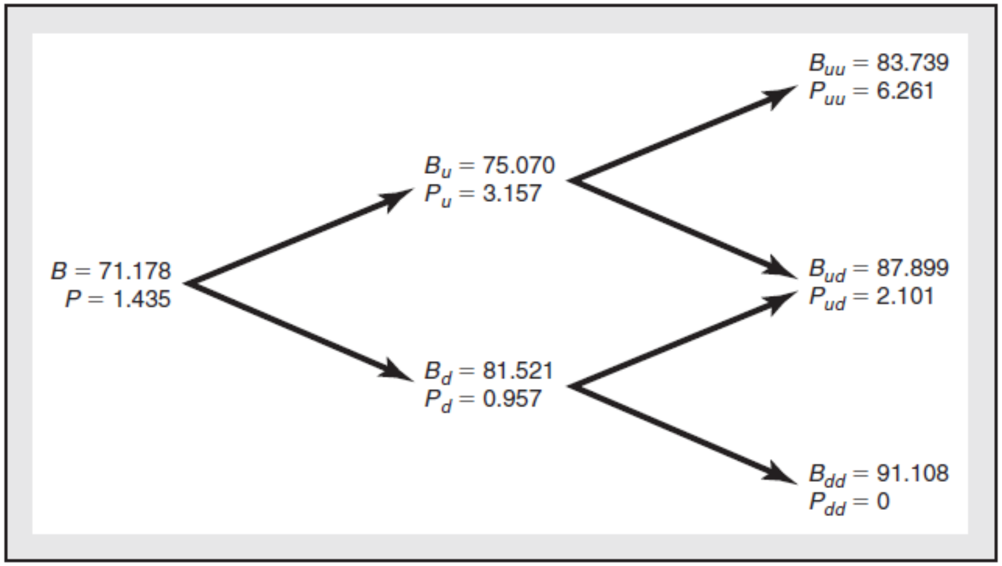
or $C = 0.4917$. The option pricing tree is presented in Figure 29.5.

Pricing example - 2

Consider a put option that gives its holder the right to sell a one-year zero for $90 in two years. The asset underlying this option is a three-year zero, since a three-year zero will be a one-year zero in two years.

We conduct the analysis in the following steps.

- Let P denote the current value of the option.
- Let P_u and P_d denote its possible values in one year.
- Let P_{uu}, P_{ud}, and P_{dd} denote its possible values after two years.



After two years, the possible prices of the three-year zero

$$B_{uu} = 83.739$$
$$B_{ud} = 87.899$$
$$B_{dd} = 91.108$$

Therefore, the option values at the end of two years are

$$P_{uu} = 6.261$$
$$P_{ud} = 2.101$$
$$P_{dd} = 0.0$$

By the usual risk-neutral pricing arguments, we must have

$$P_u = \frac{1}{r_u}\left[\frac{1}{2}P_{uu} + \frac{1}{2}P_{ud}\right]$$

$$P_d = \frac{1}{r_d}\left[\frac{1}{2}P_{ud} + \frac{1}{2}P_{dd}\right]$$

Substituting for r_u, r_d, P_{uu}, P_{ud}, and P_{dd}, we obtain

$$P_u = 3.157 \qquad P_d = 0.957$$

Finally, we must also have

$$P = \frac{1}{r}\left[\frac{1}{2}P_u + \frac{1}{2}P_d\right]$$

Substituting for r, P_u, and P_d, we finally obtain

$$P = 1.435$$

Pricing example - 3

three-year bond with a coupon rate of 10%. Let's say that this bond is callable at the end of the second year at par, i.e., the issuer can repay the bond at a price of 100.

At maturity: $$B_{uuu} = B_{uud} = B_{udd} = B_{ddd} = 110$$

On call date: $$B_{uu} = \frac{1}{1.1942}\left[\frac{1}{2}100 + \frac{1}{2}100 + 10\right] = 92.113$$

$$B_{ud} = \frac{1}{1.1377}\left[\frac{1}{2}100 + \frac{1}{2}100 + 10\right] = 96.689$$

$$B_{dd} = \frac{1}{1.0976}\left[\frac{1}{2}100 + \frac{1}{2}100 + 10\right] = 100.219$$

After one period: $$B_u = \frac{1}{1.1432}\left[\frac{1}{2}92.113 + \frac{1}{2}96.689 + 10\right] = 91.325$$

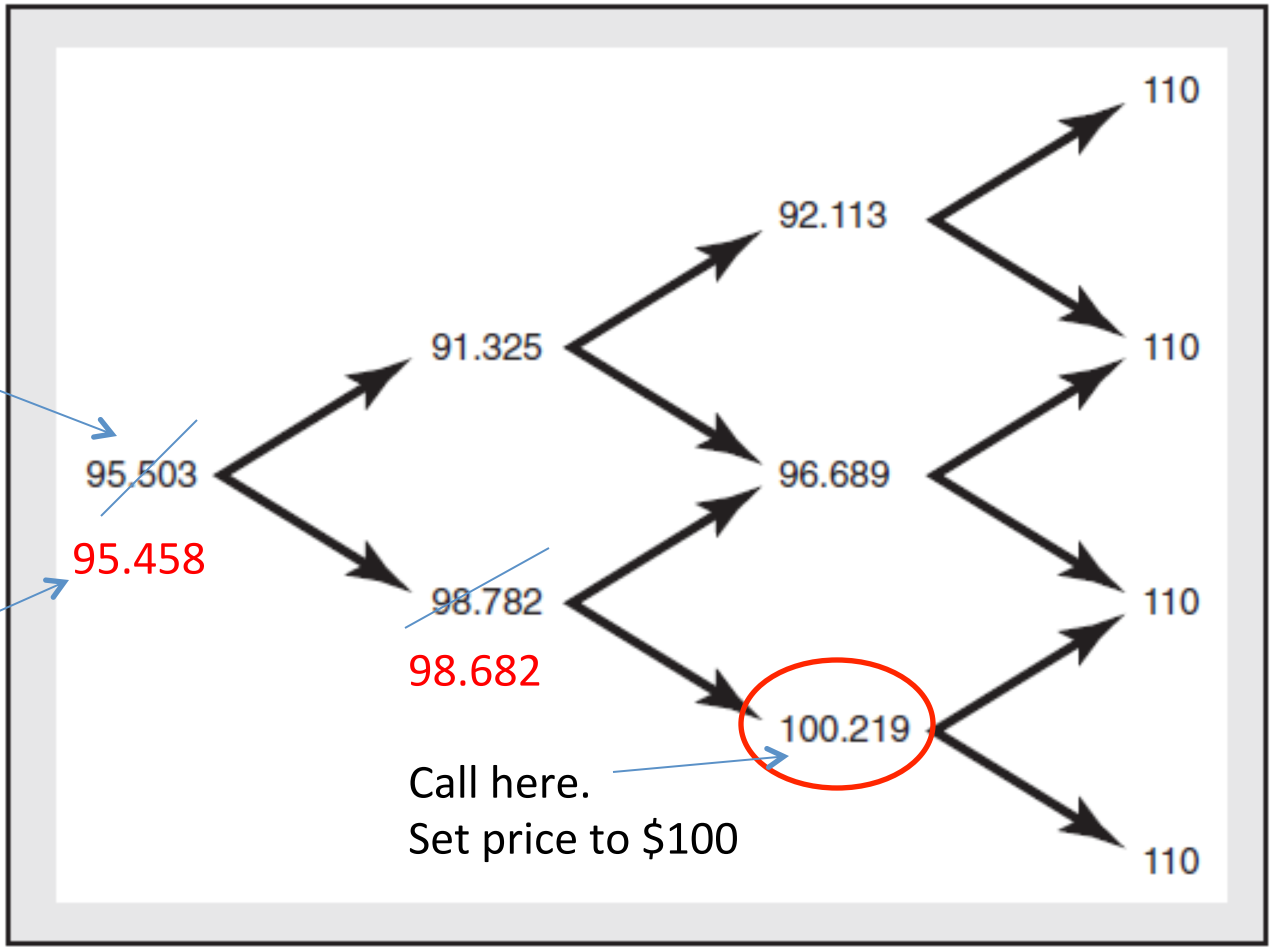
$$B_d = \frac{1}{1.0979}\left[\frac{1}{2}96.689 + \frac{1}{2}100.219 + 10\right] = 98.782$$

Current price: $$B = \frac{1}{1.10}\left[\frac{1}{2}91.242 + \frac{1}{2}98.864 + 10\right] = 95.503$$

Non-callable & callable coupon bond tree

Non-callable bond price

Callable bond price



Summary of BDT Model

Attractive features:

- Simplicity.
- Positive interest rates.
- Consistent with given term-structure of rates and volatilities.

However:

- Shares weakensses of all one-factor models.
- Only simple voaltility structures are permissible