Video lesson 0: Review of the Fluid Equations

Mittwoch, 1. Juli 2015 17:15

Review: Differential form of Fluid equations

1) Conservation of mass

For a system $\quad \dfrac{D M_{system}}{Dt} = 0$

① Rate of change of mass in C.V. ② Net rate of flow of mass across C.S.

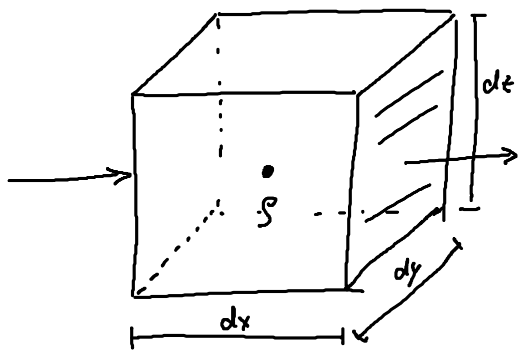
For a C.V. (Control Volume): $\quad \dfrac{\partial}{\partial t} \int_{CV} \rho \, dV + \int_{CS} \rho \vec{V} \hat{n} \, dA = 0$

Differential form. Small element $dx\, dy\, dz$

① $\quad \dfrac{\partial}{\partial t} \int_{CV} \rho \, dV = \dfrac{\partial \rho}{\partial t} dx\, dy\, dz \qquad$ (Assumption: ρ is uniform in dV)

② Rate of mass flow: In the x-direction: ρu : x-Component of the mass flow rate $\left[\frac{kg}{m^3}\right]\left[\frac{m}{s}\right]$

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u) \frac{dx}{2}\right] dy\, dz$$



$$\underbrace{\left[\rho u + \frac{\partial}{\partial x}(\rho u) \frac{dx}{2}\right]}_{\text{Taylor expansion}} dy\, dz$$

Net rate of mass outflow in x: $\boxed{m_{out} - m_{in}}$

$$\left[\cancel{\rho u} + \frac{\partial}{\partial x}(\rho u) \frac{dx}{2}\right] dy\, dz - \left[\cancel{\rho u} - \frac{\partial}{\partial x}(\rho u) \frac{dx}{2}\right] dy\, dz = \frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz$$

Similar: In the y direction: $\quad \dfrac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz$

z direction: $\quad \dfrac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Vector Notation

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Vector Notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$$

$$\frac{\partial \rho}{\partial t} + \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} \cdot \rho \begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0$$

Part 2

2) Conservation of Momentum

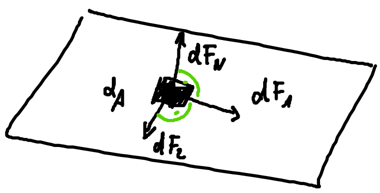
System: $$\vec{F} = \frac{D}{Dt} \int_{Syst} \vec{V} \, dm$$

C.V.: $$\sum \vec{F}_{cv} = \frac{\partial}{\partial t} \int_{CV} \vec{V} \rho \, d\mathcal{V} + \int_{CS} \vec{V} \rho \, \vec{V} \hat{n} \, dA$$

Infinitesimal fluid of mass dm $$d\vec{F} = \frac{D}{Dt}(\vec{V} dm) = dm \cdot \frac{D}{Dt} \vec{V}$$
$$= dm \cdot \vec{a}$$

Types of forces

- Body forces
  weight of the element
  $$d\vec{F}_b = dm \cdot \vec{g}$$
- Surfaces forces : normal and tangential

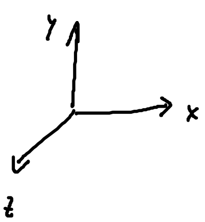


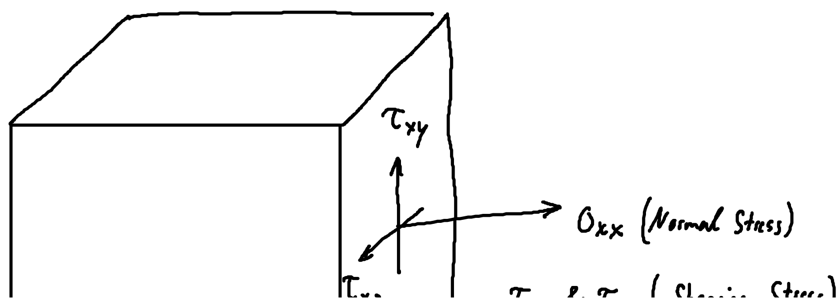
Normal Stress : $$\sigma_N = \lim_{A \to 0} \frac{dF_N}{dA}$$

Shearing Stresses: $$\tau_1 = \lim_{A \to 0} \frac{dF_1}{dA}$$

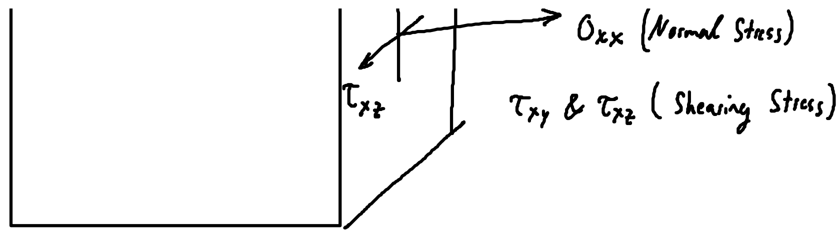
$$\tau_2 = \lim_{A \to 0} \frac{dF_2}{dA}$$

With reference to a coordinate system :

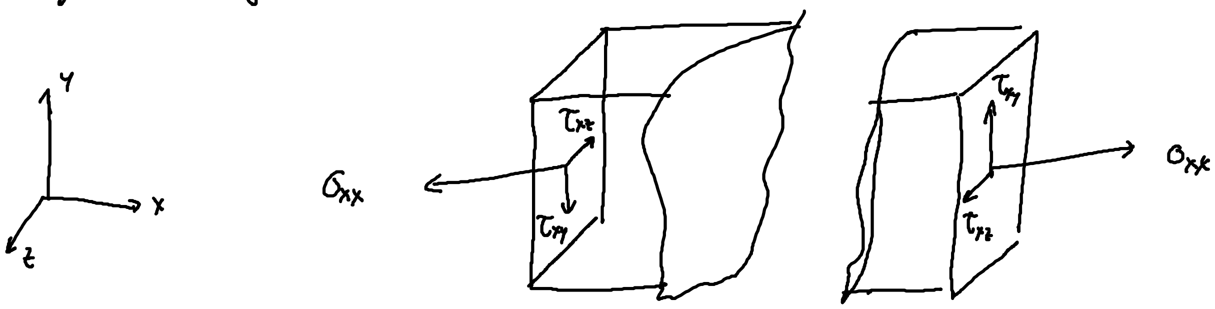




z



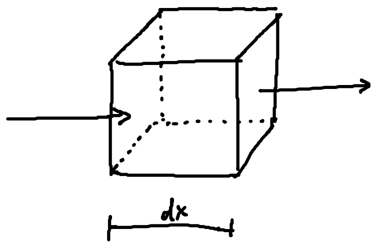
Sign convention for stresses



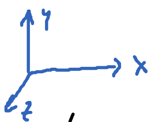
All stresses here are understood as positive. Left HS: Stress point into the negative direction
Right HS: Stress point into the positive direction

Surface forces in terms of stresses

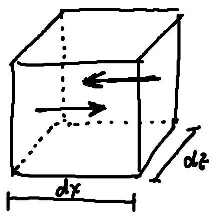
$$\left(\sigma_{xx} - \frac{\partial}{\partial x}\sigma_{xx}\frac{dx}{2}\right) dy\, dz$$



$$\left(\sigma_{xx} + \frac{\partial}{\partial x}\sigma_{xx}\frac{dx}{2}\right) dy\, dz$$

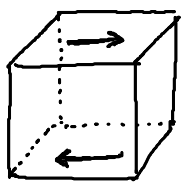


$$\left(\tau_{zx} - \frac{\partial}{\partial z}\tau_{zx}\frac{dz}{2}\right) dx\, dy$$



$$\left(\tau_{zx} + \frac{\partial}{\partial z}\tau_{zx}\frac{dz}{2}\right) dx\, dy$$

$$\left(\tau_{yx} + \frac{\partial}{\partial y}\tau_{yx}\frac{dy}{2}\right) dx\, dz$$



$$\left(\tau_{yx} - \frac{\partial}{\partial y}\tau_{yx}\frac{dy}{2}\right) dx\, dz$$

⇒ Sum of forces in x direction

$$F_{sx} = \left(\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right) dx\,dy\,dz$$

Surface

Similar for all other directions

$$F_{sy} = \left(\frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right) dx\,dy\,dz$$

$$F_{sz} = \left(\frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \sigma_{zz} \right) dx\,dy\,dz$$

<u>Equation of motion</u>: $d\vec{F} = dm \cdot \vec{a}$

$dm = \rho \; dx\,dy\,dz$

$$\rho g_x + \frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} = \rho \overbrace{\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)}^{\vec{a}}$$

$$\rho g_y + \frac{\partial}{\partial x} \tau_{yy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \tau_{zy} = \rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right)$$

$$\rho g_z + \frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \sigma_{zz} = \rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial t} + v \frac{\partial w}{\partial t} + w \frac{\partial w}{\partial t} \right)$$

3 Equations + continuity = 4 Equations

Unkown: u, v, w + all stresses

Inviscid flow: No shearing stresses; $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$

→ Euler's eqn. $\rho g_x - \frac{\partial p}{\partial x} = \rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right)$

Vector Notation $\rho \vec{g} - \nabla p = \rho \left(\frac{\partial}{\partial t} \vec{V} + (\vec{V} \nabla) \vec{V} \right)$

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3) The Navier-Stokes equations

Newtonian Fluids: linear relationship between stresses & rates of deformation

<u>Normal stresses</u>: $\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$

$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

$$\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = -3p + 2\mu \underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)}_{\nabla \cdot \vec{V} = 0 \ \text{(inkompressibel)}}$$

$$\frac{1}{3} \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz} \right) = -p$$

Shearing Stresses

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

$$\tau_{yz} = \tau_{zy} = \mu \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$

$$\tau_{zx} = \tau_{xz} = \mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)$$

All stresses are given as depending from the velocities!

Stress terms, x-direction momentum eq.

$$\underbrace{\frac{\partial}{\partial x} \sigma_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx}}$$

$$\underbrace{\frac{\partial}{\partial x} \left(-p + 2\mu \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right) + \frac{\partial}{\partial z} \left(\mu \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right)}$$

2nd Order . Rate of deformation

$$-\frac{\partial p}{\partial x} + 2\mu \frac{\partial^2 u}{\partial x^2} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) + \mu \left(\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} \right)$$

$$-\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \underbrace{\mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x \partial y} + \frac{\partial w}{\partial x \partial z} \right)}_{\mu \cdot \frac{\partial}{\partial x} \left(\underbrace{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}} \right)}$$

0 for incompressible flow

$$\Rightarrow \quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Vector Notation:

$$\rho\left(\frac{\partial}{\partial t}\vec{V} + (\vec{V}\cdot\nabla)\vec{V}\right) = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{V}$$

Navier Stokes Equation
Newtonian, incompressible fluid

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

3 eq. + continuity $\Rightarrow$ 4 eqn

unknowns $u \; v \; w \; p \; \rho \Rightarrow$ 5 unknowns $\longrightarrow$ Need an equation of state

* Nonlinear, second order PDE $\rightarrow$ Very few known solutions

$\downarrow$

$(\vec{V}\cdot\nabla)\vec{V}$

Only general approach is <u>computational</u>!

Video lesson 1 - Introduction to CFD

Donnerstag, 2. Juli 2015 15:03

Recall the Navier Stokes Equation

nonlinear

2nd order

$$\underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{unsteady term}} + \underbrace{\boxed{\vec{V} \cdot \nabla \vec{V}}}_{\text{convective acceleration}} = -\frac{\nabla p}{\rho} + \underbrace{\nu \nabla^2 \vec{V}}_{\text{Newtonian Fluid}}$$

still nonlinear

Inviscid $\nu = 0$

$$\frac{\partial V}{\partial t} + \boxed{\vec{V} \cdot \nabla \vec{V}} = -\frac{\nabla p}{\rho}$$

Euler Equation

Solution: - The velocity field (vector)
- The associated pressure

Basic ingredients of CFD

(1) Mathematical Model: Set of partial differential or integro-differential equations (and Boundary Equations)

Model ⟷ Target application

in compressible
inviscid
turbulent

2D/3D

(2) choose discretization method: Method for approximating the PDES by a system of algebraic equations

Major aspect of CFD

diff. operator

$$\mathcal{L}[u(x)] = f(x)$$

$$\Downarrow$$

$$A\underline{x} = \underline{b}$$

- Finite difference, FD
- Finite volume, FV
- Finite elements, FE
- Spectral methods
- Boundary element

"big three"

Two aspects
- Geometry → Grid, Mesh, (particle)
- Model → All mathematical operators (∂/)

Two aspects

Model → All mathematical operators ($\partial/\partial x$) transformed into arithmetic operations on the grid

3) Analyze the numerical scheme

- All the numerical schemes must satisfy certain conditions to be accepted   consitency, stability, convergence
- Must analyze accuracy

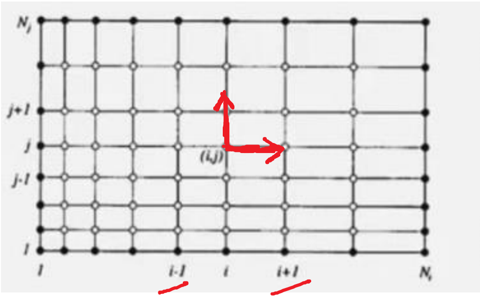
4) Solve   Obtain grid / point values of all flow variables

- time dependant : ODEs
- steady : Algebraic system of equations

- time integrators
- linear solvers

5) Post Processing

Finite Difference Method (Euler)



Basic concept: Approximate derivatives Taylor expansions

First step: Define numerical grid

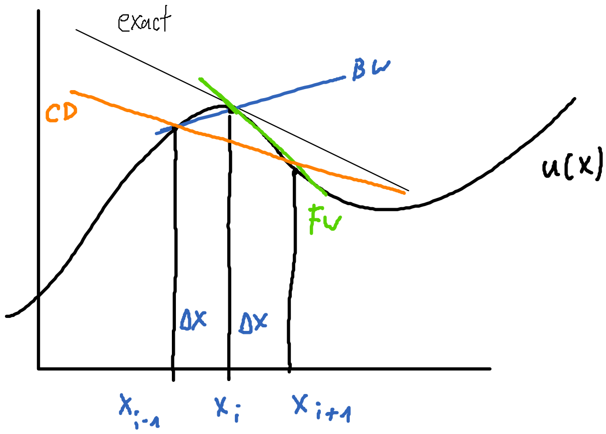
Structured grid: • Two families of lines

Node (i,j) in 2D
- Grid lines of same family do not intersect
- Grid lines of different families intersect only once

For each Node we have an unknown value of the field variable, which depends on neighbouring nodes providing one algebraic equation.

Definition of a derivative:

$$\left(\frac{\partial u}{\partial x}\right)_{x_i} = \lim_{\Delta x \to 0} \frac{u(x_i + \Delta x) - u(x_i)}{\Delta x}$$



Geometrical interpretation: Slope of tangent to curve $u(x)$

Backward difference: x_i and x_{i-1}

Forward difference: x_i and x_{i+1}

Central difference: x_{i+1} and x_{i-1}

→ Some Approximations are better than others

→ Quality of approximation improves as Δx is made smaller

Assume that Δx is uniform

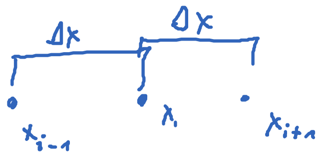
Taylor Expansions

$$u(x) = u(x_i) + (x - x_i)\left.\frac{\partial u}{\partial x}\right|_i + \underbrace{\frac{(x-x_i)^2}{2!}\frac{\partial^2 u}{\partial x^2} + \dots}_{\text{Truncation order by cutting H.O.T}}$$

x_{i+1} / x_{i-1}

$$-\frac{\partial u}{\partial x}\Big|_i = \frac{u(x_i)-u(x)}{x-x_i} + \frac{x-x_i}{2!}\frac{\partial^2 u}{\partial x^2} + \dots$$

$$\frac{\partial u}{\partial x}\Big|_i = \frac{u(x)-u(x_i)}{x-x_i} - \frac{x-x_i}{2!}\frac{\partial^2 u}{\partial x^2} - \dots$$



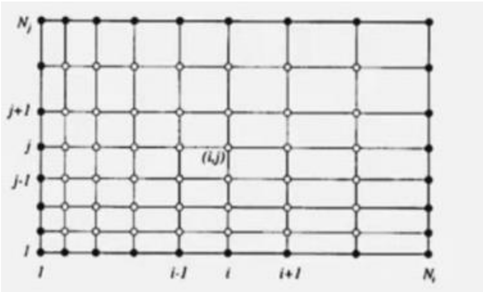
FD: $$\frac{\partial u}{\partial x}\Big|_i = \frac{\overbrace{u(x_{i+1})-u(x_i)}^{\Delta u}}{\underbrace{x-x_i}_{\Delta x}} - \frac{\overbrace{x-x_i}^{\Delta x}}{2!}\frac{\partial^2 u}{\partial x^2} - \dots$$

BD: $$\frac{\partial u}{\partial x}\Big|_i = \frac{\overbrace{u(x_{i-1})-u(x_i)}^{-\Delta u}}{\underbrace{x_{i-1}-x_i}_{-\Delta x}} - \frac{\overbrace{x_{i-1}-x_i}^{-\Delta x}}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{\overbrace{(x_{i-1}-x_i)^2}^{\Delta x^2}}{3!}\frac{\partial^3 u}{\partial x^3}$$

$$= \frac{u(x_i)-u(x_{i-1})}{x_i-x_{i-1}} + \frac{x_i-x_{i-1}}{2!}\frac{\partial^2 u}{\partial x^2} - \frac{(x_i-x_{i-1})^2}{3!}\frac{\partial^3 u}{\partial x^3} + \dots$$

Alternating signs!

The finite difference method



Structured Mesh is required for the FD Method!

$$\left(\frac{\partial u}{\partial x}\right)_{x_i} = \lim_{\Delta x \to 0} \frac{u_{x_i+\Delta x} - u_{x_i}}{\Delta x}$$

Accuracy is increased when Δx decreases

Errors are always introduced (truncation error)

Def. Order of accuracy: The power of Δx with which the truncation error tends to 0

Taylor Series: FD & BD both are 1st order / $O(\Delta x)$ („big O Notation")

Difference formulas for first order derivatives

FD & BD are called „one sided" formulas

$$FD: \quad \left.\frac{\partial u}{\partial x}\right|_i = \frac{u_{i+1} - u_i}{\Delta x} - \frac{\Delta x}{2} \left.\frac{\partial^2 u}{\partial x}\right|_i - \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} - h.o.t.$$

$$BD: \quad \left.\frac{\partial u}{\partial x}\right|_i = \frac{u_i - u_{i-1}}{\Delta x} + \frac{\Delta x}{2} \left.\frac{\partial^2 u}{\partial x}\right|_i - \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} + h.o.t.$$

truncation error

Add FD and BD to obtain 2nd order FD approximation

$$\Rightarrow \quad 2\left.\frac{\partial u}{\partial x}\right|_i = \frac{u_{i+1} - u_{i-1}}{\Delta x} - 2\,\frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} - h.o.t.$$

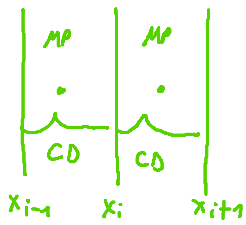
$$\underbrace{\left.\frac{\partial u}{\partial x}\right|_i = \frac{u_{i+1} - u_{i-1}}{2\Delta x}}_{\text{CD Approximation}} \underbrace{- \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3}}_{\text{Error } O(\Delta x^2)} - h.o.t.$$

Note 1D Domain $(0,1)$ with 11 mesh points

$\Delta x = 0{,}1$

1st order $O(\Delta x) \sim O(10\%)$
2nd order $O(\Delta x^2) \sim O(1\%)$ $\overset{\wedge}{=}$ 100 Divisions $\Delta x = 0{,}01$

1st order FD formula for $\left.\frac{\partial u}{\partial x}\right|_i$ can be considered as a central difference with respect to the midpoint



$$\left.\frac{\partial u}{\partial x}\right|_{i+\frac{1}{2}} = \frac{u_{i+1} - u_i}{\Delta x} + O(\Delta x^2)$$

$$\left.\frac{\partial u}{\partial x}\right|_{i-\frac{1}{2}} = \frac{u_i - u_{i-1}}{\Delta x} + O(\Delta x^2)$$

We gained an order of accuracy using the midpoints

Recall the Navier Stokes Equation $\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V} = \frac{-\nabla p}{\rho} + \nu \nabla^2 \vec{V}$

- Diffusive term appears through second order derivative term
- Convective fluxes appear as first order derivatives in space

Various modeling assumptions (eg. inviscid) $\longrightarrow$ Model equations

System of PDES
highest space derivative : 2nd order
highest time derivative : 1st order

Video lesson 4 - 12 Steps (1 to 4)

Freitag, 3. Juli 2015 10:32

12 Steps to the Navier Stokes Equation

(1) 1D linear convection

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Constant transport velocity c

Initial profile $u(x, t=0) = u_0(x)$

After time t, $u(x)$ is just the initial profile with distance $x = c \cdot t$

Solution $u(x,t) = u_0(x - ct)$

→ Wave propagation

↕

pure convection

Space - time discretization

i → index of grid in x

n → Index of time in t

Numerical scheme

FD in time
BD in space (why?)

Discrete equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Transpose equation to obtain at t_{n+1} from values at t_n

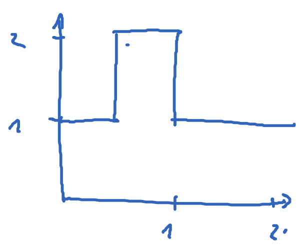
$$u_i^{n+1} = u_i^n - c\left(\frac{\Delta t}{\Delta x}\right)\left(u_i^n - u_{i-1}^n\right)$$

Initial Condition (I.C.) $u = 2$ @ $0{,}5 \le x \le 1$

$u = 1$ @ everywhere else

Boundary Condition (B.C.) $u = 0$ @ $x = 0, 2$

„square wave"



② Inviscid Burger's equation $\frac{\partial u}{\partial t} + u \cdot \frac{\partial u}{\partial x} = 0$

- Can generate discontinuous solutions from smooth I.C. (similar to shock creations)

Discretize

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \cdot \frac{u_i^n - u_{i-1}^n}{\Delta x} = 0$$

Transpose

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x} u_i^n (u_i^n - u_{i-1}^n)$$

IC.

$u = 2$ @ $0{,}5 \leq x \leq 1$
$u = 1$ @ e.e. $(0,2)$

BC

$u = 0$ @ $x = 0{,}2$

Second order derivative Geometrically: The slope of line tangent to $\frac{\partial u}{\partial x}$

→ Use approximations to $\frac{\partial u}{\partial x}$ at 2 locations

Eg. Central difference, 2nd order: Combine FD & BD

$$u_{i+1} = u_i + \Delta x \left.\frac{\partial u}{\partial x}\right|_i + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i + \frac{\Delta x^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_i + h.o.t$$

$$u_{i-1} = u_i - \Delta x \left.\frac{\partial u}{\partial x}\right|_i + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i - \frac{\Delta x^3}{6} \left.\frac{\partial^3}{\partial x^3}\right|_i + h.o.t$$

Add

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_i = \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} - O(\Delta x^2)$$

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③ 1D diffusion $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$ (heat equation)
u = Temperature

If ν is constant, exact solutions are known

Consider looking for a solution of type $u = \hat{u} e^{i(kx - \omega t)}$ $i = \sqrt{-1}$

Amplitude $\hat{u}$ $\omega = 2\pi f$

Introducing into the PDE, we obtain $i\omega = \nu k^2$ leading to solution

$$u = \hat{u}\, e^{ikt}\, e^{-\nu k^2 t}$$

exponential damping

Note $\nu > 0$ for physical diffusion

$\nu < 0$ exponentially growing phaenomena, explosives

Schemen: FD in time
CD in space

Discretize:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - u_{i-1}^n}{\Delta x}$$

Transpose

$$u_i^{n+1} = u_i^n + \left(\frac{\Delta t}{\Delta x}\right)\left(u_{i+1}^n - u_{i-1}^n\right)$$

Same I.C. / B.C. as Step 1 and 2

---

4 1D Burgers' equation $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}$

Discretize:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + u_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Transpose:

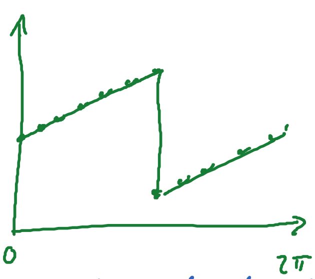
$$u_i^{n+1} = u_i^n - u_i^n \frac{\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right) + \nu \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)$$

I.C.

$$u = -2\nu \frac{\partial \varphi / \partial x}{\varphi} + 4$$

$$\varphi = \exp\left(\frac{-x^2}{4\nu}\right) + \exp\left(-\frac{(x-2\pi)^2}{4\nu}\right)$$

B.C. $u(0) = u(2\pi)$



Verify against analytical solution

Analytical solution:

$$u = \frac{-2\nu \frac{\partial \varphi}{\partial x}}{\varphi} + 4$$

$$\varphi = \exp\left[\frac{-(x-4t)^2}{4\nu(t+1)}\right] + \exp\left[\frac{-(x-4t-2\pi)^2}{4\nu(t+1)}\right]$$

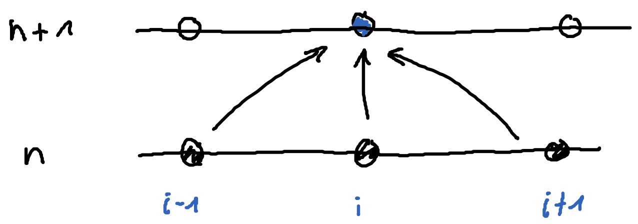
Recall the 1D diffusion equation : $\frac{\partial u}{\partial t} = r \frac{\partial^2 u}{\partial x^2}$ — parabolic PDE

Step 3 : $\left.\begin{matrix}\text{FD in time}\\ \text{CD in space}\end{matrix}\right\}$ Eqn for u_i^{n+1} : only unknown

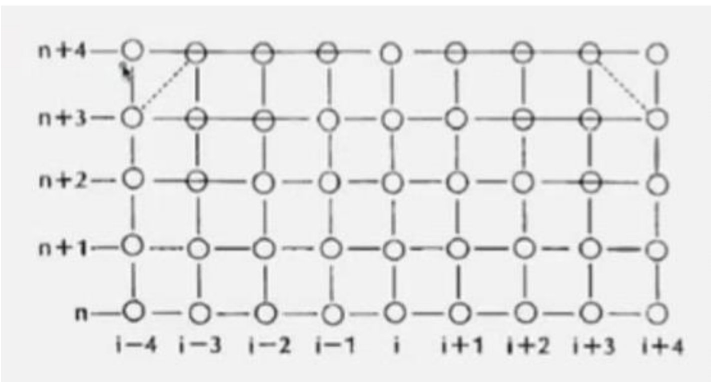
Computed values at n+1 depend only on „past history"

We need to start a solution : $\left.\begin{matrix}\text{An I.C.}\\ \text{Two B.C.}\end{matrix}\right|$ Specified

* A formulation of a continuum eqn. into a FD eqn that expresses one unknown in terms of the known values is an Explicit Method.



„Stencil" (shows dependence)



Imposed boundary conditions at i−4 and i+4

t : n+4

Note : The information at the boundaries at n+4 does not feed into the computation of unknowns at n+4

→ Contrary to the physics

→ In an explicit formula the B.C. lag behind the computation by one step

How can we have a scheme that includes the B.C. at every time level for the computation?

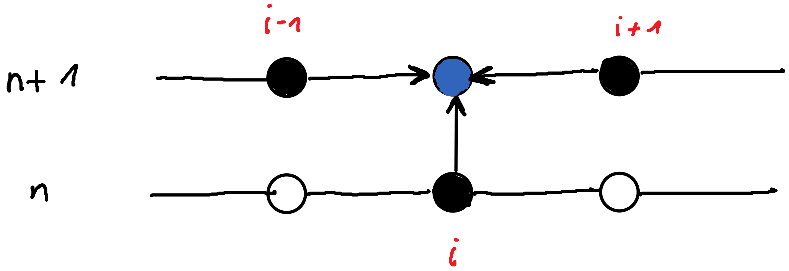
→ BD approximation for time derivative

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = r\,\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

3 unknowns

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = r \, \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2}$$

3 unknowns



„Stencil"

Need a set of coupled FD equations found by writing FD formulas for all grid points

Transpose:

$$u_{i-1}^{n+1}\left(\frac{r\Delta t}{\Delta x^2}\right) - u_i^{n+1}\left(1 - 2\frac{r\Delta t}{\Delta x^2}\right) + u_{i+1}^{n+1}\left(r\frac{\Delta t}{\Delta x^2}\right) = -u_i^n$$

→ Linear system : matrix form: tridiagonal coefficient matrix

A formulation including more than one unknown in the FD eqn is known as an implicit method.

Crank - Nicholson method

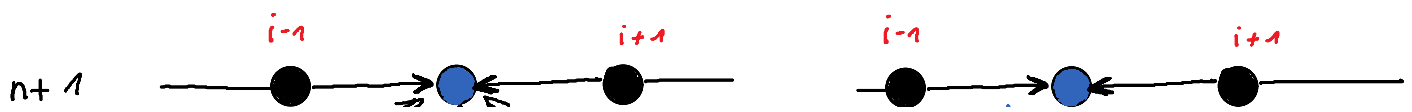
→ Average of explicit & implicit schemes

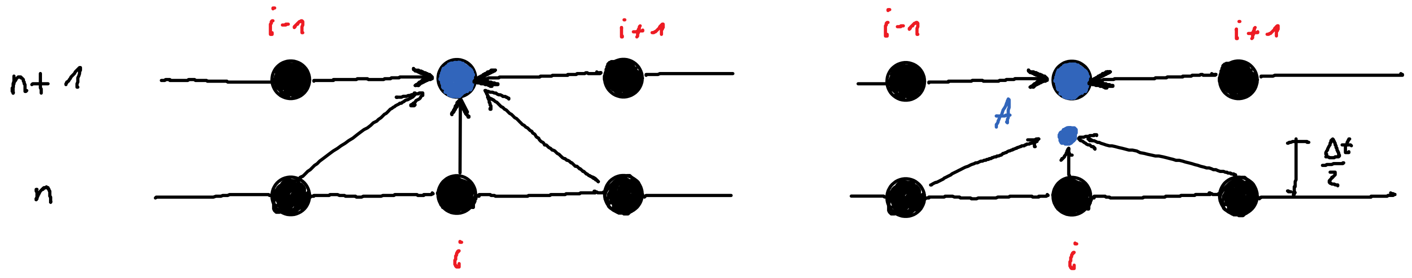
$$u_i^{n+1} = u_i^n + \frac{1}{2}\frac{r\Delta t}{\Delta x^2}\underbrace{\left(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}\right)}_{\text{implicit}} + \frac{1}{2}\frac{r\Delta t}{\Delta x^2}\underbrace{\left(u_{i+1}^{n} - 2u_i^{n} + u_{i-1}^{n}\right)}_{\text{explicit}}$$

⇒ 2nd order in time and space

implicit → tridiagonal system to solve at each time step

Note: We noted before that an expression like $\frac{u_i^{n+1} - u_i^n}{\Delta t}$ can be a CD approximation for the midpoint time $(n+\frac{1}{2})$





In terms of the grid points, we have a CD represantation of $\frac{\partial u}{\partial t}$ at point „A" and the average of the diffusion at the same point

<u>Two step computation</u>

Explicit: $$\frac{u_i^{n+\frac{t}{2}} - u_i^n}{\Delta t/2} = r \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}$$

Implicit: $$\frac{u_i^n - u^{n+\frac{t}{2}}}{\Delta t/2} = \nu \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{\Delta x^2}$$

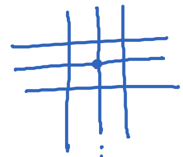
Multi-dimensional FD formulas

- Extend 1D formulas to 2D → Just apply the definition

A partial derivative with respect to x is the variation in x holding „y" constant

Build 2D grid defined by $x_i = x_0 + i\,\Delta x$

$y_i = y_0 + j\,\Delta y$



Define $u_{ij} = u(x_i, y_j)$

Point $(i+1, j+1)$, Taylor Series in 2D

$$u_{i+1,j+1} = u_{ij} + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) u \Big|_{ij}$$
$$+ \frac{1}{2}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 u \Big|_{ij} + \frac{1}{6}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^3 u \Big|_{ij}$$

1st order FD
x Direction

$$\frac{\partial u}{\partial x}\Big|_{ij} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + O(\Delta x)$$

CD
x direction

$$\frac{\partial u}{\partial x}\Big|_{ij} = \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + O(\Delta x^2)$$

2nd order
2nd derivative

$$\frac{\partial^2 u}{\partial x^2}\Big|_{ij} = \frac{u_{i+1,j} - 2_{i,j} + u_{i-1,j}}{\Delta x^2} + O(\Delta x^2)$$

---

(5) 2D - linear convection $\qquad \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0$

Discretize :

$n+1 \quad n \quad n \quad n \quad n \quad n$

Discretize :

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} + c\,\frac{u_{ij}^{n} - u_{i-1\,j}^{n}}{\Delta x} + c\,\frac{u_{ij}^{n} - u_{i\,j-1}^{n}}{\Delta x} = 0$$

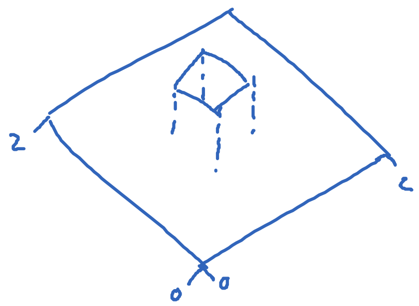
Transpose :

$$u_{ij}^{n+1} = u_{ij}^{n} - c\frac{\Delta t}{\Delta x}\left(u_{ij}^{n} - u_{i-1\,j}^{n}\right) - c\frac{\Delta t}{\Delta x}\left(u_{ij}^{n} - u_{ij-1}^{n}\right)$$

IC.: $u = 2$ @ $0.5 \leq x \leq 1$ & $0{,}5 \leq y \leq 1$

$u = 1$ @ e.e. $(0,2) \times (2,0)$

BC: $u = 1$ @ $x = 0,2$ & $y = 0,2$



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⑥ 2D convection

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0$$

Discretize :

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} + u_{ij}^{n}\frac{u_{ij}^{n} - u_{i-1\,j}^{n}}{\Delta x} + v_{ij}^{n}\frac{u_{ij}^{n} - u_{ij-1}}{\Delta y} = 0$$

$$\frac{v_{ij}^{n+1} - v_{ij}^{n}}{\Delta t} + u_{ij}^{n}\frac{v_{ij} - v_{i-1\,j}}{\Delta x} + v_{ij}^{n}\frac{v_{ij} - v_{ij-1}}{\Delta y} = 0$$

---

⑦ 2D Diffusion

$$\frac{\partial u}{\partial t} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Discretize:

$$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} = \nu \frac{u_{i-1j}^{n} - 2u_{ij}^{n} + u_{i+1j}^{n}}{\Delta x^2} + \nu \frac{u_{ij-1}^{n} - 2u_{ij}^{n} + u_{ij+1}}{\Delta y^2}$$

Transpose: $$u_{ij}^{n+1} = u_{ij}^{n} + \nu \frac{\Delta t}{\Delta x^2}\left(u_{i-1j}^{n} - 2_{ij}^{n} + u_{i+1j}^{n}\right) + \nu \frac{\Delta t}{\Delta y^2}\left(u_{ij-1}^{n} - 2u_{ij}^{n} + u_{i,j+1}^{n}\right)$$

IC. & BC. Same as before!

---

(8) 2D Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$(1)\quad \frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} + u_{ij}^{n}\frac{u_{ij}^{n} - u_{i-1j}^{n}}{\Delta x} + v_{ij}^{n}\frac{u_{ij}^{n} - u_{ij-1}^{n}}{\Delta y} = \nu\left(\frac{u_{i-1j}^{n} - 2u_{ij}^{n} + u_{i+1j}^{n}}{\Delta x^2} + \frac{u_{ij-1}^{n} - 2u_{ij}^{n} + u_{ij+1}^{n}}{\Delta y^2}\right)$$

$$(2)\quad \frac{v_{ij}^{n+1} - v_{ij}^{n}}{\Delta t} + u_{ij}^{n}\frac{v_{ij}^{n} - v_{i-1j}^{n}}{\Delta x} + v_{ij}^{n}\frac{v_{ij}^{n} - v_{ij-1}^{n}}{\Delta y} = \nu\left(\frac{v_{i-1j}^{n} - 2v_{ij}^{n} + v_{i+1j}}{\Delta x^2} + \frac{v_{ij-1}^{n} - 2v_{ij}^{n} + v_{ij+1}^{n}}{\Delta y^2}\right)$$

IC. & B.C. as in step (7)

Analysis of numerical schemes — Stability

Recall step ① FD in time, BD in space Discretized : $\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}\left(u_i^n - u_{i-1}^n\right) = 0$

Explicit schemes : Very simple & economical
But : Restrictions to get a valid solution

Alternative : 2nd order CD in space :

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0$$

$$u_i^{n+1} = u_i^n - c\frac{\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0$$ still explicit

Implicit version

CD

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right) = 0$$

→ Results in linear system of equation with tridiagonal matrix

BD

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}\left(u_i^{n+1} - u_{i-1}^{n+1}\right) = 0$$

FD

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}\left(u_{i+1}^{n+1} - u_i^{n+1}\right)$$ Implicit

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{\Delta x}\left(u_{i+1}^{n} - u_i^{n}\right)$$ Explicit

These last schemes are called upwind first order schemes for the convection equation

These last schemes are called upwind first order schemes for the convection equation

Another option: 2nd order BD in space

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x} = 0 \qquad \Big|\ \text{Explicit}$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\,\frac{3u_i^{n+1} - 4u_{i-1}^{n+1} + u_{i-2}^{n+1}}{2\Delta x} = 0 \qquad \Big|\ \text{Implicit}$$

Another option: Second order CD in space + 2nd order in time

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0$$

Example: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

$$u_0(x) = \begin{cases} 0 & x < 0{,}9 \\ 10(x - 0{,}9) & 0{,}9 \le x \le 1.0 \\ 10(1.1 - x) & 1.0 \le x \le 1.1 \\ 0 & 1.1 \ge x \end{cases}$$

a) Explicit CD scheme with the parameter $\tau = c\frac{\Delta t}{\Delta x} = 0{,}8$ $\mapsto$ unstable, useless!

b) First order upwind ("Step1"). $\sigma = 0{,}8$ Solution is ok, but significantly diffused

c) Do b) but $\sigma = 1.5$ → solutions grows erratically „Conditional stability"

→ Basic questions

- What conditions should we impose on a numerical scheme to obtain an acceptable approximation to the problem?
- Why do various schemes have such different behaviour

- How can we predict stability
- For a stable scheme, how can we obtain information on the accuracy

→ Need to find : Consitency – Stability – Convergence – truncation error

– modified differential equation – Diffusion – Dispersion

Main criteria

Define: Consistency is a condition on the numerical scheme

- The scheme must tend to the differential equation when the steps in time and space tend to zero

Define: Stability is a condition on the numerical solution

- All the errors must remain bounded when the iteration process progresses

For finite values of Δx, Δt the error has to remain bounded when the number of time steps n tends to infinity.

If the error $\bar{\varepsilon}_i^n = u_i^n - \bar{u}_\varepsilon^n$ is the difference between the computed solution $\bar{u}_\varepsilon^n$ and the exact solution of the discretized equation $\bar{u}_i^n$

Stability criterion: $\lim_{n \to \infty} |\bar{\varepsilon}_i^n| \leq K$ at fixed Δt

Note: 1) The stability criterion is a requirement on the numerical scheme only. (does not require any condition on the differential equation)

2) Stability does not ensure that the error will not become unacceptable large at intermediate time steps

3) More general definition later

Define: Convergence is a condition on the numerical solution.

The numerical solution must tend to the exact solution of the mathematical model, when steps in Δx, Δt tend to zero (i.e. mesh refine)

If the error

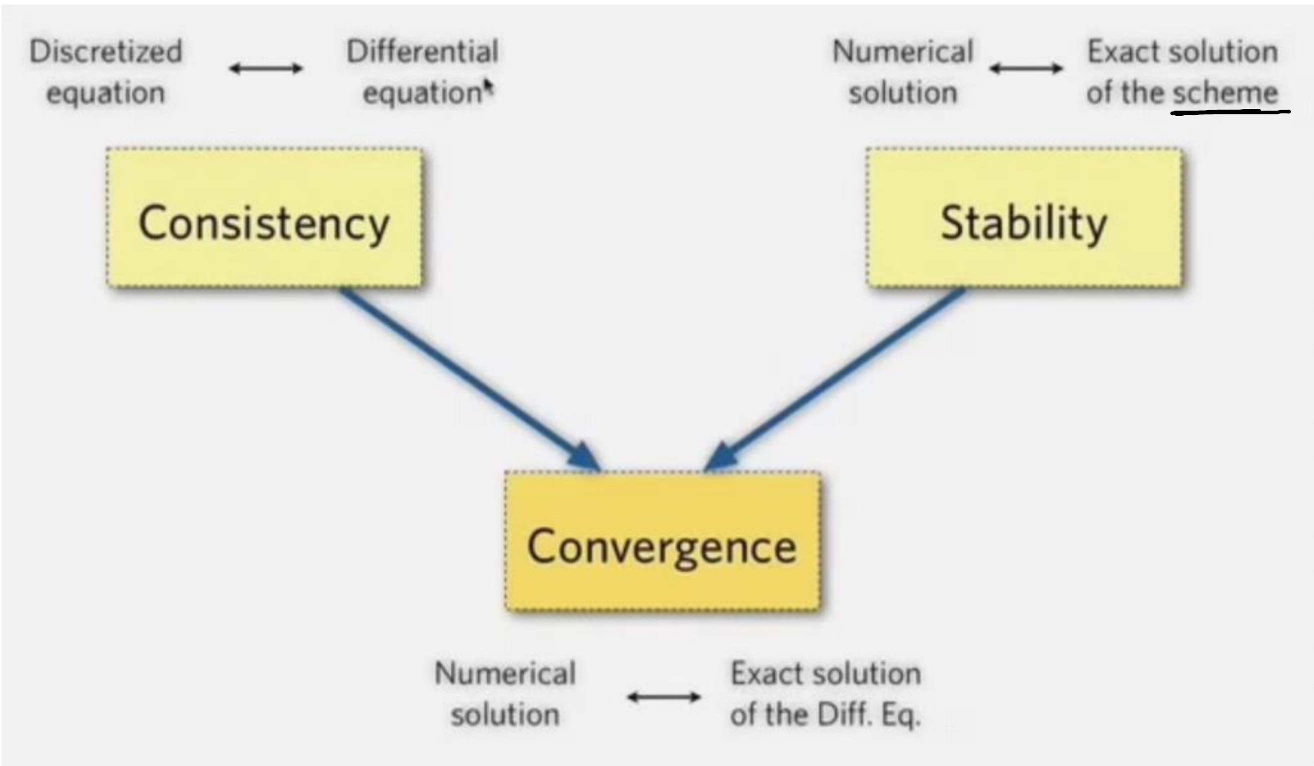
$$\tilde{\varepsilon}_i^n = u_i^n - \tilde{u}_i^n \quad , \quad \tilde{u}_i^n = \tilde{u}_i^n(i\Delta x, n\Delta t)$$

is the difference between the computed solution u_i^n and the exact solution of the analytical equation representing the mathematical model.

Convergence condition: $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} |\tilde{\epsilon}_i^n| = 0$

Equivalence Theorem of Lax

For a well-posed Initial Value Problem (IVP) and a consistent discretization scheme, stability is the necessary and sufficient condition for convergence



Consistency and the modified differential equation

↳ A consistent scheme is one in which the truncation error tends to zero for $\Delta t, \Delta x \to 0$

Eg. CD in space + FD in time for linear convection

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0 \qquad (*)$$

Taylor

$$u_i^{n+1} = u_i^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_i^n + \frac{\Delta t^2}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \frac{\Delta t^3}{6} \left.\frac{\partial^3 u}{\partial t^3}\right|_i^n + h.o.t.$$

$$u_{i+1}^n = u_i^n + \Delta x \left.\frac{\partial u}{\partial x}\right|_i^n + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n + \frac{\Delta x^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + h.o.t$$

$(-)$

$$u_{i-1}^n = u_i^n - \Delta x \left.\frac{\partial u}{\partial x}\right|_i^n + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n - \frac{\Delta x^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + h.o.t$$

Back in $(*)$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \left.\frac{\partial u}{\partial t}\right|_i^n + \frac{\Delta t}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \frac{\Delta t^2}{6} \left.\frac{\partial^3 u}{\partial t^3}\right|_i^n$$

$$\frac{u_{i+1} - u_{i-1}}{2\Delta x} = \left.\frac{\partial u}{\partial x}\right|_i^n + \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3}$$

$$\left.\frac{\partial u}{\partial t}\right|_i^n + \frac{\Delta t}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \underbrace{\cancel{\frac{\Delta t^2}{6} \left.\frac{\partial^3 u}{\partial t^3}\right|_i^n}}_{O(\Delta t^2)} + c\left(\left.\frac{\partial u}{\partial x}\right|_i^n + \frac{\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3} \right) = 0$$

$$\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} \right)_i^n + \underbrace{\frac{\Delta t}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \frac{c\,\Delta x^2}{6} \frac{\partial^3 u}{\partial x^3}}_{\text{Truncation error} \quad \epsilon_T} + O(\Delta t^2, \Delta x^4) = 0$$

$$\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} |\epsilon_T| = 0$$

Unsicher ob letzter Rechenschritt passt. Dient nur um zu zeigen dass das Schema consistent ist

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \longrightarrow \left.\begin{aligned}\boxed{\frac{u_i^{n+1} - u_i^n}{\Delta t}} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0\end{aligned}\right\} (*)$$

$$u_i^{n+1} = u_i^n + \Delta t \boxed{\left.\frac{\partial u}{\partial t}\right|_i^n} + \frac{\Delta t^2}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \frac{\Delta t^3}{6}\left.\frac{\partial^3 u}{\partial t^3}\right|_i^n + \dots$$

$$-\left\{\begin{aligned} u_{i+1}^n &= u_i^n + \Delta x \boxed{\left.\frac{\partial u}{\partial x}\right|_i^n} + \frac{\Delta x^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n + \frac{\Delta x^3}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + \dots \\ u_{i-1}^n &= u_i^n - \Delta x \boxed{\left.\frac{\partial u}{\partial x}\right|_i^n} + \frac{\Delta x^2}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n + \frac{\Delta x^3}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + \dots \end{aligned}\right.$$

In (*)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_i^n\right) - \left(\boxed{\frac{\partial u}{\partial t}} + c\boxed{\frac{\partial u}{\partial x}}\right)_i^n = \underbrace{\frac{\Delta t}{2}\left.\frac{\partial^2 u}{\partial x^2}\right|_i^n + c\frac{\Delta x^2}{6}\left.\frac{\partial^3 u}{\partial x^3}\right|_i^n + O\left(\Delta t^2, \Delta x^4\right)}_{\text{Truncation error, } \epsilon_T}$$

The truncation error is the difference between the numerical scheme and the differential equation.

$$\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)_i^n + \underbrace{\frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2}\Big|_i^n + \frac{c\,\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}}_{\text{Truncation error}} + O(\Delta t^2, \Delta x^4) = 0 \qquad (*)$$

Note: The exact solution of the numerical scheme satisfies a modified differential equation

Consider the exact solution of the discretized equation $\bar{u}_i^n$

$$\frac{\bar{u}_i^{n+1} - \bar{u}_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(\bar{u}_{i+1}^n - \bar{u}_{i-1}^n\right) \equiv 0 \qquad (V)$$

From $(*)$

$$\left(\frac{\partial \bar{u}}{\partial t} + c\frac{\partial \bar{u}}{\partial x}\right)_i^n = -\frac{\Delta t}{2}\frac{\partial^2 \bar{u}}{\partial t^2}\Big|_i^n - c\frac{\Delta x^2}{6}\frac{\partial^3 \bar{u}}{\partial t^3}\Big|_i^n + O(\Delta t^2, \Delta x^4)$$

$$\frac{\partial \bar{u}}{\partial t}\Big|_i^n = -c\frac{\partial \bar{u}}{\partial x}\Big|_i^n + O(\Delta t, \Delta x^2)$$

Take $\partial/\partial t$

$$\frac{\partial^2 \bar{u}}{\partial t^2}\Big|_i^n = -c\frac{\partial^2 \bar{u}}{\partial x \partial t}\Big|_i^n + O(\Delta t, \Delta x^2)$$

$$\frac{\partial^2 \bar{u}}{\partial t^2}\Big|_i^n = -c\left(\left(\frac{\partial \bar{u}}{\partial t}\right)\frac{\partial}{\partial x}\right)_i^n + O(\Delta t, \Delta x^2)$$

$$\frac{\partial^2 \bar{u}}{\partial t^2}\Big|_i^n = c^2\frac{\partial \bar{u}}{\partial x^2}\Big|_i^n + O(\Delta t, \Delta x^2)$$

Back in $(*)$

$$\epsilon_T = c^2\frac{\Delta t}{2}\frac{\partial^2 \bar{u}}{\partial x^2} + c\frac{\Delta x^2}{6}\frac{\partial^3 \bar{u}}{\partial x^3} + O(\Delta t^2, \Delta x^4)$$

$\Rightarrow$ The exact solution to the numerical scheme, $\bar{u}$, satisfies the following differential equation

$$\boxed{\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\frac{\Delta t}{2}c^2\frac{\partial^2 u}{\partial x^2} + O(\Delta t^2, \Delta x^2)}$$

Modified differential equation

Not a convection equation. It is a convection-diffusion equation with a numerical Diffusion coefficient of $-\frac{c^2 \Delta t}{2}$: Shows why scheme is unstable (negative diffusion → explosion)

The modified differential equation and the truncation error provide essential information about the scheme.

Method for obtaining the Modified differential equation (ModDE)

denote: $D(u) = 0$ the mathematical model we are to solve numerically and

$N(u_i^n) = 0$ the numerical scheme

1) Perform consistency analysis, obtain truncation error ϵ_T

$$\rightarrow \quad N(u_i^n) - D(u) = \epsilon_T$$

2) Consider the exact solution of the numerical scheme $\bar{u}_i^n$ defined by $N(\bar{u}_i^n) \equiv 0$ leading to the differential equation $D(\bar{u}_i^n) = -\epsilon_T$

3) Replace lowest time derivative by space derivatives in ϵ_T (by applying the differential of 2))

4) The modified differential equation is defined as an equation obtained after the replacement Step 3, restricted to the lowest order terms (contains only space derivatives)

Example: Consider the 1st order upwind scheme $\quad \dfrac{u_i^{n+1} - u_i^n}{\Delta t} + \dfrac{c}{\Delta x}\left(u_i^n - u_{i-1}^n\right) = 0 \quad (*)$

Introducing the taylor expansions

$$u_i^{n+1} = u_i^n + \Delta t \left.\frac{\partial u}{\partial t}\right|_i^n + \frac{\Delta t^2}{2} \left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \frac{\Delta t^3}{6} \frac{\partial^3 u}{\partial t^3} + \dots$$

$$u_{i-1}^n = u_i^n - \Delta x \left.\frac{\partial u}{\partial x}\right|_i^n + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n - \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + \dots$$

$$u_{i-1}^n = u_i^n - \Delta x \left.\frac{\partial u}{\partial x}\right|_i^n + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i^n - \frac{\Delta x^3}{6} \frac{\partial^3 u}{\partial x^3} + \dots$$

Into (x)

$$\left.\frac{\partial u}{\partial t}\right|_i^n + \frac{\Delta t}{2}\left.\frac{\partial^2 u}{\partial t^2}\right|_i^n + \frac{\Delta t^2}{6}\left.\frac{\partial^3 u}{\partial t^3}\right|_i^n + c\left(\left.\frac{\partial u}{\partial x}\right|_i^n - \frac{\Delta x}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^2}{6}\frac{\partial^3 u}{\partial x^3}\right) + O(\Delta x^4) = 0$$

$$\left.\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)\right|_i^n + \frac{c\Delta x}{2}\left(-\frac{\partial^2 u}{\partial x^2} + \frac{\Delta x}{3}\frac{\partial^3 u}{\partial x^3}\right) + \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} + \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3} = 0$$

$$\left.\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right)\right|_i^n = \underbrace{c\frac{\Delta x}{2}\left(\frac{\partial^2 u}{\partial x^2} - \frac{\Delta x}{3}\frac{\partial^3 u}{\partial x^3}\right) - \frac{\Delta t}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\Delta t^2}{6}\frac{\partial^3 u}{\partial t^3}}_{\epsilon_T}$$

$$\frac{\partial \bar{u}}{\partial t} = -c\frac{\partial \bar{u}}{\partial x} + o(\Delta x^2, \Delta t^2)$$

$$\frac{\partial^2 \bar{u}}{\partial t^2} = -c\frac{\partial \bar{u}}{\partial t \partial x} + o(\Delta x^2, \Delta t^2)$$

$$\frac{\partial \bar{u}}{\partial t^2} = c^2\frac{\partial \bar{u}}{\partial x^2} + o(\Delta x^2, \Delta t^2)$$

$$\left.\left(\frac{\partial \bar{u}}{\partial t} + c\frac{\partial \bar{u}}{\partial x}\right)\right|_i^n = c\frac{\Delta x}{2}\left(\frac{\partial^2 \bar{u}}{\partial x^2}\right) - \frac{\Delta t}{2}c^2\frac{\partial^2 \bar{u}}{\partial x^2} + O(\Delta x^3, \Delta t^3)$$

$$= \frac{\partial^2 \bar{u}}{\partial x^2}c\left(\frac{\Delta x}{2} - c\frac{\Delta t}{2}\right) + o(\Delta x^3, \Delta t^3)$$

$$= \underbrace{c\frac{\Delta x}{2}\left(1 - c\frac{\Delta t}{\Delta x}\right)\frac{\partial^2 \bar{u}}{\partial x^2}}_{\text{Numerical diffusion}}$$

For stability, we need $\frac{c\Delta x}{2}\left(1 - c\frac{\Delta t}{\Delta x}\right) > 0$

$$\Rightarrow 0 < c\frac{\Delta t}{\Delta x} < 1 \quad \Rightarrow c > 1$$

CFL: $\sigma = c\frac{\Delta t}{\Delta x} < 1$

CFL has a deep physical significance

For constant $\sigma < 1$, this scheme numerically diffuses of $O(\Delta x)$

which is generally excessive (the sheme has poor accuracy)

Von Neumann stability Analysis

Key: Expand the solution (or error) in a finite Fourier series

Fourier decomposition of the solution

$\bar{u}_i^n$ → exact solution of the difference equation (numerical scheme)

u_i^n → actual computed solution (roundoff error, error in I.C.)

$$u_i^n = \bar{u}_i^n + \bar{\epsilon}_i^n$$

By definition → $N(\bar{u}_i^n) = 0$

Apply $N(\,)$ to the equation above $N(u_i^n) = N(\bar{u}_i^n + \bar{\epsilon}_i^n)$

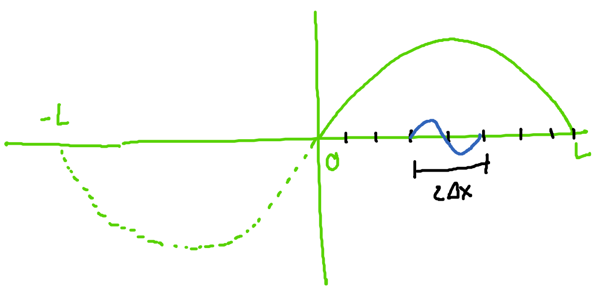
Assumption: Numerical scheme is linear $N(u_i^n) = N(\bar{u}_i^n) + N(\bar{\epsilon}_i^n)$

$$= N(\epsilon_i^n)$$

If $N(u_i^n) = 0$ represents a numerical scheme

then the errors satisfy the same equation as the numerical solution

Consider a 1D domain of $(0, L)$ reflected onto $(-L, 0)$



Create meshpoints with spacing Δx: Shortest resolveable wavelength $\lambda_{min} = 2\Delta x$

→ Maximum wave number $k_{max} = \frac{2\pi}{2\Delta x} = \frac{\pi}{\Delta x}$

→ The largest wavelength $\lambda_{max} = 2L$

→ The minimum wave number $k_{min} = \frac{\pi}{L}$

Mesh index $i = 0, \dots N$ $x_i = i\Delta x$ $\Delta x = \frac{L}{N}$

All harmonics represented in this finite mesh are

$$k_j = j\, k_{min} = j\frac{\pi}{L} = j\frac{\pi}{N\Delta x} \quad \text{with } j = (0 \dots N)$$

$j = 0$ for a constant solution

Phase angle $\varphi = k_j \Delta x = j \frac{\pi}{N}$

Covers the whole domain $(-\pi, \pi)$ in steps $\frac{\pi}{N}$ $\quad (I = \sqrt{-1})$

$$u_i^n = \sum_{j=-N}^{N} V_j^n e^{I k_j x_i} = \sum_{j=-N}^{N} V_j^n \underbrace{e^{I k_j i \Delta x}}_{e^{I i j \pi/N}}$$

Amplitude of j-th harmonic

Stability condition: Amplitude of any harmonic may not grow indefinitely in time (as $n \to \infty$)

Define amplification factor

$$G = \frac{V^{n+1}}{V^n}$$

Fraction of scheme parameters and φ (not n)

Von Neumann stability conditions: $|G| \leq 1 \quad \forall \varphi_j = j \frac{\pi}{N}; \quad j = -N \dots N$

Example (1) CD in x

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

$\sigma = \frac{c \Delta t}{\Delta x}$

Explicit $\quad u_i^{n+1} = u_i^n - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right)$

a) Replace all the terms of form u_{i+m}^{n+k} by $V^{n+k} e^{I(i+m)\varphi}$

$$V^{n+1} e^{I i \varphi} = V^n e^{I i \varphi} - \frac{\sigma}{2}\left(V^n e^{I(i+1)\varphi} - V^n e^{I(i-1)\varphi}\right)$$

b) $$V^{n+1} = V^n - V^n \frac{\sigma}{2}\left(e^{I\varphi} - e^{-I\varphi}\right)$$

Euler $\quad e^{x+iy} = e^x(\cos(y) + i\sin(y))$

c) Amplification factor $$G = \frac{V^{n+1}}{V^n} = 1 - \frac{\sigma}{2}\left(2 I \sin(\varphi)\right)$$
$$= 1 - \sigma I \sin(\varphi)$$

d) Stability $\quad G G^* = 1 + \sigma^2 \sin^2(\varphi) \geq 1$ $\quad$ Never satisfys stability $\quad$ Useless

konjugiert

Example (2) CD in x Implicit 1st order difference in time

$$u_i^{n+1} = u_i^n - \frac{\sigma}{2}\left(u_{i+1}^{n+1} - u_{i-1}^{n+1}\right)$$

$$\rightarrow V^{n+1} = V^n - V^{n+1} \frac{\sigma}{2}\left(e^{I\varphi} - e^{I\varphi}\right)$$

Amplification factor

Amplification factor

$$G = 1 - \frac{\sigma}{2} G \left(e^{I\varphi} - e^{I\varphi} \right)$$

$$G \left(1 + \frac{\sigma}{2} \left(e^{I\varphi} - e^{I\varphi} \right) \right) = 1$$

$$G = \frac{1}{1 + \frac{\sigma}{2}\left(e^{I\varphi} - e^{I\varphi} \right)} = \frac{1}{1 + \sigma I \sin(\varphi)}$$

$$|G|^2 = G\,G^* = \frac{1}{1 + I\sigma \sin(\varphi)} = \frac{1}{1 + I\sigma\sin(\varphi)} \cdot \frac{1}{1 - I\sigma\sin(\varphi)} = \frac{1}{1 + \sigma^2 \sin^2(\varphi)} \leq 1$$

konjugiert

$\boxed{\text{unconditionally stable}}$

konjugiert komplexe Zahl: $z = x + Iy$

$z^* = x - Iy$

Example ③ F.O. upwind → BD in space / Explicit FD in time

$$u_i^{n+1} = u_i^n - \sigma \left(u_i^n - u_{i-1}^n \right)$$

$$G = \frac{V^{n+1}}{V^n} = 1 - \sigma \left(1 - e^{-I\varphi} \right) = 1 - \sigma + \sigma e^{-I\varphi}$$

$$= 1 - \sigma + \sigma \cos(\varphi) - I\sigma \sin(\varphi)$$

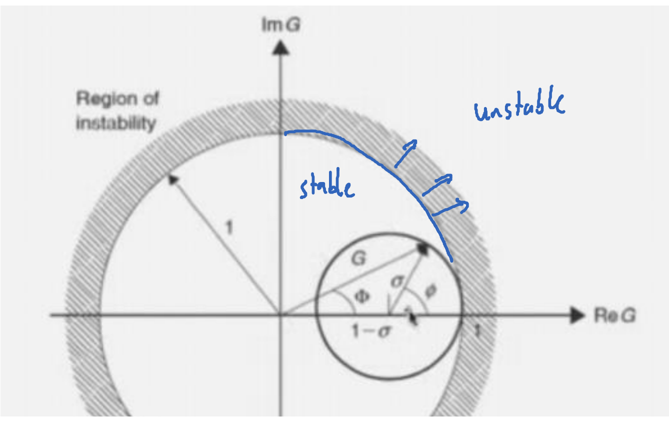
$$= 1 - 2\sigma \sin^2\left(\frac{\varphi}{2}\right) - I\sigma \sin(\varphi)$$

Seperate real and imaginary parts of G (ξ, η)

$$\xi = Re(G) = 1 - 2\sigma \sin^2\left(\frac{\varphi}{2}\right) = (1 - \sigma) + \sigma(\cos(\varphi))$$
$$\eta = Im(G) = -\sigma \sin(\varphi)$$

Parametric equations for G on complex parameter (φ parameter)

→ circle centered at the point $(1 - \sigma)$



Complex plane

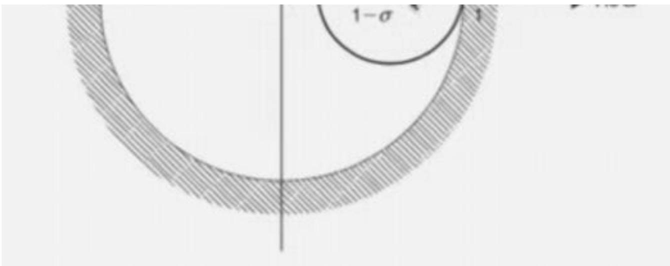
Stability criterion: $|G| < 0$

Stable for

$$0 \leq \sigma \leq 1$$

Scheme is conditionally stable

CFL < 1



Scheme is conditionally stable

$CFL < 1$

Example (4) Implicit First order upwind (BD in x)

$$u_i^{n+1} = u_i^n - \sigma \left(u_i^{n+1} - u_{i-1}^{n+1} \right)$$

$$G = \frac{1}{1 + \sigma\left(1 - e^{-I\varphi}\right)}$$

stability
$$G G^* = \frac{1}{\left(1 - \sigma + \sigma\cos(\varphi)\right)^2 + \sigma^2 \sin^2(\varphi)} < 1 \implies \text{stable } \forall \varphi$$

for all

unconditionally stable

Example (5) Diffusion Explicit First Order in time CD in space

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \rightarrow u_i^{n+1} = u_i^n + \underbrace{\frac{\nu \Delta t}{\Delta x^2}}_{\beta} \left(u_{i+1}^n - 2 u_i^n + u_{i-1}^n \right)$$

Amplification factor $G = 1 - 4\beta \sin^2\left(\frac{\varphi}{2}\right)$

Stability : $\left| 1 - 4\beta \sin^2(\varphi/2) \right| \leq 1$ satisfied for $-1 \leq 1 - 4\beta\sin^2(\varphi/2) \leq 1$

$$0 \leq \beta \leq 1/2$$

Scheme is stable for $\underbrace{\nu > 0}_{\text{stability of the physical problem}}$ and $\underbrace{\nu \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}}_{\text{Conditional stability of the scheme}}$

Video lesson 11 Computation of the Navier Stokes Equations

Dienstag, 7. Juli 2015 14:34

Recall the Navier Stokes Equations for an incompressible fluid

(1) $\nabla \vec{u} = 0$

(2) $\frac{\partial u}{\partial t} + (\vec{u}\nabla)\vec{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{u}$

(1) Conservation of mass for $\rho = \text{const}$, A kind of constraint to the equation of motion (2).

<u>Problem</u>: There is no obvious way to couple the velocity and the pressure.

(Compressible flows: Equation of state providing a relation between ρ and p)

Written out in 2D:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \Big| \; \partial/\partial x$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \qquad \Big| \; \partial/\partial y$$

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial x \partial y} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} + \nu\left(\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial y^2} = -\frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} + \nu\left(\frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 v}{\partial y^3}\right)$$

$\oplus$

LHS

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{0} + \left(\frac{\partial u}{\partial x}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \underbrace{u\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^2 v}{\partial x \partial y}} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2 + \underbrace{v\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 v}{\partial y^2}}$$

$$u\frac{\partial}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{0} + v\frac{\partial}{\partial y}\underbrace{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)}_{0}$$

$$\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial y}\right)^2$$

RHS:

RHS:

$$-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+\nu\underbrace{\left[\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+\frac{\partial}{\partial y}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\right]}$$

$$\underline{\frac{\partial^3 u}{\partial x^3}}+\underline{\frac{\partial^3 u}{\partial x\partial y^2}}+\underline{\frac{\partial^3 v}{\partial x^2\partial y}}+\underline{\frac{\partial^3 v}{\partial y^3}}$$

$$\frac{\partial^2}{\partial x^2}\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{0}+\frac{\partial^2}{\partial y^2}\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{0}$$

$$-\frac{1}{\rho}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)=\left(\frac{\partial u}{\partial x}\right)^2+2\,\frac{\partial u}{\partial y}\,\frac{\partial v}{\partial x}+\left(\frac{\partial v}{\partial y}\right)^2$$

$$\boxed{\nabla^2 p = -f}$$

Poisson equation for pressure ensures that continuity is satisfied

$$\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\vec{u}$$

First discretize in time:

$$\vec{u}^{\,n+1}=\vec{u}^{\,n}+\Delta t\left[-\vec{u}^{\,n}\cdot\nabla\vec{u}^{\,n}-\frac{1}{\rho}\nabla p^{n+1}+\nu\nabla^2\vec{u}^{\,n}\right]\qquad\Big|\ \nabla$$

$$\underset{0}{\nabla\cdot\vec{u}^{\,n+1}}\underset{=}{=}\nabla\cdot\vec{u}^{\,n}+\Delta t\left[-\nabla\left(\vec{u}^{\,n}\nabla\vec{u}^{\,n}\right)-\frac{1}{\rho}\nabla^2 p^{n+1}+\nu\nabla^2(\nabla\vec{u}^{\,n})\right]$$

In the numerical scheme, we want $\nabla\vec{u}^{\,n+1}=0$ but we have $\nabla\vec{u}^{\,n}\neq 0$ (in the discrete world).

Poisson equation for p at time $n+1$:

$$\nabla^2 p^{n+1}=\left[\rho\frac{\nabla\vec{u}^{\,n}}{\Delta t}+\left[-\rho\,\nabla(\vec{u}^{\,n}\nabla\vec{u}^{\,n})+\overset{\rho\cdot\nu\,\checkmark}{\mu}\nabla^2(\nabla\vec{u}^{\,n})\right]\right]$$

Think of velocity obtained from Navier Stokes as beeing an intermediate step:

Think of velocity obtained from Navier Stokes as beeing an intermediate step:

$$u^{n+\frac{1}{2}} \quad \text{and} \quad \nabla \vec{u}^{\,n+\frac{1}{2}} \neq 0$$

→ We need p^{n+1} so that <u>continuity is satisfied</u>

9 Laplace equation $\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0$

Discretize 2nd CD

$$\frac{p^n_{i+1,j} - 2p^n_{ij} + p_{i-1,j}}{\Delta x^2} + \frac{p_{i,j+1} - 2p_{ij} + p_{ij-1}}{\Delta y^2} = 0$$

Transpose:

$$p^n_{i,j} = \frac{\Delta y^2 (p_{i+1,j} + p_{i-1,j}) + \Delta x^2 (p_{i,j+1} + p_{i,j-1})}{2(\Delta x^2 + \Delta y^2)}$$

I.C. $p = 0$ everywhere in $(0,2) \times (0,1)$

B.C. $p = 0$ @ $x = 0$

$p = y$ @ $x = 2$

$\partial p / \partial y = 0$ @ $y = 0, 1$

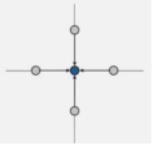
This problem has an analytical Solution

$$p(x,y) = \frac{x}{4} - 4 \sum_{\substack{n=1 \\ n\ \text{odd}}}^{\infty} \frac{1}{(n\pi)^2 \sinh(2\pi n} \sinh(n\pi x) \cos(n\pi y)$$

Note: 1) Laplace operator – typical of diffusion, it has to be discretized with CD (Physic: Isotropic phaenomena) to be consistent with the physics.

2) 2nd order CD in both x & y is the <u>most</u> widely used numerical scheme for the Laplace operator (∇^2)

Also known as the „five point difference operator"



3) This scheme is an <u>iterative method</u> for a steady state (artificial time variable)

Let $\Delta x = \Delta y$ $\quad (p_{i+1,j} - 2p_{ij} + p_{i-1,j}) + (p_{ij+1} - 2p_{ij} + p_{ij-1}) = 0$

$\Rightarrow$ Linear system of equation with $\underbrace{\text{pentadiagonal coefficient matrix}}$

(„point Jacobi method")

```
nx = 20; ny = 20; nit = 1000;
dx = 2/(nx-1); dy = 1/(ny-1);

x = 0:dx:2; y = 0:dy:1;
p = zeros(nx,ny);
p(nx,:) = y;
[y,x] = meshgrid(y,x);
pn = 0;

for iit = 1:nit
```

$$\begin{pmatrix} c_1 & d_1 & e_1 & 0 & \cdots & \cdots & 0 \\ b_1 & c_2 & d_2 & e_2 & \ddots & & \vdots \\ a_1 & b_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & a_2 & \ddots & \ddots & \ddots & e_{n-2} & 0 \end{pmatrix}.$$

```
x = 0:dx:2; y = 0:dy:1;
p = zeros(nx,ny);
p(nx,:) = y;
[y,x] = meshgrid(y,x);
pn = 0;

for iit = 1:nit
  pd = p
  for i = 2:nx-1
    for j = 2:ny-1
      p(i,j)= ( (pd(i+1,j)+pd(i-1,j) )*dy^2+ ...
              ( pd(i,j+1)+pd(i,j-1) )*dx^2)/(dx^2+dy^2)/2
    end
  end
  p(2:nx-1,1)= p(2:nx-1,2)
  p(2:nx-1,ny)= p(2:nx-1,ny-1)
end
```

$$\begin{pmatrix} b_1 & c_2 & d_2 & e_2 & \ddots & & \vdots \\ a_1 & b_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & a_2 & \ddots & \ddots & \ddots & e_{n-3} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & d_{n-2} & e_{n-2} \\ \vdots & & \ddots & a_{n-3} & b_{n-2} & c_{n-1} & d_{n-1} \\ 0 & \cdots & \cdots & 0 & a_{n-2} & b_{n-1} & c_n \end{pmatrix}.$$

---

(10) Poisson equation $\quad \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = b$

Discretite CD:

$$\frac{p^n_{i+1,j} - 2p^n_{ij} + p^n_{i-1,j}}{\Delta x^2} + \frac{p^n_{i,j+1} - 2p^n_{ij} + p^n_{i,j-1}}{\Delta y^2} = b^n_{ij}$$

$$p^n_{ij} = \frac{(p^n_{i+1,j} + p^n_{i-1,j})\,\Delta y^2 + (p^n_{i,j+1} + p^n_{i,j-1})\,\Delta x^2 - b^n_{ij}\,\Delta x^2 \Delta y^2}{2\,(\Delta x^2 + \Delta y^2)}$$

I.C. $\quad p = 0$ everywhere in $(0,2) \times (0,1)$

B.C. $\quad p = 0$ @ $x = 0, 2$ & $y = 0, 1$

Source $\quad b_{ij} = 100$ @ $i = \frac{nx}{4}$ $\quad j = \frac{ny}{4}$

$\quad b_{ij} = -100$ @ $i = \frac{3nx}{4}$ $\quad j = \frac{3ny}{4}$ $\quad \Big\}$ two spikes

$\quad b_{ij} = 0$ everywhere else

```
nx = 20; ny = 20; nit = 100;
dx = 2/(nx-1); dy = 1/(ny-1);

x = 0:dx:2; y = 0:dy:1;
p = zeros(nx,ny);
b = zeros(nx,ny);
b(nx/4, ny/4) = 100;
b(nx*3/4, ny*3/4) = -100;

[y,x] = meshgrid(y,x);

for iit = 1:nit+1
    pn = p;
    for i = 2:nx-1
        for j = 2:ny-1
            p(i,j) = ( (pn(i+1,j)+ pn(i-1,j) )*dy^2+...
                ( pn(i,j+1)+ pn(i,j-1) )*dx^2 ...
                -b(i,j)*dx^2*dy^2 )/(dx^2+dy^2)/2;
                    end
    end
end
```

---

(10) Navier Stokes equation: cavity flow

(1) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

(2) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

(2) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

(3) $$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right) + \rho\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$ <u>not sure ?!</u>

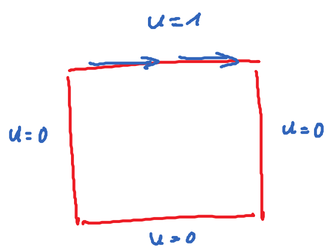
Discretized :

(1) $$\frac{u_{ij}^{n+1} - u_{ij}^{n}}{\Delta t} + u_{ij}^{n}\frac{u_{ij}^{n} - u_{i-1,j}^{n}}{\Delta x} + v_{ij}^{n}\frac{u_{ij}^{n} - u_{ij-1}}{\Delta y} = -\frac{1}{\rho}\frac{p_{i+1,j} - p_{i-1,j}}{2\Delta x} + \nu\left(\frac{u_{i+1,j}^{n} - 2u_{ij} + u_{i-1,j}}{\Delta x^2} + \frac{u_{ij+1} - 2u_{ij} + u_{ij-1}}{\Delta y^2}\right)$$

(2) ...

(3) $$\frac{p_{i+1,j}^{n} - 2p_{i,j}^{n} + p_{i-1,j}^{n}}{\Delta x^2} + \frac{p_{ij+1}^{n} - 2p_{ij}^{n} + p_{ij-1}^{n}}{\Delta y^2} =$$
$$-\rho\left\{\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right)^2 + 2\left(\frac{u_{ij+1} - u_{ij-1}}{2\Delta y}\right)\left(\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x}\right) + \left(\frac{v_{ij+1} - v_{ij-1}}{2\Delta y}\right)^2\right\}$$
$$+\rho\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{ij+1} - v_{ij-1}}{2\Delta y}\right)$$

Transpose by isolating u_{ij}^{n+1} v_{ij}^{n+1} p_{ij}^{n}



I.C. $u, v, p = 0$ everywhere

B.C. $u = 1$ @ $y = 2$

$u, v = 0$ @ $x = 0, 2$ & $y = 0$

$p = 0$ @ $y = 2$

$\partial p/\partial x = 0$ @ $x = 0, 2$

$\partial p/\partial y = 0$ @ $y = 0$

```
nx = 20; ny = 20;
nt = 100; nit = 100; dt=0.01;
vis= 0.1; rho= 1;

dx = 2/(nx-1); dy = 2/(ny-1);
x = 0:dx:2; y = 0:dy:2;
u = zeros(nx,ny);
v = zeros(nx,ny);
p = zeros(nx,ny);
[y,x] = meshgrid(y,x);

for it = 1:nt+1  % loop over time

    for i=2:nx-1
        for j=2:ny-1
            b(i,j) = rho*( (u(i+1,j)-u(i-1,j))/2/dx...
                +(v(i,j+1)-v(i,j-1))/2/dy )/dt...
                +((u(i+1,j)-u(i-1,j))/2/dx).^2 ...
            +2*(u(i,j+1)-u(i,j-1))/2/dy*(v(i+1,j)-v(i,j-1))/2/dx...
            +( (v(i,j+1)-v(i,j-1))/2/dy ).^2;
        end
    end


  for iit = 1:nit+1
      pn = p;
      for i = 2:nx-1
          for j = 2:ny-1
            p(i,j) = ((pn(i+1,j)+pn(i-1,j))*dy^2+ (pn(i,j+1)+pn(i,j-1))*dx^2 ...
                    - b(i,j)*dx^2*dy^2)/(dx^2+dy^2)/2;
          end
      end
      p(1,:)=p(2,:);  p(nx,:)=p(nx-1,:);
      p(:,1)=p(:,2);  p(:,ny)=p(:,ny-1);
  end
```

```
for iit = 1:nit+1
    pn = p;
    for i = 2:nx-1
        for j = 2:ny-1
          p(i,j) = ((pn(i+1,j)+pn(i-1,j))*dy^2+ (pn(i,j+1)+pn(i,j-1))*dx^2 ...
                  - b(i,j)*dx^2*dy^2)/(dx^2+dy^2)/2;
        end
    end
    p(1,:)=p(2,:);  p(nx,:)=p(nx-1,:);
    p(:,1)=p(:,2);  p(:,ny)=p(:,ny-1);
end

un=u; vn=v;
for i = 2:nx-1
    for j = 2:ny-1
      u(i,j) = un(i,j)-un(i,j)*dt/dx*(un(i,j)-un(i-1,j))...
            -vn(i,j)*dt/dy*(un(i,j)-un(i,j-1))...
            -1/rho*(p(i+1,j)-p(i-1,j))*dt/2/dx...
            +vis*dt/dx^2*(un(i+1,j)-2*un(i,j)+un(i-1,j))...
            +vis*dt/dy^2*(un(i,j+1)-2*un(i,j)+un(i,j-1));
      v(i,j) = vn(i,j)-un(i,j)*dt/dx*(vn(i,j)-vn(i-1,j))...
            -vn(i,j)*dt/dy*(vn(i,j)-vn(i,j-1))...
            -1/rho*(p(i,j+1)-p(i,j-1))*dt/2/dy...
            +vis*dt/dx^2*(vn(i+1,j)-2*vn(i,j)+vn(i-1,j))...
            +vis*dt/dy^2*(vn(i,j+1)-2*vn(i,j)+vn(i,j-1));
    end
end
u(1,:)=0;  u(nx,:)=0; u(:,1)=0;  u(:,ny)=1;
v(1,:)=0;  v(nx,:)=0; v(:,1)=0;  v(:,ny)=0;
end
```

Lecture 6 12

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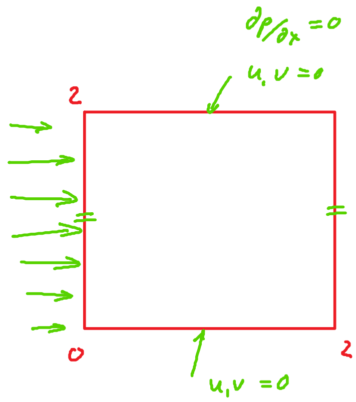
(12) Same equations as (11) !

But add 'F' to equation for u $\boxed{F = 1}$

I.C. $u, v, p = 0$ everywhere



BC. u, v, p periodic @ $x = 0, 2$

$u, v = 0$ @ $y = 0, 2$

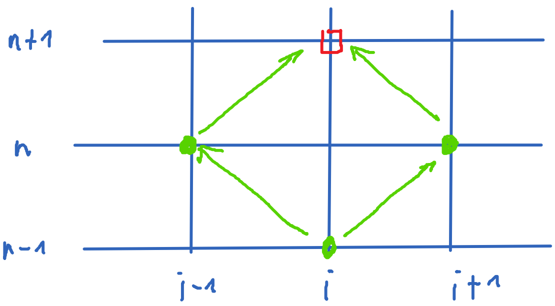
$\frac{\partial p}{\partial x} = 0$ @ $y = 0, 2$

① Leapfrog scheme

* both space and time discretized by 2nd order CD formulas

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} + \frac{c}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right) = 0 \qquad \sigma = c\frac{\Delta t}{\Delta x}$$

$$u_i^{n+1} = u_i^{n-1} - \sigma\left(u_{i+1}^n - u_{i-1}^n\right)$$



$u_i^{n+1} \rightarrow$ New solution

u_i^n does not contribute

3 time levels !!

Require a starting scheme (eg. upwind) to get values at n

Von Neumann analysis

$$V^{n+1} = V^{n-1} - \sigma V^n\left(e^{I\varphi} - e^{-I\varphi}\right) \qquad \Big| : V_n \qquad I = \sqrt{-1}$$

$$G = \frac{V^{n+1}}{V^n} = \frac{V^n}{V^{n-1}}$$

$$G - \frac{1}{G} = -\sigma\left(e^{I\varphi} - e^{-I\varphi}\right) \qquad \text{quadratic equation (solve!)}$$

$$G = I\sigma\sin(\varphi) \pm \sqrt{1 - \sigma^2\sin^2(\varphi)}$$

$\sigma > 1$: unstable (pure imaginary & $|G| > 0$)

$\sigma < 1$: term in sqrt is real and

$$GG^* = Re(G)^2 + Im(G)^2$$
$$= \left(1 - \sigma^2\sin^2(\varphi)\right) + \sigma^2\sin^2(\varphi) = \underline{\underline{1}}$$

Neutrally stable for the convection equation

---

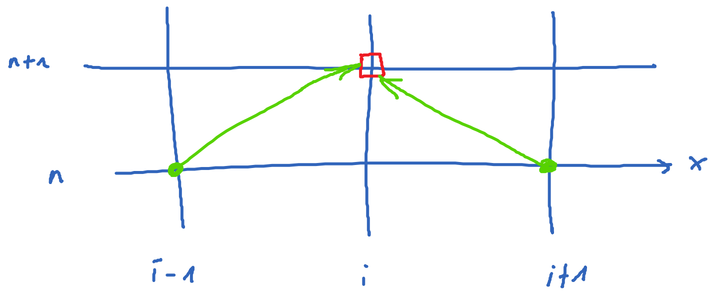
② Lax Friedrichs scheme

Replace u_i by the average $\frac{1}{2}(u_{i-1} + u_{i+1})$ to stabilize CD in x / FD in t

(Foward time, central scheme (FTCS))

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right)$$

Substitution introduces an error $O(\Delta x)$ → Reduce the order of the scheme to first order

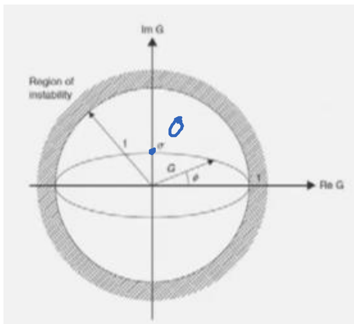


u_i^{n+1} does not depend on u_i^n

Von Neumann analysis $V^n e^{Ii\varphi}$ insert in discretized equation

$$G = \cos(\varphi) - I\,\sigma\,\sin(\varphi)$$ An ellipse

CFL condition: $\sigma < 1 \Rightarrow$ stable



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③ Lax-Wendroff scheme (1960)

First scheme with 2nd order CD with two time levels.

Taylor expansion in time:

$$u_i^{n+1} = u_i^n + \Delta t\,(u_t)_i + \frac{\Delta t^2}{2}(u_{tt})_i + \dots O(\Delta t^3) \quad (*)$$ Sub. derivative

- Keep the second time derivative in the discretization
- Replace the time derivatives by equivalent space derivatives

Use convection equation → $u_t + c\,u_x = 0$

take ∂_t →

Use convection equation $\rightarrow$ $u_t + c\, u_x = 0$

take $\partial/\partial t$ $\rightarrow$ $u_{tt} = -c(u_x)_t = -c\,(u_t)_x$

$$= c^2 u_{xx}$$

In (*)

$$u_i^{n+1} = u_i^n - c\,\Delta t\,(u_x)_i + c^2 \frac{\Delta t^2}{2}(u_{xx})_i + O(\Delta t^3)$$

Using CD is space

$$\boxed{u_i^{n+1} = u_i^n - \frac{\sigma}{2}\left(u_{i+1}^n - u_{i-1}^n\right) + \frac{\sigma^2}{2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right)}$$

Lax-Wendroff Scheme

Looking back at the modified diff. Equation for FTCS

* LW scheme is the discretization of a modified convection equation obtained by adding the lowest order truncation term

$$u_t + c\,u_x + \underbrace{\frac{\Delta t}{2} c^2 u_{xx}}_{\text{1st term of the trunc. error}} = 0$$

LW dominating truncation error is $\sim u_{xxx}$ and its modified diff. equation:

Modified DE.

$$\left\{ \bar{u}_t + c\,\bar{u}_x + \frac{\Delta t}{2} c^2 \bar{u}_{xx} = c\,\frac{\Delta x^2}{6}\,\bar{u}_{xxx} + O(\Delta t^2, \Delta x^4) \right.$$

Neumann stability analysis

$$G = 1 - \frac{\sigma}{2}\left(e^{I\varphi} - e^{-I\varphi}\right) + \frac{\sigma^2}{2}\left(e^{I\varphi} - 2 + e^{-I\varphi}\right)$$

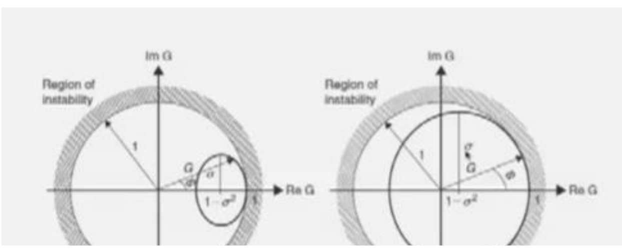
$$= 1 - I\,\sigma \sin(\varphi) - \sigma^2 (1 - \cos(\varphi))$$

$$\left.\begin{aligned} \xi &= \mathrm{Re}(G) = (1-\sigma^2) + \sigma^2 \cos(\varphi) \\ \eta &= \mathrm{Im}(G) = -\sigma \sin(\varphi) \end{aligned}\right\}$$

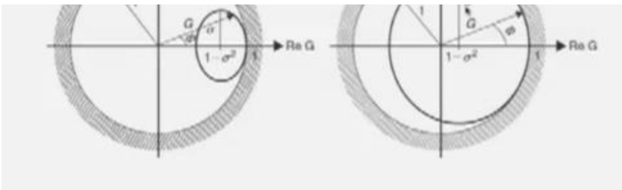
Ellipse centred on the real axis at $(1-\sigma^2)$

semi axis σ^2 (real)

σ (imaginary)



Again: CFL Condition



$\sigma^2 < 1/2$ $\sigma^2 > 1/2$

Again : CFL

$\sigma < 1 \rightarrow$ stable

FD Schemes at split time levels – Work well in nonlinear hyperbolic problems

Also called „Predictor – Corrector"

- First step, a „temporary value" for $u(x)$ is „predicted"
- Second step a „corrected value" is computed

① Richtmeyer / Lax-Wendroff
{ Variant 1 (Richtmeyer) → at point i
{ Variant 2 (2 Step L-W) → at point $i + 1/2$

(V1): Step 1 always LF method at time level $(n + \frac{1}{2})$

Lax Friedrichs

$$\frac{u_i^{n+\frac{1}{2}} - \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right)}{\Delta t/2} = -c\,\frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}$$

at $n+\frac{1}{2}$

Step 2 – Leapfrog $\dfrac{u_i^{n+1} - u_i^{n-1}}{2\Delta t}$ (with $\frac{\Delta t}{2}$)

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -c\,\frac{u_{i+1}^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}}}{2\Delta x}$$

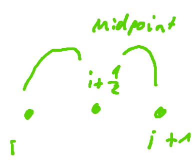
Rearrange:
$$u_i^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - c\,\frac{\Delta t}{4\Delta x}\left(u_{i+1}^n + u_{i-1}^n\right)$$

$$u_i^{n+1} = u_i^n - \frac{c\,\Delta t}{2\Delta x}\left(u_{i+1}^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}}\right)$$

→ stable for $\sigma \leq 2$

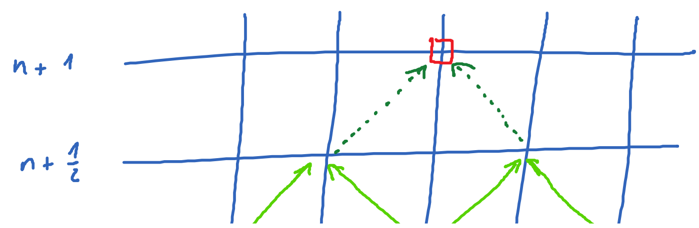
(V2) Step 1: Lax-Friedrichs at $i + \frac{1}{2}$

$$u_{i+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{c\,\Delta t}{2\Delta x}\left(u_{i+1}^n - u_i^n\right)$$



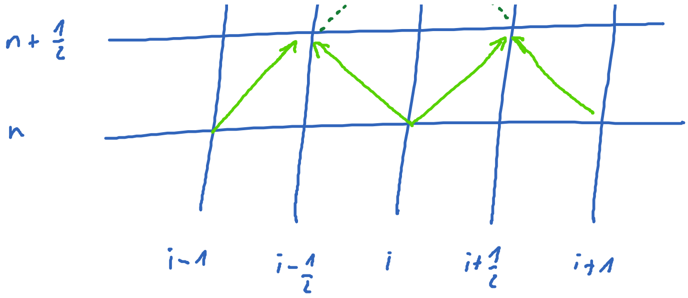
$$u_i^{n+1} = u_i^n - c\,\frac{\Delta t}{\Delta x}\left(u_{i+\frac{1}{2}}^{n+\frac{1}{2}} - u_{i-\frac{1}{2}}^{n+\frac{1}{2}}\right)$$

Stable for $c\,\frac{\Delta t}{\Delta x} \leq 1$



A bit messy in practise

- second order
- For linear PDES it is equivalent to the single step LW



② Mac Cormack method

Step 1 uses FD scheme — call u^* the temporary solution

$$\frac{u_i^* - u_i^n}{\Delta t} = -c\,\frac{u_{i+1}^n - u_{i-1}^n}{\Delta x}$$

Step 2 uses BD scheme with $\Delta t/2$

$$\frac{u^{n+1} - u^{n+\frac{1}{2}}}{\Delta t/2} = -c\,\frac{u_i^* - u_{i-1}^*}{\Delta x}$$

and replace the value $u_i^{n+\frac{1}{2}}$ by the average

$$u_i^{n+\frac{1}{2}} = \frac{1}{2}\left(u_i^n + u_i^*\right)$$

Predictor

$$u_i^* = u_i^n - c\,\frac{\Delta t}{\Delta x}\left(u_{i+1}^n - u_i^n\right)$$

Corrector

$$u_i^{n+1} = \frac{1}{2}\left[\left(u_i^n + u_i^*\right) - c\,\frac{\Delta t}{\Delta x}\left(u_i^* - u_{i-1}^*\right)\right]$$

- 2nd order
- stable for $\sigma < 1$
- For linear PDEs, equivalent to the one-step LW
- Can alternate FD / BD — BD / FD
- Works well for nonlinear problems

Non-linear Convection

$$\frac{\partial u}{\partial t} = -u\,\frac{\partial u}{\partial x} \qquad \text{(Inviscid Burgers equation)}$$

„Conservative Form" $\quad \dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(\dfrac{u^2}{2}\right)$ or $\dfrac{\partial u}{\partial t} = -\dfrac{\partial E}{\partial x} \qquad E = \dfrac{u^2}{2}$

Physical Interpretation: A wave propagating with different speeds at different points
eventually → a shock will be formed

Recall von Neumann Analysis

We introduced a Fourier decomposition of the solution $I = \sqrt{-1}$

$$u_i^n = \sum_{j=-N}^{N} V_j^n e^{I k_j x_i}$$

Amplitude

Mesh position

Sum of the modes

Single harmonic : $\rightarrow \left(u_i^n\right)_k = V^n e^{Ik(i\Delta x)}$

We defined an amplification factor $\longrightarrow G = \dfrac{V^{n+1}}{V^n}$

We defined a function of scheme parameters and of phase φ, but not of "n"

- Von Neumann Stability Condition $|G| \leq 1 \quad \forall \varphi = k_j \Delta x$ for all φ

Now - We want additional information on the errors
wish to know more of the time dependence of V^n

Consider the analytical solution of $u_t + c u_x = 0$ Sub: $t : \dfrac{\partial}{\delta t}$

Fourier decomposition $\rightarrow \tilde{u}_i^n = \hat{V} e^{Ik(x_i - ct^n)}$

Rewrite with $c = \dfrac{\tilde{\omega}}{k} \rightarrow \tilde{u}_i^n = \hat{V} e^{Ikx_i} e^{-I\tilde{\omega}t^n}$

A single harmonic is $\boxed{\left(\tilde{u}_i^n\right)_k = \hat{V}(k)\, e^{Ik(i\Delta x)}\, e^{-I\tilde{\omega}(n\Delta t)}}$ (*)

with $\hat{V}(k)$ from the initial condition $u(x, t=0) = u_0(x)$

$$\hat{V}(k) = \frac{1}{2L}\int_{-L}^{L} u_0(x)\, e^{-Ikx}\, dx$$

$\longrightarrow$ Assume that the initial condition is represented exactly on the mesh (except roundoff error)

Numerical amplitude represented to *

$$V^n = V^n(k)\, e^{-I\omega(n\Delta t)} = \hat{V}(k)\left(e^{-I\omega\Delta t}\right)^n$$

*

$$V^n = V^n(k)\, e^{-I\omega(n\Delta t)} = \hat{V}(k)\, \underbrace{\left(e^{-I\omega\Delta t}\right)^n}$$

Plane waves $\tilde{\omega} = \tilde{\omega}(k)$ → called "Dispersion relation"

Now write: $V^n = G V^{n-1} = G^2 V^{n-2} = \dots = (G)^n V^0 = G^n \cdot \hat{V}(k)$ i.c.

$$G = e^{-I\omega\Delta t} \rightarrow \text{This defines } \omega(k)$$

Similary with the analytical solution:

$$\tilde{V}_n = \left(e^{-I\omega\Delta t}\right)^n \hat{V}(k) =: \left(\tilde{G}\right)^n \hat{V}(k)$$

Exact amplification factor

Note that ω is a complex function

So $G = |G|\, e^{-I\Phi}$ and $V^n = G V^{n-1} = |G|\, e^{-I\Phi}\, V^{n-1}$

The error in amplitude is

$$\boxed{\epsilon_D = \frac{|G|}{|\tilde{G}|}}$$ Dissipation / Diffusion error

The error in phase is

$$\boxed{\epsilon_\varphi = \frac{\Phi}{\tilde{\Phi}}}$$ Dispersion

(for convection-dominated flows, bad for pure diffusion problem $\Phi = 0$, use the definition

$\epsilon_\varphi = \Phi - \tilde{\Phi}$ $\qquad \tilde{\Phi} = kc\Delta t$

Error analysis for hyperbolic problems $u_t + c\,u_x = 0$ sub: derivative

The exact solution for a wave form (*) – lecture 15

$$\tilde{\omega} = c\,t \qquad \tilde{u} = \hat{V} e^{Ikx} e^{-Ikct}$$

- Exact amplification factor $|\tilde{G}| = 1$

$$\tilde{\Phi} = c k \Delta t = c \frac{\Delta t}{\Delta x} k \Delta x = \sigma \varphi$$

$$\tilde{G} = e^{-I\sigma\varphi}$$ → Exact solution propagates without change in amplitude

- Numerical solution : Initial wave damped by a factor $|G|$ each Δt

Diffusion error is $\epsilon_D = |G|$

Phase of numerical solution defines a numerical convection speed

$$c_{num} = \frac{\Phi}{k\Delta t} \quad \text{and since} \quad \tilde{\Phi} = ck\Delta t = \sigma\varphi$$

$$c_{num} = \frac{c\,\Phi}{\sigma\varphi}$$

Dispersion error $$\epsilon_\varphi = \frac{\Phi}{ck\Delta t} = \frac{\Phi}{\sigma\varphi} = \frac{c_{num}}{c}$$

When the dispersion error is larger than 1, $\epsilon_\varphi > 1$, the phase error is a „leading error" (the numerical convection speed c_{num} is larger than the exact one c)

→ The computed solution moves faster than the physical one

When the dispersion error is less than 1, $\epsilon_\varphi < 1$, the phase error is a „lagging error"

→ The computed solution travels at lower velocity than the physical on

Note : Accuracy requires $|G|$ to be as close to 1 as possibility, but stability requires $|G| < 1$

To maintain stability —— Always diffusion error

---

① Analysis of 1st order upwind

$$G = 1 - 2\sigma \sin^2\left(\frac{\varphi}{2}\right) - I\sigma \sin(\varphi)$$

$$\xi = Re(G) = 1 - 2\sigma \sin^2\left(\frac{\varphi}{2}\right) = (1-\sigma) + \sigma \cos(\varphi)$$

$$\eta = Im(G) = -\sigma \sin(\varphi)$$

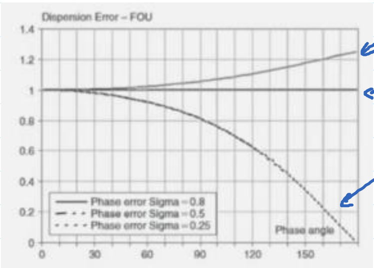
$$|G| = \sqrt{Re^2 + Im^2} = \sqrt{(1-\sigma+\sigma\cos(\varphi))^2 + \sigma \sin^2\varphi}$$

$$= \sqrt{1 - 4(\sigma(1-\sigma)\sin^2\left(\frac{\varphi}{2}\right)}$$

$$\psi = \tan^{-1}\left(-\frac{Im(G)}{Re(G)}\right)$$

Phase error $\epsilon_p = \dfrac{\tan^{-1}\left(-\frac{Im(G)}{Re(G)}\right)}{\sigma\varphi} = \dfrac{\tan^{-1}\left(\sigma\sin(\varphi) / (1-\sigma) + \sigma\cos(\varphi)\right)}{\sigma\varphi}$

ϵ_φ



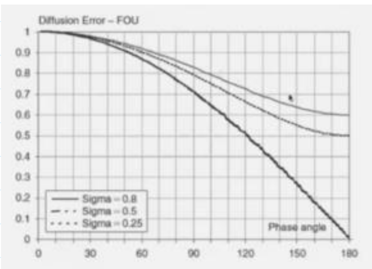
φ

For $\sigma > 0.5 \rightarrow \epsilon_\varphi > 1$: leading error

For $\sigma = 0.5 \rightarrow \epsilon_\varphi = 1$

For $0 < 0.5 \rightarrow \epsilon_\varphi < 1$: lagging error

ϵ_D



φ

$\longrightarrow$ increase in freq.

$\varphi = k\Delta x$ & $k = \frac{2\pi}{\lambda}$ so $\varphi = \frac{2\pi}{\lambda}\Delta x$

Highest frequency represented on mesh, shortest wavelength $\lambda = 2\Delta x$
$\varphi = \pi$

An error of only 1% ($|G| = 0.99$) after 100 time steps leads to an error of
$|G| = 0.36$ (0.99^{100})

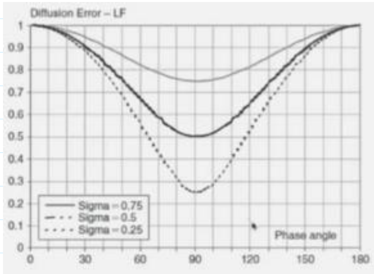
② Analysis of Lax - Friedrichs

$$|G| = \left[\cos^2(\varphi) + \sigma^2 \sin^2(\varphi)\right]^{1/2}$$

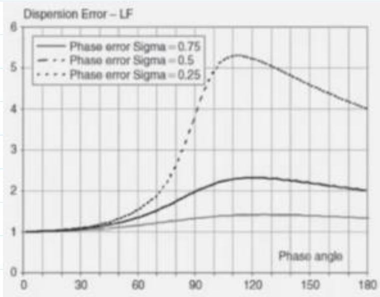
$$\Phi = \tan^{-1}\left(\sigma \tan(\varphi)\right)$$

- Diffusion error $\epsilon_D = \left[\cos^2(\varphi) + \sigma^2 \sin^2(\varphi)\right]^{1/2}$

No diffusion error for $\varphi = \pi$ (Wavelength $2\Delta x$) will not be damped



- Dispersion error $\epsilon_\varphi = \dfrac{\Phi}{\sigma\varphi} = \dfrac{\tan^{-1}(\sigma \tan(\varphi))}{\sigma\varphi}$



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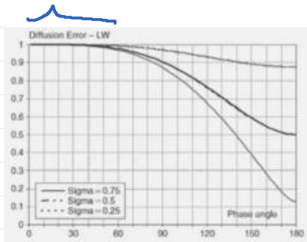
③ Lax - Wendroff

$$|G| = \sqrt{1 - 4\sigma^2(1-\sigma^2)\sin^4\left(\frac{\varphi}{2}\right)}$$

$$\epsilon_\varphi = \frac{\tan^{-1}\left(\sigma\sin(\varphi) / (1 - 2\sigma^2\sin^2(\varphi/2))\right)}{\sigma\varphi}$$

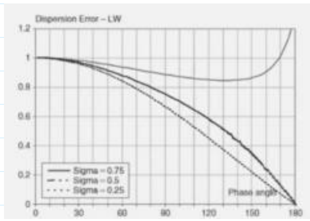
accuracy region



Test $\varphi = \dfrac{\pi}{6.25} = 28.8°$ → $\epsilon_D = 0.9985$

↓ 10%

80 steps → 0.89 $(= 0.9985^{80})$



Summary

1st order upwind: ϵ_D decreases away from 1 quickly with increasing frequencies

$\epsilon_\varphi < 1$ (lagging) for $\sigma > 0.5$

$\epsilon_\varphi > 1$ (leading) for $\sigma < 0.5$

$\epsilon_\varphi = 1$ for $\sigma = 0.5$

Lax-Friedrichs: ϵ_D shows strong damping for smaller σ. No damping $\varphi = \pi$ ($\lambda = 2\Delta x$)

$\epsilon_\varphi > 1 \longrightarrow$ leading phase error

Lax-Wendroff: ϵ_D shows „accurate region" where $\epsilon_D \approx 1$

ϵ_φ mostly $< 1 \rightarrow$ lagging

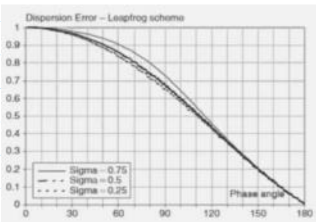
Leapfrog scheme

Note that $|G| = 1 \Longrightarrow$ No diffusion error

Leapfrog scheme is particulary useful for long-term simulations (weather forecast codes)

Dispersion error

$$\epsilon_\varphi = \frac{\pm \tan^{-1}\left[(\sigma \sin\varphi) / \sqrt{1 - \sigma^2 \sin^2\varphi} \right]}{\sigma\varphi} = \frac{\pm \sin^{-1}(\sigma \sin(\varphi))}{\sigma\varphi}$$



$\epsilon_\varphi < 1 \longrightarrow$ lagging

$\longrightarrow$ Accurate results for $u(x)$ smooth

* amplitude correctly modeled
* low frequencies, the phase error close to 1

Neutral stability $|G| = 1 \quad \forall \ \sigma < 1$

Some problems:
- high frequency errors not damping
- unstable for Burgers equation (nonlinear) !!!

Not good for any highspeed flows where shocks can occure

Note on oscillations: (LW + Leapfrog)

Have not explained the origin of oscillation

Why do they occur behind the travelling wave

- oscillations have frequencys
- ϵ_φ LW & Leapfrog predominantly < 1 (especially at higher frequencys)

$\rightarrow$ Convection speed of errors slower than physical one

Leapfrog: $\epsilon_\varphi \rightarrow 0$ for $\varphi = \pi$ and so oscillations are stronger

Consider an alternative scheme, due to Beam & Warming

Recall LW: (use BD instead of CD with LW)

$$u_i^{n+1} = u_i^n - c\Delta t\,(u_x)_i + c^2\frac{\Delta t^2}{2}(u_{xx})_i + O(\Delta t^3)$$

Discretize BD upwind

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{2\Delta x}\left(3u_i^n - 4u_{i-1}^n + u_{i-2}^n\right) + c^2\frac{\Delta t^2}{2}\left(\frac{u_i^n - 2u_{i-1}^n + u_{i-2}^n}{2\Delta x}\right)$$

Stability gives:

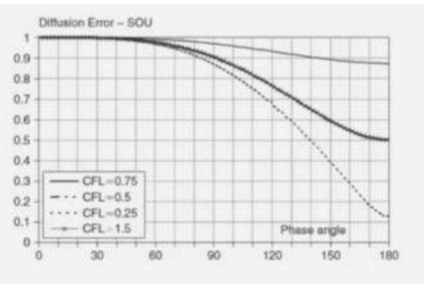
$$G = 1 - I\sigma\left[1 + 2(1-\sigma)\sin^2\left(\frac{\varphi}{2}\right)\right]\sin(\varphi) - 2\sigma\left(1-(1-\sigma)\cos(\varphi)\right)\sin^2\left(\frac{\varphi}{2}\right)$$

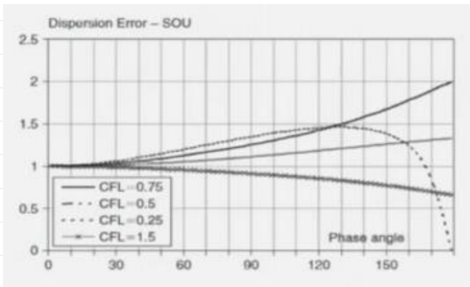
stable for $0 < \sigma < 2$

Diffusion error:

$$\epsilon_D = |G| = \sqrt{1 - \sigma(1-\sigma)^2(2-\sigma)\left(1-\cos(\varphi)\right)^2}$$

ϵ_φ : See plots





Final notes

1st order schemes : Generate large errors

2nd order schemes : Acceptable errors but should be careful especially at higher freq.

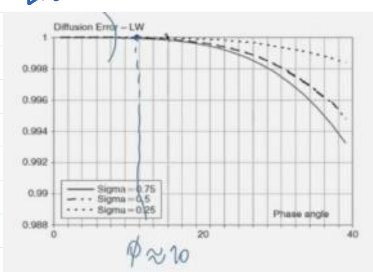
Choose a limit (φ_{lim})

$\varphi \approx 10° \approx \frac{\pi}{18} \rightarrow \epsilon_D \sim 1$

Key quantity : Number of mesh points per wavelength

$N_\lambda = \frac{\lambda}{\Delta x}$

LW:



$\varphi \approx 10$

We require $\varphi = k\Delta x = \frac{2\pi}{\lambda}\Delta x \leq \varphi_{lim}$

$$\boxed{N_\lambda = \frac{\lambda}{\Delta x} \geq \frac{2\pi}{\varphi_{lim}}}$$

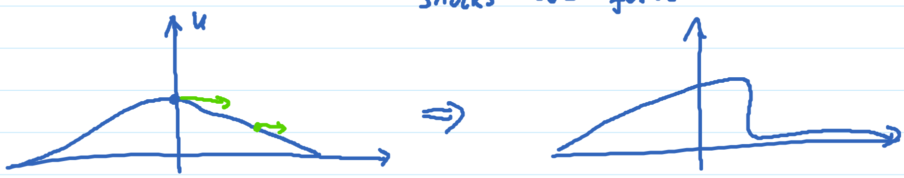
Severe

$\varphi_{lim} = \frac{\pi}{18} \Rightarrow$ at least 36 p./wavelength

Inviscid Burgers equation $\dfrac{\partial u}{\partial t} = -\underset{\text{nonlinear}}{u}\,\dfrac{\partial u}{\partial x}$

Conservative form : $\dfrac{\partial u}{\partial t} = -\dfrac{\partial}{\partial x}\left(\dfrac{u^2}{2}\right)$ or $\dfrac{\partial u}{\partial t} = -\dfrac{\partial E}{\partial x}$ $\quad \underset{\text{flux term}}{E} = \dfrac{u^2}{2}$

Physical interpretation – a wave propagating with different speeds such that shocks are formed



① Lax - Friedrichs (Explit, 1st order)

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left(E_{i+1}^n - E_{i-1}^n\right)$$

$$= \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\Delta t}{4\Delta x}\left[\left(u_{i+1}^n\right)^2 - \left(u_{i-1}^n\right)^2\right]$$

Stability :

In order to study stability → Linerize

u: average local value of the solution

$$u_i^{n+1} = \frac{1}{2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{\overset{\text{konst.}}{u}\,\Delta t}{2\Delta x}\left(u_{i+1}^n - u_{i-1}^n\right)$$

Consider at x_i → $u_i^n = V^n e^{Ik}$, recall : $\varphi = k\Delta x$

$$V^{n+1} e^{Ii\varphi} = V^n \frac{1}{2}\left(e^{I(i+1)\varphi} + e^{I(i-1)\varphi}\right) - V^n \frac{u\,\Delta t}{2\Delta x}\left(e^{I(i+1)\varphi} - e^{I(i-1)\varphi}\right)$$

$$e = V \frac{\cdot\cdot}{2} \Big| e + e \Big| - V \frac{\cdot\cdot}{2\Delta x} \Big| e - e \Big|$$

$$\cos z = \frac{e^{I\varphi} + e^{-I\varphi}}{2} \qquad \sin z = \frac{e^{I\varphi} - e^{-I\varphi}}{2}$$