



**MATHEMATICS  
LEARNING CENTRE**



THE UNIVERSITY OF  
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# The Binomial Distribution

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# 1 Tossing a Coin

## 1.1 Tossing Heads and Tails

To calculate various probabilities, we will be interested in finding the number of ways that we can obtain, as an example, three heads and two tails in five tosses of a coin.

When we say three heads and two tails, we mean that they can occur in any order.

There are many ways that this can be asked. Some examples are:

1. three heads and two tails;
2. exactly three heads in five tosses of a coin;
3. exactly two tails in five tosses of a coin;
4. two tails and three heads.

If we say three heads and **then** two tails, we mean that they must be in this order. That is, the order must be  $\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{T}}\textcircled{\text{T}}$ .

One way to calculate the number of ways we can have three  $\textcircled{\text{H}}$ 's and two  $\textcircled{\text{T}}$ 's in any order is simply to try to list all of them.

They will be:

$\textcircled{\text{T}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$
$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$
$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$
$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$
$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{T}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{H}}$	$\textcircled{\text{T}}$	$\textcircled{\text{T}}$

There are ten ways to do this.

### Exercise 1

List the number of ways of obtaining:

- i. exactly two heads in five tosses;
- ii. exactly one head in six tosses;
- iii. two tails and four heads in six tosses (there are fifteen of them);
- iv. (optional) three heads in six tosses (there are twenty ways).

The number of ways of obtaining heads and tails in, say,  $n$  tosses, gets much larger as we consider more and more tosses. There are, for example, seventy ways of obtaining four heads and four tails in any order in eight tosses of a coin.

We need a systematic method for finding how many ways there are of getting exactly  $m$  heads in  $n$  tosses of a coin in any order (this means there must be  $n - m$  tails).

To count the number of combinations let us return to our example of three heads and two tails in five tosses.

Here it is not too difficult to simply list them, though this gets cumbersome as we consider more tosses.

If we have exactly three heads in five tosses we know two tosses must come up tails. To make things simpler let's simply leave out the tails for the moment.

Our problem is then like trying to arrange the three heads in five spaces.

They are:

$\_ \mid \_ \mid H \mid H \mid H \mid$	$H \mid \_ \mid H \mid \_ \mid H \mid$
$\_ \mid H \mid \_ \mid H \mid H \mid$	$H \mid \_ \mid H \mid H \mid \_ \mid$
$\_ \mid H \mid H \mid \_ \mid H \mid$	$H \mid H \mid \_ \mid \_ \mid H \mid$
$\_ \mid H \mid H \mid H \mid \_ \mid$	$H \mid H \mid \_ \mid H \mid \_ \mid$
$H \mid \_ \mid \_ \mid H \mid H \mid$	$H \mid H \mid H \mid \_ \mid \_ \mid$

To count these, we want to know how many ways there are of putting three  $H$ 's in five spaces with, of course, just one  $H$  in each space.

Now these  $H$ 's are all considered identical - after all, a head is a head.

## 1.2 Tossing Different Heads

To make counting these combinations simpler we first regard each  $H$  as different or distinctive. Let's label these  $H_1$ ,  $H_2$ , and  $H_3$ .

Now, what are the ways of having  $H_1$  in any of the five spaces? They are:

$H_1 \mid \_ \mid \_ \mid \_ \mid \_ \mid$	$\_ \mid \_ \mid H_1 \mid \_ \mid \_ \mid$	$\_ \mid \_ \mid \_ \mid \_ \mid H_1 \mid$
$\_ \mid H_1 \mid \_ \mid \_ \mid \_ \mid$	$\_ \mid \_ \mid \_ \mid H_1 \mid \_ \mid$	

For each one of these possibilities, how many ways are there of placing  $H_2$  in the remaining spaces? For instance, for our choice of  $\_ \mid \_ \mid \_ \mid H_1 \mid \_ \mid$  we could place  $H_2$  as:

$H_2 \mid \_ \mid \_ \mid H_1 \mid \_ \mid$	$\_ \mid \_ \mid H_2 \mid H_1 \mid \_ \mid$
$\_ \mid H_2 \mid \_ \mid H_1 \mid \_ \mid$	$\_ \mid \_ \mid \_ \mid H_1 \mid H_2 \mid$

As we can see, the number of available spaces decrease as we place  $H$ 's in the spaces. There will then be four ways of placing  $H_2$  in the remaining spaces for **each** of the five placements of  $H_1$ , giving us twenty arrangements of  $H_1$  and  $H_2$  in five spaces.

For the second arrangement  $\_ \mid H_2 \mid \_ \mid H_1 \mid \_ \mid$  above, there are three places remaining in which to place  $H_3$ . They are:

$H_3 \mid H_2 \mid \_ \mid H_1 \mid \_ \mid$	$\_ \mid H_2 \mid H_3 \mid H_1 \mid \_ \mid$	$\_ \mid H_2 \mid \_ \mid H_1 \mid H_3 \mid$
--	--	--

This means for **each** of the twenty arrangements of  $H_1$  and  $H_2$  in five spaces, there are three ways of placing  $H_3$ . That is,  $20 \times 3 = 60$  ways of placing  $H_1$ ,  $H_2$ , and  $H_3$  in five spaces.

## Example

How many ways are there of placing  $H_1$ ,  $H_2$ , and  $H_3$  in six spaces?

There are six ways of placing  $H_1$ . For each of these six ways, there are five ways of placing  $H_2$ , giving  $6 \times 5 = 30$ . For each of these thirty arrangements there are four ways of placing  $H_3$ . That is, there are  $6 \times 5 \times 4 = 120$  ways of arranging  $H_1$ ,  $H_2$ , and  $H_3$  in six spaces.

## Exercise 2

1. How many ways are there of placing  $H_1$  and  $H_2$  in four spaces?
2. How many ways are there of placing  $H_1$  and  $H_2$  in six spaces?
3. How many ways are there of placing  $H_1$ ,  $H_2$ ,  $H_3$ , and  $H_4$  in four spaces?
4. How many ways are there of placing  $H_1$ ,  $H_2$ ,  $H_3$ , and  $H_4$  in five spaces?
5. How many ways are there of placing  $H_1$ ,  $H_2$ ,  $H_3$ , and  $H_4$  in eight spaces?

Let us now write down a formula for arranging distinct  $H_1, H_2, \dots$  in a number of spaces. Obviously we cannot put more than one  $H$  in each space and  $m \leq n$ . Let's say we have  $n$  spaces, and we place  $m$  different heads  $H_1, H_2, \dots, H_m$  in these spaces. There will be  $n$  places to put  $H_1$  for each of these arrangements. There will be  $(n - 1)$  spaces remaining to put  $H_2$ .

For each of the  $n \times (n - 1)$  arrangements of  $H_1$  and  $H_2$ , there will be  $(n - 2)$  ways of placing  $H_3$ .

Then there will be  $(n - 3)$  ways of placing  $H_4$ .

Then there will be  $(n - 4)$  ways of placing  $H_5$ .

Finally, there will be  $(n - (m - 1))$  ways of placing  $H_m$ .

All up, there are

$$n(n - 1)(n - 2) \dots (n - (m - 1))$$

ways of placing  $m$  distinct objects in  $n$  spaces with at most one object in each space.

## 1.3 Tossing Identical Heads

Let us remember that we have introduced different objects such as  $H_1, H_2, \dots$ , in order to make counting these arrangements easier. Let us return then to the puzzle when all the  $H$ 's are the same (we say indistinguishable).

There are six arrangements of  $H_1$  and  $H_2$  in three spaces:

$\boxed{H_1} \boxed{H_2} \boxed{\phantom{H}}$   
 $\boxed{H_2} \boxed{H_1} \boxed{\phantom{H}}$   
 $\boxed{H_1} \boxed{\phantom{H}} \boxed{H_2}$

$\boxed{H_2} \boxed{\phantom{H}} \boxed{H_1}$   
 $\boxed{\phantom{H}} \boxed{H_1} \boxed{H_2}$   
 $\boxed{\phantom{H}} \boxed{H_2} \boxed{H_1}$

There are only three ways of placing two identical  $H$ 's in three spaces:

$\boxed{H} \boxed{H} \boxed{\phantom{H}}$      $\boxed{H} \boxed{\phantom{H}} \boxed{H}$      $\boxed{\phantom{H}} \boxed{H} \boxed{H}$

As we can see, there are twice as many arrangements of  $H_1$  and  $H_2$  than there are for two identical  $H$ 's in three spaces.

We have double counted the  $H_1$  and  $H_2$  arrangement if the  $H$ 's are identical.

### Exercise 3

List all twelve ways of arranging  $H_1$  and  $H_2$  in four spaces. By pairing off these twelve, find the six ways of arranging two  $H$ 's in four spaces.

If we list all the possible arrangements of  $H_1$ ,  $H_2$ , and  $H_3$  in four spaces, we know there will be  $4 \times 3 \times 2 = 24$  of them.

Six of these will be:

$ H_1 H_2 H_3 $	$ H_2 H_3 H_1 $	$ H H H $
$ H_1 H_3 H_2 $	$ H_3 H_1 H_2 $	
$ H_2 H_1 H_3 $	$ H_3 H_2 H_1 $	

If all the  $H$ 's are identical, then all of the six arrangements to the left will correspond to:

We have then counted the arrangements of  $H_1$ ,  $H_2$ , and  $H_3$  in four spaces six times too many.

Hence there will only be  $\frac{24}{6} = 4$  arrangements of three  $H$ 's in four spaces. They are:

$ H H H $	$ H H  H $	$ H  H H $	$  H H H $
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In general, if we have  $m$  objects  $H_1, H_2, \dots, H_m$  there will be  $m \times (m-1) \times (m-2) \dots \times 2 \times 1$  ways of arranging them that will be identical if the  $H$ 's were the same.

### Exercise 4

List the  $4 \times 3 \times 2 \times 1 = 24$  ways of arranging  $H_1, H_2, H_3, H_4$  that will correspond to  $|H||H|H|H|$

In general, to find the number of ways of arranging  $m$   $H$ 's in  $n$  spaces, we need to divide the number of ways of arranging  $H_1, H_2, \dots, H_m$  in  $n$  spaces by

$$m \times (m-1) \times (m-2) \times (m-3) \times \dots \times 1.$$

The number of arrangements of  $m$  identical objects in  $n$  spaces with at most one object in each space will be

$$\frac{n(n-1)(n-2)\dots(n-(m-1))}{m(m-1)(m-2)\dots \times 2 \times 1}.$$



### Example

The number of arrangements of three  $H$ 's in four spaces will be

$$\begin{aligned}\frac{4(4-1)(4-(3-1))}{3 \times 2 \times 1} &= \frac{4 \times 3 \times 2}{3 \times 2 \times 1} \\ &= 4.\end{aligned}$$

### Example

The number of arrangements of three  $H$ 's in five spaces will be

$$\begin{aligned}\frac{5(5-1)(5-(3-1))}{3 \times 2 \times 1} &= \frac{5 \times 4 \times 3}{3 \times 2 \times 1} \\ &= 10.\end{aligned}$$

### Example

The number of arrangements of five  $H$ 's in seven spaces will be

$$\begin{aligned}\frac{7(7-1)(7-2)(7-3)(7-(5-1))}{5 \times 4 \times 3 \times 2 \times 1} &= \frac{7 \times 6 \times 5 \times 4 \times 3}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 21.\end{aligned}$$

### Exercise 5

Calculate the number of arrangements of:

- i. four  $H$ 's in five spaces;
- ii. four  $H$ 's in six spaces;
- iii. three  $H$ 's in seven spaces;
- iv. two  $H$ 's in seven spaces;
- v. one  $H$  in seven spaces.

## 2 Some Useful Notation

### 2.1 Factorial Notation

It becomes much easier to write out the formulae above by simply writing

$$5 \times 4 \times 3 \times 2 \times 1 \quad \text{as} \quad 5!.$$

This is pronounced "five factorial".

In general,  $n(n-1)(n-2)\dots \times 2 \times 1 = n!$ ,  $1! = 1$ , and  $0! = 1$ .

We also have that

$$\begin{aligned} 7 \times 6 \times 5 &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \\ &= \frac{7!}{4!}. \end{aligned}$$

Hence

$$\begin{aligned} n(n-1)(n-2)\dots(n-(m-1)) &= \frac{n(n-1)(n-2)\dots(n-(m-1))(n-m)\dots \times 2 \times 1}{(n-m)\dots \times 2 \times 1} \\ &= \frac{n!}{(n-m)!}. \end{aligned}$$

Hence

$$\frac{n(n-1)(n-2)\dots(n-(m-1))}{m(m-1)(m-2)\dots \times 2 \times 1} = \frac{n!}{(n-m)!m!}.$$

There is a factorial function on most scientific calculators (often labelled  $x!$ ).

### Example

The number of arrangements of nine identical objects in eleven spaces will be

$$\begin{aligned} \frac{11!}{(11-9)!9!} &= \frac{11!}{2!9!} \\ &= 55. \end{aligned}$$

### Exercise 6

1. Use a calculator to find the number of arrangements of:
  - i. nine identical objects in twelve spaces;
  - ii. six identical objects in ten spaces;
  - iii. seven identical objects in thirteen spaces.
2. Find the number of ways of getting exactly eight  $H$ 's in thirteen tosses of a coin.
3. Find the number of ways of getting seven  $H$ 's and four  $T$ 's in eleven tosses of a coin.
4. Find the number of ways of getting exactly four  $T$ 's in ten tosses of a coin.

## 2.2 Binomial Notation

A very much simpler notation to write is

$$\frac{n!}{(n-m)!m!} = \binom{n}{m}.$$

This is pronounced “ $n$  choose  $m$ ”. We will use this notation below.

The number of arrangements of  $m$  objects in  $n$  spaces with at most one object in each space is then

$$\begin{aligned} \frac{n(n-1)(n-2)\dots(n-(m-1))}{m(m-1)(m-2)\dots \times 2 \times 1} &= \frac{n!}{(n-m)!m!} \\ &= \binom{n}{m}. \end{aligned}$$

### 3 The Binomial Distribution

#### 3.1 Calculating Probabilities for the Binomial Distribution

We shall calculate the probabilities for a binomial distribution.

Let us return to the problem of tossing a coin five times, where say we are interested in the probability of tossing three heads exactly. That is, we must obtain three heads and two tails exactly, though we allow three heads and two tails in any order.

Now, we can calculate the probability of obtaining three heads and two tails in a specific order, which we call 3  $\textcircled{\text{H}}$ 's then 2  $\textcircled{\text{T}}$ 's. The probability of obtaining a head on the first toss with anything on the next four tosses is  $P(\textcircled{\text{H}} \sqcup \sqcup \sqcup \sqcup) = \frac{1}{2}$ .

The probability of obtaining a head on the first toss and then a head on the second toss is

$$\begin{aligned} P(\textcircled{\text{H}}\textcircled{\text{H}} \sqcup \sqcup \sqcup) &= P(\textcircled{\text{H}} \sqcup \sqcup \sqcup \sqcup) \times P(\sqcup \textcircled{\text{H}} \sqcup \sqcup \sqcup) \\ &= \frac{1}{2} \times \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^2 \\ &= 0.25. \end{aligned}$$

The probability of obtaining three heads on the first three tosses will be

$$\begin{aligned} P(\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{H}} \sqcup \sqcup) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\ &= \left(\frac{1}{2}\right)^3 \\ &= 0.125. \end{aligned}$$

The probability of obtaining three heads on the first three tosses and then two tails on the last two tosses will be

$$\begin{aligned} P(\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{T}}\textcircled{\text{T}}) &= \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)^2 \\ &= 0.0313. \end{aligned}$$

Let us just note here that we have grouped the three halves multiplied together representing the three heads, that is  $\left(\frac{1}{2}\right)^3$ . We have then grouped the two halves multiplied together representing the two tails, that is  $\left(\frac{1}{2}\right)^2$ . We shall continue to use this grouping as this simplifies our calculation when the probabilities are not the same  $\frac{1}{2}$  for both possibilities.

Now we know that three heads and then two tails is not the only way to obtain three heads and two tails in any order. In fact, there are ten ways of obtaining three  $\textcircled{\text{H}}$ 's and two  $\textcircled{\text{T}}$ 's in any order.

They are:

$$\begin{array}{cc}
 \underline{|T| |T| |H| |H| |H|} & \underline{|H| |T| |H| |T| |H|} \\
 \underline{|T| |H| |T| |H| |H|} & \underline{|H| |T| |H| |H| |T|} \\
 \underline{|T| |H| |H| |T| |H|} & \underline{|H| |H| |T| |T| |H|} \\
 \underline{|T| |H| |H| |H| |T|} & \underline{|H| |H| |T| |H| |T|} \\
 \underline{|H| |T| |T| |H| |H|} & \underline{|H| |H| |H| |T| |T|}
 \end{array}$$

We now know that we could have counted these ten ways by using

$$\begin{aligned}
 \binom{\text{number of tosses}}{\text{number of } \textcircled{H}\text{'s}} &= \binom{5}{3} \\
 &= \frac{5!}{3!(5-3)!} \\
 &= \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1} \\
 &= 10.
 \end{aligned}$$

It is important to note that each one of these other ways of obtaining three heads and two tails will have the same probability. For instance

$$\begin{aligned}
 P(\textcircled{H}\textcircled{H}\textcircled{T}\textcircled{H}\textcircled{T}) &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \\
 &= \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2.
 \end{aligned}$$

Again we group three halves multiplied together to represent the three heads. Similarly, two halves are grouped to represent the two tails.

Now, the probability of either obtaining  $\textcircled{H}\textcircled{H}\textcircled{H}\textcircled{T}\textcircled{T}$  or  $\textcircled{H}\textcircled{H}\textcircled{T}\textcircled{H}\textcircled{T}$  will be given by

$$P(\textcircled{H}\textcircled{H}\textcircled{H}\textcircled{T}\textcircled{T} \text{ or } \textcircled{H}\textcircled{H}\textcircled{T}\textcircled{H}\textcircled{T}) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2.$$

If there are ten such combinations of three  $\textcircled{H}$ 's and two  $\textcircled{T}$ 's (i.e. in any order), the probability of obtaining exactly three heads in five tosses can be calculated as

$$\begin{aligned}
 &P(\textcircled{H}\textcircled{H}\textcircled{H}\textcircled{T}\textcircled{T} \text{ or } \textcircled{H}\textcircled{H}\textcircled{T}\textcircled{H}\textcircled{T} \text{ or } \dots \text{ or } \textcircled{T}\textcircled{T}\textcircled{H}\textcircled{H}\textcircled{H}) \\
 &= \underbrace{\left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2}_{10 \text{ lots of}} \\
 &= 10 \times \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 \\
 &= 0.3125.
 \end{aligned}$$

Another way of writing this is

$$P(3\textcircled{H}\text{'s and } 2\textcircled{T}\text{'s}) = \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2.$$

In general, if we toss a fair coin  $n$  times, the probability of obtaining exactly  $m$  heads (and hence  $n - m$  tails) will be given by

$$P(m\textcircled{\text{H}}\text{'s and } (n - m)\textcircled{\text{T}}\text{'s}) = \binom{n}{m} \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{n-m}.$$

### Exercise 7

1. Calculate the probability of obtaining exactly one head in five tosses of a fair coin.
2. Calculate the probability of obtaining 3  $\textcircled{\text{H}}$ 's and 3  $\textcircled{\text{T}}$ 's (in any order). Note you can assume that the coin is fair.
3. Find the probability of obtaining exactly two tails in five tosses of a coin.
4. Find the probability of obtaining four or more heads in five tosses of a coin (i.e. we want the probability of obtaining either four  $\textcircled{\text{H}}$ 's and one  $\textcircled{\text{T}}$ , or five  $\textcircled{\text{H}}$ 's in five tosses).
5. From question 4., calculate the probability of getting three or less heads in five tosses of a coin.

## 3.2 The Binomial Distribution for Unequal Probabilities

If we roll a die, we know the probability of rolling a one is  $P(\textcircled{1}) = \frac{1}{6}$ . The probability of rolling any other number, that is, any number other than one (we will call this  $\textcircled{\cdot}$ ) is given by

$$\begin{aligned} P(\textcircled{\cdot} \text{ or } \textcircled{2} \text{ or } \textcircled{3} \text{ or } \textcircled{4} \text{ or } \textcircled{5}) &= P(\textcircled{\cdot}) \\ &= \frac{5}{6}. \end{aligned}$$

Here then let us note

$$\begin{aligned} P(\textcircled{\cdot}) &= 1 - P(\textcircled{1}) \\ &= 1 - \frac{1}{6} \\ &= \frac{5}{6} \\ &\neq P(\textcircled{1}). \end{aligned}$$

Let's say we are interested in finding the probability of exactly three ones in five rolls of a die.

To start with, we work out the probability of rolling three ones and then two of any other number (other than one, that is):

$$\begin{aligned} P(\textcircled{1}\textcircled{1}\textcircled{1}\textcircled{\cdot}\textcircled{\cdot}) &= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \\ &= \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2 \\ &\doteq 0.0032. \end{aligned}$$

Now we know there are many other ways of rolling exactly three  $\square$ 's in five rolls. They are:

$\square$	$\square$	$\square$	$\square$	$\square$		$\square$	$\square$	$\square$	$\square$	$\square$
$\square$	$\square$	$\square$	$\square$	$\square$		$\square$	$\square$	$\square$	$\square$	$\square$
$\square$	$\square$	$\square$	$\square$	$\square$		$\square$	$\square$	$\square$	$\square$	$\square$
$\square$	$\square$	$\square$	$\square$	$\square$		$\square$	$\square$	$\square$	$\square$	$\square$
$\square$	$\square$	$\square$	$\square$	$\square$		$\square$	$\square$	$\square$	$\square$	$\square$

There are ten such combinations of three  $\square$ 's and two  $\square$ 's (in any order). We could have counted these by using

$$\begin{aligned} \binom{5}{3} &= \frac{5!}{3!(5-3)!} \\ &= 10. \end{aligned}$$

The second possibility  $\square\square\square\square\square$  will have a probability given by:

$$\begin{aligned} P(\square\square\square\square\square) &= \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \\ &= \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2 \\ &\doteq 0.0032. \end{aligned}$$

Here we have grouped the three probabilities associated with rolling a  $\square$  together, that is  $(\frac{1}{6})^3$ . We have also grouped the two probabilities associated with rolling a  $\square$  together.

Here it is important to note that any one of the ten combinations of three  $\square$ 's and two  $\square$ 's will have the same probability.

$$\begin{aligned} P(\square\square\square\square\square) &= P(\square\square\square\square\square) = \dots = P(\square\square\square\square\square) \\ &= \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \\ &\doteq 0.0032. \end{aligned}$$

We can now calculate the probability of rolling exactly three  $\square$ 's in five rolls:

$$\begin{aligned} &\underbrace{P(\square\square\square\square\square \text{ or } \square\square\square\square\square \text{ or } \dots \text{ or } \square\square\square\square\square)}_{\text{all combinations of three } \square\text{'s and two } \square\text{'s}} \\ &= \underbrace{P(\square\square\square\square\square) + P(\square\square\square\square\square) + \dots + P(\square\square\square\square\square)}_{\text{ten probabilities for the ten combinations}} \\ &= \underbrace{\left\{ \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \right\} + \left\{ \frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \right\} + \dots + \left\{ \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \right\}}_{\text{ten lots of}} \\ &= \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + \dots + \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \end{aligned}$$

$$\begin{aligned}
&= 10 \times \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \\
&= \binom{5}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 \\
&\doteq 0.0322.
\end{aligned}$$

More generally, the probability of rolling exactly  $m$   $\square$ 's in  $n$  rolls (this means we must have  $(n - m)$   $\square$ 's) of a standard die will be given by:

$$P(m \text{ } \square \text{'s and } (n - m) \text{ } \square \text{'s}) = \binom{n}{m} \left(\frac{1}{6}\right)^m \left(1 - \frac{1}{6}\right)^{n-m}.$$

### Exercise 8

1. Calculate the probability of rolling exactly two  $\square$ 's in five rolls of a die.
2. Calculate the probability of rolling three or more  $\square$ 's in five rolls of a die.
3. Calculate the probability of rolling five or less  $\square$ 's in six rolls of a die by first calculating the probability of rolling exactly six  $\square$ 's in six rolls of a die.
4. Calculate the probability of rolling six or less  $\square$ 's in eight rolls of a die.

## 3.3 Binomial Probabilities

To calculate the probabilities for a binomial distribution we use some important operations.

1. We are interested in only two outcomes of a trial e.g.  $\textcircled{H}$ 's or  $\textcircled{T}$ 's in a toss of a coin;  $\square$  or anything other than one i.e.  $\square$  in the roll of a die; a component being faulty or OK, etc..
2. When we multiply probabilities we assume that the trials are independent. The probability of a  $\textcircled{H}$  on the second toss of a coin is not influenced by the outcome of the first toss.
3. The probability of an event does not change for each trial. The probability of a  $\textcircled{H}$  remains  $\frac{1}{2}$  for each toss of the coin.
4. We are interested in a fixed (finite) number of trials e.g. 5 tosses of a coin, 7 rolls of a die. All the trials occur, there being no chance that the last trial will not take place.

Mathematicians like to talk about numbers. We can simplify our discussion above by denoting certain events by simple numbers. If we toss a coin four times, we could denote the event  $\textcircled{H}\textcircled{H}\textcircled{H}\textcircled{H}$  by the (integer) number 4; the number of heads.



If we are interested in three  $\textcircled{\text{H}}$ 's and one  $\textcircled{\text{T}}$  in any order, and we are using the binomial probabilities above, we could denote all of the events

$$\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{T}} \quad \textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{T}}\textcircled{\text{H}} \quad \textcircled{\text{H}}\textcircled{\text{T}}\textcircled{\text{H}}\textcircled{\text{H}} \quad \textcircled{\text{T}}\textcircled{\text{H}}\textcircled{\text{H}}\textcircled{\text{H}}$$

by the (integer) number 3; the number of heads.

We would have six events that we would label 2 (two  $\textcircled{\text{H}}$ 's and two  $\textcircled{\text{T}}$ 's). There are four events corresponding to one  $\textcircled{\text{H}}$  and three  $\textcircled{\text{T}}$ 's labelled 1, and we would label  $\textcircled{\text{T}}\textcircled{\text{T}}\textcircled{\text{T}}\textcircled{\text{T}}$  with the integer 0.

Then for each of the sixteen possible outcomes for four tosses of a coin, we could assign the numbers 0, 1, 2, 3, 4 (note that there are five of them).

The (integer) number of  $\textcircled{\text{H}}$ 's in four tosses is called a random variable. It can take five values.

We label random variables with capital letters. This reminds us that a random variable corresponds to a process of assigning events with numbers. Let's call the number of  $\textcircled{\text{H}}$ 's (in any order) in four tosses of a coin  $X$ .

To calculate  $P(X = 4)$ , we use  $P(4\textcircled{\text{H}}\text{'s})$ .

To calculate  $P(X = 3)$ , we use  $P(3\textcircled{\text{H}}\text{'s and } 1\textcircled{\text{T}})$ .

Sometimes the event of tossing a head is called a success.

For instance, three  $\textcircled{\text{H}}$ 's in four tosses can be called three successes in four trials. Or simply  $X = 3$ .

We are now in a position to write down the general formula for the probabilities of a binomial distribution.

If we have  $n$  trials of an event where the probability of a success is  $p$  then the probability of obtaining  $m$  successes will be given by

$$P(X = m) = \binom{n}{m} (p)^m (1 - p)^{n-m}$$

Let us note that  $1 - p$  will be the probability of a failure, and  $n - m$  will be the number of failures. We also note that we assume all the conditions for a binomial distribution.

### Example

We are interested in the probability of tossing exactly 7 heads in 10 tosses. Here  $n = 10$ ,  $m = 7$ , and if we have an ideal fair coin the probability of tossing a head is  $p = \frac{1}{2}$ . We will have  $n - m = 10 - 7 = 3$  tails, and the probability of a tail will be  $1 - \frac{1}{2} = \frac{1}{2}$ .

$$\begin{aligned} P(X = 7) &= \binom{10}{7} \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^3 \\ &= 120 \times \frac{1}{128} \times \frac{1}{8} \\ &= \frac{15}{128}. \end{aligned}$$

### Example

We are interested in the probability of rolling exactly three  $\square$ 's in six rolls of a die. We have  $n = 6, m = 3$ , and if the die is fair then  $p = \frac{1}{6}$ . We require  $n - m = 3$   $\square$ 's (other than one) rolls, and the probability of rolling a number other than a one is  $1 - \frac{1}{6} = \frac{5}{6}$ .

$$\begin{aligned}P(X = 3) &= \binom{6}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^3 \\&= 20 \times \frac{1}{216} \times \frac{125}{216} \\&= \frac{625}{11664}\end{aligned}$$

### Exercise 9

1. If we are interested in the probability of rolling exactly three fours in six rolls of a fair die, write down  $n, m, n - m, p, 1 - p$ , and calculate the probability.
2. If we are interested in the probability of rolling exactly two even numbers in six rolls of a fair die, write down  $n, m, n - m, p, 1 - p$ , and calculate the probability.
3. Find the probability  $p$  of rolling a five or a six for a fair die, hence find the probability of rolling exactly two numbers larger than or equal to five in six rolls of a fair die.
4. Find the probability of rolling exactly four even numbers in five rolls of a fair die. Find the probability of rolling exactly five even numbers in five rolls of a fair die. Hence find the probability of rolling four or more even numbers in five rolls of a fair die.
5. Calculate the probability of rolling five or more numbers larger than or equal to five in six rolls of a fair die.

## 3.4 Distribution of Probabilities

Let's say we toss a coin four times, and that the coin is fair. Hence we have  $n = 4$  and the probability of tossing a head, say, is  $p = \frac{1}{2}$ . If  $X$  represents the number of heads in the four tosses, then we know the random variable  $X$  can take five (integer) values. They are  $X = 0, 1, 2, 3, 4$ .

We can now work out the probabilities associated with obtaining no heads i.e.  $X = 0$ , one head and four tails in any order  $X = 1$ , etc.. That is  $X = 0$  up to  $X = 4$ .

We have:

$$\begin{aligned}P(X = 0) &= \binom{4}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 1 \times 1 \times \frac{1}{16} = \frac{1}{16}, \\P(X = 1) &= \binom{4}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 4 \times \frac{1}{2} \times \frac{1}{8} = \frac{4}{16},\end{aligned}$$

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 6 \times \frac{1}{4} \times \frac{1}{4} = \frac{6}{16},$$

$$P(X = 3) = \binom{4}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 = 4 \times \frac{1}{8} \times \frac{1}{2} = \frac{4}{16},$$

$$P(X = 4) = \binom{4}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 = 1 \times \frac{1}{16} \times 1 = \frac{1}{16}.$$

We should note here that the sum of all possible probabilities is one. That is:

$$\begin{aligned} P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) &= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} \\ &= \frac{16}{16} \\ &= 1. \end{aligned}$$

This adds up to one because of our logical method of working out the various probabilities. We counted all the possibilities, i.e. sixteen different ways of tossing a coin four times with equal probability for each.

In fact, for any value of  $n$  and  $p$ , the sum of all the associated probabilities from  $X = 0, 1, 2, \dots, n$  will add up to 1.

We can write

$$\sum_{m=0}^n \binom{n}{m} (p)^m (1-p)^{n-m} = 1$$

where  $\sum_{m=0}^n$  means:

Add up the expression directly following the  $\sum$ ,  
starting from  $m = 0$ ,  
and finishing with  $m = n$ .

This indicates that we are certain of obtaining one of these possible values of  $X = 0$  to  $X = n$ .

## Examples

1.

$$\begin{aligned} \sum_{m=0}^5 \binom{5}{m} \left(\frac{1}{2}\right)^m \left(\frac{1}{2}\right)^{5-m} &= \binom{5}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 + \binom{5}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 + \binom{5}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + \\ &\quad \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 + \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 \\ &= \frac{1}{32} + \frac{5}{32} + \frac{10}{32} + \frac{10}{32} + \frac{5}{32} + \frac{1}{32} \\ &= 1. \end{aligned}$$

2.

$$\begin{aligned}
\sum_{m=0}^4 \binom{4}{m} \left(\frac{1}{3}\right)^m \left(\frac{2}{3}\right)^{4-m} &= \binom{4}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 + \binom{4}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 + \binom{4}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 + \\
&\quad \binom{4}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 + \binom{4}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 \\
&= \frac{16}{81} + \frac{32}{81} + \frac{24}{81} + \frac{8}{81} + \frac{1}{81} \\
&= 1.
\end{aligned}$$

As we can see the probability for each (discrete) event is a fraction of one unit. The probability of obtaining no heads ( $X = 0$ ) in four tosses ( $n = 4$ ) of a fair coin ( $p = \frac{1}{2}$ ) is  $P(X = 0) = \frac{1}{16}$ . That is, the probability of obtaining  $X = 0$  is one sixteenth of the total probability. The probability of obtaining  $X = 1$  is four sixteenths, and so on for each of  $X = 2, X = 3$ , and  $X = 4$ . In this way, the total probability of one unit is apportioned between each of the different possibilities.

We can think of each probability for each event being a proportion of the total probability, which must always be one unit. Hence the total probability taken by convention to be one unit, is divided up or distributed amongst the different probabilities for all of the possible events.

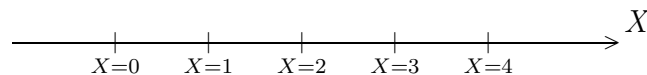
We can think of the set of probabilities for all possibilities as a way of dividing up or distributing the total probability of one. In this way, the corresponding set of probabilities for all possible events is called a probability distribution.

### 3.5 Picturing the Binomial Distribution

Let us return to our example of the number of heads  $X$  in  $n = 4$  tosses of a fair coin ( $p = \frac{1}{2}$ ). We have worked out that:

$$\begin{aligned}
P(X = 0) &= \frac{1}{16}, & P(X = 1) &= \frac{4}{16}, & P(X = 2) &= \frac{6}{16}, \\
P(X = 3) &= \frac{4}{16}, & \text{and} & & P(X = 4) &= \frac{1}{16}.
\end{aligned}$$

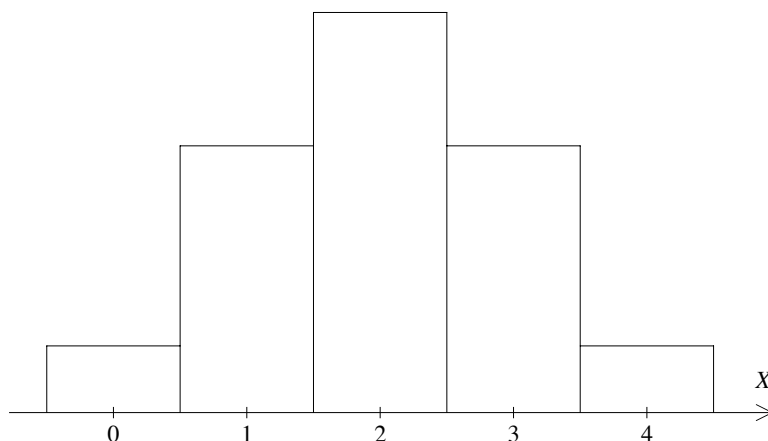
It is very useful to represent these probabilities in a diagram. We will represent the possible values that the random variable  $X$  can take along a horizontal axis:



Above each of these possible values for  $X$  we represent the corresponding probability. In this instance, since  $X$  can only take a finite number of discrete values, we represent the probability as a block above the accompanying number.

The block above  $X = 0$  is represented as one unit in width from  $-\frac{1}{2}$  to  $\frac{1}{2}$ , while the block above  $X = 1$  will be one unit wide from  $\frac{1}{2}$  to  $1\frac{1}{2}$ . In this case, each block will be the same width of one unit.

Now the area covered by each block will represent the probability for the corresponding value of  $X$ . For example, since  $P(X = 0) = \frac{1}{16}$ , we want the area of the block above  $X = 0$  to be  $\frac{1}{16}$  of the total area. The resulting probability representation will look like this:



The height of the first block must be  $\frac{1}{16}$ . Hence the area = width  $\times$  height =  $1 \times \frac{1}{16} = \frac{1}{16}$ . Each of the heights of the other blocks will (in this case) be given by the corresponding probability. The height of the block over  $X = 1$  will be  $\frac{4}{16}$ , etc..

The resulting block diagram is a representation of how the probabilities with total area equal to one are distributed over the different values that  $X$  can take.

Since  $X$  can only take a finite number of integer values, we call  $X$  a discrete random variable. This diagram then is a representation of a (discrete) probability distribution.

### Exercise 10

1. For three tosses of a fair coin, calculate all four probabilities. Hence produce a representation of the appropriate probability distribution.
2. The probability of rolling a one ( $\square$ ) on a fair die will be one sixth. Calculate all four probabilities for the number of  $\square$ 's rolled for three rolls of a fair die and sketch a representation of the probability distribution.
3. Give a representation of the probability distribution for the number of  $\square$ 's rolled in two rolls of a fair die.
4. Give a representation of the probability distribution for the number of heads tossed in two tosses of a fair coin.
5. What is the difference in the diagrams for question 3 and question 4? Why?

As we have seen above, the distribution of probabilities will be different for different values of  $n$ . The distribution of probabilities will also change if we change the probability of a success, that is, if we change the value of  $p$ .

Once we know  $n$  and  $p$  we can calculate all of the probabilities for  $X = 0, 1, \dots, n$ . This set of probabilities for  $X = 0, 1, \dots, n$  is called the binomial distribution, and we know the distribution will be determined if we specify  $n$  and  $p$ .

To say this in symbols, mathematicians use  $X \sim \mathcal{B}(n, p)$ .

### Examples

1. For four tosses of a fair coin, we can say that the number of heads  $X$  will have a binomial distribution with  $n = 4$  and  $p = \frac{1}{2}$ , or  $X \sim \mathcal{B}(4, \frac{1}{2})$ .
2. The probability distribution for five tosses of a fair coin can be denoted by  $X \sim \mathcal{B}(5, \frac{1}{2})$ , where  $X$  may represent the number of heads in five tosses.
3. The probability of rolling a six ( $\boxed{\cdot\cdot\cdot}$ ) with a fair die is  $\frac{1}{6}$ . If  $X$  represents the number of sixes ( $\boxed{\cdot\cdot\cdot}$ 's) in five rolls of a fair die, then  $X \sim \mathcal{B}(5, \frac{1}{6})$ .

### Exercise 11

1. For  $X$ , the number of heads in four tosses of a fair coin, give the probability distribution, and by calculating the probabilities for  $X = 0, 1, 2, 3, 4$ , draw a representation of the distribution.
2. Write down the probability distribution for the number of  $\boxed{\cdot\cdot\cdot}$ 's in four rolls of a fair die. Calculate all five associated probabilities and give a representation of the distribution.
3. Let  $X$  be the number of rolls that result in a number larger than or equal to five (that is  $\boxed{\cdot\cdot}$ ,  $\boxed{\cdot\cdot\cdot}$ ) in four rolls of a fair die. Write down the probability distribution. Calculate the probabilities for all possible outcomes  $X = 0, 1, 2, 3, 4$ , and give a representation.

## 3.6 Using Binomial Tables

Let's consider again four tosses of a fair coin. If  $X$  is the number of heads we know that  $X \sim \mathcal{B}(4, \frac{1}{2})$ . We have calculated the probabilities:

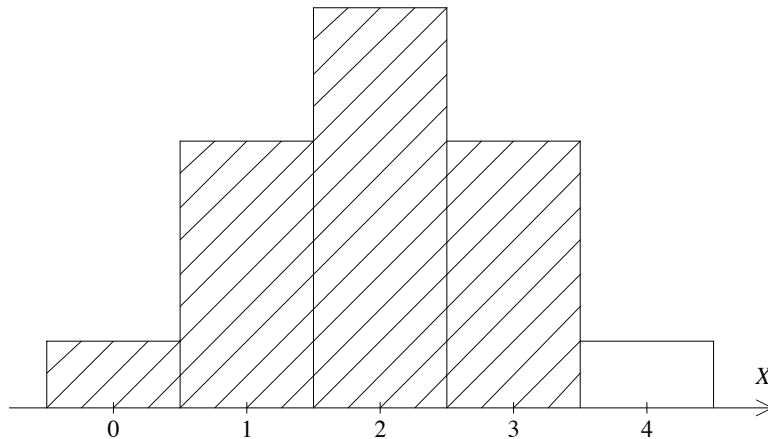
$$P(X = 0) = \frac{1}{16}, P(X = 1) = \frac{4}{16}, P(X = 2) = \frac{6}{16}, P(X = 3) = \frac{4}{16}, \text{ and } P(X = 4) = \frac{1}{16}.$$

If we wanted to calculate the probability of obtaining three heads or less, then we would have to calculate the four probabilities for  $X = 0, 1, 2, 3$  as given above, and then we would add these probabilities.

The probability of obtaining three heads or less in four tosses of a fair coin

$$\begin{aligned} P(X \leq 3) &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) \\ &= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} \\ &= \frac{15}{16} \\ &= 0.9375. \end{aligned}$$

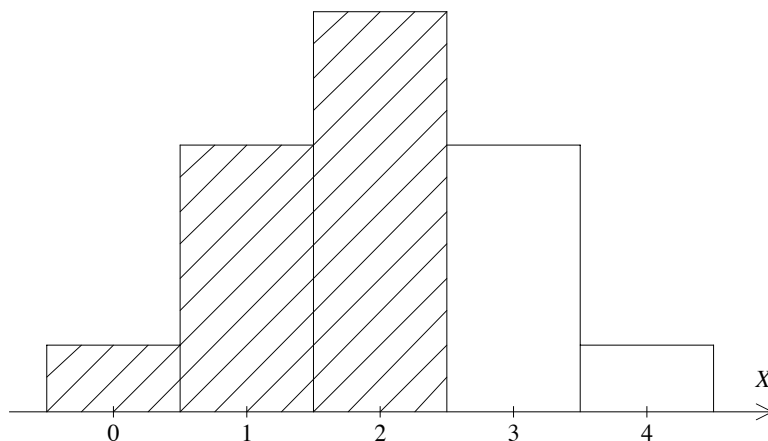
This probability can also be represented by the shaded area given below:



If we wanted to find the probability of two heads or less we would calculate:

$$\begin{aligned}
 P(X \leq 2) &= P(X = 0) + P(X = 1) + P(X = 2) \\
 &= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} \\
 &= \frac{11}{16} \\
 &= 0.6875.
 \end{aligned}$$

This probability will be represented by the shaded area:



To work out binomial probabilities, it can be very much quicker to use tables. Some tables of binomial probabilities give the probability that  $X$  is less than or equal to some given value. For instance, for  $X \sim B(4, \frac{1}{2})$ , a table might give

$$\begin{aligned}
 P(X \leq 0) &= \frac{1}{16} \\
 &= 0.0625;
 \end{aligned}$$

$$\begin{aligned}P(X \leq 1) &= \frac{1}{16} + \frac{4}{16} \\&= 0.3125;\end{aligned}$$

$$\begin{aligned}P(X \leq 2) &= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} \\&= 0.6875;\end{aligned}$$

$$\begin{aligned}P(X \leq 3) &= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} \\&= 0.9375;\end{aligned}$$

$$\begin{aligned}P(X \leq 4) &= \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} \\&= 1.000.\end{aligned}$$

There are two reasons why these cumulative probabilities are given.

Firstly, if we wish to investigate a test of significance as is performed in many medical, biological and other studies, we need to calculate the probability of obtaining less than or equal to a given number of successes.

Secondly, it is a simple procedure to calculate the probability of obtaining a single value  $X = n$ , say for  $(P(X = n))$ , from tables of the cumulative probabilities  $P(X \leq n)$ . We give examples of this below.

### Exercise 12

1. Use tables to verify that for  $X \sim B(4, \frac{1}{2})$ ,  $(n = 4, p = \frac{1}{2})$ , that  $P(X \leq 3) = 0.9375$ .
2. For  $X \sim \mathcal{B}(4, \frac{1}{2})$ , find  $P(X \leq 4)$  from tables. Explain your answer.
3. Use tables to find  $P(X \leq 0)$  for  $X \sim B(4, \frac{1}{2})$ .  
Check your answer by calculating:

$$\binom{4}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 1 \times \frac{1}{2^4}.$$

4. For  $X \sim \mathcal{B}(6, 0.5)$ , use tables to find:
  - i.  $P(X \leq 0)$ ;
  - ii.  $P(X \leq 3)$ ;
  - iii.  $P(X \leq 5)$ .

Write down the probability that  $X$  is six or less i.e.  $P(X \leq 6)$ , without using tables.

5. Use tables to find:
  - i.  $P(X \leq 3)$  for  $X \sim \mathcal{B}(6, 0.4)$ ;
  - ii.  $P(X \leq 5)$  for  $X \sim \mathcal{B}(12, 0.6)$ ;
  - iii.  $P(X \leq 7)$  for  $X \sim \mathcal{B}(7, 0.1)$ ;
  - iv.  $P(X \leq 7)$  for  $X \sim \mathcal{B}(7, 0.6)$ .



Since tables may give us the probability that  $X$  is less than some value, we may want to use them to find the probability that  $X$  is equal to some value.

For instance, above we considered four tosses of a fair coin. For  $X$ , the number of heads, we know  $X \sim \mathcal{B}(4, \frac{1}{2})$ . To find the probability that  $X = 3$ , say, we know that a table will give us:

$$P(X \leq 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3).$$

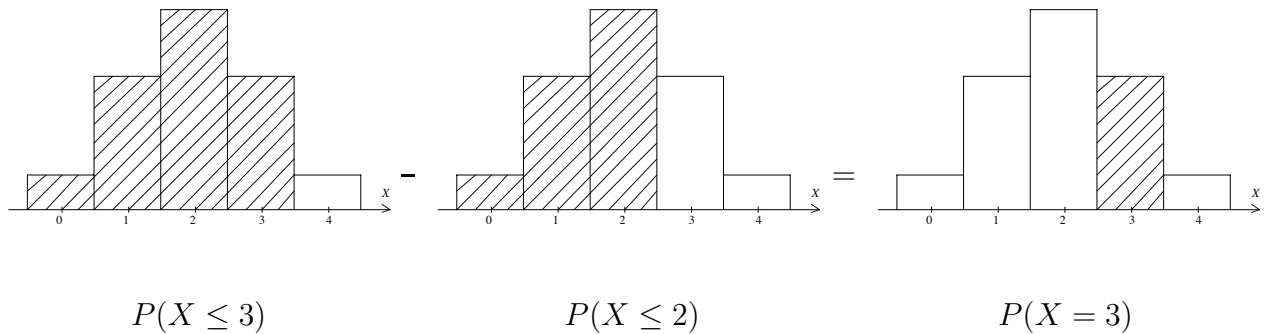
To find the last probability  $P(X = 3)$  we must subtract the first three probabilities. A table of cumulative probabilities will give us these:

$$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) - [P(X = 0) + P(X = 1) + P(X = 2)] = P(X = 3)$$

OR

$$P(X \leq 3) - P(X \leq 2) = P(X = 3)$$

This can be seen simply from the diagrams:



As we can see, it is a simple matter to retrieve single probabilities from cumulative probabilities.

## Examples

We can use the same method to find other binomial probabilities.

1. For  $X \sim \mathcal{B}(10, \frac{1}{2})$

$$\begin{aligned} P(X = 9) &= P(X \leq 9) - P(X \leq 8) \\ &\doteq 0.9990 - 0.9893 \\ &= 0.0097. \end{aligned}$$

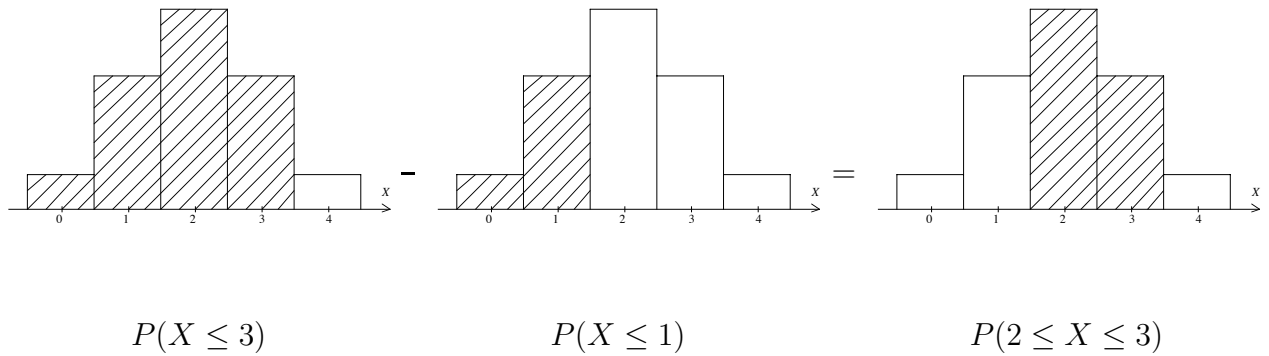
2. For  $X \sim \mathcal{B}(3, \frac{1}{2})$

$$\begin{aligned} P(X = 1) &= P(X \leq 1) - P(X \leq 0) \\ &= 3 \times \left(\frac{1}{2}\right)^3. \quad \text{Why?} \end{aligned}$$

We can use this same method to find the probability that  $X$  is between two different values. For instance, if  $X \sim \mathcal{B}\left(4, \frac{1}{2}\right)$ ,

$$\begin{aligned}
 P(2 \leq X \leq 3) &= P(X = 2) + P(X = 3) \\
 &= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) - [P(X = 0) + P(X = 1)] \\
 &= P(X \leq 3) - P(X \leq 1) \\
 &\doteq 0.9375 - 0.3125 \\
 &= 0.6250.
 \end{aligned}$$

We can see this simply from diagrams:



## Examples

1. For  $X \sim \mathcal{B}\left(9, \frac{1}{2}\right)$ ,

$$\begin{aligned}
 P(3 \leq X \leq 7) &= P(X \leq 7) - P(X \leq 2) \\
 &\doteq 0.9805 - 0.0898 \\
 &= 0.8907.
 \end{aligned}$$

2. For  $X \sim \mathcal{B}(7, 0.4)$ ,

$$\begin{aligned}
 P(1 \leq X \leq 4) &= P(X \leq 4) - P(X \leq 0) \\
 &= P(X \leq 4) - P(X = 0) \\
 &\doteq 0.9037 - 0.0280 \\
 &= 0.8757.
 \end{aligned}$$

3. For  $X \sim \mathcal{B}\left(12, \frac{1}{2}\right)$ ,

$$\begin{aligned}
 P(1 \leq X \leq 12) &= P(X \leq 12) - P(X \leq 0) \\
 &= 1 - \left(\frac{1}{2}\right)^{12}.
 \end{aligned}$$

### Exercise 13

1. For  $X \sim \mathcal{B}\left(3, \frac{1}{2}\right)$ , use tables to find  $P(X = 3)$ ,  $P(X = 2)$ ,  $P(X = 1)$ , and  $P(X = 0)$ . Use the results above to draw a representation of the  $\mathcal{B}\left(3, \frac{1}{2}\right)$  distribution.
2. For  $X \sim \mathcal{B}\left(3, \frac{1}{2}\right)$ , find  $P(1 \leq X \leq 2)$  and sketch the areas that you used to find this from question 1..
3. For  $X \sim \mathcal{B}\left(7, \frac{1}{2}\right)$ , find:
  - i.  $P(X = 5)$ ,  $P(X = 7)$ ;
  - ii.  $P(1 \leq X \leq 7)$  explain using sketches of the probability distribution why this is equal to  $1 - \left(\frac{1}{2}\right)^7$  (note a rough sketch will do, don't work out all the probabilities).
4. For  $X \sim \mathcal{B}(12, 0.8)$ , find  $P(X = 12)$  and explain why this should be  $(0.8)^{12}$ .
5. For  $X \sim \mathcal{B}(6, 0.7)$ , find
  - i.  $P(X \geq 5)$ ;
  - ii.  $P(X \geq 0)$ .
6. For  $X \sim \mathcal{B}(8, 0.2)$ , find
  - i.  $P(X \geq 4)$ ;
  - ii.  $P(X \geq 4 \text{ or } X \leq 2)$ .
7. For  $X \sim \mathcal{B}(12, 0.2)$ , the probability that  $X \leq 12$  should be one, as we know. Why is  $P(X \leq 9)$  listed as 1.0000 to four decimal places? Why is  $P(X \leq 10)$  assumed also to be 1.0000?
8. For  $X \sim \mathcal{B}(12, 0.9)$ , the  $P(X \leq 5)$  is listed as 0.0000 to four decimal places. Explain.

Since a binomial variable can only take discrete values we must be careful with inequalities. For instance, for  $X \sim \mathcal{B}(7, 0.4)$ , if  $X$  is less than five it will be less than or equal to four.

We have

$$\begin{aligned} P(X < 5) &= P(X \leq 4) \\ &= 0.9037. \end{aligned}$$

Similarly, if  $X$  is greater than five,  $X$  can take the values of six or seven:

$$\begin{aligned} P(X > 5) &= P(X \geq 6) \\ &= P(X = 6) + P(X = 7) \\ &= 1 - P(X \leq 5) \\ &= 1 - 0.9812 \\ &= 0.0188. \end{aligned}$$

This can make finding probabilities in a range of values for  $X$  more difficult.

For instance, if  $X \sim \mathcal{B}(7, 0.4)$ ,

$$\begin{aligned} P(3 \leq X < 6) &= P(3 \leq X \leq 5) \\ &= P(X \leq 5) - P(X \leq 2) \\ &= 0.9812 - 0.4199 \\ &= 0.5613. \end{aligned}$$

#### Exercise 14

For  $X \sim \mathcal{B}(9, 0.5)$ , find:

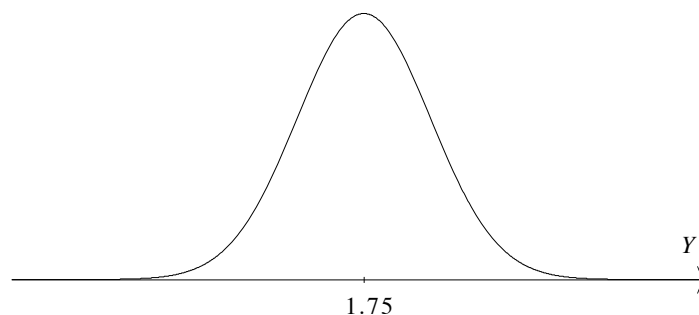
- i.  $P(X < 4)$ ;
- ii.  $P(X > 5)$ ;
- iii.  $P(3 \leq X < 5)$ ;
- iv.  $P(4 < X \leq 8)$ ;
- v.  $P(X > 5 \text{ or } X < 3)$ .

## 4 The Normal Approximation to the Binomial Distribution

### 4.1 Approximating a Binomial Distribution with a Normal Curve

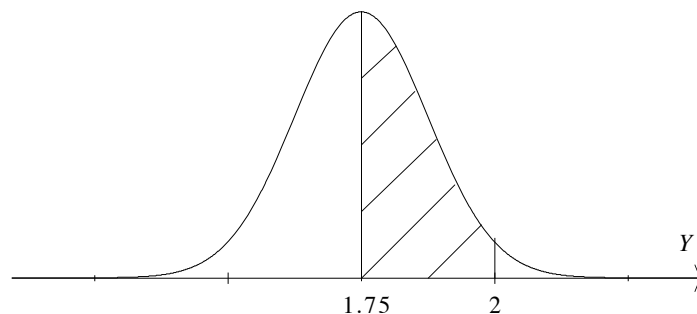
Very often natural phenomenon do not follow the binomial distribution but a normal distribution. For instance, the probability of finding people in a range of heights may have a normal distribution.

We know that a height, a weight, a length, can take any value, not just  $X = 0, 1, 2, 3, \dots, n$ . We give such a value a different name, let's say  $Y$ . Males in a lecture theatre may have heights that are well modelled by a normal distribution. A representation of the probability distribution for  $Y$  could look like



The bell shape curve is a normal curve. It is the same either side of the average height of  $Y = 1.75\text{m}$  tall. The area under the curve tells us the probability of finding people in a height range.

For instance, the probability that a person in the lecture theatre is between  $1.75\text{m}$  tall and  $2\text{m}$  tall  $P(1.75 \leq Y \leq 2)$  would be represented by the area:

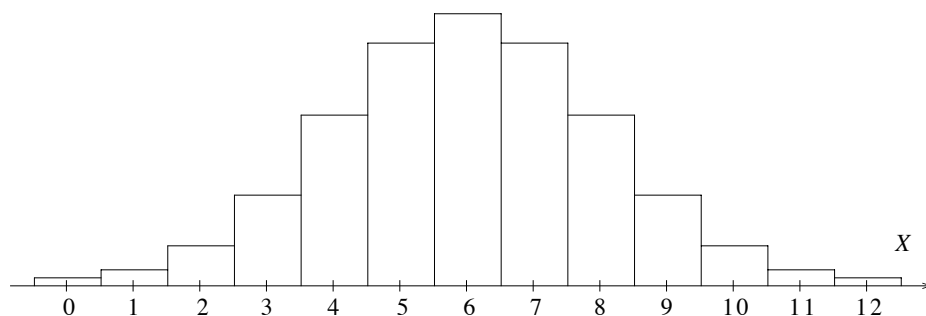


We can see that we would certainly expect to find people up to  $2\text{m}$  tall and down to  $1.5\text{m}$  tall. The heights could have a given deviation of  $0.25\text{m}$ .

We write:  $Y \sim \mathcal{N}(1.75, (0.25)^2) \sim \mathcal{N}(\mu, \sigma^2)$ .

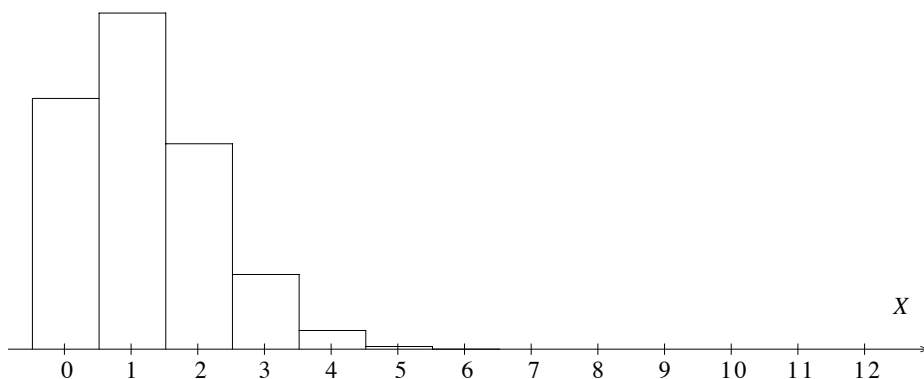
The  $\mathcal{N}$  indicates a normal distribution. The first number  $1.75$ , or  $\mu$ , in general indicates the centre of the normal curve. The  $0.25$ , or  $\sigma$ , indicates how much of a given deviation or standard deviation we expect. Since  $Y$  can take a continuum of values we call  $Y$  a continuous random variable. For more on normal distributions, see the Mathematics Learning Centre Booklet “The Normal Distribution”.

A representation for the probability distribution for  $X \sim \mathcal{B}(12, 0.5)$  is given below:



As we can see, even for  $n$  as small as twelve, the shape of our distribution is not unlike the normal bell shape curve. As  $n$  increases the distribution looks closer and closer to a normal distribution.

If we plot the probability distribution for  $X \sim \mathcal{B}(12, 0.1)$ , as shown below, we can see that the distribution is not symmetric and looks different to a normal distribution.



For large enough  $n$  though, even for  $p = 0.1$ , the distribution does approach the normal curve.

For example, the  $\mathcal{B}(50, 0.1)$  will look very similar to a normal curve.

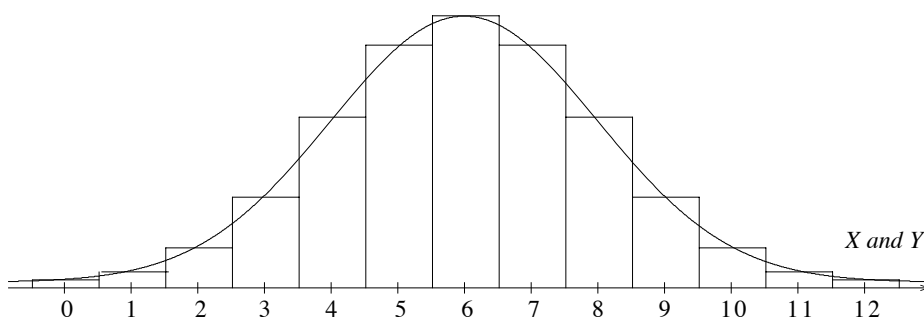
How large we need  $n$  to be in order to be able to use a normal approximation will certainly depend on how far  $p$  is from one half.

Let's return to our  $X \sim \mathcal{B}(12, 0.5)$  distribution. We can see that the distribution is symmetric about (the same either side of)  $X = 6$ .

If we are to place a normal curve on top of the  $\mathcal{B}(12, 0.5)$  distribution it should have a centre point at six. To calculate this centre point we use  $\mu = n \times p$ , where  $\mu = 12 \times \frac{1}{2} = 6$ .

As we can see we certainly expect to obtain values of  $X$  as large as nine or as small as three. The approximating normal curve for  $\mathcal{B}(12, 0.5)$  has a variance of three. To calculate this, we use the approximating variance  $\sigma^2 = np(1 - p) = 12 \times \frac{1}{2} \times \left(1 - \frac{1}{2}\right) = 3$ .

Hence the approximating normal curve for  $X \sim \mathcal{B}(12, 0.5)$  is given by  $Y \sim \mathcal{N}(6, 3)$ . These two probability distributions are shown laid on top of each other below. As we can see, they are a pretty good match:



In general the approximating normal curve for  $X \sim \mathcal{B}(n, p)$  is given by  $Y \sim \mathcal{N}(\mu, \sigma^2)$ , where  $\mu = np$ , and  $\sigma^2 = np(1 - p)$  or  $\sigma = \sqrt{np(1 - p)}$ .

### Exercise 15

1. Find the approximating normal distribution for  $X \sim \mathcal{B}(20, 0.5)$ .
2. For  $X \sim \mathcal{B}(40, 0.8)$ , find the value about which the approximating normal distribution will be symmetric. Find the variance  $\sigma^2$  for this normal curve.
3. For each of the following, find the approximating normal distributions:
  - i.  $\mathcal{B}(40, 0.3)$ ;
  - ii.  $\mathcal{B}(40, 0.4)$ ;
  - iii.  $\mathcal{B}(40, 0.5)$ ;
  - iv.  $\mathcal{B}(40, 0.6)$ ;
  - v.  $\mathcal{B}(40, 0.7)$ .

How does the spread of the normal curve  $\sigma^2$  vary as  $p$  increases up to  $p = 0.5$  and then as  $p$  increases from 0.5 upwards? How does the centre of the normal curve  $\mu$  change as  $p$  increases?

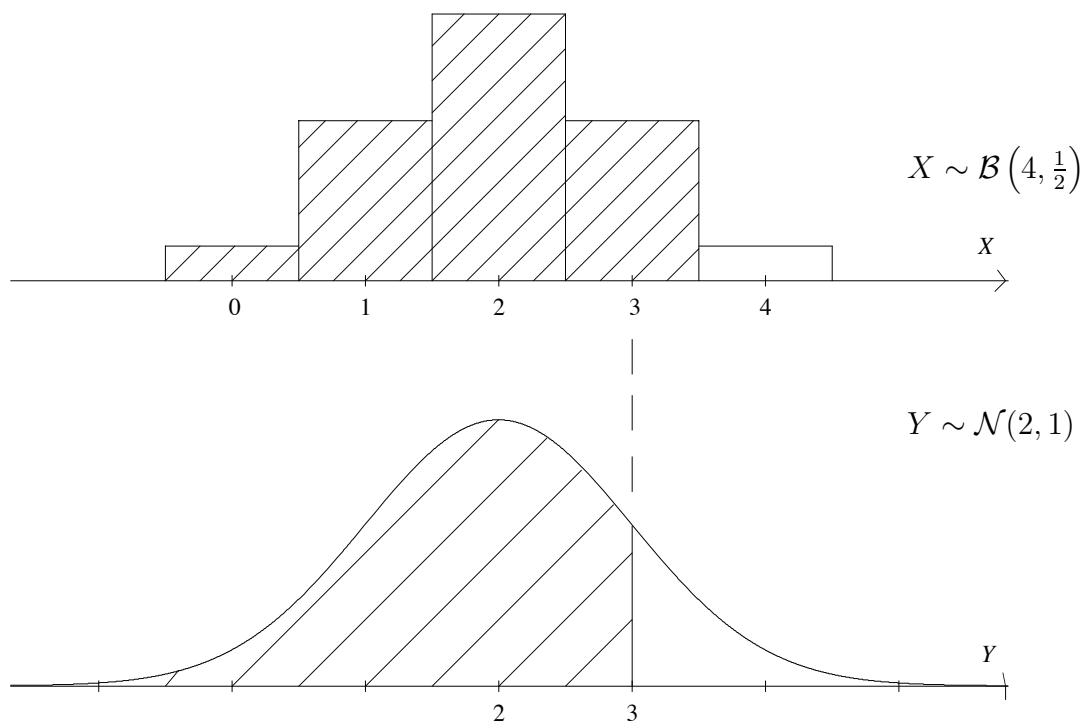
4. If we toss a fair coin fifty times, find an approximating normal distribution for the number of heads  $X$ .
5. Find the approximating normal distribution for the number of fives ( $\square$ 's) obtained in sixty throws of a die.

## 4.2 Continuity Correction

The binomial random variable  $X \sim \mathcal{B}(n, p)$  can only take discrete values  $X = 0, 1, 2, \dots, n$ . The approximating normal random variable  $Y \sim \mathcal{N}(\mu, \sigma^2)$  though can take a continuum of values from  $-\infty$  to  $+\infty$ . For this reason, we must be careful when calculating probabilities using the normal approximation to a binomial distribution. We must correct for the fact that we are using a continuous variable for a discrete one.

For  $X \sim \mathcal{B}\left(4, \frac{1}{2}\right)$ , we have an approximating normal distribution given by  $Y \sim \mathcal{N}(2, 1)$ . Even though this approximation will not be very accurate for this small  $n$  value, it is easiest to see the continuity correction for small  $n$ .

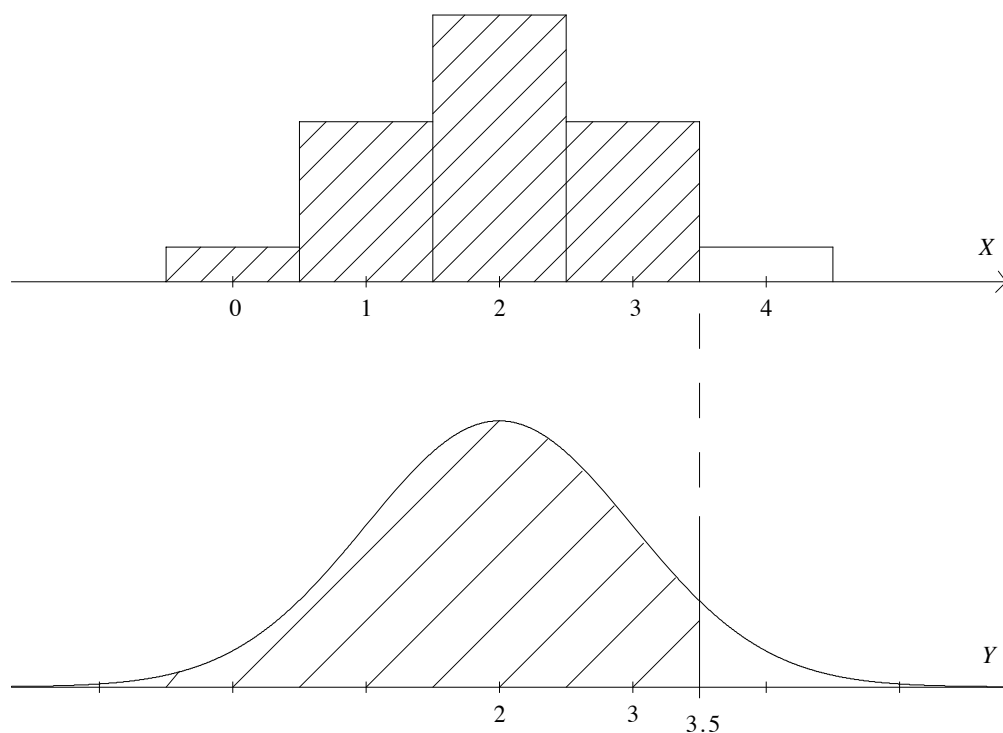
The probability  $P(X \leq 3)$  is represented by the shaded area below, the corresponding probability  $P(Y \leq 3)$  is also shown:



For the normal variable  $Y \sim \mathcal{N}(2, 1)$ ,  $P(Y \leq 3)$  is represented by the area to the left of  $Y = 3$ . For  $X \sim \mathcal{B}\left(4, \frac{1}{2}\right)$ , the  $P(X \leq 3)$  is represented by the area to the left of the point at 3.5.

As we can see, we will obtain a better normal approximation by using the area to the left of  $Y = 3.5$ .

This is illustrated in the representations below where the  $P(X \leq 3)$  and  $P(Y \leq 3.5)$  are shown.



$P(X \leq 3)$  is given by the shaded area in the top graph.



$P(Y \leq 3.5)$  is given by the shaded area in the bottom graph.

As we can see, this is a better approximation than if we had left out the area from  $Y = 3$  to  $Y = 3.5$ .

If we were interested in  $P(X \geq 3)$ , we can see that it is better to use the area to the right of  $3 - \frac{1}{2} = 2.5$  i.e. we use  $P(Y \geq 2.5)$ .

As we have noted and can see from our diagrams, for  $n$  as small as 4 the normal approximation may not be as accurate as we may need. As a rule of thumb, it is often required that  $np \geq 5$  to obtain a certain accuracy.

If we are approximating a binomial probability less than a value we add one half.

If we are approximating a binomial probability greater than a value we subtract one half.

In general:

for  $X \sim \mathcal{B}(n, p)$  and  $Y \sim \mathcal{N}(\mu, \sigma^2)$ ,

we approximate  $P(X \leq m)$  by  $P\left(Y \leq m + \frac{1}{2}\right)$ ,

and approximate  $P(X \geq m)$  by  $P\left(Y \geq m - \frac{1}{2}\right)$ .

## Examples

For  $X \sim \mathcal{B}(50, 0.4)$  we use  $Y \sim \mathcal{N}(20, 12)$ :

$P(X \leq 20)$  will be approximated by  $P(Y \leq 20.5)$ ,

$P(X \geq 15)$  will be approximated by  $P(Y \geq 14.5)$ ,

$P(X < 31)$  will be approximated by  $P(Y \leq 30.5)$ ,

$P(X > 19)$  will be approximated by  $P(Y \geq 19.5)$ , and

$P(10 \leq X \leq 40) \simeq P(9.5 \leq Y \leq 40.5)$ .

## Exercise 16

1. For  $X \sim \mathcal{B}(20, 0.4)$  find the approximating normal distribution ( $Y$ ) and write down the approximating probability interval for  $Y$  for the following:
  - i.  $P(X \leq 10)$ ;
  - ii.  $P(X \geq 5)$ ;
  - iii.  $P(13 \leq X \leq 17)$ ;
  - iv.  $P(X < 2)$ ;
  - v.  $P(X > 15 \text{ or } X \leq 5)$ .
2. Find the approximating normal probability distribution and interval if we are interested in the probability of tossing less than five heads in one hundred tosses of a fair coin.
3. In a game of dice we win if we roll either a two or a three ( $\begin{smallmatrix} \square & \cdot \\ \cdot & \square \end{smallmatrix}$  or  $\begin{smallmatrix} \cdot & \square \\ \cdot & \cdot \end{smallmatrix}$ ). Write down the binomial distribution for the number of wins in one hundred rolls of a fair die. Find the approximating normal distribution for  $Y$ . Find the approximating probability interval for  $Y$  if we are interested in the probability of rolling less than ten wins in the one hundred rolls.
4. Find the approximating normal probability distribution and interval if we are interested in the probability of rolling between one hundred and twenty and one hundred and fifty sixes ( $\begin{smallmatrix} \square & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{smallmatrix}$ 's) inclusive in two hundred rolls of a fair die.

## 5 Solutions

### Tossing Heads and Tails

#### Exercise 1

- |   |   |   |   |
|---|---|---|---|
| <p>i.    H H T T T<br/>             H T H T T<br/>             H T T H T<br/>             H T T T H<br/>             T H H T T<br/>             T H T H T<br/>             T H T T H<br/>             T T H H T<br/>             T T H T H<br/>             T T T H H</p> | <p>ii.   H T T T T T<br/>             T H T T T T<br/>             T T H T T T<br/>             T T T H T T<br/>             T T T T H T<br/>             T T T T T H</p> | <p>iii.   T T H H H H<br/>             T H T H H H<br/>             T H H T H H<br/>             T H H H T H<br/>             T H H H H T<br/>             H T T H H H<br/>             H T H T H H<br/>             H T H H T H<br/>             H T H H H T<br/>             H H T T H H<br/>             H H T H T H<br/>             H H T H H T<br/>             H H H T T H<br/>             H H H T H T<br/>             H H H H T T</p> | <p>iv.    H H H T T T<br/>             H H T H T T<br/>             H H T T H T<br/>             H H T T T H<br/>             H T H H T T<br/>             H T H T H T<br/>             H T T H H T<br/>             H T T H T H<br/>             H T T T H H<br/>             T H H H T T<br/>             T H H T H T<br/>             T H H T T H<br/>             T H T H H T<br/>             T H T H T H<br/>             T H T T H H<br/>             T T H H H T<br/>             T T H H T H<br/>             T T H T H H<br/>             T T T H H H</p> |
|---|---|---|---|

### Tossing Different Heads

#### Exercise 2

- There are four ways of placing  $H_1$ . For each of these four ways, there are three ways of placing  $H_2$ , giving  $4 \times 3 = 12$ . Therefore, there are twelve ways of arranging  $H_1$  and  $H_2$  in four spaces.
- There are six ways of placing  $H_1$ . For each of these six ways, there are five ways of placing  $H_2$ , giving  $6 \times 5 = 30$ . Therefore, there are thirty ways of arranging  $H_1$  and  $H_2$  in six spaces.
- There are four ways of placing  $H_1$ . For each of these four ways, there are three ways of placing  $H_2$ , giving  $4 \times 3 = 12$ . For each of these twelve arrangements there are two ways of placing  $H_3$ , giving  $12 \times 2 = 24$ . For each of these twenty-four arrangements there is one way of placing  $H_4$ . That is, there are  $4 \times 3 \times 2 \times 1 = 24$  ways of arranging  $H_1$ ,  $H_2$ ,  $H_3$ , and  $H_4$  in four spaces.
- There are five ways of placing  $H_1$ . For each of these five ways, there are four ways of placing  $H_2$ , giving  $5 \times 4 = 20$ . For each of these twenty arrangements there are three ways of placing  $H_3$ , giving  $20 \times 3 = 60$ . For each of these sixty arrangements there are two ways of placing  $H_4$ . That is, there are  $5 \times 4 \times 3 \times 2 = 120$  ways of arranging  $H_1$ ,  $H_2$ ,  $H_3$ , and  $H_4$  in five spaces.

5. There are eight ways of placing  $H_1$ . For each of these eight ways, there are seven ways of placing  $H_2$ , giving  $8 \times 7 = 56$ . For each of these fifty-six arrangements there are six ways of placing  $H_3$ , giving  $56 \times 6 = 336$ . For each of these three hundred and thirty-six arrangements there are five ways of placing  $H_4$ . That is, there are  $8 \times 7 \times 6 \times 5 = 1680$  ways of arranging  $H_1, H_2, H_3$ , and  $H_4$  in eight spaces.

## Tossing Identical Heads

### Exercise 3

$ H_1 H_2 _ _ $	$ H H _ _ $	$ _ H_1 H_2 _ $	$ _ H H _ $
$ H_2 H_1 _ _ $		$ _ H_2 H_1 _ $	
$ H_1 _ H_2 _ $	$ H _ H _ $	$ _ H_1 _ H_2 $	$ _ H _ H $
$ H_2 _ H_1 _ $		$ _ H_2 _ H_1 $	
$ H_1 _ _ H_2 $	$ H _ _ H $	$ _ _ H_1 H_2 $	$ _ _ H H $
$ H_2 _ _ H_1 $		$ _ _ H_2 H_1 $	

### Exercise 4

$ H_1 _ H_2 H_3 H_4 $	$ H_1 _ H_2 H_4 H_3 $	$ H_1 _ H_3 H_2 H_4 $
$ H_1 _ H_3 H_4 H_2 $	$ H_1 _ H_4 H_2 H_3 $	$ H_1 _ H_4 H_3 H_2 $
$ H_2 _ H_1 H_3 H_4 $	$ H_2 _ H_1 H_4 H_3 $	$ H_2 _ H_3 H_1 H_4 $
$ H_2 _ H_3 H_4 H_1 $	$ H_2 _ H_4 H_1 H_3 $	$ H_2 _ H_4 H_3 H_1 $
$ H_3 _ H_1 H_2 H_4 $	$ H_3 _ H_1 H_4 H_2 $	$ H_3 _ H_2 H_1 H_4 $
$ H_3 _ H_2 H_4 H_1 $	$ H_3 _ H_4 H_1 H_2 $	$ H_3 _ H_4 H_2 H_1 $
$ H_4 _ H_1 H_2 H_3 $	$ H_4 _ H_1 H_3 H_2 $	$ H_4 _ H_2 H_1 H_3 $
$ H_4 _ H_2 H_3 H_1 $	$ H_4 _ H_3 H_1 H_2 $	$ H_4 _ H_3 H_2 H_1 $

### Exercise 5

- i. The number of arrangements of four  $H$ 's in five spaces will be

$$\frac{5(5-1)(5-2)(5-(4-1))}{4 \times 3 \times 2 \times 1} = \frac{5 \times 4 \times 3 \times 2}{4 \times 3 \times 2 \times 1} = 5.$$

- ii. The number of arrangements of four  $H$ 's in six spaces will be

$$\frac{6(6-1)(6-2)(6-(4-1))}{4 \times 3 \times 2 \times 1} = \frac{6 \times 5 \times 4 \times 3}{4 \times 3 \times 2 \times 1} = 15.$$

- iii. The number of arrangements of three  $H$ 's in seven spaces will be

$$\frac{7(7-1)(7-(3-1))}{3 \times 2 \times 1} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35.$$

iv. The number of arrangements of two  $H$ 's in seven spaces will be

$$\frac{7(7 - (2 - 1))}{2 \times 1} = \frac{7 \times 6}{2 \times 1} = 21.$$

v. The number of arrangements of one  $H$  in seven spaces will be  $\frac{7}{1} = 7$ .

## Factorial Notation

### Exercise 6

1. i. The number of arrangements of nine identical objects in twelve spaces will be

$$\frac{12!}{(12 - 9)!9!} = \frac{12!}{3!9!} = 220.$$

ii. The number of arrangements of six identical objects in ten spaces will be

$$\frac{10!}{(10 - 6)!6!} = \frac{10!}{4!6!} = 210.$$

iii. The number of arrangements of seven identical objects in thirteen spaces will be

$$\frac{13!}{(13 - 7)!7!} = \frac{13!}{6!7!} = 1716.$$

2. The number of arrangements of exactly eight  $H$ 's in thirteen tosses of a coin will be

$$\frac{13!}{(13 - 8)!8!} = \frac{13!}{5!8!} = 1287.$$

3. The number of arrangements of exactly seven  $H$ 's in eleven tosses of a coin will be

$$\frac{11!}{(11 - 7)!7!} = \frac{11!}{4!7!} = 330.$$

4. The number of arrangements of exactly four  $H$ 's in ten tosses of a coin will be

$$\frac{10!}{(10 - 4)!4!} = \frac{10!}{6!4!} = 210.$$

## Calculating Probabilities for the Binomial Distribution

### Exercise 7

$$1. P(1_{\text{H}} \text{ and } 4_{\text{T}}\text{'s}) = \binom{5}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{5-1} = \binom{5}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^4 \doteq 0.1563.$$

$$2. P(3_{\text{H}}\text{'s and } 3_{\text{T}}\text{'s}) = \binom{6}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{6-3} = \binom{6}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 \doteq 0.3125.$$

$$3. \quad P(2\textcircled{\text{H}}\text{'s and } 3\textcircled{\text{T}}\text{'s}) = \binom{5}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{5-2} = \binom{5}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 \doteq 0.3125.$$

$$\begin{aligned} 4. \quad P(4 \text{ or more } \textcircled{\text{H}}\text{'s in 5 tosses}) &= P(4\textcircled{\text{H}}\text{'s and } 1\textcircled{\text{T}}) + P(5\textcircled{\text{H}}\text{'s and } 0\textcircled{\text{T}}\text{'s}) \\ &= \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{5-4} + \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{5-5} \\ &= \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 + \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 \\ &\doteq 0.1875. \end{aligned}$$

$$\begin{aligned} 5. \quad P(3 \text{ or less } \textcircled{\text{H}}\text{'s in 5 tosses}) &= 1 - P(4 \text{ or more } \textcircled{\text{H}}\text{'s in 5 tosses}) \\ &= 1 - \left( \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 + \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 \right) \\ &\doteq 0.8125. \end{aligned}$$

## The Binomial Distribution for Unequal Probabilities

### Exercise 8

$$1. \quad P(2\textcircled{\cdot}\text{'s and } 3\textcircled{\cdot}\text{'s}) = \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(1 - \frac{1}{6}\right)^{5-2} = \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 \doteq 0.1608.$$

$$\begin{aligned} 2. \quad P(3 \text{ or more } \textcircled{\cdot}\text{'s in 5 rolls}) &= P(3\textcircled{\cdot}\text{'s and } 2\textcircled{\cdot}\text{'s}) + P(4\textcircled{\cdot}\text{'s and } 1\textcircled{\cdot}) + P(5\textcircled{\cdot}\text{'s and } 0\textcircled{\cdot}\text{'s}) \\ &= \binom{5}{3} \left(\frac{1}{6}\right)^3 \left(1 - \frac{1}{6}\right)^{5-3} + \binom{5}{4} \left(\frac{1}{6}\right)^4 \left(1 - \frac{1}{6}\right)^{5-4} + \binom{5}{5} \left(\frac{1}{6}\right)^5 \left(1 - \frac{1}{6}\right)^{5-5} \\ &= \binom{5}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^2 + \binom{5}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^1 + \binom{5}{5} \left(\frac{1}{6}\right)^5 \left(\frac{5}{6}\right)^0 \\ &\doteq 0.0322 + 0.0032 + 0.0001 \\ &\doteq 0.0355. \end{aligned}$$

$$\begin{aligned} 3. \quad P(5 \text{ or less } \textcircled{\cdot}\text{'s in 6 rolls}) &= 1 - P(6\textcircled{\cdot}\text{'s and } 0\textcircled{\cdot}\text{'s}) \\ &= 1 - \binom{6}{0} \left(\frac{1}{6}\right)^6 \left(1 - \frac{1}{6}\right)^{6-6} \\ &= 1 - \binom{6}{0} \left(\frac{1}{6}\right)^6 \left(\frac{5}{6}\right)^0 \\ &\doteq 1.0000. \end{aligned}$$

$$\begin{aligned} 4. \quad P(6 \text{ or less } \textcircled{\cdot}\text{'s in 8 rolls}) &= 1 - P(7 \text{ or more } \textcircled{\cdot}\text{'s in 8 rolls}) \\ &= 1 - (P(7\textcircled{\cdot}\text{'s and } 1\textcircled{\cdot}) + P(8\textcircled{\cdot}\text{'s and } 0\textcircled{\cdot}\text{'s})) \\ &= 1 - \left( \binom{8}{7} \left(\frac{1}{6}\right)^7 \left(1 - \frac{1}{6}\right)^{8-7} + \binom{8}{8} \left(\frac{1}{6}\right)^8 \left(1 - \frac{1}{6}\right)^{8-8} \right) \\ &= 1 - \left( \binom{8}{7} \left(\frac{1}{6}\right)^7 \left(\frac{5}{6}\right)^1 + \binom{8}{8} \left(\frac{1}{6}\right)^8 \left(\frac{5}{6}\right)^0 \right) \\ &\doteq 1 - (0.0000 + 0.0000) \\ &\doteq 1.0000. \end{aligned}$$

## Binomial Probabilities

### Exercise 9

1.  $n = 6, m = 3, n - m = 6 - 3 = 3, p = \frac{1}{6}, 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$ .

$$P(X = 3) = \binom{6}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^3 = 20 \times \frac{1}{216} \times \frac{125}{216} = \frac{625}{11664} \doteq 0.0536.$$

2.  $n = 6, m = 2, n - m = 6 - 2 = 4, p = \frac{3}{6} = \frac{1}{2}, 1 - p = 1 - \frac{3}{6} = \frac{3}{6} = \frac{1}{2}$ .

$$P(X = 2) = \binom{6}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^4 = 15 \times \frac{1}{4} \times \frac{1}{16} = \frac{15}{64} \doteq 0.2344.$$

3.  $n = 6, m = 2, n - m = 6 - 2 = 4, p = \frac{2}{6} = \frac{1}{3}, 1 - p = 1 - \frac{2}{6} = \frac{4}{6} = \frac{2}{3}$ .

$$P(X = 2) = \binom{6}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4 = 15 \times \frac{1}{9} \times \frac{16}{81} = \frac{80}{243} \doteq 0.3292.$$

4.  $P(X = 4) = \binom{5}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^1 = 5 \times \frac{1}{16} \times \frac{1}{2} = \frac{5}{32} \doteq 0.1563.$

$$P(X = 5) = \binom{5}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = 1 \times \frac{1}{32} \times 1 = \frac{1}{32} \doteq 0.0313.$$

$$P(X \geq 4) = P(X = 4) + P(X = 5) = \frac{5}{32} + \frac{1}{32} = \frac{6}{32} = \frac{3}{16} \doteq 0.1875.$$

5.  $P(X = 5) = \binom{6}{5} \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1 = 6 \times \frac{1}{243} \times \frac{2}{3} = \frac{4}{243} \doteq 0.0165.$

$$P(X = 6) = \binom{6}{6} \left(\frac{1}{3}\right)^6 \left(\frac{2}{3}\right)^0 = 1 \times \frac{1}{729} \times 1 = \frac{1}{729} \doteq 0.0014.$$

$$P(X \geq 5) = P(X = 5) + P(X = 6) = \frac{4}{243} + \frac{1}{729} = \frac{12}{729} + \frac{1}{729} = \frac{13}{729} \doteq 0.0178.$$

## Picturing the Binomial Distribution

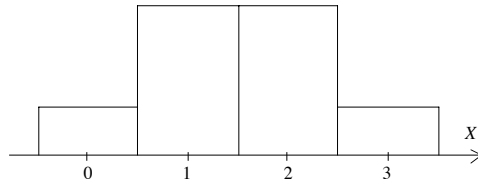
### Exercise 10

1.  $P(X = 0) = \binom{3}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^3 = 1 \times 1 \times \frac{1}{8} = \frac{1}{8}.$

$$P(X = 1) = \binom{3}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = 3 \times \frac{1}{2} \times \frac{1}{4} = \frac{3}{8}.$$

$$P(X = 2) = \binom{3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = 3 \times \frac{1}{4} \times \frac{1}{2} = \frac{3}{8}.$$

$$P(X = 3) = \binom{3}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^0 = 1 \times \frac{1}{8} \times 1 = \frac{1}{8}.$$



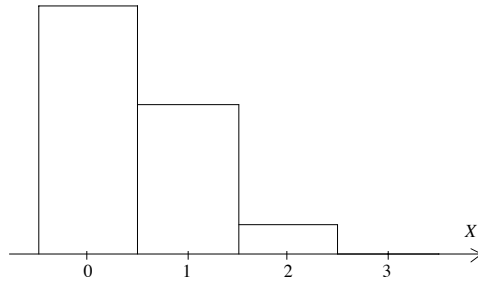
2.

$$P(X=0) = \binom{3}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^3 = 1 \times 1 \times \frac{125}{216} = \frac{125}{216}.$$

$$P(X=1) = \binom{3}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^2 = 3 \times \frac{1}{6} \times \frac{25}{36} = \frac{25}{72}.$$

$$P(X=2) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^1 = 3 \times \frac{1}{36} \times \frac{5}{6} = \frac{5}{72}.$$

$$P(X=3) = \binom{3}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^0 = 1 \times \frac{1}{216} \times 1 = \frac{1}{216}.$$

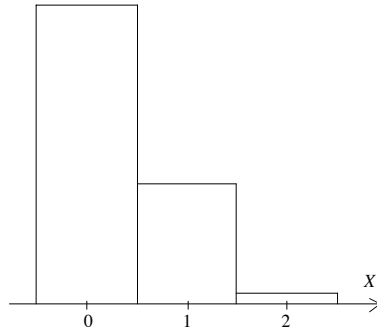


3.

$$P(X=0) = \binom{2}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^2 = 1 \times 1 \times \frac{25}{36} = \frac{25}{36}.$$

$$P(X=1) = \binom{2}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^1 = 2 \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{18}.$$

$$P(X=2) = \binom{2}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^0 = 1 \times \frac{1}{36} \times 1 = \frac{1}{36}.$$

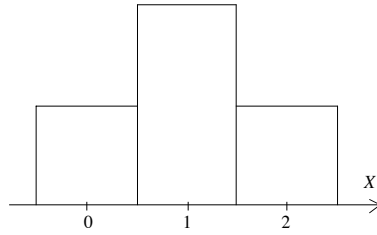


4.

$$P(X=0) = \binom{2}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^2 = 1 \times 1 \times \frac{1}{4} = \frac{1}{4}.$$

$$P(X=1) = \binom{2}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^1 = 2 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}.$$

$$P(X = 2) = \binom{2}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^0 = 1 \times \frac{1}{4} \times 1 = \frac{1}{4}.$$



5. Question 3 is skewed while question 4 is symmetric. This has occurred because question 3 involves unequal probabilities, while question 4 involves equal probabilities. The lower the probability  $p$  the more the distribution will tail off to the right. (Strangely enough we call the distribution in question 3 right skewed.)

### Exercise 11

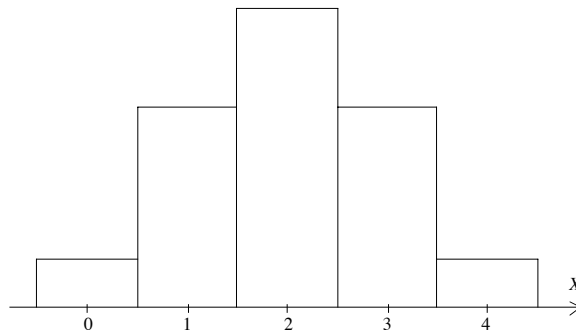
1.  $X \sim \mathcal{B}(4, \frac{1}{2})$ .  $P(X = 0) = \binom{4}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 1 \times 1 \times \frac{1}{16} = \frac{1}{16}.$

$$P(X = 1) = \binom{4}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 4 \times \frac{1}{2} \times \frac{1}{8} = \frac{1}{4}.$$

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 6 \times \frac{1}{4} \times \frac{1}{4} = \frac{3}{8}.$$

$$P(X = 3) = \binom{4}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^1 = 4 \times \frac{1}{8} \times \frac{1}{2} = \frac{1}{4}.$$

$$P(X = 4) = \binom{4}{4} \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0 = 1 \times \frac{1}{16} \times 1 = \frac{1}{16}.$$



2.  $X \sim \mathcal{B}(4, \frac{1}{6})$ .  $P(X = 0) = \binom{4}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 = 1 \times 1 \times \frac{625}{1296} = \frac{625}{1296}.$

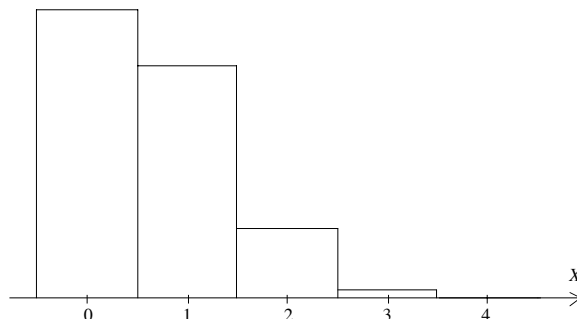
$$P(X = 1) = \binom{4}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 = 4 \times \frac{1}{6} \times \frac{125}{216} = \frac{500}{1296} = \frac{125}{324}.$$

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = 6 \times \frac{1}{36} \times \frac{25}{36} = \frac{150}{1296} = \frac{25}{216}.$$



$$P(X = 3) = \binom{4}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1 = 4 \times \frac{1}{216} \times \frac{5}{6} = \frac{20}{1296} = \frac{5}{324}.$$

$$P(X = 4) = \binom{4}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^0 = 1 \times \frac{1}{1296} \times 1 = \frac{1}{1296}.$$



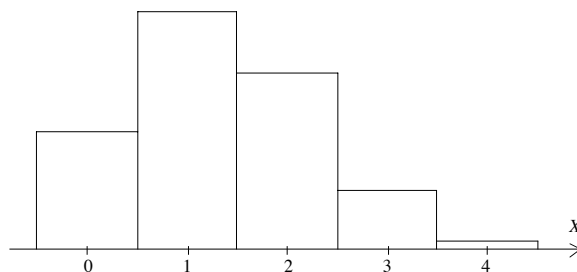
3.  $X \sim \mathcal{B}(4, \frac{1}{3})$ .  $P(X = 0) = \binom{4}{0} \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^4 = 1 \times 1 \times \frac{16}{81} = \frac{16}{81}.$

$$P(X = 1) = \binom{4}{1} \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^3 = 4 \times \frac{1}{3} \times \frac{8}{27} = \frac{32}{81}.$$

$$P(X = 2) = \binom{4}{2} \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^2 = 6 \times \frac{1}{9} \times \frac{4}{9} = \frac{24}{81} = \frac{8}{27}.$$

$$P(X = 3) = \binom{4}{3} \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^1 = 4 \times \frac{1}{27} \times \frac{2}{3} = \frac{8}{81}.$$

$$P(X = 4) = \binom{4}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^0 = 1 \times \frac{1}{81} \times 1 = \frac{1}{81}.$$



## Using Binomial Tables

### Exercise 12

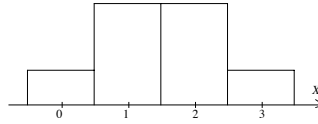
1. Using  $n = 4$  and  $p = \frac{1}{2}$ , locate 0.9375 in your tables. Note: this is the probability to 4 decimal places.
2.  $P(X \leq 4) = 1.0000$ . This is because  $P(X \leq 4)$  covers every possibility, hence the probability of every possibility added together equals 1.
3.  $P(X \leq 0) = 0.0625$  (from tables).

$$P(X = 0) = \binom{4}{0} \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = 1 \times 1 \times \frac{1}{16} = \frac{1}{16} = 0.0625.$$

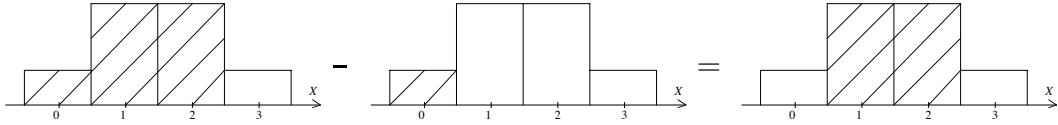
4. i.  $P(X \leq 0) = 0.0156$ .  
 iii.  $P(X \leq 5) = 0.9844$ .  
 ii.  $P(X \leq 3) = 0.6563$ .  
 iv.  $P(X \leq 6) = 1$ .
5. i.  $P(X \leq 3) = 0.8208$ .  
 iii.  $P(X \leq 7) = 1$ .  
 ii.  $P(X \leq 5) = 0.1582$ .  
 iv.  $P(X \leq 7) = 1$ .

### Exercise 13

1.  $P(X = 3) = P(X \leq 3) - P(X \leq 2) = 1.0000 - 0.8750 = 0.1250$ .  
 $P(X = 2) = P(X \leq 2) - P(X \leq 1) = 0.8750 - 0.5000 = 0.3750$ .  
 $P(X = 1) = P(X \leq 1) - P(X \leq 0) = 0.5000 - 0.1250 = 0.3750$ .  
 $P(X = 0) = P(X \leq 0) = 0.1250$ .

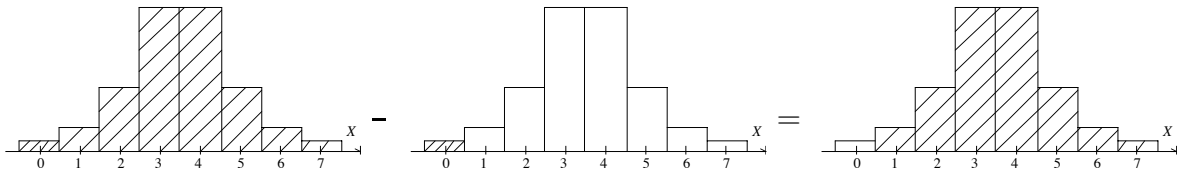


2.  $P(1 \leq X \leq 2) = P(X \leq 2) - P(X \leq 0) = 0.8750 - 0.1250 = 0.7500$ .



3. i.  $P(X = 5) = P(X \leq 5) - P(X \leq 4) \doteq 0.9375 - 0.7734 \doteq 0.1641$ .  
 $P(X = 7) = P(X \leq 7) - P(X \leq 6) \doteq 1.0000 - 0.9922 \doteq 0.0078$ .
- ii.  $P(1 \leq X \leq 7) = P(X \leq 7) - P(X \leq 0) \doteq 1.0000 - 0.0078 \doteq 0.9922$ .

$$1 - \left(\frac{1}{2}\right)^7 = 1 - \frac{1}{128} \doteq 1 - 0.0078 \doteq 0.9922.$$



4.  $P(X = 12) = P(X \leq 12) - P(X \leq 11) \doteq 1.0000 - 0.9313 \doteq 0.0687$ .

This should be  $(0.8)^{12}$  because the probability of a success is 0.8 for each of the 12 events. Mathematically,

$$P(X = 12) = \binom{12}{12} (0.8)^{12} (0.2)^0 = 1 \times (0.8)^{12} \times 1 = (0.8)^{12}.$$

5. i.  $P(X \geq 5) = P(X \leq 6) - P(X \leq 4) \doteq 1.0000 - 0.5798 \doteq 0.4202$ .
- ii.  $P(X \geq 0) = P(X \leq 6) = 1.0000$ .
6. i.  $P(X \geq 4) = P(X \leq 8) - P(X \leq 3) \doteq 1.0000 - 0.9437 \doteq 0.0563$ .
- ii. 
$$\begin{aligned} P(X \geq 4 \text{ or } X \leq 2) &= P(X \geq 4) + P(X \leq 2) \\ &= P(X \leq 8) - P(X \leq 3) + P(X \leq 2) \\ &\doteq 1.0000 - 0.9437 + 0.7969 \\ &\doteq 0.8532. \end{aligned}$$
7. 
$$\begin{aligned} P(X \geq 10) &= P(X = 10) + P(X = 11) + P(X = 12) \\ &= \binom{12}{10} (0.2)^{10} (0.8)^2 + \binom{12}{11} (0.2)^{11} (0.8)^1 + \binom{12}{12} (0.2)^{12} (0.8)^0 \\ &= 66 \times \frac{1}{9765625} \times \frac{16}{25} + 12 \times \frac{1}{48828125} \times \frac{4}{5} + 1 \times \frac{1}{244140625} \times 1 \\ &= \frac{1056}{244140625} + \frac{48}{244140625} + \frac{1}{244140625} \\ &= \frac{1105}{244140625} \\ &= \frac{221}{48828125} \\ &\doteq 0.0000045. \end{aligned}$$

The probability of  $X$  being larger than 10 i.e.  $X = 10, 11, 12$  is close to 4.5 in a million, which is very small.

The probability that  $X \leq 9$  will be one minus this very small number. To four decimal places  $P(X \leq 9) \doteq 1.0000$  even though it's not exactly one.

$P(X \leq 10)$  is also assumed to be 1.0000 since  $P(X \leq 10) > P(X \leq 9)$ .

8. Using a similar approach to that in 7, it is clear that  $P(X \leq 5) < 0.00005$ , meaning that it is given by 0.0000 to four decimal places.

#### Exercise 14

- i.  $P(X < 4) = P(X \leq 3) \doteq 0.2539$ .
- ii.  $P(X > 5) = P(X \geq 6) = P(X \leq 9) - P(X \leq 5) \doteq 1.0000 - 0.7461 \doteq 0.2539$ .
- iii.  $P(3 \leq X < 5) = P(3 \leq X \leq 4) = P(X \leq 4) - P(X \leq 2) \doteq 0.5000 - 0.0898 \doteq 0.4102$ .
- iv.  $P(4 < X \leq 8) = P(5 \leq X \leq 8) = P(X \leq 8) - P(X \leq 4) \doteq 0.9980 - 0.5000 \doteq 0.4980$
- v. 
$$\begin{aligned} P(X > 5 \text{ or } X < 3) &= P(X \geq 6 \text{ or } X \leq 2) \\ &= P(X \geq 6) + P(X \leq 2) \\ &= P(X \leq 9) - P(X \leq 5) + P(X \leq 2) \\ &\doteq 1.0000 - 0.7461 + 0.0898 \\ &\doteq 0.3437. \end{aligned}$$

## Approximating a Binomial Distribution with a Normal Curve

### Exercise 15

1.  $\mu = 20 \times 0.5 = 10$  and  $\sigma^2 = 20 \times 0.5 \times (1 - 0.5) = 5$ . Therefore,  $Y \sim \mathcal{N}(10, 5)$ .
2.  $\mu = 40 \times 0.8 = 32$  and  $\sigma^2 = 40 \times 0.8 \times (1 - 0.8) = 6.4$ .
3.
  - i.  $\mu = 40 \times 0.3 = 12$  and  $\sigma^2 = 40 \times 0.3 \times (1 - 0.3) = 8.4$ . Therefore,  $Y \sim \mathcal{N}(12, 8.4)$ .
  - ii.  $\mu = 40 \times 0.4 = 16$  and  $\sigma^2 = 40 \times 0.4 \times (1 - 0.4) = 9.6$ . Therefore,  $Y \sim \mathcal{N}(16, 9.6)$ .
  - iii.  $\mu = 40 \times 0.5 = 20$  and  $\sigma^2 = 40 \times 0.5 \times (1 - 0.5) = 10$ . Therefore,  $Y \sim \mathcal{N}(20, 10)$ .
  - iv.  $\mu = 40 \times 0.6 = 24$  and  $\sigma^2 = 40 \times 0.6 \times (1 - 0.6) = 9.6$ . Therefore,  $Y \sim \mathcal{N}(24, 9.6)$ .
  - v.  $\mu = 40 \times 0.7 = 28$  and  $\sigma^2 = 40 \times 0.7 \times (1 - 0.7) = 8.4$ . Therefore,  $Y \sim \mathcal{N}(28, 8.4)$ .

The spread of  $\sigma^2$  increases as  $p$  approaches 0.5 and decreases as  $p$  moves away from 0.5. The centre of the normal curve  $\mu$  moves to the right (i.e. increases) as  $p$  increases.
4.  $X \sim \mathcal{B}(50, \frac{1}{2})$ ,  $\mu = 50 \times 0.5 = 25$  and  $\sigma^2 = 50 \times 0.5 \times (1 - 0.5) = 12.5$ . Therefore,  $Y \sim \mathcal{N}(25, 12.5)$ .
5.  $X \sim \mathcal{B}(60, \frac{1}{6})$ ,  $\mu = 60 \times \frac{1}{6} = 10$  and  $\sigma^2 = 60 \times \frac{1}{6} \times (1 - \frac{5}{6}) = \frac{25}{3}$ . Therefore,  $Y \sim \mathcal{N}(10, \frac{25}{3})$ .

## Continuity Correction

### Exercise 16

1.  $\mu = 20 \times 0.4 = 8$  and  $\sigma^2 = 20 \times 0.4 \times (1 - 0.4) = 4.8$ . Therefore,  $Y \sim \mathcal{N}(8, 4.8)$ .
  - i.  $P(X \leq 10) \simeq P(Y \leq 10.5)$ ;
  - ii.  $P(X \geq 5) \simeq P(Y \geq 4.5)$ ;
  - iii.  $P(13 \leq X \leq 17) \simeq P(12.5 \leq Y \leq 17.5)$ ;
  - iv.  $P(X < 2) = P(X \leq 1) \simeq P(Y \leq 1.5)$ ;
  - v.  $P(X > 15 \text{ or } X \leq 5) = P(X \geq 16) + P(X \leq 5) \simeq P(Y \geq 15.5) + P(Y \leq 5.5)$ .
2.  $X \sim \mathcal{B}(100, \frac{1}{2})$ ,  $\mu = 100 \times \frac{1}{2} = 50$  and  $\sigma^2 = 100 \times \frac{1}{2} \times (1 - \frac{1}{2}) = 25$ . Therefore,  $Y \sim \mathcal{N}(50, 25)$ . Therefore,  $P(X < 5) = P(X \leq 4) \simeq P(Y \leq 4.5)$ .
3.  $X \sim \mathcal{B}(100, \frac{1}{3})$ ,  $\mu = 100 \times \frac{1}{3} = \frac{100}{3}$  and  $\sigma^2 = 100 \times \frac{1}{3} \times (1 - \frac{1}{3}) = \frac{200}{9}$ . Therefore,  $Y \sim \mathcal{N}(\frac{100}{3}, \frac{200}{9})$ . Therefore,  $P(X < 10) = P(X \leq 9) \simeq P(Y \leq 9.5)$ .
4.  $X \sim \mathcal{B}(200, \frac{1}{6})$ ,  $\mu = 200 \times \frac{1}{6} = \frac{100}{3}$  and  $\sigma^2 = 200 \times \frac{1}{6} \times (1 - \frac{1}{6}) = \frac{250}{9}$ . Therefore,  $Y \sim \mathcal{N}(\frac{100}{3}, \frac{250}{9})$ . Therefore,  $P(120 \leq X \leq 150) \simeq P(119.5 \leq Y \leq 150.5)$ .

## 6 Binomial Distribution Tables

Values are  $P(X \leq r)$  where  $X \sim \mathcal{B}(n, p)$

n = 2

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100
1	0.9900	0.9600	0.9100	0.8400	0.7500	0.6400	0.5100	0.3600	0.1900
2	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 3

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.7290	0.5120	0.3430	0.2160	0.1250	0.0640	0.0270	0.0080	0.0010
1	0.9720	0.8960	0.7840	0.6480	0.5000	0.3520	0.2160	0.1040	0.0280
2	0.9990	0.9920	0.9730	0.9360	0.8750	0.7840	0.6570	0.4880	0.2710
3	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 4

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.6561	0.4096	0.2401	0.1296	0.0625	0.0256	0.0081	0.0016	0.0001
1	0.9477	0.8192	0.6517	0.4752	0.3125	0.1792	0.0837	0.0272	0.0037
2	0.9963	0.9728	0.9163	0.8208	0.6875	0.5248	0.3483	0.1808	0.0523
3	0.9999	0.9984	0.9919	0.9744	0.9375	0.8704	0.7599	0.5904	0.3439
4	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 5

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.5905	0.3277	0.1681	0.0778	0.0313	0.0102	0.0024	0.0003	0.0000
1	0.9185	0.7373	0.5282	0.3370	0.1875	0.0870	0.0308	0.0067	0.0005
2	0.9914	0.9421	0.8369	0.6826	0.5000	0.3174	0.1631	0.0579	0.0086
3	0.9995	0.9933	0.9692	0.9130	0.8125	0.6630	0.4718	0.2627	0.0815
4	1.0000	0.9997	0.9976	0.9898	0.9688	0.9222	0.8319	0.6723	0.4095
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 6

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.5314	0.2621	0.1176	0.0467	0.0156	0.0041	0.0007	0.0001	0.0000
1	0.8857	0.6554	0.4202	0.2333	0.1094	0.0410	0.0109	0.0016	0.0001
2	0.9842	0.9011	0.7443	0.5443	0.3438	0.1792	0.0705	0.0170	0.0013
3	0.9987	0.9830	0.9295	0.8208	0.6563	0.4557	0.2557	0.0989	0.0159
4	0.9999	0.9984	0.9891	0.9590	0.8906	0.7667	0.5798	0.3446	0.1143
5	1.0000	0.9999	0.9993	0.9959	0.9844	0.9533	0.8824	0.7379	0.4686
6	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

## Binomial Distribution Tables Continued

Values are  $P(X \leq r)$  where  $X \sim \mathcal{B}(n, p)$

n = 7

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.4783	0.2097	0.0824	0.0280	0.0078	0.0016	0.0002	0.0000	0.0000
1	0.8503	0.5767	0.3294	0.1586	0.0625	0.0188	0.0038	0.0004	0.0000
2	0.9743	0.8520	0.6471	0.4199	0.2266	0.0963	0.0288	0.0047	0.0002
3	0.9973	0.9667	0.8740	0.7102	0.5000	0.2898	0.1260	0.0333	0.0027
4	0.9998	0.9953	0.9712	0.9037	0.7734	0.5801	0.3529	0.1480	0.0257
5	1.0000	0.9996	0.9962	0.9812	0.9375	0.8414	0.6706	0.4233	0.1497
6	1.0000	1.0000	0.9998	0.9984	0.9922	0.9720	0.9176	0.7903	0.5217
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 8

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.4305	0.1678	0.0576	0.0168	0.0039	0.0007	0.0001	0.0000	0.0000
1	0.8131	0.5033	0.2553	0.1064	0.0352	0.0085	0.0013	0.0001	0.0000
2	0.9619	0.7969	0.5518	0.3154	0.1445	0.0498	0.0113	0.0012	0.0000
3	0.9950	0.9437	0.8059	0.5941	0.3633	0.1737	0.0580	0.0104	0.0004
4	0.9996	0.9896	0.9420	0.8263	0.6367	0.4059	0.1941	0.0563	0.0050
5	1.0000	0.9988	0.9887	0.9502	0.8555	0.6846	0.4482	0.2031	0.0381
6	1.0000	0.9999	0.9987	0.9915	0.9648	0.8936	0.7447	0.4967	0.1869
7	1.0000	1.0000	0.9999	0.9993	0.9961	0.9832	0.9424	0.8322	0.5695
8	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 9

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.3874	0.1342	0.0404	0.0101	0.0020	0.0003	0.0000	0.0000	0.0000
1	0.7748	0.4362	0.1960	0.0705	0.0195	0.0038	0.0004	0.0000	0.0000
2	0.9470	0.7382	0.4628	0.2318	0.0898	0.0250	0.0043	0.0003	0.0000
3	0.9917	0.9144	0.7297	0.4826	0.2539	0.0994	0.0253	0.0031	0.0001
4	0.9991	0.9804	0.9012	0.7334	0.5000	0.2666	0.0988	0.0196	0.0009
5	0.9999	0.9969	0.9747	0.9006	0.7461	0.5174	0.2703	0.0856	0.0083
6	1.0000	0.9997	0.9957	0.9750	0.9102	0.7682	0.5372	0.2618	0.0530
7	1.0000	1.0000	0.9996	0.9962	0.9805	0.9295	0.8040	0.5638	0.2252
8	1.0000	1.0000	1.0000	0.9997	0.9980	0.9899	0.9596	0.8658	0.6126
9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

# Binomial Distribution Tables Continued

Values are  $P(X \leq r)$  where  $X \sim \mathcal{B}(n, p)$

n = 10

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.3487	0.1074	0.0282	0.0060	0.0010	0.0001	0.0000	0.0000	0.0000
1	0.7361	0.3758	0.1493	0.0464	0.0107	0.0017	0.0001	0.0000	0.0000
2	0.9298	0.6778	0.3828	0.1673	0.0547	0.0123	0.0016	0.0001	0.0000
3	0.9872	0.8791	0.6496	0.3823	0.1719	0.0548	0.0106	0.0009	0.0000
4	0.9984	0.9672	0.8497	0.6331	0.3770	0.1662	0.0473	0.0064	0.0001
5	0.9999	0.9936	0.9527	0.8338	0.6230	0.3669	0.1503	0.0328	0.0016
6	1.0000	0.9991	0.9894	0.9452	0.8281	0.6177	0.3504	0.1209	0.0128
7	1.0000	0.9999	0.9984	0.9877	0.9453	0.8327	0.6172	0.3222	0.0702
8	1.0000	1.0000	0.9999	0.9983	0.9893	0.9536	0.8507	0.6242	0.2639
9	1.0000	1.0000	1.0000	0.9999	0.9990	0.9940	0.9718	0.8926	0.6513
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 11

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.3138	0.0859	0.0198	0.0036	0.0005	0.0000	0.0000	0.0000	0.0000
1	0.6974	0.3221	0.1130	0.0302	0.0059	0.0007	0.0000	0.0000	0.0000
2	0.9104	0.6174	0.3127	0.1189	0.0327	0.0059	0.0006	0.0000	0.0000
3	0.9815	0.8389	0.5696	0.2963	0.1133	0.0293	0.0043	0.0002	0.0000
4	0.9972	0.9496	0.7897	0.5328	0.2744	0.0994	0.0216	0.0020	0.0000
5	0.9997	0.9883	0.9218	0.7535	0.5000	0.2465	0.0782	0.0117	0.0003
6	1.0000	0.9980	0.9784	0.9006	0.7256	0.4672	0.2103	0.0504	0.0028
7	1.0000	0.9998	0.9957	0.9707	0.8867	0.7037	0.4304	0.1611	0.0185
8	1.0000	1.0000	0.9994	0.9941	0.9673	0.8811	0.6873	0.3826	0.0896
9	1.0000	1.0000	1.0000	0.9993	0.9941	0.9698	0.8870	0.6779	0.3026
10	1.0000	1.0000	1.0000	1.0000	0.9995	0.9964	0.9802	0.9141	0.6862
11	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

n = 12

$p =$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$r = 0$	0.2824	0.0687	0.0138	0.0022	0.0002	0.0000	0.0000	0.0000	0.0000
1	0.6590	0.2749	0.0850	0.0196	0.0032	0.0003	0.0000	0.0000	0.0000
2	0.8891	0.5583	0.2528	0.0834	0.0193	0.0028	0.0002	0.0000	0.0000
3	0.9744	0.7946	0.4925	0.2253	0.0730	0.0153	0.0017	0.0001	0.0000
4	0.9957	0.9274	0.7237	0.4382	0.1938	0.0573	0.0095	0.0006	0.0000
5	0.9995	0.9806	0.8822	0.6652	0.3872	0.1582	0.0386	0.0039	0.0001
6	0.9999	0.9961	0.9614	0.8418	0.6128	0.3348	0.1178	0.0194	0.0005
7	1.0000	0.9994	0.9905	0.9427	0.8062	0.5618	0.2763	0.0726	0.0043
8	1.0000	0.9999	0.9983	0.9847	0.9270	0.7747	0.5075	0.2054	0.0256
9	1.0000	1.0000	0.9998	0.9972	0.9807	0.9166	0.7472	0.4417	0.1109
10	1.0000	1.0000	1.0000	0.9997	0.9968	0.9804	0.9150	0.7251	0.3410
11	1.0000	1.0000	1.0000	1.0000	0.9998	0.9978	0.9862	0.9313	0.7176
12	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

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