Complex Analysis

Complex number

1. Sums and Products
2. Basic Algebraic Properties
3. Further Properties
4. Vectors and Moduli
5. Complex Conjugates
6. Exponential Form
7. Products and Powers in Exponential Form
8. Arguments of Products and Quotients
9. Roots of Complex Numbers
10. Examples
11. Regions in the Complex Plane

Polar and Exponential Forms of a Complex Number (53min) by MathforThought
→ Explained basics in Details (*watch carefully*). Good, Concept with Problems. But It not solve Power of Complex with De Moivre's Formula
Write $a + bi$ in Polar form kind of problems covered. (8 Examples)

Complex Numbers : Modulus and Argument | ExamSolutions
→ *Good explaination, for signs for argiment in all different quadrants*

Finding the Principal Argument of Complex Numbers – (With Examples)

Argument of Z | Principal value of Argument by Nidhi Mishra

Quadrant	Sign of x & y	Arg(z)
I	x, y>0	$\tan^{-1}\frac{y}{x}$
II	x<0, y>0	$\pi - \tan^{-1}\left\|\frac{y}{x}\right\|$
III	x, y < 0	$\pi + \tan^{-1}\left\|\frac{y}{x}\right\|$
IV	x>0, y<0	$2\pi - \tan^{-1}\left\|\frac{y}{x}\right\|$

Complex number - Wikipedia
Complex Analysis L01: Overview & Motivation, Complex Arithmetic, Euler's Formula & Polar Coordinates by Steve Brunton
→ Arithmetic
→ Representation of Complex number in Cartisian and Polar from
→ Euler's formula - Wikipedia

Cartesian Form (Rectangular Form) :
A complex number in Cartesian form is written as:

$$z = x + yi$$

$Re(z) = x$: real part , $Im(z) = y$: imaginary part.

Polar Form
A complex number in polar form is represented using its magnitude (also called modulus) and angle (also called argument or phase):

$$z = r(\cos\theta + i\sin\theta)$$

Or using Euler's formula:

$$z = re^{i\theta}$$

Where :
$r = |z| = \sqrt{x^2 + y^2}$ is the modulus
$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ is the argument (angle in radians or degrees)

Complex Analysis L02: Euler's formula, one of the most important formulas in all of mathematics
→ Derivation of Euler's Formula, De Moivre's formula

Maclaurin (Taylor) series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \implies e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$i^0 = 0,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$$

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + \cdots\right)$$

$e^{i\theta} =$ Real part + i(Imaginary part)

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Write De Moivre's formula

$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

$r^n(\cos n\theta + i\sin n\theta)\ I\ know\ this,\ its\ Eulers\ formula$

$but\ what\ this\ is,\ How\ i\ get\ this\ ?$

$r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right)$

[Complex Analysis L03: Functions of a complex variable, f(z)]

De Moivre's Theorem

Identity theorem

The Identity Theorem is one of the most powerful results in Complex Analysis.
It tells us that if two analytic functions agree on even a small part of their domain, they must be the same everywhere on that connected region.

🧩 Statement (Standard Form):
Let $f(z)$ and $g(z)$ be analytic functions on a connected open set $D \subset C$.
If there exist a subjset $S \subset D$ such that

$$f(z) = g(z)\ \forall z \in S$$

and if S has a limit point in D, then

$$f(z) = g(z)\ \forall z \in D$$

📘 Simplified Version:
If an analytic function is zero on a set that has a limit point inside its domain,
then the function is identically zero in the entire connected region.
$f(z)$ analytic on D , $f(z_0) = 0$ for infinitely many $z_0 \in D$. and those zeros accumulate to some point $a \in D$. then $f(z) = 0$ on D.

🧠 Intuitive Meaning:
Analytic functions are rigid:
you can't "tweak" their values on even a small set without changing the entire function.
If two analytic functions agree on a dense subset or on a small arc, they must agree

everywhere (no freedom left).

Singularity

A singularity of a complex function $f(z)$ is a point where the function fails to be analytic — but is analytic everywhere else nearby.

Formally, $z = a$ is a singular point if $f(z)$ is not analytic at $z = a$,
but analytic in some punctured neighborhood around it (i.e., analytic for $0 < |z - a| < r$).

There are three main types of isolated singularities (those separated from others).
(A) Removable Singularity
(B) Pole (or Non-Removable Singularity)
(C) Essential Singularity

(A) Removable Singularity
A point where $f(z)$ is not defined or not analytic, but we can define it properly to make the function analytic.
eg : $f(z) = \dfrac{\sin z}{z},$ at $z = 0,\ f(z)$ is undefined.
But we can take a limit: $\lim\limits_{z \to 0} \dfrac{\sin z}{z} = 1.$
So if we define $f(0) = 1,$ the function becomes analytic at $z = 0.$
Hence, $z = 0$ is a removable singularity.

(B) Pole (or Non-Removable Singularity)
→ A point where $f(z) \to \infty$ as $z \to a$

Cauchy–Riemann equations

🧩 1. Context

In real analysis, for a function $f(x)$ being differentiable means the limit $\lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{}$ exist.
In complex analysis, for a function $f(z),$ where $z = x + iy$ we similarly define

$f'(z) = \lim_{\Delta z \to 0} \dfrac{f(z + \Delta z) - f(z)}{\Delta z}$ but here Δz can approach 0 from any direction in the complex plane — not just from the real axis.
So for the limit to exist, the result must be independent of the direction of approach.

$f(z)$ is differentiable everywhere → analytic (holomorphic) everywhere.

🧩 LEVEL 1 — Basic Recognition (Feel the Structure)
$f(z) = x^2 + iy^2$, $u = x^2, \ v = y^2$.

Cauchy’s Integral Formula

If f is analytic inside and on a simple closed contour C, and a is a point inside C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Here :

- $f(z)$: analytic function (smooth, differentiable everywhere in region)
- C : closed contour (like a circle)
- a : point inside C.

Assignment-1

Complex conjugate - Wikipedia

Polar coordinate system - Wikipedia
Watch Video by Nidhi Mishra

→ De Moivre's formula - Wikipedia
Learn De Moivre's Formula and this problems is nothing.

$1.\ For\ any\ z,\ w \in \mathbb{C},\ show\ that$
$(a)\ \overline{z+w} = \overline{z} + \overline{w}$

$(b)\ \overline{z \cdot w} = \overline{z} \cdot \overline{w}$
$(c)\ \overline{\overline{z}} = z$
$(d)\ |\overline{z}| = |z|$
$(e)\ |zw| = |z||w|$
Learn about complex congugates, everything is easy.

$2.\ Show\ that$
$(a)\ |z + w|^2 = |z|^2 + |w|^2 + 2Re(z\overline{w})$
$(b)\ |z + w|^2 + |z - w|^2 = 2\left(|z|^2 + |w|^2\right)$
$(c)\ |z + w| = |z| + |w|$
$if\ and\ only\ if\ either\ zw = 0\ or\ z = cw\ for\ some\ positive\ real\ number\ c$
Learn about complex congugates, everything is easy. For 3rd, you have to check for conditions.

$3.\ (a)\ Let\ w = \dfrac{-1 + i\sqrt{3}}{2}.\ Determine:\ \overline{w}, w^2, w^{-1}.\ Show\ that\ 1 + w + w^2 = 0$
$(b)\ Let\ \alpha\ be\ any\ of\ the\ n-th\ roots\ of\ unity\ except\ 1.$
$Show\ that\ 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0$

$4.\ Express\ in\ polar\ form\ :$
$(a)\ 1 + i$
$(b)\ -1 - i$
$(c)\ \sqrt{3} + i$
$(d)\ 1 + \cos\theta + i\sin\theta$
$Determine\ the\ value\ of\ \arg\left(z^2\right)\ in\ each\ of\ the\ cases.$

$5.\ Let\ z\ be\ a\ nonzero\ complex\ number\ and\ n\ a\ positive\ integer.\ If\ z = r(\cos\theta + i\sin\theta),$
$show\ that\ z^{-n} = r^{-n}(\cos n\theta - \sin n\theta)$

$6.\ Find\ the\ roots\ of\ each\ of\ the\ following\ in\ the\ form\ x + iy.$
$Indicate\ the\ principal\ root$
$(a)\ \sqrt{2i}$
$(b)\ (-1)^{1/3}$

$(c)\ (-16)^{1/4}$

Find the roots in the form $x+iy$. *Indicate the principal root* $\sqrt{2i}$

Find for $(-1)^{1/3}$

7. *Determine the values of the following*

$(a)\ (1+i)^{20}-(1-i)^{20}$

$(b)\ \cos\frac{\pi}{4}+i\cos\frac{3\pi}{4}+i^n\cos\frac{2n+1}{4}\pi+\ \cdots\ +i^{40}\cos\frac{81}{4}\pi$

→ Express each in polar (exponential) form

→ Apply De Moivre’s Theorem

→ Simplify exponents, trigonometric values

→ Substitute values and Compute

$$\cos\frac{\pi}{4}+i\cos\frac{3\pi}{4}+i^n\cos\frac{2n+1}{4}\pi+\ \cdots\ +i^{40}\cos\frac{81}{4}\pi$$

8. *Find the roots of* $z^4+4=0$. *Use these roots to factor* z^4+4 *as a product of two quadratics with real coefficients*

9. *Discuss the convergence of the following sequences*

a. $\left(z^n\right)$

b. $\left(\frac{z^n}{n!}\right)$

c. $\left(i^n\sin\frac{n\pi}{4}\right)$

d. $\left(\frac{1}{n}+i^n\right)$

$$\left(\frac{1}{n}+i^n\right)$$

Assignment 2

1. Let $z = x + iy$ and $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$.
Write $f(z)$ as a function of z and $\overline{z}$.

2. Verify Cauchy-Riemann equation for z^2 and z^3

3. Using the relations $x = \dfrac{z + \overline{z}}{2},\ y = \dfrac{z - \overline{z}}{2i}$ and the chain rule show that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right);$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

4. Let $z, w \in \mathbb{C},\ |z|,\ |w| < 1$ and $\overline{z}w = 1$. Prove that $\dfrac{|w - z|}{|1 - \overline{w}z|} < 1$. Further, show that the equality holds if either $|z| = 1$ or $|w| = 1$.

5. Determine all $z \in \mathbb{C}$ for which each of the following power series is convergent.

a. $\sum \dfrac{z^n}{n^2}$ b. $\sum \dfrac{z^n}{n!}$ c. $\sum \dfrac{z^n}{2^n}$ d. $\sum \dfrac{1}{2^n}\dfrac{1}{z^n}$

6. Show that the CR-equations in polar form are given by: $u_r = \dfrac{1}{r}v_\theta$ and $u_\theta = -rv_r$

7.
(a). The hyperbolic functions $\cosh z$ and $\sinh z$ are defined as $\cos iz$ and $-i \sin iz$, respectively.
Show that $\cosh^2 z - \sinh^2 z = 1$
(b). Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$ Conclude that $\cos z$ is not bounded in $\mathbb{C}$.
(c). Show that $\cos z = 0 \iff z = (2n+1)\dfrac{\pi}{2}$ for $n \in \mathbb{Z}$.

$Show\ that\ \cos z = 0 \iff z = (2n+1)\dfrac{\pi}{2}\ for\ n \in \mathbb{Z}$

8. Find the roots of the equation $\sin z = 2$

$Find\ the\ roots\ of\ the\ equation\ \sin z = 2$

Assignment 4

Integrate $\frac{z^2}{z^4 - 1}$ counter-clockwise around the circle $|z + 1| = 1$

$Integrate\ \frac{z^2}{z^4 - 1} counter - clockwise\ around\ the\ circle\ |z + 1| = 1$