

Linear Estimation

Book : Linear Algebra and Linear Models by R.B. Bapat (2nd Edition)

[Generalized inverse - Wikipedia](#)

[Left and Right Inverses; Pseudoinverse by Gilbert Strang](#) (super-smooth explanation, but don't go without brushing up those concepts. I understood because i already had intuition of those concepts)

[Generalized inverse of matrices by Prof. Chandra Murthy \[IISc B\] NPTELHRD \(YouTube\)](#)

→ (I don't understand it, I think its because i'm learning it for first time)

[Moore–Penrose inverse - Wikipedia](#)

[Pseudoinverses by David Shirokoff](#)

[Singular value decomposition - Wikipedia](#)

[Singular Value Decomposition \(SVD\) - Animation](#)

$A_{m \times n}$

$\text{rank}(A) = \min(m, n)$

Inverse of a Matrix :

For a square matrix A , its inverse A^{-1} satisfies :

$$AA^{-1} = A^{-1}A = I$$

Left Inverse :

Matrix $A_{m \times n}$ is tall : $m \times n$ with $m > n$

Full column rank ($\rho(A) = n$) → injective mapping ($\mathbb{R}^n \rightarrow \mathbb{R}^m$)

We want $L_{n \times m}$ such that: $LA = I_n$

Right Inverse :

Matrix A is wide : $m \times n$ with $m < n$

Full row rank ($\rho(A) = m$) → surjective mapping ($\mathbb{R}^n \rightarrow \mathbb{R}^m$)

We want R such that: $AR = I_m$

Inverse in

$((A^T A)^{-1} A^T) A = I$, where $(A^T A)^{-1} A^T$ is left inverse

$A(A^T (AA^T)^{-1}) = I$, where $A^T (AA^T)^{-1}$ is right inverse

$$A^{-1}A = AA^{-1} = I$$

Tall ($m \geq n$, full col rank) \Rightarrow injective \Rightarrow left inverse.

Wide ($m \leq n$, full row rank) \Rightarrow surjective \Rightarrow right inverse.

Definition 1 : Let A be an $m \times n$ matrix. A matrix G of order $n \times m$ is said to be a generalized inverse (or a g -inverse) of A if $AGA = A$

2. Generalised Inverses, Moore-Penrose Inverse

- i. Left and right inverses of a matrix - G-inverse
- ii. Minimum Norm and Least Squares g-Inverse
- iii. Moore-Penrose inverse. [3 Lectures]

The **pseudoinverse** (also called the **Moore–Penrose inverse**) is a generalization of the inverse of a matrix.

[Generalized inverse - Wikipedia](#)

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Theorem 2 :

Let A, G be matrices of order $m \times n$ and $n \times m$ respectively. Then the following conditions are equivalent:

I. G is a g -inverse of A .

II. For any $y \in \mathcal{C}(A)$, $x = Gy$ is a solution of $Ax = y$

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[Projection Matrix Properties - Projection, Part 1 by Sam Levey](#) (Good)

[Orthogonal Projection Formulas \(Least Squares\) - Projection, Part 2 by Sam Levey](#)

(Good+Great)

[The geometric view on orthogonal projections by Dr. Trefor Bazett](#) (untill middle)

[Projection into Subspaces by Nikola Kamburov \(MIT\)](#)

[Why a "least squares regression line" is called that by COCCmath](#)

\rightarrow One of the best visual Visual interpretation.

Assignment 1

Rank Factorization :

[Crash course on Rank Factorisation of a matrix](#) by Arnab Chakraborty

I wasn't able to understand prove of theorem. I watched this video and boom, I got very good understanding of Proof. So my guide is, If You don't understand it, try to explain yourself by simplex example.

[Example of matrix factorization](#) by MH1200

Same as Arnab Chakraborty's solution but in little detail at Factorization step.

⇒ My Notes (I'm writing what i've understood, if i am wrong, I'll have to modify it)

$$A_{m \times n} = B_{m \times r} \cdot C_{r \times n}$$

→ $\rho(A) = r$ i.e. r is rank of A and same for A and B .

[Moore Penrose pseudo inverse of a matrix](#) by Sreekanth Reddy

→ 2 examples explained in video. Watch it carefully you'll learn some concepts its just 12min long.

[Pseudo inverse/ moore Penrose inverse | inverse of rectangular matrix](#)

→ Note the 100% accurate as i have knowledge about inverses but it was good till doing derivation kind of think and also some examples. Good Girl

1. Rank Factorization Theorem

→ If you want to build intuition behind it watch [Arnab Chakraborty's](#) Video.

2. Let $A \in \mathbb{R}^{n \times n}$ be partitioned as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where A_{11} is $r \times r$ block with

$\text{rank}(A_{11}) = r$ and $\text{rank}(A) = r$.

then there exist a matrix $X \in \mathbb{R}^{r \times (n-r)}$ such that $A_{12} = A_{11}X$ and $A_{22} = A_{21}X$, and consequently $A_{22} = A_{21}A_{11}^{-1}A_{12}$.

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3. Find two different g -inverse of

$$\begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 0 & -2 & 4 \\ -1 & 1 & 1 & 3 \\ -2 & 2 & 2 & 6 \end{pmatrix}$$

[Calculating Generalized Inverse Quickly in 4 easy steps](#) by Vidur Walia

Method : non-singular submatrix of highest order

4. Find the minimum norm solution of the system of equations

$$2x + y - z = 1$$

$$x - 2y + z = -2$$

$$x + 3y - 2z = 3$$

$$\begin{pmatrix} 2 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & 3 & -2 \end{pmatrix}$$

5. Find the Moore–Penrose inverse of $\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$.

[Singular value decomposition](#) - Wikipedia

[Computing the Singular Value Decomposition](#) by Ben Harris

→ I'm not able to understand it. I'm keeping it if i understand it in future.

[Pseudoinverses](#) by David Shirokoff

→ Solved one easy problem and explained concept.

honestly I did not able to understand it right now since I think i have to understand SVD first.

Assignment 2

Assignment 3

1. Consider the linear model $\mathbb{E}(y_1) = \beta_1 + \beta_2$, $\mathbb{E}(y_2) = 2\beta_1 - \beta_2$, $\mathbb{E}(y_3) = \beta_1 - \beta_2$,

where y_1, y_2, y_3 are uncorrelated with a common variance σ^2

1.1 Find two different linear functions of y_1, y_2, y_3 that are unbiased for β_1 . Determine their variances and the covariance between the two.

1.2 Find two linear functions that are both unbiased for β_2 and are uncorrelated

1.3 Write the model in terms of the new parameters $\theta_1 = \beta_1 + 2\beta_2$, $\theta_2 = \beta_1 - 2\beta_2$

2. Consider the model $\mathbb{E}(y_1) = 2\beta_1 - \beta_2 - \beta_3$, $\mathbb{E}(y_2) = \beta_2 - \beta_4$, $\mathbb{E}(y_3) = \beta_2 - \beta_3 - 2\beta_4$ with the usual assumptions. Describe the estimable functions.

3. Consider the model $\mathbb{E}(y_1) = \beta_1 + \beta_2$, $\mathbb{E}(y_2) = \beta_1 - \beta_2$, $\mathbb{E}(y_3) = \beta_1 + 2\beta_2$ with the usual assumptions. Obtain the BLUE of $2\beta_1 + \beta_2$ and find its variance.

4. Consider the model $\mathbb{E}(y_1) = 2\beta_1 + \beta_2$, $\mathbb{E}(y_2) = \beta_1 - \beta_2$, $\mathbb{E}(y_3) = \beta_1 + \alpha\beta_2$ with the usual assumptions. Determine α such that the BLUE of β_1 and β_2 are uncorrelated.

