Complex Analysis

Final Note :
Now I'm going for End-Sem after an hour. Remember one thing, Don't Rush. My brain works solve, If i rush then i'll make mistake.
In Quiz-1 I made same mistake.
Now when i looked that paper so almost 5 marks problems i could be able to solve.
Quiz-1 : 2/20

And When i showed Quiz-2 atleast 10 marks problems i could attempt.
Quiz-2 : 0/20.

So lesson is, Don't rush.
Think deeply and then write peacfully. Don't run for 100% marks, go with 100% accuracy.
Again, My brian make silly mistake of taking $\cos 0 = 0$. so, Never-Ever Rush.

All the Best.

Complex number

1. Sums and Products
2. Basic Algebraic Properties
3. Further Properties
4. Vectors and Moduli
5. Complex Conjugates
6. Exponential Form
7. Products and Powers in Exponential Form
8. Arguments of Products and Quotients
9. Roots of Complex Numbers
10. Examples
11. Regions in the Complex Plane

Polar and Exponential Forms of a Complex Number (53min) by MathforThought
→ Explained basics in Details (*watch carefully*). Good, Concept with Problems. But It not solve Power of Complex with De Moivre's Formula
Write $a + bi$ in Polar form kind of problems covered. (8 Examples)

Complex Numbers : Modulus and Argument | ExamSolutions
→ *Good explaination, for signs for argiment in all different quadrants*

Finding the Principal Argument of Complex Numbers – (With Examples)

Argument of Z | Principal value of Argument by Nidhi Mishra

Quadrant	Sign of x & y	Arg(z)
I	x, y>0	$\tan^{-1}\frac{y}{x}$
II	x<0, y>0	$\pi - \tan^{-1}\left\|\frac{y}{x}\right\|$
III	x, y < 0	$\pi + \tan^{-1}\left\|\frac{y}{x}\right\|$
IV	x>0, y<0	$2\pi - \tan^{-1}\left\|\frac{y}{x}\right\|$

Complex number - Wikipedia
Complex Analysis L01: Overview & Motivation, Complex Arithmetic, Euler's Formula & Polar Coordinates by Steve Brunton
→ Arithmetic
→ Representation of Complex number in Cartisian and Polar from
→ Euler's formula - Wikipedia

Cartesian Form (Rectangular Form) :
A complex number in Cartesian form is written as:

$$z = x + yi$$

$Re(z) = x$: real part , $Im(z) = y$: imaginary part.

Polar Form
A complex number in polar form is represented using its magnitude (also called modulus) and angle (also called argument or phase):

$$z = r(\cos\theta + i\sin\theta)$$

Or using Euler's formula:

$$z = re^{i\theta}$$

Where :
$r = |z| = \sqrt{x^2 + y^2}$ is the modulus
$\theta = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$ is the argument (angle in radians or degrees)

[Complex Analysis L02: Euler's formula, one of the most important formulas in all of mathematics]()
→ Derivation of Euler's Formula, De Moivre's formula

Maclaurin (Taylor) series expansion

$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \implies e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$

$i^0 = 0,\ i^1 = i,\ i^2 = -1,\ i^3 = -i,\ i^4 = 1$

$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + i\frac{\theta^5}{5!} + \cdots\right)$

$e^{i\theta} =$ Real part + i(Imaginary part)

$e^{i\theta} = \cos\theta + i\sin\theta$

Write De Moivre's formula

$(r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta)$

$r^n(\cos n\theta + i\sin n\theta)\ I\ know\ this,\ its\ Eulers\ formula$

$but\ what\ this\ is,\ How\ i\ get\ this\ ?$

$r^{\frac{1}{n}}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right)$

[Complex Analysis L03: Functions of a complex variable, f(z)]()

De Moivre's Theorem

Identity theorem

The Identity Theorem is one of the most powerful results in Complex Analysis.
It tells us that if two analytic functions agree on even a small part of their domain, they must be the same everywhere on that connected region.

🧩 Statement (Standard Form):

Let $f(z)$ and $g(z)$ be analytic functions on a connected open set $D \subset C$.

If there exist a subjset $S \subset D$ such that
$$f(z) = g(z) \ \forall z \in S$$
and if S has a limit point in D, then
$$f(z) = g(z) \ \forall z \in D$$

📘 Simplified Version:
If an analytic function is zero on a set that has a limit point inside its domain,
then the function is identically zero in the entire connected region.
$f(z)$ analytic on D , $f(z_0) = 0$ for infinitely many $z_0 \in D$. and those zeros accumulate to some point $a \in D$. then $f(z) = 0$ on D.

🧠 Intuitive Meaning:
Analytic functions are rigid:
you can't "tweak" their values on even a small set without changing the entire function.
If two analytic functions agree on a dense subset or on a small arc, they must agree everywhere (no freedom left).

Singularity

A singularity of a complex function $f(z)$ is a point where the function fails to be analytic — but is analytic everywhere else nearby.

Formally, $z = a$ is a singular point if $f(z)$ is not analytic at $z = a$,
but analytic in some punctured neighborhood around it (i.e., analytic for $0 < |z - a| < r$).

There are three main types of isolated singularities (those separated from others).
(A) Removable Singularity
(B) Pole (or Non-Removable Singularity)
(C) Essential Singularity

(A) Removable Singularity
A point where $f(z)$ is not defined or not analytic, but we can define it properly to make the function analytic.
eg : $f(z) = \dfrac{\sin z}{z}$, at $z = 0$, $f(z)$ is undefined.
But we can take a limit: $\lim\limits_{z \to 0} \dfrac{\sin z}{z} = 1.$
So if we define $f(0) = 1$, the function becomes analytic at $z = 0$.
Hence, $z = 0$ is a removable singularity.

(B) Pole (or Non-Removable Singularity)
$\rightarrow$ A point where $f(z) \rightarrow \infty$ as $z \rightarrow a$

Cauchy–Riemann equations

Cauchy–Riemann equations - Wikipedia

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = v_y \qquad u_y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -v_x$$

Simple example
$f(z) = z^2$, $z = x + iy$
$f(z) = (x + iy)^2 = x^2 - y^2 + 2ixy$
The real part $u(x, y)$ and the imaginary part $v(x, y)$ are
$u(x, y) = x^2 - y^2$
$v(x, y) = 2xy$

their partial derivatives are

$$u(x, y) = x^2 - y^2$$
$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2 - y^2\right) = 2x$$
$$u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(x^2 - y^2\right) = -2y$$

$$v(x, y) = 2xy$$
$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2xy) = 2y$$
$$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2xy) = 2x$$

Verify Cauchy-Riemann equation

🧩 LEVEL 1 — Basic Recognition (Feel the Structure)
$f(z) = x^2 + iy^2$, $u = x^2,\ v = y^2$.

Cauchy's Integral Formula

If f is analytic inside and on a simple closed contour C, and a is a point inside C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}dz$$

Here :

- $f(z)$: analytic function (smooth, differentiable everywhere in region)
- C : closed contour (like a circle)
- a : point inside C.

Cauchy's Residue Theorem

Residue theorem - Wikipedia

Let $f(z)$ be analytic in a simply connected domain except at isolated singularities $z_1, z_2, z_3, \ldots, z_n$.

Let C be a positively oriented simple closed contour enclosing those points.

Then

$$\int_C f(z)dz = 2\pi i\sum_{k=1}^{n} Res(f, z_k)$$

I've sovled A4-15(b)

residues were

$Res(f, i) = 1$

$Res(f, -2i) = 2$

$$\int_C f(z)dz = 2\pi i\sum_{k=1}^{n} Res(f, z_k) = 2\pi i(1+2) = 6\pi i$$

Residue

1. Simple Pole at z_0

If z_0 has a simple pole (order 1) at z_0 :

$$Res(f, z_0) = \lim_{z \to z_0}(z-z_0)f(z)$$

Shortcut for Rational Functions

if $f(z) = \dfrac{P(z)}{Q(z)},\ Q(z_0) = 0,\ Q(z_0) \neq 0$ then, $Res(f, z_0) = \dfrac{P(z)}{Q'(z)}$

2. Pole of Order n

If z_0 is a pole of order n :

$$Res(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z) \right]$$

This is the general formula for any order pole.

3. Essential Singularity or General Case

Write the Laurent series around z_0

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k$$

Residue $Res(f, z_0) = a_{-1}$, That's the coefficient of the $\frac{1}{z - z_0}$ term.

Laurent series

Assignment-1

[Complex conjugate - Wikipedia]

[Polar coordinate system - Wikipedia]
Watch Video [by Nidhi Mishra]

→ [De Moivre's formula - Wikipedia]
Learn De Moivre's Formula and this problems is nothing.

$1.\ For\ any\ z,\ w \in \mathbb{C},\ show\ that$
$(a)\ \overline{z + w} = \overline{z} + \overline{w}$
$(b)\ \overline{z \cdot w} = \overline{z} \cdot \overline{w}$
$(c)\ \overline{\overline{z}} = z$
$(d)\ |\overline{z}| = |z|$
$(e)\ |zw| = |z||w|$
Learn about complex congugates, everything is easy.

$2.\ Show\ that$

$(a)\ |z+w|^2 = |z|^2 + |w|^2 + 2Re(z\overline{w})$
$(b)\ |z+w|^2 + |z-w|^2 = 2\left(|z|^2 + |w|^2\right)$
$(c)\ |z+w| = |z| + |w|$
$if\ and\ only\ if\ either\ zw = 0\ or\ z = cw\ for\ some\ positive\ real\ number\ c$
Learn about complex congugates, everything is easy. For 3rd, you have to check for conditions.

$3.\ (a)\ Let\ w = \dfrac{-1+i\sqrt{3}}{2}.\ Determine:\ \overline{w}, w^2, w^{-1}.\ Show\ that\ 1 + w + w^2 = 0$
$(b)\ Let\ \alpha\ be\ any\ of\ the\ n-th\ roots\ of\ unity\ except\ 1.$
$Show\ that\ 1 + \alpha + \alpha^2 + ... + \alpha^{n-1} = 0$

$4.\ Express\ in\ polar\ form\ :$
$(a)\ 1+i$
$(b)\ -1-i$
$(c)\ \sqrt{3}+i$
$(d)\ 1+\cos\theta + i\sin\theta$
$Determine\ the\ value\ of\ \arg\left(z^2\right)\ in\ each\ of\ the\ cases.$

$5.\ Let\ z\ be\ a\ nonzero\ complex\ number\ and\ n\ a\ positive\ integer.\ If\ z = r(\cos\theta + i\sin\theta),$
$show\ that\ z^{-n} = r^{-n}(\cos n\theta - \sin n\theta)$

$6.\ Find\ the\ roots\ of\ each\ of\ the\ following\ in\ the\ form\ x+iy.$
$Indicate\ the\ principal\ root$
$(a)\ \sqrt{2i}$
$(b)\ (-1)^{1/3}$
$(c)\ (-16)^{1/4}$

$Find\ the\ roots\ in\ the\ form\ x+iy. Indicate\ the\ principal\ root\ \sqrt{2i}$

$Find\ for\ (-1)^{1/3}$

$7.\ Determine\ the\ values\ of\ the\ following$
$(a)\ (1+i)^{20}-(1-i)^{20}$
$(b)\ \cos\frac{\pi}{4}+i\cos\frac{3\pi}{4}+i^n\cos\frac{2n+1}{4}\pi+\ \cdots\ +i^{40}\cos\frac{81}{4}\pi$
→ Express each in polar (exponential) form
→ Apply De Moivre’s Theorem
→ Simplify exponents, trigonometric values
→ Substitute values and Compute

$$\cos\frac{\pi}{4}+i\cos\frac{3\pi}{4}+i^n\cos\frac{2n+1}{4}\pi+\ \cdots\ +i^{40}\cos\frac{81}{4}\pi$$

$8.\ Find\ the\ roots\ of\ z^4+4=0.\ Use\ these\ roots\ to\ factor\ z^4+4\ \ as\ a\ product\ of\ two\ quadratics\ with\ real\ coefficients$

$9.\ Discuss\ the\ convergence\ of\ the\ following\ sequences$
$a.\ \ \left(z^n\right)$
$b.\ \ \left(\frac{z^n}{n!}\right)$
$c.\ \ \left(i^n\sin\frac{n\pi}{4}\right)$
$d.\ \ \left(\frac{1}{n}+i^n\right)$

$$\left(\frac{1}{n}+i^n\right)$$

Assignment-2

1. Let $z=x+iy$ and $f(z)=x^2-y^2-2y+i(2x-2xy)$.
Write $f(z)$ as a function of z and $\overline{z}$.
$\Longrightarrow$
$z=x+iy,\ \overline{z}=x-iy$

$$z+\overline{z}\Longrightarrow x=\frac{z+\overline{z}}{2}\qquad z-\overline{z}\Longrightarrow y=\frac{z-\overline{z}}{i}$$

Put value of z and $\overline{z}$ you'll get these x, y values.

$f(z) = x^2 - y^2 - 2y + i(2x - 2xy).$

$f(z) = \left(\frac{z+\overline{z}}{2}\right)^2 - \left(\frac{z-\overline{z}}{2i}\right)^2 - 2\left(\frac{z-\overline{z}}{2i}\right) + i\left(2\left(\frac{z+\overline{z}}{2}\right) - 2\left(\frac{z+\overline{z}}{2}\right)\left(\frac{z-\overline{z}}{2i}\right)\right)$

Solve it, or aks LLM to solve step by step. I'm not adding here since its just arithematics of Complexes. I'm not solving it since i think solving by myself help me to understand.

2. Verify Cauchy-Riemann equation for z^2 and z^3

3. Using the relations $x = \frac{z+\overline{z}}{2},\ y = \frac{z-\overline{z}}{2i}$ and the chain rule show that

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right);$

$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$

4. Let $z, w \in \mathbb{C},\ |z|,\ |w| < 1$ and $\overline{z}w = 1$. Prove that $\frac{|w-z|}{|1-\overline{w}z|} < 1$. Further, show that the equality holds if either $|z| = 1$ or $|w| = 1$.

5. Determine all $z \in \mathbb{C}$ for which each of the following power series is convergent.

a. $\sum \frac{z^n}{n^2}$ b. $\sum \frac{z^n}{n!}$ c. $\sum \frac{z^n}{2^n}$ d. $\sum \frac{1}{2^n}\frac{1}{z^n}$

6. Show that the CR-equations in polar form are given by: $u_r = \frac{1}{r}v_\theta$ and $u_\theta = -rv_r$

7.
(a). The hyperbolic functions $\cosh z$ and $\sinh z$ are defined as $\cos iz$ and $-i\sin iz$, respectively.
Show that $\cosh^2 z - \sinh^2 z = 1$
(b). Show that $|\cos z|^2 = \cos^2 x + \sinh^2 y$ Conclude that $\cos z$ is not bounded in $\mathbb{C}$.

(c). Show that $\cos z = 0 \iff z = (2n+1)\dfrac{\pi}{2}$ for $n \in \mathbb{Z}$.

$$Show\ that\ \cos z = 0 \iff z = (2n+1)\frac{\pi}{2}\ for\ n \in \mathbb{Z}$$

8. Find the roots of the equation $\sin z = 2$

$$Find\ the\ roots\ of\ the\ equation\ \sin z = 2$$

$\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$, put $e^{iz} = w,\ e^{-iz} = \dfrac{1}{e^{iz}} = \dfrac{1}{w}$

$$\sin z = 2 \implies \frac{w - \frac{1}{w}}{2i} = 2 \implies 2i \cdot \frac{w - \frac{1}{w}}{2i} = 2i \cdot 2 \implies w - \frac{1}{w} = 4i \implies \frac{w^2 - 1}{w} = 4i$$

$$w^2 - 1 = 4iw \implies w^2 - 4iw - 1 = 0 \implies$$

Quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$w = \frac{-(-4i) \pm \sqrt{(-4i)^2 - 4(1)(-1)}}{2(1)} = \frac{4i \pm \sqrt{-16+4}}{2} = \frac{4i \pm \sqrt{-12}}{2} = \frac{4i \pm i2\sqrt{3}}{2}$$

$$= \frac{2i\left(2 \pm \sqrt{3}\right)}{2} = i\left(2 \pm \sqrt{3}\right)$$

So

$$e^{iz} = i\left(2 + \sqrt{3}\right) \quad \text{or} \quad e^{iz} = i\left(2 - \sqrt{3}\right)$$

I did not understood after it, And i don't have time left. So I'm leaving here.

11. Let R be the radius of convergence of $\sum_n a_n z^n$. For a fixed $k \in \mathbb{N}$, find the radius of convergence of

$$(a)\ \sum a_n^k z^n \qquad\qquad (b)\ \sum a_n z^{kn}$$

$(a)\ \sum a_n^k z^n$

Use the Cauchy–Hadamard formula. Put

$\limsup_{n \to \infty} |a_n|^{\frac{1}{n}}$, so $\dfrac{1}{R} = L$.

$$\limsup_{n\to\infty}|a_n^k|^{\frac{1}{n}} = \limsup_{n\to\infty}|a_n|^{\frac{k}{n}} = \left(\limsup_{n\to\infty}|a_n^k|^{\frac{1}{n}}\right)^k = L^k \to R^k$$

$(b)\ \sum a_n z^{kn}$

Put $w = z^k$. Then

$$\sum a_n z^{kn} = \sum a_n \left(z^k\right)^n = \sum a_n w^n$$

By definition this latter power series in w converges for $|w| < R$ and diverges for $|w| > R$. Translating back to z

$$|w| < R \Longleftrightarrow |z|^k < R \Longleftrightarrow |z| < R^{\frac{1}{k}}$$

#something is missing buts enought to get ans

13. Consider the following functions

(a).

$$f(x+iy) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & if\ x+iy \neq 0 \\ 0 & if\ x+iy = 0 \end{cases}$$

(b).

$$f(x+iy) = \sqrt{|xy|}$$

Show that f satisfies the CR-equations but it is not differentiable at the origin.

$$Show\ that\ f\ satisfies\ the\ CR-equations\ but\ it\ is\ not\ differentiable\ at\ the\ origin.$$

$$f(x+iy) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & if\ x+iy \neq 0 \\ 0 & if\ x+iy = 0 \end{cases}$$

(a). $f(x+iy) = \begin{cases} \dfrac{xy(x+iy)}{x^2+y^2} & if\ x+iy \neq 0 \\ 0 & if\ x+iy = 0 \end{cases}$

for $x+iy \neq 0$

$$\frac{xy(x+iy)}{x^2+y^2} = \frac{x^2y+ixy^2}{x^2+y^2} = \frac{x^2y+}{x^2+y^2} + i\frac{xy^2}{x^2+y^2}$$

$$u(x,y) = \frac{x^2y+}{x^2+y^2},\ v(x,y) = i\frac{xy^2}{x^2+y^2}$$

Partial derivatives at the origin and the CR equations there

$$u_x(0,0) = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$
$$u_y(0,0) = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$
$$v_x(0,0) = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$
$$v_y(0,0) = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

#Ma'am

Thus the CR-equations are satisfied. However, along the x-axis, f takes the value 0.

So, $\lim_{h\to 0} \frac{f(h+i0)-f(0)}{h}$ is 0. While

$$\lim_{h(1+i)\to 0} \frac{f(h+hi)-f(0)}{h+hi} = \lim_{h(1+i)\to 0} \frac{\left(h^3 - ih^3\right)}{\left(h^2+h^2\right)(h+hi)} = \frac{1}{2}.$$

#LLM

f is not differentiable at 0

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{(x,y)\to(0,0)} \frac{f(x+iy)}{x+iy} = \frac{f(z)}{z} = \frac{\frac{xy(x+iy)}{x^2+y^2}}{x+iy}$$
$$= \frac{xy(x+iy)}{\left(x^2+y^2\right)x+iy} = \frac{xy}{x^2+y^2}$$

This limit does not exist: take two paths to the origin

along the line $y=0: \frac{xy}{x^2+y^2} = 0$

along the line $y=x: \frac{xy}{x^2+y^2} = \frac{x^2}{2x^2} = \frac{1}{2}$

$\overline{x^2+y^2}$

Assignment-3

1. Using the method of parametric representation, evaluate $\oint_C f(z) = dz$ for

$(a)\ f(z) = \overline{z}$

$(b)\ f(z) = z + \dfrac{1}{z}$

$(c)\ f(z) = \mathfrak{Re}(z)$

$(d)\ f(z) = \dfrac{\sin z}{z}$

and C is the unit circle centered at origin oriented counterclockwise.

2. Evaluate the integral $\int_{\Gamma} ze^{z^2}\,dz$ where Γ is the curve from 0 to $1+i$ along the parabola $y = x^2$.

3. (a). Assign an appropriate meaning to the integral $\int_{-i}^{i} dz$ and find its value.

(b). $\int_C \sin^2 z dz$, C is the curve from $-i\pi$ to πi along $|z| = \pi$ taken counter -clockwise.

4. Verify Cauchy's theorem for $f(z) = z^2$ over the boundary of the square with vertices $1+i,\ -1+i,\ -1-i$ and $1-i$, counterclockwise.

5. Use ML-inequality to prove the following:

(a). $\left| \int_{\gamma} \dfrac{1}{1+z^2} dz \right| \leqslant \dfrac{\pi}{3},$ γ is the arc of $|z| = 2$ from 2 to $2i$.

(b). $\left| \int_{\gamma} (1+z^2)\,dz \right| \leqslant \pi R(R^2+1),$ γ is the semicircular arc of $|z| = R$.

6. By parametrizing the curve or otherwise, evaluate:

(a). $\int_C \tan z dz,$ where C is the circle $|z| = 1$ oriented counter -clockwise.

(b). $\int_C Re\ z^2 dz,$ C is the circle $|z| = 1$ oriented counter -clockwise.

(c). $\int_C e^{4z} dz,$ C is the shortest path from $8-3i$ to $8-(3+\pi)i$

7. Use Cauchy's integral formula to find all simple closed curves C for which the following holds :

(a) $\int_C \frac{1}{z} dz = 0$ (b) $\int_C \frac{e^{1/z}}{z^2+9} dz = 0$

8. Integrate $\frac{z^2}{z^4-1}$ counter-clockwise around the circle

(a) $|z+1| = 1$ (b) $z+i = 1$

9. Integrate the functions counter-clockwise on the unit circle $|z| = 1$.

$(a)\ \frac{z^3}{2z-i}$ $(b)\ \frac{\cosh 3z}{2z}$ $(c)\ \frac{z^3 \sin z}{3z-1}$

10. Let Γ denote the positively (counter-clockwise) oriented boundary of the square whose sides lie on the lines $x = \pm 2$ and $y = \pm 2$. Using Cauchy's integral formula, evaluate the following integrals:

$(a)\ \int_\Gamma \frac{\cos z}{z(z^2+8)} dz$ $(b)\ \int_\Gamma \frac{z}{2z+1} dz$

11. Evaluate the following integrals by parametrizing the contour

(a). $\int_C Re\ z dz$ where C is the line segment joining 1 to i.

(b). $\int_C (z-1) dz$ where C is the semicircle (in the lower half plane) joining 0 to 2.

14. Use Cauchy's integral formula to find closed contours C in complex plane satisfying

$(a)\ \int_C \log(z) dz = 0$ (b) $\int_C \frac{\cos z}{z^6 - z^2} dz = 0$

15. Using Cauchy's integral formula, integrate counterclockwise:

$$\oint_C \frac{\ln(z+1)}{z^2+1}dz,\ C: |z-2i| = 2$$

Assignment 4

Integrate $\dfrac{z^2}{z^4-1}$ counter-clockwise around the circle $|z+1| = 1$

01. Evaluate the integral $\dfrac{1}{2\pi i}\displaystyle\int_C \frac{ze^{zt}}{(z+1)^3}dz$ where C s a counter-clockwise oriented simple closed contour enclosing $z = -1$

02. Write down the Taylor series centred at the given point for the following functions and find its disc of convergence:

$(i)\ f(z) = \dfrac{1}{z^2}$ at $z_0 \neq 0$ $\quad (ii)\ f(z) = \dfrac{6z+8}{(2z+3)(4z+5)}$ at $z_0 = 1$ $\quad (iii)\ f(z) = \dfrac{e^z}{z+1}$ at $z_0 = 1$

04. Derive the Taylor series representation of $\dfrac{1}{1-z}$ around i.

$\dfrac{1}{1-z} = \displaystyle\sum_{n=1}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$ where $|z-i| < \sqrt{2}$

07. Find the maximum of the function $|f|$ on the closed unit disc $\overline{\mathbb{D}}$ if

$(a)\ f(z) = z^2 - z$ $\qquad (b)\ f(z) = \sin z$

08. If $0 < |z| < 4$, Show that $\dfrac{1}{4z-z^2} = \dfrac{1}{4z} + \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$.

09. Write the two Laurent series in powers of z that represent the function $f(z) = \dfrac{1}{z(1+z^2)}$ in different domains.

$$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z}\frac{1}{(1+z^2)}$$

For $|z| < 1$

$$\frac{1}{(1+z^2)} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n z^{2n} = \sum_{n=0}^{\infty} (-1)^n z^{2n}\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n z^{2n-1} = \frac{1}{z} - z + z^3 - z^5 + \cdots$$

For $|z| > 1$

$$\frac{1}{(1+z^2)} = \frac{1}{z^2}\frac{1}{1-\frac{1}{z^2}} = \sum_{n=0}^{\infty} (-1)^n z^{-2n-2}$$

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} (-1)^n z^{-2n-2} = \sum_{n=0}^{\infty} (-1)^n z^{-2n-2}\frac{1}{z} = \sum_{n=1}^{\infty} (-1)^n z^{2n-3} = \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} \cdots$$

11. Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Let $z_0 \in \mathbb{C}$ and $r > 0$ be arbitrary. Show
that the image of f intersects the disc $B_r(z_0) = \{z : |z - z_0| < r\}$.

12. Which of the following singularities are removable/pole:

(a) $\dfrac{\sin z}{z^2 - \pi^2}$ at $z = \pi$ $\qquad$ (b) $\dfrac{\sin z}{(z-\pi)^2}$ at $z = \pi$ $\qquad$ (a) $\dfrac{z\cos z}{1 - \sin z}$ at $z = \dfrac{\pi}{2}$

$$Which\ of\ the\ following\ singularities\ are\ removable\,/\,pole:$$

$$\frac{\sin z}{z^2 - \pi^2}\ \ at\ z = \pi$$

🚩 Do it

13. Find the residue at $z = 0$ of the following functions and indicate the type of singularity they have at 0.

$$(a)\ \frac{1}{z+z^2} \qquad (b)\ z\cos\frac{1}{z} \qquad (c)\ \frac{z-\sin z}{z} \qquad (d)\ \frac{\cot z}{z^4}$$

$(a).\ \ f(z) = \dfrac{1}{z+z^2} = \dfrac{1}{z(1+z)}$

Poles at $z = 0$ (simple). For a simple pole z_0

$$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

$$\frac{1}{(1+x)} = (1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ \cdots \ = \sum_{n=0}^{\infty} (-1)^n x^n$$

At $z_0 = 0$:

$Res(f, z_0) = \lim\limits_{z \to z_0} (z - z_0) f(z)$

$$Ref\left(\frac{1}{z+z^2},\ 0\right) = \lim_{z \to 0} (z-0)\frac{1}{z+z^2} = \lim_{z \to 0} z\frac{1}{z(1+z)} = \lim_{z \to 0} \frac{1}{(1+z)} = \frac{1}{(1+0)} = 1$$

Residue at $z = 0$: 1

Type of singularity: $z = 0$ is a simple pole.

$(b)\ f(z) = z\cos\dfrac{1}{z}$ 🚩asked in Quiz-2

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{z}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n}$$

$$z\cos\left(\frac{1}{z}\right) = z\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n} z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{1-2n}$$

$1 - 2n = -1 \implies 2n = 2 \implies n = 1,$ *then* $z^{-1} = \dfrac{1}{z}$ *here when* $z = 0$ *singularitie occure.*

$$\sum_{n=0}^{\infty} \frac{(-1)^{(1)}}{(2(1))!} z^{1-2(1)} = \sum_{n=0}^{\infty} \frac{-1}{2} z^{-1} = \sum_{n=0}^{\infty} \frac{-1}{2}\frac{1}{z}$$

So the residue is $-\frac{1}{2}$

$(c)\ f(z) = \frac{z - \sin z}{z}$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\frac{z - \sin z}{z} = \frac{z}{z} - \frac{\sin z}{z} = 1 - \frac{\sin z}{z} = 1 - \frac{z - \frac{z^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \cdots}{z}$$

$$1 - \frac{z\left(1 - \frac{z^2}{3!} + \frac{x^4}{5!} - \frac{x^5}{7!} + \ \cdots\right)}{z} = 1 - \left(1 - \frac{z^2}{3!} + \frac{x^4}{5!} - \frac{x^5}{7!} + \ \cdots\right) = \frac{z^2}{3!} - \frac{x^4}{5!} + \frac{x^5}{7!} - \frac{x^7}{9!} + \ \cdots$$

There is no $z^{-1} = \frac{1}{z}$ term in the Laurent expansion about 0 hence

$Res_{z=0} \frac{z - \sin z}{z} = 0$. Also the singularity at 0 is removable

$(d)\ f(z) = \frac{\cot z}{z^4}$

$$\cot x = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{1 - \frac{x^2}{2!} + \frac{x^4}{4!}} = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945}$$

$$\frac{\cot z}{z^4} = \frac{\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945}}{z^4} = \frac{1}{z^5} - \frac{1}{3z^3} - \frac{1}{45z} - \frac{2z}{945}$$

The residue is the coefficient of $z^{-1} = \frac{1}{z}$ is $-\frac{1}{45}$

Since the principal part contains terms down to $\frac{1}{z^5}$, pole of order 5.

14. Use Cauchy's residue theorem to evaluate the integral of each of the following functions around the circle $|z| = 3$.

$$(a)\ \frac{e^{-z}}{z^2} \qquad (b)\ \frac{e^{-z}}{(z-1)^2} \qquad (c)\ z^2 e^{\frac{1}{z}} \qquad (d)\ \frac{z+1}{z^2-2z}$$

$Use\ Cauchy's\ residue\ theorem\ to\ evaluate\ the\ integral\ of\ function\ around\ the\ circle\ |z| = 3$

$\dfrac{z+1}{z^2-2z}$

$(a).\ \oint_{|z|=3} \dfrac{e^{-z}}{z^2} dz$

Inside $|z| = 3$, the only singularity is at $z = 0$.

if z_0 is a pole of order n :

$$Res(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z)\right]$$

here,

$z_0 = 0$

$f(z) = \dfrac{e^{-z}}{z^2}$

order $n = 2$ *because denominator has* z^2

$$Res\left(\frac{e^{-z}}{z^2}, 0\right) = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d^{2-1}}{dz^{2-1}} \left[(z-0)^2 \frac{e^{-z}}{z^2}\right] = \frac{1}{1!} \lim_{z \to 0} \frac{d^1}{dz^1} \left[z^2 \frac{e^{-z}}{z^2}\right] = \lim_{z \to 0} \frac{d}{dz} \left[z^2 \frac{e^{-z}}{z^2}\right]$$

$$= \lim_{z \to 0} \frac{d}{dz} \left[e^{-z}\right] = \lim_{z \to 0} \left(-e^{-z}\right) = -e^{-0} = -1$$

Cauchy's Residue Theorem

$$\int_C f(z) dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k) = 2\pi i(-1) = -2\pi i$$

$(b).\ \oint_{|z|=3} \dfrac{e^{-z}}{(z-1)^2} dz$

The only singularity inside is $z_0 = 1$, a pole of order 2

if z_0 is a pole of order n :

$$Res(f, z_0) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}}\left[(z - z_0)^n f(z)\right]$$

$$Res\left(\frac{e^{-z}}{(z-1)^2}, 1\right) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^2 \frac{e^{-z}}{(z-1)^2}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz}\left[e^{-z}\right] = \lim_{z \to 1} - e^{-z} = -e^{-1} = -\frac{1}{e}$$

Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k) = 2\pi i\left(-\frac{1}{e}\right) = -\frac{2\pi i}{e}$$

$(c).\ \oint_{|z|=3} z^2 e^{\frac{1}{z}} dz$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \cdots + \sum_{n=1}^{\infty} \frac{\left(\frac{1}{z}\right)^n}{n!}$$

$$e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots + \sum_{n=1}^{\infty} \frac{1}{n!z^n}$$

$$\oint_{|z|=3} z^2 e^{\frac{1}{z}} dz = \oint_{|z|=3} z^2 \frac{1}{n!z^n} dz = \oint_{|z|=3} \frac{z^{2-n}}{n!} dz$$

$2 - n = -1$ *then* $z^{-1} = \frac{1}{z}$ *here when* $z = 0$ *singularitie occure.*

 # Something is there which i could not understood.

$$\frac{z^{2-3}}{3!} = \frac{1}{6} z^{-1}$$

Thus the coefficient of $z^{-1} = \frac{1}{z}$ is $\frac{1}{6}$.

Cauchy's Residue Theorem

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k) = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

$(d).\ \oint_{|z|=3} \frac{z+1}{z^2-2z} dz$

$f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$ Singularities of $f(z)$ are at $z = 0$ and $z = 2$.
Both lie inside the circle $|z| = 3$, and both are simple poles.

At $z_0 = 0$
$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$
$Res\left(\frac{z+1}{z(z-2)}, 0\right) = \lim_{z \to 0} (z-0) \frac{z+1}{z(z-2)} = \lim_{z \to 0} z \frac{z+1}{z(z-2)} = \lim_{z \to 0} \frac{z+1}{(z-2)} = \frac{0+1}{(0-2)} = -\frac{1}{2}$

At $z_0 = 2$
$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$
$Res\left(\frac{z+1}{z(z-2)}, 2\right) = \lim_{z \to 2} (z-2) \frac{z+1}{z(z-2)} = \lim_{z \to 2} \frac{z+1}{z} = \lim_{z \to 2} \frac{2+1}{2} = \frac{3}{2}$

Cauchy's Residue Theorem
$\int_C f(z) dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k) = 2\pi i \left(-\frac{1}{2} + \frac{3}{2}\right) = 2\pi i \left(\frac{-1+3}{2}\right) = 2\pi i \left(\frac{2}{2}\right) = 2\pi i$

15. Find the isolated singularities and compute the residue of f:

$(a)\ \frac{e^z}{z^2-1}$ $\qquad\qquad$ $(b)\ \frac{3z}{z^2+iz+2}$

$\Longrightarrow$ How to find residue
$(a)\ \frac{e^z}{z^2-1}$

Poles at $z = \pm 1$ (simple). For a simple pole z_0

$$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

Singularities occur where the denominator vanishes : $z^2 - 1 = 0 \Longrightarrow z = \pm 1$
These are simple poles (zeros of order 1 of the denominator), hence isolated.
the singularities are where the denominator is zero: $z_0 = +1,\ -1$

At $z_0 = 1$:
$Res(f, z_0) = \lim\limits_{z \to z_0} (z - z_0) f(z)$

$$Ref\left(\frac{e^z}{z^2-1},\ 1\right) = \lim_{z \to 1}(z-1)\frac{e^z}{z^2-1} = \lim_{z \to -1}(z-1)\frac{e^z}{(z-1)(z+1)} = \lim_{z \to 1}\frac{e^z}{(z+1)} = \frac{e}{2}$$

At $z_0 = -1$:
$Res(f, z_0) = \lim\limits_{z \to z_0} (z - z_0) f(z)$

$$Ref\left(\frac{e^z}{z^2-1},\ -1\right) = \lim_{z \to -1}(z-(-1))\frac{e^z}{z^2-1} = \lim_{z \to -1}(z+1)\frac{e^z}{z^2-1}$$

$$= \lim_{z \to -1}(z+1)\frac{e^z}{(z-1)(z+1)} = \lim_{z \to -1}\frac{e^z}{(z-1)} = \frac{e^{-z}}{-1-1} = \frac{e^{-1}}{-2} = -\frac{1}{2e}$$

$(b)\ \dfrac{3z}{z^2+iz+2}$

Quadratic formula $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$z^2 + iz + 2 = 0$

$$z = \frac{-i \pm \sqrt{i^2-4(1)(2)}}{2(1)} = \frac{-i \pm \sqrt{-1-8}}{2} = \frac{-i \pm \sqrt{-9}}{2} = \frac{-i \pm \sqrt{9}\sqrt{-1}}{2} = \frac{-i \pm 3i}{2}$$

$$\frac{-i+3i}{2} = \frac{2i}{2} = i \qquad \frac{-i-3i}{2} = \frac{-4i}{2} = -2i$$

Simple poles at $z_0 = i$ and $z_0 = -2i$
At $z_0 = i$
$Res(f, z_0) = \lim\limits_{z \to z_0} (z - z_0) f(z)$

$$Ref\left(\frac{3z}{z^2+iz+2}, i\right) = \lim_{z \to i}(z-i)\frac{3z}{z^2+iz+2} = \lim_{z \to i}(z-i)\frac{3z}{(z-i)(z+2i)}$$

$$= \lim_{z \to i}\frac{3z}{(z+2i)} = \frac{3i}{(i+2i)} = \frac{3i}{3i} = 1$$

If the roots are $z = i$ and $z = -2i$ then the factorization is:
$z^2 + iz + 2 = (z-i)(z+2i)$

At $z_0 = -2i$

$Res(f, z_0) = \lim\limits_{z \to z_0} (z - z_0) f(z)$

$$Ref\left(\frac{3z}{z^2 + iz + 2}, -2i\right) = \lim_{z \to -2i} (z - (-2i)) \frac{3z}{z^2 + iz + 2} = \lim_{z \to -2i} (z + 2i) \frac{3z}{(z - i)(z + 2i)}$$

$$= \lim_{z \to -2i} \frac{3z}{(z - i)} = \frac{3(-2i)}{(-2i - i)} = \frac{-6i}{-3i} = 2$$

Quiz-1

1. Which of the following statements are true for the sequence $\{z_n\}_{n=0}^{\infty}$ where $z_n = \dfrac{i^n}{n}$:

   a) $\{z_n\}$ is absolutely convergent.
   b) $\{z_n\}$ is convergent but not absolutely convergent.
   c) The series $\displaystyle\sum_{n=1}^{\infty} z_n$ is convergent but not absolutely convergent.
   d) The series $\displaystyle\sum_{n=1}^{\infty} z_n$ is absolutely convergent

i^n : i has a 4-term cycle

$i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1.$

so its oscillate, bounded. whereas n grow infinitely.

$$z_n = \frac{i^n}{n}$$

$$z_n = \left| \frac{i^n}{n} \right| = \frac{|i^n|}{n} = \frac{1}{n}$$

But $|i^n| = 1$ always, because i^n lies on the unit circle.

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

$\Longrightarrow \{z_n\}$ is absolutely convergent.

The numerator i^n is bounded (only 4 possible values), and denominator $n \to \infty$.

so,

$$z_n = \frac{bounded}{\infty} \to 0$$

$\Longrightarrow$ Thus the sequence converges, and its limit is 0.

$$\sum_{n=1}^{\infty} z_n$$

Absolute, $\displaystyle\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \frac{1}{n}$, This is the harmonic series, which is divergent.

$\to$ So the series is NOT absolutely convergent.

use Dirichlet's test, which states:

A series $\sum a_n b_n$ converges if:

1. $\sum a_n$ has bounded partial sums
2. $b_n \to 0$

Here : $\displaystyle\sum a_n b_n = \sum_{n=1}^{\infty} i^n \frac{1}{n}$

$a_n = i^n : i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$. its oscillate, bounded

$b_n = \dfrac{1}{n} \to 0$ as $n \to \infty$.

So the conditions are satisfied. ✔ Therefore the series converges.

Quiz-2

1. Consider $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = zIm(z)$ Let S be the set of all $z \in \mathbb{C}$ such that f is differentiable at z. Which of the following are valid for the above S :

(a). $S = \mathbb{C}$

(b). $S = \mathbb{C} \setminus \{0\}$

(c). $0 \in S$

(d). $S = \emptyset$

$$f(z) = zIm(z)$$

Let $z = x + iy$

$\mathfrak{R}(z) = x, \mathfrak{I}(z) = y$

$f(z) = z\mathfrak{I}(z) \implies f(z) = (x + iy)y \implies xy + iy^2$
$u(x, y) = \mathfrak{R}(f(z)) = xy,$
$v(x, y) = \mathfrak{I}(f(z)) = y^2$

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = v_y \qquad\qquad u_y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -v_x$$

$u_x = y \quad v_x = 0$
$u_y = x \quad v_y = 2y$

$u_y = -v_x$
$x = 0$

CR equations :
$u_x = v_y \implies y = 2y = 2y - y = 0 \implies y = 0$

so, $x = 0$ and $y = 0$

Thus the CR equations hold only at $(x, y) = (0, 0)$.
i.e. only at $z = 0$ Since the partials are continuous CR are also sufficient, so f is complex-differentiable exactly at $z = 0$.

Check the derivative at 0 :
$$\frac{f(z) - f(0)}{z - 0} = \frac{z\mathfrak{I}(z)}{z} = \mathfrak{I}(z) = y \to 0.$$
so, $f'(0) = 0$.
$0 \in S$.
Conclusion $S = \{0\}$.

2. The radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ where

$$a_{2k} = \left(\frac{1 + 2i}{5}\right)^{2k}, a_{2k+1} = \left(\frac{2 - i}{\sqrt{10}}\right)^{2k+1}, \text{for } k = 0, 1, \ldots$$

is equal to

For even indices $n = 2k \to a_{2k} = \left(\frac{1 + 2i}{5}\right)^{2k}$

For odd indices $n = 2k + 1 \to a_{2k+1} = \left(\frac{2 - i}{\sqrt{10}}\right)^{2k+1}$, Even terms :

$|1+2i| = \sqrt{1^2+2^2} = \sqrt{5},$

$\left|\dfrac{1+2i}{5}\right| = \dfrac{\sqrt{5}}{5} = \dfrac{1}{\sqrt{5}}$

Odd terms :

$|2-i| = \sqrt{2^2+(-1)^2} = \sqrt{5}$

$\left|\dfrac{2-i}{\sqrt{10}}\right| = \dfrac{\sqrt{5}}{\sqrt{10}} = \dfrac{1}{\sqrt{2}}$

Use the root test : $\dfrac{1}{R} = \lim\limits_{n\to\infty} \sqrt[n]{a_n} =$

The limsup is the larger value: $\dfrac{1}{\sqrt{2}} > \dfrac{1}{\sqrt{5}}$

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \dfrac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$. *I feel something wrong here.*

3. Let $z_0 = \int_\gamma z\cos\left(\dfrac{1}{z}\right)$ where $\gamma(t) = 2e^{2\pi i t},\ 0 \leqslant t \leqslant 1.$

i. Laurent series of $\cos\left(\dfrac{1}{z}\right)$ around 0 is

ii. Residue of $z\cos\left(\dfrac{1}{z}\right)$ at 0 is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{z}\right)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n}$$

$$z\cos\left(\frac{1}{z}\right) = z\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{z}\right)^{2n}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n} z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{1-2n}$$

$1-2n = -1 \implies 2n = 2 \implies n = 1$, *then* $z^{-1} = \dfrac{1}{z}$ *here when* $z = 0$ *singularitie occure.*

$$\sum_{n=0}^{\infty} \frac{(-1)^{(1)}}{(2(1))!} z^{1-2(1)} = \sum_{n=0}^{\infty} \frac{-1}{2} z^{-1} = \sum_{n=0}^{\infty} \frac{-1}{2}\frac{1}{z}$$

So the residue is $-\frac{1}{2}$

4. Let $z_1 = e^{2(\log(1+i))}$, and r be be the maximum attained by the function $f(z) = \frac{e^z}{z^3}$ on the annulus $\{z \mid \quad 2 \leqslant |z| \leqslant 3\}$. Then,
$|z_1| =$ and $r =$
#Pending

5. Choose the correct statements
(a). e^z does not take values in $\mathbb{R}^{<0}$.
(b). $\cos z$ is never equal to 2.
(c). $\log(\exp z) = z$ for all $z \in \mathbb{C}$.
(d). $\exp(\log z) = z$ for all $z \in \mathbb{C}$.