Real Analysis

https://home.iitk.ac.in/~psraj/mth101/lecture_notes.html
https://home.iitk.ac.in/~psraj/mth101/practice-problems.html
https://home.iitk.ac.in/~psraj/mth101/assignments.html
Note : Follow Prof. P. Shunmugaraj's Notes

Power Series

Power Series Representation of a Function by Professor V
→ She explained everything like its butter. But i think it might becaues i already knew lots of about power sereis. *(Waiit... If this is case, then that's reason my classmates understand everything what professor teaches, they already know from undergrad or they study before professor teaches)*

Taylor and Maclaurin Series (Part 1) by Professor V

Taylor's Theorem

Intuition
Taylor's theorem is about approximating a complicated function $f(x)$ near some point a using polynomials.

- Near a, the function can be written as a polynomial (called the Taylor polynomial) plus a remainder.
- The more terms you take, the better the approximation (if the function is smooth enough).

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

Where $R_n(x)$ is the remainder term (error between true value and polynomial).

$f(x) = (\text{polynomial}) + (\text{error term})$

$$f(x) = \left(\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \right) + \left(\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \right)$$

Psraj :

Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ is continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Suppose $x_0 \in [a, b]$. Then, for any $x \in [a, b] \,/\, \{x_0\}$, there exist ξ between x and x_0 such that

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

Q1. Let $f : [a, b] \to \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that $f^{(n+1)}$ exists on $[a, b]$ and $f^{(n+1)}(x) = 0$ for all $x \in [a, b]$. Show that f is a polynomial of degree less than or equal to n

$\Longrightarrow$ by Taylor's theorem

Note : Understand Taylor's theorem first, then you'll get idea of this problem. *I studies function, Limits, continuity, differentiability, series etc Then i was able to understand this (somewhat)*

$f^{(n+1)}(x) = 0$ for all $x \in [a, b]$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} = \frac{0}{(n+1)!}(x-a)^{n+1} = 0$$

Thus the Taylor formula becomes exact (no error term):

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

This is a polynomial in $(x - x_0)$ of degree at most n

Therefore, f is a polynomial of degree $\leqslant n$.

Note : In Exam paper, write fromally in details. Defination Rich

Q2. Show that $1 + \frac{x}{2} - \frac{x^2}{8} \leqslant \sqrt{1+x} \leqslant 1 + \frac{x}{2}$ for $x \geqslant 0$

$$f(x) = \sqrt{1+x} = (1+x)^{1/2} \Longrightarrow f(0) = 1$$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \Longrightarrow f'(0) = \frac{1}{2}$$

$f^{''}(x) = -\frac{1}{4}(1+x)^{-3/2} \implies f^{''}(0) = -\frac{1}{4}$
$f^{'''}(x) = -\frac{3}{8}(1+x)^{-5/2} \implies f^{'''}(0) = \frac{3}{8}$
Therefore, the Taylor expansion up to the second order is:
$f(x) = f(x_0) + f^{'}(x_0)(x-x_0) + \frac{f^{''}(x_0)}{2!}(x-x_0)^2$
$\sqrt{1+x} = 1 + \frac{1}{2}(x) + \left(-\frac{1}{4}\right)\frac{1}{2!}x^2 = 1 + \frac{x}{2} - \frac{1}{8}x^2 + R_2(x)$

Right Inequality:
$\sqrt{1+x} \leqslant 1 + \frac{x}{2}$
So we are subtracting positive quantities (since for $x > 0$, $x^2 > 0$) meaning

$$\sqrt{1+x} \leqslant 1 + \frac{x}{2}$$

This proves the upper bound.

Left Inequality :
$1 + \frac{x}{2} - \frac{x^2}{8} \leqslant \sqrt{1+x}$
$R_2(x) = \frac{f^{(3)}(\xi)}{3!}x^3 = \frac{1}{16}(x+\xi)^{-5/2}x^3 > 0$ for $x > 0$

$$1 + \frac{x}{2} - \frac{x^2}{8} + R_2(x) > 1 + \frac{x}{2} - \frac{x^2}{8}$$

Q3. Show that for $x > 0$, $\left|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\right| \leqslant \frac{x^4}{4}$
Note : If you know Tayloy's Theorem, You should be able to do it.

Taylor's theorem says:

$$f(x) = P_n(x) + R_{n+1}(x) \implies P_3(x) + R_3(x)$$
$$|f(x) - P_3(x)| = |R_3(x)|$$

Here, If we substract only Taylors Polynomial from entire function we'll left with only Remiender term.

$$f(x) = f(x_0) + f^{'}(x_0)(x-x_0) + \frac{f^{''}(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x)$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1} \text{ for some } \xi \in (0, x)$$

Let $f(x) = \log(1+x)$, and expand it about $x = 0$ The Taylor polynomial of degree 3 is:

$$f(x) = \log(1+x) \implies f(0) = \log(1) = 0$$
$$f'(x) = \frac{1}{(1+x)} \implies f'(0) = 1$$
$$f''(x) = -\frac{1}{(1+x)^2} \implies f''(0) = -1$$
$$f^{(3)}(x) = \frac{2}{(1+x)^3} \implies f^{(3)}(0) = 2$$
$$f^{(4)}(x) = -\frac{6}{(1+\xi)^4} \implies$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + R_3(x)$$

$$R_3(x) = \frac{1}{4!}\left(-\frac{6}{(1+\xi)^4}\right)x^4 = -\frac{6}{24(1+\xi)^4}x^4 = -\frac{1}{4(1+\xi)^4}x^4$$

$$|f(x) - P_3(x)| = |R_3(x)| = \left|-\frac{1}{4(1+\xi)^4}x^4\right| = \frac{1}{4(1+\xi)^4}x^4$$

Since $\xi > 0 \implies (1+\xi)^4 > 1 \implies \frac{1}{(1+\xi)^4} \leqslant 1$

$$\frac{1}{4(1+\xi)^4}x^4 \leqslant \frac{x^4}{4}$$

Riemann Integration

Riemann Integration by [Bhavna Khurana]() 👑

→ Good explanation. "Love this" As many time i see this, i appriciate more and more.

Riemann Sum and Riemann Integral Explained by [Poenaru Dan]()

→ Good, help for Visualize.

Necessary and sufficient condition for Riemann integrability by Mark Arokiasamy
→ I did not watch video carefully so i can't tell how it is, but he explain problems so well that's why i'm keeping this video.

Riemann integral - Wikipedia

Real Analysis 48 | Riemann Integral - Partitions by The Bright Side of Mathematics (YouTube)
→ Good but for the time being I don't understand everything.

Real Analysis | Partitions and upper/lower sums by Micheal Penn Good explanation with example

Theorem 5 : link1

Problems

Q. Show that $f(x) = 3x + 1$ is integrable on $[1, 2]$ and $\int_1^2 (3x+1) = \frac{11}{2}$.
→ by Mark Arokiasamy : Super detailed, problem driven explanation. And since he explain in detailed manner it 30 min long (*part-1*). I will encourage you to watch this problem before other.

Q. Show that the function f defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & ; \ x \ is \ rational \\ 0 & ; \ x \ is \ irrational \end{cases}$$

is not integrable on $[0, 1]$.
→ Video

Q. Show that $f(x) = \sin x$ is Riemann integrable over $\left[0, \ \frac{\pi}{2}\right]$
→ Video

Q. Show that $f(x) = x^2$ is integrable on $[0, a]$ and $\int_0^a x^2 dx = \frac{a^3}{3}$
→ by Mark Arokiasamy
→⚠️ watch to understand process. Prof will not give questions like this.

Q. Show that $f(x) = \sin(x)$ s integrable on $\left[0, \ \frac{\pi}{2}\right]$ and $\int_0^{\frac{\pi}{2}} \sin x dx = 1$.

→ by Mark Arokiasamy
→⚠️ watch to understand process. Prof will not give questions like this.

Class Examples

Using Riemann's criterion for the integrability, show that $f(x) = \dfrac{1}{x}$ is integrable on $[1, 2]$

→ assignment 3/7 page, assignment-6, Q1.
→ Ask ChaGPT, Solve this problem as Defination step by step in details manner. (*Give bookish version not in general explanation, prof. will not allow any ans*)

→ Followed Mark Arokiasamy's steps.

$f(x) = \dfrac{1}{x}$ is bounded on $[1, 2]$. $\left(\dfrac{1}{2} \leqslant f(x) \leqslant 1\right)$ also f is strictly decreasing on $[1, 2]$.

Consider the partition $P = \{a = x_0, x_1, x_2, \ldots x_{r-1}, x_r, x_n = b\}$

First point of division = $x_0 = a = 1$

Second point of division = $x_1 = a + \dfrac{b-a}{n} = 1 + \dfrac{2-1}{n} = 1 + \dfrac{1}{n}$

$(r+1)^{th}$point of division = $x_r = a + \dfrac{r(b-a)}{n} = 1 + \dfrac{r(2-1)}{n} = 1 + \dfrac{r}{n}$

$(n+1)^{th}$point of division = $x_n = a + \dfrac{n(b-a)}{n} = 1 + \dfrac{n(2-1)}{n} = 1 + \dfrac{n}{n} = 2$

partition becomes $P = \left\{1, 1 + \dfrac{1}{n}, 1 + \dfrac{2}{n}, \ \cdots 1 + \dfrac{(r-1)}{n}, 1 + \dfrac{r}{n}, 2\right\}$

$I_r = r^{th}$ subinterval $= [x_{r-1},\ x_r] = \left[1 + \dfrac{(r-1)}{n}, 1 + \dfrac{r}{n}\right]$

Δx_r = Length of each Subinterval : $\dfrac{b-a}{n} = \dfrac{2-1}{n} = \dfrac{1}{n}$

$$M_r = \sup\{f(x) : x_{r-1} < x < x_r\} = \sup_{[x_{r-1},\ x_r]} f = f(x_{r-1}) = \frac{1}{x_{r-1}}$$

$$m_r = \inf\{f(x) : x_{r-1} < x < x_r\} = \inf_{[x_{r-1},\ x_r]} f = f(x_r) = \frac{1}{x_r}$$

$$U(P, f) = \sum_{r=1}^{n} M_r \Delta x_r = \sum_{r=1}^{n} \frac{1}{x_{r-1}} \cdot \frac{1}{n}$$

$$L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r = \sum_{r=1}^{n} \frac{1}{x_r} \cdot \frac{1}{n}$$

$$U(P,f) - L(P,f) = \frac{1}{n}\sum_{r=1}^{n}\left(\frac{1}{x_{r-1}} - \frac{1}{x_r}\right) = \frac{1}{2n} < \epsilon.$$

If f and g are continuous functions on $[a,\ b]$ and if $g(x) \geq 0$ for $a \leq x \leq b$,
then show the mean value theorem for integrals : there exists $c \in [a, b]$ such that
$\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx.$

(a). Show that there is no continuous function f on $[0, 1]$ such that $\int_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$.

(b). If f is contiunuous on $[a, b]$ then show that there exists $c \in [a,\ b]$ such that
$\int_a^b f(x)dx = f(c)(b-a)$

(c). If f and g are continuous on $[a, b]$ and $\int_a^b f(x)dx \int_a^b g(x)dx$ then show that there exist $c \in [a, b]$ such that $f(c) = g(c)$.

Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{x}$ *is integrable on* $[1, 2]$

Practice problem 15

https://home.iitk.ac.in/~psraj/mth101/practice-problems/pp15.pdf

Q1. Skiped

Q2. Let $f\colon [a, b] \to \mathbb{R}$ *be a bounded function. Suppose that there is a partition* P *of* $[a,\ b]$ *such that* $L(P, f) = L(P, f)$.
Show that f *is a constant function.*

→ Use hint. It's trivial but I can't prove it properly so remember hint. Atleast you might get partial mark.

Q3. Let $f\colon [a, b] \to \mathbb{R}$ *be a bounded function and* $f(x)\ \geq\ 0$ *for every* $x\ \in\ [a,\ b]$.
Show that $\int_{\underline{a}}^{b} f(x)dx\ \geq\ 0$ *and* $\int_{a}^{\bar{b}} f(x)dx\ \geq\ 0.$

$In\ addition,\ if\ f\ is\ integrable,\ show\ that \int_a^b f(x)dx\ \geq\ 0$

→ Follows from the definitions

$4.\ \ In\ each\ of\ the\ following\ cases,\ evaluate\ the\ upper\ and\ lower\ integrals\ of\ f\ and\ show\ that\ f\ is\ integrable.\ \ Find\ the\ integral\ of\ f.$

$(a)\ \ For\ \alpha \in R,\ \ define\ f : [a,\ \ b] \to \mathbb{R}\ by\ f(x) = \alpha\ for\ every\ x \in [a, b]$

$(b)\ \ f(x) = 0\ \ for\ 0 \leq x < \frac{1}{2},\ \ f\left(\frac{1}{2}\right) = 10\ \ and\ \ f(x) = 1\ \ for\ \frac{1}{2} < x \leq 1.$

$(c)\ \ f(x) = x\ \ for\ all\ x \in [0, 1].$

→ To solve (c) see this video by Dr. Gagendra Puhohit from 16:20.

$5.\ Let\ f : [a, b] \to \mathbb{R}\ be\ integrable\ and\ P_n\ be\ a\ partition\ such\ that$
$U(Pn, f) - L(Pn, f) \to 0.$

$Show\ that\ \lim_{n \to \infty} L(P_n, f) = \lim_{n \to \infty} U(Pn, f) = \int_a^b f(x)dx.$

→ It's easy when you undertand "Theorem-3"

$6.\ In\ each\ of\ the\ following\ cases,\ \ show\ that\ f\ is\ integrable\ using\ the\ Riemann\ criterion.$

$(a)\ \ f(x) = x\ on\ [0, 1]$

$(b)\ \ f(x) = x^2\ on\ [0, 1]$

$(c)\ \ f(x) = \frac{1}{x}\ on\ [1, 2]$

$7.\ \ Let\ f,\ \ f_1\ and\ f_2\ be\ bounded\ functions\ on\ [0, 1]\ such\ that\ f_1(x) \leq f(x) \leq f_2(x)\ for\ all\ x \in [0, 1].\ \ Suppose\ that\ f_1\ and\ f_2\ are\ integrable\ and\ \int_0^1 f_1 f_1(x)dx = \int_0^1 f_2(x)dx,\ show\ that\ f\ is\ integrable\ and\ find\ \int_0^1 f(x)dx.$

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https://home.iitk.ac.in/~psraj/mth101/assignments/assignment.pdf

1. Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{x}$ is integrable on $[1, 2]$

Problems on Riemann Integration : Part I by Mark Arokiasamy

→ Amazing Explaination (*I think this is because i already knew other problems*)

Problem 2, 3 (*I'm not following this because I don't think my prof will give these problems*)
→

$$Using\ Riemann's\ criterion\ for\ the\ integrability,\ show\ that\ f(x) = \frac{1}{x}\ is\ integrable\ on\ [1, 2]$$

Other Problems

$If\ f\ and\ g\ are\ continuous\ functions\ on\ [a,\ b]\ and\ if\ g(x)\ \geq\ 0\ for\ a\ \leq\ x\ \leq\ b,$
$then\ show\ the\ mean\ value\ theorem\ for\ integrals$
$(c)\ If\ f\ and\ g\ are\ continuous\ on\ [a,\ b]\ and\ \int_a^b f(x)dx \int_a^b g(x)dx\ hen\ show\ that\ there$
$exists\ c \in [a, b]\ such\ that\ f(c) = g(c)$

$3.\ Let\ f\colon [0, 2] \to \mathbb{R}\ be\ a\ continuous\ function\ such\ that\ \int_0^2 f(x)dx = 2.\ Find\ the\ value\ of$
$\int_0^2 \left[xf(x) + \int_0^x f(t)dt \right] dx.$
→ Split final integral in terms of A and B, You'll be B as Double integral. Solve it be changint its order and take comman some terms at final you'll get term for which you have given value as '2' put it and you'll get 2(2)=4 ans.

$4.\ Show\ that\ \int_0^x \left(\int_0^y f(t)dt \right) dt = \int_0^x f(u)(x-u)du\ assuming\ f\ to\ be\ continuous.$
→ Use "**Fundamental Theorem of Calculus**" Don't solve by change-of-order / integration-by-parts identity prof. will not give any marks
→ Define both side, LHS with (first) Fundamental Theorem of Calculus and for RHS Leibniz (differentiation under the integral sign) rule for an integral. You'll get value as 0.

$5.\ Let\ f\colon [0, 1] \to \mathbb{R}\ be\ a\ positive\ continuous\ function.\ Show\ that$

$$\lim_{n \to \infty} \left(f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \cdots f\left(\frac{n}{n}\right) \right)^{\frac{1}{n}} = e^{\int_0^1 \ln f(x)}$$

→ Riemann sum theorem

Improper Integral

An improper integral is an integral where either

1️⃣ the interval is infinite, or

2️⃣ the function becomes infinite (has a discontinuity) somewhere in the interval.

🔹 Type 1: Infinite Interval

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx$$

🔹 Type 2: Infinite Discontinuity

$$\int_0^1 \frac{1}{\sqrt{x}}dx$$

the integrand $\frac{1}{\sqrt{x}}$ is not defined at $x = 0$

So we treat that endpoint as a limit:

$$\lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}dx$$

Integral that CONVERGES:

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}dx = \lim_{t\to\infty}\left[\int_1^t x^{-2}dx\right] = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{t}-\left(-\frac{1}{1}\right)\right] = \lim_{t\to\infty}\left(1-\frac{1}{t}\right) = 1$$ ✔ Convergent (value = 1)

Integral that DIVERGES:

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x}dx = \lim_{t\to\infty}[\ln x]_1^t = \lim_{t\to\infty}[\ln t - \ln 1] = \lim_{t\to\infty}\ln t = \infty$$

✘ Divergent (grows without bound)

Blow-up at endpoint:

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \lim_{a\to 0^+}\int_a^1 x^{-\frac{1}{2}}dx = \lim_{a\to 0^+}\left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_a^1 = \lim_{a\to 0^+}\left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_a^1 = \lim_{a\to 0^+}\left[2\sqrt{x}\right]_a^1$$

$$= \lim_{a\to 0^+}\left[2\sqrt{1}-2\sqrt{a}\right] = \lim_{a\to 0^+}\left[2-2\sqrt{a}\right] = 2$$

✔ Convergent

Blow-up but DIVERGES:

$$\int_0^1 \frac{1}{x}dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{x}dx = \lim_{a\to 0^+} [\ln x]_a^1 = \lim_{a\to 0^+} [\ln 1 - \ln a] = \lim_{a\to 0^+} [-\ln a] = \infty$$

✘ Divergent

1. Test the convergence/divergence of the following improper integrals:

$$\int_0^1 \frac{1}{\log\left(1+\sqrt{x}\right)}dx \quad \int_0^1 \frac{1}{x-\log(1+x)}dx \quad \int_0^1 \frac{\log x}{\sqrt{x}}dx \quad \int_0^1 \sin\left(\frac{1}{x}\right)dx$$

$$\int_0^\infty \frac{\sin\left(\frac{1}{x}\right)}{x}dx \quad \int_0^\infty e^{-x^2}dx \quad \int_0^\infty \sin x^2 dx \quad \int_0^{\pi/2} \cot x dx$$

$$\int_0^1 \frac{1}{\log\left(1+\sqrt{x}\right)}dx$$

The interval [0,1] is finite, so the only possible issue is at x=0.

at $x = 0$ $\log\left(1+\sqrt{x}\right) \to \log(1+0) = \log(1) = 0$ so $\dfrac{1}{\log\left(1+\sqrt{x}\right)} \to \infty$

Find a simpler comparison function

$$\log\left(1+\sqrt{x}\right) \approx \sqrt{x} \to f(x) = \frac{1}{\log\left(1+\sqrt{x}\right)} \approx \frac{1}{\sqrt{x}} = g(x)$$

Set up the Limit Comparison Test

$$L = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\frac{1}{\log\left(1+\sqrt{x}\right)}}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\sqrt{x}}{\log\left(1+\sqrt{x}\right)}$$

To make it cleaner, set $u = \sqrt{x}$ $\displaystyle\lim_{u\to 0^+} \frac{u}{\log(1+u)}$

both numerator and denominator → 0 as $u \to 0^+$

So it’s a 0/0 indeterminate form, which means we can apply L’Hôpital’s Rule.

Differentiate numerator and denominator with respect to u

$$\frac{d}{du}(u) = 1, \ \frac{d}{du}(\log(1+u)) = \frac{1}{1+u}$$

$$\lim_{u\to 0^+} \frac{u}{\log(1+u)} = \lim_{u\to 0^+} \frac{1}{\frac{1}{1+u}} = \lim_{u\to 0^+} (1+u) = 1 \text{ (as } u \to 0^+\text{)}$$

$$\int_0^1 \frac{1}{x-\log(1+x)}dx$$

We only have a possible problem at $x = 0$ (the integrand is finite on $(0,1]$, so test the behaviour as $x \to 0^+$.

Use the Taylor expansion (or compute derivatives) for small x

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ so, $x - \log(1+x) = \frac{x^2}{2} - \frac{x^3}{3} + \cdots \sim \frac{x^2}{2} \ (x \to 0)$

Hence $f(x) = \frac{1}{x-\log(1+x)} \sim \frac{1}{\frac{x^2}{2}} = \frac{2}{x^2} \approx \frac{1}{x^2} = g(x) \ (x \to 0)$

$$L = \lim_{x\to 0^+} \frac{f(x)}{g(x)} = \lim_{x\to 0^+} \frac{\frac{1}{x-\log(1+x)}}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{x^2}{x-\log(1+x)} \approx \lim_{x\to 0^+} \frac{x^2}{\frac{x^2}{2}} = 2$$

but since $g(x)$ diverges so $f(x)$ also diverge

$$\int_0^1 \frac{\log x}{\sqrt{x}}dx$$

We have an improper point at $x = 0$.

Substitute $x = t^2$ so $\left(t = \sqrt{x}\right)$, $dx = 2tdt$. Then when $x \in (0,1]$ we have $t \in (0,1]$

$$\frac{\log x}{\sqrt{x}}dx = \frac{\log(t^2)}{t} \cdot 2tdt = \frac{2\log t}{t} \cdot 2tdt = 2\log t \cdot 2dt = 4\log tdt$$

$$\int_0^1 \frac{\log x}{\sqrt{x}}dx = \int_0^1 4\log tdt = 4\int_0^1 \log tdt = 4(-1) = -4$$

Solve $\int_0^1 \log tdt$ Integration by parts

$Test\ the\ convergence\ /\ divergence\ of\ the\ following\ improper\ integrals:$

$$\int_0^1 \sin\left(\frac{1}{x}\right)dx$$

2. Determine all those values of p for which the improper integral $\int_0^\infty \frac{1-e^{-x}}{x^p}dx$ converges.

Taylor expansion of e^x about $x = 0$ (also called the Maclaurin series) is:

split the improper integral at a convenient point

$$\int_0^\infty \frac{1-e^{-x}}{x^p}dx = \int_0^1 \frac{1-e^{-x}}{x^p}dx + \int_1^\infty \frac{1-e^{-x}}{x^p}dx$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

$$1 - e^{-x} = 1 - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right) = x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \cdots = x + Somthing$$

Behavior near 0

$$\frac{1-e^{-x}}{x^p} \approx \frac{x}{x^p} = x + x^{-p} = x^{1-p}$$

The integral $\int_0^1 x^{1-p}dx$ converges iff the exponent $1 - p > -1, i.e. p < 2$

Behavior as $x \to \infty$

$$\frac{1-e^{-x}}{x^p} \approx \frac{x}{x^p} \approx \frac{1}{x^p}$$

The integral $\int_1^\infty x^p dx$ converges iff the exponent $p > p$

Combine both conditions. $1 < p < 2$

→ Try by p-test (first & second) kind

3. Show that the integrals $\int_0^\infty \frac{\sin^2 x}{x^2}dx$ and $\int_0^\infty \frac{\sin x}{x}dx$ converge. Further, prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx.$$

$$Show\ that\ the\ integrals\ \int_0^\infty \frac{\sin^2 x}{x^2}dx\ and\ \int_0^\infty \frac{\sin x}{x}dx\ converge.$$

$$Further,\ prove\ that\ \int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx.$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}dx = \int_0^1 \frac{\sin^2 x}{x^2}dx + \int_1^\infty \frac{\sin^2 x}{x^2}dx$$

Near $x = 0$	As $x \to \infty$
we know $\lim_{x \to 0} \frac{\sin x}{x} = 1$ $\int_0^1 \frac{\sin^2 x}{x^2} = \lim_{x \to 0} \int_0^x \frac{\sin^2 x}{x^2} =$ $\left(\lim_{x \to 0} \int_0^x \frac{\sin x}{x}\right) = 1$	$\left\vert\frac{\sin^2 x}{x^2}\right\vert \leqslant \left\vert\frac{1}{x^2}\right\vert$ $\int_1^\infty \frac{1}{x^2}$ converges

So by comparison, the tail .

$\int_0^\infty \frac{\sin^2 x}{x^2} dx$ converges

$$\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + \int_1^\infty \frac{\sin x}{x} dx$$

Near $x = 0$	As $x \to \infty$
we know $\lim_{x \to 0} \frac{\sin x}{x} = 1$	$\int_1^\infty \frac{\sin x}{x} dx$ I dont know how but converges

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx = \int_0^\infty \frac{\sin x}{x} dx$$

Functions of several variables

1. Identify the points, if any, where the following functions fail to be continuous:

$(i)\ f(x, y) = \begin{cases} xy & if\ xy \geqslant 0 \\ -xy & if\ xy < 0 \end{cases}$

$(ii)\ f(x, y) = \begin{cases} xy & if\ xy\ is\ rationnal \\ -xy & if\ xy\ is\ irrational \end{cases}$

2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = \begin{cases} \dfrac{x^2y^2}{x^2y^2 + (x-y)^2} & if\ (x,\ y) \neq (0,\ 0) \\ 0 & if\ (x,\ y) = (0,\ 0) \end{cases}$$

Show that the function satisfy the following:

(a) The iterated limits $\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right]$ and $\lim_{y \to 0}\left[\lim_{x \to 0} f(x, y)\right]$ exist and equals 0;

(b) $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist;

(c) $f(x,\ y)$ is not continuous at $(0, 0)$

(d) the partial derivatives exist at $(0, 0)$.

3. Let $f(x, y) = \left(x^2 + y^2\right)\sin\dfrac{1}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$ and 0, otherwise.

Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at (0, 0)

4. Let $f(x, y) = |xy|$ for all $(x, y) \in \mathbb{R}^2$ Show that

(a) f is differentiable at (0, 0)

(b) $f_x(0, y_0)$ does not exist if $y_0 \neq 0$

5. Suppose f is a function with $f_x(x, y) = f_y(x, y) = 0$ for all (x, y). Then show that $f(x, y) = c$, a constant

---

Open/Closed Set

A set $U \subseteq \mathbb{R}$ is called open if: for every point $x \in U$, there exists an $\varepsilon > 0$ such that the open ball $B(x, \varepsilon) \subseteq U$. An open ball of radius ε centered at x is

$B(x, \varepsilon) = \left\{y \in \mathbb{R}^n : ||y - x|| < \varepsilon\right\}$

→ Every point in the set has some “wiggle room” around it that is still inside the set.

→ An open set does not include its boundary points.

Introduction to Open and Closed Sets by [Elliot Nicholson]

→ I did not understood till now but its very defination orriented.

Identifying Open, Closed, and Compact Sets by [WrathofMath]

→ Good for entry level understanding.

Open, closed, both and neither sets by [Joshua Helston]

→ Umm... Good since he talked about real line but i did not got much of it.

Double Integrals

Q. Integrate $f(x, y) = x^2 + y^2$ over unit disc by Dr V
→ Good explanation, Entry level.

Q. Integration in polar coordinates by David Jordan [MIT]

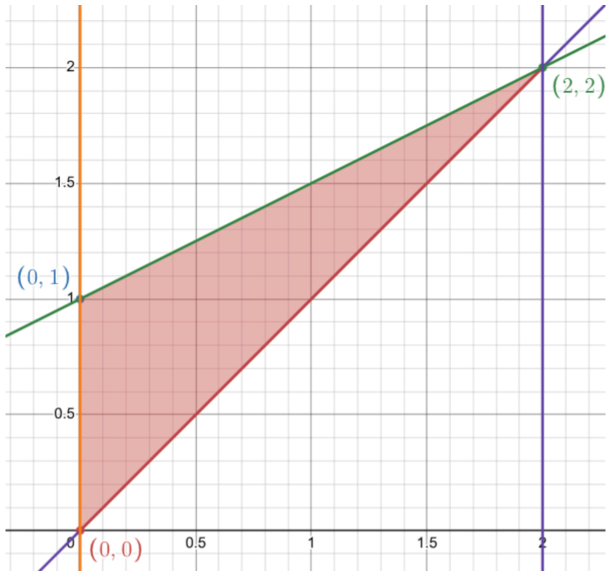
(a). $\int_{x=1}^{2}\int_{y=0}^{x} f\ dydx$ (b). $\int_{x=0}^{1}\int_{y=x^2}^{x} f\ dydx$ (c). $\int_{y=0}^{2}\int_{x-0}^{\sqrt{2y-y^2}} f\ dxdy$

→ Learn changing into polar coordinates

Q. $\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1} dydx$
→ by Trefor Bazett : Amazing explanation.

Evaluate $\iint_D (x+y)dA$ where D is the region bdd by the lines joining
$(0, 0), (0, 1)$ & $(2, 2)$.
$(x_1, y_1) = (0, 0)$ to $(x_2, y_2) = (2, 2)$
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-0}{2-0} = 1$, So the line has slope 1.
Line through $(0, 0)$ with slope 1. $(y - y_1) = m(x - x_1) = (y - 0) = 1(x - 0) \implies y = x$

$(x_1, y_1) = (0, 1)$ to $(x_2, y_2) = (2, 2)$
$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-1}{2-0} = \frac{1}{2}$, So the line has slope $\frac{1}{2}$
Line through $(0, 1)$ with slope $\frac{1}{2}$.
$(y - y_1) = m(x - x_1) = (y - 1) = \frac{1}{2}(x - 0) \implies y = 1 + \frac{1}{2}x$
Note : Don't focus on this, for exam just draw points and you'll get visual picture.



$$y : x \to \left(\frac{x}{2}+1\right)$$

$$x : 0 \to 2$$

$$\int_0^2 \int_x^{\frac{x}{2}+1} (x+y)dydx$$

→ Paste latex code of this equatio in ChaGPT and tell to solve.
I'm not adding here since its will make my note little long.
And Calculation is bit easy.

ans : $\frac{5}{3}$

Similar Example :

Evaluate the double integra $\iint_D (x+y)dA$ where D is the triangular region vertices $(0,0), (1,2)$ & $(0,3)$.
by Tam Cao

Evaluate double integral $\iint_D (x+y)dA$ where D is the triangular region with vertices $(0,0), (1,2), (0,3)$
By Academic Videos
→ Not THE best but quite similar so i'm adding. (No Audio 🔇)

Evaluate double integral $\iint_D y^3 dA$, where D is the triangular region with vertices (0,1), (7,0), (1,1)
by Julia
→ I think its one of the best Videos for getting line equation from points.

Search Help : site:youtube.com evaluate integral vertices (0,0), (0,1) and (2,2)

Q. Evaluate $\int_0^\pi \left(\int_x^\pi \frac{\sin y}{y} dy\right) dx = \iint_D \frac{\sin y}{y} dA$

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dydx \implies \int_0^\pi \int_0^y \frac{\sin y}{y} dxdy = \int_0^\pi \frac{\sin y}{y} \left(\int_0^y dx\right) dy$$

$$\int_0^y dx = [x]_0^y = [y - 0] = y$$

$$\int_0^\pi \frac{\sin y}{y}(y)dy = \int_0^\pi \sin y \ dy = [-\cos y]_0^\pi = [-\cos(\pi) - (-\cos(0))]$$

$$= [-(-1) - (-1)] = [1 + 1] = 2$$

$$\cos \pi = -1$$
$$\cos 0 = 1$$

by Deepak Subedi

→ Umm, Good but i alredy knew so its got me even i watched it by skiping

Search Help : site:youtube.com siny y dy dx

Q. Evaluate $\iint_{(x-1)^2+y^2\leqslant 1} \sqrt{x^2+y^2}dydx$

by Maksym Zubkov

→ Exactly this example explained, but not go into solution since i think he solve wrong. ChatGPT solve like this. My friend also solve like this so i think we're correct

$(x-1)^2 + (y-0)^2 \leqslant 1$, This is the standard circle form $(x-h)^2 + (y-k) \leqslant r^2$

Center: $(h, k) = (1, 0)$, Radius : $r = 1$

Left most point $(h - r, k) = (1 - 1, 0) = (0, 0)$ *(move to left i.e substract)*

Right most point $(h + r, k) = (1 + 1, 0) = (2, 0)$ *(move to right, i.e add)*

Top point $(h, r + k) = (1, 0 + 1) = (1, 1)$

Bottom point $(h, r - k) = (1, 0 - 1) = (1, -1)$

→ Draw circle out of it.

Since its circle its better to use polar coordinates

$x = r\cos\theta, \ \ y = r\sin\theta$

$x^2 + y^2 = r^2, \ \ dA = rdrd\theta$

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$(x-1)^2 + y^2 = 1 \Longrightarrow (x^2 - 2x + 1^2 + y^2 = 1 \Longrightarrow x^2 + y^2 = 2x$

$x = r\cos\theta, \ \ y = r\sin\theta$

$x^2 + y^2 = 2x \Longrightarrow r^2 = 2r\cos\theta \Longrightarrow r = 2\cos\theta$

$r \in [0, 2\cos\theta]$

$$\iint_{(x-1)^2+y^2\leqslant 1}\sqrt{x^2+y^2}dydx \Rightarrow \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\sqrt{r^2}rdrd\theta \Longrightarrow \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}r^2drd\theta$$

$$\int_0^{2\cos\theta}r^2dr=\frac{1}{3}\left[r^3\right]_0^{2\cos\theta}=\frac{1}{3}\left[(2\cos\theta)^3-0^3\right]=\frac{1}{3}\left(8\cos^3\theta\right)$$

$$\int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos\theta}r^2dr\right)d\theta=\int_{-\pi/2}^{\pi/2}\left(\frac{1}{3}\left(8\cos^3\theta\right)\right)d\theta=\frac{8}{3}\int_{-\pi/2}^{\pi/2}\cos^3\theta d\theta$$

$$\cos^3\theta=\cos\theta\cdot\cos^2\theta$$

$$\because \sin^2\theta+\cos^2\theta=1\Longrightarrow \cos^2\theta=1-\sin^2\theta$$

$$\frac{8}{3}\int_{-\pi/2}^{\pi/2}\cos\theta\left(1-\sin^2\theta\right)d\theta$$

let, $u=\sin\theta,\ \ du=\cos\theta d\theta$

When $\theta=-\dfrac{\pi}{2},u=\sin\left(-\dfrac{\pi}{2}\right)=\,-1$

When $\theta=\dfrac{\pi}{2},u=\sin\left(\dfrac{\pi}{2}\right)=1$

$$\frac{8}{3}\int_{-1}^{1}\left(1-u^2\right)du=\frac{8}{3}\left[u-\frac{u^3}{3}\right]_{-1}^{1}=\frac{8}{3}\left[1-\left(-\frac{1}{3}\right)\right]=\frac{8}{3}\left[\frac{4}{3}\right]=\frac{32}{9}$$

Area of a Circle With Double Integrals by [TheCalcSeries Guy]()

Search Help : site:youtube.com double integral unit circle polar coordinate

Q. $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}dydx=\iint_D\sqrt{1-y^2}dA$

$y:0\rightarrow\sqrt{1-x^2}$

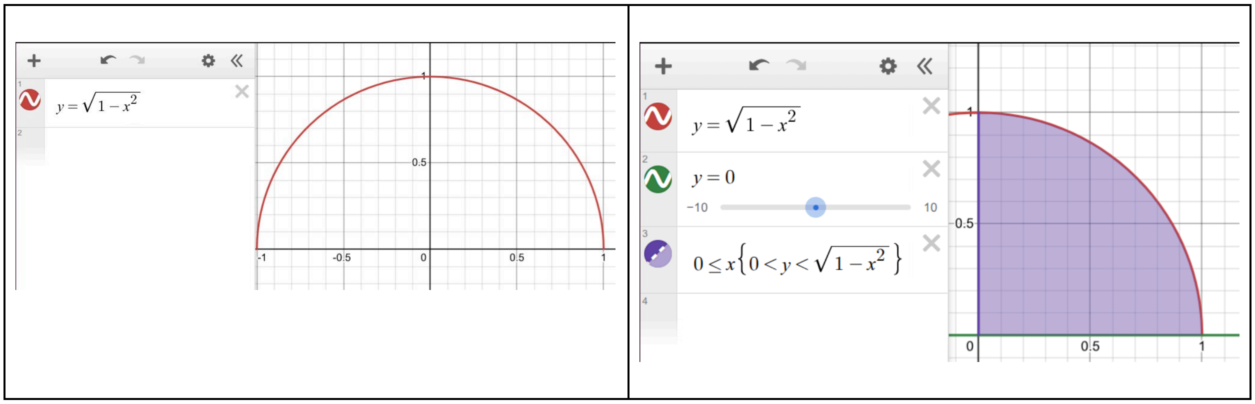
$x:0\rightarrow 1$

Plot region,

$y=\sqrt{1-x^2}\Longrightarrow y^2=1-x^2\Longrightarrow x^2+y^2=1$ *its circle.*

so, $y=\sqrt{1-x^2}$ is semicircle +ve to y-axis.

And since $0\leqslant x\leqslant 1$ so it lie in first quadrant only.



Change order of integration by Fubini's Theorem.

$x:0\rightarrow\sqrt{1-y^2}$
$y:0\rightarrow 1$

Region D is the quarter unit disk in the first quadrant:

$$D=\left\{(x,y):x\geqslant 0,\ \ y\geqslant 0,x^2+y^2\leqslant 1\right\}$$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\sqrt{1-y^2}dydx=\int_0^1\int_0^{\sqrt{1-y^2}}\sqrt{1-y^2}dxdy=\int_0^1\sqrt{1-y^2}\left(\int_0^{\sqrt{1-y^2}}dx\right)dy$$

$$\int_0^{\sqrt{1-y^2}}dx=[x]_0^{\sqrt{1-y^2}}=\left[\sqrt{1-y^2}-0\right]=\sqrt{1-y^2}$$

$$\int_0^1\sqrt{1-y^2}\left(\int_0^{\sqrt{1-y^2}}dx\right)dy=\int_0^1\sqrt{1-y^2}\left(\sqrt{1-y^2}\right)dy=\int_0^1 1-y^2dy$$

$$\int_0^1 1-y^2dy=\left[y-\frac{y^3}{3}\right]_0^1=\left[\left(1-\frac{1}{3}\right)-\left(0-\frac{0}{3}\right)\right]=\left(1-\frac{1}{3}\right)=\frac{2}{3}$$

Q. Find the volume of the solid which is common to the cylinders
$x^2+y^2=1\ \&\ x^2+z^2=1$
by Ashwani Goyal
→ Superb, Prof. solved generalized version of this problem.

$x^2+y^2=1$ is circle on xy-plan and Cylinder to z-axis in $3D$.
$x^2+z^2=1$ is circle on xz-plan and Cylinder to y-axis in $3D$.

$$\iiint_D dzdydx$$

$x^2+z^2=1\implies z^2=1-x^2\implies z=\pm\sqrt{1-x^2}$
$z:\left(-\sqrt{1-x^2}\right)\rightarrow\left(\sqrt{1-x^2}\right)$

$x^2+y^2=1\implies y^2=1-x^2\implies y=\pm\sqrt{1-x^2}$
$y:\left(-\sqrt{1-x^2}\right)\rightarrow\left(\sqrt{1-x^2}\right)$

$x: -1 \to 1$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dzdydx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\right)dydx$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz = [z]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \left[\left(\sqrt{1-x^2}\right) - \left(-\sqrt{1-x^2}\right)\right] = 2\sqrt{1-x^2}$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(2\sqrt{1-x^2}\right)dydx = \int_{-1}^{1}\left(2\sqrt{1-x^2}\right)\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right)dx$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = [y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \left[\left(\sqrt{1-x^2}\right) - \left(-\sqrt{1-x^2}\right)\right] = 2\sqrt{1-x^2}$$

$$\int_{-1}^{1}\left(2\sqrt{1-x^2}\right)\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right)dx = \int_{-1}^{1}\left(2\sqrt{1-x^2}\right)\left(2\sqrt{1-x^2}\right)dx = 4\int_{-1}^{1}\left(1-x^2\right)dx$$

$$4\int_{-1}^{1}\left(1-x^2\right)dx = 4\left[x - \frac{x^3}{3}\right]_{-1}^{1} = 4\left[\left(1-\frac{1^3}{3}\right) - \left((-1) - \frac{(-1)^3}{3}\right)\right] = 4\left[\frac{4}{3}\right] = \frac{16}{3}$$

Or, since its symmetric

$$4\int_{-1}^{1}\left(1-x^2\right)dx = 4\cdot 2\left(\int_{0}^{1} 1 - x^2 dx\right) = 8\left[x - \frac{x^3}{3}\right]_{0}^{1} = 8\left[1-\frac{1^3}{3}\right] = 8\left[\frac{2}{3}\right] = \frac{16}{3}$$

Find the volume common to the Cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$

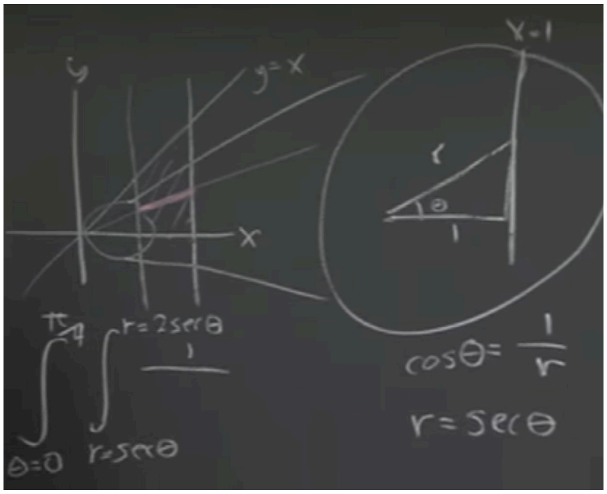
By [Jaswinder Kaur](#) → I did not understood but i'm keeping this

Search Help : (on Google.com) Find the volume of the solid which is common to the cylinders \math-container{x\power{2}+y\power{2}=1 & x\power{2}+z\power{2}=1}

Q. Evaluate $\int_{1}^{2}\int_{0}^{x}\frac{1}{\left(x^2+y^2\right)^{3/2}}dydx$

by [David Jordan [MIT]](#) at [11:40](#)

→ Good, I will recommend to watch and that way solve.



$y: 0 \to x$
$x: 1 \to 2$

$\theta: 0 \to \frac{\pi}{4}$ (π / 4 *is* 45°)
$r: \sec\theta \to 2\sec\theta$

$x^2 + y^2 = r^2$
$\left(x^2+y^2\right)^{\frac{3}{2}} = \left(r^2\right)^{\frac{3}{2}} = r^3$

$$\int_1^2\int_0^x \frac{1}{\left(x^2+y^2\right)^{3/2}}dydx = \int_0^{\frac{\pi}{4}}\int_{\sec\theta}^{2\sec\theta}\frac{1}{r^3}rdrd\theta = \int_0^{\frac{\pi}{4}}\int_{\sec\theta}^{2\sec\theta}\frac{1}{r^2}drd\theta = \int_0^{\frac{\pi}{4}}\left(\left[-\frac{1}{r}\right]_{\sec\theta}^{2\sec\theta}\right)d\theta$$

$$\frac{1}{r^2} = r^{-2} \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\left[-\frac{1}{r}\right]_{\sec\theta}^{2\sec\theta} = \left[\left(-\frac{1}{2\sec\theta}\right) - \left(-\frac{1}{\sec\theta}\right)\right] = \left[\left(-\frac{1}{2\sec\theta}\right) + \left(\frac{1}{\sec\theta}\right)\right]$$

$$\cos\theta = \frac{1}{\sec\theta}$$

$$\left[\left(-\frac{1}{2}\cos\theta\right) + \cos\theta\right] = \left(\cos\theta - \frac{1}{2}\cos\theta\right) = \frac{1}{2}\cos\theta$$

$$\int_0^{\frac{\pi}{4}}\left(\left[-\frac{1}{r}\right]_{\sec\theta}^{2\sec\theta}\right)d\theta = \int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\cos\theta\right)d\theta = \frac{1}{2}\left(\int_0^{\frac{\pi}{4}}\cos\theta d\theta\right) = \frac{1}{2}[\sin\theta]_0^{\frac{\pi}{4}}$$

$$\frac{1}{2}\left[\sin\frac{\pi}{4} - \sin 0\right] = \frac{1}{2}\left[\frac{\sqrt{2}}{2} - 0\right] = \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{4}$$

Q. $\int_0^1\int_0^{1-y}\sqrt{x+y}(y-2x)^2dxdy$

Nice integral — it simplifies cleanly with a linear change of variables.
take $u = x + y$, $v = y - 2x$

$u = x + y \Longrightarrow y = u - x$

$$v = y - 2x \implies v = (u - x) - 2x = v = u - 3x \implies 3x = u - v \implies \ x = \frac{u}{3} - \frac{v}{3}$$

$$y = u - x \implies y = u - \left(\frac{u}{3} - \frac{v}{3}\right) \implies y = \frac{3u}{3} - \frac{u}{3} + \frac{v}{3} \implies y = \frac{2u}{3} + \frac{v}{3}$$

Waiii..t........I solved this During JAM-2025, I did not knew how to choose u and v back then. But okay, i learnet that now.

Q. Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx$ using the substitution $x = \frac{u}{3} - \frac{v}{3}$, $y = \frac{2u}{3} + \frac{v}{3}$

Compute the Jacobian

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = \frac{u}{3} - \frac{v}{3},$$

$$\frac{\partial x}{\partial u} = \frac{1}{3}, \frac{\partial x}{\partial v} = -\frac{1}{3}$$

$$y = \frac{2u}{3} + \frac{v}{3}$$

$$\frac{\partial y}{\partial u} = \frac{2}{3}, \frac{\partial y}{\partial v} = \frac{1}{3}$$

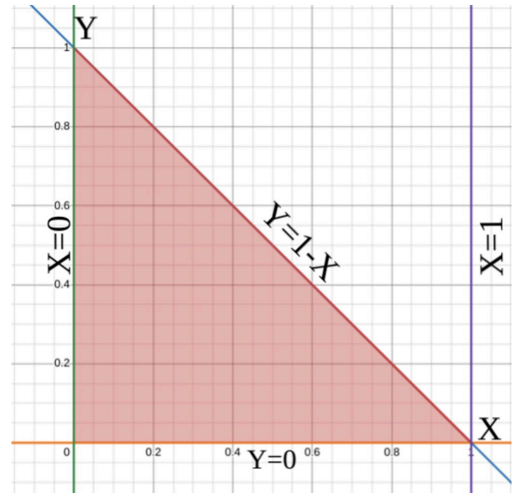
$$J(u,v) = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \left| \left(\frac{1}{3} \times \frac{1}{3}\right) - \left(\frac{2}{3} \times \left(-\frac{1}{3}\right)\right) \right| = \left| \frac{1}{9} - \left(-\frac{2}{9}\right) \right| = \left| \frac{3}{9} \right| = \left| \frac{1}{3} \right|$$

$$|\ J(u,v)\ | = \left| \frac{1}{3} \right| = \frac{1}{3}$$

Draw the origianal region and find its actual boundaries

$0 \leqslant y \leqslant (1-x)$
$0 \leqslant x \leqslant 1$

$x=0 \quad x=1$
$y=0 \quad y=1-x \Longrightarrow x+y=1$
$x+y=1,$
put $x=0$ in $x+y=1\ \ y=1$
$y=0,\ x=1$
$x=2,\ y=-1$
$y=2,\ x=-1$



Convert the boundaries into new boundaries by using substitutions

Old boundaries
$y=0,\ y=1-x$
$x=0, x=1$

I don't really need of $x=1$ because my region bounde without $x=1$.

New boundaries

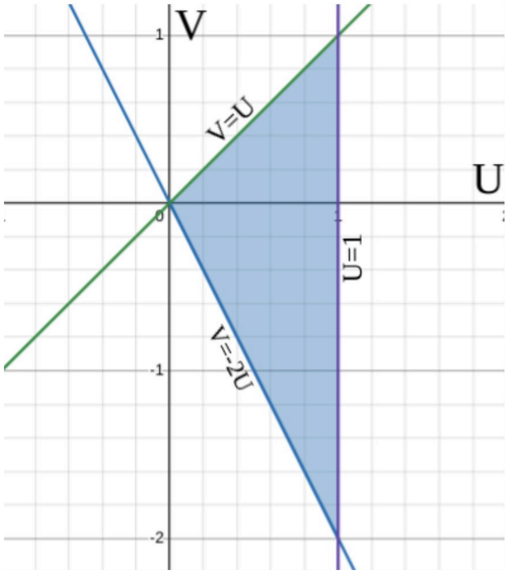
$$\boxed{x=\frac{u}{3}-\frac{v}{3}} \quad \boxed{y=\frac{2u}{3}+\frac{v}{3}}$$

$y=0,$
$$0=\frac{2u}{3}+\frac{v}{3} \Longrightarrow 2u+v \Longrightarrow u=-\frac{v}{2}$$

$x=0$
$$0=\frac{u}{3}-\frac{v}{3} \Longrightarrow u-v=0 \Longrightarrow u=v$$

$y=1-x,$
$$\left(\frac{2u}{3}+\frac{v}{3}\right)=1-\left(\frac{u}{3}-\frac{v}{3}\right)$$
$$\left(\frac{2u}{3}+\frac{v}{3}\right)+\left(\frac{u}{3}-\frac{v}{3}\right)=1$$
$$\frac{2u}{3}+\frac{u}{3}=1 \Longrightarrow \frac{3u}{3}=1 \Longrightarrow u=1$$

Step 5: Draw new region, find new limits & integrate



$u = -\dfrac{v}{2} \Longrightarrow v = -2u$

similar to $v = 2u$ but in $-ve$ sign

$u = v, \;\; u = 1$

Arrow

$v \;:\; -2u \to u$

Brush

$u \;:\; 0 \to 1$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx \Longrightarrow \int_0^1 \int_{-2u}^{u} \sqrt{u}(v)^2 \left(\frac{1}{3}\right) dvdu$$

$$\because \sqrt{x+y} = \sqrt{\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right)} = \sqrt{u}$$

$$\because y - 2x = \left(\frac{2u}{3} + \frac{v}{3}\right) - 2\left(\frac{u}{3} - \frac{v}{3}\right) = v$$

& Jacobian=$\dfrac{1}{3}$

$$\frac{1}{3}\int_0^1 \sqrt{u} \int_{-2u}^{u} v^2 dvdu$$

$$\int_{-2u}^{u} v^2 dv = \left[\frac{v^3}{3}\right]_{-2u}^{u} = \frac{1}{3}\left[v^3\right]_{-2u}^{u} = \frac{1}{3}\left(u^3 - (-2u)^3\right) = \frac{1}{3}\left(u^3 + 8u^3\right) = \frac{1}{3}9u^3 = 3u^3$$

$$\frac{1}{3}\int_0^1 \sqrt{u}\; (3u^3) du = \frac{1}{3} \cdot 3 \int_0^1 u^{\frac{1}{2}} u^3 du = \int_0^1 u^{\left(\frac{1}{2}+3\right)} du = \int_0^1 u^{\frac{7}{2}} du = \left[\frac{u^{\frac{7}{2}+1}}{\frac{7}{2}+1}\right]_0^1 = \left[\frac{u^{\frac{9}{2}}}{\frac{9}{2}}\right]_0^1$$

$$= \frac{2}{9}\left[u^{\frac{9}{2}}\right]_0^1 = \frac{2}{9}\left[(1)^{\frac{9}{2}} - (0)^{\frac{9}{2}}\right] = \frac{2}{9}$$

Triple Integrals

Q. Find volume occupied by $x^2+y^2+z^2=1$ using spherical coordinates.
$\rho\ :\ 0 \to 1$ - ρ is similer to r in 2D. it used to get radius of Sphere from core.
$\phi\ :\ 0 \to \pi$ - ϕ used to cover half most region of sphere. one end to all the way down on z axis
$\theta\ :\ 0 \to 2\pi$ - θ is as usual used to get angle.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^2\sin\phi\ d\rho\ d\phi\ d\theta=\int_0^{2\pi}\int_0^{\pi}\sin\phi\int_0^{1}\rho^2\ d\rho\ d\phi\ d\theta$$

Since they're independent eachother, I can integrate them separately

$$\int_0^1\rho^2\ d\rho=\left[\frac{\rho^3}{3}\right]_0^1=\left[\frac{(1)^3}{3}-\frac{(0)^3}{3}\right]=\frac{1}{3}$$

$$\int_0^{\pi}\sin\phi d\phi=[-\cos\phi]_0^{\pi}=[(-\cos\pi)-(-\cos 0)]=[(-(-1))-(-1)]=1+1=2$$

$$\int_0^{2\pi}d\theta=2\pi$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\rho^2\sin\phi\ d\rho\ d\phi\ d\theta=\frac{1}{3}\times 2\times 2\pi=\frac{4\pi}{3}$$

Reference : Steve Chow (BPRP),
Converting a triple integral among rectangular, cylindrical and spherical coordinates by Steve Chow (BPRP).
Rectangular, cylindrical, and spherical coordinates (introduction & conversion) by Steve Chow (BPRP),

Q. Volume bounded by the surface $x^2+y^2+z^2=36$ & $z=-\sqrt{3x^2+3y^2}$ is _____

Q. How to change the order of a triple integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y}f(x,y,z)dzdydx=\iiint f(x,y,z)dxdydz$$

by Steve Chow (BPRP)

Q. $\displaystyle\int_0^2\int_0^{\sqrt{4-y^2}}\int_{\sqrt{3(x^2+y^2)}}^{\sqrt{16-x^2-y^2}}(x^2+y^2+z^2)dzdxdy$

by Michael Penn :
→ Don't do this problem just watch it for understanding. Prof. will not give this tricky problem.

Sequence of Functions

A sequence of functions is a list

$$\{f_n(x)\}_{n=1}^{\infty} f_1(x),\ \ f_2(x),\ \ f_3(x),...$$

where each f_n is a function defined on the same domain D.

Types of Convergence

i. Pointwise Convergence

ii. Uniform Convergence

LLMs Help : Say GPT to show poinwise or uniform by plugin values and exaplin it detail. Don't ask big think, ask small thing to explain it in detail.

Pointwise Convergence

$f_n \xrightarrow{p} f_0$

Uniform convergence means:

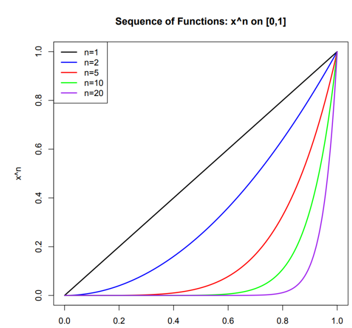
$$\sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0$$

$\forall \varepsilon > 0\ \exists N \in$ s.t. $\forall x \in [0, 1]$ and
$\forall n > N,$
$|f_n(x) - f(x)| < \varepsilon$

pointwise but not uniform convergence

sequence $f_n(x) = x^n$ on $[0, 1]$
As n increases, the curves drop lower and lower toward 0 for every $x < 1,$
but always stay 1 at $x = 1.$

$$f(x) = \begin{cases} 0, & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$



#Made in R

Pointwise & Uniform convergence of sequence $f_n(x) = x^n$ on $[0, 1]$
by Rahul Mapari : Supereb explanation. In detailed manner (even i watched it in 2x still understood)

Q. $f_n(x) = x$ on $[0, 1]$

Q. $f_n(x) = x^2(1-x)x^n$ on $[0, 1]$

When $x \neq 0, n^2(1-x)x^n = \left(n^2 - n^2x\right)x^n = n^2x^n - n^2x^{n+1} \to 0 \ \forall x \in [0,1]$

Here $f_n \xrightarrow{p} f_0$ where $f_0 = 0$

If $x = 0 : f_n(0) = n^2(1-0)0^n = 0$ for every $n,$ so the limit is 0.

If $x = 1 : f_n(1) = n^2(1-1)1^n = 0$ for every $n,$ so the limit is 0.

If $0 < x < 1 :$ where $f(x) = n^2(1-x)x^n = 0$

Q. $f_n(x) = \sqrt{x^2 + \dfrac{1}{n^2}}$ on $(-1, 1)$

as $n \to \infty$, $\dfrac{1}{n^2} \to 0.$ so $\lim\limits_{n\to\infty} \sqrt{x^2 + \dfrac{1}{n^2}} = \sqrt{x^2} = |x|$

So f_n converges pointwise to $f(x) = |x|$ on $(-1, 1)$.

Q. $f_n(x) = \dfrac{\sin 2\pi nx}{n}$ on $(-1, 1)$

$-1 \leqslant \sin x \leqslant 1 \Longrightarrow |\sin x| \leqslant 1. \ |\sin 2n\pi x| \leqslant 1$

$\left|\dfrac{\sin 2\pi nx}{n}\right| \leqslant \dfrac{1}{n}$ and as $n \to \infty$, $\dfrac{1}{n^2} \to 0$

Thus the pointwise limit is the zero function $f(x) = 0$.

Uniform convergence means: $\lim\limits_{n\to\infty} \sup\limits_{x\in(-1,1)} |f_n(x) - 0| = 0$

For any x, $|f_n(x)| = \left|\dfrac{\sin 2\pi nx}{n} - 0\right| \leqslant \dfrac{1}{n} \Longrightarrow \left|\dfrac{\sin 2\pi nx}{n}\right| \leqslant \dfrac{1}{n}$

$\lim\limits_{n\to\infty} \sup\limits_{x\in(-1,1)} |f_n(x) - f_0| \to 0$, Therefore $f_n \xrightarrow{u} f_0$ on $(-1, 1)$.

⟹ So its Pointiwise converge and Uniformally also.

What about f' ?

$f'_n(x) = \dfrac{d}{dx}\left(\dfrac{\sin 2\pi nx}{n}\right) = (\cos 2\pi nx)2\pi$

$f'_0(x) = 0 \ \forall x \in (-1, 1)$

Now, $f'_n\left(\dfrac{1}{2}\right) = (-1)^n$. $f'_n \nrightarrow f'_0$ on $(-1, 1)$ (pointwise).

Q. $f_n(x) = \begin{cases} 2n \quad on \left[\frac{1}{n}, \frac{2}{n}\right] \\ 0 \quad\; on\ [0,1] \setminus \left[\frac{1}{n}, \frac{2}{n}\right] \end{cases}$

Q. $f_n(x) = \frac{x}{1+nx^2}, \ x \in \mathbb{R}$

Show that the sequence $\{f_n\}$ where $f_n(x) = \frac{x^n}{n}, x \in [0,1]$ converges uniformally to 0

by Success only guy

→ Nice, but he explain bit more detailed. But anyways its better for entry level understanding.

$f_n(x) = \frac{x^2}{n}, \ x \in [0,1]$

$f_n(x) = x^n, \ x \in [0,1]$

by Maksym Zubkov : Nice explanation. Watch it carefully.

Uniform Convergence by Marc Renault

→ Great explanation! Great visual aids. **Must Watch**

Series of Functions

A series of functions is: $\sum_{n=1}^{\infty} f_n(x)$ Define partial sums: $S_N(x) = \sum_{n=1}^{N} f_n(x)$

The series converges if $S_N(x) \to S(x)$

Weierstrass M-test

Weierstrass M-test - Wikipedia

Suppose that $f_n(x)$ is a sequence of real- or complex-valued functions (*here, its Real*) defined on a set A, and that there is a sequence of non-negative numbers M_n satisfying the conditions

1. $|f_n(x)| \leqslant M_n$ for all $n \geqslant 1$ and $x \in A$, and
2. $\sum_{n=1}^{\infty} M_n$ converges.

Then the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on A.

Eg : Consider the series of functions: $\sum_{n=1}^{\infty} \frac{x^n}{n^2}, \;\; x \in [0,1]$

We want to check whether this series converges uniformly on $[0,1]$

$f_n(x) = \frac{x^n}{n^2}$

Find a bound M_n, We know that for $x \in [0,1] \to 0 \leqslant x^n \leqslant 1$

So, $|f_n(x)| = \left| \frac{x^n}{n^2} \right| \leqslant \frac{1}{n^2} = M_n$ ------------------------(1. condition✅)

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ This is a p-series with $p = 2 > 1$, so it converges. --(2. condition✅)

By Weierstrass M-Test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0,1]$

Q. $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2} \;\; x \in [0,1]$

$-1 \leqslant \sin x \leqslant 1 \implies |\sin x| \leqslant 1. \; |\sin nx| \leqslant 1$

$\left| \frac{\sin nx}{n^2} \right| \leqslant \frac{1}{n^2} = M_n$------------------------(1. condition✅)

$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ This is a p-series with $p = 2 > 1$, so it converges. --(2. condition✅)

By Weierstrass M-Test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0,1]$

Q. $\sum_{n=1}^{\infty} \frac{\log(1+nx)}{nx^n}, \ x \in [2, \infty)$

$\ln(1+nx) \leqslant nx$ for all $n \geqslant 1, x \geqslant 0$

$\left| \frac{\log(1+nx)}{nx^n} \right| \leqslant \frac{nx}{nx^n} = \frac{1}{x^{n-1}} \leqslant \frac{1}{2^{n-1}} = M_n$ -------------------------(1. condition✅)

On the domain $x \in [2, \infty)$ we have $x^{n-1} \geqslant 2^{n-1}$

$$\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \sum_{k=0}^{\infty} \frac{1^k}{2^k} = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

Do an index shift: set $k = n-1$

This is a geometric series with first term 1 and ration $r = \frac{1}{2}$. A geometric series $\sum_{n=1}^{\infty} r^k$ converges iff $|r| < 1$, and its sum is $\frac{1}{1-r}$. So here $\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{1-\frac{1}{2}} = 2$

So $\sum_{n=1}^{\infty} M_n$ converges (to 2) and the M-test applies.

By Weierstrass M-Test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[2, \infty)$

Q. $\sum_{n=1}^{\infty} \left(xe^{-x}\right)^n, \ x \in [0, 2]$

$\left(xe^{-x}\right)^n \leqslant \left(e^{-1}\right)^n = \frac{1}{e^n}$

Q. $\sum x^n$ on $[-M, M], \ 0 < M < 1$

Q. $\sum_{n=1}^{\infty} \frac{x}{(1+x)^n}$ on $[1, 2]$

$$\sum_{n=1}^{\infty} \frac{x}{(1+x)^n} \ on \ [1, 2]$$