

Real Analysis

https://home.iitk.ac.in/~psraj/mth101/lecture_notes/lecture15-16.pdf

<https://home.iitk.ac.in/~psraj/mth101/practice-problems/pp15.pdf>

<https://home.iitk.ac.in/~psraj/mth101/practice-problems/pp16.pdf>

[Riemann integral - Wikipedia](#)

[Real Analysis 48 | Riemann Integral - Partitions by The Bright Side of Mathematics \(YouTube\)](#)

→ Good but for the time being I don't understand everything.

Riemann Integral - Upper & Lower Darboux Sum [Part-1](#) [Part-2](#) by Dr. Gajendra Purohit

→ Good but basic level. But in my case i have to consider for n partision. Its Good for understanding.

[Real Analysis | Partitions and upper/lower sums by Micheal Penn](#) Good explanation with example

Theorem 5 : [link1](#)

Riemann Integration

Riemann Integration by [Bhavna Khurana](#) 

→ Good explanation. "Love this" As many time i see this, i appriciate more and more.

Riemann Sum and Riemann Integral Explained by [Poenaru Dan](#)

→ Good, help for Visualize.

Necessary and sufficient condition for Riemann integrability by [Mark Arokiasamy](#)

→ I did not watch video carefully so i can't tell how it is, but he explain problems so well that's why i'm keeping this video.

Problems

Q. Show that $f(x) = 3x + 1$ is integrable on $[1, 2]$ and $\int_1^2 (3x + 1) dx = \frac{11}{2}$.

→ by [Mark Arokiasamy](#) : Super detailed, problem driven explanation. And since he explain in

detailed manner it 30 min long (*part-1*). I will encourage you to watch this problem before other.

Q. Show that the function f defined on $[0, 1]$ by

$$f(x) = \begin{cases} 1 & ; x \text{ is rational} \\ 0 & ; x \text{ is irrational} \end{cases}$$

is not integrable on $[0, 1]$.

→ [Video](#)

Q. Show that $f(x) = \sin x$ is Riemann integrable over $\left[0, \frac{\pi}{2}\right]$

→ [Video](#)

Q. Show that $f(x) = x^2$ is integrable on $[0, a]$ and $\int_0^a x^2 dx = \frac{a^3}{3}$

→ by [Mark Arokiasamy](#)

→  watch to understand process. Prof will not give questions like this.

Q. Show that $f(x) = \sin(x)$ is integrable on $\left[0, \frac{\pi}{2}\right]$ and $\int_0^{\frac{\pi}{2}} \sin x dx = 1$.

→ by [Mark Arokiasamy](#)

→  watch to understand process. Prof will not give questions like this.

Class Examples

Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{x}$ is integrable on $[1, 2]$

→ [assignment](#) 3/7 page, assignment-6, Q1.

→ Ask ChaGPT, Solve this problem as Defination step by step in details manner. (*Give bookish version not in general explanation, prof. will not allow any ans*)

→ Followed [Mark Arokiasamy](#)'s steps.

$f(x) = \frac{1}{x}$ is bounded on $[1, 2]$. $\left(\frac{1}{2} \leq f(x) \leq 1\right)$ also f is strictly decreasing on $[1, 2]$.

Consider the partition $P = \{a = x_0, x_1, x_2, \dots, x_{r-1}, x_r, x_n = b\}$

First point of division = $x_0 = a = 1$

Second point of division = $x_1 = a + \frac{b-a}{n} = 1 + \frac{2-1}{n} = 1 + \frac{1}{n}$

$$(r+1)^{th} \text{ point of division} = x_r = a + \frac{r(b-a)}{n} = 1 + \frac{r(2-1)}{n} = 1 + \frac{r}{n}$$

$$(n+1)^{th} \text{ point of division} = x_n = a + \frac{n(b-a)}{n} = 1 + \frac{n(2-1)}{n} = 1 + \frac{n}{n} = 2$$

partition becomes $P = \left\{ 1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, \dots, 1 + \frac{(r-1)}{n}, 1 + \frac{r}{n}, 2 \right\}$

$$I_r = r^{th} \text{ subinterval} = [x_{r-1}, x_r] = \left[1 + \frac{(r-1)}{n}, 1 + \frac{r}{n} \right]$$

$$\Delta x_r = \text{Length of each Subinterval} : \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

$$M_r = \sup\{f(x) : x_{r-1} < x < x_r\} = \sup_{[x_{r-1}, x_r]} f = f(x_{r-1}) = \frac{1}{x_{r-1}}$$

$$m_r = \inf\{f(x) : x_{r-1} < x < x_r\} = \inf_{[x_{r-1}, x_r]} f = f(x_r) = \frac{1}{x_r}$$

$$U(P, f) = \sum_{r=1}^n M_r \Delta x_r = \sum_{r=1}^n \frac{1}{x_{r-1}} \cdot \frac{1}{n}$$

$$L(P, f) = \sum_{r=1}^n m_r \Delta x_r = \sum_{r=1}^n \frac{1}{x_r} \cdot \frac{1}{n}$$

$$U(P, f) - L(P, f) = \frac{1}{n} \sum_{r=1}^n \left(\frac{1}{x_{r-1}} - \frac{1}{x_r} \right) = \frac{1}{2n} < \epsilon.$$

If f and g are continuous functions on $[a, b]$ and if $g(x) \geq 0$ for $a \leq x \leq b$,
then show the mean value theorem for integrals : there exists $c \in [a, b]$ such that

$$\int_a^b f(x)g(x)dx = f(c) \int_a^b g(x)dx.$$

(a). Show that there is no continuous function f on $[0, 1]$ such that $\int_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}}$ for
all $n \in \mathbb{N}$.

(b). If f is continuous on $[a, b]$ then show that there exists $c \in [a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

(c). If f and g are continuous on $[a, b]$ and $\int_a^b f(x)dx \int_a^b g(x)dx$ then show that there exist

$c \in [a, b]$ such that $f(c) = g(c)$.

Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{x}$ is integrable on $[1, 2]$

Practice problem 15

<https://home.iitk.ac.in/~psraj/mth101/practice-problems/pp15.pdf>

Q1. Skipped

Q2. Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function. Suppose that there is a partition P of $[a, b]$ such that $L(P, f) = U(P, f)$.

Show that f is a constant function.

→ Use hint. It's trivial but I can't prove it properly so remember hint. Atleast you might get partial mark.

Q3. Let $f: [a, b] \rightarrow \mathbb{R}$ be a bounded function and $f(x) \geq 0$ for every $x \in [a, b]$.

Show that $\int_a^b f(x)dx \geq 0$ and $\int_a^{\frac{b}{2}} f(x)dx \geq 0$.

In addition, if f is integrable, show that $\int_a^b f(x)dx \geq 0$

→ [Follows from the definitions](#)

4. In each of the following cases, evaluate the upper and lower integrals of f and show that f is integrable. Find the integral of f .

(a) For $\alpha \in \mathbb{R}$, define $f: [a, b] \rightarrow \mathbb{R}$ by $f(x) = \alpha$ for every $x \in [a, b]$

(b) $f(x) = 0$ for $0 \leq x < \frac{1}{2}$, $f\left(\frac{1}{2}\right) = 10$ and $f(x) = 1$ for $\frac{1}{2} < x \leq 1$.

(c) $f(x) = x$ for all $x \in [0, 1]$.

→ To solve (c) see [this](#) video by Dr. Gagendra Puhohit from 16:20.

5. Let $f: [a, b] \rightarrow \mathbb{R}$ be integrable and P_n be a partition such that $U(P_n, f) - L(P_n, f) \rightarrow 0$.

Show that $\lim_{n \rightarrow \infty} L(P_n, f) = \lim_{n \rightarrow \infty} U(P_n, f) = \int_a^b f(x)dx$.

→ It's easy when you understand "Theorem-3"

6. In each of the following cases, show that f is integrable using the Riemann criterion.

- (a) $f(x) = x$ on $[0, 1]$
- (b) $f(x) = x^2$ on $[0, 1]$
- (c) $f(x) = \frac{1}{x}$ on $[1, 2]$

7. Let f , f_1 and f_2 be bounded functions on $[0, 1]$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in [0, 1]$. Suppose that f_1 and f_2 are integrable and $\int_0^1 f_1 f_1(x) dx = \int_0^1 f_2(x) dx$, show that f is integrable and find $\int_0^1 f(x) dx$.

<https://home.iitk.ac.in/~psraj/mth101/assignments/assignment.pdf>

1. Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{x}$ is integrable on $[1, 2]$

[Problems on Riemann Integration : Part I](#) by Mark Arokiasamy

→ Amazing Explanation (*I think this is because i already knew other problems*)

Problem [2](#), [3](#) (*I'm not following this because I don't think my prof will give these problems*)

→

Using Riemann's criterion for the integrability, show that $f(x) = \frac{1}{x}$ is integrable on $[1, 2]$

Other Problems

If f and g are continuous functions on $[a, b]$ and if $g(x) \geq 0$ for $a \leq x \leq b$, then show the mean value theorem for integrals

(c) If f and g are continuous on $[a, b]$ and $\int_a^b f(x) dx = \int_a^b g(x) dx$ then show that there exists $c \in [a, b]$ such that $f(c) = g(c)$

3. Let $f: [0, 2] \rightarrow \mathbb{R}$ be a continuous function such that $\int_0^2 f(x) dx = 2$. Find the value of

$$\int_0^2 \left[xf(x) + \int_0^x f(t)dt \right] dx.$$

→ Split final integral in terms of A and B, You'll be B as Double integral. Solve it by changing its order and take common some terms at final you'll get term for which you have given value as '2' put it and you'll get $2(2)=4$ ans.

4. Show that $\int_0^x \left(\int_0^y f(t)dt \right) dt = \int_0^x f(u)(x-u)du$ assuming f to be continuous.

→ Use "**Fundamental Theorem of Calculus**" Don't solve by change-of-order / integration-by-parts identity prof. will not give any marks

→ Define both sides, LHS with (first) Fundamental Theorem of Calculus and for RHS Leibniz (differentiation under the integral sign) rule for an integral. You'll get value as 0.

5. Let $f: [0, 1] \rightarrow \mathbb{R}$ be a positive continuous function. Show that

$$\lim_{n \rightarrow \infty} \left(f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \cdots f\left(\frac{n}{n}\right) \right)^{\frac{1}{n}} = e^{\int_0^1 \ln f(x) dx}$$

→ Riemann sum theorem

Open/Closed Set

A set $U \subseteq \mathbb{R}$ is called open if: for every point $x \in U$, there exists an $\varepsilon > 0$ such that the open ball $B(x, \varepsilon) \subseteq U$. An open ball of radius ε centered at x is

$$B(x, \varepsilon) = \left\{ y \in \mathbb{R}^n : \|y - x\| < \varepsilon \right\}$$

→ Every point in the set has some "wiggle room" around it that is still inside the set.

→ An open set does not include its boundary points.

Introduction to Open and Closed Sets by [Elliot Nicholson](#)

→ I did not understand till now but its very definition oriented.

Identifying Open, Closed, and Compact Sets by [WrathofMath](#)

→ Good for entry level understanding.

Open, closed, both and neither sets by [Joshua Helston](#)

→ Umm... Good since he talked about real line but I did not get much of it.

Double Integrals

Q. Integrate $f(x, y) = x^2 + y^2$ over unit disc by [Dr V](#)
 → Good explanation, Entry level.

Q. Integration in polar coordinates by [David Jordan \[MIT\]](#)

$$(a). \int_{x=1}^2 \int_{y=0}^x f \, dy \, dx \quad (b). \int_{x=0}^1 \int_{y=x^2}^x f \, dy \, dx \quad (c). \int_{y=0}^2 \int_{x=0}^{\sqrt{2y-y^2}} f \, dx \, dy$$

→ Learn changing into polar coordinates

Q. $\int_0^8 \int_{\sqrt[3]{xy^4}+1}^2 \frac{1}{dy} dx$

→ by [Trefor Bazett](#) : Amazing explanation.

Evaluate $\iint_D (x + y) dA$ where D is the region bdd by the lines joining $(0, 0), (0, 1)$ & $(2, 2)$.

$(x_1, y_1) = (0, 0)$ to $(x_2, y_2) = (2, 2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 0}{2 - 0} = 1, \text{ So the line has slope 1.}$$

Line through $(0, 0)$ with slope 1. $(y - y_1) = m(x - x_1) = (y - 0) = 1(x - 0) \Rightarrow y = x$

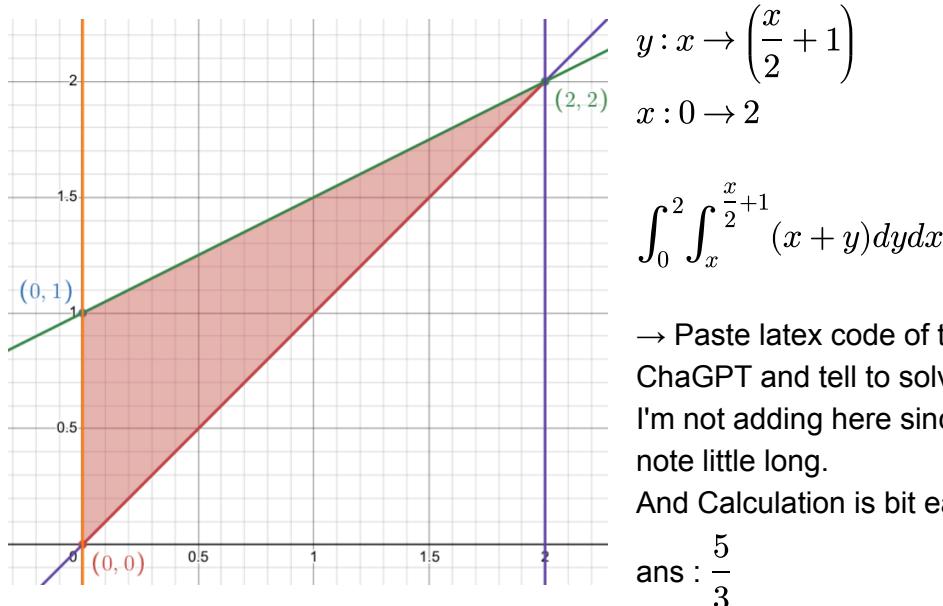
$(x_1, y_1) = (0, 1)$ to $(x_2, y_2) = (2, 2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1}{2 - 0} = \frac{1}{2}, \text{ So the line has slope } \frac{1}{2}$$

Line through $(0, 1)$ with slope $\frac{1}{2}$.

$$(y - y_1) = m(x - x_1) = (y - 1) = \frac{1}{2}(x - 0) \Rightarrow y = 1 + \frac{1}{2}x$$

Note : Don't focus on this, for exam just draw points and you'll get visual picture.



Similar Example :

Evaluate the double integral $\iint_D (x+y)dA$ where D is the triangular region vertices $(0,0), (1,2)$ & $(0,3)$.
by [Tam Cao](#)

Evaluate double integral $\iint_D (x+y)dA$ where D is the triangular region with vertices $(0,0), (1,2), (0,3)$
By [Academic Videos](#)

→ Not THE best but quite similar so i'm adding. (No Audio 

Evaluate double integral $\iint_D y^3 dA$, where D is the triangular region with vertices $(0,1), (7,0), (1,1)$
by [Julia](#)

→ I think its one of the best Videos for getting line equation from points.

Search Help :

site:youtube.com evaluate integral vertices $(0,0), (0,1)$ and $(2,2)$

Q. Evaluate $\int_0^\pi \left(\int_x^\pi \frac{\sin y}{y} dy \right) dx = \iint_D \frac{\sin y}{y} dA$

$$\int_0^\pi \int_x^\pi \frac{\sin y}{y} dy dx \Rightarrow \int_0^\pi \int_0^y \frac{\sin y}{y} dx dy = \int_0^\pi \frac{\sin y}{y} \left(\int_0^y dx \right) dy$$

$$\int_0^y dx = [x]_0^y = [y - 0] = y$$

$$\int_0^\pi \frac{\sin y}{y} dy = \int_0^\pi \sin y dy = [-\cos y]_0^\pi = [-\cos(\pi) - (-\cos(0))]$$

$$= [-(-1) - (-1)] = [1 + 1] = 2$$

$\cos \pi = -1$
 $\cos 0 = 1$

by [Deepak Subedi](#)

→ Umm, Good but i already knew so its got me even i watched it by skipping

Search Help :

site:youtube.com siny y dy dx

Q. Evaluate $\iint_{(x-1)^2+y^2 \leq 1} \sqrt{x^2 + y^2} dy dx$

by [Maksym Zubkov](#)

→ Exactly this example explained, but not go into solution since i think he solve wrong.

ChatGPT solve like this. My friend also solve like this so i think we're correct

$(x - 1)^2 + (y - 0)^2 \leq 1$, This is the standard circle form $(x - h)^2 + (y - k)^2 \leq r^2$

Center: $(h, k) = (1, 0)$, Radius : $r = 1$

Left most point $(h - r, k) = (1 - 1, 0) = (0, 0)$ (*move to left i.e subtract*)

Right most point $(h + r, k) = (1 + 1, 0) = (2, 0)$ (*move to right, i.e add*)

Top point $(h, r + k) = (1, 0 + 1) = (1, 1)$

Bottom point $(h, r - k) = (1, 0 - 1) = (1, -1)$

→ Draw circle out of it.

Since its circle its better to use polar coordinates

$$x = r \cos \theta, y = r \sin \theta$$

$$x^2 + y^2 = r^2, dA = r dr d\theta$$

$$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$(x - 1)^2 + y^2 = 1 \implies (x^2 - 2x + 1^2 + y^2 = 1 \implies x^2 + y^2 = 2x)$$

$$x = r \cos \theta, y = r \sin \theta$$

$$x^2 + y^2 = 2x \implies r^2 = 2r \cos \theta \implies r = 2 \cos \theta$$

$$r \in [0, 2 \cos \theta]$$

$$\iint_{(x-1)^2+y^2 \leq 1} \sqrt{x^2 + y^2} dy dx \Rightarrow \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \sqrt{r^2} r dr d\theta \Rightarrow \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 dr d\theta$$

$$\int_0^{2 \cos \theta} r^2 dr = \frac{1}{3} [r^3]_0^{2 \cos \theta} = \frac{1}{3} [(2 \cos \theta)^3 - 0^3] = \frac{1}{3} (8 \cos^3 \theta)$$

$$\int_{-\pi/2}^{\pi/2} \left(\int_0^{2 \cos \theta} r^2 dr \right) d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} (8 \cos^3 \theta) \right) d\theta = \frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos^3 \theta d\theta$$

$$\cos^3 \theta = \cos \theta \cdot \cos^2 \theta$$

$$\therefore \sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \cos^2 \theta = 1 - \sin^2 \theta$$

$$\frac{8}{3} \int_{-\pi/2}^{\pi/2} \cos \theta (1 - \sin^2 \theta) d\theta$$

let, $u = \sin \theta$, $du = \cos \theta d\theta$

$$\text{When } \theta = -\frac{\pi}{2}, u = \sin\left(-\frac{\pi}{2}\right) = -1$$

$$\text{When } \theta = \frac{\pi}{2}, u = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\frac{8}{3} \int_{-1}^1 (1 - u^2) du = \frac{8}{3} \left[u - \frac{u^3}{3} \right]_{-1}^1 = \frac{8}{3} \left[1 - \left(-\frac{1}{3} \right) \right] = \frac{8}{3} \left[\frac{4}{3} \right] = \frac{32}{9}$$

Area of a Circle With Double Integrals by [TheCalcSeries Guy](#).

Search Help :

site:youtube.com double integral unit circle polar coordinate

Q. $\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2} dy dx = \iint_D \sqrt{1-y^2} dA$

$$y : 0 \rightarrow \sqrt{1-x^2}$$

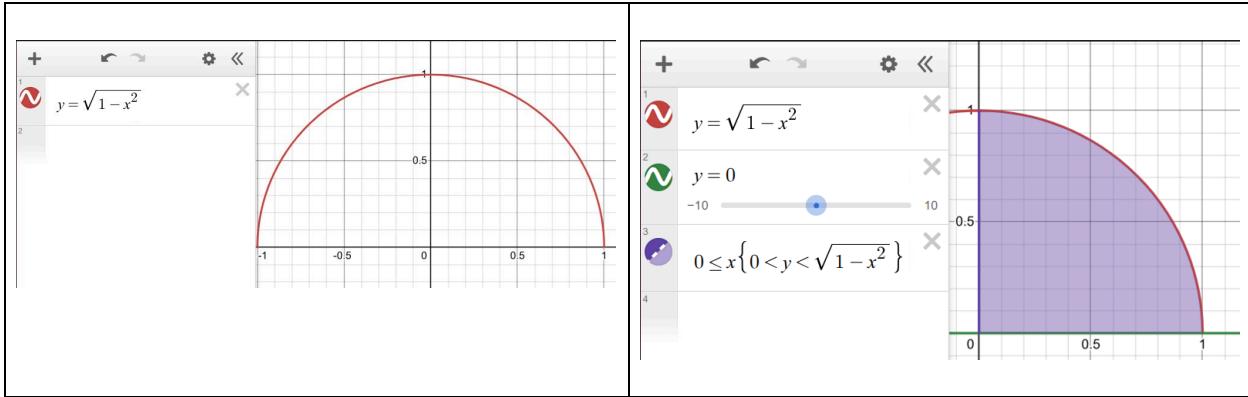
$$x : 0 \rightarrow 1$$

Plot region,

$$y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2 + y^2 = 1 \text{ its circle.}$$

so, $y = \sqrt{1-x^2}$ is semicircle +ve to y -axis.

And since $0 \leq x \leq 1$ so it lie in first quadrant only.



Change order of integration by Fubini's Theorem.

$$x : 0 \rightarrow \sqrt{1 - y^2}$$

$$y : 0 \rightarrow 1$$

Region D is the quarter unit disk in the first quadrant:

$$D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1\}$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2} dy dx = \int_0^1 \int_0^{\sqrt{1-y^2}} \sqrt{1-y^2} dx dy = \int_0^1 \sqrt{1-y^2} \left(\int_0^{\sqrt{1-y^2}} dx \right) dy$$

$$\int_0^{\sqrt{1-y^2}} dx = [x]_0^{\sqrt{1-y^2}} = [\sqrt{1-y^2} - 0] = \sqrt{1-y^2}$$

$$\int_0^1 \sqrt{1-y^2} \left(\int_0^{\sqrt{1-y^2}} dx \right) dy = \int_0^1 \sqrt{1-y^2} (\sqrt{1-y^2}) dy = \int_0^1 1-y^2 dy$$

$$\int_0^1 1-y^2 dy = \left[y - \frac{y^3}{3} \right]_0^1 = \left[\left(1 - \frac{1}{3} \right) - \left(0 - \frac{0}{3} \right) \right] = \left(1 - \frac{1}{3} \right) = \frac{2}{3}$$

Q. Find the volume of the solid which is common to the cylinders

$$x^2 + y^2 = 1 \text{ & } x^2 + z^2 = 1$$

by [Ashwani Goyal](#)

→ Superb, Prof. solved generalized version of this problem.

$x^2 + y^2 = 1$ is circle on xy -plan and Cylinder to z -axis in 3D.

$x^2 + z^2 = 1$ is circle on xz -plan and Cylinder to y -axis in 3D.

$$\iiint_D dz dy dx$$

$$x^2 + z^2 = 1 \implies z^2 = 1 - x^2 \implies z = \pm \sqrt{1 - x^2}$$

$$z : (-\sqrt{1-x^2}) \rightarrow (\sqrt{1-x^2})$$

$$x^2 + y^2 = 1 \implies y^2 = 1 - x^2 \implies y = \pm \sqrt{1 - x^2}$$

$$y : (-\sqrt{1-x^2}) \rightarrow (\sqrt{1-x^2})$$

$$x : -1 \rightarrow 1$$

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz dy dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz \right) dy dx$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz = [z]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \left[(\sqrt{1-x^2}) - (-\sqrt{1-x^2}) \right] = 2\sqrt{1-x^2}$$

$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2\sqrt{1-x^2}) dy dx = \int_{-1}^1 (2\sqrt{1-x^2}) \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \right) dx$$

$$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = [y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = \left[(\sqrt{1-x^2}) - (-\sqrt{1-x^2}) \right] = 2\sqrt{1-x^2}$$

$$\int_{-1}^1 (2\sqrt{1-x^2}) \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \right) dx = \int_{-1}^1 (2\sqrt{1-x^2}) (2\sqrt{1-x^2}) dx = 4 \int_{-1}^1 (1-x^2) dx$$

$$4 \int_{-1}^1 (1-x^2) dx = 4 \left[x - \frac{x^3}{3} \right]_{-1}^1 = 4 \left[\left(1 - \frac{1^3}{3} \right) - \left(-1 - \frac{(-1)^3}{3} \right) \right] = 4 \left[\frac{4}{3} \right] = \frac{16}{3}$$

Or, since its symmetric

$$4 \int_{-1}^1 (1-x^2) dx = 4 \cdot 2 \left(\int_0^1 1-x^2 dx \right) = 8 \left[x - \frac{x^3}{3} \right]_0^1 = 8 \left[1 - \frac{1^3}{3} \right] = 8 \left[\frac{2}{3} \right] = \frac{16}{3}$$

Find the volume common to the Cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$

By [Jaswinder Kaur](#) → I did not understand but i'm keeping this

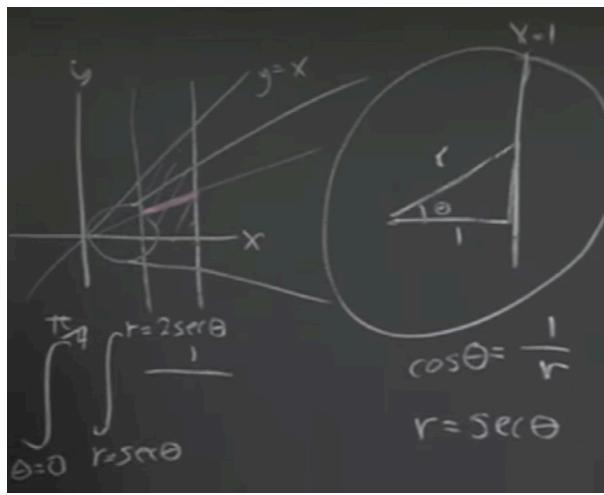
Search Help : (on Google.com)

Find the volume of the solid which is common to the cylinders $\text{math-container}\{x^2+y^2=1 \& x^2+z^2=1\}$

Q. Evaluate $\int_1^2 \int_0^x \frac{1}{(x^2 + y^2)^{3/2}} dy dx$

by [David Jordan \[MIT\]](#) at [11:40](#)

→ Good, I will recommend to watch and that way solve.



$$y : 0 \rightarrow x \\ x : 1 \rightarrow 2$$

$$\theta : 0 \rightarrow \frac{\pi}{4} (\pi/4 \text{ is } 45^\circ)$$

$$r : \sec \theta \rightarrow 2 \sec \theta$$

$$x^2 + y^2 = r^2$$

$$(x^2 + y^2)^{\frac{3}{2}} = (r^2)^{\frac{3}{2}} = r^3$$

$$\int_1^2 \int_0^x \frac{1}{(x^2 + y^2)^{3/2}} dy dx = \int_0^{\frac{\pi}{4}} \int_{\sec \theta}^{2 \sec \theta} \frac{1}{r^3} r dr d\theta = \int_0^{\frac{\pi}{4}} \int_{\sec \theta}^{2 \sec \theta} \frac{1}{r^2} dr d\theta = \int_0^{\frac{\pi}{4}} \left[\left[-\frac{1}{r} \right]_{\sec \theta}^{2 \sec \theta} \right] d\theta$$

$$\frac{1}{r^2} = r^{-2} \Rightarrow \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\left[-\frac{1}{r} \right]_{\sec \theta}^{2 \sec \theta} = \left[\left(-\frac{1}{2 \sec \theta} \right) - \left(-\frac{1}{\sec \theta} \right) \right] = \left[\left(-\frac{1}{2 \sec \theta} \right) + \left(\frac{1}{\sec \theta} \right) \right]$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\left[\left(-\frac{1}{2} \cos \theta \right) + \cos \theta \right] = \left(\cos \theta - \frac{1}{2} \cos \theta \right) = \frac{1}{2} \cos \theta$$

$$\int_0^{\frac{\pi}{4}} \left[\left[-\frac{1}{r} \right]_{\sec \theta}^{2 \sec \theta} \right] d\theta = \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} \cos \theta \right) d\theta = \frac{1}{2} \left(\int_0^{\frac{\pi}{4}} \cos \theta d\theta \right) = \frac{1}{2} [\sin \theta]_0^{\frac{\pi}{4}}$$

$$\frac{1}{2} \left[\sin \frac{\pi}{4} - \sin 0 \right] = \frac{1}{2} \left[\frac{\sqrt{2}}{2} - 0 \right] = \frac{1}{2} \left(\frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{4}$$

Q. $\int_0^1 \int_0^{1-y} \sqrt{x+y} (y-2x)^2 dx dy$

Nice integral — it simplifies cleanly with a linear change of variables.

take $u = x + y$, $v = y - 2x$

$$u = x + y \implies y = u - x$$

$$v = y - 2x \implies v = (u - x) - 2x = v = u - 3x \implies 3x = u - v \implies x = \frac{u}{3} - \frac{v}{3}$$

$$y = u - x \implies y = u - \left(\frac{u}{3} - \frac{v}{3}\right) \implies y = \frac{3u}{3} - \frac{u}{3} + \frac{v}{3} \implies y = \frac{2u}{3} + \frac{v}{3}$$

Waiii..t.....I solved this During JAM-2025, I did not knew how to choose u and v back then.
But okay, i learnt that now.

Q. Evaluate $\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx$ using the substitution $x = \frac{u}{3} - \frac{v}{3}$,

$$y = \frac{2u}{3} + \frac{v}{3}$$

Compute the Jacobian

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$x = \frac{u}{3} - \frac{v}{3},$$

$$\frac{\partial x}{\partial u} = \frac{1}{3}, \quad \frac{\partial x}{\partial v} = -\frac{1}{3}$$

$$y = \frac{2u}{3} + \frac{v}{3}$$

$$\frac{\partial y}{\partial u} = \frac{2}{3}, \quad \frac{\partial y}{\partial v} = \frac{1}{3}$$

$$J(u, v) = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{vmatrix} = \left| \left(\frac{1}{3} \times \frac{1}{3} \right) - \left(\frac{2}{3} \times \left(-\frac{1}{3} \right) \right) \right| = \left| \frac{1}{9} - \left(-\frac{2}{9} \right) \right| = \left| \frac{3}{9} \right| = \left| \frac{1}{3} \right|$$

$$|J(u, v)| = \left| \frac{1}{3} \right| = \frac{1}{3}$$

Draw the original region and find its actual boundaries

$$0 \leq y \leq (1 - x)$$

$$0 \leq x \leq 1$$

$$x = 0 \quad x = 1$$

$$y = 0 \quad y = 1 - x \implies x + y = 1$$

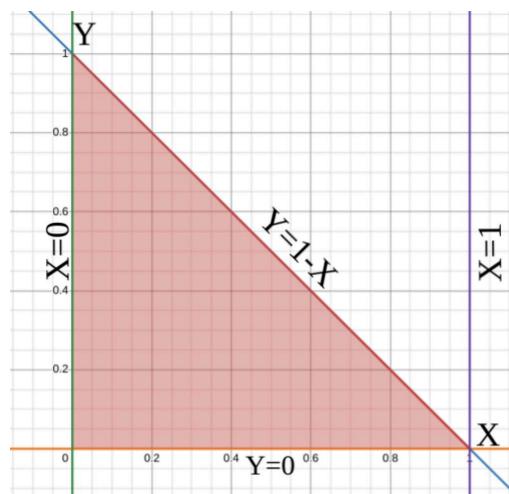
$$x + y = 1,$$

put $x = 0$ in $x + y = 1 \quad y = 1$

$$y = 0, \quad x = 1$$

$$x = 2, \quad y = -1$$

$$y = 2, \quad x = -1$$



Convert the boundaries into new boundaries by using substitutions

Old boundaries

$$y = 0, \quad y = 1 - x$$

$$x = 0, \quad x = 1$$

I don't really need of $x = 1$ because my region bounde without $x = 1$.

New boundaries

$$x = \frac{u}{3} - \frac{v}{3}$$

$$y = \frac{2u}{3} + \frac{v}{3}$$

$$y = 0,$$

$$0 = \frac{2u}{3} + \frac{v}{3} \implies 2u + v = 0 \implies u = -\frac{v}{2}$$

$$x = 0$$

$$0 = \frac{u}{3} - \frac{v}{3} \implies u - v = 0 \implies u = v$$

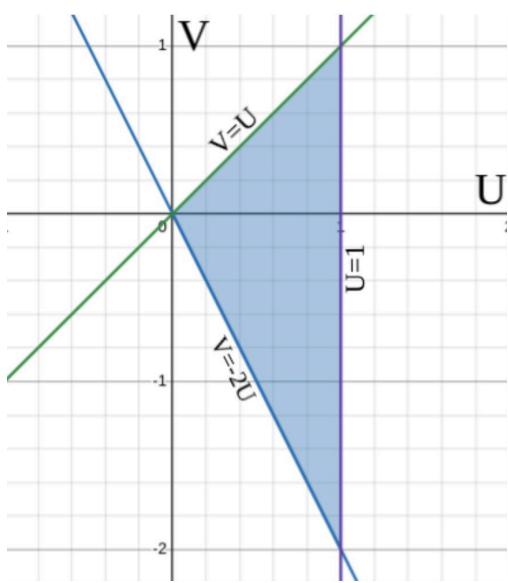
$$y = 1 - x,$$

$$\left(\frac{2u}{3} + \frac{v}{3}\right) = 1 - \left(\frac{u}{3} - \frac{v}{3}\right)$$

$$\left(\frac{2u}{3} + \frac{v}{3}\right) + \left(\frac{u}{3} - \frac{v}{3}\right) = 1$$

$$\frac{2u}{3} + \frac{u}{3} = 1 \implies \frac{3u}{3} = 1 \implies u = 1$$

Step 5: Draw new region, find new limits & integrate



$$u = -\frac{v}{2} \implies v = -2u$$

similar to $v = 2u$ but in $-ve$ sign

$$u = v, u = 1$$

Arrow

$$v : -2u \rightarrow u$$

Brush

$$u : 0 \rightarrow 1$$

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dy dx \implies \int_0^1 \int_{-2u}^u \sqrt{u}(v)^2 \left(\frac{1}{3}\right) dv du$$

$$\because \sqrt{x+y} = \sqrt{\left(\frac{u}{3} - \frac{v}{3}\right) + \left(\frac{2u}{3} + \frac{v}{3}\right)} = \sqrt{u}$$

$$\therefore y - 2x = \left(\frac{2u}{3} + \frac{v}{3}\right) - 2\left(\frac{u}{3} - \frac{v}{3}\right) = v$$

$$\text{& Jacobian} = \frac{1}{3}$$

$$\frac{1}{3} \int_0^1 \sqrt{u} \int_{-2u}^u v^2 dv du$$

$$\int_{-2u}^u v^2 dv = \left[\frac{v^3}{3} \right]_{-2u}^u = \frac{1}{3} [v^3]_{-2u}^u = \frac{1}{3} (u^3 - (-2u)^3) = \frac{1}{3} (u^3 + 8u^3) = \frac{1}{3} 9u^3 = 3u^3$$

$$\frac{1}{3} \int_0^1 \sqrt{u} (3u^3) du = \frac{1}{3} \cdot 3 \int_0^1 u^{\frac{1}{2}} u^3 du = \int_0^1 u^{\left(\frac{1}{2}+3\right)} du = \int_0^1 u^{\frac{7}{2}} du = \left[\frac{u^{\frac{7}{2}+1}}{\frac{7}{2}+1} \right]_0^1 = \left[\frac{u^{\frac{9}{2}}}{\frac{9}{2}} \right]_0^1$$

$$= \frac{2}{9} \left[u^{\frac{9}{2}} \right]_0^1 = \frac{2}{9} \left[(1)^{\frac{9}{2}} - (0)^{\frac{9}{2}} \right] = \frac{2}{9}$$

Triple Integrals

Q. Find volume occupied by $x^2 + y^2 + z^2 = 1$ using spherical coordinates.

$\rho : 0 \rightarrow 1$ - ρ is similar to r in 2D. It used to get radius of Sphere from core.

$\phi : 0 \rightarrow \pi$ - ϕ used to cover half most region of sphere. one end to all the way down on z axis

$\theta : 0 \rightarrow 2\pi$ - θ is as usual used to get angle.

$$\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\pi \sin \phi \int_0^1 \rho^2 \, d\rho \, d\phi \, d\theta$$

Since they're independent each other, I can integrate them separately

$$\int_0^1 \rho^2 \, d\rho = \left[\frac{\rho^3}{3} \right]_0^1 = \left[\frac{(1)^3}{3} - \frac{(0)^3}{3} \right] = \frac{1}{3}$$

$$\int_0^\pi \sin \phi \, d\phi = [-\cos \phi]_0^\pi = [(-\cos \pi) - (-\cos 0)] = [(-(-1)) - (-1)] = 1 + 1 = 2$$

$$\int_0^{2\pi} d\theta = 2\pi$$

$$\int_0^{2\pi} \int_0^\pi \int_0^1 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta = \frac{1}{3} \times 2 \times 2\pi = \frac{4\pi}{3}$$

Reference : [Steve Chow \(BPRP\)](#),

Converting a triple integral among rectangular, cylindrical and spherical coordinates by [Steve Chow \(BPRP\)](#).

Rectangular, cylindrical, and spherical coordinates (introduction & conversion) by [Steve Chow \(BPRP\)](#),

Q. Volume bounded by the surface $x^2 + y^2 + z^2 = 36$ & $z = -\sqrt{3x^2 + 3y^2}$ is _____

Q. How to change the order of a triple integral

$$\int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx = \iiint f(x, y, z) \, dx \, dy \, dz$$

by [Steve Chow \(BPRP\)](#)

$$\text{Q. } \int_0^2 \int_0^{\sqrt{4-y^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{16-x^2-y^2}} (x^2 + y^2 + z^2) \, dz \, dx \, dy$$

by [Michael Penn](#) :

→ Don't do this problem just watch it for understanding. Prof. will not give this tricky problem.

