Two-way Infinite Tape Turing Machines

M: Two-way infinite tape DTM

Simulate by a one-way infinite tape DTM N.

N keeps a left-end marker # not in Γ and a right-end marker \$ not in Γ.

Starts at the symbol next to #.

A right move of M is simulated by N as follows:

- Modify the current cell and move right.
- IF current symbol is \$ THEN:
  - Replace it with a $\sqcup$.
  - Move one position to the right.
  - Write \$.
  - Move left.

A left move of M is simulated by N as follows:

- Modify the current cell and move left.
- IF current symbol is # THEN:
  - Move right.
  - Shift the contents of the tape to the right by 1 position writing a $\sqcup$ in this cell.
  - Return to the left most end.
  - Move right.

Two-way Infinite Tape Turing Machines

$M = (Q_M, \Sigma, \Gamma_M, \delta_M, q_0, q_a, q_r)$.

$N = (Q_M \cup R, \Sigma, \Gamma \cup \{\#\} \cup \{\$\}, \delta_N, q_s, q_a, q_r)$, where

$R = \{r_{0,q}, r_{1,q}, r_{2,q}, r_{3,q}, r_{4,q}, r_{5,q} \mid q \in Q_M\} \cup \{s_{a,q} \mid a \in \Gamma, q \in Q_M\}$.

For all states $p, q \in Q$, for all $a, b \in \Gamma$, for all instructions of the form $\delta_M(p, a) = (q, b, R)$ include the instructions:

- $\delta_N(p, a) = (r_{0,q}, b, R)$.
- For all $a \in \Gamma$, $\delta_N(r_{0,q}, a) = (q, a, S)$
- $\delta_N(r_{0,q}, \$) = (r_{1,q}, \sqcup, R)$
- $\delta_N(r_{1,q}, \sqcup) = (q, \$, L)$

For all states $p, q \in Q$, for all $a, b \in \Gamma$, for all instructions of the form $\delta_M(p, a) = (q, b, L)$, include the instructions:

- $\delta_N(p, a) = (r_{2,q}, b, L)$
- For all $a \in \Gamma$, $\delta_N(r_{2,q}, a) = (q, a, S)$
- $\delta_N(r_{2,q}, \#) = (r_{3,q}, \#, R)$
- For all $a \in \Gamma$, $\delta_N(r_{3,q}, a) = (s_{a,q}, \sqcup, R)$
- For all $a \in \Gamma$, $\delta_N(s_{a,q}, a) = (s_{a,q}, a, R)$
- For all $a \neq b \in \Gamma$, $\delta_N(s_{a,q}, b) = (s_{b,q}, a, R)$
- For all $a \in \Gamma$, $\delta_N(s_{a,q}, \$) = (r_{4,q}, a, R)$
- $\delta_N(r_{4,q}, \sqcup) = (r_{5,q}, \$, L)$
- For all $a \in \Gamma$, $\delta_N(r_{5,q}, a) = (r_{5,q}, a, L)$
- $\delta_N(r_{5,q}, \#) = (q, \#, R)$

Multitape Deterministic Turing Machines

$M = (Q, \Sigma, \Gamma, \delta, q_0, q_a, q_r)$, where

$\delta : Q \times \Gamma^k \longrightarrow Q \times \Gamma^k \times \{L, R\}^k$

Theorem: Multitape TM has an equivalent 1-tape TM.

M: 2-tape TM; On input $w = w_1 \cdots w_n$:

1-tape machine S maintains virtual tape heads.

Change input to: $w_1{}^\circ w_2 \cdots w_n \# \sqcup^\circ$

Simulate each move of M:

Scan left to right picking the symbols under the virtual heads.

Scan right to left updating the contents of the tapes.

Scan left to right making the direction changes for virtual heads.

If a virtual head moves into a #, shift the contents of the tape right.

Simulation of a Multitape Deterministic Turing Machine by a One Tape Turing Machine

Let $M = (Q, \Sigma, \Gamma, \delta, q_s, q_a, q_r)$ be a 2-tape DTM. Define a 1-tape DTM $N = (Q_N, \Sigma, \Gamma_N, \delta_N, q_{N,s}, q_{N,a}, q_{N,r})$.
Let $\Gamma_N = \Gamma'_N \cup \{\#\}$, where $\Gamma'_N = \Gamma \cup \{a^\circ \mid a \in \Gamma\}$.
Let $Q_N = Q \times \Gamma^4 \times \{L, R\} \times \{L, R\} \times \{0, 1, 2, 3, \cdots, 17\}$.

- Convert tape to this format: $\#w_1^\circ w_2 \dots w_n \# \sqcup^\circ$. The tape head of N is on w_1°.
- Simulate each step of M by the following steps of N: (The R/W head of N is at the left most position. If the current state of M is p, the current state of N is $\langle p, \sqcup, \sqcup, \sqcup, \sqcup, L, L, 0\rangle$.)

1. Move from left to right picking up the symbols under the R/W heads on the two tapes.
   - (move right if the symbol is not one under the first virtual tape head)
     For all unmarked $X \in \Gamma'_N$, $p \in Q$,
     $$\delta_N(\langle p, \sqcup, \sqcup, \sqcup, \sqcup, L, L, 0\rangle, X) = (\langle p, \sqcup, \sqcup, \sqcup, \sqcup, L, L, 0\rangle, X, R).$$
   - (pick up the symbol under the first virtual tape head)
     For all marked $X^\circ \in \Gamma'_N$, $p \in Q$,
     $$\delta_N(\langle p, \sqcup, \sqcup, \sqcup, \sqcup, L, L, 0\rangle, X^\circ) = (\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 1\rangle, X, R).$$
   - (move right to the end of the first virtual tape)
     For all unmarked $X, U \in \Gamma'_N$, $p \in Q$,
     $$\delta_N(\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 1\rangle, U) = (\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 1\rangle, U, R).$$
   - (move right by one position to the beginning of second virtual tape)
     For all unmarked $X \in \Gamma'_N$, $p \in Q$,
     $$\delta_N(\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 1\rangle, \#) = (\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 2\rangle, \#, R).$$
   - (move right if the symbol is not one under the second virtual tape head)
     For all unmarked $X, Y \in \Gamma'_N$, $p \in Q$,
     $$\delta_N(\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 2\rangle, Y) = (\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 2\rangle, Y, R).$$
   - (pick up the symbol under the second virtual tape head)
     For all unmarked $X \in \Gamma'_N$, all marked $Y^\circ \in \Gamma_N$, $p \in Q$,
     $$\delta_N(\langle p, X, \sqcup, \sqcup, \sqcup, L, L, 2\rangle, Y^\circ) = (\langle p, X, Y, \sqcup, \sqcup, L, L, 3\rangle, Y, R).$$
2. (move to the right end of the second virtual tape and update the state (of N) to reflect the transition of M; only those transitions in which both tape heads of M move left are shown)

- For all $X, Y, Z \in \Gamma'_N$, $p \in Q$,

$$\delta_N(\langle p, X, Y, \sqcup, \sqcup, L, L, 3\rangle, Z) = \langle p, X, Y, \sqcup, \sqcup, L, L, 3\rangle, Z, R).$$

- For all $X, Y, Z, W \in \Gamma'_N$, $p \in Q$ such that $\delta_M(p, X, Y) = (q, W, Z, L, L)$

$$\delta_N(\langle p, X, Y, \sqcup, \sqcup, L, L, 3\rangle, \#) = (\langle q, X, Y, W, Z, L, L, 4\rangle, \#, L).$$

3. Scan right to left updating the contents of the virtual tapes.
   - (move left if the symbol is not one under the second virtual tape head)
     For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

     $$\delta_N(\langle q, X, Y, W, Z, L, L, 4\rangle, U) = (\langle q, X, Y, W, Z, L, L, 4\rangle, U, L)$$

   - (if the symbol is one under the second virtual tape head then replace it with the second virtual tape symbol - keep the virtual tape head on this symbol for now)
     For all unmarked $X, Y, W, Z \in \Gamma'_N$, $q \in Q$,

     $$\delta_N(\langle q, X, Y, W, Z, L, L, 4\rangle, Y^\circ) = (\langle q, X, Y, W, Z, L, L, 5\rangle, Z^\circ, L)$$

   - (move left until the end of first virtual tape)
     * For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

       $$\delta_N(\langle q, X, Y, W, Z, L, L, 5\rangle, U) = (\langle q, X, Y, W, Z, L, L, 5\rangle, U, L)$$

     * (at the end of first virtual tape move left one position into the contents of first virtual tape)
       For all unmarked $X, Y, W, Z \in \Gamma'_N$, $q \in Q$,

       $$\delta_N(\langle q, X, Y, W, Z, L, L, 5\rangle, \#) = (\langle q, X, Y, W, Z, L, L, 6\rangle, \#, L)$$

   - (move left if the symbol is not one under the first virtual tape head)
     For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

     $$\delta_N(\langle q, X, Y, W, Z, L, L, 6\rangle, U) = \langle q, X, Y, W, Z, L, L, 6\rangle, U, L)$$

   - (if the symbol is one under the first virtual tape head then replace it with the first virtual tape symbol - keep the first virtual tape head on this symbol for now)
     For all unmarked $X, Y, W, Z \in \Gamma'_N$, $q \in Q$,

     $$\delta_N(\langle q, X, Y, W, Z, L, L, 6\rangle, X^\circ) = \langle q, X, Y, W, Z, L, L, 7\rangle, W^\circ, L)$$

   - move to the left most end of the tape
     * For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

       $$\delta_N(\langle q, X, Y, W, Z, L, L, 7\rangle, U) = \langle q, X, Y, W, Z, L, L, 7\rangle, U, L)$$

* For all unmarked $X, Y, W, Z \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 7\rangle, \#) = \langle q, X, Y, W, Z, L, L, 8\rangle, \#, R)$$

4. Scan from left to right changing the directions.

   – (move right if the symbol is not one under the first virtual tape head)
   For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 8\rangle, U) = \langle q, X, Y, W, Z, L, L, 8, \rangle, U, R)$$

   – (if the symbol is one under the first virtual tape head (that is a shadow symbol) then replace it with the actual symbol and move left by one position)
   For all unmarked $X, Y, W, Z, \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 8\rangle, W^\circ) = \langle q, X, Y, W, Z, L, L, 9\rangle, W, L)$$

   – (if the current symbol is not left-end marker of the first virtual tape then replace it with the corresponding shadow symbol and move right)
   For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 9\rangle, U) = \langle q, X, Y, W, Z, L, L, 10\rangle, U^\circ, R)$$

   – (if the current symbol is left-end marker of the first virtual tape then move right by one position and replace the current symbol by its shadow symbol)
     * For all unmarked $X, Y, W, Z \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 9\rangle, \#) = \langle q, X, Y, W, Z, L, L, 11, \rangle, \#, R)$$

     * For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 11\rangle, U) = \langle q, X, Y, W, Z, L, L, 12\rangle, U^\circ, R)$$

   – (move right until the start of the second virtual tape)
     * For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 12\rangle, U) = \langle q, X, Y, W, Z, L, L, 12\rangle, U, R)$$

     * For all unmarked $X, Y, W, Z \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 12\rangle, \#) = \langle q, X, Y, W, Z, L, L, 13\rangle, \#, R)$$

   – (move right if the symbol is not one under the second virtual tape head)
   For all unmarked $X, Y, W, Z, U \in \Gamma'_N$, $q \in Q$,

$$\delta_N(\langle q, X, Y, W, Z, L, L, 13\rangle, U) = \langle q, X, Y, W, Z, L, L, 13\rangle, U, R)$$

– (if the symbol is one under the second virtual tape head (that is a shadow symbol) then replace it with the actual symbol and move left by one position)
For all unmarked $X, Y, W, Z, \ \in \ \Gamma'_N,\ q \ \in \ Q,$

$$\delta_N(\langle q, X, Y, W, Z, L, L, 13\rangle, Z^\circ) \ = \ \langle q, X, Y, W, Z, L, L, 14\rangle, Z, L)$$

– (if the current symbol is not left-end marker of the second virtual tape then replace it with the corresponding shadow symbol and move right)
For all unmarked $X, Y, W, Z, U \ \in \ \Gamma'_N,\ q \ \in \ Q,$

$$\delta_N(\langle q, X, Y, W, Z, L, L, 14\rangle, U) \ = \ \langle q, X, Y, W, Z, L, L, 15\rangle, U^\circ, R)$$

– (if the current symbol is left-end marker of the second virtual tape then move right by one position and replace the current symbol by its shadow symbol)

  * For all unmarked $X, Y, W, Z \ \in \ \Gamma'_N,\ q \ \in \ Q,$

$$\delta_N(\langle q, X, Y, W, Z, L, L, 14\rangle, \#) \ = \ \langle q, X, Y, W, Z, L, L, 16, \rangle, \#, R)$$

  * For all unmarked $X, Y, W, Z, U \ \in \ \Gamma'_N,\ q \ \in \ Q,$

$$\delta_N(\langle q, X, Y, W, Z, L, L, 16\rangle, U) \ = \ \langle q, X, Y, W, Z, L, L, 17\rangle, U^\circ, R)$$

Enumerators

- An enumerator does not have any input.
- It has a work tape and an output tape.
- At each step, the machine can choose to write a symbol from the output alphabet on the output tape. If it writes a symbol on the output tape then the output tape head moves right by one position.
- The enumerator has a distinguished state, say q_p, entering which the output tape is erased and the tape head moves to the left most position. A string w is said to be output by the enumerator if the output tape contains w at the time when the machine enters the state q_p.

Formal Definition:
$M = (Q, \Sigma, \Gamma, \delta, q_0, q_h, q_p)$, where

- Q: set of states
- Σ: output alphabet
- Γ: tape alphabet
- $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{L, R\} \times (\Sigma \cup \{\epsilon\})$
- $q_0 \in Q$: start state
- $q_p \in Q$: print state
- $q_h \in Q$: halt state

Simulating a Turing machine by an enumerator

Let $M = (Q, \Sigma, \Gamma, \delta, q_s, q_a, q_r)$ be a DTM.
Let $s_1, s_2, \cdots$ be an ordering of all the strings over Σ.
An enumerator E does the following:

- For $i = 1, 2, 3, \cdots$ do:
  - Generate the first i strings $s_1, s_2, \cdots, s_i$.
  - For $j = 1, 2, 3, \cdots i$ do:
    - Run M for i steps on s_j. If M accepts then output s_j.

Running a Turing machine for k steps

Let $M = (Q, \Sigma, \Gamma, \delta_M, q_s, q_a, q_r)$ be a DTM with one tape.
A simulating TM N has three tapes:

- Tape 1 (Input tape): contains the string on which M has to be simulated.
- Tape 2 (Work tape): used to store the contents of M's tape.
- Tape 3 (CLOCK): contains a string of k 1's.

The machine starts the simulation by copying the input w from the Input tape to the Work tape. It then simulates M step by step moving the head on the CLOCK tape right by one position for each step. Stops when it reads the first $\sqcup$ on the CLOCK tape.

For the machine N, the state set $Q_N = Q \cup \{r\}$ for some $r \notin Q$.

For all states $p \in Q$, for all $a, b \in \Gamma$:

$$\delta_N(p, a, b, 1) = (state(\delta_M(p, b)), a, R, symbol(\delta_M(p, b)), direction(\delta_M(p, b)), 1, R)$$

For all states $p \in Q$, for all $a, b \in \Gamma$, $\delta_N(p, a, b, \sqcup) = (r, a, R, b, R, \sqcup, R)$.

When N reaches state r, it has simulated M for k steps.