CS 4510: Automata and Complexity

Spring 2015

Home work 1 // Due: Friday, January 16, 2015

1. (15 points) For $k \geq 1$ and $p \geq 2$, let

$$L_{k,p} = \{w \in \{0,1,\cdots,p-1\}^*\} \mid w \text{ is a } p\text{−ary representation of a multiple of } k\}.$$

We presented a DFA for $L_{5,3}$ in class. Generalize the construction for arbitrary k and p. Give arguments to show that your construction is correct.

2. (15 points) For a fixed k, let $L_k = \{w \in \{0,1\}^* \mid \text{the } k\text{ −th symbol from the right is } 1\}$. Design a DFA for L_k.

Solution: Store the last k bits seen in the input in the state. That is, the DFA has 2^k states labeled with all possible bit strings of length k.

From a state labeled $x_1 x_2 \cdots x_k$ (where $x_i \in \{0,1\}$) there is a transition to state labeled $x_2 x_3 \cdots x_{k-1}0$ on input 0 and to state labeled $x_2 x_3 \cdots x_{k-1}1$ on input 1.

The final states are those that are labeled $1x_2 \cdots x_k$ for $x_i \in \{0,1\}$.

3. (15 points) Let A and B be two regular languages over $\Sigma = \{0,1\}$.

Show that $disjunct(A,B) = \{x \vee y \mid x \in A, y \in B, |x| = |y|\}$ is also regular. Here $x \vee y$ is the Boolean OR of the two bit strings x and y.

Solution:

Let $M_1 = (Q_1, \{0,1\}, \delta_1, q_1, F_1)$ be a DFA that accepts A. Let $M_2 = (Q_2, \{0,1\}, \delta_2, q_2, F_2)$ be a DFA that accepts B. *Construct* an NFA $M = (Q, \{0,1\}, \delta, q_0, F)$ as follows.

- $Q = Q_1 \times Q_2$.
- For all $p \in Q_1$, $q \in Q_2$, define:
  - $\delta(\langle p,q\rangle, 0) = \{\langle \delta_1(p,0), \delta_2(q,0)\rangle\}$.
  - $\delta(\langle p,q\rangle, 1) = \{\langle \delta_1(p,0), \delta_2(q,1)\rangle, \langle \delta_1(p,1), \delta_2(q,0)\rangle, \langle \delta_1(p,1), \delta_2(q,1)\rangle\}$.
- $q_0 = \langle q_1, q_2\rangle$.
- $F = F_1 \times F_2$.

If the input symbol is a 0 then the two bits whose OR is 0 must be $(0,0)$. If the input symbol is a 1 then the two bits whose OR is 1 can be either $(0,1)$ or $(1,0)$ or $(1,1)$.

4. (15 points) Let A and B be two regular languages over a finite alphabet Σ.

Define $shuffle(A,B) = \{w \mid w = a_1 b_1 \cdots a_k b_k, \text{ where each } a_i, b_i \in \Sigma^*, a_1 \cdots a_k \in A \text{ and } b_1 \cdots b_k \in B\}$.

Show that $shuffle(A,B)$ is regular.

Solution: Let $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ be a DFA that accepts A. Let $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ be a DFA that accepts B.

Define an NFA $N = (Q, \Sigma, \delta, (q_1, q_2), F)$ as follows.

- $Q = Q_1 \times Q_2$.
- For all $p \in Q_1$, $q \in Q_2$, and $a \in \Sigma$ define:

$$\delta((p,q),a) = \{(\delta_1(p,a), q), (p, \delta_2(q,a))\}.$$

- $F = F_1 \times F_2$.

5. Let L be an infinite subset of $\{1\}^*$. Show that L^* is regular.

Solution: Choose any element 1^r in L. Here r is a constant. Define $f(i)$ to be the smallest index j such that 1^j is in L^* and j is congruent to i modulo r, if such a j exists and 0 if no such j exists. Further define $A[i] := 1^{f(i)}(1^r)^*$

We claim that L^* is the union of $A[0], \ldots, A[r-1]$. By definition, every element of each $A[i]$ is is L^*. For the other direction, if 1^s is in L^*, then 1^s is in $A[i]$ where i is the residue of s modulo r.

It is easy to see that each $A[i]$ is regular and hence their union (which is L^*).