Nontrivial Property

A *property* $\mathcal{P}$ is a set of Turing machine descriptions.

Example: $NOT - REGULAR = \{e(M) \mid M$ is a TM such that $L(M)$ is not regular$\}$.

- $\mathcal{P}$ is *non-trivial*: $\mathcal{P}$ contains some but not all TM descriptions.

  There are TMs whose languages are not regular and there are TMs whose languages are regular.

- $\mathcal{P}$ is a *property of the language* of a TM: whenever $L(M_1) = L(M_2)$, either both M_1 and M_2 are in $\mathcal{P}$ or both are not in $\mathcal{P}$.

  Whenever $L(M_1) = L(M_2)$, either both M_1 and M_2 are in $\mathcal{P}$ since their languages are not regular or both are not in $\mathcal{P}$ since their languages are regular.

Rice's Theorem

Any non-trivial property $\mathcal{P}$ that is a property of the language of a TM is undecidable.

For any $M \in \mathcal{P}$, its language $L(M)$ is Turing recognizable. So, the theorem can also be stated as:

Any non-trivial property of Turing recognizable languages is undecidable.

Proof of Rice's Theorem

Reduction from A_{TM}.

- $T_\emptyset$ be a TM such that $L(T_\emptyset) = \emptyset$. Assume, without loss of generality, that $T_\emptyset$ is not in $\mathcal{P}$. (Why?)
- There is some TM T in $\mathcal{P}$ (since $\mathcal{P}$ is non-trivial).
- The reduction machine on input M, w constructs a DTM M':
  - On input x:
    - Simulate M on w.
    - M halts and rejects w: reject x.
    - M halts and accepts w: run T on x. Accept x if and only if T accepts x.
- What is the language of M'?

  $L(T)$ if M accepts w and $\emptyset$ if M does not accept w.
- That is, M' is in $\mathcal{P}$ if and only if M accepts w.

Corollaries of Rice's Theorem

The following languages are undecidable:

1. $\mathrm{N}OT - REGULAR = \{e(M) \mid M$ is a TM such that $L(M)$ is not regular$\}$.
2. $\mathrm{N}OT - EMPTY = \{e(M) \mid M$ is a TM such that $L(M) = \emptyset\}$.