Question 1

Solution

a.

(i) Total number of test conducted = $T(s)$.

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Question 2

To derive the PDF of random variable $Z = XY$, where X and Y are independent random variables with known PDFs $f_X(x)$ and $f_Y(y)$

Solution

$$P(XY \leq z) = F_Z(z)$$
$$= \int_{-\infty}^{z} f_Z(z)dz$$

Probability that variable Y lies between y and y + dy

$$P(y \leq Y \leq y + dy) = f_Y(y)dy$$

for some y. Now, for a fixed value of y, we have

$$P(XY \leq z|Y = y) = P(X \leq \frac{z}{y}) = F_X(\frac{z}{y})$$

So for y > 0,

$$P(XY \leq z|Y = y) = F_X(\frac{z}{y})$$
$$P(Xy \leq z) = F_X(\frac{z}{y})P(Y = y)$$

for y < 0,

$$P(XY \leq z|Y = y) = 1 - F_X(\frac{z}{y})$$
$$P(XY \leq z) = 1 - F_X(\frac{z}{y})P(Y = y)$$

and for y = 0,

$$P(XY \leq z|Y = 0) = \int_{0}^{\infty} f_Z(z)dz$$
$$P(XY \leq z) = \int_{0}^{\infty} f_Z(z)dzP(Y = 0)$$

So, we need to integrate on all possible values of y. We can write

$$F_Z(z) = \int_{-\infty}^{0} (1 - F_X(\frac{z}{y}))f_Y(y)dy + \int_{0}^{\infty} F_X(\frac{z}{y})f_Y(y)dy + (\int_{0}^{\infty} f_Z(z)dz)P(Y = 0)$$

Put $P(Y=0)=0$.
Now, differentiating w.r.t z, we get

$$f_Z(z) = \int_{-\infty}^{0} -\frac{1}{y} f_X(\frac{z}{y}) f_Y(y) dy + \int_{0}^{\infty} \frac{1}{y} f_X(\frac{z}{y}) f_Y(y) dy + 0$$

or

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_X(\frac{z}{y}) f_Y(y) dy$$

which is the required PDF of Z.

Conclusion : The PDF of the product of two independent random variables X and Y is given by the integral formula :

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_X(\frac{z}{y}) f_Y(y) dy$$

Question 3

Solution

The correct estimate for E(X) is: $\hat{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$
Reason:

$$E(\hat{x}) = \frac{\sum_{i=1}^{i=n} E(x_i)}{n}$$

$$= \frac{nE(x_i)}{n} = E(x_i) = E(X)$$

But when we talk about the estimate

$$\hat{x} = \frac{1}{n} \sum_{i=1}^{n} f_X(x_i) x_i$$

It does not give an expected estimate for E(X).

$$E(x f_X(x)) = \int_{-\infty}^{\infty} x f_X(x)^2 dx \neq E(X)$$

Estimate of $\hat{x}$ using the second formula will be

$$E(\hat{x}) = \frac{\sum_{i=i}^{n} E(f_X(x_i) x_i)}{n}$$

$$\neq E(X)$$

Hence, we have shown the correct estimate for E(X) will be $\hat{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$ by showing that the other formulae is estimating the expectation for some other random variable.

$\square$

Question 5

Solution

$$P(X \geq x) \leq e^{-tx}\phi_X(t)$$

for t >0 and

$$P(X \leq x) \leq e^{-tx}\phi_X(t)$$

for t <0, where $\phi_X(t)$ is the MGF of X.

For a continuous random variable X, we have

$$\phi_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)dx$$

where $f_X(x)$ is the PDF of X.

$e^t x$ is strictly increasing for $t > 0$ and strictly decreasing for $t < 0$. So, for either cases we can say.

$$P(X \geq x) = P(e^{tX} \geq e^{tx})$$
$$P(X \leq x) = P(e^{tX} \geq e^{tx})$$

.Using Markov's inequality, we can write

$$P(X \geq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E[e^{tX}]}{e^{tx}} = e^{-tx}\phi_X(t)$$

for t $>$ 0. Similarly, for t $<$ 0, we can write

$$P(X \leq x) = P(e^{tX} \geq e^{tx}) \leq \frac{E[e^{tX}]}{e^{tx}} = e^{-tx}\phi_X(t)$$

Thus, we prove the two inequalities.

Now we want to show that for $t \geq 0$,

$$\Pr\big(X > (1+\delta)\mu\big) \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}}, \quad t \geq 0,\, \delta > 0$$

where $\mu = E[X]$. For n independent Bernoulli random variable,

$$\mu = \sum_{i=1}^{n} p_i$$

Now, using the first part of the inequality derived above,

$$P(X \geq (1+\delta)\mu) \leq e^{-t(1+\delta)\mu}\phi_X(t)$$

for t $>$ 0. Now, we need to find the MGF of X. Since X is the sum of n independent Bernoulli random variables, we can write

$$\phi_X(t) = E[e^{tX}] = E[e^{t\sum_{i=1}^{n} X_i}] = \prod_{i=1}^{n} E[e^{tX_i}]$$

Now, for a Bernoulli random variable X_i with parameter p_i, we have

$$E[e^{tX_i}] = p_i e^t + (1-p_i) = 1 + p_i(e^t - 1)$$

Using the propert of MGFs that.

$$\phi_{X_1+X_2+..+X_N}(t) = \phi_{X_1}(t)\phi_{X_2}(t)...\phi_{X_N}(t)$$

we can write

$$\phi_X(t) = \prod_{i=1}^{n}(1 + p_i(e^t - 1))$$

Using the inequality $1 + x \leq e^x$, we get

$$\phi_X(t) \leq \prod_{i=1}^{n} e^{p_i(e^t-1)} = e^{(e^t-1)\sum_{i=1}^{n} p_i} = e^{\mu(e^t-1)}$$

So, we can write

$$P(X \geq (1+\delta)\mu) \leq e^{-t(1+\delta)\mu}e^{\mu(e^t-1)}$$

or

$$P(X \geq (1+\delta)\mu) \leq \frac{e^{\mu(e^t-1)}}{e^{(1+\delta)t\mu}}$$

which is the required result.Hence, proven.

Question 6

Solution

T : Trial Number until first heads

$$P(T = t) = (1-p)^{t-1}p$$

For n independent coin tosses,

$$X = \sum_{i=1}^{n} T_i$$

where T_i is the trial number until first heads for the ith coin. To find expectation of X, we use

$$E[X] = E[\sum_{i=1}^{n} T_i] = \sum_{i=1}^{n} E[T_i]$$

$$E[T_i] = i(1-p)^{t-1}p$$

$$E[X] = \sum_{i=1}^{n} i(1-p)^{i-1}p$$

This is an arithmetico-geometric series. Multiplying by 1-p,

$$(1-p)E[X] = \sum_{i=1}^{n} i(1-p)^{i}p$$

Subtracting the two equations,

$$pE[X] = p(1 + (1-p) + (1-p)^2 + ... + (1-p)^{(n-1)} - n(1-p)^n)$$

$$E[X] = \frac{1 - (1-p)^n}{p} - n(1-p)^n$$

$$E[X] = \frac{n}{p}(1 - (1-p)^n(1+np))$$

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