Pretending to factor large numbers on a quantum computer

John A. Smolin¹, Graeme Smith¹ and Alex Vargo¹

Shor's algorithm for factoring in polynomial time on a quantum computer gives an enormous advantage over all known classical factoring algorithm. We demonstrate how to factor products of large prime numbers using a compiled version of Shor's quantum factoring algorithm. Our technique can factor all products of p,q such that p,q are unequal primes greater than two, runs in constant time, and requires only two coherent qubits. This illustrates that the correct measure of difficulty when implementing Shor's algorithm is not the size of number factored, but the length of the period found.

1 Introduction

Building a quantum computer capable of factoring larger numbers than any classical computer can hope to is one of the grand challenges of computing in the 21st century. While still far off, there have already been several small-scale demonstrations of Shor's algorithm^{2–7}. Someday soon a quantum computer may factor a number hitherto unthinkably large. Such a device would most likely have to be a fully scalable fault-tolerant quantum machine, capable of carrying out any task a quantum computer could be asked to do. Thus, a large factorization would be convincing proof that one has built a practical quantum computer. Until such a time, more modest goals must suffice. The experiments mentioned above have factored numbers no larger than 21. Here we will show how current technology can demonstrate significantly larger factorizations.

We begin with a review of Shor's algorithm. Given an integer N=pq with p,q distinct primes, one proceeds as follows:

- 1. Choose (at random) an integer 0 < a < N.
- 2. Compute the greatest common divisor (GCD) of a and N. This can be found efficiently using the Euclidean algorithm⁸. If it is not 1, then GCD(a, N) is a nontrivial factor of N. Otherwise go on to the next step.
- 3. Choose $S \equiv 2^s$ such that $N^2 \leq S < 2N^2$. Construct the quantum state

$$S^{-1/2} \sum_{x=0}^{S-1} |x\rangle|0\rangle \tag{1}$$

¹IBM T.J. Watson Research Center, Yorktown Heights, NY 10598, USA

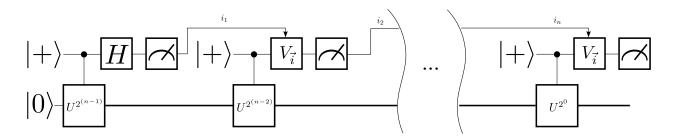


Figure 1: Circuit for Shor's algorithm using the semi-classical quantum Fourier transform. At each stage a $|+\rangle$ state is prepared. It is used as the control input on a controlled unitary $U^{2^{n-1}}$ for the nth bit of the readout, with $U|y\rangle = |ay \mod N\rangle$. Next, the gate $V_{\vec{i}} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix} H$ is applied and then the qubit is measured. $H = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is the Hadamard gate and the phase ϕ is computed as a function of all previous measurement results (see Ref.⁹). The first time there is no phase so the Hadamard is used. The process is repeated n times to read out n bits of precision of the Fourier transform.

on two quantum registers, the first is s-qubits and the second is $\log N$ qubits. Note that in the literature x and a sometimes have their meanings interchanged.

- 4. Perform a quantum computation on this state which maps $|x\rangle|0\rangle$ to $|x\rangle|a^x \mod N\rangle$. This is the slowest step, but can be done in time $O((\log N)^3)$.
- 5. Do the quantum Fourier transform on the first register, resulting in the state

$$S^{-1} \sum_{x} \sum_{y} e^{(2\pi i/S)xy} |y\rangle |a^x \mod N\rangle.$$
 (2)

This step requires $O((\log N)^2)$ time, which is much less than the modular exponentiation of the previous step.

- 6. Measure the first register to obtain classical result y. With reasonable probability, the continued fraction approximation of y/S or some y'/S for some y' near y will be an integer multiple of the period r of the function $a^x \mod N$. The GCD algorithm can then efficiently find r.
- 7. If r is odd, or if $a^{r/2} = -1 \mod N$, go back to step 1. Otherwise, $GCD(a^{r/2} \pm 1, N)$ is p or q.

Significant optimization of the basic algorithm has been achieved. As described, roughly $3 \log N$ qubits are needed. In fact, this can be reduced down to exactly $2 + 3/2 \log N$ qubits 10 . A significant part of the reduction is to replace the first "x" register with a single qubit. This was shown to be possible 11,12 and uses the fact that the bits of the quantum Fourier transform can be read out one at a time 9 . The use of this semi-classical Fourier transform has become known as qubit recycling. A circuit using qubit recycling is shown in Fig. 1.

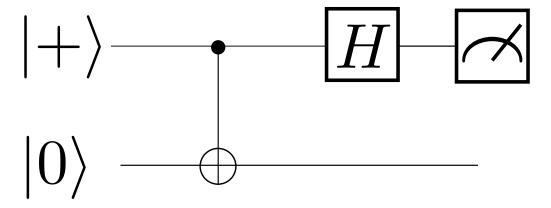


Figure 2: The circuit for the fully-compiled Shor's algorithm. The modular exponentiation is the single controlled-NOT, and the quantum Fourier transform is a Hadamard gate.

2 Compiled Shor's Algorithm

All experimental realizations of Shor's algorithm to date have relied on a further optimization, that of "compiling" the algorithm. This means employing the observation that different bases a in the modular exponentiation lead to different periods of the function $a^x \mod N$. Some of the periods are both short and lead to a factorization of the composite pq.

In 2001, the composite 15 was factored² using two different bases, an "easy" base (a=11, resulting in a period of 2), and a "difficult" base (a=7, with a period r=4). Neither is fully general and this allowed the factorization to take place on a seven bit quantum computer, when the best known uncompiled algorithm would require 8 bits $(2+3/2\log N)$ bits as per Zalka¹⁰). Other factorizations of 15 have since been performed using other architectures^{3-5,7}. More recently, 21 has been factored⁶ using just one qubit and one qutrit (a three-level system). In this case a=4 is used, resulting in a period $r=3^*$. These results are summarized in Table 1.

Recently, Zhou and Geller showed¹³ how to find a's with small periods for products of Fermat Primes¹⁴ (primes of the form $2^{2^k} + 1$). Here, we go substantially beyond this idea and show that any composite number pq has compiled versions of Shor's algorithm that can be run on a very small quantum computer. In particular, we show that there always exists a base a such that r = 2. Then, the second register need only hold two distinct states and the computation can be performed using only two qubits. In this case, the U needed in the circuit from Fig. 1 reduces to a controlled-NOT gate. Furthermore, only one stage of the circuit is required since all powers of U^{2^n} are the identity except for n = 0. The compiled circuit is shown in Fig. 2.

In order for the second register to need to hold only two distinct states, we must find a base

^{*}Note that Shor's algorithm normally fails when r is odd since $a^{r/2}$ is irrational in general. Here, since a=4 is a perfect square, this problem does not arise.

N	Qubits needed (Zalka ¹⁰)	Qubits implemented	Qubits compiled
15	8	7 [Ref. ²],4 [Refs. ^{3,4}],5 [Ref. ⁵],3 [Ref. ⁷]	2
21	10	$1 + \log 3 [\text{Ref.}^6]$	2
RSA-768	1154	2^{\dagger}	2
N-20000	30002	2^{\dagger}	2

Table 1: Number of qubits required for Shor's algorithm and experimental results. RSA-768 and N-20000 are available in the supplementary material. †Quantum version to be completed. Classical version with one random bit has been performed, see Section 3.

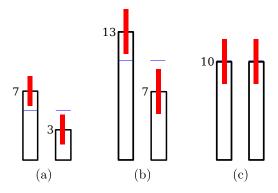


Figure 3: Experimental data from unbiased coins. (a) A 1998 US quarter was tossed 10 times to factor 15. (b) A 1968 US penny was tossed 20 times in order to factor RSA-768. (c) A 2008 US Oklahoma commemorative quarter was tossed 20 times to factor N-20000. One- σ error bars are shown.

a such that $a^2 = 1 \mod pq$. The Chinese remainder theorem¹⁵ tells us that

$$a^2 = 1 \mod pq \Leftrightarrow a^2 = 1 \mod p \text{ and } a^2 = 1 \mod q$$
 (3)

for p, q relatively prime. By construction

$$a \equiv \pm pp_q \pm qq_p \text{ has } a^2 = 1 \mod p \text{ and } a^2 = 1 \mod q$$
 (4)

where p_q is the multiplicative inverse of p, $\mod q$ and q_p is the inverse of q, $\mod p$. Then (3) tells us $a^2=1\mod pq$. These inverses can be found efficiently using the extended Euclidian algorithm. There are 4 solutions of (4) corresponding to the signs. Two of these will be trivial, ± 1 and the other two will be bases resulting in compiled Shor factorizations with a period of the function $a^x \mod N$ having period 2.

3 Experiment

A future version of this preprint will include experimental data using two superconducting transmon¹⁶ qubits. In the meantime, we perform a simpler experiment. We employ a further optimization not used in previous experiments. Observe that in the circuit in Figure 2, the second qubit is never measured. In fact, what is created by the controlled-NOT is a maximally-entangled state, half of which is simply discarded. The resulting state of the first qubit is therefore maximally mixed. Due to the unitary equivalence of purifications, if we create a maximally mixed state in any way at all, it is entangled with some system in the environment. A maximally mixed state is unaffected by the Hadamard gate, so this too is unnecessary. We can therefore produce the appropriate probability distribution at the output by tossing an unbiased coin. Fig. 3 shows the data for factoring 15, RSA-768, and N-20000 using this method.

4 Conclusions

Of course this should not be considered a serious demonstration of Shor's algorithm. It does, however, illustrate the danger in "compiled" demonstrations of Shor's algorithm. To varying degrees, *all* previous factorization experiments have benefited from this artifice. While there is no objection to having a classical compiler help design a quantum circuit (indeed, probably all quantum computers will function in this way), it is *not* legitimate for a compiler to know the answer to the problem being solved. To even call such a procedure compilation is an abuse of language.

As the cases of RSA-768 and N-20000 suggest, very large numbers can be trivially factored if we were to allow this. For this reason we stress that a factorization experiment should be judged not by the size of the number factored but by the size of the period found. Current experiments ought to be viewed instead as technology demonstrations, showing that we can manipulate small numbers of qubits. In Ref.⁶, for instance, it was shown that intentionally added decoherence reduced the contrast in the data, a hallmark of a quantum-coherent process. All the referenced experiments are important tiny steps in the direction of building a quantum computer, but actually running algorithms on such tiny experiments is a somewhat frivolous endeavor.

5 Acknowledgements

We acknowledge support from IARPA under contract no. W911NF-10-1-0324 and from the DARPA QUEST program under contract no. HR0011-09-C-0047. All statements of fact, opinion or conclusions contained herein are those of the authors and should not be construed as representing the official views or policies of the U.S. Government.

- 1. Peter W. Shor. Discrete logarithms and factoring. pages 124–134. Proceedings of the 35th Annual IEEE Symposium on the Foundations of Computer Science, 1994.
- 2. Lieven M.K. Vandersypen, Matthias Steffen, Gregory Breyta, Costantino S. Yannoni, Mark H. Sherwood, and Isaac L. Chuang. Experimental realization of Shor's quantum factoring algorithm using nuclear magnetic resonance. *Nature*, pages 883–887, 2001. arXiv:quant-ph/0112176.
- 3. B. P. Lanyon, T. J. Weinhold, N. K. Langford, M. Barbieri, D. F. V. James, A. Gilchrist, and A. G. White. Experimental demonstration of a compiled version of Shor's algorithm with quantum entanglement. *Phys. Rev. Lett.*, 99:250505, Dec 2007.
- 4. Chao-Yang Lu, Daniel E. Browne, Tao Yang, and Jian-Wei Pan. Demonstration of a compiled version of Shor's quantum factoring algorithm using photonic qubits. *Phys. Rev. Lett.*, 99:250504, Dec 2007.
- 5. Alberto Politi, Jonathan C. F. Matthews, and Jeremy L. O'Brien. Shor's quantum factoring algorithm on a photonic chip. *Science*, 325(5945):1221, 2009.
- 6. Enrique Martin-Lopez, Anthony Laing, Thomas Lawson, Xiao-Qi Zhou, and Jeremy L. O'Brien. Experimental realization of Shor's quantum factoring algorithm using qubit recycling. *Nature Photonics*, 6:773–776, 2012. arXiv:1111.4147.
- 7. Erik Lucero, R. Barends, Y. Chen, J. Kelly, M. Mariantoni, A. Megrant, P. OMalley, D. Sank, A. Vainsencher, J. Wenner, T. White, Y. Yin, A. N. Cleland, and John M. Martinis. Computing prime factors with a Josephson phase qubit quantum processor. *Nature Physics*, 8:719–723, 2012.
- 8. Euclid of Alexandria. Elements. circa 300 BCE.
- 9. Robert B. Griffiths and Chi-Sheng Niu. Semiclassical fourier transform for quantum computation. *Phys. Rev. Lett.*, 76:3228–3231, Apr 1996.
- 10. Christof Zalka. Shor's algorithm with fewer (pure) qubits. arXiv:quant-ph/0601097, 2006.
- 11. Michele Mosca and Artur Ekert. The hidden subgroup problem and eigenvalue estimation on a quantum computer. In Colin P. Williams, editor, *Quantum Computing and Quantum Communications*, volume 1509 of *Lecture Notes in Computer Science*, pages 174–188. Springer Berlin Heidelberg, 1999.
- 12. S. Parker and M. B. Plenio. Efficient factorization with a single pure qubit and $\log N$ mixed qubits. *Phys. Rev. Lett.*, 85:3049–3052, Oct 2000.
- 13. Zhongyuan Zhou and Michael R. Geller. Factoring 51 and 85 with 8 qubits. unpublished, 2012.
- 14. Pierre de Fermat. unpublished. circa 1650. see http://en.wikipedia.org/wiki/Fermat_number.

- 15. Sun Zi. The mathematical class of Sun Zi. circa 400 CE.
- Jens Koch, Terri M. Yu, Jay Gambetta, A. A. Houck, D. I. Schuster, J. Majer, Alexandre Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf. Charge-insensitive qubit design derived from the cooper pair box. *Phys. Rev. A*, 76:042319, Oct 2007.

6 Supplementary material

We have factored RSA-768:

 $RSA - 768 = 1230186684530117755130494958384962720772853569595334792197322 \\ 4521517264005072636575187452021997864693899564749427740638459 \\ 2519255732630345373154826850791702612214291346167042921431160 \\ 2221240479274737794080665351419597459856902143413 \\ = 3347807169895689878604416984821269081770479498371376856891243 \\ 388982883793878002287614711652531743087737814467999489 \\ \times 3674604366679959042824463379962795263227915816434308764267 \\ 6032283815739666511279233373417143396810270092798736308917$

which can be done using

- $a = 1029031793302493258003488818376905875264575120178567995715921117383374 \\ 0637809554762657146559655560974877155097084531342124720712415517107376 \\ 6764612501767199553731974973903504534358652759946682893508255761840004 \\ 7627481255809299529939 \text{ or }$
- $a = 2011548912276244971270061400080568455082784494167667964814013347683523\\ 3672630818125303054623423037190224096523432092963345312131577459271606\\ 3902143490245030584823163722635383870725074621773650339655236462265303\\ 792116204047602613474.$

We have also factored N-20000:

 $\begin{array}{c} N-20000 = & 3545872995518995320162216211194750088254004994975476505069782010092562\\ & 6836455218077570931827645265231608998412594091501169267869932212762643\\ & 7952340916754401101694942393398796986114303891110264606442388558651298\\ & 8969181675291250725852835615290690458331490854580449715364904882361230\\ & 3234271985470197470205533036194402979614663510505596285870603463725841\\ & 3294436223612733881541244957256787578366712932408954804647829260155941\\ \end{array}$

 $8271018129029021585609464256357290486028253978413567783992747663703346\\ 4998930738742796686640128620923272630180172545665881974962283565208378\\ 8146895814608717113146637672212361020062046644744553178935511572078813\\ 4289685668894167388898155080721001862530295759444117033443834883115100\\ 0650673799192981407882986789587653580546902745162832189506923934649727\\ 3468283522712516205990343243813973588848694703996512441662433168667879\\ 1132220512140731023345171452862842540349609656231516812698183384579682\\ 7082647120785562227006525356718270612869805415804935095162605716061024\\ 4348906073326891005641611113422562155003437331976928597724313112437132\\ 9726226175793616722507421635694278871177878800928003913785632249721075\\ 9$

=192000795467552080437885021456709626445650825395923446772711162330576733161645236543673900742063814492055178154240871961342803053595883851063357619809691615460088802326462712010390810975054420845793817460736705 80625363592770419440450498657743226310655849985633384965750995270262426259463783610590851095864429551274069878588826066040975266553379144063

 $\times 1846801200424320058688545075138064936637978107808827998771506577923856\\ 3512240821642698240746519848908888538863314146811777562735349842383990\\ 5670861632022702344791536050984503349712490164967553980037733570521519\\ 7292378714384123838303682671000232696966397317822765773532584800195449\\ 5368601567970009628561986870825780394439849598274041251373721318688499\\ 1453459939131218282383837906067367719975956305299253544862766250674360\\ 8998556300881542429404051609300094861204547432754918123186836075613988\\ 2688116428895169498745291929770561287646470297265115687387483895368013\\ 6199972574717643411232857324289391893113632565488962345764030022326064\\ 3637140144476343583887275463815046470469103711530263500124285402956246\\ 3762807770858497146737449843628571647621934352570516005139021147129387\\ 7254093410928119195530593476356748117307256152959591420794993258199178\\ 1288658087938767149531957135085795481815496115596710030791084073641958\\ 2481531196768438309249368833213399954953500171444271211952594455954004\\ 4777528236825191633941191910648017375990757253500191633238179812152213$

using

 $a = 7768655490606911685159539689674850887282976739122743910922331484127768\\ 1101598247891527093924276297376331925480726454586027525740679667172791\\ 7217555058225806037812849768665258620363753772039717327412351869425858\\ 1904615133725741922663209052524169553197120804112291503080775196150179\\ 7481370912095756084237524803358135914101301946505352067303719701391298$

 $7621042461928337773195702323394083899254945156232759245166073535525570\\0230213547360960426568078560686158288144399736900984300336175389023060\\5616437376142524810693077526916499349142045565890527378614647749168113\\6114052201940155433655070012448069659053921184482144683142113411245018\\2866313805439384828406861471681905442918834027976071244164504811135697\\3492188754652130466501198494962242554720665501900131522896323535118132\\6736727387287391416062134016119144495828759261687601071225293142451723\\2025574953555759702466842612215744635530149437609579838902237107080506\\2623093202608288457206286157404480893346053773771747450311895170916654\\6900489476391880896018139454661252438732214379026089121685763461541730\\5706473026557928820602015402938203420260179270235206050843670016447924.$