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**Assignment**$_1$

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Almaty, 2025

# Problem 1

Consider the system with two states, and the state-space model matrices given by:

$$A = \begin{bmatrix} -6 & 1 \\ -5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ K \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

where $K \in R$ is a parameter to be specified.

## (a) Finding the Transfer Function $G(s)$

The transfer function is given by:

$$G(s) = C(sI - A)^{-1}B$$

First, compute the characteristic equation:

$$sI - A = \begin{bmatrix} s+6 & -1 \\ 5 & s \end{bmatrix}$$

The determinant of $sI - A$ is:

$$\det(sI - A) = (s+6)(s) - 1(-5) = s^2 + 6s + 5$$

The inverse of $sI - A$ is given by:

$$(sI - A)^{-1} = \frac{1}{s^2 + 6s + 5} \begin{bmatrix} s & 1 \\ -5 & s+6 \end{bmatrix}$$

Multiplying by $B$:

$$(sI - A)^{-1}B = \frac{1}{s^2 + 6s + 5} \begin{bmatrix} s & 1 \\ -5 & s+6 \end{bmatrix} \begin{bmatrix} 1 \\ K \end{bmatrix}$$

$$= \frac{1}{s^2 + 6s + 5} \begin{bmatrix} s + K \\ -5 + K(s+6) \end{bmatrix}$$

Multiplying by $C$:

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s + K \\ -5 + K(s + 6) \end{bmatrix}$$

$$= \frac{s + K}{s^2 + 6s + 5}$$

The structure of $G(s)$ changes for different values of $K$, affecting the system dynamics.

## (b) Observability Analysis

The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}$$

The determinant is:

$$\det(\mathcal{O}) = (1)(1) - (0)(-6) = 1 \neq 0$$

Since the determinant is nonzero, the system is observable for all values of $K$.

## (c) Controllability Analysis

The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -6 + K \\ K & -5K \end{bmatrix}$$

The determinant is:

$$\det(\mathcal{C}) = (1)(-5K) - (K)(-6 + K) = -5K + 6K - K^2 = K - K^2$$

For the system to be controllable, $\det(\mathcal{C}) \neq 0$, meaning:

$$K - K^2 \neq 0$$

$$K(K - 1) \neq 0$$

Thus, the system is uncontrollable for $K = 0$ and $K = 1$.

## (d) Comparison of Observability and Controllability

- The system is always observable, independent of $K$. - The system is controllable except when $K = 0$ or $K = 1$. - The transfer function $G(s)$ structure varies with $K$, which aligns with the controllability analysis since the system loses control authority for certain values of $K$.

# Solution to Problem 2

## Step 1: Controllability Analysis

A system is controllable if the **controllability matrix** $\mathcal{C}$ has full rank:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

If $\text{rank}(\mathcal{C}) = n$, the system is **controllable**.

## Step 2: Observability Analysis

A system is observable if the **observability matrix** $\mathcal{O}$ has full rank:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

If $\text{rank}(\mathcal{O}) = n$, the system is **observable**.

## Step 3: Stability Analysis

To check stability, we compute the **eigenvalues** of $A$. The system is: - **Asymptotically stable** if all eigenvalues have **negative real parts**. - **Unstable** if any eigenvalue has a **positive real part**.

## Step 4: Detectability and Stabilizability

- A system is **detectable** if all **unstable** eigenvalues are observable. - A system is **stabilizable** if all **unstable** eigenvalues are controllable.

# Solution for (a)

Given matrices:

$$A = \begin{bmatrix} -1 & 3 \\ 1 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

## Step 1: Controllability

The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix}$$

Compute $AB$:

$$AB = \begin{bmatrix} -1 & 3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

Thus,

$$\mathcal{C} = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Since $\text{rank}(\mathcal{C}) = 1$, $\rightarrow rank \neq order$
$thesystemis$**uncontrollable**.

## Step 2: Observability

The observability matrix is:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}$$

Compute $CA$:

$$CA = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 0 \end{bmatrix}$$

Thus,

$$\mathcal{O} = \begin{bmatrix} 2 & -1 \\ -3 & 0 \end{bmatrix}$$

Since $\text{rank}(\mathcal{O}) = 2$, the system is **observable**.

## Step 3: Stability

Find eigenvalues of $A$:

$$\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 3 \\ 1 & 6-\lambda \end{bmatrix}$$

$$(-1-\lambda)(6-\lambda) - (3 \times 1) = \lambda^2 - 5\lambda - 9$$

Solve:

$$\lambda = \frac{5 \pm \sqrt{25+36}}{2} = \frac{5 \pm \sqrt{61}}{2}$$

One eigenvalue is **positive**, so the system is **unstable**.

### Step 4: Detectability and Stabilizability

- Since the system is **observable**, it is **detectable**. - Since the system is **controllable**, it is **stabilizable**.

### Conclusion for (a)

- **Controllable**
- **Observable**
- **Unstable** $\times$
- **Detectable**
- **Stabilizable**
- **Minimal Realization**

## Solution for (b)

Given matrices:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -3 & 0 \\ -1 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D = -1$$

### Step 1: Controllability

Computing $\mathcal{C}$:

$$AB = \begin{bmatrix} 3 \\ -3 \\ -3 \end{bmatrix}, \quad A^2B = \begin{bmatrix} -6 \\ 6 \\ 6 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 0 & 3 & -6 \\ 1 & -3 & 6 \\ 1 & -3 & 6 \end{bmatrix}$$

Since $\text{rank}(\mathcal{C}) = 3$, the system is **uncontrollable**.

### Step 2: Observability

Computing $\mathcal{O}$:

$$CA = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$$

Since $\text{rank}(\mathcal{O}) < 3$, the system is **not observable**.

## Conclusion for (b)

- **Controllable** $\times$
- **Not Observable** $\times$
- **Not Minimal Realization** $\times$
- **Stabilizable** $\times$
- **Not Detectable** $\times$

State Feedback Control Solution

# 1 Given System

$$\dot{x} = \begin{bmatrix} -5 & 1 \\ 14 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Applying full-state feedback control:

$$u = -Kx, \quad K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

# 2 (a) Open-Loop Poles and Zeros

The open-loop poles are the eigenvalues of matrix $A$, found by solving:

$$\det(sI - A) = 0$$

Computing the determinant:

$$\det \begin{bmatrix} s+5 & -1 \\ -14 & s \end{bmatrix} = (s+5)(s) - (-1)(-14)$$
$$= s^2 + 5s - 14$$

Solving for $s$:

$$s = \frac{-5 \pm \sqrt{5^2 - 4(1)(-14)}}{2(1)}$$
$$= \frac{-5 \pm \sqrt{81}}{2} = \frac{-5 \pm 9}{2}$$
$$s_1 = 2, \quad s_2 = -7$$

Thus, the open-loop poles are $s_1 = 2$ and $s_2 = -7$. The system has no finite zeros, s=-14

# 3 (b) Feedback Gain $K$ for Desired Closed-Loop Poles

The closed-loop system matrix is:

$$\begin{aligned} A - BK &= \begin{bmatrix} -5 & 1 \\ 14 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\ &= \begin{bmatrix} -5 - k_1 & 1 - k_2 \\ 14 - k_1 & -k_2 \end{bmatrix} \end{aligned}$$

The characteristic equation of $A - BK$ should match:

$$s^2 + 2\omega_n s + \omega_n^2 = 0$$

Expanding $\det(sI - (A - BK)) = 0$:

$$\det \begin{bmatrix} s + 5 + k_1 & -1 + k_2 \\ -14 + k_1 & s + k_2 \end{bmatrix} = 0$$

Expanding further:

$$(s + 5 + k_1)(s + k_2) - (-1 + k_2)(-14 + k_1) = 0$$

$$s^2 + (5 + k_1 + k_2)s + (5k_2 + k_1k_2 - 14 + k_1 + 14k_2 - k_1k_2) = 0$$

$$s^2 + (5 + k_1 + k_2)s + (5k_2 - 14 + k_1 + 14k_2) = 0$$

Matching with the desired equation gives:

$$\begin{aligned} 5 + k_1 + k_2 &= 2\omega_n \\ 5k_2 + 5k_1 - 14 + k_1 + 14k_2 &= \omega_n^2 \end{aligned}$$

# 4 (c) Effect of Large Gains on Pole Placement

From (b), shifting poles further left (more negative real parts) requires increasing $k_1$ and $k_2$, confirming that larger gains are needed for aggressive pole placement.

# 5 (d) Gain $K$ for Poles at $s = -9$ and $s = -4$

The desired characteristic equation:

$$(s + 9)(s + 4) = 0$$

Expanding:

$$s^2 + 13s + 36 = 0$$

Matching coefficients:

$$
\begin{aligned}
5 + k_1 + k_2 &= 13 \\
5k_2 + 5k_1 - 14 + k_1 + 14k_2 &= 36
\end{aligned}
$$

Solving:

$$
\begin{aligned}
k_1 + k_2 &= 8 \\
6k_1 + 19k_2 &= 50
\end{aligned}
$$

Substituting $k_1 = 8 - k_2$:

$$
\begin{aligned}
6(8 - k_2) + 19k_2 &= 50 \\
48 - 6k_2 + 19k_2 &= 50 \\
13k_2 &= 2 \\
k_2 = \frac{2}{13}, \quad k_1 &= \frac{102}{13}
\end{aligned}
$$

Thus, the feedback gain is:

$$K = \begin{bmatrix} \frac{102}{13} & \frac{2}{13} \end{bmatrix}$$

The system is described by the state-space equation:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where the state vector is:

$$x(t) = \begin{bmatrix} \theta_1(t) \\ \dot{\theta}_1(t) \\ \theta_2(t) \\ \dot{\theta}_2(t) \\ x(t) \\ \dot{x}(t) \end{bmatrix} \tag{2}$$

The system matrices are given as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 16 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -16 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3}$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

# 6 Controllability Analysis

A system is controllable if the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B & A^5B \end{bmatrix} \tag{5}$$

has full rank, which in this case must be 6.

## 6.1 Computing the Controllability Matrix

**First column:**

$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{6}$$

**Second column:**

$$AB = AB = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 16 & 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -16 & 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{7}$$

the result is:

$$AB = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{8}$$

**Third column:**

$$A^2B = A(AB) = A \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{9}$$

the result is:

$$A^2B = \begin{bmatrix} 0 \\ -16 \\ 0 \\ 16 \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

## 6.2 Rank of the Controllability Matrix

The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} 0 & -1 & 0 & \cdots \\ -1 & 0 & -16 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 16 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \end{bmatrix} \tag{11}$$

Computing the rank of $\mathcal{C}$, we find that:

$$\text{rank}(\mathcal{C}) = 4 < 6 \tag{12}$$

# 7 Problem 5

The given system equation is:

$$\dot{\Delta x}(t) = A^*\Delta x(t) + B^*\Delta i(t) \tag{13}$$

where the system matrices are:

$$A^* = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 115.2 & -0.05 & -18.6 & 0 \\ 0 & 0 & 0 & 1 \\ -37.2 & 0 & 37.2 & -0.1 \end{bmatrix} \tag{14}$$

$$B^* = \begin{bmatrix} 0 \\ -6.55 \\ 0 \\ -6.55 \end{bmatrix} \tag{15}$$

The control law is given as:

$$\Delta i(t) = -K\Delta x(t) \tag{16}$$

where:

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \tag{17}$$

Thus, the closed-loop system matrix becomes:

$$A_{cl} = A^* - B^* K \tag{18}$$

Expanding:

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 115.2 & -0.05 & -18.6 & 0 \\ 0 & 0 & 0 & 1 \\ -37.2 & 0 & 37.2 & -0.1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -6.55 \\ 0 \\ -6.55 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} \tag{19}$$

The product of $B^*K$ is:

$$B^*K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -6.55k_1 & -6.55k_2 & -6.55k_3 & -6.55k_4 \\ 0 & 0 & 0 & 0 \\ -6.55k_1 & -6.55k_2 & -6.55k_3 & -6.55k_4 \end{bmatrix} \tag{20}$$

Thus:

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 115.2 + 6.55k_1 & -0.05 + 6.55k_2 & -18.6 + 6.55k_3 & 0 + 6.55k_4 \\ 0 & 0 & 0 & 1 \\ -37.2 + 6.55k_1 & 0 + 6.55k_2 & 37.2 + 6.55k_3 & -0.1 + 6.55k_4 \end{bmatrix} \tag{21}$$

## 7.1 Characteristic Equation

$$-1 + j, \quad -1 - j, \quad -10, \quad -10 \tag{22}$$

The characteristic equation is:

$$(s + 1 - j)(s + 1 + j)(s + 10)(s + 10) = 0 \tag{23}$$

$$(s + 1)^2 - j^2 = s^2 + 2s + 2 \tag{24}$$

$$(s + 10)^2 = s^2 + 20s + 100 \tag{25}$$

$$(s^2 + 2s + 2)(s^2 + 20s + 100) = 0 \tag{26}$$

$$s^4 + 20s^3 + 100s^2 + 2s^3 + 40s^2 + 200s + 2s^2 + 20s + 200 = 0 \tag{27}$$

$$s^4 + 22s^3 + 142s^2 + 220s + 200 = 0 \tag{28}$$

The characteristic equation of $A_{cl}$ must match this polynomial.

## 7.2 $\det(sI - A_{cl})$

$$\det(sI - A_{cl}) = \begin{vmatrix} s & -1 & 0 & 0 \\ -115.2 - 6.55k_1 & s + 0.05 - 6.55k_2 & 18.6 - 6.55k_3 & -6.55k_4 \\ 0 & 0 & s & -1 \\ 37.2 - 6.55k_1 & -6.55k_2 & -37.2 - 6.55k_3 & s + 0.1 - 6.55k_4 \end{vmatrix} = \tag{29}$$

$k_1, k_2, k_3, k_4$.