JSC "Kazakh-British Technical University"
School of Information Technology and Engineering

Assignment$_2$

Done by: **Meldeshuly Sagingaly**
Checked by: **Yessenzhanov Kuanysh**

Almaty, 2025

Problem 1

Equilibrium solutions occur when $y' = 0$, so we solve:

$$y^2 - y - 6 = 0. \tag{1}$$

Factorizing:

$$(y-3)(y+2) = 0. \tag{2}$$

Thus, the equilibrium points are $y = 3$ and $y = -2$.

Step 2: Classify stability
To determine stability, we analyze $f(y) = y^2 - y - 6$ and compute its derivative:

$$f'(y) = 2y - 1. \tag{3}$$

Evaluating at equilibrium points:

$$\begin{aligned} f'(3) &= 2(3) - 1 = 5 > 0 \quad \text{(Unstable - Source)} \\ f'(-2) &= 2(-2) - 1 = -5 < 0 \quad \text{(Stable - Sink)} \end{aligned}$$

(b) $y' = (y^2 - 4)(y+1)^2$

Step 1: Find equilibrium solutions
Setting $y' = 0$:

$$(y^2 - 4)(y+1)^2 = 0. \tag{4}$$

Factorizing:

$$(y-2)(y+2)(y+1)^2 = 0. \tag{5}$$

Setting each factor to zero:

$$\begin{aligned} y - 2 = 0 \quad &\Rightarrow \quad y = 2, \\ y + 2 = 0 \quad &\Rightarrow \quad y = -2, \\ (y+1)^2 = 0 \quad &\Rightarrow \quad y = -1. \end{aligned}$$

Thus, the equilibrium points are $y = -2, -1, 2$.

Step 2: Classify stability
The derivative sign analysis depends on:

$$f(y) = (y-2)(y+2)(y+1)^2. \tag{6}$$

We analyze the sign of y' in different intervals:

Interval	$y < -2$	$-2 < y < -1$	$-1 < y < 2$	$y > 2$
$y - 2$	-	-	-	+
$y + 2$	-	+	+	+
$(y+1)^2$	+	+	+	+
y'	(+)	(-)	(-)	(+)

Classification of equilibrium points:

- $y = -2$: Changes from $+$ to $-$ $\Rightarrow$ **Stable (Sink)**.
- $y = -1$: No sign change (negative on both sides) $\Rightarrow$ **Semi-stable** (stable from above, unstable from below).
- $y = 2$: Changes from $-$ to $+$ $\Rightarrow$ **Unstable (Source)**.

1 System 1

$$\dot{x}_1 = x_2 \tag{7}$$

$$\dot{x}_2 = -x_1 + \frac{x_1^3}{6} - x_2 \tag{8}$$

1.1 Equilibrium Points

Setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$x_2 = 0 \tag{9}$$

$$-x_1 + \frac{x_1^3}{6} = 0 \tag{10}$$

Solving for x_1:

$$x_1 \left(-1 + \frac{x_1^2}{6} \right) = 0 \Rightarrow x_1 = 0, \pm\sqrt{6} \tag{11}$$

Thus, the equilibrium points are:

$$(0, 0), (\sqrt{6}, 0), (-\sqrt{6}, 0)$$

1.2 Jacobian Matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 + \frac{x_1^2}{2} & -1 \end{bmatrix} \tag{12}$$

Evaluating at equilibrium points:

- At $(0,0)$:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda^2 + \lambda + 1 = 0$ (complex roots, unstable spiral).

- At $(\pm\sqrt{6}, 0)$:

$$J(\pm\sqrt{6}, 0) = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$$

Eigenvalues: $\lambda^2 + \lambda - 2 = 0$, giving $\lambda = -2, 1$ (saddle points).

2 System 2

$$\dot{x}_1 = -x_1 + x_2 \tag{13}$$
$$\dot{x}_2 = 0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3 \tag{14}$$

2.1 Jacobian Matrix

$$J = \begin{bmatrix} -1 & 1 \\ 0.1 - 2x_1 - 0.3x_1^2 & -2 \end{bmatrix} \tag{15}$$

Evaluating at $(0,0)$:

$$J(0,0) = \begin{bmatrix} -1 & 1 \\ 0.1 & -2 \end{bmatrix}$$

Eigenvalues are determined by solving $\det(J - \lambda I) = 0$.

3 System 3

$$\dot{x}_1 = (1 - x_1)x_1 - \frac{2x_1x_2}{1 + x_1} \tag{16}$$
$$\dot{x}_2 = \left(2 - \frac{x_2}{1 + x_1}\right) x_2 \tag{17}$$

3.1 Jacobian Matrix

$$J = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} \tag{18}$$

Solution to Problem 3

The given system of differential equations is:

$$\dot{x}_1 = ax_1 - x_1x_2, \tag{19}$$
$$\dot{x}_2 = bx_1^2 - cx_2. \tag{20}$$

To find the equilibrium points, we set $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$ax_1 - x_1x_2 = 0, \tag{21}$$
$$bx_1^2 - cx_2 = 0. \tag{22}$$

From the first equation, factoring x_1:

$$x_1(a - x_2) = 0. \tag{23}$$

Thus, the solutions are:

$$x_1 = 0 \quad \text{or} \quad x_2 = a. \tag{24}$$

Substituting $x_1 = 0$ into the second equation:

$$bx_1^2 - cx_2 = 0 \Rightarrow -cx_2 = 0 \Rightarrow x_2 = 0. \tag{25}$$

Thus, one equilibrium point is:

$$(x_1, x_2) = (0, 0). \tag{26}$$

For $x_2 = a$, substituting into the second equation:

$$bx_1^2 - ca = 0 \Rightarrow x_1^2 = \frac{ca}{b} \Rightarrow x_1 = \pm\sqrt{\frac{ca}{b}}. \tag{27}$$

Thus, the equilibrium points are:

$$(0, 0), \quad \left(\sqrt{\frac{ca}{b}}, a\right), \quad \left(-\sqrt{\frac{ca}{b}}, a\right). \tag{28}$$

Solution to Problem 4

The given system is:

$$\dot{x}_1 = (u - x_1)(1 + x_2^2), \tag{29}$$
$$\dot{x}_2 = (x_1 - 2x_2)(1 + x_1^2), \tag{30}$$
$$y = x_2. \tag{31}$$

Using the feedback control $u = -Ky$, we substitute $y = x_2$:

$$u = -Kx_2. \tag{32}$$

Substituting in $\dot{x}_1$:

$$\dot{x}_1 = (-Kx_2 - x_1)(1 + x_2^2), \tag{33}$$
$$\dot{x}_2 = (x_1 - 2x_2)(1 + x_1^2). \tag{34}$$

Setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$ for equilibrium points:

$$(-Kx_2 - x_1)(1 + x_2^2) = 0, \tag{35}$$
$$(x_1 - 2x_2)(1 + x_1^2) = 0. \tag{36}$$

Solving these equations gives:

$$x_1 = 0, x_2 = 0 \quad \text{or} \quad x_1 = 2x_2, x_2 = -\frac{x_1}{K}. \tag{37}$$

article amsmath, amssymb, graphicx

Phase Portraits and Qualitative Analysis

4 Problem 5: Phase Portraits and Behavior Analysis

We are given two dynamical systems and need to analyze their phase portraits and qualitative behavior.

4.1 System (a)

The system equations are:

$$\dot{x}_1 = -x_2, \tag{38}$$
$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 + 0.1x_1^4). \tag{39}$$

4.1.1 Equilibrium Points

To find the equilibrium points, set $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$-x_2 = 0 \Rightarrow x_2 = 0, \tag{40}$$
$$x_1 - x_2(1 - x_1^2 + 0.1x_1^4) = 0. \tag{41}$$

Substituting $x_2 = 0$, we get:

$$x_1 = 0. \tag{42}$$

Thus, the only equilibrium point is $(0, 0)$.

4.1.2 Stability Analysis

The Jacobian matrix is:

$$J = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 - (-2x_1 + 0.4x_1^3)x_2 & -(1 - x_1^2 + 0.1x_1^4) \end{bmatrix}. \tag{43}$$

Evaluating at $(0, 0)$, we obtain:

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}. \tag{44}$$

The eigenvalues satisfy $\lambda^2 + \lambda + 1 = 0$, giving $\lambda = \frac{-1\pm\sqrt{-3}}{2}$, which are complex with negative real parts, indicating a stable spiral.

4.2 System (b)

The system equations are:

$$\dot{x}_1 = x_2, \tag{45}$$

$$\dot{x}_2 = x_1 + x_2 - 3\tan^{-1}(x_1 + x_2). \tag{46}$$

4.2.1 Equilibrium Points

Setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$:

$$x_2 = 0, \tag{47}$$

$$x_1 + x_2 - 3\tan^{-1}(x_1 + x_2) = 0. \tag{48}$$

Since $x_2 = 0$, the second equation simplifies to:

$$x_1 = 3\tan^{-1}(x_1). \tag{49}$$

Solving numerically, we find the equilibrium at $x_1 = 0, x_2 = 0$.

4.2.2 Stability Analysis

The Jacobian matrix is:

$$J = \begin{bmatrix} 0 & 1 \\ 1 - \frac{3}{1+(x_1+x_2)^2} & 1 - \frac{3}{1+(x_1+x_2)^2} \end{bmatrix}. \tag{50}$$

Evaluating at $(0, 0)$:

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}. \tag{51}$$

The eigenvalues satisfy $\lambda^2 + 2\lambda + 2 = 0$, which results in complex values with negative real parts, indicating a stable spiral.

5 Problem 6

5.1 Problem 1

$$y''(t) + 5 \cdot y'(t) + 6 \cdot y(t) = 0; \tag{52}$$

```
[x1, x2] = meshgrid(-2:0.2:2, -2:0.2:2);


x1dot = x2;
x2dot = -6*x1-5*x2;

figure;
quiver(x1, x2, x1dot, x2dot, 'b');
hold on;
xlabel('x_1');
ylabel('x_2');
title('Phase Portrait for System (b)');
axis tight;
grid on;
```

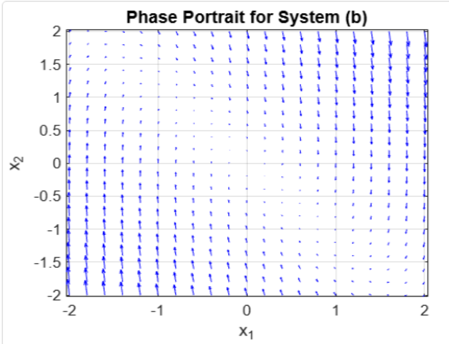


Figure 1: Stable node

5.2 Problem 2

$$y''(t) - 5 \cdot y'(t) + 6 \cdot y(t) = 0; \tag{53}$$

```
[x1, x2]=meshgrid(-1: 0.1: 1, -1: 0.1: 1);

x1dot = x2;
x2dot = 5*x2 - 6*x1;

figure;
quiver(x1,x2,x1dot,x2dot,'r');
xlabel('x-axis');
ylabel('y-axis');
title('Portrait phase for unstable node');
hold on;
axis tight;
grid on;
```

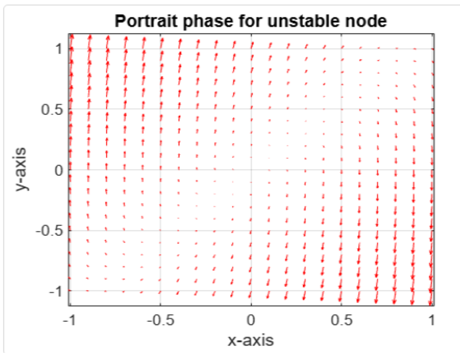


Figure 2: Unstable node

5.3 Problem 3

$$4 \cdot y''(t) + 2 \cdot y'(t) + 1 \cdot y(t) = 0; \tag{54}$$

```
[x1,x2]=meshgrid(-1: 0.1: 1,-1: 0.1: 1);

x1dot = x2;
x2dot = -0.5*x2-0.25*x1;

figure;
quiver(x1, x2, x1dot, x2dot, 'Y');
xlabel('x axis');
ylabel('y axis');
title('Phase portrait of Stable focus');
hold on;
axis tight;
grid on;
```

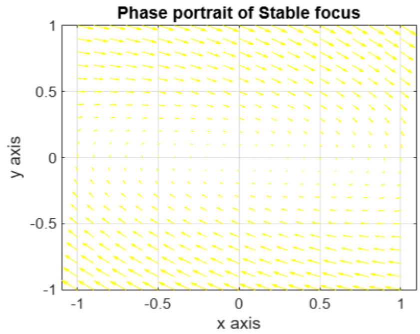


Figure 3: Stable focus

5.4 Problem 4

$$y''(t) - 4 \cdot y'(t) + 13 \cdot y(t) = 0; \tag{55}$$

```
[x1,x2]=meshgrid(-1: 0.1: 1,-1: 0.1: 1);

x1dot = x2;
x2dot = 4*x2-13*x1;

figure;
quiver(x1, x2, x1dot, x2dot, 'G');
xlabel('x axis');
ylabel('y axis');
title('Phase portrait of Unstable focus');
hold on;
axis tight;
grid on;
```

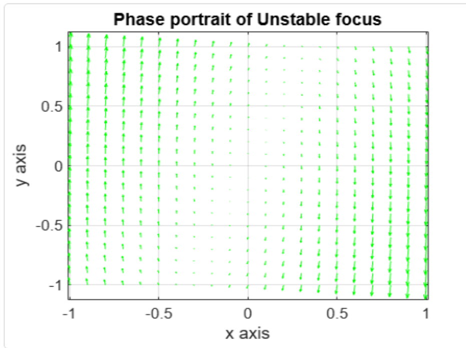


Figure 4: Unstable focus

5.5 Problem 5

$$y''(t) - y'(t) + 6 \cdot y(t) = 0; \tag{56}$$

```
[x1, x2] = meshgrid(-2:0.2:2, -2:0.2:2);

x1dot = x2;
x2dot = -x2 + 6*x1;

figure;
quiver(x1, x2, x1dot, x2dot, 'r'); % 'r' - красные стрелки
xlabel('x axis');
ylabel('y axis');
title('Phase portrait of Saddle point');
hold on;
axis tight;
grid on;
```

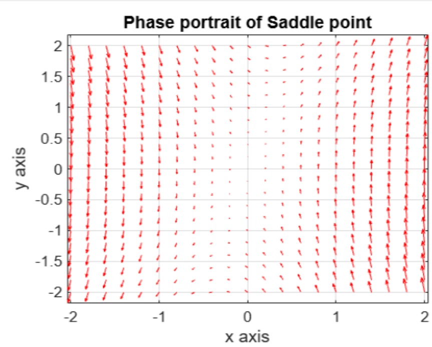
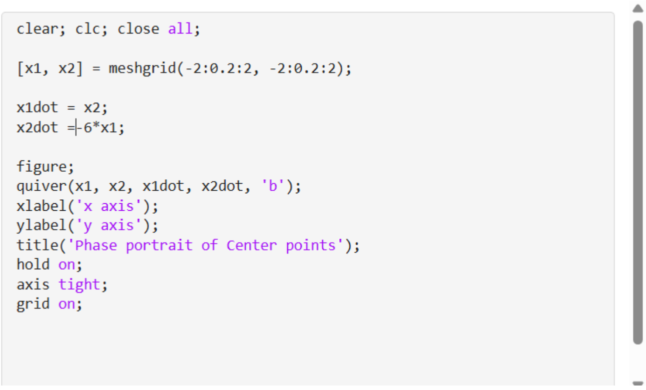


Figure 5: Saddle point

5.6 Problem 6

$$y''(t) + 6 \cdot y(t) = 0; \tag{57}$$



```
clear; clc; close all;

[x1, x2] = meshgrid(-2:0.2:2, -2:0.2:2);

x1dot = x2;
x2dot = -6*x1;

figure;
quiver(x1, x2, x1dot, x2dot, 'b');
xlabel('x axis');
ylabel('y axis');
title('Phase portrait of Center points');
hold on;
axis tight;
grid on;
```

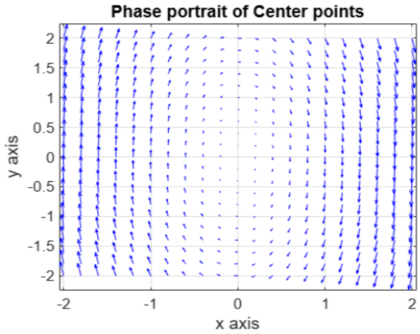


Figure 6: Center points