From spaces and continuity to categories and functoriality

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Contents

Some topology

Topological spaces

Definition: A topological space $(X, \mathcal{T})$ consists of a set X and a collection $\mathcal{T}$ of subsets of X that satisfy the following:

(i) The empty set $\emptyset$ and X are in $\mathcal{T}$.
(ii) Any union of elements in $\mathcal{T}$ is in $\mathcal{T}$.
(iii) Any finite intersection of elements in $\mathcal{T}$ is also in $\mathcal{T}$.

Some topology

Discrete topologies, indiscrete topologies, and metric spaces

Definition: For a set X, the discrete topology 2^X consists of all the subsets of X.
The indiscrete topology $\{\emptyset, X\}$ consists of the empty set and the set X itself.

Definition: A metric space (X, d) consists of a set X and a distance function d: $X \times X \to [0, \infty)$ such that:

- d(x, y) = d(y, x) for all $x, y \in X$.
- d(x, z) $\leq$ d(x, y) + d(y, z) for all $x, y, z, \in X$.
- d(x, y) = 0 if and only if x = y.

Some topology

Bases

Definition: A collection $\mathcal{B}$ of subsets of a set X is a basis for a topology on X if:

(i) for all $x \in X$, there exists some $B \in \mathcal{B}$ such that $x \in \mathcal{B}$;
(ii) if $x \in A \cap B$, where $A, B \in \mathcal{B}$, then there exists $C \in \mathcal{B}$ such that $x \in C \subseteq A \cap B$.

Some topology

Open intervals and balls

Consider the set of all finite open intervals on $\mathbb{R}$:
$\{(a, b) \mid a, b \in \mathbb{R}\}$

Similarly, consider the set of all finite open balls on $\mathbb{R}^2$:
$\{p \in \mathbb{R}^2 \mid d(x, p) < R\}$ for all $x \in \mathbb{R}^2$ and for all $R > 0$

Can we pick a more specific subset of these bases?

Some topology

Open intervals and balls around a fixed point

Consider the set of all open intervals centered at $x \in \mathbb{R}$:
for some $x \in \mathbb{R}$, $\{(x-R, x+R) \mid R>0\}$

Similarly, consider the set of all open balls centered at some $x \in \mathbb{R}^2$:
for some $x \in \mathbb{R}^2$, $\{p \in \mathbb{R}^2 \mid d(x, p) < R\}$ for all $R > 0$

However, these bases are uncountably infinite. Can we do better?

Some topology

Open intervals and balls with rational radii around rational points

Consider the set of all open intervals centered on x $\in \mathbb{Q}$ with rational radii:
for some x $\in \mathbb{Q}$, {(x-R, x+R) | R $\in \mathbb{Q}$, R>0}

Similarly, consider the set of all open balls centered on some x $\in \mathbb{Q}^2$:
for some x $\in \mathbb{Q}^2$, {p $\in \mathbb{R}^2$ | d(x, p) < R} for all R$>$ 0 rational.

Both are countably infinite by the rule that, in $\mathbb{R}^n$, we can parameterize them via $\mathbb{Q}^{n+1}$.

Isomorphisms in category theory

What are categories?

Definition: A category consists of the following:

(i) a class of objects;
(ii) for every two objects X and Y, a set C(X, Y) containing elements called morphisms denoted by arrows: X –> Y;
(iii) a composition rule defined for morphisms: C(X, Y) $\times$ C(Y, Z) $\rightarrow$ C(X, Z).

Isomorphisms in categories

Axioms for categories

(i) Composition is associative:
if f $\in$ C(X, Y), g $\in$ C(Y, Z), h $\in$ C(Z, W), then (fg)h = f(gh).

(ii) There exist identity morphisms: for every object X, there exists a morphism $1_X : X \to X$ so that $f 1_X = f = 1_Y f$ for every $f : X \to Y$.

Isomorphisms in categories

What is an isomorphism?

Definition: An isomorphism is a morphism $f : X \to Y$ that is invertible: there exists some $g : Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

Isomorphisms in categories

Examples of categories

Sets is a category with their objects as sets, morphisms as functions, and compositions as composition of functions. Its isomorphisms are called bijections and are required to be injective and surjective.

Isomorphisms in categories

Topological spaces

Top is a category with its objects being topological spaces, morphisms being continuous functions, and composition being composition of continuous functions.

Given continuous functions f: $X \to Y$ and g: $Y \to Z$, if we choose some open set U $\subset Z$, there is an open set $g^{-1}U \subset Y$. Similarly, there is an open set $(gf)^{-1}U = f^{-1}g^{-1}U \subset X$. As a result, the composite $gf : X \to Z$ is a continuous function.

Functoriality of pushforward and pullback

What is a pushforward?

Definition: For every morphism $f : X \to Y$ and object Z in a category, there is a map of sets $f_* : C(Z, X) \to C(Z, Y)$ called the pushforward of f, defined by postcomposition $f_* : g \mapsto fg$.

Functoriality of pushforward and pullback

What is a pullback?

Definition: For every morphism $f : X \to Y$ and object Z in a category, there is a map of sets $f^* : C(Y, Z) \to C(X, Z)$ called the pullback of f defined by precomposition $f^* : g \mapsto gf$.

Functoriality of pushforward and pullback

What is a functor?

Definition: A functor F from a category C to a category D consists of the following:

(i) An object FX of the category D for every object X of the category C;
(ii) A morphism $Ff : FX \to FY$ for every morphism $f : X \to Y$;

satisfying:

(iii) $(Fg)(Ff) = F(gf)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$;
(iv) $F1_X = 1_{FX}$ for every object X.

Functoriality of pushforward and pullback

Statement of proposition

Proposition: Show that both the pushforward and pullback are functorial:

(1) Given the identity morphism $1_X : X \to X$ for every object X in $\mathcal{C}$, $(1_X)_* = 1_{C(Z,X)}$.
(2) Given morphisms $f : X \to Y, g : Y \to Z$ of $\mathcal{C}$, we have $(gf)_* = g_* f_*$.
(3) Given the identity morphism $1_X : X \to X$ for every object X in $\mathcal{C}$, $(1_X)^* = 1_{C(X,Z)}$.
(4) Given morphisms $f : X \to Y, g : Y \to Z$ of $\mathcal{C}$, we have $(gf)^* = f^* g^*$.

Functoriality of pushforward and pullback

Proof of (1) and (3)

(1) Let f be some morphism in $C(Z, X)$. Then:
$(1_X)_*(f) = 1_X f = f = 1_{C(Z,X)}(f)$

(3) Let f be some morphism in $C(X, Z)$. Then:
$(1_X)^*(f) = f 1_X = f = 1_{C(X,Z)}(f)$

Functoriality of pushforward and pullback

Proof of (2) and (4)

(2) Let h be some morphism in $C(W, X)$. Then:
$(gf)_*(h) = gfh = g_*(fh) = g_*f_*(h)$

(4) Let h be some morphism in $C(X, W)$. Then:
$(gf)^*(h) = hgf = f^*(hg) = f^*g^*(h)$

References

Tai-Danae Bradley, Tyler Bryson, and John Terilla. **Topology: a categorical approach**. MIT Press, 2020.

Thank you!

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